Universität Regensburg
Mathematik

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Preprint Nr. 35/2011

THE CAUCHY PROBLEM FOR METRICS WITH PARALLEL SPINORS

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Abstract. We show that in the analytic category, given a Riemannian metric g on a hypersurface $M \subset \mathcal{Z}$ and a symmetric tensor W on M, the metric g can be locally extended to a Riemannian Einstein metric on $\mathcal{Z}$ with second fundamental form W, provided that g and W satisfy the constraints on M imposed by the contracted Codazzi equations. We use this fact to study the Cauchy problem for metrics with parallel spinors in the real analytic category and give an affirmative answer to a question raised in [15]. We also answer negatively the corresponding questions in the smooth category.

1. Introduction

This paper attempts to solve two problems: the question of existence of Riemannian Einstein metrics prescribed on a hypersurface together with their second fundamental form, and the extension problem for spinors from a hypersurface to parallel spinors on the total space. These problems are related: parallel spinors can only exist over Ricci-flat manifolds.

The Cauchy problem for Einstein metrics. In the Lorentzian setting, Ricci-flat or more generally Einstein metrics form the central objects of general relativity. Given a space-like hypersurface, a Riemannian metric, and a symmetric tensor which plays the role of the second fundamental form, there always exists a local extension to a Lorentzian Einstein metric [29], [25], provided that the local conditions given by the Gauss equation are satisfied, see (2.11), (2.12) below. One crucial step in the proof is the reduction to an evolution equation which is hyperbolic due to the signature of the metric. The corresponding equations in the Riemannian setting are elliptic and no general local existence results are available.

In the Riemannian setting, DeTurck [26] analyzed the related problem of finding a metric with prescribed nonsingular Ricci tensor. The Ricci-flat case is at the opposite spectrum of degeneracy, while the general Einstein case is reminiscent of DeTurck's setting. Despite some formal similarities with [26], the Cauchy problem for Einstein metrics studied here is in essence quite different.

Date: December 1, 2011.

2010 *Mathematics Subject Classification.* 35A10, 35J47, 53C27, 53C44, 83C05.

Key words and phrases. Cauchy problem, parallel spinors, generalized Killing spinors, Ricci-flat metrics.

Extension of generalized Killing spinors to parallel spinors. In order to introduce the second problem, we need to recall some basic facts about restrictions of spin bundles to hypersurfaces. If $\mathcal{Z}$ is a spin manifold, any oriented hypersurface $M \subset \mathcal{Z}$ inherits a spin structure and it is well-known that the restriction to M of the complex spin bundle $\Sigma\mathcal{Z}$ if n is even (resp. $\Sigma^+\mathcal{Z}$ if n is odd) is canonically isomorphic to the complex spin bundle ΣM (cf. [15]). If W denotes the Weingarten tensor of M, the spin covariant derivatives $\nabla^{\mathcal{Z}}$ and ∇^g are related by ([15, Eq. (8.1)])

$$(\nabla^{\mathcal{Z}}_X \Psi)|_M = \nabla^g_X(\Psi|_M) - \frac{1}{2} W(X)\cdot(\Psi|_M), \qquad \forall\, X \in TM, \tag{1.1}$$

for all spinors (resp. half-spinors for n odd) Ψ on $\mathcal{Z}$. We thus see that if Ψ is a parallel spinor on $\mathcal{Z}$, its restriction ψ to any hypersurface M is a *generalized Killing spinor* on M, i.e. it satisfies the equation

$$\nabla^g_X \psi = \frac{1}{2} W(X)\cdot\psi, \qquad \forall\, X \in TM, \tag{1.2}$$

and the symmetric tensor W, called the stress-energy tensor of ψ, is just the Weingarten tensor of the hypersurface M. It is natural to ask whether the converse holds:

> *(Q): If ψ is a generalized Killing spinor on M^n, does there exist an isometric embedding of M into a spin manifold $(\mathcal{Z}^{n+1}, g^{\mathcal{Z}})$ carrying a parallel spinor Ψ whose restriction to M is ψ?*

This question is precisely the Cauchy problem for metrics with parallel spinors asked in [15].

The answer is known to be positive in several special cases: if the stress-energy tensor W of ψ is the identity [12], if W is parallel [43] and if W is a Codazzi tensor [15]. Even earlier, Friedrich [30] had worked out the 2-dimensional case $n + 1 = 2 + 1$, which is also covered by [15, Thm. 8.1] since on surfaces the stress-energy of a generalized Killing spinor is automatically a Codazzi tensor. Some related embedding results were also obtained by Morel [44] and Lawn–Roth [40]. The common feature of each of these cases is that one can actually construct in an explicit way the "ambient" metric $g^{\mathcal{Z}}$ on the product $(-\varepsilon, \varepsilon) \times M$.

Our aim is to show that the same is true more generally, under the sole additional assumption that (M, g) and W are analytic.

Theorem 1.1. *Let ψ be a spinor field on a analytic spin manifold (M^n, g), and W a analytic field of endomorphisms of TM. Assume that ψ is a generalized Killing spinor with respect to W, i.e. it satisfies* (1.2). *Then there exists a unique metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ on a sufficiently small neighborhood $\mathcal{Z}$ of $\{0\} \times M$ inside $\mathbb{R} \times M$ such that $(\mathcal{Z}, g^{\mathcal{Z}})$, endowed with the spin structure induced from M, carries a parallel spinor Ψ whose restriction to M is ψ.*

In particular, the solution $g^{\mathcal{Z}}$ must be Ricci-flat. Einstein manifolds are analytic but of course hypersurfaces can lose this structure so our hypothesis is restrictive. Note that Einstein metrics with smooth initial data can be constructed for small time as constant sectional curvature metrics when the second fundamental form is a Codazzi tensor, see [15, Thm. 8.1]. In particular in dimensions $1+1$ and $2+1$ Theorem 1.1 remains valid in the smooth category since the tensor W associated to a generalized Killing spinor is automatically a Codazzi tensor in dimensions 1 and 2. The situation changes essentially if we allow arbitrary generalized Killing spinors. What we can achieve then is to solve the Einstein equation (and the parallel spinor equation) in Taylor series near the initial hypersurface. More precisely, starting from a smooth hypersurface (M, g) with prescribed Weingarten tensor W we prove that there exist formal Einstein metrics $g^{\mathcal{Z}}$ such that W is the second fundamental form at $t = 0$, i.e., we solve the Einstein equation modulo rapidly vanishing errors. Guided by the analytic and the low dimensional ($n = 1$ or $n = 2$) cases, one could be tempted to guess that actual germs of Einstein metrics do exist for any smooth initial data. However this turns out to be false. Counterexamples were found very recently in some particular cases in dimensions 3 and 7 by Bryant [20]. We give a general procedure to construct counterexamples in all dimensions in Section 4.

Note that several particular instances of Theorem 1.1 have been proved in recent years, based on the characterization of generalized Killing spinors in terms of exterior forms in low dimensions. Indeed, in dimensions 5, 6 and 7, generalized Killing spinors are equivalent to so-called *hypo*, *half-flat* and *co-calibrated* G_2 structures respectively. In [39] Hitchin proved that the cases $6+1$ and $7+1$ can be solved up to the local existence of a certain gradient flow. Later on, Conti and Salamon [22], [23] treated the cases $5+1$, $6+1$ and $7+1$ in the analytical setting, cf. also [21] [40] for further developments.

Related problems have been studied starting with the work of Fefferman-Graham [28] concerning asymptotically hyperbolic Poincaré-Einstein metrics. The initial hypersurface (M, g) is then at infinite distance from the manifold, the metric being conformal to a smooth metric $\bar{g}$ on a manifold with boundary

$$\mathcal{Z} = (0, \varepsilon) \times M, \qquad\qquad g^{\mathcal{Z}} = x^{-2}\bar{g}$$

such that the conformal factor x is precisely the distance function to the boundary $x = 0$ with respect to $\bar{g}$. The metric is required to be Einstein of negative curvature up to an error term which vanishes with all derivatives at infinity. Such a metric exists when n is odd, and its Taylor series at infinity is determined by the initial metric g and the symmetric transverse traceless tensor appearing on position $2n$, while in even dimensions some logarithmic terms must be allowed.

Let us stress that existence results of Einstein metrics with prescribed first fundamental form and Weingarten tensor clearly cannot hold globally in general (Example 2.5).

Counterexamples in the smooth setting. In a second part of the paper (Section 4) we apply the above existence results from the analytic setting to prove nonexistence of solutions for certain smooth initial data in any dimension at least 3.

The argument goes along the lines of works of the first author and his collaborators on the Yamabe problem and the mass endomorphism. We consider the functional

$$\mathcal{F}(\phi) := \frac{\langle D_0\phi, \phi\rangle_{L^2}}{\|D_0\phi\|^2_{L^{2n/(n+1)}}}$$

defined on the C^1 spinor fields ϕ on a compact connected Riemannian spin manifold (M, g_0) which are not in the kernel of the Dirac operator D_0. If the infimum of the lowest positive eigenvalue of the Dirac operator in the volume-normalized conformal class of g_0 is strictly lower than the corresponding eigenvalue for the standard sphere (Condition (4.5) below), this functional attains its supremum in a spinor ψ_0 of regularity $C^{2,\alpha}$. Moreover, ψ_0 is smooth outside its zero set.

To construct g_0 satisfying Condition (4.5) we fix $p \in M$ and we look at metrics on M which are flat near p. If the topological index of M vanishes in $KO^{-n}(pt)$, then for generic such metrics the associated Dirac operator is invertible. The mass endomorphism at p is defined as the constant term in the asymptotic expansion of the Green kernel of D near p. Again for generic metrics, this mass endomorphism is non-zero, which by a result of [7] ensures the technical Condition (4.5) for generic metrics which are flat near p. By construction this class of metrics contains metrics which are not conformally flat on some open subset of M, i.e., whose Schouten tensor (in dimension 3), resp. Weyl curvature (in higher dimensions) is nonzero on some open set. We assume g_0 was chosen with these properties.

We return now to the spinor ψ_0 maximizing the functional $\mathcal{F}$. The Euler-Lagrange equation of $\mathcal{F}$ at ψ_0 can be reinterpreted as follows: the Dirac operator with respect to the conformal metric $g := |\psi_0|^{4/(n-1)}g_0$ admits an eigenspinor of constant length 1, $\psi := \frac{\psi_0}{|\psi_0|}$.

If the dimension n equals 3, by algebraic reasons this spinor field must be a generalized Killing spinor with stress-energy tensor W of constant trace.

The metric g is defined on the complement M^* of the zero set of ψ_0. This set is open, connected and dense in M (Lemmata 4.6 and 4.9). Recall that g_0 was chosen such that its Schouten tensor vanishes identically on an open set of M and is nonzero on another open set. Then the same remains true on M^*, and therefore on M^* there exists no analytic metric in the conformal class of g_0. In particular, the metric $g = |\psi_0|^{4/(n-1)}g_0$ cannot be analytic.

Assuming now that Theorem 2.1 continues to hold for smooth initial data, we could apply it to (M^*, g, W) to get an embedding in a Ricci-flat (hence analytic) Riemannian manifold $(\mathcal{Z}, g^{\mathcal{Z}})$, with second fundamental form W. Since the trace of W is constant by construction, M would have constant mean curvature, which would imply that it were analytic (Lemma. 4.16), contradicting the non-analyticity proved above.

The above construction actually yields counterexamples to the Cauchy problem for Ricci-flat metrics in the smooth setting in any dimension $n \geq 3$, by taking products with flat spaces, see Lemma 4.28.

Applications of the main results. By similar methods, we also obtain many examples of manifolds satisfying the hypotheses of Theorem 2.1, which thus embed isometrically as hypersurfaces in Ricci-flat manifolds. More precisely, we show that for a generic analytic conformal structure c on a compact spin 3-dimensional manifold M, there exists a metric $g \in c$ in the conformal class and a dense open subset M^* such that (M^*, g) can be embedded isometrically as a hypersurface in a Ricci-flat manifold.

Acknowledgements. It is a pleasure to thank Olivier Biquard, Gilles Carron, Mattias Dahl, Paul Gauduchon, Colin Guillarmou, Christophe Margerin and Jean-Marc Schlenker for helpful discussions. We thank the DFG-Graduiertenkolleg GRK 1692 Regensburg for its support. AM was partially supported by the contract ANR-10-BLAN 0105 "Aspects Conformes de la Géométrie". SM was partially supported by the contract PN-II-RU-TE-2011-3-0053 and by the LEA "MathMode". He thanks the CMLS at the Ecole Polytechnique for its hospitality during the writing of this paper.

2. The Cauchy problem for Einstein metrics

Let $(\mathcal{Z}, g^{\mathcal{Z}})$ be an oriented Riemannian manifold of dimension $n+1$ and let M be an oriented hypersurface with induced Riemannian metric $g := g^{\mathcal{Z}}|_M$. We start by fixing some notations. Denote by $\nabla^{\mathcal{Z}}$ and ∇^g the Levi-Civita covariant derivatives on $(\mathcal{Z}, g^{\mathcal{Z}})$ and (M, g), by ν the unit normal vector field along M compatible with the orientations, and by $W \in \mathrm{End}(TM)$ the Weingarten tensor defined by

$$\nabla^{\mathcal{Z}}_X \nu = -W(X), \qquad \forall\, X \in TM. \tag{2.1}$$

Using the normal geodesics issued from M, the metric on $\mathcal{Z}$ can be expressed in a neighborhood $\mathcal{Z}_0$ of M as $g^{\mathcal{Z}} = dt^2 + g_t$, where t is the distance function to M and g_t is a family of Riemannian metrics on M with $g_0 = g$ (cf. [15]). The vector field ν extends to $\mathcal{Z}_0$ as $\nu = \partial/\partial t$ and (2.1) defines a symmetric endomorphism on $\mathcal{Z}_0$ which can be viewed as a family W_t of endomorphisms of M, symmetric with respect to g_t, and satisfying (cf. [15, Equation (4.1)]):

$$g_t(W_t(X), Y) = -\frac{1}{2}\dot{g}_t(X, Y), \qquad \forall\, X, Y \in TM. \tag{2.2}$$

By [15, Equations (4.5)–(4.8)], the Ricci tensor and the scalar curvature of $\mathcal{Z}$ satisfy for every vectors $X, Y \in TM$

$$\mathrm{Ric}^{\mathcal{Z}}(\nu, \nu) = \mathrm{tr}(W_t^2) - \frac{1}{2}\mathrm{tr}_{g_t}(\ddot{g}_t), \tag{2.3}$$

$$\mathrm{Ric}^{\mathcal{Z}}(\nu, X) = d\mathrm{tr}(W_t)(X) + \delta^{g_t}(W)(X), \tag{2.4}$$

$$\mathrm{Ric}^{\mathcal{Z}}(X, Y) = \mathrm{Ric}^{g_t}(X, Y) + 2g_t(W_t X, W_t Y) + \frac{1}{2}\mathrm{tr}(W_t)\dot{g}_t(X, Y) - \frac{1}{2}\ddot{g}_t(X, Y), \tag{2.5}$$

$$\mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^{g_t} + 3\mathrm{tr}(W_t^2) - \mathrm{tr}^2(W_t) - \mathrm{tr}_{g_t}(\ddot{g}_t). \tag{2.6}$$

Using (2.3) and (2.6) we get

$$-2\mathrm{Ric}^{\mathcal{Z}}(\nu,\nu)+\mathrm{Scal}^{\mathcal{Z}}=\mathrm{Scal}^{g_t}+\mathrm{tr}(W_t^2)-\mathrm{tr}^2(W_t), \tag{2.7}$$

where $\delta^g : \mathrm{End}(TM)\to T^*M$ is the divergence operator defined in a local g-orthonormal basis $\{e_i\}$ of TM by

$$\delta^g(A)(X)=-\sum_{i=1}^n g((\nabla^g_{e_i}A)(e_i),X). \tag{2.8}$$

A straightforward calculation yields

$$\delta^g(fA)=f\delta^g(A)-A(\nabla^g f) \tag{2.9}$$

for all functions f.

For later use, let us recall that the second Bianchi identity implies the following relation between the divergence of the Ricci tensor and the exterior derivative of the scalar curvature:

$$\delta^g(\mathrm{Ric}^g)=-\frac{1}{2}d\mathrm{Scal}^g \tag{2.10}$$

for every Riemannian metric g (cf. [16, Prop. 1.94]).

Assume now that the metric $g^{\mathcal{Z}}$ is Einstein with scalar curvature $(n+1)\lambda$, i.e. $\mathrm{Ric}^{\mathcal{Z}}=\lambda g^{\mathcal{Z}}$. Evaluating (2.4) and (2.7) at $t=0$ yields

$$d\mathrm{tr}(W)+\delta^g W=0, \tag{2.11}$$

$$\mathrm{Scal}^g+\mathrm{tr}(W^2)-\mathrm{tr}^2(W)=(n-1)\lambda. \tag{2.12}$$

If $g_t : \mathrm{End}(TM)\to T^*M\otimes T^*M$ is the isomorphism defined by $g_t(A)(X,Y):=g_t(A(X),Y)$ and $g_t^{-1} : T^*M\otimes T^*M\to\mathrm{End}(TM)$ denotes its inverse, then taking (2.3) into account, (2.5) reads

$$\ddot g_t=2\mathrm{Ric}^{g_t}+\dot g_t(g_t^{-1}(\dot g_t)\cdot,\cdot)-\mathrm{tr}(g_t^{-1}(\dot g_t))\dot g_t-2\lambda g_t, \tag{2.13}$$

which can also be written

$$\dot W_t=-g_t^{-1}\mathrm{Ric}^{g_t}+W_t\mathrm{tr}(W_t)-2\lambda\mathrm{Id}. \tag{2.14}$$

In the rest of this section we prove an existence and unique continuation result for Einstein metrics.

Theorem 2.1. *Let (M^n,g) be an analytic Riemannian manifold and let W be an analytic symmetric endomorphism field on M satisfying* (2.11) *and* (2.12)*. Then for $\varepsilon>0$, there exists a unique germ near $\{0\}\times M$ of an Einstein metric $g^{\mathcal{Z}}$ with scalar curvature $(n+1)\lambda$ of the form $g^{\mathcal{Z}}=dt^2+g_t$ on $\mathcal{Z}:=\mathbb{R}\times M$ whose Weingarten tensor at $t=0$ is W.*

Proof. In equation (2.13) the only term involving partial derivatives of the metric g_t along M is Ric^{g_t}, which is an analytic expression in g_t and its first and second order derivatives along M which does not involve any derivative with respect to t. Indeed, in local coordinates x_i on M, with the usual summation convention one has

$$\mathrm{Ric}^g(\partial_i,\partial_j)=\partial_k\Gamma^i_{jk}-\partial_j\Gamma^k_{ik}+\Gamma^k_{ij}\Gamma^l_{kl}-\Gamma^k_{il}\Gamma^l_{kj},\qquad \Gamma^i_{jk}=\frac{1}{2}g^{il}(\partial_k g_{jl}+\partial_j g_{lk}-\partial_l g_{jk}).$$

The second order Cauchy-Kowalewskaya theorem (see e.g. [24]) shows that for every $x\in M$ there exists a neighborhood $V_x\ni x$ and some $\varepsilon_x>0$ such that the Cauchy problem (2.13) with initial data

$$\begin{cases} g_0=g\\ \dot g_0=-2W\end{cases}$$

has a unique analytic solution on $(-\varepsilon_x,\varepsilon_x)\times V_x$. Let $g^{\mathcal{Z}}=dt^2+g_t$ be the metric defined on $(-\varepsilon_x,\varepsilon_x)\times V_x$ by this solution. We claim that $g^{\mathcal{Z}}$ is Einstein with scalar curvature $(n+1)\lambda$.

Consider the 1-parameter family of functions and 1-forms on M:

$$f_t:=\frac{1}{2}((n-1)\lambda-\mathrm{Scal}^{g_t}-\mathrm{tr}(W_t^2)+\mathrm{tr}^2(W_t)),\qquad \omega_t:=d\mathrm{tr}(W_t)+\delta^{g_t}W_t, \tag{2.15}$$

where W_t is defined as before by (2.2). Using (2.14) and the formula for the first variation of the scalar curvature ([16, Thm. 1.174 (e)]) we get

$$\begin{aligned}\frac{df_t}{dt}=&\Delta^{g_t}(\mathrm{tr}(W_t))+\delta^{g_t}(\delta^{g_t}W_t)-g_t(\mathrm{Ric}^{g_t},g_t(W_t))-\mathrm{tr}(W_t\circ\dot W_t)+\mathrm{tr}(W_t)\mathrm{tr}(\dot W_t)\\ =&\delta^{g_t}\omega_t-g_t(\mathrm{Ric}^{g_t},g_t(W_t))-\mathrm{tr}(W_t\circ(-g_t^{-1}\mathrm{Ric}^{g_t}+W_t\mathrm{tr}(W_t)-\lambda\mathrm{Id})\\ &+\mathrm{tr}(W_t)(-\mathrm{Scal}^{g_t}+\mathrm{tr}^2(W_t)-n\lambda)\\ =&\delta^{g_t}\omega_t+\mathrm{tr}(W_t)(-\mathrm{Scal}^{g_t}+\mathrm{tr}^2(W_t)-\mathrm{tr}(W_t^2)+(n-1)\lambda),\end{aligned}$$

whence

$$\frac{df_t}{dt}=\delta^{g_t}\omega_t+2\mathrm{tr}(W_t)f_t. \tag{2.16}$$

(note that the symmetric tensor h in [16] is $h=\dot g_t=-2g_t(W_t)$ in our notations).

In order to compute the time derivative of ω_t we need to compute the variation of δ^{g_t}. The computation being rather lengthy, we postpone it to Lemma 2.2 below. Taking $A_t=W_t$ in that lemma yields

$$\frac{d(\delta^{g_t}W_t)}{dt}=-\frac{1}{2}d\mathrm{tr}(W_t^2)+W_t(\nabla^{g_t}(\mathrm{tr}(W_t))+\delta^{g_t}(\dot W_t).$$

Using (2.9), (2.10) and (2.14) we get

$$\begin{aligned}\frac{d\omega_t}{dt} &= d\mathrm{tr}(\dot W_t) + \frac{d(\delta^{g_t} W_t)}{dt}\\ &= d(-\mathrm{Scal}^{g_t} + \mathrm{tr}^2(W_t)) - \frac{1}{2} d\mathrm{tr}(W_t^2) + W_t(\nabla^{g_t}(\mathrm{tr}(W_t)) + \delta^{g_t}(\dot W_t)\\ &= -d\mathrm{Scal}^{g_t} + d\mathrm{tr}^2(W_t) - \frac{1}{2} d\mathrm{tr}(W_t^2) + W_t(\nabla^{g_t}(\mathrm{tr}(W_t)) + \delta^{g_t}(-g_t^{-1}\mathrm{Ric}^{g_t} + W_t\mathrm{tr}(W_t))\\ &= -\frac{1}{2} d\mathrm{Scal}^{g_t} - \frac{1}{2} d\mathrm{tr}(W_t^2) + d\mathrm{tr}^2(W_t) + \mathrm{tr}(W_t)\delta^{g_t}(W_t),\end{aligned}$$

which implies

$$\frac{d\omega_t}{dt} = df_t + \mathrm{tr}(W_t)\omega_t. \tag{2.17}$$

Denoting by H the analytic function $\mathrm{tr}(W_t)$, Equations (2.16) and (2.17) show that the pair (f_t, ω_t) satisfies the first order linear system

$$\begin{cases}\partial_t f_t = \delta^{g_t}\omega_t + 2Hf_t\\ \partial_t\omega_t = df_t + H\omega_t.\end{cases} \tag{2.18}$$

Moreover, the constraints (2.11) and (2.12) show that (f_t, ω_t) vanishes at $t = 0$. By the Cauchy-Kowalewskaya theorem, (f_t, ω_t) vanishes for all t.

Using (2.4), (2.5), (2.7) and (2.13), we see that the metric $g^{\mathcal Z} := dt^2 + g_t$ constructed in this way satisfies

$$\begin{cases}\mathrm{Ric}^{\mathcal Z}(\nu, X) = 0 & \forall X \perp \nu\\ \mathrm{Ric}^{\mathcal Z}(X, Y) = \lambda g^{\mathcal Z}(X, Y) & \forall X, Y \perp \nu\\ \mathrm{Scal}^{\mathcal Z} - 2\mathrm{Ric}^{\mathcal Z}(\nu, \nu) = (n-1)\lambda. & \end{cases}$$

On the other hand we clearly have $\mathrm{Scal}^{\mathcal Z} = \mathrm{Ric}^{\mathcal Z}(\nu, \nu) + n\lambda$ and therefore $\mathrm{Ric}^{\mathcal Z} = \lambda g^{\mathcal Z}$, thus proving our claim.

To end the proof of the theorem, we note that the local metric $g_x^{\mathcal Z}$ constructed above on $(-\varepsilon_x, \varepsilon_x) \times V_x$ is unique, thus $g_x^{\mathcal Z}$ and $g_y^{\mathcal Z}$ coincide on the intersection $(-\varepsilon, \varepsilon) \times (V_x \cap V_y)$ for $\varepsilon := \min\{\varepsilon_x, \varepsilon_y\}$. Hence $g^{\mathcal Z}$ is well-defined on a neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$. □

Lemma 2.2. *If g_t is a family of Riemannian metrics on a manifold M and A_t is a family of endomorphism fields of TM symmetric with respect to g_t, then*

$$\frac{d(\delta^{g_t} A_t)}{dt}(X) = g_t(A_t(\nabla^{g_t}\mathrm{tr}(W_t)), X) - g_t(\nabla_X^{g_t} W_t, A_t) + (\delta^{g_t}\dot A_t)(X), \tag{2.19}$$

where W_t is defined by (2.2)*.*

Proof. Let vol_t denote the volume form of the metric g_t. A straightforward computation yields

$$\frac{d(\mathrm{vol}_t)}{dt} = -\mathrm{tr}(W_t)\mathrm{vol}_t. \tag{2.20}$$

In the computations below we will drop the subscripts t for an easier reading and use the dot sign for differentiation with respect to t. From [16, Thm. 1.174 (a)] we get

$$g(\dot{\nabla}_X Y, Z) = g((\nabla_Z W)X, Y) - g((\nabla_X W)Y, Z) - g((\nabla_Y W)X, Z). \tag{2.21}$$

Differentiating with respect to t the formula valid for every compactly supported vector field X on M

$$\int_M (\delta^{g_t} A_t)(X)\mathrm{vol}_t = \int_M \mathrm{tr}(A_t \circ \nabla^{g_t} X)\mathrm{vol}_t \tag{2.22}$$

and using (2.20) yields

$$\int_M (\dot{\delta} A + \delta \dot{A} - (\delta A)\mathrm{tr}(W))(X)\mathrm{vol} = \int_M \mathrm{tr}(\dot{A} \circ \nabla X + A \circ \dot{\nabla} X - \mathrm{tr}(W) A \circ \nabla X)\mathrm{vol}.$$

Subtracting (2.22) applied to $\dot{A}$ from this last equation gives

$$\int_M (\dot{\delta} A - (\delta A)\mathrm{tr}(W))(X)\mathrm{vol} = \int_M \mathrm{tr}(A \circ \dot{\nabla} X - \mathrm{tr}(W) A \circ \nabla X)\mathrm{vol}. \tag{2.23}$$

From (2.21) and the fact that A and W are symmetric with respect to g we obtain

$$\mathrm{tr}(A \circ \dot{\nabla} X) = -g(\nabla_X W, A), \tag{2.24}$$

Using (2.24) and (2.22) again, but this time applied to $-\mathrm{tr}(W)A$, (2.23) becomes

$$\int_M (\dot{\delta} A - (\delta A)\mathrm{tr}(W))(X)\mathrm{vol} = \int_M -g(\nabla_X W, A) - \delta(\mathrm{tr}(W)A)(X)\mathrm{vol},$$

so from (2.9) we get

$$\int_M (\dot{\delta} A)(X)\mathrm{vol} = \int_M -g(\nabla_X W, A) + g(A(\nabla \mathrm{tr}(W)), X)\mathrm{vol},$$

Since this holds for every compactly supported vector field X, the integrand must vanish identically, i.e.

$$(\dot{\delta} A)(X) = -g(\nabla_X W, A) + g(A(\nabla \mathrm{tr}(W)), X),$$

which is equivalent to (2.19). □

2.1. **Formal solution in the smooth case.** Without the hypothesis that g and W are analytic the nonlinear PDE system (2.13) has no solution in general, see Subsection 4.8. However, it is rather evident from (2.13) that the full Taylor series of $g^{\mathcal{Z}}$ is recursively determined by its first two coefficients, which are g and W. Let $\dot{C}^\infty(\mathcal{Z})$ denote the space of tensors vanishing at M together with all their derivatives. By the Borel lemma (see e.g. [31]), there exists a metric $g^{\mathcal{Z}}$ such that its Ricci tensor satisfies the Einstein equation in the tangential directions modulo $\dot{C}^\infty(\mathcal{Z})$. Then the system (2.18) remains valid modulo $\dot{C}^\infty(\mathcal{Z})$ and we can easily show recursively that the right-hand sides of Equations (2.4) and (2.7) vanish modulo $\dot{C}^\infty(\mathcal{Z})$. Thus $g^{\mathcal{Z}}$ is Einstein modulo $\dot{C}^\infty(\mathcal{Z})$.

Proposition 2.3. *Let (M^n, g) be a smooth Riemannian manifold and let W be a smooth symmetric field of endomorphisms of TM satisfying* (2.11) *and* (2.12)*. Then there exists on $\mathcal{Z} := (-\varepsilon, \varepsilon) \times M$ a metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ whose Weingarten tensor at $t = 0$ is W, and such that*

$$\mathrm{Ric}^{\mathcal{Z}} - \lambda g^{\mathcal{Z}} \in \dot{C}^\infty(\mathcal{Z}).$$

Moreover, $g^{\mathcal{Z}}$ is unique up to $\dot{C}^\infty(\mathcal{Z})$.

2.2. **Existence and uniqueness for smooth initial data.** The small-time uniqueness of the Ricci-flat metric, or more generally of an Einstein metric follows under milder assumptions (e.g. when the g and W are only C^∞), see [9] or [17, Thm. 4].

The small-time existence is known to fail in general for elliptic Cauchy problems with C^∞ initial data, even in the linear case. Note however that in dimension $n+1 = 2+1$ the C^3 initial value problem can always be solved for small time:

Proposition 2.4. *Let M be a surface with C^3 Riemannian metric g, and let W be a C^3 symmetric field of endomorphisms on M satisfying* (2.11) *and* (2.12) *for some $\lambda \in \mathbb{R}$. Then there exists a constant sectional curvature metric $g^{\mathcal{Z}}$ on a neighborhood of $\{0\} \times M$ inside $\mathcal{Z} := \mathbb{R} \times M$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$, whose Weingarten tensor at $t = 0$ is W.*

Proof. Direct application of [15, Theorem 7.2]. Namely, in dimension 2 the hypotheses (2.11), (2.12) are equivalent to [15, Eq. (7.3)] resp. [15, Eq. (7.4)] with $\kappa = 2\lambda$. It follows, at least in the smooth case, that g_t can be constructed explicitly in terms of g and W such that $g^{\mathcal{Z}}$ has constant sectional curvature κ. It remains to note that the proof of [15, Theorem 7.2] remains valid when g and W are of class C^3. □

Similarly, in dimension $1+1$ we can embed the curve (M, g) in a constant curvature surface with prescribed curvature function W. In this case, the constraint equations are empty, and the metric is again explicitly given by [15, Theorem 7.2].

2.3. **Global existence.** The preceding case of dimension $2+1$ hints that in general the Einstein metric $g^{\mathcal{Z}}$ cannot be extended on a complete manifold containing M as a hypersurface (or even half-complete, in the sense that geodesics pointing in one side of M can be extended until they meet again M). This sort of question is rather different from the arguments of this paper so we will only give an counterexample in dimension $1+1$ where global existence for the solution to the Cauchy problem fails. We restrict ourselves to the case of Ricci-flat metrics, which means vanishing Gaussian curvature in this dimension.

Example 2.5. Let $\mathcal{Z}$ be the incomplete flat surface obtained from $\mathbb{C}^*$ (or from the complement of a small disk in $\mathbb{C}$) by the following cut-and-paste procedure: cut along the positive real axis, then glue again after a translation of length $l > 0$. More precisely, x_+ is identified with $(x+l)_-$ for all $x > \varepsilon$. The resulting surface $\mathcal{Z}$ is clearly smooth and has a smooth flat metric including along the gluing locus. The unit circle in $\mathbb{R}^2$ gives rise to a curve in $\mathcal{Z}$ of curvature 1 and length 2π with different endpoints 1_- and $(1+l)_-$. In a complete flat surface, a curve of curvature 1 and length 2π must be closed (in fact smooth, since its lift to the universal cover must be a circle). Therefore, the surface $\mathcal{Z}$ cannot be embedded in any complete flat surface. In particular, for any closed curve in $\mathcal{Z}$ circling around the singular locus, the interior cannot be continued to a compact (or half-complete) flat surface with boundary.

3. Spinors on Ricci-flat manifolds

We keep the notations from the previous section. Our starting point is the following corollary of Theorem 2.1:

Corollary 3.1. *Assume that (M^n, g) is an analytic spin manifold carrying a non-trivial generalized Killing spinor ψ with analytic stress-energy tensor W. Then in a neighborhood of $\{0\}\times M$ in $\mathcal{Z} := \mathbb{R}\times M$ there exists a unique Ricci-flat metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ whose Weingarten tensor at $t = 0$ is W.*

Proof. We just need to check that the constraints (2.11), (2.12) are a consequence of (1.2). In order to simplify the computations, we will drop the reference to the metric g and denote respectively by ∇, R, Ric and Scal the Levi-Civita covariant derivative, curvature tensor, Ricci tensor and scalar curvature of (M, g). As usual, $\{e_i\}$ will denote a local g-orthonormal basis of TM.

We will use the following two classical formulas in Clifford calculus. The first one is the fact that the Clifford contraction of a symmetric tensor A only depends on its trace:

$$\sum_{i=1}^{n} e_i\cdot A(e_i) = -\mathrm{tr}(A). \tag{3.1}$$

The second formula expresses the Clifford contraction of the spin curvature in terms of the Ricci tensor ([11], p. 16):

$$\sum_{i=1}^{n} e_i\cdot R_{X,e_i}\psi = -\frac{1}{2}\mathrm{Ric}(X)\cdot\psi, \qquad \forall X \in TM,\ \forall\psi \in \Sigma M. \tag{3.2}$$

Let now ψ be a non-trivial generalized Killing spinor satisfying (1.2). Being parallel with respect to a modified connection on ΣM, ψ is nowhere vanishing (and actually of constant norm).

Taking a further covariant derivative in (1.2) and skew-symmetrizing yields

$$R_{X,Y}\psi = \frac{1}{4}\left(W(Y)\cdot W(X) - W(X)\cdot W(Y)\right)\cdot\psi + \frac{1}{2}\left((\nabla_X W)(Y) - (\nabla_Y W)(X)\right)\cdot\psi$$

for all $X, Y \in TM$. In this formula we set $Y = e_i$, take the Clifford product with e_i and sum over i. From (3.1) and (3.2) we get

$$\begin{aligned}
\mathrm{Ric}(X)\cdot\psi =& -\frac{1}{2}\sum_{i=1}^{n} e_i\cdot\left(W(e_i)\cdot W(X) - W(X)\cdot W(e_i)\right)\cdot\psi \\
& - \sum_{i=1}^{n} e_i\cdot\left((\nabla_X W)(e_i) - (\nabla_{e_i} W)(X)\right)\cdot\psi \\
=& \frac{1}{2}\mathrm{tr}(W)W(X)\cdot\psi + \frac{1}{2}\sum_{i=1}^{n}\big(-W(X)\cdot e_i - 2g(W(X), e_i)\big)\cdot W(e_i)\cdot\psi \\
& + \nabla_X(\mathrm{tr}(W))\psi + \sum_{i=1}^{n} e_i\cdot(\nabla_{e_i} W)(X)\cdot\psi.
\end{aligned}$$

whence

$$\mathrm{Ric}(X)\cdot\psi = \mathrm{tr}(W)W(X)\cdot\psi - W^2(X)\cdot\psi + X(\mathrm{tr}(W))\psi + \sum_{i=1}^{n} e_i\cdot(\nabla_{e_i} W)(X)\cdot\psi. \tag{3.3}$$

We set $X = e_j$ in (3.3), take the Clifford product with e_j and sum over j. Using (3.1) again we obtain

$$\begin{aligned}
-\mathrm{Scal}\,\psi &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + \nabla(\mathrm{tr}(W))\cdot\psi + \sum_{i,j=1}^{n} e_j\cdot e_i\cdot(\nabla_{e_i} W)(e_j)\cdot\psi \\
&= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + d\mathrm{tr}(W)\cdot\psi + \sum_{i,j=1}^{n}(-e_i\cdot e_j - 2\delta_{ij})\cdot(\nabla_{e_i} W)(e_j)\cdot\psi \\
&= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + 2d\mathrm{tr}(W)\cdot\psi + 2\delta W\cdot\psi,
\end{aligned}$$

which implies simultaneously (2.11) and (2.12) (indeed, if $f\psi = X\cdot\psi$ for some real f and vector X, then $-|X|^2\psi = X\cdot X\cdot\psi = X\cdot(f\psi) = f^2\psi$, so both f and X vanish). ☐

Theorem 3.2. *Let $(\mathcal{Z}, g^{\mathcal{Z}})$ be a Ricci-flat spin manifold with Levi-Civita connection $\nabla^{\mathcal{Z}}$ and let $M \subset \mathcal{Z}$ be any oriented analytic hypersurface. Assume there exists some spinor $\psi \in C^\infty(\Sigma\mathcal{Z}|_M)$ which is parallel along M:*

$$\nabla^{\mathcal{Z}}_X \psi = 0, \qquad \forall X \in TM \subset T\mathcal{Z}. \tag{3.4}$$

Assume moreover that the application $\pi_1(M) \to \pi_1(\mathcal{Z})$ induced by the inclusion is surjective. Then there exists a parallel spinor $\Psi \in C^\infty(\Sigma\mathcal{Z})$ such that $\Psi|_M = \psi$.

Proof. Any Ricci-flat manifold is analytic, cf. [27], [16], thus the analyticity of M makes sense. The proof is split in two parts.

Local extension. Let ν denote the unit normal vector field along M. Every $x \in M$ has an open neighborhood V in M such that the exponential map $(-\varepsilon, \varepsilon) \times V \to \mathcal{Z}$, $(t, y) \mapsto \exp_y(t\nu)$ is well-defined for some $\varepsilon > 0$. Its differential at $(0, x)$ being the identity, one can assume, by shrinking V and choosing a smaller ε if necessary, that it maps $(-\varepsilon, \varepsilon) \times V$ diffeomorphically onto some open neighborhood U of x in $\mathcal{Z}$. We extend the spinor ψ to a spinor Ψ on U by parallel transport along the normal geodesics $\exp_y(t\nu)$ for every fixed y. It remains to prove that Ψ is parallel on U in horizontal directions.

Let $\{e_i\}$ be a local orthonormal basis along M. We extend it on U by parallel transport along the normal geodesics, and notice that $\{e_i, \nu\}$ is a local orthonormal basis on U. More generally, every vector field X along V gives rise to a unique horizontal vector field, also denoted X, on U satisfying $\nabla_\nu X = 0$. For every such vector field we get

$$\nabla^{\mathcal{Z}}_\nu(\nabla^{\mathcal{Z}}_X \Psi) = R^{\mathcal{Z}}(\nu, X)\Psi + \nabla^{\mathcal{Z}}_{[\nu,X]}\Psi = R^{\mathcal{Z}}(\nu, X)\Psi + \nabla^{\mathcal{Z}}_{W(X)}\Psi. \tag{3.5}$$

Since $\mathcal{Z}$ is Ricci-flat, (3.2) applied to the local orthonormal basis $\{e_i, \nu\}$ of $\mathcal{Z}$ yields

$$0 = \frac{1}{2}\mathrm{Ric}^{\mathcal{Z}}(X)\cdot\Psi = \sum_{i=1}^{n} e_i\cdot R^{\mathcal{Z}}(e_i, X)\Psi + \nu\cdot R^{\mathcal{Z}}(\nu, X)\Psi\,. \tag{3.6}$$

We take the Clifford product with ν in this relation, differentiate again with respect to ν and use the second Bianchi identity to obtain:

$$\begin{aligned}\nabla^{\mathcal{Z}}_\nu(R^{\mathcal{Z}}(\nu, X)\Psi) =& \nabla^{\mathcal{Z}}_\nu\left(\nu\cdot\sum_{i=1}^{n} e_i\cdot R^{\mathcal{Z}}(e_i, X)\Psi\right) = \nu\cdot\sum_{i=1}^{n} e_i\cdot(\nabla^{\mathcal{Z}}_\nu R^{\mathcal{Z}})(e_i, X)\Psi\\ =&\nu\cdot\sum_{i=1}^{n} e_i\cdot\big((\nabla^{\mathcal{Z}}_{e_i} R^{\mathcal{Z}})(\nu, X)\Psi + (\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}})(e_i, \nu)\Psi\big),\end{aligned}$$

whence

$$\begin{aligned}\nabla^{\mathcal{Z}}_\nu(R^{\mathcal{Z}}(\nu, X)\Psi) = \nu\cdot\sum_{i=1}^{n} e_i\cdot\big(&\nabla^{\mathcal{Z}}_{e_i}(R^{\mathcal{Z}}(\nu, X)\Psi) + R^{\mathcal{Z}}(W(e_i), X)\Psi - R^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_{e_i}X)\Psi\\ &- R^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_{e_i}\Psi + \nabla^{\mathcal{Z}}_X(R^{\mathcal{Z}}(e_i, \nu)\Psi) - R^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X e_i, \nu)\Psi\\ &+R^{\mathcal{Z}}(e_i, W(X))\Psi - R^{\mathcal{Z}}(e_i, \nu)\nabla^{\mathcal{Z}}_X\Psi\big)\,.\end{aligned} \tag{3.7}$$

Let $\nu^\perp$ denote the distribution orthogonal to ν on U and consider the sections $A, B \in C^\infty((\nu^\perp)^* \otimes \Sigma U)$ and $C \in C^\infty(\Lambda^2(\nu^\perp)^* \otimes \Sigma U)$ defined for all $X, Y \in \nu^\perp$ by

$$A(X) := \nabla^{\mathcal{Z}}_X\Psi, \qquad B(X) := R^{\mathcal{Z}}(\nu, X)\Psi, \qquad C(X, Y) := R^{\mathcal{Z}}(X, Y)\Psi.$$

We have noted that the metric $g^{\mathcal{Z}}$ is analytic since it is Ricci-flat. From the assumption that M is analytic and that ψ is parallel along M it follows that Ψ, and thus the tensors A, B and C, are analytic.

Equations (3.5) and (3.7) read in our new notation:

$$(\nabla^{\mathcal{Z}}_\nu A)(X) = B(X) + A(W(X)), \tag{3.8}$$

and

$$
\begin{aligned}
(\nabla^{\mathcal{Z}}_\nu B)(X) = \nu\cdot \sum_{i=1}^n e_i\cdot\big((\nabla^{\mathcal{Z}}_{e_i} B)(X) + C(W(e_i), X) - R^{\mathcal{Z}}(\nu, X)A(e_i) \\
- (\nabla^{\mathcal{Z}}_X B)(e_i) + C(e_i, W(X)) - R^{\mathcal{Z}}(e_i, \nu)A(X)\big).
\end{aligned} \tag{3.9}
$$

Moreover, the second Bianchi identity yields

$$
\begin{aligned}
(\nabla^{\mathcal{Z}}_\nu C)(X, Y) =& (\nabla^{\mathcal{Z}}_\nu R^{\mathcal{Z}})(X, Y)\Psi = (\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}})(\nu, Y)\Psi + (\nabla^{\mathcal{Z}}_Y R^{\mathcal{Z}})(X, \nu)\Psi \\
=& \nabla^{\mathcal{Z}}_X(R^{\mathcal{Z}}(\nu, Y)\Psi) - R^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X\nu, Y)\Psi - R^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_X Y)\Psi - R^{\mathcal{Z}}(\nu, Y)\nabla^{\mathcal{Z}}_X\Psi \\
& - \nabla^{\mathcal{Z}}_Y(R^{\mathcal{Z}}(\nu, X)\Psi) + R^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_Y\nu, X)\Psi + R^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_Y X)\Psi + R^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_Y\Psi \\
=& (\nabla^{\mathcal{Z}}_X B)(Y) + C(W(X), Y) - R^{\mathcal{Z}}(\nu, Y)\nabla^{\mathcal{Z}}_X\Psi \\
& - (\nabla^{\mathcal{Z}}_Y B)(X) + C(X, W(Y)) + R^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_Y\Psi,
\end{aligned}
$$

thus showing that

$$
\begin{aligned}
(\nabla^{\mathcal{Z}}_\nu C)(X, Y) =& (\nabla^{\mathcal{Z}}_X B)(Y) + C(W(X), Y) - R^{\mathcal{Z}}(\nu, Y)(A(X)) \\
& - (\nabla^{\mathcal{Z}}_Y B)(X) + C(X, W(Y)) + R^{\mathcal{Z}}(\nu, X)(A(Y)).
\end{aligned} \tag{3.10}
$$

The hypothesis (3.4) is equivalent to $A = 0$ for $t = 0$. Differentiating this again in the direction of M and skew-symmetrizing yields $C = 0$ for $t = 0$. Finally, (3.6) shows that $B = 0$ for $t = 0$. We thus see that the section $S := (A, B, C)$ vanishes on along the hypersurface $\{0\} \times V$ of U.

The system (3.9)–(3.10) is a linear PDE for S and the hypersurfaces $t =$ constant are clearly non-characteristic. The Cauchy-Kowalewskaya theorem shows that S vanishes everywhere on U. In particular, $A = 0$ on U, thus proving our claim.

Global extension. Now we prove that there exists a parallel spinor $\Psi \in C^\infty(\Sigma\mathcal{Z})$ such that $\Psi|_M = \psi$. Take any $x \in M$ and an open neighborhood U like in Theorem 3.2 on which a parallel spinor Ψ extending ψ is defined. The spin holonomy group $\widetilde{\mathrm{Hol}}(U, x)$ thus preserves Ψ_x. Since any Ricci-flat metric is analytic (cf. [16, p. 145]), the restricted spin holonomy group $\widetilde{\mathrm{Hol}}_0(\mathcal{Z}, x)$ is equal to $\widetilde{\mathrm{Hol}}_0(U, x)$ for every $x \in \mathcal{Z}$ and for every open neighborhood U of x. By the local extension result proved above, $\widetilde{\mathrm{Hol}}_0(U, x)$ acts trivially on Ψ_x, thus showing that Ψ_x can be extended (by parallel transport along every curve in $\tilde{\mathcal{Z}}$ starting from x) to a parallel spinor $\tilde\Psi$ on the universal cover $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$. The deck transformation group acts trivially on $\tilde\Psi$ since every element in $\pi_1(\mathcal{Z}, x)$ can be represented by a curve in M (here we use the surjectivity hypothesis) and Ψ was assumed to be parallel along M. Thus $\tilde\Psi$ descends to $\mathcal{Z}$ as a parallel spinor. □

This result, together with Corollary 3.1 yields the solution to the analytic Cauchy problem for parallel spinors stated in Theorem 1.1.

4. Construction of generalized Killing spinors

The goal of this section is to describe a method which yields generalized Killing spinors on many 3-dimensional spin Riemannian manifolds. We will obtain both analytic and non-analytic generalized Killing spinors. The analytic ones will yield examples for applying Theorem 1.1. The non-analytic ones only yield a formal Taylor series in the sense of Proposition 2.3, and we will show that in general this solution is not the Taylor series of a Ricci-flat metric. Thus we see that the analyticity assumption in Theorem 1.1 cannot be removed. The method consists in combining techniques developed elsewhere. We state below the relevant results and briefly explain the underlying ideas.

Note that further examples of manifolds with generalized Killing spinors which can not be embedded as hypersurfaces in manifolds with parallel spinors were recently constructed (although not explicitly stated), by Bryant [20] in the context of K-structures satisfying the so-called *weaker torsion condition*.

4.1. **Minimizing the first Dirac eigenvalue in a conformal class.**

In [2] and [3] the following problem was studied: Suppose M is an n-dimensional compact spin manifold, $n \geq 2$ endowed with a fixed spin structure. For any metric g on M let D^g be the Dirac operator on M. The spectrum of D^g is discrete, and all eigenvalues have finite multiplicity. The first positive eigenvalue of D^g will be denoted by $\lambda_1^+(g)$. In general, the dimension of the kernel of D^g depends on g, and on many manifolds (in particular on all compact spin manifolds of dimension $n \equiv 0, 1, 3, 7 \mod 8$, $n \geq 3$) metrics g_i are known such that $g_i \to g$ in the C^∞-topology, $\dim\ker D^{g_i} < \dim\ker D^g$ and $\lambda_1^+(g_i) \to 0$. Thus $g \mapsto \lambda_1^+(g)$ is not continuous when defined on the set of all metrics.

We now fix a conformal class $[g_0]$ on M, and only consider metrics $g \in [g_0]$. Then the above properties change essentially. Due to the conformal behavior of the Dirac operator, the dimension of the kernel of D^g is constant on $[g_0]$, and furthermore $[g_0] \to \mathbb{R}_+$, $g \mapsto \lambda_1^+(g)$ is continuous in the C^1-topology. For any positive real number α one has $\lambda_1^+(\alpha^2 g) = \alpha^{-1}\lambda_1^+(g)$. The normalized first positive eigenvalue function $[g_0] \to (0, \infty)$, $g \mapsto \lambda_1^+(g)\mathrm{vol}(M, g)^{1/n}$, is thus scaling invariant and continuous in the C^1-topology. It is unbounded from above, see [8], and bounded from below by a positive constant, see [42] in the case $\ker D^{g_0} = 0$ and [1, 3] for the general case. We introduce

$$\lambda_{\min}^+(M, [g_0]) := \inf_{g\in[g_0]} \lambda_1^+(g)\mathrm{vol}(M, g)^{1/n} > 0. \tag{4.1}$$

If there is a metric of positive scalar curvature in $[g_0]$, then the Yamabe constant

$$Y(M, [g_0]) := \inf_{g\in[g_0]} \mathrm{vol}(M, g)^{(2-n)/n} \int_M \mathrm{Scal}^g \, dv^g \tag{4.2}$$

is positive, and Hijazi's inequality [35, 36] then yields

$$\lambda_{\min}^+(M, [g_0])^2 \geq \frac{n}{4(n-1)} Y(M, [g_0]). \tag{4.3}$$

Example 4.1. If $(M, g_0) = \mathbb{S}^n$ is the sphere S^n with its standard metric σ^n of volume ω_n, then Obata's theorem [47, Prop. 6.2] implies that the infimum in (4.2) is attained in $g = \sigma$, and thus $Y(\mathbb{S}^n) = n(n-1)\,\omega_n^{2/n}$. We obtain $\lambda^+_{\min}(\mathbb{S}^n) \geq \frac{n}{2}\omega_n^{1/n}$. On the other hand (M, σ) carries a Killing spinor to the Killing constant $-1/2$, thus $\lambda^+_1(\sigma) = \frac{n}{2}$. As a consequence, equality is attained in (4.3), the infimum in (4.1) is attained in $g = \sigma$ and $\lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\omega_n^{1/n}$.

Now let (M, g_0) be again arbitrary. By "blowing up a sphere" one can show that $\lambda^+_{\min}(M, [g_0]) \leq \lambda^+_{\min}(\mathbb{S}^n)$, see [1, 6]. This inequality should be seen as a spinorial analogue of Aubin's inequality between the Yamabe constants $Y(M, [g_0]) \leq Y(\mathbb{S}^n) = n(n-1)\,\omega_n^{2/n}$. For the Yamabe constants one even gets a stronger statement: If (M, g_0) is not conformal to the round sphere, then

$$Y(M, [g_0]) < Y(\mathbb{S}^n). \tag{4.4}$$

This inequality leads to a solution of the Yamabe problem, see [41]. It was proved in some cases by Aubin [10]. Later Schoen and Yau [49, 50] could solve the remaining cases, using the positive mass theorem.

It is thus natural to ask the following question which is still open in general.

Question 4.2. *Under the assumption that $(M, [g_0])$ is not conformal to $(\mathbb{S}^n)$, $n \geq 2$, does the inequality*

$$\lambda^+_{\min}(M, [g_0]) < \lambda^+_{\min}(\mathbb{S}^n). \tag{4.5}$$

always hold?

We will explain below that many Riemannian manifolds, in particular "generic" metrics on compact spin 3-dimensional manifolds, do satisfy (4.5). It is interesting to notice that using (4.3) the inequality (4.5) would imply (4.4) without referring to the positive mass theorem.

In analogy to the Yamabe problem which consists in finding a smooth metric attaining the infimum in (4.2), one can try to find a metric attaining the infimum in (4.1). If this infimum is achieved in a metric $g \in [g_0]$, then the corresponding Euler-Lagrange equation provides the existence of an eigenspinor ψ of constant length of eigenvalue $\lambda^+_1(g_0)$. In dimension $n = 3$, such constant-length eigenspinors are generalized Killing spinors, see Subsection 4.3, and – as said above – it is the goal of this section to construct generalized Killing spinors.

Unfortunately, it is unclear whether the infimum in (4.1) can be achieved by a (smooth) metric. However, if we assume that (4.5) holds, and if we allow degenerations in the conformal factor, the infimum is attained. To explain the nature of these possible degenerations precisely, we introduce the following. A generalized metric in the conformal class $[g_0]$ is a metric of the form $f^2 g_0$ where f is continuous on M and smooth on $M^* := M \setminus f^{-1}(0)$. Moreover, we only admit such generalized metrics for which M^* is dense in M. The set of all such admissible generalized metric associated to the conformal class $[g_0]$ will be denoted by $\overline{[g_0]}$.

Remark 4.3. The above definitions are slight more restrictive than in [2], but sufficient for the purpose of the present article and didactically simpler. For example, the condition that M^* is supposed to be dense, guarantees that $\overline{[g_0]} \cap \overline{[g_1]} = \emptyset$ if g_0 and g_1 are not conformal.

The functions $\lambda_+^1, \mathrm{vol} : [g_0] \to \mathbb{R}^+$ extend continuously to functions $\overline{[g_0]} \to \mathbb{R}^+$, and the infimum in (4.1) does not change when we replace $[g_0]$ by $\overline{[g_0]}$. We then have

Theorem 4.4 ([2, Theorem 1.1(B)]). *Let (M, g_0) be a compact Riemannian spin manifold of dimension $n \geq 2$. There exists a generalized metric $g \in \overline{[g_0]}$ at which the infimum in (4.1) is attained. On (M^*, g) there exists a spinor ψ of constant length with $D\psi = \lambda_+^1(g)\psi$.*

The key idea in the proof of this theorem is to reformulate the problem of minimizing (4.1) as a variational problem. For this we define

$$\mathcal{F}_q(\phi) = \frac{\int \langle D^{g_0}\phi, \phi\rangle_{g_0} \,\mathrm{dvol}^{g_0}}{\|D^{g_0}\phi\|^2_{L^q(g_0)}}, \qquad \mu_q^{g_0} := \sup \mathcal{F}_q^{g_0}(\phi), \tag{4.6}$$

where the supremum runs over all spinors ϕ of regularity C^1 which are not in the kernel of D^{g_0}. It was shown in [2, Prop. 2.3] that for $q = \frac{2n}{n+1}$ we have

$$\mu^{g_0}_{2n/(n+1)} = \frac{1}{\lambda^+_{\min}(M, g_0)}.$$

Furthermore the infimum in (4.1) is attained in a smooth metric $g \in [g_0]$ if and only if there is a nowhere vanishing spinor ψ_0 which attains the supremum in (4.6). If the infimum is attained in g and the supremum in ψ_0, then both are related via

$$g = |D^{g_0}\psi_0|^{4/(n+1)} g_0. \tag{4.7}$$

Proposition 4.5 ([2, Theorem 1.1 (A)]). *Under the condition* (4.5) *the supremum is attained in a spinor ψ_0 of regularity $C^{2,\alpha}$ for small $\alpha > 0$.*

The strategy of proof is similar to the classical approach to the Yamabe problem as e.g. in [41]. A maximizing sequence for the functional will in general not converge, due to conformal invariance. One then defines “perturbed” or “regularized” modifications of this functional such that their maximizing sequences converge to a maximizer. In a final step one shows, assuming (4.5), that the maximizers of the perturbed functionals converge to a maximizer of the unperturbed functional.

Let us now continue with the sketch of proof of Theorem 4.4. From Prop. 4.5 we know that the supremum of $\mathcal{F}$ is attained at some spinor ψ_0 which satisfies an Euler-Lagrange equation. By suitably rescaling ψ_0 and by possibly adding an element of $\ker D^{g_0}$ to ψ_0, the Euler-Lagrange equation reads

$$D^{g_0}\psi_0 = \lambda^+_{\min}(M, g_0)|\psi_0|^{2/(n-1)}\psi_0, \qquad \|\psi_0\|_{L^{2n/(n-1)}(g_0)} = 1.$$

However, it is unclear whether $D^{g_0}\psi_0$ (or equivalently ψ_0) has zeros or not, and therefore if the metric g defined in (4.7) makes sense.

We will show in the following subsection that the zero set is nowhere dense, in other words its complement is dense. Then $g := |D^{g_0}\psi_0|^{4/(n+1)}g_0$ defines a generalized metric, and by naturally extending the definition of λ_1^+ to generalized metrics, we see that the infimum in (4.1) is then attained in this generalized metric.

Consistently with the above we set $M^* := M \setminus \psi_0^{-1}(0)$. From the standard formula for the behavior of the Dirac operator under conformal change (see e.g. [37]) the spinor $\psi := \frac{\psi_0}{|\psi_0|}$ on M^* satisfies

$$D^g\psi = \lambda^+_{\min}(M,[g_0])\psi, \qquad |\psi| \equiv 1.$$

This finishes the proof for Theorem 4.4, up to the density of M^* explained below.

4.2. **The zero set of the maximizing spinor.** The goal of this subsection is to study the zero set of the maximizing spinor ψ_0 from the previous section.

Lemma 4.6. *Let (M, g_0) be a connected Riemannian spin manifold. Assume that a spinor ϕ of regularity C^1 satisfies*

$$D^{g_0}\phi = c|\phi|^r\phi \tag{4.8}$$

where $r \geq 0$ and $c \in \mathbb{R}$. If ϕ vanishes on a non-empty open set, then it vanishes on M.

Applying the lemma to $\phi := \psi_0 \not\equiv 0$ and $r := 2/(n-1)$ one obtains the density of M^* in M.

Proof. The lemma is a special case of the weak unique continuation principle [18]. More exactly we apply [18, Theorem 2.7] with $\not{D}_A = D^{g_0}$ and $\mathfrak{P}_A(\phi,x) := -|\phi(x)|^r$. As ϕ is locally bounded, we see that $x \mapsto \mathfrak{P}_A(\phi,x)$ is locally bounded as well. Thus $\mathfrak{P}_A$ is an admissible perturbation in the sense of [18], and [18, Theorem 2.7] then yields the weak unique continuation principle for this equation which is exactly the statement of the lemma. ☐

We propose two conjectures around the above lemma.

The first conjecture relies on the following remark: if r is an even integer, then $|\phi|^r\phi$ is a smooth function of ϕ, so the Main Theorem in [14] shows that the zero set of ϕ is a countably $(n-2)$-rectifiable set, and thus of Hausdorff dimension at most $n-2$. In contrast, if r is not an even integer, then Bär's method of proof does not apply, but the result seems likely to remain true.

Conjecture 4.7. The zero set of any solution of (4.8) is of Hausdorff dimension at most $n-2$.

The second conjecture is motivated from the following, cf. [34]: for generic metrics on a compact 2- or 3-dimensional spin manifold all eigenspinors, i.e. all non-trivial solutions of (4.8) with $r = 0$, do not vanish anywhere; in other words they are everywhere non-zero.

We conjecture that the same fact is true for $r := \frac{2}{(n-1)}$. This constant r is special, as then (4.8) and thus the zero set of ϕ is conformally invariant.

Conjecture 4.8. Let $r := \frac{2}{(n-1)}$, and let M be connected. For generic conformal classes on M, any solution of (4.8) with $\phi \not\equiv 0$ is everywhere non-zero.

If Conjecture 4.7 holds and if M is connected, then the manifold $M \setminus \phi^{-1}(0)$ is connected. Fortunately, for the maximizing spinor ψ_0 the following fact can be proven independently of the above conjectures:

Lemma 4.9. *Assume M to be connected. Let ψ_0 be the maximizing spinor provided by Proposition 4.5. Then $M^* = M \setminus \psi_0^{-1}(0)$ is connected.*

Proof. Assume that there exists a partition $M^* = \Omega_1 \sqcup \Omega_2$ into non-empty disjoint open sets. We define the continuous spinor ψ_1 by $\psi_1|_{\Omega_1} := \psi_0|_{\Omega_1}$ and $\psi_1|_{M\setminus\Omega_1} :\equiv 0$. Then $\|\psi_1\|_{L^{2n/(n-1)}} < \|\psi_0\|_{L^{2n/(n-1)}}$. As a first step we prove by contradiction that ψ_1 is C^1, or equivalently that $\nabla\psi_0 = 0$ on $\partial\Omega_1$.

Suppose that there existed $x \in \partial\Omega_1 \cap \partial\Omega_2$ such that $\nabla\psi_0$ is non-zero in x. Because of $(D\phi)(x) = 0$ the map $T_xM \to \Sigma_xM$, $X \mapsto \nabla_X\psi_0$ has rank at least 2. The implicit function theorem then implies that there is a connected open neighborhood U of x and a submanifold $S \subset U$ of codimension 2 such that $\psi_0^{-1}(0) \cap U \subset S$. This implies that $U \setminus S \subset \Omega_1$. One easily concludes that $S \cap \Omega_2 = \emptyset$, thus we obtain the contradiction $x \notin \partial\Omega_2$.

We have proven that ψ_1 is C^1, and thus ψ_1 is a solution to

$$D^{g_0}\psi_1 = \lambda^+_{\min}(M, [g_0])|\psi_1|^{2/(n-1)}\psi_1 \qquad 0 < \|\psi_1\|_{L^{2n/(n-1)}(g_0)} < 1.$$

A straightforward calculation then yields

$$\mathcal{F}_{2n/(n+1)}(\psi_1) > \frac{1}{\lambda^+_{\min}(M, [g_0])} = \mu^{g_0}_{2n/(n+1)}$$

which contradicts the definition of $\mu^{g_0}_{2n/(n+1)}$. □

4.3. **From eigenspinors of constant length to generalized Killing spinors.** In this section we specialize to the case $n = 3$. We will see that in this dimension any eigenspinor of constant length is a generalized Killing spinor.

Proposition 4.10. *Let ψ be a solution of $D\psi = H\psi$, $H \in C^\infty(M)$, of constant length* 1*, on a manifold of dimension $n = 3$. Then ψ is a generalized Killing spinor.*

This proposition is the natural generalization of a result in [30] from $n = 2$ to $n = 3$. We will include a simple proof here.

Proof. Let g be the metric on M and $\langle\cdot,\cdot\rangle$ the real part of the Hermitian metric on ΣM. We define $A \in \mathrm{End}(TM)$ by

$$g(A(X), Y) := \langle\nabla_X\psi, Y \cdot \psi\rangle$$

for all $X, Y \in TM$. Note that for any point $p \in M$ and any vector $X \in T_pM$ we have

$$\langle \nabla_X \psi, \psi \rangle = \frac{1}{2} \partial_X \langle \psi, \psi \rangle = 0,$$

in other words $\nabla_X \psi \in \psi^\perp = \{\phi \in \Sigma_p M \mid \langle \phi, \psi \rangle = 0\}$. Let e_1, e_2, e_3 be an orthonormal basis of T_pM. By possibly changing the order of this basis, we can achieve $e_1 \cdot e_2 \cdot e_3 = 1$ in the sense of endomorphisms of ΣM. The spinors $e_1 \cdot \psi$, $e_2 \cdot \psi$ and $e_3 \cdot \psi$ form an orthonormal system of $\psi^\perp$, and because of $\dim_{\mathbb{R}} \psi^\perp = 3$, it is a basis. It follows $\nabla_X \psi = A(X) \cdot \psi$.

Furthermore

$$\begin{aligned}
\langle A(e_2), e_1 \rangle &= \langle \nabla_{e_2} \psi, e_1 \cdot \psi \rangle = \langle e_2 \cdot \nabla_{e_2} \psi, \underbrace{e_2 \cdot e_1 \cdot \psi}_{e_3 \cdot \psi} \rangle \\
&= H \underbrace{\langle \psi, e_3 \cdot \psi \rangle}_{=0} - \langle e_1 \cdot \nabla_{e_1} \psi, e_3 \cdot \psi \rangle - \langle e_3 \cdot \nabla_{e_3} \psi, e_3 \cdot \psi \rangle \\
&= \langle e_3 \cdot e_1 \cdot \nabla_{e_1} \psi, \psi \rangle - \langle \nabla_{e_3} \psi, \psi \rangle = -\langle e_2 \cdot \nabla_{e_1} \psi, \psi \rangle \\
&= \langle A(e_1), e_2 \rangle
\end{aligned}$$

and similarly $\langle A(e_1), e_3 \rangle = \langle A(e_3), e_1 \rangle$ and $\langle A(e_2), e_3 \rangle = \langle A(e_3), e_2 \rangle$. Thus A is symmetric. □

Summarizing our knowledge until now, we have:

Corollary 4.11. *Assume that (M, g_0) is a compact connected spin manifold of dimension $n = 3$ satisfying $\lambda^+_{\min}(M, [g_0]) < \lambda^+_{\min}(\mathbb{S}^3) = \frac{3}{2}(2\pi^2)^{1/3}$. Then there is*

(1) an open, connected and dense subset M^*
(2) a metric g on M^* conformal to $g_0|_{M^*}$ and of volume 1,
(3) an eigenspinor ψ to D^g to the real eigenvalue $\lambda^1_+(g)$

such that ψ has constant length and thus is a generalized Killing spinor on (M^*, g). We obtain a selfadjoint section A of $\mathrm{End}(TM)$ such that $\nabla_X \psi = A(X) \cdot \psi$ and $\mathrm{tr} A = -\lambda^1_+(g)$.

4.4. **Analytic manifolds.**

Definition 4.12. Let g_1 be a Riemannian metric on a smooth manifold M. We say that $[g_1]$ is an *analytic conformal class* if M has a compatible structure of a (real-)analytic manifold for which one of the following equivalent statements holds:

(a) there is a (real-)analytic metric $h \in [g_1]$
(b) for any point $x \in M$ there is an open set $U \ni x$, such that there is an analytic metric g^U on U with $g^U \in [g_1|_U]$.

Lemma 4.13. *Conditions (a) and (b) in Definition 4.12 are equivalent.*

Proof. The implication from (a) to (b) is trivial. The implication from (b) to (a) is a direct consequence of uniformization in dimension $n = 2$, thus we restrict to the case $n \geq 3$.

Let g be a smooth metric in the given analytic conformal class. We have to show that the locally defined metrics g^U provided by (b) can be deformed conformally such that they match together to a globally defined metric. Let $L_p := \{\lambda g_p \,|\, \lambda > 0\} \subset T_p^*M \otimes T_p^*M$, and let $L := \bigcup L_p$. The bundle $\pi : L \to M$ is a smooth $\mathbb{R}^+$-principal bundle over M. All local Riemannian metrics g^U are local sections of $\pi : L \to M$, $g^U : U \to L$. If two local analytic metrics g^U and $g^{\tilde{U}}$ are given, then there is an analytic function $f : \tilde{U} \cap U \to \mathbb{R}^+$ such that $g^U = f g^{\tilde{U}}$ on $\tilde{U} \cap U$. Consequently, $\pi : L \to M$ carries a structure of analytic $\mathbb{R}^+$-principal bundle over M, and thus the total space L of the bundle is an analytic manifold. The smooth map $g : M \to L$ can be approximated in the strong C^1-topology by an analytic map $g^\omega : M \to L$, see [38, Chap. 2, Theorem 5.1] which is proven by Grauert and Remmert in [32]. The map $\pi \circ g^\omega : M \to M$ is a smooth analytic map, that is close to the identity in the C^1-topology, and thus (for suitably chosen g^∞) it is an analytic diffeomorphism.

As a consequence, the map $g^\omega \circ (\pi \circ g^\omega)^{-1} : M \to L$ is an analytic section of L and thus an analytic representative of the given conformal class. □

Lemma 4.14. *If an analytic conformal class is conformally flat on a non-empty open set U, and if M is connected, then the conformal class is already conformally flat on M.*

Proof. Being conformally flat on an open set U is equivalent to the vanishing of the Weyl curvature (resp. Schouten tensor) in dimension $m \geq 4$ (resp. $m = 3$). The Weyl curvature and the Schouten tensor of an analytic metric are analytic as well. Thus if they vanish on U they must vanish on all of M. □

Lemma 4.15. *Let ϕ be a smooth solution of $D\phi = c|\phi|^\alpha \phi$, $\phi \neq 0$ on a (not necessarily complete) analytic Riemannian spin manifold (U, g, χ). Then ϕ is analytic as well.*

Proof. The equation is an elliptic semi-linear equation, and has analytic coefficients on the set $M \setminus \phi^{-1}(0)$. We apply analytic regularity results for properly elliptic systems as developed by Douglis and Nirenberg and refined by Morrey, see [45] and [46]. To apply these tools it is convenient to deduce a second order equation

$$D^2\phi = \left(c^2|\phi|^{2\alpha} + c\,\mathrm{grad}(|\phi|^\alpha)\right) \cdot \phi$$

which has again analytic coefficients on $M \setminus \phi^{-1}(0)$. The linearization of this second order equation has the principal symbol of a Laplacian and is thus properly elliptic. The lemma then follows directly from [46, Theorem 6.8.1] or [45]. □

Lemma 4.16. *Constant mean curvature hypersurfaces in an analytic Riemannian manifolds are analytic. In particular, the metric and the second fundamental form of such a hypersurface are analytic.*

Proof. Let M be an n-dimensional hypersurface in an analytic Riemannian manifold $(\mathcal{Z}, h)$ of dimension $n+1$. We choose an analytic parametrization $U \times (a, b) \to \mathcal{Z}$ with U open in $\mathbb{R}^n$, such that locally the hypersurface M is the graph of a function $F : U \to (a, b)$. The standard basis of $\mathbb{R}^{n+1}$ is denoted by $e_1, \ldots, e_{n+1}$. The tangent space $T_{(x,F(x))}M$ is then spanned by $(e_i, \partial_i F)$, $i = 1, \ldots, n$.

Let $h_{ij} \in C^\omega(U \times (a,b))$ be the coefficients of the metric h, and let $g_{ij} \in C^\infty(U)$ be the coefficients of g. The inverse matrices are denoted by $(h^{ij})_{1\le i,j\le n+1}$ and $(g^{ij})_{1\le i,j\le n}$.

The first fundamental form of the hypersurface in the chart given by U is

$$g_{ij} = h_{ij} + h_{n+1,j}\partial_i F + h_{n+1,i}\partial_j F + h_{n+1,n+1}(\partial_i F)(\partial_j F).$$

The coefficients of the matrices (g_{ij}) and (g^{ij}) are thus polynomial expressions in h, F and dF. The vector field

$$X := \sum_{i=1}^{n+1}\Big(\Big(-\sum_{j=1}^{n} h^{ij}\partial_j F\Big) + h^{i,n+1}\Big)e_i$$

is normal to M, and both X and the unit normal vector field $\nu := X/|X|_h$ are analytic expressions in h, F and dF.

The second fundamental form has the coefficients

$$\begin{aligned} k_{ij} &= -\langle \nabla_{(e_i,\partial_i F)}\nu, (e_j, \partial_j F)\rangle \\ &= -\frac{1}{|X|_h}\langle \nabla_{(e_i,\partial_i F)}X, (e_j, \partial_j F)\rangle \\ &= \frac{1}{|X|_h}\left(\partial_i\partial_j F + \mathcal{F}(h, dh, F, dF)\right), \end{aligned}$$

where $1 \le i, j \le n$ and $\mathcal{F}$ is a polynomial expression in its arguments.

The mean curvature H is given as $H = \frac{1}{n}\sum_{ij} g^{ij}k_{ij}$. Thus the mean curvature operator $P : F \mapsto H$ is a quasi-linear second order differential operator with analytic coefficients.

We fix a function $\tilde F$ describing a hypersurface of constant mean curvature, the corresponding normal field will be denoted by $\tilde X$. In other words $P(\tilde F)$ is a constant.

The linearization $\hat P := T_{\tilde F}P$ of P in $\tilde F$ is a linear second order differential operator with principal symbol

$$\mathbb{R}^m \to \mathbb{R}, \qquad \xi \mapsto \frac{|\xi|^2}{|\tilde X|_h}.$$

Thus P is (properly) elliptic in a neighborhood of 0.

The analytic regularity theorem for elliptic systems of Morrey [46, Theorem 6.8.1] or [45] tells us that $\tilde F$ is analytic, and this implies the lemma. □

4.5. **Three-dimensional real projective space.** In this and in the following subsection we provide examples of compact Riemannian spin manifolds satisfying (4.5). In the present subsection we study deformations of round metrics on $\mathbb{R}P^3$ with a suitable spin structure. This already provides examples of non-analytic Riemannian manifolds with generalized Killing spinor, showing the necessity of the analyticity assumption in Theorems 1.1 and 2.1. In the following section we will then see that such examples are abundant.

Lemma 4.17. *If M is a compact spin manifold, we denote the set of metrics with invertible Dirac operator as $\mathcal{R}^{\mathrm{inv}}(M)$, equipped with the C^1-topology. Then the function*

$$\mathcal{R}^{\mathrm{inv}}(M) \to \mathbb{R}^+, \qquad g \mapsto \lambda_1^+(g)$$

is continuous.

This lemma is a special case of [13, Prop. 7.1], see also [48, Kor. 1.3.3] for more details.

Let us equip SU(2) with the unique bi-invariant metric of sectional curvature 1, hence SU(2) is isometric to $\mathbb{S}^3$. The left multiplication of SU(2) on itself lifts to an action of SU(2) on $\Sigma\,\mathrm{SU}(2)$, for any choice of orientation of SU(2) and any choice of the spinor representation. The spinor bundle is then trivialized by left-invariant spinors. A straightforward calculation, see e.g. [3], shows that

$$\nabla_X \phi = \pm\frac{1}{2} X \cdot \phi$$

for any left-invariant spinor ϕ and all $X \in T\,\mathrm{SU}(2)$. Thus all left-invariant spinors are Killing spinors to the Killing constant $\pm 1/2$. The sign depends on the choice of orientation and on the choice of spinor representation. The same discussion also applies to right-invariant spinors, and these are Killing spinors whose Killing constant have the opposite sign. We assume that these choices are made such that left-invariant spinors have Killing constant $-1/2$, and thus right-invariant ones have Killing constant $+1/2$.

If Γ is a non-trivial discrete subgroup of SU(2), we choose a spin structure on $\Gamma\backslash\,\mathrm{SU}(2)$ such that left-invariant spinors on $\mathbb{S}^3$ descend to $\Gamma\backslash\,\mathrm{SU}(2)$. Then $\Gamma\backslash\,\mathrm{SU}(2)$ carries a complex 2-dimensional space of Killing spinors with Killing constant $-1/2$, but no non-trivial Killing spinor with Killing constant $1/2$. For quotients $\mathrm{SU}(2)/\Gamma$, the role of $1/2$ and $-1/2$ have to be exchanged. All other (Riemannian) quotients of $\mathbb{S}^3$ do not carry any non-trivial Killing spinor.

In the special case $\Gamma = \{\pm\mathbb{1}\}$ both quotients $\Gamma\backslash\,\mathrm{SU}(2)$ and $\mathrm{SU}(2)/\Gamma$ are isometric to $\mathbb{R}P^3$, but they come with different spin structures. These are the 2 non-equivalent spin structures on $\mathbb{R}P^3$. We thus have obtained:

Lemma 4.18. *Let σ^3 be the standard metric on 3-dimensional real projective space $\mathbb{R}P^3$. There are two spin structures on $\mathbb{R}P^3$. For one spin structure Killing spinors to the constant $-1/2$ exist, but not for the constant $1/2$. For the other spin structure Killing spinors to the constant $1/2$ exist, but not for the constant $-1/2$.*

Thus for a suitable choice of spin structure, we have

$$\lambda^+_{\min}(\mathbb{R}P^3, [\sigma^3]) = \frac{3}{2}\left(\frac{\omega_3}{2}\right)^{1/3} = \frac{3\pi^{2/3}}{2} < \lambda^+_{\min}\mathbb{S}^3 = \frac{3}{2}\omega_3^{1/3} = \frac{3\pi^{2/3}}{2^{2/3}}.$$

Corollary 4.19. *There is a non-analytic conformal class and a spin structure on $\mathbb{R}P^3$ for which inequality (4.5) holds.*

Proof. We choose a metric g_1 close to σ^3 on $\mathbb{R}P^3$ which is conformally flat on some non-empty open set U_1 and non-conformally flat on some non-empty open set U_2. If g were an analytic metric, conformal to g_1, then g would have a vanishing Schouten tensor on U_1. By Lemma 4.14 it would be flat everywhere, thus also on U_2. This shows that $[g_1]$ is a non-analytic conformal class. Applying the previous lemmata, we obtain the corollary. □

4.6. **The mass endomorphism and application to inequality (4.5).** The goal of this subsection is to prove that inequality (4.5) holds for "generic" metrics, in a sense explained below.

In this section we assume that M is a compact connected spin manifold of dimension $n \geq 3$, and that the index of M in $KO^{-n}(pt)$ vanishes. We fix a point $p \in M$ and a flat metric g_{flat} in a neighborhood U of p, $U \neq M$. We assume that U is isometric to a convex ball and that g_{flat} can be extended to a metric on M. The set of all such extensions is denoted by $\mathcal{R}_{U,g_{\text{flat}}}(M)$. We define

$$\mathcal{R}^{\text{inv}}_{U,g_{\text{flat}}}(M) := \{g \in \mathcal{R}_{U,g_{\text{flat}}}(M) \,|\, D^g \text{ is invertible}\},$$

i.e. this is the set of all extensions of g_{flat} such that the Dirac operator is invertible. In [5] we proved that $\mathcal{R}^{\text{inv}}_{U,g_{\text{flat}}}(M)$ is open and dense in $\mathcal{R}_{U,g_{\text{flat}}}(M)$ with respect to the C^k-topology where $k \geq 1$ is arbitrary.

Definition 4.20. We say that a property (A) holds for generic metrics in $\mathcal{R}_{U,g_{\text{flat}}}(M)$ if there is a subset $\mathcal{R}' \subset \mathcal{R}_{U,g_{\text{flat}}}(M)$ that is open and dense with respect to the C^k-topology for all $k \geq 1$, such that property (A) holds for all $g \in \mathcal{R}'$.

Using this definition, the above mentioned result from [5] says that the Dirac operator with respect to a generic metric is invertible.

Given a metric $g \in \mathcal{R}^{\text{inv}}_{U,g_{\text{flat}}}(M)$, let G be the Green's function of the Dirac operator on (M, g) at the point $p \in M$, i.e. a distributional solution of

$$DG = \delta_p \mathrm{Id}_{\Sigma_p M}, \tag{4.9}$$

where δ_p is the Dirac distribution at p and G is viewed as a linear map which associates to each spinor in $\Sigma_p M$ a smooth spinor field on $M \setminus \{p\}$ defining a spinor-valued distribution on M. We write G^g and D^g for G and D to indicate their dependence on the metric g.

We also introduce the Euclidean Green's function centered at 0, defined distributionally on $\mathbb{R}^n$

$$G^{\text{eucl}}\psi = -\frac{1}{\omega_{n-1}|x|^n} x \cdot \psi.$$

It satisfies (4.9) for $G = G^{\text{eucl}}$ and $D = D^{\text{eucl}}$ on $\mathbb{R}^n$.

Identifying U with a ball in $\mathbb{R}^n$ via an isometry, both $G = G^g$ and $G = G^{\text{eucl}}$ are solutions of (4.9) on U. Thus $D^g(G^g - G^{\text{eucl}}) = 0$ on U and by elliptic regularity, $G^g - G^{\text{eucl}}$ is a smooth section, see also [7]. We obtain for any $\psi_0 \in \Sigma_p M$:

$$G^g(x)\psi_0 = -\frac{1}{\omega_{n-1}|x|^n} x \cdot \psi_0 + v^g(x)\psi_0,$$

where the spinor field $v^g(x)\psi_0$ is smooth on U and satisfies $D^g(v^g(x)\psi_0) = 0$ on U.

Definition 4.21. The *mass endomorphism* $\alpha^g : \Sigma_p M \to \Sigma_p M$ for a point $p \in U \subset M$ is defined by

$$\alpha^g(\psi_0) := v^g(p)\psi_0.$$

The mass endomorphism is thus (up to a constant) defined as the zero$^{\text{th}}$ order term in the asymptotic expansion of the Green's function in Euclidean coordinates around p. This definition is analogous to the definition of the mass in the Yamabe problem.

Theorem 4.22 ([33] for $n = 3$, [4] for $n \geq 3$)**.** *For generic metrics in $\mathcal{R}_{U,g_{\text{flat}}}(M)$ the mass endomorphism in p is non-zero.*

An important application of this theorem is inequality (4.5). The proofs in [7] yield:

Proposition 4.23. *If the mass endomorphism in a point p with flat neighborhood is non-zero, then $\lambda^+_{\min}(M,[g]) < \lambda^+_{\min}(\mathbb{S}^n)$.*

It follows:

Corollary 4.24. *For generic metrics g in $\mathcal{R}_{U,g_{\text{flat}}}(M)$ we have $\lambda^+_{\min}(M,[g]) < \lambda^+_{\min}(\mathbb{S}^n)$.*

We now deduce:

Corollary 4.25. *Let M be an n-dimensional compact spin manifold with vanishing index $\mathrm{ind}(M) \in KO^{-n}(pt)$. There there is both an analytic conformal class $[g_{\text{an}}]$ and a non-analytic, smooth conformal class $[g_{\text{non}-\text{an}}]$ on M with*

$$\lambda^+_{\min}(M,[g_{\text{an}}]) < \lambda^+_{\min}(\mathbb{S}^n), \qquad \lambda^+_{\min}(M,[g_{\text{non}-\text{an}}]) < \lambda^+_{\min}(\mathbb{S}^n).$$

In this corollary M is a priori equipped with a C^∞-structure and the "non-analyticity" means by definition that M does not carry any analytic structure in which $g_{\text{non}-\text{an}}$ is analytic.

Proof. We choose an open set U and a metric g_{flat} as above. Then choose $g \in \mathcal{R}_{U,g_{\text{flat}}}(M)$ with $\lambda^+_{\min}(M,[g]) < \lambda^+_{\min}(\mathbb{S}^n)$. Choose another smooth metric $g_{\text{non}-\text{an}}$, coinciding on U with $g = g_{\text{flat}}$, such that $g_{\text{non}-\text{an}}$ is not (everywhere) conformally flat on $M \setminus U$, and C^1-close enough to g so that $\lambda^+_{\min}(M,[g_{\text{non}-\text{an}}]) < \lambda^+_{\min}(\mathbb{S}^n)$. The metric $g_{\text{non}-\text{an}}$ is conformally flat on U but not on $M \setminus U$, hence its Schouten tensor cannot be analytic in any analytic structure. Thus as in Lemma 4.14 the conformal class $[g_{\text{non}-\text{an}}]$ cannot be analytic.

At the same time, g can be C^1-approximated by an analytic metric g_{an} so that the inequality $\lambda^+_{\min}(M,[g_{\text{an}}]) < \lambda^+_{\min}(\mathbb{S}^n)$ continues to hold. Such an analytic approximation can be done either with Abresch's smoothing technique or by using the Ricci flow: if g_t is a solution of the Ricci flow equation $\frac{d}{dt}(g_t) = -2\mathrm{Ric}^{g_t}$, defined for short times $t \in [0,t_0)$ with initial data $g_0 = g$, then g_t is analytic for all $t > 0$. We set $g_{\text{an}} := g_t$ for a sufficiently small $t > 0$. $\square$

4.7. **Analytic examples.** Summarizing the results of the preceding subsections we obtain.

Theorem 4.26. *Assume that $(M, [g_{\mathrm{an}}])$ is a compact connected analytic Riemannian spin manifold of dimension 3 with $\lambda^+_{\min}(M, [g_{\mathrm{an}}]) < \lambda^+_{\min}(\mathbb{S}^n)$. Then there is a connected, open and dense subset M^* of M carrying an analytic metric g^* and an analytic spinor field $\psi \in \Gamma(\Sigma^{g^*} M^*)$ such that*

(1) g^* *is conformal to* $g_{\mathrm{an}}|_{M^*}$,
(2) $\mathrm{vol}(M^*, g^*) = 1$,
(3) ψ *is a generalized-Killing spinor on* (M^*, g^*).

Such Riemannian metrics g_{an} exist on each compact 3-dimensional spin manifold, due to the preceding section. The corresponding endomorphism W is then analytic as well, and Theorem 1.1 can be applied. We obtain a Ricci-flat metric of the form $dt^2 + g_t$ where $g_0 = g^*$ defined on an open neighborhood of $\{0\} \times M^*$ in $\mathbb{R} \times M^*$, and carrying a parallel spinor. Further the mean curvature of $\{0\} \times M^*$ in this neighborhood is constant and equal to $(2/3)\lambda^+_{\min}(M, [g])$.

4.8. **Non-analytic examples.** Here we finally prove the existence of metrics with generalized Killing spinors on manifolds with non-analytic metrics. According to Lemma 4.16 such manifolds do not embed isometrically as constant mean curvature hypersurfaces in Ricci-flat manifolds. Thus the analyticity assumptions in Theorem 1.1 cannot be removed.

Theorem 4.27. *Any 3-dimensional compact connected spin manifold M with a fixed C^∞-structure has a connected open dense subset M^* carrying a smooth Riemannian metric g^* with a generalized Killing spinor, such that the metric is not analytic for any choice of analytic structure on M. For this manifold (M^*, g^*) the associated formal solution provided by Proposition 2.3 cannot be chosen to be Ricci-flat on a neighborhood of $\{0\} \times M$, in other words the conclusion of Theorem 1.1 does not hold. If $M = \mathbb{R}P^3$ or more generally if $M = \Gamma \backslash \mathrm{SU}(2)$ for a non-trivial subgroup Γ of $\mathrm{SU}(2)$, then we can find such a Riemannian metric g^* defined on the whole manifold M.*

Proof. By Corollary 4.25 there exists a smooth conformal class $[g_{\mathrm{non-an}}]$ whose Schouten tensor vanishes on a non-empty open set and does not vanish on another open set, and for which $\lambda^+_{\min}(M, [g_{\mathrm{non-an}}]) < \lambda^+_{\min}(\mathbb{S}^3)$. The infimum in (4.1) is then attained, according to Theorem 4.4, at a generalized metric g^*, which is a smooth Riemannian metric on a connected dense open subset M^* of M. It is clear that the restricted conformal class $[g_{\mathrm{non-an}}|_{M^*}]$ is not analytic, and thus the metric g^* cannot be analytic either. Theorem 4.4 provides moreover a Dirac eigenspinor of constant length on (M^*, g^*) which is, due to Subsection 4.3, a generalized Killing spinor. Furthermore, the trace of the associated symmetric tensor $W \in \mathrm{End}(TM)$ is constant and equal to $-(2/3)\lambda^+_{\min}(M, [g_{\mathrm{non-an}}])$.

If the formal solution provided by Proposition 2.3 (for M^* instead of M) were Ricci-flat in a neighborhood of $\{0\} \times M^*$, then M^* would be a hypersurface of constant mean curvature in a 4-dimensional Ricci-flat manifold. As Ricci-flat metrics are analytic in a suitable analytic structure, Lemma 4.16 would imply that g^* was analytic, which is a contradiction.

Now assume that $M = \mathbb{R}P^3$. We take a sequence of non-analytic metrics g_i (constructed similarly as above) converging in the C^∞-topology to the standard round metric σ^3 on $\mathbb{R}P^3$. As the functional $\mathcal{F}_q^g$ depends continuously on g in the C^1-topology, we see for $q = 2n/(n+1)$

$$\overline{\mu} := \liminf_{i\to\infty} \mu_q(\mathbb{R}P^3, g_i) \geq \mu_q(\mathbb{R}P^3, \sigma^3) = \frac{2}{3}\left(\frac{2}{\omega_3}\right)^{2/3} > \mu_q(\mathbb{S}^3) = \frac{2}{3}\left(\frac{1}{\omega_3}\right)^{2/3} \tag{4.10}$$

Now let ψ_i be a maximizing spinor on $(\mathbb{R}P^3, g_i)$ with L^p-norm 1, $p = 2n/(n-1) = 3$. These ψ_i are uniformly bounded in the C^0-norm. This uniform C^0-boundedness follows from [2, Theorem 6.1] whose proof is also valid for $p = 2n/(n-1)$ although the formulation of [2, Theorem 6.1] assumed $p < 2n/(n-1)$. Then [2, Theorem 5.2] implies that ψ_i is a bounded sequence in $C^{1,\alpha}$ for any $\alpha \in (0,1)$.

After passing to a suitable subsequence we then see that ψ_i converges to a solution $\overline{\psi}$ of

$$D^{\sigma^3}\overline{\psi} = \overline{\mu}^{-1}|\overline{\psi}|\overline{\psi}, \qquad \|\overline{\psi}\|_{L^3(\mathbb{R}P^3,\sigma^3)} = 1.$$

Calculating $\mathcal{F}_{2n/(n+1)}(\overline{\psi}) = \bar{\mu}$ we conclude $\overline{\mu} = \mu_q(\mathbb{R}P^3, \sigma^3)$. Using the regularity theorem [2, Prop. 5.1] one sees that $\overline{\psi}$ is C^2. We now apply [2, Prop. 4.1] where $\psi \in \Gamma(\Sigma\mathbb{S}^3)$ is the pullback of $\overline{\psi}$ to $\mathbb{S}^3$. One calculates $\mathcal{F}_{2n/(n+1)}(\psi) = \mu_q(\mathbb{S}^3)$, thus ψ is a maximizing spinor on $\mathbb{S}^3$. The conformal map $A : \mathbb{S}^3 \to \mathbb{S}^3$ in the conclusion of [2, Prop. 4.1] has to be an isometry as it is the lift of a map $\mathbb{R}P^3 \to \mathbb{R}P^3$. Thus [2, Prop. 4.1] implies that $\overline{\psi}$ is a Killing spinor to the Killing constant $-1/2$. As such a Killing spinor nowhere vanishes, ψ_i nowhere vanishes for large i.

The other quotients $\Gamma\backslash\mathrm{SU}(2)$ are completely analogous. $\square$

Using products with manifolds carrying parallel spinors, one can easily obtain in every dimension $n \geq 3$ examples of n-dimensional manifolds with generalized Killing spinors which do not embed isometrically as hypersurfaces in manifolds with parallel spinors. More precisely we have the following:

Lemma 4.28. *Let (M^*, g^*) be a 3-dimensional non-analytic Riemannian manifold with generalized Killing spinors given by Theorem 4.27. Then the Riemannian product $(M^*, g^*) \times (\mathbb{R}^{n-3}, g_{\mathrm{eucl}})$ carries a generalized Killing spinor Ψ but can not be embedded isometrically as a hypersurface in any manifold with parallel spinors which restrict to Ψ.*

Proof. Let $p_1^*(\Sigma M^*)$ and $p_2^*(\Sigma\mathbb{R}^{n-3})$ denote the pullbacks to $Z := M^* \times \mathbb{R}^{n-3}$ of the spin bundles of (M^*, g^*) and $(\mathbb{R}^{n-3}, g_{\mathrm{eucl}})$ with respect to the standard projections. It is a standard fact that the spin bundle ΣZ is isomorphic to $p_1^*(\Sigma M^*) \otimes p_2^*(\Sigma\mathbb{R}^{n-3})$ if n is odd and to $p_1^*(\Sigma M^*) \otimes p_2^*(\Sigma\mathbb{R}^{n-3}) \otimes \mathbb{C}^2$ if n is even, and this isomorphism preserves the spin connections. The isomorphism can be chosen such that in the first case, the Clifford product is given by

$$(X_1, X_2)\cdot(\phi \otimes \psi) = (X_1\cdot\phi) \otimes \psi + \phi \otimes (X_2\cdot\omega^{\mathbb{C}}\cdot\psi),$$

where $\omega^{\mathbb{C}}$ is the complex volume form in the Clifford algebra of $\mathbb{R}^{n-3}$. In the second case, the Clifford product is given by

$$(X_1, X_2)\cdot(\phi \otimes \psi \otimes v) = (X_1\cdot\phi) \otimes \psi \otimes a(v) + \phi \otimes (X_2\cdot\psi) \otimes b(v),$$

for every $v \in \mathbb{C}^2$, where $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The first assertion now follows immediately: take any generalized Killing spinor ϕ on M^* satisfying $\nabla_X\phi = W(X)\cdot\phi$ for all $X \in TM^*$ and let ψ be a parallel spinor on $\mathbb{R}^{n-3}$. One can of course assume that $\omega^{\mathbb{C}}\cdot\psi = \psi$ if n is odd. Then $\Psi := \phi \otimes \psi$ (resp. $\Psi := \phi \otimes \psi \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$) is a generalized Killing spinor on Z for n odd (resp. even), with associated tensor $\bar{W} = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$.

To prove the second assertion, assume that Z is a hypersurface in some spin manifold $\bar{Z}$ and that Φ is a parallel spinor on $\bar{Z}$ restricting to Ψ on Z. The second fundamental form of Z is $\bar{W}$, which has constant trace by construction. Thus Z has constant mean curvature, so is analytic by Lemma 4.16. Each factor of Z is then analytic, contradicting the non-analyticity of M^*. □

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