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On the structure of minimizers of causal variational principles in the non-compact and equivariant settings

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ON THE STRUCTURE OF MINIMIZERS OF CAUSAL VARIATIONAL PRINCIPLES IN THE NON-COMPACT AND EQUIVARIANT SETTINGS

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Abstract. We derive the Euler-Lagrange equations for minimizers of causal variational principles in the non-compact setting with constraints, possibly prescribing symmetries. Considering first variations, we show that the minimizing measure is supported on the intersection of a hyperplane with a level set of a function which is homogeneous of degree two. Moreover, we perform second variations to obtain that the compact operator representing the quadratic part of the action is positive semi-definite. The key ingredient for the proof is a subtle adaptation of the Lagrange multiplier method to variational principles on convex sets.

Contents

1. Introduction

Causal variational principles arise in the context of relativistic quantum theory [3, 7]. In [5] they were introduced in a broader mathematical context, and the existence of minimizers was proved in various situations (for previous existence results in the simpler discrete setting see [4]). The structure of minimizers was first analyzed in [8] in the compact setting without constraints. In the present paper, we turn attention to the general non-compact setting involving constraints and possibly symmetries, with the aim of getting detailed information on the structure of minimizing measures.

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Before delving into the main results, we briefly recall causal variational principles as introduced in [5, Section 2], always specializing to the class of variational principles of interest here. Let (M, μ) be a measure space normalized by $\mu(M) = 1$. For given integers k and n with $k \geq 2n$, we let $\mathcal{F}$ be the set of all Hermitian $k \times k$-matrices of rank at most $2n$, which (counting with multiplicities) have at most n positive and at most n negative eigenvalues. In a causal variational principle one minimizes an action $\mathcal{S}[F]$ under variations of a measurable function $F : M \to \mathcal{F}$, imposing suitable constraints. More specifically, for a given measurable function $F : M \to \mathcal{F}$, we let $\rho = F_* \mu$ be the push-forward measure on $\mathcal{F}$ (defined by $\rho(\Omega) = \mu(F^{-1}(\Omega))$). For any $x, y \in M$, we form the operator product

$$A_{xy} = x \cdot y \,:\, \mathbb{C}^k \to \mathbb{C}^k \tag{1.1}$$

and denote its eigenvalues counted with algebraic multiplicities by

$$\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}, \underbrace{0, \ldots, 0}_{k-2n \text{ times}} \qquad \text{with} \qquad \lambda_j^{xy} \in \mathbb{C}\,. \tag{1.2}$$

We define the *spectral weight* $|A_{xy}|$ by

$$|A_{xy}| = \sum_{j=1}^{2n} |\lambda_j^{xy}|, \tag{1.3}$$

and similarly set $|A_{xy}^2| = \sum_{j=1}^{2n} |\lambda_j^{xy}|^2$. We introduce

$$\text{the Lagrangian} \qquad \mathcal{L}[A_{xy}] = |A_{xy}^2| - \frac{1}{2n}\, |A_{xy}|^2 \tag{1.4}$$

and define the functionals $\mathcal{S}$ and $\mathcal{T}$ by

$$\mathcal{S} = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}[A_{xy}]\, d\rho(x)\, d\rho(y) \tag{1.5}$$

$$\mathcal{T} = \iint_{\mathcal{F} \times \mathcal{F}} |A_{xy}|^2\, d\rho(x)\, d\rho(y)\,. \tag{1.6}$$

We also introduce the following constraints:

(BC) The *boundedness constraint*: $\quad \mathcal{T} \leq C$

(TC) The *trace constraint*: $\quad \displaystyle\int_{\mathcal{F}} \mathrm{Tr}(x)\, d\rho(x) = k$

(IC) The *identity constraint*: $\quad \displaystyle\int_{\mathcal{F}} x\, d\rho(x) = \mathbb{1}_{\mathbb{C}^k}\,.$

Our variational principle is to minimize $\mathcal{S}$ by varying F in the class of all measurable functions from M to $\mathcal{F}$, under the constraints (BC) and either (TC) or (IC). In [5, Theorem 2.3] it is shown that the minimum of this variational principle is attained by a function $F \in L^2(M, \mathcal{F}, d\mu)$.

The measure space (M, μ) may pose constraints on the form of the push-forward measure ρ (for example, in the discrete setting one chooses μ as the normalized counting measure on $M = \{1, \ldots, m\}$; then the support of ρ necessarily consists of at most m points). In what follows, we will always be concerned with the so-called *continuous setting* where we do not want to impose any constraints on the form of the measure ρ. In technical terms, this can be achieved by assuming that the measure space (M, μ) is non-atomic; then the push-forward measure ρ can indeed be arranged to be any normalized positive regular Borel measure on $\mathcal{F}$ (see [5, Section 1.4 and Lemma 1.4]). This makes it possible to restrict attention to the measure ρ in the class

$$\rho \in \mathfrak{M} := \{\text{normalized positive regular Borel measures on } \mathcal{F}\} , \tag{1.7}$$

disregarding the measure space (M, μ) and the function F. This leads us to the variational principles to be considered here:

Definition 1.1. *For any parameter $C > 0$, the* **causal variational principle in the continuum setting** *is to minimize $\mathcal{S}$ by varying $\rho \in \mathfrak{M}$ under the constraints*

$$\text{(BC)} \qquad \textit{and either } \text{(TC)} \textit{ or } \text{(IC)} .$$

Again, the existence of minimizers is proved in [5, Theorem 2.3]. The goal of this paper is to analyze the structure of a minimizing measure ρ.

To clarify the terminology, we point out that the set $\mathcal{F}$ is a non-compact topological space; this is what we mean by the *non-compact setting.* In contrast, by prescribing the eigenvalues of the elements of $\mathcal{F}$ (see the constraint (C3) in [5, Section 2.1]), one can arrange that $\mathcal{F}$ is a compact manifold. This *compact setting* is analyzed in a more general context in [8]. Unfortunately, for most of the methods used in [8] the compactness of $\mathcal{F}$ is essential. The present paper is the first analytic work on the structure of the minimizers of causal variational principles in the non-compact setting.

The usual approach for treating variational principles with constraints is to apply the method of Lagrange multipliers. For our variational principle, this method fails, essentially because positive measures do not form a vector space (for details cf. Section 3.1 and Figure 2 below). To circumvent this difficulty, in Section 3 we will develop an alternative method which reproduces the results of Lagrange multipliers with subtle modifications.

Our main result can be understood heuristically from the standard Lagrange multiplier method as follows. We add the constraints multiplied by Lagrange parameters κ, Λ, c to the action so as to form the effective action

$$\mathcal{S}_{\text{eff}} = \mathcal{S} + \kappa \mathcal{T} - \int_{\mathcal{F}} \text{Tr}\left(\Lambda \cdot x\right) d\rho - c \int_{\mathcal{F}} d\rho , \tag{1.8}$$

where in the case of the constraint (TC), Λ is a multiple of the identity matrix, whereas in the case of (IC), it can be any Hermitian $(k \times k)$-matrix. The Lagrange multiplier c takes into account that ρ must be normalized. Note that the positivity of the measure ρ cannot be encoded in terms of Lagrange multipliers. Instead, we need to make sure in all our variations that ρ stays positive. Considering for any $x \in \mathcal{F}$ the first variation

$$\tilde{\rho}_\tau = \rho + \tau\, \delta_x , \qquad \tau \in [0, 1) \tag{1.9}$$

(where δ_x is the Dirac measure supported at x; note that τ is non-negative in order to ensure that $\tilde{\rho}_\tau$ is positive), a short formal calculation yields the Euler-Lagrange inequality

$$\Phi(x) - c \geq 0 \qquad \text{for all } x \in \mathcal{F} , \tag{1.10}$$

where

$$\Phi(x) := 2 \int_{\mathcal{F}} \left(\mathcal{L}(x, y) + \kappa\, |A_{xy}|^2 \right) d\rho(y) - \text{Tr}(\Lambda \cdot x) . \tag{1.11}$$

If the point x lies on the support of ρ, we can extend the variation (1.9) to small negative values of τ (at least heuristically; to make the argument mathematically sound, one needs to approximate the Dirac measure by a measure which is absolutely continuous with respect to ρ). When doing so, (1.10) becomes an equality,

$$\Phi(x) - c = 0 \qquad \text{for all } x \in \operatorname{supp} \rho \,. \tag{1.12}$$

Combining (1.10) with (1.12), we conclude that Φ is minimal on the support of ρ. Accordingly,

$$\frac{d}{dt} \Phi(tx)|_{t=1} = 0 \qquad \text{for all } x \in \operatorname{supp} \rho \,.$$

This implies that the parts of Φ which are homogeneous of degree two and one, denoted by

$$\Phi_2(x) := 2 \int_{\mathcal{F}} \big(\mathcal{L}(x,y) + \kappa\, |A_{xy}|^2 \big)\, d\rho(y) \tag{1.13}$$

$$\Phi_1(x) := \Phi(x) - \Phi_2(x) = -\operatorname{Tr}(\Lambda \cdot x) \,, \tag{1.14}$$

are related to each other by

$$2\Phi_2(x) + \Phi_1(x) = 0 \qquad \text{for all } x \in \operatorname{supp} \rho \,. \tag{1.15}$$

Now, combining (1.12) and (1.15) gives

$$\Phi_1(x) = 2c = -2\Phi_2(x) \,.$$

Integrating over x, one can determine the constant c.

The following theorem[1] rigorously establishes this heuristic result under the additional assumption (1.16).

Theorem 1.2. *Suppose that ρ is a minimizer of the variational principle of Definition 1.1, where the constant C satisfies the inequality*

$$C > C_{\min} := \inf \big\{ \mathcal{T}(\mu) \,|\, \mu \in \mathfrak{M} \text{ satisfies (TC) respectively (IC) } \big\} \,. \tag{1.16}$$

Then for a suitable choice of the Lagrange multipliers

$$\kappa \geq 0 \qquad \text{and} \qquad \Lambda \in \mathrm{L}(\mathbb{C}^k) \,,$$

the measure ρ is supported on the intersection of the level sets

$$\Phi_1 = -4\,(\mathcal{S} + \kappa \mathcal{T}) \qquad \text{and} \qquad \Phi_2(x) = 2\,(\mathcal{S} + \kappa \mathcal{T}) \,. \tag{1.17}$$

In the cases of the trace constraint (TC) and the identity constraint (IC), the matrix Λ is a multiple of the identity and a general Hermitian matrix, respectively. In the case $\mathcal{T}(\rho) < C$, we may choose $\kappa = 0$.

This result is illustrated in Figure 1. Note that the set $\Phi_1^{-1}(-4(\mathcal{S}+\kappa\mathcal{T}))$ is a hyperplane in $\mathrm{L}(\mathbb{C}^k)$. The set $\Phi_2^{-1}(2(\mathcal{S} + \kappa\mathcal{T}))$, on the other hand, is the level set of a function which is homogeneous of degree two. The support of ρ is contained in the intersection of these two sets. This intersection might be non-compact. It is an open problem whether the support of a minimizing measure is always compact.

The above theorem is supplemented by additional results, as we now briefly outline. Theorem 3.13 gives sufficient conditions guaranteeing that the the function Φ is indeed minimal on the support of ρ. When these conditions fail, a weaker statement can nonetheless be obtained (Theorem 3.14). In Sections 3.4 and 3.5, we consider second variations. We prove that a suitable compact operator L on a Hilbert space is

[1] For preliminary results and numerical examples see the master thesis [2], which also treats the case when the measure ρ is a counting measure. However, in this master thesis the complication discussed in Figure 2 on page 9 is disregarded.

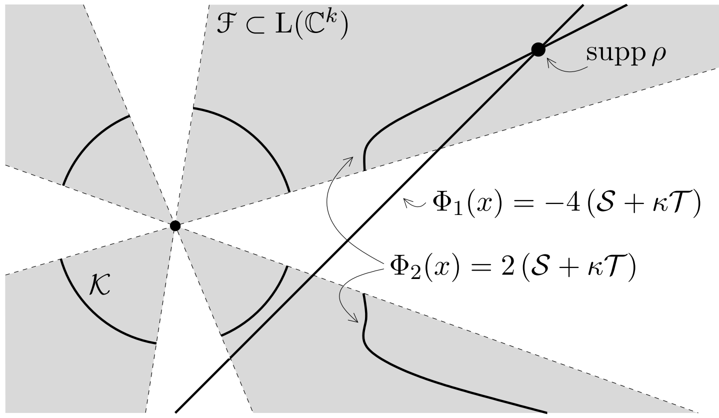

FIGURE 1. Example for the level sets of Φ_1 and Φ_2 and the support of ρ.

positive semi-definite (Theorem 3.16). This positivity results bears similarity with [8, Lemma 4.5] in the compact setting. In Theorem 3.17 we prove that the operator L stays positive when extended to the direct sum of the aforementioned Hilbert space with any one-dimensional vector space chosen within a specified class. Section 3.6 is devoted to an a-priori estimate which shows in particular that the support of ρ is compact if the Lagrange multiplier κ is strictly positive. Finally, in Section 4 we extend our results to a class of equivariant variational principles.

2. Preliminaries

2.1. **Causal Fermion Systems.**

We now briefly recall how the variational principles introduced in Definition 1.1 arise in the more general setting of causal fermion systems as introduced in [7, Section 1]. We first give the general definition.

Definition 2.1. Given a complex Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ (the *"particle space"*) and a parameter $n \in \mathbb{N}$ (the *"spin dimension"*), we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$ of finite rank, which (counting with multiplicities) have at most n positive and at most n negative eigenvalues. On $\mathcal{F}$ we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$), the so-called *universal measure*. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system in the particle representation*.

Starting from this definition, one can construct a space-time endowed with a topological, causal and metric structure, together with a collection of quantum mechanical wave functions in space-time (see [7] and [6]). We shall not enter these constructions here, but instead concentrate on the analytical aspects of the approach.

In order to get back to the setting of Section 1, we specialize the above framework in the following ways: First, we assume that particle space $\mathcal{H}$ has finite dimension k; then it can clearly be identified with the Euclidean $\mathbb{C}^k$. Moreover, we impose that ρ is in the class (1.7). Then we can consider the variational principle of Definition 1.1.

In the case when $\mathcal{H}$ is infinite dimensional, the set $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ is a topological space which is not locally compact. As a consequence, causal variational principles are in general ill-defined (the physical picture is that the limit $\dim \mathcal{H} \to \infty$ corresponds to an idealized space-time where the inherent ultraviolet regularization has been taken out). However, if one assumes a symmetry group G which is so large that $\mathcal{F}/G$ is locally compact, then causal variational principles again make mathematical sense. This is the equivariant setting which we will consider in Section 4.

2.2. **The Moment Measures.** Let us assume that the measure ρ on $\mathcal{F}$ is a minimizer of the variational principle of Definition 1.1. We recall the definition of moment measures as introduced in [5, Definition 2.10].

Definition 2.2. *Let $\mathcal{K}$ be the compact topological space*

$$\mathcal{K} = \{p \in \mathcal{F} \text{ with } \|p\| = 1\} \cup \{0\} . \tag{2.1}$$

We define the measurable sets of $\mathcal{K}$ by the requirement that the sets $\mathbb{R}^+\Omega = \{\lambda p \mid \lambda \in \mathbb{R}^+, p \in \Omega\}$ and $\mathbb{R}^-\Omega$ should be ρ-measurable in $\mathcal{F}$. We introduce the measures $\mathfrak{m}^{(0)}$, $\mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ by

$$\mathfrak{m}^{(0)}(\Omega) = \frac{1}{2}\,\rho\big(\mathbb{R}_+\Omega \setminus \{0\}\big) + \frac{1}{2}\,\rho\big(\mathbb{R}_-\Omega \setminus \{0\}\big) + \rho\big(\Omega \cap \{0\}\big) \tag{2.2}$$

$$\mathfrak{m}^{(1)}(\Omega) = \frac{1}{2}\int_{\mathbb{R}^+\Omega} \|p\|\, d\rho(p) \;-\; \frac{1}{2}\int_{\mathbb{R}^-\Omega} \|p\|\, d\rho(p) \tag{2.3}$$

$$\mathfrak{m}^{(2)}(\Omega) = \frac{1}{2}\int_{\mathbb{R}_+\Omega} \|p\|^2\, d\rho(p) \;+\; \frac{1}{2}\int_{\mathbb{R}_-\Omega} \|p\|^2\, d\rho(p)\,. \tag{2.4}$$

The measure $\mathfrak{m}^{(l)}$ is referred to as the l^{th} **moment measure***.*

Exactly as in [5, Section 2.3], the homogeneity of our functionals yields

$$1 = \rho(\mathcal{F}) = \mathfrak{m}^{(0)}(\mathcal{K}) \tag{2.5}$$

$$\int_{\mathcal{F}} x\, d\rho(x) = \int_{\mathcal{K}} x\, d\mathfrak{m}^{(1)}(x) \tag{2.6}$$

$$\mathcal{S}(\rho) = \iint_{\mathcal{K}\times\mathcal{K}} \mathcal{L}[A_{xy}]\, d\mathfrak{m}^{(2)}(x)\, d\mathfrak{m}^{(2)}(y) \tag{2.7}$$

$$\mathcal{T}(\rho) = \iint_{\mathcal{K}\times\mathcal{K}} |A_{xy}|^2\, d\mathfrak{m}^{(2)}(x)\, d\mathfrak{m}^{(2)}(y)\,, \tag{2.8}$$

making it possible to express the action as well as all the constraints in terms of the moment measures. Moreover, the moment measures have the Radon-Nikodym decomposition

$$d\mathfrak{m}^{(1)} = f\, d\mathfrak{m}^{(0)}\,, \qquad d\mathfrak{m}^{(2)} = |f|^2\, d\mathfrak{m}^{(0)} + d\mathfrak{n}\,,$$

where $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$, and $\mathfrak{n}$ is a positive measure on $\mathcal{K}$ which need not be absolutely continuous with respect to $\mathfrak{m}^{(0)}$. If $\mathfrak{n} \neq 0$, by setting $\mathfrak{n}$ to zero we can strictly decrease the action without violating our constraints (see (2.5)–(2.8)). It follows that $\mathfrak{n}$ vanishes for our minimizing measure ρ. We thus obtain the representation of the moment measures

$$d\mathfrak{m}^{(1)} = f\, d\mathfrak{m}^{(0)}\,, \qquad d\mathfrak{m}^{(2)} = |f|^2\, d\mathfrak{m}^{(0)}\,. \tag{2.9}$$

From (2.3) it is clear that f is odd,

$$f(-x) = -f(x) \quad \text{for all } x \in \mathcal{K}\,. \tag{2.10}$$

The next proposition shows that the measure ρ is uniquely determined by the moment measures.

Proposition 2.3. *For a given normalized measure $\mathfrak{m}^{(0)}$ on $\mathcal{K}$ and a given function $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$ satisfying* (2.10)*, there is a unique normalized measure ρ on $\mathcal{F}$ such that the corresponding moment measures* (2.2)-(2.4) *have the Radon-Nikodym representation* (2.9)*. The measure ρ is supported on the graph of f over $\mathcal{K}$, i.e.*

$$\operatorname{supp} \rho \subset \overline{\{f(x)\, x \quad with \quad x \in \mathcal{K}\}}\,. \tag{2.11}$$

Proof. The construction of the measure ρ is inspired by [5, Lemma 2.14]. A subset $\Omega \subset \mathcal{F}$ is called ρ-measurable if the function $\chi_\Omega\big(f(x)\, x\big)$ is $\mathfrak{m}^{(0)}$-measurable on $\mathcal{K}$ (where χ_Ω denotes the characteristic function). On the ρ-measurable sets we define the measure ρ by

$$\rho(\Omega) = \int_{\mathcal{K}} \chi_\Omega\big(f(x)\, x\big)\, d\mathfrak{m}^{(0)}(x)\,. \tag{2.12}$$

Obviously, the measure ρ is normalized and has the support property (2.11). Moreover, it is straightforward to verify that for all $l > 0$,

$$\int_{\mathbb{R}^+\Omega} \|p\|^l\, d\rho = \int_\Omega |f(x)|^l\, \chi_{\{f>0\}}(x)\, d\mathfrak{m}^{(0)}(x)\,.$$

Using this identity, a direct computation shows that the moment measures corresponding to ρ indeed satisfy (2.9).

To prove uniqueness, suppose that ρ is a measure with moment measures satisfying (2.9). Then for every $\mathfrak{m}^{(0)}$-measurable set Ω,

$$\frac{1}{2} \int_{\mathbb{R}^+\Omega} \big(\|p\| - f(p)\big)^2\, d\rho + \frac{1}{2} \int_{\mathbb{R}^-\Omega} \big(-\|p\| - f(p)\big)^2\, d\rho \;+\; f(0)^2\, \mathfrak{m}^{(0)}\big(\Omega \cap \{0\}\big) \tag{2.13}$$

$$= \mathfrak{m}^{(2)}(\Omega) - 2 \int_\Omega f\, d\mathfrak{m}^{(1)} + \int_\Omega f^2\, d\mathfrak{m}^{(0)} = 0\,, \tag{2.14}$$

where we multiplied out and used (2.9). In particular, both integrands in (2.13) must vanish almost everywhere. Now a short calculation yields that ρ coincides with the measure (2.12). □

In order to clarify the meaning of (2.11), we note that $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$ stands for an equivalence class of functions which differ on a set of measure zero. The right side of (2.11) may depend on the choice of the representative. The above proposition states that (2.11) holds for *any* choice of the function $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$.

3. The Euler-Lagrange Equations

3.1. **Treating the Constraints.** Considering on the set $\mathcal{F} \subset \mathrm{L}(\mathbb{C}^k)$ the topology induced by the sup-norm $\|.\|$ on $\mathrm{L}(\mathbb{C}^k)$, this set is a locally compact topological space. Its subset $\mathcal{K} \subset \mathcal{F}$ defined by (2.1) is compact. Let μ be a regular, locally finite Borel measure on $\mathcal{F}$ (which is real, but not necessarily positive; such measures are often called signed Radon measures). Moreover, we assume that the following integral is finite,

$$\|\mu\|_{\mathfrak{B}} := \int_{\mathcal{F}} \big(1 + \|x\|^2\big)\, d|\mu|(x) < \infty \tag{3.1}$$

(here $|\mu|$ denotes the total variation of the measure μ; see for example [9, Section 6.1]). We denote the vector space of such measures by $\mathfrak{B}$.

Lemma 3.1. $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ *is a Banach space.*

Proof. It is clear that $\|.\|_{\mathfrak{B}}$ satisfies the axioms of a norm. Thus it remains to show that this norm is complete. We first note that

$$\|\mu\|_{\mathfrak{B}} \geq |\mu|(\mathcal{F}) . \tag{3.2}$$

Accordingly, if $(\mu_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the norm $\|.\|_{\mathfrak{B}}$, then for every $\eta \in C^0_0(\mathcal{F}, \mathbb{R})$, the sequence of real numbers $(|\mu_j|(\eta))_{j\in\mathbb{N}}$ is a Cauchy sequence. A classical result on Radon measures (see for example [1, eq. (13.4.1)]) guarantees that the sequence (μ_j) converges as Radon measures to some limit measure μ. It remains to show that the limit measure satisfies the condition (3.1). We already know from the above argument that

$$\lim_{j\to\infty} \mu_j(\eta) \;=\; \mu(\eta) \qquad \forall\; \eta \in C^0_0(\mathcal{F}, \mathbb{R}) . \tag{3.3}$$

We next fix $r > 1$, and let $\eta_r : [0, \infty) \to [0, 1]$ be a continuous cut-off function satisfying

$$\eta_r(t) \;=\; \begin{cases} 1 & \text{if } t \leq r \\ 0 & \text{if } t > r + r^{-1} . \end{cases} \tag{3.4}$$

Then the function

$$x \in \mathcal{F} \;\longmapsto\; \big(1 + \|x\|^2\big)\, \eta_r\big(\|x\|\big) \tag{3.5}$$

is continuous with compact support in $B_{r+r^{-1}}$, where B_r denotes the open ball in $\mathcal{F}$,

$$B_r := \big\{x \in \mathcal{F} \text{ with } \|x\| < r\big\} \subset \mathcal{F} . \tag{3.6}$$

Whence, from (3.3), there holds

$$\lim_{j\to\infty} \int_{\mathcal{F}} \big(1 + \|x\|^2\big)\eta_r\big(\|x\|\big)\, d|\mu_j|(x) = \int_{\mathcal{F}} \big(1 + \|x\|^2\big)\, \eta_r\big(\|x\|\big)\, d|\mu|(x) . \tag{3.7}$$

It follows accordingly that

$$\begin{aligned} \int_{B_r} \big(1 + \|x\|^2\big)\, d|\mu|(x) &\leq \int_{\mathcal{F}} \big(1 + \|x\|^2\big)\, \eta_r\big(\|x\|\big)\, d|\mu|(x) \\ &= \lim_{j\to\infty} \int_{\mathcal{F}} \big(1 + \|x\|^2\big)\, \eta_r\big(\|x\|\big)\, d|\mu_j|(x) \leq \lim_{j\to\infty} \|\mu_j\|_{\mathfrak{B}} , \end{aligned}$$

and the last limit is bounded uniformly in $r > 1$. As $\mathcal{F}$ is locally compact, on the left hand side we may pass to the limit $r \nearrow \infty$ to obtain that $\|\mu\|_{\mathfrak{B}}$ is finite. This concludes the proof. ∎

The definition (1.5) and (1.6) of the functionals $\mathcal{S}$ and $\mathcal{T}$ as well as the definition of the moment measures (see Definition 2.2) can be extended in a straightforward way to a real measure $\rho \in \mathfrak{B}$. We now estimate these objects in terms of the norm $\|.\|_{\mathfrak{B}}$.

Proposition 3.2. *There is a constant $c = c(\mathcal{F}) > 0$ such that*

$$|\mathcal{S}(\mu)|,\, |\mathcal{T}(\mu)| \leq c\, \|\mu\|^2_{\mathfrak{B}} \qquad \textit{for all } \mu \in \mathfrak{B} \tag{3.8}$$

$$\|\rho\|^2_{\mathfrak{B}} \leq 2 + c\, \mathcal{T}(\rho) \qquad \textit{for all } \rho \in \mathfrak{M} . \tag{3.9}$$

Proof. Estimating the integrals in Definition 2.2 by (3.1), one readily finds that

$$|\mathfrak{m}^{(0)}|(\mathcal{K}),\, |\mathfrak{m}^{(1)}|(\mathcal{K}),\, |\mathfrak{m}^{(2)}|(\mathcal{K}) \leq \|\rho\|_{\mathfrak{B}} \qquad \text{for all } \rho \in \mathfrak{B} . \tag{3.10}$$

The functions $\mathcal{L}$ and $|A_{xy}|^2$ are clearly continuous on $\mathcal{K} \times \mathcal{K}$. As $\mathcal{K}$ is compact, they are bounded,

$$\mathcal{L}(x, y), |A_{xy}|^2 \leq c \qquad \text{for all } x, y \in \mathcal{K} .$$

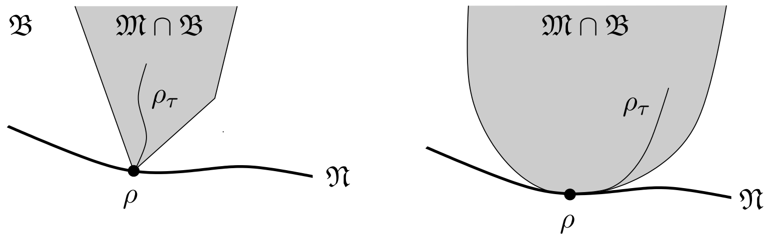

FIGURE 2. Minimizing in the convex subset $\mathfrak{M} \cap \mathfrak{B}$ with constraints.

Using these inequalities in (2.7) and (2.8), we can apply (3.10) to obtain (3.8).

In order to derive (3.9), we first note that since every measure $\rho \in \mathfrak{M}$ is normalized and positive,

$$\|\rho\|_{\mathfrak{B}} = \mathfrak{m}^{(0)}(\mathcal{K}) + \mathfrak{m}^{(2)}(\mathcal{K}) = 1 + \mathfrak{m}^{(2)}(\mathcal{K}) \, .$$

Now we can apply the lower bound on $\mathfrak{m}^{(2)}(\mathcal{K})$ in [5, Lemma 2.12]. □

The inequality (3.9) implies that a minimizer $\rho \in \mathfrak{M}$ of our variational principle will be a vector in $\mathfrak{B}$. This makes it possible to consider our variational principle on the subset $\mathfrak{M} \cap \mathfrak{B}$ of the Banach space $\mathfrak{B}$. Usually, constraints of variational principles are treated with Lagrange multipliers. We now explain in words why this method cannot be applied in our setting. Our first constraint is that we vary in the subset of positive measures. This corresponds to an infinite number of inequality constraints (namely $\rho(\Omega) \geq 0$ for all measurable $\Omega \subset \mathcal{F}$), making it impossible to apply standard Lagrange multipliers. The normalization of ρ could be treated as in (1.8) by a Lagrange multiplier. But as the normalization of ρ can always be arranged by rescaling, there is no advantage in doing so. Instead, it is preferable to consider the minimization problem on the convex subset $\mathfrak{M} \cap \mathfrak{B}$ of the Banach space $\mathfrak{B}$.

We would like to treat the constraint (BC) as well as the additional constraints (TC) or (IC) with Lagrange multipliers. The fact that (BC) is an inequality constraint does not cause difficulties, because for variations which decrease $\mathcal{T}$, we can disregard this constraint, whereas for variations which increase $\mathcal{T}$ we can impose the equality constraint $\mathcal{T} = C$. However, a general problem arises from the fact that we minimize only over a convex subset $\mathfrak{M} \cap \mathfrak{B} \subset \mathfrak{B}$. The basic difficulty is seen most easily in the examples shown in Figure 2. Assume for simplicity that we only have equality constraints and that we are in the regular setting where the measures which satisfy the constraints form a smooth Banach submanifold $\mathfrak{N} \subset \mathfrak{B}$. Then $\mathfrak{N}$ can be described locally as the zero set of a function

$$G : \mathfrak{B} \to \mathbb{R}^L \, . \tag{3.11}$$

The standard multiplier method would give parameters $\lambda_l \in \mathbb{R}$ such that

$$\frac{d}{d\tau} \Big(S(\rho_\tau) - \sum_{l=1}^{L} \lambda_l \, G_l(\rho_\tau) \Big) \Big|_{\tau=0} = 0 \tag{3.12}$$

for any variation $(\rho_\tau)_{\tau \geq 0}$. Since we are only allowed to vary in the convex subset $\mathfrak{M} \cap \mathfrak{B}$, it may happen that the minimum is attained on the boundary of $\mathfrak{M} \cap \mathfrak{B}$. In this case, we cannot expect that equality holds in (3.12). Instead, one might expect naively the corresponding inequality

$$\frac{d}{d\tau}\Big(S(\rho_\tau) - \sum_{l=1}^L \lambda_l \, G_l(\rho_\tau) \Big)\Big|_{\tau=0} \geq 0 \,, \tag{3.13}$$

which should hold for any variation $(\rho_\tau)_{\tau \in [0,1)}$ in $\mathfrak{B} \cap \mathfrak{M}$. However, this naive guess is not correct, as is illustrated in Figure 2. In the example on the left, the convex set $\mathfrak{M} \cap \mathfrak{B}$ intersects $\mathfrak{N}$ only in one point ρ. Then ρ is clearly a minimizer in $\mathfrak{M} \cap \mathfrak{B}$ subject to the constraints, simply because there are no non-trivial variations of ρ. But this fact does not give us any information on the variation $\rho_\tau \in \mathfrak{B}$. In particular, there is no reason why (3.13) should hold. In the example on the right of Figure 2, ρ is again a trivial minimizer in $\mathfrak{M} \cap \mathfrak{B}$ subject to the constraints. There is even a variation $(\rho_\tau)_{\tau \in [0,1)}$ in $\mathfrak{M} \cap \mathfrak{B}$ which is tangential to $\mathfrak{N}$, implying that the Lagrange multiplier terms in (3.13) all vanish. But one could clearly choose the action such that $\partial_\tau S(\rho_\tau)|_{\tau=0} < 0$, showing that (3.13) is indeed violated.

Our method to deal with this problem is to first derive an inequality which shows that that for any variation $\rho_\tau \in \mathfrak{M} \cap \mathfrak{B}$, the first variation of the action is bounded from below by the first variation of the constraint functions (see Proposition 3.3 below). This result is much weaker than the inequality (3.13), basically because the Lagrange multiplier terms are replaced by an estimate of their absolute values. Despite this rough estimate, Proposition (3.3) will be very useful for analyzing the minimizing measure. More precisely, in Section 3.2 we shall apply it to special variations ρ_τ for which $\partial_\tau G(\rho_\tau)|_{\tau=0}$ vanishes. Then the error term in (3.16) drops out, giving a sharp inequality. Before stating our result, we need to specify the functions which describe the constraints. The constraints (TC) or (IC) are linear in the measure; we denote their total number by L. For the constraint (TC), we choose $L = 1$ and

$$G_1(\mu) = k - \int_{\mathcal{F}} \mathrm{Tr}(x) \, d\mu(x) \,. \tag{3.14}$$

For the constraint (IC), we set $L = k(k+1)/2$. Choosing a basis $e_1, \ldots, e_L$ of the symmetric $k \times k$-matrices, we let

$$G_l(\mu) = \mathrm{Tr}\left(e_l \Big(\mathbb{1}_{\mathbb{C}^k} - \int_F x \, d\mu(x) \Big) \right) , \qquad l = 1, \ldots, L \,. \tag{3.15}$$

It is convenient to choose $e_1 = \mathbb{1}$, so that (3.14) agrees with (3.15) for $l = 1$. Moreover, it is convenient to choose the matrices $e_2, \ldots, e_L$ to be trace-free.

Proposition 3.3. *Assume that ρ is a minimizer of the variational principle of Definition 1.1, where the constant C satisfies* (1.16)*. Then there is a constant c such that for every $\mathfrak{B}$-Fréchet differentiable family of measures $(\rho_\tau)_{\tau \in [0,1)}$ in $\mathfrak{B} \cap \mathfrak{M}$ with $\rho_0 = \rho$, the first variation satisfies the inequality*

$$\begin{aligned} \frac{d}{d\tau} \mathcal{S}(\rho_\tau)\Big|_{\tau=0} \geq &- c \, \Big\| \frac{d}{d\tau} G(\rho_\tau)\Big|_{\tau=0} \Big\|_{\mathbb{R}^L} \\ &- \begin{cases} 0 & \text{if } \mathcal{T}(\rho) < C \\ c \, \max\Big(0, \frac{d}{d\tau} \mathcal{T}(\rho_\tau)\Big|_{\tau=0} \Big) & \text{if } \mathcal{T}(\rho) = C \,. \end{cases} \end{aligned} \tag{3.16}$$

The method of the proof is to construct a corresponding variation $\tilde{\rho}_\tau \in \mathfrak{M} \cap \mathfrak{B}$ which also satisfies all the constraints and then to exploit the inequality $\partial_\tau \mathcal{S}(\tilde{\rho}_\tau)|_{\tau=0} \geq 0$.

In this construction, the assumption (1.16) will be used to rule out degenerate cases as discussed in Figure 2. Unfortunately, it is impossible to write the difference of the first variations $\partial_\tau(\mathcal{S}(\rho_\tau) - \mathcal{S}(\tilde{\rho}_\tau))$ as a derivative of the constraints.

The proof of Proposition 3.3 is split up into several lemmas; it will be completed towards the end of this section.

Lemma 3.4. *The functions $\mathcal{S}$, $\mathcal{T}$ and G are continuously Fréchet differentiable.*

Proof. The inequality (3.8) implies that $\mathcal{S}$ and $\mathcal{T}$ are bounded bilinear functionals on $\mathfrak{B} \times \mathfrak{B}$. Thus they are Fréchet differentiable at any $\mu \in \mathfrak{B}$ and

$$(D\mathcal{S})_\mu(\nu) = 2 \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}[A_{xy}]\, d\mu(x)\, d\nu(y) \tag{3.17}$$

$$(D\mathcal{T})_\mu(\nu) = 2 \iint_{\mathcal{F}\times\mathcal{F}} |A_{xy}|^2\, d\mu(x)\, d\nu(y)\ . \tag{3.18}$$

More precisely, $D\mathcal{S}_\mu \in \mathfrak{B}^*$ and

$$\|D\mathcal{S}_\mu\|_{\mathfrak{B}^*} := \sup_{\nu\in\mathfrak{B},\, \|\nu\|_{\mathfrak{B}}=1} \big|(D\mathcal{S})_\mu[\nu]\big| \;\leq\; c\, \|\mu\|_{\mathfrak{B}}\ ,$$

where in the last step we used (3.8). As the functionals (3.17) and (3.18) clearly depend continuously on μ, we conclude that $\mathcal{S}$ and $\mathcal{T}$ are indeed in $C^1(\mathfrak{B})$. It remains to consider the functions (3.14) and (3.15). These are linear in μ, and the estimate

$$\int_{\mathcal{F}} \|x\|\, d|\mu|(x) \leq \int_{\mathcal{F}} (1+\|x\|^2)\, d|\mu|(x) = \|\mu\|_{\mathfrak{B}} \qquad \forall\, \mu \in \mathfrak{B}$$

readily shows that their derivative is a bounded linear functional. As this functional is continuous in μ (it is even independent of μ), it follows that $G \in C^1(\mathfrak{B})$. ☐

In the next lemma we construct measures with prescribed linear constraints but such that the value of $\mathcal{T}$ is smaller than that of a given minimizer. For the construction we rescale the argument of a measure. We denote this operation by $\mathfrak{s}$,

$$\mathfrak{s}\,:\, \mathbb{R}\times\mathfrak{B} \to \mathfrak{B}\ , \quad (\mathfrak{s}_\tau\mu)(\Omega) := \mu(\tau\Omega)\ . \tag{3.19}$$

Obviously, $\mathfrak{s}_\tau$ maps $\mathfrak{M}\cap\mathfrak{B}$ to itself.

Lemma 3.5. *For a given minimizer $\rho \in \mathfrak{M}\cap\mathfrak{B}$, there is a parameter $\delta > 0$ and a smooth mapping $\hat{\rho}\,:\, B_\delta(\rho) \subset \mathfrak{B} \to \mathfrak{M}\cap\mathfrak{B}$ such that for all $\mu \in B_\delta(\rho)$,*

$$G\big(\mu - \hat{\rho}(\mu)\big) = 0 \qquad \textit{and} \qquad \mathcal{T}(\hat{\rho}(\mu)) < C\ . \tag{3.20}$$

Moreover, the measure $\hat{\rho}$ satisfies the inequality

$$D\mathcal{T}|_\mu\, \hat{\rho} < 2C\ . \tag{3.21}$$

Proof. According to the assumption (1.16), there is a measure $\rho_1 \in \mathfrak{M}\cap\mathfrak{B}$ such that

$$\int_{\mathcal{F}} x\, d\rho_1 = \mathbb{1}_{\mathbb{C}^k} \qquad \text{and} \qquad \mathcal{T}(\rho_1) < C\ .$$

In the case of the identity constraint (IC), we choose additional measures $\rho_2, \ldots, \rho_L \in \mathfrak{M}\cap\mathfrak{B}$ such that the matrices

$$\int_{\mathcal{F}} x\, d\rho_l\ , \qquad l = 1, \ldots, L \tag{3.22}$$

are linearly independent (for example, these measures can be chosen as Dirac measures supported at certain $x \in \mathcal{F}$).

For parameters $\kappa \in (0, L^{-1})$ and $\tau \in \mathbb{R}^L$, we consider the family of measures

$$\hat{\rho}(\kappa, \tau_1, \ldots, \tau_L) = (1-\kappa L)\, \mathfrak{s}_{(1-\kappa L)^{-1}} \rho_1 + \kappa \sum_{l=1}^L \mathfrak{s}_{\tau_l}\, \rho_l \,.$$

Then the functional G depends linearly on the parameters $\tau_1, \ldots, \tau_L$, and the mapping $(\tau_1, \ldots, \tau_L) \mapsto (G_1, \ldots, G_L)$ is invertible. Moreover, by choosing the parameters κ and τ_l sufficiently small, we can arrange by continuity that $\mathcal{T}(\hat{\rho}) < C$. Finally, a direct computation shows that the measure $\hat{\rho}$ is positive and normalized.

By continuity, it suffices to derive (3.21) for $\mu = \rho$. To this end, we consider the family of measures

$$\tilde{\rho}_\tau = \tau \hat{\rho} + (1-\tau)\, \rho \,. \tag{3.23}$$

Then in view of (4.3) and (3.18),

$$\mathcal{T}(\rho_\tau) = \tau^2\, \mathcal{T}(\hat{\rho}) + \tau\, (1-\tau)\, D\mathcal{T}|_\rho\, \hat{\rho} + (1-\tau)^2\, \mathcal{T}(\rho) \,.$$

This functional is obviously quadratic in τ, and as $\lim_{\tau \to \pm\infty} \mathcal{T}(\rho_\tau) = \infty$, it is convex. Hence

$$D\mathcal{T}|_\rho\, \hat{\rho} - 2\mathcal{T}(\rho) = \frac{d}{d\tau} \mathcal{T}(\rho_\tau)\big|_{\tau=0} \leq \mathcal{T}(\hat{\rho}) - \mathcal{T}(\rho)$$

and thus

$$D\mathcal{T}|_\rho\, \hat{\rho} \leq \mathcal{T}(\hat{\rho}) + \mathcal{T}(\rho) \,.$$

Since $\mathcal{T}(\hat{\rho}) < C$ and $\mathcal{T}(\rho) \leq C$, we obtain the strict inequality (3.21). $\square$

Lemma 3.6. *Under the assumptions of Proposition 3.3, for every minimizer $\rho \in \mathfrak{M} \cap \mathfrak{B}$ there are parameters $\varepsilon, \delta > 0$ and a continuous mapping*

$$\Phi \;:\; (B_\delta(\rho) \subset \mathfrak{B}) \times (B_\varepsilon(0) \subset \mathbb{R}^L) \times [0, \varepsilon) \to \mathfrak{B}$$

with the following properties:

(a) *$\Phi(\mu, 0, 0) = \mu$ for all $\mu \in B_\delta(\rho)$.*
(b) *For every $t \in B_\varepsilon(0)$ and $\tau \in [0, \varepsilon)$, the function $\Phi(., t, \tau) : B_\delta(\rho) \to \mathfrak{B}$ maps the set $\mathfrak{M} \cap B_\delta(\rho)$ to itself.*
(c) *The composition $G \circ \Phi$ is in $C^1(B_\delta(\rho) \times B_\varepsilon(0) \times [0, \varepsilon), \mathbb{R}^L)$. Moreover, the $L \times L$-matrix $D_2(G \circ \Phi)|_{(\rho,0,0)}$ is invertible and $D_3(G \circ \Phi)|_{(\rho,0,0)} = 0$.*
(d) *The directional derivatives $u \cdot D_2(\mathcal{T} \circ \Phi)|_{(\rho,0,0)}$ (with $u \in \mathbb{R}^L$) and the partial derivative $D_3(\mathcal{T} \circ \Phi)|_{(\rho,0,0)}$ exist. They satisfy the inequalities*
$$\begin{aligned} |u \cdot D_2(\mathcal{T} \circ \Phi)(\rho, 0, 0)| &\leq c\, \|u\|_{\mathbb{R}^L} \\ D_3(\mathcal{T} \circ \Phi)|_{(\rho,0,0)} &< 2\,(C - \mathcal{T}(\rho)) \end{aligned}$$
with a constant $c = c(\rho)$.

Proof. We take the ansatz

$$\Phi(\mu, t, \tau) = \bigg(1 - \sum_{l=1}^L |t_l| - \tau \bigg)\, \mu + \tau\, \hat{\rho} \tag{3.24}$$

$$\qquad + \sum_{l=1}^L \Big(\max(t_l, 0)\, \rho_l + \max(-t_l, 0)\, \mathfrak{s}_{-1}\, \rho_l \Big) \,. \tag{3.25}$$

Obviously, Φ is trivial in the case $t = 0$ and $\tau = 0$, giving property (a). Moreover, if t and τ are sufficiently small, we have a convex combination of measures, proving property (b).

We point out that Φ is *not* differentiable in t because of the absolute values and the factors $\max(\pm t_l, 0)$. On the other hand, this property is not needed, as we only claim that $G \circ \Phi$ is differentiable. Lemma 3.5 yields that $D_3(G\circ\Phi)|_{(\rho,0,0)} = 0$. But the linear constraints depend on the parameters t_l. Our ansatz ensures that this dependence is smooth even if some of the parameters t_l vanish. Finally, as the matrices (3.22) are linearly independent, it follows immediately that $D_2(G \circ \Phi)|_{(\mu,t,\tau)}$ has maximal rank. This proves (c).

In order to prove (d), we consider the functional $\mathcal{T} \circ \Phi$. Note that, again due to the absolute values and the factors $\max(\pm t_l, 0)$, this functional is not differentiable in the parameters t_l. But clearly, the directional derivatives at $t = 0$ exist and are bounded. Finally, the derivative with respect to τ is computed with the help of (3.21). □

Proof of Proposition 3.3. Let us apply Lemma 3.6. First, as $G \circ \Phi$ is continuously differentiable, we can conclude from (c) that there is $\delta > 0$ such that the matrix $D_2\big(G\circ\Phi\big)(\mu, 0, \tau)$ is invertible for all $\mu \in B_\delta(\rho) \subset \mathfrak{B}$ and all $\tau \in [0, \varepsilon)$. Thus $(G\circ\Phi)(\mu, ., \tau)$ is a local diffeomorphism, implying that (possibly after decreasing δ) there is a mapping $h \in C^1(B_\delta(\rho) \times [0, \varepsilon), B_\varepsilon(0))$ such that $h(\rho, 0) = 0$ and

$$(G \circ \Phi)(\mu, h(\mu, \sigma), \sigma) = 0 \qquad \text{for all } \mu \in B_\delta(\rho) \text{ and } \sigma \in [0, \varepsilon) . \tag{3.26}$$

Let $(\rho_\tau)_{\tau\in[0,1)}$ be a variation in $\mathfrak{B} \cap \mathfrak{M}$ with $\rho_0 = \rho$. We choose $\sigma = \kappa\tau$ with a constant $\kappa > 0$ to be determined later. Then, using that $h(\rho, 0) = 0$ and that $D_3(G \circ \Phi)_{(\rho,0,0)} = 0$, we obtain

$$\begin{aligned} 0 &= \frac{d}{d\tau}(G \circ \Phi)(\rho_\tau, h(\rho_\tau, \kappa\tau), \kappa\tau)\Big|_{\tau=0} \\ &= D_1(G \circ \Phi)|_{(\rho,0,0)}\, \dot\rho_0 + D_2(G \circ \Phi)|_{(\rho,0,0)} \circ Dh|_{(\rho,0)}\, \dot\rho_0 . \end{aligned} \tag{3.27}$$

We now introduce for $\tau \in [0, \alpha)$ and sufficiently small $\alpha > 0$ the variation

$$\tilde\rho_\tau = \Phi\big(\rho_\tau, h(\rho_\tau, \kappa\tau), \kappa\tau\big) . \tag{3.28}$$

In view of (b) and (3.26), this variation lies in $\mathfrak{M}\cap\mathfrak{B}$ and satisfies the linear constraints. Moreover, by choosing K sufficiently large, we can arrange in view of (d) that this variation decreases $\mathcal{T}$. Thus it satisfies all the constraints and is admissible for our variational principle. The minimality of ρ implies that

$$0 \leq \frac{d}{d\tau}\mathcal{S}(\tilde\rho)_\tau)\Big|_{\tau=0} = D\mathcal{S}|_\rho\, \frac{d}{d\tau}\Phi(\rho_\tau, h(\rho_\tau, \kappa\tau), \kappa\tau)\Big|_{\tau=0} .$$

Computing the one-sided derivatives with the chain rule, we obtain

$$\frac{d}{d\tau}\Phi(\rho_\tau, h(\rho_\tau))\Big|_{\tau=0^+} = \dot\rho_0 + E ,$$

where the error term is bounded by

$$\|E\| \leq c\, \big\|Dh|_{(\rho,0)}\, (\dot\rho_0, \kappa)\big\| + c\, \kappa .$$

In the case $\mathcal{T}(\rho) < C$, we can choose $\kappa = 0$. Differentiating (3.26), we obtain

$$0 = \frac{d}{d\tau}(G \circ \Phi)|_{(\rho_\tau, h(\rho_\tau,0),0)} = DG|_\rho \dot\rho_0 + D_2(G \circ \Phi)|_{(\rho,0,0)}\, Dh|_{(\rho,0)}(\dot\rho_0, 0) ,$$

showing that Dh can be estimated in terms of the first derivatives of G. This gives the result.

In the case $\mathcal{T}(\rho) = C$, we know from (d) that $D_3(\mathcal{T} \circ \Phi)|_{(\rho,0,0)} < 0$. Thus by choosing κ sufficiently large, we can compensate the positive contribution to the variation of $\mathcal{T}$ caused by ρ_τ and by h. Clearly, the parameter κ is bounded in terms of the variation of G and the positive part of $\partial_\tau \mathcal{T}(\rho_\tau)|_{\tau=0}$. This concludes the proof. ☐

We finally show how Proposition 3.3 can be adapted to second variations.

Proposition 3.7. *Assume that ρ is a minimizer of the variational principle of Definition 1.1, where the constant C satisfies* (1.16)*. Then there is a constant c such that for every twice $\mathfrak{B}$-Fréchet differentiable family of measures $(\rho_\tau)_{\tau \in [0,1)}$ in $\mathfrak{B} \cap \mathfrak{M}$ with $\rho_0 = \rho$ and*

$$\frac{d}{d\tau}\mathcal{S}(\rho_\tau)\Big|_{\tau=0} = 0 = \frac{d}{d\tau}\mathcal{T}(\rho_\tau)\Big|_{\tau=0}\,, \qquad \frac{d}{d\tau}G(\rho_\tau)\Big|_{\tau=0} = 0\,, \tag{3.29}$$

the second variation satisfies the inequality

$$\begin{aligned} \frac{d^2}{d\tau^2}\mathcal{S}(\rho_\tau)\Big|_{\tau=0} \geq &- c\,\left\| \frac{d^2}{d\tau^2}G(\rho_\tau)\Big|_{\tau=0} \right\|_{\mathbb{R}^L} \\ &- \left\{ \begin{array}{ll} 0 & \text{if } \mathcal{T}(\rho) < C \\ c\,\max\left(0, \dfrac{d^2}{d\tau^2}\mathcal{T}(\rho_\tau)\Big|_{\tau=0}\right) & \text{if } \mathcal{T}(\rho) = C\,. \end{array} \right. \end{aligned}$$

Proof. We consider similar to (3.28) the variation

$$\tilde{\rho}_\tau = \Phi\big(\rho_\tau, h(\rho_\tau, \kappa\tau^2), \kappa\tau^2\big)\,.$$

From (3.26) one sees that the linear constraints are satisfied. Moreover, a short calculation using (3.29) shows that the first variation of $\mathcal{T}$ vanishes, and that by choosing κ sufficiently large, one can arrange that the second variation of $\mathcal{T}$ becomes negative. Now we can argue just as in the proof of Proposition 3.3. ☐

3.2. **First Variations with Fixed Support.** We now want to apply Proposition 3.3 to specific variations $(\rho_\tau)_{\tau\in[0,1]}$. Here we begin with variations keeping the support of $\mathfrak{m}$ fixed, i.e.

$$\operatorname{supp}\tilde{\mathfrak{m}}_\tau = \operatorname{supp}\mathfrak{m} \qquad \text{for all } \tau\,.$$

It turns out that it is most convenient to work in the formalism of moment measures introduced in Section 2.2. In view of (2.9) and Proposition 2.3, the moment measures corresponding to any measure $\rho \in \mathfrak{M}$ are uniquely characterized by a normalized positive regular Borel measure $\mathfrak{m}^{(0)}$ on $\mathcal{K}$ and a function $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$, being odd in the sense of (2.10). Conversely, given any positive regular Borel measure $\mathfrak{m}^{(0)}$ and any function $f \in L^2(\mathcal{K}, d\mathfrak{m}^{(0)})$ (which need not necessarily be odd), we can define a measure $\rho \in \mathfrak{M} \cap \mathfrak{B}$ by (2.12). For ease in notation, we will often omit the index $^{(0)}$. On $\mathcal{K}$ we introduce the functions

$$\ell(x) = f(x)^2 \int_{\mathcal{K}} \mathcal{L}(x,y)\, f(y)^2\, d\mathfrak{m}(y) \qquad \in L^1(\mathcal{K}, d\mathfrak{m}) \tag{3.30}$$

$$\mathfrak{t}(x) = f(x)^2 \int_{\mathcal{K}} |A_{xy}|\, f(y)^2\, d\mathfrak{m}(y) \qquad \in L^1(\mathcal{K}, d\mathfrak{m}) \tag{3.31}$$

$$g_l(x) = f(x)\ \operatorname{Tr}(e_l\, x)\,, \quad l = 1, \ldots, L\,, \qquad \in L^2(\mathcal{K}, d\mathfrak{m})\,, \tag{3.32}$$

where $(e_1, \ldots, e_L)$ again denotes the basis of the symmetric $k \times k$-matrices used in (3.15). Comparing with (2.7), (2.8) and (3.15), one sees that integrating over x with respect to $d\mathfrak{m}$ gives (up to the irrelevant additive constants $\mathrm{Tr}(e_l)$ in G_l) the functionals denoted by the corresponding capital letters. Moreover, we denote the constant function one on $\mathcal{K}$ by $1_\mathcal{K}$. We denote the scalar product on $L^2(\mathcal{K}, d\rho)$ by $\langle .|. \rangle$.

Lemma 3.8. *Under the assumptions of Proposition 3.3, there are constants $\kappa, c \in \mathbb{R}$ such that*

$$\ell(x) + \kappa\, \mathfrak{t}(x) = c \qquad \text{on } \operatorname{supp} \mathfrak{m}\, . \tag{3.33}$$

Proof. Assume conversely that the statement is false. Then there is a set $\Omega \subset \mathcal{K}$ of positive measure such that on Ω, the function ℓ is not a linear combination of $\mathfrak{t}$ and $1_\mathcal{K}$, and that moreover the restrictions $\ell|_\Omega$ and $\mathfrak{t}|_\Omega$ are bounded functions. Then $\ell|_\Omega$ is not in the span of the vectors $\mathfrak{t}|_\Omega, 1_\Omega \in L^2(\Omega, d\mathfrak{m})$. By projecting $\ell|_\Omega$ onto the orthogonal complement of these vectors, we obtain a bounded function $\psi \in L^\infty(\Omega, d\mathfrak{m})$ such that

$$\langle \psi | \ell \rangle < 0 \qquad \text{but} \qquad \langle \psi | \mathfrak{t} \rangle = 0 = \langle \psi | 1_\mathcal{K} \rangle\, . \tag{3.34}$$

Extending ψ by zero to $\mathcal{K}$, these relations again hold and $\psi \in L^\infty(\mathcal{K}, d\mathfrak{m})$.

We now consider the variation of the moment measures

$$d\tilde{\mathfrak{m}}_\tau = (1 - \tau\psi)\, d\mathfrak{m}\,, \qquad \tilde{f}_\tau = (1 + \tau\psi)\, f\,, \qquad \tau \in (-\varepsilon, \varepsilon)\, . \tag{3.35}$$

The last equation in (3.34) implies that $\tilde{\mathfrak{m}}$ is normalized, also it is positive measure for sufficiently small ε. A direct computation using (3.34) gives

$$\frac{d}{d\tau} G_l(\rho_\tau)\big|_{\tau=0} = 0\,, \quad \frac{d}{d\tau} \mathcal{T}(\rho_\tau)\big|_{\tau=0} = 2\, \langle \psi | \mathfrak{t} \rangle = 0\,, \quad \frac{d}{d\tau} \mathcal{S}(\rho_\tau)\big|_{\tau=0} = 2\, \langle \psi | \ell \rangle < 0\, .$$

Hence the first variation decreases the action without changing the constraints. This is a contradiction to Proposition 3.3. ☐

Lemma 3.9. *The parameter κ in Lemma 3.8 can be chosen to be non-negative.*

Proof. If the function ℓ is constant, we can choose $\kappa = 0$. Otherwise, as in the proof of Lemma 3.8 we can choose a function $\psi \in L^\infty(\mathcal{K}, d\mathfrak{m})$ such that

$$\langle \psi | 1_\mathcal{K} \rangle = 0 \qquad \text{and} \qquad \langle \psi | \ell \rangle = -1\, .$$

Then (3.33) implies that

$$\kappa\, \langle \psi | \mathfrak{t} \rangle = -\langle \psi | \ell \rangle = 1\, .$$

If κ were negative, by (3.35) we could vary the measure ρ in $\mathfrak{M} \cap \mathfrak{B}$ such that the first variation decreases both $\mathcal{S}$ and $\mathcal{T}$. This is a contradiction to Proposition 3.3. ☐

Lemma 3.10. *Under the assumptions of Proposition 3.3, there are real parameters $\lambda_1, \ldots, \lambda_L$ such that*

$$\sum_{l=1}^L \lambda_l\, g_l = 4\, (\mathcal{S} + \kappa \mathcal{T})\, 1_\mathcal{K} \qquad \text{on } \operatorname{supp} \mathfrak{m}\, . \tag{3.36}$$

Proof. We first want to prove that g_1 lies in the span of the other functions,

$$g_1 \in \big\langle 1_\mathcal{K}, g_2, \ldots, g_L \big\rangle\, . \tag{3.37}$$

If this were not true, just as in the proof of Lemma 3.8, we could find a function $\psi \in L^\infty(\mathcal{K}, d\mathfrak{m})$ such that

$$\langle \psi | g_1 \big\rangle = k \qquad \text{and} \qquad \langle \psi | 1_\mathcal{K} \rangle = 0 = \langle \psi | g_l \big\rangle\,, \quad l = 2, \ldots, L\, .$$

Considering the variation of the moment measures

$$d\tilde{\mathfrak{m}}_\tau = (1+2\tau\psi)\, d\mathfrak{m} \,, \qquad \tilde{f}_\tau = (1-\tau\psi-\tau)\, f \,, \qquad \tau \in (-\varepsilon, \varepsilon) \,,$$

a direct computation yields

$$\frac{d}{d\tau} G_l(\rho_\tau)\big|_{\tau=0} = 0 \,, \quad \frac{d}{d\tau} \mathcal{T}(\rho_\tau)\big|_{\tau=0} = -4\mathcal{T}(\rho) \,, \quad \frac{d}{d\tau} \mathcal{S}(\rho_\tau)\big|_{\tau=0} = -4\mathcal{S}(\rho) \,.$$

Thus the first variation decreases both $\mathcal{S}$ and $\mathcal{T}$ without changing the linear constraints. This is a contradiction, thereby proving (3.37).

According to (3.37), there are real coefficients c and $\lambda_2, \ldots, \lambda_L$ such that

$$g_1 = c\, 1_{\mathcal{K}} + \sum_{l=2}^{L} \lambda_l g_l \,. \tag{3.38}$$

From our choice of the matrices e_l (see after (3.15)), we know that

$$\int_{\mathcal{K}} g_1 \, d\mathfrak{m} = k \qquad \text{and} \qquad \int_{\mathcal{K}} g_l \, d\mathfrak{m} = 0 \quad \text{for } l = 2, \ldots, L \,. \tag{3.39}$$

Thus integrating (3.38) over $\mathcal{K}$ gives $k = c$. Hence c is non-zero, and rescaling the λ_l gives the result. □

Combining the results of the previous lemmas, we obtain the following result.

Theorem 3.11. *Assume that ρ is a minimizer of the variational principle of Definition 1.1, where the constant C satisfies* (1.16)*. Then there are Lagrange multipliers $\kappa \geq 0$ and $\lambda_1, \ldots, \lambda_L \in \mathbb{R}$ such that for almost all $x \in \operatorname{supp} \mathfrak{m} \subset \mathcal{K}$, the following identities hold*

$$\frac{1}{4} \sum_l \lambda_l \, g_l(x) = \mathcal{S} + \kappa \mathcal{T} = \ell(x) + \kappa \mathfrak{t}(x) \,. \tag{3.40}$$

If neither (TC) nor (IC) are considered, we may choose $\Lambda = 0$. In the case $\mathcal{T}(\rho) < C$, we may choose $\kappa = 0$.

Setting

$$\Lambda = \sum_{l=1}^{L} \lambda_l \, e_l \,, \tag{3.41}$$

using (3.32) and rewriting the first equation in (3.40) in terms of the measure ρ yields Theorem 1.2.

3.3. **First Variations with Varying Support.** We now consider first variations which change the support of the measure $\mathfrak{m}$. The following notion turns out to be helpful.

Definition 3.12. *A minimizing measure ρ is called* **regular** *if the following two conditions are satisfied:*

1. *In the case of the identity constraint (IC), the functions $g_1, \ldots, g_L$ must be linearly independent.*
2. *When $\mathcal{T}(\rho) = C$, the function $\mathfrak{t}$ must be non-constant on* $\operatorname{supp} \mathfrak{m}$*.*

If one of these conditions is violated, ρ is called **singular**.

Note that in the case of the trace constraint (TC), we know from the first equation in (3.39) that the function g_1 is non-zero, so that the functions $g_1, \ldots, g_L$ are automatically linearly independent. It is an open problem if or under which assumptions all minimizers are regular.

We begin with the analysis of regular minimizers (for singular minimizers see Theorem 3.14 below). Recall that, according to Theorem 1.2 the function Φ defined by (1.11) (with Λ again given by (3.41)) is constant on the support of ρ. The following result shows that Φ is minimal on the support of ρ.

Theorem 3.13. *Assume that ρ is a regular minimizer of the causal variational principle of Definition 1.1, where the constant C satisfies* (1.16). *Then*

$$\Phi(x) \geq -2\,(\mathcal{S} + \kappa \mathcal{T}) \qquad \text{for all } x \in \mathcal{F}\,.$$

Proof. We first consider $x_0 \in \operatorname{supp} \mathfrak{m}$. Then we know from Theorem 1.2 that

$$\Phi(tx_0)\big|_{t=f(x_0)} = -2(\mathcal{S} + \kappa \mathcal{T}) \qquad \text{and} \qquad \frac{d}{dt} \Phi(tx_0)\big|_{t=f(x_0)} = 0\,.$$

Since $\Phi(tx_0)$ is a quadratic polynomial in t with a non-negative quadratic term, it follows that $\Phi(tx_0$ is minimal at $t = f(x_0)$.

Next we choose $x_0 \in \mathcal{K} \setminus \operatorname{supp} \mathfrak{m}$. For given $f_0 \in \mathbb{R}$ and $\psi, \phi \in L^\infty(\mathcal{K}, d\mathfrak{m})$ with

$$\langle \phi | 1_{\mathcal{K}} \rangle = 1\,, \tag{3.42}$$

we consider the variation

$$\begin{aligned} \tilde{\mathfrak{m}}_\tau &= (1 - \tau\,\phi)\,\mathfrak{m} + \tau \delta_{x_0} \\ \tilde{f}_\tau(x) &= \begin{cases} \big(1 + \tau \psi(x) + \tau \phi(x)\big)\, f(x) & \text{if } x \in \operatorname{supp} \mathfrak{m} \\ f_0 & \text{if } x = x_0\,. \end{cases} \end{aligned}$$

Then the first variation is computed by

$$\frac{d}{d\tau} G_l \big|_{\tau=0} = g_l(x_0) + \int_{\mathcal{K}} \psi\, g_l\, d\mathfrak{m} \tag{3.43}$$

$$\frac{d}{d\tau} \mathcal{T} \big|_{\tau=0} = 2\, \mathfrak{t}(x_0) + 2 \int_{\mathcal{K}} (2\psi + \phi)\, \mathfrak{t}\, d\mathfrak{m} \tag{3.44}$$

$$\frac{d}{d\tau} \mathcal{S} \big|_{\tau=0} = 2\, \ell(x_0) + 2 \int_{\mathcal{K}} (2\psi + \phi)\, \ell\, d\mathfrak{m} \tag{3.45}$$

(where $\ell(x_0)$, $\mathfrak{t}(x_0)$ and $g_l(x_0)$ are defined according to (3.30)–(3.32)). Since the functions g_l are linearly independent, we can choose ψ such that $\partial_\tau G_l = 0$ for all $l = 1, \ldots, L$. Multiplying (3.43) by λ_l and summing over l, we can apply Lemma 3.10 to obtain

$$4\,(\mathcal{S} + \kappa \mathcal{T})\, \langle \psi | 1 \rangle = -\sum_{l=1}^L \lambda_l\, g_l(x_0)\,. \tag{3.46}$$

Next, using that the function $\mathfrak{t}$ is not constant, we can choose ϕ such that $\partial_\tau T = 0$. Applying Proposition 3.3, we conclude that $\partial_\tau \mathcal{S} \geq 0$. Hence, again using that $\partial_\tau \mathcal{T} = 0$, we obtain

$$\begin{aligned} 0 \leq \frac{1}{2} \frac{d}{d\tau} (\mathcal{S} + \kappa \mathcal{T})\,\big|_{\tau=0} &= (\ell + \kappa \mathfrak{t})(x_0) + \int_{\mathcal{K}} (2\psi + \phi)\, (\ell + \kappa \mathfrak{t})\, d\mathfrak{m} \\ &\overset{(3.40)}{=} (\ell + \kappa \mathfrak{t})(x_0) + (\mathcal{S} + \kappa \mathcal{T})\, \big\langle 2\psi + \phi \big| 1 \big\rangle\,. \end{aligned}$$

Using (3.42) and (3.46), we obtain

$$(\ell + \kappa \mathfrak{t})(x_0) + (\mathcal{S} + \kappa \mathcal{T}) - \frac{1}{2} \sum_{l=1}^{L} \lambda_l \, g_l(x_0) \geq 0 \,.$$

Applying (3.41) and rewriting the formula in $\mathcal{F}$ gives the result. □

For singular minimizers the following weaker statement holds.

Theorem 3.14. *Assume that ρ is a singular minimizer of the variational principle of Definition 1.1, where the constant C satisfies* (1.16)*. Let $\mathcal{P} \subset \mathcal{F}$ be the set*

$$\mathcal{P} = \Big\{ x \in \mathcal{F} \,|\; \text{there exist } \phi, \psi \in L^1(\mathcal{K}, d\mathfrak{m}) \text{ with } \langle \phi | 1 \rangle = 1 \,,$$
$$g_l(x_0) = -\int_{\mathcal{K}} \psi \, g_l \, d\mathfrak{m} \qquad \text{and} \qquad \mathfrak{t}(x_0) = -\int_{\mathcal{K}} (2\psi + \phi) \, \mathfrak{t} \, d\mathfrak{m} \Big\} \,,$$

where we set $x_0 = x/\|x\| \in \mathcal{K}$ and $f(x_0) = \|x\|$. Then

$$\Phi(x) \geq -2 \, (\mathcal{S} + \kappa \mathcal{T}) \qquad \text{for all } x \in \mathcal{P} \,.$$

Proof. If $x \in \mathcal{P}$, we can clearly arrange that (3.43) and (3.44) vanish. Now we can proceed exactly as in the proof of Theorem 3.13. □

We point out that if $x \in \operatorname{supp} \rho$, then x lies in $\mathcal{P}$, as can be seen by setting $x_0 = x/\|x\|$ and considering the series $\phi_n \to \delta_{x_0}$, $\psi_n \to -\delta_{x_0}$. We also remark that if $\mathfrak{t}$ is *not* constant, then the condition for $\mathfrak{t}(x)$ in the definition of $\mathcal{P}$ can clearly be satisfied. Thus in this case, $\mathcal{P}$ is defined by linear relations, thereby making it into the intersection of $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ with a plane through the origin.

3.4. **Second Variations with Fixed Support.**

For the analysis of second variations, we shall use spectral methods. To this end, we use the abbreviations

$$\mathcal{L}_{\text{eff}}(x,y) = \mathcal{L}(x,y) + \kappa \, |A_{xy}|^2 \tag{3.47}$$
$$L(x,y) = \big(\mathcal{L}(x,y) + \kappa \, |A_{xy}|^2 \big) \, f(x)^2 \, f(y)^2 \,. \tag{3.48}$$

Then the second equation in (3.40) can be expressed as

$$f(x)^2 \int_{\mathcal{K}} \mathcal{L}_{\text{eff}}(x,y) \, f(y)^2 \, d\mathfrak{m}(y) \equiv \int_{\mathcal{K}} L(x,y) \, d\mathfrak{m}(y) \equiv \mathcal{S} + \kappa \mathcal{T} \,. \tag{3.49}$$

We also consider $L(x,y)$ as the integral kernel of a corresponding operator

$$L \;:\; L^2(\mathcal{K}, d\mathfrak{m}) \to L^2(\mathcal{K}, d\mathfrak{m}) \,, \quad (L\phi)(x) := \int_{\mathcal{K}} L(x,y) \, \phi(y) \, d\mathfrak{m}(y) \,. \tag{3.50}$$

Proposition 3.15. *Under the assumptions of Theorem 3.11, the operator L is self-adjoint and Hilbert-Schmidt.*

Proof. If u is an eigenvector corresponding to a non-zero eigenvalue λ, then

$$u(x) = \frac{1}{\lambda} \int_{\mathcal{K}} L(x,y) \, u(y) \, d\mathfrak{m}(y) \,.$$

Obviously, L is formally self-adjoint. Thus it remains to show that the Hilbert-Schmidt norm is finite. Using (3.49), we obtain

$$\begin{aligned}\|L\|_2^2 &= \iint_{\mathcal{K}\times\mathcal{K}} L(x,y)^2 \, d\mathfrak{m}(x)\, d\mathfrak{m}(y) \\ &\leq \iint_{\mathcal{K}\times\mathcal{K}} \operatorname*{ess\,sup}_{y'\in\mathcal{K}} L(x,y') \; \operatorname*{ess\,sup}_{x'\in\mathcal{K}} L(x',y)\, d\mathfrak{m}(x)\, d\mathfrak{m}(y) \\ &= \Big(\int_{\mathcal{K}} \operatorname*{ess\,sup}_{x'\in\mathcal{K}} L(x',y)\, d\mathfrak{m}(y) \Big)^2 = (\mathcal{S}+\kappa\mathcal{T})^2 \,,\end{aligned}$$

concluding the proof. ☐

We remark that, similar to [5, Lemma 1.9], one could prove that the sup-norm of L is an eigenvalue of L with $1_{\mathcal{K}}$ as a corresponding eigenvector. However, it is not clear in general whether this eigenvalue is non-degenerate.

Since every Hilbert-Schmidt operator is compact, we know that L has a spectral decomposition with purely discrete eigenvalues and finite-dimensional eigenspaces.

Theorem 3.16. *Assume that ρ is a minimizer of the variational principle of Definition 1.1, where the constant C satisfies* (1.16). *If $\mathcal{T}(\rho) = C$, we assume furthermore that $\mathfrak{t}$ is not constant on* $\operatorname{supp}\mathfrak{m}$. *Then the operator L is positive semi-definite on the subspace*

$$J := \big\langle \mathfrak{t}, g_1, \ldots, g_L \big\rangle^\perp \subset L^2(\mathcal{K}, d\mathfrak{m}) \,.$$

Proof. We consider the operator $\pi_J L \pi_J$, where π_J is the orthogonal projection onto J. Assume on the contrary that this operator is not positive semi-definite. Since this operator is compact, there is a negative eigenvalue λ with corresponding eigenvector $v \in L^2(\mathcal{K}, d\mathfrak{m}) \cap J$. Let us show that there is a bounded function $u \in L^\infty(\mathcal{K}, d\mathfrak{m}) \cap J$ with $\langle u | Lu \rangle < 0$. To this end, we choose a nested sequence of measurable sets $A_i \subset \operatorname{supp}\mathfrak{m}$ such that $\mathfrak{m}(\mathcal{K} \setminus \cup_i A_i) = 0$ and the functions $v, \mathfrak{t}, g_1, \ldots, g_L$ are bounded on each A_i (this is possible by Chebycheff's inequality). We let $v_i \in L^2(A_i, d\mathfrak{m})$ be the projection of $v|_{A_i}$ onto the subspace $\big\langle \mathfrak{t}_{|A_i}, g_{1|A_i}, \ldots, g_{L|A_i} \big\rangle^\perp \subset L^2(A_i, d\mathfrak{m})$. Then the functions v_i are clearly bounded. The dominated convergence theorem shows that $\langle v_i | L v_i \rangle \to \langle v | Lv \rangle < 0$. Hence $u = v_i$ for sufficiently large i has the announced properties.

In view of Lemma 3.10, we know that $\langle u | 1_{\mathcal{K}} \rangle = 0$. Next, we choose a function $\phi \in L^\infty(\mathcal{K}, d\mathfrak{m})$ satisfying

$$\langle \phi | 1_{\mathcal{K}} \rangle = 0 \,. \tag{3.51}$$

Accordingly, the normalization of $\mathfrak{m}$ is preserved through the following variation,

$$\begin{aligned}\tilde{\mathfrak{m}}_\tau &= (1 + \tau u - \tau^2 \phi)\, \mathfrak{m} \\ \tilde{f}_\tau(x) &= \big(1 + \tau^2 \phi\big)\, f(x) \,.\end{aligned}$$

A straightforward calculation using the orthogonality relations of u and ϕ yields

$$G_l(\tau) = G_l(0) + \mathcal{O}(\tau^3) \tag{3.52}$$

$$\mathcal{T}(\tau) = \mathcal{T}(0) + \tau^2 \langle \phi | \mathfrak{t} \rangle + \tau^2 \langle u | Tu \rangle + \mathcal{O}(\tau^3) \tag{3.53}$$

$$(\mathcal{S} + \kappa \mathcal{T})(\tau) = (\mathcal{S} + \kappa\mathcal{T})(0) + \tau^2 \langle u | Lu \rangle + \mathcal{O}(\tau^3) \,, \tag{3.54}$$

where T is the operator with the integral kernel $T(x,y) = |A_{xy}|^2 f(x)^2 f(y)^2$. Since the function $\mathfrak{t}$ is not constant, by suitably choosing ϕ we can arrange that the quadratic term in (3.53) vanishes. Moreover, the term $\langle u|Lu\rangle = \lambda\,\|u\|^2$ is negative. Thus we have found a variation which preserves the constraints quadratically, but decreases the action. Per Proposition 3.7, this is a contradiction. $\square$

3.5. **Second Variations with Varying Support.** In this section we generalize Theorem 3.16 to the case where the Hilbert space $L^2(\mathcal{K}, d\mathfrak{m})$ is extended by a one-dimensional vector space consisting of functions supported on a set which is disjoint from the support of $\mathfrak{m}$. More specifically, we choose a normalized measure $\mathfrak{n}$ on $\mathcal{K}$ with

$$\operatorname{supp}\mathfrak{n} \cap \operatorname{supp}\mathfrak{m} = \varnothing\,.$$

We arbitrarily extend the function f to $\operatorname{supp}\mathfrak{n}$.

In order to consider second variations, we introduce the Hilbert space $(\mathcal{H}, \langle .|. \rangle)$ as

$$\mathcal{H} = L^2(\mathcal{K}, d\mathfrak{m}) \oplus \mathbb{R}\,.$$

We extend the operator L, (3.50), to $\mathcal{H}$ by

$$\begin{aligned}
L\,(u,a) &= (\phi, b) \qquad \text{with} \\
\phi(x) &= \int_{\mathcal{K}} L(x,y)\, u(y)\, d\mathfrak{m}(y) + a \int_{\mathcal{K}} L(x,y)\, d\mathfrak{n}(y) \\
b &= \iint_{\mathcal{K}\times\mathcal{K}} L(x,y)\, u(y)\, d\mathfrak{m}(y)\, d\mathfrak{n}(x) + a \iint_{\mathcal{K}\times\mathcal{K}} L(x,y)\, d\mathfrak{n}(x)\, d\mathfrak{n}(y)\,.
\end{aligned}$$

Then the following theorem holds.

Theorem 3.17. *Assume that ρ is a minimizer of the variational principle of Definition 1.1, where the constant C satisfies* (1.16). *If $\mathcal{T}(\rho) = C$, we assume furthermore that the function $\mathfrak{t}$ is not constant on* $\operatorname{supp}\mathfrak{m}$. *Then the operator L is positive semi-definite on the subspace*

$$J := \big\langle \mathfrak{t}, g_1, \ldots, g_L \big\rangle^\perp \subset \mathcal{H}\,.$$

Proof. Assume on the contrary that the operator $\pi_J L \pi_J$ is not positive semi-definite. Then the operator has a negative eigenvalue λ with corresponding eigenvector v. Just as in the proof of Theorem 3.16, we can choose a bounded function $w = (u,a) \in \mathcal{H} \cap J$ with $\langle w|Lw\rangle < 0$. Possibly by flipping the sign of w we can arrange that $a \geq 0$. Next, we again choose a function $\phi \in \mathcal{H}$ with $\operatorname{supp}\phi \subset \operatorname{supp}\mathfrak{m}$ satisfying (3.51). Then the variation

$$\begin{aligned}
\tilde{\mathfrak{m}}_\tau &= (1 + \tau u - \tau^2 \phi)\, \mathfrak{m} + \tau a \mathfrak{n} \\
\tilde{f}_\tau(x) &= \big(1 + \tau^2 \phi\big)\, f(x)
\end{aligned}$$

is admissible for sufficiently small positive τ. Repeating the arguments in the proof of Theorem 3.16 gives the result. $\square$

3.6. **An A-Priori Estimate.** We conclude this section with estimates under the additional assumption that

$$\inf_{x \in \operatorname{supp}\mathfrak{m}} \mathcal{L}_{\text{eff}}(x,x) > 0\,. \tag{3.55}$$

This condition is clearly satisfied in the case $\kappa > 0$. In the case $\kappa = 0$, the estimates in [4, Section 4] show that $\mathcal{L}(x,x)$ is bounded from below, provided that the trace $\operatorname{Tr}(x)$ is bounded away from zero. However, it is conceivable that for a general minimizer, $\operatorname{Tr}(x)$ might have zeros on the support of ρ, so that (3.55) could be violated.

Proposition 3.18. *Under the assumptions of Theorem 3.11 and assuming* (3.55)*, the function f is essentially bounded, $f \in L^\infty(\mathcal{K}, d\mathfrak{m}^{(0)})$. Moreover, there is a constant $c = c(\mathcal{F})$ such that for every $\varepsilon > 0$ the inequality*
$$\int_{\mathcal{K}} |f|^{4-\varepsilon}\, d\mathfrak{m} \leq \frac{c}{\inf\limits_{x \in \operatorname{supp} \mathfrak{m}} \mathcal{L}_{\mathit{eff}}(x,x)}\, \frac{\mathcal{S} + \kappa \mathcal{T}}{1 - 2^{-\varepsilon}} \tag{3.56}$$
holds.

Proof. In order to prove that $f \in L^\infty(\mathcal{K}, d\mathfrak{m})$, we proceed indirectly and assume that f is not essentially bounded. Then there is a point $x \in \mathcal{K}$ such that for every $\varepsilon > 0$,
$$\operatorname*{ess\,sup}_{B_\varepsilon(x)} |f| = \infty\,. \tag{3.57}$$
By decreasing ε, we can arrange by continuity that
$$\mathcal{L}_{\text{eff}}(y,z) \geq \delta := \frac{1}{2} \inf_{x \in \mathcal{K}} \mathcal{L}_{\text{eff}}(x,x) \qquad \text{for all } y, z \in B_\varepsilon(x)\,.$$
Using (3.30), (3.31) and (3.47), we conclude that for any $y \in B_\varepsilon(x) \cap \operatorname{supp} \rho$,
$$(\ell + \kappa \mathfrak{t})(y) \geq f(y)^2\, \delta \int_{B_\varepsilon(x)} f^2(z)\, d\mathfrak{m}(z)\,.$$
The last integral is non-zero in view of (3.57). Thus by choosing y appropriately, we can make $(\ell + \kappa\mathfrak{t})(y)$ arbitrarily large, in contradiction to Theorem 3.11.

In order to prove the inequality (3.56), for any $L > 0$ we introduce the set
$$\mathcal{K}_L = \{x \in \mathcal{K} \,|\, |f(x)| > L\}\,.$$
Integrating (3.49) over $\mathcal{K}_L$ gives
$$\iint_{\mathcal{K}_L \times \mathcal{K}} \mathcal{L}_{\text{eff}}(x,y)\, f(x)^2\, d\mathfrak{m}(x)\, f(y)^2\, d\mathfrak{m}(y) = \mathfrak{m}(\mathcal{K}_L)\, (\mathcal{S} + \kappa \mathcal{T})\,.$$
The covering argument in [5, Lemma 2.12] shows that there is a constant $c = \delta(\mathcal{F}) > 0$ such that
$$\Big(\int_{\mathcal{K}_L} f^2\, d\mathfrak{m} \Big)^2 \inf_{x \in \mathcal{K}} \mathcal{L}_{\text{eff}}(x,x) \leq c\, \mathfrak{m}(\mathcal{K}_L)\, (\mathcal{S} + \kappa \mathcal{T})\,.$$
Setting $c_1 = c / \inf_{x \in \mathcal{K}} \mathcal{L}_{\text{eff}}(x,x)$, it follows that
$$L^4\, \mathfrak{m}(\mathcal{K}_L)^2 \leq c_1\, \mathfrak{m}(\mathcal{K}_L)\, (\mathcal{S} + \kappa \mathcal{T})$$
and thus
$$\mathfrak{m}(\mathcal{K}_L) \leq c_1\, (\mathcal{S} + \kappa \mathcal{T})\, \frac{1}{L^4}\,.$$
Now we can estimate the integral by considering the sequence $L_n = 2^n$,
$$\begin{aligned} \int_{\mathcal{K}} |f^{4-\varepsilon}|\, d\mathfrak{m} &\leq \sum_{n=0}^\infty (2L_n)^{4-\varepsilon}\, \mathfrak{m}(\mathcal{K}_{L_n}) \leq c_1\, (\mathcal{S} + \kappa \mathcal{T}) \sum_{n=0}^\infty (2L_n)^{4-\varepsilon}\, L_n^{-4} \\ &\leq 16\, c_1\, (\mathcal{S} + \kappa \mathcal{T}) \sum_{n=0}^\infty 2^{-n\varepsilon} = 16\, c_1\, (\mathcal{S} + \kappa \mathcal{T})\, \frac{1}{1 - 2^{-\varepsilon}}\,. \end{aligned}$$
This gives (3.56). ∎

4. The Euler-Lagrange Equations in the Equivariant Case

In this section, we extend the previous results to the setting of a symmetry group (possibly non-compact) acting on the measures. To this end, we first replace $\mathbb{C}^k$ by a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ of possibly infinite dimension $k \in \mathbb{N} \cup \{\infty\}$. For a given parameter $n \in \mathbb{N}$, we again let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all operators of rank at most $2n$ with at most n positive and at most n negative eigenvalues. Moreover, we let G be a topological group and U a continuous unitary representation of G on $\mathcal{H}$. Then G also acts on $\mathcal{F}$ by

$$U(g) \;:\; \mathcal{F} \to \mathcal{F} \;:\; x \mapsto U(g)\, x\, U(g)^{-1} \,. \tag{4.1}$$

A Borel measure ρ on $\mathcal{F}$ is called *equivariant* if $U(g)_* \rho = \rho$ for all $g \in G$. An equivariant Borel measure ρ induces a measure on the quotient space $\mathcal{F}/G$. It is called *normalized* if $\rho(\mathcal{F}/G) = 1$. We consider the class of measures

$$\mathfrak{M}_G = \{\rho \text{ equivariant normalized regular Borel measure on } \mathcal{F}\} \,.$$

We introduce the functionals $\mathcal{S}$ and $\mathcal{T}$ by

$$\mathcal{S} = \int_{\mathcal{F}/G} \int_{\mathcal{F}} \mathcal{L}[A_{xy}]\, d\rho(x)\, d\rho(y) \tag{4.2}$$

$$\mathcal{T} = \int_{\mathcal{F}/G} \int_{\mathcal{F}} |A_{xy}|^2\, d\rho(x)\, d\rho(y) \tag{4.3}$$

and define the boundedness constraint as before,

(BC) The *boundedness constraint*: $\qquad \mathcal{T} \leq C$

In place of the trace and identity constraints, we now consider the following linear constraints. We let $h_1, \ldots, h_L \in C^0(\mathcal{F}/G)$ be continuous functions which are homogeneous of degree one, i.e.

$$h_l(\lambda x) = \lambda\, h_l(x) \qquad \text{for all } x \in \mathcal{F}/G \,.$$

For given constants $\nu_1, \ldots, \nu_L \in \mathbb{R}$ we introduce the functionals

$$G_l = \nu_l - \int_{\mathcal{F}/G} h_l(x)\, d\rho(x) \,.$$

(LC) The *linear constraints*: $\qquad G_l = 0 \quad \text{for all} \quad l = 1, \ldots, L.$

Definition 4.1. *For any parameter $C > 0$, our* **equivariant causal variational principle** *is to minimize $\mathcal{S}$ by varying $\rho \in \mathfrak{M}_G$ under the constraints* (BC) *and* (LC).

If $\mathcal{H}$ is finite dimensional, the existence of minimizers follows immediately by applying the compactness results in [5, Section 2]. Moreover, the trace and identity constraints can be reformulated in terms of (LC). In the infinite dimensional situation, the trace constraint is obviously again of the form (LC). For the identity constraint, however, it is in general not clear how by modding out the group action, the integral over $\mathcal{F}$ in (TC) can be rewritten as an integral over $\mathcal{F}/G$. Furthermore, when $\mathcal{H}$ is infinite dimensional, there are no general existence results. It is to be expected that minimizers exist only for particular choices of the symmetry group G and its unitary representation U (for a specific result in this direction see [5, Theorem 4.2]). For simplicity, we do not consider questions related to existence of minimizers. Instead, we simply assume that an equivariant minimizer ρ is given. Moreover, we only treat the case where $\mathcal{K}/G$ is compact. The case when $\mathcal{K}/G$ is non-compact remains an open problem which goes beyond the scope of the present work.

Introducing the moment measures again by (2.2)–(2.4), we can rewrite the action and the constraints in analogy to (2.6)–(2.8) and (2.9) by

$$G_l = \nu_l - \int_{\mathcal{K}/G} g_l \, d\mathfrak{m} \qquad \text{where} \qquad g_l(x) := f(x)\, h_l(x) \tag{4.4}$$

$$\mathcal{S}(\rho) = \int_{\mathcal{K}/G} \int_{\mathcal{K}} \mathcal{L}[A_{xy}]\, f(x)^2\, f(y)^2\, d\mathfrak{m}(x)\, d\mathfrak{m}(y) \tag{4.5}$$

$$\mathcal{T}(\rho) = \int_{\mathcal{K}/G} \int_{\mathcal{K}} |A_{xy}|^2\, f(x)^2\, f(y)^2\, d\mathfrak{m}(x)\, d\mathfrak{m}(y)\,, \tag{4.6}$$

where $f \in L^2(\mathcal{K}/G, d\mathfrak{m})$. Note that the integration range of the integrals in (4.5) and (4.6) is the non-compact set $\mathcal{K}$. The fact that $\mathcal{S}$ and $\mathcal{T}$ are bounded ensures that the integrals exist. However, it is not clear whether the functionals $\mathcal{S}$ and $\mathcal{T}$ are Fréchet differentiable (cf. Lemma 3.4). In order to ensure Fréchet differentiability, we impose the following condition.

Definition 4.2. *The minimizer ρ is called $\mathcal{T}$-**bounded** if*

$$\sup_{x \in \mathcal{K}/G} \int_{\mathcal{K}} |A_{xy}|^2\, f(y)^2\, d\mathfrak{m}(y) < \infty\,.$$

By straightforward adaptions of the methods used in Section 3 one derives the following result.

Theorem 4.3. *Suppose that ρ is a $\mathcal{T}$-bounded minimizer of the equivariant variational principle of Definition 4.1. Assume that $\mathcal{K}/G$ is compact and that*

$$C > C_{\min} := \inf \left\{ \mathcal{T}(\mu) \mid \mu \in \mathfrak{M}_G \textit{ satisfies (LC) } \right\}.$$

Then for a suitable choice of the Lagrange multipliers

$$\kappa \geq 0 \qquad \textit{and} \qquad \lambda_1, \ldots, \lambda_L \in \mathrm{L}(\mathbb{C}^k)\,,$$

the measure ρ is supported on the intersection of the level sets (1.17)*, where the function Φ_2 is given by* (1.13) *and*

$$\Phi_1(x) := -\sum_{l=1}^{L} \lambda_l\, h_l(x)\,.$$

In the case $\mathcal{T}(\rho) < C$, we may choose $\kappa = 0$.

Theorems 3.13, 3.14, 3.16 and 3.17 also hold in the equivariant setting for $\mathcal{T}$-bounded minimizers if we only replace the Hilbert space $L^2(\mathcal{K}, d\mathfrak{m})$ by $L^2(\mathcal{K}/G, d\mathfrak{m})$ and the integrals over $\mathcal{K}$ by integrals over $\mathcal{K}/G$.

References

[1] J. Dieudonné, *Treatise on Analysis. Vol. II*, Translated from the French by I. G. Macdonald. Pure and Applied Mathematics, Vol. 10-II, Academic Press, New York, 1970.

[2] C. Eckl, *Die Euler-Lagrange-Gleichungen von kausalen Variationsprinzipien*, Diplomarbeit, Universität Regensburg, 2010.

[3] F. Finster, *The Principle of the Fermionic Projector*, hep-th/0001048, hep-th/0202059, hep-th/0210121, AMS/IP Studies in Advanced Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2006.

[4] ———, *A variational principle in discrete space-time: Existence of minimizers*, arXiv:math-ph/0503069, Calc. Var. Partial Differential Equations **29** (2007), no. 4, 431–453.

[5] ———, *Causal variational principles on measure spaces*, arXiv:0811.2666 [math-ph], J. Reine Angew. Math. **646** (2010), 141–194.
[6] F. Finster and A. Grotz, *A Lorentzian quantum geometry*, arXiv:1107.2026 [math-ph] (2011).
[7] F. Finster, A. Grotz, and D. Schiefeneder, *Causal fermion systems: A quantum space-time emerging from an action principle*, arXiv:1102.2585 [math-ph], Quantum Field Theory and Gravity (F. Finster, O. Müller, M. Nardmann, J. Tolksdorf, and E. Zeidler, eds.), Birkhäuser Verlag, Basel, 2012, pp. 157–182.
[8] F. Finster and D. Schiefeneder, *On the support of minimizers of causal variational principles*, arXiv:1012.1589 [math-ph] (2010).
[9] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

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