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Codependent VAR Models and the Pseudo-Structural Form[*]

Carsten Trenkler[†], Enzo Weber[‡]

Abstract

This paper investigates whether codependence restrictions can be uniquely imposed on VAR and VEC models via the so-called pseudo-structural form used in the literature. Codependence of order q is given if a linear combination of autocorrelated variables eliminates the serial correlation after q lags. Importantly, maximum likelihood estimation and likelihood ratio testing are only possible if the codependence restrictions can be uniquely imposed. Applying the pseudo-structural form, our study reveals that this is not generally the case, but that unique imposition is guaranteed in several important special cases. Moreover, we discuss further issues, in particular upper bounds for the codependence order.

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1 Introduction

We investigate whether codependence restrictions can be uniquely imposed on vector autoregressive (VAR) and vector error correction models (VECMs) via their so-called pseudo-structural forms (PSFs). Based on Gourieroux & Peaucelle (1988, 1992), Vahid & Engle (1997) speak of codependence of order q if the (nonzero) impulse responses of a vector of variables are collinear after the first q periods. Thus, the according linear combination has a serial correlation structure that drops to zero after q lags. Codependence with $q = 0$ is equivalent to a serial correlation common feature (SCCF) as introduced by Engle & Kozicki (1993), where an SCCF itself is a special case of Engle & Kozicki (1993)'s more general concept of common features. Other related concepts are e.g. scalar component models (SCMs), see Tiao & Tsay (1989), or polynomial serial correlation common features (PSCCFs), see Cubadda & Hecq (2001).

We are in particular interested in imposing codependence restrictions on VAR models, which was first discussed by Vahid & Engle (1997) and by Vahid & Engle (1993) for SCCFs. If codependence can be uniquely imposed on a VAR, then efficient maximum likelihood (ML) estimation of the codependent VAR and corresponding likelihood ratio (LR) testing for codependence are possible. Moreover, the imposition of common cyclical features on a VAR can lead to higher accuracy of forecasts and of estimates of impulse-response functions as demonstrated by Vahid & Issler (2002). Therefore, it is of interest to analyze whether codependence can be uniquely imposed on a VAR.

Vahid & Engle (1993) have introduced the PSF in order to impose SCCF restrictions on a VECM. Schleicher (2007) extends this approach to the setup of codependence within a VECM. We will more generally study the usefulness of the PSF for analyzing codependence in stable finite-order VARs as well as in VEC models for variables that are integrated of order one, $I(1)$. The leading case will be a stable VAR of order p, VAR(p), since it is possible to transform (non-)cointegrated $I(1)$ systems into stable finite-order VAR processes, see e.g. Paruolo (2003) and Franchi & Paruolo (2011). We will show that the PSF does not generally allow to uniquely impose codependence restrictions on a VAR. To our knowledge, this fact has not been discussed in detail in the literature so far.

The structure of interest is a relationship which links the reduced form VAR parameters to the PSF parameters that represent the codependence restrictions, see equation (2.8) and thereafter. The motivation is to obtain the restricted reduced form parameters by transforming the PSF. This approach of rotating the system only works if the matrix of contemporaneous PSF parameters is of full rank. However, this property is not always guaranteed. A rank reduction occurs if certain additional restrictions on the moving average (MA) representation of the VAR process are present. Notwithstanding, we can separate several important cases where unique imposition of the relevant parameter restrictions is guaranteed. Moreover, we summarize upper bounds for the codependence order q within the VAR framework for various cases.

The plan for the paper is as follows. We first discuss imposing codependence restrictions for the setup of stable VAR models in the next section. Section 3 deals with codependence in case of nonstationary variables, in particular with VECMs. The last section concludes.

2 Stable VAR Models

2.1 Model Framework and Definitions

We assume that the n-dimensional time series x_t follows the reduced form VAR(p),

$$x_t = A_1 x_{t-1} + A_2 x_{t-2} + \cdots + A_p x_{t-p} + u_t, \quad t = 0, 1, 2, \ldots, \tag{2.1}$$

where A_j, $j = 1, 2, \ldots, p$, are $(n \times n)$ parameter matrices and the roots of

$$k(z) \equiv \det(A(z)) \equiv \det(I_n - A_1 z - \cdots - A_p z^p) \tag{2.2}$$

are outside the unit circle. The error terms u_t are i.i.d.$(0, \Omega)$ with positive definite covariance matrix Ω and finite fourth moments. To simplify the exposition we do not consider deterministic terms. They could be included by replacing x_t with $x_t + \mu_t$, where μ_t can contain for instance a linear trend, a constant term and seasonal dummy variables.

The initial values $x_0, x_{-1}, \ldots, x_{-p+1}$ can always be chosen such that x_t has the linear vector moving average (MA) representation $x_t = \Theta(L)u_t$ with $\Theta(L) = \sum_{i=0}^{\infty} \Theta_i L^i$, where L is the lag operator with $Lx_t = x_{t-1}$. Here, $\Theta_0 = I_n$ and $\Theta_i = \sum_{j=1}^{i} \Theta_{i-j} A_j$ for $i = 1, 2, \ldots$, with $A_j = 0$ for $j > p$, see Lütkepohl (2005).

Following Vahid & Engle (1997), codependence of order q is present in x_t if there exists a nonzero $n \times s_q$ matrix δ_0 with

$$\delta_0' \Theta_i = 0, \text{ for all } i > q \text{ and } \delta_0' \Theta_q \neq 0. \tag{2.3}$$

The s_q vectors represented by the columns of δ_0 are labeled as codependence vectors, a term introduced by Gourieroux & Peaucelle (1988, 1992). Hence, we have $\delta_0' x_t = \delta_0' \Theta(L) u_t =$

$\delta'(L)u_t$, where we assume that $\delta(z) = \Theta'(z)\delta_0 = \sum_{i=0}^{q}\delta_i z^i$ is a full column rank matrix polynomial of order q. A matrix polynomial $\delta(z) = \sum_{i=0}^{q}\delta_i z^i$, $\delta_i \in \mathbb{R}^{n\times s_q}, 0 < s_q < n$, is of full column rank if δ_0 and δ_q are of full column rank, see Franchi & Paruolo (2011). The full rank condition on δ_0 assures that the s_q codependence vectors in δ_0 are linearly independent, whereas the full rank condition on δ_q rules out that the codependence vectors can be combined such that a smaller order than q is obtained. The latter would imply that the codependence order q is not minimal.

Note that the linear combinations in $\delta_0'x_t$ can be regarded as special cases of a so-called scalar component model (SCM), see Vahid & Engle (1997). According to Tiao & Tsay (1989), a nonzero linear combination $v_0'x_t$ of an n-dimensional process x_t follows an SCM(p,q) structure if one can write

$$v_0'x_t + \sum_{j=1}^{p} v_j'x_{t-j} = v_0'u_t + \sum_{j=1}^{q} h_j'u_{t-j}$$

for a set of n-dimensional vectors $\{v_j\}_{j=1}^{p}$ and $\{h_j\}_{j=1}^{q}$ with $v_p \neq 0$ and $h_q \neq 0$. Thus, codependence of order q with respect to x_t results in an SCM$(0, q)$, where $q = 0$ represents the case of an SCCF.

In general, several codependence orders, say k, can be generated by linearly independent codependence vectors, see e.g. Schleicher (2007). In this case, we have k nonzero $n \times s_j$ matrices $\delta_{0,[j]}$ with $\delta_{0,[j]}'\Theta_i = 0$ for all $i > j$ and $\delta_{0,[j]}'\Theta_j \neq 0$, where $j = q_1, q_2, \ldots, q_k$ indicates the codependence order and s_j is the number of codependence vectors with an order of q_j. Each of the matrix polynomials $\delta_{[j]}(z) = \Theta(z)\delta_{0,[j]}$, that can be obtained analogously to $\delta(z)$ above, is assumed to be of full column rank. In total there are $s = s_{q_1} + s_{q_2} + \cdots + s_{q_k}$ codependence vectors, which we require to be linearly independent.

Analogously to the case of cointegration vectors, the linearly independent codependence vectors in $\delta_{0,[j]}$, $j = q_1, q_2, \ldots, q_k$, are only identified up to an invertible transformation. Therefore, an identification structure has to be imposed. However, one has to pay particular attention to the identification scheme applied to $D = (\delta_{0,[q_1]}, \delta_{0,[q_2]}, \ldots, \delta_{0,[q_k]})$ in order to maintain the composition of the codependence orders. The typical identification scheme $D^* = [I_s : D_{(n-s)}']'$, where D_{n-s} is an $(n-s) \times s$ matrix containing the free parameters, will generally produce a set of s linearly independent vectors generating codependence of the largest order involved. This is the case, because the columns in D^* are linear combinations of all columns of the unidentified matrix D, in general. Hence, the scheme in D^* only identifies the vector space with respect to the largest codependence order. As a consequence, the full column rank assumption imposed on the last parameter matrix of the polynomials $\delta_{[j]}(z),\ j = q_1, q_2, \ldots, q_k$, is not necessarily satisfied for a particular chosen identification structure. In the following subsections, we will comment on appropriate schemes for D for those cases for which uniquely imposing the codependence restrictions is possible.

2.2 Unique Imposition of Codependence Restrictions: Single Codependence Vector

In the following, we describe the restrictions the VAR parameters have to satisfy in case of codependence and discuss whether they can be uniquely imposed via the PSF of a VAR. To simplify the exposition we first focus on the case of a single codependence vector associated with codependence order q. Hence, δ_0 is an $n \times 1$ vector. In section 2.3, we discuss the general case of s codependence vectors that may generate $k \leq s$ different codependence orders.

Parameter restrictions and their imposition are conveniently discussed by adopting the framework of Schleicher (2007) to the case of VAR models. Schleicher's (2007) approach relies on the PSF of a state-space representation of the VECM. Here, we use the following state-space representation based on the companion form of the VAR.

$$\begin{aligned} x_t &= JX_t \\ X_t &= \boldsymbol{A}X_{t-1} + U_t, \end{aligned} \tag{2.4}$$

where

$$J = [I_n \ 0_{n\times n(p-1)}],$$

$$X_t = [x_t', x_{t-1}', \ldots, x_{t-p+1}']', \quad U_t = [u_t' \ 0_{n(p-1)\times 1}]',$$

and

$$\boldsymbol{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}$$

is an $np \times np$ companion matrix. Thus, $\boldsymbol{A}$ satisfies the companion form restrictions $R'\boldsymbol{A} = Q'$ with $R = [0_{n(p-1)\times n} : \ I_{n(p-1)}]'$ and $Q = [I_{n(p-1)} : \ 0_{n(p-1)\times n}]'$.

By iterative substitution we obtain

$$\begin{aligned} X_t &= \boldsymbol{A}X_{t-1} + U_t \\ &= \boldsymbol{A}^2X_{t-2} + U_t + \boldsymbol{A}U_{t-1} \\ &\vdots \\ &= \boldsymbol{A}^{q+1}X_{t-q-1} + \sum_{j=0}^{q} \boldsymbol{A}^jU_{t-j}. \end{aligned}$$

Hence, codependence of order q is given if

$$\delta_0' J\boldsymbol{A}^q \neq 0 \text{ and} \tag{2.5}$$

$$\delta_0' J\boldsymbol{A}^{q+1} = 0. \tag{2.6}$$

The equations (2.5)-(2.6) translate the restrictions on the MA parameter matrices given in (2.3) to (nonlinear) restrictions on the reduced form VAR parameters. Clearly, (2.6) implies that $\gamma_0'\boldsymbol{A}^i = 0$ for all $i > q+1$ with $\gamma_0 = J'\delta_0$. Thus, further restrictions on $\boldsymbol{A}^i$ for $i > q+1$ are not necessary. We define $\gamma_i' = \gamma_0'\boldsymbol{A}^i$, $i = 1, 2, \ldots$. Following Schleicher (2007), we can write the restrictions (2.5)-(2.6) as

$$\begin{aligned} \gamma_i'\boldsymbol{A} &= \gamma_{i+1}', \quad 0 \leq i < q-1, \\ \gamma_q'\boldsymbol{A} &= 0. \end{aligned} \tag{2.7}$$

Note that the vectors γ_i, $i = 0, 1, \ldots, q$, are linearly independent, see Schleicher (2007, Lemma 1). Thus, (2.7) translates the nonlinear codependence constraints on the VAR parameters into a set of linear restrictions regarding the companion form parameters in $\boldsymbol{A}$. We further define $\Upsilon = (\gamma_0, \gamma_1, \cdots, \gamma_q)$.

There exists an upper bound for the order q, say q_{max}, in the VAR framework. The upper bound q_{max} is the largest codependence order possible if y_t follows a finite-order VAR process. Such an upper bound is due to the (parameter) restrictions generated by the recursive relationship of the VAR and MA parameter matrices. For the algebraic details on this issue we refer to Franchi & Paruolo (2011). It follows from the results of Franchi & Paruolo (2011, Theorem 3.2) that q_{max} is equal to $(n-1)p$; see also Theorem 1 given in the next section and its proof in the Appendix. The upper bound q_{max} is clearly larger than the one that would be obtained if one applies the argumentation of Schleicher (2007, Lemma 1) to the setup of a VAR. The approach of Schleicher (2007) relies on the assumption that the columns of Υ, describing the codependence restrictions, and the columns of the matrix R, describing the companion restrictions, are jointly linearly independent.[1] However, codependence and companion restrictions can be linearly dependent as discussed below.

The fact that the sets of vectors describing the codependence restrictions and the vectors capturing the companion restrictions can be linearly dependent already indicates that uniquely imposing codependence restrictions may not be possible in general. In the following, we discuss this issue in more detail in relation to the PSF of the VAR.

To set up a PSF representation, let us summarize the restrictions from (2.7) by $\Upsilon'\boldsymbol{A} = \Upsilon^{0\prime}$ with $\Upsilon^0 = (\gamma_1, \gamma_2, \ldots, \gamma_q, 0_{np\times 1})$ being an $np \times (q+1)$-dimensional matrix. Remember that $\boldsymbol{A}$ satisfies $R'\boldsymbol{A} = Q'$. Moreover, one has to add, if necessary, equations representing free parameters in $\boldsymbol{A}$, see Schleicher (2007). These may be expressed by $R_P'\boldsymbol{A} = P'$, where R_P and P will be defined for appropriate cases later on. Defining $\Psi = [\Upsilon : R : R_P]'$ and $\Phi = [\Upsilon^0 : Q : R_P]'$, the system $\Psi\boldsymbol{A} = \Phi$ is obtained. In fact, it is this set of equations linking the reduced form parameters in $\boldsymbol{A}$ and the PSF parameters in Ψ and Φ that underlies the PSF presented later on. Obviously, the reduced form parameters, satisfying the codependence restrictions, can be uniquely recovered from the PSF parameters if Ψ is invertible. Put differently, if Ψ is invertible, then the parameter restrictions inducing codependence can be uniquely imposed on the reduced form VAR via the PSF. Moreover, note that we strive to uniquely obtain the restricted reduced form parameters from the PSF parameters and not the other way around. Usually, the latter constitutes the problem to be solved if structural models are of interest.

A unique and invertible Ψ requires that the columns of $M = [\Upsilon : R]$ are linearly independent. As pointed out above, this is not automatically guaranteed. In fact, if $q \geq n$, it is easily seen that the columns in M have to be linearly dependent such that the vectors γ_j, $j = 0, \ldots, q$, in Υ together with a subset of the companion restrictions generate some of the remaining companion restriction(s). Furthermore, linear dependence can also occur for $q < n$. We provide a numerical example for that case as well as for $q \geq n$ in Appendix A.

It is possible to characterize the linear dependence of the columns in M by restrictions on

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[1] The problem in Schleicher (2007, Lemma 1) is the following. Let $\theta_1, \theta_2, \ldots, \theta_k$ be a set of $1 \times n$ linearly independent vectors. Moreover, each of these vectors is (individually) linear independent from an $m \times n$ matrix M with rank $m < n$. In contrast to the assumption underlying Schleicher (2007, Lemma 1), this setup does not imply that the vectors $\theta_1, \theta_2, \ldots, \theta_k$ and the rows of M are jointly linearly independent.

the MA coefficient matrices. First, note that the last $(n-1)p$ rows of R represent an identity matrix and the first n rows of R are a zero matrix. Hence, the columns of Υ and R are linearly dependent if the columns of the first n rows of Υ linearly depend on each other. In other words, due to the structure of R one only needs to consider the first n rows of Υ to study linear dependence of the columns in M. Since the upper-left $n \times n$ block of $\boldsymbol{A}^i$ is equal to Θ_i, $i = 1, 2, \ldots$, and since $\gamma_i' = \gamma_0' \boldsymbol{A}^i$, the first n rows of Υ are given by $\Upsilon_\delta = (\delta_0, \delta_1, \ldots, \delta_q)$, where $\delta_i' = \delta_0' \Theta_i$, $i = 0, 1, \ldots, q$, are the $n \times 1$ parameter matrices of the matrix polynomial $\delta(z)$ defined above. Remember that we have $s_q = 1$ in the current setup such that $\delta(z)$ is actually a vector polynomial.

Thus, if $\delta_0' \Theta_i$, $i = 0, 1, \ldots, q$, linearly depend on each other, then the columns of M are linearly dependent. Hence, if the codependence vector δ_0 also imposes restrictions on the first q MA coefficient matrices described by linear dependence of $\delta_0' \Theta_i$, $i = 0, 1, \ldots, q$, then the companion and codependence restrictions linearly depend on each other. As a consequence, the matrix Ψ, as defined above, is not of full rank and therefore not invertible.

There emerge at least two questions. First, are there setups in which the columns of $M = [\Upsilon : R]$ cannot be linearly dependent? Second, if there is linear dependence, can one uniquely impose a dependence structure such that an adjusted full rank matrix Ψ can be obtained?

With respect to the first question it turns out that linear dependence is always ruled out for $q = 0$ and $q = 1$. In the former case of an SCCF, Υ consists only of γ_0 which is independent of R. In the latter case of codependence of order $q = 1$, we have $\Upsilon = [\gamma_0 : \gamma_1]'$, so that dependency between Υ and R would only be present if $\delta_0' A_1 = c\delta_0'$ for some $c \in \mathbb{R}$. Note that $\gamma_1 = [\delta_0' A_1 : \delta_0' A_2 : \cdots : \delta_0' A_p]'$ and $\gamma_1' \boldsymbol{A} = 0$. Using $\delta_0' A_1 = c\delta_0'$, the latter zero constraints can be expressed as $c\delta_0' + \delta_0' A_2 = c\delta_0' A_2 + \delta_0' A_3 = \cdots = c\delta_0' A_{p-1} + \delta_0' A_p = c\delta_0' A_p = 0$. From here it is easy to see that this leads to an SCCF setup, what contradicts the assumption $\gamma_0' \boldsymbol{A} \neq 0$ underlying codependence of order one. For $1 < q < n$ both scenarios with linear dependence and independence of the columns in M, i.e. of companion and codependence restrictions, are possible.

Let us assume that $q \leq 1$ such that the columns of M have to be linearly independent. Then, if $q + 1 < n$, the free parameters can be introduced by $R_P' \boldsymbol{A} = P'$ as indicated above, except for the case $n = 2$ and $q = 1$ where this is not necessary. The $np \times (n - q - 1)$ matrix R_P has to be designed such that Ψ is of full rank. This is always possible, but the choice of R_p depends on the normalization of the codependence vector. E.g. if the first element of δ_0 is normalized to one, then $R_p = [0_{(n-1)\times 1} : I_{n-1} : 0_{(n-1)\times n(p-1)}]'$ ensures full column rank of Ψ in case of an SCCF ($q = 0$). For $q = 1$, one of the second to n-th columns of the just defined R_p has to be set to zero. Full rank of Ψ is guaranteed for at least one of these choices because otherwise $\delta_1' = \delta_0' A_1$ and δ_0' are linearly dependent what is a contradiction.[2] Using the above definitions of Ψ and Φ one obtains the PSF representation for the state-space system (2.4)

$$\begin{aligned} x_t &= J X_t \\ \Psi X_t &= \Phi X_{t-1} + \Psi U_t, \end{aligned} \tag{2.8}$$

with Ψ being of full rank. Hence, the reduced form parameters are then obtained via $\boldsymbol{A} = \Psi^{-1}\Phi$ from the PSF parameters in Ψ and Φ. We also see that there are $n(p-1)+1$ restrictions underlying the PSF because $\boldsymbol{A}$ contains pn^2 reduced form parameters but there are only $[(n-1) + qnp + (n-q-1)np]$ PSF parameters in Υ, Υ_0 and P assuming that the first element in δ_0 is normalized to one.

Regarding the second question, it should be noted that linear dependence in the columns of M can be caused by different dependence structures in the columns of the first n rows of Υ. Hence, a matrix Ψ that is adjusted in order to eliminate a particular type of linear dependence structure can turn out to be of reduced rank if the specific dependence structure considered is incorrect. In other words, it is generally not possible to uniquely retrieve the (restricted) reduced form parameters from the PSF if the imposed dependence structure is wrong.

As mentioned above, for $1 < q < n$, setups with and without linear dependence of companion and codependence restrictions can occur. If one makes the explicit assumption that the columns in M are linearly independent, then the PSF uniquely delivers the appropriately restricted reduced form parameters. However, such an assumption is not harmless. If the columns of M are linear dependent, then M has no longer full column rank and Ψ in (2.8) is not invertible.

2.3 Unique Imposition of Codependence Restrictions: Multiple Codependence Vectors

For the general case of s codependence vectors with potentially $k \leq s$ different codependence orders the foregoing discussion applies accordingly. Let $\delta_{0,1}, \delta_{0,2}, \ldots, \delta_{0,s}$ be the s codependence vectors associated with the codependence orders q_j, $j = 1, 2, \ldots, s$. Hence, several

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[2] In case of inadequate normalization numerical problems during optimization are likely to occur. However, there always exists at least one appropriate variant.

codependence vectors may relate to the same codependence order. Accordingly, we do not summarize vectors with the same order in one matrix as done in section 2.1 when introducing the setup of k different codependence orders. This is done for notational convenience.

Each of the codependence vectors will satisfy a corresponding version of (2.5) and (2.6) and induces a corresponding set of restrictions as in (2.7). Regarding the latter, we define $\gamma_{0,j} = J'\delta_{0,j}$ and $\gamma'_{i,j} = \gamma'_{0,j}\boldsymbol{A}^i$, $j = 1, 2, \ldots, s$. Moreover, we now use $\Upsilon = [\Upsilon_1 : \Upsilon_2 : \cdots : \Upsilon_s]$, where $\Upsilon_j = (\gamma_{0,j}, \gamma_{1,j}, \cdots, \gamma_{q_j,j})$, $j = 1, 2, \ldots, s$.

Although $\gamma_{0,1}, \gamma_{0,2}, \ldots, \gamma_{0,s}$ do not linear depend on each other and the columns in Υ_j are linearly independent for fixed j as well, the columns in Υ are not generally linear independent. Note that this is in contrast to the claim in Schleicher (2007, Proof of Theorem 1) for the case of a VECM. Numerical examples with linear dependence can be easily found. Thus, in contrast to the case of a single codependence vector, a reduced column rank structure in $M = [\Upsilon : R]$ can also occur even without considering the companion restrictions captured by the matrix R. However, if the columns of Υ are linearly dependent, then also the upper parts of the columns, made of the first n rows, linearly depend on each other. Analogously to the case $s = 1$, this results in linear dependence of companion and codependence restrictions. The first n rows of the columns in Υ can be expressed as $\delta'_{0,j}\Theta_i$, $j = 1, 2, \ldots, s$ and $i = 0, 1, \ldots, q_j$. Hence, if $\delta'_{0,j}\Theta_i$, $j = 1, 2, \ldots, s$ and $i = 0, 1, \ldots, q_j$, are linearly dependent, then codependence and companion restrictions linearly depend on each other as claimed. Accordingly, the matrix $\Psi = [\Upsilon : R : R_P]'$ is not of full column rank and the corresponding PSF does not uniquely deliver the restricted reduced form parameters.

It is interesting to highlight one potential setup implied by linear dependence of $\delta'_{0,j}\Theta_i$, $j = 1, 2, \ldots, s$ and $i = 0, 1, \ldots, q_j$. If two codependence vectors, say $\delta_{0,1}$ and $\delta_{0,2}$, generate the same codependence order q, then linear dependence of $\delta'_{0,1}\Theta_q$ and $\delta'_{0,2}\Theta_q$ means that a linear combination of $\delta_{0,1}$ and $\delta_{0,2}$ results in codependence with an order of at most $q-1$. This case was ruled out by the assumption that the corresponding matrix polynomial $\delta_{[q]}(z)$ is of full column rank. Hence, a PSF with an invertible matrix Ψ, what, in fact, excludes linear dependence of $\delta'_{0,j}\Theta_i$, $j = 1, 2, \ldots, s$ and $i = 0, 1, \ldots, q_j$, assures the full column rank assumption regarding $\delta_{[q_j]}(z)$, $j = 1, 2, \ldots, k$.

Similar to the single codependence vector case, several dependence structures in M may exist for a particular combination of codependence orders. Hence, it is in general not possible to impose a unique dependence structure for a particular set of codependence orders. However, there are a two setups for which linear independence of the columns in M is guaranteed such that the restricted reduced form parameters are uniquely obtained from the PSF model. The first setup occurs if all $s \leq n$ codependence vectors satisfy an SCCF. In this case, the s columns of the matrix Υ are equal to $\gamma_{0,1}, \gamma_{0,2}, \ldots \gamma_{0,s}$, respectively, which are jointly independent from the columns in R. Therefore, the columns in M have to be linearly independent.

The second setup is described by one codependence vector, say the first one, generating an order of $q_1 = 1$, while the other $s - 1 < n - 1$ vectors induce SCCFs. The argument runs as for the case $s = 1$ using the additional fact that $\gamma_{0,j}' \boldsymbol{A} = 0$ for $j = 2, 3, \ldots, s$. This is the only setup with codependence of order one for which the restricted reduced form parameters can be uniquely retrieved. Consider, e.g., the case $s = 2$ with $q_1 = q_2 = 1$ and, thus, $\Upsilon = (\gamma_{0,1}, \gamma_{1,1}, \gamma_{0,2}, \gamma_{1,2})$. Define $\delta_{1,1}' = \delta_{0,1}' A_1 = \delta_{0,1}' \Theta_1$ and $\delta_{1,2}' = \delta_{0,2}' A_1 = \delta_{0,2}' \Theta_1$ as the first n rows of $\gamma_{1,1}$ and $\gamma_{1,2}$, respectively. Then, the linear combination $\delta_{1,2}' = c_1 \delta_{0,1}' + c_2 \delta_{0,2}' + c_3 \delta_{1,1}'$ can exist with a nonzero vector $c = (c_1, c_2, c_3)$ so that the columns in $M = [\Upsilon : R]$ are linearly dependent. The situation does not change if a codependence order of one is jointly considered with orders larger than one.

In order to determine the number of restrictions underlying the VARs for which codependence can be uniquely imposed, an appropriate identification scheme has to be applied to $D = (\delta_{0,1}, \delta_{0,2}, \ldots, \delta_{0,s})$. If only SCCFs or a single codependence vector associated with order one are considered, then the identifying structure $D^* = [I_s : D_{(n-s)}']'$ can be used. In contrast to the general setup discussed in subsection 2.1, only vectors related to the same codependence order, either $q = 0$ or $q = 1$, are involved. Therefore, no linear combinations of vectors of different codependence orders occur so that the full column rank assumption on the relevant finite-order matrix polynomial is satisfied. Using the definition of D^*, i.e. the corresponding identified versions of $\delta_{0,j}$, say $\delta_{0,j}^*$, $j = 1, 2, \ldots, s$, one obtains the identified vectors $\gamma_{0,j}^* = J' \delta_{0,j}^*$.

If s SCCFs are considered, then $s(n - s)$ parameters are contained in the identified (codependence) vectors $\gamma_{0,1}^*, \gamma_{0,2}^*, \ldots, \gamma_{0,s}^*$. Moreover, there are $np(n - s)$ free parameters in P such that the PSF has $n^2 p - s(n(p - 1) + s)$ parameters. Setting $s = 1$, the same number of PSF parameters is obtained in the case of a single codependence vector with $q = 1$: $n - 1$ parameters in $\gamma_{0,1}^*$, np parameters in $\gamma_{1,1}^*$ due to codependence of order one, and $np(n - 2)$ parameters in P. By contrast, the reduced form has $n^2 p$ parameters. Therefore, $s(n(p - 1) + s)$ restrictions underlie the PSF of a codependent VAR with $1 \leq s \leq n$ SCCFs or $s = 1$ vector associated with codependence order one.

If $s_0 = s - 1$ SCCF vectors $\delta_{0,1}, \delta_{0,2}, \ldots, \delta_{0,s_0}$ are combined with the codependence vector

$\delta_{0,s}$ of order one, then the identification scheme

$$D^{**} = \begin{bmatrix} I_{s_0} & 0_{s_0 \times 1} \\ D_0 & (1 : D_1')' \end{bmatrix}$$

is sufficient to ensure uniqueness, where D_0 and D_1 are $(n - s_0) \times s_0$ and $(n - s_0 - 1) \times 1$ matrices of free parameters, respectively. In fact, the first s_0 columns only have to be identified with respect to the SCCF vectors, and the last column is then chosen to be linearly independent of the first s_0 columns. D^{**} contains $(n-s_0)(s_0+1)-1 = s(n-s)+(s-1)$ parameters, $s-1$ more than in D^* above, where only a single codependence order is involved. Therefore, the PSF of a codependent VAR with $s-1$ SCCF vectors and one vector associated with codependence order one is characterized by $s(n(p-1)+s)-(s-1)$ restrictions.

To sum up, a PSF that uniquely provides the restricted reduced form parameters can only be obtained if companion and codependence restrictions are linearly independent. The codependence vectors impose additional restrictions on the MA coefficient matrices in case of linear dependence of companion and codependence restrictions. Algebraically, linear dependence of companion and codependence restrictions results in linear dependence among the columns of the first n rows of the matrix M of which the entries are nonlinear functions of the VAR parameters. Hence, such dependence introduces nonlinear constraints on the companion matrix. Accordingly, the advantage of the companion form, which lies in translating the nonlinear VAR parameter restrictions implied by codependence into linear restrictions on the companion matrix, disappears. Therefore, it is not surprising that uniqueness cannot be achieved via the pseudo-structural representation if companion and codependence restrictions are linearly dependent. In fact, the set of models where unique imposition is possible is rather limited. Only setups with SCCFs ($q = 0$), codependence of order one generated by a single codependence vector, or a combination of these two always lead to a unique set of restricted reduced form parameters. In case of SCCF, the restrictions can be directly imposed on the VAR parameters and are, therefore, linear. Accordingly, a unique imposition of the restrictions is easily achieved.

Nevertheless, from an applied point of view, the VAR framework is of limited use for analyzing general codependence restrictions since uniqueness is rarely given. Accordingly, the scope of ML estimation of codependent VARs and conventional LR testing for codependence is narrowed to a few, albeit potentially important, special cases. These facts shall inform future work in this research area.

3 VEC Models

We now assume that x_t is $I(1)$ and potentially cointegrated. Defining $\Pi = -(I_n - A_1 - \cdots - A_p)$ and $\Gamma_j = -(A_{j+1} + \cdots + A_p)$, $j = 1, \ldots, p-1$, we can re-write (2.1) in the vector error correction form

$$\Delta x_t = \Pi x_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta x_{t-j} + u_t, \quad t = 1, 2, \ldots. \tag{3.1}$$

The relationship of the VAR and VECM representations can be compactly described by $A(z) = I_n - A_1 z - \cdots - A_p z^p = I_n \Delta - \Pi z - \Gamma_1 \Delta z - \cdots - \Gamma_{p-1} \Delta z^{p-1}$. The error term assumptions of section 2 still apply. We make the following new assumption, see e.g. Johansen (1995).

Assumption 1.

(a) The roots of $k(z)$ in (2.2) are either $|z| > 1$ or $z = 1$.

(b) The matrix Π has reduced rank $r < n$, i.e. the matrix Π can be written as $\Pi = \alpha\beta'$, where α and β are $n \times r$ matrices with $\text{rk}(\alpha) = \text{rk}(\beta) = r$.

(c) The matrix $\alpha_\perp' \Gamma \beta_\perp$ has full rank, where $\Gamma = I_n - \sum_{j=1}^{p-1} \Gamma_j$ and where $\alpha_\perp$ and $\beta_\perp$ are the orthogonal complements to α and β.

Given Assumption 1, x_t is $I(1)$ and the cointegrating rank is equal to r. Hence, we obtain the Granger representation, see Johansen (1995, Theorem 4.2),

$$x_t = C \sum_{s=1}^{t} u_s + C(L) u_t + a_0,$$

where $C = \beta_\perp (\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp'$ and a_0 is the initial condition.

If the variables are not cointegrated, i.e. if $r = 0$, then (3.1) reduces to a VAR($p-1$) for Δx_t and $\alpha_\perp = \beta_\perp = I_n$. Hence, codependence can be analyzed in terms of Δx_t using the VAR($p-1$) representation. Thus, the definition in (2.3) for codependence of order q and the results on unique imposition of the codependence parameter restrictions obtained in the previous section apply accordingly. Note in this respect that $s(n(p-2)+s)$ or $s(n(p-2)+s)-(s-1)$ restrictions underlie the codependent VAR for Δx_t depending on whether only vectors associated with the same codependence order are considered or whether SCCF vectors are combined with a vector generating codependence of order one. Codependence in terms of the first differences of $I(1)$ variables has been studied e.g. in Vahid & Engle (1997).

If the variables in x_t are cointegrated with $r > 0$, then the framework of Paruolo (2003) and Franchi & Paruolo (2011) can be applied. They show that $Y_t \equiv (x_t'\beta : \Delta x_t'\beta_\perp)'$ follows the stable VAR(p) process $Y_t = \tilde{A}_1 Y_{t-1} + \tilde{A}_2 Y_{t-2} + \cdots + \tilde{A}_p Y_{t-p} + u_t^o$, with $u_t^o = (\beta : \beta_\perp)' u_t$, if Assumption 1 holds.[3] The VAR parameters in $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p$ are nonlinear functions of the VECM parameters in (3.1) as well as $\beta_\perp$, see e.g. Paruolo (2003, Appendix A).

Paruolo (2003) considers SCCFs in Y_t and provides an extensive discussion on ML inference regarding the corresponding model setup. Franchi & Paruolo (2011) characterize codependence structures with respect to Y_t. Y_t is codependent of order q if there exists a nonzero $(n \times s_q)$ matrix $\delta \equiv (\delta_{(0)}' : \delta_{(1)})'$ with $\delta' Y_t = \delta'(L) u_t^o$ and $\delta(z) = \sum_{i=0}^{q} \delta_i z^i$, $\delta_i \in \mathbb{R}^{n \times s_q}, 0 < s_q < n$, being again a full column rank matrix polynomial of order q.

As pointed out by Paruolo (2003), the matrix δ may only select elements either from $\beta' x_t$ $(\delta_{(0)} \neq 0, \delta_{(1)} = 0)$ or from $\beta_\perp' \Delta x_t$ $(\delta_{(0)} = 0, \delta_{(1)} \neq 0)$. The latter case refers to codependence in Δx_t generated by codependence vectors of the form $\delta_{\beta_\perp} = \beta_\perp \delta_{(1)}$ that are orthogonal to the cointegration matrix β. This is exactly the setup studied by Schleicher (2007) and Vahid & Engle (1993). The former case of $\delta_{(0)} \neq 0$ and $\delta_{(1)} = 0$ has been discussed in Paruolo (2003) and studied by Trenkler & Weber (2013). For the case of a single cointegration vector ($r = 1$), $\delta_{(0)}$ is a scalar and codependence is directly linked to the cointegration relation $\beta' x_t$. Thus, the cointegration vector β represents a codependence vector. If $\beta' x_t$ is codependent of order q, i.e. if it has an SCM$(0, q)$ representation, then a one-time shock to the cointegration error has no effect after q periods. Hence, codependence in $\beta' x_t$ refers to the adjustment dynamics of the system towards the cointegration equilibrium. The latter interpretation may also be applied in case of $r > 1$ since $\delta_\beta = \beta \delta_{(0)}$ also represents a set of cointegration vectors. Whether (some of) the considered cointegration vectors or linear combinations of the cointegration matrix generate codependence of a certain order q is a matter of the identification scheme applied to the cointegration matrix.

Since Y_t has a stable VAR(p) representation, one can again apply the framework of the previous section, now with respect to Y_t, in order to define and analyze codependence for cointegrated VECMs. Accordingly, only SCCF setups and the case of a single codependence vector associated with $q = 1$ can be uniquely handled via the PSF. The corresponding codependent VECMs are characterized by $s(n(p-2) + r + s)$ restrictions for setups with a single codependence order of $q = 0$ or $q = 1$, or by $s(n(p-2) + r + s) - (s - 1)$ restrictions in case

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[3] The matrix $\beta_\perp$ can be replaced by an arbitrary matrix $c_\perp$ of the same dimension as $\beta_\perp$, such that $c_\perp' \beta_\perp$ is square and of full rank, see Franchi & Paruolo (2011).

of joint consideration of codependence vectors with $q = 0$ and $q = 1$. To see this, note first that $\tilde{A}_p = (\tilde{A}_{p,0} : 0_{n\times(n-r)})$, where $\tilde{A}_p$ is partitioned according to the two components in Y_t, see Franchi & Paruolo (2011, Proposition 7.1). Hence, $\beta'_\perp \Delta x_t$ enters the process only with up to $p-1$ lags, i.e. the coefficients regarding $\beta'_\perp \Delta x_{t-p}$ are zero in A_p. Accordingly, the reduced form has $n(n-r)$ parameters less compared to an unrestricted VAR(p) model. By contrast, the number of parameters of the PSF is only reduced by $(n-s)(n-r)$ given the PSF representation of the previous section. As a consequence, one obtains the aforementioned numbers of restrictions.

If the focus is on codependence of order zero, one can use the framework of Paruolo (2003) to test for SCCFs and estimate the weights in the linear combinations of Y_t that generate the SCCFs. This can be conveniently done using reduced rank techniques. Furthermore, it is possible to test restrictions on δ, e.g. $\delta_{(0)} = 0$ or $\delta_{(1)} = 0$. Note that replacing β by a superconsistent estimate does not change the asymptotic properties of the aforementioned inference procedures, see Paruolo (2003). For the case of $q = 1$ one has to rely on nonlinear ML inference since the underlying restrictions are no longer linear in the VAR parameters.

Finally, we present in Theorem 1 the upper bounds for the codependence order q in relation to the VAR for Y_t.[4] We also consider the special cases of $r = n$ and $r = 0$ that refer to the setups of section 2 and the non-cointegrated VAR, respectively. To our knowledge, most of the upper bounds given in Theorem 1 have not been explicitly stated in the literature. A proof of Theorem 1 can be found in the Appendix B.

Theorem 1. Let x_t be an n-dimensional VAR(p) process as generated by (2.1) for which Assumption 1 holds such that $Y_t \equiv (x'_t\beta : \Delta x'_t\beta_\perp)'$ follows a stable VAR(p). Moreover, it is assumed that $\beta = 0$ and $\beta_\perp = I_n$ if $r = 0$ and that $\beta = I_n$ and $\beta_\perp = 0$ if $r = n$. Then, (i) the maximum codependence order with respect to linear combinations of Y_t is given by $q_{max} = (n-1)p-(n-r-1)$ for $r < n$ and $q_{max} = (n-1)p$ for $r = n$; (ii) the maximum codependence order with respect to linear combinations of $\beta' x_t$ is given by $q^{\beta}_{max} = (n-1)p-(n-r)$ for $r > 0$; (iii) the maximum codependence order with respect to linear combinations of $\beta'_\perp x_t$ is given by $q^{\beta_\perp}_{max} = (n-1)p-(n-r-1)$ for $r < n$. ■

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[4] Note that the upper bound does not refer to the maximum of the sum of orders generated by linearly independent codependence vectors. Instead, it states the maximum order which a single codependence order can produce if the data follow a finite-order VAR process.

4 Conclusions

This paper investigated whether codependence restrictions can be uniquely imposed on VARs and VECMs via their pseudo-structural form. Practical relevance comes from the fact that ML estimation and LR testing are only applicable if such a unique imposition is possible.

We have shown that the restricted reduced form VAR parameters cannot be uniquely obtained from the pseudo-structural form in general. We applied a linear representation of the codependence restrictions based on a companion form of the VAR. However, it was clarified that the vectors describing the codependence restrictions and the vectors capturing the restrictions on the companion matrix can be linearly dependent. This fact impairs a unique imposition of codependence restrictions.

Importantly, linear dependence is always ruled out for codependence orders zero (i.e., SCCF) and one. For models featuring multiple codependence vectors we showed that this holds only if all vectors generate SCCFs or at most one of them generates codependence of order one. Moreover, we provided upper bounds for the order of codependence both in VAR and VEC models. These facts should be recognized in future applied and theoretical work on codependence. One such example is given by Trenkler & Weber (2013) who discuss testing issues and apply the concept to US short-term interest rate data. To be precise, a GMM approach to test for general forms of codependence is presented. This approach avoids the imposition of codependence restrictions on VAR models but rather uses these models for deriving a finite set of zero correlation conditions. It should be noted in this respect that the limitations we have discussed in the current paper do not refer to the concept of codependence itself. In fact, it is a useful framework for analyzing the dynamics of multiple time series.

Appendix A: Numerical Examples

Consider a three-dimensional version of the VAR in (2.1) of order three with

$$A_1 = \begin{pmatrix} 0.00 & 0.50 & 0.00 \\ 0.00 & 0.40 & 0.00 \\ a_{31,1} & 0.00 & a_{33,1} \end{pmatrix}, A_2 = \begin{pmatrix} 0.00 & 0.36 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}, \text{ and } A_3 = \begin{pmatrix} 0.00 & a_{12,3} & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}.$$

We assume $a_{31,1} \neq 0$ and $a_{33,1} \neq 0$. If $a_{12,3} = -0.16$, then $\delta_0 = (1\ {-1}\ 0)'$ is a codependence vector which generates codependence of order $q = 2$ with respect to y_t. We have $\gamma_0 = (1\ {-1}\ 0\ 0 \cdots 0)'$, $\gamma_1 = \gamma_0' \boldsymbol{A} = (0\ 0.1\ 0\ 0\ 0.36\ 0\ 0\ -0.16\ 0)'$, and $\gamma_2 = \gamma_1' \boldsymbol{A} = (0\ 0.4\ 0\ 0$

$-0.16\ 0\ 0\ 0\ 0)'$ with $\gamma_2'\boldsymbol{A} = \mathbf{0}$. Here, $\boldsymbol{A}$ is the corresponding companion matrix of the three-dimensional VAR(3). The vectors consisting of the first three elements of γ_1 and γ_2 represent the vectors labeled by δ_1 and δ_2 in section 2.2. Obviously, δ_1 and δ_2 are linearly dependent meaning that the codependence vector δ_0 imposes additional constraints on the first two MA parameter matrices. Hence, the current setup represents an example for $q = 2 < 3 = n$ with linear dependence of codependence and companion restrictions.

Let us extend the previous VAR to the lag order of four with

$$A_4 = \begin{pmatrix} 0.00 & -0.12 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}$$

and set $a_{12,3} = 0.14$. Then, the codependence vector $\delta_0 = (1\ \ -1\ 0)'$ is associated with a codependence order $q = 3$. We obtain $\gamma_0 = (1\ -1\ 0\ 0 \cdots 0)'$, $\gamma_1 = \gamma_0'\boldsymbol{A} = (0\ 0.1\ 0\ 0\ 0.36\ 0\ 0\ 0.14\ 0\ 0\ -0.12\ 0)'$, and $\gamma_2 = \gamma_1'\boldsymbol{A} = (0\ 0.4\ 0\ 0\ 0.14\ 0\ 0\ -0.12\ 0 \cdots\ 0)'$, $\gamma_3 = \gamma_2'\boldsymbol{A} = (0\ 0.3\ 0\ 0\ -0.12\ 0 \cdots\ 0)'$ with $\gamma_3'\boldsymbol{A} = \mathbf{0}$. Now, $\boldsymbol{A}$ is the companion matrix related to the three-dimensional VAR(4). Obviously, the vectors δ_1, δ_2, and δ_3 linearly depend on each other such that we have an example for the case of $q \geq n$ with linear dependence of codependence and companion restrictions.

We make two final remarks. First, the considered VAR processes are stable if $|a_{33,1}| < 1$. Second, the values of $a_{31,1}$ and $a_{33,1}$ do not affect the properties of codependence and linear dependence with respect to the considered VAR models.

Appendix B: Proof of Theorem 1

Let us start by considering a stable n-dimensional VAR process z_t of order p,

$$A(L)z_t = u_t,$$

where u_t satisfies the same assumptions as in section 2. The autoregressive matrix polynomial is given by $A(z) = I_n - A_1 z - A_2 z^2 - \cdots - A_p z^p$. Then, let $k(z) \equiv \text{det}A(z)$ and $K(z) \equiv \text{adj}A(z)$ be respectively the characteristic and adjoint polynomials with respect to $A(z)$. As noted by Franchi & Paruolo (2011, Section 2), $k(z)$ and $K(z)$ may have common factors such that one can obtain so-called minimal characteristic and adjoint polynomials $g(z)$ and $G(z)$, respectively.

Now, let $\delta' z_t = \delta'(L)u_t$ where $\delta(L)$ is a full rank matrix polynomial of order q such that z_t is codependent of order q. Franchi & Paruolo (2011, Theorem 4.2) show that $0 \leq q \leq d_G - d_g$, where d_G and d_g are the orders of $G(z)$ and $g(z)$, respectively. Since the maximum value for d_G is $n(p-1)$ and the minimum value for d_g is zero, one obtains $q_{max} = n(p-1)$ as upper bound for the codependence order q. This confirms part (i) of Theorem 1 for $r = n$ since Y_t reduces to x_t, which is a stable VAR(p) process in case of $r = n$. Note that the upper bound $q_{max} = n(p-1)$ can only be achieved if $k(z)$ and $K(z)$ have no common factors, i.e. if $k(z) = g(z)$ and $K(z) = G(z)$, and if $A(z)$ is unimodular, i.e. $d_g = 0$.

As pointed out in section 3, Franchi & Paruolo (2011, Proposition 7.1) showed that $Y_t \equiv (x_t'\beta : \Delta x_t'\beta_\perp)'$ follows a stable VAR(p) process if Assumption 1 holds for x_t. Let the corresponding autoregressive matrix polynomial be $\tilde{A}(z) = I_n - \tilde{A}_1 z - \tilde{A}_2 z^2 - \cdots - \tilde{A}_p z^p$ and let $\tilde{G}(z)$ and $\tilde{g}(z)$ be the minimal characteristic and adjoint polynomials with respect to $\tilde{A}(z)$. As also mentioned in section 3, $\tilde{A}_p = (\tilde{A}_{p,0} : 0_{n\times(n-r)})$ such that $\beta_\perp'\Delta x_t$ enters the process only with up to $p-1$ lags. This fact has an impact on the maximum polynomial orders of the first r and last $n-r$ rows of $\tilde{G}(z)$. Let these two maximum orders be labeled as $d^r_{\tilde{G},\max}$ and $d^{n-r}_{\tilde{G},\max}$, respectively. As can be easily verified, $d^r_{G,\max} = (n-1)p - (n-r)$, assuming $r > 0$, and $d^{n-r}_{G,\max} = (n-1)p - (n-r-1)$, assuming $r < n$.

Since Y_t is a stable VAR process, we can apply the inequality $0 \leq q \leq d_{\tilde{G}} - d_{\tilde{g}}$, where $\tilde{d}_G$ and $\tilde{d}_g$ are the orders of $\tilde{G}(z)$ and $\tilde{g}(z)$, respectively. If only linear combinations of $\beta' x_t$, i.e. the first r rows of Y_t, are of interest, then it suffices to consider the maximum order of the first r rows of $\tilde{G}(z)$, i.e. $d^r_{\tilde{G},\max}$, in order to determine the maximum codependence order. Thus, the maximum codependence order with respect to linear combinations of $\beta' x_t$ is given by $q^{\beta}_{max} = (n-1)p - (n-r)$ assuming that $r > 0$. This proves part (ii) of Theorem 1. Similarly, we obtain $q^{\beta_\perp}_{max} = (n-1)p - (n-r-1)$ as the maximum codependence order for linear combinations of $\beta_\perp' x_t$ assuming that $r < n$, which shows part (iii). Since general linear combinations of Y_t may involve $\beta_\perp' x_t$, i.e. may include some of the last $n-r$ rows of Y_t, we have $q_{max} = (n-1)p - (n-r-1)$ as maximal codependence order for linear combinations of Y_t for $r < n$. This proves part (i) for $r < n$ and completes the proof. ■

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