Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives

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In §1 of this paper we review the definition and the properties of Deligne homology as given by Beilinson [Be 1] and Gillet [Gi 2]. For a complex variety X, $H_a^{\mathcal{D}}(X,A(b))$ ($A = \mathbb{Z}, \mathbb{R}, \dots$) is the extension of two groups described in terms of the Borel-Moore homology $H_\nu(X,-)$, the deRham homology $H_\nu^{DR}(X)$, and its Hodge filtration. Most of the other properties - functorialities, relative sequences, products - are summarized by the fact that Deligne cohomology and homology form a twisted Poincaré duality theory in the sense of Bloch and Ogus [BO]. Here the usual Poincaré duality in form of a non-degenerate pairing - which does not exist for the Deligne cohomology - is replaced by a duality isomorphism for smooth X .

$$H_{\mathcal{D}}^i(X,A(j)) \xrightarrow{\sim} H_{2d-i}^{\mathcal{D}}(X,A(d-j)) \quad , \quad d = \dim X \ .$$

Since homology is covariant for proper morphisms, this still suffices to define the Gysin morphisms needed for the operation of algebraic correspondences. Another application of homology is the easy definition of cycle classes, leading directly to the Abel-Jacobi map.

The construction of Deligne homology is based on currents and C^∞-chains and follows the lines of Deligne's basic papers [De 2], [De 3]: For smooth varieties one works with smooth compactifications and logarithmic singularities, and for arbitrary ones with "simplicial resolutions", i.e., by replacing a variety by a suitable simplicial one with the same

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(co)homology. By this method any situation $U \subset X \supset Y$, where X is proper, $U \subseteq X$ is open and $Y = X \setminus U$ is the closed complement, can be transformed into a simplicial situation, where $X_\bullet$ is smooth and proper, hence $U_\bullet$ is smooth, and where $Y_\bullet$ is a divisor with normal crossings, so that the technique of logarithmic singularities applies.

For the proofs and the understanding of Deligne homology one needs the Hodge theory of Borel-Moore homology. Since I could not find good references, I have given a short but complete treatment of this in §2, based on Deligne's theory of mixed Hodge complexes. These also lead naturally to Beilinson's absolute Hodge cohomology, which gives a refinement and new interpretation of Deligne cohomology, by relating it to morphisms and extensions of mixed Hodge structures.

Beilinson has not only defined Chern maps and characters

$$c, ch : K_{2j-i}(X) \to H^i_{\mathcal{D}}(X, \mathbb{Q}(j)) ,$$

but also homological counterparts

$$\tau : K'_{a-2b}(X) \to H^{\mathcal{D}}_a(X, \mathbb{Q}(b)) ,$$

which together with ch form a Riemann-Roch theorem as in [Gi 1]. Since many constructions in the K-theory, like Gysin maps or the Quillen spectral sequence, are defined via the K'-groups, this is very useful for calculations, even if one is mainly interested in the regulator maps

$$r = ch : K_{2j-i}(X)^{(j)} \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$$

for a smooth and proper variety X. Beilinson's conjecture on the surjectivity of $r \otimes \mathbb{R}$ and results of Suslin and Soulé on the Adams eigenspaces $K_{2j-i}(X)^{(j)} \subset K_{2j-i}(X) \otimes \mathbb{Q}$ lead to a conjecture on the coniveau filtration of $H^i_{\mathcal{D}}(X, \mathbb{R}(j))$. This is the Hodge-$\mathcal{D}$-conjecture. We review all this in §3 and illustrate it by an example, proving a formula of Beilinson for the regulator on $K_1(X)$.

In §4 we recall Beilinson's conjecture for motives with coefficients, i.e., for Dirichlet series with coefficients in a number field $E \neq \mathbb{Q}$, and report on some ideas of P. Deligne written down in the letter [De 5]. Among other things Deligne reformulates Beilinson's conjecture in terms of L-values in the range of convergence and a different rational structure on

$H^i_{\mathcal{D}}(X, \mathbb{R}(j))$, and moreover he interpretes the conjecture in the setting of a - conjectural - category of mixed motives.

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§1 Deligne homology

Let X be a smooth proper analytic space over $\mathbb{C}$ or over $\mathbb{R}$, of dimension d. Define

$\Omega^{p,q}_{X^\infty}$ = sheaf of C^∞-(p,q)-forms on X (often called $A^{p,q}_X$),

$'\Omega^{p,q}_{X^\infty}$ = sheaf of distributions over $\Omega^{-p,-q}_{X^\infty}$

= sheaf of currents of type (d+p,d+q) on X (often called $'\mathcal{D}^{d+p,d+q}_X$).

Thus an element of $'\Omega^{p,q}_X(U)$ for $U \subseteq X$ open is a continuous linear functional on $\Gamma_c(U, \Omega^{-p,-q}_{X^\infty})$.

1.1. Examples a) Each C^∞-(p,q) form ω gives a section of $'\Omega^{p-d,q-d}_{X^\infty}$ by

$$\omega' \mapsto \frac{1}{(2\pi\sqrt{-1})^d} \int_X \omega' \wedge \omega \quad .$$

Here the choice of $\sqrt{-1}$ determines an orientation on all complex analytic spaces X (such that on $X = \mathbb{C}$ with the coordinate $z = x + \sqrt{-1}\, y$ the differential $dx \wedge dy$ is a volume form), and the factor in front of the integral makes the expression independent of the choice of $\sqrt{-1}$ and the association compatible with the action of the involution σ in the case of a real analytic space X (see below).

b) An L^1-function f gives a section of $'\Omega^{-d,-d}_{X^\infty}$ by

$$\omega' \mapsto \frac{1}{(2\pi\sqrt{-1})^d} \int_X f \cdot \omega' \ .$$

c) A smooth oriented topological C^∞-r-chain $M \subset X$ gives an element δ_M in $\bigoplus_{p+q=r} {}'\Omega^{p,q}_{X^\infty}(X)$ by integration:

$$\omega' \mapsto \int_M \omega' .$$

$\Omega^{\cdot\cdot}_{X^\infty}$ and $'\Omega^{\cdot\cdot}_{X^\infty}$ form double complexes in a natural way; let $\Omega^{\cdot}_{X^\infty}$ and $'\Omega^{\cdot}_{X^\infty}$ be the associated simple complexes and F^i be the first filtration on these

$$F^{i}\,{}^{(')}\Omega^n_{X^\infty} = \bigoplus_{\substack{p+q=n\\ p\geq i}} {}^{(')}\Omega^{p,q}_{X^\infty} .$$

We normalize the differential on $'\Omega^{\cdot}_{X^\infty}$ such that $dD(\omega) = (-1)^{\deg D}D(d\omega)$. This differs from the convention in [GH] p. 369, but we also have interchanged ω and ω' in 1.1 a) compared with [GH]. By our choices the pairing

$$\Omega^{\cdot}_{X^\infty}(X) \otimes {}'\Omega^{\cdot}_{X^\infty}(X) \to {}'\Omega^{\cdot}_{X^\infty}(X)$$

is a morphism of complexes <u>and</u> induces a left operation of $\Omega^{\cdot}_{X^\infty}$ on $'\Omega^{\cdot}_{X^\infty}$.

<u>1.2. Lemma</u> The natural embeddings

$$(\Omega^{\cdot}_X, F^i) \hookrightarrow (\Omega^{\cdot}_{X^\infty}, F^i) \xhookrightarrow{1.1a)} ({}'\Omega^{\cdot}_{X^\infty}[-2d], F^{i-d})$$

are filtered quasi-isomorphisms.

<u>Proof</u> It is clear that the filtrations are transformed as indicated, and on the graded pieces we have quasi-isomorphisms

$$\begin{array}{ccccc}
\Omega^i_X & \to & \Omega^{i,\cdot}_{X^\infty} & \to & {}'\Omega^{i-d,\cdot-d}_{X^\infty} \\
\wr\Vert\uparrow & & \wr\Vert\uparrow & & \wr\Vert\uparrow \\
\Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{O}_X & \to & \Omega^i_X \otimes_{\mathcal{O}_X} \Omega^{0,\cdot}_{X^\infty} & \to & \Omega^i_X \otimes_{\mathcal{O}_X} {}'\Omega^{-d,\cdot-d}_{X^\infty}
\end{array} ,$$

which follow from the case $i=0$ (∂- and $\bar\partial$-lemma), since Ω^i_X is flat over $\mathcal{O}_X$.

For a subring $\mathbb{Z} \subseteq A \subseteq \mathbb{R}$ and $k \in \mathbb{Z}$ let $C.(X,A(k))$ be the complex of singular C^∞-chains on X with coefficients in $A(k) = A(2\pi\sqrt{-1})^k$, and let $'C^{\cdot}(X,A(k))$ be the associated cohomological complex: $'C^i = C_{-i}$ and $('C^i \xrightarrow{'d} {}'C^{i+1}) = (C_{-i} \xrightarrow{(-1)^i d} C_{-i-1})$. The sign is most naturally obtained by regarding a chain complex in an abelian category A as a cochain complex in the dual category A°, and then applying the sign convention of [SGA 4] XVII 1.1.5 to the contravariant functor $\mathrm{id}: A^{\circ} \to A$. Also it is necessary for making the evaluation map

$$C^{\bullet}(X,A) \otimes {}'C^{\bullet}(X,A(k)) \to A(k)$$

a morphism of complexes, where $C^{\bullet}(X,A)$ is the complex of C^{∞}-cochains with values in A and the usual differential $df(c) = f(dc)$. Finally, with our sign conventions 1.1 c) defines a morphism of complexes

$$\varepsilon : {}'C^{\bullet}(X,A(k)) \to {}'\Omega^{\bullet}_{X^{\infty}}(X) .$$

If X is an analytic space over $\mathbb{R}$, which by definition is a complex analytic space with an antiholomorphic involution $F_{\infty} : X \to X$, the sheaves F on X used above are equipped with involutions σ over F_{∞} (i.e., morphisms $\sigma: F \to (F_{\infty})_*F$ such that $F \overset{\sigma}{\to} (F_{\infty})_*F \xrightarrow{(F_{\infty})_*(\sigma)} (F_{\infty}^2)_*F = F$ is the identity), by sending a differential ω to $\overline{F_{\infty}^*\omega}$ and a distribution D to $\overline{F_{\infty *}D}$. The induced involution on the group $F(X)$ of global sections will also be denoted by σ. There is also an involution σ on ${}'C^{\bullet}(X,A(k))$, such that $\sigma(\alpha[M]) = \bar{\alpha}[F_{\infty}M]$, and all maps above then are σ-equivariant.

<u>1.3. Definition</u> With the inclusion $\iota: F^k\,{}'\Omega_{X^{\infty}}(X) \hookrightarrow {}'\Omega^{\bullet}_{X^{\infty}}(X)$ define

$${}'C^{\bullet}_{\mathcal{D}}(X,A(k)) = \mathrm{Cone}({}'C^{\bullet}(X,A(k)) \oplus F^k\,{}'\Omega^{\bullet}_{X^{\infty}}(X) \xrightarrow{\varepsilon-\iota} {}'\Omega^{\bullet}_{X^{\infty}}(X))[-1],$$

and call ${}'H^{\ell}_{\mathcal{D}}(X,A(k)) = H^{\ell}({}'C^{\bullet}_{\mathcal{D}}(X,A(k)))$ the ℓ-th Deligne homology group of X ($\ell \in \mathbb{Z}$). If X is an analytic space over $\mathbb{R}$ let ${}'H^{\ell}_{\mathcal{D}}(X,A(k)) = {}'H^{\ell}_{\mathcal{D}}(X/\mathbb{R},A(k)) = H^{\ell}(\langle\sigma\rangle, {}'C^{\bullet}_{\mathcal{D}}(X,A(k)))$ (group hypercohomology).

We now consider smooth, not necessarily proper varieties by introducing logarithmic singularities. Let $U \subset X$ be an open subspace such that the complement $Y = X\smallsetminus U$ is a divisor with normal crossings. Then the complex $\Omega^{\bullet}_X\langle Y\rangle$ of holomorphic differentials with logarithmic singularities along Y is defined and has locally free components [De 2] 3.1. Let F^i be the naive descending filtration on $\Omega^{\bullet}_X\langle Y\rangle$, as in [De 2] 3.2.2. Then the complexes $\Omega^{\bullet}_X\langle Y\rangle$ and $\Omega^{\bullet}_{X^{\infty}}$ are filtered bi-modules over the anticommutative differential graded algebra $\Omega^{\bullet}_X$ (by the exterior product), and by functoriality there is an induced bi-module structure on $'\Omega^{\cdot}_{X^\infty}$.

<u>1.4 Definition</u> Define the complexes of C^∞-forms, resp. currents, with logarithmic singularities along Y by

$$\Omega^{\cdot}_{X^\infty}\langle Y\rangle = \Omega^{\cdot}_X\langle Y\rangle \otimes_{\Omega^{\cdot}_X} \Omega^{\cdot}_{X^\infty} ,$$

resp.
$$'\Omega^{\cdot}_{X^\infty}\langle Y\rangle = \Omega^{\cdot}_X\langle Y\rangle \otimes_{\Omega^{\cdot}_X} {}'\Omega^{\cdot}_{X^\infty} ,$$

and define filtrations F^i on these by

$$F^i\, \Omega^{\cdot}_{X^\infty}\langle Y\rangle = F^i\, \Omega^{\cdot}_X\langle Y\rangle \otimes_{\Omega^{\cdot}_X} \Omega^{\cdot}_{X^\infty} ,$$

resp.
$$F^i\, {}'\Omega^{\cdot}_{X^\infty}\langle Y\rangle = F^{i+d}\Omega^{\cdot}_X\langle Y\rangle \otimes_{\Omega^{\cdot}_X} {}'\Omega^{\cdot}_{X^\infty} .$$

The grading in the tensor products is given by the sum of the degrees in $\Omega^{\cdot}_X\langle Y\rangle$ and $^{(')}\Omega^{\cdot}_{X^\infty}$, and the differentials are as in the tensor product of chain complexes: $d(\alpha\otimes\beta) = d\alpha\otimes\beta + (-1)^{\deg\alpha}\,\alpha\otimes d\beta$.

<u>1.5. Lemma</u> The inclusions $\Omega^{0,q}_{X^\infty} \subseteq \Omega^{\cdot}_{X^\infty}$ and $'\Omega^{-d,q}_{X^\infty} \subseteq {}'\Omega^{\cdot}_{X^\infty}$ induce isomorphisms of $\mathcal{O}_{X^\infty}$-modules

$$\bigoplus_{p+q=n} \Omega^p_X\langle Y\rangle \otimes_{\mathcal{O}_X} \Omega^{0,q}_{X^\infty} \xrightarrow{\sim} \Omega^n_{X^\infty}\langle Y\rangle ,$$

$$\bigoplus_{p+q=n} \Omega^{p+d}_X\langle Y\rangle \otimes_{\mathcal{O}_X} {}'\Omega^{-d,q}_{X^\infty} \xrightarrow{\sim} {}'\Omega^n_{X^\infty}\langle Y\rangle .$$

<u>Proof</u> If we forget about the differentials, 1.2 gives an isomorphism of graded modules over the graded algebra $\Omega^{\cdot}_X$

$$\Omega^{\cdot}_{X^\infty} \cong \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} \Omega^{0,\cdot}_{X^\infty} ,$$

$$'\Omega^{\cdot}_{X^\infty} \cong \Omega^{\cdot}_X \otimes_{\mathcal{O}_X} {}'\Omega^{-d,\cdot}_{X^\infty} ,$$

hence the result follows by tensoring with $\Omega^{\cdot}_X\langle Y\rangle$ over $\Omega^{\cdot}_X$.

The isomorphisms above induce natural bigradings on $\Omega^{\cdot}_{X^\infty}\langle Y\rangle$ and $'\Omega^{\cdot}_{X^\infty}\langle Y\rangle$, such that $\Omega^{p,q}_{X^\infty}\langle Y\rangle \cong \Omega^p_X\langle Y\rangle \otimes_{\mathcal{O}_X} \Omega^q_{X^\infty}$ and $'\Omega^{p,q}_{X^\infty}\langle Y\rangle = \Omega^{p+d}_X\langle Y\rangle \otimes_{\mathcal{O}_X} {}'\Omega^{-d,q}_{X^\infty}$. The differential on these can be computed by the following "twisting" formula.

1.6. Lemma If for a local section D of ${}'\Omega^{-d,q}_{X^\infty}$

$$\partial D = \sum_j \omega'_j \otimes D'_j$$

as local section of ${}'\Omega^{-d+1,q}_{X^\infty} = \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^{-d,q}_{X^\infty}$, with local sections ω'_j of Ω^1_X and D'_j of $\Omega^{-d,q}_{X^\infty}$, then for a local section ω of $\Omega^p_X\langle Y\rangle$ one has

$$d(\omega \otimes D) = \partial\omega \otimes D + (-1)^{\deg \omega}(\omega \otimes \bar{\partial} D + \sum_j (\omega \wedge \omega'_j) \otimes D'_j) .$$

(Similarly for $\Omega^{o,q}_{X^\infty}$).

This is clear from the definition.

1.7. Lemma The embeddings

$$(\Omega^\bullet_X\langle Y\rangle, F^i) \hookrightarrow (\Omega^\bullet_{X^\infty}\langle Y\rangle, F^i) \hookrightarrow ({}'\Omega^\bullet_{X^\infty}\langle Y\rangle[-2d], F^{i-d})$$

are filtered quasi-isomorphisms.

Proof Via the isomorphisms in 1.5 we have

$$Gr^i_F\, \Omega^\bullet_{X^\infty}\langle Y\rangle \cong \Omega^i_X\langle Y\rangle \otimes_{\mathcal{O}_X} \Omega^{o,\bullet-i}_{X^\infty} ,$$

$$Gr^{i-d}_F\, {}'\Omega_{X^\infty}\langle Y\rangle \cong \Omega^i_X\langle Y\rangle \otimes_{\mathcal{O}_X} {}'\Omega^{-d,\bullet-i-d}_{X^\infty} ,$$

and the formula of 1.6 shows that these are isomorphisms of complexes, if on the right we take the differentials induced by the $\bar{\partial}$-maps, which are $\mathcal{O}_X$-linear. Hence the quasi-isomorphisms follow from those in 1.2 (for $i=0$) by tensoring with the locally free $\mathcal{O}_X$-module $\Omega^i_X\langle Y\rangle$.

1.8. Remarks a) With respect to the mentioned bigradings one has

$$F^k\, {}^{(\prime)}\Omega^n_{X^\infty}\langle Y\rangle = \bigoplus_{\substack{p+q=n \\ p\geq k}} {}^{(\prime)}\Omega^{p,q}_{X^\infty}\langle Y\rangle ,$$

which is the definition of the Hodge filtrations in [Ki]. There is an inclusion

$$\Omega^\bullet_X\langle Y\rangle \otimes_{\Omega^\bullet_X} F^i\, {}^{(\prime)}\Omega^\bullet_{X^\infty} \subset F^{i(+d)}\Omega^\bullet_X\langle Y\rangle \otimes_{\Omega^\bullet_X} \Omega^\bullet_{X^\infty} = F^i\, {}^{(\prime)}\Omega^\bullet_{X^\infty}\langle Y\rangle ,$$

which however may be strict: for $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$ and $Y = \{t=0\}$ the element $\frac{dt}{t} \otimes 1$ is contained in the stalk at $t=0$ of the right hand side for $i=1$, but not in the stalk of the left hand side. Probably the definition of the Hodge filtration by the latter in [Be 1] 1.8 is a misprint.

b) For each (p,q), the sheet $\Omega^{p,q}_{X^\infty}\langle Y\rangle$ can be realized as a sheaf of differentials, namely as a subsheaf of $j_*\Omega^{p,q}_{U^\infty}$, where $j: U \hookrightarrow X$ is the open immersion. This gives an embedding of complexes $\Omega^\bullet_{X^\infty}\langle Y\rangle \hookrightarrow j_*\Omega^\bullet_{U^\infty}$, which is a quasi-isomorphism, cf. [De 2] §3.

c) By a result of King [Ki] 1.3.12, each sheaf $'\Omega^{p,q}_{X^\infty}\langle Y\rangle$ is a quotient of $'\Omega^{p,q}_{X^\infty}$, and can be realized as a sheaf of distributions. Namely, there is a certain subsheaf $\Omega^{p,q}_{X^\infty}(\text{null } Y)$ of $\Omega^{p,q}_{X^\infty}$ (the forms "vanishing holomorphically on Y") such that for the sheaf $\mathcal{D}(\Omega^{p,q}_{X^\infty}(\text{null } Y))$ of distributions over it, the obvious map $'\Omega^{p,q}_{X^\infty}\langle Y\rangle = \Omega^{d+p}_X\langle Y\rangle \otimes_{\mathcal{O}_X} {}'\Omega^{-d,q}_{X^\infty} \to \mathcal{D}(\Omega^{p,q}_{X^\infty}(\text{null}\, Y))$ is an isomorphism and the restriction map $'\Omega^{p,q}_{X^\infty} \to \mathcal{D}(\Omega^{p,q}_{X^\infty}(\text{null } Y))$ is surjective. All this is compatible with the differentials.

Let $'C^\bullet_Y(X,A(k)) \subseteq {}'C^\bullet(X,A(k))$ be the subcomplex given by the singular C^∞-chains on X with support on Y, and define $'C^\bullet(X,Y,A(k)) = {}'C^\bullet(X,A(k))/{}'C^\bullet_Y(X,A(k))$. Integration as in 1.1 c) induces a map $\varepsilon: {}'C^\bullet(X,Y,A(k)) \to {}'\Omega^\bullet_{X^\infty}\langle Y\rangle(X)$ (compare 1.8 c)).

<u>1.9. Definition</u> Let $'C^\bullet_{\mathcal{D}}(X,Y,A(k)) = \mathrm{Cone}({}'C^\bullet(X,Y,A(k)) \oplus F^k\,{}'\Omega^\bullet_{X^\infty}\langle Y\rangle(X) \xrightarrow{\varepsilon-\iota} {}'\Omega^\bullet_{X^\infty}\langle Y\rangle(X))[-1]$, and call $'H^\ell_{\mathcal{D}}(X,Y,A(k)) = H^\ell({}'C^\bullet_{\mathcal{D}}(X,Y,A(k)))$ the ℓ-th Deligne homology of the pair (X,Y).

<u>1.10.</u> Let $\tilde{\pi}_*$ be the category whose objects are pairs (X,Y) as above - X a smooth proper analytic space and $Y \subset X$ a divisor with normal crossings - and whose morphisms are proper morphisms $f: X \to X'$ with $f(Y) \subseteq Y'$ and $f(X\smallsetminus Y) \subseteq X'\smallsetminus Y'$. Then the cochain complexes in 1.9 are covariant functors on $\tilde{\pi}_*$. Hence the same is true for the Deligne homology, and moreover, we can define a complex $'C^\bullet_{\mathcal{D}}(\varphi,A(k))$ for any diagram φ in $\tilde{\pi}_*$ (= covariant functor $\varphi: I \to \tilde{\pi}_*$ from a small category I into $\tilde{\pi}_*$) by

$$'C^\bullet_{\mathcal{D}}(\varphi,A(k)) = L \varinjlim_{I} {}'C^\bullet_{\mathcal{D}}(X_i,Y_i,A(k)) .$$

Here $(X_i,Y_i) = \varphi(i)$ for $i \in ob(I)$, and $L \varinjlim_I$ is the left derivative of the direct limit $\varinjlim_I : \underline{Hom}(I,Ab) \to Ab$, making $'C^{\bullet}_{\mathcal{D}}(\varphi,A(k))$ well defined in the derived category of the category Ab of abelian groups. In certain cases we have canonical representatives for $L \varinjlim$ of a diagram of complexes and define $'C^{\bullet}_{\mathcal{D}}(\varphi,A(k))$ by these. For example, for a simplicial object $(X.,Y.)$ in $\tilde{\pi}_*$ ($I = \Delta^o$, where Δ is the category of standard simplices [De 3]§5) we have

$$'C^{\bullet}_{\mathcal{D}}((X.,Y.),A(k)) = s'N'C^{\bullet}_{\mathcal{D}}(X.,Y.,A(k)) ,$$

where $'N$ is the canonical functor transforming a simplicial abelian group $A.$ into the cochain complex $'NA.$ with $('NA.)^{-i} = A_i$ and d_i = alternating sum of the face maps, and $sC^{\bullet\bullet}$ is the associated simple complex of a bi-complex $C^{\bullet\bullet}$.
In any case the Deligne homology of a diagram φ is defined by

$$'H^{\ell}_{\mathcal{D}}(\varphi,A(k)) = H^{\ell}('C^{\bullet}_{\mathcal{D}}(\varphi,A(k))) .$$

The case of relative cohomology is not directly included in this picture. One either has to start with $I = \cdot\to\cdot$ and then pass to filtered derived categories as in [Be 1], or one may simply define

$$'C^{\bullet}_{\mathcal{D}}(f,A(k)) = \mathrm{Cone}('C^{\bullet}_{\mathcal{D}}(X,Y,A(k)) \to 'C^{\bullet}_{\mathcal{D}}(X',Y',A(k))) ,$$
$$H^{\ell}_{\mathcal{D}}(f,A(k)) = H^{\ell}('C^{\bullet}_{\mathcal{D}}(f,A(k)))$$

for a morphism $f: (X,Y) \to (X',Y')$ in $\tilde{\pi}_*$.

For analytic spaces over $\mathbb{R}$ all complexes have a σ-action, and as in 1.3 one defines the Deligne homologies over $\mathbb{R}$ by replacing the homology of the $'C^{\bullet}_{\mathcal{D}}$-complexes by their $\langle\sigma\rangle$-hypercohomology.

<u>1.11. Lemma</u> The homology of $'C^{\bullet}(X,Y,\mathbb{Z})$ is canonically isomorphic to the Borel-Moore homology of $U = X\setminus Y$.

<u>Proof</u> In both the original paper by Borel and Moore [BM] and the reformulation by Verdier [Ve] 1.2 (Borel-Moore) homology is defined as the hypercohomology

$$H^{BM}_i(X,\mathbb{Z}) = 'H^{-i}_{BM}(X,\mathbb{Z}) = H^{-i}(X,T^{\bullet}_X)$$

of a complex of sheaves $T^{\bullet}_X$ on X , called the (differential graded) homology sheaf in [BM] (and written in homological notation) and defined as the dualizing complex $Rf^! \mathbb{Z}$ for $f: X \to \operatorname{Spec} \mathbb{C}$ in [Ve]. $T_X^\bullet$ is only defined up to quasi-isomorphism, i.e., in the derived category, and from both approaches can be described as follows (loc. cit. and [Bor] V §7). For any complex of abelian groups $A^\bullet$ let $DA^\bullet = \operatorname{Hom}^\bullet(A^\bullet, [\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}])$, where $[\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}]$ is the complex with $\mathbb{Q}$ in degree 0, $\mathbb{Q}/\mathbb{Z}$ in degree 1, zero elsewhere, and $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the canonical projection. Since this is an injective resolution of the group $\mathbb{Z}$, $DA^\bullet$ represents $R\operatorname{Hom}(A^\bullet, \mathbb{Z})$ in the derived category.

Let $F^\bullet$ be a fine resolution of the constant sheaf $\mathbb{Z}$ on X, then the presheaf

$$V \mapsto T_X^\bullet(V) := D\,\Gamma_c(V, F^\bullet)$$

is a complex of sheaves. It has flabby components, hence

$$H^i(X, T_X^\bullet) = H^i(T_X^\bullet(X)) .$$

Finally, since D respects quasi-isomorphisms, $T_X^\bullet$ does not depend on the choice of $F^\bullet$ up to quasi-isomorphism.

In particular, we may take for $F^\bullet$ the complex of sheaves $\tilde{C}_X^\bullet$ associated to the presheaf

$$V \mapsto C^\bullet(V, \mathbb{Z})$$

of singular C^∞-chains with coefficients in $\mathbb{Z}$, see [Wa]5.31. We use the following three facts. The map $C_c^\bullet(V,\mathbb{Z}) \to \Gamma_c(V, \tilde{C}_X^\bullet)$ is a quasi-isomorphism [Wa] 5.32, 5.46. For a complex of abelian groups $A^\bullet$ the canonical morphism $A^\bullet \to DDA^\bullet$ is a quasi-isomorphism, if the homology groups of $A^\bullet$ are of finite type over $\mathbb{Z}$. Finally, $C_c^\bullet(V, \mathbb{Z}) \subseteq C^\bullet(X,\mathbb{Z})$ can be identified with $\operatorname{Hom}('C^\bullet(X, X\smallsetminus V, \mathbb{Z}), \mathbb{Z}) \subseteq \operatorname{Hom}('C^\bullet(X,\mathbb{Z}),\mathbb{Z})$. Altogether we have canonical quasi-isomorphisms

$$\begin{array}{ccc} T_X(V) = D\Gamma_c(V, \tilde{C}_X^\bullet) \to DC_c^\bullet(V,\mathbb{Z}) = & D\operatorname{Hom}('C^\bullet(X, X\smallsetminus V, \mathbb{Z}), \mathbb{Z}) \\ & \uparrow \phi \\ & 'C^\bullet(X, X\smallsetminus V, \mathbb{Z}) \to DD\,'C^\bullet(X, X\smallsetminus V, \mathbb{Z}) \end{array}$$

for each open $V \subset X$. The map ϕ is a quasi-isomorphism, since $\mathbb{Z}$ is quasi-isomorphic to $[\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}]$ and $'C^\bullet(X, X\smallsetminus V, \mathbb{Z})$ is a free $\mathbb{Z}$-module. Taking the homology gives the result.

1.12. Remarks a) The equality T_X (definition as above) = $Rf^!\mathbb{Z}$ is shown in [Bor] V §7. The equality of the cohomology follows immediately from Verdier duality: $H^i(X,T_X) = H^i(T_X(X))$ $= H^i(D\Gamma_c(X,F^\cdot)) = H^i(R\ \mathrm{Hom}(Rf_!\mathbb{Z},\mathbb{Z})) = H^i(R\ \mathrm{Hom}(\mathbb{Z},Rf^!\mathbb{Z})) =$ $H^i(X,Rf^!\mathbb{Z})$.

b) The finite generation of the homology of $'C^\cdot(X,X\smallsetminus V,\mathbb{Z})$ can be deduced from the proof of 1.15 below.

c) In the following we rather write $'H^i_B$ - Betti homology - instead of $'H^i_{BM}$, since these groups define the (unique) homology theory associated to the Betti (= singular) cohomology H^i_B in the sense of Bloch and Ogus [BO] .

1.13. Corollary a) The Deligne homology defined in 1.9 only depends on U and not on the pair (X,Y) in $\tilde{\pi}$ with $X\smallsetminus Y=U$, and $'H^\ell_{\mathcal{D}}(U,A(k)) := {}'H^\ell_{\mathcal{D}}(X,Y,A(k))$ is a well-defined functor on the category $\widetilde{Sch}_*$ of smooth varieties over $\mathbb{C}$ (resp. over $\mathbb{R}$) with proper morphisms.

b) There is a long exact sequence

$$\ldots\to {}'H^\ell_{\mathcal{D}}(U,A(k))\to {}'H^\ell_B(U,A(k))\oplus F^k\,{}'H^\ell_{DR}(U)\xrightarrow{\varepsilon-\iota} {}'H^\ell_{DR}(U)\to {}'H^{\ell+1}_{\mathcal{D}}(U,A(k))\to\ldots,$$

functorial for proper morphisms, where $'H^\ell_B(U,A(k))$ is the Borel-Moore homology with coefficients in $A(k)$ (which is simply $'H^\ell_B(U,\mathbb{Z})\otimes A(k)$ since $A(k)$ is flat over $\mathbb{Z}$), $'H^\ell_{DR}(U)$ is the ℓ-th deRham homology of U , and $F^k\,{}'H^\ell_{DR}(U) \subseteq {}'H^\ell_{DR}(U)$ is the k-th step of the Hodge filtration.

Proof The deRham homology of U is by definition the homology of $'\Omega^\cdot_{U^\infty}(U)$, which is isomorphic to the homology of $'\Omega^\cdot_{X^\infty}\langle Y\rangle(X)$ via the quasi-isomorphisms of fine sheaves

$$j_*{}'\Omega^\cdot_{U^\infty}\leftarrow j_*\Omega^\cdot_{U^\infty}[2d] \leftarrow \Omega^\cdot_{X^\infty}\langle Y\rangle[2d] \to {}'\Omega^\cdot_{X^\infty}\langle Y\rangle \quad .$$

The subspace $F^k\,{}'H^\ell_{DR}(U)$ is defined as the image of the map $H^\ell(F^k\,{}'\Omega^\cdot_{X^\infty}\langle Y\rangle(X)) \to H^k({}'\Omega^\cdot_{X^\infty}\langle Y\rangle(X))$, which is injective by 1.2 and the corresponding result for the deRham cohomology [De 2] 3.2.13 ii). Hence 1.11 and the definition of $'C^\cdot_{\mathcal{D}}(X,Y,A(k))$ as a cone immediately give the exact sequence in b) with

$'H^{\cdot}_{\mathcal{D}}(X,Y,A(k))$ instead of $'H^{\cdot}_{\mathcal{D}}(U,A(k))$. This implies a). In fact, by Hironaka's resolution of singularities any U in $\widetilde{Sch}_*$ can be represented as $U = X \setminus Y$ for (X,Y) in $\tilde{\pi}_*$ – where we identify smooth varieties X over $\mathbb{C}$ (resp. $\mathbb{R}$) with the smooth analytic spaces $X(\mathbb{C})$ by GAGA (resp. $X(\mathbb{C})$ with $F_\infty\colon X(\mathbb{C}) \to X(\mathbb{C})$ induced by the complex conjugation). Hence by the same arguments as in [De 2] 3.2.C we only have to show that for any morphism $f\colon (X,Y) \to (X',Y')$ in $\tilde{\pi}_*$ which is the identity on $X \setminus Y = U = X' \setminus Y'$, the induced map $'H^{\ell}_{\mathcal{D}}(X,Y,A(k)) \to {'H^{\ell}_{\mathcal{D}}}(X',Y',A(k))$ is an isomorphism. This follows from the long exact sequence proved above and the five-lemma.

<u>1.14. Remark</u> A more canonical way to define the Deligne homology of U is

$$'H^{\ell}_{\mathcal{D}}(U,A(k)) = \varprojlim {'H^{\ell}_{\mathcal{D}}}(X,Y,A(k)) ,$$

where the limit is taken over the projective system of (X,Y) in $\tilde{\pi}_*$ with $X \setminus Y = U$ (note that the limit is taken over a system where all transition morphisms are isomorphisms).

<u>1.15. Theorem</u> For a smooth connected variety U over $\mathbb{C}$ or $\mathbb{R}$ there are canonical isomorphisms between Deligne homology and cohomology

$$'H^{\ell}_{\mathcal{D}}(U,A(k)) \cong H^{2d+\ell}_{\mathcal{D}}(U,A(d+k)) ,$$

where $d = \dim U$.

<u>Proof</u> Choose (X,Y) in $\tilde{\pi}_*$ with $U = X \setminus Y$, and let $'\tilde{C}^{\cdot}_{(X,Y)}$ be the complex of sheaves on X associated to the complex of presheaves

$$V \mapsto {'C^{\cdot}}(X, Y \cup (X \setminus V), \mathbb{Z}) .$$

If $j\colon U \hookrightarrow X$ is the open immersion, one obviously has

$$'\tilde{C}^{\cdot}_{(X,Y)} = j_* j^* {'\tilde{C}^{\cdot}_{(X,Y)}} \quad \text{and} \quad j^* {'\tilde{C}^{\cdot}_{(X,Y)}} = j^* {'\tilde{C}^{\cdot}_X},$$

where $'\tilde{C}^{\cdot}_X = \tilde{C}^{\cdot}_{(X,\emptyset)}$ is the sheaf associated to

$$V \mapsto {'C^{\cdot}}(X, X \setminus V, \mathbb{Z}) .$$

If $'\tilde{C}^{\cdot}_X \to I^{\cdot}$ is a quasi-isomorphism into a complex of injective sheaves, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Gamma_Y(X,I^{\cdot}) & \to & \Gamma(X,I^{\cdot}) & \to & \Gamma(U,I^{\cdot}) & \to & 0 \\ & & \uparrow\beta_1 & & \uparrow\beta_2 & & \uparrow\beta_3 & & \\ 0 & \to & \Gamma_Y(X,'\tilde{C}^{\cdot}_X) & \to & \Gamma(X,'\tilde{C}^{\cdot}_X) & \to & \Gamma(U,'\tilde{C}^{\cdot}_X) & & \\ & & \uparrow\alpha_1 & & \uparrow\alpha_2 & & \uparrow\alpha_3 & & \\ 0 & \to & 'C^{\cdot}_Y(X,\mathbb{Z}) & \to & 'C^{\cdot}(X,\mathbb{Z}) & \to & 'C^{\cdot}(X,Y,\mathbb{Z}) & \to & 0\,. \end{array} \tag{1.15.1}$$

Since $'\tilde{C}^{\cdot}_X$ is homotopically fine (compare [Ve] 1.1.1 and [Sw] p. 88/89), β_1, β_2 and β_3 are quasi-isomorphisms, and since X is compact, α_1 and α_2 are quasi-isomorphisms (see loc. cit.). Hence α_3 is a quasi-isomorphism.

But $'\tilde{C}^{\cdot}_X$ is canonically a resolution of $\mathbb{Z}(d)[2d]$: the i-th homology has stalks

$$\varinjlim_{x\in V} H^i('C^{\cdot}(X,X\smallsetminus V,\mathbb{Z})) = H_{-i}(X,X\smallsetminus\{x\},\mathbb{Z}) \cong \begin{cases} 0 & i \neq -2d \\ \mathbb{Z} & i = -2d \end{cases}$$

at $x \in X$, and for $V \subset X$

$$H^{-2d}('C^{\cdot}(X,X\smallsetminus V,\mathbb{Q})) = H_{2d}(X,X\smallsetminus V,\mathbb{Q}) = \mathrm{Hom}(H^{2d}_c(V,\mathbb{Q}),\mathbb{Q}) \cong \mathbb{Q}(d)$$

by the canonical trace map $H^{2d}_c(V,\mathbb{Q}(d)) \underset{\sim}{\to} H^{2d}(X,\mathbb{Q}(d)) \underset{\sim}{\to} \mathbb{Q}$ mapping the cycle class of a point to 1. These normalizations make the diagram

$$\begin{array}{ccccc} \mathbb{Z}(d)[2d] & \hookrightarrow & \mathbb{C}[2d] & \xrightarrow{\text{can}} & \Omega^{\cdot}_{X^\infty}(X)[2d] \\ \downarrow & & & & \downarrow{\scriptstyle 1.1.\ a)} \\ 'C^{\cdot}(X,\mathbb{Z}) & & \xrightarrow{\quad 1.1\ c) \quad} & & '\Omega^{\cdot}_{X^\infty}(X) \end{array} \tag{1.15.2}$$

commute (the composition maps $(2\pi\sqrt{-1})^d$ to $\omega' \mapsto \int_X \omega'$, where X is oriented by the choice of $\sqrt{-1}$).

By the quasi-isomorphisms $'\tilde{C}^{\cdot}_{(X,Y)} = j_*j^*{}'\tilde{C}^{\cdot}_X \to j_*j^*I^{\cdot}$ we see that $'\tilde{C}^{\cdot}_{(X,Y)}$ represents $Rj_*(\mathbb{Z}(d)|_U)[2d]$ in the derived category $D^+(X)$ of sheaves on X, and from 1.15.2 we see that the map $'\tilde{C}^{\cdot}_{(X,Y)} \to {}'\Omega^{\cdot}_{X^\infty}\langle Y\rangle$ induced by integration over cycles (1.1 c) and 1.8 c)) can be identified with the canonical map $Rj_*\mathbb{Z}(d)|_U[2d] \to Rj_*\Omega^{\cdot}_U[2d]$ via the quasi-isomorphisms $Rj_*\Omega^{\cdot}_U \leftrightarrow \Omega^{\cdot}_X\langle Y\rangle \to {}'\Omega^{\cdot}_{X^\infty}\langle Y\rangle[-2d]$.

If we replace $\mathbb{Z}$ by the coefficients $A(k)$ in the above, so that $'\tilde{C}^{\cdot}_{(X,Y)}(A(k))$ is the sheaf associated to $V \mapsto {}'C^{\cdot}(X,X\smallsetminus(V\cap U),A(k))$, then together with 1.7. we obtain quasi-isomorphisms between

$$'\tilde{C}^{\cdot}_{\mathcal{D},(X,Y)}(A(k)) := \mathrm{Cone}({}'\tilde{C}^{\cdot}_{(X,Y)}(A(k))\oplus F^k{}'\Omega_{X^\infty}\langle Y\rangle \to {}'\Omega^{\cdot}_{X^\infty}\langle Y\rangle)[-1]$$

and

$$A(k+d)_{\mathcal{D},(X,U)}[2d] = \mathrm{Cone}(Rj_*A(k+d)\oplus F^{k+d}\Omega^{\cdot}_X\langle Y\rangle \to Rj_*\Omega^{\cdot}_U)[2d-1] .$$

By definition, the ℓ-th hypercohomology group of the latter complex of sheaves on X is $H^{2d+\ell}_{\mathcal{D}}(U,A(d+k))$. On the other hand, the ℓ-th hypercohomology of $'\tilde{C}^{\cdot}_{\mathcal{D},(X,Y)}(A(k))$ is the ℓ-th homology of $'\tilde{C}^{\cdot}_{\mathcal{D},(X,Y)}(A(k))(X)$, since the $F^i{}'\Omega^j_{X^\infty}\langle Y\rangle$ are soft and $'\tilde{C}^{\cdot}_{(X,Y)}(A(k))(X) = {}'\tilde{C}^{\cdot}_X(A(k))(U) \xrightarrow{\beta_3} \Gamma(U,I^{\cdot}(A(k)))$ is a quasi-isomorphism. Finally the map α_3 induces a quasi-isomorphism between $'C^{\cdot}_{\mathcal{D}}(X,Y,A(k))$ and $'\tilde{C}^{\cdot}_{\mathcal{D},(X,Y)}(A(k))(X)$.

<u>1.16. Remarks</u> a) In several expositions the twists $\mathbb{Z}(d)$ do not occur, e.g. in [Ve] and [Bor] V §7, but it is more canonical to introduce them. The normalizations depend on that of the cycle map, which is determined by the case of divisors. So our choice is fixed by defining the first cycle map (Chern class) to be the connecting morphism

$$\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^\times) \to H^2(X,\mathbb{Z}(1))$$

associated to the exponential sequence

$$0 \to \mathbb{Z}\cdot 2\pi\sqrt{-1} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0 ,$$

which does not depend on the choice of $\sqrt{-1}$. In terms of trace maps our normalization corresponds to the commutative diagram

$$\begin{array}{ccc} H^{2d}(X,\mathbb{Q}(d)) & \xrightarrow{tr} & \mathbb{Q} \\ \cap\!| & & \cap\!| \\ H^{2d}(X,\mathbb{C}) & \xrightarrow{tr} & \mathbb{C} \\ \| \wr & & \| \\ H^{2d}_{DR}(X,\mathbb{C}) & \xrightarrow{tr} & \mathbb{C} \end{array} \quad , \quad \omega' \mapsto \frac{1}{(2\pi\sqrt{-1})^d} \int_X \omega' \ ,$$

compare 1.1 a) and 1.15.2. Via the canonical comparison isomorphisms these are also compatible with the canonical trace maps in étale or algebraic deRham cohomology, cf [DMOS] p.22. In particular, all maps become σ-invariant for varieties over $\mathbb{R}$, and $tr\ \omega' \in \mathbb{Q}$ for X and ω' defined over $\mathbb{Q}$.

b) The proof of 1.15 shows that one has canonical isomorphisms of long exact sequences

$$\begin{array}{ccccccccc} \ldots \to {}'H^{\ell-1}_{DR}(U) & \to & {}'H^{\ell}_{\mathcal{D}}(U,A(k)) & \to & {}'H^{\ell}_{B}(U,A(k)) \oplus F^{k}\,{}'H^{\ell}_{DR}(U) & \xrightarrow{\varepsilon - \iota} & {}'H^{\ell}_{DR}(U) \to \ldots \\ \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr \\ \ldots \to H^{2d+\ell-1}_{DR}(U) & \to & H^{2d+\ell}_{\mathcal{D}}(U,A(d+k)) & \to & H^{2d+\ell}_{B}(U,A(d+k)) \oplus F^{d+k}H^{2d+\ell}_{DR}(U) & \xrightarrow{\varepsilon - \iota} & H^{2d+\ell}_{DR}(U) \to \ldots \end{array}$$

with the classical isomorphisms for the Betti and deRham theories (cf. [BO] §2).

c) I could not prove that $j^*\,{}'\tilde{C}^{\cdot}_{(X,Y)}$ and ${}'\tilde{C}^{\cdot}_{X}$ (case $Y = \emptyset$) are flabby as stated in [Be 1], proof of 1.8.5, and have some doubts whether it is true.

<u>1.17.</u> We now define the Deligne homology of arbitrary schemes and simplicial schemes by "simplicial resolutions", see [De 3] and [SGA 4] V^{bis} for the proofs of the following statements.

If Z is a scheme which is separated and of finite type over $\mathbb{C}$ or $\mathbb{R}$, there is a smooth simplicial scheme $U.$ and an augmentation $U. \xrightarrow{a} Z$ (i.e., a morphism $U_o \to Z$, or, equivalently, a morphism into the constant simplicial scheme defined by Z) such that

a) the maps $U_n \to (\operatorname{cosk}^{Z}_{n-1}\operatorname{sk}_{n-1}U.)_n$ are proper and surjective for all n (i.e., $U. \to Z$ is a proper hypercovering),

b) there is an open immersion $U. \hookrightarrow X.$ into a smooth proper simplicial scheme such that the complement $Y.$ is a divisor with normal crossings.

By a) , $U. \to Z$ has cohomological descent, i.e., the map

$$F \to Ra_*a^*F$$

is a quasi-isomorphism for all sheaves F on Z . Note that Ra_*a^*F can be represented by $s\,N(a.)_*I^{\cdot}_{\cdot}$, if $I^{\cdot}_{\cdot}$ is a resolution of a^*F with components I^m_n injective on U_n , and N is the normalization functor transforming cosimplicial objects in any abelian category A into cochain complexes in A . Since the maps $a_n\colon U_n \to Z$ are necessarily proper for all $n \geq 0$, we also have descent for cohomology with compact support, i.e.,

$$R\Gamma_c(Z,F) \to R\Gamma_c(U.,a^*F) = s\,N\Gamma_c(U.,I^{\cdot}_{\cdot})$$

is a quasi-isomorphism. By taking $F = \mathbb{Z}$ and applying $R\,Hom(-,\mathbb{Z})$ we deduce that the morphism a also has descent for the Borel-Moore homology: for $U.$ this is by definition the homology of $R\,Hom(R\Gamma_c(U.,\mathbb{Z}),\mathbb{Z})$, for Z compare remark 1.12 a) .

More generally, if $Z.$ is a simplicial scheme whose components are separated and of finite type over $\mathbb{C}$ (or over $\mathbb{R}$) , and whose face and degeneration maps are all proper, there is a smooth simplicial scheme $U.$ and a morphism $U. \to Z.$ such that

a') $U. \to Z.$ induces an isomorphism in the Borel-Moore homology,

and such that b) holds for $U.$: for example , take a smooth proper hypercovering $U.. \to Z.$ such that the analogue of b) holds for $U..$, and let $U. = \Delta U..$ be the diagonal. The first case is included in this by taking for $Z.$ a constant simplicial scheme.

In both cases we may regard $(X.,Y.)$ as a simplicial object in $\tilde{\pi}_*$, and we define

$${}'H^{\ell}_{\mathcal{D}}(Z.,A(k)) = {}'H^{\ell}_{\mathcal{D}}((X.,Y.),A(k)) \overset{1.10}{=} H^{\ell}(s\,{}'N'C^{\cdot}_{\mathcal{D}}(X.,Y.,A(k)))$$

(= ${}'H^{\ell}_{\mathcal{D}}(Z,A(k))$ in the first case).

<u>1.18. Theorem</u> a) This is well defined, i.e., independent of the choice of $(X.,Y.)$, and makes ${}'H^{\ell}_{\mathcal{D}}(Z,A(k))$ (resp.

$'H^{\ell}_{\mathcal{D}}(Z.,A(k)))$ a functor on the category Sch_* of separated schemes of finite type over $\mathbb{C}$ (or over $\mathbb{R}$) and proper morphisms (resp. on the category of simplicial objects in Sch_*).

b) There is a long exact sequence

$$\ldots \to {}'H^{\ell}_{\mathcal{D}}(Z,A(k)) \to {}'H^{\ell}_{B}(Z,A(k)) \oplus F^{k}\,{}'H^{\ell}_{DR}(Z) \xrightarrow{\varepsilon-\iota} {}'H^{\ell}_{DR}(Z) \to {}'H^{\ell+1}_{\mathcal{D}}(Z,A(k)) \to \ldots$$

functorially associated to Z in Sch_* (similarly for simplicial objects).

c) The canonical map ε induces an isomorphism $'H^{\ell}_{B}(Z,A(k)) \otimes_A \mathbb{C} \xrightarrow{\sim} {}'H^{\ell}_{DR}(Z)$. If $f\colon Z \to Z'$ induces an isomorphism in the Borel-Moore homology, it induces an isomorphism between the long exact sequences of b) (similarly for simplicial objects in Sch_*).

<u>Proof</u> If $(X.,Y.)$ is chosen for $Z.$ as indicated above, the deRham homology of $Z.$ is defined as

$$(1.18.1) \qquad {}'H^{\ell}_{DR}(Z.) = H^{\ell}(s\ {}'N\,{}'\Omega^{\cdot}_{X.}{}_{\infty}\langle Y.\rangle(X.)) ,$$

with the Hodge filtration step F^k being the image of

$$(1.18.2) \qquad \iota\colon H^{\ell}(s\,{}'NF^{k}\,{}'\Omega^{\cdot}_{X.}{}_{\infty}\langle Y.\rangle(X.)) \to H^{\ell}(s\,{}'N\,{}'\Omega^{\cdot}_{X.}{}_{\infty}\langle Y.\rangle(X.))$$

\- same definition for the case of a (constant simplicial) scheme Z, which we shall not treat separately in the following. On the other hand it follows from the descent condition a') and 1.11, that the Borel-Moore homology of $Z.$ can be computed as

$$'H^{\ell}_{B}(Z.,A(k)) = {}'H^{\ell}_{B}(U.,A(k)) = H^{\ell}(s\,{}'N\,{}'C^{\cdot}(X.,Y.,A(k))).$$

If we tensor these groups with $\mathbb{C}$, we obtain the groups in 1.18.1, by the quasi-isomorphisms

$$'C^{\cdot}(X_n,Y_n,\mathbb{Z}) \otimes \mathbb{C} \to {}'\Omega^{\cdot}_{X_n}{}_{\infty}\langle Y_n\rangle(X_n) \qquad , \qquad n \geq 0 ,$$

proved in 1.15 and applying $s'N$ to the corresponding simplicial diagram. This shows the first claim in c), and that the deRham homology groups are well-defined by 1.18.1.

By Hodge theory (see 2.9), ι is injective, hence we obtain the sequence of b) for $(X.,Y.)$ instead of $Z.$: note that $s'N\ \mathrm{Cone}(K^{\cdot}_{.} \to L^{\cdot}_{.}) \cong \mathrm{Cone}(s'NK. \to s'NL.)$ for a morphism $K^{\cdot}_{.}\to L^{\cdot}_{.}$ of complexes of simplicial abelian groups. Again by Hodge theory (see 2.9), a morphism $f.\colon (X.,Y.) \to (X'_.,Y'_.)$ induces a morphism in the deRham homology which is strictly compatible with the Hodge filtration. In particular, if $f.$ induces an isomorphism in the Borel-Moore homology, it induces an isomorphism for $'H^*_{DR}$ and $F^k{}'H^*_{DR}$, too, hence for $'H^*_{\mathcal{D}}$ by the five-lemma. For the remaining statements we now may proceed as for 1.13 - here we have to use the following fact. Any morphism $Z. \xrightarrow{g} Z!$ can be extended to a commutative diagram

$$\begin{array}{ccc} X.\smallsetminus Y. = V. & \xrightarrow{f} & V! = X!\smallsetminus Y! \\ a\downarrow & & \downarrow a' \\ Z. & \xrightarrow{g} & Z! \end{array}$$

where $(X.,Y.)$ and $(X!,Y!)$ are simplicial objects in $\tilde{\pi}_*$, f is induced by a morphism $f\colon (X.,Y.) \to (X!,Y!)$, and v and v' have descent for the Borel-Moore homology: for example, take the diagonals in suitable hypercoverings $V.. \to Z.$ and $V!. \to Z!$ (cf. [De 3] 6.28).

The following theorem summarizes a good part of the properties of Deligne cohomology and homology.

<u>1.19. Theorem</u> Let F be $\mathbb{R}$ or $\mathbb{C}$. The functors

$$Z \to X \quad \rightsquigarrow \quad H^i_{\mathcal{D},Z}(X,A(j)) \qquad , i,j \in \mathbb{Z},$$

$$X \quad \rightsquigarrow \quad H^{\mathcal{D}}_a(X,A(b)) := {}'H^{-a}_{\mathcal{D}}(X,A(-b)),\ a,b\in\mathbb{Z},$$

form a twisted Poincaré duality theory in the sense of Bloch and Ogus [BO] 1.3 on the category of all schemes which are separated and of finite type over F.

<u>Proof</u> Parts of the proof can be found in Beilinson's paper [Be 1], and the result was also announced by Gillet in [Gi 2] (for $F = \mathbb{C}$). As an illustration we shall sketch the proof for the existence of a long exact sequence

$$(1.19.1) \ldots \to H^{\mathcal{D}}_a(Y,A(b)) \to H^{\mathcal{D}}_a(X,A(b)) \to H^{\mathcal{D}}_a(U,A(b)) \to H^{\mathcal{D}}_{a-1}(Y,A(b)) \to \ldots$$

for a closed immersion $Y \hookrightarrow X$ with open complement $U = X\smallsetminus Y$ (see [Be 1] 1.8.4, but also 2.11 d) below).

First assume that X is smooth and proper and that Y is a divisor with normal crossings. Then it is shown in 2.9 that there is an exact triangle $(k \in \mathbb{Z})$

$$(1.19.2) \quad s'N'C^{\bullet}_{\mathcal{D}}(\tilde{Y}.,A(k)) \xrightarrow{\pi_{\mathcal{D}}} {}'C^{\bullet}_{\mathcal{D}}(X,A(k)) \to {}'C^{\bullet}_{\mathcal{D}}(X,Y,A(k)) \to ,$$

where the complexes compute the Deligne homology of Y,X and U, respectively, so 1.19.1 is obtained as the associated homology sequence.

Next assume that X and Y are proper but otherwise arbitrary. Then there exists a smooth proper simplicial scheme augmented to X, $X. \xrightarrow{\pi} X$, such that $Y. = \pi^{-1}(Y)$ is a divisor with normal crossings in $X.$ and π satisfies the condition 1.17 a). Then $Y. \to Y$ and $U. = X.\smallsetminus Y. \to U$ also satisfy homological descent, hence $X.$, $Y.$ and $(X.,Y.)$ can be used to compute the Deligne homology of X,Y and U, respectively, and we use the simplicial version of 1.19.2.

Now assume that X is smooth and Y is a divisor with normal crossings. There exists a smooth compactification $\overline{X}$ of X such that $\overline{X}\smallsetminus X$ and $(\overline{X}\smallsetminus X) \cup Y = Z$ are divisors with normal crossings on $\overline{X}$. Let $\overline{Y}$ be the closure of Y in $\overline{X}$, and let $\tilde{\overline{Y}}.$ and $\tilde{Y}.$ be the coskeletons of the normalizations of $\overline{Y}$ and Y, respectively (cf. 2.9). Then $s'N'C^{\bullet}_{\mathcal{D}}(\tilde{\overline{Y}}.,\tilde{\overline{Y}}.\smallsetminus\tilde{Y}.,A(k))$ computes the Deligne homology of Y, so we have to show that in the commutative diagram

$$\begin{array}{ccccc} s'N'C^{\bullet}_{\mathcal{D}}(\tilde{\overline{Y}}.,\tilde{\overline{Y}}.\smallsetminus\tilde{Y}.,A(k)) & \xrightarrow{\pi_{\mathcal{D}}} & {}'C^{\bullet}_{\mathcal{D}}(\overline{X},\overline{X}\smallsetminus X) & \overset{\alpha_3}{\twoheadrightarrow} & {}'C^{\bullet}_{\mathcal{D}}(\overline{X},Z) \\ & & \uparrow{\scriptstyle\alpha_1} & & \| \\ & & {}'C^{\bullet}_{\mathcal{D}}(\overline{X}) & \overset{\alpha_2}{\twoheadrightarrow} & {}'C^{\bullet}_{\mathcal{D}}(\overline{X},Z) \end{array}$$

the natural map $\pi_{\mathcal{D}}$ induces a quasi-isomorphism with $\ker \alpha_3$. But $\ker \alpha_1$ and $\ker \alpha_2$ compute the Deligne homology of $\overline{X}\smallsetminus X$ and Z, respectively, by step 1, so $\ker \alpha_3 =$ $\operatorname{coker}(\ker \alpha_1 \to \ker \alpha_2)$ in fact computes the Deligne homology of $Z\smallsetminus(\overline{X}\smallsetminus X) = Y$ by step 2.

For arbitrary X and Y there exists a smooth proper simplicial scheme $\overline{X}.$, an open subscheme $X. \subset \overline{X}.$, and an augmentation $X. \xrightarrow{\pi} X$ satisfying 1.17 a), such that $Y. = \pi^{-1}(Y)$, $\overline{X}.\smallsetminus X.$ and $(\overline{X}.\smallsetminus X.) \cup Y. = Z.$ are divisors with normal crossings. Thus, everything reduces to a simplicial version of the smooth situation, and we are done.

Deligne cohomology with support in a closed subscheme $Y \subseteq X$ is defined as relative Deligne cohomology of $j\colon U = X \smallsetminus Y \hookrightarrow X$, i.e.,

$$(1.19.3)\quad H^{\ell}_{\mathcal{D},Y}(X,A(k)) = H^{\ell}(\mathrm{Cone}(R\Gamma_{\mathcal{D}}(X,A(k)) \xrightarrow{j^*} R\Gamma_{\mathcal{D}}(U,A(k)))[-1]),$$

where $R\Gamma_{\mathcal{D}}(X,A(k))$ and $R\Gamma_{\mathcal{D}}(U,A(k))$ are suitable complexes of abelian groups which compute the Deligne cohomology of X and U, respectively. This immediately gives the long exact sequence

$$(1.19.4)\quad \ldots \to H^{\ell}_{\mathcal{D},Z}(X,A(k)) \to H^{\ell}_{\mathcal{D},Y}(X,A(k)) \to H^{\ell}_{\mathcal{D},Y\smallsetminus Z}(X \smallsetminus Z,A(k)) \to$$
$$\to H^{\ell+1}_{\mathcal{D},Z}(X,A(k)) \to \ldots$$

for $Z \subset Y \subset X$. The Poincaré duality isomorphism for smooth X,

$$(1.19.5)\quad H^{i}_{\mathcal{D},Y}(X,A(j)) \xrightarrow{\sim} H^{\mathcal{D}}_{2d-i}(Y,A(d-j)), \qquad d = \dim X,$$

follows from 1.15, the construction of 1.19.1, and the commutative diagram

$$\begin{tikzcd}
R\Gamma(\overline{X}_o,A(k+d)_{\mathcal{D},(\overline{X}_o,X_o\smallsetminus\pi_o^{-1}Y)})[2d] \arrow[r, "1.15", "\sim"'] & {}'C^{\bullet}_{\mathcal{D}}(\overline{X}_o,Z_o,A(k)) & s'N'C^{\bullet}_{\mathcal{D}}(\overline{X}.,Z.,A(k)) \arrow[l, "\sim"'] \\
R\Gamma(\overline{X}_o,A(k+d)_{\mathcal{D},(\overline{X}_o,X_o)})[2d] \arrow[u, "j_o^*"] \arrow[r, "1.15", "\sim"'] & {}'C^{\bullet}(\overline{X}_o,\overline{X}_o\smallsetminus X_o,A(k)) \arrow[u, "j_o^*"] \arrow[d, "(\pi_o)_*"] & s'N'C^{\bullet}_{\mathcal{D}}(\overline{X}.,\overline{X}.\smallsetminus X.,A(k)) \arrow[l] \arrow[u, "j^*"] \arrow[dl, "\sim"] \\
R\Gamma(\overline{X},A(k+d)_{\mathcal{D},(\overline{X},X)})[2d] \arrow[u, "\pi_o^*"] \arrow[r, "1.15", "\sim"'] & {}'C^{\bullet}_{\mathcal{D}}(\overline{X},\overline{X}\smallsetminus X,A(k)) &
\end{tikzcd}$$

which expresses the compatibility of 1.15 with covariance for open immersions of both sides. Here $\xrightarrow{\sim}$ denotes a quasi-isomorphism, $X. \to X$ and $\overline{X}.$ are as above, with $X_o \smallsetminus \pi_o^{-1} Y \xrightarrow{\sim} U$, and $\overline{X}$ is a smooth compactification of X such that $\overline{X} \smallsetminus X$ is a divisor with normal crossings and π_o extends to $(\overline{X}_o, \overline{X}_o \smallsetminus X_o) \to (\overline{X}, \overline{X} \smallsetminus X)$. The maps on the left and on the right induce j^* in cohomology and homology, respectively, j^* being defined like α_3 above. Note that by this construction the Poincaré duality is compatible with 1.19.1 and 1.19.4, in the obvious sense.

For the definition of the capproduct

$$H^{i}_{\mathcal{D},Y}(X,A(j)) \otimes H^{\mathcal{D}}_{a}(Y,A(b)) \to H^{\mathcal{D}}_{a-i}(Y,A(b-j))$$

we refer the reader to [Be 1] 1.8.6.

The canonical class $\eta_X \in H^{\mathcal{D}}_{2d}(X,A(d))$ for X irreducible of dimension d is by definition the image of $1 \in A$ under the isomorphism in the following lemma.

1.20 Lemma Let X be separated and of finite type over $\mathbb{C}$ or $\mathbb{R}$, and let $d = \dim X$. Then $H_a^B(X,A(b)) := {}'H_B^{-a}(X,A(-b))$, $H_a^{DR}(X) := {}'H_{DR}^{-a}(X)$ and $H_a^{\mathcal{D}}(X,A(b))$ vanish for $a > 2d$, and

$$\varepsilon_A : H_{2d}^{\mathcal{D}}(X,A(d)) \tilde{\to} H_{2d}^B(X,A(d)) \cong \bigoplus_{x \in X_{(d)}} A$$

is an isomorphism, where $X_{(d)} = \{x \in X \mid \dim \overline{\{x\}} = d\} \cong$ set of irreducible components of dimension d of X.

Proof The vanishing is known for the Borel-Moore homology, and hence follows for the other homology theories by 1.18 b) and c). For the isomorphism we may assume X to be reduced - by 1.18 c) and the fact that the Borel-Moore homology only depends on the underlying reduced complex analytic space. Then there is a smooth open subvariety $\emptyset \neq U \subset X$, which is of pure dimension d and dense in every irreducible component of dimension d, and the restriction maps $H_{2d}(X,A(d)) \to H_{2d}(U,A(d))$ are isomorphisms for the Betti and the Deligne homology by 1.19.1, the Betti analogue of it, and the above vanishing result. Hence it suffices to prove the claim for U, but by 1.15 we have canonically

$$\begin{array}{ccc} H_{2d}^{\mathcal{D}}(U,A(d)) & \xrightarrow{\varepsilon_A} & H_{2d}^B(U,A(d)) \\ \downarrow\wr & & \downarrow\wr \\ H_{\mathcal{D}}^0(U,A) & \xrightarrow[\sim]{\varepsilon_A} & H_B^0(U,A) = \bigoplus_{\text{conn.comp. of } U} A \end{array}$$

(note that $F^0H_{DR}^0(U) = H_{DR}^0(U)$) .

1.21. Let $Z_n(X)$ be the group of cycles of dimension n on X, i.e., the free abelian group on the irreducible reduced subschemes $Z \subset X$ of dimension n. As for any twisted Poincaré duality theory, the canonical class provides us with a cycle map

$$\begin{aligned} cl_{\mathcal{D}}: Z_n(X) &\to H_{2n}^{\mathcal{D}}(X,A(n)) \\ [Z] &\mapsto \text{image of } \eta_Z \text{ under } H_{2n}^{\mathcal{D}}(Z,A(n)) \to H_{2n}^{\mathcal{D}}(X,A(n)). \end{aligned}$$

Since η_Z by definition is compatible with the canonical class in the Betti homology, the composition $cl_B = \varepsilon_A\, cl_{\mathcal{D}}$ is the usual cycle map into the Betti homology. If X is smooth, then, by applying the Poincaré duality 1.15, we obtain a cycle map

$$cl_{\mathcal{D}} : Z^m(X) \to H^{2m}_{\mathcal{D}}(X,A(m))$$

on the group $Z^m(X)$ of cycles of codimension $m = \dim X-n$.

We sketch the proof that this cycle map agrees with the one defined in [EV]. Let $Z \subset X$ be of codimension m . There are canonical morphisms of functors

$$\varphi : H^i_{\mathcal{D},Z}(X, \mathbb{Z}(j)) \to H^i_Z(X, \mathbb{Z}(j)_{\mathcal{D},an}) ,$$

and we have to show that φ maps the relative class given in

$$H^{2m}_{\mathcal{D},Z}(X, \mathbb{Z}(m)) \tilde{\to} H^{\mathcal{D}}_{2n}(Z, \mathbb{Z}(n))$$

by η_Z to the relative cycle class defined in [EV] 7.1. The maps φ are compatible with the cupproducts as are the relative cycle classes ([EV] 7.4 and [Ve]3.5 for $H^{2m}_{\mathcal{D},Z}(X,\mathbb{Z}(m)) \tilde{\to}$ $H^{2m}_Z(X, \mathbb{Z}(m))$. By this and localizing (compare the argument in [EV] 6.2) we are reduced to the case of divisors. If $m = 1$,let $Z' \supset Z$ be a divisor such that $Z = \mathrm{div}(f)$ is principal on $U = X \smallsetminus Z'$, then the claim follows from the commutative diagram

$$\begin{tikzcd}
 & H^0(U,\Omega^1_U) \oplus H^1(U, \mathbb{Z}(1)) \arrow[r] & H^1_{Z'}(X,\Omega^1_X) \oplus H^2_{Z'}(X,\mathbb{Z}(1)) \\
\mathcal{O}(U)^{\times} \arrow[ur, "{(d\log,\delta)}"] & H^1(U, \mathbb{Z}(1)_{\mathcal{D},an}) \arrow[l, "\sim"'] \arrow[u] \arrow[r] & H^2_{Z'}(X,\mathbb{Z}(1)_{\mathcal{D},an}) \arrow[u] \\
\mathcal{O}(U)^{\times}_{alg} \arrow[u, phantom, "\cup" rotate=90] & H^1_{\mathcal{D}}(U, \mathbb{Z}(1)) \arrow[l, "\sim"'] \arrow[u, "\varphi"] \arrow[r] & H^2_{\mathcal{D},Z'}(X, \mathbb{Z}(1)) \arrow[u, "\varphi"]
\end{tikzcd}$$

(δ the connecting morphism for $0 \to \mathbb{Z}(1) \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^{\times} \to 0$), since f is mapped to the cycle classes in both theories (cf. [EV] 6.2 and 3.1.2 below) and $H^2_Z(X, \mathbb{Z}(1)_{\mathcal{D},an}) \to H^2_{Z'}(X, \mathbb{Z}(1)_{\mathcal{D},an})$ is injective ([EV] 6.1 b) and the corresponding result for the Betti cohomology).

Now let X be smooth and proper, and let $Z^m(X)_0 = \mathrm{Ker}\ cl_B$ $\subseteq Z^m(X)$ be the subgroup of cycles which are homologically equivalent to zero (for the Betti cohomology). By the cohomological analogue of 1.18 b) we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J^m(X) \arrow[r] & H^{2m}(X, \mathbb{Z}(m)) \arrow[r] & Hg^m(X) \arrow[r] & 0 \\
0 \arrow[r] & Z^m(X)_0 \arrow[r] \arrow[u, "cl'_{\mathcal{D}}"] & Z^m(X) \arrow[r] \arrow[u, "cl_{\mathcal{D}}"] & Z^m(X)/Z^m(X)_0 \arrow[r] \arrow[u, "cl_B"] & 0 ,
\end{tikzcd}$$

where

$$J^m(X) = H^{2m-1}(X,\mathbb{C}) / \varepsilon H^{2m-1}(X, \mathbb{Z}(m) + F^m H^{2m-1}(X,\mathbb{C})$$

is the m-th intermediate Jacobian [GH] p. 331, and

$Hg^m(X) = H^{2m}(X, \mathbb{Z}(m)) \cap \varepsilon^{-1} F^m H^{2m}(X, C) = H^{2m}(X, \mathbb{Z}(m)) \cap \varepsilon^{-1} H^{m,m}$

is the group of Hodge cycles of codimension m on X.

1.22. Lemma The map $cl'_{\mathcal{D}}$ coincides with the Abel-Jacobi map as defined by Griffiths and Weil.

Proof By definition, the complex $'C^{\bullet}_{\mathcal{D}}(X, \mathbb{Z}(k))$ is given by

$$\begin{array}{ccc} 'C^{\ell}_{\mathcal{D}}(X, \mathbb{Z}(k)) & = & 'C^{\ell}(X, \mathbb{Z}(k)) \oplus F^{k}\,'\Omega^{\ell}_{X^\infty}(X) \oplus {}'\Omega^{\ell-1}_{X^\infty}(X) \\ \Big\downarrow & & (c_B, c_F, c_\Omega) \\ & & \downarrow \\ & & (d\, c_B,\ d\, c_F,\ -\varepsilon\, c_B + \iota\, c_F - d\, c_\Omega) \\ 'C^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z}(k)) & = & 'C^{\ell+1}(X, \mathbb{Z}(k)) \oplus F^{k}\,'\Omega^{\ell+1}_{X^\infty}(X) \oplus {}'\Omega^{\ell}_{X^\infty}(X) . \end{array}$$

The class of an algebraic cycle y of codimension m is the class of $(y_B, y_F, 0)$, where $y_B = (2\pi\sqrt{-1})^{-n} y_{top}$ for a topological C^∞-claim y_{top} in $'C^{-2n}(X, \mathbb{Z})$, $n = \dim X - m$, which represents y and is oriented according to the choice of $\sqrt{-1}$ (cf. 1.1 a)), and where $y_F \in F^{-n}\,'\Omega^{-2n}_{X^\infty}(X)$ is the distribution $\omega' \mapsto (2\pi\sqrt{-1})^{-n} \int_{y_{reg}} \omega'$. This can easily be seen by choosing a resolution of singularities $\tilde{Z} \xrightarrow{\pi} Z$ for each Z in the support of y and observing that the fundamental class $\eta_{\tilde{Z}}$ of $\tilde{Z}$ is mapped to η_Z via π and hence to $cl(Z)$ via $\tilde{Z} \xrightarrow{\pi} Z \hookrightarrow X$. If y is homologous to zero, there are elements $s \in {}'C^{-2n-1}(X, \mathbb{Z}(-n))$ and $f \in F^{-n}\,'\Omega^{-2n-1}_{X^\infty}(X)$ with $ds = y_B$ and $df = y_F$. Hence $(y_B, y_F, 0)$ is homologous to $(0, 0, \varepsilon s - \iota f)$, i.e., $cl'_{\mathcal{D}}(y) \in J^m(X)$ is given by the class of $\varepsilon s - \iota f$ in $H^{-2n-1}('\Omega^{\bullet}_{X^\infty}(X)) \xrightarrow{\sim} H^{2m-1}(X, \mathbb{C})$. This is exactly the classical definition of the Abel-Jacobi map: if we evaluate this class against forms $\omega \in F^{n-1}\Omega^{2n+1}_{X^\infty}(X)$ via the Poincaré pairing, we obtain the integral of ω over s, since the product with f vanishes.

1.23. Remark Using only Betti cohomology, the definition of fundamental classes in $H^{2m}_{\mathcal{D},Z}(X, \mathbb{Z}(m)) \xrightarrow{\varepsilon} H^{2m}_{B,Z}(X, \mathbb{Z}(m))$ is easier than in $H^{2m}_Z(X, \mathbb{Z}(m)_{\mathcal{D},an})$, where non-trivial facts of deRham theory are needed. The bijectivity of ε follows from 1.19.4 and the version with support of [EV] 2.10b). Here one uses $F^m H^{2m}_{DR,Z}(X)$ and not $H^{2m}_Z(X, F^m\Omega^{\bullet}_X)$; this would also simplify the proof of [EV] 7.11.

§2 Hodge theory for homology, and absolute Hodge cohomology

We recall the following definition due to Deligne, [De 3] §8.

2.1 Definition Let A be a noetherian subring of $\mathbb{R}$, then a mixed (polarizable) A-Hodge complex $K^\cdot$ is a diagram

(2.1.1) $$K_A^\cdot \xrightarrow{\alpha} (K_{A\otimes\mathbb{Q}}^\cdot, W) \xrightarrow{\beta} (K_{\mathbb{C}}^\cdot, W, F)$$

such that

a) $K_A^\cdot$ is a complex of A-modules, bounded below, such that the homology groups are finitely generated as A-modules,

b) $(K_{A\otimes\mathbb{Q}}^\cdot, W)$ is a complex of $A\otimes\mathbb{Q}$-modules with an ascending filtration W, and α is a morphism in the derived category $D^+(A)$ of bounded below complexes of A-modules such that $K_A^\cdot \otimes \mathbb{Q} \to K_{A\otimes\mathbb{Q}}^\cdot$ is a quasi-isomorphism,

c) $(K_{\mathbb{C}}, W, F)$ is a bounded below complex of $\mathbb{C}$-vector spaces with an ascending filtration W and a descending filtration F, and β is a morphism in the derived category $D^+F(A\otimes\mathbb{Q})$ of bounded below filtered complexes of $A\otimes\mathbb{Q}$-modules such that $(K_{A\otimes\mathbb{Q}}^\cdot \otimes_A \mathbb{C}, W) \to K_{\mathbb{C}}^\cdot, W)$ is a filtered quasi-isomorphism,

d) for all $m \in \mathbb{Z}$

$$Gr_m^W K_{A\otimes\mathbb{Q}}^\cdot \to (Gr_m^W K_{\mathbb{C}}^\cdot, F)$$

is a (polarizable) $A\otimes\mathbb{Q}$-Hodge complex of weight m, i.e. the differentials of $Gr_m^W K_{\mathbb{C}}^\cdot$ are strictly compatible with the induced filtration F, and F induces a pure (polarizable) $A\otimes\mathbb{Q}$-Hodge structure of weight $m+n$ on $H^n(Gr_m^W K_{A\otimes\mathbb{Q}}^\cdot)$ for $n \in \mathbb{Z}$.

The construction of mixed A-Hodge complexes is <u>equivalent</u> to the construction of mixed A-Hodge structures, as follows from the following fundamental result of Deligne and the beautiful converse proved by Beilinson, recalled in 2.3 below.

2.2. Theorem ([De 3] 8.1.9 (ii) and (v)) If $K^\cdot$ is a mixed (polarizable) A-Hodge complex, then for all $n\in\mathbb{Z}$ $W[n]$ and F induce a mixed (polarizable) A-Hodge structure on $H^n(K_A^\cdot)$, and the spectral sequence associated to F degenerates, i.e., we see that $H^{\ell}_{\mathcal{H}}(K^{\bullet}) \to H^{\ell}_{\mathcal{H}^{W}}(K^{\bullet})$ is an isomorphism, if the weights are ≤ 0 $(W_{0}H = H)$ for the mixed A-Hodge structures $H = H^{\ell-1}, H^{\ell}$.

2.6. Finally, for $k \in \mathbb{Z}$ the k-th Tate twist $K^{\bullet}(k)$ of a mixed A-Hodge complex $K^{\bullet}$ is defined as

$$\begin{tikzcd}[column sep=small]
 & {}'K^{\bullet}_{A\otimes\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(2\pi\sqrt{-1})^{k} & & ({}'K^{\bullet}_{\mathbb{C}}, W[2k]) & \\
K^{\bullet}_{A} \otimes_{\mathbb{Z}} \mathbb{Z}(2\pi\sqrt{-1})^{k} \arrow[ur, "\alpha_{1}\otimes id"] & & (K^{\bullet}_{A\otimes\mathbb{Q}}\otimes\mathbb{Q}(2\pi\sqrt{-1})^{k}, W[2k]) \arrow[ul, "\alpha_{2}\otimes id"'] \arrow[ur, "\beta_{1}(k)"] & & (K^{\bullet}_{\mathbb{C}}, W[2k], F[k]) \arrow[ul, "\beta_{2}"']
\end{tikzcd} ,$$

where $\beta_{1}(k)$ is induced by multiplication, and $W[2k]$ and $F[k]$ are the shifted filtrations $(W[2k]_{i} = W_{i+2k}$ etc.). If $K^{\bullet} = \phi(C^{\bullet})$, this corresponds to tensoring $C^{\bullet}$ by the Hodge structure $\mathbb{Z}(k)$ ([De 2] 2.1.13).

2.7. We apply all this to the Hodge theory of algebraic varieties. It is the basis of Hodge theory that for a smooth, proper variety X over $\mathbb{C}$

$$K^{\bullet}(X, \mathbb{Z}) : \qquad R\Gamma(X, \mathbb{Z}) \to (R\Gamma(X, \Omega^{\bullet}), F)$$

with the Hodge filtration F is a pure Hodge complex of weight 0. It can be represented by $C^{\bullet}(X, \mathbb{Z}) \to (\Omega^{\bullet}_{X^{\infty}}(X), F)$ or by the mixed $\mathbb{Z}$-Hodge complex

$$K^{\bullet}(X, \mathbb{Z}): \quad \begin{tikzcd}[column sep=small]
 & C^{\bullet}(X, \mathbb{Q}) & & (C^{\bullet}(X, \mathbb{C}), W) & \\
C^{\bullet}(X, \mathbb{Z}) \arrow[ur, hook] & & (C^{\bullet}(X, \mathbb{Q}), W) \arrow[ul, "id"'] \arrow[ur, hook] & & (\Omega^{\bullet}_{X^{\infty}}(X), W, F) \arrow[ul, "integration"']
\end{tikzcd}$$

with trivial weight filtrations: $W_{-1} = 0$, $W_{0} =$ everything. If we define $K^{\bullet}(X,A)$ in an analogous way, then by 2.5 and 2.6 we have canonical isomorphisms

$$H^{\ell}_{\mathcal{D}}(X, A(k)) \cong H^{\ell}_{\mathcal{H}^{W}}(K^{\bullet}(X,A)(k)) ,$$

(note that the Tate twist changes F^{0} to F^{k}). Since $H^{\ell}(X,A)$ has a pure Hodge structure of weight ℓ , we have isomorphisms

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^{\ell-1}_{DR}(X)/\varepsilon H^{\ell-1}(X, A(k)) + F^{k} \arrow[r] & H^{\ell}_{\mathcal{D}}(X, A(k)) \arrow[r] & H^{\ell}(X, A(k)) \cap \varepsilon^{-1} F^{k} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}^{1}_{A\text{-}MH}(A, H^{\ell-1}(X,A)(k)) \arrow[r] \arrow[u, "\wr"] & H^{\ell}_{\mathcal{H}}(K^{\bullet}(X,A)(k)) \arrow[r] \arrow[u, "\wr"] & \mathrm{Hom}_{A\text{-}MH}(A, H^{\ell}(X,A)(k)) \arrow[r] \arrow[u] & 0
\end{tikzcd} \quad (2.7.1)$$

provided $\ell-2k \leq 0$.

Similarly, we define the mixed A-Hodge complex

$$
'K^{\cdot}(X,A):\quad
\begin{tikzcd}[column sep=small]
 & 'C^{\cdot}(X,A\otimes\mathbb{Q}) & & ('\Omega^{\cdot}_{X^{\infty}}(X),W) & \\
'C^{\cdot}(X,A) \arrow[ur] & & ('C^{\cdot}(X,A\otimes\mathbb{Q}),W) \arrow[ul, "id"'] \arrow[ur, "1.1\ c)"] & & ('\Omega^{\cdot}_{X^{\infty}}(X),W,F) \arrow[ul, "id"']
\end{tikzcd} ,
$$

(trivial weight filtrations), and have

$$'H^{\ell}_{\mathcal{D}}(X,A(k)) = H^{\ell}_{\mathcal{H}^W}('K^{\cdot}(X,A)(k))$$

by the canonical quasi-isomorphism

$$'C^{\cdot}_{\mathcal{D}}(X,A(k)) = \mathrm{Cone}('C^{\cdot}(X,A(k)) \oplus F^{k}\,'\Omega^{\cdot}_{X^{\infty}}(X) \xrightarrow{\varepsilon-\iota} {'\Omega^{\cdot}_{X^{\infty}}}(X))[-1]$$

$$\xrightarrow{\sim} R\Gamma_{\mathcal{H}}\,'K^{\cdot}(X,A)(k) .$$

To see that $'K^{\cdot}(X,A)$ is indeed a mixed A-Hodge complex, we may use the quasi-isomorphism

$$K^{\cdot}(X,A)(d)[2d] \xrightarrow{\sim} {'K^{\cdot}(X,A)}$$

proved in 1.15.

It turns out that one can rather simply define the mixed Hodge structure on the cohomology and homology of simplicial schemes with smooth and proper components.

<u>2.8. Proposition</u> Let $X.,Y.$ be simplicial schemes over $\mathbb{C}$ with smooth and proper components, and let $f\colon X. \to Y.$ be a morphism.

a) The map ${}^{(\prime)}H^{\ell}_{B}(X.,A) \otimes_A \mathbb{C} \to {}^{(\prime)}H^{\ell}_{DR}(X.)$ is an isomorphism, and there is a canonical mixed A-Hodge structure on ${}^{(\prime)}H^{\ell}_{B}(X.,A)$ such that

i) the Hodge filtration is given by the Hodge filtration on ${}^{(\prime)}H^{p}_{DR}(X.)$, i.e., $F^{k\,(\prime)}H^{\ell}_{DR}(X.)$ is the image of the injective map

$$
\begin{array}{ccccc}
H^{\ell}(X.,F^{k\,(\prime)}\Omega^{\cdot}_{X^{\infty}}) & \to & H^{\ell}(X.,{}^{(\prime)}\Omega^{\cdot}_{X^{\infty}}) & = & {}^{(\prime)}H^{\ell}_{DR}(X.) \\
\| & & \| & & \\
H^{\ell}(s^{(\prime)}NF^{k\,(\prime)}\Omega^{\cdot}_{X.^{\infty}}(X.)) & \to & H^{\ell}(s^{(\prime)}N^{(\prime)}\Omega^{\cdot}_{X.^{\infty}}(X.)) & , &
\end{array}
$$

ii) the weight filtrations are the ascending filtrations (those for which $Gr_r = E_{\infty}^{-,r}$) associated to the spectral sequences

(2.8.1) $$E_1^{p,q} = {'H^{q}_{B}}(X_{-p},A) \Rightarrow {'H^{p+q}_{B}}(X.,A)$$

$$(2.2.1) \qquad H^n(F^i K^\cdot_{\mathbb{C}}) \overset{\sim}{\to} F^i H^n(K^\cdot_{\mathbb{C}}) \subseteq H^n(K^\cdot_{\mathbb{C}})$$

under the map induced by the inclusion $F^i K^\cdot_{\mathbb{C}} \subseteq K^\cdot_{\mathbb{C}}$.

By the definition of morphisms in derived categories, 2.1.1 is represented by actual morphisms of complexes

$$\begin{tikzcd}
 & {}'K^\cdot_{A\otimes\mathbb{Q}} & & ({}'K^\cdot_{\mathbb{C}},W) & \\
K^\cdot_A \arrow[ur, "\alpha_1"] & & (K^\cdot_{A\otimes\mathbb{Q}},W) \arrow[ul, "\alpha_2"'] \arrow[ur, "\beta_1"] & & (K^\cdot_{\mathbb{C}},W,F) \arrow[ul, "\beta_2"']
\end{tikzcd} ,$$

where α_2 is a quasi-isomorphism, β_1 is a filtered morphism, and β_2 is a filtered quasi-isomorphism, and we rather call these diagrams mixed A-Hodge complexes, or A-$\tilde{p}$-Hodge complexes if they are polarizable (cf.[Be 2]3.9). There is an obvious notion of morphisms and quasi-isomorphisms between them, and by inverting the latter ones one obtains the category $D^+_{\mathcal{H}\tilde{p},A}$ of A-$\tilde{p}$-Hodge complexes up to quasi-isomorphism. It becomes a triangulated category by the cone construction ([Be 2]3.10).

If $C^\cdot$ is a bounded below complex in the category A-MH of mixed A-Hodge structures, there is an obvious mixed A-Hodge complex $K^\cdot=\phi(C^\cdot)$ with $K^\cdot_A=C^\cdot$(with filtration $\tilde{W}.$ defined by $\tilde{W}_m K^n = W_{m+n}K^n$).

<u>2.3 Theorem</u> ([Be 2] 3.11) The functor $C^\cdot \rightsquigarrow \phi(C^\cdot)$ induces an equivalence of triangulated categories between $D^b(\text{A-MH}^p)$, the derived category of bounded complexes of mixed, polarizable A-Hodge structures, and the subcategory $D^b_{\mathcal{H}p,A}$ of $D^+_{\mathcal{H}p,A}$ formed by bounded complexes.

<u>2.4</u> The absolute (Hodge) cohomology $H^\ell_{\mathcal{H}}(K^\cdot)$ of a mixed A-Hodge complex $K^\cdot$ is defined as the ℓ-th homology of the complex

$$R\Gamma_{\mathcal{H}}K^\cdot = \mathrm{Cone}(K^\cdot_A \oplus \hat{W}_o K^\cdot_{A\otimes\mathbb{Q}} \oplus \hat{W}_o \cap F^o K^\cdot_{\mathbb{C}} \xrightarrow{(\alpha,\beta)} {}'K^\cdot_{A\otimes\mathbb{Q}} \oplus \hat{W}'_o K^\cdot_{\mathbb{C}})[-1],$$

$(\alpha,\beta)(k_A,k_{\mathbb{Q}},k_{\mathbb{C}}) = (\alpha_1 k_A - \alpha_2 k_{\mathbb{Q}}, \beta_1 k_{\mathbb{Q}} - \beta_2 k_{\mathbb{C}})$, $\hat{W}. = (\mathrm{Dec}\ W)$. the "filtration decalée" (see [Be 2] p. 51), which up to quasi-isomorphism does not depend on the choice of $'K^\bullet_{\mathbb{Q}\otimes A}$ and $'K^\bullet_{\mathbb{C}}$, i.e., only depends on $K^\bullet$ as given in 2.1.1. If $C^\bullet$ is a complex in A-MH and $K^\bullet = \phi(C^\bullet)$, $R\Gamma_{\mathcal{H}}K^\bullet$ represents $R\mathrm{Hom}_{A\text{-}MH}(A,C^\bullet)$, where A is the trivial (pure of weight 0) A-Hodge structure, hence $H^\ell_{\mathcal{H}}(K^\bullet)$ coincides with the hyperext-group $\mathrm{Ext}^\ell_{A\text{-}MH}(A,C^\bullet)$. Moreover, via ϕ the mixed Hodge structure on $H^n(C^\bullet)$ is Deligne's mixed Hodge structure on $H^n(K^\bullet_A)$, hence the hyperext spectral sequence gives a spectral sequence

$$(2.4.1) \qquad E_2^{p,q} = \mathrm{Ext}^p_{A\text{-}MH}(A,H^q(K^\bullet_A)) \Rightarrow H^{p+q}_{\mathcal{H}}(K^\bullet)$$

where $\mathrm{Ext}^p_{A\text{-}MH}$ is the Yoneda Ext-group, see [SGA $4\frac{1}{2}$][C.D.] p. 298. On the other hand, we obtain for a mixed A-Hodge structure H

$$(2.4.2) \qquad \mathrm{Ext}^p_{A\text{-}MH}(A,H) = H^p_{\mathcal{H}}(\phi(H)) = \begin{cases} H \cap \varepsilon^{-1}(W_0 \cap F^0 H_{\mathbb{C}}) & ,p=0 , \\ \mathrm{Coker}(H \oplus W_0 H_{\mathbb{Q}} \oplus W_n \cap F^0 H_{\mathbb{C}} \to H_{\mathbb{Q}} \oplus W_0 H_{\mathbb{C}}) & ,p=1 , \\ 0 & ,p \geq 2 , \end{cases}$$

where $\varepsilon\colon H \to H_{\mathbb{C}}$ is the canonical map (in reality, Beilinson has to show this formula by other means to prove 2.3).

<u>2.5.</u> Beilinson defines the weak absolute cohomology of $K^\bullet$ by forgetting the weight filtration W, i.e., by letting $H^\ell_{\mathcal{H}^w}(K^\bullet)$ be the ℓ-th homology of

$$R\Gamma_{\mathcal{H}^w}K^\bullet = \mathrm{Cone}(K^\bullet_A \oplus K^\bullet_{A\otimes\mathbb{Q}} \oplus F^0 K^\bullet_{\mathbb{C}} \xrightarrow{(\alpha,\beta)} {'K^\bullet_{A\otimes\mathbb{Q}}} \oplus {'K^\bullet_{\mathbb{C}}})[-1].$$

In the derived category $D^+(A)$ this complex is isomorphic to

$$(2.5.1) \qquad \mathrm{Cone}(K^\bullet_A \oplus F^0 K^\bullet_{\mathbb{C}} \xrightarrow{\varepsilon-\iota} K^\bullet_{\mathbb{C}})[-1]$$

with $\varepsilon = \beta\alpha$ and ι the inclusion, so by property 2.2.1 we obtain a long exact sequence

$$(2.5.2) \quad \ldots \to H^\ell_{\mathcal{H}^w}(K^\bullet) \to H^\ell(K^\bullet_A) \oplus F^0 H^\ell(K^\bullet_{\mathbb{C}}) \xrightarrow{\varepsilon-\iota} H^\ell(K^\bullet_{\mathbb{C}}) \to H^{\ell+1}_{\mathcal{H}^w}(K^\bullet) \to \ldots ,$$

and induced short exact sequences

$$(2.5.3) \quad 0 \to H^{\ell-1}(K^\bullet_{\mathbb{C}})/\varepsilon H^{\ell-1}(K^\bullet_A) + F^0 H^{\ell-1}(K^\bullet_{\mathbb{C}}) \to H^\ell_{\mathcal{H}^w}(K^\bullet) \to$$
$$\to H^\ell(K^\bullet_A) \cap \varepsilon^{-1} F^0 H^\ell(K^\bullet_{\mathbb{C}}) \to 0$$

Comparing this with the short exact sequences

$$(2.5.4) \quad 0 \to \mathrm{Ext}^1_{A\text{-}MH}(A,H^{\ell-1}(K^\bullet_A)) \to H^\ell_{\mathcal{H}}(K^\bullet) \to \mathrm{Hom}_{A\text{-}MH}(A,H^\ell(K^\bullet_A)) \to 0$$

obtained from 2.4.1 and the vanishing of $\mathrm{Ext}^p_{A\text{-}MH}$ for $p \geq 2$,

→ p. 32c

$$(2.8.2)\qquad E_1^{p,q} = H_B^q(X_p,A) \Rightarrow H_B^{p+q}(X.,A).$$

b) After tensoring with $\mathbb{Q}$, the spectral sequences above degenerate at E_2, giving $Gr^W_{\ell+r}\,{}^{(\prime)}H_B^{\ell}(X.,A\otimes\mathbb{Q})$ as the subquotient $E_2^{-r,\ell+r}$ of $E_1^{-r,\ell+r}$.

c) There is a commutative diagram of canonical pairings

$$(2.8.3)\qquad \begin{array}{ccccccc} \psi_{DR}: & H^{\ell}_{DR}(X.) & \times & {}'H^{-\ell}_{DR}(X.) & \to & \mathbb{C} \\ & \uparrow\varepsilon & & \uparrow\varepsilon & & \cup| \\ \psi_{B,A}: & H^{\ell}_B(X.,A) & \times & {}'H^{-\ell}_B(X.,A) & \to & A \end{array}$$

the upper one perfect, the lower one perfect after tensoring with $\mathbb{Q}$, such that via these

$$(2.8.4)\qquad F^{-i}\,{}'H^{-\ell}_{DR}(X.) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(H^{\ell}_{DR}(X.)/F^{i+1},\mathbb{C}),$$

$$(2.8.5)\qquad W_{-n}\,{}'H^{-\ell}_B(X.,A\otimes\mathbb{Q}) \xrightarrow{\sim} \mathrm{Hom}_{A\otimes\mathbb{Q}}(H^{\ell}_B(X.,A\otimes\mathbb{Q})/W_{n-1},A\otimes\mathbb{Q}),$$

i.e., ${}'H^{-\ell}_B(X.,A\otimes\mathbb{Q})$ is the dual of the mixed Hodge structure $H^{\ell}_B(X.,A\otimes\mathbb{Q})$.

d) The diagram

$$\begin{array}{ccccc} H^{\ell}_B(X.,A) & \times & {}'H^{-\ell}_B(X.,A) & \to & A \\ f^*\uparrow & & \downarrow f_* & & \| \\ H^{\ell}_B(Y.,A) & \times & {}'H^{-\ell}_B(Y.,A) & \to & A \end{array}$$

is commutative, and f_* and f^* are morphisms of mixed A-Hodge structures. In particular, they are strictly compatible with the weight and the Hodge filtrations.

Proof a) For the homology we only have to prove that

$$s'N'K^\cdot(X,A): \quad \begin{array}{l} s'N'C^\cdot(X.,A) \to (s'N'C^\cdot(X.,A\otimes\mathbb{Q}),W) \to (s'N'\Omega^\cdot_{X.^\infty}(X.),W,F) \\ \quad\| 1.10 \\ {}'C^\cdot((X.),A) \end{array}$$

is a mixed A-Hodge complex, where the weight filtrations are given by the second ascending filtrations of the double complexes:

$$W_m(s'N'C^\cdot(X.,A\otimes\mathbb{Q})) = \bigoplus_{\substack{t<m \\ r\in\mathbb{Z}}} {}'C^r(X_{-t},A\otimes\mathbb{Q}),$$

$$W_m(s'N'\Omega^\cdot_{X^\infty}(X)) = \bigoplus_{\substack{t<m \\ r\in\mathbb{Z}}} {}'\Omega^r_{X^\infty_{-t}}(X_{-t}),$$

and the Hodge filtration is induced by the Hodge filtration of the $'\Omega^{\cdot}_{X_p^\infty}(X_p)$:

$$F^k s\,'N'\Omega^{\cdot}_{X.^\infty}(X.) = \bigoplus_{r,t\in\mathbb{Z}} F^k\,'\Omega^r_{X_{-t}^\infty}(X_{-t}) .$$

But this is clear, since $Gr^W_m s\,'N'K^{\cdot}(X,A)$ is the Hodge complex of weight m

$$'C^{\cdot}(X_{-m},A)[m] \to ('\Omega^{\cdot}_{X_{-m}^\infty}(X_{-m})[m],F) .$$

For the cohomology the complex considered by Deligne in [De 3] 8.1.19 is quasi-isomorphic to

$$sNK^{\cdot}(X.,A):\quad \begin{array}{ccccccc} & & sNC^{\cdot}(X.,A\otimes\mathbb{Q}) & & & (sNC^{\cdot}(X.,\mathbb{C}),W) & \\ & \nearrow & & \nwarrow^{\text{id}} & \nearrow & & \nwarrow^{\text{integration}} \\ sNC^{\cdot}(X.,A) & & & & (sNC^{\cdot}(X.,A\otimes\mathbb{Q}),W) & & (sN\Omega^{\cdot}_{X.^\infty}(X.),W,F) , \end{array}$$

where N is the cohomological analogue of $'N$ above, mapping a cosimplicial abelian group $A.$ to the cochain complex $NA.$ with $(NA.)^i = A_i$ and $d^i = \sum_{\nu=0}^{i+1}(-1)^\nu\delta_\nu$, and where

$$W_{-m}(sNC^{\cdot}(X.,A\otimes\mathbb{Q})) = \bigoplus_{\substack{t\geq m\\ r\in\mathbb{Z}}} C^r(X_t,A\otimes\mathbb{Q}) ,$$

similarly for Ω , and

$$F^k(sN\Omega^{\cdot}_{X.^\infty}(X.)) = \bigoplus_{r,t\in\mathbb{Z}} F^k\Omega^r_{X_t^\infty}(X_t) ,$$

so that $Gr^W_{-m} s\,NK^{\cdot}(X.,A)$ is quasi-isomorphic to

$$\begin{array}{ccccc} & & C^{\cdot}(X_m,\mathbb{C})[-m] & & \\ & \nearrow & & \nwarrow^{\text{integration}} & \\ C^{\cdot}(X_m,A)[-m] & & & & (\Omega^{\cdot}_{X_m^\infty}(X_m)[-m],F) . \end{array}$$

Note that $H^\ell(X.,F^{k(')}\Omega^{\cdot}_{X.^\infty}) = H^\ell(s^{(')}NF^{k(')}NF^{k(')}\Omega^{\cdot}_{X.^\infty}(X.))$,

since the sheaves $Gr_F^{k(')}\Omega^q_{X_p^\infty}$ are fine and hence acyclic.

b) By the definition of the spectral sequences 2.8.1 and 2.8.2 (cf. [De 3] 5.2.3), they agree with those given by the weight filtration, as defined above, hence the claim follows from a general property of mixed Hodge complexes ([De 3] 8.1.9(iv)).

c) For a smooth and proper scheme X over $\mathbb{C}$ we have a commutative diagram of pairings

$$(2.8.6)\qquad \begin{array}{cccccc} \Psi_\Omega : & \Omega^\bullet_{X^\infty}(X) & \times & {}'\Omega^\bullet_{X^\infty}(X) & \to & \mathbb{C} \\ & \kappa\downarrow & & \uparrow\varepsilon & & \| \\ \Psi_C : & C^\bullet(X,\mathbb{C}) & \times & {}'C^\bullet(X,\mathbb{C}) & \to & \mathbb{C} \end{array}$$

defined by evaluation and by integration $(M,\omega) \mapsto \int_M \omega$ as in 1.1 c). In the induced commutative diagram

$$\begin{array}{ccc} {}'\Omega^\bullet_{X^\infty}(X) & \xrightarrow{\psi_\Omega} & \mathrm{Hom}^\bullet_{\mathbb{C}}(\Omega^\bullet_{X^\infty}(X),\mathbb{C}) \\ \varepsilon\uparrow & & \uparrow\kappa^\vee \\ {}'C^\bullet(X,\mathbb{C}) & \xrightarrow{\psi_{\mathbb{C}}} & \mathrm{Hom}^\bullet_{\mathbb{C}}(C^\bullet(X,\mathbb{C}),\mathbb{C}) \end{array}$$

the morphisms $\psi_{\mathbb{C}}, \varepsilon$ and $\kappa^\vee$ are quasi-isomorphisms, hence the same is true for ψ_Ω .

For a morphism $f: X \to Y$ of smooth and proper schemes the diagram

$$(2.8.7)\qquad \begin{array}{ccccc} C^\bullet(X,A) & \times & {}'C^\bullet(X,A) & \to & A \\ f^*\uparrow & & \downarrow f_* & & \| \\ C^\bullet(Y,A) & \times & {}'C^\bullet(Y,A) & \to & A \end{array}$$

commutes by definition, hence $\psi_{\mathbb{C}}$ and ψ_Ω are functorial in X . This gives analogous quasi-isomorphisms $\psi_{\mathbb{C}}$ and ψ_Ω for our simplicial scheme $X.$, since $s\,N\,\mathrm{Hom}_{\mathbb{C}}(B^\bullet_\bullet,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(s\,'NB^\bullet_\bullet,\mathbb{C})$ for a simplicial complex $B^\bullet_\bullet$ of $\mathbb{C}$-vector spaces. Moreover, $\psi_{\mathbb{C}}$ comes from a pairing

$$\Psi_A : s\,N\,C^\bullet(X.,A) \times s\,N\,C^\bullet(X.,A) \to A ,$$

and we may define the pairings 2.8.3 by the latter and ψ_Ω . Since ψ_Ω is a quasi-isomorphism, ψ_{DR} is perfect, and this implies the perfectness of $\psi_{B,\mathbb{Q}}$, and then of $\psi_{B,A\otimes\mathbb{Q}}$, because the ${}^{(\prime)}H^\ell_B(X.,\mathbb{Q})$ are finite-dimensional vector spaces and we have ${}^{(\prime)}H^\ell_B(X.,\mathbb{Q}) \otimes \mathbb{C} \cong {}^{(\prime)}H^\ell_{DR}(X.)$ and ${}^{(\prime)}H^\ell_B(X.,A\otimes\mathbb{Q}) \cong {}^{(\prime)}H^\ell_B(X.,\mathbb{Q}) \otimes (A\otimes\mathbb{Q})$.

Obviously, the pairings Ψ are compatible with the filtrations F and W ($\psi(W_i \otimes W_j) \subseteq W_{i+j}$ etc.), if we endow A and $\mathbb{C}$ with the trivial filtrations , i.e., $\psi_{B,A}$ is a pairing of mixed A-Hodge structures into the trivial Hodge structure A. Hence we obtain the morphisms 2.8.4 and 2.8.5, which must be isomorphisms by the non-degeneracy and the strictness of morphisms of mixed Hodge structures.

d) The functoriality is clear from the diagram 2.8.7 and its simplicial analogue, and f_* and f^* are morphisms of mixed

A-Hodge structures since they are induced by morphisms of mixed A-Hodge complexes.

2.9. We want to apply this to prove the Hodge theoretic statements in the proof of 1.18.

Let X be smooth and proper over $\mathbb{C}$ or $\mathbb{R}$, and let $Y \subset X$ be a divisor with normal crossings. Let $\tilde{Y} = Y^1 \amalg \dots \amalg Y^r$ be the normalization of Y (Y^i the irreducible components of Y, which are smooth and proper), and let $\tilde{Y}. = \mathrm{cosk}^X \tilde{Y}$ be the coskeleton of $\tilde{Y}/X$ (see [De 3] 5.1). This is the same as the nerve of the covering $Y = \cup Y^i$, i.e., $\tilde{Y}_m$ consists of the $(m+1)$-fold intersections of the Y^i. Then $\tilde{Y}. \to Y$ is a proper hypercovering, and hence has descent for the Borel-Moore homology. Since $'C^\bullet_Y(X, \mathbb{Z})$ computes the Borel-Moore homology of Y (compare 1.14.1 and [BM] 3.5 b)), we see that the map π_C in the following commutative diagram is a quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] & {'\Omega^\bullet_{X^\infty}(\text{on } Y)(X)} \arrow[r] & {'\Omega^\bullet_{X^\infty}(X)} \arrow[r, "\alpha"] & {'\Omega_{X^\infty}\langle Y\rangle(X)} \arrow[r] & 0 \\
& {s'N'\Omega^\bullet_{\tilde{Y}.^\infty}(\tilde{Y}.)} \arrow[u, "\pi_\Omega"] & & & \\
(2.9.1) & {s'N'C^\bullet(\tilde{Y}.,A(k))} \arrow[u, "\varepsilon_{\tilde{Y}.}"] \arrow[d, "\pi_C"] & & & \\
0 \arrow[r] & {'C^\bullet_Y(X,A(k))} \arrow[r] & {'C^\bullet(X,A(k))} \arrow[r] \arrow[uuu, "\varepsilon_X"] & {'C^\bullet(X,Y,A(k))} \arrow[r] \arrow[uuu, "\varepsilon_{(X,Y)}"] & 0
\end{tikzcd}$$

On the other hand, if we let $'\Omega^\bullet_{X^\infty}(\text{on } Y)(X) = \mathrm{Ker}\,\alpha$ as indicated, π_Ω is a filtered quasi-isomorphism for the Hodge filtration by [Fu]3.7.1 (the statement for $\eta_X\langle Y\rangle$, cf. 2.11 g) below)

Together we obtain canonical quasi-isomorphisms β_C, β_F and β_Ω sitting in a commutative diagram

$$\begin{tikzcd}[column sep=small]
{\mathrm{Cone}(s'N'C^\bullet(Y.,A(k))} \arrow[r] & {s'N'C^\bullet(X,A(k)))} \arrow[r, "\beta_C"] & {'C^\bullet(X,Y,A(k))} \\
\oplus & \oplus & \oplus \\
{\mathrm{Cone}(s'NF^k{'}\Omega^\bullet_{\tilde{Y}.^\infty}(\tilde{Y}.)} \arrow[r] \arrow[d, "\varepsilon-\iota"] & {s'NF^k{'}\Omega^\bullet_{X^\infty}(X))} \arrow[r, "\beta_F"] \arrow[d] & {F^k{'}\Omega^\bullet_{X^\infty}\langle Y\rangle(X)} \arrow[d, "\varepsilon-\iota"] \\
{\mathrm{Cone}(s'N'\Omega_{\tilde{Y}.^\infty}(\tilde{Y}.)} \arrow[r] & {s'N'\Omega^\bullet_{X^\infty}(X))} \arrow[r, "\beta_\Omega"] & {'\Omega_{X^\infty}\langle Y\rangle(X)}
\end{tikzcd}$$

(2.9.2)

More generally, let $(X.,Y.)$ be a simplicial object in

$\widetilde{\pi}_*$. Then for each n we get a proper hypercovering $\widetilde{Y}_{n.} \to Y_n$ as above, hence a bi-simplicial scheme $\widetilde{Y}_{..}$ and a morphism of simplicial schemes $u : \Delta Y_{..} \to Y. \to X.$, where $Z. = \Delta\widetilde{Y}_{..}$ is the diagonal of $\widetilde{Y}_{..}$ (see [De 3]6.4). By looking at the simplicial analogue of 2.9.2, applying the normalization functor $'N$ and then the Eilenberg-Zilber quasi-isomorphism, we now obtain a commutative diagram with quasi-isomorphisms β

$$\begin{array}{ccccc}
\mathrm{Cone}(s'N'C^\cdot(Z.,A(k)) & \xrightarrow{u_*} & s'N'C^\cdot(X.,A(k))) & \xrightarrow{\beta_C} & s'N'C^\cdot(X.,Y.,A(k)) \\
\oplus & & \oplus & & \oplus \\
\mathrm{Cone}(s'NF^k{}'\Omega^\cdot_{Z.^\infty}(Z.) & \xrightarrow{u_*} & s'NF^k{}'\Omega^\cdot_{X.^\infty}(X.)) & \xrightarrow{\beta_F} & s'NF^k{}'\Omega^\cdot_{X.^\infty}\langle Y.\rangle(X.) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cone}(s'N'\Omega^\cdot_{Z.^\infty}(Z.) & \xrightarrow{u_*} & s'N'\Omega^\cdot_{X.^\infty}(X.)) & \xrightarrow{\beta_\Omega} & s'N'\Omega^\cdot_{X.^\infty}\langle Y.\rangle(X.) .
\end{array} \tag{2.9.3}$$

Here the left column is the diagram $K^\cdot_A \oplus F^oK^\cdot_{\mathbb{C}} \to K^\cdot_{\mathbb{C}}$ for the mixed A-Hodge complex

$$K^\cdot = 'K^\cdot(u) = \mathrm{Cone}(s'N'K^\cdot(Z.,A)(k) \xrightarrow{u_*} s'N'K^\cdot(X.,A)(k)) ,$$

Z. and X. having smooth and proper components, hence we can apply all properties of mixed A-Hodge to the right column. In particular, the map ι of 1.18.2 is injective by 2.2.1, and, since the above quasi-isomorphisms are functorial in $(X.,Y.)$, a morphism $f: (X.,Y.) \to (X!,Y!)$ will induce morphisms in the deRham homology, which are strictly compatible with the Hodge filtration, since they come from morphisms of mixed Hodge structures $f_* : H^*('K^\cdot(u)_A) \to H^*('K^\cdot(u')_A)$, with $u': \Delta\widetilde{Y}!. \to X!$ defined as above.

In fact, by using the canonical quasi-isomorphisms β_C, β_F and β_Ω for "transport de structure", we can put canonical mixed A-Hodge structures on the homology of $'C^\cdot((X.,Y.),A) = s'N'C^\cdot(X.,Y.,A)$, varying functorially in $(X.,Y.)$, such that the Hodge filtration is given by the Hodge filtration of $H^*(s'N'\Omega_{X.^\infty}\langle Y.\rangle(X.)) = H^*(s'N'C^\cdot(X.,Y.,A))\otimes_A\mathbb{C}$. By the same constructions and arguments as in 1.17 and the proof of 1.18, we obtain the following result.

<u>2.10 Theorem</u> a) Let Z be a separated scheme of finite type over $\mathbb{C}$, then there is a canonical mixed A-Hodge structure on the Borel-Moore homology $'H^\ell_B(X,A)$, $\ell \in \mathbb{Z}$, such that the

Hodge filtration is the Hodge filtration of $'H^{\ell}_{DR}(Z) \cong$ $'H^{\ell}_{B}(Z,A)\otimes_A \mathbb{C}$ and for a proper morphism $f\colon Z \to Z'$ the maps $f_*\colon {'H^{\ell}_{B}}(Z,A) \to {'H^{\ell}_{B}}(Z',A)$ are morphisms of mixed A-Hodge structures.

b) For the twisted Poincaré duality theory given by

$$\begin{array}{lll} Z \hookrightarrow X \rightsquigarrow H^i_{B,Z}(X,A(j)) & \left(= H^i_{B,Z}(X,A)(j) \right. & \text{as mixed A-} \\ X \rightsquigarrow H^B_a(X,A(b)) & \left. = H^B_a(X,A)(-b) \right. & \text{Hodge structures}) \end{array}$$

where $H^B_a(X,A(b)) = {'H^{-a}_B}(X,A(-b))$, all morphisms occurring in the axioms ([BO] 1.3) are morphisms of mixed A-Hodge structures.

<u>2.11. Remarks</u> a) To prove 2.10 b) for the connecting morphisms of the long exact sequences 1.1.1 and 1.2.3 in [BO], note that the latter come from exact triangles of mixed A-Hodge complexes $K_1^{\cdot} \to K_2^{\cdot} \to K_3^{\cdot} \to$, and hence the connecting morphisms come from morphisms of mixed A-Hodge complexes $K_3^{\cdot} \to K_1^{\cdot}[1]$.

b) All constructions for Deligne cohomology and homology can be made on the level of mixed Hodge complexes. Thus, Beilinson in [Be 2] defines mixed Hodge complexes $K^{\cdot}$ and $'K^{\cdot}$ for arbitrary varieties, and obtains absolute Hodge (co)homology or Deligne (co)homology by applying $R\Gamma_{\mathcal{H}}$ or $R\Gamma_{\mathcal{H}^W}$ to these (see 2.4 and 2.5). One could and perhaps should prove all statements about Deligne (co)homology along these lines. In [Be 2] Beilinson defines the homological complexes as duals of complexes with compact support $K_c^{\cdot}$, but one may also define the $'K^{\cdot}$-complexes by starting from $'K^{\cdot}(X,A)$ as defined in 2.7 and use the complex $'K^{\cdot}(u)$ in 2.9 for arbitrary schemes.

c) By only working with simplicial schemes with smooth and proper components in 2.8 and 2.9, we have avoided putting weight filtrations on (complexes quasi-isomorphic to) $'C^{\cdot}(X,Y,A)$ and $'\Omega^{\cdot}_{X}{}_{\infty}\langle Y\rangle(X)$ and then using the more difficult result of Deligne, [De 3] 8.1.15, that applying sN (resp. s'N) to a cosimplicial (resp. simplicial) mixed Hodge complex again gives a mixed Hodge complex.

d) In [Be 1] 1.8.4, Beilinson seems to assert that π_C and π_Ω in 2.9.1 above are embeddings, which obviously cannot be true.

e) The diagram 2.8.6 commutes with our sign conventions, but not with the ones in [DMOS] pp. 13,14.

f) The considerations in 2.9 show that Gillet's definition of Deligne homology [Gi 2] leads to the same groups as here.

g) The notations used for currents in the literature are quite diverse. For the convenience of the reader we compare our notation with that of Fujiki [Fu] and King [Ki], for a smooth proper variety X and a divisor with normal crossings $i: Y \to X$, with open complement $j: U \hookrightarrow X (d = \dim X)$.

here	King	Fujiki	quasi-isomorphic to
$\Omega^{\cdot}_{X^\infty}$	$A^{\cdot}_X$	$\mathcal{E}^{\cdot}_X$	$\mathbb{C}$
$\Omega^{\cdot}_{X^\infty}\langle Y\rangle$	$A^{\cdot}_X(\log Y)$	-	$Rj_*j^*\mathbb{C}$
$\Omega^{\cdot}_{X^\infty}(\text{null } Y)$	$A^{\cdot}_X(\text{null } Y)$	$\mathcal{E}\Sigma^{\cdot}_{Y/X}$	$j_!j^*\mathbb{C}$
$'\Omega^{\cdot}_{X^\infty}[-2d]$	$'\mathcal{D}^{\cdot}_X$	$'\mathcal{D}^{\cdot}_X$	$\mathbb{C}$
$'\Omega^{\cdot-2d}_{X^\infty}(\text{on } Y)$	$'\mathcal{D}^{\cdot}_X(\text{on } Y)$	$'\mathcal{D}^{\cdot}_X\langle Y\rangle$	$i_*Ri^!\mathbb{C}$
$'\Omega^{\cdot-2d}_{X^\infty}\langle Y\rangle$	$'\mathcal{D}^{\cdot}_X(\log Y)$	$'\mathcal{D}^{\cdot}_X/\mathcal{D}^{\cdot}_X\langle Y\rangle$	$Rj_*j^*\mathbb{C}$

§3 Riemann-Roch and Hodge-$\mathcal{D}$-conjecture

For a field F, let $\mathcal{V}_F$ be the category of reduced, quasi-projective schemes over F, called varieties in this section.

Recall that for a scheme X one has the Quillen spectral sequence of homological type [Qui]5.4

$$(3.0) \qquad E^1_{p,q} = \bigoplus_{x\in X_{(p)}} K_{p+q}(\kappa(x)) \Rightarrow E_{p+q} = K'_{p+q}(X) ,$$

where K' denotes Quillen's K-theory of coherent sheaves on X, $X_{(p)}$ is the set of points of X of dimension p, and $\kappa(x)$ is the residue field of $x \in X$.

<u>3.1. Lemma</u> If X is a variety of dimension d over $F = \mathbb{R}$ or $\mathbb{C}$, there is a canonical isomorphism

$$\rho: E^2_{d,1-d}(X) = \operatorname{Ker}\Big(\bigoplus_{x\in X_{(d)}} \kappa(x)^\times \xrightarrow{\text{div}} \bigoplus_{x\in X_{(d-1)}} \mathbb{Z}\Big) \xrightarrow{\sim} H^{\mathcal{D}}_{2d-1}(X/F,\mathbb{Z}(d-1)),$$

where $\text{div} = d^1_{d,1-d}$ is the divisor map [Qui]5.14, [Gray].

Proof First assume that X is smooth of pure dimension d . Then we have canonical isomorphisms

$$E^2_{d,1-d}(X) = \mathcal{O}(X)^\times_{alg} \underset{\sim}{\overset{\rho}{\rightarrow}} H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \xrightarrow[\sim]{1.15} H^{\mathcal{D}}_{2d-1}(X, \mathbb{Z}(d-1)) ,$$

see [EV] 2.12 c) for ρ .

In general, X contains a smooth open subvariety U of pure dimension d such that the complement $Y = X \setminus U$ is of smaller dimension. From the long exact homology sequence 1.19.1 and the vanishing result in 1.20 we obtain a commutative exact diagram

$$(3.1.1)\quad \begin{array}{ccccccc} 0 \rightarrow & H^{\mathcal{D}}_{2d-1}(X, \mathbb{Z}(d-1)) & \rightarrow & H^{\mathcal{D}}_{2d-1}(U, \mathbb{Z}(d-1)) & \rightarrow & H^{\mathcal{D}}_{2d-2}(Y, \mathbb{Z}(d-1)) \\ & \uparrow & & \wr\uparrow & (*) & \uparrow 1.20 \\ 0 \rightarrow & E^2_{d,1-d}(X) & \rightarrow & \mathcal{O}(U)^\times_{alg} & \xrightarrow{div} & \bigoplus_{y \in Y_{(d-1)}} \mathbb{Z} \end{array}$$

defining the dotted isomorphism we need. Since everything is functorial, it does not depend on the choice of U . The commuting of (*) can be seen as follows. By passing to the normalization and using the covariance for finite morphisms we may assume that X is normal, hence regular in codimension one. By removing subvarieties of Y of dimension < d-1 we may assume that X is smooth and hence pass to Deligne cohomology. By 1.20 we may restrict to Betti cohomology and finally have to check that for smooth X

$$(3.1.2)\quad \begin{array}{ccc} H^1_B(U, \mathbb{Z}(1)) & \rightarrow & H^2_{B,Y}(X, \mathbb{Z}(1)) \\ \uparrow & & \uparrow cl_B \\ \mathcal{O}(U)^\times_{alg} & \xrightarrow{div} & \bigoplus_{y \in Y_{(d-1)}} \mathbb{Z} \end{array}$$

is commutative, where cl_B is the relative cycle map. This is well-known, in fact, more or less the definition of the latter (compare [SGA $4\frac{1}{2}$][cycle]2.1).

3.2. By the definition of the Chern classes and characters with values in the Deligne cohomology

$$(3.2.1)\qquad c, ch : K_m(X) \rightarrow \bigoplus_{j \in \mathbb{Z}} H^{2j-m}_{\mathcal{D}}(X, A(j))$$

(see [Be 1] 2.3 and [Sch] §4), for smooth X the composite

$$(3.2.2)\qquad K_1(X) \xrightarrow{det} \mathcal{O}(X)^\times_{alg} \underset{\sim}{\overset{\rho}{\rightarrow}} H^1_{\mathcal{D}}(X, \mathbb{Z}(1))$$

coincides with $c_{1,1} = ch_{1,1}$. We now shall show that for general X of dimension d the composite

$$(3.2.3) \qquad K_1'(X) \to E^2_{d,1-d}(X) \overset{\rho}{\underset{\sim}{\to}} H^{\mathcal{D}}_{2d-1}(X, \mathbb{Z}(d-1)) ,$$

where the first map comes from the spectral sequence 3.0, has a similar interpretation.

The Riemann-Roch theorem as formulated by Grothendieck [SGA 6], using Chern classes, K_o and K_o', has been reformulated and extended by Baum, Fulton and MacPherson [BFM] by using certain natural transformations τ between K_o' and homology. Gillet has shown how to extend this to higher algebraic K-theory, if one has a twisted Poincaré duality theory satisfying certain further properties [Gi 1].

Beilinson has written down another set of axioms which assures the existence of the wanted transformations, and he has proved that the Deligne (co-)homology satisfies these axioms, the main points being the following ones.

The Deligne cohomology can be obtained as the Zariski hypercohomology of certain complexes

$$A(p)_{\mathcal{D},Zar} \qquad (p \in \mathbb{Z})$$

on the <u>big</u> Zariski site of $\mathcal{U}_{\mathbb{C}}$, see [Be 1] 1.6.5, and [EV] 5.5 for a slightly weaker statement. The Poincaré duality isomorphisms

$$H^{2d-a}_{\mathcal{D},Z}(X,A(d-b)) \overset{\sim}{\to} H^{\mathcal{D}}_a(Z,A(b)) \qquad , \qquad a,b \in \mathbb{Z} ,$$

for Z closed in a smooth variety X of pure dimension d is induced by quasi-isomorphisms

$$(3.2.4) \qquad R\Gamma_Z(X,A(d-b)_{\mathcal{D},Zar}) \overset{\sim}{\to} {}'C^{\bullet}_{\mathcal{D}}(Z,A(-b)) \qquad , \qquad b \in \mathbb{Z} ,$$

which follow from the corresponding quasi-isomorphisms for X and $j\colon X \smallsetminus Z = U \hookrightarrow X$ and the commutative diagram

$$\begin{array}{ccc} R\Gamma(X,A(d-b)_{\mathcal{D},Zar}) & \overset{\sim}{\to} & {}'C^{\bullet}_{\mathcal{D}}(X,A(-b)) \\ j^* \downarrow & & \downarrow j^* \\ R\Gamma(U,A(d-b)_{\mathcal{D},Zar}) & \overset{\sim}{\to} & {}'C^{\bullet}_{\mathcal{D}}(U,A(-b)) \end{array}$$

by the cone construction (cf. 1.15 and 1.19.3). Finally, these data satisfy the conditions a) - f) in [Be 1] 2.3; for d) compare the proof of 3.1 above.

The axioms of Beilinson and Gillet do not imply each other: While the homology groups of Gillet do not necessarily arise from (functorially given) complexes in $D^+(Ab)$, the Gysin isomorphism and its properties ([Gi 1] 1.2 vi) and vii)) are postulated by Beilinson only for the complexes of global sections $R\Gamma(x,-)$. As far as I can see, this suffices for all theorems proved by Gillet in [Gi 1], however the compatibility of the Poincaré quasi-isomorphism (cf. 3.2.4 above) with open immersions should probably be included in Beilinson's axioms to obtain a canonical restriction map in homology and property iii) of [Gi 1] 4.1 .

In any case, the Deligne (co-)homology satisfies all the axioms, hence for the universal case $A = \mathbb{Q}$ we obtain:

<u>3.3. Theorem</u> There are homomorphisms for X in $\mathcal{V}_F$, $F = \mathbb{R}$ or $\mathbb{C}$,

$$\tau\colon K'_m(X) \to \bigoplus_{b\in\mathbb{Z}} H^{\mathcal{D}}_{m-2b}(X,\mathbb{Q}(b)) \quad , \qquad m \geq 0,$$

compatible with the covariance for proper morphisms and contravariance for open immersions of both sides such that $\tau([\mathcal{O}_X]) = Td(X) \cap \eta^{\mathcal{D}}_X$, where $\eta^{\mathcal{D}}_X$ is the fundamental class in the Deligne homology and $Td(X) \in \bigoplus_{\nu\geq 0} H^{2\nu}(X,\mathbb{Q}(\nu))$ is the Todd class of X, and such that for a closed immersion $Z \hookrightarrow X$ the following diagram is commutative

$$(3.3.1)\qquad \begin{array}{ccc} \Big(\bigoplus_{j\in\mathbb{Z}} H^{2j-m}_{\mathcal{D},Z}(X,\mathbb{Q}(j))\Big) \otimes \Big(\bigoplus_{b\in\mathbb{Z}} H^{\mathcal{D}}_{n-2b}(X,\mathbb{Q}(b))\Big) & \xrightarrow{\cap} & \bigoplus_{c\in\mathbb{Z}} H^{\mathcal{D}}_{m+n-2c}(Z,\mathbb{Q}(c)) \\ \uparrow{\scriptstyle ch\otimes\tau} & & \uparrow{\scriptstyle \tau} \\ K^Z_m(X) \otimes K'_n(X) & \xrightarrow{\cap} & K'_{m+n}(Z) \end{array} ,$$

where ch is the Chern character 3.2.1 with support (cf. [Gi 1] (2.34 ii)) .

For smooth X the canonical isomorphism $K'_m(Z) \cong K^Z_m(X)$

can be obtained as capproduct with $[\mathcal{O}_X] \in K'_o(X)$, so that 3.3.1 implies a commutative diagram

$$\begin{tikzcd}
K^Z_m(X) \arrow[r] \arrow[d, "(3.3.2)\cap[\mathcal{O}_X]"', "\wr"] & \bigoplus_{j\in\mathbb{Z}} H^{2j-m}_{\mathcal{D},Z}(X,\mathbb{Q}(j)) \arrow[d, "\cap\tau(\mathcal{O}_X])"] \arrow[dr, "\cup Td(X)"] & \\
 & & \bigoplus_{j\in\mathbb{Z}} H^{2j-m}_{\mathcal{D},Z}(X,\mathbb{Q}(j)) \arrow[dl, "\sim"', "\cap\,\eta^{\mathcal{D}}_X (\text{Poincaré duality})"] \\
K'_m(Z) \arrow[r] & \bigoplus_{b\in\mathbb{Z}} H^{\mathcal{D}}_{m-2b}(X,\mathbb{Q}(b)) &
\end{tikzcd}$$

.

Conversely, the proof of the theorem consists in showing that one may define τ by this diagram .

<u>3.4. Example</u> Since $Td(U) = 1 + \text{terms in } \bigoplus_{j>0} H^{2j}_{\mathcal{D}}(U,\mathbb{Q}(j))$, we have a commutative diagram

$$\begin{tikzcd}
H^{\mathcal{D}}_{2d-1}(X,\mathbb{Q}(d-1)) \arrow[r] & H^{\mathcal{D}}_{2d-1}(U,\mathbb{Q}(d-1)) & H^1_{\mathcal{D}}(U,\mathbb{Q}(1)) \arrow[l, "1.15"', "\sim"] \\
K'_1(X) \arrow[r] \arrow[u, "\pi_{d-1}\circ\tau"] & K'_1(U) \arrow[u, "\pi_{d-1}\circ\tau"] & K_1(U) \arrow[l, "\sim"] \arrow[u, "ch_{1,1}"']
\end{tikzcd}$$

where π_{d-1} is the projection onto $H^{\mathcal{D}}_{2d-1}(-,\mathbb{Q}(d-1))$. Comparison with 3.1.1 shows that 3.2.3 coincides with $\pi_{d-1}\circ\tau$ after tensoring with $\mathbb{Q}$.

<u>3.5.</u> Recall that the $\mathbb{Q}$-rational motivic (or absolute) cohomology and homology of a variety X over a field F can be defined as

$$(3.5.1)\qquad \begin{aligned} H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) &= K_{2j-i}(X)^{(j)} , \\ H^{\mathcal{M}}_a(X,\mathbb{Q}(b)) &= {}'H^{-a}_{\mathcal{M}}(X,\mathbb{Q}(-b)) = K'_{a-2b}(X)^{(-b)} \end{aligned}$$

where $K_m(X)^{(j)}$ denotes the subspace of $K_m(X)_{\mathbb{Q}} = K_m(X)\otimes_{\mathbb{Z}}\mathbb{Q}$, on which the Adams operator ψ^k acts as multiplication by k^j for every $k \geq 0$, similarly for K' . Note that the definition of the Adams operators $'\psi^k$ on $K'_m(X)$ (called ϕ^k in [Sou] 7.2) requires the embedding of X into a smooth variety W and then changing the Adams operations on the relative K-groups $K^X_m(W) \cong K'_m(X)$ by multiplication with cannibalistic classes of $\Omega^1_{W/k}$ to make everything independent of W .

3.6. Lemma Let W be smooth of pure dimension d , $X \subset W$ be a closed immersion, and $\eta^{\mathcal{M}}_W \in H^{\mathcal{M}}_{2d}(W,\mathbb{Q}(d)) \cong CH_d(W)_{\mathbb{Q}}$ (cf. 3.8 a) below) be the fundamental class of W , corresponding to the cycle class of W . Then the following holds.

a) $ch(K_m(W)^{(j)}) \subseteq H^{2j-m}_{\mathcal{D},X}(W,\mathbb{Q}(j))$,

b) $\tau(K'_m(X)^{(b)}) \subseteq H^{\mathcal{D}}_{m-2b}(X,\mathbb{Q}(-b))$,

c) the capproduct with $\eta^{\mathcal{M}}_W$ induces a commutative diagram

$$\begin{array}{lcl} r_{\mathcal{D}} : & H^i_{\mathcal{M},X}(W,\mathbb{Q}(j)) := K^X_{2j-i}(W)^{(j)} \xrightarrow{ch} & H^i_{\mathcal{D},X}(W,\mathbb{Q}(j)) \\ (3.6.1) & \cap\, \eta^{\mathcal{M}}_W \Big\downarrow \wr \qquad\qquad & \wr\Big\downarrow \text{Poincaré duality} \\ 'r_{\mathcal{D}} : & H^{\mathcal{M}}_{2d-i}(X,\mathbb{Q}(d-i)) = K'_{2j-i}(X)^{(j-d)} \xrightarrow{\tau} & H^{\mathcal{D}}_{2d-i}(X,\mathbb{Q}(d-j)) \;. \end{array}$$

Proof The first property is proved in [Sch] §4 (the proof there also works with supports), and by [Sou] 7.2 iv) one has $K^X_m(W)^{(j)} \cap \eta^{\mathcal{M}}_W \subseteq K'_m(X)^{(j-d)}$. By 3.3.1 and [Sou] 7.2 we have a commutative diagram

$$(3.6.2)\qquad \begin{array}{ccccc} Gr^j_{F_\gamma} K_m(W)_{\mathbb{Q}} & \xleftarrow[\sim]{\varphi} & K_m(W)^{(j)} & \xrightarrow{ch} & H^{2j-m}_{\mathcal{D},X}(W,\mathbb{Q}(j)) \\ \cap[\mathcal{O}_W]\Big\downarrow\wr & & \cap\eta^{\mathcal{M}}_W \Big\downarrow & & \Big\downarrow \cap\tau(\eta_W) \\ Gr^{F^\gamma}_{d-j} K'_m(X)_{\mathbb{Q}} & \xleftarrow[\sim]{\varphi} & K'_m(X)^{(j-d)} & \xrightarrow{\tau} & \bigoplus_{\nu\in\mathbb{Z}} H^{\mathcal{D}}_{m+2\nu}(X,\mathbb{Q}(\nu)) \;, \end{array}$$

with the graded terms for the γ-filtrations on the left. The isomorphisms φ are induced by the inclusions $K_m(W)^{(j)} \subseteq F^j_\gamma K_m(W)_{\mathbb{Q}}$ and $K'_m(X)^{(\mu)} \subseteq F^\gamma_\mu K'_m(X)_{\mathbb{Q}}$, and the left square is commutative since $[\mathcal{O}_W] = \eta^\mu_W$ + terms in $F^\gamma_{d-1} K'_o(W)_{\mathbb{Q}}$. Since $\tau(\eta^{\mathcal{M}}_W) = cl_{\mathcal{D}}([W]) = \eta^{\mathcal{D}}_W \in H^{\mathcal{D}}_{2d}(W,\mathbb{Q}(d))$, see 3.8 b) below, and the capproduct with it gives the Poincaré duality isomorphism, the remaining claims follow.

3.7. The difference between 3.6.1 and 3.3.2 is explained by the fact that the capproduct with η_W^{M} does not coincide with the canonical isomorphism $K_m^X(W)_{\mathbb{Q}} \xrightarrow{\sim} K_m'(X)_{\mathbb{Q}}$. The latter is not even compatible with the Adams operators as remarked above. However, one has the commutative diagram 3.6.2, and can use the isomorphisms φ in it to describe the motivic cohomology via the γ-filtrations. This is done by Soulé in [Sou 1], and he proves that

$$Z \hookrightarrow X \rightsquigarrow H^i_{M,Z}(X,\mathbb{Q}(j)) \qquad , \qquad i,j \in \mathbb{Z} ,$$

$$X \rightsquigarrow H_a^{M}(X,A(b)) \qquad , \qquad a,b \in \mathbb{Z} ,$$

defines a twisted Poincaré duality on V_F, with the restriction that $H^i_{M,Z}(X,\mathbb{Q}(j))$ has so far only been defined for X smooth. Then 3.6 c) can be interpreted by saying that the maps $r_{\mathcal{D}}$ and $'r_{\mathcal{D}}$ in 3.6.1, which Beilinson calls the regulator maps, are morphisms between twisted Poincaré dualities, in the obvious sense.

3.8. Lemma a) After tensoring with $\mathbb{Q}$, the spectral sequence 3.0 degenerates at E^2 on the lines $p+q = 0,1$ and gives canonical isomorphisms $H^{M}_{2j}(X,\mathbb{Q}(j)) \cong E^2_{j,-j}(X)_{\mathbb{Q}} = CH_j(X)_{\mathbb{Q}}$ and $H^{M}_{2j+1}(X,\mathbb{Q}(j)) \cong E^2_{j+1,-j}(X)_{\mathbb{Q}}$.

b) Via the isomorphisms in a), $\tau: H^{M}_{2j}(X,\mathbb{Q}(j)) \to H^{\mathcal{D}}_{2j}(X,\mathbb{Q}(j))$ is given by the cycle map.

Proof a) has been proved by Soulé [Sou 1] 5.2, and b) follows from the case $j = \dim X$ by functoriality. In this case it amounts to the equality $\tau(\eta_X^{M}) = \eta_X^{\mathcal{D}}$, which can be checked in the Betti homology by 1.20, where it is well known.

3.9. In view of 3.8 b) and 3.4, 1.20 and 3.1 imply that

$$(3.9.1) \qquad \begin{aligned} 'r_{\mathcal{D}} &: H^{M}_{2d}(X,\mathbb{Q}(d)) \xrightarrow{\sim} H^{\mathcal{D}}_{2d}(X,\mathbb{Q}(d)) \\ 'r_{\mathcal{D}} &: H^{M}_{2d-1}(X,\mathbb{Q}(d-1)) \xrightarrow{\sim} H^{\mathcal{D}}_{2d-1}(X,\mathbb{Q}(d-1)) \end{aligned}$$

are isomorphisms for X of dimension d over $\mathbb{R}$ or $\mathbb{C}$. Similar isomorphisms cannot exist in general, because Beilinson has proved that for a smooth variety X the image of the regulator map

$$r_{\mathcal{D}}\colon H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X,\mathbb{Q}(j))$$

is countable for $i < j$ or $j > d+1$ ([Be 1] 2.3.4)!

Note, however, that Beilinson's conjectures in particular would imply, that for a smooth and proper variety X over $\mathbb{Q}$ the map

$$(3.9.2) \qquad r_{\mathcal{D}} = \mathrm{ch}\colon H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) \otimes \mathbb{R} \to H^i_{\mathcal{D}}(X,\mathbb{R}(j))$$

is surjective for $i < 2j$. This follows immediately from the statement of these conjectures ([Be 1] 3.4 and [Sch] §5) for $i < 2j-1$; for $i = 2j-1$ note that the image of

$$\alpha\colon Z^{j-1}(X) \xrightarrow{cl_B} H^{2j-2}(X,\mathbb{R}(j-1)) \to H^{2j-2}(X,\mathbb{C})/H^{2j-2}(X,\mathbb{R}(j))+F^j \hookrightarrow H^{2j-1}_{\mathcal{D}}(X,\mathbb{R}(j))$$

is contained in the image of the map above, by the commutative diagram

$$\begin{array}{ccccc}
H^{\mathcal{D}}_{2d-2j-1}(Z,\mathbb{Q}(d-j)) & \xleftarrow{\sim} & H^{2j-1}_{\mathcal{D},Z}(X,\mathbb{Q}(j)) & \xrightarrow{i_*} & H^{2j-1}_{\mathcal{D}}(X,\mathbb{R}(j)) \\
{\scriptstyle 3.9.1}\uparrow\wr & & \uparrow r_{\mathcal{D}} & & \uparrow r_{\mathcal{D}} \\
H^{\mathcal{M}}_{2d-2j-1}(Z,\mathbb{Q}(d-j)) & \xleftarrow{\sim} & H^{2j-1}_{\mathcal{M},Z}(X,\mathbb{Q}(j)) & \longrightarrow & H^{2j-1}_{\mathcal{M}}(X,\mathbb{Q}(j))
\end{array}$$

for $Z \overset{i}{\hookrightarrow} X$ of codimension $j-1$ and the fact, that $\alpha([Z]) \in \mathrm{Im}\, i_*$ by functoriality.

The surjectivity of 3.9.2 would have a remarkable consequence, by the following property of the motivic cohomology.

<u>3.10 Lemma</u> For a smooth variety X , $H^i_{\mathcal{M}}(X,\mathbb{Q}(j))$ has support in codimension $\geq i-j$, i.e., $N^{i-j}H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) = H^i_{\mathcal{M}}(X,\mathbb{Q}(j))$ for the coniveau filtration N^{ν} of $H^i_{\mathcal{M}}$, defined by

$$N^{\nu}H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) = \bigcup_{\substack{Z\subset X \text{ closed of}\\ \text{codimension} \geq \nu}} \mathrm{Im}(H^i_{\mathcal{M},Z}(X,\mathbb{Q}(j)) \to H^i_{\mathcal{M}}(X,\mathbb{Q}(j))) = \bigcup_{\substack{Z\subset X \text{ closed of}\\ \text{codimension} \geq \nu}} \mathrm{Ker}(H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) \to H^i_{\mathcal{M}}(X\smallsetminus Z,\mathbb{Q}(j))),$$

compare [BO] .

<u>Proof</u> For smooth X the spectral sequence 3.0 is usually written in cohomological notation ($X^{(p)} = \{x \in X \mid \mathrm{codim}\, x = p\}$)

(3.10.1) $E_1^{p,q}(X) = \bigoplus_{x\in X^{(p)}} K_{-p-q}(\kappa(x)) \Rightarrow E^{p+q} = K_{-p-q}(X)$,

via the isomorphism $K'_m(X) \cong K_m(X)$ and reindexing. Soulé has proved that 3.10.1 induces a spectral sequence

(3.10.2) $E_1^{p,q}(X)(j) = \bigoplus_{x\in X^{(p)}} K_{-p-q}(\kappa(x))^{(j-p)} \Rightarrow E^{p+q}(j) = K_{-p-q}(X)^{(j)}$,

see [Sou] 5.2. On the other hand, for a field F one has

$$K_m(F)^{(\nu)} = 0 \quad \text{for} \quad \nu > m ,$$

by results of Suslin [Su] (see [Sou] Thm. 2 and 5 ii). This implies $E_1^{p,q}(X)(j) = 0$ for $j-p > -p-q$. For $-p-q = 2j-i$ we see that $E_1^{p,q}(X)(j) = 0$ for $p < i-j$, hence the part of the spectral sequence 3.10.2 contributing to $H_M^i(X,\mathbb{Q}(j)) = K_{2j-i}(X)^{(j)}$ lives in codimension $\geq i-j$ (note that 3.10.2 is compatible with restrictions to open subvarieties and that $\varinjlim E_1^{p,q}(X-Z)(j) = 0$ where Z runs over all subvarieties of codimension $\leq p$).

Beilinson thinks that the map 3.9.2 is also surjective for a smooth and proper variety X over $\mathbb{R}$; by 3.10 this would imply

<u>3.11. Hodge-$\mathcal{D}$-conjecture</u> (Beilinson [Be 1] 1.10) If X is a smooth and proper variety over $\mathbb{C}$ and $i<2j$, then $H_{\mathcal{D}}^i(X,\mathbb{R}(j))$ has support in codimension $i-j$, i.e., for every element x in $H_{\mathcal{D}}^i(X,\mathbb{R}(j))$ there is a closed subvariety $Z \subset X$ of codimension $i-j$ such that x lies in the image of $H_{\mathcal{D},Z}^i(X, \mathbb{R}(j))$.

<u>3.12. Remarks</u> a) Note that, if we regard X as a scheme over $\mathbb{R}$ via ρ_0: $X \xrightarrow{\rho} \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$, then canonically $H_{\mathcal{D}}^*(X/_{\rho_0}\mathbb{R},\ldots) \cong H_{\mathcal{D}}^*(X/_{\rho}\mathbb{C},\ldots)$.

b) Recall that the usual Hodge conjecture is equivalent to saying that the elements in $H_B^{2j}(X,\mathbb{Q}(j)) \cap H^{j,j}$ are supported in codimension j . In fact, Beilinson's formulation [Be 1] 1.10.1 combines both conjectures and includes the case $i = 2j$.

c) In [Be 2] Beilinson generalizes the Hodge-$\mathcal{D}$-conjecture to arbitrary smooth varieties, but at least a part of this is disproved in [Ja]. Namely for non-proper varieties X there are counterexamples against the surjectivity of the map

$$K_1(X)^{(j)} \to (2\pi\sqrt{-1})^j W_{2j} H_B^{2j-1}(X,\mathbb{Q}) \cap F^j H^{2j-1}(X,\mathbb{C}) ,$$

already for dim X = 2 and j = 2. In view of this, it would be very interesting to test conjecture 3.11 for proper surfaces and (i,j) = (3,2) (see the example below).

3.13. Example Let us consider the case $i = 2j-1$. By the standard sequence for Deligne cohomology (see 2.7.1) we have isomorphisms

$$H^{2j-2}(X,\mathbb{R}(j-1)) \cap H^{j-1,j-1} \xrightarrow{\sim} H^{2j-2}(X,\mathbb{C})/H^{2j-2}(X,\mathbb{R}(j)) + F^j \xrightarrow{\sim}$$
$$\xrightarrow{\sim} H^{2j-1}_{\mathcal{D}}(X, \mathbb{R}(j)) \ ,$$

but no similar isomorphism with support in a subvariety Z , since the Hodge theory is different. Hence the conjecture does not assert that the cycles in the left group are supported in codimension $j-1$ in the Betti sense (which is false, compare 3.12 b)), but one really has to work with the Deligne (co)homology. The commutative diagram (d = dim X)

$$(3.13.1)\qquad \begin{array}{ccccc} H^{\mathcal{D}}_{2d-2j+1}(Z,\mathbb{R}(d-j)) & \xleftarrow{\sim} & H^{2j-1}_{\mathcal{D},Z}(X,\mathbb{R}(j)) & \to & H^{2j-1}_{\mathcal{D}}(X,\mathbb{R}(j)) \\ \Uparrow {}'r_{\mathcal{D}} & & \uparrow {}'r_{\mathcal{D}} & & \uparrow r_{\mathcal{D}} \\ H^{M}_{2d-2j+1}(Z,\mathbb{Q}(d-j))\otimes_{\mathbb{Q}}\mathbb{R} & \xleftarrow{\sim} & H^{2j-1}_{M,Z}(X,\mathbb{Q}(j))\otimes_{\mathbb{Q}}\mathbb{R} & \to & H^{2j-1}_{M}(X,\mathbb{Q}(j))\otimes_{\mathbb{Q}}\mathbb{R} \\ \Uparrow \wr\, 3.8 & & & & \Uparrow \wr\, 3.8 \\ E^2_{d-j+1,-d+j}(Z)\otimes\mathbb{R} & & \xrightarrow{\nu_*} & & E^2_{d-j+1,-d+j}(X)\otimes\mathbb{R} \end{array}$$

for $Z \overset{\nu}{\subseteq} X$ of codimension $j-1$, in which the surjectivity of the map $'r_{\mathcal{D}}$ on the left follows as in 3.1, shows that for $i = 2j-1$ conjecture 3.11 is actually equivalent to the surjectivity of the regulator map $r_{\mathcal{D}}$ on the right.

If Z^{sing} is the singular locus of Z , one has a commutative diagram

$$(3.13.2)\qquad \begin{array}{ccccc} H^{\mathcal{D}}_{2d-2j+1}(Z,\mathbb{R}(d-j)) & \overset{\kappa}{\hookrightarrow} & H^{\mathcal{D}}_{2d-2j+1}(Z\setminus Z^{sing},\mathbb{R}(d-j)) & \cong & H^1_{\mathcal{D}}(Z\setminus Z^{sing},\mathbb{R}(1)) \\ \uparrow {}'r_{\mathcal{D}} & & \uparrow r_{\mathcal{D}} & & \uparrow \rho \\ E^2_{d-j+1,j-d}(Z) & \hookrightarrow & E^2_{d-j+1,j-d}(Z\setminus Z^{sing}) & \cong & \mathcal{O}(Z\setminus Z^{sing})^{\times}_{alg}, \end{array}$$

similarly for $X\setminus Z^{sing}$, in which κ is injective by 1.20. By the description of ρ in [EV] 2.12 ii), 2.16 b) and 2.17 we have a commutative diagram for a smooth variety U

$$
\begin{array}{ll}
 & H^1_{\mathcal{D}}(U,\ \mathbb{R}(1)) \ \cong\ \{\varepsilon \in C^{\infty}(U,\ \mathbb{R})\ |\ \partial\varepsilon \in \Omega^1_{\bar{U}}\ \langle D\rangle(\bar{U})\} \\
 & \quad \Big\uparrow \operatorname{Re}\log f = \log|f| \qquad\qquad \Big\uparrow \varepsilon = \operatorname{Re}\varphi \\
(3.13.3)\ \rho & \qquad\qquad \{\bar{\varphi} \in \Gamma(U, G/\ \mathbb{R}(1)\ |\ d\varphi \in \Omega^1_{\bar{U}}\ \langle D\rangle(\bar{U})\} \\
 & \quad \Big\downarrow f = \exp\varphi \qquad\qquad\qquad \Big\uparrow \text{can.} \\
 & \mathcal{O}(U)^{\times}_{alg} \ \xleftarrow{\ \sim\ }\ \{\bar{\varphi} \in \Gamma(U, \mathcal{O}/\ \mathbb{Z}(1))\ |\ d\varphi \in \Omega^1_{\bar{U}}\ \langle D\rangle(\bar{U})\}\ ,
\end{array}
$$

if $\bar{U}$ is a good compactification of $U = \bar{U} \setminus D$ (i.e., $\bar{U}$ is smooth and proper, and D is adivisor with normal crossings). Similar to [EV] 2.14, there is a description of the real Deligne homology $'H^{\ell}_{\mathcal{D}}(U,\ \mathbb{R}(k))$ as the ℓ-th homology of the complex

$$(3.13.4)\quad \operatorname{Cone}(F^k\,'\Omega^{\bullet}_{\bar{U}^{\infty}}\langle D\rangle(\bar{U}) \xrightarrow{\pi_{k-1}\circ\iota} {}'S^{\bullet}_{U}(k-1)(U))[-1]\ ,$$

where $'S^{\bullet}_{U}$ is the complex of real-valued currents and π_{k-1} is the projection

$$'\Omega_{\bar{U}^{\infty}}\langle D\rangle(\bar{U}) \to {}'\Omega^{\bullet}_{U^{\infty}}(U) = {}'S^{\bullet}_{U} \otimes \mathbb{C} \to {}'S^{\bullet}_{U} \otimes \mathbb{R}(k-1) := {}'S^{\bullet}_{U}(k-1)$$

induced by the decomposition $\mathbb{C} = \mathbb{R}(k) \oplus \mathbb{R}(k-1)$. This is covariant in $(\bar{U},D) \in ob(\tilde{\pi}_*)$, and one has the Poincaré duality quasi-isomorphism $S^{\bullet}_{U} \xrightarrow{1.1a)} {}'S^{\bullet}_{U}(-d)[-2d]$. Using this for suitable good compactifications of $Z \setminus Z^{sing}$ and $X \setminus Z^{sing}$, 3.13.3 for $Z \setminus Z^{sing}$, and 3.13.1 for $Z \setminus Z^{sing}$ and $X \setminus Z^{sing}$, one obtains the following.

Via $r_{\mathcal{D}}$, a family (f_{α}) in $E^2_{d-j+1,-d+j}(X)$ (i.e., $f_{\alpha} \in \mathbb{C}(X_{\alpha})^{\times}$, $\operatorname{codim}_X(X_{\alpha}) = j-1$, and $\sum_{\alpha} \operatorname{div}(f_{\alpha}) = 0$) is mapped to the class in $H^{2j-2}(X,\ \mathbb{R}(j-1) \cap H^{j-1,j-1}$ represented by the current

$$f\colon\ \omega \mapsto \frac{1}{(2\pi\sqrt{-1})^{d-j+1}} \sum_{\alpha} \int_{Z_{\alpha} \setminus Z^{sing}_{\alpha}} \log|f_{\alpha}|\omega.$$

This has to be interpreted in an appropriate way, since this current is not closed as it stands. Via 3.13.4, the image of (f_{α}) is the class of the closed current $f + \pi_{j-1}\psi = f + \frac{1}{2}\psi + \frac{1}{2}(-1)^{j-1}\bar{\psi}$ in $H^{2j-2}(X, \mathbb{R}(j-1))/H^{2j-2}(X, \mathbb{R}(j-1)) \cap (F^j + \overline{F^j})$ $\xleftarrow{\sim} H^{2j-2}(X,\ \mathbb{R}(j-1)) \cap H^{j-1,j-1}$, where ψ is an element of $F^j\,'\Omega_{X^{\infty}}(X)$ with $df = -\pi_{j-1}d\psi$. This gives the same result as f , if we integrate against $(d-j+1, d-j+1)$-forms. Since a cycle of degree $2j-2$ in $F^j + \overline{F^j}$ is a sum of cycles in F^j and $\overline{F^j}$, the above class only depends on f and coincides with the class constructed by Beilinson in [Be 2] 6.

Concluding, Beilinson's Hodge-$\mathcal{D}$-conjecture in this case amounts to saying that $H^{2j-2}(X, \mathbb{R}(j-1) \cap H^{j-1,j-1}$ is generated by such currents. Here the singularities of $\mathrm{supp}(f)$ are essential: if $Z = \cup Z_\alpha$ is smooth, all f_α are necessarily constant, so f is an $\mathbb{R}$-linear combination of the cycle classes of the Z_α and in particular lies in $(H^{2j-2}(X,\mathbb{Q}(j-1)) \cap H^{j-1,j-1}) \otimes \mathbb{R}$.

§4 <u>Beilinson's conjectures for motives with coefficients and a reformulation</u>

Both for the introduction of coefficients and for Deligne's reformulation and interpretation of Beilinson's conjecture the language of motives is essential, so we shall briefly recall the definition.

<u>4.1.</u> Beilinson applies Grothendieck's general procedure to the full Chow groups. So for a field k and a number field E he starts with a category $C(k,E)$, whose objects are symbols EX for each smooth projective variety X over k , and whose homomorphism sets are the E-vector spaces

$$\mathrm{Hom}(EX,EY) = CH^{\dim Y}(X\times Y)\otimes E ,$$

with composition defined by the intersection product: for $f \in \mathrm{Hom}(EX_1,EX_2)$ and $g \in \mathrm{Hom}(EX_2,EX_3)$ one has $g \circ f = p_{13*}(p_{12}^* f \cdot p_{23}^* g)$ where $p_{ij}: X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projections. Note that $X \rightsquigarrow EX$ then becomes a covariant functor, by sending $f: X \to Y$ to the graph of f in $X \times Y$. Then the category $M(k,E)$ of motives over k with coefficients in E is obtained from $C(k,E)$ by formally adjoining images of projectors and inverting the Lefschetz object $E(-1)$ defined by the unique decomposition $E\mathbb{P}^1 = E\ \mathrm{Spec}\ \mathbb{Q} \oplus E(-1)$. Hence every motive $\mathbb{M}$ in $M(k,E)$ is given by a triple

$$\mathbb{M} = (X,p,r) ,$$

with X smooth and projective over k , $p \in CH^{\dim X}(X\times X)\otimes E = \mathrm{End}(EX)$ an idempotent, and $r \in \mathbb{Z}$. For $\mathbb{N} = (Y,q,t)$ one has

$$\mathrm{Hom}(\mathbb{M},\mathbb{N}) = q\,\mathrm{Hom}(EX(r),EY(t))p = qCH^{\dim Y+r-t}(X\times Y)p,$$

note that $\mathrm{Hom}(EX(r),EY(t)) = CH^{\dim Y+r-t}(X\times Y)\otimes E$ is a left $\mathrm{End}(EY)$- and a right $\mathrm{End}(EX)$-module, $EX(r) = EX\otimes E(1)^{\otimes r} = (X,id,r)$.

By replacing the Chow groups by other groups with similar formal properties, other notions of motives are possible and will in fact be needed later. Hence we call the objects of $\mathcal{M}(k,E)$ Chow motives to distinguish them.

4.2. If one has a twisted Poincaré duality theory (H^*,H_*) on $\mathcal{V}_k$ with a cupproduct in the cohomology such that the cycle map is compatible with this product structure, then one can extend H^* to $\mathcal{M}(k,E)$ as follows. Let

$$H^i(EX(r),j) = H^{i+2r}(X,j+r)\otimes_{\mathbb{Z}} E \tag{4.2.1}$$

and for $f\in \mathrm{Hom}(EX(r),EY(t))$ define

$$f^* : H^i(EY(t),j) \to H^i(EX(r),j)$$

by $f^*(\alpha) = (\pi_X)_*\,(cl(f)\cup \pi_Y^*(\alpha))$. Here the Gysin map $(\pi_X)_* : H^{i+2\dim Y}(E(X\times Y)(r-t),j+\dim Y) \to H^i(EX(r),j)$ for the projection $\pi_X\colon X\times Y\to X$ is defined via the Poincaré duality isomorphism and the covariance of homology for proper morphisms. This extends to $\mathcal{M}(k,E)$ in an obvious way by letting

$$H^i(\mathbb{M},j) = p^*H^i(EX(r),j) \quad \text{for} \quad \mathbb{M} = (X,p,r).$$

Examples for $k=\mathbb{Q}$, which we shall always assume in the following, are H_{DR} , H_{et} , $H_{\mathcal{D}}$, e.g., $H^i_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},A(j)) = p^*[H^{i+2r}_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{R},A(j+r))\otimes_{\mathbb{Z}} E]$. We can also apply this to the motivic cohomology, and by the compatibility of the Chern character with the product structure the regulator maps $r_{\mathcal{D}} : H^i_{\mathcal{M}}(X,\mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{R},\mathbb{R}(j))$ are compatible with the action of correspondences p , hence can be defined for motives.

Recall that one obtains motives for absolute Hodge cycles by replacing the Chow groups above by the groups of absolute Hodge cycles $Z^*_{AH}(-)$ of the same codimension [De 4] 0.9, and then passing to the dual category, to obtain agreement with loc. cit.. Thus the cycle maps $CH^r(X\times Y) \to Z^r_{AH}(X\times Y)$ define a contravariant functor from $\mathcal{M}(\mathbb{Q},E)$ to this category. In particular, for $i \in \mathbb{Z}$ we can define E-motives for absolute Hodge cycles $H^i_{AH}(\mathbb{M}) = p^*[H^{i+2r}_{AH}(X)(r) \otimes E]$, having realizations $H^i_{DR}(\mathbb{M})$, $H^i_{\acute{e}t}(\mathbb{M} \otimes \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ and $H^i_B(\mathbb{M} \otimes \mathbb{C}, \mathbb{Q})$ with the induced comparison isomorphisms [De 4] §0 .

The numberings are such that everywhere where classically stands an X we can write a Chow motive $\mathbb{M}$. Conversely, if a reader wants to follow the rest of this chapter without caring for Chow motives, he may read X where we have written an $\mathbb{M}$.

4.3. The i-th L-function of a Chow motive $\mathbb{M}$ is defined as an Euler product

(4.3.1) $$L^i(\mathbb{M},s) = \prod_p L^i_p(\mathbb{M},s) ,$$

where for $\ell \neq p$

(4.3.2) $$L^i_p(\mathbb{M},s) = L_p(H^i_{\acute{e}t}(\mathbb{M} \otimes \bar{\mathbb{Q}}, \mathbb{Q}_\ell), s) = \frac{1}{\det(1 - Fr_p p^{-s} \mid H^i_{\acute{e}t}(\mathbb{M} \otimes \bar{\mathbb{Q}}, \mathbb{Q}_\ell)^{I_p})},$$

I_p an inertia group at p in $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ and Fr_p a geometric Frobenius in the corresponding decomposition groups. Note that $H^i_{\acute{e}t}(\mathbb{M} \otimes \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ is indeed a free $\mathbb{Q}_\ell \otimes E$-module, as follows from the comparison isomorphisms with $H^i_B(\mathbb{M} \otimes \mathbb{C}, \mathbb{Q}) \otimes \mathbb{Q}_\ell$. By the usual conjectures on L-functions (see [Se] and [De 4]) the above determinants are in $E[p^{-s}]$, independent of $\ell \neq p$, and the Euler product converges for $Re\ s > \frac{i}{2} + 1$, giving an analytic function with values in $\mathbb{C} \otimes E$ or, equivalently, an array $(\ldots, L^i_\sigma(\mathbb{M},s), \ldots)$, indexed by $\sigma \in Hom(E,\mathbb{C})$, of $\mathbb{C}$-valued L-functions, by the isomorphism $\mathbb{C} \otimes E \cong \mathbb{C}^{Hom(E,\mathbb{C})}$ (cf. [De 4] 2.2.2). By the Weil conjectures proved by Deligne this is true for $\mathbb{M} = (X, id, r)$, if we remove the factors L^i_p for those p, where X has bad reduction. By specialization of cycles and the Lefschetz trace formula this extends to arbitrary $\mathbb{M} = (X,q,r)$.

Conjecturally, $L^i(\mathbb{M},s)$ has a meromorphic continuation and a functional equation

(4.3.3) $$(L_\infty \cdot L)^i(\mathbb{M},s) = (\varepsilon_\infty \cdot \varepsilon)^i(\mathbb{M},s) \cdot (L_\infty \cdot L)^{-i}(\mathbb{M}^\vee, 1-s) ,$$

with the Chow motive $\mathbb{M}^\vee = (X, q^t, \dim X - r)$. Here the anti-involution ^{t} on $End(EX) = CH^{\dim X}(X \times X) \otimes E$ is the map induced by transposition of the two factors of $X \times X$. The ε-

factors ε^i as the L-factors L^i only depend on the i-th ℓ-adic realizations $H^i_{\acute{e}t}(\mathbb{M}\otimes\bar{\mathbb{Q}},\mathbb{Q}_\ell)=p^*[H^{i+2r}_{\acute{e}t}(X\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Q}_\ell(r))\otimes E]$ as $\mathbb{Q}_\ell\otimes E$-$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-modules, and $L^i_\infty(\mathbb{M},s)$ and $\varepsilon^i_\infty(\mathbb{M},s)$ are determined by the i-th Betti realization $H^i_B(\mathbb{M}\otimes\mathbb{C},\mathbb{R}) = p^*[H^{i+2r}_B(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(r))\otimes E]$ as $\mathbb{R}\otimes E$-Hodge structure with "infinite Frobenius" (see [De 4] §5).

The following lemma shows the agreement with the formulation given in [De 4] §5, since by definition $L^j(\mathbb{M},s) = L(H^j_{AH}(\mathbb{M}),s)$.

<u>4 4. Lemma</u> One has canonically $H^{-i}_{AH}(\mathbb{M}^\vee) \cong H^i_{AH}(\mathbb{M})^\vee$ (E-dual).

<u>Proof</u> For a field R and finite-dimensional R-vector spaces A,B one has a commutative diagram of canonical isomorphisms

$$\begin{array}{ccc}
\mathrm{Hom}_{R\otimes E}(A\otimes E,B\otimes E) & \xrightarrow{tr} & \mathrm{Hom}_{R\otimes E}((B\otimes E)^\vee,(A\otimes E)^\vee) \\
\uparrow\wr & & \uparrow\wr \\
\mathrm{Hom}_R(A,B)\otimes_R E & \xrightarrow{tr\otimes id_E} & \mathrm{Hom}_R(B^\vee,A^\vee)\otimes_R E ,
\end{array}$$

where tr denotes the transposition and $^\vee$ the dual for $R\otimes E$ and R, respectively. Hence it suffices to show, that for varieties X and Y of dimension d and e, respectively, and a correspondence $f \in \mathrm{Hom}(\mathbb{Q}X,\mathbb{Q}Y) = CH^e(X\times Y)_{\mathbb{Q}}$ the correspondence $f^t \in CH^e(Y\times X)_{\mathbb{Q}} = \mathrm{Hom}(\mathbb{Q}Y(e),\mathbb{Q}X(d))$ induces the transposed maps via Poincaré duality in the realizations, i. e., that with the canonical trace morphisms tr_ the diagram

$$\begin{array}{ccccccc}
H^i_?(Y,R) & \times & H^{2e-i}_?(Y,R(e)) & \to & H^{2e}(Y,R(e)) & \xrightarrow[\sim]{tr_Y} & R \\
f^*\downarrow & & \uparrow(f^t)^* & & & & \| \\
H^i_?(X,R) & \times & H^{2d-i}_?(X,R(d)) & \to & H^{2d}(X,R(d)) & \xrightarrow[\sim]{tr_X} & R
\end{array}$$

commutes, ? = DR, ét and B, and $R = \mathbb{Q},\mathbb{Q}_\ell$ and $\mathbb{Q}$, respectively. This follows from the formula

$$tr_X(p_{X_*}(p_Y^*(y)\cup cl(f))\cup x) = tr_{X\times Y}(p_Y^*(y)\cup cl(f)\cup p_X^*(x))$$
$$= tr_{Y\times X}(p_Y^*(y)\cup p_X^*(x)\cup cl(f^t)) = tr_Y(y\cup p_{Y_*}(p_X^*(x)\cup cl(f^t))),$$

where we have used the relations $tr_X\, p_{X_*} = tr_{X\times Y} = tr_{Y\times X}\,\psi_*$

and $\psi_*\psi^* = \mathrm{id}$, for $\psi\colon X\times Y \to Y\times X$ the transposition.

Beilinson postulates the existence of a Chow motive $\mathbb{M}^{\circ}$ with

$$(4.4.1) \qquad L^{-i}(\mathbb{M}^{\vee},s) = L^{i}(\mathbb{M}^{\circ},i+s) ,$$
$$(4.4.2) \qquad L_{\infty}^{-i}(\mathbb{M}^{\vee},s) = L_{\infty}^{i}(\mathbb{M}^{\circ},i+s) .$$

(The equality 4.4.2 follows from [Be 1] 3.1 and the results 4.14 and 4.15 below, and the equality $(L\cdot L_{\infty})^{-i}(\mathbb{M}^{\vee},s) = (L\cdot L_{\infty})^{i}(\mathbb{M}^{\circ}\ i+s)$ follows by comparing 4.3.3 with Beilinson's functional equation). For $\mathbb{M} = EX$ one can take $\mathbb{M}^{\circ} = EX$, too, by the hard Lefschetz isomorphism

$$(4.4.3) \qquad H_{AH}^{i}(X)(i) \xrightarrow[\sim]{\cdot \ell^{d-i}} H_{AH}^{2d-i}(X)(d)$$

for $i \leq d = \dim X$ (without restriction), which depends on i and a class of an ample divisor $\ell \in CH^1(X)$. Since 4.4.3 is <u>not</u> compatible with the action of correspondences, I see no way to define $\mathbb{M}^{\circ}$ in general. Of course, we have $L^{i}(\mathbb{M}^{\circ},s) = L((q^*)^{\circ}H_{AH}^{i}(X),s)$ for $\mathbb{M} = (X,q,0)$, where $(q^*)^{\circ}$ is defined by the commutative diagram

$$(4.4.4) \qquad \begin{array}{ccc} H_{AH}^{i}(X)_E & \xrightarrow[\sim]{\ell^{d-i}} & H_{AH}^{2d-i}(X)(d-i)_E \\ \Big\downarrow{\scriptstyle (q^*)^{\circ}} & & \Big\downarrow{\scriptstyle q^*} \\ H_{AH}^{i}(X)_E & \xrightarrow[\sim]{\ell^{d-i}} & H_{AH}^{2d-i}(X)(d-i)_E \end{array} \quad ;$$

the problem is to show that $(q^*)^{\circ} = (q^{\circ})^*$ for some idempotent $q^{\circ} \in CH^d(X\times X)_E$ so that one can take $\mathbb{M}^{\circ} = (X,q^{\circ},0)$.

Since we only need the L-functions 4.4.1 and 4.4.2, which in any case can be computed from $\mathbb{M}$ via 4.4, we may and shall write everything with the more canonical motive $\mathbb{M}^{\vee}$.

<u>4.5.</u> Recall that for $n > \frac{i}{2} + 1$ we have canonical isomorphisms

$$(4.5.1) \qquad \begin{aligned} H_{\mathcal{D}}^{i+1}(X\times\mathbb{R},\mathbb{R}(n)) &\cong H_{DR}^{i}(X\times\mathbb{R})/H_{B}^{i}(X\times\mathbb{R},\mathbb{R}(n))+F^{n}H_{DR}^{i}(X\times\mathbb{R}) \\ &\cong H_{B}^{i}(X\times\mathbb{R},\mathbb{R}(n-1))/F^{n}H_{DR}^{i}(X\times\mathbb{R}) \end{aligned}$$

used in the formulation of Beilinson's conjecture for X [Sch] §5. Since then also $n+r > \frac{i+2r}{2} + 1$, and everything is compatible with the action of correspondences, we obtain the same isomorphisms with $\mathbb{M} = (X,q,r)$ instead of X. Finally, if the "integral part" $H^*_{\mathcal{M}}(X_{\mathbb{Z}},\mathbb{Q}(*)) \subseteq H^*_{\mathcal{M}}(X,\mathbb{Q}(*))$ (see [Be 1] 2.4.2.1) is compatible with Gysin maps, one can define similar subspaces $H^i_{\mathcal{M}}(\mathbb{M}_{\mathbb{Z}},\mathbb{Q}(j)) \subseteq H^i_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j))$, and then Beilinson's conjecture for a Chow motive $\mathbb{M}$ over $\mathbb{Q}$ with coefficients in E can be stated as follows

<u>4.6 Conjecture</u> Let $n > \frac{i}{2} + 1$ be an integer.

a) Let $\rho_{i,n} := \mathrm{ord}_{s=1-n} L^{-i}(\mathbb{M}^{\vee},s)$ be the multiplicity of $L^{-i}(\mathbb{M}^{\vee},s)$ at $s = 1-n$, then $\rho_{i,n} = \dim_{\mathbb{R}} \mathbb{H}^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n))$.

b) The regulator map $r_{\mathcal{D}}\otimes\mathbb{R} : H^{i+1}_{\mathcal{M}}(\mathbb{M}_{\mathbb{Z}},\mathbb{Q}(n))\otimes\mathbb{R} \to H^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n))$ is an isomorphism.

c) Let $\mathcal{R}(i,n)$ be the E-structure on $\det_{\mathbb{R}\otimes E} H^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n))$ defined by $\det_E H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n-1))\otimes[\det_E F^n H^i_{DR}(\mathbb{M})]^{-1}$, via the exact sequence

$$\begin{array}{ccccccc} 0 \to & F^nH^i_{DR}(\mathbb{M}\otimes\mathbb{R}) & \to & H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n-1)) & \to & H^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n)) \to 0 \\ & \wr & & \wr & & \\ & F^nH^i_{DR}(\mathbb{M})\otimes\mathbb{R} & & H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{Q}(n-1))\otimes\mathbb{R} & & \end{array}$$

and let $L^{-i}(\mathbb{M}^{\vee},1-n)^* = \lim_{s\to 1-n} (s-(1-n))^{-\rho_{i,n}} L^{-i}(\mathbb{M}^{\vee},s)$ be the leading coefficient of $L^{-i}(\mathbb{M}^{\vee},s)$ at $s = 1-n$. Then

$$r_{\mathcal{D}}(\det_E H^{i+1}_{\mathcal{M}}(\mathbb{M}_{\mathbb{Z}},\mathbb{Q}(n))) = L^{-i}(\mathbb{M}^{\vee},1-n)^*\cdot\mathcal{R}(i,n) .$$

For $n = \frac{i}{2} + 1$ or $\frac{i+1}{2}$ we refer the reader to [Be 1] 3.7 and 3.8. If also the Euler factors $L^i_p(\mathbb{M},s)$ at primes p of bad reduction are non-zero for $\mathrm{Re}\, s > \frac{i}{2} + 1$, conjecture 4.5 a) follows from the functional equation and the known multiplicities of the L^i_{∞}, cf. [Be 1] 3.3. and 4.15 below.

<u>4.7. Examples</u> a) Let $X = \mathrm{Spec}\, F$ for a number field F which is Galois over $\mathbb{Q}$. The Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$ acts

on F from the left and hence on X from the right. For $\sigma \in G$ denote by σ_X the corresponding automorphism of X and the induced element in $\mathrm{End}(EX) = CH^O(X\times X) \otimes E$. One easily checks that $\sigma \mapsto \sigma_X$ induces an E-algebra isomorphism

$$E[G]^{op} \xrightarrow{\sim} \mathrm{End}(EX) ,$$

$E[G]^{op}$ being the opposite algebra of the group ring E[G]. For a cohomology theory H^* the assignment $\sigma \mapsto \sigma_X^*: H^*(EX) \to H^*(EX)$ gives the usual left action of G obtained by functoriality, and $(\sigma_X)_*$ is the action of σ^{-1}. Any idempotent $p \in E[G]$ defines a motive $\mathbb{M} = (EX,p) = (EX,p,O)$, and in this manner one obtains the category of Artin motives. Let $V = E[G]p^t \cong \mathrm{Hom}_E(E[G]p,E)$, where $(\Sigma a_\sigma \sigma)^t = \Sigma a_\sigma \sigma^{-1}$. This is a left G-module, hence a left $G_{\mathbb{Q}} = \mathrm{Gal}((\bar{\mathbb{Q}}/\mathbb{Q})$-module via the canonical projection $\pi: G_{\mathbb{Q}} \twoheadrightarrow G$. I claim that

$$L^O(\mathbb{M},s) = L(V,s) ,$$

where $L(v,s) = \prod L_p(V,s)$ is formed as in 4.3.2, via the geometric Frobenius elements. In fact, if we use an embedding $\alpha: F \hookrightarrow \bar{\mathbb{Q}}$ to identify

$$H^O_{\acute{e}t}(X\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\pi_O(X\times_{\mathbb{Q}}\bar{\mathbb{Q}})} = \mathbb{Q}_\ell^{X(\bar{\mathbb{Q}})} = \mathbb{Q}_\ell^{\mathrm{Hom}(F,\bar{\mathbb{Q}})} \underset{\alpha}{\cong} \mathbb{Q}_\ell[G],$$

$$f: \pi_O(X\times_{\mathbb{Q}}\bar{\mathbb{Q}}) \to \mathbb{Q}_\ell \qquad \rightsquigarrow \qquad \Sigma f(\mathrm{Im}(\mathrm{Spec}\ \alpha\sigma \times \mathrm{id}))\sigma$$

$G_{\mathbb{Q}}$ acts via π as left multiplication, and for $\tau \in G$, τ^* acts as right multiplication by τ^{-1}. Hence $H^O(\mathbb{M}\otimes\bar{\mathbb{Q}},\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \otimes E[G]p^t$ as $G_{\mathbb{Q}}$-module, so the result follows by 4.3.2.

Now Beilinson has shown that the regulator maps $(i = O,\ n > 1)$

$$\begin{array}{ccc} r_{\mathcal{D}}\otimes\mathbb{R} : H^1_{\mathcal{M}}(EX_{\mathbb{Z}},\mathbb{Q}(n)) \otimes \mathbb{R} & \to & H^1_{\mathcal{D}}(EX\otimes\mathbb{R},\mathbb{R}(n)) \\ \wr\wr & & \wr\wr \\ K_{2n-1}(\mathcal{O}_F) \otimes E \otimes \mathbb{R} & \to & H^O_B(EX\otimes\mathbb{R},\mathbb{R}(n-1)) \cong [\mathbb{R}(n-1)^{X(\mathbb{C})}]^+\otimes E \end{array}$$

are induced by the maps $K_{2n-1}(\mathcal{O}_F) \xrightarrow{\alpha_*} K_{2n-1}(\mathbb{C}) \to \mathbb{R}(n-1)$, for $\alpha \in \mathrm{Hom}(F,\mathbb{C})$, as defined by Borel (see [Be 1] A 5.2). Here $\mathcal{O}_F$ is the ring of integers in F, and $^+$ denotes the fixed space for the involution σ acting via the complex conjugation in $\mathbb{R}(n-1)$ and $X(\mathbb{C}) = \mathrm{Hom}(F,\mathbb{C})$. Furthermore the number $c_{1-n} \in (\mathbb{R}\otimes E)^\times/E^\times$ such that

$$r_{\mathcal{D}}(\det_E K_{2n-1}(\mathcal{O}_F) \otimes E) = c_{1-n} \det_E[\mathbb{Q}(n-1)^{X(\mathbb{C})}]^+\otimes E$$

can be determined by choosing an $E[G]$-isomorphism

$$\phi\colon W := [\mathbb{Q}(n-1)^{X(\mathbb{C})}]^{+} \otimes E \xrightarrow{\sim} K_{2n-1}(\mathcal{O}_F) \otimes E =: K$$

and taking the determinant of $(r_{\mathcal{D}} \otimes \mathbb{R}) \circ (\phi \otimes \mathrm{id}_{\mathbb{R}})$ with respect to an E-basis of W. Thus, by the commutative diagram

$$\begin{array}{lccccc} & [W\otimes V]^G\otimes\mathbb{R} & \xrightarrow{\phi\otimes\mathrm{id}\otimes\mathrm{id}_{\mathbb{R}}} & [K\otimes V]^G\otimes\mathbb{R} & \xrightarrow{r_{\mathcal{D}}\otimes\mathbb{R}\otimes\mathrm{id}} & [W\otimes V]^G\otimes\mathbb{R} \\ (4.7.1) & \wr\big\downarrow \alpha_W\otimes\mathrm{id} & & \wr\big\downarrow \alpha_K\otimes\mathrm{id} & & \wr\big\downarrow \alpha_W\otimes\mathrm{id} \\ & pW\otimes\mathbb{R} & \xrightarrow{\phi\otimes\mathrm{id}_{\mathbb{R}}} & pK\otimes\mathbb{R} & \xrightarrow{r_{\mathcal{D}}\otimes\mathbb{R}} & pW\otimes\mathbb{R}, \end{array}$$

where $\alpha_W\colon [W \otimes E[G]p^t]^G \xrightarrow{\sim} \mathrm{Hom}_E(E[G]p,W)^G \xrightarrow{\sim} pW$ etc. are the canonical $E[G]$-isomorphisms, we see that Beilinson's conjecture in this case amounts to that of Gross (see [Neu]): the leading coefficient of $L_{Artin}(V,s) = L(V^\vee,s) = L^0(\mathbb{M}^\vee,s)$ at $s = 1-n$ should be the determinant of the map 4.7.1. Here $V^\vee = \mathrm{Hom}(V,E) \cong E[G]p$ is the dual of $V = E[G]p^t$, and $L_{Artin}(V,s)$ is the Artin L-series, defined via the arithmetic Frobenius elements.

b) Let $\mathbb{M} = (X,p,r)$ be a Chow motive. Say that $(\mathbb{M},i,m)$ is critical, if neither $L^i_\infty(\mathbb{M},s)$ nor $L^{-i}_\infty(\mathbb{M}^\vee,1-s)$ has a pole at $s = m$. This means that m is critical for the associated motive for absolute Hodge cycles $H^i_{AH}(\mathbb{M})$ [De 4] 2.3., and by 4.15 it implies that $H^{-i+1}_{\mathcal{D}}(\mathbb{M}^\vee\otimes\mathbb{R},\mathbb{R}(1-m))$ vanishes provided $m < \frac{i}{2}$. Then Beilinson's conjecture has to be interpreted in the way that $L^{-i}(\mathbb{M},m) \in (\mathbb{R}\otimes E)^\times/E^\times$ is the determinant of the isomorphism

$$F^{1-m}H^{-i}_{DR}(\mathbb{M}^\vee)\otimes\mathbb{R} \xrightarrow{\sim} H^{-i}_B(\mathbb{M}^\vee\otimes\mathbb{R},\mathbb{Q}(-m))\otimes\mathbb{R}$$

$$\nearrow\!\wr$$

$$H^{-i}_B(\mathbb{M}^\vee\otimes\mathbb{R},\mathbb{Q}(-m)\oplus\mathbb{Q}(1-m))\otimes\mathbb{R}/H^{-i}_B(\mathbb{M}^\vee\otimes\mathbb{R},\mathbb{Q}(1-m))$$

with respect to the indicated E-structures. By 4.4 this map is dual to

$$H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{Q}(m))\otimes\mathbb{R} \to H^i_{DR}(\mathbb{M})/F^mH^i_{DR}(\mathbb{M})\otimes\mathbb{R}.$$

Hence we may compute the determinant of the latter ([De 4] 5.1.3), which by definition is the period $c^+(H^i_{AH}(\mathbb{M})(m))$. We conclude that Beilinson's conjecture for critical $(\mathbb{M},i,m)$,

$m < \frac{i}{2}$, is equivalent to Deligne's conjecture [De 4] 2.8 for $H^i_{AH}(\mathbb{M})(m)$. It is not difficult to extend this to $m = \frac{i}{2}$ and $\frac{i+1}{2}$ (if this is entire and critical), and for $m > \frac{i+1}{2}$ one has to invoke the functional equation (compare below).

4.8. Remark The above description of motives with coefficients in E corresponds to the "language B" in [DE 4] §2. In the often used "language A" one talks of motives $\mathbb{M}$ with $\mathbb{Q}$-coefficients together with a ring morphism $E \to \mathrm{End}(\mathbb{M})$. Via the correspondence between these two (loc. cit.) EX corresponds to $X \otimes E$; note that for such objects the L-functions always have $\mathbb{Q}$-rational coefficients. One has to be careful with the dual: the relation $X^\vee \cong X(\dim X)$ does not carry over to $E \hookrightarrow \mathrm{End}(X)$ in general. For example, for an elliptic curve X with complex multiplication by a CM-field E the dual is given by the conjugate action. Of course, the relation 4.4 does not depend on the chosen language.

4.9. The E-structure $\mathcal{R}(i,n)$ of $H^{i+1}_{\mathcal{D}}(\mathbb{M},\mathbb{R}(n))$ given by the exact sequence in 4.6 c) is very convenient for the above formulation of Beilinson's conjecture, but it used the somewhat ad hoc $\mathbb{Q}$-structure $\mathbb{Q}(n-1)$ of $\mathbb{C}/\mathbb{R}(n)$. Another, perhaps more canonical E-structure $\mathcal{DR}(i,n)$ can be obtained by using the first canonical isomorphism in 4.5.1, i.e., the exact sequence

$$0 \to H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n)) \to H^i_{DR}(\mathbb{M}\otimes\mathbb{R})/F^n \to H^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n)) \to 0$$

and the E-structures $H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{Q}(n))$ and $H^i_{DR}(\mathbb{M})/F^n$.

In [De 5] Deligne has observed that this E-structure is related to a formulation of Beilinson's conjecture in terms of L-values at integers $n > \frac{i+1}{2}$, i.e., to the right of the central point.

First let us compare the E-structures. For this let $M = H^i_{AH}(\mathbb{M})(n)$ and $H_B(M)^{\pm}$ = part of $H_B(M) := H^i_B(\mathbb{M}\otimes\mathbb{C},\mathbb{Q}(n))$ where σ acts as ± 1 . Then $H_B(M)^+ = H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{Q}(n))$, $H^i_B(\mathbb{M}\otimes\mathbb{R},\mathbb{Q}(n-1)) = \frac{1}{2\pi\sqrt{-1}} H_B(M)^-$ and $F^0H_{DR}(M) = F^nH^i_{DR}(\mathbb{M})$ (note that the Tate twist shifts the filtration). We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^0H_{DR}(M)\otimes\mathbb{R} \arrow[r] & \frac{1}{2\pi\sqrt{-1}}\,H_B(M)^-\otimes\mathbb{R} \arrow[r] & H_{\mathcal{D}} \arrow[r] & 0 \\
0 \arrow[r] & (F^0H_{DR}(M)\oplus H_B(M)^+)\otimes\mathbb{R} \arrow[r] \arrow[u, twoheadrightarrow] \arrow[d, equal] & \left(\frac{1}{2\pi\sqrt{-1}}\,H_B(M)^-\oplus H_B(M)^+\right)\otimes\mathbb{R} \arrow[r] \arrow[u, twoheadrightarrow] \arrow[d, "I_{\mathbb{R}}", "\wr"'] & H_{\mathcal{D}} \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & (F^0H_{DR}(M)\oplus H_B(M)^+)\otimes\mathbb{R} \arrow[r, dash] & H_{DR}(M)\otimes\mathbb{R} \arrow[r] & H_{\mathcal{D}} \arrow[r] & 0
\end{tikzcd}$$

in which the first two rows give Beilinson's and the last one Deligne's E-structure on $H_{\mathcal{D}} = H_{\mathcal{D}}^{i+1}(\mathbb{M}, \mathbb{R}(n))$. This shows

$$(4.9.1) \qquad \mathcal{DR}(i,n) = (2\pi\sqrt{-1})^{-d^-(M)}\,\delta(M) \cdot \mathcal{R}(i,n) \ ,$$

where $d^-(M) = \dim_E H_B(M)^-$ and $\delta(M)$ is the determinant of the canonical comparison isomorphism (cf. [De 4] 0.4.1)

$$I = I(M) \ : \ H_B(M)\otimes\mathbb{C} \xrightarrow{\sim} H_{DR}(M)\otimes\mathbb{C} \ .$$

Next we relate L-values on the different sides of the functional equation, writing $a \sim b$ for two numbers $a,b \in (\mathbb{C}\otimes E)^\times$, if $ab^{-1} \in E^\times$.

<u>4.10. Lemma</u> For a Chow motive $\mathbb{M}$ and $i,n \in \mathbb{Z}$ we have

$$L^i(\mathbb{M},n)^* \sim L^{-i}(\mathbb{M}^\vee, 1-n)^*\varepsilon^i(\mathbb{M},n)(2\pi\sqrt{-1})^{-d_n^i(\mathbb{M}) - (i-2n)d^i(\mathbb{M})/2}$$

or, equivalently

$$L(M)^* \sim L(\check{M}(1))^*\varepsilon(M)(2\pi\sqrt{-1})^{-d^-(M) - w(M)d(M)/2} \ ,$$

where $M = H^i_{AH}(\mathbb{M})(n)$ and $L(M)^*$ is the leading coefficient of $L(M,s)$ at $s=0$. Here $d^i(\mathbb{M}) = \dim_E H^i_{DR}(\mathbb{M}) = \dim_E H_{DR}(M) = d(M)$, $d^i_n(\mathbb{M}) = \dim_E H^i_B(\mathbb{M}\otimes\mathbb{R}, \mathbb{Q}(n+1)) = \dim_E H_B(M)^- = d^-(M)$, and $w(M) = i-2n$ is the weight of M.

<u>Proof</u> Since $L^i(\mathbb{M},n+s) = L(H^i_{AH}(\mathbb{M}),n+s) = L(H^i_{AH}(\mathbb{M})(n),s) = L(M,s)$ and $L(\check{M}(1),s) = L(H^{-i}(\mathbb{M}^\vee)(1-n),s) = L^{-i}(\mathbb{M}^\vee, 1-n+s)$ by 4.4, and similarly for L_∞, ε and ε_∞, the equivalence is clear. Hence, as leading coefficients are multiplicative, the claim follows from the functional equation and the formulae

$$\varepsilon_\infty(M) = \prod_{\substack{p+q=w \\ p<q}} i^{(q-p+1)h^{p,q}} \cdot \left(i^{h^{p,p,(-1)^{p+1}}}\right)_{\substack{\text{if} \\ w=2p}} = \pm i^{-d^- - wd/2}$$

where $h^{p,q} = \dim H^{p,q}(M)$ and $h^{p,p,\pm} = \dim H^{p,p}(M) \cap (H_B^{2p}(M)^{\pm}\otimes\mathbb{C})$, and

$$L_\infty(\check{M}(1))^* L_\infty(M)^{*-1} \sim (2\pi)^{-d^- - wd/2} \ .$$

The first formula follows from the definitions [De 4]5.3, and the second is proved in [De 4]5.4 (the proof extends to the leading coefficients in the non-critical case).

Now we further investigate the ε-term. In [De 4]5.5 Deligne has proved

$$(4.11.1) \qquad \varepsilon(M) \sim \varepsilon(\det M)$$

where $\det M = \Lambda_E^{d(M)} M$, and in [De 4] 6.6 he has conjectured that motives of rank one are the ones we know:

<u>4.11. Conjecture</u> $\det M \cong [\chi](-w(M)d(M)/2)$ for a Dirichlet character $\chi\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to E^{\times}$ (note that $w(M)d(M)$ is the weight of $\det M$).

Here $[\chi]$ denotes the Artin motive for which $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts via χ on the ℓ-adic realizations; we do not have to distinguish between Chow motives and motives for absolute Hodge cycles here, since for Artin motives H^0 gives an anti-equivalence of categories. By [De 4] 6.5, 5.19 we have

$$(4.11.2) \quad \varepsilon([\chi](r)) \sim \varepsilon([\chi]) \sim \delta([\chi]) \sim (2\pi\sqrt{-1})^{-r}\delta([\chi](r)) .$$

Hence 4.11 would imply

$$(4.11.3) \quad \varepsilon(M) \sim (2\pi\sqrt{-1})^{w(M)d(M)/2}\delta(M) ,$$

since obviously $\delta(\det M) \sim \delta(M)$. Assuming 4.11.3 we would obtain

$$(4.11.4) \quad L(M)^* \sim L(\check{M}(1))^*(2\pi\sqrt{-1})^{-d^-(M)}\delta(M)$$

from 4.10. A comparison with 4.9.1 for $M = H^i(\mathbb{M})(n)$ gives the following reformulation of Beilinson's conjecture 4.6 c).

<u>4.12. Conjecture</u> If $\mathbb{M}$ is a Chow motive and $n > \frac{i}{2} + 1$, then

$$r_{\mathcal{D}}(\det_E H_{\mathcal{M}}^{i+1}(\mathbb{M}_{\mathbb{Z}},\mathbb{Q}(n))) = L^i(\mathbb{M},n)\cdot \mathcal{DR}(i,n) .$$

Note that n lies in the range of convergence, so that no meromorphic continuation is needed and we work with an actual L-value instead of a leading coefficient, and that moreover no dualization is involved in this formulation. For

$n = \frac{i}{2} + 1$ one has to pass to the "thickend" regulator (cf. [Be 1] 3.7) involving the group $N^{n-1}(\mathbb{M})$ of algebraic cycles of codimension n-1 modulo homological equivalence, and to the leading coefficient $L^i(\mathbb{M},n)^*$ (compare that by Tate's conjecture the pole order of $L^{2n-2}(\mathbb{M},s)$ at $s = n$ equals the rank of $N^{n-1}(\mathbb{M})$).

We now come to the motivic interpretation of Beilinson's conjecture as formulated by Deligne in the letter [De 5]. For this recall from 2.7, that for a variety over $\mathbb{C}$ one can interprete its Deligne cohomology in terms of morphisms and extensions of mixed Hodge structures. There is a similar interpretation for a variety X over $\mathbb{R}$.

4.13. Lemma Let $MH_{\mathbb{R}}$ be the category of mixed $\mathbb{R}$-Hodge structures over $\mathbb{R}$: objects are mixed $\mathbb{R}$-Hodge structures H with an "infinite Frobenius", i.e., an involution $F_\infty: H \to H$ respecting the weight filtration such that $\bar{F}_\infty: H\otimes\mathbb{C} \to H\otimes\mathbb{C}$ respects the Hodge filtration, and morphisms are morphisms of mixed $\mathbb{R}$-Hodge structures compatible with F_∞. Then for an object H of $MH_{\mathbb{R}}$ we have

a) $\mathrm{Hom}_{MH_{\mathbb{R}}}(\mathbb{R},H) \cong W_oH^+ \cap F^o(H\otimes\mathbb{C})^+$ (by $f \mapsto f(1)$),

b) $\mathrm{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R},H) \cong (W_oH\otimes\mathbb{C})^+/W_oH^+ + F^o(H\otimes\mathbb{C})^+$,

c) $\mathrm{Ext}^i_{MH_{\mathbb{R}}}(\mathbb{R},H) = 0$ for $i \geq 2$,

where $\mathbb{R}$ is the trivial Hodge structure over $\mathbb{R}$ ($F_\infty = \mathrm{id}$), and $^+$ denotes the fixed space under $\bar{F}_\infty$.

We do not prove the lemma here - it can be obtained by similar methods as for 2.4.2. We only remark that the second isomorphism sends

$$0 \to H \xrightarrow{i} E \xrightarrow{p} \mathbb{R} \to 0$$

to the class of $(r\circ s)(1)$, where s is a section of $W_oE \otimes \mathbb{C} \xrightarrow{p} \mathbb{C}$ that is compatible with the Hodge filtration and with $\bar{F}_\infty$, and where r is a section of $W_oH \xrightarrow{i} W_oE$ that is compatible with F_∞ (compare [Car], lemma 4, for the situation without F_∞).

If X is a variety over $\mathbb{R}$, then $H^\ell_R(X \times_{\mathbb{R}} \mathbb{C}, \mathbb{R})$ is a mixed $\mathbb{R}$-Hodge structure over $\mathbb{R}$, with F_∞ being the map induced by the antiholomorphic map $F_\infty\colon X \times_{\mathbb{R}} \mathbb{C} \to X \times_{\mathbb{R}} \mathbb{C}$, and we have isomorphisms

$$\begin{aligned} & H^\ell_B(X \times \mathbb{C}, \mathbb{R})(k)^+ = H^\ell_B(X/\mathbb{R}, \mathbb{R}(k)) \\ (4.13.1) \qquad & (H^\ell_B(X \times \mathbb{C}, \mathbb{R})(k) \otimes \mathbb{C})^+ = H^\ell_B(X/\mathbb{R}, \mathbb{C}) = H^\ell_{DR}(X/\mathbb{R}), \end{aligned}$$

since Tate twist (k) by definition changes F_∞ by $(-1)^k$ and since $\overline{F}_\infty$ corresponds to the deRham-complex conjugation (see [De 4] 1.4 for the smooth, projective case). If X is smooth and proper, then $H^\ell_B(X \times \mathbb{C}, \mathbb{R})$ is of weight ℓ, hence for $\ell - 2k \leq 0$ the standard sequence for Deligne cohomology ([EV] 2.10 a)) gives a short exact sequence

$$(4.13.2) \quad \begin{array}{ccccccc} 0 \to H^{\ell-1}_{DR}(X/\mathbb{R}) / H^{\ell-1}_B(X/\mathbb{R}, \mathbb{R}(k)) + F^k & \to & H^\ell_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(k)) & \to & H^\ell_B(X/\mathbb{R}, \mathbb{R}(k)) \cap F^k & \to 0 \\ \wr\!\! | & & & & \wr\!\! | & \\ \mathrm{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R}, H^{\ell-1}(X \times \mathbb{C}, \mathbb{R})(k)) & & & & \mathrm{Hom}_{MH_{\mathbb{R}}}(\mathbb{R}, H^\ell_B(X \times \mathbb{C}, \mathbb{R})(k)) & \end{array}$$

Everything can be extended to coefficients in a number field E, i.e., to the category $MH_{\mathbb{R},E}$ of pairs (H,ι) with H in $MH_{\mathbb{R}}$ and $\iota\colon E \to \mathrm{End}_{MH_{\mathbb{R}}}(H)$ a ring homomorphism. Then for a Chow motive $\mathbb{M}$ in $\mathcal{M}(\mathbb{Q},E)$ we get an exact sequence for $\ell \leq 2k$

$$\begin{aligned} (4.13.3) \quad 0 \to \mathrm{Ext}^1_{MH_{\mathbb{R},E}}(\mathbb{R} \otimes E, H^{\ell-1}_B(\mathbb{M} \otimes \mathbb{C}, \mathbb{R}(k))) \to H^\ell_{\mathcal{D}}(\mathbb{M} \otimes \mathbb{R}, \mathbb{R}(k)) \\ \to \mathrm{Hom}_{MH_{\mathbb{R},E}}(\mathbb{R} \otimes E, H^\ell_B(\mathbb{M} \otimes \mathbb{C}, \mathbb{R}(k))) \to 0 . \end{aligned}$$

Before we go on, we mention the following relation with the functions $L^i_\infty(\mathbb{M}, s) = L_\infty(H^i_B(\mathbb{M} \otimes \mathbb{R}, \mathbb{R}), s)$, which are defined for every pure $\mathbb{R}$-Hodge structure H over $\mathbb{R}$ of weight i by

$$(4.14.1) \quad L(H,s) = \prod_{\substack{p<q \\ p+q=i}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \left(\Gamma_{\mathbb{R}}(s-p)^{h^{p,p,(-1)^p}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,p,(-1)^{p+1}}}\right)_{\text{if } i=2p}$$

cf. [De 4] 5.3 and [Sch] §1. For coefficients by E these are $\mathbb{C} \otimes E$-valued functions or arrays $(\ldots, L_\infty(\sigma,H,s), \ldots)$, with $L_\infty(\sigma,H,s)$ defined via $h^{p,q}(\sigma,H) = \dim_{\mathbb{C}} H^{p,q} \otimes_{(\mathbb{C}\otimes E),\sigma} \mathbb{C}$ and $h^{p,p,\pm}(\sigma,H) = \dim_{\mathbb{C}} H^{p,p,\pm} \otimes_{(\mathbb{C}\otimes E),\sigma} \mathbb{C}$ for $\sigma \in \mathrm{Hom}(E,\mathbb{C})$.

4.14. Lemma Let H, H' be $\mathbb{R}$-Hodge structure over $\mathbb{R}$ of weight i, then

a) $\operatorname{ord}_{s=m} L_\infty(H,s) = \dim_{\mathbb{R}} \operatorname{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R}, \check{H}(1-m))$ for $m \in \mathbb{Z}$,

b) the following statements are equivalent:

i) $H \cong H'$,

ii) $L_\infty(H,s) = L_\infty(H',s)$

iii) $\operatorname{ord}_{s=m} L_\infty(H,s) = \operatorname{ord}_{s=m} L_\infty(H,s)$ for all $m \in \mathbb{Z}$, $m \leq \frac{i}{2}$.

Proof a) By the known multiplicities of $\Gamma_{\mathbb{C}}$ and $\Gamma_{\mathbb{R}}$ we have

$$(4.14.2)\quad \operatorname{ord}_{s=m} L_\infty(H,s) = \sum_{\substack{m\leq p<q \\ p+q=i}} h^{p,q} + \underset{\substack{\text{if } m\leq p, \\ 2p=i}}{(h^{p,p,(-1)^m})}$$

cf. [Sch] §1, in particular this is zero for $m > \frac{i}{2}$. On the other hand, since the weight of $K = \check{H}(1-m)$ is $-i-2+2m$, the group $\operatorname{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R},K)$ vanishes for $m \geq \frac{i}{2}+1$ (note that mixed $\mathbb{R}$-Hodge structures of weight zero are semisimple). For $i-2+2m < 0$ we have

$$K^+ \cap F^0(K \otimes \mathbb{C})^+ \subset K^+ \cap F^0(K \otimes \mathbb{C})^+ \cap \overline{F^0}(K \otimes \mathbb{C})^+ = 0,$$

hence $\operatorname{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R},K)$ has dimension

$$\begin{aligned}
& \dim_{\mathbb{R}} (K \otimes \mathbb{C})^+ - \dim_{\mathbb{R}} K^+ - \dim_{\mathbb{R}} F^0(K \otimes \mathbb{C})^+ \\
= \; & \dim_{\mathbb{C}} K \otimes \mathbb{C} - \dim_{\mathbb{C}} (K \otimes \mathbb{C})^{F_\infty} - \dim_{\mathbb{C}} F^0(K \otimes \mathbb{C}) \\
= \; & \dim_{\mathbb{R}} K - \Big(\sum_{p<q} h^{p,q}(K) + (h^{p,p,+}(K))\Big) - \sum_{p>0} h^{p,q}(K) \\
= \; & \dim_{\mathbb{R}} H - \Big(\sum_{p>q} h^{p,q}(H) + (h^{p,p,(-1)^{1-m}}(H))\Big) - \sum_{p<m} h^{p,q}(H) \\
= \; & \sum_{m\leq p<q} h^{p,q}(H) + (h^{p,p,(-1)^m}(H)),
\end{aligned}$$

where we have used $h^{p,q,\pm}(\check{H}) = h^{-p,-q,\pm}(H)$ and $h^{p,q,\pm}(H(r)) = h^{p+r,q+r,\pm(-1)^r}(H)$.

b) For the non-trivial implication iii) $\Rightarrow$ i) note that by 4.12.2 the $h^{p,q}$ and $h^{p,p,\pm}$ can be deduced from i and the multiplicities for $m \leq \frac{i}{2}$, and that these numbers determine a $\mathbb{R}$-Hodge structure with F_∞ (we remark that for both statements the knowledge of i is necessary).

4.15. Corollary For a Chow motive $\mathbb{M}$ in $\mathcal{M}(\mathbb{Q},E)$ and $m \leq \frac{i}{2}$ one has $\operatorname{ord}_{s=m} L^i_\infty(\mathbb{M},s) = \dim_{\mathbb{R}\otimes E} H_{\mathcal{D}}^{-i+1}(\mathbb{M}^\vee, \mathbb{R}(1-m))$.

Proof First we note that all considered modules over $\mathbb{R}\otimes E$ are free over $\mathbb{R}\otimes E$ by the comparison isomorphisms (compare [De 4] 2.5), and that the $L^i_\infty(\sigma,\mathbb{M},s)$ are independent of $\sigma \in \mathrm{Hom}(E,\mathbb{C})$, hence $L^i_\infty(\mathbb{M},s)$ can be regarded as a function into $\mathbb{C} \subseteq \mathbb{C}\otimes E$. The claim then follows from 4.14 a) and 4.13.3.

4.16. It was Grothendieck's idea that various cohomology groups $H^i_?(X)$ of a smooth projective variety X over a field k should just be realizations of a motive $H^i(X)$. Grothendieck's category of motives $M_{k,E}$ is constructed as in 4.2, replacing the Chow groups by their quotients modulo numerical equivalence. The existence of the Grothendieck motives $H^i(X)$ - as direct factors of the motive $(X,\mathrm{id},0)$ - then depends on the so-called standard conjectures, which also imply that numerical equivalence equals homological equivalence and that $M_{k,E}$ is an abelian and semi-simple category. The first property cannot be expected from the category $\mathcal{M}(k,E)$ of Chow motives, and the second one is definitely false: for example for a curve X the radical of $\mathrm{End}(EX) = CH^1(X\times X)_E = \mathrm{Pic}(X\times X)_E$ is $\mathrm{Pic}^0(X\times X)_E$. Deligne proposes that the Chow motives $\mathbb{M}$ should rather be regarded as objects in a derived category of motives, so that only the cohomology objects $H^i(\mathbb{M})$ give Grothendieck motives. In the realizations this corresponds to the fact that a variety $X \xrightarrow{f} \mathrm{Spec}\, k$ gives rise to objects $Rf_*\mathbb{Q}_\ell$ in $D^b(\mathrm{Spec}\, k,\mathbb{Q}_\ell) = D^b(\mathbb{Q}_\ell\text{-}\mathrm{Gal}(\bar{k}/k)\text{-modules})$ and $K^\bullet(X,A)$ in $D^b(A\text{-}MH)$ (for $k=\mathbb{C}$), whose i-th cohomology is $H^i(X \times_k \bar{k},\mathbb{Q}_\ell)$ as $\mathrm{Gal}(\bar{k}/k)$-module and $H^i_B(X,A)$ as (mixed) A-Hodge structure, respectively, and this carries over to Chow motives $\mathbb{M}$.

Now the derived category of a semi-simple abelian category is again semi-simple, so no radicals would appear. But Deligne points out that one would get a coherent picture if one believes that $M_{k,E}$ can be embedded in a certain abelian category of mixed motives $MM_{k,E}$ and that $\mathbb{M}$ can be regarded as an object in the derived category $D^b(MM_{k,E})$ of $MM_{k,E}$, denote this associated object by $R(\mathbb{M})$ (for $\mathbb{M}=X$ it has realizations $Rf_*\mathbb{Q}_\ell, K^\bullet(X,A),\ldots)$.

Mixed motives should be the motivic analogues of mixed Hodge structures, being successive extensions of pure (i.e., Grothendieck) motives. These extensions are in general non-trivial; again this could not happen in a semi-simple category. Now the motivic cohomology should be the motivic analogue of Deligne (or absolute Hodge) cohomology, so comparison with the results cited in 2.4 suggest the formula

$$\begin{aligned} H^i_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j)) &= \operatorname{Ext}^i_{D(MM_{k,E})}(E(0),R(\mathbb{M})(j)) \\ &= \operatorname{Hom}_{D(MM_{k,E})}(E(0),R(\mathbb{M})(j)[i]) , \end{aligned} \tag{4.16.1}$$

where $E(0) = R(E\mathrm{Spec}\ k)$ is the Grothendieck motive $H^0(E\mathrm{Spec}\ k)$ sitting in degree zero. By the formula

$$\operatorname{Hom}_{\mathcal{M}(k,E)}(\mathbb{M},E\ \mathrm{Spec}\ k) = CH^r(X)_E p = p^* H^{2r}_{\mathcal{M}}(X,\mathbb{Q}(r))_E = H^0_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(0))$$

for $\mathbb{M} = (X,p,r)$, we obtain agreement with the embedding of $\mathcal{M}(k,E)$ in $D^b(MM_{k,E})$, if as for the motives for absolute Hodge cycles we define $M_{k,E}$ as in 4.2 but in a dual way, so that $\mathbb{M} \rightsquigarrow R(\mathbb{M})$ is a contravariant functor (as are the functors $X \rightsquigarrow Rf_*\mathbb{Q}_\ell, K^\cdot(X,A),\ldots$; as a general rule Chow motives behave like varieties and Grothendieck motives like a (universal) cohomology theory). In fact, we can deduce the embedding from 4.16.1 in the following way. Recall that the category $\mathcal{M}(k,E)$ has duals (defined above), a tensor product $(\mathbb{M}\otimes\mathbb{N} = (X\times Y, p^*_{X\times X}p\cdot p^*_{Y\times Y}q, r+t)$ for $\mathbb{M} = (X,p,r)$ and $\mathbb{N} = (Y,q,t))$ and an internal Hom $(\underline{\mathrm{Hom}}(\mathbb{M},\mathbb{N}) = \mathbb{M}^\vee\otimes\mathbb{N})$. If we assume the same formalism for $MM_{k,E}$ and hence for $D^b(MM_{k,E})$, we get

$$\begin{aligned} \operatorname{Hom}_{\mathcal{M}(k,E)}(\mathbb{M},\mathbb{N}) &= H^0_{\mathcal{M}}(\mathbb{M}\otimes\mathbb{N}^\vee,\mathbb{Q}(0)) \\ &\overset{4.16.1}{=} \operatorname{Hom}_{D^b(MM_{k,E})}(E(0),R(\mathbb{M}\otimes\mathbb{N}^\vee)) \\ &= \operatorname{Hom}_{D^b(MM_{k,E})}(E(0),\underline{R\mathrm{Hom}}(R\mathbb{N},R\mathbb{M})) \\ &= \operatorname{Hom}_{D^b(MM_{k,E})}(R(\mathbb{N}),R(\mathbb{M})) , \end{aligned} \tag{4.16.2}$$

assuming $R(\mathbb{M}^\vee) = \underline{R\mathrm{Hom}}(\mathbb{M},E(0))$ (cf. 4.4), $R(\mathbb{M}\otimes\mathbb{N}) = R(\mathbb{M})\otimes R(\mathbb{N})$ and $\underline{R\mathrm{Hom}}(\mathbb{M},\mathbb{N}) = \underline{R\mathrm{Hom}}(\mathbb{M},E(0))\otimes\mathbb{N}$. We mention that the relation $H^i_{\mathcal{M}}(\mathbb{M}(r),\mathbb{Q}(j)) = H^{i+2r}_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j+r))$ together with 4.16.1 suggests the formula $R(\mathbb{M}(r)) = R(\mathbb{M})(r)[2r]$, in agreement with the principle 4.2.1.

For $k=\mathbb{C}$ the regulator map from motivic cohomology to Deligne cohomology should just be "passing to the realizations", in particular, it should map the hyperext spectral sequence

$$(4.16.3)\qquad E_2^{p,q} = \mathrm{Ext}^p_{MM_{k,E}}(E(0),H^q(\mathbb{M})(j)) \Rightarrow H^{p+q}_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j))$$

to the spectral sequence 2.4.1 for $K^\bullet(\mathbb{M},A)(j)$ (cf. 2.7). Now for $k=\mathbb{Q}$ Beilinson's conjecture 4.6 b) gives a close relation between $H^\ell_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j))$ and $H^\ell_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(j))$; in view of 4.13 c) this leads Deligne to assume

$$(4.16.4)\qquad \mathrm{Ext}^i_{MM_{\mathbb{Q},E}}(E(0),-) = 0 \quad \text{for} \quad i \geq 2 ,$$

so that 4.16.3 gives short exact sequences

$$(4.16.5)\quad 0\to \mathrm{Ext}^1_{MM_{\mathbb{Q},E}}(E(0),H^{\ell-1}(\mathbb{M})(j)) \to H^\ell_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j)) \to \mathrm{Hom}_{MM_{\mathbb{Q},E}}(E(0),H^\ell(\mathbb{M})(j))\to 0$$

mapping to 4.13.3 via the regulator map. Then the induced maps

$$(4.16.6)\quad \mathrm{Ext}^1_{MM_{\mathbb{Q},E}}(E(0),H^i(\mathbb{M})(n)) \to \mathrm{Ext}^1_{MH_{\mathbb{R},E}}(\mathbb{R}\otimes E,H^i_B(\mathbb{M}\otimes\mathbb{C},\mathbb{R})(n))$$

just associate to an extension of motives the extension of their real realizations, i.e., their corresponding Hodge structures over $\mathbb{R}$. For $n > \frac{i+1}{2}$ the weight of $H^{i+1}(\mathbb{M})(n)$ is $i+1-2n < 0$, so that $\mathrm{Hom}_{MM_{\mathbb{Q},E}}(E(0),H^{i+1}(\mathbb{M})(n)) =$ $\mathrm{Hom}_{M_{\mathbb{Q},E}}(E(0),H^{i+1}(\mathbb{M})(n))$ vanishes. The same is true for the real realization, hence 4.16.6 then coincides with

$$(4.16.7)\qquad H^{i+1}_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(n)) \to H^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n)) .$$

For $n = \frac{i+1}{2}$ by definition $\mathrm{Hom}_{M_{\mathbb{Q},E}}(E(0),H^{2n}(\mathbb{M})(n)) = N^n(\mathbb{M})$ is the quotient modulo numerical equivalence of $H^{2n}_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(n))$ $= CH^n(\mathbb{M}) = \mathrm{Hom}_{\mathcal{M}(\mathbb{Q},E)}(\mathbb{M}(n),E\mathrm{Spec}\,\mathbb{Q})$, and the cycle map

$$CH^n(\mathbb{M}) \to N^n(\mathbb{M}) \to \mathrm{Hom}_{MH_{\mathbb{R},E}}(\mathbb{R}\otimes E,H^{2n}_B(\mathbb{M}\otimes\mathbb{C},\mathbb{R}(n))$$

in fact maps a correspondence $\mathbb{M}\to E\mathrm{Spec}\,\mathbb{Q}$ to its real realization.

<u>4.17.</u> It is sometimes convenient to write everything in terms of cohomology - recall the formula $\mathrm{Ext}^i_X(\mathbb{Z},F) = H^i(X,F)$ for a sheaf F on a space X . Hence we may set

$$H^i_{\mathcal{M}}(M) = \mathrm{Ext}^i_{MM_{k,E}}(E(0),M) \qquad \text{for } M \text{ in } MM_{k,E} ,$$
$$H^i_{\mathcal{H}}(H) = \mathrm{Ext}^i_{MH_{\mathbb{R},E}}(\mathbb{R}\otimes E,H) \qquad \text{for } H \text{ in } MH_{\mathbb{R},E} ,$$

to obtain a commutative diagram for $i+1-2n \leq 0$ and $k=\mathbb{Q}$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^1_H(H^i(\mathbb{M}\otimes\mathbb{C},\mathbb{R}(n))) \arrow[r] & H^{i+1}_{\mathcal{D}}(\mathbb{M}\otimes\mathbb{R},\mathbb{R}(n)) \arrow[r] & H^0_H(H^{i+1}(\mathbb{M}\otimes\mathbb{C},\mathbb{R}(n)) \arrow[r] & 0 \\
0 \arrow[r] & H^1_M(H^i(\mathbb{M})(n)) \arrow[r] \arrow[u, "r'_{\mathcal{D}}"'] & H^{i+1}_M(\mathbb{M},\mathbb{Q}(n)) \arrow[r] \arrow[u, "r_{\mathcal{D}}"'] & H^0_M(H^{i+1}(\mathbb{M})(n)) \arrow[r] \arrow[u, "cl"'] & 0 .
\end{tikzcd}$$

<u>4.18. Example</u> So for there is no description of a category of mixed motives, not even a conjectural one. Only for so-called 1-motives there exists a definition [De 3] 10.1, and we illustrate the above philosophy at this example. Let A be an abelian variety over $\mathbb{Q}$, and consider the case $i=1, n=1$; then we can identify

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathrm{Ext}^1_{MM_{\mathbb{Q}}}(\mathbb{Q}(0),H^1(A)(1)) \arrow[r] & H^2_M(A,\mathbb{Q}(1)) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Hom}_{MM_{\mathbb{Q}}}(\mathbb{Q}(0),H^2(A)(1)) \arrow[r] & 0 \\
(4.18.1) & & CH^1(A)_{\mathbb{Q}} \arrow[d, phantom, "\wr"] & & \\
0 \arrow[r] & \mathrm{Pic}^0(A)_{\mathbb{Q}} \arrow[r] \arrow[uu, "\gamma", "\wr"'] & \mathrm{Pic}(A)_{\mathbb{Q}} \arrow[r] & NS(A)_{\mathbb{Q}} \arrow[r] \arrow[uu, "\wr"'] & 0
\end{tikzcd}$$

By the formula of Weil-Barsotti-Rosenlicht

$$\mathrm{Pic}^0(A) \cong \mathrm{Ext}^1_{\text{alg.grps.}}(A,\mathbb{G}_m) \tag{4.18.2}$$

every element x in $\mathrm{Pic}^0(A)$ in fact gives rise to an extension

$$0 \to \mathbb{G}_m \to E \to A \to 0 \tag{4.18.3}$$

of 1-motives. This induces the extension of mixed Hodge structures over $\mathbb{R}$

$$0 \to H^1(A\times\mathbb{C},\mathbb{R}(1)) \to H^1(E\times\mathbb{C},\mathbb{R}(1)) \to \underset{\substack{\wr\wr \\ \mathbb{R}}}{H^1(\mathbb{G}_m\times\mathbb{C},\mathbb{R}(1))} \to 0 ,$$

and one can show this extension is the image of x under the regulator map via the identification in 4.18.3. Note that the definition of 1-motives given in [De 3] 10.1 is a "homological" one; passing to the dual category and indicating this by writing H^1, 4.18.3 corresponds to an extension

$$0 \to H^1(A) \to H^1(E) \to H^1(\mathbb{G}_m) \to 0 .$$

Since one should have $H^1(\mathbb{G}_m) = \mathbb{Q}(-1)$, we get the isomorphism γ by combining 4.18.2 with

$$\mathrm{Ext}^1(A,\mathbb{G}_m) \cong \mathrm{Ext}^1_{MM_{\mathbb{Q}}}(\mathbb{Q}(-1),H^1(A)) \cong \mathrm{Ext}^1(\mathbb{Q},H^1(A)(1)) ,$$

and see that the regulator map then really becomes "passing to the Hodge realization".

4.19. Assuming that $H^i_{\mathcal{M}}(\mathbb{M}_{\mathbb{Z}},\mathbb{Q}(j)) \subseteq H^i_{\mathcal{M}}(\mathbb{M},\mathbb{Q}(j))$ can be described similarly, by using a category of mixed motives over $\mathbb{Z}$, Deligne now formulates the following principles underlying Beilinson's conjecture (Note that for $N = H^{-i}(\mathbb{M}^{\vee})(1-n)$ we have $\check{N}(1) = H^i(\mathbb{M})(n)$, and that the weight $w(N)$ of N is $-i-2+2n$).

Let N be a Grothendieck motive, let $\rho = \mathrm{ord}_{s=0} L(N,s)$ and $L(N)^* = \lim_{s\to 0} s^{-\rho} L(N,s)$ be the leading coefficient.

Principle 1 If $w(N) \geq -1$ ($\Leftrightarrow 0 \leq$ central point of the functorial equation), then $L(N)^* \bmod E^{\times}$ can be described in terms of $\check{N}(1)$.

Principle 2 If $w(N) \geq 0$ and N contains no direct factor $\mathbb{Q}(0)$, then $\rho = \dim H^1_{\mathcal{H}}(H_B(\check{N}(1))\otimes\mathbb{R})$.

Principle 3 Under the hypotheses of principle 2, $L(N)^*$ is a regulator for $H^1_{\mathcal{M}_{\mathbb{Z}}}(j_*\check{N}(1))\otimes\mathbb{R} \to H^1_{\mathcal{H}}(H_B(\check{N}(1))\otimes\mathbb{R})$, where $j\colon \mathrm{Spec}\,\mathbb{Q} \to \mathrm{Spec}\,\mathbb{Z}$ is the inclusion.

4.20. Remarks a) By Deligne's reformulation 4.12, one may also rewrite principles 1 and 3 only in terms of $M = \check{N}(1)$.
b) A consequence of principle 1 is: if an L-function only depends on the i-th cohomology of a variety X, all its properties should also be describable in terms of the i-th cohomology (i.e., the motive $H^i(X)$), perhaps involving the dual. Of course, $L(N,s)$ and hence $L(N)^*$ are defined only in terms of N, hence of $\check{N}(1)$. The non-trivial statement is that also every other description should only depend on $\check{N}(1)$ and/or N. Generally speaking, Deligne cohomology and motivic cohomology do not factorize through the category of Grothendieck motives (this can only be expected from "geometric" theories having a Künneth formula). But the kernels and cokernels in the exact sequences 4.13.3 and 4.16.5 do, and in fact, only they are involved in Beilinson's conjecture. For $w(N)=0$ the thickened regulator only uses $H^1_{\mathcal{M}_{\mathbb{Z}}}(j_*\check{N}(1))$ and $H^0_{\mathcal{M}}(N)$, and for $w(N) = -1$ (the central point) Beilinson's conjecture involves $H^1_{\mathcal{M}}(\check{N}(1))$ and $H^1_{\mathcal{M}}(N)$. However, in this case one

also needs a height pairing between these groups [Be 1]3.8. Deligne points out that so far there does not exist a motivic interpretation of the latter.

c) Let $\check{N}(1) = H^i(X)(n)$ for a smooth projective variety X over $\mathbb{Q}$. By conjectures in K-theory (see [Be 1] 2.4) one expects

$$H^1_{M_{\mathbb{Z}}}(j_*\check{N}(1)) = H^1_M(\check{N}(1)) \quad \text{for } n > i+1$$

by the above this concerns the value of $L^i(X,s)=L(H^i(X),s) = L(H^{2\dim X-i}(X)(\dim X),s) = L(N,s+n-i-1)$ at $s = i+1-n =: m < 0$. Deligne remarks that the remaining part, $0 \le m \le \frac{i}{2}$, is by conjectures on the local Euler factors $L^i_p(X,s)$ exactly the region where these can contribute to the zeroes of $L^i(X,s)$. Moreover, for $m < \frac{i}{2}$ only the $L^i_p(X,s)$ for primes p of bad reduction should contribute, and in the motivic cohomology the picture is the same: for $m<\frac{i}{2}$, i.e., $i+1\leq 2n-1$, conjecturally only the K-groups of the bad fibres contribute to the difference between $H^{i+1}_{\mathcal{M}}(X_{\mathbb{Z}},\mathbb{Q}(n))$ and $H^{i+1}_{\mathcal{M}}(X,\mathbb{Q}(n))$. In view of this, Deligne asks whether there is a Beilinson conjecture for partial L-functions $L_S(N,s) = \prod_{p\notin S} L_p(N,s)$ involving motives (or K-groups) over $\operatorname{Spec}\mathbb{Z}\smallsetminus S$ and regulators with values in a suitable modification of Deligne cohomology.

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