Continuous Étale Cohomology

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It is well known that the $l$-adic cohomology of a scheme $X$

$$H^i(X, \mathbb{Z}_l(j)) \cong \lim_{\leftarrow n} H^i(X, \mathbb{Z}/l^n(j))$$

has good properties only if the étale cohomology groups $H^i(X, \mathbb{Z}/l^n(j))$ are finite (e.g. not for varieties over number fields). In particular, these groups do not arise as derived functors, which causes problems with functoriality, already for the usual long exact cohomology sequence associated to an exact sequence of $l$-adic sheaves. The situation is similar for continuous group cohomology of $\mathbb{Z}_l$-modules defined by continuous cochains. For example there does not always exist a Hochschild-Serre spectral sequence for a closed normal subgroup. The reason for these difficulties is the non-exactness of the inverse limit.

But the inverse limit is left exact, so we may try to define a good cohomology theory $H^*(X, (F_n))$, defined for inverse systems $(F_n)$ of étale sheaves on $X$, by deriving the left exact functor

$$\{\text{inverse systems } (F_n) \text{ of étale sheaves on } X\} \rightarrow \{\text{abelian groups}\}$$

(0.1)

$$(F_n) \rightarrow \lim_{\leftarrow n} H^0(X, F_n).$$

The aim of this paper is to show that this cohomology theory exists and has all the properties we want.

It turns out, that for locally constant sheaves $F_n$ our groups coincide with the continuous étale cohomology groups $H^i_{\text{cont}}(X, (F_n))$ defined by Dwyer and Friedlander via the étale topological type of $X$ (see [3], the definition there is more general, but compare (3.28) below). Our definition, being based on derived functors, applies to arbitrary sheaves [e.g., to $\mathbb{G}_m$ in the Kummer sequence (3.27)], and is particularly suited for calculations in derived categories. In fact, this paper, where we study the groups $H^i_{\text{cont}}(X, (F_n))$ and their properties for $l$-adic sheaves $(F_n)$ by more elementary means, will be basic for a following one, in which we shall give a new approach to the derived category of $\mathbb{Z}_l$-sheaves.
The connection between continuous and usual $l$-adic cohomology is as follows. If we define the continuous $l$-adic cohomology by

$$H^i_{\text{cont}}(X, \mathbb{Z}_l(j)) = H^i(X, (\mathbb{Z}/l^n(j)))$$

($l$ invertible on $X$), the obvious spectral sequence induces short exact sequences

$$0 \to \lim_{\leftarrow n} H^{i-1}(X, \mathbb{Z}/l^n(j)) \to H^i_{\text{cont}}(X, \mathbb{Z}_l(j)) \to \lim_{\leftarrow n} H^i(X, \mathbb{Z}/l^n(j)) \to 0,$$

where $\lim_{\leftarrow n}$ is the derivative of the inverse limit [see (1.5) for the standard description].

Since $\lim_{\leftarrow n}$ is zero for systems satisfying the Mittag-Leffler condition, one has

$$H^i_{\text{cont}}(X, \mathbb{Z}_l(j)) = H^i(X, \mathbb{Z}_l(j)),$$

if the $H^i(X, \mathbb{Z}/l^n(j))$ are finite, in particular, if $X$ is a variety over an algebraically closed field (the "geometric" case). On the other hand we show

$$H^i_{\text{cont}}(\text{Spec } k, \mathbb{Z}_l(j)) = H^i_{\text{cont}}(G_k, \mathbb{Z}_l(j))$$

(continuous group cohomology, see [14, Sect. 2]) for a field $k$ with absolute Galois group $G_k = \text{Gal}(\bar{k}/k)$ (the "arithmetic" case). Connecting both cases, we show that for a variety $X$ over $k$ there is a Hochschild-Serre spectral sequence

$$H^p_{\text{cont}}(G_k, H^q(X \times_k \bar{k}, \mathbb{Z}_l(j))) \Rightarrow H^{p+q}_{\text{cont}}(X, \mathbb{Z}_l(j)),$$

which in general does not exist for the usual $l$-adic cohomology.

The paper is organized as follows. In Sect. 1 we collect the needed facts from homological algebra. In Sect. 2 we compare group cohomology of inverse systems with continuous group cohomology.

In Sect. 3 we introduce the étale cohomology groups $H^i(X, (F_n))$ of inverse systems $(F_n)$ of sheaves on $X_{et}$ and prove some of their major properties, for example the connection with continuous Galois cohomology for $X = \text{Spec } k$, the existence of a Hochschild-Serre spectral sequence, of cohomology with supports and relative cohomology sequences, and of a Gysin morphism. There is also a cycle map

$$c^{X} : CH^i(X) \to H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i))$$

for a smooth variety $X$ over a field $k$ which can be expected to have interesting properties, since it is far away from being trivial on cycles algebraically equivalent to zero, see (6.15).

In Sect. 4 we investigate some "$l$-adic" properties of abelian groups, like $l$-completeness or $l$-divisibility, and relate these to properties of inverse limits. We show that the groups $H^i(X, (F_n))$ are what we call "weakly $l$-complete" groups (called Ext-$l$-complete groups by Bousfield and Kan [1]), if $l^n F_n = 0$ for all $n$, e.g., for $(F_n) = \mathbb{Z}_l(i)$. Such groups behave very well, even if they are huge. For example, there holds a strong Nakayama lemma for them, whose application to the $H^i(X, (F_n))$ gives an extension and sharpening of Tate's results on the structure of continuous group cohomology in [14, Sect. 2].
In Sect. 5 we treat torsion-free \( l \)-adic sheaves \( F \). We introduce their \( \mathbb{Q}_l \)-cohomology by
\[
H^j_{\text{cont}}(X, F \otimes \mathbb{Q}_l) := H^j_{\text{cont}}(X, F) \otimes \mathbb{Q}_l
\]
and construct a long exact sequence
\[
\ldots \rightarrow H^j_{\text{cont}}(X, F) \rightarrow H^j_{\text{cont}}(X, F \otimes \mathbb{Q}_l) \rightarrow H^j(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^{j+1}_{\text{cont}}(X, F) \rightarrow \ldots
\]
with a certain (true) sheaf \( F \otimes \mathbb{Q}_l/\mathbb{Z}_l \) (e.g., \( \mathbb{Z}_l/(i) \otimes \mathbb{Q}_l/\mathbb{Z}_l = \mathbb{Q}_l/\mathbb{Z}_l(i) \)), such that \( \text{Im} \beta \) is the maximal divisible subgroup of its target and \( \text{Im} \delta \) is the torsion group of \( H^{j+1}_{\text{cont}}(X, F) \). This again generalizes results of Tate [14, Sect. 2].

In Sect. 6 we introduce a cupproduct for continuous étale cohomology, which often can be calculated in terms of (0.1) and allows the definition of Chern classes
\[
c_i(E) \in H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i))
\]
of vector bundles \( E \) on a scheme \( X \) (\( l \) invertible on \( X \)).

This paper was started by the search for a spectral sequence (0.3), the following investigation of Tate’s sequence (2.2) in [14], and its explanation by construction (0.1) and the implied sequence (0.2). It is clear from the introduction that the mentioned paper of Tate also influenced this work later on. I also thank Chr. Deninger for bringing my attention to the derived functors of \( \lim \) and the papers by Roos [10, 11].

Notations and Conventions: For an object \( A \) in an abelian category and \( m \in \mathbb{N} \) let \( A/m \) be the kernel and \( A/m \) be the cokernel of \( A \rightarrow A \). Functors are often only described on objects, when the functoriality is clear. The signs \( \mathbb{Z}, \mathbb{Z}/l^m \) etc. denote the abelian groups as well as the associated constant sheaves, \( \mathbb{Z}_l \) also denotes the inverse system of sheaves \( (\mathbb{Z}/l^m) \) on a scheme \( X \).

1. Inverse Systems and Right Derivatives

Let \( \mathcal{A} \) be an abelian category and \( \mathcal{A}^N \) be the category of inverse systems in \( \mathcal{A} \) indexed by the set \( N \) of natural numbers with the natural order. Thus objects in \( \mathcal{A}^N \) are inverse systems
\[
\ldots \rightarrow A_{n+1} \xrightarrow{d_n} A_n \rightarrow \ldots \rightarrow A_2 \xrightarrow{d_1} A_1
\]
in \( \mathcal{A} \) and morphism are commutative diagrams
\[
\begin{array}{ccc}
\ldots & A_{n+1} & A_n & \ldots & A_2 & A_1 \\
& \downarrow k_{n+1} & \downarrow f_n & \downarrow f_2 & \downarrow f_1 \\
\ldots & B_{n+1} & B_1 & \ldots & B_2 & B_1.
\end{array}
\]

Obviously \( \mathcal{A}^N \) is an abelian category, with kernels and cokernels taken “componentwise”.

(1.1) Proposition. a) \( \mathcal{A}^N \) has enough injectives if and only if \( \mathcal{A} \) has enough injectives.

b) \((A_n, d_n)\) is injective if and only if all \( A_n \) are injective and all \( d_n \) are split surjections.
Proof. The forgetful functor

\[
V_m : \mathcal{A}^N \to \mathcal{A}
\]

\[(A_n, d_n) \mapsto A_m
\]

has the exact, faithful left adjoint

\[
U_m : \mathcal{A} \to \mathcal{A}^N
\]

\[A \mapsto \ldots \to 0 \to 0 \to A^\text{id}_I \to A^\text{id}_I \to \ldots \to A.
\]

Therefore if \(U_mA \to I\) is a monomorphism into an injective object of \(\mathcal{A}\), the composition \(A^\text{id}_I \to V_mUA \to V_mI\) is a monomorphism into an injective object of \(\mathcal{A}\).

The exact, faithful forgetful functor

\[
V : \mathcal{A}^N \to \mathcal{A}^{|N|}
\]

\[(A_n, d_n) \mapsto (A_n)
\]

into the product category \(\mathcal{A}^{|N|}\) is left adjoint to

\[
P : \mathcal{A}^{|N|} \to \mathcal{A}^N
\]

\[(A_n) \mapsto \left(\prod_{i=1}^{n} A_i, p_n\right),
\]

where \(p_n : \prod_{i=1}^{n+1} A_i \to \prod_{i=1}^{n} A_i\) is the canonical projection. As above we conclude that \(P\) preserves monomorphisms and injective objects and that \(\mathcal{A}^N\) has enough injectives if \(\mathcal{A}^{|N|}\) has. Now an object \((A_n)\) of \(\mathcal{A}^{|N|}\) is injective if and only if all \(A_n\) are injective in \(\mathcal{A}\), so we get the other half of a).

If \((A_n, d_n)\) is injective, the \(A_m = V_m(A_n, d_n)\) are injective as remarked above, and the monomorphism \((A_n, d_n) \to PV(A_n, d_n)\) given by the adjunction has a left inverse, which immediately gives right inverses for the \(d_n\) by choosing such for the \(p_n\). Conversely, if the \(d_n\) have right inverses \(s_n\), then \((A_n, d_n)\) is isomorphic to \(P(\text{Ker} d_{n-1})\), which is injective, if \(A_n\) and hence \(\text{Ker} d_{n-1}\) is injective for every \(n \in \mathbb{N}\) (here \(d_{-1} = 0\) by definition). This proves b); for an explicit description of \(i\) see (1.5).

A left exact functor \(h : \mathcal{A} \to \mathcal{B}\) into another abelian category induces a left exact functor \(h^N : \mathcal{A}^N \to \mathcal{B}^N\) in the obvious way. If \(\mathcal{A}\) – and therefore \(\mathcal{A}^N\) – has enough injectives, we can define its right derivatives \(R^ih^N\).

(1.2) Proposition. \(R^ih^N = (R^ih)^N\) for \(i \geq 0\), i.e., \(R^ih^N(A_n, d_n) = (R^ihA_n, R^ih(d_n))\).

Proof. If \((A_n, d_n) = A \to I^I\) is an injective resolution in \(\mathcal{A}^N\), \(A_n \to I^I_n\) is an injective resolution for every \(n\), and \(R^ih(d_n)\) is induced by the transition map \(I^I_{n+1} \to I^I_n\).

If inverse limits over \(\mathbb{N}\) exist in \(\mathcal{B}\), we can define the functor

\[
\lim_{\rightarrow n} h : \mathcal{A}^N \to \mathcal{B}
\]

\[(A_n, d_n) \mapsto \lim_{\rightarrow n} (hA_n, h(d_n)),
\]
which by definition is the composition of $h^N$ with the limit functor $\lim_n: \mathcal{B}^N \to \mathcal{B}$. By the left exactness of the latter, $\lim_n h$ is left exact if and only if $h$ is, and we denote its $i$-th right derivative by $R^i(\lim_n h)$. In the following, we often omit the transition maps $d_n$ in the notation. If $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, then if $h$ is left exact and maps injectives to injectives, the same is true for $h^N$ by 1.1, and we have a Grothendieck spectral sequence [6, 2.4.1]

$E_2^{p,q} = \lim_n^p R^q h A_n \Rightarrow E_2^{p+q} = R^{p+q}(\lim_n h)(A_n, d_n),$

where $\lim_n^p$ is the $p$-th right derivative of $\lim_n$.

By Roos [10] the derivatives of $\lim_n$ also exist – at least as universal $\delta$-functors – if $\mathcal{B}$ has the property $(AB4*)$, see [6]: infinite products exist and are exact functors (it was sufficient to consider products over $\mathbb{N}$ in our case). Moreover, in this case $\lim_n^p = 0$ for $p \geq 2$, and there is a functorial exact sequence

$0 \to \lim_n B_n \to \prod_n B_n \xrightarrow{\text{id} - (d_n)} \prod_n B_n \to \lim^1_n B_n \to 0$

for $(B_n, d_n)$ in $\mathcal{B}^N$, since the canonical exact sequence

$0 \to (B_n, d_n) \to PV(B_n, d_n) \to PV(B_{n-1}, d_{n-1}) \to 0$

($B_{-1} = 0$ by definition), given by the commutative exact diagram,

\[
\begin{array}{ccc}
0 & \to & B_{n+1} \\
\downarrow d_n & & \downarrow & \downarrow \\
0 & \to & B_n \\
\end{array}
\]

and

\[
\begin{array}{ccc}
0 & \to & B_{n+1} \\
\downarrow d_n & & \downarrow & \downarrow \\
0 & \to & B_n \\
\end{array}
\]

$(\text{pr}_i$ the projections onto $B_i)$, is a $\lim$-acyclic resolution of $(B_n, d_n)$, if $\mathcal{B}$ satisfies $(AB4*)$, cf. Roos [10].

(1.6) Proposition. If $\mathcal{A}$ has enough injectives, $h: \mathcal{A} \to \mathcal{B}$ is left exact and $\mathcal{B}$ satisfies $(AB4*)$, there are functorial short exact sequences

$0 \to \lim_n^1 R^{i-1} h A_n \to R^i(\lim_n h)(A_n, d_n) \to \lim_n R^i h A_n \to 0$

for $i \geq 0$ (where $R^{-1} h = 0$ by definition and the limits are taken via the maps $R^i h (d_n)$).

Proof. If the $d_n$ have right inverses $s_n$, the last map in the exact sequence

$0 \to \lim_n B_n \to \prod_n B_n \xrightarrow{\text{id} - (d_n)} \prod_n B_n$

is surjective: a section is $(t_n)$ with $t_1 = 0$ and $t_{n+1}$ for $n \geq 1$ recursively defined by $t_{n+1} = s_n(t_n - \text{pr}_n)$, where $\text{pr}_m: \prod_n B_1 \to B_m$ is the canonical projection. In particular, this is
the case for \((hI_m, h(d_n))\) with \((I_m, e_n)\) injective in \(\mathcal{A}^n\). This shows that, under the assumption (AB 4*) for \(\mathcal{B}\), there is a functorial long exact sequence

\[
\ldots \to \prod_n R^{i-1}hA_n \xrightarrow{\text{id} - (R^{i-1}h(d_n))} \prod_n R^{i-1}hA_n \xrightarrow{\delta} R^i\left(\lim_n h\right)(A_m d_n) \\
\to \prod_n R^ihA_n \xrightarrow{\text{id} - (R^ih(d_n))} \prod_n R^ihA_n \xrightarrow{\delta} \ldots
\]

for \((A_m d_n)\) in \(\mathcal{A}^n\) inducing the claimed exact sequences.

(1.7) Remark. The exact sequences of (1.6) agree with those induced by the spectral sequence (1.3) (if it exists) and the vanishing of \(\lim^p\) for \(p \geq 2\), if above for \(\delta\) one takes the negative of the usual connecting morphism. This is also necessary to reobtain (1.4) for \(h\) the identity functor.

For the rest of this section, fix an abelian category \(\mathcal{A}\) with enough injectives and a left exact functor \(h: \mathcal{A} \to \mathcal{B}\) into another abelian category \(\mathcal{B}\), in which projective limits over \(\mathbb{N}\) exist. For \(A = (A_m d_n)\) in \(\mathcal{A}^N\) define \(A[m] = (A_{m+n} d_{n+m})\) for each \(m \geq 0\); then there are canonical maps \(A[m] \to A\) by the composition of the transition maps.

(1.8) Lemma. a) The morphism \(A[1] \to A\) induces isomorphisms

\[
R^i\left(\lim_n h\right)A[1] \cong R^i\left(\lim_n h\right)A
\]

for \(i \geq 0\).

b) If \(\mathcal{B}\) has enough injectives or satisfies (AB 4*), the morphism \(B[1] \to B\) induces isomorphisms

\[
\lim_n^p B[1] \cong \lim_n^p B
\]

for \(p \geq 0\) and \(B \in \text{Ob}(\mathcal{B}^N)\).

Proof. a) If \(A \subset I\) is an injective resolution, \(A[1] \subset I[1]\) is one, too, and we obtain a commutative diagram

\[
\begin{array}{ccc}
\text{can} \downarrow & & \text{can} \\
A & \subset & I
\end{array}
\]

The claim follows as \(\lim_n hI[1] \cong \lim_n hI\).

b) The first case follows from a) with \(h\) the identical functor. The second follows similarly, as \(\lim^1\) is effacable and the statement is obviously true for \(\lim\).

(1.9) Corollary. If \(A[m] \to A\) is zero for some \(m \geq 0\), i.e., \(A\) is AR-zero, cf. [SGA 5, V 2.2], then \(R^i\left(\lim_n h\right)A = 0\) for all \(i \geq 0\).

We want a slight generalization of this.
(1.10) **Definition.** Call a system \((A_n, d_n)\) **ML-zero** (Mittag-Leffler zero) if for any \(n \geq 1\) there is an \(m = m(n) > 0\) such that the transition map \(C_{n+m} \to C_n\) is zero.

Note that \((A_n, d_n)\) is AR-zero, if one can take on \(m\) for all \(n\).

(1.11) **Lemma.** If \(A = (A_n, d_n)\) is ML-zero, then \(R^i\left(\lim_{n} h\right)(A) = 0\) for all \(i \geq 0\).

**Proof.** By assumption there is a cofinal set \(J = \{n_1, n_2, n_3, \ldots\} \subseteq \mathbb{N}\) such that all transition maps in the inverse system \((A_{n_j}, d'_j)_{j \in \mathbb{N}}\), where the \(d'_j\) are obtained by composition of the \(d_n\), are zero. Now \(\lim_{n} h\) is also the composition of the exact forgetful functor

\[
V_j: \mathcal{A}^N \to \mathcal{A}^N, \quad (A_n, d_n) \mapsto (A_{n_j}, d'_j),
\]

which preserves injective objects, with \(\lim_{n} h\). The spectral sequence therefore reduces the question to the case where all \(d_n\) are zero, which is solved by (1.9).

By similar arguments, the statement of Lemma (1.11) is also true for \(\lim\) and \(\lim^1\) on \(\mathcal{A}\) satisfying just \((AB 4^*)\).

(1.12) **Lemma.** The objects in \(\mathcal{A}^N\) which are ML-zero, form a Serre subcategory of \(\mathcal{A}^N\).

**Proof.** It is clear that subobjects and quotients inherit the property, and for an exact sequence \(0 \to A \to B \to C \to 0\) with \(A\) and \(C\) ML-zero and \(m_A(n)\) and \(m_C(n)\) choosen for \(A\) and \(C\) as in Definition (1.10), we see that \(m_A(n) + m_C(n + m_A(n))\) works for \(B\).

We use the prefix ML, if something holds in the quotient category, i.e., up to systems being ML-zero. For example a complex \(A \Rightarrow B \Rightarrow C \Rightarrow 0\) in \(\mathcal{A}^N\) is ML-exact, if its homology \(\ker \beta / \text{Im} \alpha\) is ML-zero, and a system \((A_n, d_n)\) is called (ML)-\(l\)-adic for a prime \(l\), if it is ML-isomorphic to an \(l\)-adic system \((A'_n, d'_n)\), i.e., one with \(A'_{n+1}/l^n A'_n \xrightarrow{d'_n} A'_n\). The functor \(h^N\) respects ML-zero systems, and by Lemma (1.10) the \(R^i\left(\lim_{n} h\right)\) factorize through the quotient category. In particular we note

(1.13) **Corollary.** If \(0 \to A \to B \to C \to 0\) is ML-exact in \(\mathcal{A}^N\), there is a long exact sequence

\[
\ldots \to R^i\left(\lim_{n} h\right) A \to R^i\left(\lim_{n} h\right) B \to R^i\left(\lim_{n} h\right) C \to R^{i+1}\left(\lim_{n} h\right) A \to \ldots
\]

(1.14) **Remark.** Recall that a system \((A_n, d_n)\) is said to satisfy the Mittag-Leffler condition, if for each \(n\) the image of the transition maps \(A_{n+m} \to A_n\) is constant for \(m > 0\). It is easy to see that this is the case if and only if \((A_n, d_n)\) is ML-isomorphic to a system \((A'_n, d'_n)\) with surjective \(d'_n\). The next lemma is well known (see for example [10, Proposition 1]).

(1.15) **Lemma.** If \(\mathcal{B}\) satisfies \((AB 4^*)\) and \(B \in \text{Ob}(\mathcal{B}^N)\) satisfies the Mittag-Leffler condition, then \(\lim_{n} B_n = 0\).
Examples of categories with \((AB\,4^*)\) are the category \(\text{Ab}\) of abelian groups and the category \(\text{Mod}_R\) of \(R\)-modules over an arbitrary ring \(R\). \((AB\,4^*)\) does not hold i.e. for the category of discrete \(G\)-modules for a profinite group \(G\) or more generally for the category of sheaves on a Grothendieck topology, see Roos [11].

(1.16) Lemma. If for \(A = (A_n, d_n)\) in \(\mathcal{A}^N\) all \(A_n\) have injective resolutions of length \(N\), then \(A\) has an injective resolution of length \(N + 1\); in particular \(R^i\left(\lim_n h\right)A = 0\) for \(i > N + 1\).

Proof. If \(V(A_n, d_n) \subseteq \mathcal{I}\) is an injective resolution, then \(PV(A_n, d_n) \subseteq PI^*\) is an injective resolution, so the claim follows with the exact sequence (1.5). Note that the indicated consequence holds even when neither (1.3) nor 1.6 can be applied.

(1.17) Remark. a) The functor \(\lim_n \mathcal{B}^N \to \mathcal{B}\) preserves injectives, since it is right adjoint to the constant functor \(\mathcal{B} \to \mathcal{B}^N, B \to \langle B \rangle\), which is exact and faithful.

b) For any \(m \in \mathbb{N}\) there is a unique morphism of \(\delta\)-functors \(\pi_m : R^i\left(\lim_n h\right)A \to R^i h V_m(A_n, d_n) = R^i h A_m\)

coincides with the canonical projection \(\lim_n hA_n \to hA_m\) for \(i = 0\). The \(\pi_m^i\) are compatible for different \(m\), and the induced morphisms

\[
R^i\left(\lim_n h\right)A \to \lim_n R^i h A_n
\]

agree with those occurring in (1.3) or (1.6).

c) If \(\mathcal{A}_1 \hookrightarrow \mathcal{A}_2\) and \(\mathcal{A}_2 \hookrightarrow \mathcal{A}_1\) are adjoint functors, then \(\mathcal{H}^N\) and \(\mathcal{K}^N\) are adjoint in an obvious way: this holds in general for functor categories \(\text{Hom}(I^0, \mathcal{A}_i), \mathbb{N}\) being replaced by any small category \(I\).

While the considerations of Mittag-Leffler properties are specific for the case \(I = \mathbb{N}\) (made into a category as usual, with morphisms \(j \leq i\)), let us mention how the other results carry over to the case of arbitrary small categories \(I\).

(1.18) Proposition. Let \(\mathcal{A}\) be an abelian category.

a) If \(\mathcal{A}\) has enough injectives and possesses products indexed by the sets \(I/i = \{\text{morphisms } j \to i \text{ in } I\}\) for all \(i \in I\), then \(\mathcal{A}^I = \text{Hom}(I^0, \mathcal{A})\) has enough injectives.

b) If \(\mathcal{A}\) possesses arbitrary products, then \(\lim_I : \mathcal{A}^I \to \mathcal{A}\), the right adjoint of the constant functor \(\mathcal{A} \to \mathcal{A}^I\), exists, is left exact, and maps injectives to injectives.

The proofs are similar, the functor \(P\) in (1.1.4) now becomes

\[
P : (A_i)_{i \in I} \mapsto \left\{ i \mapsto \prod_{j \mapsto i} A_j \right\},
\]

where for \(i \mapsto i'\) the morphism \(\prod_{j \mapsto i'} \to \prod_{j \mapsto i} A_j\) has components \(\text{pr}_{j \mapsto i, i'} : \prod_{j \mapsto i'} A_j \to A_j\) for \(j \mapsto i\). Proposition (1.2) holds similarly, and for \(\mathcal{A}, \mathcal{B}\) and \(h\) with the properties
assumed there, (1.3) becomes
\[ E_2^{p,q} = \lim_{\rightarrow} R^q h A_i \Rightarrow R^{p+q} \left( \lim_{\rightarrow} h \right) (A_i), \]
where as before we have suppressed the transition maps in the notation. For the description of \( \lim_{\rightarrow} \) in categories satisfying \((AB4^*)\) we refer to the paper of Roos [10].

2. Connections with Continuous Group Cohomology

Let \( G \) be a profinite group, then the category \( M(G) \) of discrete \( G \)-modules is an abelian category with enough injectives, and we let\[ (M_n, d_n) \mapsto H^i(G, (M_n, d_n)) \]
be the \( i \)-th right derived functor of the left exact functor\[ M(G) \rightarrow Ab \]
\[ (M_n, d_n) \mapsto H^0(G, (M_n, d_n)) = \lim_{\rightarrow} H^0(G, M_n). \]
Then by 1.6 we have short exact sequences
\[ 0 \rightarrow \lim_{\rightarrow} H^{i-1}(G, M_n) \rightarrow H^i(G, (M_n, d_n)) \rightarrow \lim_{\rightarrow} H^i(G, M_n) \rightarrow 0 \]
for each \( i \geq 0 \).

(2.2) **Theorem.** Let \( (T_n, d_n) \) be an inverse system of discrete \( G \)-modules satisfying the Mittag-Leffler condition and let \( T = \lim_{\rightarrow} T_n \) as a topological \( G \)-module. Then there are canonical isomorphisms\[ H^i_{cont}(G, T) \cong H^i(G, (T_n, d_n)) \]
for \( i \geq 0 \), functorial in \( (T_n, d_n) \), where \( H^i_{cont}(G, T) \) denote the continuous cohomology groups defined by Tate in [14]. If \( 0 \rightarrow (R_n) \rightarrow (S_n) \rightarrow (T_n) \rightarrow 0 \) is a ML-exact sequence of Mittag-Leffler systems with limits \( R, S \) and \( T \), respectively, the above isomorphisms induce an isomorphism of long exact sequences
\[
\begin{align*}
\cdots & \rightarrow H^i(G, (R_n)) \rightarrow H^i(G, (S_n)) \rightarrow H^i(G, (T_n)) \rightarrow H^{i+1}(G, (R_n)) \rightarrow \cdots \\
& \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots & \rightarrow H^i_{cont}(G, R) \rightarrow H^i_{cont}(G, S) \rightarrow H^i_{cont}(G, T) \rightarrow H^{i+1}_{cont}(G, R) \rightarrow \cdots,
\end{align*}
\]
the lower one associated to the exact sequence \( 0 \rightarrow R \rightarrow S \rightarrow T \rightarrow 0 \) (Note that Tate’s existence criterion is fulfilled).

**Proof.** We may assume that all \( d_n \) are surjective. By definition \( H^i_{cont}(G, T) \) is the \( i \)-th homology group of the complex\[ C(G, T) \]
of continuous cochains in \( T \), and by the definition of the topology of \( T \), this is the inverse limit of the complexes \( C(G, T_n) \) of continuous cochains in \( T_n \), where the \( T_n \)
have the discrete topology. It is well-known (using the homogeneous bar resolution, see [4, 1. and 2.]), that \( C'_{(G, T_n)} \) is the complex of fixed modules of a canonical complex

\[
D'(G, T_n)
\]

of discrete \( G \)-modules, which is an acyclic resolution of the discrete \( G \)-module \( T_n \). Moreover, this resolution is functorial, and we get a resolution

\[
(T_n, d_n) \prec (D^1(G, T_n), d^1_n) \rightarrow (D^2(G, T_n), d^2_n) \rightarrow \ldots
\]

of \( (T_n, d_n) \) in \( M(G)^N \), from which \( H_i^{\text{cont}}(G, T) \) is obtained by applying \( \lim_n H^0(G, -) \) and taking homology. So we only have to show that the systems \( (D^i(G, T_n), d^i_n) \) are acyclic for \( \lim_n H^0(G, -) \), which follows at once from (2.1) and (1.15), since the

\[
C^i(G, T_n) = H^0(G, D^i(G, T_n))
\]

form a Mittag-Leffler system. The functoriality is obvious.

(2.3) **Remark.** The theorem in particular applies to the case of finitely generated \( \mathbb{Z}_l \)-modules \( T \) with continuous \( G \)-action and the inverse system of the \( T_n = T/l^n T \). Note that for an exact sequence

\[
0 \rightarrow R \rightarrow S \rightarrow T \rightarrow 0
\]

of \( \mathbb{Z}_l \)-modules, the sequence

\[
0 \rightarrow (R/l^n R) \rightarrow (S/l^n S) \rightarrow (T/l^n T) \rightarrow 0
\]

is \( ML \)-exact.

3. Continuous Étale Cohomology

Let \( X \) be a scheme and \( (F_n, d_n) \) be an inverse system of sheaves on the small étale site \( X_{\text{et}} \) of \( X \), then we define

\[
H^i(X, (F_n, d_n)) = R^i \left( \lim_n \Gamma \right) (F_n, d_n),
\]

for \( i \geq 0 \), where \( \Gamma = H^0(X, -) \) is the section functor on the category \( S(X_{\text{et}}) \) of sheaves on \( X_{\text{et}} \), i.e., \( H^i(X, -) \) is the \( i \)-th derivative of the functor \( S(X_{\text{et}})^N \rightarrow \text{Ab} \), \( (F_n, d_n) \mapsto \lim_n H^0(X, F_n) \). By (1.6) we have short exact sequences

\[
0 \rightarrow \lim_n H^{i-1}(X, F_n) \rightarrow H^i(X, (F_n, d_n)) \rightarrow \lim_n H^i(X, F_n) \rightarrow 0.
\]

For example this applies to an étale \( l \)-adic sheaf \( F = (F_n) \) on \( X \) for a prime \( l \) (mostly assumed to be invertible on \( X \)), and we call

\[
H^i_{\text{cont}}(X, F) := H^i(X, (F_n))
\]

the continuous cohomology groups of \( F \). In particular we have continuous \( l \)-adic cohomology groups \( H^i_{\text{cont}}(X, \mathbb{Z}(j)) \) associated to the sheaves \( \mathbb{Z}(j) = (\mathbb{Z}/l^n(\mathbb{Z})) \) (for a definition of these sheaves and the Tate twist \( F(j) \) in general see for example [9, pp. 163, 164]). The name is justified by

(3.2) **Theorem.** Let \( X = \text{Spec} K \) for a field \( K \) with separable closure \( K_s \) and let \( F = (F_n) \) be a Mittag-Leffler system of sheaves on \( X_{\text{et}} \) (for example an \( l \)-adic sheaf).
Let \( F_\mathfrak{x} = \lim \limits_{\leftarrow n} (F_n)_\mathfrak{x} \) be the inverse limit of the stalks at the geometric point \( \mathfrak{x} = \text{Spec} K_s \) of \( X \), considered as a topological group with continuous action of \( G_K = \text{Gal}(K_s/K) \). Then there are canonical isomorphisms for \( i \geq 0 \)

\[
H^i(X, (F_n)) \cong H^i_{\text{cont}}(G_K, F_\mathfrak{x}),
\]

functorial in \( F \) and with respect to short ML-exact sequences.

**Proof.** Since the functor \( \text{G} \to \text{G}_x \) gives an equivalence between \( S(X_{et}) \) and \( M(G_\mathfrak{x}) \), and \( \Gamma(X, G) = G^G_\mathfrak{x} \), the statement immediately follows from the definitions and Theorem 2.2.

(3.3) **Theorem.** Let \( \pi : X' \to X \) be a possibly infinite Galois covering of schemes with Galois group \( G \) and let \( (F_n) \) be a system of sheaves on \( X_{et} \). Then there is a spectral sequence

\[
E^{p, q}_2 = H^p(G, (H^q(X', \pi^*F_n))) \Rightarrow E^{p+q}_\infty = H^{p+q}(X, (F_n)).
\]

**Proof.** This is just the Grothendieck spectral sequence for the composition of functors

\[
S(X_{et})^N \to M(G)^N \to Ab \quad (F_n) \mapsto (\Gamma(X', \pi^*F_n)) \quad (M_n) \mapsto (\lim M_n)^G = \lim M_n^G
\]

since \( \Gamma(X', \pi^*F_n)^G = \Gamma(X, F_n) \).

(3.4) **Corollary.** Let \( X = \text{a scheme of finite type over a field } K \) with separable closure \( K_s \), let \( \bar{X} = X \times_K K_s \) and \( G_K = \text{Gal}(K_s/K) \). If \( (F_n) \) is an inverse system of sheaves on \( X_{et} \) such that \( H^i(\bar{X}, F_n) \) is finite for every \( n \) and \( i \), then there is a spectral sequence

\[
E^{p, q}_2 = H^p_{\text{cont}}(G_K, \lim H^q(\bar{X}, F_n)) \Rightarrow E^{p+q}_\infty = H^{p+q}(X, (F_n)).
\]

**Proof.** If the \( H^i(\bar{X}, F_n) \) are finite, then the system \( (H^i(\bar{X}, F_n)) \) satisfies the Mittag-Leffler condition, and we can apply (2.2).

(3.5) **Remark.**

a) The corollary applies, if \( X \) is proper over \( K \) and the \( F_n \) are constructible (by the proper base change theorem, see [9, VI 2.1]).

b) By Deligne's finiteness theorem, cf. [SGA 4\_1, p. 236], we get another example: if \( X \) is of finite type over \( K \) and \( l \neq \text{char} K \), then there are spectral sequences

\[
H^p_{\text{cont}}(G_K, H^q(\bar{X}, \mathbb{Z}_l(j))) \Rightarrow H^{p+q}_{\text{cont}}(X, \mathbb{Z}_l(j))
\]

where \( H^q(\bar{X}, \mathbb{Z}_l(j)) = \lim H^q(\bar{X}, \mathbb{Z}_l^l(j)) \) and this group has the l-adic topology by (4.5) below and Jouanolou's theorem, cf. [SGA 5, V 5.3.1].

c) If moreover \( G_K \) has finite cohomology for all finite \( l \)-torsion \( G_K \)-modules (for example if \( K \) is finite or a local field), then there are isomorphisms

\[
H^i_{\text{cont}}(G_K, H^q(\bar{X}, \mathbb{Z}_l(j))) \cong \lim H^i(G_K, H^q(\bar{X}, \mathbb{Z}_l^l(j)))
\]

\[
H^i_{\text{cont}}(X, \mathbb{Z}_l(j)) \cong \lim H^i(X, \mathbb{Z}_l^l(j))
\]
by (1.15), (2.1), and (3.1), because the $H^i(X, \mathbb{Z}/l^m(j))$ are finite (by the finiteness of the $H^i(X, \mathbb{Z}/l^m(j))$ and the Hochschild-Serre spectral sequence—note that this argument and especially Corollary 2.8 in [9, VI] is false without the assumption on $G_K$). Then the spectral sequence in b) is just the limit of the Hochschild-Serre spectral sequences for $\mathbb{Z}/l^m(j)$.

Much of the formalism of étale cohomology carries over to the groups $H^i(X, (F_n, d_n))$; on the other hand the cohomology $H^i(X, F)$ of a sheaf $F$ can be identified with the cohomology of the constant system $F = (F_n = F, d_n = \text{id}_F)$.

For a closed subscheme $Z \hookrightarrow X$ and an inverse system $(F_n)$ of sheaves on $X_{et}$ we have the cohomology groups with support

$$H^i_Z(X, (F_n)) := R^i \left( \lim_{\to n} \Gamma_Z(X, -) \right)(F_n),$$

where $\Gamma_Z(X, F) = \ker(\Gamma(X, F) \rightarrow \Gamma(U, F))$ for $U = X - Z$, and $-\text{ since the sequence of functors}$

$$0 \to \lim_{\to n} \Gamma_Z(X, -) \to \lim_{\to n} \Gamma(X, -) \to \lim_{\to n} \Gamma(U, -) \to 0$$

is exact on injectives—a long exact sequence

$$(3.6) \quad \cdots \to H^{i-1}(U, (F_n)) \to H^i_Z(X, (F_n)) \to H^i(X, (F_n)) \to H^i(U, (F_n)) \to \cdots$$

There is also a cohomology with compact support for a scheme $X$ separated and of finite type over a field $k$ by setting

$$H_c(X, (F_n)) := H^i(X, ((F_n)))$$

for a compactification $\overline{X} \to X$, and a long exact sequence

$$(3.7) \quad \cdots \to H^{i-1}_c(Z, (F_n)) \to H^i_c(U, (F_n)) \to H^i_c(X, (F_n)) \to H^i_c(Z, (F_n)) \to \cdots$$

for $X, Z, U, and (F_n)$ as above (same argument as in [9, III 1.30]).

The main difference to usual étale cohomology is that $H^i(X, (F_n))$ in general does not commute with inverse limits in $X$ and direct limits in $(F_n)$. However, we still have

$$(3.8) \text{ Proposition. Let } z \text{ be a closed point of } X \text{ and } \mathcal{O}_{X, z}^h \text{ be Henselization of the local ring at } z. \text{ Then}$$

$$H^i_Z(X, (F_n)) \cong H^i_Z(\text{Spec } \mathcal{O}_{X, z}^h, (F_n))$$

for any inverse system $(F_n)$ of sheaves on $X_{et}$.

Proof. The morphisms $\text{Spec } \mathcal{O}_{X, z}^h \to X$ induces a commutative exact diagram

$$0 \to \lim_{\to n} H^{i-1}_Z(\text{Spec } \mathcal{O}_{X, z}^h, F_n) \to H^i_Z(\text{Spec } \mathcal{O}_{X, z}^h, F_n) \to \lim_{\to n} H^i_Z(\text{Spec } \mathcal{O}_{X, z}^h, F_n) \to 0$$

$$\alpha \downarrow \quad \downarrow \beta \quad \downarrow \gamma$$

$$0 \to \lim_{\to n} H^{i-1}(X, F_n) \quad \to \quad H^i(X, (F_n)) \quad \to \quad \lim_{\to n} H^i(X, F_n) \quad \to \quad 0$$

in which $\alpha$ and $\beta$ are isomorphisms [9, III 1.28].
Since inverse limits exist in $S(X_{et})$, we can define relative versions of all these functors and we get many Grothendieck spectral sequences by (1.17) and the fact that $H^0(X_{et}, -)$ and $\pi_*^N$ for a morphism $\pi: X \to Y$ commute with inverse limits. For example, we already obtained (3.1) from $\lim H^0(X, -) = H^0(Y, -)^N$, and

$$H^p \left( X, \lim_n F_n \right) \Rightarrow H^{p+q}(X, (F_n)).$$

(3.9)

Note however, that $\lim^n F_n$ does in general not vanish for $n \geq 2$, see [11]. From $\lim H^0(X, -) = \lim H^0(Y, -) \circ \pi_*^N$ and $\lim \pi_* = \lim \circ \lim$ we get

$$\lim \pi_* \circ \lim \pi_*^N = \lim \pi_* \circ \lim \pi_*^N,$$

(3.10)

$$H^p (Y, (R^n \pi_* F_n)) \Rightarrow H^{p+q}(X, (F_n)).$$

(3.11)

As for the computation we have

(3.12) **Lemma.** For a morphism $\pi: X \to Y$ and a system $(F_n)$ of sheaves on $X_{et}$, $R^i \left( \lim \pi_* \right) (F_n)$ is the sheaf associated to the pre-sheaf $V \mapsto H^i(V \times_X Y, (F_n)_{|V \times_X Y})$ on $Y_{et}$. In particular, $\lim^q F_n$ is the sheaf associated to the presheaf $U \mapsto H^q(U, (F_n))$ on $X_{et}$.

**Proof.** Let $i: S(X_{et}) \to P(X_{et})$ be the embedding into the category of presheaves on $X_{et}$, and $\alpha: P(X_{et}) \to S(X_{et})$ be the left-adjoint, $\alpha(P) = $ associated sheaf. With the canonical map $\pi_p: P(X_{et}) \to P(Y_{et})$ we have $\pi_* = \alpha \pi_i$. Let $(F_n) = F \otimes I^*$ be an injective resolution of $F$ in $S(X_{et})$. Then

$$R^i \left( \lim \pi_* \right) (F_n) = \mathcal{H}^i \left( \lim \pi_*^N I^* \right) = \mathcal{H}^i \left( \pi_* \lim I^* \right) = \alpha \pi_i \mathcal{H}^i \left( \lim I^* \right),$$

as $\pi_*^N$ commutes with $\lim$, and by definition $\mathcal{H}^i \left( \lim I^* \right)$ is the presheaf $U \mapsto H^p(U, (F_n))$ on $X_{et}$.

Like in any abelian category with enough injectives, the Ext-functors $\mathrm{Ext}_X^i((G_n, g_n), -)$ on $S(X_{et})^N$ are the $i$-th right derivatives of the left exact functor $S(X_{et})^N \to \text{Ab}$ given by

$$(F_n, f_n) \mapsto \text{Hom}_X((G_n, g_n), (F_n, f_n)).$$

(3.13) **Proposition.** There is a long exact sequence

$$\cdots \to \mathrm{Ext}_X^i((G_n, g_n), (F_n, f_n)) \to \prod_n \mathrm{Ext}_X^i(G_n, F_n) \xrightarrow{\alpha} \prod_n \mathrm{Ext}_X^i(G_{n+1}, F_n) \to \cdots,$$

$$\alpha = (g_n^*) - (f_n^*),$$

which is functorial in $(G_n, g_n)$ and $(F_n, f_n)$.
Proof. There is a functorial exact sequence
\[
0 \to \operatorname{Hom}_X((G_n, g_n), (F_n, f_n)) \to \prod_n \operatorname{Hom}_X(G_n, F_n) \to \prod_n \operatorname{Hom}_X(G_{n+1}, F_n)
\]
in which the last map is surjective for injective \((F_n, f_n)\): if the \(f_n\) have right inverses \(s_n\), a preimage for \((v_n) \in \prod_n \operatorname{Hom}_X((G_{n+1}, F_n)\) is \((u_n)\) with \(u_1 = 0\) and recursively \(u_{n+1} = s_n(u_n g_n - u_n)\). This implies the claim by standard arguments in homological algebra, as \(AB\) satisfies \((AB 4^*)\).

\((3.14)\) Remark. There are canonical isomorphism \(\operatorname{Ext}^i_X(\mathbb{Z}, (F_n, f_n)) \cong H^i(X, (F_n, f_n))\), by extension of the obvious one for \(i = 0\) to the derived functors. Via these and the analogous isomorphisms \(\operatorname{Ext}^i_X(\mathbb{Z}, F_n) \cong H^i(X, F_n)\), the sequence in \((3.13)\) for \((G_n, g_n) = \mathbb{Z}\) coincides with the sequence constructed in the proof of \((1.6)\).

Whenever purity or semi-purity holds for a closed immersion \(i: Z \hookrightarrow X\) (cf. [SGA 5, 3.1.4], [SGA 4\(\Xi\), (cycle) 2.2]), it can be extended to our setting – in the absolute case by the spectral sequence
\[
E_2^{p, q} = H^n(Z, (R^q i_* F_n)) \Rightarrow H^{p+q}(X, (F_n))
\]
and the short exact sequences
\[
0 \to \lim_n H^{i-1}_Z(X, F_n) \to H^i_Z(X, (F_n)) \to \lim_n H^i(X, F_n) \to 0
\]
for a pro-sheaf \((F_n)\) on \(X\). The next theorem gives an example.

\((3.15)\) Theorem. Let \(S\) be a scheme and \((Z, X)\) be a smooth \(S\)-pair of codimension \(c\), i.e., a closed immersion \(i: Z \hookrightarrow X\) of smooth \(S\)-schemes such that each fibre over \(S\) has codimension \(c\). Let \((F_n)\) be an inverse system of locally constant torsion sheaves on \(X_{et}\) with torsion prime to \(\operatorname{char}(S)\).

Then there are canonical isomorphisms for \(j \in \mathbb{Z}\)
\[
H^j(Z, (R^{2c + j} i_* F_n)) \cong H^{j+2c}(X, (F_n)).
\]
If \(F_n\) is annihilated by \(l^n\), \(l\) a prime invertible on \(S\), there is a canonical isomorphism of inverse systems of sheaves
\[
(R^{2c + j} i_* F_n) \cong (i^* F_n \otimes \mathbb{Z} R^{2c + j} \mathbb{Z}/l^n).
\]
Moreover, there is a canonical isomorphism of \(l\)-adic sheaves
\[
R^{2c + j} i_* \mathbb{Z}(c) = (R^{2c + j} \mu_{l^n}) \cong (\mathbb{Z}/l^n) = \mathbb{Z}_l
\]
given by the local cycle class, thus by combining \((3.18)\), \((3.19)\), and \((3.20)\) one has the Gysin isomorphisms
\[
H^j(Z, (i^* F_n(-c)) \cong H^{j+2c}(X, (F_n)), \quad j \in \mathbb{Z},
\]
and, by composing with \(H^{j+2c}(X, (F_n)) \to H^{j+2c}(X, (F_n))\), the Gysin morphisms
\[
i_* : H^j(Z, (i^* F_n(-c))) \to H^{j+2c}(X, (F_n)), \quad j \in \mathbb{Z}.
\]
Proof. \((3.18)\) follows from \((3.15)\), since \(R^q i_* F_n = 0\) for \(q \geq 2c\) see [SGA 4, XVI 3.7]. For \((3.19)\) and \((3.20)\) see [SGA 4, XV 3.8, 3.10], and [SGA 4\(\Xi\), (cycle) 2.2]; one only has to check that the isomorphisms there, of which the second one is given by the
local cycle (or fundamental) class (loc. cit.), are compatible with the (obvious)
transition maps. The noetherian assumption in [SGA 4\frac{1}{2}] can be avoided by using
[SGA 4, XVI 3.8.1], the separatedness by localization.
As usual there are analogs of (3.15), (3.18), (3.21), and (3.22) with supports in a
closed subscheme \( W \) of \( Z \).
We end this section by defining a global cycle class in the continuous l-adic
cohomology. It will be clear from the construction that this can be done whenever
one has a cycle class in the usual l-adic cohomology, but to fix ideas we stick to
varieties over a field \( k \).
(3.23) Theorem. Let \( X \) be a smooth variety over a field \( k \) and let \( \ell = \text{char}(k) \)
be a prime. If \( Z \) is a closed subscheme of codimension \( c \), then
\[
H^i_{Z, \text{cont}}(X, \mathbb{Z}(j)) = 0 \quad \text{for } i < 2c \quad \text{and} \quad j \in \mathbb{Z},
\]
\[
H^{2c}_{Z, \text{cont}}(X, \mathbb{Z}(j)) \simeq \lim_{\rightarrow} H^{2c}_{Z}(X, \mathbb{Z}/\ell^m(j))
\]
In particular, for \( Z \) irreducible and reduced there is a canonical cycle class (with
support)
\[
\text{cl}(Z) = \text{cl}^X_2(Z) \in H^{2c}_{Z, \text{cont}}(X, \mathbb{Z}(c)),
\]
which is the "limit" of the cycle classes \( \text{cl}(Z) \in H^{2c}_{Z}(X, \mathbb{Z}/\ell^m(c)) \) defined in [SGA 4\frac{1}{2},
cycle 2.2.10]. The global cycle class \( \text{cl}^X(Z) \) is defined as the image of \( \text{cl}^X_2(Z) \) in
\( H^{2c}_{Z, \text{cont}}(X, \mathbb{Z}(c)) \).
Proof. The first statement follows from (3.16) and the vanishing of \( H^i_Z(X, \mathbb{Z}/\ell^m(j)) \)
for \( i < 2c \), see [SGA 4\frac{1}{2}, (cycle) 2.2.8]. The rest is clear, since \( \text{cl}^{n+1}_Z(Z) \) is mapped to
\( \text{cl}_Z(Z) \) under the canonical morphism \( H^{2c}_Z(X, \mathbb{Z}/\ell^{n+1}(c)) \to H^{2c}_Z(X, \mathbb{Z}/\ell^m(c)) \).
(3.24) Remark. If \( Z \) is a smooth prime cycle, then, directly by definition, \( \text{cl}^X_2(Z) \) and
\( \text{cl}^X(Z) \) are the images of \( 1 \in H^0(Z, \mathbb{Z}) \) under the Gysin isomorphism and
morphism, respectively. In particular, if \( k \) is perfect, then for any prime cycle \( Z \) of
codimension \( c \) there is a canonical isomorphism
\[
H^{2c}_{Z, \text{cont}}(X, \mathbb{Z}(c)) \cong \mathbb{Z}_t
\]
such that \( \text{cl}(Z) \) is the image of \( 1 \) (use excision as in [6, VI, pp. 268, 269]).
For a cycle \( Z = \sum n_j Z_j \) of codimension \( c \), with irreducible and reduced \( Z_j \), we let
as usual
\[
\text{cl}(Z) = \sum n_j \text{cl}(Z_j) \in H^{2c}_{Z, \text{cont}}(X, \mathbb{Z}(c)) \cong \bigoplus_j H^{2c}_{Z_j, \text{cont}}(X, \mathbb{Z}(c))
\]
and
\[
\text{cl}^X(Z) = \sum n_j \text{cl}^X(Z_j).
\]
Then we have the following functorial properties [see also (6.13) below].
(3.25) Proposition. Let \( f : Y \to X \) be a morphism of smooth varieties over the field \( k \).
\[\begin{align*}
a) \text{ If } Z \text{ is a cycle on } X \text{ and } f^*Z \text{ is defined, then } \text{cl}^Y(f^*Z) &= f^*\text{cl}^X(Z). \\
b) \text{ If } f \text{ is a closed immersion and } W \text{ is a cycle on } Y, \text{ then } \text{cl}^X(W) &= f_*\text{cl}^Y(W), \text{ where } f_* \text{ is the Gysin map.}
\end{align*}\]
Proof. This is implied by the corresponding statements for the cycle classes with support, which follow by passing to the limit from the properties of the classes $\text{cl}_n(Z)$ proved in [SGA 4 1/2, (cycle) 2.3].

(3.26) Lemma. For any scheme $X$ on which $l$ is invertible let

$$\delta = \delta_X : \text{Pic}(X) = H^1(X, \mathcal{G}_m) \to H^2_{\text{cont}}(X, \mathbb{Z}_l(1))$$

be the connecting morphism for the exact sequence

\[
\begin{array}{c}
0 \to \mu_{l^n} \to \mathcal{G}_m \xrightarrow{r} \mathcal{G}_m \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \text{id} \\
0 \to \mu_{l^n} \to \mathcal{G}_m \xrightarrow{r} \mathcal{G}_m \to 0 \\
\end{array}
\]

(3.27)

of inverse systems of sheaves on $X$ (called the Kummer sequence).

a) If $i : Z \hookrightarrow X$ is a smooth $S$-pair of codimension 1 for some scheme $S$, defined by the invertible ideal $\mathcal{J} \subseteq \mathcal{O}_X$, then $\delta_x(-[\mathcal{J}]) = i_*([\mathcal{J}])$, where $[\mathcal{J}]$ is the class of $\mathcal{J}$ in $\text{Pic}(X)$ and $i_* : H^0_{\text{cont}}(X, \mathbb{Z}_l(1)) \to H^2_{\text{cont}}(X, \mathbb{Z}_l(1))$ is the Gysin morphism.

b) If $X$ is smooth over a field and $D$ is a divisor, then $\delta_x(\mathcal{O}_X(D)) = \text{cl}_X(D)$.

Proof. Since $J = \mathcal{O}(-D)$ for the associated Cartier divisor $D$ in a), this follows from [SGA 4 1/2, (cycle) 2.1.2] and the commutative diagram

\[
\begin{array}{ccc}
\text{cl}(D) & \xrightarrow{\delta} & H^2_{\text{cont}}(X, \mathcal{G}_m) \\
\downarrow \quad \downarrow \text{id} & & \downarrow \text{id} \\
\text{cl}(D) & \xrightarrow{\delta} & H^2_{\text{cont}}(X, \mathbb{Z}_l(1)) \\
\end{array}
\]

for any Cartier divisor $D$ in $X$, where the $\delta$ above is the connecting morphism for the sequence (\#).

(3.28) Generalization. Let $I, J$ be small categories, and let $X : I \to \text{Schemes}$ be a functor into the category of schemes, i.e., a diagram of schemes. A sheaf $F$ on $X$ is given by sheaves $F_i$ on $X_i$ for each $i \in I$ and morphisms $F_i \to X(\alpha) \cdot F_j$ for each $\alpha : j \to i$ in $I$, functorial in the obvious sense. Then the category $\mathcal{S}(X)$ of sheaves on $X$ is an abelian category with enough injectives and arbitrary products (in fact, a topos). The groups $H^0(X, F)$ form an $I^0$-diagram of abelian groups via the transition maps $H^0(X, F) \to H^0(X, \mathcal{O}_X(D)) = H^0(X, F)$ for $\alpha : j \to i$, and we define the cohomology $H^\bullet(X, F)$ by deriving the left exact functor

$$F \to \varprojlim_I H^0(X, F_i)$$

on $\mathcal{S}(X)$. We get a spectral sequence

(3.29) $E_2^{pq} = \varprojlim_I H^p(X, F_i) \Rightarrow H^{p+q}(X, F)$. 
For \( J^0 \)-diagrams of sheaves on \( X \), \((F_j) \in S(X)^I\), we define the cohomology by deriving the functor
\[
(F_j) \mapsto \lim_{\to} H^0(X, F_j) = \lim_I \lim_j H^0(X, F_j),
\]
compare 1.18. We obtain the same result, if we extend \( X \) to a diagram \( X^J : I \times J \to \text{Schemes} \) ("constant in \( J \)-direction"), regard \((F_j)\) as sheaf on this, and take the cohomology (note the equivalence \( S(X)^J \simeq S(X^J) \)).

The case treated above is \( I = \text{pt} \), \( J = \mathbb{N} \); for \( I = \text{arbitrary} \) and \( J = \text{pt} \) one obtains usual étale cohomology on diagrams of schemes, and for \( I = \text{arbitrary} \) and \( J = \mathbb{N} \) we get, for example, continuous \( l \)-adic cohomology \( H^*(X, \mathbb{Z}(j)) \) of diagrams \( X \) as above, e.g., of simplicial schemes \( (I = \Delta^0) \). For relative cohomology \( H^*_\text{cont}(f, \mathbb{Z}(j)) \) of morphisms \( f : X_1 \to X_2 \) \((I = 1 \to 2)\) one rather derives the functor
\[
\lim_n \ker(H^0(X_2, -) \to H^0(X_1, -))
\]
to get an exact sequence
\[
\ldots \to H^i_\text{cont}(f, \mathbb{Z}(j)) \to H^i_\text{cont}(X_2, \mathbb{Z}(j)) \to H^i_\text{cont}(X_1, \mathbb{Z}(j)) \to H^{i+1}_\text{cont}(f, \mathbb{Z}(j)) \to \ldots.
\]
All results of this paper have counterparts for diagrams of schemes.

(3.30) Lemma. Let \( X \) be a locally noetherian simplicial scheme and let \((F_j)_{j \in J}\) be a diagram of locally constant étale sheaves on \( X \). Then for \( r \geq 0 \) there are canonical isomorphisms between the continuous étale cohomology groups \( H^r_\text{cont}(X, (F_j)) \) defined by Dwyer and Friedlander [3, 2.8] and the groups defined above, functorial in \( X \) and \((F_j)\).

Proof. With the notations in [3, 2.8], one has for \( r \leq m \)
\[
H^r_\text{cont}(X, (F_j)) = \pi_{m-r} \lim_n \limf \limf \limf \text{Hom}(\pi \cdot U, K(\mathcal{M}_j, m) \langle n \rangle)_{NG_j},
\]
where \( \mathcal{M}_j = (M_j, G_j) \) is the coefficient system determined by \( F_j \), and \( \limf \) indicates limit only over those hypercoverings \( U \) of \( X \) for which \( F_j \) is trivial on \( U_{00} \). Since \( K(\mathcal{M}_j, m) \to K(\mathcal{M}_j, m) \langle n \rangle \) is a weak equivalence for \( n > m \) we may omit \( \lim_n \) and \( \langle n \rangle \).

There are natural homomorphisms of cochain complexes
\[
N \text{Hom}(\pi \cdot U, K(\mathcal{M}_j, m))_{NG_j} \to \tau_{\leq m} C(\pi \cdot U, L_j)[m] \to \tau_{\leq m} F_j(\text{diag } U)[m],
\]
functorial in \( F_j \), where \( N \) is the normalization functor giving an equivalence between simplicial abelian groups and cochain complexes, \( \tau_{\leq m} \) is the canonical truncation, and \( C(\pi \cdot U, L_j) \) is the usual cochain complex for the local system \( L_j \) associated to \( F_j \) on \( \pi \cdot U \). The first map is a quasi-isomorphism.

Let \((F_j) \hookrightarrow (I_j)\) be an injective resolution in \( S(X)^I \), then there are canonical quasi-isomorphisms of \( J \)-diagrams of complexes
\[
\lim_{\to} (F_j(\text{diag } U)) \to \text{tot} \lim_{\to} (I_j(\text{diag } U)) \to (H^0(X, I_j))_J,
\]
since the left functor is defined on all systems \((F_j)\) of sheaves and its homology gives a universal \( \delta \)-functor computing the cohomology of \((F_j)\). Since \( \text{holim} \) corresponds to \( R\lim \) and \( \pi_i \) to \( H^{-i} \) via \( N \), the result follows – the functoriality is clear.
4. Inverse Limits and Properties of Abelian Groups

Let $A$ be an abelian group and $l$ a prime number, then we have a short exact sequence of inverse systems

$$
\begin{array}{ccccccc}
0 & \to & l_{n+1}A & \to & A & \to & l_n A & \to 0 \\
\downarrow & & \downarrow & & \circlearrowright & & \downarrow \\
0 & \to & l_n A & \to & A & \to & l_n A & \to 0 \\
\end{array}
$$

(4.1)

By definition, $\varprojlim_n l_n A = T_l(A)$ is the $l$-Tate module of $A$, while $\varprojlim l^n A = \bigcap_n l^n A = l\text{-}\text{div}(A)$ is the subgroup of $l$-divisible elements. Denote by $(A, l)$ the system $\ldots \to A \xrightarrow{l} A \xrightarrow{l} \ldots$, then $\varprojlim (A, l)$ and $\varprojlim^1 (A, l)$ are uniquely $l$-divisible by 1.8.

Finally, by the exact sequence

$$
\begin{array}{ccccccc}
0 & \to & l_{n+1} A & \to & A & \to & A/l^n A & \to 0 \\
\downarrow & & \downarrow & & \circlearrowright & & \downarrow \\
0 & \to & l_n A & \to & A & \to & A/l^n A & \to 0 \\
\end{array}
$$

we get the exact sequence (note that $\varprojlim_1 (A, id) = 0$)

$$
0 \to \bigcap_n l^n A \to A \to \hat{A} \to \varprojlim_n l^n A \to 0,
$$

where $\hat{A} = \varprojlim A/l^n A$ is the $l$-adic completion of $A$, and therefore a canonical isomorphism $\varprojlim l^n A \cong \text{coker}(A \to \hat{A})$. So by (4.1) we get a "long" exact sequence

$$
\begin{array}{ccccccc}
0 & \to & \varprojlim_n l_n A & \to & \varprojlim_n (A, l) & \to & \bigcap_n l^n A & \to & \varprojlim^1 l_n A \\
\| & & \| & & \| & & \| \\
& & T_l(A) & & \text{uniquely} & & l\text{-}\text{div}(A) \ \\
\end{array}
$$

(4.2)

(4.3) \textbf{Lemma.} a) The image of $\varprojlim (A, l)$ in $l\text{-}\text{div}(A)$ is $l\text{-}\text{Div}(A)$, the maximal $l$-divisible subgroup of $A$.

b) $\text{Tor} \left( \varprojlim l^n A \right) = l-D(\text{Tor}(A)) \cong \text{Tor}(l-D(A))$, where $\text{Tor}(B)$ is the torsion subgroup and $l-D(B) = l\text{-}\text{div}(B)/l\text{-}\text{Div}(B)$ for an abelian group $B$. 


c) \( l\text{-Div}\left(\lim_{n}^{\text{l}}A\right) = 0 \).

**Proof.** a) can be checked directly, and b) follows from a) and the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & T_l(A) & \to & \lim_{n}^{\text{l}}(A,l) & \to & l\text{-div}(A) \\
& | & | & | \ & \ | & \\
& 0 & \to & T_l(Tor(A)) & \to & \lim_{n}^{\text{l}}(Tor(A),l) & \to & l\text{-div}(Tor(A)) \\
& & & & & \downarrow & & \\
& & & & & \lim_{n}^{\text{l}}(Tor(A),l) & \to & \lim_{n}^{\text{l}}(Tor(A),l) \\
\end{array}
\]

since \( \lim_{n}^{\text{l}}(Tor(A),l) \) is \( l \)-torsion free and \( l\text{-div}(Tor(A)) = Tor(l\text{-div}(A)) \) is \( l \)-torsion.

Note that \( A \) is the direct sum of \( l\text{-Div}(A) \) and another group \( A' \) with \( l\text{-Div}(A') = 0 \).

For c) consider the exact sequence

\[
0 \to \lim_{n}^{\text{l}}A \to \prod_{n}^{\text{l}}A \to \lim_{n}^{\text{l}}\prod_{n}^{\text{l}}A \to 0
\]

with \( x = \text{id} - (d_n) \), where \( \prod_{n}^{\text{l}}A \to \prod_{n}^{\text{l}}A \) is the canonical map. Suppose given \( a^{(m)} = (a_n^{(m)}) \in \prod_{n}^{\text{l}}A \) with \( a^{(m)} \equiv la^{(m+1)} \mod \text{Im}x \), i.e.,

\[
a_n^{(m)} = la_n^{(m+1)} + u_n^{(m)} - d_n u_{n+1}^{(m)}
\]

for \( (u_n^{(m)}) \in \prod_{n}^{\text{l}}A \). Then

\[
a_0^{(0)} = \sum_{m=0}^{\infty} l^m u_n^{(m)} - d_n u_{n+1}^{(m)} = \sum_{m=0}^{\infty} l^m u_n^{(m)} - d_n \sum_{m=0}^{\infty} l^m u_{n+1}^{(m)} ,
\]

i.e., \( a_0^{(0)} = x(v_n) \) with \( v_n = \sum_{m=0}^{\infty} l^m u_n^{(m)} \) (note that these sums really are finite, since \( l^m u_n^{(m)} = 0 \)). q.e.d.

Recall that an abelian group is called \( l \)-complete, if the canonical map \( A \to \lim_{n}^{\text{l}}A = \tilde{A} \) is an isomorphism.

(4.4) **Proposition.** For an abelian group \( A \) are equivalent:

a) \( A \) is \( l \)-complete.

b) \( A = \lim_{n} A_n \), where the \( A_n \) are \( \mathbb{Z}_l \)-modules with finite exponents.

c) All groups in the sequence (4.2) are zero.

**Proof.** Clearly a) implies b), and for \( A \) like in b) we have \( l\text{-div}(A) = 0 \), as \( \lim_{n}^{\text{l}}A_n \subseteq \prod_{n}^{\text{l}}A_n \); so we only have to show that \( A \to \tilde{A} \) is surjective. Take a sequence
\( a^{(m)} = (a_n^{(m)}) \in \lim A_n \subseteq \prod A_n \) with \( a^{(m+1)} \equiv a^{(m)} \mod l^m (\lim \frac{A_n}{l^n A_n}) \). Then \( a_n^{(m+1)} - a_n^{(m)} \in l^m A_n \) and thus \( a_n^{(m)} = a_n^{(m(n))} \) for \( m \geq m(n) := \max(m(n-1), \text{exponent of } A_n) \).

Let \( a = (a_n^{(m(n))}) \in \prod A_n \). Then \( a \in \lim A_n n \) as

\[
\begin{align*}
d_n a_n^{(m(n+1))} &= a_n^{(m(n+1))} = a_n^{(m(n))}
\end{align*}
\]

for the transition maps \( d_n : A_{n+1} \to A_n \). Furthermore we have

\[
\begin{align*}
d_n a_n^{(m(n))} - a_n^{(m)} &= 0 \quad m \geq m(n) \\
&= l^m A_n \quad m \leq m(n),
\end{align*}
\]

i.e., \( a \mapsto (a^{(n)}) \in \lim A/l^n A \).

It remains to show that a) implies c). Since \( l\text{-div}(A) = 0 = \lim l^n A \), we get

\[
\lim l^n A \simeq \lim (A, l).
\]

But as \( \lim l^n (A, l) \) is \( l \)-divisible, both groups must be zero by (4.3c). Finally it is clear that \( \lim (A, l) = 0 \) for \( l\text{-Div}(A) = 0 \). q.e.d.

The inverse limit \( A = \lim A_n \) carries a natural topology, namely the one induced by the product topology on \( \prod A_n \), where the \( A_n \) are endowed with the discrete topology. In the situation of (4.4b), this "limit topology" is always coarser than the \( l \)-adic topology of \( A \). It might be different, as the example \( A = \prod Z/l = \lim \prod \frac{Z}{l} \) shows. However we have

\[
(4.5) \text{Lemma. If } (A_n) \text{ is } ML-l\text{-adic, the limit topology of } A = \lim A_n \text{ is the } l \text{-adic one.}
\]

\[
\text{Proof. Because the topology does not change under a } ML\text{-isomorphism, we only have to consider an } l \text{-adic system } (A_n). \text{ But then}
\]

\[
A \cap \left( \prod A_n \times \prod \{1\} \right) \subseteq l^n \cdot \prod A_n .
\]

\[
(4.6) \text{Definition (cf. [1, VI, Sect. 3]). Call an abelian group } A \text{ weakly } l \text{-complete or Ext-}l\text{-complete, if } \lim (A, l) = 0 = \lim l^n (A, l).
\]

\[
(4.7) \text{Remark. This property can be expressed without inverse limits, because for an abelian group (resp. } Z_l\text{-module) } A \text{ there are canonical isomorphisms (compare 5.5)}
\]

\[
\lim (A, l) \simeq \text{Hom} \left( \frac{Z}{l} , A \right) \quad \left( \simeq \text{Hom}_{Z_l}(Q_l, A) \right),
\]

\[
\lim l^n (A, l) \simeq \text{Ext}^1 \left( \frac{Z}{l} , A \right) \quad \left( \simeq \text{Ext}^1_{Z_l}(Q_l, A) \right).
\]

The vanishing of the first group is equivalent to \( l\text{-Div}(A) = 0 \).
(4.8) **Proposition.** a) If two groups in an exact sequence \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) are weakly \(l\)-complete, then also the third.

b) The weakly \(l\)-complete groups form a full abelian subcategory of \(\text{Ab}\).

c) If \((A_n)\) is an inverse system of abelian groups such that each \(A_n\) has a finite \(l\)-power exponent, then \(\lim_n^{-1} A_n\) is weakly \(l\)-complete.

**Proof.** a) The exact sequence \(0 \rightarrow (A, l) \rightarrow (B, l) \rightarrow (C, l) \rightarrow 0\) induces an exact sequence

\[
0 \rightarrow \lim_n (A, l) \rightarrow \lim_n (B, l) \rightarrow \lim_n (C, l) \rightarrow \lim_n^{-1} (A, l)
\]

\[
\rightarrow \lim_n^{-1} (B, l) \rightarrow \lim_n^{-1} (C, l) \rightarrow 0.
\]

b) We have to show that for a morphism \(f: A \rightarrow B\) of weakly \(l\)-complete groups \(\text{Ker} f\) and \(\text{Coker} f\) are again weakly \(l\)-complete. But from the short exact sequences

\[
0 \rightarrow \text{Ker} f \rightarrow A \rightarrow \text{Im} f \rightarrow 0
\]

\[
0 \rightarrow \text{Im} f \rightarrow B \rightarrow \text{Coker} f \rightarrow 0
\]

we obviously have \(\lim_n (\text{Im} f, l) = 0 = \lim_n^{-1} (\text{Im} f, l)\). Thus \(\text{Im} f\) is weakly \(l\)-complete, and we can proceed by a).

c) follows by applying a) to the sequences

\[
0 \rightarrow \lim_n A_n \rightarrow \prod_n A_n \rightarrow X \rightarrow 0
\]

\[
0 \rightarrow X \rightarrow \prod_n A_n \rightarrow \lim_n^{-1} A_n \rightarrow 0,
\]

because \(\lim_n A_n\) and \(\prod_n A_n\) are \(l\)-complete by (4.4).

We apply this to extend Tate’s results for continuous group cohomology in [14, Sect. 2] to continuous étale cohomology.

(4.9) **Corollary.** Let \(X\) be a scheme and \((F_n)\) be an inverse system of sheaves on \(X_{et}\) such that each \(F_n\) is killed by some finite power of \(l\). Then \(H^i(X, (F_n))\) is weakly \(l\)-complete for each \(i \geq 0\); in particular \(l\)-Div\((H^i(X, (F_n))) = 0\).

**Proof.** This follows by applying (4.4) and (4.8a) and (4.8c) to the sequences (3.1).

(4.10) **Corollary.** In the situation of (4.9), \(H^i(X, (F_n))\) is a finitely generated \(\mathbb{Z}_l\)-module if and only if \(H^i(X, (F_n))/lH^i(X, (F_n))\) is finite.

**Proof.** This holds for any \(\mathbb{Z}_l\)-module \(M\) with \(l\)-Div\((M) = 0\): Let \(Y \subset M\) be a finitely generated \(\mathbb{Z}_l\)-module with \(Y + lM = M\). Then \(M/Y\) is \(l\)-divisible, hence zero by the exact sequence

\[
0 = \lim_n (M, l) \rightarrow \lim_n (M/Y, l) \rightarrow \lim_n^{-1} (Y, l) = 0.
\]

Since the vanishing of \(\lim_n^{-1} (A, l)\) carries over to quotients, we get the following “Nakayama lemma” by the same arguments.

(4.11) **Lemma.** Let \(\phi: Y \rightarrow M\) be a homomorphism of abelian groups, which induces a surjection \(Y/lY \rightarrow M/lM\). If \(\lim_n^{-1} (Y, l) = 0\) and \(l\)-Div\((M) = 0\), then \(\phi\) is surjective.
(4.12) **Corollary.** Let \( \varphi : Y \to M \) be a homomorphism of weakly \( l \)-complete groups. If \( \varphi \) induces isomorphisms \( Y/lY \cong M/lM \) and \( _1Y \cong _1M \), it is an isomorphism itself.

**Proof.** By (4.11) there is an exact sequence \( 0 \to X \to Y \to M \to 0 \), and the induced exact sequence

\[
0 \to _1X \to _1Y \to _1M \to X/lX \to Y/lY \to M/lM \to 0
\]

shows \( X/lX = 0 \), i.e., \( X \) is \( l \)-divisible. But \( X \) is weakly \( l \)-complete by (4.8a), hence \( X = 0 \).

In (4.10), the \( H^i(X, (F_n)) \) have a (canonical) \( \mathbb{Z}_l \)-structure, since we may derive in the category of \( \mathbb{Z}_l \)-modules. It is not clear a priori that a weakly \( l \)-complete group admits a \( \mathbb{Z}_l \)-structure, especially not a unique one; however we have

(4.13) **Lemma.** a) The weakly \( l \)-complete groups have a unique structure of \( \mathbb{Z}_l \)-modules such that any homomorphism between them is a \( \mathbb{Z}_l \)-morphism.

b) If \( A \) is weakly \( l \)-complete, there is an exact sequence

\[
0 \to T_1 \to T_0 \to A \to 0
\]

with torsion-free \( l \)-complete groups \( T_0 \) and \( T_1 \).

**Proof.** a) Give any weakly \( l \)-complete group \( A \) the \( \mathbb{Z}_l \)-structure induced by the \( \mathbb{Z}_l \)-structure of \( \mathbb{Q}_l/\mathbb{Z}_l \) and the isomorphism

\[
a = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \xrightarrow{\phi} \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}_l/\mathbb{Z}_l, A)
\]

coming from the Ext-sequence for \( 0 \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{1} \mathbb{Q}_l/\mathbb{Z}_l \to 0 \), then the functoriality is clear. The uniqueness follows from b), since any \( l \)-complete group has a unique \( \mathbb{Z}_l \)-structure (the \( l \)-adic topology is Hausdorff).

b) Choosing an exact sequence \( 0 \to X_1 \to X_0 \to A \to 0 \) with torsion-free abelian groups \( X_0 \) and \( X_1 \), the sequence

\[
0 \to \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}_l/\mathbb{Z}_l, X_1) \to \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}_l/\mathbb{Z}_l, X_0) \to \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}_l/\mathbb{Z}_l, A) = A \to 0
\]

is exact since \( \text{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, A) = 0 \), and \( T_1 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}_l/\mathbb{Z}_l, X_i) = \lim_{\leftarrow n} X_i/l^nX_i \) [compare (5.3) below] is torsion-free \( l \)-adic.

(4.14) **Remark.** a) From (4.13b) we see that the category of weakly \( l \)-complete groups is the smallest abelian subcategory of \( \text{Ab} \) containing the \( l \)-complete groups.

b) If \( A \) is weakly \( l \)-complete, the connected component of the unit element w.r.t. the \( l \)-adic topology is \( \text{Ker}(A \to \hat{A}) = l\text{-div}(A) \), which is also isomorphic to \( \lim_{\leftarrow n} A \) by (4.2). This is compatible with the fact that for any inverse system \( (A_n) \) of topological abelian groups the topology of \( \lim_{\leftarrow n} A_n \) induced by the canonical surjection \( \prod_n A_n \to \lim_{\leftarrow n} A_n \) is the indiscrete one.

c) There is an interesting structure theory for weakly (Ext-)\( l \)-complete groups, see [8] and [1, Sect. 2-4], also for further literature.
When is $\lim_{n} H^i(X, F_n) = 0$ for a scheme $X$ and an inverse system of sheaves $(F_n)$ on $X_{et}$? In view of the fact that "very often" the cohomology groups $H^i(X, F)$ are countable, e.g. if $F$ is constructible and $X$ is of finite type over a field $k$ with separable absolute Galois group $G_k = \text{Gal}(k_s/k)$, the answer is in many cases given by the following result of Gray [5, Proposition]:

(4.15) **Lemma.** Let $(A_n)$ be an inverse system of countable abelian groups. Then $\lim_{n} A_n = 0$ if and only if $(A_n)$ satisfies the Mittag-Leffler condition.

(4.16) **Corollary.** Let $(A_n)$ be an inverse system of countable abelian groups. Then $\lim_{n} A_n = 0 = \lim_{n} A_n$ if and only if $(A_n)$ is ML-zero.

**Proof.** A Mittag-Leffler system $(A_n)$ with $\lim_{n} A_n = 0$ is ML-zero, compare (1.14).

(4.17) **Corollary.** A countable group $A$ is weakly l-complete if and only if it has finite exponent $l^m$ for some $m \in \mathbb{N}$.

5. Torsion-Free l-Adic Sheaves and $\mathbb{Q}_l$-Cohomology

Often l-adic sheaves arise by the following construction.

(5.1) **Definition and Proposition.** For a scheme $X$ define the left exact functor

$$T_l : S(X_{et}) \rightarrow S(X_{et})^{\mathbb{N}}$$

by

$$F \mapsto (\Pi, F),$$

where the transition maps $\Pi_{n+1} F \rightarrow \Pi_n F$ are induced by the multiplication with $l$.

- a) $R^i T_l F = 0$ for $i \geq 2$ and $R^1 T_l F = T_l^1 F = (F/\Pi F)$, where the transition maps $F/\Pi^{n+1} F \rightarrow F/\Pi^n F$ are the canonical surjections.
- b) $T_l F$ is l-adic if and only if $\lim_{n} \Pi_n F = l\text{-Tor}(F)$ is l-divisible.

**Proof.** We use the following fact.

(5.2) **Lemma.** Let $I$ be an injective sheaf on $X_{et}$. Then for every $m \in \mathbb{N}$

- a) $I$ is $m$-divisible,
- b) $H^0(X, I)$ is $m$-divisible,
- c) $H^i(X, mI) = 0$ for $i \geq 1$.

**Proof.** a) follows from the statement b) for all schemes (étale) over $X$, b) follows from the Ext-sequence

$$0 \rightarrow \text{Hom}_X(\mathbb{Z}/m, I) \rightarrow \text{Hom}_X(\mathbb{Z}, I)^{\mathbb{N}} \rightarrow \text{Hom}_X(\mathbb{Z}, I) \rightarrow \text{Ext}_X^1(\mathbb{Z}/m, I)$$

associated to $0 \rightarrow \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/m \rightarrow 0$, and c) follows from a), b and the cohomology sequence for $0 \rightarrow mI \rightarrow I \rightarrow I \rightarrow 0$.

For the proof of (5.1a) note that $(T_l, T_l^1, 0, \ldots)$ is an exact $\delta$-functor on $S(X_{et})$, and $T_l^1(I) = 0$ for injective $I$ by (5.2a). The statement (5.1b) is clear, as both assertions are equivalent to the equality $\Pi_n F = l_{\Pi_n+1} F$ for all $n \in \mathbb{N}$.
(5.3) **Corollary.** For a sheaf $F$ on $X_{et}$ there is a long exact sequence

$$0 \to H^1(X, (\mu_F)) \to \text{Ext}^1_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to H^0(X, (F/\mu F))$$

$$\to H^2(X, (\mu_F)) \to \text{Ext}^2_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to H^1(X, (F/\mu F)) \to \ldots,$$

functorial in $F$ and $X$.

**Proof.** The functor $S(X_{et}) \to \text{Ab}$, $F \mapsto \text{Hom}_X(\mathbb{Q}_l/\mathbb{Z}_l, F) = \lim H^n(X, \mu F)$ is the composition of $T_l$ with $\lim H^n(X, -)$, and $T_l$ carries injectives to $\lim H^0(X, -)$-acyclics by (5.2c) and (3.1); note that $H^n(X, \mu F)$ is surjective by (5.2b). The statement thus follows from (5.1a) and the Grothendieck spectral sequence

$$H^n(X, R^qT_lF) \Rightarrow \text{Ext}^{n+q}_X(\mathbb{Q}_l/\mathbb{Z}_l, F).$$

The functoriality of the latter in $F$ and $X$ is clear.

(5.4) **Corollary.** If $F$ is $l$-divisible, there are canonical isomorphisms

$$H^i(X, (\mu_F)) \Rightarrow \text{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l, F).$$

(5.5) **Definition and Proposition.** For a sheaf $F$ on $X_{et}$ denote by $V_F$ or $(F, l)$ the inverse system $\ldots \to F \to F \to F \to \ldots$. Then there are canonical isomorphisms

$$H^i(X, (F, l)) \Rightarrow \text{Ext}^i_X\left(\mathbb{Z}\left[\frac{1}{l}\right], F\right),$$

functorial in $F$ and $X$.

**Proof.** $\text{Hom}_X\left(\mathbb{Z}\left[\frac{1}{l}\right], F\right) = \text{Hom}_X(\lim F, F) = \lim (H^0(X, F), l)$ is the composition of $\lim H^0(X, -)$ with the exact functor $F \mapsto (F, l)$, which carries injectives to $\lim H^0(X, -)$-acyclics by (5.2b) and (3.1).

(5.6) **Corollary.** For any $l$-divisible sheaf $F$ on $X_{et}$ the isomorphisms above induce an isomorphism of long exact sequences

$$\ldots \to \text{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to \text{Ext}^i_X\left(\mathbb{Z}\left[\frac{1}{l}\right], F\right) \to \text{Ext}^i_X(\mathbb{Z}, F)$$

$$\uparrow$$

$$\ldots \to H^i(X, T_lF) \to H^i(X, (F, l)) \to H^i(X, F)$$

$$\to \text{Ext}^{i+1}_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to \ldots$$

$$\uparrow$$

$$\to H^{i+1}(X, T_lF) \to \ldots,$$

coming from the short exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z}\left[\frac{1}{l}\right] \to \mathbb{Q}_l/\mathbb{Z}_l \to 0$$
and
\[ 0 \rightarrow T_l F \rightarrow (F, l) \rightarrow F \rightarrow 0, \]
respectively.

Proof. Let \( F \subset I \cdot \) be an injective resolution, then by (5.2a) there is an exact sequence
\[ 0 \rightarrow T_l I \cdot \rightarrow (I, l) \rightarrow I \cdot \rightarrow 0, \]
and the claimed isomorphism is obtained by taking homology in the commutative exact diagram
\[
\begin{align*}
0 & \rightarrow \text{Hom}_X(Q_L/Z_L, I) \rightarrow \text{Hom}_X \left( \mathbb{Z} \left[ \frac{1}{l} \right], I \right) \rightarrow \text{Hom}_X(\mathbb{Z}, I) \rightarrow 0 \\
& \uparrow \quad \uparrow \\
0 & \rightarrow H^0(X, T_l I) \rightarrow H^0(X, (I, l)) \rightarrow H^0(X, I) \rightarrow 0,
\end{align*}
\]
as \( T_l F \subset T_l I \cdot \) is an \( \lim_{n} H^0(X, \cdot) \)-acyclic resolution for \( l \)-divisible \( F \). It follows from the definitions that the induced isomorphisms are those of (5.4) and (5.5).

(5.7) Remark. If \( F \) is an \( l \)-primary torsion sheaf [e.g. the sheaves \( Q_L/Z_L(j) \) on \( X_{et} \)], one may replace \( T./// \) by \( \text{all} l = Q_L \cdot \) and \( 7 \) by \( Z_L \cdot \) in the statements of (5.5) and (5.6), as \( Z_L/Z \) and \( Q_L/Z \left[ \frac{1}{l} \right] \) are uniquely \( l \)-divisible.

(5.8) Proposition. For \( l \)-torsion sheaves \( F \) on \( X_{et} \), the homomorphisms \( \text{Ext}^i_X(Q_L/Z_L, F) \rightarrow \text{Ext}^i_X(Q_L, X) \) induce functorial isomorphisms
\[
\text{Ext}^i_X(Q_L/Z_L, F) \otimes_{Z_L} Q_L \cong \text{Ext}^i_X(Q_L, F).
\]

Proof. As \( Q_L \) is flat over \( Z_L \), this is implied by the case \( i = 0 \), which follows from the commutative diagram
\[
\begin{align*}
\text{Hom}_X(Q_L/Z_L, F) \otimes_{Z_L} Q_L & \rightarrow \text{Hom}_X(Q_L, F) \\
& \downarrow \quad \downarrow \\
T_l H^0(X, F) \otimes_{Z_L} Q_L & \cong \lim_{n} (H^0(X, F), l)
\end{align*}
\]
in which the lower maps is an isomorphism by the following lemma, whose proof is left to the reader.

(5.9) Lemma. If \( A \) is an abelian \( l \)-torsion group, the map
\[
T_l A \otimes_{Z_L} Q_L \rightarrow \lim_{n} (A, l) \\
(a_n) \otimes \frac{1}{l_m} \mapsto (a_n + m),
\]
is an isomorphism. If \( A \) is \( l \)-divisible, it is a topological isomorphism.
For $\text{ML}$-isomorphic inverse systems $(F_n)$ and $(G_n)$ on $X_{\text{et}}$ and an $m \in \mathbb{N}$, the systems $(F_n/l^mF_n)$ and $(G_n/l^mG_n)$ are also $\text{ML}$-isomorphic, so $\lim_{n \to \infty} F_n/l^mF_n$ and $\lim_{n \to \infty} G_n/l^mG_n$ are isomorphic. In particular, for a $\text{ML}$-$l$-adic system $F = (F_n)$ the system $(F_n/l^mF_n)$ is $\text{ML}$-constant, and we define the sheaf $F \otimes \mathbb{Z}/l^m$ on $X_{\text{et}}$ by

$$F \otimes \mathbb{Z}/l^m = \lim_{n \to \infty} F_n/l^mF_n$$

($\cong F'_n/l^mF'_n$ for $n \gg m$ where $(F'_n) \subseteq (F_n)$ is the subsystem of universal images).

(5.10) **Definition and Proposition.** For any $\text{ML}$-$l$-adic system $F = (F_n) \in \text{Ob}(S(X_{\text{et}}))$ define the sheaf $F \otimes \mathbb{Q}_l/\mathbb{Z}_l \in \text{Ob}(\text{S}(X_{\text{et}}))$ by

$$F \otimes \mathbb{Q}_l/\mathbb{Z}_l = \lim_{m \to \infty} F \otimes \mathbb{Z}/l^m,$$

where $F \otimes \mathbb{Z}/l^m \rightarrow F \otimes \mathbb{Z}/l^{m+1}$ is induced by $F \rightarrow F$.

a) The functor $F \otimes F \otimes \mathbb{Q}_l/\mathbb{Z}_l$ only depends on the $\text{ML}$-isomorphism classes of objects and morphisms.

b) $F \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is $l$-divisible.

**Proof.** a) is clear from the remarks above, and b) follows from the exact commutative diagrams

$$\begin{array}{c}
F \otimes \mathbb{Z}/l^{m+1} \rightarrow F \otimes \mathbb{Z}/l^{m+1} \rightarrow F \otimes \mathbb{Z}/l \rightarrow 0 \\
\phi^{m} \rightarrow \phi^{m} \rightarrow 0 \\
F \otimes \mathbb{Z}/l^{m} \rightarrow F \otimes \mathbb{Z}/l^{m} \rightarrow F \otimes \mathbb{Z}/l \rightarrow 0,
\end{array}$$

as the inductive limit is an exact functor for étale sheaves.

(5.11) **Lemma.** Let $F = (F_n)$ be an $l$-adic sheaf on $X_{\text{et}}$.

a) $F$ is $\text{ML}$-torsion-free if and only if $\phi^{m}(F_{n+1}) = \phi^{m}F_{n+1}$ for all $n \in \mathbb{N}$.

b) If $F$ is $\text{ML}$-torsion-free, there is a canonical isomorphism

$$F \approx \mathcal{T}_l(F \otimes \mathbb{Q}_l/\mathbb{Z}_l).$$

**Proof.** a) By definition, $F$ is $\text{ML}$-torsion-free if and only if $(F_n)$ is $\text{ML}$-zero. If this is the case, then for any $n \in \mathbb{N}$ there is an $m = m(n) > 0$ such that $\phi^{m}(F_{n+m}) \subseteq \text{Ker}(F_{n+m} \rightarrow F_n) = l^nF_{n+m}$. By the commutative exact diagram

$$\begin{array}{c}
0 \rightarrow F_{n+m} + l^nF_{n+m} \rightarrow F_{n+m} + l^nF_{n+m} \rightarrow F_{n+m}/l^{n+1}F_{n+m} \\
\phi^{m} \rightarrow \phi^{m} \rightarrow 0 \\
0 \rightarrow F_{n+1} + l^nF_{n+1} \rightarrow F_{n+1} + l^nF_{n+1} \rightarrow F_{n+1},
\end{array}$$

this is the case if and only if $\phi^{m}(F_{n+1}) = l^nF_{n+1}$, i.e., if all maps in $(F_n)$ are zero.

b) By induction on $r$ we get $l^rF_{n} = l^{n-r}F_{n}$ for all $1 \leq r \leq n$, and from the commutative diagrams

$$\begin{array}{c}
F \otimes \mathbb{Z}/l^{n+1} \approx F_{n+1} \approx F_{n+1} \\
\phi^{n} \rightarrow \phi^{n} \rightarrow 0 \\
F \otimes \mathbb{Z}/l^{n} \approx F_{n+1}/l^nF_{n+1} \approx F_{n}.
\end{array}$$
we see that the \( \varphi_n \) are injective and identify \( F \otimes \mathbb{Z}/l^n \mathbb{Z} \) with \( \mu(F \otimes \mathbb{Z}/l^{n+1}) \). As this holds for all \( n \in \mathbb{N} \), we get canonical isomorphisms

\[
\mu(F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \cong F \otimes \mathbb{Z}/l^n \cong F_n,
\]

and via these the transition maps \( F_{n+1} \rightarrow F_n \) coincide with the maps \( \mu(F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow \mu(F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \) induced by \( l \)-multiplications.

(5.12) **Remark.** a) If \( F = (F_n) \) is ML-l-adic, the projections \( F \otimes \mathbb{Z}/l^n \rightarrow F_n \) induce a canonical morphism of the \( l \)-adicsystem \( (F \otimes \mathbb{Z}/l^n) \) – with obvious transition maps – into \( F \). This morphism is a ML-isomorphism and an AR-isomorphism if \( F \) is AR-l-adic. So by (5.10a) and (5.11b) we get a canonical ML-isomorphism

\[
T_l(F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow F,
\]

if \( F \) is ML-torsion-free, which is an AR-isomorphism if \( F \) is AR-l-adic.

b) In particular, AR-l-adic and ML-torsion-free implies AR-torsion-free, so that we may simply talk of torsion-freeness.

(5.13) **Definition.** For a ML-l-adic sheaf \( F = (F_n) \) on \( X_{et} \) define

\[
H^i_{\text{cont}}(X, F \otimes \mathbb{Q}_l) := H^i((X, (F_n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l).
\]

(5.14) **Theorem.** a) If \( F \) is a torsion-free ML-l-adic sheaf on \( X_{et} \), there is a canonical exact sequence

\[
\ldots \rightarrow H^i_{\text{cont}}(X, F) \rightarrow H^i_{\text{cont}}(X, F \otimes \mathbb{Q}_l) \rightarrow H^i(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^{i+1}_{\text{cont}}(X, F) \rightarrow \ldots,
\]

functorial in \( F \) and \( X \).

b) In this sequence one has

\[
\text{Tor}(H^i_{\text{cont}}(X, F)) = \text{Im}(H^{i-1}(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow H^i_{\text{cont}}(X, F)),
\]

\[
\text{l-Div}(H^i(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l)) = \text{Im}(H^i_{\text{cont}}(X, F \otimes \mathbb{Q}_l) \rightarrow H^i(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l)).
\]

**Proof.** a) Using the ML-isomorphism \( F \leftarrow T_l(F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \), this is the sequence from (5.6) for \( F \otimes \mathbb{Q}_l/\mathbb{Z}_l \):

\[
\ldots \rightarrow H^i_{\text{cont}}(X, T_l(F \otimes \mathbb{Q}_l/\mathbb{Z}_l)) \rightarrow H^i(X, (F \otimes \mathbb{Q}_l/\mathbb{Z}_l, l)) \rightarrow H^i(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow \ldots
\]

combined with the isomorphism

\[
H^i_{\text{cont}}(X, T_l(F \otimes \mathbb{Q}_l/\mathbb{Z}_l)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \cong H^i(X, (F \otimes \mathbb{Q}_l/\mathbb{Z}_l, l))
\]

implied by (5.8) and the commutative diagram

\[
\begin{array}{ccc}
H^i_{\text{cont}}(X, T_l(F \otimes \mathbb{Q}_l/\mathbb{Z}_l)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \rightarrow & H^i(X, (F \otimes \mathbb{Q}_l/\mathbb{Z}_l, l)) \\
\downarrow^{5.4} \downarrow & & \downarrow^{5.5} \\
\text{Ext}^i_l(\mathbb{Q}_l/\mathbb{Z}_l, F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \rightarrow & \text{Ext}^i_l(\mathbb{Q}_l, F \otimes \mathbb{Q}_l/\mathbb{Z}_l)
\end{array}
\]

b) This is clear, since the \( H^i_{\text{cont}}(X, F \otimes \mathbb{Q}_l) \) are uniquely divisible, the \( H^i(X, F \otimes \mathbb{Q}_l/\mathbb{Z}_l) \) are torsion groups, and \( \text{l-Div}(H^i_{\text{cont}}(X, F)) = 0 \) for all \( i \geq 0 \) by (4.9).

(5.15) **Theorem.** Let \( X = \text{Spec} k \) for a field \( k \) with separable closure \( k_s \) and \( F = (F_n) \) be a ML-l-adic sheaf on \( X_{et} \). Let \( T = \lim_{\longrightarrow \leftarrow n} (F_n) \) be its "stalk" at the geometric point
\(\tilde{x} = \text{Spec} \, \mathbb{Z}_l\) with the \(l\)-adic topology and give \(T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\) the induced topology w.r.t. \(T \to T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\).

a) The stalk of \(F \otimes_{\mathbb{Q}_l} / \mathbb{Z}_l\) at \(\tilde{x}\) is isomorphic to \(T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l\) as module under \(G_k = \text{Gal}(k_s/k)\).

b) \(G_k\) acts continuously on \(T\) and \(T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\), and there are functorial isomorphisms for \(i \geq 0\)

\[H^i_{\text{cont}}(X, F) \simeq H^i_{\text{cont}}(G_k, T).\]

c) If \(\text{Tor}(T)\) has a finite exponent, these induce isomorphisms for \(i \geq 0\)

\[H^i_{\text{cont}}(X, F \otimes \mathbb{Q}_l) \simeq H^i_{\text{cont}}(G_k, T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)\]

d) If \(F\) is torsion free, these induce isomorphisms of the long exact sequences

\[\ldots \to H^i_{\text{cont}}(X, F) \to H^i_{\text{cont}}(X, F \otimes \mathbb{Q}_l) \to H^i(X, F \otimes \mathbb{Q}_l / \mathbb{Z}_l) \downarrow \downarrow \downarrow \ldots \]

\[\ldots \to H^i_{\text{cont}}(G_k, T) \to H^i_{\text{cont}}(G_k, T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \to H^i(G_k, T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l) \to H^{i+1}(X, F) \to \ldots \downarrow \downarrow \downarrow \ldots \]

\[\to H^{i+1}_{\text{cont}}(G_k, T) \to \ldots \]

given by (5.14) and the short exact sequence

\[0 \to T \to T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l \to 0,\]

respectively.

**Proof.** a) By passing to an \(l\)-adic sheaf we easily see that \((F \otimes \mathbb{Z}_l / l^m)_x\) is canonically isomorphic to \(T / l^m T\) as \(G_k\)-module and that these isomorphism induce an isomorphism

\[(F \otimes \mathbb{Q}_l / \mathbb{Z}_l)_x = \lim_{\longrightarrow} (F \otimes \mathbb{Z}_l / l^m)_x \simeq \lim_{\longrightarrow} T \otimes_{\mathbb{Z}_l} \mathbb{Z}_l / l^m = T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l.\]

b) By (4.5) the limit topology on \(T\) is the \(l\)-adic one, which implies the continuity of the action and the claimed isomorphisms by (3.2).

c) We have to show that the canonical map

\[H^i_{\text{cont}}(G_k, T) \to H^i_{\text{cont}}(G_k, T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)\]

induces an isomorphism

\[H^i_{\text{cont}}(G_k, T) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq H^i_{\text{cont}}(G_k, T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l).\]

As \(\mathbb{Q}_l\) is flat over \(\mathbb{Z}_l\), it suffices to show that we have isomorphisms

\[C^i(G_k, T) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq C^i(G_k, T \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)\]

for the groups of continuous cochains. But if \(\varphi : G_k^l \to T\) has its image in \(\text{Tor}(T) = \text{Ker}(T \to T \otimes \mathbb{Q}_l)\), it is by assumption annihilated by \(l^r\) for some \(r \in \mathbb{N}\), which shows the injectivity of the above maps. The surjectivity is implied by the fact that every cochain \(\psi : G_k^l \to T \otimes \mathbb{Q}_l\) has a compact image, which therefore is contained in \(l^{-s} \text{Im}(T \to T \otimes \mathbb{Q}_l)\) for some \(s \in \mathbb{N}\).

d) In view of (3.2) and the construction of the upper sequence we have to show that the exact sequence

\[0 \to (T / l^m T) \to (T \otimes \mathbb{Q}_l / \mathbb{Z}_l) \to (T \otimes \mathbb{Q}_l / \mathbb{Z}_l, \text{id}) \to 0\]
induces a commutative exact diagram

\[
0 \to \lim_n T/l^n T \to \lim_n (T \otimes \mathbb{Q}_l/\mathbb{Z}_l, l) \to T \otimes \mathbb{Q}_l/\mathbb{Z}_l \to 0
\]

\[
0 \to T \to T \otimes \mathbb{Z}_l \mathbb{Q}_l \to T \otimes \mathbb{Q}_l/\mathbb{Z}_l \to 0
\]

in which the vertical arrows are topological isomorphisms. This follows from (4.5) and (5.9). q.e.d.

We leave it to the reader to formulate and prove the $\mathbb{Q}_l$-analogues of the results in Sect. 2 and Sect. 3, by "tensoring everything with $\mathbb{Q}_l"$ over $\mathbb{Z}_l$ or with $\mathbb{Q}$ over $\mathbb{Z}$. I am indebted to the referee for pointing out a possible source of confusion: Like for any abelian category the above means to work modulo (the Serre subcategory of) objects of finite exponent, not modulo torsion objects. Note that

\[\text{nr}(X, F| = \text{nr}(X, F)\mid\text{for } F = (F_i) \text{ in } \mathbb{S}(X_{et}) \mid \text{[see (6.9)] in fact only depends on } F \otimes \mathbb{Q}_l \text{ read as } \text{"F up to systems of finite exponent"}, \text{ while for } F = \left( \bigoplus_{i=1}^n \mathbb{Z}/l^i \right) = \lim_m F \text{ in } Ab^\mathbb{Z}_l \text{ one has } \mathbb{Q}_l \otimes \lim_n F_i \neq 0. \]

The spectral sequences (3.10) etc. carry over since only finitely many terms are involved in the limit groups.

(5.16) **Proposition.** Let $F = (F_n)$ be a torsion-free ML-$l$-adic sheaf on $X_{et}$ and $Z \subseteq X$ be a closed subscheme. Then the exact sequence (3.16)

\[0 \to \lim_{n} H^{1}_{Z}^{-1}(X, F_n) \to H^{1}_{Z, \text{cont}}(X, F) \to \lim_{n} H^{1}_{Z}(X, F_n) \to 0\]

identifies $\lim_{n} H^{1}_{Z}^{-1}(X, F_n)$ with $l\text{-div}(H^{1}_{Z, \text{cont}}(X, F))$.

Proof. We may assume $F$ to be $l$-adic; then the ML-exact sequences

\[0 \to F \to F \to F_n \to 0\]

induce sequences

\[0 \to H^{1}_{Z, \text{cont}}(X, F)/l^n \to H^{1}_{Z}(X, F_n) \to \mu H^{1}_{Z, \text{cont}}(X, F) \to 0,\]

and, by passing to the inverse limit, an isomorphism

\[\lim_{n} H^{1}_{Z, \text{cont}}(X, F)/l^n \cong \lim_{n} H^{1}_{Z}(X, F_n)\]

since $T_l(H^{1}_{Z, \text{cont}}(X, F)) = 0$. This implies that in the sequence (3.16) the cokernel can be identified with the $l$-adic completion of $H^{1}_{Z, \text{cont}}(X, F)$, hence the statement for the kernel.

6. Cupproducts and Chern Classes

It will be useful and clearer to introduce cupproducts in the general setting of derived functors. This is certainly well-known and included only for lack of a suitable reference.
(6.1) Definition. Let $f : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume that $\mathcal{A}$ possesses enough injectives and a tensor product.

a) Call an exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ admissible, if $0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ is exact for each object $B$ of $\mathcal{A}$.

b) Say that $\mathcal{A}$ has the property $C1(f)$, if any object $A$ of $\mathcal{A}$ can be embedded into an $f$-acyclic object $A'$ such that $0 \to A \to A' \to A/A' \to 0$ is admissible.

c) Say that $\mathcal{A}$ has the property $C2(f)$, if there is an exact functor $C$ associating to each object $A$ of $\mathcal{A}$ an $f$-acyclic resolution

$$C'(A) : C^0(A) \xrightarrow{\delta} C^1(A) \rightarrow \cdots \rightarrow C^i(A) \xrightarrow{\delta} C^{i+1}(A) \rightarrow \cdots$$
on of $A$, which is admissible, i.e., such that $0 \to \ker \delta \to C^i(A) \to \text{Im} \delta \to 0$ is admissible for every $i \geq 0$.

(6.2) Proposition. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with tensor products. Assume that $\mathcal{B}$ has enough injectives, and let $f_1$, $f_2$, and $f_3$ be left exact functors $\mathcal{A} \to \mathcal{B}$.

a) If $\mathcal{A}$ has the property $C1(f_i)$ for $i=1,2$, then every morphism of bi-functors

$$f_1(A) \otimes f_2(B) \to f_3(A \otimes B)$$

can uniquely be extended to a family of morphisms of bi-functors

$$R^p f_1 A \otimes R^q f_2 B \to R^{p+q} f_3(A \otimes B)$$

called cupproduct such that

i) for $p=q=0$ it is the given morphism;

ii) if $0 \to A' \to A \to A'' \to 0$ is admissible and $B$ is an object of $\mathcal{A}$, then the diagram

$$\begin{array}{ccc}
R^p f_1 A'' & \otimes & R^q f_2 B
\downarrow \delta \\
R^{p+1} f_1 A' & \otimes & R^q f_2 B
\end{array}$$

is commutative;

iii) if $0 \to B' \to B \to B'' \to 0$ is admissible, then for each object $A$ of $\mathcal{A}$ the diagram

$$\begin{array}{ccc}
R^p f_1 A & \otimes & R^q f_2 B''
\downarrow (-1)^p \text{id} \otimes \delta \\
R^{p+1} f_1 A & \otimes & R^q f_2 B'
\end{array}$$

is commutative.

b) If $A'$ and $B'$ are $f_1$-acyclic and $f_2$-acyclic admissible resolutions of $A$ and $B$, respectively, there is a morphism

$$A' \otimes B' \to J'$$

into an $f_3$-acyclic resolution $J'$ of $A \otimes B$ inducing the identity on $A \otimes B$. For any such the morphisms induced by

$$f_1 A' \otimes f_2 B' \to f_3(A' \otimes B') \to f_3 J'$$
in the homology give the above cupproduct (here as usual, $A' \otimes B'$ is the complex with $(A' \otimes B')_p = \bigoplus_{p+q=n} A^p \otimes B^q$ and differential $d = d_A \otimes \text{id} + (-1)^p \text{id} \otimes d_B$ on $A^p \otimes B^q$).
c) Let \( g_1, g_2, \text{ and } g_3 \) be left exact functors from \( \mathcal{B} \) to another abelian category \( \mathcal{C} \) with tensor products. Assume that \( \mathcal{B} \) has enough injectives and that \( f_i \) carries injectives to \( g_i \)-acyclics for \( i = 1, 2, 3 \). Let \( \mathcal{A} \) have the property \( C1(f_i) \) for \( i = 1, 2 \) and \( \mathcal{B} \) have the property \( C2(g_i) \) for \( i = 1, 2, 3 \). Then for couplings

\[
\begin{align*}
&f_1(A_1) \otimes f_2(A_2) \to f_3(A_1 \otimes A_2), \\
g_1(B_1) \otimes g_2(B_2) \to g_3(B_1 \otimes B_2),
\end{align*}
\]

inducing cupproducts \( \cup_f \) and \( \cup_g \) respectively, there is a functorial cupproduct

\[
E^{p,q}_r \otimes E^{p',q'}_{r'} \to E^{p+p',q+q'+r}_{r}, \quad r \geq 2,
\]

of the Grothendieck spectral sequences

\[
iE^{p,\nu}_r = R^p g_1 R^\nu f_1 A_i \Rightarrow iE^{p+\nu}_r = R^p g_1 f_1 A_i, \quad i = 1, 2, 3,
\]

(where \( A_3 = A_1 \otimes A_2 \)), which coincides with

\[
R^p g_1 R^q f_1 A_1 \otimes R^q g_2 R^q f_2 A_2 \xrightarrow{(-1)^{pq}} R^p g_3 (R^q f_1 A_1 \otimes R^q f_2 A_2) \xrightarrow{R^q g_3 R^q f_3} R^p g_3 R^q f_3 (A_1 \otimes A_2)
\]

for \( r = 2 \) and is compatible on the limit terms \( iE^{\infty,\nu}_r \) with the cupproduct

\[
R^m(g_1 f_1) A_1 \otimes R^m(g_2 f_2) A_2 \xrightarrow{\cup_f} R^m(g_3 f_3) (A_1 \otimes A_2)
\]

induced by

\[
g_1 f_1 A_1 \otimes g_2 f_2 A_2 \to g_3 (f_1 A_1 \otimes f_2 A_2) \to g_3 f_3 (A_1 \otimes A_2).
\]

**Proof.** a) This is proved by induction and dimension shifting in a standard way, compare Bredon [2, II 6.2].

b) Since \( A' \) and \( B' \) are admissible, \( A' \otimes B' \) is a resolution of \( A \otimes B \), as easily follows from the first or second spectral sequence for the double complex \((A^p \otimes B^q, d_A \otimes \text{id}, (-1)^p \text{id} \otimes d_B)\), whose total complex is \( A \otimes B \). Therefore we get the wanted morphism into an injective resolution \( I \) of \( A \otimes B \). The second claim again follows by induction and dimension shifting; note that \( A' \otimes B' \to J' \) induces \( A'[1] \otimes B'[1] \to J'[1] \) by taking the same morphisms and \( A' \otimes B'[1] \to J'[1] \) by multiplying \( A^p \otimes B^{q+1} \to J^{p+q+1} \) with \((-1)^p\), and that canonically \( H^p(A'[1]) = H^{p+1}(A') \) etc. (here \( A'[1] = A'^{p+1} \) and \( d_{A[1]} = -d_A \) as usual).

c) Let \( A_i \) be admissible \( f_i \)-acyclic resolutions of \( A_i \) for \( i = 1, 2 \) and \( C_i \) be the functorial admissible \( g_i \)-acyclic resolutions for \( i = 1, 2 \). Let \( J' \) be an injective resolution of \( A_1 \otimes A_2 \) and let \( f_3 J' \subseteq I' \) be an injective resolution of the complex \( f_3 J' \).

By b) we have a morphism \( A_1 \otimes A_2 \to J' \) such that

\[
f_1 A_1 \otimes f_2 A_2 \to f_3 J'.
\]

induces \( \cup_f \). Similarly, \( C_1(f_1 A_1) \otimes C_2(f_2 A_2) \) is a resolution of the complex \( f_1 A_1 \otimes f_2 A_2 \), therefore we get a morphism into an injective resolution of \( f_1 A_1 \otimes f_2 A_2 \) and by composition a morphism into \( I' \). This means we have pairings

\[
C_1^p(f_1 A_1) \otimes C_2^p(f_2 A_2) \xrightarrow{h} I^{p+s, q+t}
\]
with
\[ \rho^d \circ h = h \circ (c_1 \otimes \text{id}) + (-1)^p h \circ (\text{id} \otimes c_2) \]
and
\[ \rho^{1d} \circ h = h \circ (d_1 \otimes \text{id}) + (-1)^q h \circ (\text{id} \otimes d_2). \]
Here \( \rho^d \) and \( \rho^{1d} \) are the first and second differentials of \( I^* \), respectively, \( c_i \) is the differential of \( C_i \), and \( d_i \) is the differential induced by the differential \( d_i \) of \( A_i \) \((i = 1, 2)\). Taking homology w.r.t. the second differentials we get pairings
\[ C^q_i(\mathbb{R}^q f_1 A_1) \otimes C^q_i(\mathbb{R}^q f_2 A_2) \rightarrow h \otimes \mathbb{H}^q + r(I^{p+q}), \]
as the \( C_i \) are exact functors. By assumption on \( I \) the complex \( \mathbb{H}^q + r(I^{p+q}) \) is an injective resolution of \( H^q + r(f_3) = \mathbb{R}^q f_3(A_1 \otimes A_2) \), so by b) we see that \( (-1)^q h \) induces the claimed pairing on the \( E_2 \)-terms.

On the other hand \(-1)^p h \) induces pairings of the associated double complexes (with second differentials \(-1)^p \rho^{d_1}, (-1)^q \rho^{d_2}, \) and \(-1)^p \rho^{d_3}, \) respectively, whose first spectral sequences (after applying \( g_1, g_2, \) and \( g_3 \) respectively) are just the Leray spectral sequences. So we get the pairing on these in an obvious way, compare Bredon [2, Appendix 3]. To see the compatibility with the cup product on the limit term, one may argue with the second spectral sequences like in [2, IV 6.5].

If \( \mathcal{A} \) is an abelian category with tensor product, then \( \mathcal{A}^\mathbb{N} \) has the tensor product given by
\[ (A_n, d_n) \otimes (B_n, e_n) = (A_n \otimes B_n, d_n \otimes e_n). \]

(6.3) Proposition. a) If \( \mathcal{A} \) has the property \( C_1(f) \) (resp. \( C_2(f) \)) for \( f: \mathcal{A} \rightarrow \mathcal{B} \), then \( \mathcal{A}^\mathbb{N} \) has the property \( C_1(f^{\mathbb{N}}) \) (resp. \( C_2(f^{\mathbb{N}}) \)), and \( \mathcal{A}^\mathbb{N} \) has the property \( C_1(\lim_n f) \) (resp. \( C_2(\lim_n f) \)), if \( \mathcal{B} \) satisfies \( AB 4^* \).

b) If \( G \) is a profinite group, then \( M(G) \) has the property \( C_2(H^0(G, -)) \).

c) If \( \mathcal{A} = \mathcal{S}(X_{et}) \) for a scheme \( X \) and every induced sheaf is \( f \)-acyclic (e.g., \( f = H^0_Z(X, -) \) for a closed subscheme \( Z \subseteq X \)), then \( \mathcal{A} \) has the property \( C_2(f) \). If moreover \( \mathcal{B} = \mathcal{S}(Y_{et}) \) for a scheme \( Y \) and \( f \) carries induced sheaves to \( H^0(Y, -) \)-acyclics (e.g., \( f = \pi_* \) for a morphism \( \pi: X \rightarrow Y \) or \( f = i^* \) for a closed immersion \( i: Y \rightarrow X \)), then \( \mathcal{A}^\mathbb{N} \) has the property \( C_2(\lim_n f) \).

Proof. b) follows with the resolutions by induced modules, cf. Serre [12, I 2.5], and the first claim of c) with the Godement resolution, cf. Milne [9, V 1.15].

In a), the statements for \( f^{\mathbb{N}} \) are clear by (1.2). For \( \lim_n f \) let \( (A_n, d_n) \) be an object of \( \mathcal{A}^\mathbb{N} \) and let \( 0 \rightarrow A_n \rightarrow A_n^0 \rightarrow A_n^0/A_n \rightarrow 0 \) be admissible sequences for \( n \in \mathbb{N} \), with \( f \)-acyclic objects \( A_n^0 \). Then
\[ 0 \rightarrow (A_n, d_n) \rightarrow (P(A_n^0), P(A_n^0)/(A_n, d_n)) \rightarrow 0, \]
with \( i \) induced by the adjunction of \( P \) and \( V \) [see the proof of (1.1)], is an admissible sequence, and functorial, if the \( A_n^0 \) can be defined functorially. So it suffices to show that \( P(A_n^0) \) is acyclic for \( \lim_n f \). If \( \mathcal{B} \) satisfies \( AB 4^* \), this follows from (1.3), since
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\( f^N P(A_n^0) = P(fA_n^0) \) is acyclic for \( \lim_n \). Now assume that \( \mathcal{A} = \mathcal{S}(X_{et}), \mathcal{B} = \mathcal{S}(Y_{et}) \), and that the \( A_n^0 \) are induced sheaves on \( X_{et} \). If such sheaves are carried to \( H^0(Y, -) \)-acyclic sheaves, the same is true for injective sheaves, so there is a spectral sequence

\[
\lim_n R^q f^N P(A_n^0) \Rightarrow R^{p+q} \left( \lim_n f \right) P(A_n^0)
\]

by the lemma below. Therefore it suffices to show that \( P(fA_n^0) \) is acyclic for \( \lim_n \), which again follows from the lemma.

The examples given in c) are well-known, note that \( i^* \) carries induced sheaves to induced sheaves.

(6.4) **Lemma.** If \( X \) is a scheme and \( (F_n) \) is a family of \( H^0(X, -) \)-acyclic sheaves on \( X_{et} \), then \( P(F_n) \) is acyclic for \( \lim_n \) and \( \lim H^0(X, -) \).

**Proof.** By (3.12) we only have to show the second statement, which follows with (3.1) and (1.15).

If there is a natural cupproduct for functors \( f_1, f_2, \) and \( f_3 \) on \( \mathcal{A} \), we always take the "componentwise" cupproduct for \( f_1^N, f_2^N, \) and \( f_3^N \) on \( \mathcal{A}^N \). If \( \lim f_i \) exists for \( i = 1, 2, 3 \), we always take the cupproduct for these functors that is induced by the functorial morphisms

\[
\lim_n f_1 A_n \otimes \lim_n f_2 B_n \to \lim_n f_3 (A_n \otimes B_n)
\]

which are the limit of the bilinear maps \( f_1 A_n \times f_2 B_n \to f_3 (A_n \otimes B_n) \). In particular, there are canonical cupproducts (compare [6, V 1.17(c)])

(6.5) \[
H^p Z_1(X, (F_n)) \otimes H^q Z_2(X, (G_n)) \to H^{p+q} Z_1 \otimes Z_2(X, (F_n \otimes G_n))
\]

for étale pro-sheaves \( (F_n) \) and \( (G_n) \) on a scheme \( X \), \( Z_1 \) and \( Z_2 \) being closed subschemes, and a canonical cupproduct

(6.6) \[
H^p(G, (M_n))^\otimes H^q(G, (N_n)) \to H^{p+q}(G, (M_n \otimes N_n))
\]

for inverse systems of discrete \( G \)-modules \( (M_n) \) and \( (N_n) \) for a profinite group \( G \).

(6.7) **Proposition.** Via the isomorphism in (2.2), the cupproduct (6.6) is compatible with the cupproduct for continuous group cohomology defined by Tate in [14, Sect. 2].

**Proof.** The cupproduct for continuous group cohomology, w.r.t. the continuous pairing \( \lim M_n \times \lim N_n \to \lim (M_n \otimes N_n) \), is induced by a morphism of complexes

\[
C^*(G, \lim M_n) \otimes C^*(G, \lim N_n) \to C^*(G, \lim (M_n \otimes N_n))
\]

[notations as in the proof of (2.2)], which is obtained by passing to the limit over \( n \) from morphisms

\[
C^*(G, M_n) \otimes C^*(G, N_n) \to C^*(G, M_n \otimes N_n).
\]
These are in turn induced by morphisms

$$D'(G, M_n) \otimes D'(G, N_n) \to D'(G, M_n \otimes N_n)$$

via taking fixed modules under $G$, compare [4, 4]. Since $D'(G, (M_n))$ is an admissible $\lim H^0(G, -)$-acyclic resolution for $(M_n)$, for any $(M_n)$ satisfying the Mittag-Leffler condition, the claim follows from (6.2b).

(6.8) Proposition. If $\mathcal{B} = \mathbb{A}_b$ in the situation of (6.2a), then there are commutative diagrams for systems $(A_n)$ and $(B_n)$ in $\mathcal{A}^N$

$$R^p \left( \lim_{n} f_1 \right)(A_n) \times R^q \left( \lim_{n} f_2 \right)(B_n) \to R^{p+q} \left( \lim_{n} f_3 \right)(A_n \otimes B_n)$$

$$\xymatrix{ \lim_{n} R^{p-1} f_1 A_n \times \lim_{n} R^q f_1 B_n \ar[r]^-{\cup} & \lim_{n} R^{p+q-1} f_3 (A_n \otimes B_n) }$$

where $(\overline{z})$ is the image of $(z_n) \in \prod_n C_n$ in $\lim_{n}^1 C_n$ via the canonical sequence (1.4), and $(t_n) \in \lim_{n} C_n$ is regarded as element of $\prod_n C_n$ via the same sequence, and where the vertical arrows are those from (1.6). In particular, the cup product vanishes on $\lim_{n} R^{p-1} f_1 A_n \times \lim_{n} R^q f_1 B_n$.

Proof. By Remark (1.7), the vertical maps are the edge morphisms of the Leray spectral sequence for $\lim_{n} f_i$, $i = 1, 2, 3$. The claim then follows from (6.2c) (case $s = 0$), once we have checked that the cup product

$$\lim_{n}^1 C_n \times \lim_{n}^1 D_n \to \lim_{n}^1 (C_n \otimes D_n)$$

for inverse systems $(C_n)$ and $(D_n)$ of abelian groups is given by mapping $((c_n), (d_n))$ to $(c_n \otimes d_n)$. This follows at once from (6.2b) and the standard resolution for $\lim_{n}$ on $\mathbb{A}_b^N$, see (1.5).

Remark. If one interchanges the terms on the left side, one has to add a sign $(-1)^q$ below to have a commutative diagram, as we change by $(-1)^{pq}$ above and $(-1)^{p-1}q$ below. This agrees perfectly with the sign $(-1)^{qs}$ in (6.2c).

We now want to compare the cup product with the Yoneda-pairing. For later applications it is convenient to consider the Ext-groups not in $S(X_{et})^N$, but in a certain subcategory. (6.9) Definition. For an abelian category $\mathcal{A}$ and a prime $l$, denote by $\mathcal{A}^Z_l$ the full subcategory of $\mathcal{A}^N$ consisting of the systems $(A_n, d_n)$ with $l^n A_n = 0$ for all $n \in \mathbb{N}$ (called pro-$l$-systems).

(6.10) Proposition. Let $X$ be a scheme and $Z$ be a closed subscheme.

a) $S(X_{et})^Z_l$ has enough injectives.

b) The cohomology functors $H^*_Z(X, -)$ are the same, whether computed as derived functors of $H^*_Z(X, -)$ in $S(X_{et})^N$ or $S(X_{et})^Z_l$. 
c) Let \( \text{Ext}^i_{-X}((F_n), (G_n)) := \text{Ext}^i_{(\mathcal X_\text{et})^\text{Z}}((F_n), (G_n)) \) for objects \( (F_n) \) and \( (G_n) \) of \( \mathcal X_\text{et}^\text{Z} \).

i) There are canonical isomorphisms of \( \delta \)-functors
\[
\text{Ext}^i_{-X}(\mathbb Z, G) \xrightarrow{\psi} H^i(X, G), \quad i \geq 0,
\]
where \( \mathbb Z \) stands for the system \( (\mathbb Z/l^n) \) with obvious transition maps.

ii) There are canonical functorial homomorphisms
\[
H^i(X, G) \rightarrow \text{Ext}^i_{-X}(F, F \otimes G)
\]
such that the diagram
\[
\begin{array}{ccc}
H^i(X, F) \times H^i(X, G) & \rightarrow & H^{i+s}(X, F \otimes G) \\
\psi^\dagger & & \downarrow \psi^\star \\
\text{Ext}^i_{-X}(\mathbb Z, F) \times \text{Ext}^i_{-X}(F, F \otimes G) & \rightarrow & \text{Ext}^i_{-X}(\mathbb Z, F \otimes G)
\end{array}
\]
commutes, where the upper map is the cupproduct and the lower the Yoneda pairing.

Proof. a) This follows as in the proof of (1.1a), since \( S(X_\text{et}, \mathbb Z/l^n) \) has enough injectives and for a family \((I_n)\), with \( I_n \) injective in \( S(X_\text{et}, \mathbb Z/l^n) \), the inverse system \( P(I_n) \) is an injective object of \( S(X_\text{et})^{Z*} \).

b) For \((I_n)\) as above, the object \( P(I_n) \) is acyclic for \( H_2^i(X, -) \) on \( S(X_\text{et})^{N} \), since the inclusion \( S(X_\text{et}, \mathbb Z/l^n) \rightarrow S(X_\text{et}) \) maps injective sheaves to flabby sheaves.

c) i) The isomorphisms are obtained by deriving the obvious one for \( i=0 \).

ii) Let \( G \rightarrow C'(G) \) be the “canonical resolution” obtained by repeated embedding \( G \rightarrow C^0(G) \rightarrow PVC^0(G) \), where \( C^0 \) is the Godement functor (cf. [9, III 120(c)] and the proof of (6.3c) above), and let \((F \otimes G)^* \) be an injective resolution of \( F \otimes G \). Then there is a morphism \( F \otimes C'(G) \rightarrow (F \otimes G)^* \) of resolutions of \( F \otimes G \), unique up to homotopy, and the \( \psi^i \) are induced by the obvious morphisms
\[
\text{Hom}_{-X}(\mathbb Z, C'(G)) \rightarrow \text{Hom}_{-X}(F, F \otimes C'(G)) \rightarrow \text{Hom}_{-X}(F, (F \otimes G)^*)
\]
With this, the commutativity is proved literally as Proposition 1.20 in V Sect. 1 of Milne’s book [9].

Using (6.10c) ii), one now can proceed exactly as Milne in [9, VI 6.5] to prove the wanted projection formula for the cupproduct:

(6.11) Proposition. Let \( i: Z \rightarrow X \) be a smooth \( S \)-pair of co-dimension \( c \) for a scheme \( S \), and let \( l \) be a prime invertible on \( S \). Then
\[
H^i_{\text{cont}}(Z, \mathbb Z_l) \times H^i_{\text{cont}}(Z, \mathbb Z_l) \xrightarrow{\omega} H^{i+s}_{\text{cont}}(Z, \mathbb Z_l)
\]
\[
\text{Hom}_{-X}(\mathbb Z_l, C'(G)) \rightarrow \text{Hom}_{-X}(F, F \otimes C'(G)) \rightarrow \text{Hom}_{-X}(F, (F \otimes G)^*)
\]
is commutative, where \( i_* \) is the Gysin map.

We now can prove the main result of this section.

(6.12) Theorem. Let \( l \) be a prime. Then there is a unique theory of Chern classes
\[
c_i(E) \in H^{2i}_{\text{cont}}(S, \mathbb Z_l(i)) \quad i \geq 0,
\]
for locally free \( \mathcal O_S \)-modules \( E \) on schemes \( S \) on which \( l \) is invertible such that for \( c(E) = (c_i(E)) \in \prod_i H^{2i}_{\text{cont}}(S, \mathbb Z_l(i)) \)
a) \(c(L) = (1, \delta_S([L]), 0, \ldots)\) for \(L\) invertible,

b) \(c(E) = c(E') \cdot c(E'')\) for an exact sequence of locally free \(O_S\)-modules \(0 \to E' \to E \to E'' \to 0\).

c) \(f^*c(E) = c(f^*E)\) for morphisms \(f : S' \to S\).

**Proof.** This is provided by the following data and axioms, see Grothendieck [7] and also Jouanolou [SGA 5, VII 3].

i) Via the cupproduct, \(H^*(S) = \bigoplus_{i \geq 0} H^{2i}_{cont}(S, \mathbb{Z}(i))\) is a graded ring, functorially associated to \(S\) in a contravariant way.

ii) \(\delta_S : \text{Pic}(S) \to H^2_{cont}(S, \mathbb{Z}(1)),\) defined in (3.26) via the Kummer sequence \(0 \to \mathbb{Z}(1) \to \mathbb{G}_m \to \mathbb{G}_m \to 0,\) is a morphism of functors.

iii) For any smooth \(S\)-pair \(i : Z \subset X\) of codimension \(c, S\) a scheme over \(\mathbb{Z}[1/\ell]\), the Gysin morphism \(i_* : H^*(Z) \to H^*(X)\) is a group morphism raising degrees by \(2c\).

A1) For any vector bundle \(E\) of rank \(m\) on \(S/\mathbb{Z}[1/\ell]\) there is a Dold-Thom isomorphism \(H^*(\mathbb{P}_S(E)) \cong \bigoplus_{i \geq 0} H^*(S) \mathbb{Z}_E^{-i},\) which holds more generally by the proposition below.

A2) If \(i : Z \subset X\) is a smooth \(S\)-pair of codimension 1, then \(\delta_X([\mathcal{O}(Z)]) = i_*(1),\) see (3.26a).

A3) The Gysin morphisms behave transitively. This is immediately clear from the description of the local cycle class in [SGA 4\(_2\), (cycle) 2.2.2c].

A4) For a smooth \(S\)-pair \(i : Z \subset X\) the projection formula holds by 6.11: \(i_*(i^!(x) \cup z) = x \cup i_*(z)\) for \(x \in H^*(X), z \in H^*(Z)\).

In fact, then the Chern classes \(c_i(E)\) of a vector bundle \(E\) of rank \(m\) on \(S\) are the unique elements in \(H^*_{cont}(S, \mathbb{Z}(i))\) with

\[
\sum_{i=0}^{m} c_i(E) \cup \xi_E^{m-1-i} = 0 \quad \text{in} \quad H^*_{cont}(\mathbb{P}_S(\tilde{E}), \mathbb{Z}(m))
\]

\[
c_0(E) = 1, \quad c_i(E) = 0 \quad \text{for} \quad i > m,
\]

where \(\tilde{E}\) is the dual bundle and the notations are as in (6.13). (In [9, p. 273] one either should take \(\tilde{E}\) as above or \(\xi\) the class of \(\mathcal{O}(-1)\).) For general locally free \(O_S\)-modules one has to proceed as in [SGA 5, VII 3.2].

(6.13) **Proposition.** Let \(S\) be a scheme and \(\ell\) a prime invertible on \(S\). Let \(E\) be a locally free \(O_S\)-module of rank \(m,\) and \(P = \mathbb{P}_S(\mathcal{O})\) be the associated projective fibre bundle. Let \(\mathcal{O}_P(1)\) be the canonical invertible sheaf on \(P\) and \(\xi = \xi_E\) be the image of its class \([\mathcal{O}_P(1)]\) in \(\text{Pic}(P)\) under the morphism

\[
\delta_P : \text{Pic}(P) \to H^2_{cont}(P, \mathbb{Z}(1))
\]

defined in (3.26). Then via \(p^*\) the bi-graded ring \(\bigoplus_{i,j} H^i_{cont}(P, \mathbb{Z}(j))\) is a free module over \(\bigoplus_{r,s} H^r_{cont}(S, \mathbb{Z}(s))\) with basis \(1, \xi, \ldots, \xi^{m-1}\), i.e., the map

\[
\bigoplus_{r,s} H^{r-2v}_{cont}(S, \mathbb{Z}(j-v)) \to H^r_{cont}(P, \mathbb{Z}(j))
\]

\[
(x_0, \ldots, x_{m-1}) \mapsto \sum_{v=0}^{m-1} p^*x_v \cup \xi^v
\]

are isomorphisms for every \(i\) and \(j\).
Proof. By (6.8), this follows, by passing to $\lim$ and $\lim^1$, from the corresponding result with $\mathbb{Z}_l(j)$ replaced by the finite coefficients $\mathbb{Z}/l^n(j)$ for any $n$, which holds by [SGA 5, VII 2.2.4].

We finish this section with two properties of the cycle map.

(6.14) Lemma. Let $X$ be a smooth variety over a field $k$, $l \neq \text{char}(k)$.

i) If two cycles $Z_1$ and $Z_2$ on $X$ are linearly equivalent, then $\text{cl}^X(Z_1) = \text{cl}^X(Z_2)$. Therefore the cycle classes induce homomorphisms for $i \geq 1$

$$\text{cl}^X = \text{cl}^{X, i} : \text{CH}^i(X) \rightarrow H^2i_{\text{cont}}(X, \mathbb{Z}_l(i)),$$

where $\text{CH}^i(X)$ is the Chow group of cycles of codimension $i$ modulo linear equivalence.

ii) If $Z_1$ and $Z_2$ are two cycles on $X$ and $Z_1 \cdot Z_2$ is their intersection product, then

$$\text{cl}^X(Z_1 \cdot Z_2) = \text{cl}^X(Z_1) \cdot \text{cl}^X(Z_2).$$

Proof. i) By assumption, there are a cycle $Z$ on $\mathbb{P}^1_k \times X$ and two points $t_1, t_2$ of $\mathbb{P}^1_k$ such that for the induced morphisms $\varphi_i : X \rightarrow \mathbb{P}^1_k \times X$, $i = 1, 2$ one has $\varphi_i^*(Z) = Z_1$ and $\varphi_2^*(Z) = Z_2$. Choosing an automorphism $\alpha$ of $\mathbb{P}^1_k$ with $\alpha(t_1) = t_2$ we have $(\alpha \times \text{id}_X)^* \varphi_1 = \varphi_2$ and therefore

$$\text{cl}(Z_2) = \varphi_2^*(\text{cl}(Z)) = \varphi_2^*((\alpha \times \text{id}_X)^* \text{cl}(Z)) = \varphi_2^*(\text{cl}(Z)) = \text{cl}(Z_1),$$

since $(\alpha \times \text{id}_X)^*$ acts trivially on $H^*(\mathbb{P}^1_k \times X)$ by (6.12): it fixes $H^*(X)$ and the canonical ample class which comes from $\mathbb{P}^1_k$.

ii) Once the cupproduct is defined, this follows as in (3.25) by passing to classes with support and then to finite coefficients, where the statement holds by [SGA 4, (cycle) 2.3.9]. Note that by i) and Chow's moving lemma we may assume that $Z_1$ and $Z_2$ intersect properly.

(6.15) Remark. a) It is not true in general, that the cycle map factorizes through algebraic equivalence, unless $k$ is algebraically closed. Indeed by the theorem of Mordell-Weil the morphism

$$\text{cl}^{X, 1} = \delta_X : \text{Pic}(X) \rightarrow H^2\text{cont}(X, \mathbb{Z}_l(1))$$

is injective for $k$ of finite type over the prime field and $X$ projective over $k$: by the Kummer sequence (3.27)

$$0 \rightarrow \mathbb{Z}_l(1) \rightarrow (G_m, l) \rightarrow G_m \rightarrow 0$$

and the functoriality of (3.1), the kernel of the above map is $l\text{-Div}(\text{Pic}(X))$ for any scheme $X$. If moreover $H^2(X, G_m)$ is finite, the induced map $\text{Pic}(X) \otimes \mathbb{Z}_l \rightarrow H^2\text{cont}(X, \mathbb{Z}_l(1))$ is an isomorphism. It would be very interesting to get similar results on $\text{cl}^{X, i}$ for $i \geq 2$.

b) It follows from the definitions that the cycle map

$$\overline{\text{cl}}^X : \text{CH}^i(X) \rightarrow H^2i(X, \mathbb{Z}_l(i))^G_k$$

considered by Tate in [15] is the composition of $\text{cl}^X$ and the restriction map

$$H^2i_{\text{cont}}(X, \mathbb{Z}_l(i)) \rightarrow H^2i(X, \mathbb{Z}_l(i))^G_k$$
where \( \bar{X} = X \times_k \bar{k} \) for an algebraic closure \( \bar{k} \) of \( k \). Tate conjectures that for a finitely generated field \( k \) and \( X/k \) smooth projective \( \text{cl}^X \) becomes surjective after tensoring with \( \mathbb{Q}_l \) [15, (7)]. A necessary condition for this to be true therefore is that \( \text{res} \) is surjective up to torsion or, equivalently, that all differentials
\[
d_{r,0}^{0,2i} : E_{r,0}^{0,2i} \rightarrow E_{r,0}^{2i-r+1}
\]
in the spectral sequence of (3.5b) have torsion images. In a following paper we shall in fact show this property for all differentials \( d_{r,p}^{p,q} \), \( p, q \geq 0 \).

c) Let \( \text{CH}^i(X)_0 \) be the kernel of \( \text{cl}^X \), i.e., the subgroup generated by cycles homologous to zero (perhaps depending on \( l \)). Then the commutative diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & \text{CH}^i(X)_0 & \rightarrow & \text{CH}^i(X) & \rightarrow & \text{CH}^i(X) / \text{CH}^i(X)_0 & \rightarrow & 0 \\
\downarrow & & \downarrow \text{cl}^X & & & & \downarrow \text{cl}^X & & \\
0 & \rightarrow & F^1 H_{\text{cont}}^{2i}(X, \mathbb{Z}_l(i)) & \rightarrow & H_{\text{cont}}^{2i}(X, \mathbb{Z}_l(i)) & \rightarrow & H_{\text{cont}}^{2i}(X, \mathbb{Z}_l(i))^{\text{res}} E_2 = E_2^{0,2i} & \rightarrow & 0 \\
\downarrow & & & & & & \downarrow & & \\
E_2^{1,2i-1} = H_{\text{cont}}^{1}(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))) & & & & & & & & \\
\end{array}
\]
induced by the Hochschild-Serre spectral sequence (3.5b) defines an "Abel-Jacobi map"
\[
\text{cl}^X_0 : \text{CH}^i(X)_0 \rightarrow H_{\text{cont}}^{1}(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))),
\]
and it can be shown that for a cycle \( Z \) on \( X \) homologous to zero
\[
\text{cl}^X_0(Z) \in H_{\text{cont}}^{1}(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))) = \text{Ext}^1_{G_k}(\mathbb{Z}_l, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)))
\]
is the class of the extension of continuous \( G_k \)-modules
\[
0 \rightarrow H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)) \rightarrow H^{2i-1}(\bar{U}, \mathbb{Z}_l(k)) \rightarrow H^{2i}(\bar{X}, \mathbb{Z}_l(i)) \rightarrow H^{2i}(\bar{X}, \mathbb{Z}_l(i))
\]
\[
\downarrow \quad \text{res} \quad \downarrow \quad \text{res} \quad \downarrow
\]
\[
0 \rightarrow H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)) \rightarrow E \rightarrow \mathbb{Z}_l \cdot \text{cl}^X_0(Z) \rightarrow 0
\]
obtained by pull-back, where \( U = X \setminus \{Z\} \). This will be discussed in another paper.

References
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Received December 31, 1986