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Justification of per-unit risk capital allocation in portfolio credit risk models

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Abstract

Risk capital allocation is based on the assumption that the risk of a homogeneous portfolio is scaled up and down with the portfolio size. In this article we show that this assumption is true for large portfolios, but has to be revised for small ones. On basis of numerical examples we calculate the minimum portfolio size that is necessary to limit the error of gradient risk capital allocation and the resulting error in a portfolio optimization algorithm or pricing strategy. We show the dependence of this minimum portfolio size on different parameters like the probability of default and on the credit risk model that is used.

Keywords: risk capital, captial allocation, risk contribution, per-unit risk, portfolio credit risk

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1. Introduction

Calculation and allocation of risk capital is one of the major tasks of risk management in banks. Risk capital became an even more important issue during the financial crisis. Risk management departments continue to grow and to gather more influence on business decisions. When talking about risk capital, one has to differentiate between regulatory capital and economic risk capital. Regulatory capital is necessary to fulfill regulatory requirements and is meant to ensure that the bank is able to meet all its obligations. Economic risk capital is calculated by using a more flexible internal model that does not underly regulatory rules and can therefore represent bank's specifics in a more accurate way. It is often used to steer managerial decisions. In this paper we will restrict ourselves to economic risk capital because we want to focus on internal portfolio steering. From a risk point of view, a business decision like the calculation of interest rates or an investment in a new obligor or an increase or decrease of existing portfolio segments, should be based on return per risk. A large number of performance figures (ratios, quantities, trademarks, etc.) have been produced and discussed over the last years and decades, the most well-known ones being RORAC (return on risk adjusted capital, also known or slightly differently defined as RAROC or RARORAC) and EVA® (Economic Value Added). Based on these measures, several methods have been developed to optimize portfolio compositions. The basic idea of portfolio optimization in general is that a subportfolio or asset class with an above-average performance should be increased, while other subportfolio sizes should be reduced. All calculation methods or algorithms assume and imply that by increasing one homogeneous asset class the risk is simply scaled up. In asset classes with perfect dependence this becomes obvious. Consider e.g., a subportfolio consisting of 100 shares of the same company. Obviously, risk is scaled up linearly with the subportfolio size. But there is no indication of whether this assumption is still justifiable in a portfolio of debt instruments in which each obligor has individual characteristics and is not perfectly correlated to the other obligors.

To determine the return per risk of an asset class or a single obligor, it is necessary to allocate risk or respectively risk capital to the asset class or obligor. There are three options to determine risk contributions: stand-alone contribution, incremental contribution or marginal contribution (Mausser and Rosen (2007)). Stand-alone contribution calculates the risk of one asset class without considering the rest of the portfolio. Diversification effects are ignored. Incremental contribution is calculated by comparing the risk of the total portfolio with the risk of the portfolio without one asset class. Incremental risk then becomes the resulting delta. This approach is useful for portfolios consisting of few large deals. Marginal risk contribution is calculated through an allocation principle like gradient allocation. There the allocated risk contribution of an asset class or subportfolio is based on the derivative of the risk measure with respect to the number of obligors. Tasche (2004a) and Tasche (2008) demonstrate that the gradient allocation (also called Euler allocation) is a tool well-suited to measuring the risk of single asset classes or single obligors in portfolios with homogeneous asset classes. The axiomatic framework behind capital allocation principles is provided from a mathematical perspective in Kalkbrener (2005) and from another viewpoint by Tasche (2004a) and Buch and Dorfleitner (2008), while Merton and Perold (1993) and Stoughton and Zechner (2007) explain the principles with a focus on a more economical outlook. Each allocation method is connected with a risk measure that is typically chosen coherently as introduced in Artzner et al. (1999) and Acerbi (2002), e.g., expected shortfall. Nevertheless, value-at-risk (VaR) is used in many cases, although it is not coherent, because it is the most common risk measure in practice. Various literary contributions contain several techniques on the application of capital allocation to credit portfolios. The target here is to develop an analytical formula that calculates the risk contribution of one subportfolio. Mausser and Rosen (2007) give an overview of calculation methods for risk contributions based on gradient allocation in credit portfolios. Gouriéroux et al. (2003) introduce kernel estimators to estimate value-at-risk, which Tasche (2009) uses to deduce a formula for value-at-risk contributions in credit portfolios. Kalkbrener et al. (2004) consider gradient allocation specifically for expected shortfall. The results are transferred to the specific situation of the CreditRisk+ model by Tasche (2004b).

Based on gradient allocation, Buch et al. (2011) introduce an algorithm that allows the calculation of the optimal amount of capital that should be invested to each subportfolio. Since Buch et al. (2011) base the risk measurement on gradient allocation, the profit or loss fluctuations of subportfolios are supposed to have a linear structure. By doing that, the authors implicitly assume a specific loss distribution per obligor that is multiplied with the number of obligors in the subportfolio. Application of the algorithm leads to the optimum amount of businesses per subportfolio, optimal in a sense of the maximization of RORAC. The same idea is the basis for portfolio optimization following Rockafellar and Uryasev (2000). The authors state that the portfolio consists of different asset classes and that the complete portfolio can be composed of these asset classes by giving each asset class a weight. They optimize conditional value-at-risk by changing these weights. This approach again assumes that risk scales linearly with the portfolio size. The approach is extended by Krokhmal et al. (2001) to an approach with conditional value-at-risk constraints. Hallerbach (2004) develops a portfolio optimization approach via RAROC, too. The author adds the constraint of limited capital or budget for business ventures and optimizes return. Thus, he reaches an optimal portfolio composition with given side conditions. Finally, Gaivoronski and Pflug (2005) develop a numerical approach of value-at-risk optimization with given return random variables per asset class.

In this paper we apply gradient-based capital allocation to loan portfolios and analyze the conditions under which this approach is justifiable. Credit portfolios are typically characterized by the individuality of the single deals or obligors. For each obligor default is a binary event. The loss distribution of the complete subportfolio can be calculated by weighting the loss with its probability and exposure. The loss distribution of the complete portfolio then differs from the loss distribution per obligor whenever there is no perfect dependence. We will show that under a number of reasonable conditions, each asset class has a limit loss distribution, so that even in loan portfolios the incremental risk of an obligor can be approximated by the marginal risk for any asset class with a minimum number of obligors. We base the discussion on the results of McNeil et al. (2005) and Schoenbucher (2006), who prove that limit-loss distributions exist for a number of credit risk models. We generalize the results for a setting with more than one asset classes and calculate the error of an application of gradient allocation on asset classes of finite size for several examples. Furthermore, we provide evidence that portfolio optimization based on gradient allocation is justifiable in both cases, when several asset classes are scaled up or down proportionally or non-proportionally.

This paper is structured as follows: In Section 2 the notation is introduced and the target of the following sections is defined. Furthermore, we motivate the discussion through an example. This example shows how per-unit capital allocation can trigger wrong business decisions in an inadequate business environment. In Section 3 we provide the mathematical background and show that portfolio optimization based on gradient capital allocation rules makes sense for large portfolios. To broaden the theoretical results we perform different simulations in Section 4. There, we give evidence that per-unit risk allocation is justifiable even for portfolios with less strict conditions so that we veer towards real world scenarios. In Section 5 we conclude with a discussion of our findings.

2. Motivation

2.1. Problem statement and notation

Suppose that a bank's credit portfolio consists of n subportfolios or asset classes. We will use these two expressions equivalently. In practice, one asset class can be defined by common characteristics of the obligors like the industry, the country or a specific range of ratings. An asset class $i \in \{1, ..., n\}$ consists of $u_i \in \mathbb{N}$ obligors. Loss occurs when an obligor k_i ($k_i = 1, ..., u_i$) defaults within a given time period. Typically, a period of one year is chosen. This event is described by the random variable $X_{i,k_i} \in \{0, 1\}$ for each obligor in asset class i, where $X_{i,k_i} = 1$ indicates default and $X_{i,k_i} = 0$ indicates no default. For obligor k_i we denote the exposure at default $EaD_{i,k_i} \in [0, 1]$ and loss given default $LGD_{i,k_i} \in [0, 1]$. The loss of the bank due to one obligor k_i is therefore given by $L_{i,k_i} = X_{i,k_i} \cdot EaD_{i,k_i} \cdot LGD_{i,k_i}$ and the loss of an asset class by $L_i := L_i(u_i) = \sum_{k=1}^{u_i} L_{i,k_i}$. The total loss of the portfolio then is calculated as follows:

$$L(u) = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \sum_{k=1}^{u_i} X_{i,k_i} \cdot EaD_{i,k_i} \cdot LGD_{i,k_i}, \tag{1}$$

with $u = (u_1, ..., u_n)$. If obligor k_i defaults, the bank suffers a loss L_{i,k_i}; if the obligor does not default it gains a fixed return. Traditionally for credit risk only losses are considered. Given a risk measure ρ, the risk of the portfolio can be calculated as $\rho(L)$. Formally ρ is a mapping from the set of random variables to the positive real numbers. ρ can be chosen coherent (Artzner et al. (1999)). Furthermore, in the following we denote by $X_i := \frac{1}{u_i} \sum_{k=1}^{u_i} X_{i,k_i}$ the fraction of defaults in the asset class i. An asset class is differentiated from the other asset classes by a number of characteristics. As long as not stated differently we assume that within one asset class i all obligors have:

- the same probability of default $P(X_{i,k_i} = 1) = PD_i$,
- the same correlation $\mathrm{corr}(X_{i,k_i}; X_{i,l_i}) = \varrho_i$ $(k_i, l_i = 1, ..., u_i)$ between each other,
- the same correlation $\mathrm{corr}(X_{i,k_i}; X_{j,l_j}) = \varrho_{ij}$ $(k_i = 1, ..., u_i, l_j = 1, ..., u_j)$ to obligors of another asset class j,
- the same exposure at default $EaD_{i,k_i} = EaD_i \in [0, 1]$,
- the same distribution of loss given defaults $LGD_{i,k_i} = LGD_i \in [0, 1]$.

Section 2.2 will show that in this setting gradient allocation will not necessarily lead to identical risk for identical obligors within one asset class due to the missing linearity of losses. To apply gradient allocation the following condition is necessary: There exists a random variable $\tilde{X}_i$, such that

$$\sum_{i=1}^{n} \sum_{k=1}^{u_i} L_{i,k_i} \sim \sum_{i=1}^{n} u_i \cdot \tilde{X}_i \cdot EaD_i \cdot LGD_i, \tag{2}$$

where $\sim$ is equality in distribution or a close enough approximation. $\tilde{X}_i$ represents the average fluctuation of losses in asset class i. The existence and form of $\tilde{X}_i$ has to be determined. Under the assumption that EaD_i and LGD_i are fixed real numbers, one can set $EaD_i = LGD_i = 1$ without loss of generality. We will assume this for the following sections as long as not stated otherwise. This shortens condition (2) as follows:

$$\sum_{i=1}^{n} \sum_{k=1}^{u_i} X_{i,k_i} \sim \sum_{i=1}^{n} u_i \cdot \tilde{X}_i. \tag{3}$$

This condition can be decomposed for large portfolios into two steps: Let l_{i,u_i} be the distribution function of $L_i(u_i)$ for $i = 1, ..., n$. Firstly, for any single asset class, proof has to be given that there is an $\tilde{X}_i$ with distribution function $\tilde{l}_i$, for which

$$\textbf{Step 1: } \frac{1}{u_i} l_{i,u_i} \to \tilde{l}_i,$$

for $u_i \to \infty$ as a weak convergence on the space of univariate distribution functions. Secondly, the dependency structure of the asset classes has to be considered, i.e., the convergence of the copula of the loss distribution functions of any pair of asset classes i, j with $i \neq j$ has to be proven.

$$\textbf{Step 2: } C_{i,j}^{u_i,u_j}(l_{i,u_i}, l_{j,u_j}) \to C_{i,j} \text{ pointwise},$$

for all $u_i \to \infty$ and $u_j = q \cdot u_i$, q constant, where $C_{i,j}$ and $C_{i,j}^{u_i,u_j}$ are copulas. The convergence for any proportion follows if step two is true for all q. By putting these two steps together, one can use the following lemma.

Lemma 1. *Let $\{l_{i,u_i} : u_i \in \mathbb{Z}_+\}$ and $\{l_{j,u_j} : u_j = q \cdot u_i,\ q\ const\}$ be two sequences of univariate distribution functions and let $\{C_{i,j}^{u_i,u_j} : u_i \in \mathbb{Z}_+, u_j = q \cdot u_i\}$ be a sequence of copulas; then, for every $u_i \in \mathbb{Z}_+$, a bivariate distribution function is defined through*

$$l_{i,j}^{u_i,u_j}(x,y) := C_{i,j}^{u_i,u_j}\left(l_{i,u_i}(x); l_{j,u_j}(y)\right).$$

If the sequences $\{l_{i,u_i}\}$ and $\{l_{j,u_j}\}$ converge to $\tilde{l}_i$ and to $\tilde{l}_j$ respectively in the weak convergence on the space of univariate distribution functions, and if the sequence of copulas $\{C_{i,j}^{u_i,u_j}\}$ converges to the copula $C_{i,j}$ pointwise in $[0,1]^2$, then the sequence $\{l_{i,j}^{u_i,u_j}\}$ converges in the weak topology of the space of bivariate distribution functions against $C_{i,j}(\tilde{l}_i(x); \tilde{l}_j(y))$.

A proof of this lemma can be found in Sempi (2004).

With this lemma, one can show by induction that the joint distribution function of the losses in the asset classes converges weakly. With this result, the convergence of the sum of losses can be concluded, or, alternatively, the convergence of the total loss.

Theorem 1. *Let l_u be the distribution function of total portfolio losses $L(u)$ with $u = (u_1, ..., u_n)$. Assume the limit distribution function of losses $\tilde{l}_i$ for each asset class i, $i = 1, ..., n$, exists and is piecewise continuous. If the limit copula $C_{i,j}(\tilde{l}_i(x); \tilde{l}_j(y))$ of any pair of distribution functions exists and is piecewise continuous, the total loss distribution function l_u of the portfolio converges for $u_i \to \infty$ for any given proportion $u_1 : u_2 : ... : u_n$ of asset class sizes and the limit per-unit risks per asset class exists.*

A proof of this theorem can be found in Appendix A.

Note that the assumption of piecewise continuity of losses is not a significant restriction in a real world loan portfolio.

Under the assumption that approximation (3) is valid, gradient allocation can be used to calculate the risk contribution of each asset class or obligor and truly measures the additional necessary risk capital of any additional obligor of that kind. We denote the risk contribution of an asset class as $\rho(L_i|L)$, so that $\sum_i \rho(L_i|L) = \rho(L(u))$. An application of gradient allocation according to Tasche (2008) then states that for the risk contribution of obligor k_i we have:

$$\rho^{p.u.}(X_{i,k_i}) = \frac{1}{u_i}\rho(L_i|L) = \frac{1}{u_i}\frac{\partial \rho\left(L(u)\right)}{\partial u_i}(u_1, ..., u_n). \tag{4}$$

According to the Euler Theorem, the sum of all per-unit risks then adds up to the total risk of the portfolio. Based upon the existence of a per-unit risk $\rho^{p.u.}(X_{i,k_i})$ all theoretical results that use gradient allocation can be applied. In particular, the following approximation can be used:

$$\rho\left(\sum_{i=1}^{n}\sum_{k=1}^{u_i} X_{i,k_i}\right) \simeq \rho\left(\sum_{i=1}^{n} u_i \cdot \tilde{X}_i\right). \tag{5}$$

2.2. Motivating example

In order to motivate the discussion we demonstrate the potential pitfalls of capital allocation models in small portfolios by presenting a short example. We show that capital allocation rules can lead to an erroneous calculation of the necessary risk capital whenever there is no perfect dependence of the single assets within each subportfolio.

We consider a Bernoulli mixture model, or more specifically the Beta Binomial approach as explained in Moraux (2010). We assume that the portfolio consists of ten obligors and is divided into two subportfolios of equal size. Each subportfolio consists of $u_1 = u_2 = 5$ obligors of identical exposure (measured as $EaD \cdot LGD$) equal to 1. In this model, the PD is random and hence the probability of r defaults in one subportfolio is:

$$P[L = r] = \int_0^1 \binom{5}{r} PD^r (1 - PD)^{5-r} f(PD) dPD, \quad r \in \{0, ..., 5\},$$

where f is the so-called mixing distribution and we choose a Beta distribution for both subportfolios'PDs. The density function of this distribution is given by:

$$f(PD; \alpha, \beta) := \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} PD^{\alpha-1}(1 - PD)^{\beta-1},$$

where α, $\beta \in \mathbb{R}^+$. According to Moraux (2010), the default correlation between two obligors within one subportfolio is:

$$\varrho_i := \frac{1}{1 + \alpha_i + \beta_i}.$$

For simplicity we choose $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 31$, so that $\varrho_1 = \varrho_2 = 3\%$. Furthermore, the two subportfolios are assumed to be independent of each other, i.e., the correlation between the two asset classes is $\varrho_{12} = 0$. For this assumption, we can calculate the probability of r defaults in the total portfolio:

$$\begin{aligned} P[L=r] &= \sum_{i=0}^{r} P[L_1=i]\cdot P[L_2=r-i], \qquad (6)\\ \text{where } P[L_j=i] &= \binom{5}{i}\frac{B(\alpha_j+i,\beta_j+5-i)}{B(\alpha_j,\beta_j)}, \qquad j=1,2. \end{aligned}$$

Here, $B(\alpha,\beta) := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ for $\alpha,\beta>0$ and $B(\alpha,\beta)=0$ for $\alpha,\beta\le 0$ is the beta function. Formula (6) describes the loss of the total portfolio. Hence, we can calculate the risk, defined as value-at-risk at the 95% confidence level of the total portfolio previously and subsequently adding the new obligor by

$$\mathrm{VaR}_{0.95}(L)=1, \qquad \mathrm{VaR}_{0.95}(L+X_{1,6})=2.$$

This significant effect of an increase of risk by 1 or 100% partly results through the missing continuity of VaR in this example. We can reduce the effect by switching to a more conservative risk measure called expected shortfall or conditional value-at-risk. The definition and a formula for expected shortfall in the case of discrete distribution functions can be found in Acerbi (2002). The main two advantages of expected shortfall are that it is continuous and coherent. Using the same loss distribution as above, we obtain

$$\mathrm{ES}_{0.95}(L)=2.0803, \qquad \mathrm{ES}_{0.95}(L+X_{1,6})=2.2001.$$

Therefore, the incremental expected shortfall for $X_{1,6}$ is $2.2001-2.0803=0.1198$. We can compare this result with the result according to gradient risk capital allocation. The per-unit allocated capital is expressed as $\mathrm{ES}_{\eta}^{p.u.}$.

$$\mathrm{ES}_{0.95}^{p.u.}(X_{1,6}) = \frac{\partial}{\partial u_1}\mathrm{ES}_{0.95}\left(\sum_{k=0}^{\infty} P[L(u)=k]\right).$$

The expression cannot be solved analytically since the portfolio is discrete and the derivative is not defined. But because we chose the two subportfolios with the exact same parameters we know that gradient allocation will allocate the same risk to each obligor[1]. This results in

$$\mathrm{ES}_{0.95}^{p.u.}(X_{1,6}) = \mathrm{ES}_{0.95}(L)/10 = 2.0803/10 = 0.2080.$$

[1]For alternative assumptions, i.e., different parameters in the two subportfolios, one can use capital allocation based on one-sided moments as introduced in Fischer (2003).

We see that a pure application of the allocation principle leads to an error in the charged risk capital. This can result in erroneous business decisions wherever portfolio sizes are steered by capital driven performance indicators like RORAC.

As an example, consider the case that the pricing of loans is based on the calculation above. The interest rate excluding operating expenses is calculated by refinancing cost plus expected losses that have to be compensated for by the share of non-defaulting loans plus risk capital charge. The expected loss EL can be calculated as $EL = PD{\cdot}EaD{\cdot}LGD$ via the values of α_1 and β_1 with:

$$PD = \frac{B(\alpha_1+1,\beta_1)}{B(\alpha_1,\beta_1)} = \frac{\alpha_1}{\alpha_1+\beta_1} = 3.13\%.$$

For this example, we assume that each loan has an exposure of $EaD \cdot LGD = 10{,}000$ and refinancing costs of 2%. Then, the additional risk capital for the new obligor is calculated by:

$$(\mathrm{ES}_{0.95}(L+X_{1,6}) - \mathrm{EL}(L+X_{1,6})) - (\mathrm{ES}_{0.95}(L) - \mathrm{EL}(L)) = (0.1198-0.0313)\cdot 10{,}000 = 886,$$

where EL means expected loss. Under the assumption of cost of capital (in the sense of opportunity costs for the bank) of 10%, this means a per-obligor capital charge of 88.6. With the equivalent capital calculation for the larger portfolio, the total interest rate in the case of the correct additional risk capital calculation is **6.12%**, and if the additional risk is calculated according to gradient allocation, it reaches **7.00%**. This means that we make an error of **88 bp** when using gradient capital allocation[2]. This can lead to a disadvantage in a price war with a competitor.

There is an additional important conclusion: The per-unit risk in this case is obviously not constant, i.e., the new obligor adds a lower risk to the portfolio than the existing obligors, even if it has the exact same characteristics. Under the assumption of a constant profit margin, the new obligor increases a performance indicator like RORAC, while an algorithm based on gradient allocation would assume positive homogeneity of risk and therefore lead to a constant RORAC and ultimately to an incorrect business decision.

[2] The same calculation can be conducted for value-at-risk as risk measure and would lead to an error of 969 bp due to the strong discreteness of the loss distribution.

3. Theoretical results

The example of Section 2.2 highlights that there are cases in which gradient allocation leads to significant errors in the calculation of interest rates. This section will prove that under some restrictions this error is small enough to be ignored. The analysis is based on existing results of asymptotic loss distributions, that are put into the context of capital allocation and per-unit risk. We show that there exists a per-unit risk per obligor so that up- and downscaling of risk as it is used in portfolio optimization, based on risk capital allocation is justifiable, i.e., approximation (5) is valid.

3.1. Prerequisites

We start with analyzing factor models (also called static structural models, see McNeil et al. (2005)) in the next subsection and then extend this view to mixture models.

Following Rosen and Saunders (2010) or Dorfleitner et al. (2012), we identify each obligor with a so called creditworthiness index, which is an obligor specific random variable. In general, the creditworthiness index is based on the Merton model, which was originally formulated for asset values. In the context of portfolio credit risk modeling it is a hidden variable (see e.g., Crouhy et al. (2000)). The obligor defaults if its CWI_{i,k_i} falls below a given barrier S_i within a given time period (usually one year). Therefore, X_{i,k_i} is expressed as:

$$X_{i,k_i} = 1_{\{CWI_{i,k_i} < S_i\}}.$$

In the factor model, we use CWI_{i,k_i} as a weighted sum of systematic risk factors M_j, which are common between all obligors, and an idiosyncratic factor E_{i,k_i}, which is specific for each obligor. All idiosyncratic risks E_{i,k_i} are independent of one another and independent of the systematic factors M_j. The vector of systematic factors is denoted as $\mathbf{M} = (M_j)_j$.

$$CWI_{i,k_i} = \sum_{j=1}^{m} \alpha_{i,j} M_j + \alpha_{i,E} E_{i,k_i}, \tag{7}$$

where M_j for $j = 1, ..., m$ and E_{i,k_i} for $k_i = 1, ..., u_i$ are standard normally distributed. Moreover, $\alpha_{i,E}$ is chosen in a way that CWI_{i,k_i} itself is standard normally distributed. To prevent the calculations from becoming too technical we will focus on a one-factor model, i.e., $m = 1$ and we write $M_1 = M$.

In this case it follows that $\text{corr}(CWI_{i,k_i}, CWI_{i,l_i}) = \alpha_{i,j}^2 = \alpha_i^2$.[3] This model is very similar to the CreditMetrics model of JP Morgan or the KMV model (see e.g., Crouhy et al. (2000)).[4]

3.2. One asset class

We begin with considering one asset class. In this case we will omit the index i indicating the number of the asset class. As first step, we prove that there exists an $\tilde{X}$ which satisfies

$$\sum_{k=1}^{u} X_k \sim u \cdot \tilde{X}. \tag{8}$$

If (8) holds, then for every homogeneous risk measure we have:

$$\rho\left(\sum_{k=1}^{u} X_k\right) = u \cdot \rho(\tilde{X}). \tag{9}$$

We will discover that there exists an $\tilde{X}$ so that equation (9) is approximated with an error ϵ, where ϵ is small for large u. From now on we will refer to $\rho(\tilde{X})$ as per-unit risk of an obligor.

To calculate the loss distribution, one has to look at the credit risk model, in this case the one-factor model as introduced above. The systematic factor M materializes at one point in time and takes a value c. The probability of default of one given obligor k is conditional on the state of the factor $M = c$:

$$PD(c) = P[CWI_k < S | M = c] = \Phi\left(\frac{S - \alpha\ c}{\sqrt{1-\alpha^2}}\right) \tag{10}$$

for all k. With this equation we conclude:

Theorem 2. *Assume we have a portfolio consisting of one asset class. Let S be the default threshold and α^2 the correlation between the obligors' CWIs. Then the loss distribution of $X := 1/u \sum_{k=1}^{u} X_k$ representing the default proportion of the complete portfolio based on a one-factor model as defined above converges against a limit distribution function $\tilde{l}$ and*

$$\tilde{l}(x) = \Phi\left(\frac{1}{\alpha}\left(\sqrt{1-\alpha^2}\ \ \Phi^{-1}(x) - S\right)\right), \qquad x \in [0,1]. \tag{11}$$

[3] If we choose a one-factor model, we determine the correlation between two asset classes by choosing the correlation within the asset classes ($\text{corr}(CWI_{i,k_i}, CWI_{j,l_i}) = \alpha_i \cdot \alpha_j$). In a multi-factor model all correlation can be chosen individually.

[4] In CreditMetrics the probability of default is given by rating tables and rating transition matrices which we ignore for our discussion.

For a proof see Schoenbucher (2006).

With this loss distribution function, the risk (measured as a function only depending on $\tilde{l}$) obviously converges against a limit $\rho(X) = \rho(\frac{1}{u}\sum_{k=1}^{u} X_k) \to \rho(\tilde{X})$. Thus, $\tilde{X}$ can be defined by this limit and for any fixed u the total portfolio risk can be approximated by $u \cdot \rho(\tilde{X})$ for every homogeneous risk measure. Nevertheless we are likely to make an error for a small number of obligors u.

These results can be generalized in various ways. Schoenbucher (2001) describes the model with volatility uncertainty. We will extend the results and supplement a generalization for more asset classes in Section 3.3 and for alternative risk models in Section 3.4.

3.3. More than one asset classes

Allocation of risk capital to subportfolios or asset classes only makes sense if there are at least two asset classes or subportfolios to which the capital can be allocated. Thus, we will now consider portfolios of two asset classes. The results can be easily translated into more than two asset classes by induction. We again assume that each asset class is homogeneous as defined in Section 2, but the asset classes differ from one another. We still assume for simplicity that all assets have the same exposure at default and loss given default equals 1, but the probability of default and correlation can be different.

In a general setting with the notation introduced in Section 3.1 we use the following lemma.

Lemma 2. *Assume a portfolio of n asset classes. Let $\boldsymbol{M}$ be a vector of systematic factors, $(c_j)_j \in \mathbb{R}^m$ a vector of constants and let X be the fraction of defaults in the portfolio (i.e., $0 \leq X \leq 1$).*
Under the assumption that $\frac{u_i}{\sum_{k=1}^{n} u_k}$ converges for all i, and conditional on $\boldsymbol{M} = (c_j)_j$ the convergence

$$X - \frac{1}{\sum_{i=1}^{n} u_i} \left(\sum_{i=1}^{n} \Big(u_i PD_i\big((c_j)_j\big) \Big) \right) \overset{a.s.}{\to} 0$$

holds, where $PD_i\big((c_j)_j\big) = P[CWI_i < S_i | \boldsymbol{M} = (c_j)_j]$.

This lemma is an extension of the law of large numbers and follows from the work of Lucas et al. (2001).

For two asset classes formula (7) implies:

$$\begin{aligned}
CWI_{1,k_1} &= \alpha_1 M + \sqrt{1-\alpha_1^2} E_{1,k_1} \qquad \text{for all } k = 1, ..., u_1 \text{ from asset class 1,} \\
CWI_{2,k_2} &= \alpha_2 M + \sqrt{1-\alpha_2^2} E_{2,k_2} \qquad \text{for all } l = 1, ..., u_2 \text{ from asset class 2,} \\
\alpha_i^2 &= \text{corr}(CWI_{i,k_i}, CWI_{i,l_i}), \text{ for } i = 1, 2, \\
\alpha_1\alpha_2 &= \text{corr}(CWI_{1,k_1}, CWI_{2,l_2}).
\end{aligned}$$

We denote the probability of default of assets from the two asset classes PD_1 and PD_2. We can now consider two cases: Case 1 assumes an asset class with a fixed number of obligors while the second asset class is scaled up. Case 2 considers a proportional upscaling of both asset classes.

Theorem 3. *Let X be the fraction of defaults in the portfolio (i.e., $0 \leq X \leq 1$). Then the following holds:*

1. *If we fix the number of obligors u_2 of the second asset class and only increase the number of obligors u_1 of asset class 1, we get*

$$P\Big[|X - PD_1(c)| > \epsilon | M = c\Big] \overset{a.s.}{\to} 0 \text{ as } u_1 \to \infty,$$

2. *If we increase the number of obligors of both asset classes simultaneously, whilst retaining a fixed proportion ($u_1 : u_2 = a : b$, with $a, b > 0$), we obtain*

$$P\Big[|X - \underbrace{\frac{a}{a+b}}_{=:a'} PD_1(c) - \underbrace{\frac{b}{a+b}}_{=:b'} PD_2(c)| > \epsilon | M = c\Big] \overset{a.s.}{\to} 0 \text{ as } u_1, u_2 \to \infty.$$

The proof of this theorem follows directly from Lemma 2, when we set $n = 2, m = 1$. Based on this and the one asset class case of Schoenbucher (2006) we can deduce the limit loss distribution for more than one asset class.

Theorem 4. *Assume we have a portfolio consisting of two asset classes or subportfolios. Let S_1 and S_2 be the default thresholds for the two subportfolios, and α_1^2 and α_2^2 the correlation within the obligors of the subportfolios. Then the loss distribution of the complete portfolio based on a one-factor model as defined before converges against a limit distribution function $\tilde{l}$, and $\tilde{l}$ is given as follows:*

1. *For a fix number of obligors in the second subportfolio u_2:*

$$\tilde{l}(x) = \Phi\left(\frac{1}{\alpha_1}\left(\sqrt{1-\alpha_1^2}\ \Phi^{-1}(x) - S_1\right)\right), \qquad x \in [0, 1].$$

2. *For fixed proportion between the number of obligors of the two subportfolios ($u_1 : u_2 = a : b$, with $a, b > 0$ and $a' = \frac{a}{a+b}, b' = \frac{b}{a+b}$):*

$$\begin{aligned}\tilde{l}(x) &= \int\limits_{x'=s_1}^{s_2} \min\Big[\Phi\left(\frac{1}{\alpha_1}\Big(\sqrt{1-\alpha_1^2}\;\Phi^{-1}\Big(\frac{x-x'}{a'}\Big) - S_1\Big)\right); \\ &\qquad\qquad \Phi\left(\frac{1}{\alpha_2}\Big(\sqrt{1-\alpha_2^2}\;\Phi^{-1}\Big(\frac{x'}{b'}\Big) - S_2\Big)\right)\Big]dx' \\ &= \int\limits_{x'=s_1}^{s_2} C^{FH}\left(\tilde{l}_1\Big(\frac{x-x'}{a'}\Big), \tilde{l}_2\Big(\frac{x'}{b'}\Big)\right) dx', \qquad x \in [0,1],\end{aligned}$$

with $s_1 = max(0; x - a'), s_2 = min(x; b')$, C^{FH} Frechét-Hoeffding upper bound copula, $\tilde{l}_i$ limit loss distribution of asset class i (i = 1,2).

A proof of this theorem can be found in Appendix B.

Again, the loss distribution converges against a limit distribution. We can calculate the per-unit risk of one obligor in the two cases by

1. $u_2 =: c$ fix and $u_1 \gg u_2$.
   $\sum_1^{u_2} X_{2,k_2}$ is bounded by a constant c, so $\rho(X) \leq \rho(\frac{1}{u_1+c}\sum X_{1,k_1} + \frac{c}{u_1+c})$. Hence, the second term in the brackets converges to zero if u_1 gets larger, so $\rho(\tilde{X})$ is an approximation for the average risk contribution for one obligor from the first asset class.
2. $u_1 : u_2 = q$ fix $\Rightarrow u_1 + u_2 = u_2 \cdot (q+1)$, where $q \in \mathbb{Q}_+$ and $u_2 \to \infty$.
   When define the risk of the limit loss distribution function as follows:
   $$R_q := \lim_{\substack{u_1,u_2\to\infty,\\ u_1/u_2=q}} \rho\left(\frac{1}{u_1+u_2}\left(\sum_{k_1=1}^{u_1} X_{1,k_1} + \sum_{k_2=1}^{u_2} X_{2,k_2}\right)\right). \tag{12}$$
   R_q now describes one "package" consisting of $\frac{q}{q+1}$ obligors of asset class 1 and $\frac{1}{q+1}$ obligors of asset class 2. To use this for portfolio optimization, one then has to split the risk of the package to the single obligors.

A more general way of modeling two asset classes is achieved through increasing the number of systematic factors. This approach has the advantage of a better presentation of concentration risks. In a two-factor model, the two asset classes are described as follows:

$$\begin{aligned}
CWI_{1,k_1} &= \frac{1}{\sqrt{\alpha_{11}^2+\alpha_{12}^2+1}} \quad (\alpha_{11}M_1+\alpha_{12}M_2+E_{1,k_1}),\\
CWI_{2,k_2} &= \frac{1}{\sqrt{\alpha_{21}^2+\alpha_{22}^2+1}} \quad (\alpha_{21}M_1+\alpha_{22}M_2+E_{2,k_2}),
\end{aligned}$$

where $\mathbf{M} = (M_1, M_2)$ is a two-dimensional random vector of systematic factors with $\mathbf{M} \sim N_2(0, \Omega)$ normally distributed with a given covariance matrix Ω. M_1, M_2 and the idiosyncratic factors E_{1,k_1}, E_{2,k_2} are standard normally distributed. It follows for the correlations:

$$\begin{aligned}
\mathrm{corr}(CWI_{1,k}; CWI_{1,l}) &= \frac{\alpha_{11}^2+\alpha_{12}^2}{\alpha_{11}^2+\alpha_{12}^2+1}, \qquad (k \neq l)\\
\mathrm{corr}(CWI_{2,k}; CWI_{2,l}) &= \frac{\alpha_{21}^2+\alpha_{22}^2}{\alpha_{21}^2+\alpha_{22}^2+1}, \qquad (k \neq l)\\
\mathrm{corr}(CWI_{1,k}; CWI_{2,l}) &= \frac{\alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}}{\sqrt{\alpha_{11}^2+\alpha_{12}^2+1}\sqrt{\alpha_{21}^2+\alpha_{22}^2+1}}.
\end{aligned}$$

For the conditional probabilities of default in this case we obtain

$$\begin{aligned}
PD_i(c_1, c_2) &= P[CWI_i < S_i|(M_1, M_2) = (c_1, c_2))] =\\
&= P\Big[E_{1,k_1} < \sqrt{\alpha_{i1}^2+\alpha_{i2}^2+1}\ \ S_i - \alpha_{i1}c_1 - \alpha_{i2}c_2\Big]\\
&= \Phi\left(\sqrt{\alpha_{i1}^2+\alpha_{i2}^2+1}\ \ S_i - \alpha_{i1}c_1 - \alpha_{i2}c_2\right).
\end{aligned}$$

The loss distribution function can be calculated via two-dimensional integration over all values that can be realized by M_1 and M_2. This is analytically complex. Under the assumption of independent systematic factors M_1 and M_2 we obtain for every single asset class:

$$\tilde{l}_i(x) = \Phi_{0,\sqrt{\alpha_{i1}^2+\alpha_{i2}^2}}\left(\Phi^{-1}(x) - \sqrt{\alpha_{i1}^2+\alpha_{i2}^2}S_i\right), \qquad x \in [0,1]$$

where $\Phi_{\mu,\sigma}$ is the normal distribution with mean μ and standard deviation σ. In the general case we have to solve the following integral.

$$\tilde{l}(x) = \int_{\mathbb{R}^2} P[X \leq x|\mathbf{M} = (c_1, c_2)] f(c_1, c_2) dc_1 dc_2,$$

where $f : \mathbb{R}^2 \to [0,1]$ denotes the density function of $\mathbf{M}$. From Lemma 2 we deduce the existence of a limit distribution function of the complete portfolio for any fix limit proportion of the two asset classes, i.e., for $\frac{u_i}{u_1+u_2}$ converges for $i = 1, 2$. The limit distribution then only depends on the proportion of the asset classes, the probabilities of default and the factor loadings defined by the choice of $\alpha_{i,j}$ for $i, j = 1, 2$.

$$\tilde{l}(x) = \int_{\mathbb{R}^2} 1_{\{a'PD_1(c_1,c_2)+b'PD_2(c_1,c_2)\}} f(c_1, c_2) dc_1 dc_2.$$

We conclude that even in a more-factor threshold model, the limit of the loss distribution exists under a number of reasonable assumptions. This allows us to use gradient allocation and consequently portfolio optimization tools in this setting as well. Again, we have based the results on some restrictions, namely the assumption of homogeneous asset classes as well as the condition of a proportional up-scaling of the number of obligors in the asset classes.

3.4. Mixture Models

So far we have discussed factor models for which defaults occur when the creditworthiness index (CWI) falls below a threshold. Mixture models are a more general class of models (McNeil et al. (2005)). In these models, the systematic factors still form a base for calculating the probability of default, but the precise mechanism of how default is calculated can be defined in various ways.

For an asset class i, let X_{i,k_i}, $k_i = 1, .., u_i$ be a random variable. One can choose a binomial random variable as is used in a factor model. In this case, the model is called Bernoulli mixture model. Then, the probability of default for obligor k_i is defined by

$$P[X_{i,k_i} = 1 | \mathbf{M} = (c_j)_j] = p_{i,k_i}(\mathbf{M}), \text{ with } j = 1, ..., m.$$

The probability $p_{i,k_i} \in [0, 1]$ for $k_i = 1, ..., u_i$ is a random variable itself. The distribution of p_{i,k_i} describes the approach in a closer way. For example, if p_{i,k_i} is beta distributed then we obtain the Beta Binomial approach described in Section 2.2. A very common model is CreditRisk+, which was proposed by Credit Suisse in 1997. It is a Poisson mixture model, and thus, p_{i,k_i} is Poisson distributed and it follows for an asset class in a Poisson mixture model:

$$P[L_i = r | \mathbf{M} = (c_j)_j] = exp\Big(- \sum_{k_i=1}^{u_i} \lambda_{k_i}\big((c_j)_j\big)\Big) \frac{\Big(\sum_{k_i=1}^{u_i} \lambda_{k_i}\big((c_j)_j\big)\Big)^r}{r!}.$$

In particular concerning the case of CreditRisk+, there is only one factor $\mathbf{M} = M$ and the function $\lambda_{k_i}(M)$ is chosen as $\lambda_{k_i}(M) = c_{k_i} M$, where $c_{k_i} > 0$ is a constant, and M is assumed to be $\Gamma(\alpha, \beta)$-distributed. For further details see Crouhy et al. (2000) and McNeil et al. (2005). Asset classes differ from one another by their distributions of default probabilities and the correlation within the asset class and to another asset class. We additionally release the definition of an homogeneous asset class by allowing different exposures per obligor. For a given obligor k_i ($k_i \in \{1, ..., u_i\}$) the exposure at default EaD_{i,k_i} is deterministic with values in $(0, 1]$, and the loss given default LGD_{i,k_i} is a random variable with values in $(0, 1]$ that is independent of the default indicator X_{i,k_i}.

We focus on one asset class according to step one in Section 2.1 and omit index i. For the further discussion we make the following assumptions.

1. There are functions $l_u : \mathbb{R}^m \to [0, 1]$ such that conditional on $\mathbf{M}$, the losses $(L(u))_{u \in \mathbb{N}}$ form a sequence of independent random variables with mean $l_u((c_j)_j) = E[L(u)|\mathbf{M} = (c_j)_j]$.
2. There exists a function $\tilde{l} : \mathbb{R}^u \to \mathbb{R}$ such that
$$\lim_{u \to \infty} \frac{1}{u} E[L(u)|\mathbf{M} = (c_j)_j] = \tilde{l}\big((c_j)_j\big).$$
3. There is a constant $c < \infty$ such that $\sum_{k=1}^{u} (EaD_k/k)^2 < c$ for all u.

These assumptions require scrutinizing more carefully. First of all we demand independence of the obligors (or their losses) at a given state of economy. The second assumption states that the expected loss for a given state of economy converges, which means the essential composition of the asset class, in terms of PD, EaD and LGD, remains the same when the number of obligors is increased. To be precise, the composition does not have to remain fixed but it must converge to a fixed constant. Finally, the third assumption prevents the exposure from growing with the number of obligors approaching ∞. Thus far we have obtained the result by giving each exposure a weight of $1/u$ for u obligors in the portfolio. The theorem shows that for every Bernoulli mixture model under a few basic assumptions the loss distribution converges against a limiting distribution. Once this becomes certain, the desired approximative equality (3) is valid for every risk measure. Based on these assumptions, we can draw a conclusion for the limit loss distribution.

Theorem 5. *Let $u \in \mathbb{N}$ be the number of obligors in the portfolio. If the above assumptions 1.-3. hold, then*

$$\lim_{u \to \infty} \frac{1}{u} L(u) = \tilde{l}((c_j)_j), \qquad P(\cdot | \boldsymbol{M} = (c_j)_j) - a.s.$$

A proof of this theorem can be found in Frey and McNeil (2003).

In the special case of a one-factor Bernoulli mixture model, we obtain a stronger result:

Theorem 6. *Let $\boldsymbol{M} = M$ be a one-dimensional random variable with distribution function G. Assume that the conditional asymptotic loss function $\tilde{l}(c)$ is strictly increasing and right continuous and that G is strictly increasing at $q_\eta(M)$, i.e., $G(q_\eta(M) + \delta) > \eta$ for every $\delta > 0$. Thus, if assumptions 1.-3. hold, then*

$$\lim_{u \to \infty} \frac{1}{u} q_\eta(L(u)) \to \tilde{l}(q_\eta(M)).$$

A proof of this theorem can be found in Frey and McNeil (2003). This theorem proves that under the given conditions the tail of the limit loss distribution only depends on the tail of the factor M. Hence, for any quantile-based risk measure, there exists a limit per-unit risk.

At first glance the definition of a mixture model appears to be different from the threshold model we previously discussed. However, McNeil et al. (2005) prove that every multi-factor threshold model can be equivalently described by a Bernoulli mixture model. With this equivalence, we can apply all results in this section to the setting we have considered so far in Section 3.2 and 3.3. Nevertheless, sections 3.2 and 3.3 provide additional information through the analytically calculated limit distribution functions. Furthermore, we can mathematically prove the convergence of the distribution function of the complete portfolio and hence the copula function (see Theorem 1).

3.5. Summary of theoretical results

For factor models of the discussed form, we have seen that the loss distribution of the total portfolio converges if the number of obligors increases. The same holds true for Bernoulli mixture models under the condition of convergence of the copula of the loss distribution functions of the asset classes. This result was based on some economically reasonable assumptions. This means that there always exists a limit loss distribution function $\tilde{l}$, which describes the losses of large portfolio. Based on the limit distribution function $\tilde{l}$ for large portfolios the per-unit risk $\rho(\tilde{X})$ is constant, meaning it is independent of the portfolio size.

In the case of one asset class, the total portfolio risk can be calculated via $u \cdot \rho(\tilde{X})$. This implies that a portfolio consisting of a sum of u obligors can be represented as $\sum_{k=1}^{u} X_k \overset{d}{\sim} u \cdot \tilde{X}$. With this approximative equality capital allocation and portfolio optimization based on capital allocation are acceptable. In the case of two or more asset classes the limit loss distribution also exists as long as the asset classes are up-scaled in a fixed proportion. The risk calculated from it describes the risk of a package consisting of a specific proportion of obligors of the asset classes.

Summarizing, we have obtained several theoretical results. Firstly, in large portfolios we can allocate a per-unit risk to every obligor for factor and Bernoulli mixture models, which can be used to estimate the risk of a new obligor of the same characteristics. Secondly, the per-unit risk exists for any risk measure we choose. Thirdly, under the used assumptions gradient allocation is justifiable. Nevertheless, the theoretical discussion opens up the following questions: Which error do we make in small portfolios? What happens if the portfolio is not perfectly homogeneous? What happens if we scale two or more asset classes up or down and the proportion is not fixed? The next section will deal with these questions based on Monte Carlo simulation.

4. Evidence from simulation

In this section we supplement the analytically derived results from the previous section through simulation. In particular, we investigate the questions left unanswered in the previous section. This includes the speed of convergence, the dependence on input variables and the effect of an increase of asset classes in a non-fixed proportion.

4.1. General model assumptions

To make all results comparable we fix some assumptions and input parameters for the simulations for all following sections. All assumptions hold as long as not stated otherwise. We consider a one-factor credit risk model as introduced in Section 3, formula (7). We will show how the distribution function changes if we increase the portfolio size and analyze how many obligors are necessary to gain a constant per-unit risk. In order to do this, we compare portfolios with identical characteristics but a different number of obligors. For this reason, we will indicate the number of obligors of the first asset in the two scenarios by u_1 and u'_1.

- number of obligors

  case 1: $u_1 = 100$, $u'_1 = 1{,}000$,

  case 2: $u_1 = 1{,}000$, $u'_1 = 1{,}500$,
- exposure: $EaD_1 = 1/u_1$ and accordingly $EaD'_1 = 1/u'_1$ in the case of one asset class, and respectively $EaD_1 = 1/(u_1 + u_2)$ and $EaD'_1 = 1/(u'_1 + u_2)$ for two asset classes,
- loss given default: LGD_1 random variable equal 50% or 100% with probability 0.5.

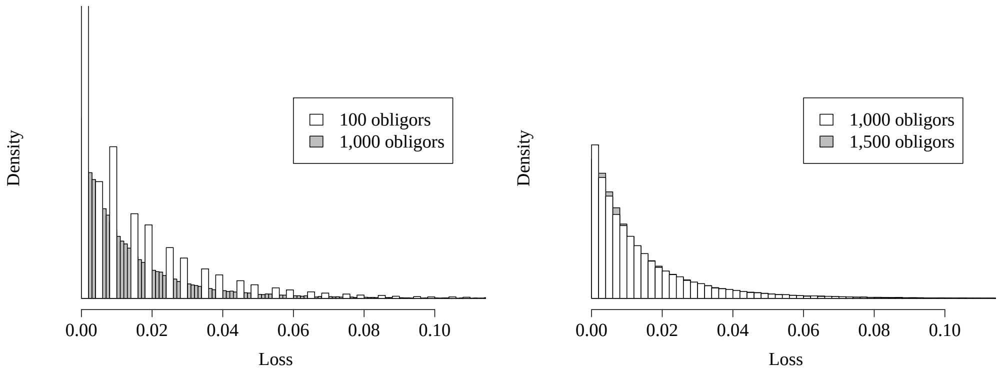
As risk measure we choose the VaR. Note that all considerations in the theoretical discussion were made pertaining to the loss distribution function, and thus, any quantile-based, homogeneous risk measure could be chosen. The setting is more flexible than the one we chose in the theoretical part because we allow random LGDs that were not part of the theoretical discussion for factor models. Furthermore, the setting allows us to analyze the influence of the asset class size on characteristics of the loss distribution. We compare the VaR or the quantile of loss distributions between portfolios of different sizes. We weight the exposures so that the calculated VaR corresponds to the per-unit risk of one obligor; see also equation (9).

Based on these assumptions, we simulate different scenarios, analyze the output graphically and draw conclusions for the per-unit risk.

4.2. One asset class - factor model

As above, we consider a single asset class first. Even if this case is not relevant for capital allocation, it can nevertheless produce results that can be transferred to more asset classes. In addition to the assumptions of Section 4.1 we choose the following case specific assumptions: $PD = 2\%$, $\alpha^2 \approx 0.16$.[5]

[5]To use a realistic input parameter we choose the correlation according to the Basel II formula for big corporations: $\alpha^2 = 0.12\frac{1-e^{-50PD}}{1-e^{-50}} + 0.24(1 - \frac{1-e^{-50PD}}{1-e^{-50}})$; see e.g., the Basel Basel Committee on Banking Supervision (2010 (rev 2011)).



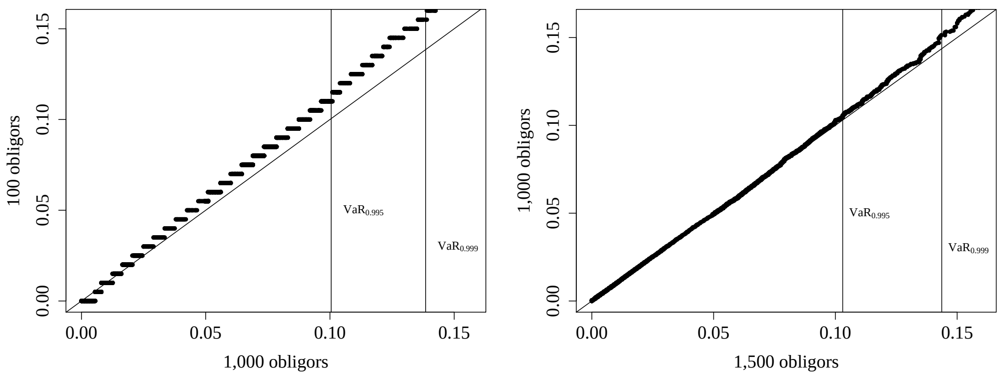
(a) 100 vs. 1,000 obligors

(b) 1,000 vs. 1,500 obligors

Figure 1: Comparison of density functions of losses for one asset class simulated with 100,000 model runs in a one-factor model as described in formula (7). Both figures describe asset classes with $PD = 2\%$ and $\varrho = 0.16$.

The density function of losses for the smaller portfolio (100 obligors) differs significantly from the density function of the larger portfolio (1,000 obligors). The difference is partly due to the high discreteness of losses in the smaller portfolio, i.e., there are fewer possible outcomes or loss values than in the larger portfolio. Another part of the effect is due to the fatter tails in the loss density, meaning a higher per-unit risk as can be seen in Figure 1a. Figure 1b shows that the per-unit risk in a portfolio consisting of 1,000 obligors or 1,500 obligors is almost identical. The per-unit risk calculated with VaR as risk measure is clearly higher in a small portfolio. The Q-Q-plot of the two portfolios in Figure 2a bends to the left, while the plot in Figure 2b is straight. At a confidence level of $\eta = 0.995$ the per-unit risk in the small portfolio is 0.115, while it is 0.106 in a portfolio of 1,000 obligors, which corresponds to a decrease by 8.3% for larger portfolios. Figure 2b shows that this effect disappears for large u.

Since this result directly follows from the convergence of the distribution function, the same behavior can be expected from alternative risk measures such as expected shortfall. For expected shortfall as risk measure, we display the results in Figure 3. The effects are similar to the VaR results. Again, in a portfolio of 100 obligors the per-unit risk is higher than in a portfolio of 1,000 obligors, but then remains constant for an even



(a) 100 vs. 1,000 obligors

(b) 1,000 vs. 1,500 obligors

Figure 2: Comparison of Q-Q-plots for one asset class simulated with 100,000 model runs in a one-factor model as described in formula (7). Both figures describe asset classes with $PD = 2\%$ and $\varrho = 0.16$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class size on the x-axis.

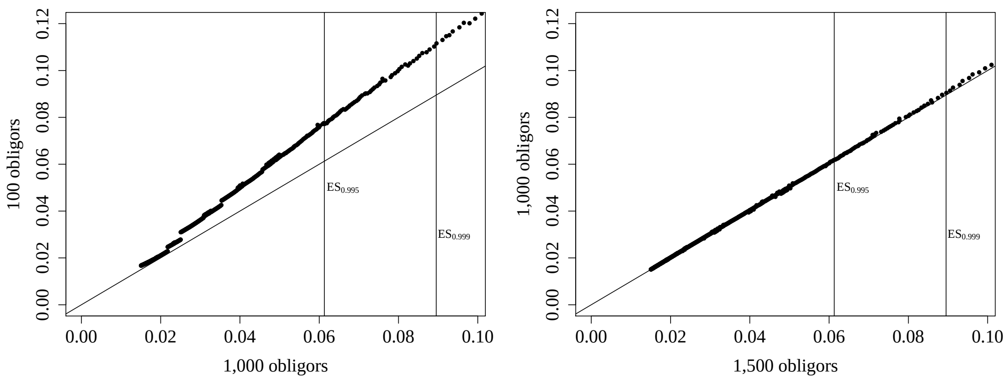
higher number of obligors. It is interesting to see that Figure 3a differs from Figure 1a with regard to two characteristics: First of all, the expected shortfall curve has no steps because even in small portfolios, expected shortfall is continuous. Secondly, the curve has a higher slope. This shows that expected shortfall punishes little diversification more than VaR.

In Figure 4 we fix the confidence level η for the VaR at 0.99 and look at the per-unit risk depending on the number of obligors. Per-unit risk is larger for small portfolios but then converges. From the simulation result we calculate the minimum $\bar{u}_\epsilon$ for a given ϵ to obtain:

$$\frac{1}{\bar{u}_\epsilon}\rho\left(\sum_{k=1}^{\bar{u}_\epsilon} X_k\right) - \rho(\tilde{X}) \leq \epsilon.$$

If we choose, for example, a maximum error of $\epsilon = 20$ bp $= 0.002$, we obtain $\bar{u}_\epsilon = \bar{u}_{20} = 361$. That means that the per-unit risk is overrated by maximal 20 bp as long as the portfolio has a minimum size of 361 obligors. Notice that, for example, with an assumed capital charge of 10%, this equals an error of only 2 bp $= 0.0002$ in the calculation

[6] The calculation is based on simulation for a one-factor model. $\bar{u}_\epsilon$ is calculated via non-linear regression: $y = a + b/x$, limit = per-unit risk for a portfolio size of 2,000 obligors



(a) 100 vs. 1,000 obligors

(b) 1,000 vs. 1,500 obligors

Figure 3: Comparison of expected shortfall contribution for one asset class simulated with 100,000 model runs in a one-factor model as described in formula (7). Both figures describe asset classes with $PD = 2\%$ and $\varrho = 0.16$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class size on the x-axis.

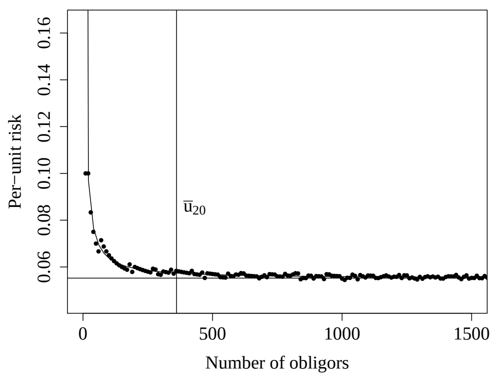


Figure 4: Per-unit risk vs. portfolio size simulated with 100,000 model runs for a portfolio size of 10 to 2,000 obligors ($PD = 1\%$, $\eta = 0.99$, correlation according to Basel II formula for big corporations). $\bar{u}_{20}$ marks the number of necessary obligors to reach a maximum delta between per-unit risk and limit per-unit risk[6]of $\epsilon = 20bp$.

of interest rates, as calculated in the motivating example. As a comparison we choose $\epsilon = 10$ bp and obtain an obviously higher $\bar{u}_{10} = 612$.

As a next step we analyze how sensitive this result is to the input parameters. Table 1

$\eta\backslash PD$	0.5%	1%	2%	5%	10%
0.95	204	250	305	355	382
0.97	246	275	341	424	461
0.99	316	361	450	516	518
0.999	467	478	569	668	722

Table 1: Number of necessary obligors to achieve constant per-unit risk with a maximum error of 20 bp (risk measure = VaR) simulated with 100,000 model runs in a one-factor model as described in formula (7).

shows the number of obligors necessary to gain a constant per-unit risk with a maximum error of $\epsilon = 20$ bp depending on the choice of η and PD. The simulation shows that the number of necessary obligors depends on η and PD. We see that a higher confidence level demands a larger portfolio to gain a constant per-unit risk. This is due to the fact that the limit loss distribution function $\tilde{l}(x)$, given in equation (11), is approximated in a better way for a larger number of obligors. However, there is a high dependence on the PD of the obligors. If we look at $\tilde{l}(x)$ in equation (11), we see that the probability of defaults shifts the argument in the normal distribution function to the left. This means that a higher probability of defaults leads to a movement of the data points out of the tails. In this way, the observations match the mathematical derivation.

$\eta\backslash PD$	0.5%	1%	2%	5%	10%
0.95	578	615	712	830	902
0.97	598	633	749	851	957
0.99	673	696	843	944	1,075
0.999	819	883	998	1,174	1,194

Table 2: Number of necessary obligors to achieve constant per-unit risk with a maximum error of 20 bp (risk measure = expected shortfall) simulated with 100,000 model runs in a one-factor model as described in formula (7).

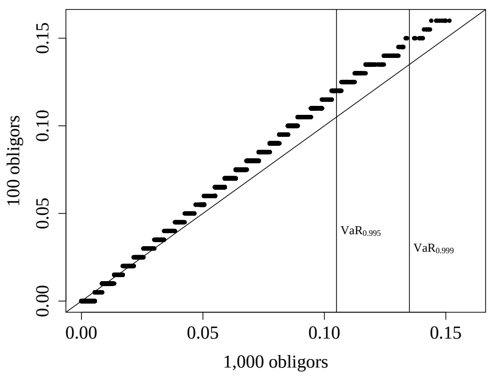
If we examine the same parameters for expected shortfall, we obtain the results displayed in Table 2. As expected, the results are similar to the results we calculated for VaR in terms of dependency on PD and confidence level. However, we recognize two effects: The number of necessary obligors is generally higher then for VaR. This can be explained by the higher sensitivity towards concentration risks (see Bonti et al. (2006)). Additionally, the sensitivity with respect to the input parameters is not as strong as in the case of VaR.

4.3. One asset class - mixture model

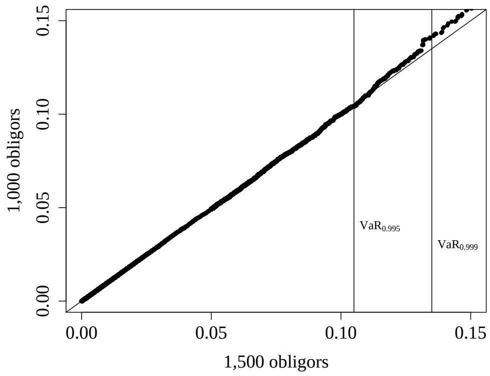
We conduct the same discussion for an alternative credit risk model, namely CreditRisk+ as introduced in Section 3.4. As additional input parameters to Section 4.1 for CreditRisk+ we choose the following:

- unconditional PD of the obligor: $PD = 2\%$,
- shape and scale of common factor: $\gamma_1 = 0.8$[7], $\gamma_2 = 1/\gamma_1$. These parameters guarantee that correlation corresponds to the correlation chosen for the factor model (see Section 4.2).

It is evident from Figure 5 and Table 3 that the main result for a mixture model is similar to the one for a factor model. In small portfolios the per-unit risk is higher than in larger portfolios. However, if the portfolio size is high enough the per-unit risk converges to the same limits as in a factor model.



(a) 100 vs. 1,000 obligors



(b) 1,000 vs. 1,500 obligors

Figure 5: Comparison of Q-Q-plots modeled in CreditRisk+ simulated with 100,000 model runs. Both figures describe asset classes with $PD = 2\%$ and $\varrho = 0.16$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class size on the x-axis.

The speed of convergence as well as the range of minimum obligors are in the same order of magnitude as for the one-factor model, as shown in Table 3. There is one

[7] This value is calculated via simulation, using
$\varrho = \frac{1}{PD(1-PD)}\left(\int_{-\infty}^{\infty}\Phi^2\left(\frac{S-\alpha c}{\sqrt{1-\alpha^2}}\right)d\phi(c) - PD^2\right) = \frac{1}{PD(1-PD)}\left(\binom{1+\gamma_1}{2}\left(\frac{2PD}{2PD+\gamma_1}\right)^2\left(\frac{\gamma_1}{2PD+\gamma_1}\right)^{\gamma_1} - PD^2\right)$

small difference in the results: The number of necessary obligors for high probabilities of default is slightly lower for high default probabilities. As mentioned before, this effect is expectable due to the increasing correlation between the obligors. Apparently, this effect is slightly stronger in a mixture model than in a factor model.

$\eta \backslash PD$	0.5%	1%	2%	5%	10%
0.95	182	203	299	397	420
0.97	219	286	352	474	473
0.99	340	374	480	596	546
0.999	507	609	646	740	668

Table 3: Number of necessary obligors necessary to achieve constant per-unit risk with a maximum error of 20 bp, simulated with 100,000 model runs in in CreditRisk+ as introduced in Section 3.4 with ϱ according to Basel II formula for big corporations.

4.4. More than one asset class

Next we discuss the most relevant scenario, namely the case of more than one asset class and consider an example with two asset classes. From the previous sections we know that the marginal distributions, meaning the loss distributions of the single asset classes, converge for a large number of obligors. In this section we will give evidence of the convergence of the copula. With the existence of a limit copula we know that, additional to the limit loss distributions per asset class, there is a limit dependency structure for all combinations of asset class sizes. As explained in Section 2.1, this gives evidence of the existence of a limit loss distribution for the total portfolio and allows the conclusion of the existence of per-unit risks. For any error ϵ and any ratio of asset class sizes the per-unit risks can therefore be calculated. In the case of two asset classes this means: For every pair of large number of obligors in two asset classes (u_1, u_2) we can approximate the following equality with an error ϵ:

$$\rho\left(\sum_{k_1=1}^{u_1} X_{1,k_1} + \sum_{k_2=1}^{u_2} X_{2,k_2}\right) = \rho\left(u_1\tilde{X}_1 + u_2\tilde{X}_2\right).$$

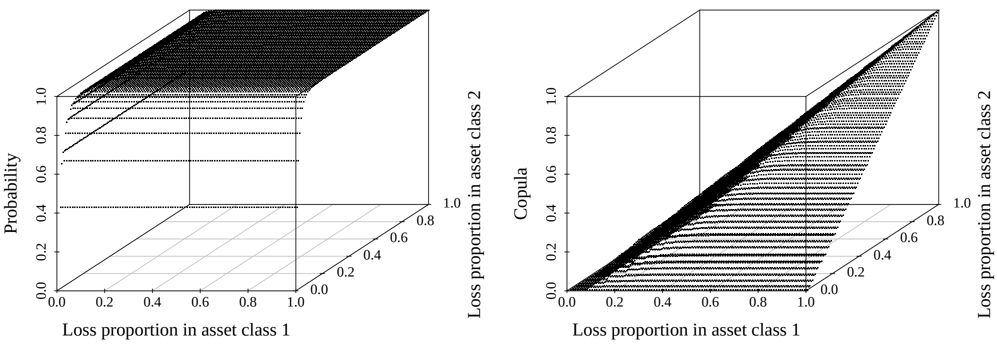
This result drawn from simulation is very powerful and more general than theoretical proof in Theorems 4 and 5.

We choose the input parameters for the model as follows:

- $PD_1 = 1\%$,

- $PD_2 = 2\%$,
- $\alpha_1^2 = 0.19$, so $\varrho_1 = 2.28\%$[8],
- $\alpha_2^2 = 0.16$, so $\varrho_2 = 2.73\%$[8],
- $\alpha_1\alpha_2 = 0.12$, so $\varrho_{12} = 2.49\%$,
- LGD_1, LGD_2 random variables equal 50% or 100% with probability 0.5.

The resulting loss distribution function and empirical copula for a one-factor model are shown in Figure 6.



(a) Joint distribution function

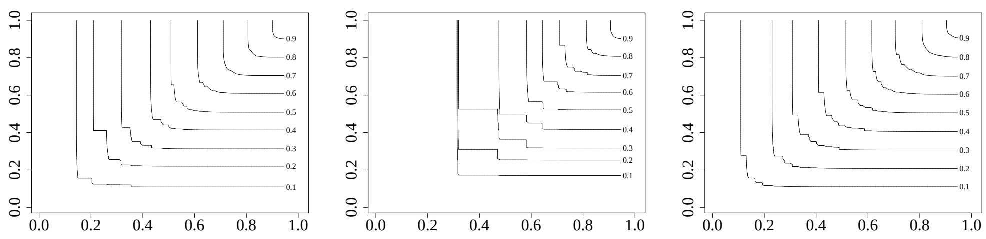
(b) Copula

Figure 6: Simulated joint distribution function and empirical copula of loss variables of two asset classes for $u_1 = u_2 = 2{,}000$ with input parameters as introduced in Section 4.4. Simulated with 10,000 model runs for the loss distribution and 10,000 nodes.

In order to draw initial conclusions about the limit loss distribution function one can look at the plot of the contour lines of the copula based on a one-factor model, and additionally, on a two-factor model as well as a mixture model as shown in Figure 7. The copulas in Figure 7a and 7c show similarity with the Frechét-Hoeffding upper bond. This result is in concordance with the theoretical result in Theorem 4 for the one-factor model.

When simulating the copula for alternative pairs of u_1 and $u_2 \in \{1, ..., 2{,}000\}$ the average distance of the copula functions to the copulas in Figure 7 decreases. As an

[8]To use a realistic input parameter we choose the correlation according to the Basel II formula for big corporations; see e.g., the Basel Basel Committee on Banking Supervision (2010 (rev 2011)).



(a) One-factor model (b) Two-factor model (c) CreditRisk+

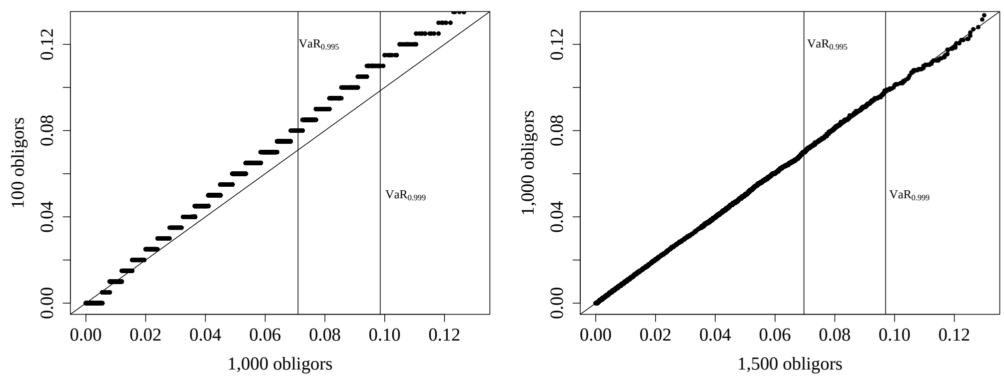
Figure 7: Contour lines of empirical copulas of loss variables for two asset classes for $u_1 = u_2 = 2{,}000$ with input parameters as introduced in Section 4.4. Simulated with 10,000 model runs for the loss distribution and 10,000 nodes.

u_2 \ u_1	10	100	500	1,000	1,500
One-factor model					
10	0.3434	0.2889	0.2683	0.2616	0.2610
100	0.3256	0.1723	0.1138	0.0864	0.0846
500	0.3090	0.1351	0.0558	0.0297	0.0196
1,000	0.3082	0.1301	0.0503	0.0205	0.0142
1,500	0.3079	0.1297	0.0367	0.0152	0.0107
Two-factor model					
10	0.2930	0.2848	0.2654	0.2612	0.2576
100	0.2726	0.2457	0.1781	0.1629	0.1487
500	0.2546	0.2103	0.0935	0.0681	0.0421
1,000	0.2530	0.2061	0.0858	0.0546	0.0310
1,500	0.2522	0.2055	0.0820	0.0525	0.0277
Mixture model					
10	0.3505	0.2946	0.2731	0.2694	0.2681
100	0.3228	0.1651	0.0955	0.0783	0.0762
500	0.3164	0.1302	0.0457	0.0276	0.0231
1,000	0.3150	0.1239	0.0392	0.0167	0.0127
1,500	0.3146	0.1218	0.0344	0.0145	0.0073

Table 4: Convergence of copula measured as average distance of the copula with $u_1 = u_2 = 2{,}000$ with input parameters as introduced in Section 4.4. Simulated with 10,000 model runs for each loss distribution and 10,000 nodes per copula.

example, the results of this simulation are shown in Table 4 for a one-factor model, and respectively, for a two-factor model and mixture model. The convergence seems slow since errors smaller than 1% are only produced with approximately 1,500 obligors per asset class. However, the error is clearly smaller if we focus on the cases of the high number of defaults that are relevant for risk measurement. If we only consider the highest 10% of occuring default numbers per asset class, for example in the case of 100 obligors per asset class, the average error reduces from 0.1723 to 0.0246. Hence, through simulation, we provide evidence of the convergence of the copula function. Based on the convergence of the copula and respectively of the joint distribution function of the two asset classes, we can deduce the convergence of the per-unit risk, independently of the chosen risk measure.

To visualize the results we consider one specification of the model by choosing a specific proportion of asset class sizes according to case 2 of Theorem 4 in Section 3.3. We will see how the per-unit risk changes with the number of obligors and also ascertain how many obligors are necessary to reach a constant per-unit risk with a maximum error of 20 bp. In case 2 we used the following assumption: $u_1 : u_2 = 1$, i.e., both asset classes are the same size.



(a) 100 vs. 1,000 obligors

(b) 1,000 vs. 1,500 obligors

Figure 8: Comparison of Q-Q-plots for two asset classes with fix proportion of number of obligors simulated with 100,000 model runs in a one-factor model as described in formula (7). Both figures describe asset classes with $PD_1 = 1\%, PD_2 = 2\%$ and $\varrho_1 = 0.19, \varrho_2 = 0.16$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class sizes on the x-axis.

Looking at the Q-Q-plot we see that the line in Figure 8a has a higher slope than the bisecting line. This shows that the quantiles of small portfolios are higher, meaning that the VaR contribution per obligor is higher. In Figure 8b, the line almost equals the angle bisector, meaning that the per-unit risk in a 1,000 obligor portfolio is the same as that in a 1,500 obligor portfolio. In Figure 8a, the VaR line bends a little less to the left than in the case of one asset class. This is due to the lower average probability of default of the portfolio. As seen before, a lower PD leads to a lower number of necessary obligors for a constant per-unit risk.

Table 5 shows the minimum number of necessary packages to achieve a constant R_q and respectively a constant per-unit risk. In the case of $PD_1 = 1\%$ and $PD_2 = 2\%$ and with VaR at a confidence level of 95% as risk measure 277 obligors in total, i.e., 139 obligors per asset class, are necessary to achieve convergence of the per-unit risk. The comparatively low number is due to the proportional up-scaling. If, on the other hand, the number of obligors of each asset class is changed individually, each asset class must be in a region of constant per-unit risk.

$\eta \backslash PD$	1%/2%	2%/5%	1%/10%	5%/10%
0.95	277	318	328	361
0.97	321	369	404	394
0.99	405	470	475	505
0.999	545	594	594	692

Table 5: Number of necessary obligors to achieve constant per-unit risk with a maximum error of 20 bp; two asset classes with a fix proportion of number of obligors, correlation according to Basel II formula for big corporations; simulated with 100,000 model runs per asset class size.

The results are in the same order of magnitude as the results we observed with only one asset class. One conspicuous feature needs to be pointed out though. When comparing the case $PD_1 = 2\%, PD_2 = 5\%$ with the case of $PD_1 = 1\%, PD_2 = 10\%$ yields a similar number of obligors even if the average PD differs. This is due to the fact that the number of obligors necessary does not increase linearly with the PD.

5. Conclusion

In this paper we show under which conditions it is justifiable to use the assumption of constant per-unit risk in portfolio credit risk models. This result is especially relevant in portfolio optimization or performance measurement. We study the asymptotic behavior of loss distributions in order to show that, irrespective of the risk measure we use, for a large homogeneous asset class the risk per obligor converges to a limit per-unit risk. We supplement this result through several simulations, showing the effect of the error being made by assuming constant per-unit risk to be limited, as long as each asset class has a minimum number of obligors. In the simulated examples, on average, this minimum portfolio size was approximately 400 obligors per asset class. Simulations show that the exact number is highly dependent on input parameters such as probability of default or the risk measure.

We prove for a one- and two-factor model and give Monte Carlo evidence for other models, that the copula of the loss distributions of two asset classes converges as well. By putting these results together, we can conclude that in all common credit risk models portfolio optimization based on gradient allocation is justified as long as the single asset classes are a minimum size. However, if this minimum size is not achieved, gradient capital allocation could lead to false business decisions. In most cases, the risk of a new obligor in a small asset class might be overestimated. Notice that all results are based on the assumption of homogeneous asset classes that can be in- or decreased without changing the asset class characteristics. Furthermore, only one time period was considered.

For further research it will be crucial to consider portfolios of certain inhomogeneity, in order to gain proximity to real world scenarios. It is necessary to examine what happens when increasing one asset class leads to a change in the asset class characteristics. Furthermore a number of additional constraints or stress scenarios can be added in order to challenge a business decision that is based on the purely mathematical optimization algorithm.

Appendix A. Proof of Theorem 1

Proof. From Lemma 1 follows for any pair of asset classes that the limit of the joint distribution function $\{l_{i,j}^{u_i,u_j}\}$ exists. If the marginal distributions $\tilde{l}_i$ and the copula functions are piecewise continuous, it follows that the joint distribution function as composition of piecewise continuous functions is also piecewise continuous and bounded by $f(x) \equiv 1$. It follows that the integral of the function exists and consequently the loss distribution function of the two asset classes i and j. With induction, the existence of the total loss function of the complete portfolio can be concluded.
The per-unit risk can be calculated via gradient allocation; see approximation (3). □

Appendix B. Proof of Theorem 4

Proof. The first claim follows directly from Theorem 2. If only the number of obligors in the first asset class is increased, the share of obligors in the second asset class converges to zero. The term for the second subportfolio converges to 0, because X describes the fraction of defaults and with the first asset class increasing, the share of the second asset class becomes smaller. To prove the second claim we calculate:

$$\begin{aligned}
\tilde{l}(x) &= P[X \leq x] = \int_{-\infty}^{\infty} P[X \leq x | M = c]\phi(c)dc \\
&= \int_{-\infty}^{\infty} 1_{\{a'PD_1(c)+b'PD_2(c)\leq x\}}\phi(c)dc \\
&= \int_{-\infty}^{\infty} \left(\int_{x'=0}^{x} \left(1_{\{PD_1(c)\leq \frac{x-x'}{a'}\}} \cdot 1_{\{PD_2(c)\leq \frac{x'}{b'}\}} \right) dx' \right) \phi(c)dc \\
&= \int_{x'=0}^{x} \int_{-y}^{\infty} \phi(c)dc \;\; dx',
\end{aligned}$$

where

$$\begin{aligned}
0 &\leq \frac{x-x'}{a'} \leq 1, \;\; 0 \leq \frac{x'}{b'} \leq 1, \text{ i.e., } x - a' \leq x' \leq b'; \\
y &= min\left(\frac{1}{\alpha_1}\Big(\sqrt{1-\alpha_1^2}\;\; \Phi^{-1}\left(\frac{x-x'}{a'}\right) - S_1\Big); \frac{1}{\alpha_2}\Big(\sqrt{1-\alpha_2^2}\;\; \Phi^{-1}\left(\frac{x'}{b'}\right) - S_2\Big)\right).
\end{aligned}$$

For the second line we used part 2 of Theorem 3.
By using that Φ is the antiderivative of ϕ, we obtain the formula in the theorem. □

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