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On the Galois cohomology of ℓ-adic representations attached to varieties over local or global fields

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The title refers to the following problem. For a smooth, projective variety X over a local or global field k, we study Galois cohomological properties of the étale cohomology groups $H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$, where $\bar{X} = X \times_k \bar{k}$ for a separable closure $\bar{k}$ of k. In particular, we are interested in the coranks of the groups

$$H^\nu(\mathrm{Gal}(\bar{k}/k), H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))).$$

For $\nu = 2$ the global question can be reduced to the local one, but for $\nu = 1$ the situation is more difficult, and there are very few results. In the local case one knows a lot more, especially in the case of good reduction, but the general question is unsolved here, too.

In [Ja 3] we discuss the known results and some conjectures, which would imply that the coranks in question have a simple description in a certain "stable range", in particular, for $|n| \gg 0$. Such a description for almost all $n \in \mathbb{Z}$ was announced in [Sou 3], but there is a gap in the proof. Also, we are interested in the precise bounds for the stable range.

The conjectures about this are motivated partly by the function field case, where they can proved to a big extent, partly by conjectures about K–theory and ℓ-adic cohomology, which are related to those of Quillen and Beilinson. For certain subspaces $K_m(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{(n)}$ of the K–groups with coefficients, Quillen's conjecture would imply that the ℓ-adic Chern class maps

$$K_{2n-\nu}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{(n)} \longrightarrow H^\nu(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

have kernels and cokernels of finite exponent for $n \gg 0$, while Beilinson's conjecture suggests a formula for the corank of $K_{2n-\nu}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{(n)}$ for $n \gg 0$ and a number field k. Again we are interested in the precise bounds for n, thereby sharpening Quillen's conjecture in the number field case.

In the following we review the main points of the paper [Ja 3], to which we refer for more details and results, for example the relation with Iwasawa theory.

During this work I had helpful discussions with several people, in particular I am indebted to K. Kato, P. Schneider and K. Wingberg.

Notations : For a pro–finite group G and a topological G–module A let A^G and A_G be the modules of invariants and co–invariants, respectively. For an ℓ–primary torsion group A let A^* be its Pontrjagin dual. Recall that the corank (or dimension) of A can be defined as the $\mathbb{Z}_\ell$–rank of A^*.

1.– Conjectures and questions

For a field k we always let $\bar{k}$ be a separable closure and $G_k = \mathrm{Gal}(\bar{k}/k)$ be its absolute Galois group. Let X be a smooth, projective variety over k and let $\bar{X} = X \times_k \bar{k}$. We are interested in the G_k–modules $H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \varinjlim_r H^i(\bar{X},\mathbb{Z}/\ell^r(n))$, $H^i(\bar{X},\mathbb{Z}_\ell(n)) = \varprojlim_r H^i(\bar{X},\mathbb{Z}/\ell^r(n))$ and $H^i(\bar{X},\mathbb{Q}_\ell(n)) = H^i(\bar{X},\mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, where $H^i(\bar{X},\mathbb{Z}/\ell^r(n))$ is the étale cohomology group of the sheaf $\mathbb{Z}/\ell^r(n) = \mu_{\ell^r}^{\otimes n}$, μ_{ℓ^r} being the sheaf of ℓ^r–th roots of unity (ℓ a prime different from the characteristic of k), cf. [Mi 1] p. 164.

Conjecture 1.

Let k be a number field and let S be a finite set of places of k, containing all places above ℓ and ∞, and all primes where X has bad reduction. Let $G_S = \mathrm{Gal}(k_S/k)$, where k_S is the maximal S–ramified extension of k. Then

$$H^2(G_S,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \text{ is finite if } \begin{array}{l} \text{a) } i+1 < n \\ \text{or} \\ \text{b) } i+1 > 2n \end{array}.$$

This can be reformulated in various ways. Since G_S has the property (F_ℓ) : $H^\nu(G_S,A)$ is finite for every finite ℓ–primary G_S–module A, one easily deduces that $H^\nu(G_S,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ is of cofinite type (i.e. its Pontrjagin dual is of finite type over $\mathbb{Z}_\ell$) and the continuous cohomology $H^\nu(G_S,H^i(\bar{X},\mathbb{Z}_\ell(n)))$ is of finite type over $\mathbb{Z}_\ell$, and that

$$\operatorname{corank} H^\nu(G_S,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) = \operatorname{rank}_{\mathbb{Z}_\ell} H^\nu(G_S,H^i(\bar{X},\mathbb{Z}_\ell(n))).$$

Moreover, this is the same as the $\mathbb{Q}_\ell$–dimension of the continuous cohomology $H^\nu(G_S,H^i(\bar{X},\mathbb{Q}_\ell(n))) \cong H^\nu(G_S,H^i(\bar{X},\mathbb{Z}_\ell(n))) \otimes \mathbb{Q}_\ell$, and an equivalent formulation of conjecture 1 is the vanishing of these numbers for $\nu = 2$ and (i,n) as in a) or b). Also, one can reformulate everything in terms of the first cohomology group :

Lemma 1.

For an ℓ-primary G_S–module N of cofinite type let $\chi(G_S,N) = \sum_{\nu=0}^{3} (-1)^\nu \operatorname{corank} H^\nu(G_S,N)$. *Then*

$$\chi(G_S,N) = -[k:\mathbb{Q}].\operatorname{corank} N + \operatorname{corank}(\operatorname{Ind}_k^{\mathbb{Q}} N)^{G_\infty},$$

where $G_\infty \subset G_\mathbb{Q}$ is a decomposition group at ∞. In particular,

$$\chi(G_S,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))= - \dim_\mathbb{R} H^i(X(\mathbb{C}),\mathbb{R})^{(n-1)} =: \chi_{i,n}(X),$$

where $H^i(X(\mathbb{C}),\mathbb{R})^{(n-1)}$ is the subspace of the singular cohomology of $X(\mathbb{C})$, where the involution F_∞ induced by the complex conjugation acts by $(-1)^{n-1}$.

This follows from Tate's computation of the Euler–Poincaré characteristic of G_S as in Soulé's paper [Sou 3] 1.4.

Lemma 2.

If $i \neq 2n$, *then* $H^i(\bar{X},\mathbb{Q}_\ell(n))^{G_k} = 0 = H^i(\bar{X},\mathbb{Q}_\ell(n))_{G_K}$ *and, moreover,* $H^i(\bar{X},\mathbb{Q}_\ell(n))^{G_p} = 0 = H^i(\bar{X},\mathbb{Q}_\ell(n))_{G_p}$ *for every decomposition group* $G_p \subset G_k$ *at a prime* $p \notin S$.

The first statement is implied by the second one, which follows via smooth and proper base change from the Weil conjectures, cf. [Sou 3] lemma 1. Hence conjecture 1 is equivalent to saying that $H^1(G_S,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ has corank $-\chi_{i,n}(X)$ for $i+1<n$ or $i+1>2n$. Equivalent is also that $H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ has this corank, by the following result, also observed by P. Schneider and W. Raskind.

Lemma 3.

a) *If* $i \neq 2(n-1)$, *then via the inflation*

$$H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \xrightarrow{\text{inf}} H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

the maximal ℓ*-divisible subgroups of these groups coincide.*

b) *The inflation* $H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell(n))) \xrightarrow{\text{inf}} H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell(n)))$ *is an isomorphism for* $i \neq 2(n-1)$.

The first statement follows from the sequence on p. 117 of [Sou 3] and the Weil conjectures (in form of lemma 1 loc. cit.). The proof of the second one is similar.

Conjecture 1 is partly motivated by another one on the ℓ-adic Chern character maps from Quillen's higher algebraic K-groups $K_r(X)$ into continuous étale cohomology

$$K_{2n-\nu}(X) \longrightarrow H^\nu_{\text{cont}}(X,\mathbb{Q}_\ell(n))$$

(see [Sou 2] for the definition of these maps). Following Beilinson, we define the motivic cohomology by $H^\nu_{\mathcal{M}}(X,\mathbb{Q}(n)) = K_{2n-\nu}(X)^{(n)}$, the subspace of

$K_{2n-\nu}(X)\otimes\mathbb{Q}$, where the Adams operators ψ^k act like multiplication by k^n. Then by the Hochschild–Serre spectral sequence ([Ja 1] 3.4)

$$E_2^{p,q} = H^p(G_k, H^q(\bar{X},\mathbb{Q}_\ell(n))) \Longrightarrow H^{p+q}_{cont}(X,\mathbb{Q}_\ell(n))$$

and lemma 2 we get maps

$$(1) \qquad r : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n)) \longrightarrow H^1(G_k, H^i(\bar{X},\mathbb{Q}_\ell(n))),\ i+1 \neq 2n.$$

These can be considered as ℓ-adic analogues of Beilinson's regulator maps $(i+1 \neq 2n)$

$$(2) \qquad r : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n)) \longrightarrow H^i(X(\mathbb{C}),\mathbb{C})^+/(H^i(X(\mathbb{C}),\mathbb{R}(2\pi\sqrt{-1})^n)^+ + F^n),$$

where $F^{\cdot}$ is the Hodge filtration and $H^i(X(\mathbb{C}),-)^+$ is the space fixed by $F_\infty\otimes c$, c the complex conjugation ([Bei]). In fact, it is easy to show that the target of (1) describes extensions of the G_k-representations $\mathbb{Q}_\ell$ and $H^i(\bar{X},\mathbb{Q}_\ell(n))$, while the target of (2) describes extensions of the "real Hodge structures with F_∞" $\mathbb{R}$ and $H^i(X(\mathbb{C}),\mathbb{R})(n)$. Beilinson's conjectures (loc. cit. 2.4.2 and 3.4) would imply that

$$(3) \qquad r : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))\otimes\mathbb{R} \longrightarrow H^i(X(\mathbb{C}),\mathbb{C})^+/H^i(X(\mathbb{C}),\mathbb{R}(2\pi\sqrt{-1})^n)^+$$

is an isomorphism for $i+1<n$ (note that $F^n = 0$ in this range).

Conjecture 2.
$r : H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))\otimes\mathbb{Q}_\ell \longrightarrow H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell(n)))$ *is an isomorphism for* $i+1<n$.

The relation between these conjectures is as follows : Beilinson's conjecture would imply

$$\dim_{\mathbb{Q}} H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n)) = -\chi_{i,n}(X) \quad \text{for } i+1<n,$$

since this is the dimension of the target of (3). By lemmas 1, 2, 3 this is the dimension of $H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell(n)))$ if and only if $H^2(G_S,H^i(\bar{X},\mathbb{Q}_\ell(n)))=0$. Moreover, if the map r in conjecture 2 is surjective, Beilinson's conjecture would imply its bijectivity, since always $\dim_{\mathbb{Q}_\ell} H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell(n))) \geq -\chi_{i,n}(X)$ (lemma 1). There are strong arguments for the surjectivity of r, since by results of Thomason [Tho] (generalizing work of his with Dwyer, Friedlander and Snaith) the analogous Chern character maps

$$K_{2n-i-1}(X,\mathbb{Q}_\ell)^{(n)} \longrightarrow H^1(G_k,H^i(\bar{X},\mathbb{Q}_\ell(n)))$$

on the K–groups with coefficients are surjective for $n \gg 0$. The question, whether the natural map

$$K_{2n-i-1}(X)^{(n)} \otimes \mathbb{Q}_\ell \longrightarrow K_{2n-i-1}(X,\mathbb{Q}_\ell)^{(n)}$$

is an isomorphism, is related to conjectures of Bass on the finite generation of K–groups.

We now come to the local conjecture.

Conjecture 3.

If k *is a local field, then* $H^i(\bar{X},\mathbb{Q}_\ell(n))^{G_k} \neq 0$ *at most for* $0 \leq n \leq \frac{i}{2}$ *in general, and at most for* $i = 2n$, *if* X *has good reduction* (*over the ring of integers of* k).

Tate's local duality theorem [Ta] gives a duality

$$\text{(4)} \qquad H^2(G_k,H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))^* \cong H^{2d-i}(\bar{X},\mathbb{Z}_\ell(d+1-n))^{G_k},$$

where we have used Poincaré duality

$$H^i(\bar{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^* \cong H^{2d-i}(\bar{X},\mathbb{Z}_\ell(d-n)),$$

assuming (without restriction) that X has pure dimension d. Hence an equivalent formulation of conjecture 3 is that $H^2(G_k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ is finite for $i+1 < n$ or $i+1 \geq 2n$. If X is defined over a number field, the local conjecture is implied by the global one –except for the case $n = \frac{i+1}{2}$. This follows from the exact sequence for $N = H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$

$$H^2(G_S, N) \longrightarrow \bigoplus_{\mathfrak{p} \in S} H^2(G_\mathfrak{p}, N) \longrightarrow N(-1)_{G_S} \longrightarrow 0$$

coming from Tate's global duality theorem (loc. cit.). Here $G_\mathfrak{p}$ is a decomposition group at $\mathfrak{p}$ in G_k; hence $G_\mathfrak{p} = G_{k_\mathfrak{p}}$ for the $\mathfrak{p}$–adic completion $k_\mathfrak{p}$ of k. Conversely, the global conjecture 1 would follow from the local one, if the following question had a positive answer.

Question A. Is the kernel of the localization map

$$H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \longrightarrow \bigoplus_{\mathfrak{p} \in S} H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

finite for $i \neq 2n-1$?

Since G_S and the $G_\mathfrak{p}$ satisfy property (F_ℓ), we get an equivalent question if $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$ is replaced by $\mathbb{Z}_\ell(n)$. Still equivalent, by Tate's duality theorem, is the question whether the localization maps

$$H^1(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \longrightarrow \bigoplus_{\mathfrak{p} \in S} H^1(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

have finite kernels for $n \neq \frac{i+1}{2}$, or the same question for $\mathbb{Z}_\ell(n)$.

Let k be a finite extension of $\mathbb{Q}_p$, and let $\ell \neq p$. If X has potentially good reduction, then conjecture 3 is true by base change and the Weil conjectures (same argument as for lemma 2). Otherwise, it is equivalent to the well–known conjecture that the local L–factor

$$\frac{1}{\det(1-\mathrm{Fr}\, q^{-s} \mid H^i(\bar{X},\mathbb{Q}_\ell)^I)}$$

($I \subset G_k$ the inertia group, Fr a geometric Frobenius in G_k/I, and $q = p^f$ the cardinality of the residue field of k) considered at integral places $s = m \in \mathbb{Z}$ has poles at most for $0 \leq m \leq \frac{i}{2}$. This would follow from the more precise conjecture that the quotients $\mathrm{Gr}_r^M H^i(\bar{X},\mathbb{Q}_\ell)$ of the monodromy filtration $M.$ on $H^i(\bar{X},\mathbb{Q}_\ell)$ are pure of weight $r + i$ (see [De 1]), and that $\mathrm{Gr}_r^M H^i(\bar{X},\mathbb{Q}_\ell) = 0$ for $|r| > i$ as in the geometric case, cf. below (note that $H^i(\bar{X},\mathbb{Q}_\ell)^I \subseteq M_0 H^i(\bar{X},\mathbb{Q}_\ell)$ by definition of $M.$).

Now let $\ell = p$. If X has good reduction, then Fontaine's crystalline conjecture ([Fo] appendix) would imply a canonical isomorphism

$$H^i(\bar{X},\mathbb{Q}_\ell(m))^{G_k} \cong \{v \in H^i_{\mathrm{cris}}(X_s/W(\kappa)) \otimes k^0 \mid \phi v = p^m v\} \cap F^m.$$

Here $H^i_{\mathrm{cris}}(X_s/W(\kappa))$ is the crystalline cohomology of the reduction X_s of X with values in the ring of Witt vectors of the residue field κ of k, $k^0 = \mathrm{Quot}(W(\kappa))$, ϕ is the crystalline Frobenius, and $F^\cdot$ is the Hodge filtration, induced by the canonical isomorphism

$$H^i_{\mathrm{cris}}(X_s/W(\kappa)) \otimes k \cong H^i_{\mathrm{DR}}(X/k).$$

By the Weil conjectures for the crystalline Frobenius (see [KM]) this would prove conjecture 3 in this case. The crystalline conjecture has been proved by Fontaine and Messing for $k = k^0$ and $p > \min(i, \dim X)$ [FM], and has been announced by Faltings for $k = k^0$ and arbitrary p.

If X has bad reduction, then $H^i(\bar{X},\mathbb{Q}_p)$ can neither be expected to be crystalline, nor to have a suitable monodromy filtration, as was observed by Mazur, Tate and Teitelbaum ([MTT] II § 15). Moreover, while the dimensions of $H^i(\bar{X},\mathbb{Q}_\ell(m))^{G_k}$, for $\ell \neq p$, should all be the same, there are examples where $\dim H^i(\bar{X},\mathbb{Q}_p(m))^{G_k}$ is smaller, see [Ja 3] § 5.

This suggests looking for a monodromy filtration on the crystalline side, and in [Ja 3] I propose a conjecture which would imply a formula

$$H^i(\bar{X},\mathbb{Q}_p(m))^{G_{k'}} \cong \{v \in D \mid \phi v = p^m v,\ Nv = 0\} \cap F^m$$

for a finite extension k' of k. Here D is a certain filtered module over k' ([Fo] 5.1), equipped with a nilpotent homomorphism $N : D \longrightarrow D$ satisfying $\phi^{-1}N\phi = pN$ (no compatibility with $F^{\cdot}$ assumed). If $M.$ is the monodromy filtration associated to N, then $\phi^{f'}$ should have the same characteristic polynomial on $Gr_r^M D$ as Fr' has on $Gr_r^M H^i(\bar{X},\mathbb{Q}_\ell)$, $\ell \neq p$ (f' and Fr' being defined for k' like f and Fr are for k). Together with the conjecture on the ℓ-adic monodromy this would imply conjecture 3 for $H^i(\bar{X},\mathbb{Q}_\ell)$. I do not have a precise definition for D yet, but I would expect a relation with the crystalline cohomology of the special fibre in the case of semi-stable reduction.

2.– Results

For $X = \operatorname{Spec} k$, where we consider $H^0(\operatorname{Spec} \bar{k}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$, conjecture 1 claims that

$$H^2(G_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \quad \text{is finite for} \quad \begin{matrix} \text{a)} & n > 1 \\ \text{or} & \\ \text{b)} & n < 1 \end{matrix} .$$

For $\ell \neq 2$ or k totally imaginary this means that the considered groups actually vanish for $n \neq 1$, and this has been conjectured by P. Schneider [Sch]. In fact, conjecture 1 can be regarded as a generalization of his conjecture to higher-dimensional varieties.

Theorem 1.

Let $X = \operatorname{Spec} k$, *or a form of the standard cellular varieties* ($\mathbb{P}^n_k$, $Gr_{k,N}$,...). *Then conjecture* 1a), *conjecture* 2 *and conjecture* 3 *are true for* X.

In fact, all conjectures are true, if they are true over a finite extension of k. Hence we may assume that $H^i(\bar{X},\mathbb{Q}_\ell)$ is zero for i odd and consists of copies of

$\mathbb{Q}_p(-\frac{i}{2})$ for i even. Thus conjecture 1 and 3 are immediately reduced to the case $X = \operatorname{Spec} k$. Conjecture 2 can also be reduced to $\operatorname{Spec} k$, by considering stratifications by affine spaces and formulating and proving conjecture 2 also for these, see [Ja 2]. For $X = \operatorname{Spec} k$, conjecture 3 is trivial, and the other conjectures follow directly from work of Soulé, see [Sou 1].

Remark 1.

a) Quite generally, part b) of conjecture 1 seems to be more mysterious than part a), partly because of the lacking connection with K–theory. For $X = \operatorname{Spec} k$ it contains the Leopoldt conjecture, which is the case $n = 0$, cf. [Sch] § 7.

b) Schneider's conjecture amounts to determinating the coranks of

$$H^r(\operatorname{Spec} k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)).$$

There are two obvious ways to generalize this to higher–dimensional varieties X. First, one may replace $\operatorname{Spec} k$ by X and study

$$H^r(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)),$$

the cohomology of X over k. This is done in [Sou 3] (but see [Ja 3] for a necessary correction). In our approach, we have replaced the coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n) = H^0(\operatorname{Spec} \bar{k}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ by $H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ and studied

$$H^r(\operatorname{Spec} k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))).$$

Of course, both approaches are closely connected by the Hochschild–Serre spectral sequence, but the second one is better suited for the considerations of motives and their L–functions – for example the L–function of the G_k–representation $H^i(\bar{X}, \mathbb{Q}_\ell)$. Also the bounds for the vanishing of the cohomology groups can be made more precise. In fact, the basic idea is that the vanishing is governed by "motivic" properties of the representations $H^i(\bar{X}, \mathbb{Q}_\ell)$, like purity, weight, entireness, etc., compare the function field case below.

Theorem 2.

a) *Conjecture* 3 *is true for* $i = 1$.

b) *Let* E *be an elliptic curve over an imaginary quadratic field* k *with complex multiplication by* k. *If* ℓ *is a regular prime for* E, *then* $H^2(G_S, H^1(\bar{E}, \mathbb{Q}_\ell(n))) = 0$ *for all* $n \in \mathbb{Z}$.

Since $H^1(\bar{X}, \mathbb{Z}_\ell(1)) \cong T_\ell A$, the Tate module of the abelian variety $A = \underline{\mathrm{Pic}}^0_{X/k}$, part a) follows from the results in SGA 7 I and IX. Namely, by loc. cit. the monodromy conjecture is true for $\ell \neq p$, and there is even a p–adic monodromy filtration $M.$ on $H^1(\bar{X}, \mathbb{Q}_p(1)) = T_p(A) \otimes \mathbb{Q}_p$, such that the quotients are crystalline ([Fo]) and such that the crystalline Frobenius on the associated filtered modules has the same characteristic polynomial as the geometric Frobenius has on the ℓ–adic counterparts $\mathrm{Gr}^M_r H^1(\bar{X}, \mathbb{Q}_\ell(1))$ for $\ell \neq p$. Moreover, it follows from SGA 7 IX 5.8 that $H^1(\bar{X}, \mathbb{Q}_p(m))^{G_k} \subseteq M_0 H^1(\bar{X}, \mathbb{Q}_p(m))$, so one can proceed as in the case of good reduction, by using the Weil conjectures.

Part b) has been proved by K. Wingberg [Wi].

Remark 2. Since $H^i(\bar{A}, \mathbb{Q}_\ell) \cong \Lambda^i H^1(\bar{A}, \mathbb{Q}_\ell)$ for an abelian variety A over a field k and all primes $\ell \neq$ char k, the monodromy conjecture – and hence conjecture 3 – is also true for this cohomology group, if $\ell \neq p$ = residue characteristic of the local field. For $H^i(\bar{A}, \mathbb{Q}_p)$ one also gets a filtration, with crystalline quotients of a known nature, but the missing thing is the proof that $H^i(\bar{A}, \mathbb{Q}_p))^{G_k} \subseteq M_0\, H^i(\bar{A}, \mathbb{Q}_p)$. Of course, this would follow from the generalized crystalline conjecture mentioned above.

Theorem 3.

Let k *be a global field of characteristic* $\neq \ell$ *and let* V *be a finite-dimensional* $\mathbb{Q}_\ell$*–representation of* G_k *which is pure of weight* $\neq 0$ (*see* [De 2] 6.1.1, *for example this is the case for* $H^i(\bar{X}, \mathbb{Q}_\ell(n))$, $i \neq 2n$). *Let* $\Lambda \subset V$ *be a* $\mathbb{Z}_\ell$*–lattice respected by* G_k, *and for every place* v *of* k *let* $G_v \subset G_k$ *be a decomposition group at* v. *Then*

a) $\alpha_1(\Lambda) \quad : \quad H^1(G_k,\Lambda) \xrightarrow{\text{res}} \prod_v H^1(G_v,\Lambda)$ *is injective,*

b) $\alpha_1(\Lambda') \quad : \quad H^2(G_k,\Lambda') \xrightarrow{\text{res}} \bigoplus_v H^2(G_v,\Lambda')$ *is injective,*

for $\Lambda' = \mathrm{Hom}\,(\Lambda,\mu_{\ell^\infty}) = \Lambda^*(1)$,

c) $\varprojlim_r H^2(G_k,\Lambda^+/\ell^r) \longrightarrow \prod_v H^2(G_v,\Lambda^+)$ *has a finite kernel, for*

$\Lambda^+ = \mathrm{Hom}(\Lambda,\mathbb{Z}_\ell(1)) = \mathrm{Hom}(\Lambda,\mathbb{Z}_\ell)(1)$.

Proof. Let $\mathcal{G} = \mathrm{Im}(G_k \longrightarrow \mathrm{Aut}\,\Lambda)$, then, by an argument of Serre ([Se] 2), $H^\nu(\mathcal{G},\Lambda)$ is finite for all $\nu \geq 0$. From the long exact cohomology sequence associated to the exact sequence $0 \longrightarrow \Lambda \xrightarrow{\ell^r} \Lambda \longrightarrow \Lambda/\ell^r \longrightarrow 0$ we see that $H^1(\mathcal{G},\Lambda/\ell^r)$ is finite, of an order bounded independently of r. Hence the same is true for $H^1(\mathcal{G},\Lambda/\ell^r)$, where $\mathcal{G}_r = \mathrm{Im}(G_k \longrightarrow \mathrm{Aut}(\Lambda/\ell^r))$. Consider the commutative exact diagram

$$\begin{array}{ccccccc}
0 \to & \prod_v H^1(\mathcal{G}_{r,v},\Lambda/\ell^r) & \to & \prod_v H^1(\mathcal{G}_v,\Lambda/\ell^r) & \to & \prod_v H^1(H_{r,v},\Lambda/\ell^r) \\
& \uparrow & & \uparrow & & \uparrow \\
0 \to & H^1(\mathcal{G}_r,\Lambda/\ell^r) & \to & H^1(G_k,\Lambda/\ell^r) & \to & H^1(H_r,\Lambda/\ell^r)
\end{array}$$

where $H_r = \ker(G_k \to \mathrm{Aut}(\Lambda/\ell^r))$, and $\mathcal{G}_{r,v} = \mathrm{Im}(G_v \to \mathcal{G}_r)$ and $H_{r,v} = H_r \cap G_v$ are the decomposition groups in $\mathcal{G}_r$ and H_r, respectively. By definition, H_r acts trivially on Λ/ℓ^r, so the right vertical map is injective by Čeboratev's theorem. Hence the kernels of the left vertical maps are the same, and by passing to the limit over r we get

$$\ker \alpha_1(\Lambda) \subseteq H^1(\mathcal{G},\Lambda).$$

On the other hand, there is a decomposition group G_v with $\Lambda^{G_v} = 0$, since Λ is of weight $\neq 0$ (same argument as for lemma 2). The commutative diagram

$$\begin{tikzcd}
0 = \Lambda^{G_v} \arrow[r] & (\Lambda/\ell^r)^{G_v} \arrow[r] & H^1(G_v,\Lambda) \arrow[r, "\ell^r"] & H^1(G_v,\Lambda) \\
0 = \Lambda^{G_k} \arrow[r] \arrow[u] & (\Lambda/\ell^r)^{G_k} \arrow[r] \arrow[u, hook] & H^1(G_k,\Lambda) \arrow[r, "\ell^r"] \arrow[u] & H^1(G_k,\Lambda) \arrow[u]
\end{tikzcd}$$

then shows that $\ker \alpha_1(\Lambda)$ is torsion free, hence zero.

Statement b) follows from a) via the duality theorem of Tate and Poitou. So does c) : Since $\Lambda^+/\ell^r \cong \mathrm{Hom}(\Lambda/\ell^r,\mu_{\ell^r})$, the duality theorem gives an isomorphism $\ker \alpha_2(\Lambda^+/\ell^r) \cong \ker \alpha_1(\Lambda/\ell^r)^*$, and the latter group has an order bounded independently of r by the above.

Remark 3. In contrast to the local case, $\varprojlim_r H^2(G_k,\Lambda^+/\ell^r)$ is not in general the continuous cohomology of Λ^+. Via the exact sequence

$$0 \longrightarrow \varprojlim_r{}^1 H^1(G_k,\Lambda^+/\ell^r) \longrightarrow H^2(G_k,\Lambda^+) \longrightarrow \varprojlim_r H^2(G_k,\Lambda^+/\ell^r) \longrightarrow 0,$$

the group $\varprojlim_r{}^1 H^1(G_k,\Lambda^+/\ell^r)$ is the subgroup of ℓ–divisible elements in $H^2(G_k,\Lambda^+)$ ([Ja 1] 5.16). It is mapped to zero under the localization map since $H^2(G_v,\Lambda^+) = \varprojlim_r H^1(G_k,\Lambda^+/\ell^r)$ is ℓ–complete.

The following result provides part of the motivation for the conjectures of chapter 1.

Theorem 4.

a) *If* k *is a global function field, then the analogue of conjecture 1 is true, and the analogue of question* A *has a positive answer for* $i \neq 2n-1, 2n-2$.

b) *Conjecture* 3 *is true for globally defined varieties over local fields of positive characteristic.*

Let us indicate the proof of the theorem and at the same time explain what "analogue" and "globally defined" means. Let $\mathbb{F}_q$ be a finite field, let Y be a smooth, projective curve over $\mathbb{F}_q$, let $j : U \hookrightarrow Y$ be an open, affine part, and let $\ell \neq \mathrm{char}(\mathbb{F}_q)$ be a prime. Let $\eta = \mathrm{Spec}\, k$ be the generic point of U and let $\pi_1(U,\bar\eta)$ be the fundamental group of U with base point $\bar\eta = \mathrm{Spec}\, \bar k$. Then U and $\pi_1(U,\bar\eta)$ are the analogues respectively, of the spectrum of $\mathcal{O}_S$ (the ring of S–integers) and $G_S = \pi_1(\mathrm{Spec}\, \mathcal{O}_S, \mathrm{Spec}\, \bar k)$ in the number field case.

A smooth (= twisted–constant) $\mathbb{Q}_\ell$–sheaf F on U corresponds to the $\pi_1(U,\bar\eta)$–representation $V = F_{\bar\eta}$ (stalk in $\bar\eta$), and we have a long exact sequence and identifications

$$
\begin{array}{ccccccc}
 & & H^r(\pi_1(U,\bar\eta),V) & \xrightarrow{\alpha_r(V)} & \bigoplus\limits_{y\in Y\setminus U} H^r(G_y,V) & & \\
(5) & & \wr\big| & & \wr\big| & & \\
\ldots \to H^r(Y,j_!F) & \xrightarrow{\beta_r} & H^r(U,F) & \longrightarrow & \bigoplus\limits_{y\in Y\setminus U} H^r_y(Y,j_!F) & \to H^{r+1}(Y,j_!F) \to & \ldots,
\end{array}
$$

where $G_y \subset G_k$ is a decomposition group at $y \in Y$, and α_r is the localization map, induced by the restriction (see [Mi 2] II 1.1 and 2.9). Recall that F is called pure of weight i, if for every closed point $x \in U$ the eigenvalues α of the geometric Frobenius Fr_x on the stalk $F_{\bar x}$ are algebraic numbers with $|\alpha| = (Nx)^{\frac{i}{2}}$ for every archimedean absolute value $|\;|$, where Nx is the cardinality of the residue field $\kappa(x)$ of x. F is called entire, if the α are algebraic integers. Then the following is a reformulation of results of Deligne in [De 2] 1.8 and 1.10.

Lemma 4.

Fix $y \in Y \setminus U$, *and let* $M.$ *be the monodromy filtration on the* G_y*–representation* V (*loc. cit.* 1.7). *If* F *is pure of weight* i, *then* $Gr^M_r V = M_r V/M_{r-1}V$ *is pure of weight* $r+i$ (*in the sense of loc. cit.* 1.7). *If in addition* F *is entire, then* $Gr^M_r V$ *is entire, in particular,* $Gr^M_r V = 0$ *for* $|r| > i$.

Corollary 1.

If F *is entire of weight* i, *then* $V^{I_y} \subseteq M_0V$ *is mixed, with weights* $w \in \{0,1,\dots,i\}$ *(here* $I_y \subset G_y$ *is the inertia group). In particular,*

$$V(m)^{G_y} = 0 \quad \text{for } m < 0 \text{ or } i < 2m,$$

where $V(m) = V \otimes \mathbb{Z}_\ell(1)^{\otimes m}$ *is the Tate twist of* V.

We may apply this to $V = H^i(\bar{X},\mathbb{Q}_\ell)$, for a smooth and proper variety X over k. In fact, for a suitable $U \longrightarrow Y$ there is a smooth and proper model $f : \mathscr{X} \longrightarrow U$ of X, and then $F_{\bar{x}} \cong H^i(\bar{\mathscr{X}}_x,\mathbb{Q}_\ell)$ for $F = R^if_*\mathbb{Q}_\ell$ and $x \in U$ by smooth and proper base change (see, e.g., [Mi 1] VI 4.2), $\mathscr{X}_x$ being the fibre of f at x. Hence $F_{\bar{\eta}} = V$, and F is entire of weight i by Deligne's proof of the Weil conjectures. Similarly, $V \cong H^i(\overline{X \times_k k_y},\mathbb{Q}_\ell)$ for the completion k_y of k at y, so lemma 4 and corollary 1 prove the analogues of the monodromy conjecture and conjecture 3 for $X \times_k k_y$, i.e., for smooth and proper varieties over k_y that are defined over global function fields k (i.e., are obtained via base change to the completion).

For question A, we have to consider the vanishing of the map β_r in (5), which factorizes through $H^r(Y,j_*F)$. The Hochschild–Serre spectral sequence gives short exact sequences

$$0 \longrightarrow H^{r-1}(\bar{Y},j_*F)_\Gamma \longrightarrow H^r(Y,j_*F) \longrightarrow H^r(\bar{Y},j_*F)^\Gamma \longrightarrow 0,$$

where $\Gamma = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, and Deligne has proved that $H^r(\bar{Y},j_*F)$ is pure of weight $w+r$, if F is pure of weight w ([De 2] 3.2.2). Therefore

$$H^2(Y,j_*F) = 0 \quad \text{if} \quad w \neq -1,-2.$$

Corollary 2.

The localization map

$$H^2(\pi_1(U,\bar{\eta}),V) \xrightarrow{\text{res}} \bigoplus_{y\in Y\setminus U} H^2(G_y,V)$$

is injective for every $\pi_1(U,\bar{\eta})$*–representation* V *which is pure of weight* $w \neq -1,-2$. *In particular, this is the case for* $V = H^i(\bar{X},\mathbb{Q}_\ell(n))$, X *smooth and proper over* k *and having good reduction over* U, *provided* $i \neq 2n-1, 2n-2$.

Since Tate's local duality (4) is also valid for local fields of positive characteristic $p \neq \ell$, corollaries 1 and 2 together imply

$$H^2(\pi_1(U,\bar{\eta}), H^i(\bar{X},\mathbb{Q}_\ell(n))) = 0 \text{ if } \begin{array}{l} \text{a) } i+1 < n \\ \text{or} \\ \text{b) } i+1 > 2n, \end{array}$$

which is the function field analogue of conjecture 1.

Corollary 2 is also true for $w = -2$, if $V(-1)_{G_k} = 0$ or if V is semi–simple, see [Ja 3] § 6, but, by looking at the example of abelian varieties, one can show that $w = -1$ (and, similarly, $i = 2n-1$ in question A) has definitively to be excluded.

Added in proof: In a letter to the author (November 1987), J.–M. Fontaine proposed a precise formulation for the p–adic monodromy conjecture stated at the end of § 1, and proved it for abelian varieties.

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