

Universität Regensburg Mathematik



Equivariant Iwasawa theory and non-abelian Stark-type conjectures

Andreas Nickel

Preprint Nr. 31/2011

Equivariant Iwasawa theory and non-abelian Stark-type conjectures

Andreas Nickel*

Abstract

We discuss three different formulations of the equivariant Iwasawa main conjecture attached to an extension \mathcal{K}/k of totally real fields with Galois group \mathcal{G} , where k is a number field and \mathcal{G} is a p -adic Lie group of dimension 1 for an odd prime p . All these formulations are equivalent and hold if Iwasawa's μ -invariant vanishes. Under mild hypotheses, we use this to prove non-abelian generalizations of Brumer's conjecture, the Brumer-Stark conjecture and a strong version of the Coates-Sinnott conjecture provided that $\mu = 0$.

Introduction

Let K/k be a finite Galois CM-extension of number fields with Galois group G . To each finite set S of places of k which contains all the infinite places, one can associate a so-called "Stickelberger element" $\theta_S(K/k)$ in the center of the group ring algebra $\mathbb{C}G$. This Stickelberger element is defined via L -values at zero of S -truncated Artin L -functions attached to the (complex) characters of G . Let us denote the roots of unity of K by μ_K and the class group of K by cl_K . Assume that S contains the set S_{ram} of all finite primes of k which ramify in K/k . Then it was independently shown in [Ca79], [DR80] and [Ba77] that for abelian G one has

$$\text{Ann}_{\mathbb{Z}G}(\mu_K)\theta_S(K/k) \subset \mathbb{Z}G. \quad (1)$$

Now Brumer's conjecture asserts that $\text{Ann}_{\mathbb{Z}G}(\mu_K)\theta_S(K/k)$ annihilates cl_K . There is a large body of evidence in support of Brumer's conjecture (cf. the expository article [Gr04]); in particular, Greither [Gr07] has shown that the appropriate special case of the equivariant Tamagawa number conjecture (ETNC) implies the p -part of Brumer's conjecture for an odd prime p if the p -part of μ_K is a c.t. (short for cohomologically trivial) G -module. A similar result for arbitrary G was proven by the author [Ni10], improving an unconditional annihilation result due to Burns and Johnston [BJ11]. Note that the assumptions made in loc.cit. are adapted to ensure the validity of the strong Stark conjecture. Moreover, in [Nia], the author has introduced non-abelian generalizations of Brumer's conjecture, the Brumer-Stark conjecture and of the so-called strong Brumer-Stark property. The extension K/k fulfills the latter if certain Stickelberger elements are contained in the (non-commutative) Fitting invariants of corresponding ray class groups; but it does not hold in general, even if

*I acknowledge financial support provided by the DFG

2010 *Mathematics Subject Classification*: 11R23, 11R42

Keywords: Iwasawa theory, main conjecture, equivariant L -values, Stark conjectures

G is abelian, as follows from the results in [GK08]. But if this property happens to be true, this also implies the validity of the (non-abelian) Brumer-Stark conjecture and Brumer's conjecture. We will prove the p -part of a dual version of the strong Brumer-Stark property for an arbitrary CM-extension of number fields and an odd prime p under the only restriction that S contains all the p -adic places of k and that Iwasawa's μ -invariant vanishes. In particular, this implies the (non-abelian) Brumer-Stark conjecture and Brumer's conjecture under the same hypotheses.

For this, we have to discuss three different versions of the equivariant Iwasawa main conjecture (EIMC). The first formulation is due to Ritter and Weiss [RW04], the second follows the framework of [CFKSV05] and was used by Kakde [Ka] in his proof of the EIMC. Finally, Greither and Popescu [GrP] have formulated an EIMC via the Tate module of a certain Iwasawa-theoretic abstract 1-motive; but they restrict their formulation to abelian extensions. So one of our first tasks is to give a formulation of their conjecture in the non-abelian situation as well. In fact, it will be this formulation which will lead to the above mentioned proof of the (dual) strong Brumer-Stark property. All variants of the EIMC hold if Iwasawa's μ -invariant vanishes. This follows from the recent result of Ritter and Weiss [RWa] on the EIMC for p -adic Lie groups of dimension 1. In fact, this can be generalized to Lie groups of higher dimension as shown by Kakde [Ka] and, independently, by Burns [Bub].

Finally, we will introduce a (non-abelian) analogue of the strong Brumer-Stark property for higher étale cohomology groups. For abelian extensions, this property implies the Coates-Sinnott conjecture, and for arbitrary extensions it implies a non-abelian analogue of this conjecture which is closely related to the central conjecture in [Ni11c]. In contrast to the strong Brumer-Stark property, we conjecture that its higher analogue holds in general and we consequently will call this conjecture the (non-abelian) strong Coates-Sinnott conjecture. We provide several reduction steps which under certain mild hypotheses allows us to assume that K/k is a Galois CM-extension. In this situation, we show that the strong Coates-Sinnott conjecture is (nearly) equivalent to an appropriate special case of the ETNC. We may conclude that the strong Coates-Sinnott conjecture holds provided that $\mu = 0$, since these special cases of the ETNC have been proven by Burns [Bub] under this assumption. We also give a direct proof of our conjecture, still assuming that $\mu = 0$, using our new formulation of the EIMC. This will also provide a new proof of Burns' result on the ETNC.

This article is organized as follows. In section 1, we provide the necessary background material. In particular, we discuss the notion of non-commutative Fitting invariants which have been introduced by the author [Ni10], and how we may define them for certain perfect complexes. In section 2, we give the formulation of the EIMC due to Ritter and Weiss, but using Fitting invariants rather than the Hom description. We show that the canonical complex which occurs in the construction of Ritter and Weiss is isomorphic in the derived category of Iwasawa modules to $R\mathrm{Hom}(R\Gamma_{\acute{e}t}(\mathrm{Spec}(\mathfrak{o}_K[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$. This will explain the relation of the first two above mentioned formulations of the EIMC in more detail than it is available in the literature so far. In section 3, we recall the notion of abstract 1-motives as formulated in [GrP] and show, how to use the Iwasawa-theoretic abstract

1-motive of [GrP] to formulate an EIMC in the non-abelian situation as well. Assuming the vanishing of μ , we deduce this conjecture from the result on the EIMC due to Ritter and Weiss [RWa]. In fact, the argument can be reversed such that both conjectures are equivalent. In section 4, we use our new formulation of the EIMC to prove the above mentioned cases of the (dual) strong Brumer-Stark property. Finally, we introduce and discuss the strong Coates-Sinnott conjecture in section 5.

The author would like to thank Cornelius Greither for several discussions concerning the article [GrP].

1 Preliminaries

1.0.1 K -theory

Let Λ be a left noetherian ring with 1 and $\text{PMod}(\Lambda)$ the category of all finitely generated projective Λ -modules. We write $K_0(\Lambda)$ for the Grothendieck group of $\text{PMod}(\Lambda)$, and $K_1(\Lambda)$ for the Whitehead group of Λ which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center of Λ which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S -torsion Λ -modules of finite projective dimension by $K_0S(\Lambda)$. Writing Λ_S for the ring of quotients of Λ with denominators in S , we have the following Localization Sequence (cf. [CR87], p. 65)

$$K_1(\Lambda) \rightarrow K_1(\Lambda_S) \xrightarrow{\partial} K_0S(\Lambda) \xrightarrow{\rho} K_0(\Lambda) \rightarrow K_0(\Lambda_S). \quad (2)$$

In the special case where Λ is an \mathfrak{o} -order over a commutative ring \mathfrak{o} and S is the set of all nonzerodivisors of \mathfrak{o} , we also write $K_0T(\Lambda)$ instead of $K_0S(\Lambda)$. Moreover, we denote the relative K -group corresponding to a ring homomorphism $\Lambda \rightarrow \Lambda'$ by $K_0(\Lambda, \Lambda')$ (cf. [Sw68]). Then we have a Localization Sequence (cf. [CR87], p. 72)

$$K_1(\Lambda) \rightarrow K_1(\Lambda') \xrightarrow{\partial_{\Lambda, \Lambda'}} K_0(\Lambda, \Lambda') \rightarrow K_0(\Lambda) \rightarrow K_0(\Lambda').$$

It is also shown in [Sw68] that there is an isomorphism $K_0(\Lambda, \Lambda_S) \simeq K_0S(\Lambda)$. For any ring Λ we write $\zeta(\Lambda)$ for the subring of all elements which are central in Λ . Let L be a subfield of either \mathbb{C} or \mathbb{C}_p for some prime p and let G be a finite group. In the case where Λ' is the group ring LG the reduced norm map $\text{nr}_{LG} : K_1(LG) \rightarrow \zeta(LG)^\times$ is always injective. If in addition $L = \mathbb{R}$, there exists a canonical map $\hat{\partial}_G : \zeta(\mathbb{R}G)^\times \rightarrow K_0(\mathbb{Z}G, \mathbb{R}G)$ such that the restriction of $\hat{\partial}_G$ to the image of the reduced norm equals $\partial_{\mathbb{Z}G, \mathbb{R}G} \circ \text{nr}_{\mathbb{R}G}^{-1}$. This map is called the extended boundary homomorphism and was introduced by Burns and Flach [BF01].

For any ring Λ we write $\mathcal{D}(\Lambda)$ for the derived category of Λ -modules. Let $\mathcal{C}^b(\text{PMod}(\Lambda))$ be the category of bounded complexes of finitely generated projective Λ -modules. A complex of Λ -modules is called perfect if it is isomorphic in $\mathcal{D}(\Lambda)$ to an element of $\mathcal{C}^b(\text{PMod}(\Lambda))$. We denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ consisting of perfect complexes by $\mathcal{D}^{\text{perf}}(\Lambda)$. For any $C \in \mathcal{C}^b(\text{PMod}(\Lambda))$ we define Λ -modules

$$C^{\text{ev}} := \bigoplus_{i \in \mathbb{Z}} C^{2i}, \quad C^{\text{odd}} := \bigoplus_{i \in \mathbb{Z}} C^{2i+1}.$$

Similarly, we define $H^{ev}(C^\cdot)$ and $H^{odd}(C^\cdot)$ to be the direct sum over all even (resp. odd) degree cohomology groups of C^\cdot .

For the following let R be a Dedekind domain of characteristic 0, K its field of fractions, A a finite dimensional K -algebra and Λ an R -order in A . A pair (C^\cdot, t) consisting of a complex $C^\cdot \in \mathcal{D}^{\text{perf}}(\Lambda)$ and an isomorphism $t : H^{odd}(C^\cdot_K) \rightarrow H^{ev}(C^\cdot_K)$ is called a trivialized complex, where C^\cdot_K is the complex obtained by tensoring C^\cdot with K . We refer to t as a trivialization of C^\cdot . One defines the refined Euler characteristic $\chi_{\Lambda, A}(C^\cdot, t) \in K_0(\Lambda, A)$ of a trivialized complex as follows: Choose a complex $P^\cdot \in \mathcal{C}^b(\text{PMod}(R))$ which is quasi-isomorphic to C^\cdot . Let $B^i(P^\cdot_K)$ and $Z^i(P^\cdot_K)$ denote the i^{th} coboundaries and i^{th} cocycles of P^\cdot_K , respectively. We have the obvious exact sequences

$$B^i(P^\cdot_K) \twoheadrightarrow Z^i(P^\cdot_K) \twoheadrightarrow H^i(P^\cdot_K), \quad Z^i(P^\cdot_K) \twoheadrightarrow P^\cdot_K \twoheadrightarrow B^{i+1}(P^\cdot_K).$$

If we choose splittings of the above sequences, we get an isomorphism

$$\phi_t : P^\cdot_K \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P^\cdot_K) \oplus H^{odd}(P^\cdot_K) \simeq \bigoplus_{i \in \mathbb{Z}} B^i(P^\cdot_K) \oplus H^{ev}(P^\cdot_K) \simeq P^\cdot_K,$$

where the second map is induced by t . Then the refined Euler characteristic is defined to be

$$\chi_{\Lambda, A}(C^\cdot, t) := (P^{odd}, \phi_t, P^{ev}) \in K_0(\Lambda, A)$$

which indeed is independent of all choices made in the construction. For further information concerning refined Euler characteristics we refer the reader to [Bu03].

Denote the full triangulated subcategory of $\mathcal{D}(\Lambda)$ consisting of perfect complexes whose cohomologies are R -torsion by $\mathcal{D}_{\text{tor}}^{\text{perf}}(\Lambda)$. For any complex $C^\cdot \in \mathcal{D}_{\text{tor}}^{\text{perf}}(\Lambda)$ there is a unique trivialization, namely $t = 0$; hence C^\cdot defines a class

$$[C^\cdot] := \chi_{\Lambda, A}(C^\cdot, 0) \in K_0(\Lambda, A) = K_0T(\Lambda).$$

In fact, $K_0(\Lambda, A)$ identifies with the Grothendieck group whose generators are $[C^\cdot]$, where C^\cdot is an object of the category $\mathcal{C}_{\text{tor}}^b(\text{PMod}(\Lambda))$ of bounded complexes of finitely generated projective Λ -modules whose cohomologies are R -torsion, and the relations are as follows: $[C^\cdot] = 0$ if C^\cdot is acyclic, and $[C_2^\cdot] = [C_1^\cdot] + [C_3^\cdot]$ for any short exact sequence

$$C_1^\cdot \twoheadrightarrow C_2^\cdot \twoheadrightarrow C_3^\cdot$$

in $\mathcal{C}_{\text{tor}}^b(\text{PMod}(\Lambda))$ (cf. [We]). Moreover, if M is a finitely generated R -torsion Λ -module of finite projective dimension, then the class of M in $K_0T(\Lambda)$ agrees with the class $[M] \in K_0(\Lambda, A)$, where M is considered as a perfect complex concentrated in degree 1.

1.0.2 Non-commutative Fitting invariants

For the following we refer the reader to [Ni10]. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m \times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_{n \times n}(R)$ by $\text{Gl}_n(R)$. Let A be a separable K -algebra and Λ be an \mathfrak{o} -order in A , finitely generated as \mathfrak{o} -module, where \mathfrak{o} is a complete commutative noetherian local ring with field

of quotients K . Moreover, we will assume that the integral closure of \mathfrak{o} in K is finitely generated as \mathfrak{o} -module. The group ring $\mathbb{Z}_p G$ of a finite group G will serve as a standard example. Let N and M be two $\zeta(\Lambda)$ -submodules of an \mathfrak{o} -torsionfree $\zeta(\Lambda)$ -module. Then N and M are called $\text{nr}(\Lambda)$ -equivalent if there exists an integer n and a matrix $U \in \text{Gl}_n(\Lambda)$ such that $N = \text{nr}(U) \cdot M$, where $\text{nr} : A \rightarrow \zeta(A)$ denotes the reduced norm map which extends to matrix rings over A in the obvious way. We denote the corresponding equivalence class by $[N]_{\text{nr}(\Lambda)}$. We say that N is $\text{nr}(\Lambda)$ -contained in M (and write $[N]_{\text{nr}(\Lambda)} \subset [M]_{\text{nr}(\Lambda)}$) if for all $N' \in [N]_{\text{nr}(\Lambda)}$ there exists $M' \in [M]_{\text{nr}(\Lambda)}$ such that $N' \subset M'$. Note that it suffices to check this property for one $N_0 \in [N]_{\text{nr}(\Lambda)}$. We will say that x is contained in $[N]_{\text{nr}(\Lambda)}$ (and write $x \in [N]_{\text{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\text{nr}(\Lambda)}$ such that $x \in N_0$.

Now let M be a finitely presented (left) Λ -module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \quad (3)$$

be a finite presentation of M . We identify the homomorphism h with the corresponding matrix in $M_{a \times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geq b$. In the case $a = b$ we call (3) a quadratic presentation. The Fitting invariant of h over Λ is defined to be

$$\text{Fitt}_\Lambda(h) = \begin{cases} [0]_{\text{nr}(\Lambda)} & \text{if } a < b \\ [\langle \text{nr}(H) | H \in S(h) \rangle_{\zeta(\Lambda)}]_{\text{nr}(\Lambda)} & \text{if } a \geq b. \end{cases}$$

We call $\text{Fitt}_\Lambda(h)$ a Fitting invariant of M over Λ . One defines $\text{Fitt}_\Lambda^{\max}(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with respect to the partial order “ \subset ”. If M admits a quadratic presentation h , one also puts $\text{Fitt}_\Lambda(M) := \text{Fitt}_\Lambda(h)$ which is independent of the chosen quadratic presentation.

Assume now that \mathfrak{o} is an integrally closed commutative noetherian ring, but not necessarily complete or local. We denote by $\mathcal{I} = \mathcal{I}(\Lambda)$ the $\zeta(\Lambda)$ -submodule of $\zeta(A)$ generated by the elements $\text{nr}(H)$, $H \in M_{b \times b}(\Lambda)$, $b \in \mathbb{N}$. We choose a maximal order Λ' containing Λ . We may decompose the separable K -algebra A into its simple components

$$A = A_1 \oplus \cdots \oplus A_t,$$

i.e. each A_i is a simple K -algebra and $A_i = Ae_i = e_i A$ with central primitive idempotents e_i , $1 \leq i \leq t$. For any matrix $H \in M_{b \times b}(\Lambda)$ there is a unique matrix $H^* \in M_{b \times b}(\Lambda')$ such that $H^* H = H H^* = \text{nr}(H) \cdot 1_{b \times b}$ and $H^* e_i = 0$ whenever $\text{nr}(H) e_i = 0$ (cf. [Ni10], Lemma 4.1; the additional assumption on \mathfrak{o} to be complete local is not necessary). If $\tilde{H} \in M_{b \times b}(\Lambda)$ is a second matrix, then $(H \tilde{H})^* = \tilde{H}^* H^*$. We define

$$\mathcal{H} = \mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) | x H^* \in M_{b \times b}(\Lambda) \forall b \in \mathbb{N} \forall H \in M_{b \times b}(\Lambda)\}.$$

Since $x \cdot \text{nr}(H) = x H^* H$, we have in particular

$$\mathcal{H} \cdot \mathcal{I} = \mathcal{H} \subset \zeta(\Lambda). \quad (4)$$

We put $\mathcal{H}_p(G) := \mathcal{H}(\mathbb{Z}_p G)$ and $\mathcal{H}(G) := \mathcal{H}(\mathbb{Z} G)$. The importance of the $\zeta(\Lambda)$ -module \mathcal{H} will become clear by means of the following result which is [Ni10], Th. 4.2.

Theorem 1.1. *If \mathfrak{o} is an integrally closed complete commutative noetherian local ring and M is a finitely presented Λ -module, then*

$$\mathcal{H} \cdot \text{Fitt}_{\Lambda}^{\max}(M) \subset \text{Ann}_{\Lambda}(M).$$

Now let $C^{\cdot} \in \mathcal{D}_{\text{tor}}^{\text{perf}}(\Lambda)$. If $\rho([C^{\cdot}]) = 0$, we choose $x \in K_1(A)$ such that $\partial(x) = [C^{\cdot}]$ and define

$$\text{Fitt}_{\Lambda}(C^{\cdot}) := [\langle \text{nr}_A(x) \rangle_{\zeta(\Lambda)}]_{\text{nr}(\Lambda)}.$$

It is straightforward to show that

$$\text{Fitt}_{\Lambda}(C_2) = \text{Fitt}_{\Lambda}(C_1) \cdot \text{Fitt}_{\Lambda}(C_3)$$

for any short exact sequence $C_1^{\cdot} \rightarrow C_2^{\cdot} \rightarrow C_3^{\cdot}$ in $\mathcal{C}_{\text{tor}}^b(\text{PMod}(\Lambda))$, provided that all Fitting invariants are defined. Finally, if C^{\cdot} is isomorphic in $\mathcal{D}(\Lambda)$ to a complex $P^{-1} \rightarrow P^0$ concentrated in degree -1 and 0 such that P^i are finitely generated \mathfrak{o} -torsion Λ -modules of finite projective dimension, $i = -1, 0$, then

$$\text{Fitt}_{\Lambda}(C^{\cdot}) = \text{Fitt}_{\Lambda}(P^0 : P^{-1}),$$

where the righthand side denotes the relative Fitting invariant of [Ni10], Def. 3.6.

Now let $p \neq 2$ be a prime and let $\Lambda(\mathcal{G})$ be the complete group algebra $\mathbb{Z}_p[[\mathcal{G}]]$, where \mathcal{G} is a profinite group which contains a finite normal subgroup H such that $\mathcal{G}/H \simeq \Gamma$ for a pro- p -group Γ , isomorphic to \mathbb{Z}_p ; thus \mathcal{G} can be written as a semi-direct product $H \rtimes \Gamma$. We fix a topological generator γ of Γ and choose a natural number n such that γ^{p^n} is central in \mathcal{G} . Since also $\Gamma^{p^n} \simeq \mathbb{Z}_p$, there is an isomorphism $\mathbb{Z}_p[[\Gamma^{p^n}]] \simeq \mathbb{Z}_p[[T]]$ induced by $\gamma^{p^n} \mapsto 1 + T$. Here, $R := \mathbb{Z}_p[[T]]$ denotes the power series ring in one variable over \mathbb{Z}_p . If we view $\Lambda(\mathcal{G})$ as an R -module, there is a decomposition

$$\Lambda(\mathcal{G}) = \bigoplus_{i=0}^{p^n-1} R\gamma^i[H].$$

Hence $\Lambda(\mathcal{G})$ is finitely generated as an R -module and an R -order in the separable $\text{Quot}(R)$ -algebra $\mathcal{Q}(\mathcal{G}) := \bigoplus_i \text{Quot}(R)\gamma^i[H]$. Note that $\mathcal{Q}(\mathcal{G})$ is obtained from $\Lambda(\mathcal{G})$ by inverting all non-zero elements in R . For any ring Λ and any Λ -module M , we write $\text{pd}_{\Lambda}(M)$ for the projective dimension of M over Λ . For any finitely generated $\Lambda(\mathcal{G})$ -module M , we write $\mu(M)$ for the Iwasawa μ -invariant of M .

Proposition 1.2. *Let C^{\cdot} be a complex in $\mathcal{D}_{\text{tor}}^{\text{perf}}(\Lambda(\mathcal{G}))$. Assume that C^{\cdot} is isomorphic in $\mathcal{D}(\Lambda)$ to a bounded complex P^{\cdot} such that $\text{pd}_{\Lambda(\mathcal{G})}(P^j) \leq 1$, $\mu(P^j) = 0$ and P^j is R -torsion for all $j \in \mathbb{N}$. Assume that the Fitting invariant $\text{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})}(\mathbb{Q}_p \otimes^L C^{\cdot})$ of $\mathbb{Q}_p \otimes^L C^{\cdot}$ over $\mathbb{Q}_p\Lambda(\mathcal{G})$ is generated by an element $\Phi \in \text{nr}(K_1(\Lambda_{(p)}(\mathcal{G})))$, where the subscript (p) means localization at the prime (p) . Then also*

$$\text{Fitt}_{\Lambda(\mathcal{G})}(C^{\cdot}) = [\langle \Phi \rangle_{\zeta(\Lambda(\mathcal{G}))}]_{\text{nr}(\Lambda(\mathcal{G}))}.$$

Proof. We first observe that the homomorphism $\partial : K_1(\Lambda(\mathcal{G})) \rightarrow K_0T(\Lambda(\mathcal{G}))$ is surjective (cf. [Ni10], Lemma 6.2 or more directly [Ka], Lemma 5). Hence $\text{Fitt}_{\Lambda(\mathcal{G})}(C')$ is defined for any complex in $\mathcal{D}_{\text{tor}}^{\text{perf}}(\Lambda(\mathcal{G}))$. Our assumptions on C' imply that we have an equality

$$[C'] = [P^{\text{odd}}] - [P^{\text{ev}}] \in K_0T(\Lambda(\mathcal{G})).$$

Then P^{odd} and P^{ev} are two finitely generated R -torsion $\Lambda(\mathcal{G})$ -modules of projective dimension less or equal to 1 and trivial μ -invariant. Let Ψ be a generator of $\text{Fitt}_{\Lambda(\mathcal{G})}(P^{\text{ev}})$. Since P^{ev} vanishes after localization at (p) , we have $\Psi \in \text{nr}(K_1(\Lambda_{(p)}(\mathcal{G})))$. But then $\Phi \cdot \Psi$ also belongs to $\text{nr}(K_1(\Lambda_{(p)}(\mathcal{G})))$ and is a generator of

$$\text{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})}(\mathbb{Q}_p \otimes^L C') \cdot \text{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})}(\mathbb{Q}_p \otimes P^{\text{ev}}) = \text{Fitt}_{\mathbb{Q}_p\Lambda(\mathcal{G})}(\mathbb{Q}_p \otimes P^{\text{odd}}).$$

Now [Ni11b], Prop. 3.2 implies that $\Phi \cdot \Psi$ is actually a generator of $\text{Fitt}_{\Lambda(\mathcal{G})}(P^{\text{odd}})$ such that Φ is a generator of

$$\text{Fitt}_{\Lambda(\mathcal{G})}(C') = \text{Fitt}_{\Lambda(\mathcal{G})}(P^{\text{odd}}) \cdot \text{Fitt}_{\Lambda(\mathcal{G})}(P^{\text{ev}})^{-1}.$$

□

1.0.3 Equivariant L -values

Let us fix a finite Galois extension K/k of number fields with Galois group G . For any place v of k we fix a place w of K above v and write G_w resp. I_w for the decomposition group resp. inertia subgroup of K/k at w . Moreover, we denote the residual group at w by $\overline{G}_w = G_w/I_w$ and choose a lift $\phi_w \in G_w$ of the Frobenius automorphism at w . For a (finite) place w we sometimes write \mathfrak{P}_w for the associated prime ideal in K and ord_w for the associated valuation.

If S is a finite set of places of k containing the set S_∞ of all infinite places of k , and χ is a (complex) character of G , we denote the S -truncated Artin L -function attached to χ and S by $L_S(s, \chi)$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi \in \text{Irr}(G)} \mathbb{C}$, where $\text{Irr}(G)$ denotes the set of irreducible characters of G . We define the equivariant Artin L -function to be the meromorphic $\zeta(\mathbb{C}G)$ -valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \text{Irr}(G)}.$$

If T is a second finite set of places of k such that $S \cap T = \emptyset$, we define $\delta_T(s) := (\delta_T(s, \chi))_{\chi \in \text{Irr}(G)}$, where $\delta_T(s, \chi) = \prod_{v \in T} \det(1 - N(v)^{1-s} \phi_w^{-1} | V_\chi^{I_w})$ and V_χ is a G -module with character χ . We put

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\sharp,$$

where we denote by $\sharp : \mathbb{C}G \rightarrow \mathbb{C}G$ the involution induced by $g \mapsto g^{-1}$. These functions are the so-called (S, T) -modified G -equivariant L -functions and, for $r \in \mathbb{Z}_{\leq 0}$, we define Stickelberger elements

$$\theta_S^T(K/k, r) = \theta_S^T(r) := \Theta_{S,T}(r) \in \zeta(\mathbb{C}G).$$

If T is empty, we abbreviate $\theta_S^T(r)$ by $\theta_S(r)$, and if $r = 0$, we write θ_S^T for $\theta_S^T(0)$. Now a result of Siegel [Si70] implies that

$$\theta_S^T(r) \in \zeta(\mathbb{Q}G) \quad (5)$$

for all integers $r \leq 0$. Let us fix an embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$; then the image of $\theta_S(r)$ in $\zeta(\mathbb{Q}_p G)$ via the canonical embedding

$$\zeta(\mathbb{Q}G) \hookrightarrow \zeta(\mathbb{Q}_p G) = \bigoplus_{\chi \in \text{Irr}_p(G)/\sim} \mathbb{Q}_p(\chi),$$

is given by $\sum_{\chi} L_S(r, \tilde{\chi}^{\iota^{-1}})^{\iota}$ and similarly for $\theta_S^T(r)$. Here, the sum runs over all \mathbb{C}_p -valued irreducible characters of G modulo Galois action. Note that we will frequently drop ι and ι^{-1} from the notation. Finally, for an irreducible character χ with values in \mathbb{C} (resp. \mathbb{C}_p) we put $e_{\chi} = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$ which is a central idempotent in $\mathbb{C}G$ (resp. $\mathbb{C}_p G$).

1.0.4 Ray class groups

For any set S of places of k , we write $S(K)$ for the set of places of K which lie above those in S . Now let T and S be as above. We write $\text{cl}_{K,T}$ for the ray class group of K to the ray $\mathfrak{M}_T := \prod_{w \in T(K)} \mathfrak{P}_w$ and \mathfrak{o}_S for the ring of $S(K)$ -integers of K . Let S_f be the set of all finite primes in $S(K)$; then there is a natural map $\mathbb{Z}S_f \rightarrow \text{cl}_{K,T}$ which sends each prime $w \in S_f$ to the corresponding class $[\mathfrak{P}_w] \in \text{cl}_{K,T}$. We denote the cokernel of this map by $\text{cl}_{S,T,K} =: \text{cl}_{S,T}$. Further, we denote the $S(K)$ -units of K by E_S and define $E_S^T := \{x \in E_S : x \equiv 1 \pmod{\mathfrak{M}_T}\}$. All these modules are equipped with a natural G -action and we have the following exact sequences of G -modules

$$E_{S_{\infty}}^T \twoheadrightarrow E_S^T \xrightarrow{\nu} \mathbb{Z}S_f \rightarrow \text{cl}_{K,T} \twoheadrightarrow \text{cl}_{S,T}, \quad (6)$$

where $\nu(x) = \sum_{w \in S_f} \text{ord}_w(x)w$ for $x \in E_S^T$, and

$$E_S^T \twoheadrightarrow E_S \rightarrow (\mathfrak{o}_S/\mathfrak{M}_T)^{\times} \xrightarrow{\nu} \text{cl}_{S,T} \twoheadrightarrow \text{cl}_S, \quad (7)$$

where the map ν lifts an element $\bar{x} \in (\mathfrak{o}_S/\mathfrak{M}_T)^{\times}$ to $x \in \mathfrak{o}_S$ and sends it to the ideal class $[(x)] \in \text{cl}_{S,T}$ of the principal ideal (x) . Note that the G -module $(\mathfrak{o}_S/\mathfrak{M}_T)^{\times}$ is c.t. if no prime in T ramifies in K/k . If $S = S_{\infty}$, we also write E_K^T instead of $E_{S_{\infty}}^T$. Finally, we suppress the superscript T from the notation if T is empty. If M is a finitely generated \mathbb{Z} -module and p is a prime, we put $M(p) := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$. In particular, we will be interested in $\text{cl}_{K,T}(p)$ for odd primes p ; we will abbreviate this module by $A_{K,T}$ if p is clear from the context.

2 On different formulations of the equivariant Iwasawa main conjecture

The following reformulation of the EIMC was given in [Ni11b], §2.

Let $p \neq 2$ be a prime and let \mathcal{K}/k be a Galois extension of totally real fields with Galois group \mathcal{G} , where k is a number field, \mathcal{K} contains the cyclotomic \mathbb{Z}_p -extension k_{∞} of k and

$[\mathcal{K} : k_\infty]$ is finite. Hence \mathcal{G} is a p -adic Lie group of dimension 1 and there is a finite normal subgroup H of \mathcal{G} such that $\mathcal{G}/H = \text{Gal}(k_\infty/k) =: \Gamma_k$. Here, Γ_k is isomorphic to the p -adic integers \mathbb{Z}_p and we fix a topological generator γ_k . If we pick a preimage γ of γ_k in \mathcal{G} , we can choose an integer m such that γ^{p^m} lies in the center of \mathcal{G} . Hence the ring $R := \mathbb{Z}_p[[\Gamma^{p^m}]]$ belongs to the center of $\Lambda(\mathcal{G})$, and $\Lambda(\mathcal{G})$ is an R -order in the separable $\text{Quot}(R)$ -algebra $Q(\mathcal{G})$. Let S be a finite set of places of k containing all the infinite places S_∞ and the set S_p of all places of k above p . Moreover, let M_S be the maximal abelian pro- p -extension of \mathcal{K} unramified outside S , and denote the Iwasawa module $\text{Gal}(M_S/\mathcal{K})$ by X_S . If S additionally contains all places which ramify in \mathcal{K}/k , there is a canonical complex

$$C_S(\mathcal{K}/k) : \cdots \rightarrow 0 \rightarrow C^{-1} \rightarrow C^0 \rightarrow 0 \rightarrow \cdots \quad (8)$$

of R -torsion $\Lambda(\mathcal{G})$ -modules of projective dimension at most 1 such that $H^{-1}(C_S(\mathcal{K}/k)) = X_S$ and $H^0(C_S(\mathcal{K}/k)) = \mathbb{Z}_p$. For the moment we are insistent that this complex is the one constructed by Ritter and Weiss in [RW02]. We will see later that we can work with $R\text{Hom}(R\Gamma_{\text{ét}}(\text{Spec}(\mathfrak{o}_{\mathcal{K}}[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$ as well. We put (cf. [RW04], §4)

$$\mathfrak{U}_S = \mathfrak{U}_S(\mathcal{K}/k) := (C^{-1}) - (C^0) \in K_0T(\Lambda(\mathcal{G})).$$

Since $\rho(\mathfrak{U}_S) = 0$, there is a well defined Fitting invariant of \mathfrak{U}_S ; more precisely,

$$\text{Fitt}_{\Lambda(\mathcal{G})}(\mathfrak{U}_S) := \text{Fitt}_{\Lambda(\mathcal{G})}(C^{-1} : C^0) = \text{Fitt}_{\Lambda(\mathcal{G})}(C_S(\mathcal{K}/k))^{-1}.$$

We recall some results concerning the algebra $Q(\mathcal{G})$ due to Ritter and Weiss [RW04]. Let \mathbb{Q}_p^c be an algebraic closure of \mathbb{Q}_p and fix an irreducible (\mathbb{Q}_p^c -valued) character χ of \mathcal{G} with open kernel. Choose a finite field extension E of \mathbb{Q}_p such that the character χ has a realization V_χ over E . Let η be an irreducible constituent of $\text{res}_H^{\mathcal{G}}\chi$ and set

$$\text{St}(\eta) := \{g \in \mathcal{G} : \eta^g = \eta\}, \quad e_\eta = \frac{\eta(1)}{|H|} \sum_{g \in H} \eta(g^{-1})g, \quad e_\chi = \sum_{\eta | \text{res}_H^{\mathcal{G}}\chi} e_\eta.$$

For any finite field extension K of \mathbb{Q}_p with ring of integers \mathfrak{o} , we set $Q^K(\mathcal{G}) := K \otimes_{\mathbb{Q}_p} Q(\mathcal{G})$ and $\Lambda^{\mathfrak{o}}(\mathcal{G}) = \mathfrak{o}[[\mathcal{G}]]$. By [RW04], corollary to Prop. 6, e_χ is a primitive central idempotent of $Q^E(\mathcal{G})$. By loc.cit., Prop. 5 there is a distinguished element $\gamma_\chi \in \zeta(Q^E(\mathcal{G})e_\chi)$ which generates a procyclic p -subgroup Γ_χ of $(Q^E(\mathcal{G})e_\chi)^\times$ and acts trivially on V_χ . Moreover, γ_χ induces an isomorphism $Q^E(\Gamma_\chi) \xrightarrow{\simeq} \zeta(Q^E(\mathcal{G})e_\chi)$ by loc.cit., Prop. 6. For $r \in \mathbb{N}_0$, we define the following maps

$$j_\chi^r : \zeta(Q^E(\mathcal{G})) \rightarrow \zeta(Q^E(\mathcal{G})e_\chi) \simeq Q^E(\Gamma_\chi) \rightarrow Q^E(\Gamma_k),$$

where the last arrow is induced by mapping γ_χ to $\kappa^r(\gamma_\chi)\gamma_k^{w_\chi}$, where $w_\chi = [\mathcal{G} : \text{St}(\eta)]$ and κ denotes the cyclotomic character of \mathcal{G} . Note that $j_\chi := j_\chi^0$ agrees with the corresponding map j_χ in loc.cit. It is shown that for any matrix $\Theta \in M_{n \times n}(Q(\mathcal{G}))$ we have

$$j_\chi(\text{nr}(\Theta)) = \det_{Q^E(\Gamma_k)}(\Theta | \text{Hom}_{EG}(V_\chi, Q^E(\mathcal{G})^n)). \quad (9)$$

Here, Θ acts on $f \in \text{Hom}_{EG}(V_\chi, Q^E(\mathcal{G})^n)$ via right multiplication, and γ_k acts on the left via $(\gamma_k f)(v) = \gamma_k \cdot f(\gamma_k^{-1}v)$ for all $v \in V_\chi$. Hence the map

$$\begin{aligned} \text{Det}(\cdot)(\chi) : K_1(Q(\mathcal{G})) &\rightarrow Q^E(\Gamma_k)^\times \\ [P, \alpha] &\mapsto \det_{Q^E(\Gamma_k)}(\alpha | \text{Hom}_{EG}(V_\chi, E \otimes_{\mathbb{Q}_p} P)), \end{aligned}$$

where P is a projective $Q(\mathcal{G})$ -module and α a $Q(\mathcal{G})$ -automorphism of P , is just $j_\chi \circ \text{nr}$. If ρ is a character of \mathcal{G} of type W , i.e. $\text{res}_H^{\mathcal{G}} \rho = 1$, then we denote by ρ^\sharp the automorphism of the field $Q^c(\Gamma_k) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\Gamma_k)$ induced by $\rho^\sharp(\gamma_k) = \rho(\gamma_k)\gamma_k$. Moreover, we denote the additive group generated by all \mathbb{Q}_p^c -valued characters of \mathcal{G} with open kernel by $R_p(\mathcal{G})$; finally, $\text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_k)^\times)$ is the group of all homomorphisms $f : R_p(\mathcal{G}) \rightarrow Q^c(\Gamma_k)^\times$ satisfying

$$\begin{aligned} f(\chi \otimes \rho) &= \rho^\sharp(f(\chi)) && \text{for all characters } \rho \text{ of type } W \text{ and} \\ f(\chi^\sigma) &= f(\chi)^\sigma && \text{for all Galois automorphisms } \sigma \in \text{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p). \end{aligned}$$

We have an isomorphism

$$\begin{aligned} \zeta(Q(\mathcal{G}))^\times &\simeq \text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_k)^\times) \\ x &\mapsto [\chi \mapsto j_\chi(x)]. \end{aligned}$$

By loc.cit., Th. 5 the map $\Theta \mapsto [\chi \mapsto \text{Det}(\Theta)(\chi)]$ defines a homomorphism

$$\text{Det} : K_1(Q(\mathcal{G})) \rightarrow \text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_k)^\times)$$

such that we obtain a commutative triangle

$$\begin{array}{ccc} & K_1(Q(\mathcal{G})) & \\ \text{nr} \swarrow & & \searrow \text{Det} \\ \zeta(Q(\mathcal{G}))^\times & \xrightarrow{\sim} & \text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_k)^\times). \end{array} \quad (10)$$

We put $u := \kappa(\gamma_k)$ and fix a finite set S of places of k containing S_∞ and all places which ramify in \mathcal{K}/k . Each topological generator γ_k of Γ_k permits the definition of a power series $G_{\chi,S}(T) \in \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \text{Quot}(\mathbb{Z}_p[[T]])$ by starting out from the Deligne-Ribet power series for abelian characters of open subgroups of \mathcal{G} (cf. [DR80]). One then has an equality

$$L_{p,S}(1-s, \chi) = \frac{G_{\chi,S}(u^s - 1)}{H_\chi(u^s - 1)},$$

where $L_{p,S}(s, \chi)$ denotes the p -adic Artin L -function, and where, for irreducible χ , one has

$$H_\chi(T) = \begin{cases} \chi(\gamma_k)(1+T) - 1 & \text{if } H \subset \ker(\chi) \\ 1 & \text{otherwise.} \end{cases}$$

Now [RW04], Prop. 11 implies that

$$L_{k,S} : \chi \mapsto \frac{G_{\chi,S}(\gamma_k - 1)}{H_\chi(\gamma_k - 1)}$$

is independent of the topological generator γ_k and lies in $\text{Hom}^*(R_p(\mathcal{G}), Q^c(\Gamma_k)^\times)$. Diagram (10) implies that there is a unique element $\Phi_S \in \zeta(Q(\mathcal{G}))^\times$ such that

$$j_\chi(\Phi_S) = L_{k,S}(\chi).$$

The EIMC as formulated in [RW04] now states that there is a unique $\Theta_S \in K_1(Q(\mathcal{G}))$ such that $\text{Det}(\Theta_S) = L_{k,S}$ and $\partial(\Theta_S) = \mathcal{U}_S$. The EIMC without its uniqueness statement hence asserts that there is $x \in K_1(Q(\mathcal{G}))$ such that $\partial(x) = \mathcal{U}_S$ and $\text{Det}(x) = L_{k,S}$; now diagram (10) implies that $\text{nr}(x) = \Phi_S$, and thus Φ_S is a generator of $\text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{U}_S)$. Conversely, if Φ_S is a generator of $\text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{U}_S)$, then there is an element $x \in K_1(Q(\mathcal{G}))$ such that $\partial(x) = \mathcal{U}_S$ and $\langle \text{nr}(x) \rangle_{\zeta(\Lambda(\mathcal{G}))}$ is $\text{nr}(\Lambda(\mathcal{G}))$ -equivalent to $\langle \Phi_S \rangle_{\zeta(\Lambda(\mathcal{G}))}$, i.e. there is an $u \in K_1(\Lambda(\mathcal{G}))$ such that $\text{nr}(x) = \text{nr}(u) \cdot \Phi_S$. But then $\Theta_S := x \cdot u^{-1}$ has $\partial(\Theta_S) = \partial(x) = \mathcal{U}_S$ and $\text{Det}(\Theta_S) = L_{k,S}$, since $\text{nr}(\Theta_S) = \Phi_S$. We have shown that the following conjecture is equivalent to the EIMC without the uniqueness of Θ_S .

Conjecture 2.1. *The element $\Phi_S \in \zeta(Q(\mathcal{G}))^\times$ is a generator of $\text{Fitt}_{\Lambda(\mathcal{G})}(\mathcal{U}_S)$.*

The following theorem is due to Ritter and Weiss [RWa]:

Theorem 2.2. *Conjecture 2.1 is true provided that Iwasawa's μ -invariant vanishes.*

We also discuss Conjecture 2.1 within the framework of the theory of [CFKSV05], §3. For this, let

$$\pi : \mathcal{G} \rightarrow \text{Gl}_n(\mathfrak{o}_E)$$

be a continuous homomorphism, where \mathfrak{o}_E denotes the ring of integers of E and n is some integer greater or equal to 1. There is a ring homomorphism

$$\Phi_\pi : \Lambda(\mathcal{G}) \rightarrow M_{n \times n}(\Lambda^{\mathfrak{o}_E}(\Gamma_k)) \quad (11)$$

induced by the continuous group homomorphism

$$\begin{aligned} \mathcal{G} &\rightarrow (M_{n \times n}(\mathfrak{o}_E) \otimes_{\mathbb{Z}_p} \Lambda(\Gamma_k))^\times = \text{Gl}_n(\Lambda^{\mathfrak{o}_E}(\Gamma_k)) \\ \sigma &\mapsto \pi(\sigma) \otimes \bar{\sigma}, \end{aligned}$$

where $\bar{\sigma}$ denotes the image of σ in $\mathcal{G}/H = \Gamma_k$. By loc.cit., Lemma 3.3 the homomorphism (11) extends to a ring homomorphism

$$\Phi_\pi : Q(\mathcal{G}) \rightarrow M_{n \times n}(Q^E(\Gamma_k))$$

and this in turn induces a homomorphism

$$\Phi'_\pi : K_1(Q(\mathcal{G})) \rightarrow K_1(M_{n \times n}(Q^E(\Gamma_k))) = Q^E(\Gamma_k)^\times.$$

Let $\text{aug} : \Lambda^{\mathfrak{o}_E}(\Gamma_k) \rightarrow \mathfrak{o}_E$ be the augmentation map and put $\mathfrak{p} = \ker(\text{aug})$. Writing $\Lambda^{\mathfrak{o}_E}(\Gamma_k)_{\mathfrak{p}}$ for the localization of $\Lambda^{\mathfrak{o}_E}(\Gamma_k)$ at \mathfrak{p} , it is clear that aug naturally extends to a homomorphism $\text{aug} : \Lambda^{\mathfrak{o}_E}(\Gamma_k)_{\mathfrak{p}} \rightarrow E$. One defines an evaluation map

$$\begin{aligned} \phi : Q^E(\Gamma_k) &\rightarrow E \cup \{\infty\} \\ x &\mapsto \begin{cases} \text{aug}(x) & \text{if } x \in \Lambda^{\mathfrak{o}_E}(\Gamma_k)_{\mathfrak{p}} \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

If Θ is an element of $K_1(Q(\mathcal{G}))$, we define $\Theta(\pi)$ to be $\phi(\Phi'_\pi(\Theta))$. We need the following lemma.

Lemma 2.3. *If $\pi = \pi_\chi$ is a representation of \mathcal{G} with character χ and $r \in \mathbb{N}_0$, then*

$$\begin{array}{ccc} K_1(Q(\mathcal{G})) & \xrightarrow{\Phi'_{\pi_\chi \kappa^r}} & K_1(M_{n \times n}(Q^E(\Gamma_k))) \\ \downarrow \text{nr} & & \downarrow \simeq \text{nr} \\ \zeta(Q(\mathcal{G}))^\times & \xrightarrow{j_\chi^r} & Q^E(\Gamma_k)^\times \end{array}$$

commutes. In particular, we have $\text{nr} \circ \Phi'_{\pi_\chi} = \text{Det}(\)(\chi)$.

Proof. This is [Ni11b], lemma 2.3. □

Conjecture 2.1 now implies that there is an element $\Theta_S \in K_1(Q(\mathcal{G}))$ such that $\partial(\Theta_S) = \mathfrak{U}_S$ and for any $r \geq 1$ divisible by $p - 1$ we have

$$\Theta_S(\pi_\chi \kappa^r) = \phi(j_\chi^r(\Phi_S)) = L_S(1 - r, \chi).$$

The following result explains, why we may replace the complex (8) by the complex $R\text{Hom}(R\Gamma_{\text{ét}}(\text{Spec}(\mathfrak{o}_K[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$. Though it might be no surprise to experts, the author is not aware of any reference for this result.

Theorem 2.4. *With the notation as above, there is an isomorphism*

$$C_S(K/k) \simeq R\text{Hom}(R\Gamma_{\text{ét}}(\text{Spec}(\mathfrak{o}_K[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$$

in $\mathcal{D}(\Lambda(\mathcal{G}))$. In particular, there is an equality

$$\mathfrak{U}_S = -[R\text{Hom}(R\Gamma_{\text{ét}}(\text{Spec}(\mathfrak{o}_K[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)] \in K_0T(\Lambda(\mathcal{G})).$$

Proof. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is a direct limit of finite abelian groups of p -power order, we have an isomorphism with Galois cohomology

$$R\Gamma_{\text{ét}}(\text{Spec}(\mathfrak{o}_K[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p) \simeq R\Gamma(X_S, \mathbb{Q}_p/\mathbb{Z}_p). \quad (12)$$

We put $G_S := \text{Gal}(M_S/k)$. Now for any compact (right) $\Lambda(G_S)$ -modules M and discrete (left) $\Lambda(G_S)$ -module N (considered as complexes in degree zero), there is an isomorphism

$$M \otimes_{\Lambda(X_S)}^L R\text{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p) \simeq R\text{Hom}(R\text{Hom}_{\Lambda(X_S)}(M, N), \mathbb{Q}_p/\mathbb{Z}_p)$$

in $\mathcal{D}(\Lambda(\mathcal{G}))$ (cf. [NSW00], Cor. 5.2.9 or [We94], Th. 10.8.7). Noting that $R\text{Hom}_{\Lambda(X_S)}(\mathbb{Z}_p, N)$ identifies with $R\Gamma(X_S, N)$ we specialize $M = \mathbb{Z}_p$ and $N = \mathbb{Q}_p/\mathbb{Z}_p$ which yields an isomorphism

$$\mathbb{Z}_p \otimes_{\Lambda(X_S)}^L \mathbb{Z}_p \simeq R\text{Hom}(R\Gamma(X_S, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p) \quad (13)$$

in $\mathcal{D}(\Lambda(\mathcal{G}))$. Now we consider the short exact sequence

$$\Delta G_S \twoheadrightarrow \Lambda(G_S) \twoheadrightarrow \mathbb{Z}_p$$

of left $\Lambda(G_S)$ -modules, where the surjection is the augmentation map and $\Delta(G_S)$ denotes its kernel. We now apply $\mathbb{Z}_p \hat{\otimes}_{\Lambda(G_S)} -$ to this sequence and obtain the following exact homology sequence:

$$\begin{array}{ccccccc}
 H_1(X_S, \mathbb{Z}_p) \hookrightarrow & H_0(X_S, \Delta(G_S)) & \longrightarrow & H_0(X_S, \Lambda(G_S)) & \twoheadrightarrow & H_0(X_S, \mathbb{Z}_p) \\
 \downarrow \simeq & \parallel & & \parallel & & \parallel \\
 X_S \hookrightarrow & \mathbb{Z}_p \hat{\otimes}_{\Lambda(G_S)} \Delta(G_S) & \longrightarrow & \Lambda(\mathcal{G}) & \twoheadrightarrow & \mathbb{Z}_p
 \end{array}$$

In particular, we find that

$$H_i(X_S, \Delta(G_S)) = H_i(X_S, \Lambda(G_S)) = 0 \text{ for all } i > 0.$$

Hence the exact triangle

$$\mathbb{Z}_p \otimes_{\Lambda(G_S)}^L \Delta(G_S) \rightarrow \mathbb{Z}_p \otimes_{\Lambda(G_S)}^L \Lambda(G_S) \rightarrow \mathbb{Z}_p \otimes_{\Lambda(G_S)}^L \mathbb{Z}_p \rightarrow$$

implies that the complex

$$\mathbb{Z}_p \hat{\otimes}_{\Lambda(G_S)} \Delta(G_S) \rightarrow \Lambda(\mathcal{G}) \tag{14}$$

of the above homology sequence is isomorphic to $\mathbb{Z}_p \otimes_{\Lambda(G_S)}^L \mathbb{Z}_p$ in $\mathcal{D}(\Lambda(\mathcal{G}))$, and hence also to $R\mathrm{Hom}(R\Gamma_{\acute{e}t}(\mathrm{Spec}(\mathfrak{o}_K[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$ using (12) and (13). Let $\Delta(G_S, X_S)$ be the closure of the right $\Lambda(G_S)$ -ideal generated by $x - 1$, $x \in X_S$. Then

$$\mathbb{Z}_p \hat{\otimes}_{\Lambda(G_S)} \Delta(G_S) = \Delta(G_S) / \Delta(G_S, X_S) \Delta(G_S) =: Y_S$$

and the map in (14) is induced by mapping $g - 1$ to $\bar{g} - 1$, where \bar{g} denotes the image of an element $g \in G_S$ in \mathcal{G} under the canonical projection $G_S \rightarrow \mathcal{G}$. But the translation functor of Ritter and Weiss transfers the exact sequence $X_S \hookrightarrow G_S \rightarrow \mathcal{G}$ into

$$X_S \hookrightarrow Y_S \rightarrow \Delta(\mathcal{G}),$$

where the projection is induced in exactly the same way. Hence if we glue this sequence with the natural augmentation sequence $\Delta(\mathcal{G}) \hookrightarrow \Lambda(\mathcal{G}) \rightarrow \mathbb{Z}_p$, we obtain exactly the homology sequence above. The result now follows, once we observe that the complex (8) of Ritter and Weiss is achieved by a commutative diagram

$$\begin{array}{ccccccc}
 & & \Lambda(\mathcal{G}) & \xlongequal{\quad} & \Lambda(\mathcal{G}) & & \\
 & & \downarrow & & \downarrow & & \\
 X_S \hookrightarrow & Y_S & \longrightarrow & \Lambda(\mathcal{G}) & \twoheadrightarrow & \mathbb{Z}_p \\
 \parallel & \downarrow & & \downarrow & & \parallel \\
 X_S \hookrightarrow & C^{-1} & \longrightarrow & C^0 & \twoheadrightarrow & \mathbb{Z}_p
 \end{array}$$

□

3 Another reformulation of the equivariant Iwasawa main conjecture

We first recall the notion of abstract 1-motives and their basic properties as formulated in [GrP]. For an arbitrary abelian group J and a positive integer m , we denote by $J[m]$ the maximal m -torsion subgroup of J . For a prime p the p -adic Tate module of J is defined to be

$$T_p(J) := \varprojlim_n J[p^n],$$

where the projective limit is taken with respect to multiplication by p . An abelian, divisible group J is of finite local corank if for any prime p there is an integer $r_p(J)$ and a \mathbb{Z}_p -isomorphism

$$J[p^\infty] \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{r_p(J)},$$

where $J[p^\infty] = \bigcup_n J[p^n]$.

Definition 3.1. *An abstract 1-motive $\mathcal{M} := [L \xrightarrow{\delta} J]$ consists of the following data.*

- a free \mathbb{Z} -module L of finite rank;
- an abelian, divisible group J of finite local corank;
- a group morphism $\delta : L \rightarrow J$.

Now let $n \geq 1$ be an integer and consider the fiber product $J \times_n^n L$ with respect to the map δ and the multiplication by n map $J \xrightarrow{n} J$.

Definition 3.2. *The group $\mathcal{M}[n] := (J \times_n^n L) \otimes \mathbb{Z}/n\mathbb{Z}$ is called the group of n -torsion points of \mathcal{M} . Moreover, if $n \mid m$, there are canonical surjective multiplication by m/n maps $\mathcal{M}[m] \rightarrow \mathcal{M}[n]$ and we define the p -adic Tate module of \mathcal{M} to be*

$$T_p(\mathcal{M}) := \varprojlim_n \mathcal{M}[p^n].$$

In this way, we obtain for every prime p an exact sequence of free \mathbb{Z}_p -modules

$$T_p(J) \twoheadrightarrow T_p(\mathcal{M}) \twoheadrightarrow \mathbb{Z}_p \otimes L. \quad (15)$$

We are now going to define an Iwasawa theoretic abstract 1-motive. Again, we follow the treatment in [GrP]. For this, let p be an odd prime and \mathcal{K} be the cyclotomic \mathbb{Z}_p -extension of a number field K and let K_n denote its n -th layer, $n \in \mathbb{N}$. We denote the set of p -adic places of \mathcal{K} by \mathcal{S}_p and fix two finite sets \mathcal{S} and \mathcal{T} of places of \mathcal{K} such that $\mathcal{T} \cap (\mathcal{S} \cup \mathcal{S}_p) = \emptyset$. The divisor group of \mathcal{K} is given by

$$\mathrm{Div}_{\mathcal{K}} := \bigoplus_v \Gamma_v \cdot v,$$

where the direct sum runs over all finite primes of \mathcal{K} and $\Gamma_v = \mathbb{Z}$ (resp. $\Gamma_v = \mathbb{Z}[\frac{1}{p}]$) if $v \notin \mathcal{S}_p$ (resp. $v \in \mathcal{S}_p$). Note that in both cases Γ_v identifies with the value group of an appropriate chosen valuation ord_v corresponding to v . We let

$$\mathrm{Div}_{\mathcal{K}, \mathcal{T}} := \bigoplus_{v \notin \mathcal{T}} \Gamma_v \cdot v, \quad \mathcal{K}_{\mathcal{T}}^\times := \{x \in \mathcal{K}^\times \mid \mathrm{ord}_v(x-1) > 0 \ \forall v \in \mathcal{T}\}.$$

The usual divisor map induces a group morphism

$$\mathrm{div}_{\mathcal{K}} : \mathcal{K}_{\mathcal{T}}^{\times} \rightarrow \mathrm{Div}_{\mathcal{K}, \mathcal{T}}, \quad x \mapsto \sum_v \mathrm{ord}_v(x)$$

and we define a generalized ideal class group by

$$C_{\mathcal{K}, \mathcal{T}} := \frac{\mathrm{Div}_{\mathcal{K}, \mathcal{T}}}{\mathrm{div}_{\mathcal{K}}(\mathcal{K}_{\mathcal{T}}^{\times})}.$$

Then the Iwasawa μ -invariant vanishes if and only if $\mathcal{A}_{\mathcal{K}, \mathcal{T}} := C_{\mathcal{K}, \mathcal{T}} \otimes \mathbb{Z}_p$ is divisible of finite local corank. Note that the vanishing of μ only depends upon \mathcal{K} and p , but not on the number field K . We will henceforth assume that $\mu = 0$ and associate to the data $(\mathcal{K}, \mathcal{S}, \mathcal{T})$ the abstract 1-motive

$$\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}} := [\mathrm{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \xrightarrow{\delta} \mathcal{A}_{\mathcal{K}, \mathcal{T}}],$$

where $\mathrm{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p)$ is the group of divisors of \mathcal{K} supported on $\mathcal{S} \setminus \mathcal{S}_p$ and δ is induced by the usual divisor class map. In particular, the exact sequence (15) now reads

$$T_p(\mathcal{A}_{\mathcal{K}, \mathcal{T}}) \hookrightarrow T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}}) \twoheadrightarrow \mathrm{Div}_{\mathcal{K}}(\mathcal{S} \setminus \mathcal{S}_p) \otimes \mathbb{Z}_p. \quad (16)$$

Now assume that \mathcal{K} is a Galois extension of a totally real number field k , and that \mathcal{K} is the cyclotomic \mathbb{Z}_p -extension of a CM-number field K . Let $\mathcal{G} = \mathrm{Gal}(\mathcal{K}/k)$ and let $j \in \mathcal{G}$ denote the unique central automorphism in \mathcal{G} which is induced by complex conjugation. Let \mathcal{K}^+ be the maximal real subfield of \mathcal{K} and $\mathcal{G}^+ = \mathcal{G}/\langle j \rangle$ its Galois group over k . We fix two finite, non-empty, disjoint sets S and T of places in k , such that S contains $S_{\mathrm{ram}}(\mathcal{K}/k) \cup S_{\infty}$. Let \mathcal{S} and \mathcal{T} be the sets of finite primes in \mathcal{K} sitting above primes in S and T , respectively. Then by [GrP], Th. 4.6 the Tate module $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-$ is a \mathbb{Z}_p -free torsion $\Lambda(\mathcal{G})_- := \Lambda(\mathcal{G})/(1+j)$ -module of projective dimension at most 1. For $v \in T$ we put

$$\xi_v := \mathrm{nr}(\kappa(\phi_w) - \phi_w),$$

where $\phi_w \in \mathcal{G}$ denotes the Frobenius at a chosen prime w in \mathcal{K} above v . Let $x \mapsto \hat{x}$ be the automorphism on $\Lambda(\mathcal{G})$ induced by $g \mapsto \kappa(g)g^{-1}$ for $g \in \mathcal{G}$. We put

$$\Psi_{S, T} = \Psi_{S, T}(\mathcal{K}/k) := \prod_{v \in T} \xi_v \cdot \hat{\Phi}_S.$$

For any positive integer n , we denote by ζ_n a primitive n -th root of unity. We are now ready to prove the following variant of the equivariant Iwasawa main conjecture which generalizes [GrP], Th. 5.6 to the non-abelian situation.

Theorem 3.3. *Let $(\mathcal{K}/k, S, T, p)$ be as above. If Iwasawa's μ -invariant vanishes, then $\Psi_{S, T}$ is a generator of $\mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-)$.*

Proof. We first remark that $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-$ admits a quadratic presentation by [Ni10], Lemma 6.2. Hence $\mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-)$ is well defined.

If M is an Iwasawa torsion module, we write $\alpha(M)$ for the Iwasawa adjoint of M . We will need the following proposition.

Proposition 3.4. *Assume that C is a finitely generated R -torsion $\Lambda(\mathcal{G}^+)$ -module of projective dimension at most 1 which has no nontrivial finite submodule and that Φ is a generator of $\text{Fitt}_{\Lambda(\mathcal{G}^+)}(C)$; then $\text{Fitt}_{\Lambda(\mathcal{G})}(\alpha(C)(1))$ is generated by $\dot{\Phi}e^- + e^+$, where $e^\pm = \frac{1 \pm j}{2}$.*

Proof. Let $\psi : \Lambda(\mathcal{G}^+)^m \rightarrow \Lambda(\mathcal{G}^+)^m$ be a quadratic presentation of C such that $\text{nr}(\psi) = \Phi$. By [Ni10], Prop. 6.3 (i) resp. its proof it follows that $\psi^{T,\sharp}$ is a finite presentation of $\alpha(C)$ and $\text{nr}(\psi^{T,\sharp}) = \Phi^\sharp$ is a generator of $\text{Fitt}_{\Lambda(\mathcal{G}^+)}(\alpha(C))$, where ψ^T denotes the transpose of ψ . Now $\Lambda(\mathcal{G}^+) \simeq \Lambda(\mathcal{G})e_+$ and the involution $g \mapsto \kappa(g^{-1})g$ induces an isomorphism between the first Tate twist of $\Lambda(\mathcal{G}^+)$ and $\Lambda(\mathcal{G})e_-$. We obtain a quadratic presentation $\dot{\psi}^T : (\Lambda(\mathcal{G})e_-)^m \rightarrow (\Lambda(\mathcal{G})e_-)^m$ of $\alpha(C)(1)$ regarded as $\Lambda(\mathcal{G})e_-$ -module. Since $\text{nr}(\dot{\psi}^T) = \dot{\Phi}$ and $\alpha(C)(1)$ is trivial on plus parts, we are done. \square

Returning to the proof of Theorem 3.3, we let

$$\Delta_{\mathcal{K},\mathcal{T}} := \bigoplus_{w \in \mathcal{T}} \kappa(w)^\times,$$

where $\kappa(w)$ denotes the residue field at w . We first assume that ζ_p lies in \mathcal{K} . Then there is an exact sequence of $\Lambda(\mathcal{G})$ -modules (cf. [GrP], remark 3.10)

$$\mathbb{Z}_p(1) \mapsto T_p(\Delta_{\mathcal{K},\mathcal{T}})^- \rightarrow T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^- \rightarrow T_p(\mathcal{M}_{\mathcal{S},\emptyset}^\mathcal{K})^-. \quad (17)$$

Applying $\alpha(_)(1)$ to this sequence yields a complex

$$\tilde{C}(\mathcal{K}^+/k) : \alpha(T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^-)(1) \rightarrow \alpha(T_p(\Delta_{\mathcal{K},\mathcal{T}})^-)(1)$$

whose non-trivial cohomology groups are given by $H^{-1}(\tilde{C}(\mathcal{K}^+/k)) = X_{\mathcal{S}}^+$ and $H^0(\tilde{C}(\mathcal{K}^+/k)) = \mathbb{Z}_p$; here, we have used [GrP], Lemma 3.9. Most likely, this complex is isomorphic in $\mathcal{D}(\Lambda(\mathcal{G}^+))$ to the canonical complex $C(\mathcal{K}^+/k)$, but we will not need this. Since we assume that $\mu = 0$, we have

$$\text{Fitt}_{\mathcal{R}}(\mathcal{R} \otimes_{\Lambda(\mathcal{G}^+)}^L \tilde{C}(\mathcal{K}^+/k)) = \text{Fitt}_{\mathcal{R}}(\mathcal{R} \otimes_{\Lambda(\mathcal{G}^+)}^L C(\mathcal{K}^+/k)),$$

where \mathcal{R} is either $\mathbb{Q}_p \otimes \Lambda(\mathcal{G}^+)$ or $\Lambda_{(p)}(\mathcal{G}^+)$. But then Proposition 1.2 implies that the two Fitting invariants also agree for $\mathcal{R} = \Lambda(\mathcal{G}^+)$. Hence

$$\begin{aligned} \text{Fitt}_{\Lambda(\mathcal{G}^+)}(\alpha(T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^-)(1)) &= \text{Fitt}_{\Lambda(\mathcal{G}^+)}(\alpha(T_p(\Delta_{\mathcal{K},\mathcal{T}})^-)(1)) \cdot \text{Fitt}_{\Lambda(\mathcal{G}^+)}(\tilde{C}(\mathcal{K}^+/k))^{-1} \\ &= \text{Fitt}_{\Lambda(\mathcal{G}^+)}(\alpha(T_p(\Delta_{\mathcal{K},\mathcal{T}})^-)(1)) \cdot \text{Fitt}_{\Lambda(\mathcal{G}^+)}(C(\mathcal{K}^+/k))^{-1} \\ &= \prod_{v \in \mathcal{T}} \dot{\xi}_v \cdot \Phi_{\mathcal{S}} \\ &= \dot{\Psi}_{\mathcal{S},\mathcal{T}}, \end{aligned}$$

where we have used Theorem 2.2 and Proposition 3.4. The result follows by applying Proposition 3.4 again. If $\zeta_p \notin \mathcal{K}$, let $\tilde{\mathcal{K}} := \mathcal{K}(\zeta_p)$ and let $\tilde{\mathcal{S}}$ (resp. $\tilde{\mathcal{T}}$) be the set of places of $\tilde{\mathcal{K}}$ above those in \mathcal{S} (resp. \mathcal{T}). Let $\tilde{\mathcal{G}}$ denote the Galois group $\text{Gal}(\tilde{\mathcal{K}}/k)$. By [GrP], Cor. 4.8 we have an isomorphism of $\Lambda(\mathcal{G})_-$ -modules

$$T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^\mathcal{K})^- \simeq \Lambda(\mathcal{G})_- \otimes_{\Lambda(\tilde{\mathcal{G}})_-} T_p(\mathcal{M}_{\tilde{\mathcal{S}},\tilde{\mathcal{T}}}^{\tilde{\mathcal{K}}})^-.$$

If π denotes the canonical epimorphism $\Lambda(\tilde{\mathcal{G}})_- \rightarrow \Lambda(\mathcal{G})_-$, the natural behavior of Fitting invariants thus gives an equality

$$\mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(T_p(\mathcal{M}_{S,T}^{\mathcal{K}})^-) = \pi \left(\mathrm{Fitt}_{\Lambda(\tilde{\mathcal{G}})_-}(T_p(M_{\tilde{S},\tilde{T}}^{\tilde{\mathcal{K}}})^-) \right).$$

This suffices, since the natural behavior of p -adic L -functions gives

$$\pi(\Psi_{S,T}(\tilde{\mathcal{K}}/k) = \Psi_{S,T}(\mathcal{K}/k).$$

□

Remark 3.5. *Note that the argument can be reversed to show that $\Psi_{S,T}$ is a generator of $\mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(T_p(\mathcal{M}_{S,T}^{\mathcal{K}})^-)$ if and only if the EIMC (Conjecture 2.1) holds (provided that $\mu = 0$).*

4 The non-abelian Brumer-Stark conjecture

Let K/k be a Galois CM-extension with Galois group G . Let S and T be two finite sets of places of k such that

- S contains all the infinite places of k and all the places which ramify in K/k , i.e. $S \supset S_{\mathrm{ram}} \cup S_{\infty}$.
- $S \cap T = \emptyset$.
- E_S^T is torsionfree.

We refer to the above hypotheses as $\mathit{Hyp}(S, T)$. We put $\Lambda = \mathbb{Z}G$ and choose a maximal order Λ' containing Λ . For a fixed set S we define \mathfrak{A}_S to be the $\zeta(\Lambda)$ -submodule of $\zeta(\Lambda')$ generated by the elements $\delta_T(0)$, where T runs through the finite sets of places of k such that $\mathit{Hyp}(S, T)$ is satisfied. The following conjecture has been formulated in [Nia] and is a non-abelian generalization of Brumer's conjecture.

Conjecture 4.1 ($B(K/k, S)$). *Let S be a finite set of places of k containing $S_{\mathrm{ram}} \cup S_{\infty}$. Then $\mathfrak{A}_S \theta_S \subset \mathcal{I}(G)$ and for each $x \in \mathcal{H}(G)$ we have*

$$x \cdot \mathfrak{A}_S \theta_S \subset \mathrm{Ann}_{\Lambda}(\mathrm{cl}_K).$$

Remark 4.2. • *If G is abelian, [Ta84], Lemma 1.1 p. 82 implies that the module \mathfrak{A}_S equals $\mathrm{Ann}_{\mathbb{Z}G}(\mu_K)$. In this case the inclusion $\mathfrak{A}_S \theta_S \subset \mathcal{I}(G) = \Lambda = \mathbb{Z}G$ holds by (1) and, since $\mathcal{H}(G) = \Lambda$ in this case, Conjecture 4.1 recovers Brumer's conjecture.*

- *Replacing the class group cl_K by its p -parts $\mathrm{cl}_K(p)$ for each rational prime p , Conjecture $B(K/k, S)$ naturally decomposes into local conjectures $B(K/k, S, p)$. Note that it is possible to replace $\mathcal{H}(G)$ by $\mathcal{H}_p(G)$ by [Nia], Lemma 1.4.*
- *Burns [Bua] has also formulated a conjecture which generalizes many refined Stark conjectures to the non-abelian situation. In particular, it implies our generalization of Brumer's conjecture (cf. loc.cit., Prop. 3.5.1).*

For $\alpha \in K^\times$ we define

$$S_\alpha := \{v \text{ finite place of } k : \mathfrak{p}_v | N_{K/k}(\alpha)\}$$

and we call α an *anti-unit* if $\alpha^{1+j} = 1$. Let $\omega_K := \text{nr}(|\mu_K|)$. The following is a non-abelian generalization of the Brumer-Stark conjecture (cf. [Nia], Conj. 2.6).

Conjecture 4.3 (*BS(K/k, S)*). *Let S be a finite set of places of k containing $S_{\text{ram}} \cup S_\infty$. Then $\omega_K \cdot \theta_S \in \mathcal{I}(G)$ and for each $x \in \mathcal{H}(G)$ and each fractional ideal \mathfrak{a} of K, there is an anti-unit $\alpha = \alpha(x, \mathfrak{a}, S) \in K^\times$ such that*

$$\mathfrak{a}^{x \cdot \omega_K \cdot \theta_S} = (\alpha)$$

and for each finite set T of primes of k such that $\text{Hyp}(S \cup S_\alpha, T)$ is satisfied there is an $\alpha_T \in E_{S_\alpha}^T$ such that

$$\alpha^{z \cdot \delta_T(0)} = \alpha_T^{z \cdot \omega_K} \quad (18)$$

for each $z \in \mathcal{H}(G)$.

Remark 4.4. • If G is abelian, we have $\mathcal{I}(G) = \mathcal{H}(G) = \mathbb{Z}G$ and $\omega_K = |\mu_K|$. Hence it suffices to treat the case $x = z = 1$. Then [Ta84], Prop. 1.2, p. 83 states that the condition (18) on the anti-unit α is equivalent to the assertion that the extension $K(\alpha^{1/\omega_K})/k$ is abelian.

• As above, we obtain local conjectures $BS(K/k, S, p)$ for each prime p .

For any $\mathbb{Z}_p G$ -module M , we denote the Pontryagin dual $\text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of M by M^\vee which is endowed with the contravariant G -action $(gf)(m) = f(g^{-1}m)$ for $f \in M^\vee$, $g \in G$ and $m \in M$. As a consequence of Theorem 3.3 we now prove the following non-abelian generalization of [GrP], Th. 6.5.

Theorem 4.5. *Let K/k be a Galois CM-extension with Galois group G and p an odd prime. Fix two finite sets S and T of primes of k such that $\text{Hyp}(S, T)$ is satisfied. If Iwasawa's μ -invariant vanishes, then*

$$(\theta_S^T)^\sharp \in \text{Fitt}_{\mathbb{Z}_p G_-}^{\max}((A_{K,T}^-)^\vee)$$

whenever $S_p \subset S$.

Corollary 4.6. *Let K/k be a Galois CM-extension and p an odd prime. Then $B(K/k, S, p)$ and $BS(K/k, S, p)$ hold whenever $S_p \subset S$ and Iwasawa's μ -invariant vanishes.*

Proof. Since $BS(K/k, S, p)$ implies $B(K/k, S, p)$ by [Nia], Lemma 2.9, we only have to treat the case of the Brumer-Stark conjecture. But $BS(K/k, S, p)$ is implied by the so-called strong Brumer-Stark property by [Nia], Prop. 3.8. Recall that this property is fulfilled if $\theta_S^T \in \text{Fitt}_{\mathbb{Z}_p G_-}^{\max}(A_{K,T}^-)$. But in fact the proof of [Nia], Prop. 3.8 carries over unchanged once we observe that

$$\text{Ann}_{\mathbb{Z}_p G_-}(M) = \text{Ann}_{\mathbb{Z}_p G_-}(M^\vee)^\sharp$$

for any finite $\mathbb{Z}_p G_-$ -module M . □

Proof of Theorem 4.5. Let \mathcal{K} be the cyclotomic \mathbb{Z}_p -extension of K and $\mathcal{G} = \text{Gal}(\mathcal{K}/k)$. There is an isomorphism of $\Lambda(\mathcal{G})_-$ -modules

$$\mathcal{A}_{\mathcal{K},\mathcal{T}}^- \simeq \varinjlim_n \mathcal{A}_{K_n, T_n}^-$$

where $T_n := T(K_n)$ and the transition maps in the inductive limit are injective (cf. [GrP], Lemma 2.9). Hence we may choose a sufficiently large integer n such that $\mathcal{A}_{\mathcal{K},\mathcal{T}}^- \subset \mathcal{A}_{\mathcal{K},\mathcal{T}}^-[p^n]$ which induces a natural epimorphism

$$\mathcal{A}_{\mathcal{K},\mathcal{T}}^-[p^n]^\vee \twoheadrightarrow (\mathcal{A}_{\mathcal{K},\mathcal{T}}^-)^\vee. \quad (19)$$

For any $\Lambda(\mathcal{G})$ -module M let $M^* := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ endowed with the contravariant \mathcal{G} -action. We have an isomorphism $\mathcal{A}[p^m]^\vee \simeq T_p(\mathcal{A})^* \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m\mathbb{Z}$ of $\Lambda(\mathcal{G})$ -modules for any positive integer m and any \mathbb{Z}_p -torsion, divisible $\Lambda(\mathcal{G})$ -module \mathcal{A} of finite local corank (cf. [GrP], Lemma 6.6; the assumption on \mathcal{G} being abelian is not necessary). This together with (19) and the dual of sequence (16) leads to the following epimorphisms

$$(T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})^-)^* \twoheadrightarrow T_p(\mathcal{A}_{\mathcal{K},\mathcal{T}}^-)^* \twoheadrightarrow \mathcal{A}_{\mathcal{K},\mathcal{T}}^-[p^n]^\vee \twoheadrightarrow (\mathcal{A}_{\mathcal{K},\mathcal{T}}^-)^\vee.$$

Since $\Gamma = \text{Gal}(\mathcal{K}/K)$ acts trivially on $(\mathcal{A}_{\mathcal{K},\mathcal{T}}^-)^\vee$, we obtain an epimorphism of $\mathbb{Z}_p G_-$ -modules

$$(T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})^-)_\Gamma^* \twoheadrightarrow (\mathcal{A}_{\mathcal{K},\mathcal{T}}^-)^\vee.$$

But $(\Psi_S^T)^\sharp$ is a generator of the Fitting invariant of $(T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})^-)_\Gamma^*$ by Theorem 3.3 and [Ni10], Prop. 6.3 (i). Consequently,

$$(\theta_S^T)^\sharp = (\Psi_S^T)^\sharp(0) \in \text{Fitt}_{\mathbb{Z}_p G_-}^{\max}((T_p(\mathcal{M}_{\mathcal{S},\mathcal{T}}^{\mathcal{K}})^-)_\Gamma^*) \subset \text{Fitt}_{\mathbb{Z}_p G_-}^{\max}((\mathcal{A}_{\mathcal{K},\mathcal{T}}^-)^\vee)$$

by [Ni10], Th. 6.4 and Prop. 3.5 (i). \square

5 The non-abelian Coates-Sinnott conjecture

In this section, we discuss an analogue of the strong Brumer-Stark property for higher étale cohomology. We once more recall that a Galois extension K/k with Galois group G fulfills this property at an odd prime p if

$$\theta_S^T \in \text{Fitt}_{\mathbb{Z}_p G}^{\max}(A_{K,T}) \quad (20)$$

for any two finite sets S and T of places of k such that $\text{Hyp}(S, T)$ is satisfied. Note that this does not hold in general, even if G is abelian, as follows from the results in [GK08]. We will see that its higher analogue should behave much better.

Let K/k be a Galois extension of number fields with Galois group G and p an odd prime. We fix an integer $n > 1$ and two finite non-empty sets S and T of places of k such that S contains $S_{\text{ram}} \cup S_\infty$ and $S \cap T = \emptyset$. We also assume that no p -adic place of k lies in T . For a finite place w of K , we write $K(w)$ for the residue field of K at w . To these data we associate the complex

$$C_S^T(K/k, \mathbb{Z}_p(n)) = C_S^T(\mathbb{Z}_p(n)) := \text{cone}(R\Gamma(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n)) \rightarrow \bigoplus_{w \in T(K)} R\Gamma(K(w), \mathbb{Z}_p(n)))[-1]).$$

We now state the following conjecture.

Conjecture 5.1 ($SCS(K/k, S, T, p, n)$). *Let the data $(K/k, S, T, p, n)$ be as above. Then*

$$\theta_S^T(1-n) \in \text{Fitt}_{\mathbb{Z}_p G}^{\max}(H^2(C_S^T(\mathbb{Z}_p(n)))).$$

We will refer to this conjecture as the (non-abelian) strong Coates-Sinnott conjecture. To see the analogy to the strong Brumer-Stark property (20), we assume that $Hyp(S, T)$ is satisfied and look at the associated cohomology sequence in the case $n = 1$:

$$\begin{aligned} H^1(C_S^T(K/k, \mathbb{Z}_p(1))) &\hookrightarrow E_{S \cup S_p} \otimes \mathbb{Z}_p \rightarrow (\mathfrak{o}_{S \cup S_p} / \mathfrak{M}_T)^\times \rightarrow \\ &H^2(C_S^T(K/k, \mathbb{Z}_p(1))) \rightarrow \text{cl}_{S \cup S_p} \otimes \mathbb{Z}_p. \end{aligned}$$

In fact, this sequence coincides with sequence (7) and we have canonical identifications

$$H^1(C_S^T(K/k, \mathbb{Z}_p(1))) \simeq E_{S \cup S_p}^T \otimes \mathbb{Z}_p, \quad H^2(C_S^T(K/k, \mathbb{Z}_p(1))) \simeq \text{cl}_{S \cup S_p, T, K} \otimes \mathbb{Z}_p.$$

We now study Conjecture 5.1 in some detail.

Lemma 5.2. *The complex $C_S^T(\mathbb{Z}_p(n))$ is acyclic outside degrees 1 and 2. Moreover, $H^1(C_S^T(\mathbb{Z}_p(n)))$ is torsion-free and $H^2(C_S^T(\mathbb{Z}_p(n)))$ is finite.*

Proof. As p is odd, the complex $R\Gamma(\mathfrak{o}_{K, S}[\frac{1}{p}], \mathbb{Z}_p(n))$ is acyclic outside degrees 1 and 2. For any $w \in T(K)$, the cohomology of $R\Gamma(K(w), \mathbb{Z}_p(n))$ is concentrated in degree 1. Hence the associated cohomology sequence is

$$0 \rightarrow H^1(C_S^T(K/k, \mathbb{Z}_p(n))) \rightarrow H_{\text{ét}}^1(\mathfrak{o}_{K, S}[\frac{1}{p}], \mathbb{Z}_p(n)) \rightarrow \quad (21)$$

$$\bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n)) \rightarrow H^2(C_S^T(K/k, \mathbb{Z}_p(n))) \rightarrow H_{\text{ét}}^2(\mathfrak{o}_{K, S}[\frac{1}{p}], \mathbb{Z}_p(n)) \rightarrow 0.$$

We see that $C_S^T(\mathbb{Z}_p(n))$ is acyclic outside degrees 1 and 2 and that $H^2(C_S^T(\mathbb{Z}_p(n)))$ is finite, as $H_{\text{ét}}^2(\mathfrak{o}_{K, S}[\frac{1}{p}], \mathbb{Z}_p(n))$ and $H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))$, $w \in T(K)$ are. Moreover there are isomorphisms

$$H_{\text{ét}}^1(\mathfrak{o}_{K, S}[\frac{1}{p}], \mathbb{Z}_p(n))_{\text{tor}} \simeq (\mathbb{Q}_p/\mathbb{Z}_p(n))^{G_K}, \quad H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n)) \simeq (\mathbb{Q}_p/\mathbb{Z}_p(n))^{G_{K, w}},$$

where G_K and $G_{K, w}$ denote the absolute Galois group of K and the absolute decomposition group at w , respectively; note that the inertia subgroup acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p(n)$ for all $w \in T(K)$. Since T is not empty, the natural map

$$(\mathbb{Q}_p/\mathbb{Z}_p(n))^{G_K} \rightarrow \bigoplus_{w \in T(K)} (\mathbb{Q}_p/\mathbb{Z}_p(n))^{G_{K, w}}$$

is injective, i.e. $H_{\text{ét}}^1(\mathfrak{o}_{K, S}[\frac{1}{p}], \mathbb{Z}_p(n))_{\text{tor}}$ injects into $\bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))$. This shows that $H^1(C_S^T(\mathbb{Z}_p(n)))$ is in fact torsion-free. \square

The relation to the classical Coates-Sinnott conjecture is the following.

Proposition 5.3. *Assume that $SCS(K/k, S, T, p, n)$ holds. Then*

$$\mathcal{H}_p(G) \cdot \theta_S^T(1-n) \subset \text{Ann}_{\mathbb{Z}_p G}(H_{\text{ét}}^2(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))).$$

In particular, if G is abelian and $SCS(K/k, S, T, p, n)$ holds for all admissible sets T , then

$$\text{Ann}_{\mathbb{Z}_p G}(H_{\text{ét}}^1(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))_{\text{tor}}) \cdot \theta_S(1-n) \subset \text{Ann}_{\mathbb{Z}_p G}(H_{\text{ét}}^2(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))).$$

In fact, it suffices to consider sets T which only consist of one place.

Proof. The surjection $H^2(C_S^T(K/k, \mathbb{Z}_p(n))) \rightarrow H_{\text{ét}}^2(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))$ of sequence (21) implies an inclusion of Fitting invariants

$$\text{Fitt}_{\mathbb{Z}_p G}^{\max}(H^2(C_S^T(\mathbb{Z}_p(n)))) \subset \text{Fitt}_{\mathbb{Z}_p G}^{\max}(H_{\text{ét}}^2(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))).$$

The first assertion now follows from Theorem 1.1. For the second assertion, we observe that, for abelian G , one has $\mathcal{H}_p(G) = \mathbb{Z}_p G$ and $\text{Ann}_{\mathbb{Z}_p G}(H_{\text{ét}}^1(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))_{\text{tor}})$ is generated by the elements $\delta_{\{v\}}(1-n)$, where v runs through the finite places of k not above p , $v \notin S_{\text{ram}}$ by a Lemma of Coates [Co77]. \square

Lemma 5.4. *The finite G -module $\bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))$ is cohomologically trivial and $\delta_T(1-n)$ is a generator of $\text{Fitt}_{\mathbb{Z}_p G}(\bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n)))$.*

Proof. Let $v \in T$ and $w \in T(K)$ be a place above v . The complex $R\Gamma(K(w), \mathbb{Z}_p(n))$ is $\mathbb{Z}_p G_w$ -perfect and its cohomology is concentrated in degree 1. Hence $H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))$ is c.t. as G_w -module and hence $\bigoplus_{w|v} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n)) = \text{ind}_{G_w}^G H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))$ is c.t. as G -module. More precisely, $H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))$ is the cokernel of the injective map

$$\mathbb{Z}_p G_w \rightarrow \mathbb{Z}_p G_w, \quad x \mapsto x \cdot (1 - N(v)^n \phi_w^{-1}).$$

Hence, its Fitting invariant is generated by $\text{nr}(1 - N(v)^n \phi_w^{-1})$. Since $\delta_T(1-n) = \prod_{v \in T} \text{nr}(1 - N(v)^n \phi_w^{-1})$, we are done. \square

Proposition 5.5. *Let U be a normal subgroup of G and put $F := K^U$ and $\bar{G} = G/U$. Then we have an isomorphism*

$$\mathbb{Z}_p \bar{G} \otimes_{\mathbb{Z}_p G}^L C_S^T(K/k, \mathbb{Z}_p(n)) \simeq C_S^T(F/k, \mathbb{Z}_p(n))$$

in $\mathcal{D}(\mathbb{Z}_p \bar{G})$. In particular, $SCS(K/k, S, T, p, n)$ implies $SCS(F/k, S, T, p, n)$.

Proof. The first assertion follows, since the corresponding statement holds for the complexes $R\Gamma(\mathfrak{o}_{K,S}[\frac{1}{p}], \mathbb{Z}_p(n))$ and $\bigoplus_{w \in T(K)} R\Gamma(K(w), \mathbb{Z}_p(n))$ (the latter follows easily from Lemma 5.4 above). In particular we have canonical isomorphisms

$$\begin{aligned} H^1(C_S^T(K/k, \mathbb{Z}_p(n)))^U &\simeq H^1(C_S^T(F/k, \mathbb{Z}_p(n))) \\ H^2(C_S^T(K/k, \mathbb{Z}_p(n)))_U &\simeq H^2(C_S^T(F/k, \mathbb{Z}_p(n))). \end{aligned} \tag{22}$$

Since $\theta_S^T(K/k, 1-n)$ is mapped to $\theta_S^T(F/k, 1-n)$ by the canonical projection $\mathbb{Z}_p G \rightarrow \mathbb{Z}_p \overline{G}$, the natural behavior of Fitting invariants implies that if $\theta_S^T(K/k, 1-n)$ lies in $\text{Fitt}_{\mathbb{Z}_p \overline{G}}^{\max}(H^2(C_S^T(K/k, \mathbb{Z}_p(n))))$, then

$$\theta_S^T(F/k, 1-n) \in \text{Fitt}_{\mathbb{Z}_p \overline{G}}^{\max}(H^2(C_S^T(K/k, \mathbb{Z}_p(n))))_U = \text{Fitt}_{\mathbb{Z}_p \overline{G}}^{\max}(H^2(C_S^T(F/k, \mathbb{Z}_p(n))))$$

as desired. \square

Corollary 5.6. *It suffices to prove Conjecture 5.1 under the additional assumption that k is totally real and K is totally imaginary.*

Proof. Since $\theta_S^T(1-n) = 0$ if k is not totally real, Conjecture 5.1 holds trivially in this case. So we may assume that k is totally real. Moreover, Proposition 5.5 implies that we may assume that K is totally imaginary. \square

We also provide another useful reduction step.

Lemma 5.7. *It suffices to prove Conjecture 5.1 under the assumption $S_p \subset S$.*

Proof. We have an equality

$$\theta_{S \cup S_p}^T(1-n) = \theta_S^T(1-n) \cdot \prod_{v \in S_p, v \notin S} \text{nr}(1 - N(v)^{1-n} \phi_w^{-1}),$$

where w is a place of K above v . But the product on the righthand side lies in $\text{nr}(\mathbb{Z}_p G^\times) = \text{nr}(K_1(\mathbb{Z}_p G))$ such that the claim follows by $\text{nr}(\mathbb{Z}_p G)$ -equivalence. \square

We will henceforth assume that k is totally real, K is totally imaginary and $S_p \subset S$. For any $w \in S_\infty(K)$, the decomposition group G_w is cyclic of order two and we denote its generator by j_w . Consider the normal subgroup

$$U := \langle j_w \cdot j_{w'} \mid w, w' \in S_\infty(L) \rangle$$

of G . The fixed field $K^{CM} := K^U$ is the maximal CM-subfield of K and is Galois over k with group $\overline{G} := G/U$. We already know by Proposition 5.5 that $SCS(K/k, S, T, p, n)$ implies $SCS(K^{CM}/k, S, T, p, n)$. We now show that under a mild hypothesis the converse is also true.

Lemma 5.8. *Assume that U is a 2-group. Then*

$$SCS(K/k, S, T, p, n) \iff SCS(K^{CM}/k, S, T, p, n).$$

Proof. If U is a 2-group, then the idempotent $\varepsilon_U := |U|^{-1} \sum_{u \in U} u$ lies in $\mathbb{Z}_p G$. Hence we have decompositions

$$\mathbb{Z}_p G = \varepsilon_U \mathbb{Z}_p G \oplus (1 - \varepsilon_U) \mathbb{Z}_p G,$$

$$\begin{aligned} \text{Fitt}_{\mathbb{Z}_p G}^{\max}(H^2(C_S^T(\mathbb{Z}_p(n)))) &= \text{Fitt}_{\varepsilon_U \mathbb{Z}_p G}^{\max}(\varepsilon_U H^2(C_S^T(\mathbb{Z}_p(n)))) \oplus \\ &\quad \text{Fitt}_{(1-\varepsilon_U) \mathbb{Z}_p G}^{\max}((1 - \varepsilon_U) H^2(C_S^T(\mathbb{Z}_p(n)))). \end{aligned}$$

The first term of the latter decomposition naturally identifies with $\text{Fitt}_{\mathbb{Z}_p \overline{G}}^{\max}(H^2(C_S^T(\mathbb{Z}_p(n))))_U$ and the result follows from (22) once we observe that $\theta_S^T(K/k, 1-n) = \varepsilon_U \cdot \theta_S^T(K/k, 1-n)$ maps to $\theta_S^T(K^{CM}/k, 1-n)$ under the natural identification $\varepsilon_U \mathbb{Q}_p G \simeq \mathbb{Q}_p \overline{G}$. \square

Remark 5.9. *Note that U is a 2-group if G has a unique 2-Sylow subgroup. This in particular applies to nilpotent groups. Moreover, if our primary interest was in an extension K/k of totally real fields, then we may enlarge K to a CM-field.*

We will now focus on CM-extensions K/k with Galois group G . We denote the maximal real subfield of K by K^+ and let $j \in G$ be complex conjugation. We put $e_n := \frac{1+(-1)^n j}{2}$ which is a central idempotent in G . For any $\mathbb{Z}_p G$ -module M we have natural isomorphisms

$$e_n \cdot M = \begin{cases} M^+ & \text{if } n \text{ is even} \\ M^- & \text{if } n \text{ is odd.} \end{cases}$$

Since $e_n \theta_S^T(1-n) = \theta_S^T(1-n)$, Conjecture 5.1 is true if and only if $\theta_S^T(1-n)$ belongs to $\text{Fitt}_{e_n \mathbb{Z}_p G}^{\max}(e_n H^2(C_S^T(\mathbb{Z}_p(n))))$. Let

$$C_{p,r} := R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(1-n)), \mathbb{Z}_p[-2]),$$

where $R\Gamma_c(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(1-n))$ is the complex of $\mathbb{Z}_p G$ -modules given by the cohomology with compact support as defined in [BF01], p. 522. Then the complex $C_{p,r}$ belongs to $\mathcal{D}^{\text{perf}}(\mathbb{Z}_p G)$ and fits into an exact triangle in $\mathcal{D}(\mathbb{Z}_p G)$ (cf. [BF98], Prop. 4.1):

$$\bigoplus_{w \in S_\infty(K)} R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_\Delta(K(w), \mathbb{Z}_p(1-n)), \mathbb{Z}_p)[-3] \longrightarrow R\Gamma(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n)) \longrightarrow C_{p,r}[-1] \longrightarrow$$

where $R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_\Delta(K(w), \mathbb{Z}_p(1-n)), \mathbb{Z}_p)$ is given by $\mathbb{Z}_p(n-1)$ (placed in degree zero) if w is complex, and by

$$\mathbb{Z}_p \xrightarrow{\delta_1} \mathbb{Z}_p \xrightarrow{\delta_0} \mathbb{Z}_p \xrightarrow{\delta_1} \dots$$

if w is real (which does not occur here) and δ_i is multiplication with $1 - (-1)^{i-n}$ for $i = 0, 1$; here, the first \mathbb{Z}_p is placed in degree 0. Hence the only non-trivial term of $R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_\Delta(L(w), \mathbb{Z}_p(1-n)), \mathbb{Z}_p)$ is $\bigoplus_{w \in S_\infty(K)} \mathbb{Z}_p(n-1)$ which is annihilated by e_n . Hence we have an isomorphism

$$e_n R\Gamma(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n)) := e_n \mathbb{Z}_p G \otimes_{\mathbb{Z}_p G}^L R\Gamma(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n)) \simeq e_n C_{p,r}[-1]$$

in $\mathcal{D}^{\text{perf}}(e_n \mathbb{Z}_p G)$. In fact

$$e_n H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n)) = H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}} \quad (23)$$

such that we obtain an exact triangle

$$e_n C_S^T(\mathbb{Z}_p(n)) \rightarrow e_n C_{p,r}[-1] \rightarrow e_n \bigoplus_{w \in T(K)} R\Gamma(K(w), \mathbb{Z}_p(n)) \rightarrow \quad (24)$$

in $\mathcal{D}_{\text{tor}}^{\text{perf}}(e_n \mathbb{Z}_p G)$. The following theorem gives the relation to the ETNC as formulated by Burns and Flach [BF01].

Theorem 5.10. *Assume that K/k is a Galois CM-extension. Then $e_n H^1(C_S^T(\mathbb{Z}_p(n)))$ vanishes and $e_n H^2(C_S^T(\mathbb{Z}_p(n)))$ is a cohomologically trivial G -module. Moreover, the following assertions are equivalent.*

1. $\theta_S^T(1-n) \in \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n H^2(C_S^T(\mathbb{Z}_p(n))))$.
2. $\theta_S^T(1-n)$ is a generator of $\text{Fitt}_{e_n \mathbb{Z}_p G}(e_n H^2(C_S^T(\mathbb{Z}_p(n))))$.
3. The p -part of the ETNC for the pair $(\mathbb{Q}(1-n)_K, e_n \mathbb{Z}[\frac{1}{2}]G)$ holds.

In particular, (1) and (2) are independent of the sets S and T .

Corollary 5.11. *Assume that K/k is a Galois extension of number fields with k totally real and let p be an odd prime. If there exists a totally imaginary field \tilde{K} containing K such that \tilde{K}/k is Galois and \tilde{K}/\tilde{K}^{CM} is a 2-extension, then $SCS(K/k, S, T, p, n)$ holds for all admissible sets S and T provided that Iwasawa's μ -invariant attached to the cyclotomic \mathbb{Z}_p -extension of $\tilde{K}^{CM,+}$ vanishes.*

Proof. By Proposition 5.5 and Lemma 5.8 we are reduced to the case $K = \tilde{K}^{CM}$, i.e. K is actually a CM-field. Now [Bub], Cor. 2.10 shows that the relevant part of the ETNC holds if $\mu = 0$. \square

Remark 5.12. *We will sketch a second proof of Corollary 5.11 below, using our results of section 2 and 3.*

The following is a non-abelian analogue of [GrP], Th. 6.11 and also reproves [Ni11c], Cor. 4.2.

Corollary 5.13. *Assume that K/k is a Galois CM-extension and let p be an odd prime. If Iwasawa's μ -invariant vanishes, then*

$$\text{Fitt}_{\mathbb{Z}_p G}^{\max}(H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}}^{\vee})^{\sharp} \cdot \theta_S(1-n) = e_n \text{Fitt}_{\mathbb{Z}_p G}^{\max}(H_{\text{ét}}^2(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))).$$

Proof. We will make use of the validity of the above special case of the ETNC under the assumption $\mu = 0$. The e_n -part of the exact sequence (21) is a four term sequence of finite $e_n \mathbb{Z}_p G$ -modules. The two middle terms are c.t. with generator $\delta_T(1-n)$ and $\theta_S^T(1-n)$ by Lemma 5.4 and Theorem 5.10, respectively. The result now follows by applying [Ni10], Prop. 5.3 (ii) to this four term sequence. \square

Remark 5.14. *Since $H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}}$ is a cyclic module, we always have an inclusion*

$$\text{nr}(\text{Ann}_{\mathbb{Z}_p G}(H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}})) \subset \text{Fitt}_{\mathbb{Z}_p G}^{\max}(H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}}^{\vee})^{\sharp}$$

with equality if G is abelian (and one can drop nr as it is just the identity on $\mathbb{Z}_p G$ for abelian G). For arbitrary G , equality seems to be likely, but is not clear.

Proof of Theorem 5.10. Since $e_n H^1(C_S^T(\mathbb{Z}_p(n)))$ is torsion-free by Lemma 5.2 and is a submodule of $e_n H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))$ which is the torsion submodule of $H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))$ by (23), it must be trivial. The triangle (24) shows that $C_S^T(\mathbb{Z}_p(n))e_n$ is a perfect complex. But the only non-trivial cohomology group is $e_n H^2(C_S^T(\mathbb{Z}_p(n)))$ which is thus of

finite projective dimension, hence c.t. as G -module. In particular, it admits a quadratic presentation and, using the exact triangle (24) and Lemma 5.4,

$$\begin{aligned} \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n H^2(C_S^T(\mathbb{Z}_p(n)))) &= \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n C_S^T(\mathbb{Z}_p(n)))^{-1} \\ &= \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n \bigoplus_{w \in T(K)} R\Gamma(K(w), \mathbb{Z}_p(n))) \\ &\quad \cdot \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n C_{p,r}[-1])^{-1} \\ &= \delta_T(1-n) \cdot \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n C_{p,r}). \end{aligned}$$

But $\text{Fitt}_{e_n \mathbb{Z}_p G}(e_n C_{p,r})$ is generated by $\theta_S(1-n)$ if and only if the refined Euler characteristic $\chi_{e_n \mathbb{Z}_p G, e_n \mathbb{Q}_p G}(e_n C_{p,r}, 0)$ equals $\hat{\partial}_G(\theta_S(1-n))$, i.e. if and only if the ETNC for the pair $(\mathbb{Q}(1-n)_K, e_n \mathbb{Z}[\frac{1}{2}]G)$ holds; this reformulation of the ETNC is due to Burns [Bu10], Prop. 4.2.6, but see [Ni11c], Prop. 2.15 which applies more directly. This shows the equivalence of (2) and (3). Since clearly (2) implies (1), we are left with the proof of (1) \implies (2). For this, let E be a splitting field of $\mathbb{Q}_p G$; then

$$E \otimes \zeta(\mathbb{Q}_p G) = \zeta(EG) = \bigoplus_{\chi \in \text{Irr}_p(G)} E e_\chi$$

and we may write $1 \otimes \theta_S^T(1-n) = \sum_\chi \theta_S^T(1-n)_\chi e_\chi$. By [Ni10], Prop. 5.4 it suffices to show that

$$\prod_\chi \theta_S^T(1-n)_\chi^{\chi(1)} \sim |e_n H^2(C_S^T(\mathbb{Z}_p(n)))|,$$

where the product runs through all irreducible odd (resp. even) characters of G if n is odd (resp. even) and \sim means “equal up to a p -adic unit”. By the same proposition and Lemma 5.4, we have

$$\prod_\chi \delta_T(1-n, \chi)^{\chi(1)} \sim |e_n \bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n))|.$$

Using the fact that $e_n H^1(C_S^T(\mathbb{Z}_p(n)))$ vanishes and sequence (21), we are left to show (in obvious notation)

$$\prod_\chi \theta_S(1-n)_\chi^{\chi(1)} \sim \frac{|e_n H_{\text{ét}}^2(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))|}{|e_n H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))|}. \quad (25)$$

If n is even, the left hand side equals $\zeta_{K^+}(1-n)$, where ζ_{K^+} denotes the Dedekind zeta function of the number field K^+ . Moreover,

$$\frac{|e_n H_{\text{ét}}^2(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))|}{|e_n H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))|} = \frac{|H_{\text{ét}}^2(\mathfrak{o}_{K^+,S}[1/p], \mathbb{Z}_p(n))|}{|H_{\text{ét}}^1(\mathfrak{o}_{K^+,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}}|}$$

such that (25) is equivalent to the cohomological version of Lichtenbaum’s conjecture which is a theorem due to Wiles [Wi90] in this case. If n is odd, a similar argument shows that (25) is equivalent to a higher relative class number formula as formulated and proved by Kolster [Ko02], Prop. 1.1. \square

Finally, we briefly illustrate how to use the results of section 2 and 3 to give an alternative proof of Corollary 5.11. Since this will not lead to a new result, some of the details

are left to the reader. As before, we may reduce the problem to the case, where K/k is a CM-extension. In fact by Theorem 5.10, the following provides a new proof of the p -part of the ETNC for the pair $(\mathbb{Q}(1-n)_K, e_n \mathbb{Z}[\frac{1}{2}]G)$ if $\mu = 0$.

By Lemma 5.7, we may assume that S contains the p -adic places, and by Proposition 5.5 we may assume that $\zeta_p \in K$. Let \mathcal{K} be the cyclotomic \mathbb{Z}_p -extension of K and denote by \mathcal{S} and \mathcal{T} the places of \mathcal{K} above the places in S and T , respectively. As before, let $\mathcal{G} = \text{Gal}(\mathcal{K}/k)$ and $\Gamma = \text{Gal}(\mathcal{K}/K)$. The exact sequence (17) tensored with $\mathbb{Z}_p(n-1)$ leads to an exact sequence of $\Lambda(\mathcal{G})$ -modules

$$\mathbb{Z}_p(n) \rightarrow T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1) \rightarrow T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-(n-1) \rightarrow X_S^+(-n)^*.$$

A spectral sequence argument leads to natural isomorphisms of $\mathbb{Z}_p G$ -modules (cf. [GrP], Prop. 6.17; the assumption on G to be abelian is not necessary)

$$e_n H_{\text{ét}}^2(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n)) \simeq X_S^+(-n)_{\Gamma}^*, \quad e_n H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}} \simeq \mathbb{Z}_p(n)_{\Gamma}.$$

Since also

$$T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1)_{\Gamma} \simeq e_n \bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n)),$$

taking Γ -coinvariants yields an exact sequence

$$H_{\text{ét}}^1(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))_{\text{tor}} \rightarrow e_n \bigoplus_{w \in T(K)} H_{\text{ét}}^1(K(w), \mathbb{Z}_p(n)) \rightarrow$$

$$T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-(n-1)_{\Gamma} \rightarrow e_n H_{\text{ét}}^2(\mathfrak{o}_{K,S}[1/p], \mathbb{Z}_p(n))$$

which is rather similar to sequence (21) times e_n . In fact, if the extension class of $\alpha_{\text{ét}}(_)(1)$ applied to sequence (17) matches the extension class of the Ritter-Weiss sequence (8), i.e. if the complex which consists of the two middle terms is isomorphic to $R\text{Hom}(R\Gamma_{\text{ét}}(\text{Spec}(\mathfrak{o}_{\mathcal{K}}[\frac{1}{S}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$ in $\mathcal{D}(\Lambda(\mathcal{G}))$, then $T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-(n-1)_{\Gamma}$ naturally identifies with $e_n H^2(C_S^T(\mathbb{Z}_p(n)))$. If not, one can construct a four term exact sequence

$$\mathbb{Z}_p(n) \rightarrow T_p(\Delta_{\mathcal{K}, \mathcal{T}})^-(n-1) \rightarrow Y_S^T(n-1) \rightarrow X_S^+(-n)^*$$

which has the correct extension class, and a proof similar to that of Theorem 3.3 shows that $\Psi_{S,T}$ is a generator of $\text{Fitt}_{\Lambda(\mathcal{G})_-}(Y_S^T)$. In any case, we have an equality of Fitting invariants

$$\text{Fitt}_{e_n \mathbb{Z}_p G}(T_p(\mathcal{M}_{\mathcal{S}, \mathcal{T}}^{\mathcal{K}})^-(n-1)_{\Gamma}) = \text{Fitt}_{e_n \mathbb{Z}_p G}(e_n H^2(C_S^T(\mathbb{Z}_p(n)))).$$

But Theorem 3.3 implies that the left hand side is generated by $t_{n-1}(\Psi_S^T)(0) = \theta_S^T(1-n)$ as desired; here, for $m \in \mathbb{Z}$ we denote by t_m the continuous \mathbb{Z}_p -algebra endomorphism of $\Lambda(\mathcal{G})$ induced by $t_m(g) = \kappa(g)^m \cdot g$ and we have used the following fact. Let M be a finitely generated torsion $\Lambda(\mathcal{G})$ -module of projective dimension at most 1 which has no non-trivial finite submodule. Then $t_m(\Psi)$ is a generator of the Fitting invariant of $M(m)$ if Ψ is a generator of the Fitting invariant of M ; this follows from the proof of Proposition 3.4.

References

- [Ba77] Barsky, D.: *Fonctions zêta p -adique d'une classe de rayon des corps de nombres totalement réels*, Groupe d'Etude d'Analyse Ultramétrique (1977/78), Exp. No. **16**
- [Bu03] Burns, D.: *Equivariant Whitehead Torsion and Refined Euler Characteristics*, CRM Proc. Lect. Notes **36** (2003), 35-59
- [Bu10] Burns, D.: *On leading terms and values of equivariant motivic L -functions*, Pure Appl. Math. Q. **6** (1) (2010), 83-171
- [Bua] Burns, D.: *On derivatives of Artin L -series*, to appear in Invent. Math., see http://www.mth.kcl.ac.uk/staff/dj_burns/newdbpublist.html
- [Bub] Burns, D.: *On main conjectures in non-commutative Iwasawa theory and related conjectures*, preprint - see http://www.mth.kcl.ac.uk/staff/dj_burns/newdbpublist.html
- [BF98] Burns, D., Flach, M.: *On Galois structure invariants associated to Tate motives*, Amer. J. Math. **120** (1998), 1343-1397
- [BF01] Burns, D., Flach, M.: *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501-570
- [BJ11] Burns, D., Johnston, H.: *A non-abelian Stickelberger Theorem*, Compos. Math. **147**, No. 1 (2011), 35-55
- [Ca79] Cassou-Noguès, P.: *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p -adiques*, Invent. Math. **51** (1979), 29-59
- [Co77] Coates, J.: *p -adic L -functions and Iwasawa's theory*, Algebr. Number Fields (Proc. Symp. London math. Soc., Univ. Durham 1975) (1977), 269-353
- [CFKSV05] Coates, J., Fukaya, T., Kato, K., Sujatha, R., Venjakob, O.: *The GL_2 main conjecture for elliptic curves without complex multiplication*, Publ. Math. IHES **101** (2005), 163-208
- [CR87] Curtis, C. W., Reiner, I.: *Methods of Representation Theory with applications to finite groups and orders*, Vol. 2, John Wiley & Sons, (1987)
- [DR80] Deligne, P., Ribet, K.: *Values of abelian L -functions at negative integers over totally real fields*, Invent. Math. **59** (1980), 227-286
- [Gr04] Greither, C.: *Arithmetic annihilators and Stark-type conjectures*, in Burns, D., Popescu, C., Sands, J., Solomon, D. (eds.): *Stark's Conjectures: Recent work and new directions*, Papers from the international conference on Stark's Conjectures and related topics, Johns Hopkins University, Baltimore, August 5-9, 2002, Contemporary Math. **358** (2004), 55-78

- [Gr07] Greither, C.: *Determining Fitting ideals of minus class groups via the Equivariant Tamagawa Number Conjecture*, Compos. Math. **143**, No. 6 (2007), 1399-1426
- [GK08] Greither, C., Kurihara, M.: *Stickelberger elements, Fitting ideals of class groups of CM fields, and dualisation*, Math. Z. **260** (2008), 905-930
- [GrP] Greither, C., Popescu, C.: *An equivariant main conjecture in Iwasawa theory and applications*, preprint
- [Ka] Kakde, M.: *The main conjecture of Iwasawa theory for totally real fields*, preprint, see arXiv:1008.0142v3
- [Ko02] Kolster, M.: *Higher relative class number formulae*, Math. Ann. **323** (2002), 667-692
- [NSW00] Neukirch, J., Schmidt, A., Wingberg, K.: *Cohomology of number fields*, Springer (2000)
- [Ni10] Nickel, A.: *Non-commutative Fitting invariants and annihilation of class groups*, J. Algebra **323** (10) (2010), 2756-2778
- [Ni11a] Nickel, A.: *On the equivariant Tamagawa number conjecture in tame CM-extensions*, Math. Z. **268** (1) (2011), 1-35
- [Ni11b] Nickel, A.: *On the equivariant Tamagawa number conjecture in tame CM-extensions, II*, Compos. Math. **147** (4) (2011), 1179-1204
- [Ni11c] Nickel, A.: *Leading terms of Artin L-series at negative integers and annihilation of higher K-groups*, Math. Proc. Camb. Philos. Soc. **151** (1) (2011), 1-22
- [Nia] Nickel, A.: *On non-abelian Stark-type conjectures*, to appear in Ann. Inst. Fourier
- [RW02] Ritter, J., Weiss, A.: *Toward equivariant Iwasawa theory*, Manuscr. Math. **109** (2002), 131-146
- [RW04] Ritter, J., Weiss, A.: *Toward equivariant Iwasawa theory, II*, Indag. Math. **15** (2004), 549-572
- [RWa] Ritter, J., Weiss, A.: *On the 'main conjecture' of equivariant Iwasawa theory*, preprint, see arXiv:1004.2578v2
- [Si70] Siegel, C.L.: *Über die Fourierschen Koeffizienten von Modulformen*, Nachr. Akad. Wiss. Gött., II. Math.-Phys. Kl. 1970 (1970), 15-56
- [Sw68] Swan, R.G.: *Algebraic K-theory*, Springer Lecture Notes **76** (1968)
- [Ta84] Tate, J.: *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$* , Birkhäuser, (1984)

- [We94] Weibel, C.: *An introduction to homological algebra*, Cambridge Studies in advanced mathematics **38**, Cambridge University Press (1994)
- [We] Weibel C.: *An introduction to algebraic K-theory* (online) see - <http://www.math.rutgers.edu/~weibel/Kbook.html>
- [Wi90] Wiles, A.: *The Iwasawa conjecture for totally real fields*, Ann. Math. **131**, 493-540 (1990)

Andreas Nickel andreas.nickel@mathematik.uni-regensburg.de
Universität Regensburg, Fakultät für Mathematik, Universitätsstr. 31, 93053 Regensburg,
Germany