Universität Regensburg Mathematik

Points of general relativistic shock wave interaction are “regularity singularities” where spacetime is not locally flat

Moritz Reintjes and Blake Temple

Preprint Nr. 28/2011

POINTS OF GENERAL RELATIVISITIC SHOCK WAVE INTERACTION ARE "REGULARITY SINGULARITIES" WHERE SPACETIME IS NOT LOCALLY FLAT

MORITZ REINTJES AND BLAKE TEMPLE

APRIL 2011

Abstract. We show that the regularity of the gravitational metric tensor in spherically symmetric spacetimes cannot be lifted from $C^{0,1}$ to $C^{1,1}$ within the class of $C^{1,1}$ coordinate transformations in a neighborhood of a point of shock wave interaction in General Relativity, without forcing the determinant of the metric tensor to vanish at the point of interaction. This is in contrast to Israel's Theorem [5] which states that such coordinate transformations always exist in a neighborhood of a point on a smooth *single* shock surface. The results thus imply that points of shock wave interaction represent a new kind of singularity for perfect fluids evolving in spacetime, singularities that make perfectly good sense physically, that can form from the evolution of smooth initial data, but at which the spacetime is not locally Minkowskian under any coordinate transformation. In particular, at such singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems of the $C^{1,1}$ atlas, but due to cancelation, the curvature tensor remains uniformly bounded.

1. Introduction

The guiding principle in Albert Einstein's pursuit of general relativity was the principle that spacetime should be *locally inertial* [1]. That is, an observer in freefall through a gravitational field should observe all of the physics of special relativity, except for the second order acceleration effects due to spacetime curvature (gravity). But the assumption that spacetime is locally inertial is equivalent to assuming the gravitational metric tensor g has a certain level of smoothness around every point. That is, the assumption that spacetime is locally inertial at a spacetime point p assumes the gravitational metric tensor g is smooth enough so that one can pursue the construction of Riemann Normal Coordinates at p, coordinates in which g is exactly the Minkowski metric at p, such that all first order derivatives of g vanish at p, with all second order derivatives of g bounded in a neighborhood of p. The nonzero second derivatives are then a measure of spacetime curvature. However, the Einstein equations are a system of partial differential equations (PDE's) for the metric tensor g and the PDE's by themselves determine the smoothness of the gravitational metric tensor by the evolution they impose. Thus the condition on spacetime that it be locally inertial at every

The ideas and methods presented here are the creation of Moritz Reintjes. The detailed proofs are due to him and can be found in his doctoral thesis [8], which was supervised by Blake Temple. Both authors were partially supported by second author's NSF Grant, where the problem was first proposed.

[1]Also referred to as *locally Lorentzian* or *locally Minkowskian*.

point cannot be assumed at the start, but must be determined by regularity theorems for the Einstein equations.

This issue becomes all the more interesting when the sources of matter and energy are modeled by a perfect fluid, and the resulting Einstein-Euler equations form a system of PDE's for the metric tensor g coupled to the density, velocity and pressure of the fluid. It is well known that the evolution of a perfect fluid governed by the compressible Euler equations leads to shock wave formation from smooth initial data whenever the flow is sufficiently compressive.[2] At a shock wave, the fluid density, pressure and velocity are discontinuous, and when such discontinuities are assumed to be the sources of spacetime curvature, the Einstein equations imply that the curvature must also become discontinuous at shocks. But discontinuous curvature by itself is not inconsistent with the assumption that spacetime be locally inertial. For example, if the gravitational metric tensor were $C^{1,1}$, (differentiable with Lipschitz continuous first derivatives, [11]), then second derivatives of the metric are at worst discontinuous, and the metric has enough smoothness for there to exist coordinate transformations which transform g to the Minkowski metric at p, with zero derivatives at p, and bounded second derivatives as well, [11]. Furthermore, Israel's theorem, [5], (see also [11]) asserts that a spacetime metric need only be $C^{0,1}$, i.e., Lipschitz continuous, across a smooth *single* shock surface, in order that there exist a $C^{1,1}$ coordinate transformation that lifts the regularity of the gravitational metric one order to $C^{1,1}$, and this again is smooth enough to ensure the existence of locally inertial coordinate frames at each point. In fact, when discontinuities in the fluid are present, $C^{1,1}$ coordinate transformations are the natural atlas of transformations that are capable of lifting the regularity of the metric one order, while still preserving the weak formulation of the Einstein equations, [11]. It is common in GR to assume the gravitational metric tensor is at least $C^{1,1}$. For example, the $C^{1,1}$ regularity of the gravitational metric is assumed at the start in singularity theorems of Hawking and Penrose, [4]. However, in Standard Schwarzschild Coordinates[3] (SSC) the gravitational metric will be no smoother than $C^{0,1}$, if a discontinuous energy momentum tensor in the Einstein equations is present, c.f. [3].

In this paper we prove there do not exist $C^{1,1}$ coordinate transformations that can lift the regularity of a gravitational metric tensor from $C^{0,1}$ to $C^{1,1}$ at a point of a shock wave interaction in a spherically symmetric spacetime in GR, without forcing the determinant of the metric tensor to vanish at the point of interaction. This is in contrast to Israel's Theorem [5] which states that such coordinate transformations always exist in a neighborhood of a point on a smooth single shock surface. It follows that solutions of the Einstein equations containing single smooth shock surfaces can solve the Einstein equations strongly, (in fact, pointwise almost everywhere in Gaussian normal coordinates), but this fails to be the case at points of shock wave interaction, where the Einstein equations can only hold weakly in the sense of the theory of distributions. The results thus imply that points of shock wave interaction represent a new kind of singularity in General Relativity that can form from the evolution of smooth initial

[2]Since the Einstein curvature tensor G satisfies the identity $Div\, G = 0$, the Einstein equations $G = \kappa T$ imply $Div\, T = 0$, and so the assumption of a perfect fluid stress tensor T automatically implies the coupling of the Einstein equations to the compressible Euler equations $Div\, T = 0$.

[3]It is well known that a general spherically symmetric metric of form $ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + E(t,r)dtdr + C(t,r)^2 d\Omega^2$ can be transformed to SSC in a neighborhood of a point where $\frac{\partial C}{\partial r} \neq 0$, c.f. [15]

data, that correctly reflects the physics of the equations, but at which the spacetime is not locally Minkowskian under any $C^{1,1}$ coordinate transformation. At such singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems of the $C^{1,1}$ atlas, but due to cancelation, the curvature tensor remains uniformly bounded.

To state the main result precisely, we consider spherically symmetric spacetime metrics $g_{\mu\nu}$ which solve the Einstein equations in SSC, where the metric takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2, \tag{1.1}$$

where either t or r can be taken to be timelike, and $d\Omega^2 = d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$ is the line element on the unit 2-sphere, c.f. [12]. In Section 2 we make precise the definition of a point of regular of shock wave interaction in SSC. Essentially, this is a point in (t,r)-space where two distinct shock waves enter or leave the point p at distinct speeds, such that the metric is Lipschitz continuous, the Rankine Hugoniot (RH) jump conditions hold across the shocks [10], and the SSC Einstein equations hold weakly in a neighborhood of p and strongly away from the shocks. The main result of the paper is the following theorem, (c.f. Definition 3.1 and Theorem 7.1 below):

Theorem 1.1. *Assume p is a point of regular shock wave interaction in SSC. Then there does not exist a $C^{1,1}$ regular coordinate transformation*[4]*, defined in a neighborhood of p, such that the metric components are C^1 functions of the new coordinates and such that the metric has a nonzero determinant at p.*

The proof of Theorem 1.1 is constructive in the sense that we characterize the Jacobians of coordinate transformations that smooth the components of the gravitational metric in a deleted neighborhood of a point p of regular shock wave interaction, and then prove that any such Jacobian must have a vanishing determinant at p itself. Because the metric becomes singular at p whenever C^1 regularity is forced upon it, we refer to points of regular shock wave interaction as *regularity singularities*. The numerical explorations in [14] strongly indicate that such singularities can form out of smooth initial data within a finite time just as with fluids governed by the special relativistic Euler equations, but we know of no complete mathematical proof of this fact.

Our assumptions in Theorem 1.1 apply to the upper half ($t \geq 0$) and the lower half ($t \leq 0$) of a shock wave interaction (at $t = 0$) separately, suitable for the initial value problem, and also general enough to include the case of two timelike[5] interacting shock waves of opposite families[6] that cross at the point p, but also general enough to include the cases of two outgoing shock waves created by the focusing of compressive

[4]The atlas of $C^{1,1}$ coordinate transformations is generic for lifting metric regularity at shock waves in General Relativity, because C^2 coordinate transformations preserve the metric regularity, (c.f. Section 5), while a $C^{1,\alpha}$ atlas seems to be appropriate only for metric tensors in $C^{0,\alpha}$. The $C^{1,1}$ atlas is natural because it preserves the weak formalism that derives from the quasilinear structure of the Einstein-Euler equations, a property we expect cannot be met by any atlas less regularity than $C^{1,1}$, (e.g., a $C^{0,1}$ atlas with resulting discontinuous metric components). Given this, points of regular shock wave interaction in SSC represent *regularity singularities* in the sense that they are points where the gravitational metric is less regular than C^1 in any coordinate system that can be reached within the $C^{1,1}$ atlas.

[5]In fact the theorem applies to non-null surfaces that can be regularly parameterized by the SSC time or radial variable, c.f. Theorem 7.1 below.

[6]Two shock waves typically change their speeds discontinuously at the point of interaction, c.f. [10]

rarefaction waves, or two incoming shock waves of the same family that interact at p to create an outgoing shock wave of the same family and an outgoing rarefaction wave of the opposite family, c.f. [10]. In particular, our framework and our theorems are general enough to incorporate and apply to the shock wave interaction which was numerically simulated in [14].

Historically, the issue of the smoothness of the gravitational metric tensor across interfaces began with the $C^{0,1}$ matching of the interior Schwarzschild solution to the vacuum across an interface, followed by the celebrated work of Oppenheimer and Snyder [9] who gave the first dynamical model of gravitational collapse. In [11], Smoller and Temple extended the Oppenheimer-Snyder model to a $C^{0,1}$ shock wave interface that allowed for nonzero pressure. In his celebrated 1966 paper [5], Israel gave the definitive conditions for $C^{0,1}$ matching of gravitational metrics at smooth interfaces, by showing that if the second fundamental form is continuous across a single smooth interface, then the RH jump conditions also hold, and Gaussian normal coordinates smooth the metric to $C^{1,1}$ and thereby provides a locally inertial coordinate system. In [3] Groah and Temple addressed these issues rigorously in the first general existence theory for shock wave solutions of the Einstein-Euler equations. In coordinates where their analysis is feasible, SSC, the gravitational metric was Lipschitz continuous at shock waves, but could be no smoother, and it has remained an open problem as to whether the weak solutions constructed by Groah and Temple could be smoothed to $C^{1,1}$ by coordinate transformation, like the single shock surfaces addressed by Israel. The results in this paper resolve this issue by proving definitively that the weak solutions constructed by Temple and Groah *cannot* be smoothed within the class of $C^{1,1}$ coordinate transformations when they contain points of shock wave interaction.

As a final comment, we note that although points of shock wave interaction are straightforward to construct for the relativistic compressible Euler equations in flat spacetime, to our knowledge there is no rigorous mathematical construction of an exact solution of the Einstein equations containing a point of shock wave interaction, where two shock waves cross in spacetime. But all the evidence indicates that points of shock wave interaction exist, have the structure we assume in SSC, and in fact cannot be avoided in solutions consisting of, say, an outgoing spherical shock wave (the blast wave of an an explosion) evolving inside an incoming spherical shock wave (the leading edge of an implosion). The existence theory of Temple and Groah [3] lends strong support to this claim by establishing existence of weak solutions of the Einstein-Euler equations in spherically symmetric spacetimes. The theory applies to arbitrary numbers of initial shock waves of arbitrary strength, existence is established beyond the point of shock wave interaction, and the regularity assumptions of our theorem are within the regularity class to which the Groah-Temple theory applies. All of this is substantiated by the recent work of Vogler and Temple which gives a numerical simulation of a class of solutions in which two shock waves emerge from a point of interaction, and the numerics demonstrate that the structure of the emerging shock waves meets all of the assumptions of our theorem at the point of interaction. Taken on whole, we interpret this as definitive physical proof that points of shock wave interaction exist in GR, and meet the regularity assumptions of our theorem. Given this, the conclusion of our theorem is that points must exist where the gravitational metric tensor cannot be smoothed from $C^{0,1}$ to $C^{1,1}$ by coordinate transformation.

In Section 2 we begin with preliminaries. In section 3 we set out the framework of shock waves in GR, and define what we call a point of *regular shock wave interaction* in SSC. In Section 4 we give a precise sense in which a function is said to be $C^{0,1}$ across a hypersurface, and we introduce a canonical form for such functions, which isolates the Lipschitz regularity from the C^1 regularity in a neighborhood of the hypersurface.

In Section 5 we derive conditions on the Jacobians of general $C^{1,1}$ coordinate transformations necessary and sufficient to lift the regularity of a metric tensor from $C^{0,1}$ to C^1 at points on a shock surface. The result is a canonical form for the Jacobians of all coordinate transformations that can possibly lift the regularity of the gravitational metric tensor to C^1.

In Section 6, we give a new constructive proof of Israel's theorem for spherically symmetric spacetimes, by showing directly that the Jacobians expressed in our canonical form do indeed smooth the gravitational metric to $C^{1,1}$ at points on a single shock surface. The essential difficulty is to prove that the freedom to add an arbitrary C^1-function to our canonical form, is sufficient to guarantee that we can meet the integrability condition on the Jacobian required to integrate it up to an actual coordinate system. The main point is that this is achievable within the required C^1 gauge freedom if and only if the RH jump conditions and the Einstein equations hold at the shock interface, [10].

The main step towards Theorem 1.1 is then achieved in Section 7 where we prove that at a point of regular shock wave interaction in SSC there exists no coordinate transformations of the (t,r)-plane that lift the metric regularity to C^1. The essential point is that the C^1 gauge freedom in our canonical forms cannot satisfy the integrability condition on the Jacobians, without forcing the determinant of the Jacobian to vanish at the point of interaction. In section 8 we extend this result to the full spacetime atlas of coordinate transformations that allow changes of angular variables as well as (t,r), thereby proving Theorem 1.1. In the final section 9, we we show that Theorem Theorem 1.1 implies the non-existence of locally inertial frames at points of regular shock wave interaction. Since we do not know how to make mathematical and physical sense of coordinate transformations less regular than $C^{1,1}$ in general relativity, we conclude that points of shock wave interaction represent a new kind of *regularity singularity* in spacetime.

2. Preliminaries

In General Relativity, the gravitational field is described by a Lorentzian metric g of signature $(-1,1,1,1)$ on a four dimensional spacetime manifold M. We call M a C^k-manifold if it is endowed with a C^k-atlas, a collection of four dimensional local diffeomorphisms that are C^k regular from M to $\mathbb{R}^4$. A composition of two local diffeomorphisms x and y of the form $x \circ y^{-1}$ is refered to as a coordinate transformation. In this paper we consider $C^{1,1}$-manifolds, since this low level of regularity offers a generic framework to address shock wave solutions of the Einstein-Euler equations.

We use standard index notation for tensors whereby indices determine the coordinate system, (e.g., T^μ_ν denotes a $(1,1)$-tensor in coordinates x^μ and T^i_j denotes the same tensor in coordinates x^j), and repeated up-down indices are assumed summed from 0 to 3. Under coordinate transformation, tensors transform by contraction with the Jacobian

$$J^\mu_j = \frac{\partial x^\mu}{\partial x^j}\,, \tag{2.1}$$

J^j_ν denotes the inverse Jacobian, and indices are raised and lowered with the metric and its inverse g^{ij}, which transform as bilinear forms,

$$g_{\mu\nu} = J^i_\mu J^j_\nu g_{ij}, \tag{2.2}$$

c.f. [15]. We use the fact that a matrix of functions J^μ_j is the Jacobian of a regular local coordinate transformation if and only if the curls vanish, i.e.,

$$J^\mu_{i,j} = J^\mu_{j,i} \quad \text{and} \quad Det\left(J^\mu_j\right) \neq 0, \tag{2.3}$$

where $f_{,j} = \frac{\partial f}{\partial x^j}$ denotes partial differentiation with respect to the coordinate x^j and $Det\left(J^\mu_j\right)$ denotes the determinant of the Jacobian.

The time evolution of a gravitational field in general relativity is governed by the *Einstein equations* [1]

$$G^{ij} + \Lambda g^{ij} = \kappa T^{ij}, \tag{2.4}$$

a system of 10 second order partial differential equations that relate the metric tensor g_{ij} to the undifferentiated sources T^{ij} through the Einstein curvature tensor

$$G^{ij} = R^{ij} - \frac{1}{2} R g^{ij}, \tag{2.5}$$

a tensor involving second derivatives of g. Here Λ is the cosmological constant, (our results apply to a vanishing as well as a non-vanishing cosmological constant), $\kappa = \frac{8\pi}{3}\mathcal{G}$ is the coupling constant which incorporates Newton's gravitational constant $\mathcal{G}$ and the speed of light c, and T^{ij} is the stress energy tensor.

We assume throughout that T^{ij} is the stress tensor for a perfect fluid,

$$T^{ij} = (p+\rho)u_i u_j + p g_{ij}, \tag{2.6}$$

where ρ is the energy density, u_i the 4-velocity, and p the pressure. Conservation of energy and momentum enter the Einstein equations through,

$$T^{ij}{}_{;j} = 0\,. \tag{2.7}$$

which reduces to the relativistic compressible Euler equations in flat spacetime, and follows from the divergence free property of the Einstein equations, $G^{ij}{}_{;j} = 0$ a property built into G at the start as an identity following from the Bianchi identities of geometry. Here as usual, semicolon denotes covariant differentiation $v_{;j} = v_{,j} + \Gamma^i_{lj} v^l$, where Γ^i_{lj} denote the Christoffel symbols or connection coefficients associated with metric g, c.f. [15].

System (2.4) and (2.7) forms the coupled Einstein-Euler equations, a system of second order differential equations for the unknown metric g_{ij}, and unknown fluid variables ρ, p and u^j . For example, imposing an equation of state $p = p(\rho)$ closes the system, yielding a set of fourteen differential equations in fourteen unknowns. In special relativity the spacetime metric is taken to be $g_{ij} \equiv \eta_{ij}$ where $\eta_{ij} = diag(-1,1,1,1)$ is the Minkowski metric, in which case (2.7) reduces to the relativistic compressible Euler equations, a system of conservation laws in which it is well known that shock waves form out of smooth initial data whenever the flow is sufficiently compressive, (see [10] or [7]). Shock waves are discontinuous solutions that only solve the Euler equations weakly, in a distributional sense. Across a smooth shock surface Σ, the Rankine-Hugoniot jump conditions hold,

$$[T^{\mu\nu}]n_\nu = 0, \tag{2.8}$$

where $[f] = f_L - f_R$ denotes the jump from right to left (wrt r) in function f across Σ, and n_ν is the surface normal. In particular, for smooth shock surfaces, the jump conditions (2.8) are equivalent to the shock wave solution satisfying the weak formulation of (2.7) across Σ, c.f. [10].

For many astrophysical processes it makes sense to assume the spacetime is spherically symmetric, by which we mean a Lorentz manifold admitting two Killing vector fields that give rise to coordinates in which the metric takes the simplified form

$$ds^2 = -Adt^2 + Bdr^2 + 2Edtdr + Cd\Omega^2\,, \tag{2.9}$$

where the coefficients A, B, C and E depend only on t and r, and $d\Omega^2$ is the line element on the unit 2-sphere, [15] . In this case we often suppress the dependence on ϕ and θ, and refer to the spacetime parameterized by the variables t and r as the (t, r)-plane.

In a spherically symmetric spacetime with $\frac{\partial C}{\partial r} \neq 0$, one can always transform to Standard Schwarzschild Coordinates (SSC), where the metric takes the form [15],

$$ds^2 = -A(t, r)dt^2 + B(t, r)dr^2 + r^2 d\Omega^2. \tag{2.10}$$

The Einstein equations for a metric in SSC are given by

$$\begin{aligned}
B_r + B\frac{B-1}{r} &= \kappa AB^2 rT^{00} && (2.11)\\
B_t &= -\kappa AB^2 rT^{01} && (2.12)\\
A_r - A\frac{1+B}{r} &= \kappa AB^2 rT^{11} && (2.13)\\
B_{tt} - A_{rr} + \Phi &= -2\kappa ABr^2 T^{22}\,, && (2.14)
\end{aligned}$$

with

$$\Phi = -\frac{BA_tB_t}{2AB} - \frac{B_t^2}{2B} - \frac{A_r}{r} + \frac{AB_r}{rB} + \frac{A_r^2}{2A} + \frac{A_rB_r}{2B}\,.$$

The first three Einstein equations in SSC play a crucial role in the method we develop in this paper, and it is straightforward to read off from the first three equations that the metric cannot be any smoother than Lipschitz continuous if the matter source is discontinuous, for example, $T^{ij} \in L^\infty$, and in this paper we make the assumption throughout that the gravitational metric is Lipschitz continuous. This provides a consistent framework to address shock waves in GR, in agreement with the theory and examples of shock wave solutions to the coupled Einstein-Euler equations, see for instance [11] or [3].

Lipschitz continuity arises naturally in the general problem of matching two spacetimes across a hypersurface, as first considered by Israel in [5]. Israel proved the rather remarkable result that whenever a metric is Lipschitz continuous ($C^{0,1}$) across a smooth single shock surface Σ, there always exists a coordinate transformation defined in a neighborhood of Σ, that smooths the components of the gravitational metric to $C^{1,1}$. The precise result is that the gravitational metric is smoothed to $C^{1,1}$ in Gaussian Normal Coordinates if and only if the second fundamental form of the metric is continuous across the surface. The latter is an invariant condition meaningful for metrics Lipschitz continuous across a hypersurface, and is often referred to in the literature as the *junction condition*, c.f. [15]. In [11], Smoller and Temple showed that in spherically symmetric spacetimes, the junction conditions hold across radial surfaces if and only if the single $\left[T^{ij}\right] n_i n_j = 0$, implied by (2.8), holds. Thus, for example, single radial shock surfaces can be no smoother than Lipschitz continuous in SSC coordinates, but can be smoothed to $C^{1,1}$ by coordinate transformation. However, it has remained an open problem whether or not such a theorem applies to the more complicated $C^{0,1}$ SSC solutions proven to exist [3]. Our purpose here is to show that such solutions cannot be smoothed to C^1 in a neighborhood of a point of *regular shock wave interaction*, a notion we now make precise.

3. A point of regular shock wave interaction in SSC

In this paper we restrict attention to radial shock waves, hypersurfaces Σ locally parameterized by

$$\Sigma(t,\vartheta,\varphi) = (t, x(t), \vartheta, \varphi), \tag{3.1}$$

and across which T in (2.6) is discontinuous. [7] For such hypersurfaces in SSC, the angular variables play a passive role, and the essential issue regarding smoothing the metric components by $C^{1,1}$ coordinate transformation lies in the atlas of coordinate transformations acting on the (t,r)-plane alone. (The main issue is to prove theorems 6.1 and 7.1 for (t,r)-transformations. In section 8 we discuss the straightforward extension to the full atlas of transformations that include the angular variables.) Thus we introduce γ, the restriction of a shock surface Σ to the (t,r)-plane,

$$\gamma(t) = (t, x(t)), \tag{3.2}$$

with normal 1-form

$$n_\sigma = (\dot{x}, -1). \tag{3.3}$$

For radial shock surfaces (3.1) in SSC, the RH jump conditions (2.8) take the simplified form

$$\left[T^{00}\right]\dot{x} = \left[T^{01}\right], \tag{3.4}$$

$$\left[T^{10}\right]\dot{x} = \left[T^{11}\right]. \tag{3.5}$$

Now suppose two timelike shock surfaces Σ_i are parameterized in SSC by

$$\Sigma_i(t,\theta,\phi) = (t, x_i(t), \theta, \phi) \quad i = 1, 2. \tag{3.6}$$

Let $\gamma_i(t)$ denote their corresponding restrictions to the (t,r)-plane,

$$\gamma_i(t) = (t, x_i(t)), \tag{3.7}$$

with normal 1-forms

$$(n_i)_\sigma = (\dot{x}_i, -1), \tag{3.8}$$

and use the notation that $[f]_i(t) = [f(\gamma_i(t))]$ denotes the jump in the quantity f across the surface $\gamma_i(t)$.

For our theorem it suffices to restrict attention to the lower or upper part of a shock wave interaction that occurs at $t = 0$. That is, in either the lower or upper half plane

$$\mathbb{R}^2_- = \{(t,r) : t < 0\},$$

or

$$\mathbb{R}^2_+ = \{(t,r) : t > 0\},$$

[7] Note that if t is timelike, then all timelike shock surfaces in SSC can be so parameterized. Our subsequent methods apply to spacelike and timelike surfaces alike, (inside or outside a black hole, c.f. [12]), in the sense that t can be timelike or spacelike, but without loss of generality and for ease of notation, in the remainder of this paper we restrict to timelike surfaces parameterized as in (3.1).

respectively, whichever half plane contains two shock waves that intersect at p with distinct speeds. (We denote with $\overline{\mathbb{R}^2_\pm}$ the closure of $\mathbb{R}^2_\pm$.) Thus, without loss of generality, let $\gamma_i(t) = (t, x_i(t))$, $(i = 1, 2)$, be two shock curves in the lower (t, r)-plane that intersect at a point $(0, r_0)$, $r_0 > 0$, i.e.

$$x_1(0) = r_0 = x_2(0). \tag{3.9}$$

With this notation, we can now give a precise definition of what we call a point of regular shock wave interaction in SSC. By this we mean a point p where two distinct shock waves enter or leave the point p with distinct speeds. The structure makes precise what one would generically expect, namely, that the metric is a smooth solution of the Einstein equations away from the interacting shock curves, the metric is Lipschitz continuous and the RH jump conditions hold across each shock curve, and derivatives are continuous up to the boundary on either side. In particular, the definition reflects the regularity of shock wave solutions of the coupled Einstein-Euler equations consistent with the theory in [3] and confirmed by the numerical simulation in [14]. Without loss of generality we assume a lower shock wave interaction in $\mathbb{R}^2_-$.

Definition 3.1. *Let $r_0 > 0$, and let $g_{\mu\nu}$ be an SSC metric in $C^{0,1}\left(\mathcal{N} \cap \overline{\mathbb{R}^2_-}\right)$, where $\mathcal{N} \subset \mathbb{R}^2$ is a neighborhood of a point $p = (0, r_0)$ of intersection of two timelike shock curves $\gamma_i(t) = (t, x_i(t)) \in \mathbb{R}^2_-$, $t \in (-\epsilon, 0)$. Assume the shock speeds $\dot{x}_i(0) = \lim_{t\to 0} \dot{x}_i(t)$ exist and are distinct, and let $\hat{\mathcal{N}}$ denote the neighborhood consisting of all points in $\mathcal{N} \cap \mathbb{R}^2_-$ not in the closure of the two intersecting curves $\gamma_i(t)$. Then we say that p is a point of* regular shock wave interaction *in SSC if:*

(i) The pair (g, T) is a strong solution of the SSC Einstein equations (2.11)-(2.14) *in $\hat{\mathcal{N}}$, with $T^{\mu\nu} \in C^0(\hat{\mathcal{N}})$ and $g_{\mu\nu} \in C^2(\hat{\mathcal{N}})$.*

(ii) The limits of T and of metric derivatives $g_{\mu\nu,\sigma}$ exist on both sides of each shock curve $\gamma_i(t)$ for all $-\epsilon < t < 0$.

(iii) The jumps in the metric derivatives $[g_{\mu\nu,\sigma}]_i(t)$ are C^1 function with respect to t for $i = 1, 2$ and for $t \in (-\epsilon, 0)$.

(iv) The limits

$$\lim_{t\to 0}[g_{\mu\nu,\sigma}]_i(t) = [g_{\mu\nu,\sigma}]_i(0)$$

exist for $i = 1, 2$.

(v) The metric g is continuous across each shock curve $\gamma_i(t)$ separately, but no better than Lipschitz continuous in the sense that, for each i there exists μ, ν such that

$$[g_{\mu\nu,\sigma}]_i(n_i)^\sigma \neq 0$$

at each point on γ_i, $t \in (-\epsilon, 0)$ and

$$\lim_{t\to 0}[g_{\mu\nu,\sigma}]_i(n_i)^\sigma \neq 0.$$

(vi) The stress tensor T is bounded on $\mathcal{N} \cap \overline{\mathbb{R}^2_-}$ and satisfies the RH jump conditions

$$[T^{\nu\sigma}]_i(n_i)_\sigma = 0$$

at each point on $\gamma_i(t)$, $i = 1, 2$, $t \in (-\epsilon, 0)$, and the limits of these jumps exist up to p as $t \to 0$.

4. Functions $C^{0,1}$ across a hypersurface

In this section we give a precise definition which isolates the $C^{0,1}$ regularity of a function to a hypersurface, and use this to derive a canonical form for such functions.

Definition 4.1. *Let Σ be a smooth (timelike) hypersurface in some open set $\mathcal{N} \subset \mathbb{R}^d$. We call a function f "Lipschitz continuous across Σ", (or $C^{0,1}$ across Σ), if $f \in C^{0,1}(\mathcal{N})$, f is smooth[8] in $\mathcal{N} \setminus \Sigma$, and limits of derivatives of f exist and are smooth functions on each side of Σ separately. We call a metric $g_{\mu\nu}$ Lipschitz continuous across Σ in coordinates x^μ if all metric components are $C^{0,1}$ across Σ.*

The main point of the above definition is that we assume smoothness of f, (or $g_{\mu\nu}$), away and tangential to the hypersurface Σ. Note that the continuity of f across Σ implies the continuity of all derivatives of f tangent to Σ, i.e.,

$$[f_{,\sigma}]v^\sigma = 0, \tag{4.1}$$

for all v^σ tangent to Σ. Moreover, Definition 4.1 allows for the normal derivative of f to be discontinuous, that is,

$$[f_{,\sigma}]n^\sigma \neq 0, \tag{4.2}$$

where n^σ is normal to Σ with respect to some (Lorentz-) metric $g_{\mu\nu}$ defined on $\mathcal{N}$.

We can now clarify the connections between the Einstein equations and the RH jump conditions (3.4), (3.5) for SSC metrics only $C^{0,1}$ across a hypersurface. So consider a spherically symmetric spacetime metric (1.1) given in SSC, assume that the first three Einstein equations (2.11)-(2.13) hold, and assume that the stress tensor T is discontinuous across a smooth radial shock surface described in the (t,r)-plane by $\gamma(t)$ as in (3.1)-(3.3). To this end, condition (4.1) across γ applied to each metric component $g_{\mu\nu}$ in SSC (2.10) reads

$$[B_t] = -\dot{x}[B_r], \tag{4.3}$$
$$[A_t] = -\dot{x}[A_r]. \tag{4.4}$$

On the other hand, the first three Einstein equations in SSC (2.11)-(2.13) imply

$$[B_r] = \kappa A B^2 r [T^{00}], \tag{4.5}$$
$$[B_t] = -\kappa A B^2 r [T^{01}], \tag{4.6}$$
$$[A_r] = \kappa A B^2 r [T^{11}]. \tag{4.7}$$

Now, using the jumps in Einstein equations (4.5)-(4.7), we find that (4.3) is equivalent to the first RH jump condition (3.4),[9] while the second condition (4.4) is independent of equations (4.5)-(4.7), because A_t does not appear in the first order SSC equations (2.11)-(2.13). The result, then, is that in addition to the assumption that the metric be $C^{0,1}$ across the shock surface in SSC, the RH conditions (3.4) and (3.5) together with the Einstein equations (4.5)-(4.7), yield only one additional condition over and above (4.3) and (4.4), namely,

$$[A_r] = -\dot{x}[B_t]\ . \tag{4.8}$$

The RH jump conditions together with the Einstein equations will enter our method in Section 5-7 only through the three equations (4.8), (4.3) and (4.4).

[8] For us, "smooth" means enough continuous derivatives so that smoothness is not an issue. Thus here, $f \in C^2(\mathcal{N} \setminus \Sigma)$ suffices.

[9] This observation is consistent with Lemma 9, page 286, of [11], where only one jump condition need be imposed to meet the full RH relations.

The following lemma provides a canonical form for any function f that is Lipschitz continuous across a *single* shock curve γ in the (t,r)-plane, under the assumption that the vector n^μ, normal to γ, is obtained by raising the index in (3.3) with respect to a Lorentzian metric g that is $C^{0,1}$ across γ. (Note that by Definition 4.1, n^μ varies C^1 in directions tangent to γ, and we suppress the angular coordinates.)

Lemma 4.2. *Suppose f is $C^{0,1}$ across a smooth curve $\gamma(t)=(t,x(t))$ in the sense of Definition 4.1, $t\in(-\epsilon,\epsilon)$, in an open subset $\mathcal{N}$ of $\mathbb{R}^2$. Then there exists a function $\Phi\in C^1(\mathcal{N})$ such that*

$$f(t,r)=\frac{1}{2}\varphi(t)\,|x(t)-r|+\Phi(t,r), \tag{4.9}$$

where

$$\varphi(t)=\frac{[f_{,\mu}]n^\mu}{n^\sigma n_\sigma}\in C^1(-\epsilon,\epsilon), \tag{4.10}$$

and $n_\mu(t)=(\dot{x}(t),-1)$ is a 1-form normal to the tangent vector $v^\mu(t)=\dot{\gamma}^\mu(t)$. In particular, it suffices that indices are raised and lowered by a Lorentzian metric $g_{\mu\nu}$ which is $C^{0,1}$ across γ.

In words, the canonical form (4.9) separates off the $C^{0,1}$ kink of f across γ from its more regular C^1 behavior away from γ: The kink is incorporated into $|x(t)-r|$, φ gives the smoothly varying strength of the jump, and Φ encodes the remaining C^1 behavior of f. I.e., φ gives the strength of the jump because upon taking the jump in the normal derivative of f across γ, the dependence on Φ cancels out. Note finally that the regularity assumption on the metric across γ is required for $\varphi(t)$ to be well defined in (4.10), and also to get the C^1 regularity of $n^\sigma n_\sigma$ tangent to γ. In Section 6 below we prove Israel's Theorem for a single shock surface by constructing a $C^{1,1}$ coordinate transformation using 4.9 of Lemma 4.2 as a canonical form for the Jacobian derivatives of the transformation.

In Section 7 we need a canonical form analogous to (4.9) for two shock curves, but such that it allows for the Jacobian to be in the weaker regularity class $C^{0,1}$ away from the shock curves. To this end, suppose two timelike shock surfaces described in the (t,r)-plane by, $\gamma_i(t)$, such that (3.6)-(3.9) applies. To cover the generic case of shock wave interaction, we assume each $\gamma_i(t)$ is smooth away from $t=0$ with continuous tangent vectors up to $t=0$, and it suffices to restrict to lower shock wave interactions in $\mathbb{R}^2_-$.

Corollary 4.3. *Let $\gamma_i(t)=(t,x_i(t))$ be two smooth curves defined on $I=(-\epsilon,0)$, some $\epsilon>0$, such that the limits $\lim_{t\to0-}\gamma_i(t)=(0,r_0)$ and $\dot{x}_i(0)=\lim_{t\to0-}\dot{x}_i(t)$ both exist for $i=1,2$. Let f be a function in $C^{0,1}(\mathcal{N}\cap\mathbb{R}^2_-)$ for $\mathcal{N}$ a neighborhood of $(0,r_0)$ in $\mathbb{R}^2$, so that f meets condition (4.1) on each γ_i. Then there exists a $C^{0,1}$ function Φ defined on $\mathcal{N}\cap\mathbb{R}^2_-$, such that*

$$[\Phi_t]_i\equiv0\equiv[\Phi_r]_i,\quad i=1,2, \tag{4.11}$$

and

$$f(t,r)=\frac{1}{2}\sum_{i=1,2}\varphi_i(t)\,|x_i(t)-r|+\Phi(t,r), \tag{4.12}$$

for all (t,r) in $\mathcal{N}\cap\mathbb{R}^2_-$, where

$$\varphi_i(t)=\frac{[f_{,\mu}]_i n_i^\mu}{n_i^\mu n_{i\mu}}\in C^{0,1}(I). \tag{4.13}$$

In particular, φ_i has discontinuous derivatives wherever $f\circ\gamma_i$ does, and again it suffices that indices are raised and lowered by a Lorentzian metric $g_{\mu\nu}$ which is $C^{0,1}$ across each γ_i.

5. A Necessary and Sufficient Condition for Smoothing Metrics

In this section we derive a necessary and sufficient pointwise condition on the Jacobians of a coordinate transformation that it lift the regularity of a $C^{0,1}$ metric tensor to $C^{1,1}$ in a neighborhood of a point on a single shock surface Σ. In the next section we use this condition to prove that such transformations exist in a neighborhood of a point on a single shock surface, and in the section following that we use this pointwise condition on each of two intersecting shock surfaces to prove that no such coordinate transformation exists in a neighborhood of a point of shock wave interaction.

We begin with the transformation law

$$g_{\alpha\beta} = J^\mu_\alpha g_{\mu\nu} J^\nu_\beta, \tag{5.1}$$

for the metric components at a point on a hypersurface Σ for a general $C^{1,1}$ coordinate transformation $x^\mu \to x^\alpha$, where, as customary, the indices indicate the coordinate system. Let J^μ_α denote the Jacobian of the transformation

$$J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}.$$

Assume now, that the metric components $g_{\mu\nu}$ are only Lipschitz continuous with respect to x^μ across Σ. Then differentiating (5.1) in the direction $w = w^\sigma \frac{\partial}{\partial x^\sigma}$ we obtain

$$[g_{\alpha\beta,\gamma}]w^\gamma = J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\sigma}]w^\sigma + g_{\mu\nu} J^\mu_\alpha [J^\nu_{\beta,\sigma}]w^\sigma + g_{\mu\nu} J^\nu_\beta [J^\mu_{\alpha,\sigma}]w^\sigma\,, \tag{5.2}$$

where $[f]$ denotes the jump in the quantity f across the shock surface Σ. Thus, since both g and J^μ_α are in general Lipschitz continuous across Σ, the jumps appear only on the derivatives. Equation (5.2) gives a necessary and sufficient condition for the metric g to be $C^{1,1}$ in x^α coordinates. Namely, taking $w = \frac{\partial}{\partial x^\sigma}$, (5.2) implies that $[g_{\alpha\beta,\gamma}] = 0$ for every $\alpha, \beta, \gamma = 0, ..., 3$ if and only if

$$[J^\mu_{\alpha,\gamma}] J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}] J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}] = 0\ . \tag{5.3}$$

Note that if the coordinate transformation is C^2, so that J^μ_α is C^1, then the jumps in J vanish, and (5.2) reduces to

$$[g_{\alpha\beta,\gamma}]w^\gamma = J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\sigma}]w^\sigma,$$

which is tensorial because the non-tensorial terms cancel out in the jump $[g_{\alpha\beta,\gamma}]$. Since tensor transformations preserve the zero tensor, it is precisely the lack of covariance in (5.2) for $C^{1,1}$ transformations that provides the necessary degrees of freedom, (the jumps $[J^\mu_{\alpha,\gamma}]$ in the first derivatives of the Jacobian), that make it possible for a Lipschitz metric to be smoothed by coordinate transformation at points on a single shock surface, illustrating that there is no hope of lifting the metric regularity by coordinate transformations that are C^2.

Equation (5.3) is linear in the jumps in the derivatives of the Jacobians, and our intention is to use this to solve for the $[J^\mu_{\alpha,\gamma}]$ associated with a given $C^{1,1}$ coordinate transformation. To this end, suppose we are given a single radial shock surface Σ in SSC locally parameterized by

$$\Sigma(t,\theta,\phi) = (t, x(t), \theta, \phi)\,. \tag{5.4}$$

For such a hypersurface in Standard Schwarzschild Coordinates (SSC), the angular variables play a passive role, and the essential issue regarding smoothing the metric components by $C^{1,1}$ coordinate transformation, lies in the atlas of (t,r)-coordinate transformations. Thus we restrict to the atlas of (t,r)-coordinate transformations for a general $C^{0,1}$ metric in SSC, c.f. (2.10). The following lemma gives the unique solution $[J^\mu_{\alpha,\gamma}]$ of (5.3) for (t,r)-transformations of $C^{0,1}$ metrics g in SSC.

Lemma 5.1. *Let*

$$g_{\mu\nu} = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2\,,$$

be a given metric expressed in SSC, let Σ denote a single radial shock surface (5.4) *across which g is only Lipschitz continuous. Then the unique solution $[J^\mu_{\alpha,\gamma}]$ of* (5.3) *which satisfies the integrability condition,* (*c.f.* (2.3)),

$$[J^\mu_{\alpha,\beta}] = [J^\mu_{\beta,\alpha}]\,, \tag{5.5}$$

is given by:

$$\begin{aligned}
[J^t_{0,t}] &= -\frac{1}{2}\left(\frac{[A_t]}{A}J^t_0 + \frac{[A_r]}{A}J^r_0\right); \qquad [J^t_{0,r}] = -\frac{1}{2}\left(\frac{[A_r]}{A}J^t_0 + \frac{[B_t]}{A}J^r_0\right)\\
[J^t_{1,t}] &= -\frac{1}{2}\left(\frac{[A_t]}{A}J^t_1 + \frac{[A_r]}{A}J^r_1\right); \qquad [J^t_{1,r}] = -\frac{1}{2}\left(\frac{[A_r]}{A}J^t_1 + \frac{[B_t]}{A}J^r_1\right)\\
[J^r_{0,t}] &= -\frac{1}{2}\left(\frac{[A_r]}{B}J^t_0 + \frac{[B_t]}{B}J^r_0\right); \qquad [J^r_{0,r}] = -\frac{1}{2}\left(\frac{[B_t]}{B}J^t_0 + \frac{[B_r]}{B}J^r_0\right)\\
[J^r_{1,t}] &= -\frac{1}{2}\left(\frac{[A_r]}{B}J^t_1 + \frac{[B_t]}{B}J^r_1\right); \qquad [J^r_{1,r}] = -\frac{1}{2}\left(\frac{[B_t]}{B}J^t_1 + \frac{[B_r]}{B}J^r_1\right).
\end{aligned} \tag{5.6}$$

(*We use the notation $\mu,\nu \in \{t,r\}$ and $\alpha,\beta \in \{0,1\}$, so that t,r are used to denote indices whenever they appear on the Jacobian J.*)

Proof: Equation (5.3) as an inhomogeneous 6×6 linear system in eight unknowns $[J^\mu_{\alpha,\gamma}]$. Imposing the integrability condition in the form of (5.5) gives two additional equations which complete (5.3) to an 8×8 system which is uniquely solvable for $[J^\mu_{\alpha,\gamma}]$. The result is a purely algebraic system whose unique solution (5.6) we obtain by a lengthy calculation aided by MAPLE, (c.f. [8] for details.) □

Condition (5.3) is a necessary and sufficient condition for $[g_{\alpha\beta,\gamma}] = 0$ at a point on a smooth single shock surface. Because Lemma 5.1 tells us that we can uniquely solve (5.3) for the Jacobian derivatives, it follows that a necessary and sufficient condition for $[g_{\alpha\beta,\gamma}] = 0$ is also that the jumps in the Jacobian derivatives be exactly the functions of the jumps in the original SSC metric components recorded in (5.6). In light of this, Lemma 5.1 immediately implies the following corollary:

Corollary 5.2. *Let p be a point on a single smooth shock curve γ, and let $g_{\mu\nu}$ be a metric tensor in SSC, which is $C^{0,1}$ across γ in the sense of Definition 4.1. Suppose J^μ_α is the Jacobian of an actual coordinate transformation defined on a neighborhood $\mathcal{N}$ of p. Then the metric in the new coordinates $g_{\alpha\beta}$ is in $C^{1,1}(\mathcal{N})$ if and only if J^μ_α satisfies* (5.6).[10]

[10] Note that to lift the metric regularity, the Jacobian must mirror the regularity of the metric in order to compensate for all discontinuous first order derivatives of the metric by its own discontinuous first order derivatives. This explains why only $C^{1,1}$ transformations can possibly lift the metric regularity from $C^{0,1}$ to $C^{1,1}$, and $C^{1,\alpha}$ does not suffice for $\alpha \neq 1$.

We conclude that (5.6) is a necessary and sufficient condition for a coordinate transformation to lift the regularity of an SSC metric from $C^{0,1}$ to $C^{1,1}$ at a point on a single smooth shock surface. The condition relates the jumps in the derivatives of the Jacobian to the jumps in the metric derivatives across the shock. This establishes the rather remarkable result that there is no algebraic obstruction to lifting the regularity in the sense that the jumps in the Jacobian derivatives can be *uniquely* solved for in terms of the jumps in the metric derivatives, precisely when the integrability condition (5.5) is imposed. The condition is a statement purely about spherically symmetric spacetime metrics in SSC coordinates because neither the RH conditions nor the Einstein equations have yet been imposed. But we know by Israel's theorem that the RH conditions must be imposed to conclude that smoothing transformations exist. The point then, is that to prove the existence of coordinate transformations that lift the regularity of SSC metrics to $C^{1,1}$ at $p \in \Sigma$, we must prove that there exists a set of functions J^μ_α defined in a neighborhood of p, such that (5.6) holds at p, and such that the integrability condition (2.3), (required for J^μ_α to be the Jacobian of a coordinate transformation), holds in a whole neighborhood containing p. In the next section we give an alternative proof of Israel's Theorem by showing that such J^μ_α always exist in a neighborhood of a point p on a smooth single shock surface, and the following section we prove that no such functions exist in a neighborhood of a point p of shock wave interaction, unless $Det\,(J^\mu_\alpha) = 0$ at p.

6. Metric Smoothing on Single Shock Surfaces and a Constructive Proof of Israel's Theorem

We have shown in Corollary 5.2 that (5.6) is a necessary and sufficient condition on a Jacobian derivative J^μ_α for lifting the SSC metric regularity to $C^{1,1}$ in a neighborhood of a shock curve. We now address the issue of how to obtain such Jacobians of actual coordinate transformations defined on a whole neighborhood of a shock surface. For this we need to find a set of functions J^μ_α that satisfies (5.6), and also satisfies the integrability condition (2.3) in a whole neighborhood. In this section we show that this can be accomplished in the case of single shock surfaces, thereby giving an alternative constructive proof of Israel's Theorem for spherically symmetric spacetimes:

Theorem 6.1. (Israel's Theorem) *Suppose $g_{\mu\nu}$ is an SSC metric that is $C^{0,1}$ across a radial shock surface γ in the sense of Definition 4.1, such that it solves the Einstein equations* (2.11) - (2.14) *strongly away from γ, and assume $T^{\mu\nu}$ is everywhere bounded and in C^0 away from γ. Then around each point p on γ there exists a $C^{1,1}$ coordinate transformation of the (t,r)-plane, defined in a neighborhood $\mathcal{N}$ of p, such that the transformed metric components $g_{\alpha\beta}$ are $C^{1,1}$ functions of the new coordinates, if and only if the RH jump conditions* (3.4)*,* (3.5) *hold on γ in a neighborhood of p.*

The main step is to construct Jacobians acting on the (t,r)-plane that satisfy the smoothing condition (5.6) on the shock curve, the condition that guarantees $[g_{\alpha\beta,\gamma}] = 0$. The following lemma gives an explicit formula for functions J^μ_α satisfying (5.6). The main point is that, in the case of single shock curves, both the RH jump conditions and the Einstein equations are necessary and sufficient for such functions J^μ_α to exist.

Lemma 6.2. *Let p be a point on a single shock curve γ across which the SSC metric $g_{\mu\nu}$ is Lipschitz continuous in the sense of Definition 4.1 in a neighborhood $\mathcal{N}$ of p. Then there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N})$ satisfying the smoothing condition*

(5.6) *on* $\gamma \cap \mathcal{N}$ *if and only if* (4.8) *holds on* $\gamma \cap \mathcal{N}$*. Furthermore, all* J^μ_α *that are in* $C^{0,1}(\mathcal{N})$ *and satisfy* (5.6) *on* $\gamma \cap \mathcal{N}$ *are given by*

$$\begin{aligned} J^t_0(t,r) &= \frac{[A_r]\phi(t) + [B_t]\omega(t)}{4A \circ \gamma(t)} \, |x(t) - r| + \Phi(t,r) \\ J^t_1(t,r) &= \frac{[A_r]\nu(t) + [B_t]\zeta(t)}{4A \circ \gamma(t)} \, |x(t) - r| + N(t,r) \\ J^r_0(t,r) &= \frac{[B_t]\phi(t) + [B_r]\omega(t)}{4B \circ \gamma(t)} \, |x(t) - r| + \Omega(t,r) \\ J^r_1(t,r) &= \frac{[B_t]\nu(t) + [B_r]\zeta(t)}{4B \circ \gamma(t)} \, |x(t) - r| + Z(t,r)\,, \end{aligned} \tag{6.1}$$

for arbitrary functions Φ, Ω, Z, $N \in C^{0,1}(\mathcal{N})$, *where*

$$\phi = \Phi \circ \gamma, \quad \omega = \Omega \circ \gamma, \quad \nu = N \circ \gamma, \quad \zeta = Z \circ \gamma\,. \tag{6.2}$$

Moreover, each arbitrary function $U = \Phi, \Omega, Z$ *or* N *satisfies*

$$[U_r] = 0 = [U_t]. \tag{6.3}$$

Proof: Suppose there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N})$ satisfying (5.6), then their continuity implies that tangential derivatives along γ match across γ, that is

$$[J^\mu_{\alpha,t}] = -\dot{x}[J^\mu_{\alpha,r}] \tag{6.4}$$

for all $\mu \in \{t,r\}$ and $\alpha \in \{0,1\}$. Imposing (6.4) in (5.6) and using (4.3) - (4.4) yields (4.8).

To prove the opposite direction it suffices to show that all t and r derivatives of J^μ_α, defined in (6.1), satisfy (5.6) for all $\mu \in \{t,r\}$ and $\alpha \in \{0,1\}$. This follows directly from (4.3), (4.4) and (4.8), upon noting that (6.2) implies the identities

$$\phi = J^t_0 \circ \gamma, \quad \nu = J^t_1 \circ \gamma, \quad \omega = J^r_0 \circ \gamma, \quad \zeta = J^r_1 \circ \gamma\,. \tag{6.5}$$

This proves the existence of functions J^μ_α satisfying (5.6). Applying (the one shock version of) Corollary 4.3 (which allows Φ to have the lower regularity $\Phi \in C^{1,1}$ but imposes the jumps (7.4) along γ), confirms that all such functions can be written in the canonical form (6.1). $\Box$

To complete the proof of Israel's Theorem, we must prove the existence of coordinate transformations $x^\mu \to x^\alpha$ that lift the $C^{0,1}$ regularity of $g_{\mu\nu}$ to $C^{1,1}$. It remains, then, to show that the functions J^μ_α defined above in ansatz (6.1) can be integrated to coordinate functions, i.e., that they satisfy the integrability condition (2.3) in a whole neighborhood. This is accomplished in the following two lemmas.

Lemma 6.3. *The functions* J^μ_α *defined in* (6.1) *satisfy the integrability condition (2.3) if an only if the free functions* Φ, Ω, N *and* Z *satisfy the following system of two PDE's:*

$$\begin{aligned} (\dot{\alpha}|X| + \Phi_t)\,(\beta|X| + N) + \Phi_r\,(\epsilon|X| + Z) - (\alpha|X| + \Phi)\left(\dot{\beta}|X| + N_t\right) & \\ -N_r\,(\delta|X| + \Omega) + fH(X) &= 0 \end{aligned} \tag{6.6}$$

$$\begin{aligned} \left(\dot{\delta}|X| + \Omega_t\right)(\beta|X| + N) + \Omega_r\,(\epsilon|X| + Z) - (\dot{\epsilon}|X| + Z_t)\,(\alpha|X| + \Phi) & \\ -Z_r\,(\delta|X| + \Omega) + hH(X) &= 0\,, \end{aligned} \tag{6.7}$$

where $X(t,r)=x(t)-r$, $H(\cdot)$ denotes the Heaviside step function,

$$\begin{aligned}\alpha &= \frac{[A_r]\phi(t)+[B_t]\omega(t)}{4A\circ\gamma(t)}; \qquad \beta=\frac{[A_r]\nu(t)+[B_t]\zeta(t)}{4A\circ\gamma(t)};\\ \delta &= \frac{[B_t]\phi(t)+[B_r]\omega(t)}{4B\circ\gamma(t)}; \qquad \epsilon=\frac{[B_t]\nu(t)+[B_r]\zeta(t)}{4B\circ\gamma(t)};\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}f &= (\beta\delta-\alpha\epsilon)\,|X|+\alpha\dot{x}N-\beta\dot{x}\Phi+\beta\Omega-\alpha Z\\ h &= (\beta\delta-\alpha\epsilon)\,\dot{x}|X|+\delta\dot{x}N-\epsilon\dot{x}\Phi+\epsilon\Omega-\delta Z\,,\end{aligned} \tag{6.9}$$

where, α,β,δ and ϵ are C^1 functions of t and f and h are in $C^{0,1}$.

The proof of Lemma 6.3 follows by substituting ansatz (6.1) into the integrability condition (2.3)and identifying the terms in the resulting first order differential equations for J^μ_α. (For details see [8].)

The proof of Israel's Theorem is complete once we prove the existence of solutions Φ,Ω,N and Z of (6.6), (6.7) that are $C^{0,1}$, such that they satisfy (6.3). For this it suffices to choose N and Z arbitrarily, so that (6.6), (6.7) reduces to a system of 2 linear first order PDE's for the unknown functions Φ and Ω. The condition (6.3) essentially imposes that Φ,Ω,N and Z be C^1 across the shock γ. Since (6.6), (6.7) are linear equations for Φ and Ω, they can be solved along characteristics, and so the only obstacle to solutions Φ and Ω with the requisite smoothness to satisfy the condition (6.3), is the presence of the Heaviside function $H(X)$ on the right hand side of (6.6), (6.7). Lemma 6.3 thus isolates the discontinuous behavior of equations (6.6), (6.7) in the functions f and h, the coefficients of H. Israel's theorem is now a consequence of the following lemma which states that these coefficients of $H(X)$ vanish precisely when the RH jump conditions hold on γ. (See [8] for details.)

Lemma 6.4. *Assume the SSC metric $g_{\mu\nu}$ is $C^{0,1}$ across γ and solves the first three Einstein equations strongly away from γ. Then the coefficients f and g of $H(X)$ in (6.6), (6.7) vanish on γ if and only if the RH jump conditions (2.8) hold on γ.*

We can now complete the proof of Israel's Theorem. Assuming that the Einstein equations hold strongly away from the shock curve (in fact, it suffices to assume that only the first three equations hold), we have that there exist functions J^μ_α satisfying the smoothing condition (5.6) if and only if the RH jump conditions hold (c.f. Lemma 6.2). Furthermore, by lemmas 6.3 and 6.4, a solution to the integrability condition with the required regularity holds if and only if the RH jump conditions hold (in the sense of (4.8)). Thus, under the assumption that the Einstein equations hold strongly away from γ, we can integrate the Jacobians J^μ_α to coordinate functions that smooth the metric g to $C^{1,1}$ if and only if the RH jump conditions hold. This completes the proof of Theorem 6. $\Box$

7. Shock Wave Interactions as Regularity Singularities in GR - Transformations in the (t,r)-Plane

The main step in the proof of Theorem 1.1 is to prove that there do not exist $C^{1,1}$ coordinate transformations of the (t,r)-plane in a neighborhood of a point p of regular shock wave interaction in SSC that lifts the regularity of the metric g from $C^{0,1}$ to $C^{1,1}$ in a neighborhood of p. We then prove in Section 8 that no such transformation can exist within the full $C^{1,1}$ atlas that transforms all four variables of the spacetime, i.e., including the angular variables. We formulate the main step precisely for lower shock wave interactions in $\mathbb{R}^2_-$ in the following theorem, which is the topic of this section. A corresponding result applies to upper shock wave interactions in R^2_+, as well as two wave interactions in a whole neighborhood of p.

Theorem 7.1. *Suppose that p is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a $C^{1,1}$ coordinate transformation $x^\alpha \circ (x^\mu)^{-1}$ of the (t,r)-plane, defined on $\mathcal{N} \cap \mathbb{R}^2_-$ for a neighborhood $\mathcal{N}$ of p in $\mathbb{R}^2$, such that the metric components $g_{\alpha\beta}$ are C^1 functions of the coordinates x^α in $\mathcal{N} \cap \mathbb{R}^2_-$ and such that the metric has a non-vanishing determinant at p, (that is, such that $\lim_{q\to p} Det\,(g_{\alpha\beta}(q)) \neq 0$).*[11]

In the remainder of this section we outline the proof of Theorem 7.1, which mirrors the constructive proof of Israel's Theorem 6.1 in that it uses the extension (7.1) of ansatz (6.1) to construct all $C^{1,1}$ coordinate transformations that can smooth the gravitational metric to $C^{1,1}$ in a neighborhood of a point p of regular shock wave interaction. The negative conclusion is then reached by proving that any such coordinate transformation must have a vanishing Jacobian determinant at p. But now, to prove non-existence, we must show the ansatz (7.1) is general enough to include all $C^{0,1}$ Jacobians that could possibly lift the regularity of the metric. For this we use condition (5.6) to construct a canonical form for the Jacobians in a neighborhood of p, that generalizes (6.1) to the case of two shock curves, with the weaker assumption of $C^{0,1}$ regularity on the functions Φ, Ω, Z, N. We conclude the proof by showing that this canonical form is inconsistent with the assumption that $Det\,(g_{\alpha\beta}) \neq 0$ at p, by using the continuity of the Jacobians up to p.

To implement these ideas, the main step is to show that the canonical form 4.12 of Corollary 4.3 can be applied to the Jacobians J^μ_α in the presence of a shock wave interaction. The result is recorded in the following lemma:

Lemma 7.2. *Let p be a point of regular shock wave interaction in SSC in the sense of Definition 3.1, corresponding to the SSC metric $g_{\mu\nu}$ defined on $\mathcal{N} \cap \overline{\mathbb{R}^2_-}$. Then there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ satisfying the smoothing condition (5.6) on $\gamma_i \cap \mathcal{N}$, $i = 1, 2$, if and only if (4.8) holds on each shock curve $\gamma_i \cap \mathcal{N}$. In this case, all J^μ_α in $C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ assume the canonical form*

$$\begin{aligned}
J^t_0(t,r) &= \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r),\\
J^t_1(t,r) &= \sum_i \beta_i(t)\,|x_i(t) - r| + N(t,r),\\
J^r_0(t,r) &= \sum_i \delta_i(t)\,|x_i(t) - r| + \Omega(t,r),\\
J^r_1(t,r) &= \sum_i \epsilon_i(t)\,|x_i(t) - r| + Z(t,r)\,,
\end{aligned} \tag{7.1}$$

where

$$\alpha_i(t) \quad = \quad \frac{[A_r]_i\;\phi_i(t) + [B_t]_i\;\omega_i(t)}{4A\circ\gamma_i(t)},$$

[11] Note that Theorem 1.1 states the non-existence of coordinates on an entire neighborhood $\mathcal{N}$ of p in $\mathbb{R}^2$, but here we have prove the stronger result that such coordinates do not exist on the upper or lower half planes separately.

$$\begin{aligned}
\beta_i(t) &= \frac{[A_r]_i\ \nu_i(t) + [B_t]_i\ \zeta_i(t)}{4A \circ \gamma_i(t)}, \\
\delta_i(t) &= \frac{[B_t]_i\ \phi_i(t) + [B_r]_i\ \omega_i(t)}{4B \circ \gamma_i(t)}, \\
\epsilon_i(t) &= \frac{[B_t]_i\ \nu_i(t) + [B_r]_i\ \zeta_i(t)}{4B \circ \gamma_i(t)}\ ,
\end{aligned} \tag{7.2}$$

with

$$\phi_i = \Phi \circ \gamma_i, \quad \omega_i = \Omega \circ \gamma_i, \quad \zeta_i = Z \circ \gamma_i, \quad \nu_i = N \circ \gamma_i\,, \tag{7.3}$$

and where $\Phi, \Omega, Z, N \ \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ *have matching derivatives on each shock curve* $\gamma_i(t)$,

$$[U_r]_i = 0 = [U_t]_i\,, \tag{7.4}$$

for $U = \Phi, \Omega, Z, N,\ t \in (-\epsilon, 0)$.

The essence of the canonical form (7.1) is that the jumps in derivatives across the shock waves have been taken out of the functions Φ, Ω, Z, N in (7.4). We now have a canonical form for all functions J^μ_α that meet the necessary and sufficient condition (5.6) for $[g_{\alpha\beta,\gamma}] = 0$. However, for J^μ_α to be proper Jacobians that can be integrated to a coordinate system, we must use the free functions Φ, Ω, Z, N to meet the integrability condition (2.3). To finish the proof of Theorem 7.1, we show that, as a consequence of (7.4), (that is, the free functions are C^1 regular at the shocks), the Jacobian determinant $Det\, J^\mu_\alpha$ must vanish at the point of shock interaction, which then implies $Det\,(g_{\alpha\beta}) = 0$.

Thus, using the canonical form (7.1) restricted to the shock curve and taking the determinant of the resulting J^μ_α leads directly to

$$Det\ (J^\mu_\alpha \circ \gamma_i(t)) = \left(J^t_0 J^r_1 - J^t_1 J^r_0\right)|_{\gamma_i(t)} = \phi_i(t)\zeta_i(t) - \nu_i(t)\omega_i(t). \tag{7.5}$$

Since J^μ_α is continuous, we obtain the same limit $t \to 0$ for $i = 1, 2$,

$$\lim_{t \to 0^+} Det\ (J^\mu_\alpha \circ \gamma_i(t)) = \phi_i(0)\zeta_i(0) - \nu_i(0)\omega_i(0) = \phi_0\zeta_0 - \nu_0\omega_0. \tag{7.6}$$

Therefore, the final step in the proof of Theorem 7.1 is the following lemma:

Lemma 7.3. *Let* $p \in \mathcal{N}$ *be a point of regular shock wave interaction in SSC in the sense of Definition 3.1. Then if the integrability condition*

$$J^\mu_{\alpha,\beta} = J^\mu_{\beta,\alpha} \tag{7.7}$$

holds in $\mathcal{N} \cap \mathbb{R}^2_-$ *for the functions* J^μ_α *defined in (7.1), (so that* Φ, Ω, N *and* Z *satisfy* (7.4)), *then*

$$\frac{1}{4B}\left(\frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B}\right)[B_r]_1 [B_r]_2\,(\dot{x}_1 - \dot{x}_2)\,(\phi_0\zeta_0 - \nu_0\omega_0) = 0. \tag{7.8}$$

Proof: Substituting the J^μ_α in (7.1) into (7.7) gives equations (6.6), (6.7) except that we now sum over two shock curves instead of one. The difference is the appearance of additional mixed terms in the coefficients f and h of the discontinuous terms multiplying the Heaviside function $H(X)$. The proof is accomplished by showing that, unlike f and g in (6.6), (6.7), these mixed terms do not vanish by the jump conditions for the Einstein equations alone. Finally, a lengthy calculation to evaluate the limit $t \to 0$ demonstrates that imposing the condition that these additional mixed terms should vanish, which is necessary for (7.4) to hold, implies the final equation (7.8). (See [8] for details.) □

To finish the proof of Theorem 7.1, observe that the first three terms in (7.8) are nonzero by our assumption that shock curves are non-null, and have distinct speeds at $t = 0$. Thus (7.8) implies

$$Det\, J^{\mu}_{\alpha}(p) = (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0, \tag{7.9}$$

as claimed. □

In summary, we remark that at first there appears to be more than enough freedom to choose the free functions Φ, Ω, Z, N in the canonical form of Lemma 7.2, to arrange for the discontinuous term in the integrability condition to vanish, (just as in Lemma 6.2, the main step leading to Israel's Theorem). This together with the fact that the derivatives of J^{μ}_{α} are uniquely solvable in condition (5.6), lead us to believe until the very end that one *could* construct coordinates in which $g_{\alpha\beta}$ was $C^{1,1}$. But at the very last step, taking the limit to the point p of shock wave interaction, we find that the condition (7.4), expressing that $[g_{\alpha\beta,\gamma}]$ vanishes at shocks, has the effect of freezing out all the freedom in Φ, Ω, Z, N, thereby forcing condition (7.9), implying that the determinant of the Jacobian must vanish at p. The answer was not apparent until the very last step, and thus we find the result quite remarkable and surprising.

8. Shock Wave Interactions as Regularity Singularities in GR - the Full Atlas

For the proof of Theorem 1.1, we have established the nonexistence of $C^{1,1}$ coordinate transformations in the (t, r)-plane that can map a $C^{0,1}$ regular SSC metric $g_{\mu\nu}$ over to a $C^{1,1}$ metric $g_{\alpha\beta}$. It remains to extend this result to the full atlas of coordinate transformations that depend on all four coordinate variables, including the SSC angular variables. In this section we outline the proof, (see [8] for details).

So assume for contradiction there exist coordinates x^j in which the metric g_{ij} is C^1. In general g_{ij} is not of the box diagonal form (2.9), however, one can always transform back to a metric $g_{\alpha\beta}$ in box diagonal form (c.f. [15] chapter 13). The point is now that this transformation is in C^2, since Killing's equation

$$X_{i,j} + X_{j,i} = \Gamma^k_{ij} X_k,$$

yields a C^1 regular Killing vector field X, provided g_{ij} is in C^1, which if integrated up to coordinates yields a C^2 coordinate transformation. Therefore the resulting metric in box diagonal form $g_{\alpha\beta}$ is C^1 regular. Now $g_{\alpha\beta}$ can be taken over to SSC by a $C^{1,1}$ coordinate transformation, which, together with its inverse, act only on the (t, r) variables. This contradicts Theorem 7.1, and completes the proof of Theorem 1.1.

9. The Loss of Locally Inertial Frames

Finally we discuss the non-existence of locally inertial frames around a point of regular shock wave interaction. This is in surprising contrast to the case of points on single shock surfaces for which locally inertial coordinate frames always exist, (c.f. [11]). To start, we clarify what we mean by a locally inertial frame:

Definition 9.1. *Let p be a point in a Lorentz manifold and let x^j be a coordinate system defined in a neighborhood of p. We call x^j a locally inertial frame around p if the metric g_{ij} in those coordinates satisfies:*

(i) $g_{ij}(p) = \eta_{ij}$, $\eta_{ij} = diag(-1, 1, 1, 1)$,
(ii) $g_{ij,l}(p) = 0$ *for all* $i, j, l \in \{0, ..., 3\}$,
(iii) $g_{ij,kl}$ *are bounded in every compact neighborhood of* p. [12]

When the metric components g_{ij} satisfy (1)-(3) at point p, we say g_{ij} is *locally Minkowskian* (or locally flat or locally inertial) at p. By Theorem 1.1, there exist second order derivatives of the metric which are pointwise unbounded in every neighborhood of p. Therefore, the following Corollary is a straightforward consequence of Theorem 1.1:

Corollary 9.2. *Let* p *be a point of regular shock wave interaction in SSC in the sense of Definition 3.1. Then there does not exist a* $C^{1,1}$ *coordinate transformation such that the resulting metric* g_{ij} *is locally Minkowskian around* p.

10. Conclusion

Our results show that points of shock wave interaction give rise to a new kind of singularity which is different from the well known singularities of GR. The famous examples of singularities in GR are either non-removable singularities beyond physical spacetime, (for example the center of the Schwarzschild and Kerr metrics, and the Big Bang singularity in cosmology where the curvature cannot be bounded), or else they are removable in the sense that they can be transformed to locally inertial points of a regular spacetime under coordinate transformation, (for example, the apparent singularity at the Schwarzschild radius, the interface at vacuum in the interior Schwarzschild, Oppenheimer-Snyder [9], Smoller-Temple shock wave solutions [13, 12], and any apparent singularity at smooth shock surfaces that become regularized by Israel's Theorem, [5, 11]). In contrast, points of regular shock wave interaction are non-removable singularities that propagate in physically meaningful spacetimes in GR, such that the curvature is uniformly bounded, but the spacetime is essentially *not* locally inertial at the singularity. For this reason we call these *regularity singularities.*

Since the gravitational metric tensor is not locally inertial at points of shock wave interaction, it begs the question as to whether there are general relativistic gravitational effects at points of shock wave interaction that cannot be predicted from the compressible Euler equations in special relativity alone. Indeed, even if there are dissipativity terms, like those of the Navier Stokes equations,[13] which regularize the gravitational metric at points of shock wave interaction, our results assert that the steep gradients in the derivative of the metric tensor at small viscosity cannot be removed uniformly while keeping the metric determinant uniformly bounded away from zero, so one would expect the general relativistic effects at points of shock wave interaction to persist. We thus wonder whether shock wave interactions might provide a physical regime where new general relativistic effects might be observed. Said differently, a regularity singularity is not hidden behind an event horizon, so it is a sort of counterexample to the

[12] This condition ensures that the physical equations in GR (which are tensorial) differ from the corresponding equations in flat Minkowski space by only *gravitational effects*, i.e., effects that are second order in the metric derivatives. In most of the literature on GR, (c.f. [4]), the gravitational metric is assumed to be at least $C^{1,1}$, (c.f. [4]), which then directly implies condition (3) of Definition 9.1. At this stage it is not clear to the authors whether or not there exist coordinates that could play a (physically satisfying) role as locally inertial frames by satisfying (1)-(2), but not (3).

[13] The issue of how to incorporate a relativistic viscosity that meets the speed of light bound is problematic, [15].

cosmic censorship conjecture in the sense that it gives rise to unbounded second order metric derivatives, which by themselves might yield physically measurable effects that resemble some effects of unbounded curvature.

References

[1] A. Einstein, *Die Feldgleichungen der Gravitation*, Preuss. Akad. Wiss., Berlin, Sitzber. 1915b, pp. 844-847. **314**(1970), pp. 529-548.

[2] L. C. Evans, *Partial Differential Equations*, Vol 3A, Berkeley Mathematics Lecture Notes, 1994.

[3] J. Groah and B. Temple, *Shock-Wave Solutions of the Einstein Equations with Perfect Fluid Sources: Existence and Consistency by a Locally Inertial Glimm Scheme*, Memoirs AMS, Vol. 172, Number 813, November 2004, ISSN 0065-9266.

[4] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge University Press, 1973.

[5] W. Israel, "Singular hypersrfaces and thin shells in general relativity", Il Nuovo Cimento, Vol. XLIV B, N. 1, 1966, pp. 1-14.

[6] F. John, *Partial Differential Equations*, Applied Mathematical Sciences 1, Springer-Verlag, New York, 1981.

[7] P. Lax, "Hyperbolic systems of conservation laws, II", Comm. Pure Appl. Math., 10(1957), pp. 537-566.

[8] M. Reintjes, *Shock Wave Interactions in General Relativity and the Emergence of Regularity Singularities*, Dissertation, University of California - Davis, to appear (May 2011).

[9] J.R. Oppenheimer and J.R. Snyder, *On continued gravitational contraction*, Phys. Rev., **56**, pp. 455-459 (1939).

[10] J. Smoller, *Shock Waves and Reaction Diffusion Equations*, Springer-Verlag, 1983.

[11] J. Smoller and B. Temple, "Shock wave Solutions of the Einstein equations: The Oppenheimer-Snyder model of gravitational collapse extended to the case of non-zero pressure", Archive Rational Mechanics and Analysis, 128 (1994), pp. 249-297, Springer-Verlag 1994.

[12] J. Smoller and B. Temple, "Shock wave cosmology inside a black hole", PNAS, Vol. 100, no. 20, 2003, pp. 11216-11218.

[13] J. Smoller and B. Temple, "Astrophysical shock wave solutions of the Einstein equations", Phys. Rev. D, Vol. 51, No. 6, 1995, pp. 2733-2743.

[14] Z. Vogler and B. Temple, "Simulation of General Relativistic Shock Waves by a Locally Inertial Godunov Method featuring Dynamical Time Dilation", Davis Preprint.

[15] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York, 1972.

Department of Mathematics, University of California, Davis, CA 95616, USA
E-mail address: moritzreintjes@googlemail.com

NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: moritzreintjes@googlemail.com

Department of Mathematics, University of California, Davis, CA 95616, USA
E-mail address: temple@math.ucdavis.edu