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Michael Hellus and Peter Schenzel

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NOTES ON LOCAL COHOMOLOGY AND DUALITY

MICHAEL HELLUS AND PETER SCHENZEL

ABSTRACT. We provide a formula (see Theorem 1.5) for the Matlis dual of the injective hull of R/\mathfrak{p} where \mathfrak{p} is a one dimensional prime ideal in a local complete Gorenstein domain (R, \mathfrak{m}) . This is related to results of Enochs and Xu (see [4] and [3]). We prove a certain 'dual' version of the Hartshorne-Lichtenbaum vanishing (see Theorem 2.2). There is a generalization of local duality to cohomologically complete intersection ideals I in the sense that for $I = \mathfrak{m}$ we get back the classical Local Duality Theorem. We determine the exact class of modules to which a characterization of cohomologically complete intersection from [6] generalizes naturally (see Theorem 4.4).

In this paper we prove a Matlis dual version of Hartshorne-Lichtenbaum Vanishing Theorem and generalize the Local Duality Theorem. The latter generalization is done for ideals which are cohomologically complete intersections, a notion which was introduced and studied in [6]. The generalization is such that local duality becomes the special case when the ideal I is the maximal ideal \mathfrak{m} of the given local ring (R, \mathfrak{m}) . We often use formal local cohomology, a notion which was introduced and studied by the second author in [13]. Formal local cohomology is related to Matlis duals of local cohomology modules (see [5, Sect. 7.1 and 7.2] and Corollary 3.4).

We start in Section 1 with the study of the Matlis duals of local cohomology modules $H_I^{n-1}(R)$, where $n = \dim R$. The latter is also the formal local cohomology module $\varprojlim H_{\mathfrak{m}}^1(R/I^\alpha)$ provided R is a Gorenstein ring. We describe this module as the cokernel of a certain canonical map. As a consequence we derive a formula (see Theorem 1.5) for the Matlis dual of $E_R(R/\mathfrak{p})$, where $\mathfrak{p} \in \text{Spec } R$ is a 1-dimensional prime ideal. In some sense this is related to results by Enochs and Xu (see [4] and [3]).

In Section 3 we generalize the Local Duality (see Theorem 3.1). The canonical module in the classical version is replaced by the dual of $H_I^c(R)$ where I is a cohomologically complete intersection ideal of grade c (the case $I = \mathfrak{m}$ specializes to the classical local duality). See also [5, Theorem 6.4.1]. It is a little bit surprising that the d -th formal local cohomology occurs as the duality module for the duality of cohomologically complete intersections in a Gorenstein ring (see Corollary 3.4).

In Theorem 4.4 we generalize the main result [6, Theorem 3.2]. This provides a characterization of the property of 'cohomologically complete intersection' given for ideals to finitely generated modules. Finally, in Section 5 we fill a gap in our proof of [6, Lemma 1.2]. To this end we use a result on inverse limits as it was shown by the second author (see [11]). Some of the results of Section 4 are obtained independently by W. Mahmood (see [9]).

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1. ON FORMAL LOCAL COHOMOLOGY

Let (R, \mathfrak{m}) be a local ring, let $I \subset R$ be an ideal. In the following let \hat{R}^I denote the I -adic completion of R . Let $0 = \bigcap_{i=1}^r \mathfrak{q}_i$ denote a minimal primary decomposition of the zero ideal. Then we denote by $u(I)$ the intersections of those $\mathfrak{q}_i, i = 1, \dots, r$, such that $\dim R/(\mathfrak{p}_i + I) > 0$, where $\text{Rad } \mathfrak{q}_i = \mathfrak{p}_i, i = 1, \dots, r$.

For the definition and basic properties of $\varprojlim H_{\mathfrak{m}}^i(R/I^\alpha)$, the so-called formal local cohomology, we refer to [13]. We denote the functor of global transform by $T(\cdot) = \varprojlim \text{Hom}_R(\mathfrak{m}^\alpha, \cdot)$, in order to distinguish it from Matlis duality

$$D(M) = \text{Hom}_R(M, E_R(R/\mathfrak{m})),$$

where $E_R(R/\mathfrak{m})$ is a fixed R -injective hull of $k := R/\mathfrak{m}$.

Lemma 1.1. *Let $I \subset R$ denote an arbitrary ideal. Then there is a short exact sequence*

$$0 \rightarrow \hat{R}^I / u(I\hat{R}^I) \rightarrow \varprojlim T(R/I^\alpha) \rightarrow \varprojlim H_{\mathfrak{m}}^1(R/I^\alpha) \rightarrow 0.$$

Proof. For each $\alpha \in \mathbb{N}$ there is the following canonical exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(R/I^\alpha) \rightarrow R/I^\alpha \rightarrow T(R/I^\alpha) \rightarrow H_{\mathfrak{m}}^1(R/I^\alpha) \rightarrow 0.$$

It splits up into two short exact sequences

$$\begin{aligned} 0 \rightarrow H_{\mathfrak{m}}^0(R/I^\alpha) \rightarrow R/I^\alpha \rightarrow R/I^\alpha : \langle \mathfrak{m} \rangle \rightarrow 0 \text{ and} \\ 0 \rightarrow R/I^\alpha : \langle \mathfrak{m} \rangle \rightarrow T(R/I^\alpha) \rightarrow H_{\mathfrak{m}}^1(R/I^\alpha) \rightarrow 0. \end{aligned}$$

Now the inverse systems at the left side of both of the exact sequences satisfy the Mittag-Leffler condition. That is, by passing to the inverse limits it provides two short exact sequences. Putting them together there is an exact sequence

$$0 \rightarrow \varprojlim H_{\mathfrak{m}}^0(R/I^\alpha) \rightarrow \hat{R}^I \rightarrow \varprojlim T(R/I^\alpha) \rightarrow \varprojlim H_{\mathfrak{m}}^1(R/I^\alpha) \rightarrow 0.$$

Now it follows that $\varprojlim H_{\mathfrak{m}}^0(R/I^\alpha) = u(I\hat{R}^I)$, see [13, Lemma 4.1]. This finally proves the statement. \square

Of a particular interest in the above Corollary is the case where $I \subset R$ is an ideal such that $\dim R/I = 1$.

Corollary 1.2. *Suppose that $\dim R/I = 1$. Then there is a short exact sequence*

$$0 \rightarrow \hat{R}^I / u(I\hat{R}^I) \rightarrow \bigoplus_{i=1}^s \widehat{R}_{\mathfrak{p}_i} \rightarrow \varprojlim H_{\mathfrak{m}}^1(R/I^\alpha) \rightarrow 0,$$

where $\mathfrak{p}_i, i = 1, \dots, s$, denote those prime ideals \mathfrak{p} of $\text{Ass } R/I$ such that $\dim R/\mathfrak{p} = 1$.

Proof. Since $\dim R/I = 1$ there is an element $x \in \mathfrak{m}$ that is a parameter for R/I^α for all $\alpha \in \mathbb{N}$. Then there is an isomorphism $T(R/I^\alpha) \simeq R_x/I^\alpha R_x$ for all $\alpha \in \mathbb{N}$.

Now let $S = \bigcap_{i=1}^s (R \setminus \mathfrak{p}_i)$. Since $x \in S$ there is a natural isomorphism (by the local-global-principle)

$$R_x/I^\alpha R_x \simeq R_S/I^\alpha R_S \text{ for all } \alpha \in \mathbb{N}.$$

Then R_S is a semi local ring. The Chinese Remainder Theorem provides isomorphisms

$$R_S/I^\alpha R_S \simeq \bigoplus_{i=1}^s R_{\mathfrak{p}_i}/I^\alpha R_{\mathfrak{p}_i} \text{ for all } \alpha \in \mathbb{N}.$$

Now $\text{Rad } IR_{\mathfrak{p}_i} = \mathfrak{p}_i R_{\mathfrak{p}_i}$, $i = 1, \dots, s$. So by passing to the inverse limit we get the isomorphism

$$\varprojlim T(R/I^\alpha) \simeq \bigoplus_{i=1}^s \widehat{R}_{\mathfrak{p}_i}.$$

Therefore 1.1 finishes the proof of the statement. \square

Remark 1.3. In the case that R/I is one dimensional it follows that

$$H_{\mathfrak{m}}^1(R/I^\alpha) \simeq H_x^1(R/I^\alpha) \simeq H_x^1(R) \otimes R/I^\alpha \simeq (R_x/R) \otimes R/I^\alpha$$

for all $\alpha \in \mathbb{N}$, where $x \in \mathfrak{m}$ denotes a parameter of R/I . This finally implies that

$$\varprojlim H_{\mathfrak{m}}^1(R/I^\alpha) \simeq \widehat{R_x/R}^I.$$

Corollary 1.4. *Suppose that $I \subset R$ is a one dimensional ideal in a local Gorenstein ring (R, \mathfrak{m}) with $n = \dim R$. Then there is a short exact sequence*

$$0 \rightarrow \widehat{R}^I / u(I\widehat{R}^I) \rightarrow \bigoplus_{i=1}^s \widehat{R}_{\mathfrak{p}_i} \rightarrow \text{Hom}_R(H_I^{n-1}(R), E) \rightarrow 0,$$

where $\mathfrak{p}_i, i = 1, \dots, s$, denote those prime ideals \mathfrak{p} of $\text{Ass } R/I$ such that $\dim R/\mathfrak{p} = 1$.

Proof. This is clear because of $\varprojlim H_{\mathfrak{m}}^1(R/I^\alpha) \simeq \text{Hom}_R(H_I^{n-1}(R), E)$ as it is a consequence of the Local Duality Theorem for Gorenstein rings (the Hom-functor in the first place transforms a direct limit into an inverse limit). \square

In particular the Matlis dual of $H_I^{n-1}(R)$ is exactly the cokernel of the canonical map $\widehat{R}^I \rightarrow \bigoplus_{i=1}^s \widehat{R}_{\mathfrak{p}_i}$. This generalizes [5, Lemma 3.2.1] (see also [7, Lemma 1.5]).

If we assume in addition that $I = \mathfrak{p}$ is a one dimensional prime ideal and that R is a complete domain, then by [6, Theorem 3.2] the fact $H_{\mathfrak{p}}^n(R) = 0$ (as follows by the Hartshorne-Lichtenbaum Vanishing Theorem) is equivalent to: The minimal injective resolution of $H_{\mathfrak{p}}^{n-1}(R)$ looks as follows:

$$0 \rightarrow H_{\mathfrak{p}}^{n-1}(R) \rightarrow E_R(R/\mathfrak{p}) \rightarrow E_R(R/\mathfrak{m}) \rightarrow 0.$$

On the other hand we have (see Corollary 1.4) a short exact sequence

$$0 \rightarrow R \rightarrow \widehat{R}_{\mathfrak{p}} \rightarrow D(H_{\mathfrak{p}}^{n-1}(R)) \rightarrow 0.$$

Note that the natural map $R \rightarrow \widehat{R}_{\mathfrak{p}}$ is injective because R is a complete domain. Therefore, $u(\mathfrak{p}) = u(\mathfrak{p}\widehat{R}^{\mathfrak{p}}) = 0$. The comparison of the two exact sequences has the following consequence:

Applying the functor D to the first exact sequence it induces a natural homomorphism

$$R = D(E_R(R/\mathfrak{m})) \rightarrow D(E_R(R/\mathfrak{p})).$$

Since the latter is an $R_{\mathfrak{p}}$ -module, we get a map $R_{\mathfrak{p}} \rightarrow D(E_R(R/\mathfrak{p}))$ and therefore a family of homomorphisms

$$R_{\mathfrak{p}}/\mathfrak{p}^\alpha R_{\mathfrak{p}} \rightarrow D(E_R(R/\mathfrak{p}))/\mathfrak{p}^\alpha D(E_R(R/\mathfrak{p}))$$

for any integer $\alpha \in \mathbb{N}$. But now we have the isomorphisms

$$\begin{aligned} D(E_R(R/\mathfrak{p})) &= \text{Hom}_R(\varinjlim_{\alpha} \text{Hom}_R(R/\mathfrak{p}^\alpha, E_R(R/\mathfrak{p})), E_R(R/\mathfrak{m})) = \\ &= \varprojlim_{\alpha} D(E_R(R/\mathfrak{p}))/\mathfrak{p}^\alpha D(E_R(R/\mathfrak{p})). \end{aligned}$$

Therefore the above inverse systems induce a homomorphism $f : \widehat{R}_{\mathfrak{p}} \rightarrow D(E_R(R/\mathfrak{p}))$. Clearly the natural homomorphism $R = D(E_R(R/\mathfrak{m})) \rightarrow D(E_R(R/\mathfrak{p}))$ factors through f . So the above two short exact sequences induce a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & \widehat{R}_{\mathfrak{p}} & \longrightarrow & D(H_{\mathfrak{p}}^{n-1}(R)) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D(E_R(R/\mathfrak{m})) = R & \longrightarrow & D(E_R(R/\mathfrak{p})) & \longrightarrow & D(H_{\mathfrak{p}}^{n-1}(R)) & \longrightarrow & 0 \end{array}$$

All maps in this commutative diagram are canonical and it is easy to see that the vertical homomorphism on the right is the identity. Therefore f is an isomorphism too. We conclude with the following result:

Theorem 1.5. *Let \mathfrak{p} be a prime ideal of height $n - 1$ in an n -dimensional local, complete Gorenstein domain (R, \mathfrak{m}) . Then the Matlis dual of $E_R(R/\mathfrak{p})$ is $\widehat{R}_{\mathfrak{p}}$.*

This is related to results from Enochs and Xu: $D(E_R(R/\mathfrak{p}))$ is flat and cotorsion by [4, Lemma 2.3] (see also [8, Theorem 1.5]), therefore (see [4, Theorem]), it is isomorphic to a direct product of modules $T_{\mathfrak{q}}$ (over $\mathfrak{q} \in \text{Spec } R$) where each $T_{\mathfrak{q}}$ is the completion of a free module over $R_{\mathfrak{q}}$. It was also proved in [4] that in this direct product the ranks of these free modules are uniquely determined. By [3, Theorem 2.2] these ranks are

$$\pi_0(\mathfrak{q}, D(E_R(R/\mathfrak{p}))) = \dim_{k(\mathfrak{q})} k(\mathfrak{q}) \otimes_{R_{\mathfrak{q}}} \text{Hom}_R(R_{\mathfrak{q}}, D(E_R(R/\mathfrak{p})))$$

(all higher π_i for $i > 0$ vanish since $D(E_R(R/\mathfrak{q}))$ is flat: Its minimal flat resolution is trivial). For each \mathfrak{q} different from \mathfrak{p} the latter rank is zero: In case $\mathfrak{p} \neq \mathfrak{m}$ this follows from

$$\text{Hom}_R(R_{\mathfrak{q}}, D(E_R(R/\mathfrak{p}))) = D(R_{\mathfrak{q}} \otimes_R E_R(R/\mathfrak{p})) = 0,$$

and in case $\mathfrak{q} = \mathfrak{m}$ we have

$$(R/\mathfrak{m}) \otimes_R D(E_R(R/\mathfrak{p})) = D(\text{Hom}_R(R/\mathfrak{m}, E_R(R/\mathfrak{p}))) = 0.$$

Therefore, the use of those results from [4], [3] leads us to

$$D(E_R(R/\mathfrak{p})) = T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is the completion of a free $R_{\mathfrak{p}}$ -module.

Our Theorem 1.5 gives the more precise information that the rank of this free module is exactly 1, i. e. $T_{\mathfrak{p}} \cong \widehat{R}_{\mathfrak{p}}$.

Question 1.6. *Is it possible to deduce the fact that this rank is 1 directly from [3, Theorem 2.2], i. e., without using our theorem 1.5?*

2. A REMARK ON THE HARTSHORNE-LICHTENBAUM VANISHING THEOREM

In this Section there is a comment on the Hartshorne-Lichtenbaum Vanishing Theorem in view to the previous investigations. Let I denote an ideal in a local Noetherian ring (R, \mathfrak{m}) with $\dim R = n$. As above let \widehat{R}^I denote the I -adic completion of R . Let $0 = \bigcap_{i=1}^r \mathfrak{q}_i$ denote a minimal primary decomposition of the zero ideal. Then we denote by $v(I)$ the intersection of those $\mathfrak{q}_i, i = 1, \dots, r$, such that $\dim R/(\mathfrak{q}_i + I) > 0$ and $\dim R/\mathfrak{q}_i = n$.

Recall that $v(I) = u(I)$ (for the ideal $u(I)$ as introduced at the beginning of Section 1) if R is equi-dimensional.

The following result provides a variant of the Hartshorne Lichtenbaum Vanishing Theorem.

Theorem 2.1. ([11, Theorem 2.20]) *Let $I \subset R$ denote an ideal and $n = \dim R$. Then*

$$H_I^n(R) \cong \text{Hom}_R(v(I\hat{R}), E_R(R/\mathfrak{m})).$$

That is $H_I^n(R)$ is an Artinian R -module and $H_I^n(R) = 0$ if and only if $\dim \hat{R}/(I\hat{R} + \mathfrak{p}) > 0$ for all $\mathfrak{p} \in \text{Ass } \hat{R}$ with $\dim \hat{R}/\mathfrak{p} = n$.

For an ideal I of a Noetherian ring R let $\text{As } I$ denote the ultimate constant (see [1]) value of $\text{Ass } R/I^\alpha$ for $\alpha \gg 0$. We define the multiplicatively closed set $S = \bigcap_{\mathfrak{p} \in \text{As } I \setminus \{\mathfrak{m}\}} R \setminus \mathfrak{p}$. Then there is an exact sequence

$$0 \rightarrow I^\alpha : \langle \mathfrak{m} \rangle / I^\alpha \rightarrow R/I^\alpha \rightarrow R_S / I^\alpha R_S \text{ for all } \alpha \gg 0.$$

Since the modules on the left are of finite length the corresponding inverse system satisfies the Mittag-Leffler condition. By passing to the inverse limit it induces an exact sequence

$$0 \rightarrow u(I\hat{R}^I) \rightarrow \hat{R}^I \rightarrow \widehat{R}_S^I$$

(see [13, Lemma 4.1]). Now let R denote a complete equidimensional local ring. Then the natural homomorphism $R \rightarrow \widehat{R}_S^I$ is injective if and only if $H_I^n(R) = 0$. This follows since $v(I) = u(I) = 0$ if and only if $H_I^n(R) = 0$ under the additional assumption on R .

If in addition $\dim R/I = 1$ we have as above that $R_S / I^\alpha R_S \simeq \bigoplus_{i=1}^s R_{\mathfrak{p}_i} / I^\alpha R_{\mathfrak{p}_i}$. Therefore, if I is a one dimensional ideal in an equidimensional complete local ring (R, \mathfrak{m}) . Then the natural homomorphism $R \rightarrow \bigoplus_{i=1}^s \widehat{R}_{\mathfrak{p}_i}$ is injective if and only if $H_I^n(R) = 0$ (the \mathfrak{p}_i are defined as above). In case R is in addition a domain then our map $R \rightarrow \bigoplus_{i=1}^s \widehat{R}_{\mathfrak{p}_i}$ is clearly injective and, therefore, $H_I^n(R) = 0$.

In the following we shall continue with this series of ideas in the case of (R, \mathfrak{m}) a Gorenstein ring. To this end we put $V(I)_1 = \{\mathfrak{p} \in V(I) \mid \dim R/\mathfrak{p} = 1\}$.

Theorem 2.2. *Let (R, \mathfrak{m}) denote a Gorenstein ring with $n = \dim R$. Let $I \subset R$ denote an ideal. Then $D(H_I^n(R))$ is isomorphic to the kernel of the natural map $\hat{R} \rightarrow \prod_{\mathfrak{p} \in V(I)_1} \widehat{R}_{\mathfrak{p}}$. In particular, this homomorphism is injective if and only if $H_I^n(R) = 0$.*

Proof. By applying the section functor $\Gamma_I(\cdot)$ to the minimal injective resolution of the Gorenstein ring R it provides an exact sequence

$$\bigoplus_{\mathfrak{p} \in V(I)_1} E_R(R/\mathfrak{p}) \rightarrow E_R(R/\mathfrak{m}) \rightarrow H_I^n(R) \rightarrow 0.$$

Now apply the Matlis duality functor $D(\cdot)$ to the sequence. It provides the exact sequence

$$0 \rightarrow D(H_I^n(R)) \rightarrow \hat{R} \rightarrow \prod_{\mathfrak{p} \in V(I)_1} D(E_R(R/\mathfrak{p})).$$

By virtue of Theorem 1.5 it follows that $D(E_R(R/\mathfrak{p})) \cong \widehat{R}_{\mathfrak{p}}$. Therefore $D(H_I^n(R))$ is isomorphic to the kernel of the natural map $\hat{R} \rightarrow \prod_{\mathfrak{p} \in V(I)_1} \widehat{R}_{\mathfrak{p}}$. Matlis duality provides the claim. \square

If in addition $\dim R/I = 1$, then $V(I)_1$ is a finite set. Therefore the direct product in Theorem 2.2 is in fact a direct sum. Hence, the result in Theorem 2.2 is a generalization of the considerations above for the case $\dim R/I = 1$.

Moreover, in a certain sense Theorem 2.2 is a dual version to [2, Proposition] shown by Call and Sharp.

3. ON A DUALITY FOR COHOMOLOGICALLY COMPLETE INTERSECTIONS

As above let (R, \mathfrak{m}) denote a local Noetherian ring. An ideal $I \subset R$ is called a cohomologically complete intersection whenever $H_I^i(R) = 0$ for all $i \neq c$ for some c (see [6] for the definition and a characterization). If I is a cohomologically complete intersection, then in the paper of Zargar and Zakeri (see [15]) the ring R is called Cohen-Macaulay with respect to I .

The main aim of the present section is to prove a generalized local duality for a cohomologically complete intersection I . A corresponding result was already obtained by W. Mahmood (see [9]) resp. by the second author in [5, Theorem 6.4.1] by different means.

Theorem 3.1. *Let $I \subset R$ denote a cohomologically complete intersection with $c = \text{grade } I$. Let X denote an arbitrary R -module. Then there are the following functorial isomorphisms*

- (a) $\text{Tor}_{c-i}^R(X, H_I^c(R)) \cong H_I^i(X)$ and
- (b) $\text{Ext}_R^{c-i}(X, \text{Hom}_R(H_I^c(R), E_R(k))) \cong \text{Hom}_R(H_I^i(X), E_R(k))$

for all $i \in \mathbb{Z}$.

Proof. First of all choose $\underline{x} = x_1, \dots, x_r$ a system of elements of R such that $\text{Rad } \underline{x}R = \text{Rad } I$. Then we consider the Čech complex $\check{C}_{\underline{x}}$. This is a bounded complex of flat R -modules with $H^i(\check{C}_{\underline{x}}) = 0$ for all $i \neq c$ and $H^c(\check{C}_{\underline{x}}) \cong H_I^c(R)$. Moreover, $H^i(X \otimes_R \check{C}_{\underline{x}}) \cong H_I^i(X)$ (see e.g. [12]). In order to compute the cohomology $\text{Tor}_i(\check{C}_{\underline{x}}, X)$ there is the following spectral sequence

$$E_2^{i,j} = \text{Tor}_{-i}^R(H_I^j(R), X) \Rightarrow E_\infty^{i+j} = \text{Tor}_{-i-j}^R(\check{C}_{\underline{x}}, X) :$$

Since I is a cohomologically complete intersection we get a degeneration to the following isomorphisms

$$\text{Tor}_{c-i}^R(H_I^c(R), X) \cong \text{Tor}_i^R(\check{C}_{\underline{x}}, X) \cong H_I^i(X)$$

for all $i \in \mathbb{Z}$. This proves the isomorphisms of the statement in (a).

For the proof of (b) note that

$$\text{Hom}_R(\text{Tor}_{c-i}^R(H_I^c(R), X), E_R(k)) \cong \text{Ext}_R^{c-i}(X, \text{Hom}_R(H_I^c(R), E_R(k)))$$

as follows by adjunction since $E_R(k)$ is an injective R -module. \square

The Matlis dual $\text{Hom}_R(H_I^c(R), E_R(k)) = D(H_I^c(R))$ plays a central rôle in the above generalized duality. It allows to express the Matlis dual of $H_I^i(X)$ in terms of an Ext module.

Definition 3.2. Let $I \subset R$ denote a cohomologically complete intersection with $c = \text{grade } I$. Then we call $D_I(R) = \text{Hom}_R(H_I^c(R), E_R(k)) = D(H_I^c(R))$ the duality module of I .

In general the structure of $D_I(R)$ is difficult to determine. In the following we want to discuss a few particular cases of cohomologically complete intersections and their duality module. To this end let $K(R)$ denote the canonical module of R , provided it exists.

Corollary 3.3. *Let (R, \mathfrak{m}) denote a local ring such that \mathfrak{m} is a cohomologically complete intersection. There are natural isomorphisms*

$$H_{\mathfrak{m}}^{n-i}(M) \cong \text{Hom}_R(\text{Ext}_R^i(M, K(\hat{R})), E_R(k)), n = \dim R,$$

for a finitely generated R -module M and any $i \in \mathbb{Z}$. Note that R is Cohen-Macaulay ring and $D_{\mathfrak{m}}(R) \cong K(\hat{R})$.

Proof. In case \mathfrak{m} is a cohomologically complete intersection, then $\text{depth } R = \text{grade } \mathfrak{m} = \dim R$ since $H_{\mathfrak{m}}^c(R)$ is the onliest non-vanishing local cohomology module. This follows by the non-vanishing of $H_{\mathfrak{m}}^i(R)$ for $i = \text{depth } R$ and $i = \dim R$ (see e.g. [11]). Therefore R is a Cohen-Macaulay ring.

Moreover, $\text{Hom}_R(H_{\mathfrak{m}}^d(R), E_R(k)) \cong \text{Hom}_{\hat{R}}(H_{\mathfrak{m}}^d(\hat{R}), (E_{\hat{R}}(k)))$ since $E_R(k)$ is an Artinian R -module. Then \hat{R} admits a canonical module and $K(\hat{R}) \cong \text{Hom}_{\hat{R}}(H_{\mathfrak{m}}^d(\hat{R}), (E_{\hat{R}}(K)))$ (see also [11] for more details).

It is known (and easy to see) that $H_{\mathfrak{m}}^i(M)$ is an Artinian R -module for any i and a finitely generated R -module M . Then the isomorphisms follow by Theorem 3.1 (b) by the aid of Matlis duality. \square

In the particular case of a Gorenstein ring it follows that $K(\hat{R}) \cong \hat{R}$. So Corollary 3.3 provides the classically known Local Duality Theorem for a Gorenstein ring. In the following we shall consider the case of an arbitrary cohomologically complete intersection I in a Gorenstein ring.

Corollary 3.4. *Let $I \subset R$ denote a cohomologically complete intersection in a Gorenstein ring (R, \mathfrak{m}) . Then there is the isomorphism $D_I(R) \cong \varprojlim H_{\mathfrak{m}}^d(R/I^\alpha)$, where $d = \dim R/I$. That is, there are natural isomorphisms*

$$\text{Hom}_R(H_I^{c-i}(X), E_R(k)) \simeq \text{Ext}_R^i(X, \varprojlim H_{\mathfrak{m}}^d(R/I^\alpha))$$

for any R -module X and all $i \in \mathbb{Z}$.

Proof. By the definition of local cohomology there are the following isomorphisms $H_I^i(R) \cong \varinjlim \text{Ext}_R^i(R/I^\alpha, R)$ for all $i \in \mathbb{Z}$. By the duality we get the isomorphisms

$$\text{Hom}_R(H_I^c(R), E_R(k)) \cong \text{Hom}_R(\varinjlim \text{Ext}_R^c(R/I^\alpha, R), E_R(k)) \cong \varprojlim H_{\mathfrak{m}}^d(R/I^\alpha).$$

Note that the Hom-functor in the first place transforms a direct limit into an inverse limit. \square

Note that $\varprojlim H_{\mathfrak{m}}^i(R/I^\alpha)$ was studied in [13] under the name formal local cohomology. See also [13] for more details. It is a little bit surprising that the d -th formal local cohomology occurs as the duality module for the duality of cohomologically complete intersections in a Gorenstein ring.

Now we consider the particular case of a one dimensional cohomologically complete intersection in a Gorenstein ring.

Corollary 3.5. *Let $I \subset R$ denote a one dimensional cohomologically complete intersection in a Gorenstein ring R with $n = \dim R$. Let $x \in \mathfrak{m}$ be a parameter of R/I and let X denote an arbitrary R -module. Then for all $i \in \mathbb{Z}$ there are natural isomorphisms*

$$\mathrm{Hom}_R(H_I^{c-i}(X), E_R(k)) \cong \mathrm{Ext}_R^i(X, D),$$

where D denotes the cokernel of the natural homomorphism $\hat{R}^I \rightarrow \hat{R}_x^I$.

Proof. The proof is an obvious consequence of Corollary 3.4 by the aid of the results from section 1. \square

Another interpretation of the duality module D in Corollary 3.5 can be done as the cokernel of the natural map $\hat{R}^I \rightarrow \bigoplus_{i=0}^s \widehat{R}_{\mathfrak{p}_i}$ as done in Corollary 1.2.

4. COHOMOLOGICALLY COMPLETE INTERSECTIONS: A GENERALIZATION TO MODULES

In this section let I be an ideal of a local ring (R, \mathfrak{m}) . Let M denote a finitely generated R -module. Let $E_R(M)$ denote a minimal injective resolution of the R -module M . The cohomology of the complex $\Gamma_I(E_R(M))$ is by definition the local cohomology $H_I^i(M)$, $i \in \mathbb{N}$. Suppose that $c = \mathrm{grade}(I, M)$. Then $\Gamma_I(E_R^i(M)) = 0$ for all $i < c$. Therefore $H_I^c(M) = \mathrm{Ker}(\Gamma_I(E_R^c(M)) \rightarrow \Gamma_I(E_R^{c+1}(M)))$ and there is an embedding $H_I^c(M)[-c] \rightarrow \Gamma_I(E_R(M))$ of complexes.

Definition 4.1. The cokernel of the embedding $H_I^c(M)[-c] \rightarrow \Gamma_I(E_R(M))$ is defined by $C_M(I)$, the truncation complex of M with respect to I . So there is a short exact sequence

$$0 \rightarrow H_I^c(M)[-c] \rightarrow \Gamma_I(E_R(M)) \rightarrow C_M(I) \rightarrow 0$$

of complexes of R -modules. In particular $H^i(C_M(I)) = 0$ for all $i \leq c$ and $H^i(C_M(I)) \cong H_I^i(M)$ for all $i > c$.

Note that the definition of the truncation complex was used in the case of $M = R$ a Gorenstein ring in [6]. This construction is used in order to obtain certain natural homomorphisms.

Lemma 4.2. *Let M denote a finitely generated R -module with $c = \mathrm{grade}(I, M)$. Then there are natural homomorphisms*

$$H_{\mathfrak{m}}^{i-c}(H_I^c(M)) \rightarrow H_{\mathfrak{m}}^i(M)$$

for all $i \in \mathbb{N}$. These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $H_{\mathfrak{m}}^i(C_M(I)) = 0$ for all $i \in \mathbb{Z}$.

Proof. Take the short exact sequence of the truncation complex (cf. 4.1) and apply the derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}(\cdot)$. In the derived category this provides a short exact sequence of complexes

$$0 \rightarrow \mathrm{R}\Gamma_{\mathfrak{m}}(H_I^c(M))[-c] \rightarrow \mathrm{R}\Gamma_{\mathfrak{m}}(\Gamma_I(E_R(M))) \rightarrow \mathrm{R}\Gamma_{\mathfrak{m}}(C_M(I)) \rightarrow 0.$$

Since $\Gamma_I(E_R(M))$ is a complex of injective R -modules we might use $\Gamma_{\mathfrak{m}}(\Gamma_I(E_R(M)))$ as a representative of $\mathrm{R}\Gamma_{\mathfrak{m}}(\Gamma_I(E_R(M)))$. But now there is an equality for the composite of section functors $\Gamma_{\mathfrak{m}}(\Gamma_I(\cdot)) = \Gamma_{\mathfrak{m}}(\cdot)$. Therefore $\Gamma_{\mathfrak{m}}(E_R(M))$ is a representative of $\mathrm{R}\Gamma_{\mathfrak{m}}(\Gamma_I(E_R(M)))$ in the derived category.

First of all it provides the natural homomorphisms of the statement. Then the long exact cohomology sequence provides that these maps are isomorphisms if and only if $H_{\mathfrak{m}}^i(C_M(I)) = 0$ for all $i \in \mathbb{Z}$. \square

Definition 4.3. The finitely generated R -module M is called cohomologically complete intersection with respect to I in case there is an integer $c \in \mathbb{N}$ such that $H_I^i(M) = 0$ for all $i \neq c$. Clearly $c = \text{grade}(I, M)$.

This notion extends those of a cohomologically complete intersection $I \subset R$ in a Gorenstein ring R as it was studied in [6].

It is our intention now to generalize part of [6, Theorem 5.1] to the situation of a module M and an ideal $I \subset R$ satisfying the requirements of Definition 4.3. See also [9] for similar results.

Theorem 4.4. *Let (R, \mathfrak{m}) be a local ring, let M be a finitely generated R -module, I an ideal of R . Let $c := \text{grade}(I, M)$. Then the following conditions are equivalent:*

- (i) $H_I^i(M) = 0$ for all $i \neq c$.
- (ii) The natural map

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^i(H_{IR_{\mathfrak{p}}}^c(M_{\mathfrak{p}})) \rightarrow H_{\mathfrak{p}R_{\mathfrak{p}}}^{i+c}(M_{\mathfrak{p}})$$

is an isomorphism for all $\mathfrak{p} \in V(I) \cap \text{Supp } M$ and all $i \in \mathbb{Z}$.

Proof. We begin with the proof of the implication (i) \implies (ii). By the assumption it follows easily that $c = \text{grade}(IR_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all $\mathfrak{p} \in V(I) \cap \text{Supp } M$. That is we might reduce the proof to the case of the maximal ideal. By the assumption in (i) it follows (see Definition 4.1) that $C_M(I)$ is an exact bounded complex. Therefore $H_{\mathfrak{m}}^i(C_M(I)) = 0$ for all $i \in \mathbb{Z}$. So the claim follows by virtue of Lemma 4.2.

For the proof of (ii) \implies (i) we proceed by an induction on $\dim V(I) \cap \text{Supp } M =: t$. In the case of $t = 0$, i.e. $V(I) \cap \text{Supp } M = \{\mathfrak{m}\}$, it follows that $\text{R}\Gamma_{\mathfrak{m}}(C_M(I)) \cong C_M(I)$. Then the claim is true by virtue of Lemma 4.2. Now suppose that $t > 0$ and by induction hypothesis the statement holds for all smaller dimensions. Then it follows that $\text{Supp } H_I^i(M) \subseteq \{\mathfrak{m}\}$ for all $i \neq c$. By the definition of $C_M(I)$ we get that $\text{Supp } H^i(C_M(I)) \subseteq \{\mathfrak{m}\}$. Therefore it follows that $H_{\mathfrak{m}}^i(C_M(I)) \cong H^i(C_M(I))$ for all $i \in \mathbb{Z}$. By the assumption in (i) for $\mathfrak{p} = \mathfrak{m}$ it implies (see Lemma 4.2) that

$$H_{\mathfrak{m}}^i(C_M(I)) \cong H^i(C_M(I)) = H_I^i(M) = 0$$

for all $i \neq c$. This completes the proof. \square

We remark that Theorem 4.4 works without the hypothesis " R is Gorenstein ". In the paper [6] the authors considered only the case of a Gorenstein ring.

5. A NOTE ON DIRECT AND INVERSE LIMITS

In the proof of [6, Lemma 1.2(a)] it is claimed that Ext of a direct limit in the first variable is the projective limit of the corresponding Ext 's. In general, this is not true: E. g. it is well-known and not very difficult to see that $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z})$ is non-zero (it is actually uncountable), while \mathbb{Q} can be written as a direct limit of copies of \mathbb{Z} 's and each copy of $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z})$ is of course zero. We explain how this problem can be overcome (literally all results from [6] are valid – apart from lemma 1.2 (a)).

The general result is the following:

Theorem 5.1. ([14, Lemma 2.6]) *Let $\{M_\alpha\}$ be a direct system of R -modules. Let N denote an arbitrary R -module. Then there is a short exact sequence*

$$0 \rightarrow \varprojlim^1 \mathrm{Ext}_R^{i-1}(M_\alpha, N) \rightarrow \mathrm{Ext}_R^i(\varinjlim M_\alpha, N) \rightarrow \varprojlim \mathrm{Ext}_R^i(M_\alpha, N) \rightarrow 0$$

for all $i \in \mathbb{Z}$. In particular, $\mathrm{Hom}_R(\varinjlim M_\alpha, N) \cong \varprojlim \mathrm{Hom}_R(M_\alpha, N)$.

The previous Lemma 5.1 gives the corrected version of [6, Lemma 1.2 (a)]. In the following we shall explain how to derive the other results of [6, Lemma 1.2].

Lemma 5.2. *Let (R, \mathfrak{m}, k) be an n -dimensional local Gorenstein ring. For each R -module X there are canonical isomorphisms*

$$\mathrm{Ext}_R^{n-i}(X, \hat{R}) \simeq \mathrm{Hom}_R(H_{\mathfrak{m}}^i(X), E)$$

for all $i \in \mathbb{Z}$. Here \hat{R} denotes the completion of R .

Proof. The proof follows immediately from Corollary 3.4 or from [5, Theorem 6.4.1]. We put $I := \mathfrak{m}$. Recall that $D_I(R) = \hat{R}$. \square

Having established this general version of Local Duality, it is easy to produce the statement of [6, lemma 1.2 (b)].

Corollary 5.3. *Let I be a proper ideal of height c in a n -dimensional local Gorenstein ring (R, \mathfrak{m}) . There are canonical isomorphisms*

$$\mathrm{Ext}_R^{n-i}(H_I^j(R), \hat{R}) \simeq \mathrm{Hom}_R(H_{\mathfrak{m}}^i(H_I^j(R)), E) \simeq \varprojlim \mathrm{Ext}_R^{n-i}(\mathrm{Ext}_R^j(R/I^\alpha, R), \hat{R})$$

for all $i, j \in \mathbb{Z}$.

Proof. The first of the isomorphisms is a consequence of Lemma 5.2 applied to $H_I^j(R)$. Lemma 5.2 applied to $\mathrm{Ext}_R^j(R/I^\alpha, R)$ provides a family of isomorphisms

$$\mathrm{Hom}_R(H_{\mathfrak{m}}^i(\mathrm{Ext}_R^j(R/I^\alpha, R)), E) \simeq \mathrm{Ext}_R^{n-i}(\mathrm{Ext}_R^j(R/I^\alpha, R), \hat{R}), \text{ for all } \alpha \in \mathbb{N},$$

which are compatible with the inverse systems induced by the natural surjections. So, it induces an isomorphism

$$\varprojlim \mathrm{Hom}_R(H_{\mathfrak{m}}^i(\mathrm{Ext}_R^j(R/I^\alpha, R)), E) \simeq \varprojlim \mathrm{Ext}_R^{n-i}(\mathrm{Ext}_R^j(R/I^\alpha, R), \hat{R}),$$

for all i and j . Since the inverse limit commutes with the direct limit under Hom in the first place (see Theorem 5.1) it induces an isomorphism

$$\varprojlim \mathrm{Hom}_R(H_{\mathfrak{m}}^i(\mathrm{Ext}_R^j(R/I^\alpha, R)), E) \simeq \mathrm{Hom}_R(\varinjlim H_{\mathfrak{m}}^i(\mathrm{Ext}_R^j(R/I^\alpha, R)), E).$$

This finally completes the proof since $H_I^j(R) \cong \varinjlim \mathrm{Ext}_R^j(R/I^\alpha, R)$ and because local cohomology commutes with direct limits. \square

With these results in mind the proof of [6, Lemma 1.2 (c)] follows the same line of arguments as in the original paper. In the proof of [6, Corollary 2.9] there is a reference to [6, Lemma 1.2(b)]: However 2.9 can be easily deduced from the minimal injective resolution $0 \rightarrow H_I^c(R) \rightarrow \Gamma_I(E^c) \rightarrow \Gamma_I(E^{c+1}) \rightarrow \dots$ (where $0 \rightarrow R \rightarrow E^\bullet$ is a minimal injective resolution of R) of $H_I^c(R)$; note that we know what indecomposable injective modules occur in the complex E^\bullet since R is Gorenstein.

In the proof of (iii) \iff (iv) of [6, Theorem 3.1] there is a reference to [6, lemma 1.2(b)]: But this equivalence (iii) \iff (iv) follows from Lemma 5.2.

Remark 5.4. Combining the statements in Theorem 5.1 with those of Corollary 5.3 it follows that

$$\varprojlim^1 \text{Ext}_R^i(\text{Ext}_R^j(R/I^\alpha, R), \hat{R}) = 0$$

for all $i, j \in \mathbb{Z}$.

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT REGENSBURG, D — 93040 REGENSBURG, GERMANY
E-mail address: michael.hellus@mathematik.uni-regensburg.de

MARTIN-LUTHER-UNIVERSITÄT HALLE-WITTENBERG, INSTITUT FÜR INFORMATIK, D — 06 099 HALLE (SAALE), GERMANY
E-mail address: peter.schenzel@informatik.uni-halle.de