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NONCOMMUTATIVE FITTING INVARIANTS AND IMPROVED ANNIHILATION RESULTS (PRELIMINARY VERSION)

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ABSTRACT. To each finitely presented module M over a commutative ring R one can associate an R-ideal $\mathrm{Fit}_R(M)$ which is called the (zeroth) Fitting ideal of M over R and which is always contained in the R-annihilator of M. In an earlier article, the second author generalised this notion by replacing R with a (not necessarily commutative) $\mathfrak{o}$-order Λ in a finite dimensional separable algebra, where $\mathfrak{o}$ is an integrally closed complete commutative noetherian local domain. To obtain annihilators, one has to multiply the Fitting invariant of a (left) Λ-module M by a certain ideal $\mathcal{H}(\Lambda)$ of the centre of Λ. In contrast to the commutative case, this ideal can be properly contained in the centre of Λ. In the present article, we determine explicit lower bounds for $\mathcal{H}(\Lambda)$ in many cases. Furthermore, we define a class of 'nice' orders Λ over which Fitting invariants have several useful properties such as good behaviour with respect to direct sums of modules, computability in a certain sense, and $\mathcal{H}(\Lambda)$ being the best possible.

1. Introduction

Let R be a commutative ring (with identity) and let M be a finitely presented R-module. If we choose a presentation

$$R^a \xrightarrow{h} R^b \twoheadrightarrow M \tag{1.1}$$

we may identify the homomorphism h with an $a \times b$ matrix with entries in R. If $a \geq b$, the (zeroth) Fitting ideal of M over R, denoted by $\mathrm{Fit}_R(M)$, is defined to be the R-ideal generated by all $b \times b$ minors of the matrix corresponding to h. If $a < b$ then $\mathrm{Fit}_R(M)$ is defined to be the zero ideal of R. A key point is that this definition is independent of the choice of presentation h. This notion was introduced by H. Fitting [Fit36] and is now a very important tool in commutative algebra thanks to several useful properties. In particular, $\mathrm{Fit}_R(M)$ is always a subset of $\mathrm{Ann}_R(M)$, the R-annihilator of M. Furthermore, $\mathrm{Fit}_R(M)$ is often computable, thanks to being independent of the choice of presentation h and, for example, good behaviour with respect to quotients of R, as well as epimorphisms and direct sums of R-modules. A summary of the properties of Fitting ideals can be found in [MW84, Appendix]; for a full account of the theory, we refer the reader to [Nor76].

It is natural to ask whether analogous invariants can be defined for modules over noncommutative rings; indeed, there have been several attempts to overcome the technical obstacles involved in order to do this. In [Sus88] and [Sus89], J. Susperregui considered two particular cases: skewcommutative graded rings and rings of differential operators satisfying the left Ore property. In his Ph.D. thesis [Gri02], P. Grime considered several

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cases including matrix rings over commutative rings, as well as certain hereditary orders and (twisted) group rings. We say that a (left) R-module M has a quadratic presentation if one can take $a = b$ in (1.1). In the case where G is a finite group and R is a group ring $\mathbb{Z}[G]$, $\mathbb{Z}_{(p)}[G]$, or $\mathbb{Z}_p[G]$ for some prime p, A. Parker in his Ph.D. thesis [Par07] defined noncommutative Fitting invariants for modules with a quadratic presentation.

Let A be a finite dimensional separable algebra over a field F and Λ an $\mathfrak{o}$-order in A, where $\mathfrak{o}$ is an integrally closed complete commutative noetherian local domain with field of quotients F. We call such an order Λ a Fitting order; a standard example is the group ring $\mathbb{Z}_p[G]$ where p is a prime and G is a finite group. We denote by $\zeta(A)$ and $\zeta(\Lambda)$ the centres of A and Λ, respectively. All modules are henceforth assumed to be left modules unless otherwise stated. Let M be a Λ-module admitting a finite presentation

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M.$$

In [Nic10], the Fitting invariant $\mathrm{Fitt}_\Lambda(h)$ is defined to be an equivalence class of a certain $\zeta(\Lambda)$-submodule of $\zeta(A)$ generated by reduced norms. In the case that Λ is commutative, the reduced norm is the same as the usual determinant and this notion is compatible with the classical definition of Fitting ideal described above. In contrast to the commutative case, $\mathrm{Fitt}_\Lambda(h)$ does in general depend on h; however, for a given M there exists a distinguished Fitting invariant $\mathrm{Fitt}^{\max}_\Lambda(M)$ that is maximal among all $\mathrm{Fitt}_\Lambda(h)$. Moreover, if M admits a quadratic presentation h, then $\mathrm{Fitt}_\Lambda(h)$ is independent of the choice of h (as long as h is quadratic) and the definition is compatible with that given by A. Parker in his thesis [Par07]. It is also shown in [Nic10] that $\mathrm{Fitt}^{\max}_\Lambda(M)$ enjoys many of the useful properties of the commutative case (see Theorems 3.2 and 3.5). To obtain annihilators from $\mathrm{Fitt}^{\max}_\Lambda(M)$, one has to multiply by a certain ideal $\mathcal{H}(\Lambda)$ of $\zeta(\Lambda)$; if Λ is commutative or maximal, then $\mathcal{H}(\Lambda) = \zeta(\Lambda)$, but in general $\mathcal{H}(\Lambda)$ is a proper ideal of $\zeta(\Lambda)$. Though much progress is made in [Nic10], several questions remain:

(i) Can $\mathcal{H}(\Lambda)$ be computed or approximated explicitly?
(ii) Does $\mathrm{Fitt}^{\max}_\Lambda(M)$ behave well with respect to direct sums of Λ-modules?
(iii) For a left ideal I of Λ, can we give an explicit formula for $\mathrm{Fitt}^{\max}_\Lambda(\Lambda/I)$?
(iv) Are there certain Fitting orders Λ for which $\mathrm{Fitt}^{\max}_\Lambda(M)$ can be computed from a presentation h of M, independently of the choice of h?

The present article goes some way towards answering these questions. We now describe the contents and main results in more detail. In §2 we consider the case of a matrix ring Λ over an arbitrary commutative ring R (with identity). We use explicit Morita equivalence of Λ and R to define an ideal of R (the definition is essentially equivalent to that of [Gri02, §5.2]), and go on to establish a number of useful properties. This ideal is equal to the usual Fitting ideal in the commutative case (i.e. $\Lambda = R$). We also give a slight sharpening of an existing result on classical Fitting ideals. In §3 we review background material and the main results of [Nic10]. We return to the situation in which Λ is a Fitting order contained in A and introduce $\mathrm{Fit}_\Lambda(h)$ as an alternative to $\mathrm{Fitt}_\Lambda(h)$. The former is a $\zeta(\Lambda)$-submodule of $\zeta(A)$ whereas the latter (originally introduced in [Nic10]) is an equivalence class of such modules; the two definitions are closely related. We define $\mathrm{Fit}^{\max}_\Lambda(M)$ analogously to $\mathrm{Fitt}^{\max}_\Lambda(M)$. Furthermore, we show that $\mathrm{Fit}^{\max}_\Lambda(M)$ is equal to the ideal defined in §2 when Λ is both a Fitting order and a matrix ring over a commutative ring. In §4 we introduce the notion of a 'nice' Fitting order. A Fitting order is defined to be nice if it is a finite direct product of maximal orders and matrix rings over commutative rings. Such an order has particularly useful properties; indeed, the answer to each of questions (i)-(iv) above is affirmative in this case. In particular, if Λ is nice then $\mathcal{H}(\Lambda) = \zeta(\Lambda)$ and so $\mathrm{Fit}_\Lambda^{\max}(M)$ is always a subset of $\mathrm{Ann}_{\zeta(\Lambda)}(M)$. We show that if p is a prime and G is a finite group then the group ring $\mathbb{Z}_p[G]$ is a nice Fitting order if and only if p does not divide the order of the commutator subgroup G'. Moreover, we show a similar result for completed group algebras $\mathbb{Z}_p[[G]]$, where G is a p-adic Lie group of dimension 1. In §5 we explicitly compute the maximal Fitting invariant of the quotient of a Fitting order Λ by a left ideal I when either Λ is nice or I is principal; we give a containment in other cases. In §6 we compute certain conductors and thereby give explicit bounds for $\mathcal{H}(\Lambda)$ in the case that Λ is not nice; we also give further annihilation results relating to change of order. In the Appendix we generalise many of the results of §2 by considering the case where Λ is any ring that is Morita equivalent to a commutative ring R (with identity).

Notation and conventions. All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. We denote the set of all $m \times n$ matrices with entries in a ring R by $M_{m\times n}(R)$ and in the case $m = n$ the group of all invertible elements of $M_{n\times n}(R)$ by $\mathrm{GL}_n(R)$. We write $\zeta(R)$ for the centre of R and $K_1(R)$ for the Whitehead group (see [CR87, §40]).

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2. Matrix rings over commutative rings

Let R be a commutative ring and fix $n \in \mathbb{N}$. Let $\Lambda = M_{n\times n}(R)$ and for $1 \le i, j \le n$ let $e_{ij} \in \Lambda$ be the matrix with 1 in position (i, j) and 0 everywhere else. Then

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1. Let M be a finitely presented Λ-module. Then define

$$\mathrm{Fit}_\Lambda(M) := \mathrm{Fit}_R(e_{11}M),$$

where the right hand side denotes the usual Fitting ideal over a commutative ring.

Remark 2.2. In the case $n = 1$ we have $\Lambda = R$ and $e_{11} = 1$, so Definition 2.1 is just the standard definition in this case and hence our notation is consistent.

Lemma 2.3. *Let M be a Λ-module. For $1 \le i, j \le n$ we have $e_{ii}M \simeq e_{jj}M$ as R-modules.*

Proof. Define an R-module homomorphism $\alpha_{ij} : e_{ii}M \to e_{jj}M$ by $x \mapsto e_{ji}x$. Note that this is in fact well-defined since $e_{ji}M = e_{jj}e_{ji}M \subset e_{jj}M$. Define α_{ji} symmetrically. Then

$$\alpha_{ji} \circ \alpha_{ij}(x) = e_{ij}e_{ji}x = e_{ii}x = x.$$

So by symmetry α_{ij} and α_{ji} are mutually inverse and hence are isomorphisms. □

We give some of the important properties of Fitting ideals over Λ.

Theorem 2.4. *Let M, M_1, M_2 and M_3 be finitely presented Λ-modules.*

(i) *For any $1 \le i \le n$, we have* $\mathrm{Fit}_\Lambda(M) = \mathrm{Fit}_R(e_{ii}M)$.
(ii) *We have* $\mathrm{Fit}_\Lambda(M) \subset \mathrm{Ann}_R(M)$.
(iii) *If $M_1 \twoheadrightarrow M_2$ is an epimorphism then* $\mathrm{Fit}_\Lambda(M_1) \subset \mathrm{Fit}_\Lambda(M_2)$.
(iv) *If $M_2 = M_1 \oplus M_3$ then* $\mathrm{Fit}_\Lambda(M_2) = \mathrm{Fit}_\Lambda(M_3) \cdot \mathrm{Fit}_\Lambda(M_1)$.

(v) *If* $M_1 \xrightarrow{\iota} M_2 \twoheadrightarrow M_3$ *is an exact sequence* (ι *need not be injective*) *then*

$$\mathrm{Fit}_\Lambda(M_1) \cdot \mathrm{Fit}_\Lambda(M_3) \subset \mathrm{Fit}_\Lambda(M_2).$$

(vi) *If* $M_1 \hookrightarrow M_2 \twoheadrightarrow M_3$ *is an exact sequence and* M_3 *has a quadratic presentation (i.e. of the form* $\Gamma^k \to \Gamma^k \twoheadrightarrow M_3$ *for some* $k \in \mathbb{N}$*) then*

$$\mathrm{Fit}_\Lambda(M_1) \cdot \mathrm{Fit}_\Lambda(M_3) = \mathrm{Fit}_\Lambda(M_2).$$

(vii) *For any map* $R \to S$ *of commutative rings we have*

$$\mathrm{Fit}_{S\otimes_R\Lambda}(S \otimes_R M) = S \cdot \mathrm{Fit}_\Lambda(M).$$

(viii) *We have* $\mathrm{Fit}_R(M) = \mathrm{Fit}_\Lambda(M)^n$.

(ix) *If* I *is a finitely generated two-sided ideal of* Λ *then* $I = M_{n\times n}(J)$ *for some ideal* J *of* R *and so* $\Lambda/I = M_{n\times n}(R/J)$*; hence we have* $\mathrm{Fit}_\Lambda(\Lambda/I) = J^n$.

Remark 2.5. If R is a Dedekind domain then factorisation of ideals in R is unique and so Theorem 2.4(viii) shows that $\mathrm{Fit}_\Lambda(M)$ is completely determined by $\mathrm{Fit}_R(M)$ in this case.

Remark 2.6. We note that $\mathrm{Ann}_\Lambda(M) := \{x \in \Lambda \mid xM = 0\}$ is always a two-sided ideal of Λ and from this it is straightforward to show that $\mathrm{Ann}_\Lambda(M) = M_{n\times n}(\mathrm{Ann}_R(M))$. Thus nothing is lost by computing or approximating $\mathrm{Ann}_R(M)$ rather than $\mathrm{Ann}_\Lambda(M)$.

Proof. Definition 2.1 and Lemma 2.3 give (i). For (ii), note that $e_{11} + \cdots + e_{nn}$ is the identity matrix in Λ and that $e_{ii}M \cap e_{jj}M = 0$ for $i \neq j$. Hence as R-modules

$$M = (e_{11} + \cdots + e_{nn})M = e_{11}M \oplus \cdots \oplus e_{nn}M. \tag{2.1}$$

By (i) and the annihilation property of Fitting ideals over R, we have $\mathrm{Fit}_\Lambda(M) = \mathrm{Fit}_R(e_{ii}M) \subset \mathrm{Ann}_R(e_{ii}M)$ for each i and therefore $\mathrm{Fit}_\Lambda(M) \subset \mathrm{Ann}_R(M)$.

Equation (2.1) shows that $M \mapsto e_{11}M$ is an exact covariant functor from the category of (left) Λ-modules to R-modules. (Note that this functor takes a Λ-homomorphism $M \to N$ to its restriction $e_{11}M \to e_{11}N$ considered as an R-homomorphism.) Furthermore, $e_{11}\Lambda \simeq R^n$ as R-modules, so free (resp. finitely presented) Λ-modules map to free (resp. finitely presented) R-modules. Therefore (iii)-(vii) follow from the corresponding properties for Fitting ideals over R. Proofs of (iii) and (iv) in the case $\Lambda = R$ can be found in [Nor76, Chapter 3]; for (vii) see [Eis95, Corollary 20.5]. Properties (v) and (vi) follow from Lemma 2.13 below. Note that for (v), we first reduce to the case that ι is injective: as M_1 surjects onto $\ker(M_2 \twoheadrightarrow M_3)$ by exactness, we can assume by (iii) that in fact $M_1 = \ker(M_2 \twoheadrightarrow M_3)$. Property (viii) follows from equation (2.1), Lemma 2.3, and (iv) in the case $\Lambda = R$. The first part of (ix) is well-known; the second part now follows from the R-module isomorphism $e_{11}(\Lambda/I) \simeq (R/J)^n$, the fact that $\mathrm{Fit}_R(R/J) = J$ (see [Nor76, §3.1, Exercise 4]; solution on p.93), and parts (i) and (iv). ☐

Example 2.7. Let $n = 2$ and $R = \mathbb{Z}$ so that $\Lambda = M_{2\times 2}(\mathbb{Z})$. Consider $M = M_{2\times 2}(\mathbb{Z}/2\mathbb{Z})$ as a Λ-module. Then $\mathrm{Fit}_\mathbb{Z}(M) = 16\mathbb{Z}$, $\mathrm{Fit}_\Lambda(M) = 4\mathbb{Z}$, and $\mathrm{Ann}_\mathbb{Z}(M) = 2\mathbb{Z}$. Now let $N = Me_{11}$. Then $\mathrm{Fit}_\mathbb{Z}(N) = 4\mathbb{Z}$ and $\mathrm{Fit}_\Lambda(N) = \mathrm{Ann}_\mathbb{Z}(N) = 2\mathbb{Z}$.

Remark 2.8. The key fact we have used is that R and Λ are Morita equivalent rings (for background on Morita equivalence see [CR81, §3D], [Rei03, Chapter 4] or [Lam99, Chapter 7]). Let ${}_R\mathfrak{M}$ and ${}_\Lambda\mathfrak{M}$ denote the categories of (left) R modules and left Λ-modules, respectively. Fix $1 \leq i \leq n$. Then we have mutually inverse category equivalences

$$F : {}_\Lambda\mathfrak{M} \longrightarrow {}_R\mathfrak{M} \quad \text{and} \quad G : {}_R\mathfrak{M} \longrightarrow {}_\Lambda\mathfrak{M}$$

given explicitly by

$$(2.2)\qquad \begin{aligned} F(M) &= e_{ii}\Lambda \otimes_\Lambda M \simeq e_{ii}M \simeq \mathrm{Hom}_\Lambda(\Lambda e_{ii}, M),\\ G(N) &= \Lambda e_{ii} \otimes_R N \simeq \mathrm{Hom}_R(e_{ii}\Lambda, N). \end{aligned}$$

The R-module isomorphisms of (2.2) can be used to give definitions equivalent to Definition 2.1. In fact, in his PhD thesis [Gri02, §5.2], Peter Grime essentially defines the Fitting ideal of a Λ-module M to be $\mathrm{Fit}_R(\mathrm{Hom}_\Lambda(\Lambda e_{11}, M))$. However, most of his results are quite different to those given here.

Remark 2.9. In the Appendix, Definition 2.1 and most of Theorem 2.4 are extended to the case where Λ is any ring that is Morita equivalent to a commutative ring R. The advantages of the more specific case described in this section are that it is very explicit, and thus is easier to understand and more results can be obtained. Note that if R is a ring over which every finitely generated projective module is in fact free (for example, a principal ideal domain or a local ring) then we must have $\Lambda \simeq M_{n\times n}(R)$ for some n, and so this case is covered by Definition 2.1. In fact, for most of this article we shall work over a ring Λ whose centre $\zeta(\Lambda)$ is a product of local rings; we can without loss of generality suppose that $\zeta(\Lambda)$ is in fact local. Since Λ is Morita equivalent to R, we have $\zeta(\Lambda) \simeq \zeta(R) = R$; therefore $\Lambda \simeq M_{n\times n}(R)$ for some n. Thus the more general argument given in the Appendix is not needed for most of this article.

The following technical lemma is essentially equivalent to [Gri02, Lemma 5.1].

Lemma 2.10. *Fix $1 \le i \le n$ and note that $\mathcal{B}_i := \{e_{ij}\}_{1\le j\le n}$ is an R-basis of $e_{ii}\Lambda$. For any $r, s \in \mathbb{N}$ and any Λ-homomorphism $\alpha : \Lambda^r \longrightarrow \Lambda^s$, let $\alpha' : (e_{ii}\Lambda)^r \longrightarrow (e_{ii}\Lambda)^s$ be the restriction of α considered as an R-homomorphism. Let $h : \Lambda^a \longrightarrow \Lambda^b$ be a Λ-homomorphism represented by $H \in M_{a\times b}(\Lambda)$ with respect to the standard basis. Let $H' \in M_{na\times nb}(R)$ be the matrix representing h' with respect to the bases of $(e_{ii}\Lambda)^a$ and $(e_{ii}\Lambda)^b$ obtained from $\mathcal{B}_i$ in the obvious way. Let $\tilde{H} \in M_{na\times nb}(R)$ be the same matrix as H but with entries considered in R rather than Λ. Then $H' = \tilde{H}$.*

Proof. Fix $1 \le k \le a$ and $1 \le \ell \le b$. Let $\iota_k : \Lambda \longrightarrow \Lambda^a$ be the obvious injection and $\pi_\ell : \Lambda^b \longrightarrow \Lambda$ be the obvious projection. Then ι_k' (resp. π_ℓ') is also the obvious injection (resp. projection). Let $h_{k\ell} = \pi_\ell \circ h \circ \iota_k : \Lambda \longrightarrow \Lambda$. Then $h_{k\ell}' = \pi_\ell' \circ h' \circ \iota_k'$. Hence we can and do assume without loss of generality that $a = b = 1$.

Write $\tilde{H} = (r_{pq}) \in M_{n\times n}(R) = \Lambda$. Then for $1 \le j \le n$ we have

$$h'(e_{ij}) = e_{ij}H = e_{ij}\sum_{p,q=1}^{n} e_{pq}r_{pq} = \sum_{p,q=1}^{n} e_{ij}e_{pq}r_{pq} = \sum_{q=1}^{n} e_{iq}r_{jq}.$$

Hence H' is the matrix $(r_{jq})_{j,q} = \tilde{H}$, as required. □

Remark 2.11. Lemma 2.10 can be used to give an alternative proof of Theorem 2.4(i).

Proposition 2.12. *Let I be a finitely generated left ideal of Λ. Then*

$$\mathrm{Fit}_\Lambda(\Lambda/I) = \langle \det(x) \mid x \in I \rangle_R.$$

Proof. We adopt the notation and assume the result of Lemma 2.10. Let $\{x_1, \ldots, x_{r-1}\}$ be a fixed set of generators of I and let x_r be an arbitrary element of I. Then there exists a presentation of Λ/I of the form

$$\Lambda^r \xrightarrow{h} \Lambda \twoheadrightarrow \Lambda/I,$$

where $H := (x_1, \dots, x_r)^t \in M_{r\times 1}(\Lambda)$ is the matrix representing h. Let S denote the set of all $n \times n$ submatrices of $H' = \tilde{H} \in M_{nr\times n}(R)$. Since h' is an R-module presentation of $e_{11}(\Lambda/I)$ and $\mathrm{Fit}_R(e_{11}(\Lambda/I))$ is independent of the choice of presentation, we have

$$\mathrm{Fit}_\Lambda(\Lambda/I) = \mathrm{Fit}_R(e_{11}(\Lambda/I)) = \langle \det(T) \mid T \in S\rangle.$$

However, one of the elements of S is equal to x_r, and so we see that $\det(x_r) \in \mathrm{Fit}_\Lambda(\Lambda/I)$. We therefore have $\langle \det(x) \mid x \in I\rangle_R \subset \mathrm{Fit}_\Lambda(\Lambda/I)$.

Now let $T \in S$. Fix i with $1 \le i \le n$. Then the ith row of T is a row of $H' = \tilde{H}$, which in turn is the jth row of x_k for some k, j with $1 \le k \le r$ and $1 \le j \le n$. Hence $e_{ii}T = e_{ij}x_k$. Since $x_k \in I$, $e_{ij} \in \Lambda$, and I is a left ideal of Λ, we thus have that $e_{ii}T \in I$. Therefore $T = (e_{11} + \cdots + e_{nn})T = e_{11}T + \cdots + e_{nn}T \in I$, and so $\mathrm{Fit}_\Lambda(\Lambda/I) \subset \langle \det(x) \mid x \in I\rangle_R$. $\square$

2.1. **Auxiliary result on Fitting ideals over commutative rings.** Let R be a commutative ring. We provide a proof of the following result as the second part is slightly stronger than similar results that the authors were able to locate in the literature.

Lemma 2.13. *Let M_1, M_2 and M_3 be finitely presented R-modules.*

(i) *If $M_1 \overset{\iota}{\hookrightarrow} M_2 \twoheadrightarrow M_3$ is an exact sequence then*
$$\mathrm{Fit}_R(M_1) \cdot \mathrm{Fit}_R(M_3) \subset \mathrm{Fit}_R(M_2).$$

(ii) *If in addition M_3 has a quadratic presentation (i.e. of the form $R^k \to R^k \twoheadrightarrow M_3$ for some $k \in \mathbb{N}$) then in fact*
$$\mathrm{Fit}_R(M_1) \cdot \mathrm{Fit}_R(M_3) = \mathrm{Fit}_R(M_2).$$

Remark 2.14. Lemma 2.13(i) is well-known (see [Nor76, Exercise 2, Chapter 3]; solution on p.90-91). Proofs of slightly weaker versions of Lemma 2.13(ii) can be found in [Nor76, p.80-81] or [CG98, Lemma 3]); these assume that M_3 has a presentation of the form $R^k \overset{h}{\to} R^k \twoheadrightarrow M$ with h injective, whereas Lemma 2.13(ii) does not require h to be injective.

Proof. We choose presentations $R^{a_i} \overset{h_i}{\longrightarrow} R^{b_i} \overset{\pi_i}{\twoheadrightarrow} M_i$ for $i = 1, 3$ and construct a finite presentation of M_2 in the following way. Since R^{b_3} is projective, π_3 factors through M_2 via a map $f_1 : R^{b_3} \to M_2$. We define $\pi_2 = (\iota \circ \pi_1 \mid f_1) : R^{b_1} \oplus R^{b_2} \twoheadrightarrow M_2$. In a similar manner we construct $h_2 = (h_1 \mid f_2) : R^{a_1} \oplus R^{a_3} \to R^{b_1} \oplus R^{b_3}$, where f_2 realizes the factorization of h_3 through $\ker(\pi_2)$. Let $a_2 = a_1 + a_3$ and $b_2 = b_1 + b_3$. We identify each h_i with multiplication on the right by a matrix in $M_{a_i\times b_i}(R)$ in the obvious way. Then h_2 is of the form

$$\begin{pmatrix} h_1 & 0 \\ * & h_3 \end{pmatrix}.$$

Since Fitting ideals over R are independent of the chosen presentation, this gives the desired inclusion of part (i).

Now suppose that M_3 has a quadratic presentation; then we can choose $a_3 = b_3$. Without loss of generality we can assume that $a_1 \ge b_1$ and so $a_2 \ge b_2$. Let H_2 be a $b_2 \times b_2$ submatrix of h_2. Then H_2 is obtained from h_2 by deleting rows. If none of the last a_3 rows are deleted, then H_2 is of the form

$$\begin{pmatrix} H_1 & 0 \\ * & h_3 \end{pmatrix},$$

where H_1 is some $b_1 \times b_1$ submatrix of h_1. Otherwise, H_2 is of the form

$$\begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

where A and B are square matrices (B is a submatrix of h_3) and the last column of A consists only of zeros; hence $\det(H_2) = \det(A)\det(B) = 0$. In either case, we have the reverse of the inclusion of part (i) and thus have the desired equality of part (ii). □

3. Noncommutative Fitting invariants

3.1. **Reduced norms.** Let $\mathfrak{o}$ be a noetherian integral domain with field of quotients F and let A be a finite dimensional semisimple F-algebra. If $e_1, \ldots, e_t$ are the central primitive idempotents of A then

$$A = A_1 \oplus \cdots \oplus A_t$$

where $A_i := Ae_i = e_iA$. Each A_i is isomorphic to an algebra of $n_i \times n_i$ matrices over a skewfield D_i, and $F_i := \zeta(A_i) = \zeta(D_i)$ is a finite field extension of F; hence each A_i is a central simple F_i-algebra. We denote the Schur index of D_i by s_i so that $[D_i : F_i] = s_i^2$. The reduced norm map

$$\mathrm{nr} = \mathrm{nr}_A : A \longrightarrow \zeta(A) = F_1 \oplus \cdots \oplus F_t$$

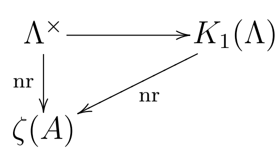
is defined componentwise (see [Rei03, §9]) and extends to matrix rings over A in the obvious way; hence this induces a map $K_1(A) \to \zeta(A)^\times$ which we also denote by nr.

Now suppose further that A is a separable F-algebra and that $\mathfrak{o}$ is integrally closed. Let Λ be an $\mathfrak{o}$-order in A. Then Λ is noetherian and so any finitely generated Λ-module is in fact finitely presented; we shall use this fact repeatedly without further mention. By [Rei03, Corollary (10.4)] we may choose a maximal order Λ' containing Λ and there is a decomposition

$$\Lambda' = \Lambda'_1 \oplus \cdots \oplus \Lambda'_t$$

where $\Lambda'_i = \Lambda' e_i$. Let $\mathfrak{o}'_i$ be the integral closure of $\mathfrak{o}$ in F_i. Then each Λ'_i is a maximal $\mathfrak{o}'_i$-order with centre $\mathfrak{o}'_i$ (see [Rei03, Theorem (10.5)]). A key point is that the reduced norm maps Λ into $\zeta(\Lambda') = \mathfrak{o}'_1 \oplus \cdots \oplus \mathfrak{o}'_t$, but not necessarily into $\zeta(\Lambda)$. As above, the reduced norm induces a map $K_1(\Lambda) \to \zeta(\Lambda')^\times$ which we again denote by nr.

Remark 3.1. Suppose that $\mathfrak{o}$ is local. Then Λ is semilocal and by [CR87, Theorem (40.31)] the natural map $\Lambda^\times \to K_1(\Lambda)$ is surjective. Furthermore, the diagram



commutes and therefore $\mathrm{nr}(\Lambda^\times) = \mathrm{nr}(K_1(\Lambda)) = \mathrm{nr}(\mathrm{GL}_n(\Lambda))$ for all $n \in \mathbb{N}$.

3.2. **Fitting domains and Fitting orders.** We shall now specialize to the following situation. Let $\mathfrak{o}$ be an integrally closed complete commutative noetherian local domain with field of quotients F. We shall refer to $\mathfrak{o}$ as a *Fitting domain*. For example, one can take $\mathfrak{o}$ to be a complete discrete valuation ring or a power series ring in one variable over a complete discrete valuation ring. Let A be a separable F-algebra (i.e. a finite dimensional semisimple F-algebra, such that the centre of each simple component of A is a separable field extension of F) and let Λ be an $\mathfrak{o}$-order in A. We shall refer to Λ as a *Fitting order* over $\mathfrak{o}$. A standard example of Λ is the group ring $\mathbb{Z}_p[G]$ where p is a prime and G is a finite group.

3.3. **Reduced norm equivalence.** We recall the following definition from [Nic10, §1.0.2]. Let N and M be two $\zeta(\Lambda)$-submodules of an $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-module. Then N and M are called $\mathrm{nr}(\Lambda)$-*equivalent* if there exists an integer n and a matrix $U \in \mathrm{GL}_n(\Lambda)$ such that $N = \mathrm{nr}(U) \cdot M$. (Note that by Remark 3.1, we can in fact replace $\mathrm{GL}_n(\Lambda)$ by $\Lambda^\times$ in this definition.) We say that N is $\mathrm{nr}(\Lambda)$-contained in M (and write $[N]_{\mathrm{nr}(\Lambda)} \subset [M]_{\mathrm{nr}(\Lambda)}$) if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subset M'$. Note that it suffices to check this property for one $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$. We will say that x is contained in $[N]_{\mathrm{nr}(\Lambda)}$ (and write $x \in [N]_{\mathrm{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$.

Let $e \in A$ be a central idempotent. Suppose that N and M are two $\mathfrak{o}$-torsionfree $\zeta(\Lambda)$-modules that are $\mathrm{nr}(\Lambda)$-equivalent. Then eN and eM are $\mathrm{nr}(\Lambda e)$-equivalent $\zeta(\Lambda e)$-modules, since for $U \in \Lambda^\times$ we have $Ue \in (\Lambda e)^\times$ and $\mathrm{nr}_A(U)e = \mathrm{nr}_{Ae}(Ue)$. Hence $e[N]_{\mathrm{nr}(\Lambda)} := [eN]_{\mathrm{nr}(\Lambda e)}$ is well-defined.

3.4. **Noncommutative Fitting invariants.** We recall the following definitions and results from [Nic10] and [Nic11, §1.0.3]. Let M be a Λ-module with finite presentation

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M. \tag{3.1}$$

We identify the homomorphism h with the corresponding matrix in $M_{a\times b}(\Lambda)$ and define $S_b(h)$ to be the set of all $b \times b$ submatrices of h if $a \geq b$. In the case $a = b$ we call (3.1) a quadratic presentation. The Fitting invariant of h over Λ is defined to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}(H) \mid H \in S_b(h)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geq b. \end{cases} \tag{3.2}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a Fitting invariant of M over Λ. If M admits a quadratic presentation h, we put $\mathrm{Fitt}_\Lambda(M) := \mathrm{Fitt}_\Lambda(h)$ which is independent of the chosen quadratic presentation. We define $\mathrm{Fitt}_\Lambda^{\max}(M)$ to be the unique Fitting invariant of M over Λ which is maximal among all Fitting invariants of M with respect to the partial order "$\subset$". Finally, we define a $\zeta(\Lambda)$-submodule of $\zeta(A)$ by

$$\mathcal{I} = \mathcal{I}(\Lambda) := \langle \mathrm{nr}(H) \mid H \in M_{b\times b}(\Lambda), b \in \mathbb{N}\rangle_{\zeta(\Lambda)}$$

and note that this is in fact an $\mathfrak{o}$-order in $\zeta(A)$ contained in $\zeta(\Lambda')$.

Theorem 3.2. *Let M, M_1, M_2 and M_3 be finitely generated Λ-modules.*

(i) *If $M_1 \twoheadrightarrow M_2$ is an epimorphism then* $\mathrm{Fitt}_\Lambda^{\max}(M_1) \subset \mathrm{Fitt}_\Lambda^{\max}(M_2)$.

(ii) *If $M_1 \to M_2 \twoheadrightarrow M_3$ is an exact sequence, then*
$$\mathrm{Fitt}_\Lambda^{\max}(M_1) \cdot \mathrm{Fitt}_\Lambda^{\max}(M_3) \subset \mathrm{Fitt}_\Lambda^{\max}(M_2).$$

(iii) *Let $M_1 \hookrightarrow M_2 \twoheadrightarrow M_3$ be an exact sequence. If M_1 and M_3 admit quadratic presentations, so does M_2 and*
$$\mathrm{Fitt}_\Lambda(M_1) \cdot \mathrm{Fitt}_\Lambda(M_3) = \mathrm{Fitt}_\Lambda(M_2).$$

(iv) *If $\theta \in \mathrm{Fitt}_\Lambda^{\max}(M)$ and $\lambda \in \mathcal{I}$ then $\lambda \cdot \theta \in \mathrm{Fitt}_\Lambda^{\max}(M)$.*

(v) *If M admits a quadratic presentation, then* $\mathrm{Fitt}_\Lambda^{\max}(M) = \mathcal{I} \cdot \mathrm{Fitt}_\Lambda(M)$.

(vi) *Let $e \in A$ be a central idempotent. Then* $e\mathrm{Fitt}_\Lambda^{\max}(M) = \mathrm{Fitt}_{\Lambda e}^{\max}(\Lambda e \otimes_\Lambda M)$.

(vii) *Set $M_F := F \otimes_{\mathfrak{o}} M$ and $\Upsilon(M) := \{i \in \{1, \dots, t\} \mid e_i M_F = 0\}$. Then*
$$\mathrm{Fitt}_\Lambda^{\max}(M) = e\mathrm{Fitt}_\Lambda^{\max}(M) = \mathrm{Fitt}_{\Lambda e}^{\max}(\Lambda e \otimes_\Lambda M)$$
where $e = e(M) := \sum_{i \in \Upsilon(M)} e_i$.

Proof. For (i), (ii) and (iii), see [Nic10, Proposition 3.5]. For (iv) and (v) see [Nic11, Proposition 1.1]. For (vi) and (vii) see [Nic10, Lemma 3.4]. ∎

3.5. **An alternative definition of noncommutative Fitting invariants.** We define

$$\mathcal{U} = \mathcal{U}(\Lambda) := \langle \mathrm{nr}(H) \mid H \in \mathrm{GL}_b(\Lambda), b \in \mathbb{N} \rangle_{\zeta(\Lambda)} = \langle \mathrm{nr}(H) \mid H \in \Lambda^{\times} \rangle_{\zeta(\Lambda)},$$

where the last equality is due to Remark 3.1. This is an $\mathfrak{o}$-order in $\zeta(A)$ contained in $\mathcal{I}(\Lambda)$. Let M be a Λ-module with finite presentation

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M.$$

An alternative definition to (3.2) is

$$\mathrm{Fit}_{\Lambda}(h) = \begin{cases} \langle 0 \rangle_{\mathcal{U}(\Lambda)} & \text{if} \quad a < b \\ \langle \mathrm{nr}(H) \mid H \in S_b(h) \rangle_{\mathcal{U}(\Lambda)} & \text{if} \quad a \geq b. \end{cases} \tag{3.3}$$

(Note that $\mathrm{Fitt}_{\Lambda}(h)$ of (3.2) has two t's whereas $\mathrm{Fit}_{\Lambda}(h)$ of (3.3) has one t.) We define $\mathrm{Fit}^{\max}_{\Lambda}(M)$ to be the unique Fitting invariant of M over Λ which is maximal with respect to inclusion among all $\mathrm{Fit}_{\Lambda}(h')$ where h' is a presentation of M. An argument analogous to that given for Theorem 3.2(iv) shows that $\mathrm{Fit}^{\max}_{\Lambda}(M)$ is in fact a module over $\mathcal{I}(\Lambda)$.

The two definitions are explicitly related as follows. Consider the category $\mathcal{N}$ with $\mathrm{nr}(\Lambda)$-equivalence classes of finitely generated $\zeta(\Lambda)$-submodules of $\zeta(A)$ as objects and inclusions as morphisms. Let $\mathcal{M}$ be the category of finitely generated $\mathcal{I}(\Lambda)$-submodules of $\zeta(A)$ with inclusions as morphisms. Then

$$\begin{array}{rcl} \iota : \mathcal{N} & \longrightarrow & \mathcal{M} \\ [X]_{\mathrm{nr}(\Lambda)} & \mapsto & X \cdot \mathcal{I}(\Lambda) \end{array} \tag{3.4}$$

is a covariant functor. Note that ι is well-defined: If X' is $\mathrm{nr}(\Lambda)$-equivalent to X, then there is a $U \in \Lambda^{\times}$ such that $X' = \mathrm{nr}(U) \cdot X$; but $\mathrm{nr}(U) \in \mathcal{I}(\Lambda)^{\times}$ and hence $X' \cdot \mathcal{I}(\Lambda) = X \cdot \mathcal{I}(\Lambda)$. In the special case $\zeta(\Lambda) = \mathcal{I}(\Lambda)$ (e.g. Λ is commutative or maximal), the equivalence class $[X]_{\mathrm{nr}(\Lambda)}$ contains precisely one element and we have $\iota([X]_{\mathrm{nr}(\Lambda)}) = X$. In the general case, it is straightforward to see that we have

$$\iota(\mathrm{Fitt}^{\max}_{\Lambda}(M)) = \mathrm{Fit}^{\max}_{\Lambda}(M). \tag{3.5}$$

It follows that $\mathrm{Fit}^{\max}_{\Lambda}(M)$ has the properties analogous to those of $\mathrm{Fitt}^{\max}_{\Lambda}(M)$ given in Theorems 3.2 and 3.5.

The advantage of $\mathrm{Fit}^{\max}_{\Lambda}(M)$ is that $\mathrm{nr}(\Lambda)$-equivalence classes are not required and, as we shall see, it is compatible with Definition 2.1; the advantage of $\mathrm{Fitt}^{\max}_{\Lambda}(M)$ is that it can be directly related to Fitting invariants of quadratic presentations which in turn can be used to do computations in relative K-groups. For instance, the application in [Nic10, §7] shows how to compute annihilators of the class group of a number field via this notion of Fitting invariants from an appropriate special case of the equivariant Tamagawa number conjecture (which asserts a certain equality in a relative K-group). Moreover, it can be used to define relative Fitting invariants (see [Nic10, p.2764]). However, in most cases it does not really matter which definition we work with, as they are explicitly related as above. For the rest of this article, the reader may almost always think in terms of $\mathrm{Fit}^{\max}_{\Lambda}(M)$ rather than $\mathrm{Fitt}^{\max}_{\Lambda}(M)$.

3.6. **Generalised adjoint matrices.** Choose $n \in \mathbb{N}$ and let $H \in M_{n \times n}(\Lambda)$. Then recalling the notation of §3.1, decompose H into

$$H = \sum_{i=1}^{t} H_i \in M_{n \times n}(\Lambda') = \bigoplus_{i=1}^{t} M_{n \times n}(\Lambda'_i),$$

where $H_i := He_i$. Let $m_i = n_i \cdot s_i \cdot n$. The reduced characteristic polynomial $f_i(X) = \sum_{j=0}^{m_i} \alpha_{ij} X^j$ of H_i has coefficients in $\mathfrak{o}_i'$. Moreover, the constant term α_{i0} is equal to $\mathrm{nr}(H_i) \cdot (-1)^{m_i}$. We put

$$H_i^* := (-1)^{m_i+1} \cdot \sum_{j=1}^{m_i} \alpha_{ij} H_i^{j-1}, \quad H^* := \sum_{i=1}^{t} H_i^*.$$

Lemma 3.3. *We have* $H^* \in M_{n\times n}(\Lambda')$ *and* $H^*H = HH^* = \mathrm{nr}_A(H) \cdot 1_{n\times n}$.

Proof. The first assertion is clear by the above considerations. Since $f_i(H_i) = 0$, we find that

$$H_i^* \cdot H_i = H_i \cdot H_i^* = (-1)^{m_i+1}(-\alpha_{i0}) = \mathrm{nr}(H_i),$$

as desired. □

Remark 3.4. Note that the above definition of H^* differs slightly from the definition in [Nic10, §4]. However, the only properties of H^* needed are those stated in Lemma 3.3. Moreover, if H is invertible (over A), then H^* is uniquely determined by the equation in Lemma 3.3, and hence the two definitions agree in this case. The new definition has the advantage that it is precisely the adjoint matrix if Λ is commutative, and the assignment $H \mapsto H^*$ is often continuous (e.g. with respect to the p-adic topology if $\mathfrak{o} = \mathbb{Z}_p$).

We define

$$\mathcal{H} = \mathcal{H}(\Lambda) := \{x \in \zeta(\Lambda) \mid xH^* \in M_{b\times b}(\Lambda)\, \forall H \in M_{b\times b}(\Lambda)\, \forall b \in \mathbb{N}\}.$$

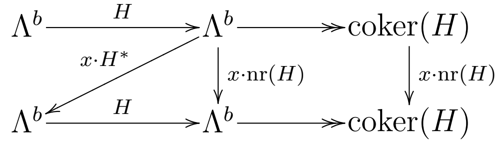
Since $x \cdot \mathrm{nr}(H) = xH^*H \in \zeta(\Lambda)$, in particular we have $\mathcal{H} \cdot \mathcal{I} = \mathcal{H} \subset \zeta(\Lambda)$. Hence $\mathcal{H}$ is an ideal in the $\mathfrak{o}$-order $\mathcal{I}(\Lambda)$.

3.7. **Fitting invariants and annihilation.**

Theorem 3.5. *Let* Λ *be a Fitting order and let* M *be a finitely generated* Λ*-module. Then*

$$\mathcal{H}(\Lambda) \cdot \mathrm{Fit}_\Lambda^{\max}(M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M).$$

Proof. (Also see [Nic10, Theorem 4.2].) Let $\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M$ be a finite presentation of M. Then it suffices to show that $\mathcal{H}(\Lambda) \cdot \mathrm{Fit}_\Lambda(h) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M)$. Fix $H \in S_b(h)$ and $x \in \mathcal{H}(\Lambda)$. As $\mathrm{Fit}_\Lambda(h)$ is generated by elements of the form $\mathrm{nr}(H)$, we are further reduced to showing that $x \cdot \mathrm{nr}(H)$ annihilates M. The cokernel of H surjects onto M and hence the assertion follows from the commutative diagram



once one notes that the right most map is zero. □

3.8. **Fitting invariants of matrix rings over commutative rings.** Fix $n \in \mathbb{N}$ and let $\Lambda = M_{n\times n}(R)$ where R is a commutative $\mathfrak{o}$-order. Hence Λ is both a Fitting order and a matrix ring over a commutative ring. The aim of this section is to show that Definition 2.1 is compatible with (3.3) in this case, thereby justifying the similar notation.

Proposition 3.6. *Let* M *be a finitely generated* Λ*-module. Then* $\mathrm{Fit}_\Lambda(M) = \mathrm{Fit}_\Lambda^{\max}(M)$.

Proof. First note that $R = \zeta(\Lambda) = \mathcal{U}(\Lambda) = \mathcal{I}(\Lambda)$. Let $\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M$ be a finite presentation of Λ. We can and do assume without loss of generality that $a \geq b$. Let $H \in M_{a\times b}(\Lambda)$ and $H', \tilde{H} \in M_{na\times nb}(R)$ be the matrices corresponding to h as in Lemma 2.10; then $H' = \tilde{H}$. Hence we have

$$\begin{aligned} \mathrm{Fit}_{\Lambda}(h) := \langle \mathrm{nr}(T) \mid T \in S_b(H)\rangle_R \;&\subset\; \langle \mathrm{nr}(\tilde{T}) \mid \tilde{T} \in S_{nb}(\tilde{H})\rangle_R \\ &=\; \langle \mathrm{nr}(\tilde{T}) \mid \tilde{T} \in S_{nb}(H')\rangle_R \\ &=\; \mathrm{Fit}_R(e_{11}M) =: \mathrm{Fit}_{\Lambda}(M). \end{aligned}$$

It follows that $\mathrm{Fit}^{\max}_{\Lambda}(M) \subset \mathrm{Fit}_{\Lambda}(M)$.

Now let $\tilde{T} \in S_{nb}(H')$. Then by swapping rows of H' appropriately, there exists $\tilde{E} \in \mathrm{GL}_{na}(R)$ with $\det_R(\tilde{E}) = \pm 1$ such that the $nb \times nb$ submatrix of $\tilde{E}H'$ formed by taking the first nb rows is equal to $\tilde{T}$. Let $E \in M_{a\times a}(\Lambda)$ (resp. $T \in M_{b\times b}(\Lambda)$) be the same matrix as $\tilde{E}$ (resp. $\tilde{T}$) but with entries considered in Λ rather than R. Then $E \in \mathrm{GL}_a(\Lambda)$ and the diagram

$$\begin{array}{ccccc} \Lambda^a & \xrightarrow{EH} & \Lambda^b & \twoheadrightarrow & \mathrm{coker}(EH) \\ \simeq\big\downarrow E & & \big\| & & \big\downarrow\simeq \\ \Lambda^a & \xrightarrow{H} & \Lambda^b & \twoheadrightarrow & M \end{array}$$

commutes. (Note that the order of function composition and corresponding matrix multiplication are reversed since we consider left Λ-modules and so functions are represented by multiplying by their corresponding matrices on the right.) Since T is a $b\times b$ submatrix of EH we therefore have

$$\mathrm{nr}(\tilde{T}) = \mathrm{nr}(T) \in \langle \mathrm{nr}(V) \mid V \in S_b(EH)\rangle_R \subset \mathrm{Fit}^{\max}_{\Lambda}(\mathrm{coker}(EH)) = \mathrm{Fit}^{\max}_{\Lambda}(M).$$

Since $\tilde{T} \in S_{nb}(H')$ was arbitrary, we have shown that

$$\mathrm{Fit}_{\Lambda}(M) := \mathrm{Fit}_R(e_{11}M) = \langle \mathrm{nr}(\tilde{V}) \mid \tilde{V} \in S_{nb}(H')\rangle_R \subset \mathrm{Fit}^{\max}_{\Lambda}(M).$$

Therefore we have $\mathrm{Fit}^{\max}_{\Lambda}(M) = \mathrm{Fit}_{\Lambda}(M)$, as required. ☐

4. Nice Fitting orders

Definition 4.1. Let Λ be a Fitting order over $\mathfrak{o}$. Suppose that $\Lambda = \oplus_{j=1}^{k}\Lambda_j$ where each Λ_j is either a maximal $\mathfrak{o}$-order or is of the form $M_{a_j\times a_j}(\Gamma_j)$ for some commutative ring Γ_j. Then we say that Λ is a *nice* Fitting order.

Remark 4.2. If a Fitting order Λ is either maximal or commutative then it is immediate from the definition that Λ is nice.

Proposition 4.3. *Let Λ be a nice Fitting order. Then $\mathcal{U}(\Lambda) = \mathcal{I}(\Lambda) = \mathcal{H}(\Lambda) = \zeta(\Lambda)$.*

Proof. Fix $n \in \mathbb{N}$ and let $H \in M_{n\times n}(\Lambda)$. Write $H = \sum_{j=1}^{k} H_j$ corresponding to the decomposition $\Lambda = \oplus_{j=1}^{k}\Lambda_j$. If Λ_j is a maximal order then it is clear from the definition of H_j^* that $H_j^* \in M_{n\times n}(\Lambda_j)$. If $\Lambda_j \simeq M_{a_j\times a_j}(\Gamma_j)$ for some commutative ring Γ_j, then H_j^* is the usual adjoint matrix if considered as a matrix in $M_{na_j\times na_j}(\Gamma_j)$, and so $H_j^* \in M_{n\times n}(\Lambda_j)$. Therefore $H^* = \sum_{j=1}^{k} H_j^*$ lies in $M_{n\times n}(\Lambda)$. Since n was arbitrary, it follows that $\zeta(\Lambda) \subset \mathcal{H}(\Lambda)$. In particular, $1 \in \mathcal{H}(\Lambda)$ so must have $\mathcal{H}(\Lambda) = \mathcal{I}(\Lambda)$ since $\mathcal{H}(\Lambda)$ is an ideal of $\mathcal{I}(\Lambda)$. Thus $\zeta(\Lambda) = \mathcal{I}(\Lambda) = \mathcal{H}(\Lambda)$. The desired result now follows from the inclusions $\zeta(\Lambda) \subset \mathcal{U}(\Lambda) \subset \mathcal{I}(\Lambda)$. ☐

Corollary 4.4. *Suppose Λ is a Fitting order that is an intersection of nice Fitting orders or is such that $\zeta(\Lambda)$ is maximal. Then $\mathcal{U}(\Lambda) = \mathcal{I}(\Lambda) = \mathcal{H}(\Lambda) = \zeta(\Lambda)$. In particular, this is the case if Λ is a hereditary or graduated order over a complete discrete valuation ring.*

Proof. Suppose $\Lambda = \cap_i \Lambda_i$ where each Λ_i is a nice Fitting order. Fix $n \in \mathbb{N}$ and let $H \in M_{n\times n}(\Lambda)$. Then the argument above shows that $H^* \in \Lambda_i$ for each i and so $H^* \in \Lambda$. The rest of the argument follows as before. If $\zeta(\Lambda)$ is maximal, then the result follows directly from the definitions in §3.6.

Let Λ be a graduated order over a complete discrete valuation ring. (Recall that an order is graduated if there exist orthogonal primitive idempotents $e_1, \dots, e_t \in \Lambda$ with $1 = e_1 + \cdots + e_t$ such that $e_i \Lambda e_i$ is a maximal order for $i = 1, \dots, t$. In particular, maximal and hereditary orders are graduated. See [Ple83, §II] for further details.) The result now follows from the observation that $\zeta(\Lambda)$ is maximal. □

Definition 4.5. Let $\mathfrak{o}$ be a Fitting domain and let G be a finite group with commutator subgroup G'. Let Λ' be a maximal order containing the group ring $\mathfrak{o}[G]$ and let $e = |G'|^{-1}\mathrm{Tr}_{G'}$ where $\mathrm{Tr}_{G'} := \sum_{g' \in G'} g'$. Define $\Lambda'_G := \mathfrak{o}[G]e \oplus \Lambda'(1-e)$.

Proposition 4.6. *In the setting above, Λ'_G is a nice Fitting order containing $\mathfrak{o}[G]$.*

Proof. Note that $\mathfrak{o}[G]e$ is commutative and $\Lambda'(1-e)$ is maximal; hence Λ'_G is nice. The second assertion follows from the observation that $\Lambda'_G = \mathfrak{o}[G] + \Lambda'(1-e)$. □

Remark 4.7. Of course, Λ'_G depends on the choice of Λ'. However, for many applications this choice does not matter. For explicit examples, see Examples 4.11 and 6.11.

Proposition 4.8. *Let $\mathfrak{o}$ be a Fitting domain with residue field of characteristic $p > 0$ and let G be a finite group with commutator subgroup G'. Then the group ring $\Lambda := \mathfrak{o}[G]$ is a nice Fitting order if and only if $p \nmid |G'|$.*

Remark 4.9. Note that $p \nmid |G'|$ if and only if G has an abelian p-Sylow subgroup P and a normal p-complement N, in which case G is isomorphic to a semi-direct product $N \rtimes P$.

Proof. If $p \nmid |G'|$, then a special case of [DJ83, Corollary] shows that Λ is a finite direct product of matrix rings over commutative rings and hence is nice. Suppose conversely that Λ is a nice Fitting order. Let $H = 0 \in \Lambda = M_{1\times 1}(\Lambda)$. Recall the notation of §3.6 and write $H = \sum_{i=1}^t H_i \in \oplus_{i=1}^t \Lambda'_i$. Then the reduced characteristic polynomial of H_i is $f_i(X) = X^{n_i s_i}$ and so H_i^* is $h_i(0)$ where $h_i(X) := X^{n_i s_i - 1}$. Hence $H_i^* = 1$ if $n_i s_i = 1$ and $H_i^* = 0$ if $n_i s_i > 1$. Therefore $H^* = |G'|^{-1}\mathrm{Tr}_{G'}$. However, $H^* \in \Lambda = \mathfrak{o}[G]$ by Proposition 4.3 since Λ is nice. But then $|G'|$ must be invertible in $\mathfrak{o}$ and so $p \nmid |G'|$ since the residue field of $\mathfrak{o}$ has characteristic p. □

Corollary 4.10. *We have $\mathcal{H}(\mathfrak{o}[G]) = \zeta(\mathfrak{o}[G])$ if and only if $p \nmid |G'|$.*

Example 4.11. Let A_4 be the alternating group on 4 letters. Then $\mathbb{Z}_3[A_4]$ is neither commutative nor maximal, yet is a nice Fitting order by an application of Proposition 4.8. In fact, one can show that $\mathbb{Z}_3[A_4] = \Lambda'_{A_4}$ where Λ' is the unique maximal order in $\mathbb{Q}_3[A_4]$ containing $\mathbb{Z}_3[A_4]$.

Example 4.12. Let p, q be distinct primes with p odd such that $q|(p-1)$. Let r be a primitive q-th root of 1 mod p. Let $F_{p,q} := \langle x, y \mid x^p = y^q = 1, yxy^{-1} = y^r \rangle$. Then $F_{p,q}$ is a metacyclic group of order pq and in the special case $q = 2$, $F_{p,q}$ is the dihedral group of order $2p$. One can show that $\mathbb{Z}_q[F_{p,q}]$ is a nice Fitting order by either applying Proposition 4.8 or following the explicit computation of [CR81, §34E].

Remark 4.13. Let L/K be a finite Galois CM-extension of number fields with Galois group G. Let p be an odd prime and let cl_L denote the class group of L. Under mild technical hypotheses on p, [BJ11, Theorem 1.2] gives annihilators of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{cl}_L$ in terms of special values of a truncated Artin L-function of L/K. Building on this result, [Nic10, Corollary 7.2] uses noncommutative Fitting invariants to predict similar annihilators under the assumption of the relevant special case of the p-part of the Equivariant Tamagawa Number Conjecture (ETNC) (see [BF01], [Bur01]). Now Corollary 4.10 can be used to give explicit examples in which [Nic10, Corollary 7.2] predicts strictly more annihilators than the unconditional annihilators of [BJ11, Theorem 1.2] (e.g. one can use a minor variant of Example 4.11 in the case $p = 3$ and $G = A_4 \times C_2$, where C_2 is the group of order 2.) Note that the results of §6 can be used to give further examples in the case that p divides $|G'|$.

Proposition 4.14. *Let $\mathfrak{o}$ be a Fitting domain with residue field of characteristic $p > 0$. Let G be a profinite group containing a finite normal subgroup H such that $G/H \simeq \Gamma$, where Γ is a pro-p group isomorphic to $\mathbb{Z}_p$. Then the commutator subgroup G' is finite and the complete group algebra $\Lambda := \mathfrak{o}[[G]]$ is a nice Fitting order if and only if $p \nmid |G'|$.*

Proof. Let $\mathfrak{O} := \mathfrak{o}[[T]]$ be the power series ring in one variable over $\mathfrak{o}$. We fix a topological generator γ of Γ and choose a natural number n such that γ^{p^n} is central in G. Since $\Gamma^{p^n} \simeq \mathbb{Z}_p$, there is an isomorphism $\mathfrak{o}[[\Gamma^{p^n}]] \simeq \mathfrak{O}$ induced by $\gamma^{p^n} \mapsto 1 + T$. Note that G can be written as a semi-direct product $H \rtimes \Gamma$; hence if we view Λ as an $\mathfrak{O}$-module, there is a decomposition

$$\Lambda = \bigoplus_{i=0}^{p^n-1} \mathfrak{O}\gamma^i[H].$$

Hence Λ is finitely generated as an $\mathfrak{O}$-module and is an $\mathfrak{O}$-order in the separable $F := Quot(\mathfrak{O})$-algebra $A = \mathcal{Q}(G) := \bigoplus_i F\gamma^i[H]$. Note that A is obtained from Λ by inverting all regular elements. Since $\mathfrak{O}$ is again a Fitting domain, Λ is a Fitting order over $\mathfrak{O}$.

Let $\mathfrak{p}$ (resp. $\mathfrak{P}$) be the maximal ideal of $\mathfrak{o}$ (resp. $\mathfrak{O}$). Then $\mathfrak{P}$ is generated by $\mathfrak{p}$ and T. Since $\gamma^{p^n} = 1 + T \equiv 1 \mod \mathfrak{P}$, we have

$$\overline{\Lambda} := \Lambda/\mathfrak{P}\Lambda = \bigoplus_{i=0}^{p^n-1} k\gamma^i[H] = k[H \rtimes C_{p^n}],$$

where C_{p^n} denotes the cyclic group of order p^n and $k := \mathfrak{O}/\mathfrak{P} = \mathfrak{o}/\mathfrak{p}$ is the residue field of characteristic p. Since G/H is abelian, the commutator subgroup G' of G is actually a subgroup of H and thus is finite. Moreover, G' identifies with the commutator subgroup of $H \rtimes C_{p^n}$.

If Λ is a nice Fitting order, then the same reasoning as that in the proof of Proposition 4.8 shows that $p \nmid |G'|$. Suppose conversely that $p \nmid |G'|$. Then $\mathfrak{o}[H \rtimes C_{p^n}]$ is a separable $\mathfrak{o}$-algebra by [DJ83, Theorem 1]. Since $k = \mathfrak{o}/\mathfrak{p}$, [AG60, Theorem 4.7] implies that $k[H \rtimes C_{p^n}]$ is a separable k-algebra. However, $\overline{\Lambda} = k[H \rtimes C_{p^n}]$ and $k = \mathfrak{O}/\mathfrak{P}$, so the same theorem also shows that $\Lambda = \mathfrak{o}[[G]]$ is a separable $\mathfrak{O}$-algebra. Now [AG60, Theorem 2.3] shows that Λ is also separable over its centre, i.e., Λ is an Azumaya algebra. However, $\zeta(\Lambda)$ is semiperfect by [Lam01, Example 23.3] and thus a direct product of local rings by [Lam01, Theorem 23.11], say

$$\zeta(\Lambda) = \bigoplus_{i=1}^{r} \mathfrak{O}_i,$$

where each $\mathfrak{O}_i$ contains $\mathfrak{O}$. By [CR81, Proposition 6.5(ii)] each $\mathfrak{O}_i$ is in fact a complete local ring. Let $\mathfrak{P}_i$ be the maximal ideal of $\mathfrak{O}_i$ and $k_i := \mathfrak{O}_i/\mathfrak{P}_i$ be the residue field. Since

$\mathfrak{P} \subset \mathfrak{P}_i$, the natural projection $\mathfrak{O}_i \twoheadrightarrow k_i$ factors through $\mathfrak{O}_i \twoheadrightarrow \mathfrak{O}_i/\mathfrak{P} = \mathfrak{O}_i \otimes_{\mathfrak{O}} k$. Hence we have the corresponding homomorphisms of Brauer groups

$$\mathrm{Br}(\mathfrak{O}_i) \to \mathrm{Br}(\mathfrak{O}_i/\mathfrak{P}) \to \mathrm{Br}(k_i).$$

Now $\mathrm{Br}(\mathfrak{O}_i) \to \mathrm{Br}(k_i)$ is injective by [AG60, Corollary 6.2] and hence $\mathrm{Br}(\mathfrak{O}_i) \to \mathrm{Br}(\mathfrak{O}_i/\mathfrak{P})$ must also be injective. This yields an embedding

$$\mathrm{Br}(\zeta(\Lambda)) = \bigoplus_{i=1}^{r} \mathrm{Br}(\mathfrak{O}_i) \hookrightarrow \bigoplus_{i=1}^{r} \mathrm{Br}(\mathfrak{O}_i \otimes_{\mathfrak{O}} k) = \mathrm{Br}(\zeta(\Lambda) \otimes_{\mathfrak{O}} k).$$

Since Λ is Azumaya, it defines a class $[\Lambda] \in \mathrm{Br}(\zeta(\Lambda))$ which is mapped to $[\overline{\Lambda}]$ via this embedding. However, $\overline{\Lambda}$ is a group ring of a finite group over a field of positive characteristic and such a group ring is Azumaya if and only if it is a direct product of matrix rings over commutative rings (see [Pas77, p. 232] or the remark after [DJ83, Corollary, p. 390].) Hence $[\overline{\Lambda}]$ is trivial and thus so is $[\Lambda]$. Therefore Λ is a direct product of matrix rings over commutative rings and hence is a nice Fitting order. $\square$

Corollary 4.15. *We have $\mathcal{H}(\mathfrak{o}[[G]]) = \zeta(\mathfrak{o}[[G]])$ if and only if $p \nmid |G'|$.*

Remark 4.16. Let Λ be a nice Fitting order and let X be a finitely generated Λ-module. Then $\mathcal{I}(\Lambda) = \zeta(\Lambda)$ and so, as noted in §3.5, the equivalence class $[X]_{\mathrm{nr}(\Lambda)}$ contains precisely one element and we have $\iota([X]_{\mathrm{nr}(\Lambda)}) = X$. Hence we need not distinguish between $\mathrm{Fit}_{\Lambda}^{\max}$ and $\mathrm{Fitt}_{\Lambda}^{\max}$ in the proof and statement of Theorem 4.17 and Lemma 4.18 below.

Theorem 4.17. *Let Λ be a nice Fitting order over the Fitting domain $\mathfrak{o}$. Let M, M_1, M_2 and M_3 be finitely generated Λ-modules.*

(i) *We have $\mathrm{Fit}_{\Lambda}^{\max}(M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M)$.*

(ii) *Suppose that Λ is a direct product of matrix rings over commutative rings or that $\mathfrak{o}$ is a complete discrete valuation ring. If $M_2 = M_1 \oplus M_3$, then*

$$\mathrm{Fit}_{\Lambda}^{\max}(M_2) = \mathrm{Fit}_{\Lambda}^{\max}(M_1) \cdot \mathrm{Fit}_{\Lambda}^{\max}(M_3).$$

(iii) *If Λ is a maximal order over a complete discrete valuation ring $\mathfrak{o}$, and $M_1 \hookrightarrow M_2 \twoheadrightarrow M_3$ is an exact sequence, then*

$$\mathrm{Fit}_{\Lambda}^{\max}(M_2) = \mathrm{Fit}_{\Lambda}^{\max}(M_1) \cdot \mathrm{Fit}_{\Lambda}^{\max}(M_3). \tag{4.1}$$

Proof. Property (i) follows from combining Proposition 4.3 and Theorem 3.5. For (ii) it suffices to treat the cases where Λ is a matrix ring over a commutative ring or a maximal order over a complete discrete valuation ring. In the former case, (ii) is Theorem 2.4 (iv); in the latter, (ii) follows from (iii) applied to the tautological exact sequence $M_1 \hookrightarrow M_1 \oplus M_3 \twoheadrightarrow M_3$. So it suffices to prove (iii). We shall need the following lemma.

Lemma 4.18. *Let Λ be a maximal order over a complete discrete valuation ring $\mathfrak{o}$ such that the F-algebra A is simple. Let M be a finitely generated Λ-module. Then either $F \otimes_{\mathfrak{o}} M \neq 0$ and $\mathrm{Fit}_{\Lambda}^{\max}(M) = 0$ or M admits a quadratic presentation.*

Proof. Since A is simple, it is isomorphic to a matrix ring $M_{n\times n}(D)$, where D is a skewfield of finite dimension over its centre L, and L is a finite field extension of F. Let $\mathfrak{o}_L$ be the integral closure of $\mathfrak{o}$ in L. Then $\mathfrak{o}_L$ is the centre of Λ and M is also an $\mathfrak{o}_L$-module. If $L \otimes_{\mathfrak{o}_L} M = F \otimes_{\mathfrak{o}} M \neq 0$, then there is no nonzero element in $\mathfrak{o}_L$ annihilating M. This implies that $\mathrm{Fit}_{\Lambda}^{\max}(M) = 0$ by (i) of the Theorem.

Now suppose that $F \otimes_{\mathfrak{o}} M = 0$ and choose an epimorphism $\pi : \Lambda^k \twoheadrightarrow M$. Since maximal orders are hereditary by [CR81, Theorem 26.12], $\ker(\pi)$ is projective by [CR81, Proposition 4.3]. But as $F \otimes_{\mathfrak{o}} M = 0$, we have $F \otimes_{\mathfrak{o}} \ker(\pi) \simeq A^k$; thus $\ker(\pi) \simeq \Lambda^k$ by [Rei03, Theorem 18.10]. □

We return to the proof of Theorem 4.17 (iii). Since the reduced norm is computed component-wise, we may assume that A is simple. If $F \otimes_{\mathfrak{o}} M_2 \neq 0$, then also $F \otimes_{\mathfrak{o}} M_1 \neq 0$ or $F \otimes_{\mathfrak{o}} M_3 \neq 0$ and both sides in (4.1) are zero by Lemma 4.18. If $F \otimes_{\mathfrak{o}} M_2 = 0$, then also $F \otimes_{\mathfrak{o}} M_1 = F \otimes_{\mathfrak{o}} M_3 = 0$. Hence M_1, M_2 and M_3 admit quadratic presentations by Lemma 4.18 and the result follows from Theorem 3.2 (iii) and (v). □

Remark 4.19. Note that Theorem 4.17 may be applied to the nice Fitting orders considered in Propositions 4.8 and 4.14, as their proofs show that these are direct products of matrix rings over commutative rings.

Remark 4.20. It is useful to be able to determine whether or not a given presentation of a finitely generated Λ-module M can be used to compute $\mathrm{Fit}_{\Lambda}^{\max}(M)$. If Λ is a direct product of matrix rings over commutative rings, this problem is solved by Proposition 3.6; recall that Fitting invariants over commutative rings do not depend on the chosen presentation. If Λ is a maximal order over a complete discrete valuation ring, we may apply Lemma 4.18. Hence we have solved this question for maximal Fitting invariants over arbitrary nice Fitting orders over complete discrete valuation rings. However, we note that if Λ is isomorphic to a nice Fitting order, then it may be necessary to compute this isomorphism explicitly, though in many cases it is possible to get away with less.

Example 4.21. Let G be a finite group and let $\mathfrak{o}$ be a complete discrete valuation ring with field of fractions F. Suppose the group algebra $F[G]$ decomposes into a (finite) direct product of matrix rings over a field, i.e., the Schur indices of all F-irreducible characters of G are equal to 1. (This happens, for exampe, if G is dihedral or symmetric, or if G is a p-group where p is an odd prime not necessarily equal to the residue characteristic of $\mathfrak{o}$; see [CR87, §74] for more on this topic.) Let $\Lambda = \Lambda'_G$ as in Definition 4.5; an explicit example is $\Lambda = \mathbb{Z}_3[A_4]$ as discussed in Example 4.11. Now one only needs to compute the central idempotent $e = |G'|^{-1}\mathrm{Tr}_{G'}$. Indeed, $\Lambda(1-e)$ is a finite direct product of matrix rings over complete discrete valuation rings; thus Remark 2.5 shows that $\mathrm{Fit}_{\Lambda(1-e)}((1-e)M)$ is completely determined by $\mathrm{Fit}_{\zeta(\Lambda(1-e))}((1-e)M)$. Since Λe is commutative, we therefore see that $\mathrm{Fit}_{\Lambda}(M)$ is completely determined by $\mathrm{Fit}_{\zeta(\Lambda)}(M)$ in this case.

5. Quotients by left ideals

We compute the maximal Fitting invariant of the quotient of a Fitting order by a left ideal in several cases.

Theorem 5.1. *Let Λ be a Fitting order and let I be a left ideal of Λ. Then*

(i) *We have $\langle \mathrm{nr}(x) \mid x \in I \rangle_{\mathcal{I}(\Lambda)} \subset \mathrm{Fit}_{\Lambda}^{\max}(\Lambda/I)$.*
(ii) *If I is a principal left ideal generated by α then $\mathrm{Fit}_{\Lambda}(\Lambda/I) \cdot \mathcal{I}(\Lambda) = \mathrm{nr}(\alpha) \cdot \mathcal{I}(\Lambda)$.*
(iii) *If Λ is a direct product of matrix rings over commutative rings, or Λ is a nice Fitting order over a complete discrete valuation ring, then*

$$\mathrm{Fit}_{\Lambda}^{\max}(\Lambda/I) = \langle \mathrm{nr}(x) \mid x \in I \rangle_{\zeta(\Lambda)}.$$

Proof. (i) Let $\{x_1, \ldots, x_{r-1}\}$ be a fixed set of generators of I and let x_r be an arbitrary element of I. Then there exists a presentation of the form

$$\Lambda^r \xrightarrow{h} \Lambda \twoheadrightarrow \Lambda/I,$$

where $(x_1, \dots, x_r)^t \in M_{r\times 1}(\Lambda)$ is the matrix representing h. Then we have $\mathrm{nr}(x_r) \in \mathrm{Fit}_\Lambda(h) \subset \mathrm{Fit}^{\max}_\Lambda(\Lambda/I)$. Since x_r was arbitrary, this gives the desired containment.

(ii) Let $\Lambda \xrightarrow{h} \Lambda \twoheadrightarrow \Lambda/I$ be the presentation given by right multiplication by α. Then since h is a quadratic presentation we have

$$\mathrm{Fit}^{\max}_\Lambda(\Lambda/I) = \mathrm{Fit}_\Lambda(h) \cdot \mathcal{I}(\Lambda) = \mathrm{nr}(\alpha) \cdot \mathcal{I}(\Lambda),$$

where the first equality follows from Theorem 3.2 and equation (3.5).

(iii) If Λ is a direct product of matrix rings over commutative rings then the result follows from Proposition 2.12. Thus it remains to consider the case where Λ is a maximal order over a complete discrete valuation ring; the result follows from Lemma 4.18 and part (ii) above. □

6. Annihilation and change of order

6.1. **Conductors and annihilation.** We give annihilation results in terms of conductors. For background material on conductors, we refer the reader to [CR81, §27].

Definition 6.1. Let $\Lambda \subset \Gamma$ be $\mathfrak{o}$-orders in A. Then we define

$$\begin{aligned} (\Gamma : \Lambda)_l &= \{x \in \Gamma \mid x\Gamma \subset \Lambda\} = \text{largest right } \Gamma\text{-module in } \Lambda, \\ (\Gamma : \Lambda)_r &= \{x \in \Gamma \mid \Gamma x \subset \Lambda\} = \text{largest left } \Gamma\text{-module in } \Lambda, \end{aligned}$$

and say that $(\Gamma : \Lambda)_l$ (resp. $(\Gamma : \Lambda)_r$) is the *left* (resp. *right*) *conductor of* Γ *into* Λ. We define the *central conductor* of Γ over Λ to be

$$\mathcal{F}(\Gamma, \Lambda) = \{x \in \zeta(\Gamma) \mid x\Gamma \subset \Lambda\} = \zeta(\Gamma) \cap (\Gamma : \Lambda)_l = \zeta(\Gamma) \cap (\Gamma : \Lambda)_r.$$

Proposition 6.2. *Let $\Lambda \subset \tilde{\Lambda}$ be Fitting orders such that $\mathcal{H}(\tilde{\Lambda}) = \zeta(\tilde{\Lambda})$. (In particular, this is the case if $\tilde{\Lambda}$ is nice.) Then $\mathcal{F}(\tilde{\Lambda}, \Lambda) \subset \mathcal{H}(\Lambda)$ and so for any finitely generated Λ-module M we have*

$$\mathcal{F}(\tilde{\Lambda}, \Lambda) \cdot \mathrm{Fit}^{\max}_\Lambda(M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M).$$

Proof. Let $x \in \mathcal{F}(\tilde{\Lambda}, \Lambda)$. Fix $b \in \mathbb{N}$ and let $H \in M_{b\times b}(\Lambda)$. Then $H \in M_{b\times b}(\tilde{\Lambda})$ so $H^* \in M_{b\times b}(\tilde{\Lambda})$ since $1 \in \zeta(\tilde{\Lambda}) = \mathcal{H}(\tilde{\Lambda})$ by hypothesis (in the case that $\tilde{\Lambda}$ is nice, this follows from Proposition 4.3). By definition of $\mathcal{F}(\tilde{\Lambda}, \Lambda)$ we have $xH^* \in M_{b\times b}(\Lambda)$. Since b and H were arbitrary, we have shown that $x \in \mathcal{H}(\Lambda)$. Therefore $\mathcal{F}(\tilde{\Lambda}, \Lambda) \subset \mathcal{H}(\Lambda)$ and the result now follows from Theorem 3.5. □

Corollary 6.3. *Let Λ be a Fitting order contained in a maximal order Λ'. Let M be a finitely generated Λ-module. Then $\mathcal{F}(\Lambda', \Lambda) \cdot \mathrm{Fit}^{\max}_\Lambda(M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M)$.*

In fact we can improve this slightly:

Proposition 6.4. *Let Λ be a Fitting order contained in a maximal order Λ'. Then $\mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda)) \subset \mathcal{H}(\Lambda)$ and so for any finitely generated Λ-module M we have*

$$\mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda)) \cdot \mathrm{Fit}^{\max}_\Lambda(M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M).$$

Proof. Let $n \in \mathbb{N}$ and let $H \in M_{n\times n}(\Lambda)$. Then recalling the notation of §3.1 and §3.6, the generalised adjoint matrix H^* was defined to be

$$H^* = \sum_{i=1}^{t} (-1)^{m_i - 1} \sum_{j=1}^{m_i} \alpha_{ij} H_i^{j-1},$$

where $m_i = n_i \cdot s_i \cdot n \in \mathbb{N}$, $H_i = He_i$ and $\alpha_{ij} \in \mathfrak{o}_i'$, $1 \le i \le t$, $1 \le j \le m_i$. We put $m = \max_{1\le i\le t}(m_i)$ and for $1 \le i \le t$, $1 \le j \le m$ we define

$$\tilde{\alpha}_{ij} = \begin{cases} \alpha_{ij} & \text{if} \quad j \le m_i \\ 0 & \text{if} \quad j > m_i. \end{cases}$$

Then we may write

$$H^* = \sum_{j=1}^{m} H^{j-1} \sum_{i=1}^{t} (-1)^{m_i+1} \tilde{\alpha}_{ij} e_i = \sum_{j=1}^{m} H^{j-1} \cdot \lambda_j',$$

where $\lambda_j' = \sum_{i=1}^{t} (-1)^{m_i+1} \tilde{\alpha}_{ij} e_i$ belongs to $\oplus_{i=1}^{t} \mathfrak{o}_i' = \zeta(\Lambda')$. Now it is clear that for any $x \in \mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda))$ we have

$$x \cdot H^* = \sum_{j=1}^{m} H^{j-1} \cdot x \cdot \lambda_j' \in M_{n\times n}(\Lambda)$$

as desired. □

Remark 6.5. Let Λ be a Fitting order. Then as noted in §3.6 we have $\mathcal{I}(\Lambda)\cdot\mathcal{H}(\Lambda) \subset \zeta(\Lambda)$, and so $\mathcal{H}(\Lambda) \subset \mathcal{F}(\mathcal{I}(\Lambda), \zeta(\Lambda))$. In particular, if $\mathcal{I}(\Lambda) = \zeta(\Lambda')$ for a maximal order Λ' containing Λ, then $\mathcal{H}(\Lambda) = \mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda))$ by Proposition 6.4.

Remark 6.6. Let p be prime and G be a finite group; then $\mathcal{A}_p(G)$ in [BMC11, §2.1.2] is defined to be equal to $\mathcal{H}(\mathbb{Z}_p[G])$. Hence Corollary 4.10 shows that $\mathcal{A}_p(G) = \zeta(\mathbb{Z}_p[G])$ in the case $p \nmid |G'|$ and Proposition 6.2 can be used to compute a subset of $\mathcal{A}_p(G)$ otherwise. Thus several of the annihilation results of [BMC11] can be made more explicit. Similar remarks apply to $\mathcal{A}(R[G])$ in [Bur11, §2.3].

6.2. **Conductors in the group ring case.** Let G be a finite group and let $\mathfrak{o}$ be a complete discrete valuation ring. Let $\tilde{\Lambda}$ be a nice Fitting order containing the group ring $\Lambda := \mathfrak{o}[G]$. We may write

$$\tilde{\Lambda} = \bigoplus_{i=1}^{k} \tilde{\Lambda}_i,$$

where $\tilde{\Lambda}_i$ is isomorphic to either a matrix ring $M_{n_i\times n_i}(\mathfrak{o}_i)$ over a commutative ring $\mathfrak{o}_i$ (not necessarily integrally closed) or a matrix ring $M_{n_i\times n_i}(\mathfrak{o}_{D_i})$ over the valuation ring $\mathfrak{o}_{D_i}$ of a skewfield D_i. In the latter case, we put $\mathfrak{o}_i := \zeta(\tilde{\Lambda}_i) = \zeta(\mathfrak{o}_{D_i})$ and denote the Schur index of D_i by s_i. In the former case, we put $s_i = 1$. In both cases, $\mathfrak{o}_i$ is a commutative noetherian complete local ring and we may assume that it is indecomposable. As usual, let F be the fraction field of $\mathfrak{o}$ and put $A_i := F \otimes_{\mathfrak{o}} \tilde{\Lambda}_i$ so that $A := F[G] = \bigoplus_{i=1}^{k} A_i$. For convenience, we also put $F_i = F \otimes_{\mathfrak{o}} \mathfrak{o}_i$ so that $\zeta(A) = \bigoplus_{i=1}^{k} F_i$; note that F_i is not necessarily a field.

We denote the reduced trace from A_i to F by tr_i; then we have

$$\mathrm{tr}_i = \mathrm{Tr}_{F_i/F} \circ \mathrm{tr}_{A_i/F_i},$$

where $\mathrm{Tr}_{F_i/F}$ is the ordinary trace from F_i to F, and tr_{A_i/F_i} is the reduced trace from A_i to F_i. For the ordinary trace $\mathrm{Tr}_{A/F}$ from A to F we thus have

$$\mathrm{Tr}_{A/F}(x) = \sum_{i=1}^{k} n_i s_i \mathrm{tr}_i(x_i)$$

for $x = \sum_{i=1}^{k} x_i \in A = \bigoplus_{i=1}^{k} A_i$. Abusing notation, we define the inverse different of $\tilde{\Lambda}_i$ with respect to the reduced trace tr_i to be

$$\mathfrak{D}_i^{-1} := \left\{ x \in A_i \mid \mathrm{tr}_i(x\tilde{\Lambda}_i) \subset \mathfrak{o} \right\}.$$

In the case where $\tilde{\Lambda}_i$ is a matrix ring over the valuation ring $\mathfrak{o}_{D_i}$ of a skewfield D_i, this is in fact an invertible $\tilde{\Lambda}_i$-lattice, and $\mathfrak{D}_i$ is called the different of $\tilde{\Lambda}_i$ with respect to tr_i. However, we note that $\mathfrak{D}_i^{-1}$ is *not* invertible in general.

Proposition 6.7. *With notation as above, we have*

$$(\tilde{\Lambda} : \Lambda)_l = (\tilde{\Lambda} : \Lambda)_r = \bigoplus_{i=1}^{k} \frac{|G|}{n_i s_i} \mathfrak{D}_i^{-1}.$$

Proof. This is essentially the same proof as that of [CR81, Theorem 27.8]. ☐

Corollary 6.8. *With the notation as above, we have*

$$\bigoplus_{i=1}^{k} \frac{|G|}{n_i s_i} \mathfrak{D}^{-1}(\mathfrak{o}_i/\mathfrak{o}) \subset \mathcal{F}(\tilde{\Lambda}, \Lambda),$$

where

$$\mathfrak{D}^{-1}(\mathfrak{o}_i/\mathfrak{o}) = \left\{ x \in F_i \mid \mathrm{Tr}_{F_i/F}(x\mathfrak{o}_i) \subset \mathfrak{o} \right\},$$

which is the usual inverse different if F_i is a field with ring of integers $\mathfrak{o}_i$.

Proof. For each i, we have an inclusion

$$\frac{|G|}{n_i s_i} \mathfrak{D}^{-1}(\mathfrak{o}_i/\mathfrak{o}) \subset \frac{|G|}{n_i s_i} \mathfrak{D}_i^{-1} \cap \mathfrak{o}_i. \tag{6.1}$$

The result now follows since $\zeta(\tilde{\Lambda}) = \bigoplus_{i=1}^{k} \mathfrak{o}_i$ and $\mathcal{F}(\tilde{\Lambda}, \Lambda) = \zeta(\tilde{\Lambda}) \cap (\tilde{\Lambda} : \Lambda)_l$. ☐

Remark 6.9. If $\tilde{\Lambda}$ is a maximal order and $\mathfrak{o}$ is the ring of integers in a local field of characteristic zero, Jacobinski's central conductor formula [Jac66, Theorem 3] (also see [CR81, Theorem 27.13]) implies that the inclusion (6.1) is an equality for each i; thus we have also an equality in Corollary 6.8. However, the argument that shows equality can not be extended to the more general situation of nice Fitting orders, since our notion of the inverse different does not lead to invertible lattices in general.

We now specialise to the following situation. Let $\mathfrak{o}$ be the ring of integers in a local field F of characteristic zero and let G be a finite group. Let Λ' be a maximal order containing $\Lambda := \mathfrak{o}[G]$. We have a natural decomposition

$$\zeta(\Lambda') \simeq \bigoplus_{i=1}^{k} \mathfrak{o}_i', \tag{6.2}$$

where k is the number of irreducible $\mathbb{C}_p$-valued characters of G modulo Galois action and each $\mathfrak{o}_i'$ corresponds to an irreducible $\mathbb{C}_p$-valued character χ_i. Note that the quotient field F_i of $\mathfrak{o}_i'$ equals $F_i = F(\chi_i(g) \mid g \in G)$.

Proposition 6.10. *Let $\Lambda_G' = \mathfrak{o}[G]e \oplus \Lambda'(1-e)$ where G' is the commutator subgroup of G and $e = |G'|^{-1}\mathrm{Tr}_{G'}$ (as in Definition 4.5). Then with the notation as above, we have*

$$\mathcal{F}(\Lambda_G', \Lambda) = \mathfrak{o}[G] \cdot \mathrm{Tr}_{G'} \oplus \mathcal{F}(\Lambda', \Lambda)(1-e) = \mathfrak{o}[G] \cdot \mathrm{Tr}_{G'} \oplus \bigoplus_{\chi(1) \neq 1} \frac{|G|}{\chi(1)} \mathfrak{D}^{-1}(\mathfrak{o}_i'/\mathfrak{o}).$$

Proof. First observe that $\mathfrak{o}[G]e$ is commutative and so $\zeta(\Lambda'_G) = \mathfrak{o}[G]e \oplus \zeta(\Lambda'(1-e))$. Moreover, $\mathcal{F}(\Lambda'_G, \Lambda)$ is an ideal $\mathcal{I} \oplus \mathcal{J}$ of $\zeta(\Lambda'_G)$, so we may compute $\mathcal{I}$ and $\mathcal{J}$ separately. Since $\Lambda'(1-e)$ is maximal and (6.1) is an equality in this case (see Remark 6.9), we see that $\mathcal{J}$ is of the desired form. Now observe that

$$\mathcal{F}(\Lambda'_G, \mathfrak{o}[G])e = \mathcal{I} = ((\Lambda'_G : \mathfrak{o}[G])_l)e = \mathfrak{o}[G]e \cap \mathfrak{o}[G] = \mathfrak{o}[G] \cdot \mathrm{Tr}_{G'}.$$

We explain the last two equalities. By definition, $\mathcal{I}$ is the largest ideal of $\mathfrak{o}[G]e$ contained in $\mathfrak{o}[G]$, so $\mathcal{I} \subset \mathfrak{o}[G]e \cap \mathfrak{o}[G]$. If $xe \in \mathfrak{o}[G]$ with $x \in \mathfrak{o}[G]$, then for any ye with $y \in \mathfrak{o}[G]$ we have $(xe)(ye) = (xe)y \in \mathfrak{o}[G]$, giving the reverse inclusion. Let $x_1, \ldots, x_r$ be a set of representatives in G of the quotient group G/G'; then $\{x_1e, \ldots, x_re\}$ is an $\mathfrak{o}$-basis for $\mathfrak{o}[G]e$. Write $G' = \{h_1, \ldots, h_s\}$; then $G = \{h_ix_j\}_{i,j}$ is an $\mathfrak{o}$-basis for $\mathfrak{o}[G]$. Let $x \in \mathfrak{o}[G]e$. Then we can write $x = \lambda_1x_1e + \cdots + \lambda_rx_re$ where each $\lambda_k \in \mathfrak{o}$. Since $e = |G'|^{-1}\mathrm{Tr}_{G'} = |G'|^{-1}\sum_{i=1}^{s} h_i$, we see that $x \in \mathfrak{o}[G]$ if and only if $|G'|$ divides each λ_k if and only if $x \in \mathfrak{o}[G] \cdot \mathrm{Tr}_{G'}$. Therefore $\mathfrak{o}[G]e \cap \mathfrak{o}[G] = \mathfrak{o}[G] \cdot \mathrm{Tr}_{G'}$. □

Example 6.11. Let $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group of order 8, let $\Lambda = \mathbb{Z}_2[D_8]$, and let Λ' be a maximal order containing Λ. Let $\chi_1, \ldots, \chi_5$ be the $\mathbb{Q}_2$-irreducible characters of D_8, where $\chi_1(1) = \cdots = \chi_4(1) = 1$ and $\chi_5(1) = 2$. Let e_i be the primitive central idempotent associated to χ_i. Then $\{8e_1, 8e_2, 8e_3, 8e_4, 4e_5\}$ is a $\mathbb{Z}_2$-basis of $\mathcal{F}(\Lambda', \Lambda)$ and $\{1+a^2, a+a^3, b+a^2b, ab+a^3b, 4e_5\}$ is a $\mathbb{Z}_2$-basis of $\mathcal{F}(\Lambda'_{D_8}, \Lambda)$. By using the character table of D_8 to express one basis in terms of the other and then computing the appropriate determinant, one can show that $[\mathcal{F}(\Lambda'_{D_8}, \Lambda) : \mathcal{F}(\Lambda', \Lambda)]_{\mathbb{Z}_2} = 2^4$. Thus using $\mathcal{F}(\Lambda'_{D_8}, \Lambda)$ instead of $\mathcal{F}(\Lambda', \Lambda)$ in Proposition 6.2 gives an improved annihilation result. Almost identical reasoning applies in the case $\Lambda = \mathbb{Z}_2[Q_8]$, where Q_8 is the quaternion group of order 8.

We now define an $\mathfrak{o}'_i$-ideal $\mathfrak{A}_i$ by

$$\mathfrak{A}_i := \langle \chi_i(g) \mid g \in G\rangle_{\mathfrak{o}'_i}.$$

Note that $\mathfrak{A}_i = \mathfrak{o}'_i$ if the degree $\chi_i(1)$ of the character χ_i is invertible in $\mathfrak{o}'_i$; this in particular applies to all linear characters of G.

Proposition 6.12. *With notation as above, we have an equality*

$$\mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda)) = \bigoplus_{i=1}^{k} \frac{|G|}{\chi_i(1)} \mathfrak{A}_i^{-1}\mathfrak{D}^{-1}(\mathfrak{o}'_i/\mathfrak{o}).$$

Proof. Let $\alpha = \sum_{i=1}^{k} \alpha_i$ and $\beta = \sum_{i=1}^{k} \beta_i$ be elements in $\bigoplus_{i=1}^{k} \mathfrak{o}'_i$. Then the above isomorphism (6.2) maps $\alpha\beta$ to the group ring element

$$\sum_{g \in G} \sum_{i=1}^{k} \sum_{\sigma \in \mathrm{Gal}(F_i/F)} \frac{\chi_i(1)}{|G|} \alpha_i^\sigma \beta_i^\sigma \chi_i^\sigma(g^{-1})g \in \zeta(\Lambda').$$

We see that $\alpha\zeta(\Lambda') \subset \zeta(\Lambda)$ if and only if for all $1 \le i \le k$, $\beta_i \in \mathfrak{o}'_i$ and all $g \in G$ we have

$$\sum_{\sigma \in \mathrm{Gal}(F_i/F)} \frac{\chi_i(1)}{|G|} \alpha_i^\sigma \beta_i^\sigma \chi_i^\sigma(g^{-1}) \in \mathfrak{o}.$$

The latter condition is equivalent to $\frac{\chi_i(1)}{|G|}\mathrm{Tr}_{F_i/F}(\alpha_i\mathfrak{A}_i) \subset \mathfrak{o}$ for all $1 \le i \le k$, i.e., $\alpha_i \in \frac{|G|}{\chi_i(1)}\mathfrak{A}_i^{-1}\mathfrak{D}^{-1}(\mathfrak{o}'_i/\mathfrak{o})$. □

Corollary 6.13. *Let $\mathfrak{o}$ be the ring of integers in a local field F of characteristic zero and residue field of characteristic $p > 0$. Let G be a finite group and let Λ' be a maximal order containing $\Lambda := \mathfrak{o}[G]$. If the degrees of all irreducible characters of G are prime to p, then $\mathcal{F}(\Lambda', \Lambda) = \mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda))$.*

Proof. As noted above, we have $\mathfrak{A}_i = \mathfrak{o}_i'$ for all $1 \leq i \leq k$ in this case. Hence the result follows from Proposition 6.12 and Jacobinski's central conductor formula (see Remark 6.9). ☐

Corollary 6.14. *Let $\mathfrak{o}$ be the ring of integers in a local field F and let G be a finite group. Let Λ' be a maximal order containing $\Lambda := \mathfrak{o}[G]$. Then*

$$\mathfrak{o}[G] \cdot \mathrm{Tr}_{G'} \oplus \bigoplus_{\substack{i=1 \\ \chi_i(1) \neq 1}}^{k} \frac{|G|}{\chi_i(1)} \mathfrak{A}_i^{-1} \mathfrak{D}^{-1}(\mathfrak{o}_i'/\mathfrak{o}) \subset \mathcal{H}(\Lambda).$$

Proof. This is an immediate consequence of Propositions 6.4, 6.10 and 6.12. ☐

Example 6.15. Let p be an odd prime and let $D_{2p} = \langle x, y \mid x^p = y^2 = 1, yx = x^{-1}y \rangle$ be the dihedral group of order $2p$. Let $\Lambda = \mathbb{Z}_p[D_{2p}]$ and let Λ' be a maximal $\mathbb{Z}_p$-order containing Λ. Following [CR81, Example 7.39], there is a decomposition

$$\mathbb{Q}_p[D_{2p}] \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p \oplus A_p, \tag{6.3}$$

where A_p is the twisted group algebra $\mathbb{Q}_p(\zeta_p) \oplus \mathbb{Q}_p(\zeta_p)y$; here, ζ_p denotes a primitive pth root of unity and multiplication in A_p is given by $y^2 = 1$ and $\alpha y = y\tau(\alpha)$ for $\alpha \in \mathbb{Q}_p(\zeta_p)$, where τ denotes the unique element in $\mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ of order 2. The surjection $\mathbb{Q}_p[D_{2p}] \twoheadrightarrow A_p$ is given by $x \mapsto \zeta_p$ and $y \mapsto y$. The idempotents corresponding to (6.3) are

$$e_1 = \frac{1}{2p} \sum_{g \in D_{2p}} g, \quad e_2 = \frac{1}{2p}(1-y) \cdot \sum_{i=0}^{p-1} x^i, \quad e_3 = 1 - e_1 - e_2.$$

Since A_p is not a skewfield, there must be an isomorphism $A_p \simeq M_{2\times 2}(E_p)$, where $E_p = \mathbb{Q}_p(\zeta_p + \zeta_p^{-1})$ is the unique subfield of $\mathbb{Q}_p(\zeta_p)$ such that $[\mathbb{Q}_p(\zeta_p) : E_p] = 2$. To compute the reduced norms, however, it is more convenient to work with the irreducible matrix representation of A_p over $\mathbb{Q}_p(\zeta_p)$ given by

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha \in \mathbb{Q}_p(\zeta_p).$$

It is now easy to check that

$$\mathrm{nr}(y) = e_1 - e_2 - e_3, \quad \mathrm{nr}(-y) = -e_1 + e_2 - e_3.$$

Since $\mathrm{nr}(1) = 1$ and $2 \in \mathbb{Z}_p^\times$, we conclude that $e_i \in \mathcal{U}(\Lambda)$ for $i = 1, 2, 3$.

For $r \in \mathbb{N}$ we have

$$e_3 \mathrm{nr}(x^r + x^{-r}) = \det \begin{pmatrix} \zeta_p^r + \zeta_p^{-r} & 0 \\ 0 & \zeta_p^r + \zeta_p^{-r} \end{pmatrix} = (\zeta_p^r + \zeta_p^{-r})^2 = \zeta_p^{2r} + \zeta_p^{-2r} + 2.$$

As p is odd we can choose $r \in \mathbb{N}$ such that $2r \equiv 1 \mod p$. Since we already know that $e_1, e_2, e_3 \in \mathcal{U}(\Lambda) \subset \mathcal{I}(\Lambda)$, this shows that $e_3(\zeta_p + \zeta_p^{-1}) \in \mathcal{I}(\Lambda)$. But $\mathcal{I}(\Lambda)$ is a $\mathbb{Z}_p$-order and $\zeta(\Lambda') \simeq \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathfrak{o}_{E_p}$, so we conclude that $\mathcal{I}(\Lambda) = \zeta(\Lambda')$. (In fact, with more work one can show that $x^r + x^{-r} \in (\mathbb{Z}_p[D_{2p}])^\times$ and so $\mathcal{U}(\Lambda) = \mathcal{I}(\Lambda) = \zeta(\Lambda')$.) Since all irreducible characters have degree 1 or 2, Remark 6.5 and Corollary 6.13 imply that $\mathcal{H}(\Lambda)$ is worst possible in this case, i.e., $\mathcal{H}(\Lambda) = \mathcal{F}(\Lambda', \Lambda)$.

Example 6.16. We continue with Example 6.11, where $G = D_8$ is the dihedral group of order 8 and $\Lambda = \mathbb{Z}_2[D_8]$. There is only one $\mathbb{Q}_2$-irreducible non-linear character of D_8 which was denoted by χ_5. This character is of degree two, and a computation shows that $\chi_5(g)$ either equals 0 or 2 for any $g \in D_8$; hence $\mathfrak{A}_5 = 2 \cdot \mathbb{Z}_2$. If Λ' denotes a maximal order containing Λ then Proposition 6.12 and Remark 6.9 (respectively) imply that

$$\begin{aligned} \mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda)) &= 2^3(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus 2\mathbb{Z}_2, \\ \text{and} \quad \mathcal{F}(\Lambda', \Lambda) &= 2^3(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus 4\mathbb{Z}_2. \end{aligned}$$

By Corollary 6.14 we find that

$$\mathbb{Z}_2[D_8] \cdot \mathrm{Tr}_{D_8} \oplus 2\mathbb{Z}_2 \subset \mathcal{H}(\Lambda).$$

Thus by the index computation in Example 6.11 we have

$$[\mathbb{Z}_2[D_8] \cdot \mathrm{Tr}_{D_8} \oplus 2\mathbb{Z}_2 : \mathcal{F}(\Lambda', \Lambda)]_{\mathbb{Z}_2} = 2^5,$$

and so the annihilation result given therein can be further improved slightly. More generally, if $\Lambda = \mathbb{Z}_2[D_{2^a}]$ with $a \geq 3$, then one can show that

$$[\mathcal{F}(\zeta(\Lambda'), \zeta(\Lambda)) : \mathcal{F}(\Lambda', \Lambda)]_{\mathbb{Z}_2} = 2^{a-2}.$$

Remark 6.17. Conductors for completed group algebras are considered in [Nic12].

6.3. **Change of order.** Let p be prime and let G be a finite group. Let $\tilde{\Lambda}$ be a nice Fitting order containing $\Lambda := \mathbb{Z}_p[G]$. We adopt the notation of §6.2 (with $\mathfrak{o} = \mathbb{Z}_p$).

Theorem 6.18. *Let M be a finitely generated Λ-module. We have*

(i) $(\bigoplus_{i=1}^k \frac{|G|}{n_i s_i} \mathfrak{D}^{-1}(\mathfrak{o}_i/\mathbb{Z}_p)) \cdot \mathrm{Fit}_\Lambda^{\max}(M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M)$*; and*
(ii) $(\bigoplus_{i=1}^k \frac{|G|}{n_i s_i} \mathfrak{D}^{-1}(\mathfrak{o}_i/\mathbb{Z}_p)) \cdot \mathrm{Fit}_{\tilde{\Lambda}}^{\max}(\tilde{\Lambda} \otimes_\Lambda M) \subset \mathrm{Ann}_{\zeta(\Lambda)}(M)$.

Remark 6.19. Part (i) is just the combination of Proposition 6.2 and Corollary 6.8. The advantage of part (ii) over part (i) is that $\mathrm{Fit}_{\tilde{\Lambda}}^{\max}(\tilde{\Lambda} \otimes_\Lambda M)$ may be easier to compute than $\mathrm{Fit}_\Lambda^{\max}(M)$ since $\tilde{\Lambda}$ is a nice Fitting order.

We shall first prove the following auxiliary lemma.

Lemma 6.20. *Let G be a finite group and let $R \subset S$ be commutative rings, where S is flat as an R-module. Then for any finitely generated $R[G]$-module M we have*

$$\mathrm{Ann}_{S[G]}(S \otimes_R M) = S \otimes_R \mathrm{Ann}_{R[G]}(M).$$

Proof. We first remark that for any two left ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R[G]$ one has

$$S \otimes_R (\mathfrak{a} \cap \mathfrak{b}) = (S \otimes_R \mathfrak{a}) \cap (S \otimes_R \mathfrak{b}).$$

This follows immediately if we tensor the left exact sequence

$$\mathfrak{a} \cap \mathfrak{b} \hookrightarrow \mathfrak{a} \oplus \mathfrak{b} \to R[G]$$

with S. If M is a cyclic $R[G]$-module, the result follows from the exact sequences

$$\mathrm{Ann}_{R[G]}(M) \hookrightarrow R[G] \twoheadrightarrow M, \qquad S \otimes_R \mathrm{Ann}_{R[G]}(M) \hookrightarrow S[G] \twoheadrightarrow S \otimes_R M.$$

If M is finitely generated, then we may write $M = \sum_{i=1}^{r} M_i$, where each M_i is cyclic. Hence we may compute

$$\begin{aligned} S \otimes_R \mathrm{Ann}_{R[G]}(M) &= S \otimes_R \bigcap_{i=1}^{r} \mathrm{Ann}_{R[G]}(M_i) = \bigcap_{i=1}^{r} (S \otimes_R \mathrm{Ann}_{R[G]}(M_i)) \\ &= \bigcap_{i=1}^{r} \mathrm{Ann}_{S[G]}(S \otimes_R M_i) = \mathrm{Ann}_{S[G]}(\sum_{i=1}^{r} S \otimes_R M_i) \\ &= \mathrm{Ann}_{S[G]}(S \otimes_R M) \end{aligned}$$

as desired. □

Proof of Theorem 6.18. By Remark 6.19, we need only prove part (ii). We may treat each $\tilde{\Lambda}_i$ separately. If $\tilde{\Lambda}_i$ is maximal, the result follows from [BJ11, Lemmas 11.1 and 11.2] as shown in [Nic10, Proposition 5.1]. Hence we can and do assume that each $s_i = 1$, i.e., $\tilde{\Lambda}_i$ is a matrix ring over a commutative ring. Let f_i be an indecomposable idempotent of $\tilde{\Lambda}_i$ and define $T_i := f_i \cdot \tilde{\Lambda}_i$ which is an $\mathfrak{o}_i$-free (of rank $n_i s_i = n_i$) right $\mathfrak{o}_i[G]$-module. Then the inclusion $\Lambda \hookrightarrow \tilde{\Lambda}$ is induced by

$$\Lambda \to \tilde{\Lambda}_i \simeq \mathrm{End}_{\mathfrak{o}_i}(T_i), \quad \lambda \mapsto \rho_i(\lambda),$$

where $\rho_i(\lambda)$ is right multiplication by λ. Moreover, $\tilde{\Lambda}_i = \tilde{\Lambda} e_i$, where

$$e_i = \frac{n_i}{|G|} \sum_{g \in G} \psi_i(g^{-1}) g \in F_i[G],$$

and $\psi_i = \mathrm{Tr} \circ \rho_i$ is the character afforded by the $\mathbb{Q}_p[G]$-module A_i. Conversely, if $x \in \mathfrak{o}_i$ is considered as an element of $\zeta(\tilde{\Lambda}_i) \subset \tilde{\Lambda}_i \subset \mathbb{Q}_p[G]$, then the image of x in $\mathbb{Q}_p[G]$ is given by $\mathrm{Tr}_{F_i/\mathbb{Q}_p}(x \cdot e_i)$, where we extend $\mathrm{Tr}_{F_i/\mathbb{Q}_p}$ to $F_i[G]$ by linearity.

We now give two more auxiliary lemmas; the first is proven in essentially the same way as [BJ11, Lemma 11.1], and the second is a generalisation of [BJ11, Lemma 11.2].

Lemma 6.21. *Let M be a finitely generated $\mathbb{Z}_p[G]$-module. If $x \in \mathrm{Ann}_{\mathfrak{o}_i}((T_i \otimes_{\mathbb{Z}_p} M)^G)$, then $x \cdot \frac{|G|}{n_i} e_i \in \mathrm{Ann}_{\mathfrak{o}_i[G]}(\mathfrak{o}_i \otimes_{\mathbb{Z}_p} M)$.*

Lemma 6.22. *Let M be a finitely generated $\mathbb{Z}_p[G]$-module. If $x \in \mathfrak{o}_i$ such that $x \cdot \frac{|G|}{n_i} e_i \in \mathrm{Ann}_{\mathfrak{o}_i[G]}(\mathfrak{o}_i \otimes_{\mathbb{Z}_p} M)$, then for every $y \in \mathfrak{D}^{-1}(\mathfrak{o}_i/\mathbb{Z}_p)$ we have $\mathrm{Tr}_{F_i/\mathbb{Q}_p}(yx \cdot \frac{|G|}{n_i} e_i) \in \mathrm{Ann}_{\mathbb{Z}_p[G]}(M)$.*

Proof. By Lemma 6.20, the element $yx \cdot \frac{|G|}{n_i} e_i$ belongs to

$$\mathfrak{D}^{-1}(\mathfrak{o}_i/\mathbb{Z}_p) \cdot \mathrm{Ann}_{\mathfrak{o}_i[G]}(\mathfrak{o}_i \otimes_{\mathbb{Z}_p} M) = \mathfrak{D}^{-1}(\mathfrak{o}_i/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{Ann}_{\mathbb{Z}_p[G]}(M).$$

Hence $\mathrm{Tr}_{F_i/\mathbb{Q}_p}(yx \cdot \frac{|G|}{n_i} e_i)$ belongs to $\mathrm{Tr}_{F_i/\mathbb{Q}_p}(\mathfrak{D}^{-1}(\mathfrak{o}_i/\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathrm{Ann}_{\mathbb{Z}_p[G]}(M) \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(M)$. □

We continue with the proof of Theorem 6.18. Put $T_i^* := \mathrm{Hom}_{\mathfrak{o}_i}(T_i, \mathfrak{o}_i)$ which is again a right $\mathfrak{o}_i[G]$-module via $(fg)(t) = f(tg^{-1})$ for $f \in T_i^*$, $g \in G$ and $t \in T_i$. It is again $\mathfrak{o}_i$-free of the same rank as T_i and

$$\tilde{\Lambda}^* := \bigoplus_{i=1}^{k} \mathrm{End}_{\mathfrak{o}_i}(T_i^*)$$

is a nice Fitting order containing Λ. In fact, if we denote by $^\sharp : \mathbb{Q}_p[G] \to \mathbb{Q}_p[G]$ the involution induced by $g \mapsto g^{-1}$, then $^\sharp$ induces a bijective map

$$^\sharp : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}^*.$$

For any finitely generated $\mathbb{Z}_p[G]$-module M, we denote the Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ of M by $M^\vee$ which is endowed with the contravariant G-action $(gf)(m) = f(g^{-1}m)$ for $f \in M^\vee$, $g \in G$ and $m \in M$. By Theorem 3.2(vii) we can and do assume that M is finite. Then we have a canonical isomorphism of $\mathfrak{o}_i$-modules

$$(T_i^* \otimes_{\mathbb{Z}_p} M^\vee)^G \simeq ((T_i \otimes_{\mathbb{Z}_p} M)_G)^\vee.$$

Let $x \in \mathrm{Fit}^{\max}_{\tilde{\Lambda}}(\tilde{\Lambda} \otimes_\Lambda M)$. We may write $x = \sum_{i=1}^k x_i$, where each x_i belongs to

$$\mathrm{Fit}^{\max}_{\tilde{\Lambda}_i}(\tilde{\Lambda}_i \otimes_\Lambda M) = \mathrm{Fit}_{\mathfrak{o}_i}(T_i \otimes_{\mathbb{Z}_p[G]} M)$$

by Proposition 3.6. Hence we have

$$x_i^\sharp \in \mathrm{Ann}_{\mathfrak{o}_i}((T_i \otimes_{\mathbb{Z}_p} M)_G)^\sharp = \mathrm{Ann}_{\mathfrak{o}_i}\left(((T_i \otimes_{\mathbb{Z}_p} M)_G)^\vee\right) = \mathrm{Ann}_{\mathfrak{o}_i}((T_i^* \otimes_{\mathbb{Z}_p} M^\vee)^G).$$

Since $\mathrm{Ann}_{\mathbb{Z}_p[G]}(M) = \mathrm{Ann}_{\mathbb{Z}_p[G]}(M^\vee)^\sharp$, the result follows if we apply Lemmas 6.21 and 6.22 to T_i^* and $M^\vee$. ∎

Appendix A. Fitting invariants and Morita equivalence

We generalise the construction and many of the results of §2. For background on Morita equivalence we refer the reader to [CR81, §3D], [Rei03, Chapter 4] or [Lam99, Chapter 7]. Let Λ and R be any two rings (with identity and not necessarily commutative) that are Morita equivalent to one another. Denote by ${}_R\mathfrak{M}$ the category of left R-modules, by ${}_R\mathfrak{M}_\Lambda$ the category of (R, Λ)-bimodules, and so on. Now choose a progenerator $P \in \mathfrak{M}_R$ such that the functors

$$G : {}_R\mathfrak{M} \longrightarrow {}_\Lambda\mathfrak{M}, \quad N \mapsto P \otimes_R N$$
$$F : {}_\Lambda\mathfrak{M} \longrightarrow {}_R\mathfrak{M}, \quad M \mapsto P^* \otimes_\Lambda M$$

induce the Morita equivalence, where $P^* = \mathrm{Hom}_R(P, R) \in {}_R\mathfrak{M}$. (Note that in fact $P \in {}_\Lambda\mathfrak{M}_R$ and $P^* \in {}_R\mathfrak{M}_\Lambda$ via the map $\Lambda^{\mathrm{op}} = \mathrm{End}_\Lambda(\Lambda) \longrightarrow \mathrm{End}_R(P^*)$ induced by F.) Then we have isomorphisms

$$\Lambda \simeq \mathrm{End}_R(P) \quad \text{and} \quad R \simeq \mathrm{End}_\Lambda(P)^{\mathrm{op}}. \tag{A.1}$$

Lemma A.1. *If M is a finitely presented left Λ-module, then $F(M)$ is a finitely presented left R-module.*

Proof. We note that this result is surely standard, but the authors were unable to locate it in the literature. Since M is a finitely presented Λ-module, there is an epimorphism $\pi : \Lambda^b \twoheadrightarrow M$ for some $b \in \mathbb{N}$ such that $M' := \ker(\pi)$ is finitely generated. Since P^* is projective over Λ, it is flat; thus applying F yields a short exact sequence $F(M') \hookrightarrow (P^*)^b \twoheadrightarrow F(M)$. As P^* is a finitely generated R-module, $F(M)$ is also finitely generated over R. By the same argument, $F(M')$ is a finitely generated R-module. We now choose a finitely generated projective R-module Q such that $(P^*)^b \oplus Q$ is free and extend the epimorphism $F(\pi) : (P^*)^b \twoheadrightarrow F(M)$ by zero, thereby obtaining an epimorphism $(P^*)^b \oplus Q \twoheadrightarrow F(M)$. The kernel of this latter epimorphism equals $F(M') \oplus Q$, which is finitely generated over R. Hence $F(M)$ is a finitely presented R-module. ∎

We now specialise to the situation where R is commutative. Hence the $^{\mathrm{op}}$ can be dropped from (A.1). Since Morita equivalent rings have isomorphic centres, we can and do assume that $R = \zeta(\Lambda)$.

Definition A.2. Let M be a finitely presented (left) Λ-module. Then we define the Fitting invariant of M over Λ to be the R-ideal

$$\mathrm{Fit}_\Lambda(M) := \mathrm{Fit}_R(F(M)) = \mathrm{Fit}_R(P^* \otimes_\Lambda M).$$

Proposition A.3. *The Fitting invariant* $\mathrm{Fit}_\Lambda(M)$ *is well-defined.*

Proof. We first note that $F(M)$ is a finitely presented R-module by Lemma A.1. Now let P' be a second progenerator of R which induces a Morita equivalence between ${}_R\mathfrak{M}$ and ${}_\Lambda\mathfrak{M}$ via the functors G' and F' defined as above with P replaced by P'. Then the compositions of functors $F' \circ G$ and $F \circ G'$ are mutually inverse category auto-equivalences of ${}_R\mathfrak{M}$. In fact, the functors

$$F' \circ G : {}_R\mathfrak{M} \longrightarrow {}_R\mathfrak{M}, \quad N \mapsto ((P')^* \otimes_\Lambda P) \otimes_R N$$

$$F \circ G' : {}_R\mathfrak{M} \longrightarrow {}_R\mathfrak{M}, \quad N \mapsto (P^* \otimes_\Lambda P') \otimes_R N$$

give a Morita auto-equivalence of ${}_R\mathfrak{M}$ with R-progenerators $(P')^* \otimes_R P$ and $P^* \otimes_R P'$. Note that F induces an isomorphism

$$\mathrm{End}_R(P^* \otimes_\Lambda P') \simeq \mathrm{End}_\Lambda(P') \simeq R$$

and similarly for $(P')^* \otimes_R P$. Now the result follows from Lemma A.4 below with $Q = (P')^* \otimes_R P$, $U = F' \circ G$ and $T = F \circ G'$. □

Lemma A.4. *Let Q be an R-progenerator such that* $\mathrm{End}_R(Q, Q) \simeq R$*, that is the functors*

$$U : {}_R\mathfrak{M} \longrightarrow {}_R\mathfrak{M}, \quad N \mapsto Q \otimes_R N$$

$$T : {}_R\mathfrak{M} \longrightarrow {}_R\mathfrak{M}, \quad M \mapsto Q^* \otimes_R M$$

induce a Morita auto-equivalence. Then we have equalities

$$\mathrm{Ann}_R(M) = \mathrm{Ann}_R(T(M)) \quad \text{and} \quad \mathrm{Fit}_R(M) = \mathrm{Fit}_R(T(M)).$$

Proof. Let $x \in \mathrm{Ann}_R(M)$. Then

$$x \cdot T(M) = x \cdot (Q^* \otimes_R M) = Q^* \otimes_R x \cdot M = 0.$$

Hence we have an inclusion $\mathrm{Ann}_R(M) \subset \mathrm{Ann}_R(T(M))$ and, by symmetry, $\mathrm{Ann}_R(T(M)) \subset \mathrm{Ann}_R(UT(M)) = \mathrm{Ann}_R(M)$. This proves the first equality.

For the second equality choose a finite presentation $R^a \xrightarrow{h} R^b \twoheadrightarrow M$. As $\mathrm{Fit}_R(M)$ is generated by the $b \times b$ minors of h, we can and do assume that $a = b$. Hence we may view h as an element of $M_{b\times b}(R)$. Applying T yields an endomorphism

$$T(h) = 1 \otimes h \in \mathrm{End}_R((Q^*)^b) \simeq M_{b\times b}(R),$$

where the last isomorphism is induced by $\mathrm{End}_R(Q^*) \simeq R$. Under this isomorphism, we have

$$\det(T(h)) = \det(1 \otimes h) = \det(h).$$

This shows $\mathrm{Fit}_R(M) \subset \mathrm{Fit}_R(T(M))$ and we again obtain equality by symmetry. □

Remark A.5. We obtain the analoguous statements of (iii), (iv), (v) and (vi) of Theorem 2.4 for Λ, since these hold over R. We obtain the analogue of (vii) after observing that $S \otimes_R P$ is an S-progenerator if P is an R-progenerator.

Proposition A.6. *We have* $\mathrm{Fit}_\Lambda(M) \subset \mathrm{Ann}_R(M)$.

Proof. Let $x \in \mathrm{Fit}_\Lambda(M) = \mathrm{Fit}_R(F(M))$. Then x annihilates $F(M) = P^* \otimes_\Lambda M$ by the respective property of Fitting ideals over the commutative ring R. Hence x annihilates also $F(M)^k$ for any positive integer k. But as P^* is also a Λ-progenerator by [CR87, Theorem 3.54(i)], there exists a positive integer k such that Λ is a direct summand of $(P^*)^k$ (see [CR87, Lemma 3.45(iii)]). Hence $M = \Lambda \otimes_\Lambda M$ occurs as a direct summand of $F(M)^k$ and thus is annihilated by x, as desired. □

Finally, we formulate an analogue of Theorem 2.4(ix).

Proposition A.7. *Let I be a two-sided ideal of Λ. Under the identification $\Lambda \simeq \mathrm{End}_R(P)$ we have $I = \mathrm{Hom}_R(P, \mathfrak{a}\cdot P)$ for a uniquely determined R-ideal $\mathfrak{a}$. Then we have an equality*

$$\mathrm{Fit}_\Lambda(\Lambda/I) = \mathrm{Fit}_R(\mathrm{Hom}_R(P, R/\mathfrak{a})).$$

In particular, if P is free of rank n over R, then we recover Theorem 2.4(ix).

Proof. That I is of the given form is [Rei03, Theorem (16.14)(v)]. The short exact sequence $I \hookrightarrow \Lambda \twoheadrightarrow \Lambda/I$ yields a short exact sequence

$$P^* \otimes_\Lambda I \hookrightarrow P^* \twoheadrightarrow P^* \otimes_\Lambda \Lambda/I.$$

We claim that the image of $P^* \otimes_\Lambda I$ in P^* equals $\mathrm{Hom}_R(P, \mathfrak{a})$. In fact, the map

$$P^* \otimes_\Lambda I = \mathrm{Hom}_R(P, R) \otimes_\Lambda \mathrm{Hom}_R(P, \mathfrak{a} \cdot P) \longrightarrow P^* = \mathrm{Hom}_R(P, R)$$

is given by $f \otimes g \mapsto f \circ g$, where $f \in \mathrm{Hom}_R(P, R)$ and $g \in \mathrm{Hom}_R(P, \mathfrak{a} \cdot P)$. As the image of g lies in $\mathfrak{a} \cdot P$ and f is R-linear, the image of $f \circ g$ actually lies in $\mathrm{Hom}_R(P, \mathfrak{a})$. Thus we in fact have a map

$$\alpha : P^* \otimes_\Lambda I = \mathrm{Hom}_R(P, R) \otimes_\Lambda \mathrm{Hom}_R(P, \mathfrak{a} \cdot P) \longrightarrow \mathrm{Hom}_R(P, \mathfrak{a})$$

To show that α is surjective (and thus an isomorphism), it suffices to show the respective statement after localization at each prime ideal $\mathfrak{p}$ of R. However, a projective module over a local ring is free, so there exists a positive integer n such that $P \simeq R^n$. Via this isomorphism, both the domain and codomain of α identify naturally with $\oplus_{i=1}^n \mathfrak{a}$ and α becomes the identity map. Hence α is an isomorphism.

We have shown that we have an isomorphism of R-modules

$$P^* \otimes_\Lambda \Lambda/I \simeq \mathrm{Hom}_R(P, R)/\mathrm{Hom}_R(P, \mathfrak{a}) \simeq \mathrm{Hom}_R(P, R/\mathfrak{a}),$$

where the last isomorphism holds by projectivity of P. The result now follows. □

References

[AG60] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409. MR 0121392 (22 #12130)

[BF01] D. Burns and M. Flach, *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501–570 (electronic). MR 1884523 (2002m:11055)

[BJ11] D. Burns and H. Johnston, *A non-abelian Stickelberger theorem*, Compos. Math. **147** (2011), no. 1, 35–55. MR 2771125

[BMC11] D. Burns and D. Macias Castillo, *Organising matrices for arithmetic complexes*, preprint (2011).

[Bur01] D. Burns, *Equivariant Tamagawa numbers and Galois module theory. I*, Compositio Math. **129** (2001), no. 2, 203–237. MR 1863302 (2002g:11152)

[Bur11] ———, *On derivatives of Artin L-series*, Invent. math. **186** (2011), 291–371.

[CG98] P. Cornacchia and C. Greither, *Fitting ideals of class groups of real fields with prime power conductor*, J. Number Theory **73** (1998), no. 2, 459–471. MR 1658000 (99j:11137)

[CR81] C. W. Curtis and I. Reiner, *Methods of representation theory. Vol. I*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1981, With applications to finite groups and orders, A Wiley-Interscience Publication. MR 632548 (82i:20001)

[CR87] ———, *Methods of representation theory. Vol. II*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1987, With applications to finite groups and orders, A Wiley-Interscience Publication. MR 892316 (88f:20002)

[DJ83] F. R. DeMeyer and G. J. Janusz, *Group rings which are Azumaya algebras*, Trans. Amer. Math. Soc. **279** (1983), no. 1, 389–395. MR 704622 (85a:16006)

[Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)

[Fit36] H. Fitting, *Die Determinantenideale eines Moduls.*, Jahresber. Dtsch. Math.-Ver. **46** (1936), 195–229 (German).

[Gri02] P. Grime, *Fitting ideals and module structure*, Ph.D. thesis, University of Durham, 2002.

[Jac66] H. Jacobinski, *On extensions of lattices*, Michigan Math. J. **13** (1966), 471–475. MR 0204538 (34 #4377)

[Lam99] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR 1653294 (99i:16001)

[Lam01] ———, *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR 1838439 (2002c:16001)

[MW84] B. Mazur and A. Wiles, *Class fields of abelian extensions of* **Q**, Invent. Math. **76** (1984), no. 2, 179–330. MR 742853 (85m:11069)

[Nic10] A. Nickel, *Non-commutative Fitting invariants and annihilation of class groups*, J. Algebra **323** (2010), no. 10, 2756–2778. MR 2609173

[Nic11] ———, *On non-abelian stark-type conjectures*, Ann. Inst. Fourier (to appear) (2011).

[Nic12] ———, *A conductor formula for completed group algebras*, preprint, 2012.

[Nor76] D. G. Northcott, *Finite free resolutions*, Cambridge University Press, Cambridge, 1976, Cambridge Tracts in Mathematics, No. 71. MR 0460383 (57 #377)

[Par07] A. Parker, *Equivariant Tamagawa numbers and non-commutative Fitting invariants*, Ph.D. thesis, King's College London, 2007.

[Pas77] D. S. Passman, *The algebraic structure of group rings*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1977. MR 470211 (81d:16001)

[Ple83] W. Plesken, *Group rings of finite groups over p-adic integers*, Lecture Notes in Mathematics, vol. 1026, Springer-Verlag, Berlin, 1983. MR 724074 (85c:16019)

[Rei03] I. Reiner, *Maximal orders*, London Mathematical Society Monographs. New Series, vol. 28, The Clarendon Press Oxford University Press, Oxford, 2003, Corrected reprint of the 1975 original, With a foreword by M. J. Taylor. MR 1972204 (2004c:16026)

[Sus88] J. Susperregui, *On determinantal ideals over certain noncommutative rings*, Ring theory (Granada, 1986), Lecture Notes in Math., vol. 1328, Springer, Berlin, 1988, pp. 269–282. MR 959761 (89j:16045)

[Sus89] ———, *Fitting invariants for modules over anticommutative graded rings*, Comm. Algebra **17** (1989), no. 8, 2035–2054. MR 1013481 (90h:16006)

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