Branched microstructures in a single-slip model in finite crystal plasticity

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Tobias Schubert
aus
Lauingen

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Die Arbeit wurde angeleitet von: Prof. Dr. Georg Dolzmann

Prüfungsausschuss:

Vorsitzender:	Prof. Dr. Klaus Künnemann
1. Gutachter:	Prof. Dr. Georg Dolzmann
2. Gutachter:	Prof. Dr. Sergio Conti
weiterer Prüfer:	Prof. Dr. Harald Garcke

Contents

Abstract

In this thesis we consider a geometrically nonlinear model of crystal elastoplasticity with one active slip system in dimension two. We use a time-discretization of the corresponding evolution problem and are interested in the variational problem of one single time step, which is not necessarily the first. Thereby we assume that the crystal initially has no defects such that the self-energy of the crystal is equal to the core energy of the dislocations. To compute this energy a description of the dislocations in the continuous setting, namely the geometrical dislocation tensor, is needed. There are several different tensors appearing in the literature. The most famous ones are compared in this thesis or in references therein. The core energy can be expressed by a small parameter δ times a term depending on the curl of the plastic part of the deformation and thus depending on derivatives of an internal slip variable. Thus one has to solve a variational problem depending on the deformation and the internal variable, where they depend on each other. This is one of the main difficulties of the problem.

We assume that the elastic energy density penalizes elastic deformations, which are not rotations, by a factor $\frac{1}{\varepsilon}$ and thus obtain the model of rigid elasticity for vanishing ε. We want to know if this model can be well approximated, for vanishing δ and ε, by the model without self-energy and which is additionally based on the assumptions of rigid elasticity. An answer to this question is already known for the first time step, where we do not have a self-energy part. In this case the answer depends on the fact if one includes hardening or not.

If not, then the relaxed energy density vanishes for a large class of applied loads. We show that this is no more true for an other time step. Beyond that we prove some lower and upper scaling relations for the energy using a branching construction and a proof by contradiction, respectively. For a more simple model we show a lower relation that fits together with the upper relation given by the branching construction.

In the case of linear hardening, it was shown for the first time step that the model of rigid elasticity without self-energy is a good approximation. If we choose the self-energy parameter δ small in comparison to ε this is also true for another time step, which was confirmed by a partially Γ-convergence result.

1 Introduction

Some cars produced by the industry only live for one reason, to be destroyed in a crash test. To reduce this waste, the automotive industry started to simulate crash tests by high-performance computers. In order to achieve realistic results one needs an adequate model for the elastoplastic deformation of a metallic body under external forces. For metal plasticity, or in general crystal plasticity, this is usually done by multiscale models, which describe the movements in the lattice structure, in the subgrain structure and for polycrystals. In this work we restrict ourselves to the lattice structure.

There are several other reasons, which legitimate the huge research concerning plasticity models, which was done in the last decades. For example one can predict the time until a damaged building collapses or one can detect and eliminate the regions, where the material tends to rupture. Furthermore, one might be able to evolve new types of material, by changing the microstructure. Thereby, we denote each structure on a scale between the atomic and the macroscopic level as microstructure.

These microstructures are responsible for many astonishing material properties related to ductility, strength, hardness, corrosion resistance, the temperature behavior and the resistance to wear. They arise due to an inhomogeneous arrangement of material components, for example grains in polycrystals, or due to a lack of convexity in the relevant energy density. Such microstructures cannot be resolved exactly by numerical calculations of the relevant energy, since they are to expensive to calculate. Thus one needs to find a way to capture the influence of microstructures to the macroscopic material response without knowing every single detail of the behavior on fine scales. This can be obtained by using the theory of relaxation, which was established by Morrey [55] and Dacorogna [28]. Technically speaking, one has to compute the quasiconvex envelope of the corresponding energy density, which means that one has to optimize locally over all possible microstructures. The quasiconvex envelope is usually difficult to compute, since one has to solve an infinite dimensional minimizing problem. Therefore one commonly introduces the notion of rank-one convexity and polyconvexity, which are necessary and sufficient condition, respectively, at least in the case of a finite energy density. Nevertheless the analytic computation of quasiconvex envelopes is only known for a few specific cases, see for example [3, 22, 23, 27, 28].

There is also some research to the numerical computation. For example, a quasiconvex envelope was approximated numerically in [18], the numerical polyconvexification and rank-one convexification were discussed in [11] and [30], respectively. Another theory, which helps to reduce the complexity of the problem, is the theory of Γ-convergence, see

[15, 29]. If one part of the energy is multiplied by a very small or big parameter, one can compute its Γ-limit if the parameter vanishes or tends to infinity. Then, the limit energy might capture the relevant behavior of minimizers and a solution could be more easily obtained. In the following we pay attention to the model.
In 1934 the physicists E. Orowan [58, 59], M. Polanyi [62] and G.I. Taylor [70] found out, almost simultaneously, that plastic deformation can be best explained by the movement of dislocations. Namely in the case of an edge dislocation, the plastic deformation can be explained by the movement of atomic half planes through the crystal. This is affected by defects in the crystal. For example by zero dimensional defects, i.e., point defects, like interstitial impurity atoms or self interstitial atoms or vacancies, by one dimensional defects like other dislocations, by two dimensional defects like stacking faults, grain boundaries or phase boundaries and finally by three dimensional defects like precipitates or voids. These crystal defects play also an important role for the ductility, the strength, the stiffness and the hardness of the material. For example an increasing ratio of impurity carbon atoms in steel leads to more hardness, but also to a more brittle material. Thereby, the impurity atoms hinder the dislocations to move through the crystal and thus impede the body to deform plastically. The movement of dislocations does only occur on specific slip planes, which are given by the crystalline structure, e.g. a body-centered cubic (bcc) structure, a face-centered cubic (fcc) structure or a hexagonal structure [8].
In the discrete theory a dislocation may be quantified by the Burgers vector [17, 20, 71], a glide vector associated with the dislocation. Whereas, in the continuum theory the dislocations are characterized through a tensor field G, called geometric dislocation tensor, that measures the local Burgers vector per unit area. The problem is that many different tensors appear in the literature and one needs to find out, which one is the right one. Cermelli and Gurtin studied this question in [20] by introducing some physically reasonable requirements, which are stated in Section 2.4. Throughout, their preferred tensor is used in this thesis. But one has to remark that there are also objections towards the reasonability of their requirements, see for example the counterpoint of Acharya [1].
Fundamental aspects of modeling finite-strain deformations of elastoplastic material, as the multiplicative decomposition of the deformation gradient, trace back to Kröner [44], Lee [46] and Rice [63]. These were later extended and improved by Aubry and Ortiz [9], Carstensen, Hackl and Mielke [19], Ortiz and Repetto [60], and Miehe, Schotte and Lambrecht [48] et al.
In this thesis we restrict ourselves mainly to a single-slip model for a two dimensional single crystal, in particular we neglect grain-boundary effects. Our model is based on the flow rule. For approaches based on dissipation distances we refer to [49, 50].
Time dependent evolution of elastoplastic bodies is commonly done by a time discrete variational approach. Thus one has to solve a minimizing problem in each time step. We restrict ourselves to one single time step, not necessarily the first, and assume that the crystal initially has no defects. Thus the defects in the crystal are caused only by

the dislocation movement along the slip direction. The model we are interested in, is the model, which was investigated in the thesis of Carolin Kreisbeck [42], which is mainly based on [19] and [60], plus the self-energy of the dislocation. The self-energy is needed due to the fact that the linear constitutive relation is not satisfying close to the dislocations cores, where the strains are too large such that the linear approximation is not valid, see [65, 71] for more details.
Using the geometric dislocation tensor, one can compute the dislocation self-energy per unit length of a simple dislocation loop and derive the dislocation self-energy, see Ortiz and Repetto [60], by using the dislocation line tension [45, 71]. This dislocation self-energy enhances the model used in the thesis of Carolin Kreisbeck [42], where the theory was purely local. The self-energy hinders the microstructure to get fine and thus includes an intrinsic length scale.

Overview of the model

Next, we state the variational problem mainly investigated in this thesis, without deriving it here. For its justification the reader is referred to Chapter 2 and Chapter 4. Let $\Omega \subseteq \mathbb{R}^2$ be the reference configuration of a two-dimensional elastoplastic body and let $u : \Omega \to \mathbb{R}^2$ describe the deformation of the sample at a fixed time. We use a multiplicative decomposition of the deformation gradient $F = \nabla u = F_{el} F_{pl}$, into an elastic part F_{el} and a plastic part F_{pl}, and we assume that plastic deformations are volume preserving, i.e., $\det(F_{pl}) = 1$. Next, we assume that the plastic deformation occurs only on one slip system, which is characterized by the slip direction $s \in \mathbb{S}^1$ and the slip plane normal $m \in \mathbb{S}^1$, with $s \cdot m = 0$. For simplicity we choose in the following $s := \vec{e}_1$ and $m := \vec{e}_2$. Using that the crystal initially has no defects, we get that the plastic deformation is given by $F_{pl} = \mathbb{1} + \gamma \vec{e}_1 \otimes \vec{e}_2$, for a function $\gamma : \Omega \to \mathbb{R}$ commonly denoted as slip strain. Then the energy E, which has to be minimized in one single time step, reads $E_{\varepsilon,\delta}[u] = \inf_{\gamma} I_{\varepsilon,\delta}(u, \gamma)$, where

$$I_{\varepsilon,\delta}(u, \gamma) = \fint_{\Omega} \left(\frac{1}{\varepsilon} W_e \left(\nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right) \right) + |\gamma|^p \right) \, d\lambda_2 + \frac{\delta}{|\Omega|} V_x \left(\gamma, \Omega \right).$$

The first part is the elastic energy and the last one is the self-energy, where $\delta > 0$ is a small parameter comparable to the distances in the lattice structure. In the case $p = 1$, i.e., without hardening, the second part in the integral results from the principle of maximal dissipation and in the case $p = 2$ it is the hardening energy density. This is a further simplification, since the part resulting from the principle of maximal dissipation exists in the case of hardening too. Thereby, the deformation u is assumed to be Lipschitz continuous and the slip strain γ is assumed to be a function of bounded variation on Ω. Furthermore, the deformation is assumed to have affine boundary values, i.e., it exists an $F^* \in \mathbb{R}^{2\times 2}$ with $u = F^*$ on $\partial\Omega$, namely $u(x) = F^* x$ for all $x \in \partial\Omega$. The elastic energy density $W_e : \mathbb{R}^{2\times 2} \to [0,\infty]$ is assumed to be frame indifferent and to have rigid body motions as minimizers. Furthermore it has polynomial growth of order $q \geq 0$. In some results proved in this thesis, we choose the explicit formula

$$W_e(F) := \operatorname{dist}^q (F, SO(2)).$$

The parameter $\varepsilon > 0$ was introduced in order to penalize deformations, which are not rotations, and to compare it with the model of rigid elasticity for small ε.
Define now the set of matrices $F \in \mathbb{R}^{2\times 2}$, whose elastic part F_{el} is a rotation, by $\mathcal{M}^{(2)}$, i.e., $\mathcal{M}^{(2)} = \left\{F \in \mathbb{R}^{2\times 2} \,:\, F(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2) \in SO(2),\ \gamma \in \mathbb{R}\right\}$. The lamination convex hull of $\mathcal{M}^{(2)}$ coincides with the polyconvex hull and is calculated as

$$\mathcal{N}^{(2)} := \left(\mathcal{M}^{(2)}\right)^{lc} = \left(\mathcal{M}^{(2)}\right)^{pc} = \left\{F \in \mathbb{R}^{2\times 2} \,:\, \det(F) = 1,\ |F\vec{e}_1| \leq 1\right\}.$$

This was determined in [27].

Main results

Next, we state the main results proven in this thesis. First we consider the case without hardening, i.e., $p = 1$. Then, we can show that for a rectangle Ω and boundary conditions $F^* \in \mathcal{M}^{(2)}$ the value $\inf_{(u,\gamma)\,:\,u=F^* \text{ on } \partial\Omega} I(u,\gamma)$ has an upper bound, which scales like $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, for small ε, δ. Thereby, q is the growth exponent of the elastic energy density.

Theorem 1.1. *Let $\Omega := [-L, L] \times [0, H] \subset \mathbb{R}^2$, $L > 0$, $H > 0$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $F^* \in \mathcal{M}^{(2)}$, $W_e(F) := \operatorname{dist}^q(F, SO(2))$. Then we have that*

$$\inf_{u\in W^{1,\infty}(\Omega;\mathbb{R}^2)\,:\,u=F^* \text{ on } \partial\Omega} E_{\varepsilon,\delta}[u] \leq C\left(\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}} L^{\frac{q}{q+1}}} + \frac{\delta}{H}\right),$$

where $C = C(F^*, q)$ is independent of $\varepsilon, \delta, p, L, H$.

Thereby we use the same laminate as in the proof of [23, Theorem 1.1]. This construction ensures that the energy without self-energy part, i.e.,

$$\int_\Omega \frac{1}{\varepsilon} W_e\left(\nabla u\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right)\right) + |\gamma|^p \; d\lambda_2,$$

is small. The self-energy part $\delta V_x(\gamma, \Omega)$ hinders the the laminate structure to get finer, as it was possible in [23, Theorem 1.1], and thus one needs to take care of the right cutting method to achieve the boundary values. We obtained the result by using a branching construction similar as in [21, 26, 38, 39]. Unfortunately, we are only able to show a lower bound, which scales like δ and not like $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$. This lower bound is the second main result, which is shown in this thesis. We can prove that the infimum of the energy $E_{\varepsilon,\delta}[u]$ over all configurations u, with bounded L^∞-norm and boundary values $u = F^* \in \mathcal{N}^{(2)} \setminus SO(2)$ on $\partial\Omega$, has a lower bound which scales as δ, for small δ. Physically, the additional assumption that the L^∞-norm of u is bounded is no real restriction if one thinks about finite deformations of finite bodies.

Theorem 1.2. *Let $F^* \in \mathbb{R}^{2\times 2} \setminus \{\lambda B \in \mathbb{R}^{2\times 2} \ : \ \lambda \in [0,1], B \in SO(2)\}$ with $F^*\vec{e}_1 \neq 0$, $\Omega \subseteq \mathbb{R}^2$ open, bounded, with C^1-boundary. Let $q, p \geq 1$, $K_1, K_2 > 0$ then we have: $\exists \eta = \eta(K_1, K_2, p, q) > 0 \ : \ \forall \delta, \varepsilon \leq K_1$*

$$\inf_{\substack{u \in W^{1,\infty}(\Omega;\mathbb{R}^2) \,:\, u = F^* \text{ on } \partial\Omega \\ \|u\|_{L^\infty(\Omega;\mathbb{R}^2)} < K_2}} E_{\varepsilon,\delta}[u] \geq \eta\delta$$

If we have additionally $q \geq 2$ then we get the above statement for all $F^* \in \mathbb{R}^{2\times 2} \setminus SO(2)$.

Up to now we are not able to close the gap between the lower and upper bound, which might be possible. Therefore, we have simplified the model by changing the self-energy part $V_x(\gamma, \Omega)$ into $V_y\left(\chi_{\{\gamma=0\}}, \Omega\right)$, where we have to ensure that γ does not achieve small non-zero values. For a motivation of this simplification the reader is referred to the Sections 5.3, 6.3 and 6.4. We consider the energy $\widetilde{E}_{\varepsilon,\delta}[u] = \inf_\gamma \widetilde{I}_{\varepsilon,\delta}(u,\gamma)$, where

$$\widetilde{I}_{\varepsilon,\delta}(u,\gamma) = \fint_\Omega \frac{1}{\varepsilon} \operatorname{dist}^q\left(\nabla u\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + |\gamma|^p \ d\lambda_2 + \frac{\delta}{|\Omega|} V_y\left(\chi_{\{\gamma=0\}}, \Omega\right)$$

and γ does not take small non zero values. Then one can show that the infimum of the simplified energy scales like $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$ from above and below, for small ε, δ. This was proven in the Sections 5.3 and 6.4.
Considering now the case of linear hardening, i.e., $p = 2$. For $\delta = \varepsilon^\kappa$, where κ we choose big enough, one suspects that the Γ-limit of the energy $E_{\varepsilon,\delta}[u]$ converge to the Γ-limit of the energy $E_{\varepsilon,0}[u]$, for $\varepsilon \to 0$. Unfortunately, we can only show this for more regular u, namely for $u \in C^{1,\alpha}\left(\Omega; \mathbb{R}^2\right)$. This was proven in Chapter 7.

Outline of the thesis

Straight after this introductory words we give a brief overview of modelling finite crystal plasticity. We consider a time discrete variational approach of the rate independent evolution of elastoplastic bodies and apply this for a single-slip model of a single crystal. Thereby our focus lies on the self-energy of the dislocations, since the other energy parts have been investigated in the thesis of Carolin Kreisbeck [42] already in a more detailed way. The main part of Chapter 2 is the following. We compare the various geometric dislocation tensors appearing in the literature and decide which is the right one in order to derive the self-energy of the dislocation.

In Chapter 3 we summarize the mathematical preliminaries used in this thesis. Starting with some useful properties about Null Lagrangians we give a short overview of functions of bounded variation. They are needed since we assume that the slip strain γ is of bounded variation. Subsequently we state the definition and some simple properties of Γ-convergence. Particulary we point out that in order to get convergence of minima one must not construct the recover sequence for the hole space, if the limit function has an additional assumption. This is used in Chapter 7. Finally we summarize the notions of convexity and introduce the method of convex integration, see [53, 54].

In this thesis all statement are proven for the two-dimensional case only. In Chapter 4 we describe the two-dimensional model from a mathematical point of view. Namely we assign functions spaces for the deformation and the slip strain and we use growth exponent for the elastic and plastic energy parts. In this chapter, we summarize the statements for the model without self-energy, see [42]. Then we point out a simple corollary of the convex integration method and explain the problems appearing using this method. Afterwards we show a partially relaxation result for the case of two slip systems with infinite latent hardening and without self-energy. At the end of the chapter we show a scaling behavior of the energy.

In the fifth and sixth chapter we investigate the case without hardening and in the seventh chapter the case of linear hardening is examined. In Chapter 5 we prove upper bounds. We start with a construction on a unit square, which is improved later on by a branching construction. In between we make a construction for more general regions. Due to the bad scaling property of the energy there is some work to be done. These upper bounds are valid for affine boundary values in $\mathcal{M}^{(2)}$, namely for boundary values whose elastic part is a rotation. We close this chapter by a double laminate construction for boundary values in the polyconvex hull of $\mathcal{M}^{(2)}$, i.e. in $\mathcal{N}^{(2)}$.

In the sixth chapter we prove lower bounds. Straight after presenting some useful algebraic estimates we show a lower bound using a proof by contradiction. Unfortunately the obtained scaling relation does not fit together with the upper bound obtained by the branching construction. In order to get an idea how to close the gap we simplify our model. This is done in such way that the scaling relation of the upper bound remains the same and is obtained by the same branching construction. We do also present a motivation for the simplification in the case of simple laminate constructions. For the simplified model we are able to close the gap in the scaling relations of the upper and lower bound.

The case of linear hardening is examined in Chapter 7. If δ is small in comparison to ε, we can show that the model of rigid elasticity is a good approximation. This is done using a partially Γ-convergence result. Thereby one needs to construct the recovery

sequence constructed for affine functions only, in order to get the required convergence of minima.
The thesis closes by a short outlook, where we list some possible further research projects. Finally some calculations of the Section 2.4 can be found in Appendix A. The notations and conventions were written down in Appendix B and in Appendix C, where one can found the symbols used in this thesis.

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2 An outline of finite crystal plasticity

Modeling finite crystal plasticity goes back to the work of Kröner [44], Lee [46] and Rice [63]. The key ingredients are the following. First of all the plastic deformations can be best explained by the movement of dislocations, which was found out almost simultaneously by E. Orowan [58, 59], M. Polanyi [62] and G.I. Taylor [70]. Secondly one introduces a configuration between the reference and deformed configuration, and we consider a multiplicative split of the deformation gradient into a plastic and an elastic part. Therefore this theory is often called geometrically nonlinear plasticity in contrast to the theory of infinitesimal strains, which is essentially a linear theory.
This chapter is mostly a summary of the model introduced in the thesis of Carolin Kreisbeck [42, Chapter 2], which relies on [19, 60]. In Section 2.4, where we have used parts of [20] and [51], the self-energy was computed, which was not yet included in the model used in [42]. In this chapter all functions are chosen differentiable enough such that the occurring rates and derivatives exist in a suitable way.

2.1 Fundamental assumptions

The reference configuration of an elastoplastic body is modeled by a set $\Omega \subset \mathbb{R}^n$ with space dimension $n = 2$ or $n = 3$. The time-dependent total deformation of the sample is described by a smooth function $u : [0, \widehat{T}] \times \Omega \to \mathbb{R}^n,\ (t, x) \mapsto u(t, x)$, with time variable $\widehat{T} > 0$. Its gradient with respect to the space variable x, namely $F = \nabla_x u$, is called deformation gradient and has positive determinant. The image $u\,(t, \Omega)$ is called the deformed configuration at time t or short deformed configuration. The local reference configuration, which is also called microstuctural or lattice configuration, is chosen as in [20], [33], [44] or [51] and will be defined later in Section 2.4 from a mathematical point of view. In the following we omit the dependence in time and space in the appearing definitions. Modern treatments in finite plasticity are based on a multiplicative decomposition of F into an elastic part F_{el} and a plastic one F_{pl}, i.e.,

$$F = F_{el}F_{pl}.$$

This consideration goes back to the work by Kröner [44] and Lee [46]. Thereby this decomposition is not unique e.g. due to rotations. Furthermore one usually takes F_{pl} together with a vector $\mathbf{p} \in \mathbb{R}^M$ with $M \in \mathbb{N}$, which is related to the mechanical properties of the material, such as hardening, see [19, 42]. In the following we assume that $\det(F_{pl}) = 1$, i.e., the plastic deformation is volume preserving. The pair

$\left(F_{pl}^{-1}, \mathbf{p}\right) \in Sl(n) \times \mathbb{R}^M$ denotes the internal variables of the system and $\mathbf{p}$ is called hardening variable.

2.2 Rate independent evolution of elastoplastic bodies, based on the flow rule

Now we want to investigate the evolution of elastoplastic bodies under a time-dependent external loading. This is done by a time-discrete variational approach, which leads to an approximate solution of the underlying time-continuous problem. We apply the incremental method for rate-independent processes, delivering a sequence of minimization problems [19, 27]. There are different ways of dealing with dissipation in the energy formulation. For a detailed investigation, we refer to [42] and the references therein. The concept, we restrict ourselves to, results from the plastic flow rule, which can be derived from the fundamental principle of maximal plastic dissipation [19, 66, 67].
Define $\mathbf{P} = F_{pl}^{-1}$ and remark that we omit the dependence on $x \in \Omega$ and $t \in [0, \widehat{T}]$. It is assumed that the total free energy density ψ_{total} can be written as $\psi_{total} = \psi + \psi_{self}$, where the free energy density $\psi = \psi(F, \mathbf{P}, \mathbf{p})$ accounts for the long-range elastic distortions of the lattice and suffices to compute the dislocation interaction energy. The self-energy density ψ_{self} contains the energy caused by the highly distorted region near the dislocation core, observe [60, 71] and Section 2.4 for more details. The free energy density $\psi : Gl(n) \times Sl(n) \times \mathbb{R}^M \to [0, \infty]$ is supposed to depend on F_{el} and $\mathbf{p}$ only, namely $\psi(F, \mathbf{P}, \mathbf{p}) = \widehat{\psi}(F\mathbf{P}, \mathbf{p}) = \widehat{\psi}\,(F_{el}, \mathbf{p})$. Furthermore it is assumed to be frame-indifferent, i.e., $\widehat{\psi}\,(RF_{el}, \mathbf{p}) = \widehat{\psi}\,(F_{el}, \mathbf{p})$ for all $R \in \mathrm{SO}(n)$. Next, we assume that $\widehat{\psi} : Gl(n) \times \mathbb{R}^M \to [0, \infty]$ is continuous and satisfies the coercivity assumption $\widehat{\psi}(F_{el}, \mathbf{p}) \to \infty$ for $\|F_{el}\| + \left\|F_{el}^{-1}\right\| + |\mathbf{p}| \to \infty$. Later on we will see that the self-energy density ψ_{self} depends only on F_{pl}, see Equation (2.4).
The thermo-mechanical dual variables corresponding to $F, \mathbf{P}$ and $\mathbf{p}$ are the first Piola-Kirchhoff stress tensor

$$\mathbf{T} = \frac{\partial}{\partial F}\psi(F, \mathbf{P}, \mathbf{p}) = \frac{\partial}{\partial F_{el}}\widehat{\psi}\,(F_{el}, \mathbf{p})\,\mathbf{P}^T,$$

the conjugate plastic stresses

$$\mathbf{Q} = -\frac{\partial}{\partial \mathbf{P}}\psi(F, \mathbf{P}, \mathbf{p}) = -F^T\frac{\partial}{\partial F_{el}}\widehat{\psi}\,(F_{el}, \mathbf{p})\,,$$

and the conjugate hardening forces

$$\mathbf{q} = -\frac{\partial}{\partial \mathbf{p}}\psi(F, \mathbf{P}, \mathbf{p}) = -\frac{\partial}{\partial \mathbf{p}}\widehat{\psi}\,(F_{el}, \mathbf{p})\,.$$

Then we derive that $\overline{\mathbf{Q}} = \mathbf{P}^T\mathbf{Q} = -\mathbf{P}^T F^T \frac{\partial}{\partial F_{el}}\widehat{\psi}\,(F_{el}, \mathbf{p}) = -F_{el}^T \frac{\partial}{\partial F_{el}}\widehat{\psi}\,(F_{el}, \mathbf{p})$ is independent of $\mathbf{P}$. In order to describe the evolution of $(\mathbf{P}, \mathbf{p})$ an appropriate quantity for the characterization of the threshold between plastic and elastic material behavior is needed. Therefore we choose a yield function $\phi = \phi(\mathbf{T}, \mathbf{Q}, \mathbf{P}, \mathbf{q})$ and postulate as in [19] that ϕ only depends on $\overline{\mathbf{Q}}$ and $\mathbf{q}$, i.e., $\phi(\mathbf{T}, \mathbf{Q}, \mathbf{P}, \mathbf{q}) = \widehat{\phi}(\overline{\mathbf{Q}}, \mathbf{q})$. The yield function defines the set of admissible stresses

$$\mathbb{Q} = \left\{ \left(\overline{\mathbf{Q}}, \mathbf{q}\right) \in \mathbb{R}^{n \times n} \times \mathbb{R}^M \ : \ \widehat{\phi}(\overline{\mathbf{Q}}, \mathbf{q}) \leq 0 \right\},$$

where we assume that $\mathbb{Q}$ is a closed and convex set, which contains 0. The principle of maximal plastic dissipation [66, 67] postulates that the plastic dissipation

$$-D_{\mathbf{P}}\psi(F, \mathbf{P}, \mathbf{p}) : \dot{\mathbf{P}} - D_{\mathbf{p}}\psi(F, \mathbf{P}, \mathbf{p}) \cdot \dot{\mathbf{p}} = \overline{\mathbf{Q}} : \left(\mathbf{P}^{-1}\dot{\mathbf{P}}\right) + \mathbf{q} \cdot \dot{\mathbf{p}}$$

is maximal if $\mathbf{P}^{-1}\dot{\mathbf{P}}$ and $\dot{\mathbf{p}}$ are kept fix. Maximization of the plastic dissipation in the set of all admissible stresses, namely under the inequality constraint $\widehat{\phi}\left(\overline{\mathbf{Q}}, \mathbf{q}\right) \leq 0$, gives the necessary condition for an optimum, see [6, Chapter 7],

$$\left(\mathbf{P}^{-1}\dot{\mathbf{P}}, \dot{\mathbf{p}}\right) = \lambda \left(\frac{\partial \widehat{\phi}}{\partial \overline{\mathbf{Q}}} \left(\overline{\mathbf{Q}}, \mathbf{q}\right), \frac{\partial \widehat{\phi}}{\partial \mathbf{q}} \left(\overline{\mathbf{Q}}, \mathbf{q}\right) \right)$$

for $\widehat{\phi}$ and λ satisfying the complementarity condition $\widehat{\phi} \leq 0 \leq \lambda$ and $\lambda\widehat{\phi} = 0$, thereby we have used that $\mathbb{Q}$ is a convex set. This necessary condition is also well known as flow rule. Next, the dissipation can be described by the function $U : \mathbb{R}^{n \times n} \times \mathbb{R}^M \to \mathbb{R}$ defined by $U(S, s) := \sup_{(\overline{\mathbf{Q}}, \mathbf{q}) \in \mathbb{Q}} \left\{ \overline{\mathbf{Q}} : S + \mathbf{q} \cdot s \right\}$. Then U is non-negative, since $0 \in \mathbb{Q}$, and positively 1-homogeneous, i.e., $U(\alpha S, \alpha s) = \alpha U(S, s)$ for all $\alpha > 0$ and $(S, s) \in \mathbb{R}^{n \times n} \times \mathbb{R}^M$. The dissipation then reads $U(\mathbf{P}^{-1}\dot{\mathbf{P}}, \dot{\mathbf{p}})$.

Next, we discretize the problem at the time steps $0 = t^0 < t^1 < \ldots < t^N = \widehat{T},\ N \in \mathbb{N}$. Let $\left(u^0, \mathbf{P}^0, \mathbf{p}^0\right)$ be a stable initial state and $\left(u^k, \mathbf{P}^k, \mathbf{p}^k\right)$ with $k \in \{1, \ldots, N\}$ the state variables at time t^k. We describe the boundary condition $u^k = u_b\left(t^k, .\right)$ on $\partial\Omega$ through a function $u_b : [0, \widehat{T}] \times \partial\Omega \to \mathbb{R}^n$. Further the time-dependent external loading is modeled by l through $< l(t), u >= \int_\Omega f(t)u \ dx + \int_{\partial\Omega} g(t)u \ dS$, where f and g stand for the applied body forces and the applied surface forces, respectively. The discretization of $\dot{\mathbf{p}}$ in the k-th time step is done by $\frac{\mathbf{p}^k - \mathbf{p}^{k-1}}{t^k - t^{k-1}}$ and $\mathbf{P}^{-1}\dot{\mathbf{P}}$ is approximated by $\frac{1}{t^k - t^{k-1}} \left(\mathbb{1} - \left(\mathbf{P}^k\right)^{-1} \mathbf{P}^{k-1}\right)$. Now we can formulate the functional to be minimized in the k-th time step by $E^k[u, \mathbf{P}, \mathbf{p}]$ defined by

$$\int_\Omega \psi_{total}\left(\nabla u, \mathbf{P}, \mathbf{p}\right) + \frac{1}{t^k - t^{k-1}} U \left(\mathbb{1} - \mathbf{P}^{-1}\mathbf{P}^{k-1}, \mathbf{p} - \mathbf{p}^{k-1}\right) \ dx - < l\left(t^k\right), u >, \quad (2.1)$$

see [19] for more details. The incremental problem is then formulated as follows:

For $k = 1, \ldots, N$ find $u^k : \Omega \to \mathbb{R}^n$ with $u^k = u_b\left(t^k, .\right)$ on $\partial\Omega$

and $\left(\mathbf{P}^k, \mathbf{p}^k\right) : \Omega \to Sl(n) \times \mathbb{R}^M$ which minimize $E^k[u, \mathbf{P}, \mathbf{p}]$.

2.3 Single-slip model

Now we want to apply the above model to crystal plasticity. Basic constitutive relations about multi-slip systems were examined in [9, 60, 68]. The concrete slip system of several kinds of crystals, like fcc or bcc crystals was examined in [71]. In the following we restrict our attention to one active slip system $(s, m, \tau) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times (0, \infty)$, where $s \cdot m = 0$, and a scalar hardening parameter $\mathbf{p} \in \mathbb{R}$. Thereby s denotes the slip direction, while m is the unit normal of the slip plane and τ is the critical resolved shear stress of the slip system. Then according to Carstensen, Hackl and Mielke [19], the yield function is given by

$$\widehat{\phi}\left(\overline{\mathbf{Q}}, \mathbf{q}\right) = \begin{cases} \left|\overline{\mathbf{Q}} : s \otimes m\right| - \tau - \mathbf{q} & \text{if} \quad \mathbf{q} \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

and the corresponding flow rule is

$$\left(\mathbf{P}^{-1}\dot{\mathbf{P}}, \dot{\mathbf{p}}\right) = \dot{\sigma}\left(\text{sign}\left(s \cdot \overline{\mathbf{Q}}m\right) s \otimes m, -1\right)$$

for $\widehat{\phi} \leq 0 \leq \dot{\sigma}$ with $\dot{\sigma}\widehat{\phi} = 0$, see [6]. One can interchange the value one in the definition of $\widehat{\phi}$ by an arbitrary strictly positive real number. It was introduced to make sure that each admissible stress $(\overline{\mathbf{Q}}, \mathbf{q}) \in \mathbb{Q}$ has a non-negative conjugate hardening force, i.e., $\mathbf{q} \geq 0$. The parameter $\dot{\sigma} \geq 0$ can be interpreted as slip rate of the system. Next, we define $\gamma : \left[0, \widehat{T}\right] \to \mathbb{R}$ by $\dot{\gamma} = -\dot{\sigma}\text{sign}\left(s \cdot \overline{\mathbf{Q}}m\right)$ and $\gamma\left(t^0\right) = 0$. We write again $\gamma^k = \gamma\left(t^k\right)$ for $k \in [0, N] \cap \mathbb{Z}$. In view of $\dot{\mathbf{P}} = -\mathbf{P}\dot{\gamma}s \otimes m$ we get $\dot{\mathbf{P}}s = 0$, which leads to $\mathbf{P}(t)s = \mathbf{P}^{k-1}s$ for all t. Integration of $\dot{\mathbf{P}} = -\dot{\gamma}\mathbf{P}^{k-1}s \otimes m$ from t^{k-1} to $t > t^{k-1}$ gives $\mathbf{P}(t) - \mathbf{P}^{k-1} = -\left(\gamma(t) - \gamma^{k-1}\right)\mathbf{P}^{k-1}s \otimes m$, i.e., $\mathbf{P}(t) = \mathbf{P}^{k-1}\left(\mathbb{I} - \left(\gamma(t) - \gamma^{k-1}\right) s \otimes m\right)$ and thus iteratively $\mathbf{P}(t) = \mathbf{P}^0\left(\mathbb{1} - \gamma(t)s \otimes m\right)$ for all $t \geq 0$. Using this we can compute the appearing dissipation by

$$\begin{aligned}
&U\left(\mathbb{1} - \mathbf{P}^{-1}\mathbf{P}^{k-1}, \mathbf{p} - \mathbf{p}^{k-1}\right) = U\left(-\left(\gamma - \gamma^{k-1}\right) s \otimes m, \mathbf{p} - \mathbf{p}^{k-1}\right) \\
&= \sup_{(\overline{\mathbf{Q}}, \mathbf{q}) \in \mathbb{R}^{n \times n} \times \mathbb{R}} \left\{-\overline{\mathbf{Q}} : \left(\left(\gamma - \gamma^{k-1}\right) s \otimes m\right) + \mathbf{q}\left(\mathbf{p} - \mathbf{p}^{k-1}\right) : \widehat{\phi}\left(\overline{\mathbf{Q}}, \mathbf{q}\right) \leq 0\right\} \\
&= \sup_{(\overline{\mathbf{Q}}, \mathbf{q}) \in \mathbb{Q}} \left\{\overline{\mathbf{Q}} : \left(\left(\gamma^{k-1} - \gamma\right) s \otimes m\right) + \mathbf{q}\left(\mathbf{p} - \mathbf{p}^{k-1}\right) : \left|\overline{\mathbf{Q}} : s \otimes m\right| \leq \mathbf{q} + \tau, \mathbf{q} \geq 0\right\} \\
&= \sup_{\mathbf{q} \in \mathbb{R}} \left\{\left|\gamma - \gamma^{k-1}\right| (\mathbf{q} + \tau) + \mathbf{q}\left(\mathbf{p} - \mathbf{p}^{k-1}\right) : \mathbf{q} \geq 0\right\} \\
&= \begin{cases} \tau\left|\gamma - \gamma^{k-1}\right| & \text{if} \quad \left|\gamma - \gamma^{k-1}\right| + \mathbf{p} - \mathbf{p}^{k-1} \leq 0 \\ \infty & \text{otherwise} \end{cases} .
\end{aligned}$$

Inserting this into Equation (2.1) we get

$$E^k[u,\mathbf{P},\mathbf{p}] = E^k[u,\gamma,\mathbf{p}] = \int_\Omega \Big[\widehat{\psi}\left(\nabla u \left(\mathbf{P}^0\left(\mathbb{1} - \gamma s \otimes m\right)\right), \mathbf{p}\right)$$
$$+\psi_{self}\left(\left(\mathbb{1} + \gamma s \otimes m\right)\left(\mathbf{P}^0\right)^{-1}\right) + \frac{\tau}{t^k - t^{k-1}}\left|\gamma - \gamma^{k-1}\right|\Big]\, dx - < l\left(t^k\right), u >,$$

if $\left|\gamma - \gamma^{k-1}\right| + \mathbf{p} - \mathbf{p}^{k-1} \leq 0$ for almost every $x \in \Omega$ and $E^k[u,\mathbf{P},\mathbf{p}] = \infty$ else. Next, we suppose that the energy density $\widehat{\psi}$ consists of a purely elastic component ψ_{el} depending only on F_{el} and a hardening energy density ψ_h, i.e., $\widehat{\psi}\left(F_{el},\mathbf{p}\right) = \psi_{el}\left(F_{el}\right) + \psi_h\left(\mathbf{p}\right)$. Thereby it is assumed that rigid body motions are minimizers of the elastic energy $\psi_{el} : Gl(n) \to [0,\infty]$, i.e., $\psi_{el}\left(R\right) = 0$ for all $R \in SO(n)$. For linear hardening, we choose $\psi_h(\mathbf{p}) = \frac{1}{2}a\mathbf{p}^2$, where $a > 0$ is the hardening modulus, while $\psi_h = 0$ in a model neglecting hardening effects. In the model with linear hardening the conjugate hardening force of an admissible stress $\left(\overline{\mathbf{Q}},\mathbf{q}\right)$ is non-negative, i.e., $0 \leq \mathbf{q} = -\frac{\partial}{\partial \mathbf{p}}\widehat{\psi}\left(F_{el},\mathbf{p}\right) = -a\mathbf{p}$, which implies $\mathbf{p} \leq 0$. Since the above energy is independent of derivatives of $\mathbf{p}$, we can minimize pointwise in $\mathbf{p}$ under the side condition $\mathbf{p} \leq \mathbf{p}^{k-1} - \left|\gamma - \gamma^{k-1}\right| \leq 0$ and get the reduced energy density

$$\psi_{red}\left(F,\gamma\right) = \psi_{el}\left(F\left(\mathbf{P}^0\left(\mathbb{1} - \gamma s \otimes m\right)\right)\right) + \psi_h\left(\mathbf{p}^{k-1} - \left|\gamma - \gamma^{k-1}\right|\right)$$
$$+\psi_{self}\left(\left(\mathbb{1} + \gamma s \otimes m\right)\left(\mathbf{P}^0\right)^{-1}\right) + \frac{\tau}{t^k - t^{k-1}}\left|\gamma - \gamma^{k-1}\right|.$$

This implies that $\mathbf{p}^k = \mathbf{p}^{k-1} - \left|\gamma^k - \gamma^{k-1}\right| = \mathbf{p}^0 - \sum_{l=1}^k \left|\gamma^l - \gamma^{l-1}\right|$ for $k \geq 1$. In the following we want to specify the self-energy density. For this purpose we introduce a geometric dislocation tensor, which is the basic ingredient in developing a self-energy density.

2.4 Notes about the geometric dislocation tensor

One challenge of the Kröner-Lee decomposition $F = F_{el}F_{pl}$ is its non-uniqueness. In the single-slip model we have achieved an explicit description of the plastic deformation, which depends on γ, namely $F_{pl} = \left(\mathbb{1} + \gamma s \otimes m\right)\left(P^0\right)^{-1}$. By minimization over γ, we will achieve later on the right decomposition. There is another problem of the Kröner-Lee decomposition, we do not have accounted for yet. While F is the gradient of a vector field, this is in general not true for F_{el} and F_{pl}, namely they are incompatible. This property is related to the formation of dislocations. Such dislocations are termed geometrically necessary, as they arise solely from the underlying kinematics, and their intrinsic characterization is basic to general theories of plasticity. In crystal physics dislocations may by quantified by the Burgers vector, which represents the closure deficit of circuits deformed from a perfect lattice [17, 20, 35]. We repeat its definition in the following subsection. A detailed investigation of the various modes of dislocations in the discrete setting, such as edge or screw dislocations can be found in [36, 71, 72]. In the following we adopt their definition of a dislocation loop and a dislocation line. In the continuum theory one characterizes the dislocations through a tensor field G that measures the Burgers vector per unit area. The problem is that there have appeared many different tensors in the literature.

2.4.1 Overview of the different dislocation tensors

As in the work of Cermelli and Gurtin [20], we repeat now the frequently used definitions of the geometric dislocation tensor in three dimensions, stated by Acharya and Bassani [2], Bilby, Bullough and Smith [13], Fox [34], Kondo [40, 41] and Noll [56]. They are based on Nye's ideas [57] and are up to a sign, which we will ignore in the following, equal to

$$\operatorname{curl}\left(F_{el}^{-1}\right),\ F_{el}\operatorname{curl}\left(F_{el}^{-1}\right),\ F_{pl}^{-1}\operatorname{curl}\left(F_{pl}\right),\ F_{el}^{-1}\operatorname{curl}\left(F_{el}\right),\ \det\left(F_{el}\right)\operatorname{curl}\left(F_{el}^{-1}\right)F_{el}^{-T},$$

where the j-th entry belongs to the j-th group or person. Thereby, the third tensor is given in the reference configuration, the fifth tensor is given in the lattice configuration and the rest are given in the deformed configuration, see Table 9.1. In addition to the work of Bilby, Bullough and Smith [13] the reader is referred to [14], where some assertions of the former article are proven, like the equivalence of the definitions of the local Burgers vector. The tensor introduced by Bilby, Bullough and Smith can be also found in the work of Eshelby [31], Fox [33] and Kröner [44]. A comparison of these tensors to the one defined by Kondo [40, 41] was done in the work of Kröner [43].
Later on, in Subsection 2.4.2, we will see that there are only these two approaches, namely except for Kondo's tensor they can be obtained by the continuous description of the discretely defined true and local Burgers vector.
Cermelli and Gurtin have investigated, which of the above tensors have an intrinsic physical meaning. This was done by postulating three physically reasonable requirements for the characterization of such a measure, namely:

(*i*) G should measure the local Burgers vector in the microstructural configuration, per unit area in that configuration;

(*ii*) G should, at any point, be expressible in terms of the field F_{pl} in a neighborhood of the point;

(*iii*) G should be invariant under superposed compatible elastic deformations and also under compatible local changes in the reference configuration.

Thereby we call the Burgers vector in the microstructural configuration later on true Burgers vector. According to Cermelli and Gurtin the dislocation tensor

$$G^{el} = \det(F_{el}) \operatorname{curl} \left(F_{el}^{-1}\right) \left(F_{el}^{-1}\right)^T ,$$

defined by Noll [56], where curl denotes the curl with respect to a point $y = u(x,t)$ in the deformed configuration, seemed to be the best choice. One has to remark that according to Cermelli and Gurtin [20] this tensor was first introduced in Kondo's work [40, 41]. Following Teodosiu [71] we ascribe it to Noll [56], since in this article this tensor first appears, to the best of our knowledge, in the explicit formula stated above. By the transformation formula of hypersurfaces in $\mathbb{R}^3$ this is equivalent to

$$G^{pl} = \frac{1}{\det(F_{pl})} \operatorname{Curl}(F_{pl}) F_{pl}^T,$$

where Curl denotes the Curl with respect to a point in the reference configuration, refer to [20, Chapter 4], and this tensor is again defined on the lattice configuration.
The slight differences to the formulas in the work of Cermelli and Gurtin occur from two facts. On the one hand we define the curl (resp. Curl) by applying the vectorial curl $\nabla\times$ to each row separately and thus obtain a matrix in $\mathbb{R}^{3\times 3}$, which is the transpose of theirs and on the other hand we require that G^{pl}, and not $\left(G^{pl}\right)^T$, provides the desired measure.
There are objections against this choice of the dislocation tensor, namely Acharya wrote an interesting counterpart against Cermelli and Gurtin's work [1]. He criticizes that one could choose instead of (i), even though it is a reasonable physical requirement, another criterion, namely that the dislocation tensor should measure the local, undeformed Burgers vector, per unit area in the deformed configuration. The second one is, in his view, of dubious physical origin and the third one is again a reasonable physical requirement, but not one that can be used to rule out other dislocation tensors.

There are some reasons, why we do use the tensor G^{pl} in this thesis. For simplicity, we assume that the crystal has initially no defects, i.e., $P^0 = \mathbb{1}$ and the slip direction and slip plane normal are given by $(s,m) = (\vec{e}_1, \vec{e}_2)$. First of all, we will use the tensor to compute the dislocation length only, which must be the same for each reasonable tensor, if one computes the length in the same configuration. The explicit description of the plastic deformation in the single slip model, i.e., $F_{pl} = \mathbb{1} + \gamma \vec{e}_1 \otimes \vec{e}_2$, implies that the dislocation length can be computed more easily, if one uses a tensor depending on the plastic deformation only. Secondly, we are mainly interested in a two dimensional single-slip model. In this case the dislocation tensors defined in [2, 13, 34, 56] transformed to the reference configuration are identical to the corresponding ones in the lattice configuration. Furthermore the three tensors defined by Acharya and Bassani, Fox, Noll [2, 34, 56] are equal and by pre-multiplication with the deformation gradient F we obtain the tensor defined by Bilby, Bullough and Smith [13], see Appendix A.

2.4.2 The true and local Burgers vector

First, we repeat the definitions of the local and true Burgers vector in the discrete setting, which can be found in Teodosiu [71, Chapter 7] using the ideas of Frank [35]. Afterwards, we derive its corresponding definitions in the continuous setting. In our model we will assume later on that the crystal has initially no defects and that the lattice defects only occur from the dislocation motion along the slip direction. Thus we exclude "Moebius crystals" and the occurring Burgers circuits can be defined in regions of "good" crystals, where a one to one correspondence of lattice directions in the deformed state to the ones in a perfect crystal can be established, refer to Frank [35]. We restrict ourselves to an edge dislocation with a straight dislocation line in direction of the third standard basis vector $\vec{e}_3$. This suffices to get an idea of the definition and furthermore the two dimensional model excludes screw dislocations. In order to define the true Burgers vector $\vec{b}$ we draw a closed circle from atom to atom in the deformed configuration, which encircles the dislocation core counterclockwise. Such a circle is also called Burgers circuit. Choose a starting point P_1 and a final point $Q_1 = P_1$ on this circle. Repeat this circle in the same sense in a perfect crystal starting from P_1' and ending in Q_1', then it does no more close. The vector from P_1' to Q_1', needed to close the circle, is defined as true Burgers vector $\vec{b}$, refer to Figure 2.1. Therefore the true Burgers vector is a lattice vector in the perfect crystal and it is independent of the starting point P_1'.

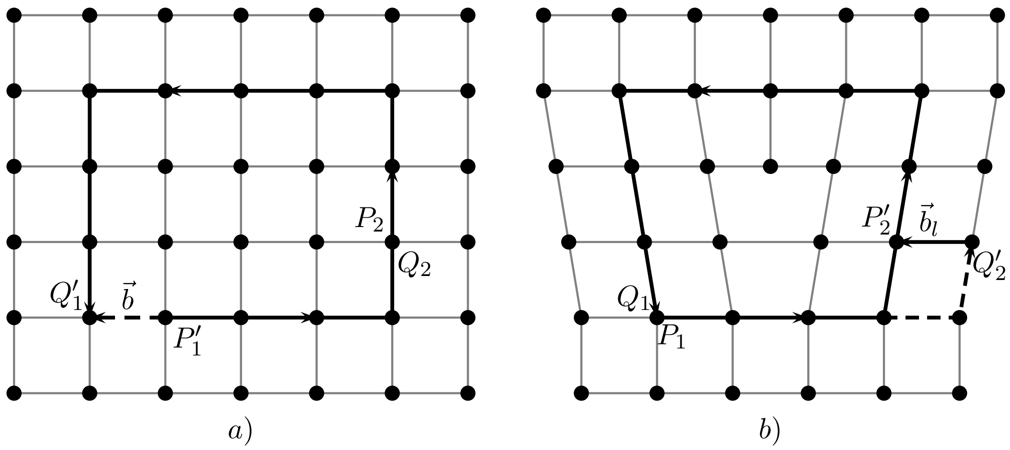

Figure 2.1: $a)$ Perfect lattice, $b)$ Deformed configuration

In order to define the local Burgers vector, we proceed the other way round. Namely we choose a closed circuit in the perfect lattice, with starting point P_2 and endpoint $Q_2 = P_2$. Then repeat this circuit in the deformed configuration starting in P_2' and ending in Q_2'. The vector from Q_2' to P_2' needed to close this circle is denoted by local

Burgers vector $\vec{b}_l$, refer to Figure 2.1. It is a lattice vector in the deformed configuration and thus depends on the starting point P_2'. For sake of completeness, we call the vector $\vec{b}_r = F^{-1}\vec{b}_l$ the reference Burgers vector. One can also define $-\vec{b}$, $-\vec{b}_l$ or $-\vec{b}_r$ as true, local or reference Burgers vector, which is only due to convention.
We have obtained the Burgers vector by identification of corresponding lattice vectors. For the continuous case we repeat the definition made by Bilby, Bullough and Smith [13], which can also be found in [12, 14, 31, 44]. Assume that we have an underlying lattice structure in the three dimensional deformed crystal. By refinement of the lattice structure one achieves the continuous case as limit of the discrete setting. Choosing at each point P in the deformed configuration three linearly independent basis vectors $\vec{v}_j(P)$, $j \in \{1,2,3\}$, such that $\vec{v}_j(P)$ corresponds to the same lattice vector for each P. Comparing these vectors with the corresponding vectors $\vec{a}_i \in \mathbb{R}^3$, $i \in \{1,2,3\}$ of an ordinary triclinic lattice, which describes the perfect crystal, gives

$$\vec{v}_j(P) = \sum_{i=1}^{3} D_{ij}(P)\vec{a}_i \text{ for } j \in \{1,2,3\}\,. \tag{2.2}$$

Thereby $\vec{v}_j$ is the same as $\mathbf{e}_j \in \mathbb{R}^3$ used in [13] and $\vec{a}_i \in \mathbb{R}^3$ matches $\mathbf{a}_j$ in [13]. Furthermore the lattice vectors $\vec{a}_i$ are independent of P. Since the vectors $\vec{v}_j(P)$, $j \in \{1,2,3\}$, are linearly independent the matrix $D(P) = \left(D_{ij}(P)\right)_{ij} \in \mathbb{R}^{3\times 3}$ is non-singular and one can denote its inverse by $E(P)$. Define the matrices $V = V(P) = \sum\limits_{i=1}^{3} \vec{v}_i(P) \otimes \vec{e}_i$ and $A = \sum\limits_{i=1}^{3} \vec{a}_i \otimes \vec{e}_i$, where $\vec{e}_j \in \mathbb{R}^3$, $j \in \{1,2,3\}$ is the standard orthonormal basis of $\mathbb{R}^3$. Then the equations in (2.2) can be summarized to

$$V = AD \text{ or equivalently } V = VDV^{-1}A.$$

Since locally the lattice configuration can be identified with the triclinic lattice of the perfect crystal we get that $F_{el} = VDV^{-1}$. Denote now by $F_{el,A}$ the elastic distortion in the basis A, i.e. $F_{el,A} = A^{-1}F_{el}A$ and analogously we set $F_{el,A}^{-1} = A^{-1}F_{el}^{-1}A$. Then we get using $V = AD$ that

$$F_{el,A} = A^{-1}VDV^{-1}A = D \text{ and analogously } F_{el,A}^{-1} = E.$$

Next, we want to compute an expression for the Burgers vector density at a point P in the deformed configuration. Draw now a closed, smooth circuit Γ, which encircles P, in the deformed configuration. Let $\gamma : [0,1] \to \mathbb{R}^3$, with $\gamma(0) = \gamma(1)$ and $\gamma([0,1]) = \Gamma$ be an orientation preserving parametrization of Γ. Then we have

$$0 = \int_\Gamma d\vec{x} = \int_0^1 \gamma'(t)dt.$$

In the following we use the Einstein notation, namely we add up indices appearing twice, where all appearing indices are subscripts. Write $\gamma'(t) = \widetilde{\gamma}'_j(t)\vec{a}_j = \widetilde{\gamma}'_j(t)E_{kj}(Q)\vec{v}_k(Q)$ for each $Q \in \Gamma$. Thus the coefficient of $\gamma'(t)$ in the basis $\vec{v}_k(Q)$ is given by $\widetilde{\gamma}'_j(t)E_{kj}(Q)$. As in the discrete case one can identify the vector $\gamma'(t)$ in the deformed configuration with the corresponding vector in the perfect lattice, which has the same components in the $\vec{a}_k$-basis as $\gamma'(t)$ has in the $\vec{v}_k$-basis. Then the true Burgers vector associated with Γ is given by

$$\int_0^1 \widetilde{\gamma}'_j(t)E_{kj}(\gamma(t))\vec{a}_k dt. \tag{2.3}$$

Write the vector-valued one-form $d\vec{x}$ in the basis $\vec{a}_k$, namely $d\vec{x} = \widetilde{dx}_k\vec{a}_k$, where $\widetilde{dx}_k$ for $k \in \{1, 2, 3\}$ are real-valued one-forms. Then we get

$$\widetilde{dx}_j\vec{a}_j = d\vec{x} = \gamma'(t)dt = \widetilde{\gamma}'_j(t)\vec{a}_j dt$$

and by comparison of coefficients it follows that $\widetilde{\gamma}'_j(t)dt = \widetilde{dx}_j$. Using this we can write Equation (2.3) as

$$\int_0^1 \widetilde{\gamma}'_j(t)E_{kj}(\gamma(t))\vec{a}_k dt = \int_\Gamma E_{kj}\vec{a}_k\widetilde{dx}_j = \left\{\int_\Gamma E_{kj}\widetilde{dx}_j\right\}\vec{a}_k.$$

Writing $d\vec{x}$ in the basis $\vec{e}_r$ we get $\widetilde{dx}_j\vec{a}_j = dx_r\vec{e}_r = dx_r\left(A^{-1}\right)_{jr}\vec{a}_j$ and thus we have $\widetilde{dx}_j = \left(A^{-1}\right)_{jr}dx_r$. Using this and $\vec{a}_k = A_{ik}\vec{e}_i$ we conclude

$$\left\{\int_\Gamma E_{kj}\widetilde{dx}_j\right\}\vec{a}_k = \left\{\int_\Gamma E_{kj}\left(A^{-1}\right)_{jr}dx_r\right\}A_{ik}\vec{e}_i = \left\{\int_\Gamma \left(AEA^{-1}\right)_{ir}dx_r\right\}\vec{e}_i.$$

Let $\Sigma \subseteq \mathbb{R}^3$ be a two-dimensional, compact and orientable submanifold, with boundary Γ, which is oriented by the unit normal field $\vec{n} : \Sigma \to \mathbb{S}^2$. Then the true Burgers vector associated with the circuit $\Gamma = \partial\Sigma$ is given by

$$\int_\Gamma F_{el}^{-1}\,d\vec{x} = \int_\Sigma \operatorname{curl}\left(F_{el}^{-1}\right)\vec{n}\,dS,$$

where we have used Stokes' theorem and $F_{el}^{-1} = AEA^{-1}$. The limit $\Sigma \to 0$ delivers that the density of the true Burgers vector measured per unit area in the deformed configuration is given by $\operatorname{curl}\left(F_{el}^{-1}\right)$. This is the tensor preferred by Acharya and Bassani [2]. As in the discrete case one can define the local Burgers vector as the closure failure associated with a closed circuit in the perfect lattice, which is repeated in the deformed configuration. According to [12] and [13] the coefficients of the true Burgers vector in the $\vec{a}_k$ basis are the same as those of the local Burgers vector at P in the $\vec{v}_k(P)$ basis and thus one can obtain the local Burgers vector from the true Burgers vector by pre-multiplication by $D(P)$ from the left. In our case the true Burgers vector density reads

$$\text{curl}\left(F_{el}^{-1}\right)\vec{n} = \left(\text{curl}\left(F_{el}^{-1}\right)\vec{n}\right)_i \left(A^{-1}\right)_{ki}\vec{a}_k$$

and thus the local Burgers vector density is given by

$$\begin{aligned}\left(\text{curl}\left(F_{el}^{-1}\right)\vec{n}\right)_i \left(A^{-1}\right)_{ki}\vec{v}_k &= \left(\text{curl}\left(F_{el}^{-1}\right)\vec{n}\right)_i \left(A^{-1}\right)_{ki} D_{lk}\vec{a}_l \\ &= A_{rl}D_{lk}\left(A^{-1}\right)_{ki}\left(\text{curl}\left(F_{el}^{-1}\right)\vec{n}\right)_i \vec{e}_r = F_{el}\,\text{curl}\left(F_{el}^{-1}\right)\vec{n}.\end{aligned}$$

Thus the tensor $F_{el}\,\text{curl}\left(F_{el}^{-1}\right)$ describes the local Burgers vector measured per unit area in the deformed configuration. This tensor is preferred by Bilby, Bullough and Smith [13]. By replacing F_{el}^{-1} by F_{pl}, we get the tensor preferred by Fox [34], namely $F_{pl}^{-1}\,\text{curl}\,(F_{pl})$. This tensor can be obtained with the same argumentation, where we use the reference configuration instead of the deformed configuration. Thus it computes the reference Burgers vector measured per unit area in the reference configuration. Next we show that the tensor defined by Noll [56] describes the true Burgers vector measured per unit area in the lattice configuration and it can be obtained by transforming the tensor of Acharya and Bassani to the lattice configuration. This was done in Subsection 2.4.3 from a mathematical viewpoint.

2.4.3 Transformation rule for the true Burgers vector

The following relies on the work of Mielke and Müller [51]. For simpler notations we choose $\Omega = \mathbb{R}^n$. As in [20] we distinguish between a material point $x \in \mathbb{R}^n$ and the tangent space at x, in order to capture the incompatibility of F_{pl}. This means we work with the tangent bundle $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ instead of $\mathbb{R}^n$. Following [20, 51] we can define the reference configuration, the lattice configuration, and the deformed configuration as the tangent bundle $T\mathbb{R}^n$. For a given deformation $u : \mathbb{R}^n \to \mathbb{R}^n$ we fix a decomposition $\nabla u(x) = F_{el}(x)F_{pl}(x)$. The mapping from the reference configuration to the lattice configuration is then given by

$$u_{pl} : T\mathbb{R}^n \to T\mathbb{R}^n, \text{ where } u_{pl}(x,v) = (x, F_{pl}(x)v),$$

and the mapping from the lattice configuration to the deformed configuration is given by

$$u_{el} : T\mathbb{R}^n \to T\mathbb{R}^n, \text{ where } u_{el}(x,v) = (u(x), F_{el}(x)v).$$

Their composition gives the usual deformation u extended to the tangent bundle, i.e.,

$$u_{el} \circ u_{pl} = du, \text{ where } du(x,v) = (u(x), \nabla u(x)v).$$

Consider now the vector-valued one-forms $\widehat{\alpha}_{pl},\ \widehat{\alpha}_{el} : T\mathbb{R}^n \to L\left(T\mathbb{R}^n, \mathbb{R}^n\right)$ defined by $\widehat{\alpha}_{pl}(x,w)[(y,v)] = F_{pl}(x)v$ and $\widehat{\alpha}_{el}(x,w)[(y,v)] = F_{el}^{-1}(x)v$. Thereby $\widehat{\alpha}_{pl}$ is defined on the reference configuration and $\widehat{\alpha}_{el}$ is defined on the deformed configuration. We identify them in the following by the vector-valued one-forms $\alpha_{pl},\ \alpha_{el} : \mathbb{R}^n \to L\left(\mathbb{R}^n, \mathbb{R}^n\right)$ given by

$$\alpha_{pl}(x)[v] = F_{pl}(x)v \text{ and } \alpha_{el}(x)[v] = F_{el}^{-1}(x)v.$$

Adapted to this identification, we define the pullback of α_{pl} under a smooth map $f : T\mathbb{R}^n \to T\mathbb{R}^n$ by $f^*\alpha_{pl}(x)[v] := f^*\widehat{\alpha}_{pl}\,(x,0)\,[(0,v)]$ and analogously for $d\alpha_{pl}$ by $f^*d\alpha_{pl}(x)[v,w] := f^*d\widehat{\alpha}_{pl}\,(x,0)\,[(0,v),(0,w)]$, with $x, v \in \mathbb{R}^n$. If $\Gamma \subset \mathbb{R}^n$ is a closed and smooth curve, which is the boundary of a two-dimensional, compact and orientable submanifold $S \subset \mathbb{R}^n$, and if we assume that α_{pl} is a smooth one-form, then we can measure the incompatibility of F_{pl} by the true Burgers vector

$$b^{pl}\left(\Gamma\right) = \int_\Gamma \alpha_{pl} = \int_S d\alpha_{pl},$$

where we used Stokes' theorem. Hence $d\alpha_{pl}$ measures true Burgers vector, and thus the incompatibility, per unit reference area. To obtain a measure per unit area in the lattice configuration we consider the pullback of $d\alpha_{pl}$ under the map u_{pl}^{-1}, i.e., we define the geometric dislocation tensor $G = G\left(F_{pl}\right)$ as vector-valued two form by

$$\begin{aligned}
&G\left(F_{pl}\right)(x)\left[w_1, w_2\right] := \left(u_{pl}^{-1}\right)^* \left(d\alpha_{pl}\right)(x)\left[w_1, w_2\right] \\
&= \left(u_{pl}^{-1}\right)^* d\widehat{\alpha}_{pl}(x,0)\left[\left(0,w_1\right),\left(0,w_2\right)\right] = d\widehat{\alpha}_{pl}(x,0)\left[\left(0, F_{pl}^{-1}(x)w_1\right), \left(0, F_{pl}^{-1}(x)w_2\right)\right].
\end{aligned}$$

In the following we write $A = F_{pl}$. Next, we get, using Einstein's notation, that

$$\begin{aligned}
&G\left(A\right)(x)\left[w_1, w_2\right] = d\alpha_{pl}(x)\left[A^{-1}w_1, A^{-1}w_2\right] \\
&= \left(d\left(\alpha_{pl}(.)\left[A^{-1}w_2\right]\right)\right)(x)\left[A^{-1}w_1\right] - \left(d\left(\alpha_{pl}(.)\left[A^{-1}w_1\right]\right)\right)(x)\left[A^{-1}w_2\right] \\
&= \left[\partial_k\left(A_{ij}\right)\left(A^{-1}w_2\right)_j\left(A^{-1}w_1\right)_k - \partial_k\left(A_{ij}\right)\left(A^{-1}w_1\right)_j\left(A^{-1}w_2\right)_k\right]\vec{e}_i \\
&= DA(x)\left[A^{-1}w_1\right]A^{-1}w_2 - DA(x)\left[A^{-1}w_2\right]A^{-1}w_1,
\end{aligned}$$

which is equal to the definition of the geometric dislocation tensor in [51, Chapter 2]. Thereby $DA(x) \in \mathbb{R}^{3\times3}\times\mathbb{R}^3$ denotes the Jacobian of $A : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ evaluated at $x \in \mathbb{R}^3$. For $n = 2$, respectively $n = 3$, the tensor $G\left(A\right)$ can be identified with $\widehat{G}_2\left(A\right) \in \mathbb{R}^2$, respectively $\widehat{G}_3\left(A\right) \in \mathbb{R}^{3\times3}$, given by

$$\widehat{G}_2\left(A\right) := \frac{1}{\det(A)}\begin{pmatrix}\partial_1 A_{12} - \partial_2 A_{11} \\ \partial_1 A_{22} - \partial_2 A_{21}\end{pmatrix} \in \mathbb{R}^2 \text{ and } \widehat{G}_3\left(A\right) := \frac{1}{\det(A)}\mathrm{Curl}\left(A\right)A^T,$$

and for $A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ one can identify $\widehat{G}_2(A)$ with $\widehat{G}_3(\text{diag}\,(A,1))$. This is shown in the Appendix A. Due to this abstract definition of the geometric dislocation tensor, one can easily show that G is invariant under compatible local changes in the reference configuration, refer to [51, Chapter 5] and the references therein. For this purpose we define a smooth bijective map $f : \mathbb{R}^n \to \mathbb{R}^n$ and want to evaluate $G(A\nabla f)$. Denote with $\widehat{f}$ the map f extended to the tangent bundle, i.e., $\widehat{f}(x,v) = (f(x), \nabla f(x)v)$. Therefore we have to use the transformed one-form $\delta : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$ with

$$\delta(x)[v] = F_{pl}(f(x))\nabla f(x)v = f^*\alpha_{pl}(x)[v]$$

instead of α_{pl} and we have to consider the pullback of $d\delta$ under the map $\widehat{f}^{-1} \circ u_{pl}^{-1}$ instead of u_{pl}^{-1}. This gives the asserted invariance, i.e.,

$$\begin{aligned} G(A\nabla f)(x)\,[w_1,w_2] &= \left(\widehat{f}^{-1} \circ u_{pl}^{-1}\right)^* d\delta(x)\,[w_1,w_2] \\ &= \left(\widehat{f}^{-1} \circ u_{pl}^{-1}\right)^* \left(\widehat{f}^* d\alpha_{pl}\right)(x)\,[w_1,w_2] = G(A)(x)\,[w_1,w_2]\,, \end{aligned}$$

where we have used that δ can be identified with a one-form in the tangent bundle. Furthermore one can show that G is invariant under compatible change in the deformed configuration, refer to [51, Chapter 5] and the references therein. For this purpose we have to define a geometric dislocation tensor in the lattice configuration with respect to the elastic part of the deformation, i.e.,

$$\begin{aligned} G_{el}\left(F_{el}^{-1}\right)(x)\,[w_1,w_2] &:= u_{el}^*\left(d\alpha_{el}\right)(x)\,[w_1,w_2] = u_{el}^*\left(d\left(u^{-1}\right)^*\alpha_{pl}\right)(x)\,[w_1,w_2] \\ &= \left(u^{-1} \circ u_{el}\right)^*\left(d\alpha_{pl}\right)(x)\,[w_1,w_2] = G\,(A)\,(x)\,[w_1,w_2]\,. \end{aligned}$$

Since G^{el} is equivalent to G^{pl}, which was shown in [20, Chapter 4], one can identify the tensor $G_{el}\left(F_{el}^{-1}\right)$ with $\widehat{G}_{el,3}\left(F_{el}^{-1}\right) := G^{el} = \det\left(F_{el}\right)\text{curl}\left(F_{el}^{-1}\right)\left(F_{el}^{-1}\right)^T$ for $n = 3$ and $\widehat{G}_{el,2}\left(F_{el}^{-1}\right) := \widehat{G}_{el,3}\left(\text{diag}\;\left(F_{el}^{-1}, 1\right)\right)$ for $n = 2$. Otherwise if one computes the identified forms explicitly, as it was done for the plastic tensor, then one gets a simple proof that the tensor G^{el} is equivalent to G^{pl}. Analogously as above, one can show that for a smooth, bijective map $g : \mathbb{R}^n \to \mathbb{R}^n$ the equation $G_{el}\left(F_{el}^{-1}\nabla g\right) = G_{el}\left(F_{el}^{-1}\right)$ holds as requested. A straightforward computation for this invariance in the identified case can be found in the Appendix. Summarized we have obtained a transformation rule for the dislocation tensors, which measures the true Burgers vector [2, 56]. The tensors, which measure the local and reference Burgers vector [13, 34], transform in the same way. The transformed forms of these tensors were stated in Table 9.1 in the Appendix. Next, we want to derive the corresponding energy density of the dislocation tensor.

2.4.4 The self-energy density of the dislocations

This paragraph relies on the work of Ortiz, Repetto [60] and Teodosiu [71]. Even though we are interested in a continuum theory, the dislocation density is in fact a finite sum of single dislocations, which occur in the discrete setting, like edge or screw dislocations or in general curvilinear dislocations. The true Burgers vector of these dislocations is still constant, refer to Teodosiu [71, Chapter 7.2].

Therefore it suffices to investigate the self-energy of a single planar dislocation loop or line with constant true Burgers vector $b \in \mathbb{R}^3$. Suppose that the crystal has slipped by $b \in \mathbb{R}^3$ over an open subset Σ of a slip plane with normal $m \in \mathbb{R}^3$ orthogonal to b, and with smooth boundary $C = \partial\Sigma$. W.l.o.g. we can assume that the slip plane is a linear space. Define the delta distribution $\delta_U \in \mathcal{D}'(\Omega)$ supported on $U \subseteq \Omega$ by $< \delta_U, \varphi >:= \int_U \varphi \; dx$ for all $\varphi \in \mathcal{D}(\Omega) = C_c^\infty(\Omega)$. Then the corresponding plastic deformation is given by

$$F_{pl} = A = \mathbb{1} + b \otimes m \chi_\Sigma \delta_{\{x \in \Omega \,:\, x \cdot m = 0\}},$$

where $\chi_\Sigma(x) = 1$ for $x \in \Sigma$ and $\chi_\Sigma(x) = 0$ else. Let $V := \{x \in \Omega \;:\; x \cdot m = 0\}$, then we can calculate in the sense of distributions,

$$\begin{aligned}
&< (\mathrm{Curl}\, A)_{ij} \,, \varphi >=< \epsilon_{jlk} b_i m_k \partial_l \chi_\Sigma \delta_V \,, \varphi >= - < \epsilon_{jlk} b_i m_k \chi_\Sigma \delta_V \,, \partial_l \varphi > \\
&= b_i \int_\Sigma \epsilon_{klj} m_k \partial_l \varphi \; dx = b_i \int_\Sigma \nabla \times (\varphi \vec{e}_j) \cdot m \; dx = b_i \int_C (\varphi \vec{e}_j) \cdot d\vec{s} \\
&= b_i \int_C \varphi \, (v \times m)_j \; ds =< (b \otimes (v \times m))_{ij} \, \delta_C, \varphi >
\end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$, where we used Stokes' theorem and defined $v \in m^\perp$ as the unit normal to C pointing in the inside of Σ. Next, we want to compute the dislocation tensor G. For the two-dimensional case we extend the deformation to dimension three by leaving the third component unchanged, i.e., we consider $A := \mathrm{diag}\,(\widehat{A}, 1) \in \mathbb{R}^{3\times 3}$ instead of $\widehat{A} = \widehat{A}\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2\times 2}$, then we can identify the dislocation tensor $\widehat{G}_2(\widehat{A})$ with $\frac{1}{\det(A)}\mathrm{Curl}\;(A)\, A^T$. In the three dimensional case we can compute

$$\begin{aligned}
\widehat{G}_3(A) &= \frac{1}{\det(A)} \mathrm{Curl}\,(A) A^T = \frac{1}{\det(A)} b \otimes (v \times m)\, \delta_C \,(\mathbb{1} + m \otimes b \chi_\Sigma \delta_V) \\
&= \frac{1}{\det(A)} b \otimes (v \times m)\, \delta_C.
\end{aligned}$$

Therefore one can evaluate the dislocation length L of the above single planar dislocation loop contained in a volume $U \subset \Omega \subset \mathbb{R}^n$ for dimension $n = 2$ or 3 through

$$\mathcal{L} := \int_U \frac{\left\| \widehat{G}_n(A) \right\|}{|b|} \; d\lambda_n.$$

For the evaluation of the total energy stored per unit length of a dislocation one usually splits the region into a part with distance less than r_0, called core radius, to the dislocation loop and the rest. This is done because in the region near the dislocation loop, which is called the dislocation core, even the non-linear theory proves to be inappropriate, refer to [71, Chapter 7.3, Chapter 16] or [65]. In Chapter 8 and 10 Teodosiu [71] proves that the total energy stored per unit length of a straight dislocation, in a cylinder $B(0,R) \times \mathbb{R} \subseteq \mathbb{R}^3$ with $R > r_0$, is

$$W_t(R) = \frac{K\mu\,|b|^2}{4\pi} \ln\left(\frac{R}{r_0}\right) + \mathcal{T} = \frac{K\mu\,|b|^2}{4\pi} \ln\left(\frac{R}{r_1}\right),$$

where $\mathcal{T}$ is the core or self-energy per unit length, which is also called dislocation line tension, μ is an average shear modulus and K is a constant, refer also to [36, 45, 57, 61]. Experiments reveal that $r_1 = \frac{|b|}{\alpha}$, with α varying between 1 and 2 for most metals, and $r_0 = c\,|b|$ with a constant $c > 1$, refer to [71, Chapter 16]. Therefore one can evaluate the self-energy per unit length with

$$\mathcal{T} = \frac{K\mu\,|b|^2}{4\pi} \left(\ln(r_0) - \ln(r_1)\right) = \frac{K\mu\,|b|^2}{4\pi} \left(\ln(c) + \ln(\alpha)\right) = C\mu\,|b|^2\,,$$

with a constant $C > 0$, refer also to [45]. The assumption of a well defined line tension permits writing the self-energy of the above dislocation inside U as

$$\widetilde{E}_{self} = \mathcal{T}\mathcal{L} = \int_U \frac{\mathcal{T}\left\|\widehat{G}_n(A)\right\|}{|b|}\,d\lambda_n.$$

Thus we have derived the self-energy of a single planar dislocation loop. For an arbitrary distribution of dislocation loops one can argue as follows. Since one can assume that the set of the points which belong to more than one dislocation loop is a discrete point set, the dislocation length inside U can be evaluated by

$$\mathcal{L} = \int_U \frac{\left\|\widehat{G}_n(A)\right\|}{|b^{pl}(\Gamma_x)|}\,d\lambda_n$$

and the self-energy of an arbitrary dislocation by

$$\widetilde{E}_{self} = \int_U \frac{\mathcal{T}\left\|\widehat{G}_n(A)\right\|}{|b^{pl}(\Gamma_x)|}\,d\lambda_n,$$

where Γ_x is a suitable small Burgers circuit around x. The magnitude of a non zero Burgers vector has a lower bound $l > 0$, which arise due to the lattice structure, and an upper bound $d > l$, which is related to the grain size. Thus the self-energy of an arbitrary dislocation, inside U, is comparable with

$$E_{self} = \int_U \delta \left\| \widehat{G}_n(A) \right\| \, d\lambda_n,$$

where $\delta > 0$ is a small constant, which depends on μ and an average of the magnitude of all Burgers vectors. Then we can define the self-energy density ψ_{self} through

$$\psi_{self}\left(F_{pl}\right) = \psi_{self,\delta}\left(F_{pl}\right) = \delta \left\| \widehat{G}_n(F_{pl}) \right\|. \tag{2.4}$$

2.5 Variational formulation

In this section we choose again $\mathbf{P} := \left(F_{pl}\right)^{-1}$ and consider $n = 2$. Choose initial conditions $\mathbf{P}^0 = \mathbb{1} \in \mathbb{R}^{2\times 2}$ and $\mathbf{p}^0 = 0 \in \mathbb{R}$, i.e., we consider a perfect crystal at initial time $t^0 = 0$. Then we can compute that

$$\left|\widehat{G}_2\left(\mathbb{1} + \gamma s \otimes m\right)\right| = \left|\begin{pmatrix} \partial_x\left(\gamma m_2 s_1\right) - \partial_y\left(\gamma m_1 s_1\right) \\ \partial_x\left(\gamma m_2 s_2\right) - \partial_y\left(\gamma m_1 s_2\right) \end{pmatrix}\right| = \left|\begin{pmatrix} m_2\left(s_1 \partial_x \gamma + s_2 \partial_y \gamma\right) \\ m_1\left(s_1 \partial_x \gamma + s_2 \partial_y \gamma\right) \end{pmatrix}\right| = \left|\partial_s \gamma\right|,$$

where we have defined $\partial_s \gamma = \nabla \gamma \cdot s$ and we have used $m \cdot s = 0$. Furthermore define $\mathbf{P}^k \in Sl(2)$ through $\mathbf{P}^k = \mathbb{1} - \gamma^k s \otimes m$, then our incremental problem reads:

$$\begin{aligned} \text{For } k = 1, \ldots, N \quad & \text{find} \quad u^k : \Omega \to \mathbb{R}^2 \text{ with } u^k = u_b\left(t^k, .\right) \text{ on } \partial\Omega \\ & \text{and} \quad \gamma^k : \Omega \to \mathbb{R} \text{ which minimize } E_\delta^k[u, \gamma], \end{aligned}$$

where

$$\begin{aligned} E_\delta^k[u,\gamma] \quad = \quad & \int_\Omega \left[\psi_{el}\left(\nabla u\left(\mathbb{1} - \gamma s \otimes m\right)\right) + \psi_h \left(\sum_{l=1}^{k-1} \left|\gamma^l - \gamma^{l-1}\right| + \left|\gamma - \gamma^{k-1}\right| \right) \right. \\ & \left. + \delta \left|\partial_s \gamma\right| + \frac{\tau}{t^k - t^{k-1}} \left|\gamma - \gamma^{k-1}\right| \right] \, dx - < l\left(t^k\right), u > . \end{aligned} \tag{2.5}$$

3 Mathematical preliminaries

In this section we collect a few well known statements, which are used in this thesis. Proofs can be found in [5, 7, 15, 28, 29, 32, 52].

3.1 Traces and Null Lagrangians

Lemma 3.1. *[32, p. 258]. Let $n, K \in \mathbb{N}$, $p \in [1, \infty)$ and assume that $\Omega \subseteq \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}\left(\Omega; \mathbb{R}^K\right) \to L^p\left(\partial\Omega; \mathbb{R}^K\right)$$

such that

a) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}\left(\Omega; \mathbb{R}^K\right) \cap C\left(\overline{\Omega}; \mathbb{R}^K\right)$,

b) and

$$\|Tu\|_{L^p(\partial\Omega;\mathbb{R}^K)} \leq C\,\|u\|_{W^{1,p}(\Omega;\mathbb{R}^K)}\,,$$

for each $u \in W^{1,p}(\Omega)$, with a constant C depending only on p and Ω.

An operator with this properties is called trace operator and Tu denotes the trace of u.

Proof: The case $K = 1$ was proven in [32, p. 258-259]. Assume T is a trace operator for $K = 1$ then we get that $u = (u_1, \ldots, u_K)^T \mapsto (T(u_1), \ldots T(u_K))^T$, with $u_i \in W^{1,p}(\Omega)$ for $i \in \{1, \ldots, K\}$, is a trace operator for $K > 1$.

□

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded with C^1-boundary $\partial\Omega$, $u \in W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ with $1 \leq p \leq \infty$, then*

$$\int_\Omega \operatorname{div} u \, d\lambda_N = \int_{\partial\Omega} < Tu, \vec{n} > d\mathcal{H}^{N-1}$$

for a trace operator

$$T : W^{1,1}\left(\Omega; \mathbb{R}^N\right) \to L^1\left(\partial\Omega; \mathbb{R}^N\right),$$

as in Lemma 3.1 and $\vec{n}$ denotes the outer unit normal on $\partial\Omega$.

Proof: Since Ω is bounded and $\partial\Omega$ is C^1 it exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq C^\infty\left(\overline{\Omega};\mathbb{R}^N\right)$ with $u_k \to u \in W^{1,1}\left(\Omega;\mathbb{R}^N\right)$, refer to [32, Chapter 5.3.3]. Thus we have $D_i u_{k,i} \to D_i u_i$ in $L^1(\Omega)$ for $i = 1 \dots N$. Furthermore the continuity of the trace operator T implies

$$\|Tu_k - Tu\|_{L^1(\partial\Omega;\mathbb{R}^N)} \leq C\,\|u_k - u\|_{W^{1,1}(\Omega;\mathbb{R}^N)}\,,$$

with a constant $C > 0$ depending only on Ω. This implies $Tu_k \to Tu$ in $L^1\left(\partial\Omega;\mathbb{R}^N\right)$ and finally we can conclude that

$$\begin{aligned}\int_\Omega \operatorname{div} u\; d\lambda_N = \lim_{k\to\infty}\int_\Omega \operatorname{div} u_k\; d\lambda_N = \lim_{k\to\infty}\int_{\partial\Omega} &< Tu_k, \vec{n} > d\mathcal{H}^{N-1}\\ = \int_{\partial\Omega} &< Tu, \vec{n} > d\mathcal{H}^{N-1}.\end{aligned}$$

□

Lemma 3.3. *[28, Corollary 5.22] or [52, Theorem 2.3]. Let $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set, M be an $r\times r$ sub-determinant for $r \in \{1,2\}$, $F \in \mathbb{R}^{2\times 2}$.*

(i) Let $q \geq r$ and $u, v \in W^{1,q}\left(\Omega;\mathbb{R}^2\right)$ with $u - v \in W_0^{1,q}\left(\Omega;\mathbb{R}^2\right)$ then

$$\int_\Omega M(Du)\; d\lambda_2 = \int_\Omega M(Dv)\; d\lambda_2.$$

In particular

$$\int_\Omega M(Du)\; d\lambda_2 = \int_\Omega M(F)\; d\lambda_2 \text{ if } u = F \text{ on } \partial\Omega.$$

(ii) Let $p > r$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}\left(\Omega;\mathbb{R}^2\right)$, with

$$u_j \rightharpoonup u \text{ in } W^{1,p}\left(\Omega;\mathbb{R}^2\right)$$

then

$$M\left(Du_j\right) \rightharpoonup M\left(Du\right) \text{ in } L^{\frac{p}{r}}\left(\Omega\right).$$

Remark 3.4. *[28, 52]. A function f, for which $\int_\Omega f\left(Du\right)\,dx$ only depends on the boundary values of u is called null Lagrangian, since the Euler-Lagrange equations are automatically satisfied for all smooth functions u. Affine combinations of minors are the only null Lagrangians.*

Lemma 3.5. *[52, Theorem 2.4] or [37, p. 231] Let Ω be an open, bounded and connected set with Lipschitz boundary and $Du \in SO(2)$ a.e. in Ω. Then there exists a $Q \in SO(2)$, so that $Du(x) = Q$ a.e. in Ω.*

3.2 Functions of bounded variation

In this section we assume $m, N \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^N$ be an open set and $\mathcal{E}$ be a σ-algebra in a nonempty set X. For a topological space X we denote its Borel σ-algebra by $\mathcal{B}(X)$.

Definition 3.6. *[7, p. 3]. Let $(X, \mathcal{E})$ be a measure space.*

a) We say that $\mu : \mathcal{E} \to \mathbb{R}^N$(resp. $[0, \infty]$) is an $\mathbb{R}^N$-valued measure, or short a measure, (resp. positive measure), if $\mu(\emptyset) = 0$ and for any sequence $(E_k)_{k\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{E}$

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

A (positive) measure is called (positive) Borel measure if $\mathcal{E} = \mathcal{B}(X)$.

b) If μ is a measure, we define its total variation $|\mu|$ for every $E \in \mathcal{E}$ by

$$|\mu|(E) := \sup\left\{\sum_{k=1}^{\infty} |\mu(E_k)| \;:\; E_k \in \mathcal{E} \text{ pairwise disjoint}, E = \bigcup_{k=1}^{\infty} E_k\right\}.$$

c) If μ is an $\mathbb{R}$-valued measure, we define its positive and negative parts respectively by

$$\mu^+ := \frac{|\mu| + \mu}{2} \text{ and } \mu^- := \frac{|\mu| - \mu}{2}.$$

d) All the above measures are tacitly understood to be extended to the completion of $\mathcal{E}$ denoted by $\mathcal{E}_\mu$, see [7, Definition 1.11].

Lemma 3.7. *[7, p. 4] or [64, p. 118/119]. Let $(X, \mathcal{E})$ be a measure space and μ be an $\mathbb{R}^N$-valued measure, then $|\mu|$ is a positive measure and it is finite, i.e., $|\mu|(X) < \infty$. If μ is an $\mathbb{R}$-valued measure the same is true for μ^+ and μ^-.*

Definition 3.8. *[7, p. 7-10] Let $(X, \mathcal{E})$ be a measure space.*

a) Let $\mu : \mathcal{E} \to [0, \infty]$ be a positive measure. A function $u : X \to \overline{\mathbb{R}}$ is μ-measurable if it is $\mathcal{E}_\mu$-measurable, i.e., $u^{-1}(A) \in \mathcal{E}_\mu$ for every open set $A \subseteq \overline{\mathbb{R}}$.

b) Let $\mu : \mathcal{E} \to [0, \infty]$ be a positive measure. A μ-measurable map $u : X \to \overline{\mathbb{R}}$ is called μ-summable if

$$\int_X |u| \, d\mu < \infty.$$

We say that a μ-measurable map $u : X \to \overline{\mathbb{R}}$ is μ-integrable if either

$$\int_X u^+ \, d\mu < \infty \text{ or } \int_X u^- \, d\mu < \infty,$$

where $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := -\min\{u(x), 0\}$ for all $x \in X$. If u is μ-integrable, we set

$$\int_X u \, d\mu := \int_X u^+ \, d\mu - \int_X u^- \, d\mu.$$

c) Let μ be an $\mathbb{R}^N$-valued measure and $u : X \to \overline{\mathbb{R}}$ a $|\mu|$-measurable function. We say that u is μ-summable if u is $|\mu|$-summable and, if $N = 1$, we set

$$\int_X u \, d\mu := \int_X u \, d\mu^+ - \int_X u \, d\mu^-,$$

and if $N > 1$, i.e., $\mu = (\mu_1, \ldots, \mu_N) : \mathcal{E} \to \mathbb{R}^N$, we set

$$\int_X u \, d\mu := \left(\int_X u \, d\mu_1, \ldots, \int_X u \, d\mu_N \right).$$

d) Let μ be a positive measure $f : X \to \mathbb{R}$ be μ-measurable and μ-summable, then we define

$$f\mu(B) := \int_B f \, d\mu \quad \forall B \in \mathcal{E}.$$

Lemma 3.9. *[7, p. 10/11]. Let $(X, \mathcal{E})$ be a measure space and let $f\mu$ be the measure introduced in the previous definition, then*

$$|f\mu|(B) = \int_B |f| \, d\mu \quad \forall B \in \mathcal{E}.$$

Lemma 3.10. *Let $(X, \mathcal{E})$ be a measure space, μ be an $\mathbb{R}^N$-valued measure. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $|\mu|$-measurable functions $f_n : X \to \overline{\mathbb{R}}$ with $0 \leq f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$, $x \in X$, and let $\left\{\int_X f_n \, d|\mu|\right\}_{n\in\mathbb{N}}$ be a bounded sequence. Then we get*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu.$$

Proof: W.l.o.g. we can choose $N = 1$. For $A \in \mathcal{E}$ we get $\mu^+(A), \mu^-(A) \leq |\mu|(A)$ and because of $f_n(x) \geq 0$ for all $n \in \mathbb{N}$, $x \in X$ the sequences $\left\{\int_X f_n \, d\mu^+\right\}_{n\in\mathbb{N}}$ and $\left\{\int_X f_n \, d\mu^-\right\}_{n\in\mathbb{N}}$ are also bounded and thus convergent. Then we get

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu^+ - \lim_{n\to\infty} \int_X f_n \, d\mu^- = \int_X \lim_{n\to\infty} f_n d\mu^+ - \int_X \lim_{n\to\infty} f_n d\mu^-,$$

where we used Lebesgue's monotone convergence theorem for positive measures, which was proven e.g. in [64, p. 21].

□

Definition 3.11. *[64, p. 41,47]. Let X be a locally compact and separable metric space, short l.c.s. metric space. A positive Borel measure $\mu : \mathcal{B}(X) \to [0, \infty]$, with the properties that,*

a) for every $E \in \mathcal{B}(X)$, we have

$$\mu(E) = \inf \{\mu(V) \ : \ E \subseteq V, \ V \subseteq X \ \textit{open}\},$$

b) for every open set E, and for every $E \in \mathcal{B}(X)$ with $\mu(E) < \infty$, we have

$$\mu(E) = \sup \{\mu(K) \ : \ K \subseteq E, \ K \subseteq X \ \textit{compact}\},$$

is called regular.
A regular positive Borel measure μ is called Radon measure if for every $K \subseteq X$ compact $\mu(K) < \infty$. An $\mathbb{R}^N$-valued measure is called Radon measure if $|\mu|$ is a Radon measure.

Lemma 3.12. *[64, p. 48]. Let X be an l.c.s. metric space in which every open set U is σ-compact, i.e., $U = \bigcup_{i=1}^{\infty} K_i$, where K_i is a compact set for each $i \in \mathbb{N}$. Then a positive Borel measure $\mu : \mathcal{B}(X) \to [0, \infty]$, with $\mu(K) < \infty$ for every compact set K, is regular. Let $\Omega \subseteq \mathbb{R}^N$ be open, then each $\mathbb{R}^N$- valued measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^N$ is a Radon measure.*

Theorem 3.13. *[64, p. 131]. Let X be an l.c.s. metric space. To each bounded and linear functional $\Phi : C_0(X) \to \mathbb{R}$, there corresponds a unique Radon measure $\mu : \mathcal{B}(X) \to \mathbb{R}$ such that*

$$\Phi(u) = \int_X u \, d\mu \ \textit{for all} \ u \in C_0(X).$$

Moreover we have

$$\|\Phi\| = |\mu|(X),$$

where $\|\Phi\| := \sup\limits_{u \in C_0(X) - \{0\}} \frac{|\Phi(u)|}{\|u\|_\infty}$.

Proof: Regard $\Phi : C_0(X) \to \mathbb{R}$ as linear functional on $\mathbb{C}$ and use [64, p. 131].

□

In order to extend this to the vector-valued case one needs the following lemma.

Lemma 3.14. *[7, p. 21]. Let X be an l.c.s. metric space and $\mu : \mathcal{B}(X) \to \mathbb{R}^N$ an $\mathbb{R}^N$-valued Radon measure. Then for every open set $U \subseteq X$ the following equality holds:*

$$|\mu|(U) = \sup \left\{ \sum_{i=1}^{N} \int_U u_i \, d\mu_i \ : \ u \in [C_c(U)]^N, \ \|u\|_\infty \leq 1 \right\}.$$

Corollary 3.15. *[7, p. 25]. Let X be an l.c.s. metric space. To each bounded and linear functional $\Phi : [C_0(X)]^N \to \mathbb{R}$, there corresponds a unique $\mathbb{R}^N$-valued Radon measure $\mu : \mathcal{B}(X) \to \mathbb{R}^N$ such that*

$$\Phi(u) = \sum_{k=1}^{N} \int_X u_k \, d\mu_k \text{ for all } u = (u_1, \ldots, u_N) \in [C_0(X)]^N .$$

Moreover we have

$$\|\Phi\| = |\mu|(X).$$

Proof: The functions $\Phi_i : C_0(X) \to \mathbb{R}$, $\Phi_i(v) = \Phi(v \cdot \vec{e}_i)$, where $\vec{e}_i$ denotes the i-th unit vector in $\mathbb{R}^N$, are bounded and linear functions, for $i \in \{1, \ldots, N\}$. Using Theorem 3.13 we get $\mathbb{R}$-valued Radon measures $\mu_i : \mathcal{B}(X) \to \mathbb{R}$, $i \in \{1, \ldots, N\}$ with $\Phi_i(v) = \int_X v \, d\mu_i$ for all $v \in C_0(X)$. Since Φ is linear we conclude for $u = (u_1, \ldots, u_N) \in [C_0(X)]^N$ that

$$\Phi(u) = \Phi\left(\sum_{i=1}^{N} u_i \vec{e}_i\right) = \sum_{i=1}^{N} \Phi_i(u_i) = \sum_{i=1}^{N} \int_X u_i \, d\mu_i.$$

Further $\mu := (\mu_1, \ldots, \mu_N) : \mathcal{B}(X) \to \mathbb{R}^N$ is an $\mathbb{R}^N$-valued Radon measure and we get by Lemma 3.14 that

$$\begin{aligned} |\mu|(X) &= \sup\left\{\sum_{i=1}^{N} \int_X u_i \, d\mu_i \ : \ u \in [C_c(X)]^N, \ \|u\|_\infty \le 1\right\} \\ &= \sup\left\{\Phi(u) \ : \ u \in [C_c(X)]^N, \ \|u\|_\infty \le 1\right\} = \|\Phi\| . \end{aligned}$$

□

The following definitions and lemmata can be found in [7, p. 117-121].

Definition 3.16. *Let $u \in L^1(\Omega)$, then u is a function of bounded variation in Ω if*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, dD_i u \quad \forall \varphi \in C_c^\infty(\Omega), \ i = 1 \ldots N$$

for a Radon measure $Du = (D_1 u, \ldots, D_N u) : \mathcal{B}(\Omega) \to \mathbb{R}^N$ in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

A smoothing argument shows that the above N-formulas are still true for any Lipschitz-continuous φ with compact support in Ω.

Definition 3.17. *Let $u \in \left[L^1_{loc}(\Omega)\right]^m$. The variation $V(u, \Omega)$ of u in Ω is defined by*

$$V(u, \Omega) := \sup\left\{\sum_{k=1}^{m} \int_\Omega u_k \operatorname{div} \varphi_k \, dx \ : \ \varphi \in \left[C_c^1(\Omega)\right]^{mN}, \|\varphi\|_\infty \le 1\right\} \in [0, \infty] .$$

Using Lemma 3.14 and Corollary 3.15 one can show the following.

Lemma 3.18. *Let* $u \in \left[L^1(\Omega)\right]^m$ *then we have*

$$u \in [BV(\Omega)]^m \Leftrightarrow V(u,\Omega) < \infty.$$

Additionally, we have for $u \in [BV(\Omega)]^m$ *that* $V(u,\Omega) = |Du|(\Omega)$.

Remark 3.19. *In other words* $u \in \left[L^1(\Omega)\right]^m$ *is in* $[BV(\Omega)]^m$ *iff*

$$\begin{aligned} \Phi : \left[C_c^1(\Omega)\right]^{mN} &\rightarrow \mathbb{R} \\ \varphi &\mapsto \sum_{k=1}^m \int_\Omega u_k \operatorname{div} \varphi_k \, dx \end{aligned}$$

is a linear and bounded functional, with respect to the $\|.\|_\infty$*-norm. If* $u \in C^1(\Omega)$ *then we have* $V(u,\Omega) = \int_\Omega |\nabla u| \ dx$.

Lemma 3.20. *[7, p. 118]. Assume that* $u \in [BV_{loc}(\Omega)]^m$. *For any locally Lipschitz-continuous function* $\psi : \Omega \rightarrow \mathbb{R}$ *we have* $\psi u \in [BV_{loc}(\Omega)]^m$ *and*

$$D(\psi u) = \psi Du + (u \otimes \nabla\psi)\, \mathcal{L}^N.$$

Thereby we call a function $\psi : \Omega \rightarrow \mathbb{R}$ locally Lipschitz-continuous, if $\psi|_K$ is Lipschitz-continuous for each compact set $K \subset \Omega$.

Corollary 3.21. *Assume that* Ω *is bounded and* $u \in [BV(\Omega)]^m$. *For a Lipschitz-continuous function* $\psi : \Omega \rightarrow \mathbb{R}$ *we have* $\psi u \in [BV(\Omega)]^m$ *and*

$$D(\psi u) = \psi Du + (u \otimes \nabla\psi)\, \mathcal{L}^N.$$

Proof: W.l.o.g. we can choose $m = 1$. Since Ω is bounded we get $\|\psi\|_\infty < \infty$ and thus we get $\psi u \in L^1(\Omega)$. Let φ be Lipschitz-continuous with compact support in Ω and thus also $\psi\varphi$ is Lipschitz-continuous with compact support in Ω. Thus we get

$$\begin{aligned} \int_\Omega u\psi \frac{\partial\varphi}{\partial x_i}\, dx = \int_\Omega u \left[\frac{\partial}{\partial x_i}(\psi\varphi) - \varphi\frac{\partial\psi}{\partial x_i}\right] dx &= -\int_\Omega \psi\varphi \, dD_i u - \int_\Omega \varphi u \frac{\partial\psi}{\partial x_i}\, dx \\ &= -\int_\Omega \varphi \, d(\psi D_i u) - \int_\Omega \varphi \, d\left(u\frac{\partial\psi}{\partial x_i}\mathcal{L}^N\right), \end{aligned}$$

where we have used Definition 3.8 d) in the last equality. The Lipschitz continuity of ψ and $\mathcal{L}^N(\Omega) < \infty$ imply that $\psi Du + (u\nabla\psi)\,\mathcal{L}^N$ is a Radon measure and thus we get $u\psi \in BV(\Omega)$ with the desired product rule.

□

Lemma 3.22. *[7, p. 130/131]. Assume that* Ω *has a compact Lipschitz boundary. Then we get for each open set* A *with* $\bar{\Omega} \subseteq A$ *and any* $m \geq 1$ *that there exists a linear and continuous operator* $E : [BV(\Omega)]^m \to \left[BV\left(\mathbb{R}^N\right)\right]^m$ *satisfying*

- $Eu|_\Omega = u$;
- $Eu = 0$ *a.e. in* $\mathbb{R}^N \setminus A$ *for all* $u \in [BV(\Omega)]^m$;
- $|DEu|(\partial\Omega) = 0$ *for all* $u \in [BV(\Omega)]^m$;
- *for all* $p \in [1, \infty]$ *the restriction of* E *to* $\left[W^{1,p}(\Omega)\right]^m$ *induces a linear and continuous map between this space and* $\left[W^{1,p}\left(\mathbb{R}^N\right)\right]^m$.

Such E *is called extension operator.*

Using Lemma 3.22 and Lemma 3.20 one immediately gets Corollary 3.21 for a set Ω, which has compact Lipschitz boundary.
In the one-dimensional case there is a simpler way to define the variation. The following definitions and the results can be found in [7, p. 134-136].

Definition 3.23. *Let* $a, b \in \overline{\mathbb{R}}$, *with* $a < b$ *and* $I = (a, b)$. *For a function* $f : I \to \mathbb{R}$ *the pointwise variation* $pV(f, I)$ *of* f *in* I *is defined by*

$$pV(f, I) := \sup\left\{\sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| \ : \ a < x_1 < \ldots < x_n < b\right\} \in [0, \infty].$$

If $\omega \subseteq \mathbb{R}$ *is open, the pointwise variation* $pV(f, \omega)$ *is defined by* $\sum_I pV(f, I)$, *where the sum runs along all the connected components of* ω.
Furthermore we define, for $\omega \subseteq \mathbb{R}$ *open, the essential variation* $eV(f, \omega)$ *with*

$$eV(f, \omega) := \inf\left\{pV(g, \omega) : g = f \ \mathcal{L}^1\text{-a.e in } \omega\right\}.$$

Lemma 3.24. *For any* $f \in L^1_{loc}(\omega)$, *with* $\omega \subseteq \mathbb{R}$ *open, we get*

$$V(f, \omega) = eV(f, \omega).$$

Next, we investigate one dimensional restrictions of BV-functions, see [7, p. 194-195].

Definition 3.25. *For* $\nu \in \mathbb{R}^N \setminus \{0\}$ *we define* $\pi_\nu := \left\{x \in \mathbb{R}^N \ : < x, \nu >= 0\right\}$ *and denote the orthogonal projection of* Ω *on* π_ν *by* Ω_ν. *We define the section of* Ω *corresponding to* $y \in \pi_\nu$ *in* ν*-direction by* $\Omega^\nu_y := \{t \in \mathbb{R} \ : \ y + t\nu \in \Omega\}$ *and conclude that for any* $y \in \Omega_\nu$ *this set is not empty, see Figure 3.1.*
Accordingly, for any function $u : \Omega \to \mathbb{R}$ *the function* $u^\nu_y : \Omega^\nu_y \to \mathbb{R}$ *is defined by* $u^\nu_y(t) = u(y + t\nu)$. *For simplicity we write* $u^x_y := u^{\vec{e}_1}_y$, $\Omega^x_y := \Omega^{\vec{e}_1}_y$ *and* $\Omega_x := \Omega_{\vec{e}_1}$.

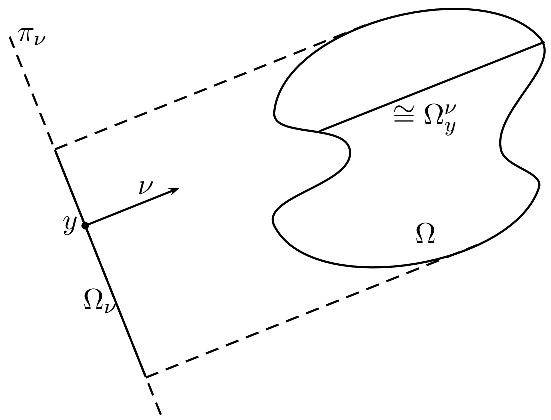

Figure 3.1:

Definition 3.26. *Let $u \in \left[L^1_{loc}(\Omega)\right]^m$ and $\nu \in \mathbb{R}^N \setminus \{0\}$. Then we say that the distributional derivative of u along ν is a measure if there exists an $\mathbb{R}^m$-valued measure $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^m$ such that*

$$\int_\Omega u \frac{\partial \varphi}{\partial \nu}\, dx = -\int_\Omega \varphi \, d\mu \quad \forall \varphi \in C_c^\infty(\Omega).$$

This measure μ is uniquely determined by this condition and will be denoted by $D_\nu u$.

Definition 3.27. *For $u \in \left[L^1_{loc}(\Omega)\right]^m$ and $\nu \in \mathbb{S}^{N-1}$ we define the variation of u along ν by*

$$V_\nu(u,\Omega) := \sup \left\{ \sum_{k=1}^m \int_\Omega u_k \frac{\partial \varphi_k}{\partial \nu}\, dx \ : \ \varphi \in \left[C_c^1(\Omega)\right]^m, \ \|\varphi\|_\infty \leq 1 \right\}.$$

We write short $V_x(u,\Omega) := V_{\vec{e}_1}(u,\Omega)$ and $V_y(u,\Omega) := V_{\vec{e}_2}(u,\Omega)$.

Lemma 3.28. *Let $u \in \left[L^1_{loc}(\Omega)\right]^m$ and $\nu \in \mathbb{S}^{N-1}$. Then*

$$V_\nu(u,\Omega) = \int_{\Omega_\nu} V(u_y^\nu, \Omega_y^\nu)\, d^{N-1}y.$$

If $u \in \left[C^1(\Omega)\right]^m$, then we have $V_\nu(u,\Omega) = \int_\Omega \left|\frac{\partial u}{\partial \nu}\right| \, dx$.

Lemma 3.29. *Let $u \in [BV(\Omega)]^m$ then $V_\nu(u,\Omega) = |D_\nu u|(\Omega) < \infty$ for all $\nu \in \mathbb{S}^{N-1}$.*

Remark 3.30. *Let $\Omega \subseteq \mathbb{R}^2$ be open, $\delta > 0$, and $B_\delta(A) := \{x \in \Omega \ : \ \operatorname{dist}(x,A) < \delta\}$, for a set $A \subseteq \mathbb{R}^2$. Let $M \in \mathbb{N}$, $A_i \subseteq \Omega$ for $i \in \{1,\dots,M\}$ be open and convex sets. Further* *we assume that there exists a* $\rho > 0$ *such that* $B_\rho(A_i) \cap B_\rho(A_j) = \emptyset$ *and* $B_\rho(A_i) \subseteq \Omega$ *for* $i, j \in \{1, \dots, M\}$, *with* $i \neq j$. *Let* $a_i \in \mathbb{R}, i \in \{1, \dots, M\}$ *and*

$$\begin{aligned} \gamma : \Omega &\to \mathbb{R} \\ x &\mapsto \begin{cases} a_i & \text{if} \quad x \in A_i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Then we have

$$V_x(\gamma, \Omega) = \int_{\Omega_x} V\left(\gamma_y^x, \Omega_y^x\right) \, dy = \sum_{i=1}^{M} \int_{(A_i)_x} V\left(\gamma_y^x, (A_i)_y^x\right) \, dy = \sum_{i=1}^{M} 2a_i \mathcal{L}^1\left((A_i)_x\right).$$

Finally we summarize some properties of sets of finite perimeter, which were taken from [7, p. 143-145,153].

Definition 3.31. *Let* E *be an* $\mathcal{L}^N$*-measurable subset of* $\mathbb{R}^N$. *For any open set* $\Omega \subseteq \mathbb{R}^N$ *the perimeter of* E *in* Ω, *denoted by* $P(E, \Omega)$, *is the variation of* χ_E *in* Ω, *i.e.,*

$$P(E, \Omega) := \sup\left\{ \int_E \operatorname{div} \varphi \, dx \ : \ \varphi \in \left[C_c^1(\Omega)\right]^N, \ \|\varphi\|_\infty \leq 1 \right\}.$$

We say that E *is a set of finite perimeter in* Ω *if* $P(E, \Omega) < \infty$.

Lemma 3.32. *(Coarea formula in BV) For any open set* $\Omega \subseteq \mathbb{R}^N$ *and* $u \in L^1_{loc}(\Omega)$ *one has*

$$V(u, \Omega) = \int_{-\infty}^{\infty} P\left(\{x \in \Omega \ : \ u(x) > t\}, \Omega\right) \, dt.$$

In particular, if $u \in BV(\Omega)$ *the set* $\{x \in \Omega \ : \ \overline{u}(x) > t\}$ *has finite perimeter in* Ω *for* $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}$ *and every representative* $\overline{u} \in u$.

Remark 3.33. *Let* $E, F \subseteq \mathbb{R}^N$ *be* $\mathcal{L}^N$*-measurable and* $\Omega \subseteq \mathbb{R}^N$ *be an open set, then we get* $P(E, \Omega) = P\left(\mathbb{R}^N \setminus E, \Omega\right)$ *and*

$$P(E \cup F, \Omega) + P(E \cap F, \Omega) \leq P(E, \Omega) + P(F, \Omega).$$

Lemma 3.34. *Let* $E \subseteq \mathbb{R}$ *be* $\mathcal{L}^1$*-measurable. If the set* E *has finite perimeter in* (a, b), *where* $a, b \in \overline{\mathbb{R}} = [-\infty, \infty]$, *and* $|E \cap (a, b)| > 0$, *then there exist an* $N \in \mathbb{N}$ *and* N *pairwise disjoint and nonempty intervals* $I_l = [a_l, b_l] \subseteq \mathbb{R}$ *with* $l \in \{1, \dots, N\}$, *such that* $E \cap (a, b)$ *is equal to* $\bigcup_{l=1}^{N} I_l$ *except for an* $\mathcal{L}^1$*-null set.*

Corollary 3.35. *Let $\omega \subseteq \mathbb{R}$ be open,*

$$\gamma \in \{v \in BV(\omega) \;:\; \exists \overline{v} : \omega \to \mathbb{R},\ \overline{v} \in v, \forall x \in \omega : \overline{v}(x) = 0 \text{ or } |\overline{v}(x)| \geq \mu\}$$

and assume that there exists a representative $\overline{\gamma}$ of γ such that $|\{x \in \omega \;:\; \overline{\gamma}(x) = 0\}| > 0$, where $\mu > 0$. Then there exists an $N \in \mathbb{N}$ and N pairwise disjoint and nonempty intervals $I_l = [a_l, b_l] \subseteq \mathbb{R}$ with $l \in \{1, \dots, N\}$, such that $\{x \in \omega \;:\; \overline{\gamma}(x) = 0\}$ is equal to $\bigcup_{l=1}^{N} I_l$ except for an $\mathcal{L}^1$-null set.

Proof: According to Lemma 3.32 there exists a $t \in \left[0, \frac{\mu}{2}\right]$ such that the sets $E = \{x \in \omega \;:\; \overline{\gamma}(x) > t\}$ and $F = \{x \in \omega \;:\; -\overline{\gamma}(x) > t\}$ have finite perimeter, simultaneously. For $\Gamma := \{x \in \omega \;:\; \overline{\gamma}(x) = 0\} = \{x \in \omega \;\; |\overline{\gamma}(x)| \leq t\} = \mathbb{R} \setminus (E \cup F)$ we get with Remark 3.33

$$P\left(\Gamma, \omega\right) = P\left(\mathbb{R} \setminus \left(E \cup F\right), \omega\right) + P\left(E \cap F, \omega\right) \leq P(E, \omega) + P(F, \omega) < \infty.$$

Finally Lemma 3.34, applied to $\Gamma \cap (-\infty, \infty)$, gives the desired result.

□

3.3 Γ-convergence

Many mathematical problems depend on some model parameters, which are small or large in comparison to the other parameters. In particular, in the model studied in this thesis, we do have a small parameter $\delta > 0$ in front of the self-energy part, which is comparable to the lattice size, and a large parameter $\frac{1}{\varepsilon}$, which penalizes elastic deformations, which are not rotations. In general one has to solve a variational problem of the form

$$\min\left\{f_\varepsilon(x) \;:\; x \in X\right\},$$

for a small $\varepsilon > 0$. One may ask, whether it is possible to replace this family of variational problems by a single variational problem

$$\min\left\{f(x) \;:\; x \in X\right\},$$

which captures the relevant behaviour of minimizers and for which a solution can be more easily obtained. This can be achieved using Γ-convergence.

Definition 3.36. *[15, p. 22,37]. Let X be a metric space with metric d, $f_\varepsilon : X \to \overline{\mathbb{R}}$ for $\varepsilon > 0$ and $f : X \to \overline{\mathbb{R}}$ be given functionals. Then we say that the sequence $\{f_\varepsilon\}_{\varepsilon > 0}$ Γ-converges to f in X as $\varepsilon \to 0$, in symbols $f_\varepsilon \xrightarrow{\Gamma} f$, if the following properties are fulfilled for all $x \in X$:*

(i) Lower bound: For every sequence $\{x_\varepsilon\}_{\varepsilon>0} \subseteq X$ *converging to* $x \in X$*, it holds*

$$f(x) \leq \liminf_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon).$$

(ii) Recovery sequence: There is a sequence $\{x_\varepsilon\}_{\varepsilon>0} \subseteq X$ *converging to* x*, such that*

$$f(x) = \lim_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon).$$

Then, f *is called* Γ*-limit of* f_ε*, in symbols* $f = \Gamma - \lim_{\varepsilon \to 0} f_\varepsilon$*.*
We say that the sequence $\{f_\varepsilon\}_{\varepsilon>0}$ Γ*-converges to* f *at* $x \in X$*, if (i) and (ii) hold for* x*. In this case we write* $f(x) = \Gamma - \lim_{\varepsilon \to 0} f_\varepsilon(x)$*. Let* $Y \subseteq X$*, then* f_ε *converges to* f *in the sense of pointwise* Γ*-convergence on* Y*, if* $\{f_\varepsilon\}_{\varepsilon>0}$ Γ*-converges to* f *at each* $y \in Y$*.*

Remark 3.37. *Below some of the well-known properties of* Γ*-convergence are summarized, proofs can be found in [15], [16] and [29].*

(LS) Lower semicontinuity: Any Γ*-limit* $f : X \to \overline{\mathbb{R}}$ *is lower semicontinuous in* X*.*

(S) Stability under continuous perturbations: If $g : X \to \mathbb{R}$ *is continuous and* $f_\varepsilon \xrightarrow{\Gamma} f$*, then* $f_\varepsilon + g \xrightarrow{\Gamma} f + g$*.*

(M) Convergence of minima: Let $Y \subseteq X$*, so that* $\inf_{y \in Y} f(y) = \inf_{x \in X} f(x)$*,* f_ε *converges to* f *in the sense of pointwise* Γ*-convergence on* Y *and the part (i) in Definition 3.36 is fulfilled for all* $x \in X$*. Suppose that* $\overline{x}_\varepsilon \in X$ *are minimizers of* f_ε *for* $\varepsilon > 0$*. Then each cluster point of* $\{\overline{x}_\varepsilon\}_{\varepsilon>0}$ *is a minimum of* f*.*

To avoid that (M) is an empty statement, i.e., the sequence $\{\overline{x}_\varepsilon\}_{\varepsilon>0}$ *possesses no accumulation point, one commonly shows a compactness result related to the following:*

(C) Every sequence $\{x_\varepsilon\}_{\varepsilon>0} \subseteq X$ *of bounded energy, i.e., a sequence with* $f_\varepsilon(x_\varepsilon) \leq C$ *for some constant* $C > 0$*, is relatively compact in* X*.*

(LE) Relaxation for low energy states: Let $Y \subseteq X$*, so that* $\inf_{y \in Y} f(y) = \inf_{x \in X} f(x)$*,* f_ε *converges to* f *in the sense of pointwise* Γ*-convergence on* Y *and the part (i) in Definition 3.36 is fulfilled for all* $x \in X$*, then one gets under the assumption (C),*

$$\inf_{y \in Y} f(y) = \lim_{\varepsilon \to 0} \inf_{x \in X} f_\varepsilon(x),$$

where the infimum on the left hand side is attained if $Y = X$*.*

Proof of (LE) and (M): The proof is inspired by Theorem 1.21 of [15], which can also be found in Theorem 7.2 of [16]. Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence with $\varepsilon_n \overset{n\to\infty}{\longrightarrow} 0$. Suppose first that for all $C > 0$, there exists a natural number $N \in \mathbb{N}$, so that $\inf\limits_{x\in X} f_{\varepsilon_n}(x) > C$ for all $n \geq N$. Let $\{\widetilde{x}_{\varepsilon_n}\}_{n\in\mathbb{N}} \subseteq X$ be a recovery sequence for fixed $\widetilde{y} \in Y$, then we get $f(\widetilde{y}) = \lim\limits_{n\to\infty} f_{\varepsilon_n}(\widetilde{x}_{\varepsilon_n}) \geq \liminf\limits_{n\to\infty} \inf\limits_{x\in X} f_{\varepsilon_n}(x) \geq C$, and thus $f(\widetilde{y}) = \infty = \lim\limits_{n\to\infty} \inf\limits_{x\in X} f_{\varepsilon_n}(x)$, for all $\widetilde{y} \in Y$, which shows assertion (LE) and (M). Thus we can assume that there exist a constant $C > 0$ and a sub-sequence $\{\varepsilon_{k_n}\}_{n\in\mathbb{N}}$, such that $\inf\limits_{x\in X} f_{\varepsilon_{k_n}}(x) \leq C$ for all $n \in \mathbb{N}$. Furthermore we can assume that $\inf\limits_{x\in X} f_{\varepsilon_l}(x) > C$ for each $l \in \mathbb{N} \setminus \bigcup\limits_{n\in\mathbb{N}} \{k_n\}$, which implies $\liminf\limits_{n\to\infty} \left(\inf\limits_{x\in X} f_{\varepsilon_{k_n}}(x) \right) = \liminf\limits_{n\to\infty} \left(\inf\limits_{x\in X} f_{\varepsilon_n}(x) \right)$. Consider first the assumptions in (LE) are fulfilled. Let $\left\{\widehat{x}_{\varepsilon_{k_n}}\right\}_{n\in\mathbb{N}} \subseteq X$ be a sequence with $f_{\varepsilon_{k_n}}\left(\widehat{x}_{\varepsilon_{k_n}}\right) \leq \inf\limits_{x\in X} f_{\varepsilon_{k_n}}(x) + \varepsilon_{k_n}$. By assumption (C) there exists a convergent subsequence, which is denoted again by $\left\{\widehat{x}_{\varepsilon_{k_n}}\right\}_{n\in\mathbb{N}}$ and its limit is called $x^* = \lim\limits_{n\to\infty} \widehat{x}_{\varepsilon_{k_n}} \in X$. Define the sequence $\{\overline{x}_{\varepsilon_n}\}_{n\in\mathbb{N}} \subseteq X$ by

$$\overline{x}_{\varepsilon_n} = \begin{cases} \widehat{x}_{\varepsilon_n} & \text{if} \quad n \in \bigcup\limits_{l\in\mathbb{N}} \{k_l\} \\ x^* & \text{otherwise} \end{cases},$$

then we get $x^* = \lim\limits_{n\to\infty} \overline{x}_{\varepsilon_n}$. Let again $\{\widetilde{x}_{\varepsilon_n}\}_{n\in\mathbb{N}} \subseteq X$ be a recovery sequence of a fixed $\widetilde{y} \in Y$, then we have

$$\begin{aligned} f(x^*) &\leq \liminf_{n\to\infty} f_{\varepsilon_n}(\overline{x}_{\varepsilon_n}) \leq \liminf_{n\to\infty} f_{\varepsilon_{k_n}}\left(\overline{x}_{\varepsilon_{k_n}}\right) \leq \liminf_{n\to\infty} \left(\inf_{x\in X} f_{\varepsilon_{k_n}}(x) + \varepsilon_{k_n} \right) \\ &= \liminf_{n\to\infty} \left(\inf_{x\in X} f_{\varepsilon_n}(x) \right) \leq \limsup_{n\to\infty} \left(\inf_{x\in X} f_{\varepsilon_n}(x) \right) \leq \limsup_{n\to\infty} \left(f_{\varepsilon_n}(\widetilde{x}_{\varepsilon_n}) \right) = f(\widetilde{y}), \end{aligned}$$

for all $\widetilde{y} \in Y$, which gives $\min\limits_{x\in X} f(x) = \lim\limits_{n\to\infty} \inf\limits_{x\in X} f_{\varepsilon_n}(x)$ in the case $Y = X$. Using $\inf\limits_{y\in Y} f(y) = \inf\limits_{x\in X} f(x)$, we get

$$f(x^*) \leq \inf_{y\in Y} f(y) = \inf_{x\in X} f(x) \leq f(x^*)$$

and thus we obtain (LE), since the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ was chosen arbitrary. In the case that the statement (M) is not empty, we have a convergent sequence $\{\overline{x}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ of minimizers of f_{ε_n}. Let $x^* = \lim\limits_{n\to\infty} \overline{x}_{\varepsilon_n} \in X$, then we get as above that

$$f(x^*) \leq \liminf_{n\to\infty} f_{\varepsilon_n}(\overline{x}_{\varepsilon_n}) = \liminf_{n\to\infty} \left(\inf_{x\in X} f_{\varepsilon_n}(x) \right) \leq \inf_{y\in Y} f(y) = \inf_{x\in X} f(x),$$

which shows (M).

□

3.4 Notions of convexity

Next, we introduce the notion of quasiconvexity, which is the fundamental notion of convexity for vector valued variational problems. This definition was first introduced by Morrey in 1952 [55]. It is closely related to lower semicontinuity of integral functionals, existence and regularity of minimizers, see [52].

Definition 3.38 (Quasiconvexity)**.** *A function $f : \mathbb{R}^{m\times n} \to (-\infty, \infty]$, is called quasiconvex if for every bounded, open and nonempty set $\Omega \subseteq \mathbb{R}^n$ with $|\partial\Omega| = 0$ one has*

$$f(F) \leq \frac{1}{|\Omega|} \int_\Omega f\left(F + \nabla\varphi(x)\right)\, dx \text{ for all } F \in \mathbb{R}^{m\times n} \text{ and } \varphi \in W_0^{1,\infty}\left(\Omega; \mathbb{R}^m\right),$$

whenever the integral on the right-hand side exists.

It is in general hard to verify if a function is quasiconvex or not. For finite valued functions one can introduce other notions of convexity, which are necessary and sufficient conditions for a quasiconvex function and are easier to check. First we formulate a stronger property named polyconvexity [10, 28, 52].

Definition 3.39 (Polyconvexity)**.** *A function $f : \mathbb{R}^{m\times n} \to (-\infty, \infty]$ is called polyconvex if there exists a convex function $g : \mathbb{R}^{\tau(n,m)} \to (-\infty, \infty]$, such that*

$$f(F) = g(M(F)) \text{ for all } F \in \mathbb{R}^{m\times n},$$

where $M(F)$ is the vector, which consists of all minors, i.e., subdeterminants, of F, and $\tau(n, m) = \sum_{k=1}^{\min\{n,m\}} \binom{n}{k}\binom{m}{k}$ denotes the length of $M(F)$.
In the case $m = n = 2$, we are interested in, one gets $M(F) = (F, det(F)) \in \mathbb{R}^5$.

Next, we define the notion of rank-one convexity, a property, which is weaker in comparison to quasiconvexity.

Definition 3.40 (Rank-one convexity)**.** *A function $f : \mathbb{R}^{m\times n} \to (-\infty, \infty]$ is called rank-one convex if*

$$f\left(\lambda A + (1-\lambda)B\right) \leq \lambda f(A) + (1-\lambda)f(B)$$

for every $\lambda \in (0, 1)$, $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A - B) = 1$.

If $n = 1$ or $m = 1$ then all these notions of convexity are equivalent. Otherwise, one can show the following implications [28, 52]. For finite valued f we have

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank-one convex}$$

and for a extended valued $f : \mathbb{R}^{m\times n} \to (-\infty, \infty]$ one can only show

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ rank-one convex and quasiconvex.}$$

The reverse implications are not true in general. The question whether rank-one convexity implies quasiconvexity is the most difficult one. For $m \geq 3$ and $n \geq 2$ there is a famous counterexample by Šverák [28, 69], but the case $m = 2$ is still open.
For a non convex function f, one defines its convexification f^c as largest convex function below f. The same can be made with polyconvex, quasiconvex and rank-one convex functions.

Definition 3.41. *For a function $f : \mathbb{R}^{m\times n} \to (-\infty, \infty]$ we define its quasiconvex envelope f^{qc} as largest quasiconvex function smaller or equal to f. This means*

$$f^{qc}(F) = \sup\left\{g(F) \ : \ g : \mathbb{R}^{m\times n} \to (-\infty, \infty] \text{ quasiconvex and } g \leq f\right\}$$

for all $F \in \mathbb{R}^{m\times n}$. We define the convex, polyconvex and the rank-one convex envelopes of f in the same way and denote them by f^c, f^{pc} and f^{rc}, respectively.

There are several different possibilities of defining convex, polyconvex, quasiconvex and rank-one convex hulls in the literature [28, 52]. We restrict ourselves to the following.

Definition 3.42. *A set $K \subseteq \mathbb{R}^{m\times n}$ is called convex if $\lambda A + (1-\lambda)B \in K$ for all $\lambda \in (0, 1)$ and all $A, B \in K$. The smallest closed, convex set containing K is called its convex hull and is denoted by K^c.*

Alternatively, it was shown by Dacorogna [28, Prop. 2.36] that one can define K^c by separation, namely

$$K^c = \left\{F \in \mathbb{R}^{m\times n} \ : \ f(F) \leq \sup_{G\in K} f(G) \text{ for every convex } f : \mathbb{R}^{m\times n} \to \mathbb{R}\right\}.$$

Motivated by this, we define the poly-, quasi- and rank-one convex hull in the same way.

Definition 3.43. *The quasiconvex hull of a set $K \subseteq \mathbb{R}^{m\times n}$ is defined by*

$$K^{qc} = \left\{F \in \mathbb{R}^{m\times n} \ : \ f(F) \leq \sup_{G\in K} f(G) \text{ for every quasiconvex } f : \mathbb{R}^{m\times n} \to \mathbb{R}\right\}.$$

The polyconvex hull K^{pc} and the rank-one convex hull K^{rc} are defined in the same way. A set $K \subseteq \mathbb{R}^{m\times n}$ is called quasiconvex, if $K = K^{qc}$. Analogously, we define polyconvex and rank-one convex hulls.

In contrast to this, one can define the lamination convex as follows.

Definition 3.44. *For $K \subseteq \mathbb{R}^{m\times n}$ we define $K^{(0)} := K$,*

$$K^{(i)} := \left\{ \lambda A + (1-\lambda)B \ : \ A, B \in K^{(i-1)}, \ \text{rank } (B-A) \leq 1, \ \lambda \in (0,1) \right\},$$

for $i \in \mathbb{N}$ and $K^{(lc)} := \bigcup\limits_{i=0}^{\infty} K^{(i)}$.
A set $K \subseteq \mathbb{R}^{2\times 2}$ is called lamination convex if

$$\lambda A + (1-\lambda)B \in K$$

for all $\lambda \in (0,1)$ and all $A, B \in K$ with $\text{rank}\,(B-A) = 1$. *One can show that the lamination convex hull of K called K^{lc}, i.e., the smallest lamination convex set containing K, is equal to $K^{(lc)}$.*

Lemma 3.45. *[52, Section 4.4]. Let $K \subseteq \mathbb{R}^{m\times n}$, then we have*

$$K \subseteq K^{lc} \subseteq K^{rc} \subseteq K^{qc} \subseteq K^{pc} \subseteq K^{c}.$$

3.5 Convex integration

Assume now that Ω is open, bounded and has a Lipschitz-continuous boundary $\partial\Omega$. Examine the variational problem

$$\inf \left\{ \int_{\Omega} f(Du)\, dx, \ u \in W^{1,\infty}(\Omega; \mathbb{R}^m), \ u = F \text{ on } \partial\Omega \right\},$$

where $F \in \mathbb{R}^{m\times n}$ and $f : \mathbb{R}^{m\times n} \to [0,\infty)$ is a function, with $0 \in f(\mathbb{R}^{m\times n})$. Sometimes it suffices to search the minimum in the class of all functions, which minimize the integrand pointwise. This means, if there is a Lipschitz-continuous function $\widehat{u}$, with

$$\begin{aligned} D\widehat{u} &\in K := f^{-1}(0) \text{ a.e. in } \Omega, \\ \widehat{u}(x) &= Fx \text{ on } \partial\Omega, \end{aligned} \tag{3.1}$$

then this $\widehat{u}$ is a minimizer of the variational problem. Problems as in Equation (3.1), are also called partial differential inclusions. This inclusions, can be solved by the method of convex integration, see [53, 54]. Next, we state a famous result proved by Müller and Šverák, which helps to solve many partial differential inclusions.

Theorem 3.46. *[54, Theorem 1.3]. Let* $\Sigma = \{F \in \mathbb{R}^{n\times n} : \det(F) = 1\}$ *and* $K \subset \Sigma$. *Suppose that* $\{U_j\}_{j\in\mathbb{N}}$ *is an in-approximation of* K*, i.e.,*

(*i*) U_j *are open in* Σ *and uniformly bounded,*

(*ii*) $U_j \subset (U_{j+1})^{rc}$,

(*iii*) $U_j \to K$ *in the following sense: if* $F_j \in U_j$ *and* $F_j \to F$ *then* $F \in K$.

Then for any $F \in U_1$ *and for any open domain* $\Omega \subset \mathbb{R}^n$ *there exists a Lipschitz-continuous solution of the partial differential inclusion*

$$\begin{aligned} Du &\in K \text{ a.e. in } \Omega, \\ u(x) &= Fx \text{ on } \partial\Omega. \end{aligned}$$

This abstract result is useful for problems, where one does not have to know the explicit shape of the construction. Conti and Theil [27] have shown a solution of a special partial differential inclusion, which fits in our setting and is equal to a simple laminate on a big part of the region. Furthermore the hole construction is explicitly known.

Definition 3.47. *By a simple laminate of period* $h > 0$ *between two given matrices* $A, B \in \mathbb{R}^{2\times 2}$ *with* $A - B = a \otimes \nu$*, where* $a, \nu \in \mathbb{R}^2$*, we mean a function* $l : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$l(x) = (\lambda A + (1-\lambda)B)\, x + h\chi_\lambda \left(\frac{\nu \cdot x + c}{h}\right) a.$$

Thereby $\lambda \in [0,1]$ *and* $(1-\lambda)$ *are the weights of the laminate,* $c \in \mathbb{R}$ *describes the shift and* χ_λ *is a Lipschitz-continuous, one-periodic and real-valued function of one variable with* $\chi_\lambda(0) = \chi_\lambda(1)$ *and*

$$\chi'_\lambda(t) = \begin{cases} 1-\lambda & \text{for } t \in (0,\lambda) \\ -\lambda & \text{for } t \in (\lambda, 1) \end{cases} .$$

Lemma 3.48. *[27, p. 136]. Let* $\Omega \subseteq \mathbb{R}^2$ *be open,* $A, B \in \mathbb{R}^{2\times 2}$ *with* $\det(A) = \det(B) = 1$ *and* $\operatorname{rank}(A - B) = 1$. *Further let* $v \in \mathbb{R}^2$ *be such that* $|Av| = |Bv|$ *and* $Av \neq Bv$. *For any* $\lambda \in (0,1)$ *and any* $\eta > 0$ *there are* $h_0 > 0$ *and* $\Omega_\eta \subset \Omega$ *with* $|\Omega \setminus \Omega_\eta| \leq \eta$ *such that the restriction to* Ω_η*, of any simple laminate between the gradients* A *and* B *with weights* λ *and* $1-\lambda$ *and period* $h < h_0$ *can be extended to a finitely piecewise affine function* $u : \Omega \to \mathbb{R}^2$ *so that* $u(x) = (\lambda A + (1-\lambda)B)\, x$ *for* $x \in \partial\Omega$, $\det(\nabla u) = 1$, $|(\nabla u)\, v| \leq |Av| = |Bv|$ *and* $\operatorname{dist}(\nabla u, [A, B]) \leq \eta$ *on* Ω.

Remark 3.49. *By finitely piecewise affine we mean that the domain can be decomposed into finitely many open sets such that the function is affine on each of them.*

3.6 Consequence of the div-curl lemma

Definition 3.50. *(Equi-integrability in L^q). Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \leq q < \infty$. A sequence $\{f_k\}_{k\in\mathbb{N}} \subseteq L^q(\Omega;\mathbb{R}^m)$ is called equi-integrable in L^q, if for every $\varepsilon > 0$ there exists a $\lambda = \lambda(\varepsilon)$, such that*

$$\int_E \left|f^k(x)\right|^q \, dx < \varepsilon$$

for all $k \in \mathbb{N}$ and all measurable sets $E \subseteq \Omega$, with $|E| < \lambda$.

Lemma 3.51. *[42, Corollary 6.24]. Let $q \geq 2$ and $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set with Lipschitz boundary. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,1}(\Omega;\mathbb{R}^2)$, so that $\nabla u_n = A_n + B_n$, with $B_n \to 0$ in $L^1(\Omega;\mathbb{R}^{2\times 2})$ and $A_n \rightharpoonup A$ in $L^q(\Omega;\mathbb{R}^{2\times 2})$ for $n \to \infty$. If the sequence $\{\det(A_n)\}_{n\in\mathbb{N}}$ is equi-integrable in $L^1(\Omega)$, then*

$$\det(A_n) \rightharpoonup \det(A) \ \text{ in } L^{\frac{q}{2}}(\Omega) \ \text{ as } n \to \infty.$$

4 Mathematical model

In contrast to the engineering literature mathematicians commonly denote densities of energy functionals by W instead of ψ. Therefore we write below W_{el} instead of ψ_{el}. In the following we restrict ourselves to the two-dimensional case, i.e., $n = 2$. In order to handle the variational formulation one simplifies the energy defined in Equation (2.5). First of all we fix the slip direction $s := \vec{e}_1$ and the slip plane normal $m := \vec{e}_2$. Next, we simplify the hardening part ψ_h and the part resulting by the principle of maximal dissipation by writing $|\gamma|^p$ instead of them. Thereby $p = 1$ corresponds to the case with no hardening and $p = 2$ corresponds to the one with linear hardening. In some parts of this thesis we have shown our results also for the term $|\gamma| + |\gamma|^2$, which gives a more realistic model for the case of linear hardening. Besides we are only interested in one single time step.

Consider now the energy resulting from body and surface forces, i.e., $\int_\Omega fu\, dx$ and $\int_{\partial\Omega} gu\, dS$. They are continuous in $u \in W^{1,1}(\Omega)$, if one takes $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$ and one assumes that Ω is a bounded set with C^1-boundary. One can omit them first, due to the stability of the Γ-limit under continuous perturbations, and add them at the end. A more detailed argumentation can be found in [26, Chapter 3]. Furthermore we assume that we have affine boundary conditions. Bringing all together we want to investigate the following variational problem for $F \in \mathbb{R}^{2\times 2}$:

$$\begin{gathered}\text{Find } u : \Omega \to \mathbb{R}^2 \text{ with } u(x) = Fx \text{ for all } x \in \partial\Omega \\ \text{and } \gamma : \Omega \to \mathbb{R} \text{ which minimize } \widetilde{E}_\delta[u,\gamma],\end{gathered}$$

where

$$\widetilde{E}_\delta[u,\gamma] = \int_\Omega W_{el}\left(\nabla u\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right)\right) + |\gamma|^p + \delta\left|\partial_x \gamma\right|\; d\lambda_2. \tag{4.1}$$

It was assumed that rigid body motions are minimizers of the elastic energy W_{el}, i.e., $W_{el}(R) = 0$ for all $R \in SO(2)$. In order to be able to show lower bounds on the energy we assume that there are no other minimizers, i.e., $W_{el}(R) = 0$ iff $R \in SO(2)$. Then we can penalize elastic deformations, which are not rotations, by introducing a small parameter $\varepsilon > 0$ and write $W_{el} =: \frac{1}{\varepsilon} W_e$, where $W_e : \mathbb{R}^{2\times 2} \to [0,\infty]$ is independent of ε. This function W_e is denoted again as elastic energy density or short elastic energy and it is assumed to satisfy the following hypotheses:

(H1) $W_e(RF) = W_e(F)$ for all $R \in \mathrm{SO}(2)$, $F \in \mathbb{R}^{2\times 2}$ (frame indifference);

(H2) $W_e(R) = 0$ for all $R \in \mathrm{SO}(2)$ (rigid body motions are minimizers);

(H3) There exist $N,\ M \geq 0,\ q > 0$ and $c_1, c_2 \geq 0$, such that

$$W_e(F) \leq c_1 \|F\|^q + c_2,$$

for all $F \in \mathbb{R}^{2\times 2}$, with $\det F > M$ and $\|F\| \geq N$ (growth condition).

An example of an energy, which satisfies $(H1) - (H3)$ is $W_e(F) = \mathrm{dist}^q\,(F, SO(2))$.

4.1 Results for the model without self-energy

In this section we want to give a short review of the case $\delta = 0$. This was investigated in the dissertation of Carolin Kreisbeck [42]. If $\delta = 0$, then one gets that the energy formula (4.1) no longer depends on derivatives of γ and one can minimize it out pointwise and get the condensed energy

$$E_{cond,\varepsilon}[u] = \int_\Omega W_{cond,\varepsilon}\,(\nabla u)\ d\lambda_2, \tag{4.2}$$

if $u \in W^{1,\frac{pq}{p+q}}\left(\Omega; \mathbb{R}^2\right)$ and $E_{cond,\varepsilon}[u] := \infty$ otherwise. Thereby we defined the condensed energy density $W_{cond,\varepsilon}(F) = \inf_{\gamma\in\mathbb{R}} \left\{\frac{1}{\varepsilon} W_e\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right)\right) + |\gamma|^p\right\}$ for $F \in \mathbb{R}^{2\times 2}$. Define now the set of matrices F, whose elastic part F_{el} is a rotation and whose plastic part is a single slip in the slip system $(\vec{e}_1, \vec{e}_2)$ by $\mathcal{M}^{(2)}$, i.e.,

$$\begin{aligned}\mathcal{M}^{(2)} &:= \left\{F \in \mathbb{R}^{2\times 2}\ :\ F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right) \in SO(2),\ \gamma \in \mathbb{R}\right\} \\ &= \left\{F \in \mathbb{R}^{2\times 2}\ :\ \det(F) = 1,\ |F\vec{e}_1| = 1\right\}.\end{aligned}$$

The lamination convex hull of $\mathcal{M}^{(2)}$, coincides with the polyconvex hull [28, 52] and is calculated as

$$\mathcal{N}^{(2)} := \left(\mathcal{M}^{(2)}\right)^{lc} = \left(\mathcal{M}^{(2)}\right)^{pc} = \left\{F \in \mathbb{R}^{2\times 2}\ :\ \det(F) = 1,\ |F\vec{e}_1| \leq 1\right\}.$$

This was determined in [27], where it is shown that already $\left(\mathcal{M}^{(2)}\right)^{(1)}$ is equal to $\mathcal{N}^{(2)}$, see Definition 3.44. Since we use this frequently in the thesis we give a short proof.

Lemma 4.1. *Let $F \in \mathcal{N}^{(2)}$, then there exist a $\lambda \in [0,1]$ and matrices $F_+, F_- \in \mathcal{M}^{(2)}$, with* rank $(F_+ - F_-) \leq 1$*, such that $F = \lambda F_+ + (1-\lambda) F_-$. One can choose F_+, F_- such that $F_+\vec{e}_2 = F_-\vec{e}_2$ and $F_\pm = R_\pm \begin{pmatrix} 1 & \pm\sqrt{|F\vec{e}_2|^2 - 1} \\ 0 & 1 \end{pmatrix}$ for rotations $R_\pm \in SO(2)$.*

Proof: W.l.o.g. we can choose $F \in \mathcal{N}^{(2)} - \mathcal{M}^{(2)}$. Define $F_t := F + tF\vec{e}_2 \otimes \vec{e}_1$, then we have $F_0 = F$, $\det(F_t) = 1$ and rank $(F_t - F_s) \le 1$ for all $t, s \in \mathbb{R}$. Since $|F\vec{e}_1| < 1$ the quadratic equation $|F_t\vec{e}_1|^2 - 1 = 0$ has two solutions $t_- < 0 < t_+$. Define $F_+ := F_{t_+}$, $F_- := F_{t_-}$, then we get the assertion.

□

It was shown in the dissertation of Carolin Kreisbeck [42] that the relaxation of the energy density $W_{cond,\varepsilon}$ strictly depends on the growth exponent p. Namely, one has to distinguish between the case $\frac{1}{p} + \frac{1}{q} > 1$, which corresponds to the case without hardening, i.e., $p = 1$, and its opposite $\frac{1}{p} + \frac{1}{q} \le 1$, which belongs to the case with linear hardening, i.e., $p = 2$, if one takes $q \ge 2$.

Theorem 4.2. *[42, Theorem 7.35] and [23, Theorem 1.1]. Let $\frac{1}{p} + \frac{1}{q} > 1$ and suppose that W_e satisfies the hypotheses $(H1) - (H3)$ and is continuous at the identity. Then, $W^{pc}_{cond,\varepsilon}(F) = W^{qc}_{cond,\varepsilon}(F) = W^{rc}_{cond,\varepsilon}(F) = 0$ for all $F \in \mathcal{N}^{(2)}$ and all $\varepsilon > 0$.*

Let $p \ge 2$, then we define

$$E_{rigid}[u] := \int_\Omega W^{qc}_{rigid,p}(\nabla u)\ d\lambda_2, \tag{4.3}$$

if $u \in W^{1,p}(\Omega; \mathbb{R}^2)$ and $E_{rigid}[u] := \infty$ otherwise. Thereby $W^{qc}_{rigid,p}$ is the quasiconvex envelope of

$$W_{rigid,p}(F) := \begin{cases} |\gamma|^p & \text{if} \quad F \in \mathcal{M}^{(2)} \\ \infty & \text{otherwise} \end{cases},$$

which denotes the condensed energy of rigid elasticity. This means the corresponding rigid elastic energy $W_{rigid,el}(F)$ is zero for rotations $F \in SO(2)$ and infinity otherwise. The quasiconvex envelope $W^{qc}_{rigid,p}$ of $W_{rigid,p}$ is

$$W^{qc}_{rigid,1}(F) = \begin{cases} \sqrt{|F|^2 - 2} & \text{for} \quad F \in \mathcal{N}^{(2)} \\ \infty & \text{otherwise} \end{cases}$$

for $p = 1$ and

$$W^{qc}_{rigid,p}(F) = \begin{cases} \left(|F\vec{e}_2|^2 - 1\right)^{\frac{p}{2}} & \text{for} \quad F \in \mathcal{N}^{(2)} \\ \infty & \text{otherwise} \end{cases}$$

for $p \ge 2$, refer to [42, Theorem 4.1] and [22].

Theorem 4.3. *[42, Theorem 7.18, Theorem 7.28]. Suppose $\Omega \subseteq \mathbb{R}^2$ is a bounded and open set with Lipschitz boundary and $p \ge 2$, $2 \le q \le 2p$. For $\varepsilon > 0$ let $E_{cond,\varepsilon}$ and E_{rigid} be the functionals defined in (4.2) and (4.3). Then, $E_{cond,\varepsilon}$ converge in the sense* *of Γ-convergence to the functional E_{rigid} with respect to strong $L^{\frac{pq}{p+q}}$ convergence as ε tends to zero. Moreover, any bounded energy sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of $E_{cond,\varepsilon}$ is relatively compact in $L^{\frac{pq}{p+q}}$.*

A similar result in the three-dimensional case was shown by Conti, Dolzmann and Kreisbeck [24].
The result in Theorem 4.2 is obtained by the limit $t \to \infty$ of a laminate construction depending on t. Thereby, for fixed t the function γ only attains two values, namely zero and a value, which modulus strictly increases for increasing t. One may ask if a similar result as in Theorem 4.2 can be shown, if we restrict ourselves to functions γ, which only attains two values. The following lemma gives a negative answer to this question for a simplified model.

Lemma 4.4. *Let $\Gamma, p, q \geq 1$, $F^* = R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(2)}$, with $R \in SO(2)$ and $\gamma_0 > 0$, and let $\Omega \subseteq \mathbb{R}^2$ be open and bounded with C^1-boundary. Further we define the functional*

$$J(u,\gamma) = J_\varepsilon(u,\gamma) := \frac{1}{|\Omega|} \int_\Omega \frac{1}{\varepsilon} \left\| \nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right) - \mathbb{1} \right\|^q + |\gamma|^p \; d\lambda_2$$

and the function space

$$Y_\Gamma = Y_{\Gamma;p,q} := \{ \gamma \in BV(\Omega) \cap L^\infty(\Omega) \ : \ \exists\, \overline{\gamma} : \Omega \to \mathbb{R} \ : \ \overline{\gamma} \in \gamma, \ \overline{\gamma}(\Omega) \subseteq \{0, \Gamma\} \} .$$

Then, there exists a constant $C = C(\gamma_0, q) > 0$ independent of ε, Γ, p such that for all $\varepsilon > 0$

$$\inf_{u \in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2),\ \gamma \in Y_\Gamma} J(u,\gamma) \geq C \min \left\{ \min \left\{ \frac{1}{\varepsilon}, 1 \right\} \frac{1}{\Gamma^q}, 1 \right\} .$$

Furthermore we get for fixed $\varepsilon > 0$ that there exists a $\widetilde{\gamma}_0 > 0$ such that for all $0 < \gamma_0 < \widetilde{\gamma}_0$ the constant C can be chosen as $\frac{\gamma_0^q}{2^{\frac{q}{2}}}$.

Proof:
First of all we consider the case $\varepsilon \geq \widehat{\varepsilon} := q 2^{-\frac{q}{2}} \gamma_0^{q-1}$. Choose $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ with $u = F^*$ on $\partial\Omega$ and $\widetilde{\gamma} \in Y_\Gamma$. Let $\gamma : \Omega \to \mathbb{R}$ with $\gamma \in \widetilde{\gamma}$ such that $\gamma(\Omega) \subseteq \{0, \Gamma\}$. In order to abbreviate the following formulas we write $\Omega_0 = \Omega_0(\gamma) = \{x \in \Omega \ : \ \gamma(x) = 0\}$ and $\Omega_\Gamma = \Omega_\Gamma(\gamma) = \{x \in \Omega \ : \ \gamma(x) = \Gamma\}$. Then using $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$ for $a, b \in \mathbb{R}$

and Jensen's inequality we conclude, if $|\Omega_0| \neq 0$ and $|\Omega_\Gamma| \neq 0$, that

$$
\begin{aligned}
&J(u,\widetilde{\gamma}) - \frac{1}{|\Omega|}\int_\Omega |\gamma|^p \ d\lambda_2 \geq \frac{1}{\varepsilon\,|\Omega|}\int_\Omega \left[(\partial_x u_1 - 1)^2 + (\partial_y u_1 - \gamma\partial_x u_1)^2\right]^{\frac{q}{2}} \ d\lambda_2 \\
&\geq \frac{1}{\varepsilon 2^{\frac{q}{2}}\,|\Omega|}\int_\Omega [|\partial_x u_1 - 1| + |\partial_y u_1 - \gamma\partial_x u_1|]^q \ d\lambda_2 \\
&\geq \frac{1}{\varepsilon 2^{\frac{q}{2}}\,|\Omega|^q}\left(\int_\Omega |\partial_x u_1 - 1| + |\partial_y u_1 - \gamma\partial_x u_1| \ d\lambda_2\right)^q. \\
&\geq \frac{1}{\varepsilon 2^{\frac{q}{2}}}\left[\frac{|\Omega_0|}{|\Omega|}\left|\frac{1}{|\Omega_0|}\int_{\Omega_0}\partial_x u_1 \ d\lambda_2 - 1\right| + \frac{|\Omega_\Gamma|}{|\Omega|}\left|\frac{1}{|\Omega_\Gamma|}\int_{\Omega_\Gamma}\partial_x u_1 \ d\lambda_2 - 1\right|\right. \\
&\left. + \frac{|\Omega_0|}{|\Omega|}\left|\frac{1}{|\Omega_0|}\int_{\Omega_0}\partial_y u_1 \ d\lambda_2\right| + \frac{|\Omega_\Gamma|}{|\Omega|}\left|\frac{1}{|\Omega_\Gamma|}\int_{\Omega_\Gamma}\partial_y u_1 - \Gamma\partial_x u_1 \ d\lambda_2\right|\right]^q,
\end{aligned}
$$

thereby the first three inequalities are also valid if $|\Omega_0| = 0$ or $|\Omega_\Gamma| = 0$. Consider now the two cases that $|\Omega_0|$ is equal to $|\Omega|$ or zero. The first case leads to

$$
J(u,\widetilde{\gamma}) \geq \frac{1}{\varepsilon 2^{\frac{q}{2}}\,|\Omega|^q}\left|\int_\Omega \partial_y u_1 \ d\lambda_2\right|^q = \frac{\gamma_0^q}{2^{\frac{q}{2}}\varepsilon},
$$

where we have used Lemma 3.3 and $u_1\begin{pmatrix}x\\y\end{pmatrix} = x + \gamma_0 y$ for $\begin{pmatrix}x\\y\end{pmatrix} \in \partial\Omega$. In the second case, namely $|\Omega_\Gamma| = |\Omega|$, we get

$$
J(u,\widetilde{\gamma}) \geq \frac{1}{|\Omega|}\int_{\Omega_\Gamma} |\gamma|^p \ d\lambda_2 = \Gamma^p.
$$

For the remaining cases we define $\lambda := \frac{|\Omega_0|}{|\Omega|} \in (0,1)$ and write for simplicity

$$
\begin{aligned}
x_1 &:= \frac{1}{|\Omega_0|}\int_{\Omega_0}\partial_x u_1 \ d\lambda_2, \qquad & x_2 &:= \frac{1}{|\Omega_\Gamma|}\int_{\Omega_\Gamma}\partial_x u_1 \ d\lambda_2 \\
y_1 &:= \frac{1}{|\Omega_0|}\int_{\Omega_0}\partial_y u_1 \ d\lambda_2 \qquad \text{and} & y_2 &:= \frac{1}{|\Omega_\Gamma|}\int_{\Omega_\Gamma}\partial_y u_1 \ d\lambda_2.
\end{aligned}
$$

We get $\lambda x_1 + (1-\lambda)x_2 = \frac{1}{|\Omega|}\int_\Omega \partial_x u_1 \ d\lambda_2 = 1$ and $\lambda y_1 + (1-\lambda)y_2 = \frac{1}{|\Omega|}\int_\Omega \partial_y u_1 \ d\lambda_2 = \gamma_0$. Using this we have $(1-\lambda)\,|x_2 - 1| = |1 - \lambda x_1 - (1-\lambda)| = \lambda\,|x_1 - 1|$ and further we have $(1-\lambda)\,y_2 - \Gamma\,(1-\lambda)\,x_2 = \gamma_0 - \lambda y_1 - \Gamma\,(1 - \lambda x_1)$. Thus we can conclude

$$
\begin{aligned}
&\varepsilon 2^{\frac{q}{2}} J(u,\widetilde{\gamma}) \geq [\lambda\,|x_1 - 1| + (1-\lambda)\,|x_2 - 1| + \lambda\,|y_1| + (1-\lambda)\,|y_2 - \Gamma x_2|]^q + (1-\lambda)\Gamma^p \\
&= [2\lambda\,|x_1 - 1| + \lambda\,|y_1| + |\gamma_0 - \lambda y_1 - \Gamma\,(1-\lambda x_1)|]^q + (1-\lambda)\Gamma^p \\
&\geq [2\lambda\,|x_1 - 1| + |\gamma_0 - \Gamma\,(1-\lambda x_1)|]^q + (1-\lambda)\Gamma,
\end{aligned}
$$

since $\Gamma \geq 1$. The function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto 2\lambda |x-1| + |\gamma_0 - \Gamma + \lambda\Gamma x|$ has a minimum for $x-1=0$ or $\gamma_0 - \Gamma + \lambda\Gamma x = 0$. Thus we get using again $\Gamma \geq 1$ that

$$\begin{aligned} f(x) &\geq \min\left\{ |\gamma_0 - (1-\lambda)\Gamma|, 2\lambda\left|\frac{\Gamma-\gamma_0}{\lambda\Gamma} - 1\right| \right\} \\ &= \min\left\{1, \frac{2}{\Gamma}\right\} |\gamma_0 - (1-\lambda)\Gamma| \geq \frac{1}{\Gamma} |\gamma_0 - (1-\lambda)\Gamma|, \end{aligned}$$

for all $x \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{1}{\varepsilon 2^{\frac{q}{2}}} \left[\frac{1}{\Gamma}|\gamma_0 - x|\right]^q + x$, for $x \neq \gamma_0$, is convex since $q \geq 1$ and fulfills $g'(0) = 1 - \frac{q}{\varepsilon 2^{\frac{q}{2}}} \frac{1}{\Gamma^q} \gamma_0^{q-1} \geq 0$, since we have chosen $\varepsilon \geq q 2^{-\frac{q}{2}} \gamma_0^{q-1}$ and $\gamma_0 > 0$. Compound we get for the case $\lambda \in (0,1)$ that

$$J(u, \widetilde{\gamma}) \geq \frac{1}{\varepsilon 2^{\frac{q}{2}}} \left(\frac{\gamma_0}{\Gamma}\right)^q.$$

For $\varepsilon < \widehat{\varepsilon} = q 2^{-\frac{q}{2}} \gamma_0^{q-1}$ we get $J_\varepsilon(u, \widetilde{\gamma}) \geq J_{\widehat{\varepsilon}}(u, \widetilde{\gamma})$ and we can use the estimates for $J_{\widehat{\varepsilon}}(u, \widetilde{\gamma})$. Bringing all together one gets

$$\begin{aligned} J(u, \widetilde{\gamma}) &\geq \min\left\{ \min\left\{\frac{1}{\varepsilon}, \frac{2^{\frac{q}{2}}}{q\gamma_0^{q-1}}\right\} \cdot \frac{\gamma_0^q}{2^{\frac{q}{2}}\Gamma^q}, \Gamma^p \right\} \\ &\geq \min\left\{ \frac{2^{\frac{q}{2}}}{q\gamma_0^{q-1}}, 1 \right\} \cdot \min\left\{ \min\left\{\frac{1}{\varepsilon}, 1\right\} \frac{\gamma_0^q}{2^{\frac{q}{2}}\Gamma^q}, 1 \right\} \\ &\geq \min\left\{ \frac{2^{\frac{q}{2}}}{q\gamma_0^{q-1}}, 1 \right\} \cdot \min\left\{ \frac{\gamma_0^q}{2^{\frac{q}{2}}}, 1 \right\} \cdot \min\left\{ \min\left\{\frac{1}{\varepsilon}, 1\right\} \frac{1}{\Gamma^q}, 1 \right\}. \end{aligned}$$

Since we choose $u \in W^{1,\infty}_{F^*}(\Omega; \mathbb{R}^2)$ and $\widetilde{\gamma} \in Y_\Gamma$ arbitrary one gets the desired result, where $C = C(\gamma_0, q) = \min\left\{\frac{2^{\frac{q}{2}}}{q\gamma_0^{q-1}}, 1\right\} \cdot \min\left\{\frac{\gamma_0^q}{2^{\frac{q}{2}}}, 1\right\}$, which is equal to $\frac{\gamma_0^q}{2^{\frac{q}{2}}}$ for γ_0 small enough. □

4.2 Application of the convex integration method

A first approach to find a solution, or at least an upper bound, of the variational problem (4.1) is to look for a minimum in the class of all functions, which pointwise minimizes the elastic energy. Namely, we look for solutions of the partial differential inclusion

$$\begin{aligned} Du &\in \mathcal{M}^{(2)} \text{ a.e. in } \Omega, \\ u(x) &= Fx \text{ on } \partial\Omega, \end{aligned} \tag{4.4}$$

where $F \in \mathcal{N}^{(2)}$, γ is defined by $Du(x) = R(x)\left(\mathbb{1} + \gamma(x)\vec{e}_1 \otimes \vec{e}_2\right)$ and $R(x) \in \mathrm{SO}(2)$ for a.e. $x \in \Omega$. A solution was computed in [27, Lemma 2], see also [42, Lemma 5.30]. We give here an improved version of these lemmas with a better control of $|Du(x)\vec{e}_2|$ and thus of $\gamma^2(x) = |Du(x)\vec{e}_2|^2 - 1$.

Lemma 4.5. *Let* $U^k := \left\{G \in \mathbb{R}^{2\times 2} \ : \ \det(G) = 1, |G\vec{e}_1| < 1, \ |G\vec{e}_2|^2 < k^2\right\}$ *for* $k > 0$, $\Omega \subset \mathbb{R}^2$ *be an open and bounded set with Lipschitz boundary. Then we get that for each* $F \in U^k$ *there exists a Lipschitz continuous map* $u : \Omega \to \mathbb{R}^2$ *such that* $u(x) = Fx$ *for every* $x \in \partial\Omega$ *and the gradient fulfills* $\nabla u \in K^k := \left\{G \in \mathcal{M}^{(2)} \ : \ |G\vec{e}_2|^2 = k^2\right\}$ *almost everywhere in* Ω.

Proof:
In the following we write $\langle \mathcal{A}, \mathcal{B}\rangle := (\mathcal{A} \cup \mathcal{B})^{(1)}$ for sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^{2\times 2}$. Define for $j \in \mathbb{N}$,

$$U_j^k := \left\{G \in \mathbb{R}^{2\times 2} \ : \ \det(G) = 1, 1 - \frac{1}{2^{j-1}} < |G\vec{e}_1| < 1, \ k^2\left(1 - \frac{1}{2^{j-1}}\right) < |G\vec{e}_2|^2 < k^2\right\}.$$

Now we have to show that $\left\{U_j^k\right\}$ is an in-approximation for K^k. Obviously each U_j^k is open in $\Sigma = \left\{G \in \mathbb{R}^{2\times 2} \ : \ \det(G) = 1\right\}$ and uniformly bounded by U^k. Moreover U_j^k converge to K^k in the desired way, see Theorem 3.46. Finally we have to verify that $U_j^k \subset \left(U_{j+1}^k\right)^{rc}$. Choose $G \in U_j^k \setminus U_{j+1}^k$ then we get as in the proof of [42, Lemma 5.30] that there exist $G_{t^+}, G_{t^-} \in \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) = 1, |F\vec{e}_1| = \alpha, \ |F\vec{e}_2| = |G\vec{e}_2|\right\}$, with $\alpha \in \left(1 - \frac{1}{2^j}, 1\right)$, rank $(G_{t^+} - G_{t^-}) = 1$ and $G \in \langle G_{t^+}, G_{t^-}\rangle$. W.l.o.g. we can assume $|G_{t^\pm}\vec{e}_2| \in k^2\left(1 - \frac{1}{2^{j-1}}, 1 - \frac{1}{2^j}\right]$, otherwise $G \in \left(U_{j+1}^k\right)^{rc}$. Define for $r \in \mathbb{R}$

$$H_r^+ := G_{t^+} + r\left(G_{t^+}\vec{e}_1 \otimes \vec{e}_2\right).$$

Then we have $\det\left(H_r^+\right) = \det\left(G_{t^+}\right) = \det(G) = 1$, $|H_r^+\vec{e}_1| = |G_{t^+}\vec{e}_1|$ and we get $|H_r^+\vec{e}_2|^2 = |G_{t^+}\vec{e}_2|^2 + 2rG_{t^+}\vec{e}_2 \cdot G_{t^+}\vec{e}_1 + r^2|G_{t^+}\vec{e}_1|^2$. Choose $\beta \in \left(k^2\left(1 - \frac{1}{2^j}\right), k^2\right)$ then we get because of $|G_{t^+}\vec{e}_2| \in k^2\left(1 - \frac{1}{2^{j-1}}, 1 - \frac{1}{2^j}\right]$ that $|G_{t^+}\vec{e}_2|^2 - \beta^2 < 0$. Thus the equation

$$|G_{t^+}\vec{e}_2|^2 + 2rG_{t^+}\vec{e}_2 \cdot G_{t^+}\vec{e}_1 + r^2|G_{t^+}\vec{e}_1|^2 - \beta^2 = 0$$

has two solutions $r_- < 0 < r_+$. Consequently $G_{t^+} \in \langle H_{r_+}^+, H_{r_-}^+\rangle$ with $H_{r_+}^+, H_{r_-}^+ \in U_{j+1}^k$ and with the same argumentation there exist $H_{r_+}^-, H_{r_-}^- \in U_{j+1}^k$ with $G_{t^+} \in \langle H_{r_+}^-, H_{r_-}^-\rangle$. Summarized we get $G \in \langle\langle H_{r_+}^+, H_{r_-}^+\rangle, \langle H_{r_+}^-, H_{r_-}^-\rangle\rangle \subset \left(U_{j+1}^k\right)^{rc}$. Thus the family $\left\{U_j^k\right\}_{j\in\mathbb{N}}$ is an in-approximation of K^k. Now we can apply Theorem 3.46 and the lemma is shown. □

Remark 4.6. *Unfortunately this is the best we can show by using [54, Theorem 1.3]. Namely, we are only able to control* γ *up to a sign and in fact* γ *will jump between these two values and it is hard to evaluate the self-energy part for this* γ*. Therefore we need a more explicit construction to obtain an upper bound.*

4.3 Results for two slip systems

We digress now to the case of two slip systems and without self-energy. The case of two orthogonal slip systems was investigated in [4, 25]. We want to generalize this for the case of two slip systems, which are not orthogonal. In this section we compute the rank-one convex envelope of the condensed energy density of rigid elasticity for two slip systems with linear hardening and infinite latent hardening on a strict subset of all 2×2 matrices. Thereby infinite latent hardening means that the crystal deforms in single slip for all material points only [26, 60]. Furthermore we show that the polyconvex envelope is equal to the rank-one envelope for those special matrices. In the following we write $\vec{a}^{\perp} = J\vec{a}$, for the counterclockwise rotation of a vector $\vec{a} \in \mathbb{R}^2$ by the angle $\frac{\pi}{2}$, this means we have $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One of the slip directions can be chosen again as $\vec{e}_1$ and the other slip direction $\vec{v} \in \mathbb{S}^1$ fulfills $\vec{v} = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$ for $\varphi \in (-\pi, \pi) \setminus \{0\}$. Then we define the set of matrices, whose elastic part is a rotation and whose plastic part is a single slip in one of the slip systems $(\vec{e}_1, \vec{e}_2)$ or $\left(\vec{v}, \vec{v}^{\perp}\right)$ by $\mathcal{K}_1 \cup \mathcal{K}_2$, where

$$\begin{aligned} &\mathcal{K}_1 := \left\{ F \in \mathbb{R}^{2\times 2} \;:\; \det(F) = 1,\ |F\vec{e}_1| = 1 \right\} \\ \text{and} \qquad &\mathcal{K}_2 := \left\{ F \in \mathbb{R}^{2\times 2} \;:\; \det(F) = 1,\ |F\vec{v}| = 1 \right\}. \end{aligned}$$

We will see later on that the set $\mathcal{K}_1 \cup \mathcal{K}_2$ can be visualized as the curve painted in Figure 4.1 on page 54. We prove this rigorously in Corollary 4.8. Moreover the proof of Lemma 4.7 implies that each point of this curve represents exactly one element of $\mathcal{K}_1$ and one element of $\mathcal{K}_2$. The condensed energy density of rigid elasticity for two slip systems with linear and infinite latent hardening is given by

$$W_{two}(F) := \begin{cases} |F\vec{e}_2|^2 - 1 & \text{if } F \in \mathcal{K}_1 \\ |F\vec{v}^{\perp}|^2 - 1 & \text{if } F \in \mathcal{K}_2 \\ \infty & \text{otherwise} \end{cases}.$$

For $F \in \mathcal{K}_1 \cap \mathcal{K}_2$, we get $\underbrace{|F\vec{e}_1|^2}_{=1} + |F\vec{e}_2|^2 = \|F\|^2 = \underbrace{|F\vec{v}|^2}_{=1} + |F\vec{v}^{\perp}|^2$, which shows that the function W_{two} is well-defined. Next, we give some notations used in this section. We set $\alpha := \frac{\varphi}{2}$ and $\vec{r} := \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \in \mathbb{S}^1$. Define the functions $a_{\pm},\ d_{\pm} : (0, \infty) \to (0, \infty)$, which occur in the proofs of the following two lemmata and an intercalated corollary,

$$\begin{aligned}
a_\pm(x) &:= \frac{1}{|\cos(\alpha)|}\left[\left(1\pm\sqrt{\sin^2(\alpha)\left(x^2-\cos^2(\alpha)\right)_+}\right)^2+\cos^2(\alpha)\sin^2(\alpha)\right]^{\frac12}\\
&= \frac{1}{|\cos(\alpha)|}\left[1+\sin^2(\alpha)\max\left\{x^2,\cos^2(\alpha)\right\}\pm 2\sqrt{\sin^2(\alpha)\left(x^2-\cos^2(\alpha)\right)_+}\right]^{\frac12},\\
d_\pm(x) &:= \frac{1}{|\sin(\alpha)|}\left[\left(1\pm\sqrt{\cos^2(\alpha)\left(x^2-\sin^2(\alpha)\right)_+}\right)^2+\sin^2(\alpha)\cos^2(\alpha)\right]^{\frac12}\\
&= \frac{1}{|\sin(\alpha)|}\left[1+\cos^2(\alpha)\max\left\{x^2,\sin^2(\alpha)\right\}\pm 2\sqrt{\cos^2(\alpha)\left(x^2-\sin^2(\alpha)\right)_+}\right]^{\frac12},
\end{aligned}$$

where $(x)_+ := \max\{x,0\}$ for $x\in\mathbb{R}$. Next, we show that the lamination convex hull of $\mathcal{K}_1\cup\mathcal{K}_2$ is equal to $\left\{F\in\mathbb{R}^{2\times 2} : \det(F)=1\right\}$. The idea is that for a matrix $F\in\mathbb{R}^{2\times2}$ with $\det(F)=1$ one can show that either on the rank-one line $t\mapsto F_t := F\left(\mathbb{1}+t\vec r\otimes\vec r^{\perp}\right)$ or on the rank-one line $t\mapsto \widetilde F_t := F\left(\mathbb{1}+t\vec r^{\perp}\otimes\vec r\right)$ we find $t_-\le 0\le t_+$ such that $F_{t_\pm}\in\mathcal{K}_1\cup\mathcal{K}_2$ or $\widetilde F_{t_\pm}\in\mathcal{K}_1\cup\mathcal{K}_2$.

Lemma 4.7. *Let $\vec v,\alpha,\vec r$, $\mathcal{K}=\mathcal{K}_1\cup\mathcal{K}_2$, be defined as above, then we have*

$$\mathcal{K}^{lc}=\mathcal{K}^{(1)}=\left\{F\in\mathbb{R}^{2\times2} : \det(F)=1\right\}.$$

Proof:
Since $|F(-\vec v)|=|F\vec v|$ we can choose $\varphi\in(0,\pi)$. Thus we have that $\cos(\alpha)>0$ and $\sin(\alpha)>0$ for $\alpha=\frac{\varphi}{2}\in\left(0,\frac{\pi}{2}\right)$. For $\vec r=\begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\end{pmatrix}\in\mathbb{S}^1$ we have

$$\vec v=\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}=\begin{pmatrix}\cos(\alpha)\cos(\alpha)-\sin(\alpha)\sin(\alpha)\\ \sin(\alpha)\cos(\alpha)+\sin(\alpha)\cos(\alpha)\end{pmatrix}=\cos(\alpha)\vec r+\sin(\alpha)\vec r^{\perp}$$

and $\vec e_1=\cos(\alpha)\vec r-\sin(\alpha)\vec r^{\perp}$.
Let $F_i\in\mathbb{R}^{2\times2}$, $i\in\{1,2\}$, be defined by $F_i\vec r=R\begin{pmatrix}a\\0\end{pmatrix}$, $F_i\vec r^{\perp}=R\begin{pmatrix}b_i\\ \frac1a\end{pmatrix}$, where $R\in SO(2)$, $a\ge\sin(\alpha)$ and $b_i\in\mathbb{R}$. Next, we look for $b_i=b_i(a)\in\mathbb{R}$ such that $F_i\in\mathcal{K}_i$. We have

$$|F_1\vec e_1|^2=\left|\begin{pmatrix}a\cos(\alpha)-b_1\sin(\alpha)\\ -\frac{\sin(\alpha)}{a}\end{pmatrix}\right|^2=a^2\cos^2(\alpha)-2a\cos(\alpha)\sin(\alpha)b_1+\sin^2(\alpha)b_1^2+\frac{\sin^2(\alpha)}{a^2},$$

which is one iff

$$\begin{aligned}
b_{1,\pm}(a) &= \frac{a\cos(\alpha)\sin(\alpha)\pm\sqrt{a^2\cos^2(\alpha)\sin^2(\alpha)-\sin^2(\alpha)\left(a^2\cos^2(\alpha)+\frac{\sin^2(\alpha)}{a^2}-1\right)}}{\sin^2(\alpha)}\\
&= \frac{a\cos(\alpha)}{\sin(\alpha)}\pm\frac{1}{\sin(\alpha)}\sqrt{1-\frac{\sin^2(\alpha)}{a^2}},
\end{aligned}$$

where we have used $\sin(\alpha) > 0$ and $a \geq \sin(\alpha)$. Analogously, we get that $|F_2\vec{v}|^2$ is one iff

$$b_{2,\pm}(a) = -\frac{a\cos(\alpha)}{\sin(\alpha)} \pm \frac{1}{\sin(\alpha)}\sqrt{1-\frac{\sin^2(\alpha)}{a^2}}.$$

Let $F \in \mathbb{R}^{2\times 2}$ with $F\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $F\vec{r}^{\perp} = R\begin{pmatrix} b \\ \frac{1}{a} \end{pmatrix}$, where $R \in SO(2)$, $a \geq |\sin(\alpha)|$ and

$$|b| \leq \frac{a\cos(\alpha)}{\sin(\alpha)} + \frac{1}{\sin(\alpha)}\sqrt{1-\frac{\sin^2(\alpha)}{a^2}} = b_{1,+}(a)$$

or equivalently

$$|F\vec{r}^{\perp}| \leq \sqrt{\frac{a^2\cos^2(\alpha)}{\sin^2(\alpha)} + \frac{2a\cos(\alpha)}{\sin^2(\alpha)}\sqrt{1-\frac{\sin^2(\alpha)}{a^2}} + \frac{1}{\sin^2(\alpha)}} = d_+(a).$$

Especially we have shown

$$d_{\pm}(a) = \sqrt{b_{1,\pm}^2(a) + \frac{1}{a^2}} = \sqrt{b_{2,\mp}^2(a) + \frac{1}{a^2}}. \tag{4.5}$$

Furthermore the computation above implies that there exist $F_i \in \mathcal{K}_i$, $i \in \{1,2\}$, with rank $(F_2 - F_1) = 1$ and a $\lambda \in [0,1]$ such that $F = \lambda F_1 + (1-\lambda)F_2$. Thus we get $F \in \mathcal{K}^{(1)}$.
Let $G_i \in \mathbb{R}^{2\times 2}$, $i \in \{1,2\}$, be defined by $G_i\vec{r} = Q\begin{pmatrix} \frac{1}{d} \\ f_i \end{pmatrix}$, $G_i\vec{r}^{\perp} = Q\begin{pmatrix} 0 \\ d \end{pmatrix}$, whereupon $Q \in \mathrm{SO}(2)$, $d \geq \cos(\alpha)$ and $f_i \in \mathbb{R}$ to be chosen such that $G_i \in \mathcal{K}_i$. We have as above that

$$|G_1\vec{e}_1|^2 = \left|Q\begin{pmatrix} \frac{\cos(\alpha)}{d} \\ \cos(\alpha)f_i - \sin(\alpha)d \end{pmatrix}\right|^2$$

is one iff

$$f_{1,\pm}(d) = \frac{d\sin(\alpha)}{\cos(\alpha)} \pm \frac{1}{\cos(\alpha)}\sqrt{1-\frac{\cos^2(\alpha)}{d^2}}$$

and $|G_2\vec{v}|^2$ is one iff

$$f_{2,\pm}(d) = -\frac{d\sin(\alpha)}{\cos(\alpha)} \pm \frac{1}{\cos(\alpha)}\sqrt{1-\frac{\cos^2(\alpha)}{d^2}}.$$

Let $G \in \mathbb{R}^{2\times 2}$ with $G\vec{r} = Q\begin{pmatrix} \frac{1}{d} \\ f \end{pmatrix}$, $G\vec{r}^{\perp} = Q\begin{pmatrix} 0 \\ d \end{pmatrix}$, where $Q \in SO(2)$, $d \geq \cos(\alpha)$ and

$$|f| \leq \frac{d\sin(\alpha)}{\cos(\alpha)} + \frac{1}{\cos(\alpha)}\sqrt{1-\frac{\cos^2(\alpha)}{d^2}} = f_{1,+}(d)$$

or equivalently

$$|G\vec{r}| \leq \sqrt{\frac{d^2\sin^2(\alpha)}{\cos^2(\alpha)} + \frac{2d\sin(\alpha)}{\cos^2(\alpha)}\sqrt{1-\frac{\cos^2(\alpha)}{d^2}} + \frac{1}{\cos^2(\alpha)}} = a_+(d),$$

where we have shown

$$a_\pm(d) = \sqrt{f_{1,\pm}^2(d) + \frac{1}{d^2}} = \sqrt{f_{2,\mp}^2(d) + \frac{1}{d^2}}. \tag{4.6}$$

Then we get as above that $G \in \mathcal{K}^{(1)}$. Using $d_+(x) \geq x\frac{\cos(\alpha)}{\sin(\alpha)}$ and $a_+(x) \geq x\frac{\sin(\alpha)}{\cos(\alpha)}$ for $x > 0$ we get that $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{K}^{(1)}$, where

$$\mathcal{A}_1 := \left\{A \in \mathbb{R}^{2\times 2} \ : \ \det(A) = 1,\ |A\vec{r}^\perp| \leq |A\vec{r}|\frac{\cos(\alpha)}{\sin(\alpha)},\ |A\vec{r}| \geq \sin(\alpha)\right\}$$

and $$\mathcal{A}_2 := \left\{A \in \mathbb{R}^{2\times 2} \ : \ \det(A) = 1,\ |A\vec{r}| \leq |A\vec{r}^\perp|\frac{\sin(\alpha)}{\cos(\alpha)},\ |A\vec{r}^\perp| \geq \cos(\alpha)\right\}.$$

Let $H \in \mathbb{R}^{2\times 2} \setminus \mathcal{A}_1$ with $\det(H) = 1$, then we get $|H\vec{r}^\perp| > |H\vec{r}|\frac{\cos(\alpha)}{\sin(\alpha)}$ or $|H\vec{r}| < \sin(\alpha)$. In the case $|H\vec{r}| < \sin(\alpha)$ we have $|H\vec{r}^\perp| \geq \frac{1}{\sin(\alpha)} \geq \cos(\alpha)$ and thus we conclude $|H\vec{r}| < \sin(\alpha) \leq |H\vec{r}^\perp|\frac{\sin(\alpha)}{\cos(\alpha)}$, which shows $H \in \mathcal{A}_2$. In the case $|H\vec{r}^\perp| > |H\vec{r}|\frac{\cos(\alpha)}{\sin(\alpha)}$ we can choose $|H\vec{r}| \geq \sin(\alpha)$ and get $|H\vec{r}^\perp| > \cos(\alpha)$, which shows $H \in \mathcal{A}_2$.
This means $\widetilde{\mathcal{K}} := \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) = 1\right\} \subseteq \mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{K}^{(1)} \subseteq \mathcal{K}^{(lc)}$ and since $\widetilde{\mathcal{K}}$ is lamination convex we get $\widetilde{\mathcal{K}} = \mathcal{K}^{(1)} = \mathcal{K}^{lc}$.

□

We note a simple corollary of the above computations, which we use in the subsequent lemma.

Corollary 4.8. *Let $\mathcal{K}_1$, $\mathcal{K}_2$, $\vec{r}$, d_-, a_- be defined as above, then we get*

$$\mathcal{K}_1 \cup \mathcal{K}_2 = \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) = 1,\ |F\vec{r}| \geq 1,\ |F\vec{r}^\perp| = d_-(|F\vec{r}|)\right\}$$
$$\cup \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) = 1,\ |F\vec{r}^\perp| \geq 1,\ |F\vec{r}| = a_-(|F\vec{r}^\perp|)\right\}, \tag{4.7}$$

see Figure 4.1.

Proof:
Let $\widehat{\mathcal{K}}$ be the right-hand side in Equation (4.7). It was shown in the proof of Lemma 4.7, refer to (4.5) and (4.6), that

$$\mathcal{K}_1 \cup \mathcal{K}_2 = \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) = 1,\ |F\vec{r}| \geq 1,\ |F\vec{r}^\perp| \in \{d_-(|F\vec{r}|), d_+(|F\vec{r}|)\}\right\}$$
$$\cup \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) = 1,\ |F\vec{r}^\perp| \geq 1,\ |F\vec{r}| \in \{a_-(|F\vec{r}^\perp|), a_+(|F\vec{r}^\perp|)\}\right\},$$

where we used that for $A \in \mathbb{R}^{2\times 2}$, with $\det(A) = 1$, we have $\max\left\{|A\vec{r}|, |A\vec{r}^{\perp}|\right\} \geq 1$. Let $F \in \mathbb{R}^{2\times 2}$ with $\det(F) = 1$, $|F\vec{r}| \geq 1$ and $|F\vec{r}^{\perp}| = d_+(|F\vec{r}|)$. Since d_+ is monotone increasing we get

$$|F\vec{r}^{\perp}| = d_+(|F\vec{r}|) \geq d_+(1) = \frac{1}{|\sin(\alpha)|}\left[1 + 3\cos^2(\alpha)\right]^{\frac{1}{2}} \geq 1.$$

Next, we get $a_-\left(|F\vec{r}^{\perp}|\right)$ is equal to

$$a_-\left(d_+(|F\vec{r}|)\right) = \frac{1}{|\cos(\alpha)|}\left[\left(1 - \sqrt{\sin^2(\alpha)\left(d_+^2(|F\vec{r}|) - \cos^2(\alpha)\right)}\right)^2 + \cos^2(\alpha)\sin^2(\alpha)\right]^{\frac{1}{2}}$$

$$= \frac{1}{|\cos(\alpha)|}\left[\left(1 - \left(1 + \sqrt{\cos^2(\alpha)\left(|F\vec{r}|^2 - \sin^2(\alpha)\right)}\right)\right)^2 + \cos^2(\alpha)\sin^2(\alpha)\right]^{\frac{1}{2}} = |F\vec{r}|.$$

Thus we get $F \in \widehat{\mathcal{K}}$. Analogously, we get for $G \in \mathbb{R}^{2\times 2}$, with $\det(G) = 1$, $|G\vec{r}^{\perp}| \geq 1$ and $|G\vec{r}| = a_+\left(|G\vec{r}^{\perp}|\right)$ that $|G\vec{r}| \geq 1$ and $|G\vec{r}^{\perp}| = d_-\left(a_+\left(|G\vec{r}^{\perp}|\right)\right)$ and thus $G \in \widehat{\mathcal{K}}$, which implies $\mathcal{K}_1 \cup \mathcal{K}_2 = \widehat{\mathcal{K}}$ as asserted.

□

For $\vec{r} := \begin{pmatrix}\cos(\frac{\varphi}{2})\\ \sin\left(\frac{\varphi}{2}\right)\end{pmatrix}$, we define $\mathcal{B} := \bigcup_{i=1}^{3} \mathcal{B}_i$, where

$$\begin{aligned}
\mathcal{B}_1 = \mathcal{B}_1(\vec{r}) &:= \left\{F \in \mathcal{K}^{lc} \ : \ |F\vec{r}| \geq 1,\ |F\vec{r}^{\perp}| \leq d_-(|F\vec{r}|)\right\}\\
\mathcal{B}_2 = \mathcal{B}_2(\vec{r}) &:= \left\{F \in \mathcal{K}^{lc} \ : \ |F\vec{r}^{\perp}| > 1,\ |F\vec{r}| \leq a_-\left(|F\vec{r}^{\perp}|\right)\right\}\\
\text{and} \qquad \mathcal{B}_3 &:= \mathbb{R}^{2\times 2} \setminus \mathcal{K}^{lc} = \left\{F \in \mathbb{R}^{2\times 2} \ : \ \det(F) \neq 1\right\},
\end{aligned}$$

refer to Figure 4.1.

Remark 4.9. *The set* $\mathcal{B}_2$ *is the same as*

$$\widetilde{\mathcal{B}}_2 = \left\{F \in \mathcal{K}^{lc} \ : \ \left(|F\vec{r}^{\perp}| \geq d_+(|F\vec{r}|)\right) \vee \left(|F\vec{r}^{\perp}| \leq d_-(|F\vec{r}|) \wedge |F\vec{r}| < 1\right)\right\}.$$

Proof:
We use the notation $a := |F\vec{r}|$ and $d := |F\vec{r}^{\perp}|$, for an $F \in \mathbb{R}^{2\times 2}$. Let $F \in \mathcal{B}_2$, then we get using $a_-^2(d) \geq a^2$ that $1 + \sin^2(\alpha)d^2 - a^2\cos^2(\alpha) \geq 2\sqrt{\sin^2(\alpha)\left(d^2 - \cos^2(\alpha)\right)} > 0$. For $F \in \widetilde{\mathcal{B}}_2$, we get $1 + \sin^2(\alpha)d^2 - a^2\cos^2(\alpha) > 0$ in the case $a < 1$ and because of $d^2 \geq d_+^2(a)$ we get $1 + \sin^2(\alpha)d^2 - a^2\cos^2(\alpha) \geq 2 + 2\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} > 0$ in the case $a \geq 1$. Thus we get for $F \in \mathcal{B}_2 \cup \widetilde{\mathcal{B}}_2$ that $d > \max\left\{1, \sqrt{\max\left\{\frac{a^2\cos^2(\alpha)-1}{\sin^2(\alpha)}, 0\right\}}\right\}$,

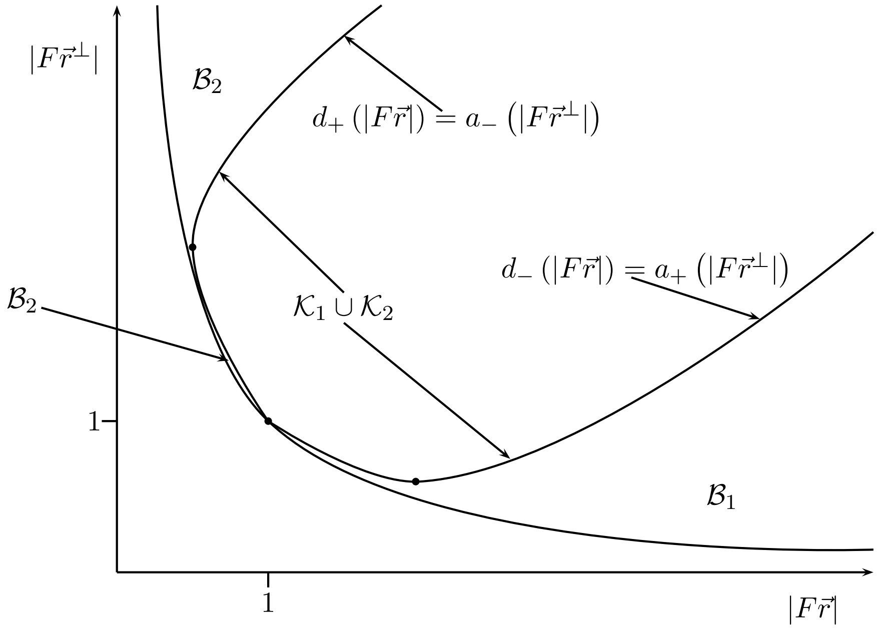

Figure 4.1: $\alpha = \frac{\pi}{6}$, $\det(F) = 1$

which implies

$$
\begin{aligned}
a \le a_-(d) \quad &\Leftrightarrow \quad 2\sqrt{\sin^2(\alpha)\left(d^2 - \cos^2(\alpha)\right)} \le 1 + \sin^2(\alpha)d^2 - a^2\cos^2(\alpha) \\
&\Leftrightarrow \quad 0 \le d^4 \sin^4(\alpha) + d^2\left(-2\sin^2(\alpha) - 2a^2 \sin^2(\alpha)\cos^2(\alpha)\right) \\
&\qquad +1 + a^4\cos^4(\alpha) - 2a^2\cos^2(\alpha) + 4\sin^2(\alpha)\cos^2(\alpha).
\end{aligned}
$$

The formula on the right hand side is zero iff $a \ge |\sin(\alpha)|$ and d^2 is equal to

$$
\begin{aligned}
d_{1,2}^2(a) &= \frac{1}{s^2}\left[s + a^2 sk \pm \frac{1}{2}\sqrt{4s^2 + 4a^4 s^2 k^2 + 8a^2 s^2 k - 4s^2\left(1 + a^4k^2 - 2a^2k + 4sk\right)}\right] \\
&= \frac{1}{s}\left[1 + a^2 k \pm 2\sqrt{a^2k - sk}\right] = d_\pm^2(a),
\end{aligned}
$$

where we have used for shorter notation $k := \cos^2(\alpha)$ and $s := \sin^2(\alpha)$. If $a < |\sin(\alpha)|$, this means that $a \le a_-(d)$ is no restriction in the definition of $\mathcal{B}_2$. Furthermore we get for $a < |\sin(\alpha)|$ that '$d \ge d_+(a) \vee d \le d_-(a)$' is no restriction in the definition of $\widetilde{\mathcal{B}}_2$, since $d_+(a) = d_-(a)$ for $a < |\sin(\alpha)|$. Next, we get for $a \ge 1$ that

$$
\left(d_-^2(a)\right)' = \frac{\cos^2(\alpha)}{\sin^2(\alpha)}\left[2a - \frac{2a}{\sqrt{\cos^2(\alpha)(a^2 - \sin^2(\alpha))}}\right] \ge 0 \Leftrightarrow a^2 \ge \frac{1}{\cos^2(\alpha)} + \sin^2(\alpha) \tag{4.8}
$$

and since $d(1) = 1$ we get $d_-(a) \leq 1$ for $a^2 < \frac{1}{\cos^2(\alpha)} + \sin^2(\alpha)$. For $a^2 \geq \frac{1}{\cos^2(\alpha)} + \sin^2(\alpha)$ we get

$$\begin{aligned} \sqrt{\max\left\{\frac{a^2\cos^2(\alpha)-1}{\sin^2(\alpha)}, 0\right\}} > d_-(a) \quad &\Leftrightarrow \quad 2\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} > 2 \\ &\Leftrightarrow \quad a^2 > \frac{1}{\cos^2(\alpha)} + \sin^2(\alpha) \end{aligned}$$

and we can conclude for $a \geq 1$ that

$$d_-(a) \leq \max\left\{1, \sqrt{\max\left\{\frac{a^2\cos^2(\alpha)-1}{\sin^2(\alpha)}, 0\right\}}\right\} < d.$$

This means

$$a \leq a_-(d) \Leftrightarrow (d \geq d_+(a) \vee (a < 1 \wedge d \leq d_-(a))),$$

and thus we get $\mathcal{B}_2 = \widetilde{\mathcal{B}}_2$ as asserted.

□

Next, we show the main theorem of this section. We derive the rank-one and polyconvex envelope of W_{two} on the set $\mathcal{B}_1 \cup \mathcal{B}_2$. For $F \in \mathcal{B}_1$ this is done using the rank-one line $t \mapsto F\left(\mathbb{1} + t\vec{r} \otimes \vec{r}^{\perp}\right)$ and for $G \in \mathcal{B}_2$ with help of the rank-one line $t \mapsto G\left(\mathbb{1} + t\vec{r}^{\perp} \otimes \vec{r}\right)$, refer to Figure 4.2. These rank-one lines connect an element of $\mathcal{K}_1$ with one of $\mathcal{K}_2$ and they have the same energy.

Theorem 4.10. *Let* $\vec{v} := \begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}$, $\vec{r} := \begin{pmatrix}\cos(\alpha)\\ \sin(\alpha)\end{pmatrix} \in \mathbb{S}^1$, $\varphi \in (-\pi, \pi) \setminus \{0\}$, $\alpha := \frac{\varphi}{2}$ *and* $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$, $\mathcal{B} = \bigcup_{i=1}^{3} \mathcal{B}_i$ *be defined as above. Then we get for all* $F \in \mathcal{B}$ *that*

$$W^{rc}_{two}(F) = W^{pc}_{two}(F) = \begin{cases} f\left(|F\vec{r}|\right) & \text{if } F \in \mathcal{B}_1 \\ g\left(|F\vec{r}^{\perp}|\right) & \text{if } F \in \mathcal{B}_2 \\ +\infty & \text{if } F \in \mathcal{B}_3 \end{cases}, \tag{4.9}$$

where $f, g := (0, \infty) \to \mathbb{R}$ *are defined by* $f(x) = g(x) = 0$ *for all* $x \in (0, 1)$ *and*

$$\begin{aligned} f(x) &:= \frac{1}{\sin^2(\alpha)}\left[x^2 + \cos^2(\alpha) - \sin^2(\alpha) - 2\sqrt{\cos^2(\alpha)\left(x^2 - \sin^2(\alpha)\right)}\right], \\ g(x) &:= \frac{1}{\cos^2(\alpha)}\left[x^2 + \sin^2(\alpha) - \cos^2(\alpha) - 2\sqrt{\sin^2(\alpha)\left(x^2 - \cos^2(\alpha)\right)}\right] \end{aligned}$$

for all $x \geq 1$.

Remark 4.11. *Considering now the case of two orthogonal slip-systems, namely* $\varphi = \frac{\pi}{2}$ *and thus* $\alpha = \frac{\pi}{4}$*, then we have* $f(x) = g(x) = 2\left(x^2 - 2\sqrt{\frac{1}{2}x^2 - \frac{1}{4}}\right) = 2x^2 - 2\sqrt{2x^2-1}$ *for all* $x \geq 1$*. Let* $\psi(t) = \left(\sqrt{(t^2-1)_+} - 1\right)_+^q$ *defined as in [25, Theorem 1.1], where* $(a)_+^q := \max\{a, 0\}^q$ *for* $a \in \mathbb{R}$*. Then we get for* $q = 2$ *and* $t \geq \sqrt{2}$ *that we have* $\psi(t) = t^2 - 2\sqrt{t^2-1}$*, which implies* $f(x) = g(x) = \psi\left(\sqrt{2}x\right)$ *for all* $x \geq 1$*. Thus the case we investigate here corresponds to the third case in [25, Theorem 1.1]. The situation for the remaining matrices* $\mathcal{B}_4 := \mathbb{R}^{2\times 2} \setminus \bigcup_{i=1}^{3} \mathcal{B}_i$ *is more complicated, see Remark 4.12.*

Proof of Theorem 4.10:
Since $|F(-\vec{v})| = |F\vec{v}|$ and $|F(-\vec{v})^{\perp}|^2 - 1 = |F\vec{v}^{\perp}|^2 - 1$ we can choose $\varphi \in (0, \pi)$. This implies again that $\cos(\alpha) > 0$ and $\sin(\alpha) > 0$, for $\alpha := \frac{\varphi}{2}$. For simplicity we set $a := |F\vec{r}|$ and $d := |F\vec{r}^{\perp}|$ for a matrix $F \in \mathbb{R}^{2\times 2}$. Let $\widetilde{W}$ be the formula on the right-hand side in Equation (4.9), defined for $F \in \mathcal{B}$. For a given set $E \subseteq \mathbb{R}^4$, we denote by I_E the indicator function on the set E, i.e.,

$$I_E(F) = \begin{cases} 0 & \text{if } F \in E \\ +\infty & \text{otherwise} \end{cases},$$

which is convex iff the set E is convex. Define now $\widehat{W} : \mathbb{R}^{2\times 2} \to \mathbb{R}$, by

$$\widehat{W}(F) := \max\left\{f\left(|F\vec{r}|\right), g\left(|F\vec{r}^{\perp}|\right)\right\} + I_{\mathcal{K}^{lc}}(F).$$

Below we show the following four steps. First of all we show that $\widehat{W}(F)$ is equal to $\widetilde{W}(F)$ for all $F \in \mathcal{B}$. In the next step we verify that $\widehat{W}(F) \leq W_{two}(F)$ for all $F \in \mathbb{R}^{2\times 2}$. In the third part we prove that $W_{two}^{rc}(F) \leq \widehat{W}(F)$ for all $F \in \mathcal{B}$. Finally we show that $\widehat{W} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is polyconvex, which implies $\widehat{W}(F) \leq W_{two}^{pc}(F) \leq W_{two}^{rc}(F) \leq \widehat{W}(F)$ for all $F \in \mathcal{B}$ as asserted.
Step 1: $\widehat{W}(F) = \widetilde{W}(F)$ for all $F \in \mathcal{B}$. In the following we choose $F \in \mathcal{B}_1 \cup \mathcal{B}_2$, since if $F \in \mathcal{B}_3$ both functions take the value $+\infty$. First of all we show that g is a monotone increasing function. It is continuous, constant in $(0, 1)$ and we have for $x \in (1, \infty)$ that

$$\begin{aligned} g'(x) &= \frac{1}{\cos^2(\alpha)}\left[2x - \frac{2\sin^2(\alpha)x}{\sqrt{\sin^2(\alpha)\left(x^2 - \cos^2(\alpha)\right)}}\right] \\ &= \frac{2x}{\cos^2(\alpha)\sqrt{\sin^2(\alpha)\left(x^2-\cos^2(\alpha)\right)}}\left[\sqrt{\sin^2(\alpha)\left(x^2-\cos^2(\alpha)\right)} - \sin^2(\alpha)\right] \geq 0. \end{aligned}$$

Consider now the case $a = |F\vec{r}| < 1$, i.e., $F \in \mathcal{B}_2$. Since $f(x) = 0$ for $x < 1$ and $g(x) \geq 0$ for $x > 0$ we get $\widehat{W}(F) = g(d) = \widetilde{W}(F)$.

In the case $a \geq 1$ we show subsequently that $g(r(a)) = f(a)$, where

$$r(a) := \frac{1}{\sin(\alpha)} \sqrt{\left(\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} + \sin^2(\alpha) - \cos^2(\alpha)\right)^2 + \sin^2(\alpha)\cos^2(\alpha)},$$

refer to Figure 4.2. We get, by using the shorter notations $k := \cos^2(\alpha)$ and $s := \sin^2(\alpha)$

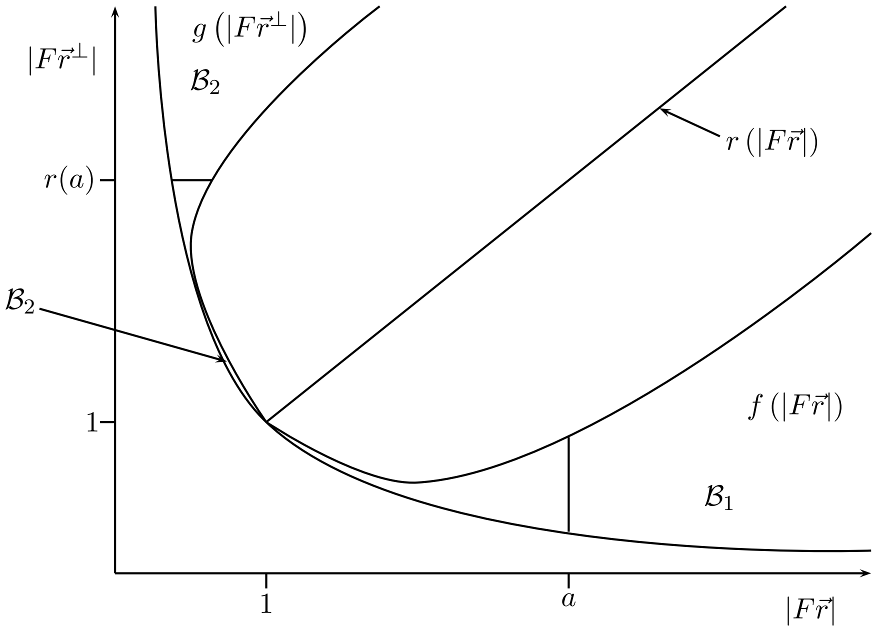

Figure 4.2: $\alpha = \frac{\pi}{6}$

that

$$\begin{aligned}
g(r(a)) &= \frac{1}{\cos^2(\alpha)}\left[r^2(a) + \sin^2(\alpha) - \cos^2(\alpha) - 2\sqrt{\sin^2(\alpha)\left(r^2(a) - \cos^2(\alpha)\right)}\right] \\
&= \frac{1}{sk}\left[ka^2 + 2(s-k)\sqrt{k\left(a^2 - s\right)} + (s-k)^2 + s^2 - sk - 2s\left(\sqrt{k\left(a^2 - s\right)} + s - k\right)\right] \\
&= \frac{1}{sk}\left[ka^2 + (s-k)^2 - s(s-k) - 2k\sqrt{k\left(a^2 - s\right)}\right] \\
&= \frac{1}{\sin^2(\alpha)}\left[a^2 + \cos^2(\alpha) - \sin^2(\alpha) - 2\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)}\right] = f(a).
\end{aligned}$$

Next we show that $d_+(a) \geq r(a) \geq d_-(a)$, for $a \geq 1$. The first inequality is an immediate consequence of the definitions, since $d_+(a) \geq r(a)$ is equivalent to

$$\left(\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} + 1\right)^2 \geq \left(\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} + \sin^2(\alpha) - \cos^2(\alpha)\right)^2$$

and thus we have $d_+(a) \geq r(a) \Leftrightarrow 1 \geq \sin^2(\alpha) - \cos^2(\alpha)$. For the second inequality we get $d_-(a) \leq r(a)$ is equivalent to

$$\left(\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} - 1\right)^2 \leq \left(\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} + \sin^2(\alpha) - \cos^2(\alpha)\right)^2.$$

Thus we get in the case $\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} \geq 1$ that

$$d_-(a) \leq r(a) \Leftrightarrow \sin^2(\alpha) - \cos^2(\alpha) \geq -1$$

and in the case $\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} < 1$ that

$$d_-(a) \leq r(a) \Leftrightarrow 2\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} + \sin^2(\alpha) - \cos^2(\alpha) \geq 1,$$

which is true for $a \geq 1$. Summarized we get $d_+(a) \geq r(a) \geq d_-(a)$ as asserted. With help of these foregoing consideration we can show now step one.
Let $F \in \mathcal{B}_1$, then we get $d \leq d_-(a) \leq r(a)$ by definition and thus $f(a) = g(r(a)) \geq g(d)$, since g is monotone increasing in d.
With the same reason we get for $F \in \mathcal{B}_2 = \widetilde{\mathcal{B}}_2$, with $a = |F\vec{r}| \geq 1$, that $d \geq d_+(a) \geq r(a)$ and finally $f(a) = g(r(a)) \leq g(d)$. Thus we have shown

$$\widehat{W}(F) = \widetilde{W}(F)$$

for all $F \in \mathcal{B}$.
Step 2: $\widehat{W}(F) \leq W_{two}(F)$ for all $F \in \mathbb{R}^{2\times 2}$.
Since $W_{two}(F)$ is infinite for $F \in \mathbb{R}^{2\times 2} \setminus (\mathcal{K}_1 \cup \mathcal{K}_2)$, we can choose $F \in \mathcal{K}_1 \cup \mathcal{K}_2$. Compute now $W_{two}(F)$ for a matrix $F \in \mathbb{R}^{2\times 2}$, with $F\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$, $F\vec{r}^\perp = R\begin{pmatrix} b \\ \frac{1}{a} \end{pmatrix}$, $R \in SO(2)$, $a \geq 1$, $b \in \mathbb{R}$ and $W_{two}(G)$ for a matrix $G \in \mathbb{R}^{2\times 2}$, with $G\vec{r} = Q\begin{pmatrix} \frac{1}{d} \\ f \end{pmatrix}$, $G\vec{r}^\perp = Q\begin{pmatrix} 0 \\ d \end{pmatrix}$, $Q \in SO(2)$, $f \in \mathbb{R}$, $d \geq 1$. Due to Corollary 4.8 it suffices to compute these two cases. We have $\vec{e}_2 = \sin(\alpha)\vec{r} + \cos(\alpha)\vec{r}^\perp$ and thus

$$\begin{aligned} |F\vec{e}_2|^2 - 1 &= \left|R\begin{pmatrix} \sin(\alpha)a + \cos(\alpha)b \\ \frac{\cos(\alpha)}{a} \end{pmatrix}\right|^2 - 1 \\ &= \sin^2(\alpha)a^2 + \cos^2(\alpha)b^2 + 2ab\sin(\alpha)\cos(\alpha) + \frac{\cos^2(\alpha)}{a^2} - 1 =: F(a,b). \end{aligned} \tag{4.10}$$

Since $\vec{v}^\perp = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix} = \begin{pmatrix} -\sin(\alpha)\cos(\alpha) - \sin(\alpha)\cos(\alpha) \\ \cos(\alpha)\cos(\alpha) - \sin(\alpha)\sin(\alpha) \end{pmatrix} = -\sin(\alpha)\vec{r} + \cos(\alpha)\vec{r}^\perp$, we get

$$|F\vec{v}^\perp|^2 - 1 = \left|R\begin{pmatrix} -\sin(\alpha)a + \cos(\alpha)b \\ \frac{\cos(\alpha)}{a} \end{pmatrix}\right|^2 - 1 = F(a,-b). \tag{4.11}$$

For the matrix G we get with the same calculations

$$|G\vec{e}_2|^2 - 1 = \cos^2(\alpha)d^2 + \sin^2(\alpha)f^2 + 2df\sin(\alpha)\cos(\alpha) + \frac{\sin^2(\alpha)}{d^2} - 1 =: G(d,f)$$

and analogously

$$|G\vec{v}^\perp|^2 - 1 = G(d,-f).$$

Define as in the proof of Lemma 4.7

$$\begin{aligned} b_\pm(a) = b_{1,\pm} &:= \frac{a\cos(\alpha)}{\sin(\alpha)} \pm \frac{1}{\sin(\alpha)}\sqrt{1-\frac{\sin^2(\alpha)}{a^2}} \\ \text{and} \qquad f_\pm(d) = f_{1,\pm} &:= \frac{d\sin(\alpha)}{\cos(\alpha)} \pm \frac{1}{\cos(\alpha)}\sqrt{1-\frac{\cos^2(\alpha)}{d^2}}, \end{aligned}$$

then we get by the Equations (4.5) and (4.6) that $b_-^2(a) + \frac{1}{a^2} = d_-^2(a)$ and analogously $a_-^2(d) = f_-^2(d) + \frac{1}{d^2}$. It was shown in Corollary 4.8 that

$$\begin{aligned} \mathcal{K}_1 \cup \mathcal{K}_2 = \quad & \left\{ F \in \mathcal{K}^{(lc)} \;:\; \det(F) = 1,\ |F\vec{r}| \geq 1,\ |F\vec{r}^\perp| = d_-\left(|F\vec{r}|\right) \right\} \\ \cup \quad & \left\{ F \in \mathcal{K}^{(lc)} \;:\; \det(F) = 1,\ |F\vec{r}^\perp| \geq 1,\ |F\vec{r}| = a_-\left(|F\vec{r}^\perp|\right) \right\}. \end{aligned}$$

Let $F \in \mathcal{B}_1 \cap \mathcal{K}_1$, then we get by the proof of Lemma 4.7 that there exist a $R \in SO(2)$ such that $F\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$, $F\vec{r}^\perp = R\begin{pmatrix} b_-(a) \\ \frac{1}{a} \end{pmatrix}$, with $a \geq 1$. Thus we have

$$W_{two}(F) = |F\vec{e}_2|^2 - 1 = F\left(a, b_-(a)\right).$$

For a matrix $F \in \mathcal{B}_1 \cap \mathcal{K}_2$ we get $F\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$, $F\vec{r}^\perp = R\begin{pmatrix} -b_-(a) \\ \frac{1}{a} \end{pmatrix}$ for $R \in SO(2)$, $a \geq 1$ and thus we have $W_{two}(F) = |F\vec{v}^\perp|^2 - 1 = F\left(a, -(-b_-(a))\right) = F\left(a, b_-(a)\right)$. With the same argumentation we get for a $F \in \overline{\mathcal{B}}_2 \cap (\mathcal{K}_1 \cup \mathcal{K}_2)$ that $W_{two}(F) = G\left(d, f_-(d)\right)$. To finish step two, it remains to show that $F\left(a, b_-(a)\right) = f(a)$ and $G\left(d, f_-(d)\right) = g(d)$. We have for $a \geq 1$

$$\begin{aligned} &F\left(a, b_-(a)\right) = \sin^2(\alpha)a^2 + \cos^2(\alpha)d_-^2(a) + 2ab_-(a)\sin(\alpha)\cos(\alpha) - 1 \\ &= \frac{1}{\sin^2(\alpha)}\Big[\sin^4(\alpha)a^2 + \cos^2(\alpha) + a^2\cos^4(\alpha) - 2\cos^2(\alpha)\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} \\ &+ 2a^2\cos^2(\alpha)\sin^2(\alpha) - 2\cos(\alpha)\sin^2(\alpha)\sqrt{a^2 - \sin^2(\alpha)} - \sin^2(\alpha)\Big] \\ &= \frac{1}{\sin^2(\alpha)}\left[a^2 - 2\sqrt{\cos^2(\alpha)\left(a^2 - \sin^2(\alpha)\right)} + \cos^2(\alpha) - \sin^2(\alpha)\right] = f(a). \end{aligned}$$

With an analog computation we get $G\left(d, f_-(d)\right) = g(d)$. This means $\widehat{W}(F) = W_{two}(F)$ for all $F \in \mathcal{K}_1 \cup \mathcal{K}_2 \subseteq \mathcal{B}_1 \cup \overline{\mathcal{B}}_2$ and thus we have shown step two.

Step 3: $W_{two}^{rc}(F) \le \widehat{W}(F)$ *for all* $F \in \mathcal{B}$.

Since $\widehat{W}(F) = +\infty$ for $F \in \mathcal{B}_3$, it suffices to investigate $F \in \mathcal{B}_1 \cup \mathcal{B}_2$. Let $F \in \mathcal{B}_1$, i.e., $F\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$, $F\vec{r}^{\perp} = R\begin{pmatrix} b \\ \frac{1}{a} \end{pmatrix}$, with $a \ge 1$ and $|F\vec{r}^{\perp}| = \sqrt{b^2 + \frac{1}{a^2}} \le d_-(a)$, i.e., $|b| \le |b_-(a)|$. Then we get as in the proof of Lemma 4.7 that for $F_{+,-} \in \mathbb{R}^{2\times 2}$, defined by $F_{+,-}\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$, $F_+\vec{r}^{\perp} = R\begin{pmatrix} b_-(a) \\ \frac{1}{a} \end{pmatrix}$ and $F_-\vec{r}^{\perp} = R\begin{pmatrix} -b_-(a) \\ \frac{1}{a} \end{pmatrix}$, we have $F_+ \in \mathcal{K}_1$ and $F_- \in \mathcal{K}_2$. Furthermore the matrices F_+ and F_- fulfill rank $(F_+ - F_-) \le 1$ and it exists a $\lambda \in [0,1]$ such that $F = \lambda F_+ + (1-\lambda)F_-$. As in step two we get that $W(F_+) = F(a, b_-(a)) = W(F_-) = f(a) = \widehat{W}(F)$. This implies

$$W_{two}^{rc}(F) \le \lambda W(F_+) + (1-\lambda)\, W(F_-) = \widehat{W}(F).$$

Let $F \in \mathcal{B}_2$, i.e., $F\vec{r} = R\begin{pmatrix} \frac{1}{d} \\ f \end{pmatrix}$, $F\vec{r}^{\perp} = R\begin{pmatrix} 0 \\ d \end{pmatrix}$, $d > 1$, $|f| \le |f_-(d)|$, $R \in SO(2)$, then we get with the same arguments as above that

$$W_{two}^{rc}(F) \le G\left(d, f_-(d)\right) = \widehat{W}(F).$$

Step 4: $\widehat{W}$ is polyconvex.

In the following we use that the concatenation $h \circ \widetilde{h} : \mathbb{R}^k \to \overline{\mathbb{R}}$ of a convex and non decreasing function $h : I \to \overline{\mathbb{R}}$, with a convex function $\widetilde{h} : \mathbb{R}^k \to I$ is again a convex function, where $I \subseteq \mathbb{R}$ is an interval. Notice that the function $\widehat{W}$ was defined by $\widehat{W}(F) = \max\left\{f\left(|F\vec{r}|\right), g\left(|F\vec{r}^{\perp}|\right)\right\} + I_{\mathcal{K}^{lc}}(F)$. The function $I_{\mathcal{K}^{lc}}$ is polyconvex since $I_{\mathcal{K}^{lc}}(F) = I_{\{1\}}\left(\det(F)\right)$. Thus we only have to show that f and g are nondecreasing and convex functions. In step one it was shown that g is monotone increasing and analogously we get that f is monotone increasing. Furthermore we get for $x > 1$ that

$$\begin{aligned} g''(x) &= \frac{2}{\cos^2(\alpha)} - \frac{2\sin^2(\alpha)}{\cos^2(\alpha)} \left[\frac{\sqrt{\sin^2(\alpha)\left(x^2 - \cos^2(\alpha)\right)} - x\frac{\sin^2(\alpha)x}{\sqrt{\sin^2(\alpha)(x^2-\cos^2(\alpha))}}}{\sin^2(\alpha)\left(x^2 - \cos^2(\alpha)\right)} \right] \\ &= \frac{2}{\cos^2(\alpha)} + \frac{2\sin^4(\alpha)}{\left(\sin^2(\alpha)\left(x^2 - \cos^2(\alpha)\right)\right)^{\frac{3}{2}}} \ge 0 \end{aligned}$$

and since g is continuous, increasing and constant on $(0,1)$, we get the convexity of g. The convexity of f, can be shown in the same way. Let $\vec{a} \in \mathbb{R}^2$, then the function $F \mapsto |F\vec{a}|$ is convex. This leads to the convexity of the functions $F \mapsto f(|F\vec{r}|)$ and $F \mapsto g(|F\vec{r}^{\perp}|)$. Since the maximum of two convex functions is convex, and the sum of a convex and a polyconvex function is polyconvex we get the polyconvexity of $\widehat{W}$. □

Remark 4.12. *The proof of Lemma 4.7 implies that the set $\mathcal{B}_4$, which was defined in Remark 4.11 fulfills $\mathcal{B}_4 \subseteq \mathcal{K}_1^{lc} \cup \mathcal{K}_2^{lc}$. Using the case of one slip-system, see [22], we obtain $W_{two}^{rc}(F) \leq |F\vec{e}_2|^2 - 1$ for $F \in \mathcal{K}_1^{lc}$ and $W_{two}^{rc}(G) \leq |G\vec{v}^{\perp}|^2 - 1$ for $G \in \mathcal{K}_2^{lc}$. Next, we will show that there are matrices $F \in \mathcal{K}_1^{lc} \cap \mathcal{B}_4$ and $G \in \mathcal{K}_2^{lc} \cap \mathcal{B}_4$, such that these inequalities are strict, if $\varphi \in (0,\pi) \setminus \left\{\frac{\pi}{2}\right\}$ and thus $\sin(\alpha) \neq \cos(\alpha)$, where $\alpha := \frac{\varphi}{2}$. Consider first the case $\sin(\alpha) > \cos(\alpha)$. Let $F_b \in \mathbb{R}^{2\times 2}$, with $F_b\vec{r} = R\begin{pmatrix} a \\ 0 \end{pmatrix}$, $F_b\vec{r}^{\perp} = R\begin{pmatrix} b \\ \frac{1}{a} \end{pmatrix}$, $R \in SO(2)$, $a \geq \sin(\alpha)$, $b \in \mathbb{R}$. Then we have $F_{b_{1,\pm}(a)} \in \mathcal{K}_1$ and $F_{-b_{1,\pm}(a)} \in \mathcal{K}_2$, see Lemma 4.7. Next, we get*

$$b_{1,-}(a) < 0 \Leftrightarrow \frac{a\cos(\alpha)}{\sin(\alpha)} < \frac{1}{\sin(\alpha)}\sqrt{1 - \frac{\sin^2(\alpha)}{a^2}} \Leftrightarrow h(a^2) := a^4\cos^2(\alpha) - a^2 + \sin^2(\alpha) < 0.$$

The quadratic function $h : \mathbb{R} \to \mathbb{R}$ is zero for

$$a_{\pm}^2 := \frac{1 \pm \sqrt{1 - 4\left(\sin^2(\alpha)\cos^2(\alpha)\right)}}{2\cos^2(\alpha)} = \frac{1 \pm \left(1 - 2\cos^2(\alpha)\right)}{2\cos^2(\alpha)},$$

since $\sin(\alpha) > \cos(\alpha)$ implies $\cos^2(\alpha) < \frac{1}{2}$. Since $a \geq \sin(\alpha) > 0$ we have $b_{1,-}(a) < 0$ if

$$1 < a < \frac{\sin(\alpha)}{\cos(\alpha)}.$$

It was shown in Equation (4.10) that $b \mapsto F(a,b) = |F_b\vec{e}_2|^2 - 1$ is a quadratic function with angular point $b_{SP} = -a\frac{\sin(\alpha)}{\cos(\alpha)} < 0$. This implies $F(a,b) > F(a,-b)$ for all $b > 0$. Choose an $a \in \mathbb{R}$ with $1 < a < \frac{\sin(\alpha)}{\cos(\alpha)}$. Since $b_{1,+}(a) \geq |b_{1,-}(a)| \geq 0$ and $b_{1,-}(a) < 0$ we get that $-b_{1,-}(a) \in (b_{1,-}(a), b_{1,+}(a)]$. This implies that the matrix $F_{-b_{1,-}(a)} \in \mathcal{K}_2$ belongs to $\mathcal{K}_1^{lc}$. Bringing all together we conclude

$$W_{two}^{rc}(F_{-b_{1,-}(a)}) \leq W_{two}(F_{-b_{1,-}(a)}) = F\left(a, b_{1,-}(a)\right) < F\left(a, -b_{1,-}(a)\right) = \left|F_{-b_{1,-}(a)}\vec{e}_2\right|^2 - 1,$$

where we have used Equation (4.11). For the case $\sin(\alpha) < \cos(\alpha)$ we consider the matrix $F_f \in \mathbb{R}^{2\times 2}$, with $F_f\vec{r} = Q\begin{pmatrix} \frac{1}{d} \\ f \end{pmatrix}$, $F_f\vec{r}^{\perp} = Q\begin{pmatrix} 0 \\ d \end{pmatrix}$, $Q \in SO(2)$, $f \in \mathbb{R}$, $d \geq \cos(\alpha)$. Replacing in the proof above a by d, $b_{1,\pm}(a)$ by $f_{1,\pm}(d)$ and $\sin(\alpha)$ by $\cos(\alpha)$, then we get $W_{two}^{rc}(F_{-f_{1,-}(d)}) < \left|F_{-f_{1,-}(d)}\vec{e}_2\right|^2 - 1$. Finally the case $G \in \mathcal{K}_2^{lc} \cap \mathcal{B}_4$ follows by a change in the sign of b and f.

4.4 Energy functionals

In this section we write down the mathematical framework used to analyze the variational problem (4.1) and show a simple scaling behavior of the energy.

Definition 4.13. *Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded, $p \in [1, \infty)$. Define the functional $I = I_{\varepsilon,\delta} : W^{1,1}\left(\Omega; \mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right) \to [0, \infty]$ by*

$$I\left(u, \gamma\right) = I_\Omega\left(u, \gamma, \varepsilon, \delta, p, q\right) := \fint_\Omega \frac{1}{\varepsilon} W_e(\nabla u(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2)) + |\gamma|^p \ d\lambda_2 + \frac{\delta}{|\Omega|} V_x(\gamma, \Omega),$$

if the integral exists and $I\left(u, \gamma\right) := \infty$ otherwise. Thereby $V_x\left(\gamma, \Omega\right)$ denotes the variation of γ along $\vec{e}_1$-direction, see Definition 3.27 and the elastic energy $W_e : \mathbb{R}^{2\times 2} \to [0, \infty]$ satisfies the hypotheses $(H1)-(H3)$, see page 42. Finally we define the energy functionals $E_{\varepsilon,\delta} = E_{\varepsilon,\delta;q,p;\Omega} : W^{1,1}\left(\Omega; \mathbb{R}^2\right) \to [0, \infty]$ by

$$E_{\varepsilon,\delta}[u] := \inf_{\gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)} I\left(u, \gamma\right) \tag{4.12}$$

and $E = E_{p;\Omega} : W^{1,1}\left(\Omega; \mathbb{R}^2\right) \to [0, \infty]$ for $p \geq 2$ by

$$E[u] := \frac{1}{|\Omega|} \int_\Omega W^{qc}_{rigid,p}\left(\nabla u\right) \ d\lambda_2 = \frac{1}{|\Omega|} \int_\Omega \left(|\nabla u \vec{e}_2|^2 - 1\right)^{\frac{p}{2}} \ d\lambda_2, \tag{4.13}$$

if the integral exists and $E[u] := \infty$ otherwise. In Chapter 7 we extend these functionals to $L^1\left(\Omega; \mathbb{R}^2\right)$ by $E[u] = E_{\varepsilon,\delta}[u] := \infty$ for $u \in L^1\left(\Omega; \mathbb{R}^2\right) \setminus W^{1,1}\left(\Omega; \mathbb{R}^2\right)$.

Scaling behavior

Consider now the scaling behavior of $I(u, \gamma)$. Set $\Omega_L := [0, L]^2$ and define for the functions $u \in W^{1,\infty}\left(\Omega_L; \mathbb{R}^2\right)$ and $\gamma \in BV\left(\Omega_L\right) \cap L^{\max\{p,q\}}\left(\Omega_L\right)$ the scaled versions by

$$\widetilde{u}(x) = \frac{1}{L} u(Lx) \text{ and } \widetilde{\gamma}(x) = \gamma(Lx),$$

for $x \in \Omega_1$. Then we have $\nabla \widetilde{u}(x) = \nabla u(Lx)$ and by Lemma 3.28 that

$$\begin{aligned} \frac{1}{|\Omega_1|} V_x\left(\widetilde{\gamma}, \Omega_1\right) &= \int_{(0,1)} V(\widetilde{\gamma}^{\vec{e}_1}_y, (0,1))\, dy = \int_{(0,1)} V\left(\gamma^{L\vec{e}_1}_{Ly}, (0,1)\right) \ dy \\ &= \frac{1}{L} \int_{(0,L)} V\left(\gamma^{L\vec{e}_1}_y, (0,1)\right) \ dy = \frac{1}{L} \int_{(0,L)} V\left(\gamma^{\vec{e}_1}_y, (0,L)\right) \ dy = L \frac{1}{|\Omega_L|} V_x\left(\gamma, \Omega_L\right). \end{aligned}$$

This leads to

$$I_{\Omega_L}(u, \gamma, \varepsilon, \delta) = I_{\Omega_1}(\widetilde{u}, \widetilde{\gamma}, \varepsilon, \frac{\delta}{L})$$

and using the scaling invariance of the function spaces one gets for $F \in \mathbb{R}^{2\times 2}$ that

$$\inf_{u \in W^{1,\infty}_F(\Omega_L; \mathbb{R}^2)} E_{\varepsilon,\delta;q,p;\Omega_L}[u] = \inf_{u \in W^{1,\infty}_F(\Omega_1; \mathbb{R}^2)} E_{\varepsilon,\frac{\delta}{L};q,p;\Omega_1}[u], \tag{4.14}$$

which increases for decreasing L. This illustrates the non-local behavior of our problem.

5 Upper bounds

In this chapter we show results for $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} > 1$ and thus especially for the case of no hardening, i.e. $p = 1$.

5.1 Upper bound for $\Omega = (0, L)^2$ - basic construction

In this section we show that the infimum of the energy $E_{\varepsilon,\delta}[u]$ over all configuration u, with boundary values $u = F^* \in \mathcal{M}^{(2)}$ on $\partial\Omega$ has an upper bound, which scales as $\sqrt{\frac{\delta}{\varepsilon}}$ for small δ. This scaling behavior is not optimal as one can see in Section 5.3. There we will show an upper bound, which scales as $\frac{\delta^{\frac{2}{3}}}{\varepsilon^{\frac{1}{3}}}$ for small δ and elastic growth exponent $q = 2$. Nonetheless, we give a complete proof of the scaling $\sqrt{\frac{\delta}{\varepsilon}}$, since the ideas of the construction were used in Section 5.3 again in a more complicated way and it helps to understand the computation of the variational part $V_x(\gamma, \Omega)$. In the proof we use the same laminate construction as in [42, Theorem 7.35], since this construction gives a minimizing sequence for the model without self-energy part, i.e., $\delta = 0$.

Lemma 5.1. *Let $F^* \in \mathcal{M}^{(2)}$, $\Omega := (0, L)^2$ and $\delta, \varepsilon, q, L > 0$. Suppose the elastic energy density $W_e : \mathbb{R}^{2\times 2} \to [0, \infty]$ satisfies hypotheses $(H1) - (H3)$ and additionally:*

(H4) W_e is continuous at the identity;

(H5) $\exists \mu > 0,\ c_3 > 0\ :\ W_e(G) < c_3$ for all $G \in B_\mu(F^*)$.

Then we have for all $p \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$, which is equivalent to $q < \frac{p}{p-1}$ and $p < \frac{q}{q-1}$ if $p > 1$ and $q > 1$, that

$$\inf_{u \in W^{1,\infty}(\Omega;\mathbb{R}^2)\,:\, u = F^* \text{ on } \partial\Omega} E_{\varepsilon,\delta}[u] \leq C\sqrt{\frac{\delta}{L\varepsilon}} + \frac{2\delta}{L},$$

with $C = C(F^*, c_1, c_2, c_3, \mu, N, M) > 0$ independent of $\delta, \varepsilon, L, p, q$.

Proof:
Using the scaling relation (4.14), it suffices to investigate the case $L = 1$. If $F^* = \mathbb{1}$ then choose $u\begin{pmatrix} x \\ y \end{pmatrix} = F^*\begin{pmatrix} x \\ y \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega$ and $\gamma = 0$. Then, we get $E_{\varepsilon,\delta}[u] = I(u, \gamma) = 0$.

Because of the frame indifference of W_e we can assume that $F^* = \begin{pmatrix} 1 & \sigma\gamma_0 \\ 0 & 1 \end{pmatrix}$, with $\sigma \in \{-1, 1\}$, and $\gamma_0 > 0$. Choose $\eta \in \left(q, \frac{p}{p-1}\right) \cap (1, \infty)$, where we define $\frac{p}{p-1} := \infty$ if $p = 1$. This is possible since $\frac{1}{p} + \frac{1}{q} > 1$.
The following construction was developed on the basis of [23, Theorem 1.1.], which can also be found in [42, Theorem 7.35]. We adopt largely the notation in these references, but use $\eta - 1$ instead of b. Define now $t > t_1 := \max\left\{1, \gamma_0^{\frac{1}{\eta}}\right\}$, $\lambda_t := \frac{\gamma_0}{t^\eta} \in (0,1)$, $\alpha_t := 1 + \gamma_0 \frac{t-1}{t^\eta} = 1 + (t-1)\lambda_t \overset{t\to\infty}{\longrightarrow} 1$,

$$F_t(0) := \frac{1}{\alpha_t} \begin{pmatrix} 1 & \sigma\gamma_0 (\alpha_t - 1) \\ 0 & \alpha_t \end{pmatrix} \text{ and } F_t(1) := \frac{1}{\alpha_t} \begin{pmatrix} t & \sigma t^\eta + \sigma\gamma_0 (\alpha_t - 1) \\ 0 & \alpha_t \end{pmatrix}.$$

Then we have $\operatorname{rank}(F_t(0) - F_t(1)) = 1$ and $F^* = F_t(\lambda_t) = (1 - \lambda_t) F_t(0) + \lambda_t F_t(1)$, where $F_t(s) := (1 - s) F_t(0) + s F_t(1)$. The matrices $F_t(0)$, $F_t(1)$ are the same as in [23] and can be obtained by a simple calculation from their original definition in [23, Theorem 1.1.]. By definition one gets $F_t(0) \overset{t\to\infty}{\longrightarrow} \mathbb{1}$. The definitions of $F_t(0)$ and $F_t(1)$ are illustrated in Figure 5.1, where $H := F_t(1) (\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2)$. It also provides an indication that the modulus of $F_t(1) (\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2)$ grows as t, for $\gamma \approx t^{\eta-1}$, which is the crucial idea of the following construction. The qualitative shape of the rank-one line $s \mapsto F_t(s)$ is displayed

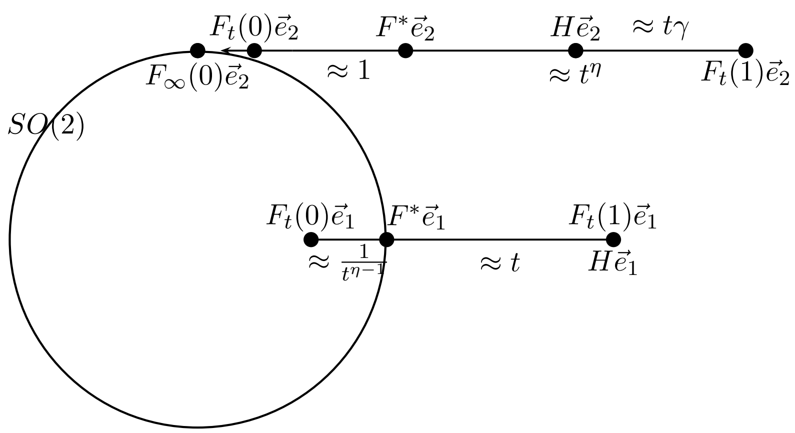

Figure 5.1:

in Figure 5.2. Moreover we have

$$F_t(1) - F_t(0) = \frac{1}{\alpha_t} \begin{pmatrix} t-1 & \sigma t^\eta \\ 0 & 0 \end{pmatrix}$$

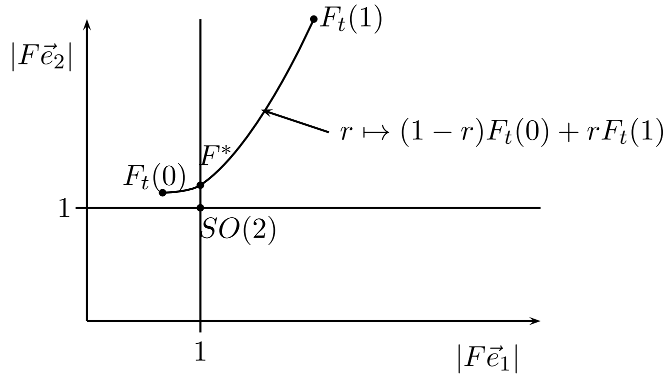

Figure 5.2:

and thus we get

$$(F_t(1) - F_t(0)) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in < \begin{pmatrix} \sigma t^\eta \\ 1-t \end{pmatrix} >_{\mathbb{R}} .$$

Next, we show that it suffices to investigate the case $\sigma = 1$. To accept this we make a distinction of cases.
Case $\sigma = 1$:
Define

$$\vec{a}_t^1 := \frac{1}{\sqrt{(t^\eta)^2 + (t-1)^2}} \begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \text{ and } \vec{a}_t^2 := \left(\vec{a}_t^1\right)^\perp = \frac{1}{\sqrt{(t^\eta)^2 + (t-1)^2}} \begin{pmatrix} -t^\eta \\ t-1 \end{pmatrix},$$

then we have $\vec{a}_t^1 \overset{t\to\infty}{\longrightarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{a}_t^2 \overset{t\to\infty}{\longrightarrow} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Thus we get

$$F_t(1) - F_t(0) = \frac{1}{\alpha_t} \left\| \begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \right\| \vec{e}_1 \otimes \vec{a}_t^1. \tag{5.1}$$

We require with the notation in Figure 5.3, where $a_t, b_t \geq 0$, that $a_t + b_t = 1$ and $\frac{b_t}{a_t} = \frac{t-1}{t^\eta}$. This leads to $a_t = \frac{1}{1+\frac{t-1}{t^\eta}} \in (0,1)$ and $c_t = \sqrt{a_t^2 + b_t^2} = a_t \sqrt{1 + \left(\frac{t-1}{t^\eta}\right)^2} \in (0,1)$, since $t > 1$. Furthermore we get $a_t,\ c_t \overset{t\to\infty}{\longrightarrow} 1$ and $b_t \overset{t\to\infty}{\longrightarrow} 0$. Let

$$\widetilde{\Omega}_1 = \widetilde{\Omega}_1^t := \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ \mu_1(1-\lambda_t)c_t\vec{a}_t^1 + \mu_2 c_t \vec{a}_t^2 \ : \ \mu_1, \mu_2 \in [0,1] \right\}$$

and

$$\widetilde{\Omega}_2 = \widetilde{\Omega}_2^t := \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ (1-\lambda_t)c_t\vec{a}_t^1 + \mu_1 \lambda_t c_t \vec{a}_t^1 + \mu_2 c_t \vec{a}_t^2 \ : \ \mu_1, \mu_2 \in [0,1] \right\},$$

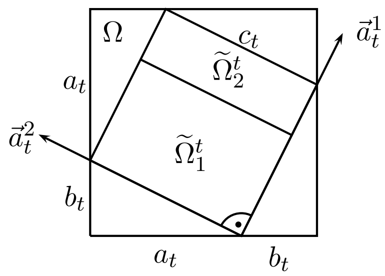

Figure 5.3: $\lambda_t = \frac{1}{3}$

refer to Figure 5.3.
Case $\sigma = -1$**:**
We define

$$\widehat{\Omega}_1^t := \begin{pmatrix} b_t \\ 0 \end{pmatrix} + \left\{ \mu_1(1-\lambda_t)c_t\vec{b}_1 + \mu_2 c_t \vec{b}_2 \;:\; \mu_1, \mu_2 \in [0,1] \right\}$$

and

$$\widehat{\Omega}_2^t := \begin{pmatrix} b_t \\ 0 \end{pmatrix} + \left\{ (1-\lambda_t)c_t\vec{b}_1 + \mu_1\lambda_t c_t\vec{b}_1 + \mu_2 c_t \vec{b}_2 \;:\; \mu_1, \mu_2 \in [0,1] \right\},$$

where $\vec{b}_1 := \frac{1}{\sqrt{(t^\eta)^2+(t-1)^2}} \begin{pmatrix} 1-t \\ t^\eta \end{pmatrix}$ and $\vec{b}_2 := \frac{1}{\sqrt{(t^\eta)^2+(t-1)^2}} \begin{pmatrix} t^\eta \\ t-1 \end{pmatrix}$, see Figure 5.4.
Hence the case given by $\sigma = -1$, is symmetric to the one with $\sigma = 1$. So we only have

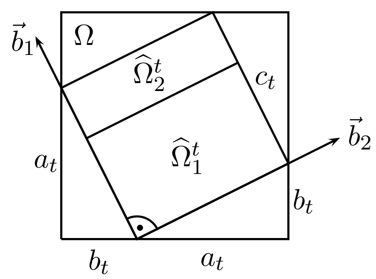

Figure 5.4: $\lambda_t = \frac{1}{3}$

to examine the one with $\sigma = 1$.
With help of this definitions we can begin now with the construction of a minimizing sequence $\{(u_t^n, \gamma_t^n)\}_{(n,t)\in\mathbb{N}\times\mathbb{R}} \subseteq W_{F^*}^{1,\infty}\left(\Omega;\mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right)$.

Step 1: In the first step we define a laminate l_t on $\widetilde{\Omega} := \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ with gradients $F_t(0)$ in $\widetilde{\Omega}_1$ and $F_t(1)$ in $\widetilde{\Omega}_2$. The laminate is chosen in such way that $(l_t - F^*)(x) = \begin{pmatrix} v(x) \\ 0 \end{pmatrix}$, for $x \in \widetilde{\Omega}$ and a non positive function v. This means that the sawtooth lies below the linear map F^*. The definition implies that this laminate is equal to F^* on the top and bottom part of the boundary $\partial\widetilde{\Omega}$, namely on $\begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ c_t \vec{a}_t^1 + \mu c_t \vec{a}_t^2 \ : \ \mu \in [0,1] \right\}$ and $\begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ \mu c_t \vec{a}_t^2 \ : \ \mu \in [0,1] \right\}$. Afterwards we scale this laminate in $\vec{a}_t^1$- direction by a factor $n \in \mathbb{N}$ and extend it then periodically to $\widetilde{\Omega}$. This gives a laminate with n saw teeth and gradients $F_t(0)$ and $F_t(1)$, which is denoted by $\overline{l}_t^n$ and fulfills the same boundary condition as l_t.
Define the laminate

$$l_t(p) = F^* p + c_t \chi_{\lambda_t} \left(\frac{(p - a_t \vec{e}_1) \cdot \vec{a}_t^1}{c_t} \right) \frac{1}{\alpha_t} \left\| \begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \right\| \vec{e}_1 \text{ for } p \in \mathbb{R}^2,$$

where $\chi_\lambda : \mathbb{R} \to \mathbb{R}$ is the one-periodic extension of the Lipschitz continuous function $\widetilde{\chi}_\lambda$ given by $\widetilde{\chi}_\lambda(0) = \widetilde{\chi}_\lambda(1) = 0$ and

$$\widetilde{\chi}_\lambda'(t) = \begin{cases} -\lambda \text{ for } t \in (0, 1-\lambda) \\ (1-\lambda) \text{ for } t \in (1-\lambda, 1) \end{cases} .$$

Thus we get for $z \in \mathbb{R}^2$ with $\frac{1}{c_t}(z - a_t \vec{e}_1) \cdot \vec{a}_t^1 \in \{k + \mu \in \mathbb{R} \ : \ k \in \mathbb{Z}, \ \mu \in (0, 1-\lambda_t)\}$ that

$$\nabla l_t(z) = F^* - \frac{\lambda_t}{\alpha_t} \left\| \begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \right\| \vec{e}_1 \otimes \vec{a}_t^1 = (1-\lambda_t) F_t(0) + \lambda_t F_t(1) - \lambda_t (F_t(1) - F_t(0)) = F_t(0),$$

where we have used Equation (5.1).
Similar we get for $z \in \mathbb{R}^2$ with $\frac{1}{c_t}(z - a_t \vec{e}_1) \cdot \vec{a}_t^1 \in \{k + \mu \in \mathbb{R} \ : \ k \in \mathbb{Z}, \ \mu \in (1-\lambda_t, 1)\}$ that $\nabla l_t(z) = F_t(1)$. In particular we have for $x \in \widetilde{\Omega}$ that

$$l_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} F^* \begin{pmatrix} a_t \\ 0 \end{pmatrix} + F_t(0) \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right] & \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{\Omega}_1 \\ F_t(1) \begin{pmatrix} x \\ y \end{pmatrix} + (F^* - F_t(1)) \left[\begin{pmatrix} a_t \\ 0 \end{pmatrix} + c_t \vec{a}_t^1 \right] & \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{\Omega}_2 \end{cases} .$$

Next, we get

$$(l_t(p) - F^* p) \cdot \vec{e}_1 \geq -c_t \lambda_t (1-\lambda_t) \frac{1}{\alpha_t} \left\| \begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \right\| \overset{t\to\infty}{\longrightarrow} -\gamma_0, \tag{5.2}$$

which means that the height of sawtooth converges to a non zero value. In the following we choose $n \in \mathbb{N}$. Then we can define $\Omega_n = \Omega_n^t := \Omega_{n,1} \cup \Omega_{n,2}$, where

$$\Omega_{n,i} = \Omega_{n,i}^t := \bigcup_{k=0}^{n-1} \left(\frac{1}{n} \left[\widetilde{\Omega}_i - \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right] + \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \frac{k}{n} c_t \vec{a}_t^2 \right) \text{ for } i \in \{1, 2\}.$$

Define the laminate

$$\begin{aligned} l_t^n : \Omega_n &\to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \frac{1}{n} (l_t - F^*) \left(n \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right] + \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right) + F^* \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Then we have $l_t^n \begin{pmatrix} x \\ y \end{pmatrix} = F^* \begin{pmatrix} x \\ y \end{pmatrix}$ for all

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ \frac{\mu_1 c_t}{n} \vec{a}_t^1 + \mu_2 c_t \vec{a}_t^2 \ : \ \mu_1 \in \{0, 1\}, \ \mu_2 \in [0, 1] \right\}$$

and l_t^n is Lipschitz continuous since this was true for l_t. Finally we have

$$\nabla l_t^n \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{n} \nabla (l_t - F^*) \left(n \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right] + \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right) n + F^*.$$

Thus we have for $\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega_{n,1}$ that $\nabla l_t^n \begin{pmatrix} x \\ y \end{pmatrix} = F_t(0)$ and for a $\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega_{n,2}$ that $\nabla l_t^n \begin{pmatrix} x \\ y \end{pmatrix} = F_t(1)$. Let $\overline{l}_t^n - F^*$ be the periodic extension from $l_t^n - F^*$ to $\widetilde{\Omega}$, this means $\left(\overline{l}_t^n - F^* \right) \left(z + \frac{k}{n} \vec{a}_t^1 \right) = \left(l_t^n - F^* \right) (z)$ for all $k \in [0, n-1] \cap \mathbb{Z}$ and all $z \in \Omega_n$.

Step 2: Next, we want to cut $\overline{l}_t^n$ on the right and left boundary of $\partial\widetilde{\Omega}$ such that the resulting function, denoted by u_t^n, fulfills $u_t^n = F^*$ on $\partial\widetilde{\Omega}$ and can therefore be linearly extended to Ω such that $u_t^n = F^*$ on $\partial\Omega$. The cutting is done with the help of affine functions G_1 and G_2. Next, we define the corresponding γ_t^n in the same way as it was done in [23, Theorem 1.1]. This ensures a small value of $\|\gamma_t^n\|_{L^p(\Omega)}^p$ and one gets a small elastic energy in the regions where $\nabla u_t^n \in \{F_t(0), F_t(1)\}$. This gives a sequence $\{(u_t^n, \gamma_t^n)\}_{(n,t) \in \mathbb{N} \times \mathbb{R}} \subseteq W^{1,\infty}\left(\Omega, \mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right)$, which leads to the asserted upper bound.

For $\beta \in (0, \infty)$ we define

$$\begin{aligned} G_1 = G_{1,\beta,t} : \Omega &\to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto F^* \begin{pmatrix} x \\ y \end{pmatrix} - \left[\begin{pmatrix} \beta \\ 0 \end{pmatrix} \otimes \vec{a}_t^2 \right] \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right], \end{aligned}$$

$$
\begin{aligned}
G_2 = G_{2,\beta,t} : \Omega &\rightarrow \mathbb{R}^2 \\
\begin{pmatrix} x \\ y \end{pmatrix} &\mapsto F^* \begin{pmatrix} x \\ y \end{pmatrix} + \left[\begin{pmatrix} \beta \\ 0 \end{pmatrix} \otimes \vec{a}_t^2 \right] \left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_t \\ 0 \end{pmatrix} \right] - \begin{pmatrix} c_t \beta \\ 0 \end{pmatrix}
\end{aligned}
$$

and denote the gradient of these affine mappings by $\widehat{G}_1$, respectively $\widehat{G}_2$. By definition we have $G_1 = F^*$ on the right boundary of $\partial\widetilde{\Omega}$, i.e., for

$$
\begin{pmatrix} x \\ y \end{pmatrix} \in \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ \mu_1 c_t \vec{a}_t^1 \ : \ \mu_1 \in [0,1] \right\}.
$$

Furthermore we get $G_2 = F^*$ on the left boundary of $\partial\widetilde{\Omega}$, i.e., for

$$
\begin{pmatrix} x \\ y \end{pmatrix} \in \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{ \mu_1 c_t \vec{a}_t^1 + c_t \vec{a}_t^2 \ : \ \mu_1 \in [0,1] \right\}.
$$

Next, we define

$$
\begin{aligned}
u_t^n : \Omega &\rightarrow \mathbb{R}^2 \\
\begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{cases} \begin{pmatrix} \max\left\{ \left(\bar{l}_t^n \begin{pmatrix} x \\ y \end{pmatrix} \right)_1, \left(G_1 \begin{pmatrix} x \\ y \end{pmatrix} \right)_1, \left(G_2 \begin{pmatrix} x \\ y \end{pmatrix} \right)_1 \right\} \\ y \end{pmatrix} & \text{for} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{\Omega} \\ F^* \begin{pmatrix} x \\ y \end{pmatrix} & \text{else} \end{cases},
\end{aligned}
$$

see Figure 5.5 for $n = 3$. Thereby one has to remark that this figure is painted for sake of clarity for $\lambda_t \approx 0.4$, which is misleading. Namely we show in step three that for big enough t and thus λ_t small enough, the slope of the line, with respect to the $\vec{e}_1$-direction, connecting the points z_1 and z_2, denoted in Figure 5.5, is in fact negative, refer to Equation (5.3). A more realistic picture for the case $n = 1$ is drawn in Figure 5.6. Finally we define

$$
\begin{aligned}
\gamma_t^n : \Omega &\rightarrow \mathbb{R} \\
\begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{cases} t^{\eta - 1} & \text{if} \quad \nabla u_t^n \begin{pmatrix} x \\ y \end{pmatrix} = F_t(1) \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

By construction we have $u_t^n \in W_{F^*}^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ and $\gamma_t^n \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)$, for all $n \in \mathbb{N}$ and $t > t_1$.

Step 3: Here we want to evaluate the self-energy part $V_x(\gamma_t^n, \Omega)$. Since γ_t^n is piecewise constant one only has to know the shape of the boundary of $\{\nabla u_t^n = F_t(1)\}$. We consider first the case $n = 1$. The computation implies then immediately that we get

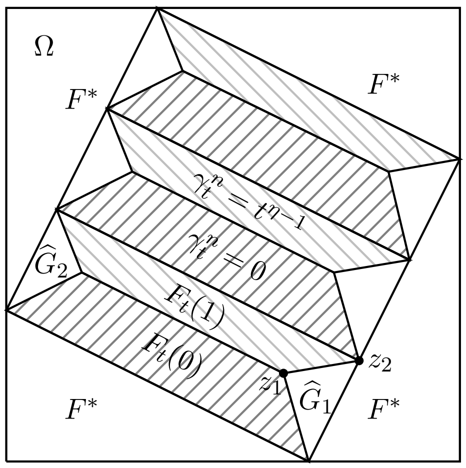

Figure 5.5: $\nabla u_t^n, \gamma_t^n$ for $n = 3,\ \lambda_t \approx 0.4$

$V_x(\gamma_t^n, \Omega) = n \cdot V_x(\gamma_t^1, \Omega)$ for all $n \in \mathbb{N}$. Therefore we only have to consider the affine sets $\{(l_t)_1 = (G_1)_1\} \cap \widetilde{\Omega}_2$ and $\{(l_t)_1 = (G_2)_1\} \cap \widetilde{\Omega}_2$. For $\begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{\Omega}_2$ we compute

$$[l_t - G_1]\begin{pmatrix} x \\ y \end{pmatrix} = [F_t(1) - F^*]\left[\begin{pmatrix} x - a_t \\ y \end{pmatrix} - c_t \vec{a}_t^1\right] + \left[\begin{pmatrix} \beta \\ 0 \end{pmatrix} \otimes \vec{a}_t^2\right]\left[\begin{pmatrix} x - a_t \\ y \end{pmatrix}\right]$$
$$= \begin{pmatrix} \frac{t}{\alpha_t} - 1 & \frac{t^\eta - \gamma_0}{\alpha_t} \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} x - a_t \\ y \end{pmatrix} - c_t \vec{a}_t^1\right] + \begin{pmatrix} \beta\left(\vec{a}_t^2\right)_1 & \beta\left(\vec{a}_t^2\right)_2 \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} x - a_t \\ y \end{pmatrix}\right].$$

Thus we have for $\begin{pmatrix} x \\ y \end{pmatrix} \in \widetilde{\Omega}_2$ that

$$[l_t - G_1]\begin{pmatrix} x \\ y \end{pmatrix} = 0$$
$$\Leftrightarrow \quad (x - a_t)\left(\frac{t}{\alpha_t} - 1 + \beta\left(\vec{a}_t^2\right)_1\right) - c_t\left(\vec{a}_t^1\right)_1\left(\frac{t}{\alpha_t} - 1\right) - c_t\left(\vec{a}_t^1\right)_2\left(\frac{t^\eta - \gamma_0}{\alpha_t}\right)$$
$$= y\left(\frac{\gamma_0 - t^\eta}{\alpha_t} - \beta\left(\vec{a}_t^2\right)_2\right)$$
$$\Leftrightarrow \quad y = \frac{\frac{t}{\alpha_t} - 1 + \beta\left(\vec{a}_t^2\right)_1}{\frac{\gamma_0 - t^\eta}{\alpha_t} - \beta\left(\vec{a}_t^2\right)_2}(x - a_t) - c_t\frac{\left(\vec{a}_t^1\right)_1\left(\frac{t}{\alpha_t} - 1\right) + \left(\vec{a}_t^1\right)_2\left(\frac{t^\eta - \gamma_0}{\alpha_t}\right)}{\frac{\gamma_0 - t^\eta}{\alpha_t} - \beta\left(\vec{a}_t^2\right)_2}. \tag{5.3}$$

This leads to $y \overset{t\to\infty}{\longrightarrow} 1$, uniformly in x. Because of $t^\eta > \gamma_0$, we get $\frac{\gamma_0 - t^\eta}{\alpha_t} - \beta\left(\vec{a}_t^2\right)_2 \leq 0$ and thus

$$\frac{\frac{t}{\alpha_t} - 1 + \beta\left(\vec{a}_t^2\right)_1}{\frac{\gamma_0 - t^\eta}{\alpha_t} - \beta\left(\vec{a}_t^2\right)_2} \leq 0 \Leftrightarrow t \geq \alpha_t\left(1 - \beta\left(\vec{a}_t^2\right)_1\right), \tag{5.4}$$

which is true for all $t \geq t_2 > t_1$, t_2 big enough. The same is true for arbitrary $n \in \mathbb{N}$. This means that the slope of the line connecting z_1 and z_2, denoted in Figure 5.5, is negative for t big enough and we get $\{(l_t)_1 = (G_1)_1\} \cap \widetilde{\Omega}_2 \subseteq \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;:\; y = h(x)\right\}$ for a decreasing affine map h. The same is obviously true for $\{(l_t)_1 = (G_2)_1\} \cap \widetilde{\Omega}_2$ and thus by construction also for $\{(l_t^n)_1 = (G_1)_1\} \cap \Omega_{n,2}$ and $\{(l_t^n)_1 = (G_2)_1\} \cap \Omega_{n,2}$.
To evaluate the variation $V_x(\gamma_t^1, \Omega)$ we have to compute the distance in the second variable of p_i and p_j with $i, j \in \{1,2,3,4\}$, where $p_1 = \begin{pmatrix} 1 \\ a_t \end{pmatrix}$, $p_4 = \begin{pmatrix} b_t \\ 1 \end{pmatrix}$, $\widetilde{p}_2$ (resp. $\widetilde{p}_3$) is the intersection point of the line segments $\left\{\begin{pmatrix} a_t \\ 0 \end{pmatrix} + \mu c \vec{a}_t^2 + (1-\lambda_t)\, c_t \vec{a}_t^1 \;:\; \mu \in [0,1]\right\}$ and $S_1 := \overline{\left\{\left(l_t^1\right)_1 = (G_1)_1\right\} \cap \Omega_{1,2}}$ (resp. $S_2 := \overline{\left\{\left(l_t^1\right)_1 = (G_2)_1\right\} \cap \Omega_{1,2}}$), see Figure 5.6. If now $(\widetilde{p}_3)_2 < (\widetilde{p}_2)_2$ then we define $p_2 = p_3$ as intersection of S_1 and S_2, otherwise

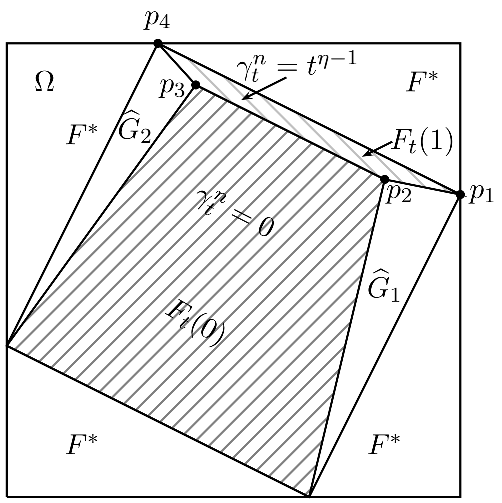

Figure 5.6: $\nabla u_t^1, \gamma_t^1,\ \lambda_t \approx 0.06,\ p_2 = \widetilde{p}_2,\ p_3 = \widetilde{p}_3$

we define $p_2 := \widetilde{p}_2$ and $p_3 := \widetilde{p}_3$, refer to Figure 5.6. Due to Equation (5.4) we have

$(p_2)_2 \geq (p_1)_2$, and since $(p_4)_2 \geq (p_3)_2$ by construction, we get

$$|(p_1 - p_2)_2| + |(p_2 - p_3)_2| + |(p_3 - p_4)_2| = |(p_1 - p_4)_2| = |b_t| \,.$$

Using Remark 3.30 we get the equation

$$V_x\left(\gamma_t^1, \Omega\right) = \int_{(0,1)} V\left((\gamma_t^1)_y^x, (0,1)\right) dy = 2 \cdot b_t \cdot t^{\eta - 1}.$$

For an arbitrary $n \in \mathbb{N}$ the open region $\{\nabla u_t^n = F_t(1)\}$ can be split into n connected components A_i, $i \in \{1, \ldots, n\}$, which are separated from each other by a strict positive distance, i.e., it exists a $\rho > 0$ such that $B_\rho(A_i) \cap B_\rho(A_j) \neq \emptyset$ for $i, j \in \{1, \ldots, n\}$, with $i \neq j$. Thereby we define for a set $A \subseteq \mathbb{R}^2$, the set $B_\rho(A) := \left\{x \in \mathbb{R}^2 \ : \ \mathrm{dist}\,(x, A) < \rho\right\}$. For each connected component we can derive as above that $V_x\left(\gamma_t^n, B_{\frac{\rho}{2}}(A_i)\right) = 2 \cdot b_t \cdot t^{\eta-1}$. Compound we get with Remark 3.30 that

$$V_x\left(\gamma_t^n, \Omega\right) = 2 \cdot n \cdot b_t \cdot t^{\eta - 1}. \tag{5.5}$$

Step 4: Next, we show that the energy part $\int_\Omega |\gamma_t^n|^p \ d\lambda_2$ vanishes for $t \to \infty$. Since we have $|\{z \in \Omega \ : \ \nabla u_t^n(z) = F_t(1)\}| \leq \lambda_t \, |\Omega|$ we can conclude

$$\int_\Omega |\gamma_t^n|^p \ d\lambda_2 = \int_{\{\nabla u_t^n = F_t(1)\}} t^{(\eta-1)p} \ d\lambda_2 \leq \lambda_t \, |\Omega| \, t^{(\eta-1)p} = \gamma_0 t^{(\eta-1)p - \eta} \overset{t \to \infty}{\longrightarrow} 0,$$

where we used for $p > 1$ that $\eta < \frac{p}{p-1}$ by definition and thus we get $(\eta - 1)\, p - \eta < 0$.
Step 5: In this step we want to show that the elastic energy part in the laminate region $\{\nabla u_t^n = F_t(1)\} \cup \{\nabla u_t^n = F_t(0)\}$ vanishes for $t \to \infty$. The special choose of γ_t^n implies that the modulus of the elastic deformation $F_t(1)\,(\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2)$ has only t-growth, which implies by $(H3)$ that the energy $W_e\,(F_t(1)\,(\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2))$ has t^q-growth. In formulas this means

$$\begin{aligned} W_e\left(F_t(1)\left(\mathbb{1} - t^{\eta-1}\vec{e}_1 \otimes \vec{e}_2\right)\right) &\leq c_1 \left\| \begin{pmatrix} \frac{t}{\alpha_t} & \gamma_0\left(1 - \frac{1}{\alpha_t}\right) \\ 0 & 1 \end{pmatrix} \right\|^q + c_2 \\ &\leq c_1 \left(t^2 + 1 + \gamma_0^2\right)^{\frac{q}{2}} + c_2 \leq 2^{\max\left\{0, \frac{q}{2} - 1\right\}} c_1 \left(t^q + \left(y_0^2 + 1\right)^{\frac{q}{2}}\right) + c_2 \leq C\left(t^q + 1\right), \end{aligned} \tag{5.6}$$

for $t \geq t_3$, where $t_3 \geq t_2 \in \mathbb{R}$ is chosen such that $\det\left(F_t(1)\,(\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2)\right) > M$ and $\|F_t(1)\,(\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2)\| \geq N$, for all $t \geq t_3$, where M, N are defined in $(H3)$. Thus we have

$$\int_{\{\nabla u_t^n = F_t(1)\}} W_e\left(F_t(1)\,(\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2)\right) \ d\lambda_2 \leq C\left(t^q + 1\right) \lambda_t \, |\Omega| = C\left(t^q + 1\right) \frac{\gamma_0}{t^\eta} \overset{t \to \infty}{\longrightarrow} 0,$$

where we used $\eta > \max\{q, 1\}$.
By construction of $F_t(0)$ we get $F_t(0) \to \mathbb{1}$ for $t \to \infty$. Due to hypotheses $(H4)$ and $(H2)$ the elastic energy W_e is continuous at the identity and $W_e(\mathbb{1}) = 0$. Thus we get $W_e(F_t(0)) \overset{t\to\infty}{\longrightarrow} 0$, which leads to

$$\int_{\{\nabla u_t^n = F_t(0)\}} W_e(F_t(0))\ d\lambda_2 \leq |\Omega|\,|W_e(F_t(0))| \overset{t\to\infty}{\longrightarrow} 0.$$

Step 6: In this step we show that the elastic energy part in the regions where $\nabla u_t^n \in \widehat{G}_1$ or $\nabla u_t^n \in \widehat{G}_2$ fulfills

$$\sum_{i=1}^{2} \int_{\{\nabla u_t^n = \widehat{G}_i\}} \frac{1}{\varepsilon} W_e\left(\widehat{G}_i\right)\ d\lambda_2 \leq \frac{C}{\varepsilon n}, \tag{5.7}$$

for a constant $C > 0$ independent of n, t and ε.

Since

$$\widehat{G}_{1,\beta,t} = \begin{pmatrix} 1-\beta\left(\vec{a}_t^2\right)_1 & \gamma_0 - \beta\left(\vec{a}_t^2\right)_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\beta\frac{t^\eta}{\sqrt{(t^\eta)^2+(t-1)^2}} & \gamma_0 - \beta\frac{t-1}{\sqrt{(t^\eta)^2+(t-1)^2}} \\ 0 & 1 \end{pmatrix},$$

we can choose $\beta = \beta_1$ big enough and independent of t, such that $\det\left(\widehat{G}_1\right) > M$ and $\left|\widehat{G}_1\right| \geq N$ for each $t \geq t_3$. Because of $(H3)$ we get $W_e\left(\widehat{G}_1\right) \leq C_{\widehat{G}_1} < \infty$, for $C_{\widehat{G}_1} = C_{\widehat{G}_1}(N, M, c_1, c_2, F^*)$. Since

$$\widehat{G}_2 = \widehat{G}_{2,\beta,t} = \begin{pmatrix} 1-\beta\frac{t^\eta}{\sqrt{(t^\eta)^2+(t-1)^2}} & \gamma_0 + \beta\frac{t-1}{\sqrt{(t^\eta)^2+(t-1)^2}} \\ 0 & 1 \end{pmatrix} \overset{t\to\infty}{\longrightarrow} F^* - \beta\vec{e}_1 \otimes \vec{e}_1,$$

we get for $\beta = \beta_2 < \mu$ that $W_e\left(\widehat{G}_{2,\beta_2}\right) \leq c_3 < \infty$, where μ and c_3 are defined by the hypothesis $(H5)$, for all $t \geq t_4$, where $t_4 \geq t_3 \in \mathbb{R}$ is chosen big enough. Because of Equation (5.2) we get

$$\left(\bar{l}_t^n\begin{pmatrix} x \\ y \end{pmatrix} - F^*\begin{pmatrix} x \\ y \end{pmatrix}\right)_1 \overset{t\to\infty}{\longrightarrow} -\frac{\gamma_0}{n}$$

and thus the left hand side is strictly bigger than $-\frac{2\gamma_0}{n}$ for $t \geq t_5$, with $t_5 \geq t_4$ big enough. Furthermore we have

$$\left(G_{1,\beta_1}\begin{pmatrix} x \\ y \end{pmatrix} - F^*\begin{pmatrix} x \\ y \end{pmatrix}\right)_1 \leq -\frac{2\gamma_0}{n}$$

for $\begin{pmatrix} x \\ y \end{pmatrix} \in \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \left\{\mu_1 c_t \vec{a}_t^1 + \mu_2 \vec{a}_t^2\ : \mu_1 \in [0,1],\ \mu_2 \in [\frac{2\gamma_0}{n\beta_1}, c_t]\right\}$. Thus we have for $t \geq t_5$ that $\left|\left\{\nabla u_t^n = \widehat{G}_{1,\beta_1}\right\}\right| \leq \frac{2\gamma_0}{n\beta_1}$ and analogously we get $\left|\left\{\nabla u_t^n = \widehat{G}_{2,\beta_2}\right\}\right| \leq \frac{2\gamma_0}{n\beta_2}$. Since we

can choose β_1 and β_2 independent of n and t this implies Equation (5.7).
Step 7: Bringing all together we will show that for each $\zeta > 0$ there exists a t such that $I(u_t^n, \gamma_t^n) \le \zeta + C\left(\frac{1}{\varepsilon n} + n\delta\right)$. This will prove the lemma, since this bound is optimal for $n \approx \frac{1}{\sqrt{\delta\varepsilon}}$.
We get using the statements in step four and five and Equations (5.5) and (5.7) that

$$\begin{aligned}
I(u_t^n, \gamma_t^n) &= \int_\Omega \frac{1}{\varepsilon} W_e\left(\nabla u_t^n (\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2)\right) + |\gamma_t^n|^p \; d\lambda_2 + \delta V(\gamma_t^n, \Omega) \\
&= \int_{\{\nabla u_t^n = F^*\}} \frac{1}{\varepsilon} W_e(F^*) \, d\lambda_2 + \int_{\{\nabla u_t^n = F_t(0)\}} \frac{1}{\varepsilon} W_e(F_t(0)) \, d\lambda_2 \\
&+ \sum_{i \in \{1,2\}} \int_{\{\nabla u_t^n = \widehat{G}_i\}} \frac{1}{\varepsilon} W_e\left(\widehat{G}_i\right) d\lambda_2 + \int_{\{\nabla u_t^n = F_t(1)\}} \frac{1}{\varepsilon} W_e\left(F_t(1)\left(\mathbb{1} - t^{\eta-1} \vec{e}_1 \otimes \vec{e}_2\right)\right) d\lambda_2 \\
&+ \int_{\{\nabla u_t^n = F_t(1)\}} t^{p(\eta-1)} \; d\lambda_2 + \delta V_x(\gamma_t^n, \Omega) \qquad (5.8)\\
&\le \frac{1}{\varepsilon} 2 a_t b_t W_e(F^*) + |\{\nabla u_t^n = F_t(0)\}| \frac{1}{\varepsilon} W_e(F_t(0)) \\
&+ \sum_{i=1}^{2} \left|\left\{\nabla u_t^n = \widehat{G}_i\right\}\right| \frac{1}{\varepsilon} W_e\left(\widehat{G}_i\right) + \frac{1}{\varepsilon} \lambda_t C(t^q + 1) + \lambda_t t^{p(\eta-1)} + 2n\delta b_t t^{\eta-1} \\
&\le \zeta + \frac{C}{\varepsilon n} + 2n\delta,
\end{aligned}$$

where the first inequality in the last row hold for all $t \ge t_6$ with $t_6 = t_6(\zeta) \ge t_5$ big enough. The derivative of the map $h : (0, \infty) \to \mathbb{R}$, $x \mapsto \frac{C}{\varepsilon x} + 2\delta x$ fulfills the equation

$$h'(x) = -\frac{C}{\varepsilon x^2} + 2\delta \ge 0 \Leftrightarrow x \ge \sqrt{\frac{C}{2\delta\varepsilon}}.$$

Using this and Equation (5.8) we get

$$I(u_t^n, \gamma_t^n) \le \zeta + \frac{C}{\varepsilon\left(\sqrt{\frac{C}{2\delta\varepsilon}} + 1\right)} + 2\left(\sqrt{\frac{C}{2\delta\varepsilon}} + 1\right)\delta \le \zeta + \sqrt{8C}\sqrt{\frac{\delta}{\varepsilon}} + 2\delta, \qquad (5.9)$$

for all $\zeta > 0$. This shows the asserted upper bound.

□

Remark 5.2. *The proof immediately implies that the statement of Lemma 5.1 is true if one exchanges the energy $\|\gamma\|_{L^p(\Omega)}^p$ by the more realistic one $\|\gamma\|_{L^1(\Omega)} + \|\gamma\|_{L^p(\Omega)}^p$.*

5.2 Upper bound for more general Ω - basic construction

Below we want to generalize Lemma 5.1, namely we prove it for more general Ω. In fact we can generalize it to well-scaling and line-$\vec{e}_1$ connected regions Ω, see Definition 5.3 and Definition 5.6.

Definition 5.3. *For an open set $\Omega \subseteq \mathbb{R}^2$ we define $\Omega_{\text{dist},\varepsilon} := \{z \in \Omega \ : \ \text{dist}(z, \partial\Omega) < \varepsilon\}$. If Ω is additionally bounded, then it is called well-scaling region, if there exists a constant $C > 0$ independent of ε, such that for all $\varepsilon > 0$*

$$|\Omega_{\text{dist},\varepsilon}| \leq C \cdot \varepsilon.$$

Remark 5.4. *Let $\Omega \subseteq \mathbb{R}^2$ open, bounded and convex, then Ω is a well-scaling region.*

In the following lemma we show that being a well scaling region is not a very strong restriction.

Lemma 5.5. *Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded, $\mathcal{H}^1(\partial\Omega) < \infty$ and $\partial\Omega = \gamma([0,1])$ for a continuous function $\gamma : [0,1] \to \mathbb{R}^2$. Then Ω is a well-scaling region.*

Proof:
W.l.o.g. it suffices to show that it exists a constant $C > 0$, such that we have for all $0 < \varepsilon \leq 1$ that $|\Omega_{\text{dist},\varepsilon}| \leq C\varepsilon$. In the following we will show that one can cover $\partial\Omega$ by $N_\varepsilon \leq \widehat{N} := \left\lfloor \frac{\mathcal{H}^1(\partial\Omega)}{\varepsilon} + 1 \right\rfloor$ balls $B_\varepsilon(x_i),\ x_i \in \partial\Omega,\ i \in \{1, \ldots, N_\varepsilon\}$ with radius $\varepsilon > 0$, where $\lfloor x \rfloor := \max\{k \in \mathbb{Z} \ : \ k \leq x\}$ denotes the floor function of x. This is shown by an iterative construction. Afterwards, we will show that $\Omega_{\text{dist},\varepsilon}$ is a subset of the union of the expanded balls $B_{2\varepsilon}(x_i),\ i \in \{1, \ldots, N_\varepsilon\}$. Since $N_\varepsilon \leq \frac{C}{\varepsilon}$, for a constant $C > 0$ independent of ε, one can conclude $|\Omega_{\text{dist},\varepsilon}| \leq 4\pi\varepsilon^2 N_\varepsilon \leq C\varepsilon$.
Let $0 < \varepsilon \leq 1$ and $t_0 := 0$. Choose now, if it exists, a minimal $t_1 > t_0$, with $t_1 \leq 1$, such that $\gamma(t_1) \in \partial B_\varepsilon(\gamma(t_0))$. If this is not possible then the continuity of γ implies $\gamma([t_0, 1]) \subseteq B_\varepsilon(\gamma(t_0))$. Thus we get $\Omega_{\text{dist},\varepsilon} \subseteq \Omega \subseteq B_\varepsilon(\gamma(t_0))$, which leads to the desired inequality, i.e., $|\Omega_{\text{dist},\varepsilon}| \leq \pi\varepsilon^2 \leq \pi\varepsilon$. If such a t_1 exists then we get $\gamma([t_0, t_1]) \subseteq \overline{B_\varepsilon(\gamma(t_0))}$ and $\mathcal{H}^1(\gamma([t_0, t_1])) \geq \varepsilon$.
Assume $0 = t_0 < t_1 < \ldots < t_N < 1$ are chosen such that $\gamma(t_{i+1}) \in \partial B_\varepsilon(\gamma(t_i))$ and $t_{i+1} > t_i$ minimal for all $i \in \{0, \ldots, N-1\}$. Thus we have $\gamma([t_i, t_{i+1}]) \subseteq \overline{B_\varepsilon(\gamma(t_i))}$ and $\mathcal{H}^1(\gamma([t_i, t_{i+1}])) \geq \varepsilon$ for all $i \in \{0, \ldots, N-1\}$.
Because of $\mathcal{H}^1(\gamma([t_i, t_{i+1}])) \geq \varepsilon$ for all $i \in \{0, \ldots, N-1\}$ we get

$$\mathcal{H}^1(\partial\Omega) \geq \mathcal{H}^1(\gamma([t_0, t_N])) \geq N\varepsilon.$$

Using this and $\widehat{N}\varepsilon = \left\lfloor \frac{\mathcal{H}^1(\partial\Omega)}{\varepsilon} + 1 \right\rfloor \varepsilon > \mathcal{H}^1(\partial\Omega)$ we get that it exists an $N_\varepsilon \in \mathbb{N}$ with $N_\varepsilon \leq \widehat{N}$ such that $\gamma([t_{N_\varepsilon}, 1]) \subseteq B_\varepsilon(\gamma(t_{N_\varepsilon}))$ and finally $\partial\Omega = \gamma([0,1]) \subseteq \bigcup\limits_{k=0}^{N_\varepsilon} \overline{B_\varepsilon(\gamma(t_k))}$.

This gives that for each $x \in \Omega_{\text{dist},\varepsilon}$ there exists a $\widetilde{x} \in \partial\Omega$ with $|x - \widetilde{x}| < \varepsilon$ and an $i \in \{0, \ldots, N_\varepsilon\}$ such that $\widetilde{x} \in \overline{B_\varepsilon(\gamma(t_i))}$. This implies $x \in B_{2\varepsilon}(\gamma(t_i))$ and thus we get $\Omega_{\text{dist},\varepsilon} \subseteq \bigcup_{k=0}^{N_\varepsilon} B_{2\varepsilon}(\gamma(t_k))$. Finally we get

$$|\Omega_{\text{dist},\varepsilon}| \leq (N_\varepsilon + 1)\,\pi\,(2\varepsilon)^2 \leq \left(\left\lfloor \frac{\mathcal{H}^1(\partial\Omega)}{\varepsilon} + 1 \right\rfloor + 1\right) 4\pi\varepsilon^2 \leq 4\pi\mathcal{H}^1(\partial\Omega)\,\varepsilon + 8\pi\varepsilon^2 \leq C\varepsilon,$$

where $C = 4\pi\left(\mathcal{H}^1(\partial\Omega) + 2\right)$. This finalizes the proof.

□

Definition 5.6. *Let $\vec{a} \in \mathbb{R}^2 \setminus \{0\}$. An open set $\Omega \subseteq \mathbb{R}^2$ is called a line-$\vec{a}$ connected region, if there exists an $\varepsilon > 0$ such that for all $\vec{b} \in \mathbb{R}^2$ with $\angle\left(\vec{a}, \vec{b}\right) < \varepsilon$ the inclusion*

$$\bigcup_{p\in\mathbb{R}^2} \left(\left(p + < \vec{b} >_{\mathbb{R}}\right) \cap \Omega\right)^c \subseteq \Omega$$

holds. Thereby S^c denotes the convex hull of a set $S \subseteq \mathbb{R}^2$, which is defined in this section as smallest convex set containing S, i.e., S^c need not be closed.

Remark 5.7. *Let $\Omega \subseteq \mathbb{R}^2$ open, bounded and convex, then Ω is a line-$\vec{a}$ connected region for each $\vec{a} \in \mathbb{R}^2 \setminus \{0\}$.*

Lemma 5.8. *Let $F^* \in \mathcal{M}^{(2)}$, Ω be an open, bounded, connected set, and a well-scaling and line-$\vec{e}_1$ connected region, $\varepsilon, \delta, q > 0$ and $p \geq 1$. Suppose the elastic energy density $W_e : \mathbb{R}^{2\times 2} \to [0, \infty]$ satisfies hypotheses $(H1) - (H3)$ and additionally:*

(H4) W_e is continuous at the identity;

(H6) there exist $c_3 > 0$ and $\beta > 0$ with $\max_{F\in\mathcal{K}_{Q,\beta}} W_e(F) \leq c_3 < \infty$, where

$$\mathcal{K}_Q = \mathcal{K}_{Q,\beta} := \left\{F^* + \beta\vec{e}_1 \otimes \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \,:\, d_1, d_2 \in [-1, 1]\right\}.$$

Then we have for all q, p with $\frac{1}{p} + \frac{1}{q} > 1$ that

$$\inf_{u\in W^{1,\infty}(\Omega):u=F^* \text{ on } \partial\Omega} E_{\varepsilon,\delta}[u] \leq C\left(\sqrt{\frac{\delta}{\varepsilon}} + \delta\right),$$

where $C = C(F^*, \beta, \Omega, c_3) > 0$ is independent of δ, ε, p, q, c_1 and c_2.

Remark 5.9. *If one uses the laminate construction, defined in the proof of Lemma 5.1, then there are at least two different cutting methods, which would lead to a proof of Lemma 5.8. On the one hand one can inscribe in* Ω *a disjoint union of squares* $B^{\infty}_{d_n}(z)$, $z \in \mathbb{R}^2$, *i.e.,* $U_n := \bigcup_{z\in\Gamma} B^{\infty}_{d_n}(z) \subseteq \Omega$, *with side length* $d_n > 0$ *fulfilling* $d_n \overset{n\to\infty}{\longrightarrow} 0$, *where* $\Gamma \subseteq d_n\mathbb{Z} \times d_n\mathbb{Z}$ *is chosen such that* U_n *is line-*$\vec{e}_1$*connected and* $|\Omega - U_n| \leq Cd_n$. *Then one can assume w.l.o.g. that* $\Omega = U_n$ *and by a similar construction as in the proof of Lemma 5.1, we can show Lemma 5.8.*
On the other hand one can use the dist*-function to achieve the boundary values. We use the second method, since it might help to show the result of Theorem 5.11 for more general* Ω*, which is obviously not possible if one uses the first method. Furthermore the ideas of the first method will be used in detail later on in the proof of Lemma 5.14.*

Proof of Lemma 5.8:
Because of the frame indifference we can assume that $F^* = \left(F^*_{ij}\right)_{i,j\in\{1,2\}} = \begin{pmatrix} 1 & \sigma\gamma_0 \\ 0 & 1 \end{pmatrix}$, with $\sigma \in \{-1,1\}$ and $\gamma_0 \geq 0$. If $\gamma_0 = 0$, then choose $u(z) = z$ for all $z \in \Omega$ and $\gamma = 0$, and obtain $E_{\varepsilon,\delta}[u] = I(u,\gamma) = 0$.
The following construction was developed on the basis of [23, Theorem 1.1], which was used already in Lemma 5.1. The notation used there is repeated below, for the sake of completeness. Choose $\eta \in \left(q, \frac{p}{p-1}\right) \cap (1,\infty)$ and let $t > \max\left\{1, \gamma_0^{\frac{1}{\eta}}\right\}$, $\lambda_t := \frac{\gamma_0}{t^\eta}$, $\alpha_t := 1 + (t-1)\lambda_t$,

$$F_t(0) := \frac{1}{\alpha_t}\begin{pmatrix} 1 & \sigma\gamma_0\,(\alpha_t - 1) \\ 0 & \alpha_t \end{pmatrix} \text{ and } F_t(1) := \frac{1}{\alpha_t}\begin{pmatrix} t & \sigma t^\eta + \sigma\gamma_0\,(\alpha_t - 1) \\ 0 & \alpha_t \end{pmatrix}.$$

Then we have $\operatorname{rank}\left(F_t(0) - F_t(1)\right) = 1$ and $F^* = (1-\lambda_t)\,F_t(0) + \lambda_t F_t(1)$, with $\lambda_t \in (0,1)$, refer to Figure 5.1. W.l.o.g. we can assume that β defined in $(H6)$ fulfills $\beta \leq 1$ and as in Lemma 5.1 it suffices to investigate the case $\sigma = 1$. More precisely, for a line-$\vec{e}_1$ connected region, the region mirrored on the $\vec{e}_2$-axis is also a line-$\vec{e}_1$ connected region. Moreover we have

$$F_t(1) - F_t(0) = \frac{1}{\alpha_t}\begin{pmatrix} t-1 & t^\eta \\ 0 & 0 \end{pmatrix} = \vec{e}_1 \otimes \frac{1}{\alpha_t}\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} = \vec{v} \otimes \vec{a}^1_t,$$

where $\vec{a}^1_t := \frac{1}{\sqrt{(t^\eta)^2 + (t-1)^2}}\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \overset{t\to\infty}{\longrightarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\| \vec{e}_1$.
Furthermore we define $\vec{a}^2_t := \vec{a}^{\perp}_1 = \frac{1}{\sqrt{(t^\eta)^2+(t-1)^2}}\begin{pmatrix} -t^\eta \\ t-1 \end{pmatrix}$. The above definitions are motivated in [23, Theorem 1.1] and in the proof of Lemma 5.1.
Choose points $\widehat{q} = \begin{pmatrix} \widehat{q}_1 \\ \widehat{q}_2 \end{pmatrix}$, $\widehat{p} = \begin{pmatrix} \widehat{p}_1 \\ \widehat{p}_2 \end{pmatrix} \in \partial\Omega$ with $\widehat{q}_2 = \max\left\{y \in \mathbb{R} \,:\, \exists x \in \mathbb{R}\ \begin{pmatrix} x \\ y \end{pmatrix} \in \partial\Omega\right\}$

and $\widehat{p}_2 = \min \left\{ y \in \mathbb{R} \ : \ \exists\, x \in \mathbb{R} \begin{pmatrix} x \\ y \end{pmatrix} \in \partial\Omega \right\}$, refer to Figure 5.7. Furthermore we choose $\widehat{q}_1$ minimal and $\widehat{p}_1$ maximal. After translation we can assume that $\widehat{p} = 0$. Next, we define for $t \in (0, \infty)$ the sets

$$\widehat{\Omega}_t := \left\{ \mu_1 \vec{a}_t^1 + \mu_2 \vec{a}_t^2 \ : \ \mu_1 \in (0, d_t), \mu_2 \in \mathbb{R} \right\} \text{ and } \Omega_t := \widehat{\Omega}_t \cap \Omega,$$

with $d_t := \operatorname{dist}\left(0, \widehat{q} + < \vec{a}_t^2 >_{\mathbb{R}}\right)$, refer to Figure 5.7.

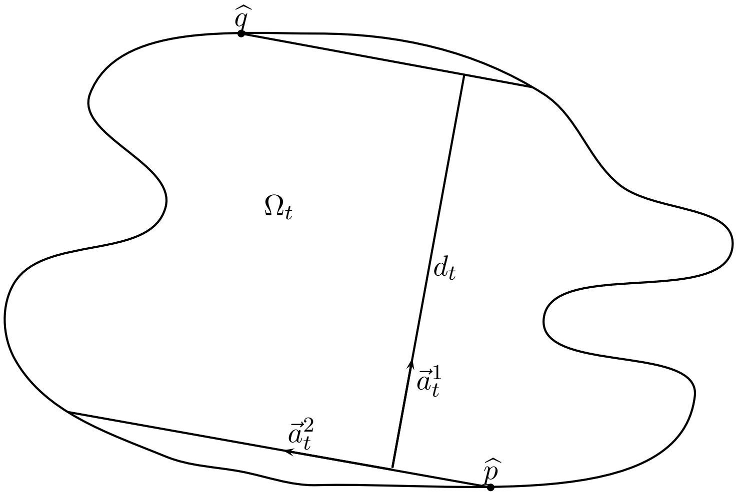

Figure 5.7:

Step 1: As in Lemma 5.1 we define a laminate l_t^n on Ω_t with n saw teeth and gradients changing between $F_t(1)$ and $F_t(0)$. In contrast as it was defined there, we choose it now in such way that the saw teeth lie above of the linear map F^*, see step one in the proof of Lemma 5.1 for a more precise definition. This has no further reason. Next, we want to cut the laminate near the boundary of Ω_t, to achieve a Lipschitz continuous function u_t^n on Ω_t with $u_t^n = F^*$ on $\partial\Omega_t$. In the remaining region $\Omega \setminus \Omega_t$ we choose $u_t^n = F^*$ and thus obtain a Lipschitz continuous function u_t^n with $u_t^n = F^*$ on $\partial\Omega$. Thereby one has to remark that $|\Omega \setminus \Omega_t| \overset{t\to\infty}{\longrightarrow} 0$ by construction and this implies, in consideration of $W_e(F^*) < \infty$, that the energy part of the region $\Omega \setminus \Omega_t$ vanishes for $t \to \infty$, if one chooses $\gamma = 0$ in $\Omega \setminus \Omega_t$. The cutting is done with the help of the functions $G = G_\beta : \Omega_t \to \mathbb{R}$ and $\widetilde{G}_\beta : \Omega_t \to \mathbb{R}^2$, where β is defined by hypothesis $(H6)$. Finally we choose again $\gamma_t^n = t^{\eta-1}$ if $\nabla u_t^n = F_t(1)$ and zero else. The resulting sequence $\{(u_t^n, \gamma_t^n)\}_{(n,t)\in\mathbb{N}\times\mathbb{R}} \subseteq W^{1,\infty}_{F^*}\left(\Omega; \mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right)$ gives the asserted upper bound.

Let $n \in \mathbb{N}$ and $h = h(t) := \frac{d_t}{n}$, then define on $\widehat{\Omega}_t$ the laminate $l_t^n \in W^{1,\infty}\left(\widehat{\Omega}_t; \mathbb{R}^2\right)$ by

$$l_t^n(z) = F^* z + h\chi_{\lambda_t}\left(\frac{\vec{a}_t^1 \cdot z}{h}\right)\vec{v},$$

where $\chi_\lambda : \mathbb{R} \to \mathbb{R}$, with $\lambda \in (0,1)$ is a Lipschitz continuous, one-periodic real-valued function, with $\chi_\lambda(0) = \chi_\lambda(1) = 0$, $\chi'_\lambda(s) = 1 - \lambda$ for $s \in (0,\lambda)$ and $\chi'_\lambda(s) = -\lambda$ for $s \in (\lambda, 1)$. Define for $k \in [0, n-1] \cap \mathbb{Z}$ the open set

$$\Omega_{k,n,t} = \Omega_{k,\frac{d_t}{h},t} := \left\{ z \in \mathbb{R}^2 \ : \ \left\lfloor \frac{\vec{a}_t^1 \cdot z}{h} \right\rfloor = k \wedge \frac{\vec{a}_t^1 \cdot z}{h} \notin \mathbb{N} \right\}.$$

In order to fulfill the boundary condition we define the auxiliary functions

$$\begin{aligned} G = G_\beta : \Omega_t &\to \mathbb{R} \\ z &\mapsto \begin{cases} (F^* z)_1 + \beta \operatorname{dist}(z, \partial\Omega \cap \Omega_{k,n,t}) & \text{if} \quad z \in \Omega_{k,n,t} \\ (F^* z)_1 & \text{otherwise} \end{cases} \end{aligned}$$

and $\widetilde{G} = \widetilde{G}_\beta : \Omega_t \to \mathbb{R}^2$, $\widetilde{G}_\beta(z) = \begin{pmatrix} G(z) \\ z_2 \end{pmatrix}$. Finally we set

$$\begin{aligned} u_t^n : \Omega &\to \mathbb{R}^2 \\ z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\mapsto \begin{cases} \begin{pmatrix} \min\{G(z), (l_t^n(z))_1\} \\ z_2 \end{pmatrix} & \text{if} \quad z \in \Omega_t \\ F^* z & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma_t^n : \Omega &\to \mathbb{R} \\ z &\mapsto \begin{cases} t^{\eta - 1} & \text{if} \quad \nabla u_t^n(z) = F_t(1) \text{ and } u_t^n(z) = l_t^n(z) \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

see Figure 5.8. As in the proof of Lemma 5.1 at the end of step two this figure is painted for sake of clarity for $\lambda_t \approx 0.44$, which is misleading. A more realistic figure is obtained if λ_t is very small, see Figure 5.6 in comparison to Figure 5.5.

Step 2: In this step we want to evaluate the energy $I(u_t^n, \gamma_t^n)$ without the variation part $\delta V_x(\gamma_t^n, \Omega)$. This is similar to the calculation of the analogous part in Lemma 5.1. We will show that, by definition of $\widetilde{G}$ and hypothesis $(H6)$, $W_e\left(\nabla\widetilde{G}(z)\right) \leq c_3 < \infty$ for almost every $z \in \Omega_t$. The difficulty is that we have to know the shape of the set $\left\{\nabla u_t^n = \nabla\widetilde{G}\right\}$, or at least of a larger set, in order to estimate the energy in this region. We will show that $\left\{\nabla u_t^n = \nabla\widetilde{G}\right\} \subseteq \Omega_{\text{dist},\frac{2d_t\gamma_0}{\beta n}}$. Since Ω is a well-scaling region one can

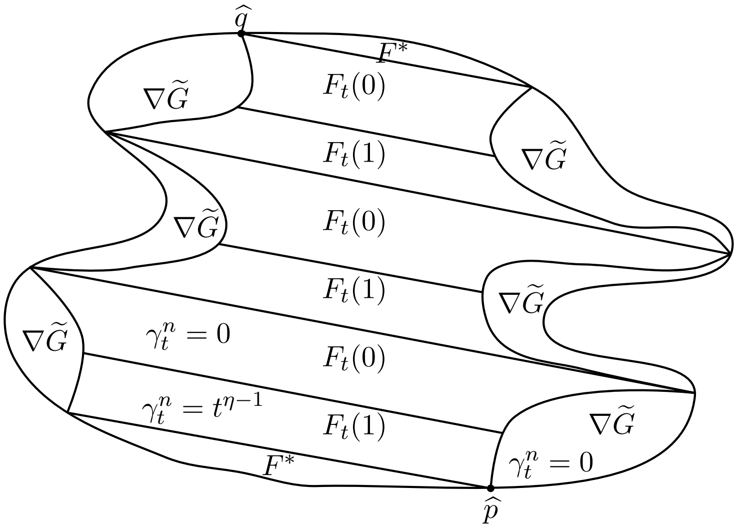

Figure 5.8: ∇u_t^n for $n = 3,\ \lambda_t \approx 0.44$

conclude $\left|\left\{\nabla u_t^n = \nabla\widetilde{G}\right\}\right| \leq \frac{C}{n}$, for a constant $C > 0$ independent of n.
Let M, N be defined in $(H3)$, which was defined at the beginning of Chapter 4. Since $\|F_t(1)\| \geq \det(F_t(1)) = \frac{t}{\alpha_t} \overset{t\to\infty}{\longrightarrow} \infty$, we get for t big enough that $\det(F_t(1)) > M$ and $\|F_t(1)\| \geq N$. As in Lemma 5.1 Equation (5.6) we get that

$$W_e\left(F_t(1)\left(\mathbb{1} - t^{\eta-1}\vec{e}_1 \otimes \vec{e}_2\right)\right) \leq C\left(t^q + 1\right),$$

with a constant $C > 0$ independent of t. We define now the sets

$$\mathcal{K} := \left\{\nabla\widetilde{G}\begin{pmatrix} x \\ y \end{pmatrix} \,:\, \begin{pmatrix} x \\ y \end{pmatrix} \in \Omega_t \text{ and } \nabla\widetilde{G}\begin{pmatrix} x \\ y \end{pmatrix} \text{ exists}\right\} \setminus \{F^*, F_t(0)\},$$

$U_t := \{z \in \Omega_t \,:\, (l_t^n(z))_1 \leq G(z)\}$, $V_t := \{\nabla u_t^n = F_t(1)\} \cap U_t$ and finally we denote

$Z_t := \{\nabla u_t^n \in \mathcal{K}\} \cap (\Omega_t \setminus U_t)$. Then we can deduce

$$
\begin{aligned}
&\varepsilon|\Omega| I\left(u_t^n, \gamma_t^n\right) - \varepsilon\delta V_x\left(\gamma_t^n, \Omega\right) = \int\limits_{\Omega} W_e(\nabla u_t^n(\mathbb{1} - \gamma_t^n \vec{e}_1 \otimes \vec{e}_2)) + \varepsilon \left|\gamma_t^n\right|^p \, d\lambda_2 \\
&= \int_{\{\nabla u_t^n = F^*\}} W_e\left(F^*\right) d\lambda_2 + \int_{\{\nabla u_t^n = F_t(0)\}} W_e\left(F_t(0)\right) d\lambda_2 + \int_{Z_t} W_e\left(\nabla u_t^n\right) d\lambda_2 \\
&+ \int_{V_t} W_e\left(F_t(1)\left(\mathbb{1} - t^{\eta-1}\vec{e}_1 \otimes \vec{e}_2\right)\right) \, d\lambda_2 + \varepsilon \int_{V_t} t^{p(\eta-1)} d\lambda_2 \\
&\leq \underbrace{\left|\{\nabla u_t^n = F^*\}\right|}_{\overset{t\to\infty}{\longrightarrow} 0} \underbrace{W_e\left(F^*\right)}_{<\infty} + \underbrace{\left|\{\nabla u_t^n = F_t(0)\}\right|}_{\leq|\Omega|<\infty} \underbrace{W_e(F_t(0))}_{\overset{t\to\infty}{\longrightarrow} 0} \\
&+ \left|Z_t\right| \sup_{\nabla u_t^n \in \mathcal{K}} W_e\left(\nabla u_t^n\right) + \left|V_t\right| C\left(t^q + 1\right) + \varepsilon \left|V_t\right| t^{p(\eta-1)}.
\end{aligned}
$$

Since Ω is bounded, we find $a < b$, $c < d$ with $a, b, c, d \in \mathbb{R}$, such that Ω is contained in $Q_t := \left\{\lambda_1 \vec{a}_t^1 + \lambda_2 \vec{a}_t^2 \ : \ \lambda_1 \in [a, b], \ \lambda_2 \in [c, d]\right\}$. Using this we can conclude that we have $|V_t| = |\{\nabla u_t^n = F_t(1)\} \cap U_t| \leq |Q_t| \lambda_t \leq C\lambda_t$, with a constant $C > 0$ independent of t. This implies $|V_t| C\left(t^q + 1\right) \leq C\frac{\gamma_0}{t^\eta}\left(t^q + 1\right) \overset{t\to\infty}{\longrightarrow} 0$ and $\left|V_t\right| t^{p(\eta-1)} \leq C\frac{\gamma_0}{t^\eta} t^{p(\eta-1)} \overset{t\to\infty}{\longrightarrow} 0$, because of $\eta < \frac{p}{p-1} \Leftrightarrow p(\eta - 1) < \eta$ and $\eta > q$, where we used again $\frac{p}{p-1} := \infty$ if $p = 1$. In the following we examine the set $\mathcal{K}$. Since the dist-function is 1-Lipschitz we have $\partial_x \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) \in [-1, 1]$ and $\partial_y \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) \in [-1, 1]$ for all $z \in \Omega$ for which the partial derivatives exists.
Thus we have

$$
\begin{aligned}
\nabla \widetilde{G}\left(z\right) &= \begin{pmatrix} F_{11}^* + \beta\partial_x \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) & F_{12}^* + \beta\partial_y \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) \\ 0 & 1 \end{pmatrix} \\
&= F^* + \begin{pmatrix} \beta \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \partial_x \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) \\ \partial_y \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) \end{pmatrix}
\end{aligned}
$$

for all $z \in \Omega_t$, for which $\nabla\widetilde{G}\left(z\right)$ exists, and finally

$$
\mathcal{K} \subseteq \mathcal{K}_Q = \left\{F^* + \beta\vec{e}_1 \otimes \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \ : \ d_1, d_2 \in [-1, 1]\right\},
$$

which implies using hypothesis $(H6)$ that $\sup_{\nabla u_t^n \in \mathcal{K}} W_e\left(\nabla u_t^n\right) \leq c_3 < \infty$. In the following we show $Z_t \subseteq \Omega_{\operatorname{dist}, \frac{2d_t\gamma_0}{\beta n}}$, for t big enough. Let $z \in Z_t$ then we have $G(z) < \left(l_t^n\left(z\right)\right)_1$, which leads to

$$
\begin{aligned}
\left(F^* z\right)_1 + \beta \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) &< \left(F^* z\right)_1 + h\chi_{\lambda_t}\left(\frac{\vec{a}_t^1 \cdot z}{h}\right) v_1 \\
\Leftrightarrow \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) &< \frac{1}{\beta\alpha_t} h\chi_{\lambda_t}\left(\frac{\vec{a}_t^1 \cdot z}{h}\right) \left\|\begin{pmatrix} t - 1 \\ t^\eta \end{pmatrix}\right\|
\end{aligned}
$$

and thus

$$\operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) < \frac{1}{\beta\alpha_t}\frac{d_t}{n}\left(1-\lambda_t\right)\lambda_t \left\| \begin{pmatrix} t-1 \\ t^{\eta} \end{pmatrix} \right\| \leq \frac{d_t\gamma_0}{\beta n}\sqrt{1+\left(\frac{t-1}{t^{\eta}}\right)^2} \leq \frac{2d_t\gamma_0}{\beta n},$$

for t big enough. Since Ω is a well-scaling region we get $|Z_t| \leq \frac{C}{n}$, with a constant $C = C(\beta, F^*, \Omega) > 0$ independent of n, t, ε and δ. Summarized we have shown that for each $\zeta > 0$, there exists a $t > 0$, so that

$$I\left(u_t^n, \gamma_t^n\right) \leq \zeta + \frac{C}{n\varepsilon} + \frac{\delta}{|\Omega|} V_x\left(\gamma_t^n, \Omega\right), \tag{5.10}$$

for a constant $C = C(\beta, F^*, c_3, \Omega) > 0$ independent of $n, t, p, q, c_1, c_2, \varepsilon$ and δ.
Step 3: Next, we examine the variation term $V_x\left(\gamma_t^n, \Omega\right)$. Therefore we have to know the shape of $\partial\left\{\gamma_t^n = t^{\eta-1}\right\}$. This boundary consists of the lines $kh\vec{a}_t^1 + < \vec{a}_t^2 >_{\mathbb{R}}$, with $k \in [0, n-1] \cap \mathbb{Z}$ restricted to $\overline{\Omega}$ and an additional part. The variation caused by this line segments is bounded from above by n times the maximal width, i.e., the maximal length of Ω in $\vec{e}_1$-direction. It suffices to investigate the additional part only restricted to $\Omega_{k,n,t}$, called $S_{\mathrm{dist},\lambda_t}$, for one $k \in [0, n-1] \cap \mathbb{Z}$. The crucial point is now that $S_{\mathrm{dist},\lambda_t}$ can be written as image of a decreasing function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, for an adequate interval $I \subseteq \mathbb{R}$ and a big enough t. Thereby we have to remind the reader that Figure 5.8 is misleading, since one can observe this effect for very big t and thus small λ_t only. Here we need essentially that Ω is line-$\vec{e}_1$ connected.
Define a family of lines parallel to $< \vec{a}_t^2 >_{\mathbb{R}}$, i.e., $S_{k,\nu} := \left\{\frac{k+\nu}{n} d_t \vec{a}_t^1 + \mu_2 \vec{a}_t^2 \ : \ \mu_2 \in \mathbb{R}\right\}$, for $k \in [0, n-1] \cap \mathbb{Z}$ and $\nu \in [0,1]$. For fixed $k \in [0, n-1] \cap \mathbb{Z}$ the line $S_{k,\nu}$ intersects $\partial\Omega$ in at least two points. Denote the point with maximal value in the second component by $w_{1,\nu} = \arg\sup_{v \in \partial\Omega \cap S_{k,\nu}} v_2 \in \partial\Omega$ and the point with the minimal value in the second component by $w_{2,\nu} = \arg\min_{v \in \partial\Omega \cap S_{k,\nu}} v_2 \in \partial\Omega$, refer to Figure 5.9. Since Ω is a line-$\vec{e}_1$ connected region, we have for t big enough that $S\left(w_{1,\nu}, w_{2,\nu}\right) \subseteq \Omega$ for all $\nu \in (0,1)$, where $S(x,y) := \{\lambda x + (1-\lambda) y \ : \ \lambda \in (0,1)\}$ for $x, y \in \mathbb{R}^2$. Choose now, if it exists, the biggest $\nu \in (0, \lambda_t]$ called $\nu_{\max}$ for which

$$\sup_{z \in \bar{S}\left(w_{1,\nu}, w_{2,\nu}\right)} \left\{G(z) - \left(F^* z\right)_1\right\} \geq \left(l_t^n\left(w_{1,\nu}\right) - F^* w_{1,\nu}\right)_1,$$

where we have to remark that $l_t^n(z) - F^* z = l_t^n\left(w_{1,\nu}\right) - F^* w_{1,\nu}$ for all $z \in \overline{S}\left(w_{1,\nu}, w_{2,\nu}\right)$. If such ν does not exist we get $\gamma_t^n = 0$ in $\Omega \cap \Omega_{k,n,t}$ and thus $V_x\left(\gamma_t^n, \Omega \cap \Omega_{k,n,t}\right) = 0$. Let $\nu_0 \in (0, \nu_{\max}]$, then we define

$$u_1 := \arg \sup_{v \in \bar{S}\left(w_{1,\nu_0}, w_{2,\nu_0}\right) \, : \, G(v) = \left(l_t^n(v)\right)_1} v_2 \ \text{ and } \ u_2 := \arg \inf_{v \in \bar{S}\left(w_{1,\nu_0}, w_{2,\nu_0}\right) \, : \, G(v) = \left(l_t^n(v)\right)_1} v_2.$$

We consider now the set

$$S_{\mathrm{dist},\nu_0} = S_{\mathrm{dist}} := \left\{ z \in \bigcup_{v \in (0,\nu_0)} S\left(w_{1,v}, w_{2,v}\right) \ : \ G(z) = \left(l_t^n(z)\right)_1 \right\},$$

refer to Figure 5.9. Using these definitions, we can try to understand the shape of S_{dist,ν_0}

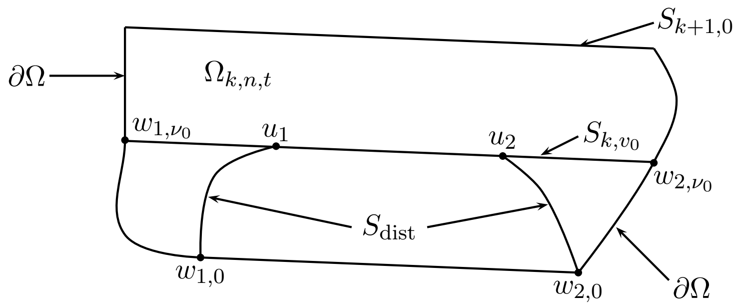

Figure 5.9:

for $\nu_0 \in (0, \nu_{\max}]$ and thus of $S_{\mathrm{dist},\lambda_t}$. This is done with the help of the following three assertions. Thereby we choose fixed $k \in [0, n-1] \cap \mathbb{Z}$ and $\nu_0 \in (0, \nu_{\max})$.

(i) For each $z \in \bar{S}\,(u_1, u_2) \subseteq S_{k,\nu_0}$ with $G(z) \geq (l_t^n(z))_1$ we have $S_{\mathrm{dist},\nu_0} \subset T_t(z)$, where

$$T_t(z) := \left\{ z + \beta_1 \vec{a}_t^1 + \beta_2 \vec{a}_t^2 \ : \ \beta_2 \in \mathbb{R},\ \beta_1 \in \left[-\sqrt{\frac{\beta^2 d_t^2}{C_*^2 - \beta^2 d_t^2}}\, |\beta_2| \,, 0 \right] \right\},$$

with $C_{*,t} = C_* := \frac{d_t}{\alpha_t} \left\| \begin{pmatrix} t-1 \\ t^\eta \end{pmatrix} \right\| (1 - \lambda_t) > \beta d_t$ for t big enough.

(ii) For each $z \in S\,(u_1, u_2)$ we have $G\,(z) > (l_t^n\,(z))_1$.

(iii) For each $\rho > 0$ we have

$$B_\rho\,(u_1) \cap S_{\mathrm{dist},\nu_0} \cap \left\{ u_1 + \beta_1 \vec{a}_1 + \lambda \vec{a}_t^2 \ : \ \beta_1 \in (-\infty, 0)\,,\ \lambda \in [0, \infty) \right\} \neq \emptyset$$
$$\text{and } B_\rho\,(u_2) \cap S_{\mathrm{dist},\nu_0} \cap \left\{ u_2 + \beta_1 \vec{a}_1 + \lambda \vec{a}_t^2 \ : \ \beta_1 \in (-\infty, 0)\,,\ \lambda \in (-\infty, 0] \right\} \neq \emptyset.$$

Assume we have proven this, we can conclude the following. If we consider now the sets $M := \{ z \in \Omega_{k,n,t} \ : \ \nabla u^n\,(z) = F_t(1) \ : \ l_t^n(z) \leq G(z) \}$ and $M_1 := (\partial M) \setminus S_{k,0}$, then our

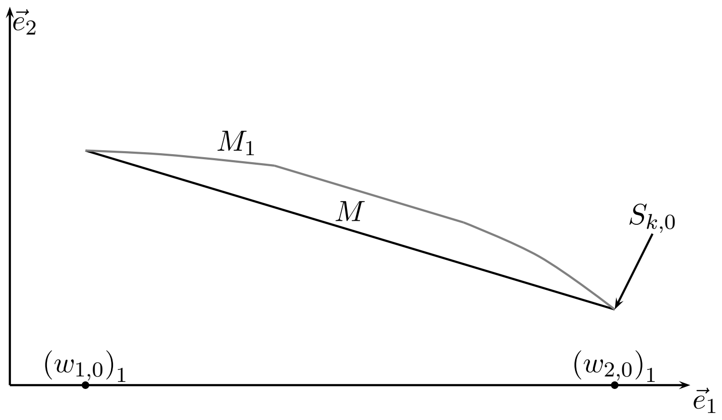

Figure 5.10:

problem looks like the case, sketched in Figure 5.10. The slope of the line $S_{k,0}$, with respect to the $-\vec{e}_1$-direction, evaluates to $\frac{t-1}{t^\eta}$. Then we have for t big enough that

$$\frac{t-1}{t^\eta} > \sqrt{\frac{\beta^2 d_t^2}{C_*^2 - \beta^2 d_t^2}} = h(t),$$

since $C^* \in \mathcal{O}\left(t^\eta\right)$ and thus $h(t) \in \mathcal{O}\left(\frac{1}{t^\eta}\right)$ for $t \to \infty$. Thus $M_1 \subseteq \mathbb{R}^2$ can be written as the image of a strictly monotonic decreasing function $f : \left((w_{1,0})_1, (w_{2,0})_1\right) \to \mathbb{R}$. Define for $U \subseteq \mathbb{R}^2$ the width of U by $\text{width}(U) := \sup\left\{|< x-y, \vec{e}_1 >|,\ x, y \in U\right\}$. This gives us

$$V_x\left(\gamma_t^n, \Omega_{k,n,t}\right) \leq \text{width}\left(\Omega_{k,n,t} \cap \Omega\right) \cdot \frac{t-1}{t^\eta} \cdot t^{\eta-1}.$$

Finally we get

$$\delta V_x\left(\gamma_t^n, \Omega\right) \leq \delta \cdot \text{width}(\Omega) \cdot \frac{t-1}{t^\eta} \cdot t^{\eta-1} \cdot 2n \leq C\delta n, \tag{5.11}$$

where $C = C(\Omega) > 0$ is independent of n, δ and t.
Next we show the assertions $(i) - (iii)$, we start with part (i).
Choose $z \in \overline{S}\left(u_1, u_2\right) \subseteq S_{k,\nu_0}$ with $G(z) \geq \left(l_t^n(z)\right)_1$ and let $\widetilde{z} = z + \beta_1 \vec{a}_t^1 + \beta_2 \vec{a}_t^2 \in S_{\text{dist},\nu_0}$. Then, there exists a $\widetilde{\nu} \in (0, \nu_0)$ so that $\widetilde{z} \in S\left(w_{1,\widetilde{\nu}}, w_{2,\widetilde{\nu}}\right) \subseteq S_{k,\widetilde{\nu}}$ and furthermore we

have $G(\widetilde{z}) = \left(l_t^n\left(\widetilde{z}\right)\right)_1$, which implies

$$\beta \operatorname{dist}\left(\widetilde{z}, \partial\Omega \cap \Omega_{k,n,t}\right) = h\chi_{\lambda_t}\left(\frac{\vec{a}_t^1 \cdot \widetilde{z}}{h}\right) v_1 = \frac{d_t}{n}\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\| \chi_{\lambda_t}\left(\widetilde{\nu}\right)$$
$$= \frac{d_t}{n}\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\| \left(1-\lambda_t\right)\widetilde{\nu} = \frac{C_*\widetilde{\nu}}{n}.$$

Because of

$$\operatorname{dist}\left(\widetilde{z}, \partial\Omega \cap \Omega_{k,n,t}\right) \geq \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) - \left|\widetilde{z} - z\right|$$

and $G(z) \geq \left(l_t^n\left(z\right)\right)_1$, which means

$$\beta \operatorname{dist}\left(z, \partial\Omega \cap \Omega_{k,n,t}\right) \geq \frac{d_t}{n}\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\| \left(1-\lambda_t\right)\nu_0,$$

we get that $\frac{1}{n}\nu_0 C_* \leq \frac{1}{n}\widetilde{\nu}C_* + \beta\left|\widetilde{z} - z\right|$ and thus we obtain

$$0 \geq \beta_1 = \left(\widetilde{z} - z\right)\cdot \vec{a}_t^1 = \frac{\widetilde{\nu} - \nu_0}{n} d_t \geq -\beta\frac{d_t}{C_*}\sqrt{\beta_1^2 + \beta_2^2}.$$

This leads to

$$\left(1 - \frac{\beta^2 d_t^2}{C_*^2}\right)\beta_1^2 \leq \frac{\beta^2 d_t^2}{C_*^2}\beta_2^2 \text{ and thus } \beta_1 \geq -\sqrt{\frac{\beta^2 d_t^2}{C_*^2 - \beta^2 d_t^2}}\left|\beta_2\right|.$$

Therefore we get

$$S_{\text{dist},\nu_0} \subset T_t(z).$$

Next we proceed with (ii).
W.l.o.g. we can choose $u_1 \neq u_2$. Assume there is a $z = u_3 \in S\left(u_1, u_2\right) \subseteq S_{k,\nu_0}$ with $G\left(u_3\right) \leq \left(l_t^n\left(u_3\right)\right)_1$, then it is sufficient to investigate the case $G\left(u_3\right) = \left(l_t^n\left(u_3\right)\right)_1$, since the first assertion of the following argumentation leads to an obvious conflict for the case $G\left(u_3\right) < \left(l_t^n\left(u_3\right)\right)_1$. Since $l_t^n(z) - F^*z = l_t^n\left(u_1\right) - F^*u_1$ for all $z \in S\left(u_1, u_2\right) \subseteq S_{k,\nu_0}$, we get

$$G\left(u_3\right) - \left(F^*u_3\right)_1 = l_t^n\left(u_3\right) - \left(F^*u_3\right)_1 = l_t^n\left(u_1\right) - \left(F^*u_1\right)_1 = G\left(u_1\right) - \left(F^*u_1\right)_1,$$

and thus there is an $\widetilde{u}_3 \in \partial\Omega \cap \bar{\Omega}_{k,n,t}$, with $\left\|\widetilde{u}_3 - u_3\right\| = \operatorname{dist}\left(u_1, \partial\Omega \cap \Omega_{k,n,t}\right) =: \operatorname{dist}_u$. Since $B_u := \bigcup_{i\in\{1,2,3\}} B_{\operatorname{dist}_u}\left(u_i\right) \subseteq \Omega$ and Ω is line-$\vec{e}_1$ connected region we can conclude that the convex hull of B_u, namely B_u^c, fulfills $B_u^c \subseteq \Omega$. This leads to $\widetilde{u}_3 - u_3 || \vec{a}_t^1$, w.l.o.g. we choose $\widetilde{u}_3 - u_3 = \lambda\vec{a}_t^1$ with $\lambda = \operatorname{dist}_u \in [0, \infty)$. Define $\widetilde{u}_1$ and $\widetilde{u}_2$ with $\operatorname{dist}_u = \left\|\widetilde{u}_1 - u_1\right\| = \left\|\widetilde{u}_2 - u_2\right\|$ and $\widetilde{u}_i - u_i = \lambda\vec{a}_t^1$ for $i \in \{1, 2\}$. Assume there is an $u_4 \in \bar{S}\left(\widetilde{u}_1, \widetilde{u}_2\right)$ with $u_4 \in \Omega$. Since Ω is open there exists a $\rho > 0$, such that $B_\rho\left(u_4\right) \subseteq \Omega$. For t big enough we can choose $u_5 \in B_\rho\left(u_4\right)$, $u_5 \notin B_u^c$ so that the line

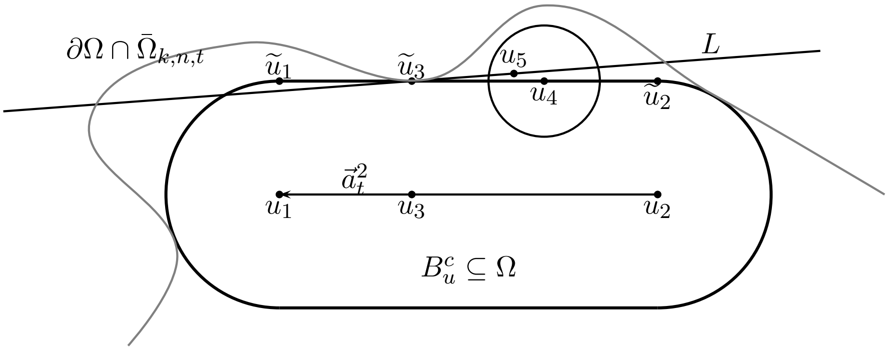

Figure 5.11:

$L := \{\widetilde{u}_3 + \lambda (u_5 - \widetilde{u}_3) \ : \ \lambda \in \mathbb{R}\}$ intersect the set B_u^c and $\angle (u_5 - \widetilde{u}_3, \vec{e}_1)$ is small enough, in contradiction to the assumption that Ω is line-$\vec{e}_1$ connected, see Figure 5.11. This leads to $\overline{S}(\widetilde{u}_1, \widetilde{u}_2) \subseteq \partial\Omega$. Define $u_6 \in A := \{\widetilde{u}_1 + \mu (\widetilde{u}_2 - \widetilde{u}_1) \ \mu \in \mathbb{R}\}$, so that

$$\overline{S}(u_6, \widetilde{u}_2) \subseteq \partial\Omega \tag{5.12}$$

and the second component of u_6 is maximal. With the same argumentation as above we get that there is no $z \in A \cap \Omega$ so that $\overline{S}(u_6, \widetilde{u}_2) \subseteq \overline{S}(z, \widetilde{u}_2)$. Thus, we are in the case $k = n-1$ and get by construction $u_6 = \widehat{q}$, which was defined independently of t. The Equation (5.12) depends continuously on t. To be more exact, the slope of $S(u_6, \widetilde{u}_2)$ in $\vec{e}_1$-direction, namely $\frac{1-t}{t^\eta}$, is continuous and strict monotone increasing for t big enough. Thus there is a $\rho > 0$ and a $v \in \mathbb{R}^2$ with $B_\rho(v) \subseteq \partial\Omega$. This gives a conflict to the definition of $\partial\Omega$. Therefore, for t big enough, there is no $u_3 \in S(u_1, u_2)$ with $Gu_3 \leq (l_t^n(u_3))_1$.
In the following we show part (iii).
We only prove the first formula, namely

$$B_\rho(u_1) \cap S_{\mathrm{dist},\nu_0} \cap \left\{u_1 + \beta_1\vec{a}_1 + \lambda\vec{a}_t^2 \ : \ \beta_1 \in (-\infty, 0), \ \lambda \in [0,\infty)\right\} \neq \emptyset \text{ for all } \rho > 0,$$

since the second follows analogously. W.l.o.g. we can choose $\rho > 0$ small enough, so that $B_\rho(u_1) \subseteq \Omega$, which is possible since $u_1 \in \Omega$. Furthermore, we have defined $u_1 \in S_{k,\nu_0}$, with $\nu_0 \in (0, \lambda_t]$. Let $0 < \kappa_1 < \rho$ small enough such that $u_1 - \kappa_1\vec{a}_t^1 \in S_{k,\widetilde{\nu}}$, with

$\widetilde{\nu} \in (0, \lambda_t]$. Using $\left(l_t^n\left(u_1\right)\right)_1 = G\left(u_1\right)$ and the definition of l_t^n, then we get

$$\left(l_t^n\left(u_1 - \kappa_1 \vec{a}_t^1\right)\right)_1 = \left(l_t^n\left(u_1\right)\right)_1 - \kappa_1\left(F^* \vec{a}_t^1\right)_1 - \left(1-\lambda_t\right)\kappa_1 v_1$$
$$= G\left(u_1\right) - \kappa_1\left(F^* \vec{a}_t^1\right)_1 - \left(1-\lambda_t\right)\kappa_1 v_1 < G\left(u_1\right) - \kappa_1\left(F^* \vec{a}_t^1\right)_1 - \beta\kappa_1 \leq G\left(u_1 - \kappa_1 \vec{a}_t^1\right),$$

for t big enough so that $\left(1-\lambda_t\right)v_1 > \beta$, where we have used in the last inequality that the dist-function is 1-Lipschitz. Next, there exists a $0 < \kappa_2 < \rho$ so that

$$\left(l_t^n\left(u_1 + \kappa_2 \vec{a}_t^2\right)\right)_1 = \left(l_t^n\left(u_1\right)\right)_1 + \kappa_2\left(F^* \vec{a}_t^2\right)_1 = G\left(u_1\right) + \kappa_2\left(F^* \vec{a}_t^2\right)_1 > G\left(u_1 + \kappa_2 \vec{a}_t^2\right),$$

because if not, namely $0 \leq G\left(u_1\right) - \left(F^*\left(u_1\right)\right)_1 \leq G\left(u_1 + \kappa_2 \vec{a}_t^2\right) - \left(F^*\left(u_1 + \kappa_2 \vec{a}_t^2\right)\right)_1$, then we get, since G is continuous and $G(z) - \left(F^*(z)\right)_1 = 0$ for $z \in \partial\Omega$, that there would exists a $\kappa_3 \geq \kappa_2$ with $G\left(u_1 + \kappa_3 \vec{a}_t^2\right) = G\left(u_1\right) + \kappa_3\left(F^* \vec{a}_t^2\right)_1 = \left(l_t^n\left(u_1 + \kappa_3 \vec{a}_t^2\right)\right)_1$ in contradiction to the definition of u_1. Since l_t^n and G are continuous there exists a $z \in S\left(u_1 - \kappa_1 \vec{a}_t^1, u_1 + \kappa_2 \vec{a}_t^2\right) \subseteq B_\rho\left(u_1\right)$ with $\left(l_t^n(z)\right)_1 = G(z)$. Because of $u_1 \in S_{k,\nu_0}$ and $u_1 - \kappa_1 \vec{a}_t^1 \in S_{k,\widetilde{\nu}}$, there exists a $\widehat{\nu} \in \mathbb{R}$, with $\widetilde{\nu} < \widehat{\nu} < \nu_0$ so that $z \in S_{k,\widehat{\nu}}$ and thus $z \in S_{\mathrm{dist},\nu_0}$. Let $\lambda \in (0,1)$ so that $z = \lambda\left(u_1 - \kappa_1 \vec{a}_t^1\right) + (1-\lambda)\left(u_1 + \kappa_2 \vec{a}_t^2\right)$, then we get the assertion, since $\kappa_1, \kappa_2 > 0$.
Thus we have proven the three sub-assertions $(i) - (iii)$. Now we can continue with the estimation of $I\left(u_t^n, \gamma_t^n\right)$. Combining now Equation (5.10) and (5.11), we get for a each $\zeta > 0$ that there exists a $t = t(\zeta)$, so that

$$I\left(u_t^n, \gamma_t^n\right) \leq \zeta + C\left(\frac{1}{\varepsilon n} + n\delta\right)$$

for all $n \in \mathbb{N}$, where $C > 0$ is independent of n, δ and ε. The function $g : (0,\infty) \to \mathbb{R}$, $x \mapsto \frac{1}{\varepsilon x} + x\delta$ is minimal for $x = \frac{1}{\sqrt{\varepsilon\delta}}$, decreasing on $(0, \frac{1}{\sqrt{\varepsilon\delta}})$, and increasing on $(\frac{1}{\sqrt{\varepsilon\delta}}, \infty)$. So it exists an $n_0 \in \mathbb{N}$ with

$$\frac{1}{\varepsilon n_0} + n_0\delta \leq \frac{1}{\varepsilon\left(\frac{1}{\sqrt{\varepsilon\delta}} + 1\right)} + \left(\frac{1}{\sqrt{\varepsilon\delta}} + 1\right)\delta \leq 2\sqrt{\frac{\delta}{\varepsilon}} + \delta.$$

Therefore we have

$$I\left(u_t^n, \gamma_t^n\right) \leq \zeta + C\left(\sqrt{\frac{\delta}{\varepsilon}} + \delta\right),$$

with $C = C\left(F^*, \beta, \Omega, c_3\right) > 0$ is independent of ζ, δ, ε, p, q, c_1 and c_2. This completes the proof.

□

Remark 5.10. *In order to be able to penalize or prohibit deformations with negative determinants, namely* $W_e(F) = \infty$ *for* $F \in \mathbb{R}^{2\times 2}$ *with* $\det(F) < 0$*, one has to choose the above* β *in the hypothesis* $(H6)$ *small enough.*

5.3 Upper bound for a rectangle Ω - branching construction

In the following we choose a rectangle $\Omega = [-L, L] \times [0, H]$, $L, H > 0$ and want to show that $\inf_{u \in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2)} E_{\varepsilon,\delta}[u]$ has an upper bound which scales like $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, for small $\varepsilon, \delta > 0$. Thereby we choose in this section an explicit formula for the elastic energy, namely $W_e(F) := \operatorname{dist}^q(F, SO(2))$. This scaling relation does not depend on p, since the construction is made in such way that $\int_\Omega |\gamma|^p \; d\lambda_2$ becomes small independently of δ and ε. Additionally the slip strain γ becomes large in a small part of the region. This effect is also called slip concentration, see [26, Section 3.6]. Considering $q = 2$ we obtain $\frac{\delta^{\frac{2}{3}}}{\varepsilon^{\frac{1}{3}}}$, which is the same scaling relation as in [38, Section 4.1], where α and ε defined therein equates to $\frac{1}{\varepsilon}$ and δ in our model, respectively. We give now a short sketch of the proof, the details can be found in the proof of Theorem 5.11. We adopt the definitions of $\vec{a}^1_t$, $\vec{a}^2_t$, λ_t, $F_t(0)$, $F_t(1)$, F^*, η from Lemma 5.1. As in Lemma 5.1 or Lemma 5.8 the construction in the second component can be chosen very easy by $u_2\begin{pmatrix} x \\ y \end{pmatrix} = y$, where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. In the first component we make a construction on a rotated rectangle $\Omega_t := R\left([-L_t, L_t] \times [0, H_t]\right) \subset \Omega$ with $|\Omega \setminus \Omega_t| \overset{t\to\infty}{\longrightarrow} 0$ and the rotation R is given by $R := -\vec{a}^2_t \otimes \vec{e}_1 + \vec{a}^1_t \otimes \vec{e}_2 \in SO(2)$. In the set $\Omega \setminus \Omega_t$ we choose $\gamma = 0$ and $u = F^*$. This leads to the energy $\frac{|\Omega \setminus \Omega_t|}{\varepsilon} \operatorname{dist}^q\left(F^*, SO(2)\right)$, which vanishes for $t \to \infty$. Finally, the construction on Ω_t relies on a branching-construction similar to the one used in [21, 26, 38, 39].

Let $d = d_z$ be the distance from $z \in \Omega_t$ in $\vec{a}^2_t$-direction to the boundary of Ω_t. For $d = L_t$ we define $u_1 = u_1(t)$ as laminate with gradients $\begin{pmatrix} 1 & 0 \end{pmatrix} F_t(0)$ and $\begin{pmatrix} 1 & 0 \end{pmatrix} F_t(1)$. Let $N \in \mathbb{N}$ be the number of saw teeth for $d = L_t$ and let $C > 0$ be a generic constant not depending on N, d, ε and δ. Next, we frequently use the sign $\sim$, which means that a generic constant C is hidden in the relation and additionally the relation is only true approximately.

The branching-construction is made in such way that for decreasing d, the slope in $\vec{a}^1_t$-direction is still jumping between $\begin{pmatrix} 1 & 0 \end{pmatrix} F_t(0)\vec{a}^1_t$ and $\begin{pmatrix} 1 & 0 \end{pmatrix} F_t(1)\vec{a}^1_t$, but one gets an increasing number of saw teeth $\sim d^\alpha N$, where $\alpha = \frac{\ln(2)}{\ln(\theta)} < 0$ for a $0 < \theta < 1$ to be chosen later. The price we have to pay is that we get $\nabla u(x) \sim F_t(0) \pm \frac{C}{N d d^\alpha} \vec{e}_1 \otimes \vec{a}^2_t$ or $\nabla u(x) = F_t(1)$ for $x \in \Omega_t$, if we want to ensure that u is continuous. Due to this one can show that $|\{\nabla u = F_t(1)\}| \sim \lambda_t \sim \frac{1}{t^\eta}$. Next, we define γ by $\gamma = t^{\eta - 1}$ if $\nabla u = F_t(1)$ and $\gamma = 0$ else. This gives as in Lemma 5.1 that the energy parts $\int_\Omega |\gamma|^p \; d\lambda_2$ and $\int_{\{\nabla u = F_t(1)\}} \frac{1}{\varepsilon} \operatorname{dist}^q\left(\nabla u(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2), SO(2)\right) \; d\lambda_2$ vanish for $t \to \infty$. Thus we get for

$\nabla u(x) \neq F_t(1)$, namely $\nabla u(x) \sim F_t(0) + \frac{C}{Ndd^\alpha}\vec{e}_1 \otimes \vec{a}_t^2$, that

$$\int_{\{\nabla u \neq F_t(1)\}} \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla u(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2), SO(2)\right) \, d\lambda_2 \sim \int_0^C \frac{1}{\varepsilon} \left(\frac{1}{Nxx^\alpha}\right)^q \, dx$$
$$\underbrace{\sim}_{\text{for } 1-q(1+\alpha)\neq 0} \frac{1}{\varepsilon N^q} x^{1-q(1+\alpha)}|_{x=0}^{x=C} \underbrace{\sim}_{\text{for } \theta > \frac{1}{2^{\frac{q}{q-1}}}} \frac{1}{\varepsilon N^q},$$

since $F_t(0) \to \mathbb{1}$ for $t \to \infty$ and $1 - q(1+\alpha) > 0 \Leftrightarrow \frac{\ln(2)}{\ln(\theta)} = \alpha < \frac{1}{q} - 1 \Leftrightarrow \theta > \frac{1}{2^{\frac{q}{q-1}}}$. Unfortunately the u constructed in this way is not Lipschitz continuous, for small d. Therefore one can use the above construction only in $D = D_{d_{k(t)}} := \left\{z \, : \, d_z \geq d_{k(t)}\right\}$ for a t dependent lower bound $d_{k(t)} := \theta^{k(t)}$, with $k(t) \overset{t\to\infty}{\longrightarrow} \infty$. In the remaining region $D^C := \Omega_t \setminus D$ one uses linear interpolation to fulfill the boundary condition. The definition of γ remains the same. One can compute for this construction

$$\int_{D^C} \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla u(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2), SO(2)\right) \, d\lambda_2 \sim \frac{1}{\varepsilon} \left(\frac{1}{d_{k(t)} d_{k(t)}^\alpha N}\right)^q d_{k(t)},$$

which vanishes for $t \to \infty$, if we choose again $\theta > \frac{1}{2^{\frac{q}{q-1}}}$, i.e., $1 - q(1+\alpha) > 0$, and if we choose $d_{k(t)}$ such that $d_{k(t)} \overset{t\to\infty}{\longrightarrow} 0$. The $\int_{D^C} |\gamma|^p \, dx$ part vanishes for $t \to \infty$. The $V_x\left(\gamma, D^C\right)$ part vanishes analogously, if we take for technical reasons $k(t) \sim -\frac{\ln(t)}{\ln(2\theta)}$, which is the same relation needed for the $V_x(\gamma, D)$ part. The construction in D is made in such a way that the jump set of γ is a union of finitely many affine parts and their slope is $\sim \frac{1}{t^{\eta-1}}$ or $\sim \frac{1}{t^{\eta-1}} + \frac{C}{t^\eta d^\alpha dN}$. Thereby we need again for technical reasons $k(t) \sim -\frac{\ln(t)}{\ln(2\theta)}$. This choice of $k(t)$ implies

$$d_{k(t)} d_{k(t)}^\alpha = \theta^{k(t)} 2^{k(t)} \sim \exp\left(-\ln(t)\right) = \frac{1}{t}$$

and thus $\frac{C}{t^\eta d_{k(t)}^\alpha d_{k(t)} N} \sim \frac{C}{t^{\eta-1} N}$. Then we get

$$\delta V_x(\gamma, D) \sim \delta \int_{d_{k(t)}}^C \underbrace{x^\alpha N}_{= \text{ number of jumps}} \underbrace{\left(\frac{1}{t^{\eta-1}} + \frac{C}{t^\eta x^\alpha x N}\right)}_{= \text{ slope of jump line}} \underbrace{t^{\eta-1}}_{= \text{ jump height}}$$
$$\sim \delta N x^{\alpha+1}|_{x=d_{k(t)}}^{x=C} + \delta \frac{C}{t} \ln(x)|_{x=d_{k(t)}}^{x=C} \underbrace{\sim}_{\text{for } 0<\theta<\frac{1}{2}} N\delta.$$

Here we need $k(t) \leq t$ to ensure that $\ln(d_{k(t)}) \sim k(t) \leq t$ and we need, $\alpha + 1 > 0 \Leftrightarrow -\frac{\ln(2)}{\ln(\theta)} < 1 \underbrace{\Leftrightarrow}_{\theta<1} \theta < \frac{1}{2}$. Summarized we need $\frac{1}{2^{\frac{q}{q-1}}} < \theta < \frac{1}{2}$ to show the desired upper bound U of $\inf_{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2)} E_{\varepsilon,\delta}[u]$, which scales like

$$U \sim \frac{1}{\varepsilon N^q} + N\delta \sim \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}},$$

where we have chosen $N \sim \frac{1}{(\varepsilon\delta)^{\frac{1}{q+1}}} \to \infty$ for $\varepsilon,\ \delta \to 0$. Thus one can show the following.

Theorem 5.11. *Let $\Omega := [-L, L] \times [0, H] \subset \mathbb{R}^2,\ L > 0,\ H > 0$ and $p, q \geq 1,$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $F^* \in \mathcal{M}^{(2)}$, $W_e(F) := \operatorname{dist}^q(F, SO(2))$, for $F \in \mathbb{R}^{2\times 2}$. Then we have for $\varepsilon, \delta > 0$ that*

$$\inf_{u\in W^{1,\infty}(\Omega;\mathbb{R}^2)\,:\, u=F^* \text{ on } \partial\Omega} E_{\varepsilon,\delta}[u] \leq C\left(\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}} L^{\frac{q}{q+1}}} + \frac{\delta}{H}\right),$$

where $C = C(F^*, q) > 0$ is independent of $\varepsilon, \delta, p, L, H$.

Proof:
Let $\vec{a}_t^1,\ \vec{a}_t^2,\ \alpha_t,\ F_t(0),\ F_t(1),\ \lambda_t$ be as in Lemma 5.1. Choose again $\eta \in \left(q, \frac{p}{p-1}\right)$ for $p > 1$ and $\eta \in (q, \infty)$ for $p = 1$, which is possible since $\frac{1}{p} + \frac{1}{q} > 1$. Because of the frame-indifference of $\operatorname{dist}^q\left(.\cdot(\mathbb{1} - \gamma\vec{e}_1 \otimes \vec{e}_2), SO(2)\right)$ we can assume that $F^* = \begin{pmatrix} 1 & \sigma\gamma_0 \\ 0 & 1\end{pmatrix}$ with $\gamma_0 > 0$ and $\sigma \in \{1, -1\}$. The case $\gamma_0 = 0$ is excluded, since in this case one can choose $\gamma = 0$ and $u(x) = x$ and get $E_{\varepsilon,\delta}[u] = 0$. Due to the symmetry of Ω we can choose $\sigma = 1$. For $a, b, c, d \in \mathbb{R}$ we define $[a, b]_{a_i} := [a, b]\vec{a}_t^i = \left\{\lambda\vec{a}_t^i : \lambda \in [a, b]\right\}$ for $i \in \{1, 2\}$ and $[a, b]_{a_1} \times [c, d]_{a_2} := \left\{\lambda\vec{a}_t^1 + \mu\vec{a}_t^2\ :\ \lambda \in [a, b],\ \mu \in [c, d]\right\}$. Analogously, we define this for open or half-open intervals. Choose a sequence of closed rectangles $\{\Omega_t\}_{t>0}$, whereupon $\Omega_t = p_t + [0, H_t]_{a_1} \times [-L_t, L_t]_{a_2}$ with $p_t \in \mathbb{R}^2$, $\Omega_t \subseteq \Omega$ and $\{L_t\}_{t>0}$, $\{H_t\}_{t>0}$ are monotonically increasing sequences with $L_t \to L$ and $H_t \to H$ for $t \to \infty$. This definition implies $|\Omega \setminus \Omega_t| \overset{t\to\infty}{\longrightarrow} 0$. W.l.o.g. we can translate our problem such that $p_t = 0$. Let $\theta \in (0, 1)$ to be chosen later, where we will see that it suffices to take an arbitrary $\theta < \frac{1}{2}$, which fulfills additionally $\theta > \frac{1}{2^{\frac{q}{q-1}}}$ in the case $q > 1$. Next, we define the matrices $\widehat{G}_0 := F_t(0) - F^* = \begin{pmatrix} \frac{1}{\alpha_t} - 1 & -\frac{\gamma_0}{\alpha_t} \\ 0 & 0\end{pmatrix}$ and $\widehat{G}_1 := F_t(1) - F^* = \begin{pmatrix} \frac{t}{\alpha_t} - 1 & \frac{t^\eta}{\alpha_t} - \frac{\gamma_0}{\alpha_t} \\ 0 & 0\end{pmatrix}$, and we denote the first row of $\widehat{G}_i,\ i \in \{0, 1\}$ by G_i. Then we get

$$G_0 = \begin{pmatrix} \frac{1}{\alpha_t} - 1 & -\frac{\gamma_0}{\alpha_t}\end{pmatrix} = \begin{pmatrix} -\frac{\lambda_t}{\alpha_t}(t-1) & -\frac{\lambda_t}{\alpha_t}t^\eta\end{pmatrix} = -\frac{\lambda_t}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta\end{pmatrix}\right\| \otimes \vec{a}_t^1 := m_0 \otimes \vec{a}_t^1,$$

which converges to $-\gamma_0 \otimes \vec{e}_2$ for $t \to \infty$ and

$$G_1 = \begin{pmatrix} \frac{t}{\alpha_t} - 1 & \frac{t^\eta}{\alpha_t} - \frac{\gamma_0}{\alpha_t}\end{pmatrix} = \begin{pmatrix} \frac{1-\lambda_t}{\alpha_t}(t-1) & \frac{1-\lambda_t}{\alpha_t}t^\eta\end{pmatrix} = \frac{1-\lambda_t}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta\end{pmatrix}\right\| \otimes \vec{a}_t^1 := m_1 \otimes \vec{a}_t^1.$$

This definition implies

$$m_1\lambda_t = m_0\left(\lambda_t - 1\right). \tag{5.13}$$

Divide now Ω_t into several regions, namely let $x_k = x_k(t) := L_t - \theta^k L_t$, which implies $x_{k+1} - x_k = \theta^k L_t(1-\theta)$, then we define the sets $A_k := [0, H_t]_{a_1} \times [x_k, x_{k+1}]_{a_2}$ and $B_k := [0, H_t]_{a_1} \times [-x_{k+1}, -x_k]_{a_2}$ and conclude

$$\Omega_t = \bigcup_{k=0}^{\infty} A_k \cup \bigcup_{k=0}^{\infty} B_k.$$

This decomposition is sketched in Figure 5.12. Choose k, $N \in \mathbb{N}$ fixed. We give now a

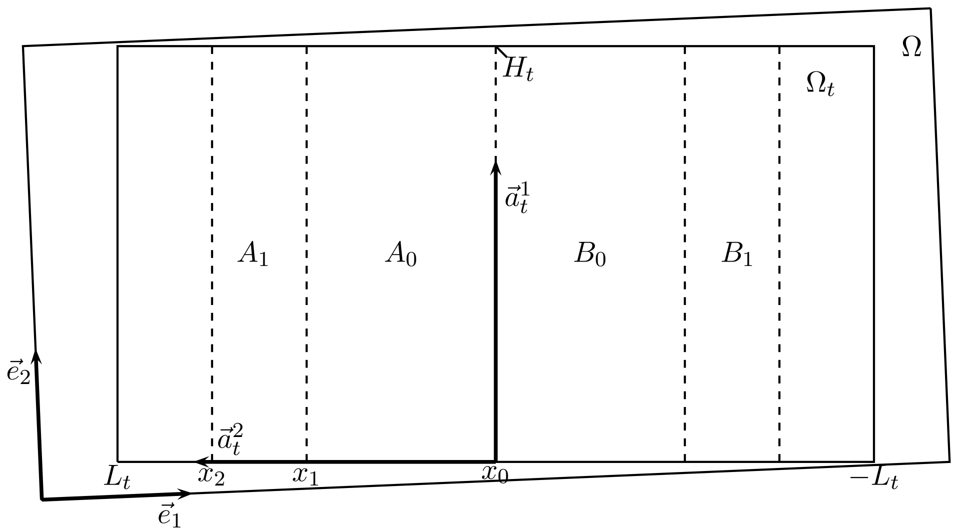

Figure 5.12:

motivation for the role of N and k. The basic part of the following construction is made in such way that we start in the middle of the region Ω_t, namely on $x_0 + [0, H_t]_{a_1}$, with $2N$ saw teeth and double them from the line $x_l + [0, H_t]_{a_1}$ to the line $x_{l+1} + [0, H_t]_{a_1}$, with $l \in \mathbb{N} \cup \{0\}$.

For simplicity we write $\widetilde{m} := \frac{H_t}{4(1-\theta)L_t}$ and $y_{N,k} := \frac{H_t}{2N2^k}$, then we can show that $\frac{\frac{y_{N,k}}{2} - y_{N,k}}{x_{k+1} - x_k} = -\frac{\widetilde{m}}{2^k\theta^k N}$. Next, we define functions $v_{N,k,t}$ on $[0, y_{N,k}]_{a_1} \times [x_k, x_{k+1}]_{a_2}$ and connect them afterwards to get a function on Ω_t. Define now subregions

$$
\begin{aligned}
R_1 &= \left\{ x\vec{a}_t^2 + y\vec{a}_t^1 \in \mathring{A}_k \colon 0 < y < \lambda_t y_{N,k} - \lambda_t \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right\}, \\
R_2 &= \left\{ x\vec{a}_t^2 + y\vec{a}_t^1 \in \mathring{A}_k \colon \lambda_t y_{N,k} - \lambda_t \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) < y < y_{N,k} - \lambda_t \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right\}, \\
R_3 &= (0, y_{N,k})_{a_1} \times (x_k, x_{k+1})_{a_2} \setminus \overline{R_1 \cup R_2}
\end{aligned}
$$

see Figure 5.13. Thereby we define for a set $A \subseteq \mathbb{R}^2$ the inner set of A by $\mathring{A}$, i.e., $\mathring{A} := \overline{A} \setminus \partial A$. We define the map $v = v_{N,k,t}$ by

$$
\begin{aligned}
v : [0, y_{N,k}]_{a_1} \times [x_k, x_{k+1}]_{a_2} &\to \mathbb{R} \\
z = y\vec{a}_t^1 + x\vec{a}_t^2 &\mapsto \begin{cases} m_1 y & \text{if} \quad z \in R_1 \\ m_0 \left(y - \left(y_{N,k} - \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right) \right) & \text{if} \quad z \in R_2 \\ m_1 (y - y_{N,k}) & \text{if} \quad z \in R_3 \end{cases},
\end{aligned}
$$

and in the remaining points such that it becomes continuous, see Figure 5.13. This is possible, since on the line

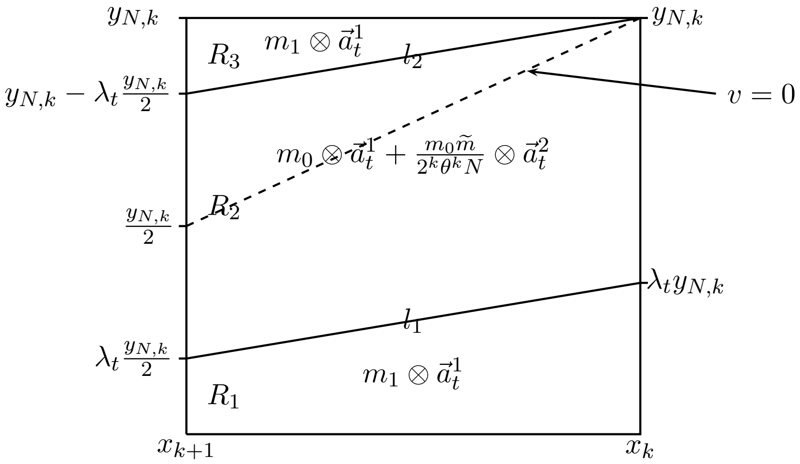

Figure 5.13: $\nabla v_{N,k,t}$

$$
l_1 := \left\{ y\vec{a}_t^1 + x\vec{a}_t^2 \in \Omega_t \ : x \in (x_k, x_{k+1}), \ y = \lambda_t y_{N,k} - \lambda_t \tfrac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right\}
$$

we have using Equation (5.13) that

$$
m_0 \left(y - \left(y_{N,k} - \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right) \right) = m_0 (\lambda_t - 1) \left(y_{N,k} - \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right) = m_1 y
$$

and on the line $l_2 := \left\{ y\vec{a}_t^1 + x\vec{a}_t^2 \in \Omega_t \ : x \in (x_k, x_{k+1}) \,,\ y = y_{N,k} - \lambda_t \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right\}$ we get again with help of Equation (5.13) that

$$m_0 \left(y - \left(y_{N,k} - \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) \right) \right) = m_0(1 - \lambda_t) \frac{\widetilde{m}}{2^k \theta^k N} (x - x_k) = m_1 (y - y_{N,k}) \,.$$

Next, we compute the gradient of v in the regions R_i, with $i \in \{1, 2, 3\}$. We have for $z \in R_1 \cup R_3$ that $\nabla v(z) = m_1 \otimes \vec{a}_t^1$ and for $z \in R_2$ that $\nabla v(z) = m_0 \otimes \vec{a}_t^1 + \frac{m_0 \widetilde{m}}{2^k \theta^k N} \otimes \vec{a}_t^2$, which implies that the map $v_{N,k,t}$ is Lipschitz continuous. Define now $\widehat{l_1}$ (resp. $\widehat{l_2}$) by reflecting l_1 (resp. l_2) on the line $y_{N,k} + < \vec{a}_t^2 >_{\mathbb{R}}$ and extend $v_{N,k,t}$ in an anti-symmetric way to $[y_{N,k}, 2y_{N,k}]_{a_1} \times [x_k, x_{k+1}]_{a_2}$ by $v_{N,k,t} \left(x\vec{a}_t^2 + y\vec{a}_t^1 \right) := -v_{N,k,t} \left(x\vec{a}_t^2 + (2y_{N,k} - y)\, \vec{a}_t^1 \right)$. Next, we can extend the function $v_{N,k,t}$ periodically to $[0, H_t]_{a_1} \times [x_k, x_{k+1}]_{a_2}$ through $v_{N,k,t} \left(x\vec{a}_t^2 + (y + 2y_{N,k})\, \vec{a}_t^1 \right) := v_{N,k,t} \left(x\vec{a}_t^2 + y\vec{a}_t^1 \right)$. Connecting these function we can define $\widehat{v}_N = \widehat{v}_{N,t} : [0, H_t]_{a_1} \times [-L_t, L_t]_{a_2} \to \mathbb{R}$ through $\widehat{v}_N(x\vec{a}_t^2 + y\vec{a}_t^1) := v_{N,k,t}(x\vec{a}_t^2 + y\vec{a}_t^1)$ if $x \in [x_k, x_{k+1}]$ and $\widehat{v}_N(x\vec{a}_t^2 + y\vec{a}_t^1) := v_{N,k,t}(-x\vec{a}_t^2 + y\vec{a}_t^1)$ if $-x \in [x_k, x_{k+1}]$. By definition we have for $y \in \left[0, \frac{y_{N,k}}{2}\right]$ that

$$\frac{1}{2} v_{N,k,t} \left(2y\vec{a}_t^1 + x_k\vec{a}_t^2 \right) = v_{N,k,t} \left(y\vec{a}_t^1 + x_{k+1}\vec{a}_t^2 \right) = v_{N,k+1,t} \left(y\vec{a}_t^1 + x_{k+1}\vec{a}_t^2 \right).$$

Using this and the antisymmetry of $v_{N,k,t}$ we get the same for $y \in \left(\frac{y_{N,k}}{2}, y_{N,k} \right]$. This implies that the function $\widehat{v}_N$ is well defined and thus continuous. Otherwise it is not globally Lipschitz continuous in $[0, H_t]_{a_1} \times [-L_t, L_t]_{a_2}$ for $\theta < \frac{1}{2}$, since the gradient of $v_{N,k,t}$ in R_2 cannot be bounded uniformly in k because of $\frac{1}{2^k \theta^k} \overset{k \to \infty}{\to} \infty$. We ignore this gap first and compute the energy for the non Lipschitz function

$$\begin{aligned} \widehat{u} = \widehat{u}_{t,N} : \Omega_t \ &\to\ \mathbb{R}^2 \\ z \ &\mapsto\ F^* z + \begin{pmatrix} \widehat{v}_N(z) \\ 0 \end{pmatrix}, \end{aligned}$$

with corresponding $\widehat{\gamma}_t$ defined by

$$\begin{aligned} \widehat{\gamma} = \widehat{\gamma}_t : \Omega_t \ &\to\ \mathbb{R} \\ z \ &\mapsto\ \begin{cases} t^{\eta - 1} & \text{if} \quad \nabla \widehat{u}(z) = F_t(1) \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

This gives an idea how to modify this, such that it becomes Lipschitz continuous and minimizes the energy simultaneously.
The gradients of $\widehat{u}$ on the reference region $R = R_{N,k} = [0, 2y_{N,k}]_{a_1} \times [x_k, x_{k+1}]_{a_2}$ are sketched in Figure 5.14. Furthermore we have

$$|R_{N,k}| = \theta^k L_t (1 - \theta) \frac{H_t}{N 2^k} \leq LH \frac{\theta^k}{N 2^k}. \tag{5.14}$$

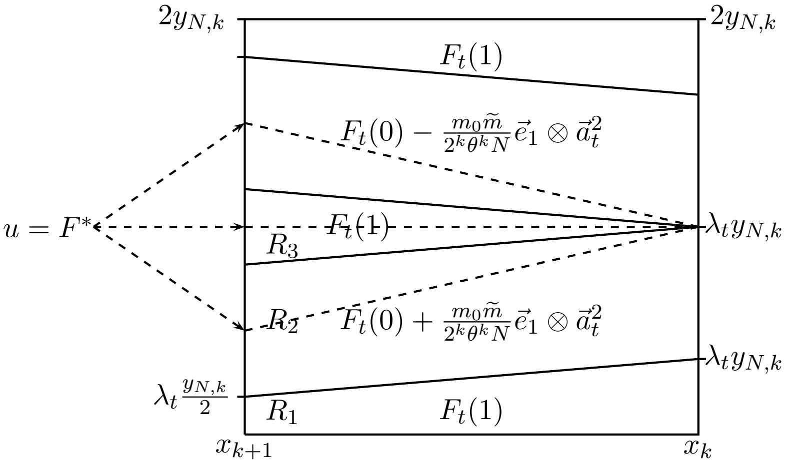

Figure 5.14: $\nabla\widehat{u}$

Compute now the energy of the function $\widehat{u}$, $\widehat{\gamma}$ in the reference region R. For the computation of the $\delta V_x(\widehat{\gamma}, R)$-term we need the following fact,

$$\exists \mu > 0\ \forall \varphi, \psi \in (-\mu, \mu)\ :\ |\tan(\varphi + \psi)| \leq |\tan(\varphi)| + 2|\psi| \leq |\tan(\varphi)| + 2|\tan(\psi)|\,, \tag{5.15}$$

which is true since $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is continuously differentiable with $\tan'(0) = 1$ and $\tan'(x) > 1$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$. By definition the slope of the lines $l_1,\ l_2, \widehat{l}_1,\ \widehat{l}_2$ with respect to the $\vec{a}_t^2$-direction are up to a sign equal to $\left|\lambda_t \frac{\widetilde{m}}{2^k\theta^k N}\right|$. Thus we get for fixed $k \in \mathbb{N}$ and t big enough that we can use Equation (5.15). This means that the slope of the lines $l_1,\ l_2, \widehat{l}_1,\ \widehat{l}_2$, with respect to the $\vec{e}_1$- direction, is bounded from above by

$$\left|\frac{t-1}{t^\eta}\right| + 2\left|\lambda_t \frac{\widetilde{m}}{2^k\theta^k N}\right|, \tag{5.16}$$

for big enough t. Since $m_0 \overset{t\to\infty}{\longrightarrow} -\gamma_0$ and $\widetilde{m} \overset{t\to\infty}{\longrightarrow} \frac{H}{4(1-\theta)L}$ we get

$$F_t(0) \pm \frac{m_0\widetilde{m}}{2^k\theta^k N}\vec{e}_1 \otimes \vec{a}_t^2 \overset{t\to\infty}{\longrightarrow} \mathbb{1} \mp \frac{\gamma_0 H}{4(1-\theta)L2^k\theta^k N}. \tag{5.17}$$

Using the Equation (5.6) in the proof of Lemma 5.1 and Equations (5.14), (5.16) and

(5.17), then we get, for a fixed $\zeta > 0$ and for $t = t(\zeta, \varepsilon) > 0$ large enough that

$$\begin{aligned}
&\int_{R_{N,k}} \left(\frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla \widehat{u} \left(\mathbb{1} - \widehat{\gamma} \vec{e}_1 \otimes \vec{e}_2 \right), SO(2) \right) + |\widehat{\gamma}|^p \right) \, d\lambda_2 + \delta V_x \left(\widehat{\gamma}, R_{N,k} \right) \\
&\leq \frac{1}{\varepsilon} |R_{N,k}| \operatorname{dist}^q \left(F_t(0) \pm \frac{m_0 \widetilde{m}}{2^k \theta^k N} \vec{e}_1 \otimes \vec{a}_t^2, SO(2) \right) \\
&+ \frac{1}{\varepsilon} \lambda_t |R_{N,k}| \underbrace{\operatorname{dist}^q \left(F_t(1) \left(\mathbb{1} - t^{\eta-1} \vec{e}_1 \otimes \vec{e}_2 \right), SO(2) \right)}_{\leq C(t^q+1)} + \lambda_t \left| t^{\eta-1} \right|^p |R_{N,k}| \\
&+ \delta \cdot \underbrace{4}_{= \text{ number of jump-lines}} \cdot \underbrace{|x_{k+1} - x_k| \left(\frac{1}{t^{\eta-1}} + 2\lambda_t \frac{\widetilde{m}}{2^k \theta^k N} \right)}_{\geq \text{ jump-region}} \cdot \underbrace{t^{\eta-1}}_{= \text{jump height}} \\
&\leq C \frac{1}{\varepsilon} \frac{\theta^k HL}{2^k N} \frac{H^q}{2^{kq} \theta^{kq} N^q L^q} + \zeta \frac{\theta^k HL}{2^k N} + \zeta \frac{\theta^k HL}{2^k N} + C\delta \theta^k L + C\delta \frac{H}{2^k N t},
\end{aligned} \tag{5.18}$$

since $\lambda_t t^q \to 0$ and $\lambda_t t^{(\eta-1)p} \to 0$ for $t \to \infty$ by definition of η. Thereby $C = C(F^*, q, \theta)$ is a constant independent of $k, \varepsilon, \delta, t, N, L, H$, which is also independent of θ for the additional assumption $\theta < \frac{1}{2}$. The above estimate is valid for fixed k and t big enough. If we took the limit $k \to \infty$ we would get $\left| \lambda_t \cdot \frac{\widetilde{m}}{2^k \theta^k N} \right| \overset{k \to \infty}{\longrightarrow} \infty$, for $\theta < \frac{1}{2}$ and fixed t. Therefore we have to choose a t dependent k. Define $k(t) := \left\lfloor -\frac{\ln(t)}{\ln(2\theta)} \right\rfloor$, with the floor function $\lfloor x \rfloor := \max \{ z \in \mathbb{Z} \, : \, z \leq x \}$, then we have for $\theta < \frac{1}{2}$ that $k(t) \ln(2\theta) \geq -\ln(t)$ and thus we get

$$\frac{1}{t^\eta 2^{k(t)} \theta^{k(t)}} = \frac{1}{t^\eta \exp\left(k(t) \ln(2\theta)\right)} \leq \frac{1}{t^\eta \exp\left(-\ln(t)\right)} = \frac{t}{t^\eta} \overset{t \to \infty}{\longrightarrow} 0. \tag{5.19}$$

This implies that we can use Equation (5.15), which gives Equation (5.16). Next, we have to guarantee for $q > 1$ that $\frac{1}{2^q \theta^{q-1}} < 1$, which is equivalent to $\theta > \frac{1}{2^{\frac{q}{q-1}}}$ and for $q \geq 1$ that $2\theta < 1$, i.e., $\theta < \frac{1}{2}$. Choose now a fixed θ with these properties. Then we can conclude for $\widetilde{\Omega}_t := \bigcup_{k=0}^{k(t)-1} A_k \cup \bigcup_{k=0}^{k(t)-1} B_k$ that

$$\begin{aligned}
&\int_{\widetilde{\Omega}_t} \left(\frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla \widehat{u} \left(\mathbb{1} - \widehat{\gamma} \vec{e}_1 \otimes \vec{e}_2 \right), SO(2) \right) + |\widehat{\gamma}|^p \right) \, d\lambda_2 + \delta V_x \left(\widehat{\gamma}, \widetilde{\Omega}_t \right) \\
&\leq 2 \sum_{k=0}^{k(t)-1} 2^k N \left[C \frac{\theta^k HL}{\varepsilon 2^k N} \frac{H^q}{2^{kq} \theta^{kq} N^q L^q} + \zeta \frac{\theta^k HL}{2^k N} + \zeta \frac{\theta^k HL}{2^k N} + C\delta \theta^k L + C\delta \frac{H}{2^k N t} \right] \\
&\leq \frac{CH^{q+1}}{\varepsilon N^q L^{q-1}} \sum_{k=0}^{\infty} \left(\frac{1}{2^q \theta^{q-1}} \right)^k + 4\zeta \sum_{k=0}^{\infty} \theta^k HL + CN\delta L \sum_{k=0}^{\infty} (2\theta)^k + CH\delta \sum_{k=0}^{k(t)} \frac{1}{t} \\
&\leq 8HL\zeta + C \left(\frac{H^{q+1}}{\varepsilon N^q L^{q-1}} + NL\delta \right),
\end{aligned} \tag{5.20}$$

where we have used $\frac{k(t)}{t} \overset{t\to\infty}{\longrightarrow} 0$ by definition, and $C = C(F^*, q)$ is a constant independent of $\varepsilon, \delta, t, N, L, H$. The k dependence of t is introduced by the Lipschitz continuous auxiliary function,

$$\begin{aligned} w_{t,N,k} : Q_{k,N} &\to \mathbb{R} \\ x\vec{a}_t^2 + y\vec{a}_t^1 &\mapsto \begin{cases} m_1 y & \text{if} \quad 0 \le y \le \lambda_t y_{N,k} \frac{(x-L_t)}{x_k - L_t} \\ m_0 (y - y_{N,k}) & \text{if} \quad y \ge y_{N,k} + (\lambda_t - 1) y_{N,k} \frac{(x-L_t)}{x_k - L_t} \\ m_1 \lambda_t y_{N,k} \frac{(x-L_t)}{x_k-L_t} & \text{otherwise} \end{cases}, \end{aligned}$$

where we write short $Q_{k,N} := [0, y_{N,k}]_{a_1} \times [x_k, L_t]_{a_2}$. The gradients of $w_{t,N,k}$ are sketched in Figure 5.15, where we have used $\frac{m_1 \lambda_t y_{N,k}}{x_k - L_t} = \frac{m_0(1-\lambda_t)H_t}{2^k \theta^k 2N L_t}$. Extend this w in the same

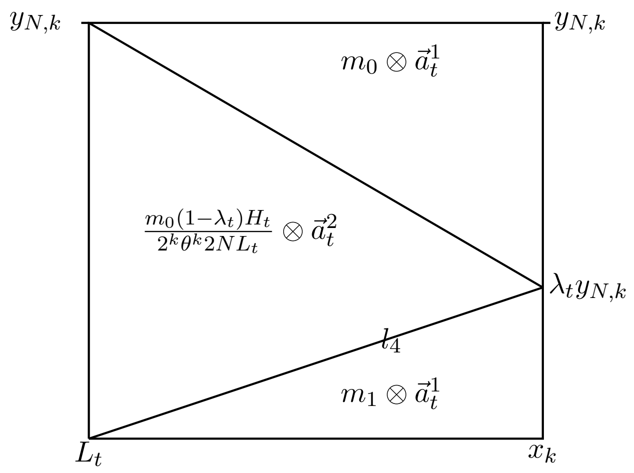

Figure 5.15: $\nabla w_{t,N,k}$

way to $[0, H_t]_{a_1} \times [x_k, L_t]_{a_2}$ as it was done before with $v_{N,k,t}$. Furthermore we define it on $[0, H_t]_{a_1} \times [-L_t, -x_k]_{a_2}$ by $w_{t,N,k}\left(x\vec{a}_t^2 + y\vec{a}_t^1\right) = w_{t,N,k}\left(-x\vec{a}_t^2 + y\vec{a}_t^1\right)$. For the region $C_k := \bigcup_{l=k}^{\infty} A_l \cup \bigcup_{l=k}^{\infty} B_l = ([0, H_t]_{a_1} \times [x_k, L_t]_{a_2}) \cup ([0, H_t]_{a_1} \times [-L_t, -x_k]_{a_2})$, where $w_{t,N,k}$ is defined, we have $|C_k| \le 2HL\theta^k$. With help of the function w we define

$$\begin{aligned} u_t : \Omega &\to \mathbb{R}^2 \\ z &\mapsto \begin{cases} F^* z + \begin{pmatrix} \widehat{v}_N(z) \\ 0 \end{pmatrix} & \text{if} \quad z \in \widetilde{\Omega}_t = \Omega_t - C_{k(t)} \\ F^* z + \begin{pmatrix} w_{t,N,k(t)}(z) \\ 0 \end{pmatrix} & \text{if} \quad z \in C_{k(t)} \\ F^* z & \text{otherwise} \end{cases}. \end{aligned}$$

Finally, we define

$$\gamma_t : \Omega \to \mathbb{R} \ ; \ z \mapsto \begin{cases} t^{\eta-1} & \text{if} \quad \nabla u_t(z) = F_t(1) \\ 0 & \text{otherwise} \end{cases} .$$

The gradients of u_t, and the values of γ_t in the region $[0, 2y_{N,k(t)}]_{a1} \times [x_{k(t)-1}, L_t]_{a2}$ are illustrated in Figure 5.16. By construction we obtain the Lipschitz continuity of u_t and we get $\{(u_t, \gamma_t)\}_{t\in\mathbb{R}} \subseteq W^{1,\infty}_{F^*}\left(\Omega;\mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right)$. Since $F^* = \mathbb{1} + \gamma_0 \vec{e}_1 \otimes \vec{e}_2$,

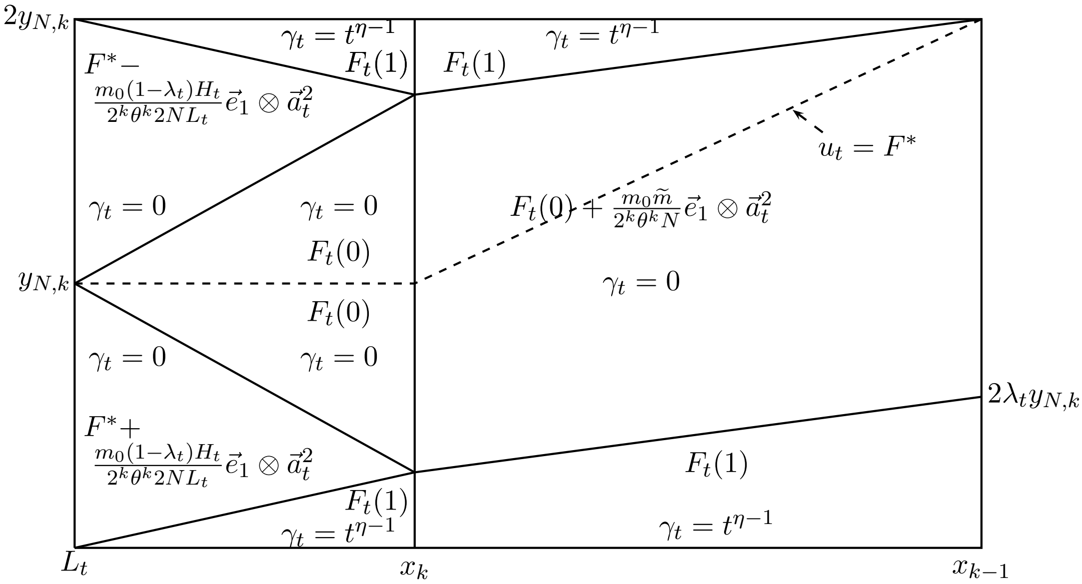

Figure 5.16: $\theta = \frac{3}{8},\ \lambda_t = \frac{2}{7},\ k = k(t) = \left\lfloor \frac{-\ln(t)}{\ln(2\theta)} \right\rfloor$

we have

$$\begin{aligned} \operatorname{dist}^q\left(F^* \pm \frac{m_0 H_t\,(1-\lambda_t)}{2^{k(t)}\theta^{k(t)} 2NL_t}\vec{e}_1 \otimes \vec{a}_t^2, SO(2)\right) &\le \left\|\gamma_0 \vec{e}_1 \otimes \vec{e}_2 \pm \frac{m_0 H_t\,(1-\lambda_t)}{2^{k(t)}\theta^{k(t)} 2NL_t}\vec{e}_1 \otimes \vec{a}_t^2\right\|^q \\ &\le \left(|\gamma_0| + \left|\frac{m_0 H_t\,(1-\lambda_t)}{2^{k(t)}\theta^{k(t)} 2NL_t}\right|\right)^q \le 2^{q-1}\left(|\gamma_0|^q + \left|\frac{m_0 H_t\,(1-\lambda_t)}{2^{k(t)}\theta^{k(t)} 2NL_t}\right|^q\right). \end{aligned}$$

Define

$$l_4 = l_{4,k} := \left\{ y\vec{a}_t^1 + x\vec{a}_t^2 \in \Omega_t \ : x \in (x_k, L_t)\,,\ y = \lambda_t y_{N,k}\frac{(x - L_t)}{x_k - L_t} \right\},$$

see Figure 5.15, and $\widehat{l}_4$ by reflecting l_4 on the line $y_{N,k}+<\vec{a}_t^2>_{\mathbb{R}}$. The slop of the lines l_4 and $\widehat{l}_4$ with respect to the $\vec{a}_t^2$-direction are up to a sign equal to $\frac{\gamma_0 H_t}{t^\eta 2^{k(t)}\theta^{k(t)}L_t 2N}$ and due to Equation (5.19) this converges to zero for $t\to\infty$. Therefore we can use Equation (5.16) in order to compute the variation part $V_x\left(\gamma_t, C_{k(t)}\right)$ in the region $C_{k(t)}$. Using this and the previous estimations, we can show that for each $\zeta>0$ there exists a $t=t\left(\zeta,\varepsilon,\delta\right)$ large enough so that

$$
\begin{aligned}
&\int_{C_{k(t)}} \frac{1}{\varepsilon}\operatorname{dist}^q\left(\nabla u_t\left(\mathbb{1}-\gamma_t\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right)+|\gamma_t|^p\ d\lambda_2+\delta V_x\left(\gamma_t, C_{k(t)}\right)\\
&\le \frac{1}{\varepsilon}\underbrace{\operatorname{dist}^q\left(F_t(0), SO(2)\right)}_{\stackrel{t\to\infty}{\longrightarrow}0}\underbrace{\left|C_{k(t)}\right|}_{\le|\Omega|}+\frac{1}{\varepsilon}\underbrace{\operatorname{dist}^q\left(F_t(1)\left(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right)}_{\le C(t^q+1)}\underbrace{\lambda_t}_{\le C\frac{1}{t^\eta}}\left|C_{k(t)}\right|\\
&+\frac{1}{\varepsilon}\operatorname{dist}^q\left(F^*\pm\frac{m_0H_t\left(1-\lambda_t\right)}{2^{k(t)}2N\theta^{k(t)}L_t}\vec{e}_1\otimes\vec{a}_t^2, SO(2)\right)\left|C_{k(t)}\right|+\left|t^{\eta-1}\right|\lambda_t\left|C_{k(t)}\right|\\
&+\delta\underbrace{2N\cdot 2^{k(t)}}_{=\text{ number of jump lines}}\underbrace{\left(\frac{1}{t^{\eta-1}}+\frac{\gamma_0H_t}{t^\eta 2^{k(t)}\theta^{k(t)}L_t2N}\right)\theta^{k(t)}L_t}_{\ge\text{ jump length}}\cdot\underbrace{t^{\eta-1}}_{=\text{ jump height}}\\
&\le\widetilde{\zeta}+\frac{1}{\varepsilon}C\left(|\gamma_0|^q+\left|\frac{m_0H_t(1-\lambda_t)}{2^{k(t)}\theta^{k(t)}2NL_t}\right|^q\right)\underbrace{\left|C_{k(t)}\right|}_{\le\theta^{k(t)}HL}+\widetilde{\zeta}\delta\le C(\delta+1)\widetilde{\zeta}\le\zeta
\end{aligned}
\tag{5.21}
$$

since $\left(\frac{1}{2^q\theta^{q-1}}\right)^{k(t)}\stackrel{t\to\infty}{\longrightarrow}0$ for $q\ge 1$, where $\widetilde{\zeta}$ can be chosen arbitrary small. Since we have $\gamma_t=0$ in $\Omega\setminus\Omega_t$, $\operatorname{dist}^q\left(\nabla u_t(x), SO(2)\right)=\operatorname{dist}^q\left(F^*, SO(2)\right)<\infty$, uniformly in x for $x\in\Omega\setminus\Omega_t$, $|\Omega\setminus\Omega_t|\stackrel{t\to\infty}{\longrightarrow}0$ the energy part in $\Omega\setminus\Omega_t$, without the variational part, vanishes for $t\to\infty$. For the variational part we have $V_x\left(\gamma_t,\Omega\right)\le V_x\left(\gamma_t,\Omega_t\right)+2L_t$. Using this, Equation (5.20) and Equation (5.21), we get that for each $\zeta>0$ there is a $t=t(\zeta,\varepsilon,\delta)$, such that

$$
\begin{aligned}
\int_\Omega \frac{1}{\varepsilon}\operatorname{dist}^q\left(\nabla u_t\left(\mathbb{1}-\gamma_t\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right)+|\gamma_t|^p\ d\lambda_2+\delta V_x\left(\gamma_t,\Omega\right)&\\
\le\zeta+C\left(\frac{H^{q+1}}{q\varepsilon N^qL^{q-1}}+N\delta L\right)&.
\end{aligned}
$$

Define the function $f:(0,\infty)\to\mathbb{R}$, $f(x)=x\delta L+\frac{H^{q+1}}{q\varepsilon x^qL^{q-1}}$, then its derivative fulfills $f'(x)=\delta L-\frac{H^{q+1}}{\varepsilon x^{q+1}L^{q-1}}\ge 0\Leftrightarrow x\ge\frac{H}{(\varepsilon\delta L^q)^{\frac{1}{q+1}}}$. For $N\in\left[\frac{H}{(\varepsilon\delta L^q)^{\frac{1}{q+1}}},\frac{H}{(\varepsilon\delta L^q)^{\frac{1}{q+1}}}+1\right)\cap\mathbb{N}$ we get that

$$\int_\Omega \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla u_t \left(\mathbb{1} - \gamma_t \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + |\gamma_t|^p \, d\lambda_2 + \delta V_x\left(\gamma_t, \Omega\right)$$

$$\leq \zeta + C\left(\frac{H^{q+1}}{\varepsilon L^{q-1}} \frac{\left(\varepsilon\delta L^q\right)^{\frac{q}{q+1}}}{H^q} + \delta L \frac{H}{\left(\varepsilon\delta L^q\right)^{\frac{1}{q+1}}} + \delta L\right) = \zeta + C\left(\frac{\delta^{\frac{q}{q+1}} H L^{\frac{1}{q+1}}}{\varepsilon^{\frac{1}{q+1}}} + \delta L\right),$$

where $C = C(F^*, q)$ is independent of $\varepsilon, \delta, p, L, H$ as asserted. Divide this by $|\Omega| = LH$, then we get the assertion, since $\zeta > 0$ was chosen arbitrary.

□

Simplified model

Unfortunately, we are not able to prove a lower bound which scales as $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$. Therefore one has to make some simplifications on the energy. In the proof of Lemma 5.1 we have seen that computing the variational part $V_x(\gamma, \Omega)$ is the same as counting the number of jumps in y-direction and multiplying it by height of the jump and by the length of one jump-line in x-direction. To be more precisely we get, for this special construction, that

$$V_x(\gamma, \Omega) = V_y\left(\chi_{\{\gamma=0\}}, \Omega\right),$$

and it is not astonishing, see Corollary 5.12, that they do not differ much in the branching construction. For a more rigorous justification, of this simplification one needs to show that the laminate construction and the corresponding γ used in the proof of Lemma 5.1 is in some sense an optimal construction. This has not yet been done. Since small perturbations of γ in the region where γ is equal to zero do not affect the variation $V_x(\gamma, \Omega)$ much, but essentially change the variation $V_y\left(\chi_{\{\gamma=0\}}, \Omega\right)$, we have to ensure, additionally, that γ does not achieve small, non zero values. Consider now the simplified energy $\widetilde{I} = \widetilde{I}_{\varepsilon,\delta} : W^{1,\infty}\left(\Omega; \mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right) \to [0, \infty]$, defined by

$$\widetilde{I}(u, \gamma) = \fint_\Omega \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + |\gamma|^p \, d\lambda_2 + \frac{\delta}{|\Omega|} V_y\left(\chi_{\{\gamma=0\}}, \Omega\right). \quad (5.22)$$

We will see in Theorem 6.15, that this energy has a lower bound, which scales as $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, at least if the upper bound has the same scaling. The proof relies on the fact that it suffices to consider the energy $\widetilde{I}(u, \gamma)$ on a suitable one dimensional line in y-direction, which is not helpful in the model, which uses the variational energy in x-direction. Next, we show that the infimum of the simplified energy $\widetilde{I}(u, \gamma)$ has an upper bound, which scales as $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, for small δ.

Corollary 5.12. *Let $\Omega := [-L, L] \times [0, H] \subset \mathbb{R}^2$, $L > 0$, $H > 0$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $F^* \in \mathcal{M}^{(2)}$. Then we have for $\varepsilon, \delta > 0$, that*

$$\inf_{\substack{u \in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2) \\ \gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)}} \widetilde{I}_{\varepsilon,\delta}(u,\gamma) \leq C \left(\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}} L^{\frac{q}{q+1}}} + \frac{\delta}{H} \right),$$

where $C = C(F^*, q) > 0$ is independent of $\varepsilon, \delta, p, L, H$.

Proof:
We use the same branching-construction as in the proof of Theorem 5.11. The slight difference is the computation of the variation part $V_y\left(\chi_{\{\gamma=0\}}, \Omega\right)$. In the following we use the same notations and definitions as in the proof of Theorem 5.11. Consider first $V_y\left(\chi_{\{\gamma_t=0\}}, R_{N,k}\right)$, see Figure 5.14 and Figure 5.16. The length in x-direction of one jump line in $R_{N,k}$ can be bounded from above by $|x_{k+1} - x_k| + \lambda_t \frac{y_{N,k}}{2}$. Then we can bound the variational part $V_y\left(\chi_{\{\gamma=0\}}, R_{N,k}\right)$ form above by

$$\begin{aligned} &\underbrace{4}_{=\text{number of jump lines}} \cdot \left(|x_{k+1} - x_k| + \lambda_t \frac{y_{N,k}}{2} \right) = 4 \left(\theta^k L_t (1-\theta) + \frac{\gamma_0 H_t}{4N 2^k t^\eta} \right) \\ &\leq C \left(\theta^k L + \frac{H}{t^\eta N 2^k} \right) \leq C \left(\theta^k L + \frac{H}{2^k N t} \right), \end{aligned}$$

since we have chosen $\eta > 0$ and $t \geq 1$. This is exactly the same term we get in the computation of Equation (5.18). Next, we have to estimate the variation part in the region $C_{k(t)}$, i.e., $V_y\left(\chi_{\{\gamma_t=0\}}, C_{k(t)}\right)$. As in the proof of Theorem 5.11 we get, that we have $2N \cdot 2^{k(t)}$ jump lines and thus the part $V_y\left(\chi_{\{\gamma=0\}}, C_{k(t)}\right)$ has the upper bound

$$2N \cdot 2^{k(t)} \cdot (|L_t - x_k| + \lambda_t y_{N,k}) = 2N 2^{k(t)} \left(\theta^{k(t)} L_t + \frac{H_t \gamma_0}{2N 2^{k(t)} t^\eta} \right) \stackrel{t\to\infty}{\longrightarrow} 0,$$

since $k(t) \stackrel{t\to\infty}{\longrightarrow} \infty$ and $\theta < \frac{1}{2}$. Thus, we get the same estimate as in Equation (5.21), which implies that we can show the same assertion as in Theorem 5.11.

□

In order to prove the lower bound for $\widetilde{I}$, it is useful to investigate first a more simple energy, namely for $F^* = R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(2)}$ with $R \in SO(2)$ and $\gamma_0 \in \mathbb{R}$ we consider the energy $\widehat{I} = \widehat{I}_{F^*} = \widehat{I}_{\varepsilon,\delta;F^*} : W^{1,\infty}\left(\Omega;\mathbb{R}^2\right) \times \left(BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right) \to [0,\infty]$, defined by

$$\widehat{I}(u,\gamma) = \fint_\Omega \frac{1}{\varepsilon} \left\| \nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right) - R \right\|^q + |\gamma|^p \; d\lambda_2 + \frac{\delta}{|\Omega|} V_y\left(\chi_{\{\gamma=0\}}, \Omega\right). \tag{5.23}$$

As a simple corollary one can obtain, that the infimum of the energy $\widehat{I}\,(u,\gamma)$ has also an upper bound, which scales as $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, for small δ.

Corollary 5.13. *Let $\Omega := [-L,L]\times[0,H]\subset\mathbb{R}^2$, $L>0$, $H>0$ and $p,q\geq 1$ with $\frac{1}{p}+\frac{1}{q}>1$ and $F^*\in\mathcal{M}^{(2)}$. Then we have for $\varepsilon,\delta>0$, that*

$$\inf_{\substack{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2)\\ \gamma\in BV(\Omega)\cap L^{\max\{p,q\}}(\Omega)}} \widehat{I}(u,\gamma)\leq C\left(\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}L^{\frac{q}{q+1}}}+\frac{\delta}{H}\right),$$

where $C=C(F^*,q)>0$ is independent of ε,δ,p,L,H.

Proof:
Because of the frame indifference of the part $\|\nabla u\,(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2)-R\|^q$ we can choose again $F^*=\begin{pmatrix}1&\sigma\gamma_0\\0&1\end{pmatrix}$ and we use the same branching construction as in the proof of Theorem 5.11. Then the assertion is an immediate consequence of Corollary 5.12 and the fact that all estimates in the proof of Theorem 5.11 are still true if we write $\|\nabla u\,(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2)-\mathbb{1}\|^q$ instead of $\operatorname{dist}^q\left(\nabla u\,(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2)\,,SO(2)\right)$.

□

5.4 Upper bound for boundary values in $\mathcal{N}^{(2)}$

In the following we choose $\Omega = B(0,L)$ for $L > 0$. Until now we have only shown an upper bound for configurations u, with affine boundary values $u = F^* \in \mathcal{M}^{(2)}$, namely on the set of all matrices F^*, for which the rigid elasticity density $W_{rigid,p}$ is finite, i.e., whose elastic part is a rotation. One might be interested in affine boundary values in the set of all matrices, for which the quasiconvex envelope $W_{rigid,p}^{qc}$ is finite, namely on $\mathcal{N}^{(2)}$. For these boundary values we will show an upper bound for the infimum of the energy, which scales as $\frac{\delta^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}}$. This was shown by a simple double laminate construction as in [47], simple means that we use linear interpolation in order achieve the boundary values. Presumably this result might be improved using a branching construction, which should give a q-dependent scaling relation. As in Section 5.3 this scaling relation does not depend on p, since the construction is made so that $\int_\Omega |\gamma|^p \ d\lambda_2$ becomes small independently of δ and ε.

Lemma 5.14. *Let $F = (F_{ij})_{i,j\in\{1,2\}} \in \mathcal{N}^{(2)}$, $\zeta \in (0,1)$, $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $L > 0$. The elastic energy density fulfills the hypotheses $(H1) - (H4)$, refer to the pages 42 and 63, and is additionally finite valued, i.e., $W_e : \mathbb{R}^{2\times 2} \to [0,\infty)$. Then exists a $\gamma \in BV(B(0,L)) \cap L^{\max\{p,q\}}(B(0,L))$ and a $z \in W^{1,\infty}\left(\overline{B(0,L)};\mathbb{R}^2\right)$ with $z(x) = Fx$ for every $x \in \partial B(0,L)$, so that for all $\varepsilon, \delta > 0$,*

$$I(z,\gamma) \leq \zeta + C\left(\frac{\delta^{\frac{1}{3}}}{L^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}\right),$$

where $C = C(F) > 0$ is independent of ε, δ, L, p, q and ζ.

Proof:
In the case $F \in SO(2)$ we can choose $z(x) = Fx$ for all $x \in \overline{B(0,L)}$, $\gamma = 0$ and conclude $I(z,\gamma) = 0$. Assume first, that $L \leq \varepsilon\delta$, then one can define $z(x) = Fx$ for all $x \in \overline{B(0,L)}$ and $\gamma = 0$. This choice implies

$$I(z,\gamma) \leq \frac{W_e(F)}{\varepsilon} = \frac{W_e(F)}{\varepsilon^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}} \leq C\frac{\delta^{\frac{1}{3}}}{L^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}},$$

as asserted. Consider now the case $L > \varepsilon\delta$. Let $F \in \mathcal{N}^{(2)} - SO(2)$, then we get by Lemma 4.1, that there are matrices $F_{-1}, F_1 \in \mathcal{M}^{(2)}$ and a $\mu \in [0,1]$, so that $F = \mu F_{-1} + (1-\mu)F_1$ and $F_{-1} - F_1 = \vec{b} \otimes \vec{e}_1$, for a vector $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \overline{B(0,2)}$, since $F_1\vec{e}_1, F_{-1}\vec{e}_1 \in \mathbb{S}^1$. According to Lemma 4.1 there exist $R_{-1}, R_1 \in SO(2)$ such that, after a potential change in the role of F_{-1} and F_1, we get $R_\sigma F_\sigma = \begin{pmatrix} 1 & \sigma\gamma_0 \\ 0 & 1 \end{pmatrix}$ for $\sigma \in \{-1,1\}$ and $\gamma_0 \in (0,\infty)$. Analogously

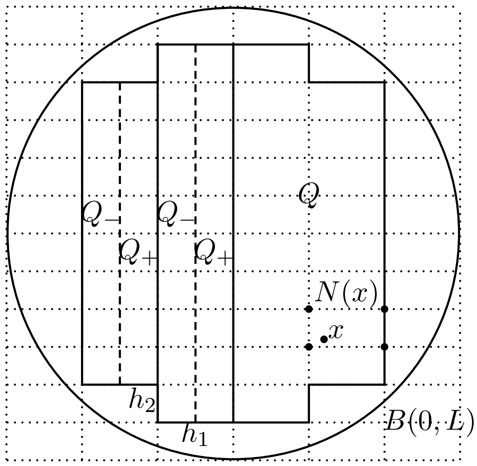

Figure 5.17: $Q_- = Q_{-1},\ Q_+ = Q_{+1}$

as in the proof of Lemma 5.1 we can write now $R_\sigma F_\sigma = (1-\lambda_t)\, F_{t,\sigma}(0) + \lambda_t F_{t,\sigma}(1)$, where we define again $\lambda_t = \frac{\gamma_0}{t^\eta}$, $\alpha_t = 1 + (t-1)\lambda_t$, $F_{t,\sigma}(0) = \frac{1}{\alpha_t}\begin{pmatrix} 1 & \sigma\gamma_0\,(\alpha_t - 1) \\ 0 & \alpha_t \end{pmatrix}$, $F_{t,\sigma}(1) = \frac{1}{\alpha_t}\begin{pmatrix} t & \sigma t^\eta + \sigma\gamma_0\,(\alpha_t - 1) \\ 0 & \alpha_t \end{pmatrix}$, for a $t > \max\left\{\gamma_0^{\frac{1}{\eta}}, 1\right\}$. Furthermore we have $F_{t,\sigma}(1) - F_{t,\sigma}(0) = \frac{1}{\alpha_t}\vec{e}_1 \otimes \vec{a}^1_{t,\sigma}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\|$, where $\vec{a}^1_{t,\sigma} := \frac{1}{\sqrt{(t-1)^2 + (t^\eta)^2}}\begin{pmatrix} t-1 \\ \sigma t^\eta \end{pmatrix}$.

Define the lattice $\Gamma = \Gamma_{h_1,h_2} := h_1\mathbb{Z} \times h_2\mathbb{Z} = \left\{x \in \mathbb{R}^2 \ :\ \exists\, k,l \in \mathbb{Z},\ x = kh_1\vec{e}_1 + lh_2\vec{e}_2\right\}$ for $h_1, h_2 > 0$, and denote the indices which correspond to an element in $B(0,L)$ by $\Delta = \Delta_{h_1,h_2} := \left\{(k,l) \in \mathbb{Z}^2 \ :\ kh_1\vec{e}_1 + lh_2\vec{e}_2 \in B(0,L)\right\}$. Next, we define for a point $x = \lambda_1 h_1\vec{e}_1 + \lambda_2 h_2\vec{e}_2 \in \mathbb{R}^2$, with $\lambda_1, \lambda_2 \in \mathbb{R}$, the indices of its neighboring lattice points by

$$N(x) := \{(\lfloor\lambda_1\rfloor, \lfloor\lambda_2\rfloor), (\lfloor\lambda_1\rfloor, \lceil\lambda_2\rceil), (\lceil\lambda_1\rceil, \lfloor\lambda_2\rfloor), (\lceil\lambda_1\rceil, \lceil\lambda_2\rceil)\},$$

see Figure 5.17, where $\lfloor x\rfloor := \max\{m \in \mathbb{Z} \ :\ m \le x\}$ and $\lceil x\rceil := \min\{m \in \mathbb{Z} \ :\ m \ge x\}$ for an $x \in \mathbb{R}$. Finally we define the set

$$Q = Q_{h_1,h_2} := \left\{x = \lambda_1 h_1\vec{e}_1 + \lambda_2 h_2\vec{a}^1_t \in B(0,L) \ :\ N(x) \subseteq \Delta\right\},$$

on which we want to define the nontrivial part of our construction, see Figure 5.17. In the following we will show that $|B(0,L) \setminus Q| \le 4\pi L \max\{h_1,h_2\} = 4\pi L h_{\max}$, for $h_{\max} := \max\{h_1,h_2\}$. Using this, we can define $z = F$ in $\overline{B(0,L)} \setminus Q$ and $\gamma = 0$ in

$B(0,L) \setminus Q$, and get that the energy-part caused by the region $\overline{B(0,L)} \setminus Q$ is bounded from above by $\frac{4\pi L h_{\max} W_e(F)}{\varepsilon}$, which suggests, that we only need to define an adequate $z \in W^{1,\infty}\left(\overline{Q};\mathbb{R}^2\right)$ with $z = F$ on ∂Q.
For a point $x = \lambda_1 h_1 \vec{e}_1 + \lambda_2 h_2 \vec{e}_2 \in B(0,L) \setminus Q$, there exist a $(k,l) \in N(x)$, with $(k,l) \notin \Delta$ and $|\lambda_1 - k| \leq 1,\ |\lambda_2 - l| \leq 1$. Then we get for $y := k h_1 \vec{e}_1 + l h_2 \vec{e}_2$ that $y \notin B(0,L)$ and thus $\operatorname{dist}(x, \partial B(0,L)) \leq \|x-y\| = \|(\lambda_1 - k)\, h_1 \vec{e}_1 + (\lambda_2 - l)\, h_2 \vec{e}_2\| \leq h_1 + h_2 \leq 2h_{\max}$. Using this we get $|B(0,L) \setminus Q| \leq \pi\left(L^2 - (L - 2h_{\max})^2\right) \leq 4\pi L h_{\max}$, as asserted.
Next, we define on Q the function

$$v_{h_1}(x) := Fx + h_1 \chi_\mu\left(\frac{\vec{e}_1 \cdot x}{h_1}\right)\vec{b},$$

which is a laminate for $F \in \mathcal{N}^{(2)} \setminus \mathcal{M}^{(2)}$. Thereby we define for $\kappa \in (0,1)$ the continuous function $\chi_\kappa : \mathbb{R} \to \mathbb{R}$ as periodic extension of the piecewise affine function $\widetilde{\chi}_\kappa : [0,1] \to \mathbb{R}$ given by $\widetilde{\chi}_\kappa(0) = 0 = \widetilde{\chi}_\kappa(1)$ and

$$\widetilde{\chi}_\kappa'(t) = \begin{cases} 1-\kappa & \text{if} \quad t \in (0,\kappa) \\ -\kappa & \text{if} \quad t \in (\kappa,1) \end{cases}.$$

Defining the sets

$$Q_{-1} := \left\{x \in Q \ : \ \frac{\vec{e}_1 \cdot x}{h_1} \in (0,\mu) + \mathbb{Z}\right\},\ Q_1 := \left\{x \in Q \ : \ \frac{\vec{e}_1 \cdot x}{h_1} \in (\mu,1) + \mathbb{Z}\right\},$$

see Figure 5.17, then one gets for each $x \in Q_{-1}$, for which $\nabla v_{h_1}(x)$ exists, that

$$\nabla v_{h_1}(x) = F + (1-\mu)\vec{b} \otimes \vec{e}_1 = \mu F_{-1} + (1-\mu)F_1 + (1-\mu)\left(F_{-1} - F_1\right) = F_{-1}$$

and analogously for $x \in Q_1$ that $\nabla v_{h_1}(x) = F_1$ if the gradient exists. By construction of Q we get $v_{h_1}(x) = F(x)$ for $x \in Q$ with $\frac{\vec{e}_1 \cdot x}{h_1} \in \mathbb{Z}$, and thus on the vertical parts of the boundary of Q. In order to achieve the boundary conditions on the boundary ∂Q one introduces the auxiliary function

$$\begin{aligned} H_i : Q &\to \mathbb{R} \\ x &\mapsto \begin{pmatrix} F_{i1} & F_{i2} \end{pmatrix} x + \operatorname{dist}_{\vec{e}_2}(x, \partial Q)\, b_i, \end{aligned}$$

for $i \in \{1,2\}$, where $F = (F_{ij})_{i,j \in \{1,2\}}$ and we define for a set $A \subseteq \mathbb{R}^2$, $\nu \in \mathbb{S}^1$ and $x \in A$, $\operatorname{dist}_\nu(x, \partial A) := \inf\{\lambda \in [0,\infty) \ : \ x + \lambda\nu \notin A \text{ or } x - \lambda\nu \notin A\}$. Define now

$$\begin{aligned} \widetilde{z}_i : Q &\to \mathbb{R} \\ x &\mapsto \begin{cases} \min\left\{H_i(x), \left(v_{h_1}(x)\right)_i\right\} & \text{if } b_i \geq 0 \\ \max\left\{H_i(x), \left(v_{h_1}(x)\right)_i\right\} & \text{if } b_i < 0 \end{cases} \end{aligned}$$

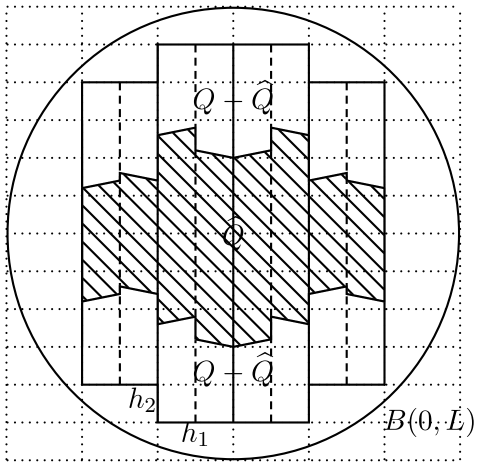

Figure 5.18: $\mu = \frac{1}{2}$

for $i \in \{1,2\}$ and $\widetilde{z} : Q \to \mathbb{R}^2$, $x \mapsto \begin{pmatrix} \widetilde{z}_1(x) \\ \widetilde{z}_2(x) \end{pmatrix}$, then we get

$$\begin{aligned} \{x \in Q \,:\, \widetilde{z}_1(x) = H_1(x)\} &= \left\{x \in Q \,:\, \operatorname{dist}_{\vec{e}_2}(x, \partial Q) \le h_1 \chi_\mu \left(\frac{\vec{e}_1 \cdot x}{h_1}\right)\right\} \\ &= \{x \in Q \,:\, \widetilde{z}_2(x) = H_2(x)\}\,. \end{aligned} \tag{5.24}$$

Define for $x \in Q_\sigma,\ \sigma \in \{-1,1\}$ the closed parallelogram around x, by

$$P(x) = P_{\sigma,t}(x) := \overline{\left\{y \in Q_\sigma \;:\; \left\lfloor \frac{\vec{a}^1_{t,\sigma} \cdot y}{h_2} \right\rfloor = \left\lfloor \frac{\vec{a}^1_{t,\sigma} \cdot x}{h_2} \right\rfloor \text{ and } \left\lfloor \frac{\vec{e}_1 \cdot y}{h_1} \right\rfloor = \left\lfloor \frac{\vec{e}_1 \cdot x}{h_1} \right\rfloor \right\}}$$

and for $x \in Q \setminus (Q_{-1} \cup Q_1)$ by $P(x) := \{x\}$. Furthermore we define the set

$$\widehat{Q} = \widehat{Q}_t := \{x \in Q \;:\; \forall y \in P(x) \;:\; \operatorname{dist}_{\vec{e}_2}(y, \partial Q) > h_1\}\,,$$

see Figure 5.18. Next, we show that the set $\widehat{Q}$ was defined in such way that we have $\{x \in Q \,:\, \widetilde{z}_1(x) = H_1(x)\} \cap \widehat{Q} = \emptyset$ and $\left|Q \setminus \widehat{Q}\right| \le 12 L h_{\max}$. Using Equation (5.24) and $h_1 \chi_\mu\left(\frac{\vec{e}_1 \cdot x}{h_1}\right) \le h_1 \mu(1-\mu) < h_1$ we get for an $x \in Q$ with $\widetilde{z}_1(x) = H_1(x)$, that $\operatorname{dist}_{\vec{e}_2}(x, \partial Q) < h_1$. Since $\widehat{Q} \subseteq \{x \in Q \,:\, \operatorname{dist}_{\vec{e}_2}(x, \partial Q) > h_1\}$, we get the first assertion. The second is true because for $x = \lambda_1 h_1 \vec{e}_1 + \lambda_2 h_2 \vec{a}^1_{t,\sigma} \in Q \setminus \widehat{Q}$, with $\lambda_1,\ \lambda_2 \in \mathbb{R}$ there exists a $y=\mu_1 h_1 \vec{e}_1+\mu_2 h_2 \vec{a}^1_{t,\sigma} \in P(x)$, $\mu_1,\ \mu_2 \in \mathbb{R}$ with $|\mu_1-\lambda_1|\leq 1$, $|\mu_2-\lambda_2|\leq 1$ and $\operatorname{dist}_{\vec{e}_2}(y,\partial\Omega)\leq h_1$ and thus we get

$$\operatorname{dist}_{\vec{e}_2}(x,\partial Q)\leq \operatorname{dist}_{\vec{e}_2}(y,\partial Q)+\|x-y\|\leq h_1+h_1+h_2\leq 3h_{\max},$$

since $\|x-y\| = \left\|(\lambda_1-\mu_1)\,h_1\vec{e}_1+(\lambda_2-\mu_2)\,h_2\vec{a}^1_{t,\sigma}\right\| \leq h_1+h_2$. Finally, we can show $\left|Q\setminus\widehat{Q}\right|\leq 2\cdot 2L\cdot 3h_{\max}=12Lh_{\max}$, as asserted. Furthermore the set $\widehat{Q}$ was defined in such way that $\partial\widehat{Q}\cap Q_\sigma\subseteq\left\{x\in Q_\sigma\ :\ \left\lfloor\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}\right\rfloor\in\mathbb{Z}\right\}$. Thus we have for each $x\in Q_\sigma\cap\widehat{Q}$, that the laminate

$$\widetilde{w}_{\sigma,t}(x):=v_{h_1}(x)+h_2\chi_{\lambda_t}\left(\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}\right)\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1\\ t^\eta\end{pmatrix}\right\|R_\sigma^{-1}\vec{e}_1$$

fulfills $\widetilde{w}_{\sigma,t}(x)=v_{h_1}(x)$ for $x\in\partial\widehat{Q}\cap Q_\sigma$. Then we can compute for $x\in Q_\sigma\cap\widehat{Q}$ with $\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}-\left\lfloor\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}\right\rfloor\in(0,\lambda_t)$, that

$$\begin{aligned}&\nabla\widetilde{w}_{\sigma,t}(x)=F_\sigma+(1-\lambda_t)\,\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1\\ t^\eta\end{pmatrix}\right\|R_\sigma^{-1}\vec{e}_1\otimes\vec{a}^1_{t,\sigma}\\ &=R_\sigma^{-1}\left[(1-\lambda_t)\,F_{t,\sigma}(0)+\lambda_t F_{t,\sigma}(1)+(1-\lambda_t)\,(F_{t,\sigma}(1)-F_{t,\sigma}(0))\right]=R_\sigma^{-1}F_{t,\sigma}(1).\end{aligned}$$

Analogously we get $\nabla\widetilde{w}_{\sigma,t}(x)=R_\sigma^{-1}F_{t,\sigma}(0)$ for $x\in Q_\sigma\cap\widehat{Q}$ with $\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}-\left\lfloor\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}\right\rfloor\in(\lambda_t,1)$. In order to get a continuous function on the vertical lines $\partial Q_{-1}\cap\partial Q_1\cap\widehat{Q}$ we define for $i\in\{1,2\}$ the auxiliary functions

$$\begin{aligned}G_{\sigma,i}:Q_\sigma\cap\widehat{Q}\ &\to\ \mathbb{R}\\ x\ &\mapsto\ v_{h_1}(x)+\operatorname{dist}_{\vec{e}_1}(x,\partial Q_\sigma)\left(R_\sigma^{-1}\vec{e}_1\right)_i,\end{aligned}$$

and we write

$$\begin{aligned}\widehat{z}_{\sigma,i,t}:Q_\sigma\cap\widehat{Q}\ &\to\ \mathbb{R}\\ x\ &\mapsto\ \begin{cases}\min\left\{G_{\sigma,i}(x),(\widetilde{w}_{\sigma,t}(x))_i\right\} & \text{if } \left(R_\sigma^{-1}\vec{e}_1\right)_i\geq 0\\ \max\left\{G_{\sigma,i}(x),(\widetilde{w}_{\sigma,t}(x))_i\right\} & \text{if } \left(R_\sigma^{-1}\vec{e}_1\right)_i< 0\end{cases}.\end{aligned}$$

Finally we define $\widehat{z}_t:\widehat{Q}\to\mathbb{R}^2$ by $\widehat{z}_t(x)=\begin{pmatrix}\widehat{z}_{\sigma,1,t}(x)\\ \widehat{z}_{\sigma,2,t}(x)\end{pmatrix}$ for $x\in Q_\sigma\cap\widehat{Q}$ and in the remaining points such that it is continuous, which is possible by construction of $G_{\sigma,i}$. In particular we get $\widehat{z}_t(x)=v_{h_1}(x)$ for $x\in\partial\overline{Q}_{-1}\cap\partial\overline{Q}_1$. As in Equation (5.24) we can show, that $\left\{x\in Q_\sigma\cap\widehat{Q}\ :\ G_{\sigma,1}(x)=\widehat{z}_{\sigma,1,t}(x)\right\}=\left\{x\in Q_\sigma\cap\widehat{Q}\ :\ G_{\sigma,2}(x)=\widehat{z}_{\sigma,2,t}(x)\right\}$.

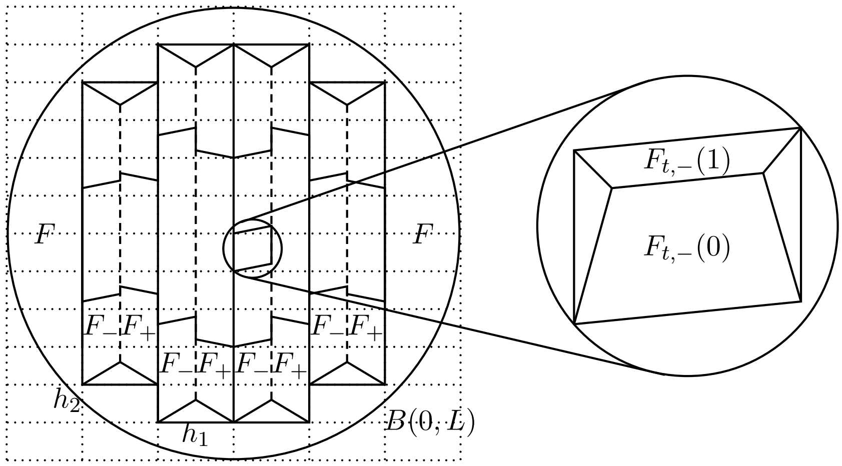

Figure 5.19: $F_+ = F_{+1},\ F_- = F_{-1},\ F_{t,-} = F_{t,-1}$

Bringing all together we can define the Lipschitz continuous function

$$\begin{aligned} z_t : B(0,L) &\to \mathbb{R}^2 \\ x &\mapsto \begin{cases} \widehat{z}_t(x) & \text{for } x \in \widehat{Q} \\ \widetilde{z}(x) & \text{for } x \in Q \setminus \widehat{Q} \\ Fx & \text{for } x \in B(0,L) \setminus Q \end{cases} \end{aligned}$$

and the corresponding γ_t is defined by

$$\begin{aligned} \gamma_t : B(0,L) &\to \mathbb{R} \\ x &\mapsto \begin{cases} \sigma t^{\eta-1} & \text{for } \nabla z_t(x) = R_\sigma^{-1} F_{t,\sigma}(1),\ \sigma \in \{-1,1\} \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

One can see, that $z \in W_F^{1,\infty}\left(\overline{B(0,L)};\mathbb{R}^2\right)$ and $\gamma_t \in BV\left(B(0,L)\right) \cap L^{\max\{p,q\}}\left(B(0,L)\right)$. By construction of γ_t we get as in Equation (5.6) in the proof of Lemma 5.1, that

$$W_e\left(F_{t,\sigma}(1)\left(\mathbb{1} - \sigma t^{\eta-1}\vec{e}_1 \otimes \vec{e}_2\right)\right) \le C\left(t^q + 1\right),$$

for a constant $C > 0$ independent of t. Let $A_\sigma := \left\{x \in Q_\sigma \cap \widehat{Q}\ :\ G_{\sigma,1}(x) = \widehat{z}_{\sigma,1,t}(x)\right\}$, then we get with similar computations as in Lemma 5.1, that

$$|B(0,L)|\, I\left(z_t,\gamma_t\right) \leq \zeta\, |B(0,L)| + |B(0,L)\setminus Q|\, \frac{W_e\left(F\right)}{\varepsilon} + \delta V_x\left(\gamma_t,Q\right) \\ + \sum_{\sigma\in\{-1,1\}} \int_{A_\sigma} \frac{1}{\varepsilon} W_e\left(\nabla \widehat{z}_t\right)\, d\lambda_2 + \int_{Q-\widehat{Q}} \frac{1}{\varepsilon} W_e\left(\nabla \widetilde{z}\right)\, d\lambda_2, \tag{5.25}$$

for t big enough. Furthermore we can compute

$$V_x\left(\gamma_t,Q\right) \leq \underbrace{2\cdot\frac{2L}{h_2}}_{=\text{number of jumps}} \cdot 2L\frac{t-1}{t^\eta}\cdot t^{\eta-1} \leq 8\frac{L^2}{h_2}. \tag{5.26}$$

Next, we have for $x \in \widehat{Q}\cap Q_\sigma$, that

$$\widehat{z}_{\sigma,1,t}(x) = G_{\sigma,1}(x) \Leftrightarrow \operatorname{dist}_{\vec{e}_1}\left(x,\partial Q_\sigma\right) \leq h_2 \chi_{\lambda_t}\left(\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}\right)\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\|$$

and

$$h_2 \chi_{\lambda_t}\left(\frac{\vec{a}^1_{t,\sigma}\cdot x}{h_2}\right)\frac{1}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\| \leq h_2\frac{\left(1-\lambda_t\right)\lambda_t}{\alpha_t}\left\|\begin{pmatrix} t-1 \\ t^\eta \end{pmatrix}\right\| \leq \sqrt{2}h_2\gamma_0$$

for t big enough. Thus, we get for $x\in A_\sigma$ that $\operatorname{dist}_{\vec{e}_1}\left(x,\partial Q_\sigma\right) \leq \sqrt{2}\gamma_0 h_2$, which implies

$$|A_\sigma| = \left|\left\{x\in Q_\sigma\cap\widehat{Q} \;:\; \widehat{z}_{\sigma,1,t}(x) = G_{\sigma,1}(x)\right\}\right| \leq \sqrt{2}\gamma_0 h_2\cdot 2\cdot\frac{2L}{h_1}\cdot 2L = 8\sqrt{2}\gamma_0\frac{h_2 L^2}{h_1}.$$

For $x,y\in A_\sigma$ we get $W_e\left(\nabla\widehat{z}_t(x)\right) = W_e\left(\nabla\widehat{z}_t(y)\right)$ if the gradients exist, and since W_e is finite valued we get

$$\int_{A_\sigma} W_e\left(\nabla\widehat{z}_t(x)\right)\, d\lambda_2 \leq C\frac{L^2 h_2}{h_1}, \tag{5.27}$$

for a suited constant $C = C(\gamma_0) > 0$. We have already shown, that the measure of the set $Q\setminus\widehat{Q}$ fulfills the inequality $\left|Q\setminus\widehat{Q}\right| \leq 12Lh_{\max}$ and as above we get, using the definition $B_\sigma := \left\{x\in Q\setminus\widehat{Q} \;:\; \nabla\widetilde{z}_t(x) \text{ exists}\right\}$, that

$$\sup_{x\in B_\sigma}\left\{W_e\left(\nabla\widetilde{z}_t(x)\right)\right\} = \sup_{x\in B_\sigma}\left\{W_e\left(F_{-1}\right),\ W_e\left(F_1\right),\ W_e\left(\nabla\begin{pmatrix} H_1(x) \\ H_2(x)\end{pmatrix}\right)\right\} < \infty.$$

This implies

$$\int_{Q\setminus\widehat{Q}} W_e\left(\nabla\widetilde{z}(x)\right)\, d\lambda_2 \leq CLh_{\max}, \tag{5.28}$$

for a suited constant $C = C(F) > 0$. At the beginning of the proof we have shown, that $|B(0, L) \setminus Q| \leq CLh_{\max}$. Using this and the Equations (5.25) – (5.28), we get

$$|B(0,L)|\, I(z_t, \gamma_t) \leq \zeta\, |B(0,L)| + C \left(\frac{h_{\max} L}{\varepsilon} + \frac{L^2}{h_2}\delta + \frac{h_2 L^2}{h_1 \varepsilon} \right).$$

Choose now $L > h_1 > 0$ and $h_2 := \frac{h_1^2}{L} < h_1$, then we get

$$|B(0,L)|\, I(z_t, \gamma_t) \leq \zeta\, |B(0,L)| + C \left(\frac{h_1 L}{\varepsilon} + \frac{L^3}{2h_1^2}\delta \right).$$

The function $f : (0, \infty) \to \mathbb{R}\ x \mapsto \frac{L}{\varepsilon x} + \frac{1}{2}L^3 x^2 \delta$ is minimal for $x = \frac{1}{(L^2 \varepsilon \delta)^{\frac{1}{3}}}$, since $f'(x) = \delta L^3 x - \frac{L}{\varepsilon x^2} \geq 0 \Leftrightarrow x^3 \geq \frac{1}{\delta \varepsilon L^2}$. We have to ensure that $L > h_1$, which means, that we are looking for the minimum of $f|_{(\frac{1}{L}, \infty)}$. This minimum is attained again in $x = \frac{1}{(L^2 \varepsilon \delta)^{\frac{1}{3}}}$, since $\varepsilon \delta < L$ implies $\frac{1}{(L^2 \varepsilon \delta)^{\frac{1}{3}}} > \frac{1}{L}$. Thus we get

$$|B(0,L)|\, I(z_t, \gamma_t) \leq \zeta\, |B(0,L)| + C \left(\frac{L^{\frac{5}{3}} \delta^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}} \right),$$

for a constant $C = C(F) > 0$ independent of $\varepsilon,\ \delta,\ L,\ p,\ q$ as asserted.

□

6 Lower bounds

In this chapter we show results for $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} > 1$ and thus especially for the case of no hardening, i.e. $p = 1$.

6.1 Algebraic estimates

Next, we state some simple inequalities based on convexity, which are frequently used in this chapter.

Remark 6.1. *For all $x, a \in \mathbb{R}$ and $q \geq 1$ we have*

$$|x - a|^q \geq \frac{1}{2^{q-1}} |x|^q - |a|^q , \tag{6.1}$$

since $|.|^q$ is convex.

Definition 6.2. *Denote by*

$$Con(2) := \left\{ M \in \mathbb{R}^{2\times 2} \ : \ M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} , \ a, b \in \mathbb{R} \right\}$$

the set of the conformal matrices and by

$$Anticon(2) := \left\{ M \in \mathbb{R}^{2\times 2} \ : \ M = \begin{pmatrix} c & d \\ d & -c \end{pmatrix} , \ c, d \in \mathbb{R} \right\} \setminus \{0\}$$

the set of the anti-conformal matrices.

Remark 6.3. *Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, then there exists a unique decomposition $X = X^+ + X^-$ with the conformal part $X^+ = \frac{1}{2} \begin{pmatrix} X_{11} + X_{22} & -(X_{21} - X_{12}) \\ X_{21} - X_{12} & X_{11} + X_{22} \end{pmatrix} \in Con(2)$ of X and the anti-conformal part $X^- = \frac{1}{2} \begin{pmatrix} X_{11} - X_{22} & X_{21} + X_{12} \\ X_{21} + X_{12} & -(X_{11} - X_{22}) \end{pmatrix} \in Anticon(2)$ of X. Let $A \in Con(2)$ and $B \in Anticon(2)$, then we have $A : B = tr(A^T B) = 0$ and thus $\mathbb{R}^{2\times 2} = Con(2) \oplus Anticon(2)$. We have*

$$2 \left\| X^- \right\|^2 = \|X\|^2 - 2\det(X) \quad \text{or} \quad \left\| X^- \right\| = \frac{1}{\sqrt{2}} \sqrt{\|X\|^2 - 2\det(X)},$$

$$2 \left\| X^+ \right\|^2 = \|X\|^2 + 2\det(X) \quad \text{or} \quad \left\| X^+ \right\| = \frac{1}{\sqrt{2}} \sqrt{\|X\|^2 + 2\det(X)},$$

and thus

$$2\det(X) = \left\|X^+\right\|^2 - \left\|X^-\right\|^2.$$

Furthermore we get for $M \in Con(2)$, that

$$\operatorname{dist}^2(M, SO(2)) = \inf_{A\in SO(2)} \|M - A\|^2 = \inf_{A\in SO(2)} \left(2\|M\vec{e}_1 - A\vec{e}_1\|^2\right)$$
$$= \inf_{A\in SO(2)} \left(2\|M\vec{e}_1\|^2 - 4M\vec{e}_1\cdot A\vec{e}_1\right) + 2 = \|M\|^2 - 2\sqrt{2}\|M\| + 2 = \left(\|M\| - \sqrt{2}\right)^2,$$

since $\|M\| = \sqrt{\|M\vec{e}_1\|^2 + \|M\vec{e}_2\|^2} = \sqrt{2}\|M\vec{e}_1\|$. *For an arbitrary $X \in \mathbb{R}^{2\times 2}$ we get*

$$\operatorname{dist}^2(X, SO(2)) = \inf_{A\in SO(2)} \left\|X^+ + X^- - A\right\|^2$$
$$= \inf_{A\in SO(2)} \left[\left\|X^+ - A\right\|^2 + 2X^- : \left(X^+ - A\right) + \left\|X^-\right\|^2\right]$$
$$= \operatorname{dist}^2\left(X^+, SO(2)\right) + \left\|X^-\right\|^2 = \left(\left\|X^+\right\| - \sqrt{2}\right)^2 + \left\|X^-\right\|^2$$
$$= \|X\|^2 + 2 - 2\sqrt{\|X\|^2 + 2\det(X)}$$

and thus we have

$$\operatorname{dist}^2(X, SO(2)) \geq \left\|X^-\right\|^2 = \frac{1}{2}\|X\|^2 - \det(X) \geq 0. \tag{6.2}$$

Next, we summarise some useful lower bounds of the special elastic energy density $\operatorname{dist}^q(F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2), SO(2))$. Thereby some ideas of the proofs have already been used in the thesis of Carolin Kreisbeck [42, Chapter 7].

Lemma 6.4. *Let $F \in \mathbb{R}^{2\times 2}$ and $\gamma \in \mathbb{R}$, then we have*

i) for $q \geq 1$ that $\operatorname{dist}^q(F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2), SO(2)) \geq \frac{1}{2^{q-1}}\operatorname{dist}^q(F, SO(2)) - \|\gamma F\vec{e}_1\|^q$

ii) for $q \geq 2$ that $\operatorname{dist}^q(F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2), SO(2)) \geq \frac{1}{2^{q-1}}\|F\vec{e}_1\|^q - 1$,

iii) for $q \geq 2$ that $\frac{1}{4^{q-1}}\|F\vec{e}_2\|^q \leq (|\gamma|^q + 1)(\operatorname{dist}^q(F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2), SO(2)) + 1)$,

iv) and for $q \geq 2$ there exists a constant $C = C(q) > 0$, so that

$$|\det(F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2))|^{\frac{q}{2}} = |\det(F)|^{\frac{q}{2}} \leq C\left(\operatorname{dist}^q(F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2), SO(2)) + 1\right).$$

Proof:
i) Let $A \in SO(2)$ then we have

$$\|F(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2) - A\| \geq |\|F - A\| - \|\gamma F\vec{e}_1\otimes\vec{e}_2\||.$$

Equation (6.1) implies

$$\|F(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2)-A\|^q \geq \frac{1}{2^q}\|F-A\|^q - \|\gamma F\vec{e}_1\|^q,$$

and since $A\in SO(2)$ was chosen arbitrary we get i).
ii) Let $G := F(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2)$, then we have

$$\begin{aligned}
&\operatorname{dist}^q(G,SO(2)) = \inf_{A\in SO(2)}\|G-A\|^q = \inf_{a\in\mathbb{S}^1}\left(\left\|G-a\otimes\vec{e}_1-a^\perp\otimes\vec{e}_2\right\|^2\right)^{\frac{q}{2}}\\
&= \inf_{a\in\mathbb{S}^1}\left(\|F\vec{e}_1-a\|^2+\left\|F\vec{e}_2-\gamma F\vec{e}_1-a^\perp\right\|^2\right)^{\frac{q}{2}}\\
&\geq \inf_{a\in\mathbb{S}^1}\left(\|F\vec{e}_1-a\|^q+\left\|F\vec{e}_2-\gamma F\vec{e}_1-a^\perp\right\|^q\right)\\
&= \|F\vec{e}_1-a_{\text{inf}}\|^q+\left\|F\vec{e}_2-\gamma F\vec{e}_1-a_{\text{inf}}^\perp\right\|^q, \qquad (6.3)
\end{aligned}$$

where $a_{\text{inf}}\in\mathbb{S}^1$ is defined by the last equality. Now we have

$$|\|F\vec{e}_1\|-1| = |\|F\vec{e}_1\|-\|a_{\text{inf}}\|| \leq \|F\vec{e}_1-a_{\text{inf}}\|$$

and with Equation (6.1) we get

$$\operatorname{dist}^q(F(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2),SO(2)) \geq |\|F\vec{e}_1\|-1|^q \geq \frac{1}{2^{q-1}}\|F\vec{e}_1\|^q-1.$$

This is ii).
iii) With the notations from above we get

$$\operatorname{dist}^q(G,SO(2)) \geq \left\|F\vec{e}_2-\gamma F\vec{e}_1-a_{\text{inf}}^\perp\right\|^q \geq \frac{1}{4^{q-1}}\|F\vec{e}_2\|^q-\frac{1}{2^{q-1}}\|\gamma F\vec{e}_1\|^q-1,$$

where we used Equation (6.1) twice and thus we get

$$\frac{1}{4^{q-1}}\|F\vec{e}_2\|^q \leq \operatorname{dist}^q(G,SO(2))+\frac{1}{2^{q-1}}\|\gamma F\vec{e}_1\|^q+1 \leq (|\gamma|^q+1)(\operatorname{dist}^q(G,SO(2))+1),$$

where we have used ii) in the last inequality.
iv) We have $\det(F(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2))=\det(F)$, since $\det(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2)=1$ for all $\gamma\in\mathbb{R}$. Let $J=\vec{e}_2\otimes\vec{e}_1-\vec{e}_1\otimes\vec{e}_2$ be the counterclockwise rotation by $\pi/2$ in the plane. Then one calculates, for $\gamma\in\mathbb{R}$ and $a_{\text{inf}}\in\mathbb{S}^1$ defined in ii), that

$$\begin{aligned}
\det(F) &= JF\vec{e}_1\cdot F\vec{e}_2 = JF\vec{e}_1\cdot(F\vec{e}_2-\gamma F\vec{e}_1)\\
&= J(F\vec{e}_1-a_{\text{inf}})\cdot(F\vec{e}_2-\gamma F\vec{e}_1)+Ja_{\text{inf}}\cdot(F\vec{e}_2-\gamma F\vec{e}_1)\\
&= J(F\vec{e}_1-a_{\text{inf}})\cdot\left(F\vec{e}_2-\gamma F\vec{e}_1-a_{\text{inf}}^\perp\right)+J(F\vec{e}_1-a_{\text{inf}})\cdot a_{\text{inf}}^\perp\\
&\quad +Ja_{\text{inf}}\cdot\left(F\vec{e}_2-\gamma F\vec{e}_1-a_{\text{inf}}^\perp\right)+1.
\end{aligned}$$

Using Equation (6.3), we get

$$\|F\vec{e}_1 - a_{\inf}\| + \left\|F\vec{e}_1 - \gamma F\vec{e}_1 - a_{\inf}^{\perp}\right\| \leq 2\,\mathrm{dist}\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right),$$

where the factor 2 can be improved to $\sqrt{2}$, if one uses the concavity of $|.|^{\frac{1}{2}}$. Next, we get

$$\|F\vec{e}_1 - a_{\inf}\| \left\|F\vec{e}_1 - \gamma F\vec{e}_1 - a_{\inf}^{\perp}\right\| \leq \frac{1}{2}\,\mathrm{dist}^2\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right),$$

where we have used $a^2 + b^2 \geq 2ab$ for $a, b \in \mathbb{R}$. Finally, we get

$$\begin{aligned} |\det(F) - 1| &\leq \frac{1}{2}\,\mathrm{dist}^2\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + 2\,\mathrm{dist}\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \\ &\leq \frac{3}{2}\,\mathrm{dist}^2\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + 1 \qquad (6.4) \end{aligned}$$

and thus by Equation (6.1), that

$$\begin{aligned} \frac{1}{2^{\frac{q}{2}-1}}\,|\det(F)|^{\frac{q}{2}} &\leq \left(\frac{3}{2}\,\mathrm{dist}^2\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + 1\right)^{\frac{q}{2}} + 1 \\ &\leq \frac{3^{\frac{q}{2}}}{2}\,\mathrm{dist}^q\left(F\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + 2^{\frac{q}{2}-1} + 1, \qquad (6.5) \end{aligned}$$

where we have used the inequality $(a+b)^{\alpha} \leq 2^{\alpha-1}\left(a^{\alpha} + b^{\alpha}\right)$, for $\alpha \geq 1$ and $a, b \in \mathbb{R}$. For $C := \max\left\{\frac{6^{\frac{q}{2}}}{4}, 2^{q-2} + 2^{\frac{q}{2}-1}\right\}$ we obtain the statement.

□

6.2 Lower bound for affine boundary values in $\mathcal{N}^{(2)} \setminus SO(2)$

In the following we will show, that the infimum of the energy $E_{\varepsilon,\delta}[u]$ over all configurations u, with uniformly bounded L^∞-norm and boundary values $u = F^* \in \mathcal{N}^{(2)} \setminus SO(2)$ on $\partial\Omega$, has a lower bound which scales as δ, for small δ. The proof is done by contradiction. If one does not have this scaling property one can show, that a minimizing sequence $\{(u_n, \gamma_n)\}_{n\in\mathbb{N}}$ with $u_n = F^*$ on $\partial\Omega$ has a vanishing elastic energy for $n \to \infty$. In the case of at least quadratic growth $q \geq 2$, this implies that $\det(F^*) = 1$. Using the property of Null Lagrangian's and the boundary values of u_n, one can show that the q-th radical of the elastic part of the energy is bounded from below by

$$C \operatorname{dist}\left(F^* - \frac{1}{|\Omega|}\int_\Omega \gamma_n \nabla u_n \vec{e}_1 \otimes \vec{e}_2 \, d\lambda_2, \mu_n SO(2)\right),$$

for a fixed $\mu_n \in [0,1]$, with $n \in \mathbb{N}$, and a constant $C > 0$. Since, the following lemma implies that $\int_\Omega \gamma_n \left(\nabla u_n \vec{e}_1\right) \, d\lambda_2$ vanishes for $n \to \infty$, we obtain, that $F^* \in \mathcal{N}^{(2)}$ has to be in $\{\mu R \,:\, \mu \in [0,1],\ R \in SO(2)\}$, which implies $F^* \in SO(2)$. Because of $\det(F^*) = 1$, in the case of at least quadratic growth $q \geq 2$, we can show the lower bound in this case for each $F^* \in \mathbb{R}^{2\times 2} \setminus SO(2)$.

Lemma 6.5. *Let $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set with C^1-boundary and $F \in \mathbb{R}^{2\times 2}$. Let $p \geq 1$, $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$ with $u_n = F$ on $\partial\Omega$ and $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq BV(\Omega) \cap L^p(\Omega)$ a sequence with*

$$\|\gamma_n\|_{L^p(\Omega)} \overset{n\to\infty}{\longrightarrow} 0 \text{ and } V_x(\gamma_n, \Omega) = |D_x\gamma_n|(\Omega) = \int_\Omega d\,|D_x\gamma_n| \overset{n\to\infty}{\longrightarrow} 0,$$

where we have used Lemma 3.29. If there exists a constant $K > 0$ with $\|u_n\|_{L^\infty(\Omega;\mathbb{R}^2)} \leq K$ for all $n \in \mathbb{N}$, then we have

$$\left|\int_\Omega \gamma_n \left(\nabla u_n \vec{e}_1\right) \, d\lambda_2\right| \to 0 \text{ for } n \to \infty.$$

Proof:
Let $v_n := u_n - F$, then we have $v_n \in W_0^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$ and we get $\|v_n\|_{L^\infty(\Omega;\mathbb{R}^2)} \leq K + C\|F\|$, where $C = C(\Omega) > 0$ is independent of n. Next, we have

$$\left|\int_\Omega \gamma_n \left(\nabla u_n \vec{e}_1\right) \, d\lambda_2\right| \leq \sum_{i=1}^{2}\left|\int_\Omega \gamma_n \left(\nabla v_n \vec{e}_1\right)_i \, d\lambda_2\right| + \underbrace{\int_\Omega |\gamma_n| \, d\lambda_2 \, \|F\vec{e}_1\|}_{\overset{n\to\infty}{\longrightarrow} 0},$$

since Ω is bounded and $\|\gamma_n\|_{L^1(\Omega)} \leq \|\gamma_n\|_{L^p(\Omega)} |\Omega|^{1-\frac{1}{p}} \overset{n\to\infty}{\longrightarrow} 0$. Choose now $i \in \{1, 2\}$, then Corollary 3.21 implies, using the notation $v_{n,i} := (v_n)_i$,

$$\left|\int_\Omega \gamma_n \left(\nabla v_n \vec{e}_1\right)_i \; d\lambda_2\right| = \left|\gamma_n \partial_x v_{n,i} \mathcal{L}^2(\Omega)\right| = \left|D_x\left(\gamma_n v_{n,i}\right)(\Omega) - v_{n,i} D_x \gamma_n(\Omega)\right|$$

$$\leq \left|D_x\left(\gamma_n v_{n,i}\right)(\Omega)\right| + \left|\int_\Omega v_{n,i} \; dD_x\gamma_n\right| \leq \left|D_x\left(\gamma_n v_{n,i}\right)(\Omega)\right| + (K + C\left\|F\right\|) \underbrace{\int_\Omega d\left|D_x\gamma_n\right|}_{\overset{n\to\infty}{\longrightarrow} 0}.$$

In order to compute the first part, namely $\left|D_x\left(\gamma_n v_{n,i}\right)(\Omega)\right|$, one defines for $k \in \mathbb{N}$ the sets $\Omega_k := \left\{x \in \Omega \;:\; \operatorname{dist}(x, \partial\Omega) > \frac{1}{k}\right\}$ and the functions $\varphi_k : \Omega \to [0,1]$ by

$$\varphi_k(x) := \begin{cases} 1 & \text{if} \quad x \in \Omega_k \\ k \operatorname{dist}(x, \partial\Omega) & \text{if} \quad x \in \Omega \setminus \Omega_k \end{cases}.$$

The function φ_k is Lipschitz continuous with Lipschitz-constant k, since the distance-function is 1-Lipschitz. By Corollary 3.21 we get $\gamma_n v_{n,i} \in BV(\Omega)$ and we conclude, that for each $k \in \mathbb{N}$ the integral $\int_\Omega \varphi_k \; d\left|D_x\left(\gamma_n v_{n,i}\right)\right|$ is bounded from above by the value $\int_\Omega \; d\left|D_x\left(\gamma_n v_{n,i}\right)\right| = \left|D_x\left(\gamma_n v_{n,i}\right)\right|(\Omega) < \infty$, where we used Lemma 3.29. Thus we get with help of Lemma 3.10 and Definition 3.16, that

$$\left|D_x\left(\gamma_n v_{n,i}\right)(\Omega)\right| = \left|\int_\Omega \lim_{k\to\infty} \varphi_k \; dD_x(\gamma_n v_{n,i})\right| = \left|\lim_{k\to\infty} \int_\Omega \varphi_k \; dD_x(\gamma_n v_{n,i})\right|$$

$$= \left|-\lim_{k\to\infty} \int_\Omega \gamma_n v_{n,i} \partial_x \varphi_k \; d\lambda_2\right| \leq \lim_{k\to\infty} \int_{\Omega - \Omega_k} \left|\gamma_n\right| \underbrace{\left|v_{n,i}\right|}_{\leq L_{n,i}\frac{1}{k}} \underbrace{\left|\partial_x \varphi_k\right|}_{\leq k} \; d\lambda_2 = 0,$$

where $L_{n,i}$ is the Lipschitz-constant of $v_{n,i}$.
Thus we get

$$\left|\int_\Omega \gamma_n \left(\nabla u_n \vec{e}_1\right) \; d\lambda_2\right| \overset{n\to\infty}{\longrightarrow} 0$$

as asserted.

□

Next, we can prove a lower bound for the energy, which scales like δ, if we have affine boundary values $F^* \in \mathcal{N}^{(2)} \setminus SO(2)$.

Theorem 6.6. *Let $F^* \in \mathbb{R}^{2\times 2} \setminus \left\{\lambda B \in \mathbb{R}^{2\times 2} \;:\; \lambda \in [0,1], B \in SO(2)\right\}$, $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set with C^1-boundary. Let $q, p \geq 1$, $K_1, K_2 > 0$ and the elastic energy density is given by $W_e(F) = \operatorname{dist}^q(F, SO(2))$, for $F \in \mathbb{R}^{2\times 2}$, then we have: $\exists \eta = \eta(K_1, K_2, p, q) > 0$, $\forall \delta, \varepsilon \leq K_1$,*

$$\inf_{\substack{u \in W^{1,\infty}(\Omega;\mathbb{R}^2) \,:\, u = F^* \text{ on } \partial\Omega \\ \|u\|_{L^\infty(\Omega;\mathbb{R}^2)} < K_2}} E_{\varepsilon,\delta}[u] \geq \eta\delta.$$

If we have additionally $q \geq 2$, then we get the above statement for all $F^* \in \mathbb{R}^{2\times 2} \setminus SO(2)$.

Proof:
Assume this is not true, namely: $\forall \eta > 0$, $\exists \delta, \varepsilon \leq K_1$, $\exists u, \gamma$: $I(u,\gamma) < \eta\delta \leq \eta K_1$. This means, that we have sequences $\{\eta_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$, $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2)$ and $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)$ with $\lim_{n\to\infty} \eta_n = 0$, $\|u_n\|_{L^\infty(\Omega;\mathbb{R}^2)} < K_2$,

$$\frac{1}{|\Omega|}\int_\Omega \operatorname{dist}^q(\nabla u_n(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2), SO(2))\, d\lambda_2 < \eta_n K_1^2,$$
$$\frac{1}{|\Omega|}\int_\Omega |\gamma_n|^p \; d\lambda_2 < \eta_n K_1 \text{ and } \frac{1}{|\Omega|} V_x(\gamma_n,\Omega) < \eta_n.$$

Since Ω is bounded and $q \geq 1$ we get by Hölder-inequality

$$\fint_\Omega \operatorname{dist}(\nabla u_n(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2), SO(2))\, d\lambda_2 \leq \left(\fint_\Omega \operatorname{dist}^q(\nabla u_n(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2), SO(2))\, d\lambda_2\right)^{\frac{1}{q}}.$$

This implies

$$\begin{aligned}
&\left(\eta_n K_1^2\right)^{\frac{1}{q}} |\Omega| \geq \int_\Omega \operatorname{dist}(\nabla u_n(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2), SO(2))\, d\lambda_2 \\
&= \int_\Omega \inf_{a\in\mathbb{S}^1} \left(\|\nabla u_n \vec{e}_1 - a\|^2 + \left\|\nabla u_n \vec{e}_2 - \gamma_n \nabla u_n \vec{e}_1 - a^\perp\right\|^2\right)^{\frac{1}{2}} d\lambda_2 \\
&= \int_\Omega \left(\|\nabla u_n \vec{e}_1 - a_{n,\inf}\|^2 + \left\|\nabla u_n \vec{e}_2 - \gamma_n \nabla u_n \vec{e}_1 - a^\perp_{n,\inf}\right\|^2\right)^{\frac{1}{2}} d\lambda_2 \qquad (6.6)\\
&\geq \frac{1}{\sqrt{2}} \int_\Omega \|\nabla u_n \vec{e}_1 - a_{n,\inf}\| + \left\|\nabla u_n \vec{e}_2 - \gamma_n \nabla u_n \vec{e}_1 - a^\perp_{n,\inf}\right\| d\lambda_2 \\
&\geq \frac{1}{\sqrt{2}} \left(\left\|\int_\Omega \nabla u_n \vec{e}_1 - a_{n,\inf}\, d\lambda_2\right\| + \left\|\int_\Omega \nabla u_n \vec{e}_2 - \gamma_n \nabla u_n \vec{e}_1 - a^\perp_{n,\inf}\, d\lambda_2\right\|\right),
\end{aligned}$$

where we have used $\sqrt{\frac{a+b}{2}} \geq \frac{\sqrt{a}}{2} + \frac{\sqrt{b}}{2}$ for $a, b \geq 0$, fixed a representative of γ_n and defined $a_{n,\inf} : \Omega \to \mathbb{S}^1 \cup \{0\}$ as the pointwise infimum, if it is unique and as $\vec{e}_1$ otherwise, for all $x \in \Omega$ for which u_n is differentiable and zero else. Thereby we will see later on that if the infimum is not unique, then each point in $\mathbb{S}^1$ is an infimum. Next, we will show that

$$a_{n,\inf}(x) = \begin{cases} \frac{\nabla u_n(x)\vec{e}_1 - J(\nabla u_n(x)\vec{e}_2 - \gamma_n(x)\nabla u_n(x)\vec{e}_1)}{\|\nabla u_n(x)\vec{e}_1 - J(\nabla u_n(x)\vec{e}_2 - \gamma_n(x)\nabla u_n(x)\vec{e}_1)\|} & \text{if} \quad \nabla u_n(x) \text{ exists} \\ 0 & \text{otherwise} \end{cases},$$

where we define $\frac{\nabla u_n(x)\vec{e}_1 - J(\nabla u_n(x)\vec{e}_2 - \gamma_n(x)\nabla u_n(x)\vec{e}_1)}{\|\nabla u_n(x)\vec{e}_1 - J(\nabla u_n(x)\vec{e}_2 - \gamma_n(x)\nabla u_n(x)\vec{e}_1)\|} := \vec{e}_1$ if the denominator is zero. This implies that the function $a_{n,\inf} : \Omega \to \mathbb{S}^1 \cup \{0\}$ is measurable. For $\vec{b}, \vec{c} \in \mathbb{R}^2$ we get, that the map $A : \mathbb{S}^1 \to \mathbb{R}$, $\vec{a} \mapsto \|\vec{b} - \vec{a}\|^2 + \|\vec{c} - \vec{a}^\perp\|^2$ can be written as $A(\vec{a}) = \|\vec{b}\|^2 - 2 < \vec{b}, \vec{a} > + \|\vec{a}\|^2 + \|\vec{c}\|^2 - 2 < \vec{c}, \vec{a}^\perp > + \|\vec{a}^\perp\|^2$ and is therefore minimal if $<\vec{b},\vec{a}> + <\vec{c},\vec{a}^\perp> = <\vec{b}-J\vec{c},\vec{a}>$ is maximal. Thus the map A is constant if $\vec{b}-J\vec{c}=0$ and if $\vec{b}-J\vec{c}\neq 0$ it is minimal for $\vec{a}=\frac{\vec{b}-J\vec{c}}{\|\vec{b}-J\vec{c}\|}$, which leads to the above formula for $a_{n,\inf}(x)$.

Therefore one can define $\vec{a}_{n,\Omega} := \int_\Omega a_{n,\inf}\ d\lambda_2 \in \mathbb{R}^2$ and get $\|\vec{a}_{n,\Omega}\| \leq |\Omega|$. This means we can choose a convergent subsequence, denoted again by $\{\vec{a}_{n,\Omega}\}_{n\in\mathbb{N}}$, with $\|\vec{a}_{n,\Omega}-\vec{a}_\Omega\| \overset{n\to\infty}{\longrightarrow} 0$ for an $\vec{a}_\Omega\in\mathbb{R}^2$. Summarized we get by Equation (6.6), that

$$\left(\eta_n K_1^2\right)^{\frac{1}{q}}|\Omega|\sqrt{2} \geq \left|\int_\Omega \nabla u_n\vec{e}_1\ d\lambda_2 - \vec{a}_{n,\Omega}\right| + \left|\int_\Omega \nabla u_n\vec{e}_2\ d\lambda_2 - \vec{a}^\perp_{n,\Omega}\right| - \left|\int_\Omega \gamma_n\nabla u_n\vec{e}_1\ d\lambda_2\right|.$$

Using Lemma 6.5 we get

$$\left|\int_\Omega \gamma_n\nabla u_n\vec{e}_1\ d\lambda_2\right| \overset{n\to\infty}{\longrightarrow} 0.$$

The frame indifference implies, that we can choose w.l.o.g. $F^* = \begin{pmatrix} a & b\\ 0 & c\end{pmatrix}$ and furthermore at least one of the following statements is true, $b\neq 0$ or $a\neq c$ or $|a|>1$. Because of Lemma 3.3 we get for $n\to\infty$, that

$$|\Omega|\begin{pmatrix} a\\ 0\end{pmatrix} = \vec{a}_\Omega \text{ and } |\Omega|\begin{pmatrix} b\\ c\end{pmatrix} = \vec{a}^\perp_\Omega.$$

This implies $b=0$, $a=c$ and $|a|\leq 1$, which is a contradiction to the choice of F^*, namely to $F^*\in\mathbb{R}^{2\times 2}\setminus\left\{\lambda B\in\mathbb{R}^{2\times 2}\ :\ \lambda\in[0,1], B\in SO(2)\right\}$. Thus the first part of the lemma is proven.

Choose now $q\geq 2$.

Then we get by Equation (6.4) in the proof of Lemma 6.4 *iv*), that

$$|\det(F)-1| \leq \frac{1}{2}\operatorname{dist}^2\left(F\left(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right) + 2\operatorname{dist}\left(F\left(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right)$$

for $F\in\mathbb{R}^{2\times 2}$ and thus

$$\int_\Omega |\det(\nabla u_n)-1|\,d\lambda_2 \leq \left(\frac{1}{2}\left(\eta_n K_1^2\right)^{\frac{2}{q}} + 2\left(\eta_n K_1^2\right)^{\frac{1}{q}}\right)|\Omega| \overset{n\to\infty}{\longrightarrow} 0.$$

Next, Lemma 3.3 implies $\int_\Omega \det(\nabla u_n)\,d\lambda_2 = \det(F^*)\,|\Omega|$. Thus $1=\det(F^*)=a\cdot c$, which implies $a=c=1$ and gives a contradiction to $F^*\in\mathbb{R}^{2\times 2}\setminus SO(2)$ and the second part of the lemma is proven.

□

Lemma 6.7. *Let $q \geq 1$ and $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set and let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of open sets, with $\Omega_n \subseteq \Omega$. Let $\{X_n\}_{n\in\mathbb{N}} \subseteq L^q\left(\Omega_n, \mathbb{R}^{2\times 2}\right)$, with*

$$\int_{\Omega_n} \operatorname{dist}^q\left(X_n, SO(2)\right)\, d\lambda_2 \to 0 \text{ for } n \to \infty, \tag{6.7}$$

then there exists a $\widetilde{C} > 0$ independent of n, so that $\|X_n\|_{L^q(\Omega_n,\mathbb{R}^{2\times 2})} \leq \widetilde{C}$ for all $n \in \mathbb{N}$.
Let $q \geq 2$ and assume that the set $\widehat{\Omega} := \bigcap_{n\in\mathbb{N}} \Omega_n$ is open.
If $X_n = Du_n\left(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2\right) \in L^q\left(\Omega_n, \mathbb{R}^{2\times 2}\right)$ fulfills the Equation (6.7), whereupon $u_n \in W^{1,\infty}\left(\Omega;\mathbb{R}^2\right)$ and it exists a constant $K > 0$, so that $\|u_n\|_{L^q(\widehat{\Omega};\mathbb{R}^2)} \leq K$ for all $n \in \mathbb{N}$. Let $\gamma_n \in L^\infty(\Omega_n)$ with $\|\gamma_n\|_{L^\infty(\Omega_n)} < \alpha_n$, where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence with $\lim_{n\to\infty} \alpha_n = 0$, then it exists an $u \in W^{1,q}\left(\widehat{\Omega}, \mathbb{R}^2\right)$ such that $u_n \rightharpoonup u \in W^{1,q}\left(\widehat{\Omega};\mathbb{R}^2\right)$ and $Du(x) \in Con(2)$ for a.e. $x \in \widehat{\Omega}$.
If $\widehat{\Omega}$ is connected has Lipschitz boundary, then there exists a rotation $Q \in SO(2)$, so that $Du(x) = Q$ for a.e. $x \in \widehat{\Omega}$.

Proof: Due to Equation (6.1) and Equation (6.2) we get for $F \in \mathbb{R}^{2\times 2}$, that

$$\operatorname{dist}^q\left(F, SO(2)\right) = \left|\operatorname{dist}^2\left(F, SO(2)\right)\right|^{\frac{q}{2}} \geq \left|\frac{1}{2}\|F\|^2 - \det(F)\right|^{\frac{q}{2}} \geq \frac{1}{2^{q-1}}\|F\|^q - |\det(F)|^{\frac{q}{2}}.$$

Using Lemma 6.4 part *iv*), and $\gamma = 0$, we get

$$\frac{1}{2^{q-1}}\|F\|^q \leq \operatorname{dist}^q\left(F, SO(2)\right) + C\left(\operatorname{dist}^q\left(F, SO(2)\right) + 1\right),$$

for a constant $C = C(q) > 0$. This leads to

$$\int_{\Omega_n} \|X_n\|^q\, d\lambda_2 \leq 2^{q-1}\left[(C+1)\int_{\Omega_n} \operatorname{dist}^q\left(X_n, SO(2)\right)\, d\lambda_2 + C\right]$$

and thus $\|X_n\|_{L^q(\Omega_n,\mathbb{R}^{2\times 2})} \leq \widetilde{C}$ for an adequate $\widetilde{C} > 0$. This can be also proved by the formulas (*ii*) and (*iii*) in Lemma 6.4, for $\gamma = 0$.
In the following we assume $X_n = Du_n\left(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2\right)$ and $q \geq 2$.
Using Lemma 6.4 *i*) and *ii*) we get

$$\begin{aligned}\frac{1}{2^{q-1}}\int_{\Omega_n} \operatorname{dist}^q\left(Du_n, SO(2)\right)\, d\lambda_2 \quad &\leq \quad \int_{\Omega_n} \operatorname{dist}^q\left(Du_n\left(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right)\, d\lambda_2 \\ &\quad + \int_{\Omega_n} \|\gamma_n Du_n \vec{e}_1\|^q\, d\lambda_2\end{aligned}$$

and

$$\int_{\Omega_n} \|\gamma_n Du_n \vec{e}_1\|^q\, d\lambda_2 \leq \alpha_n^q 2^{q-1}\left(\int_{\Omega_n} \operatorname{dist}^q\left(Du_n\left(\mathbb{1} - \gamma_n \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right)\, d\lambda_2 + |\Omega_n|\right).$$

This leads to

$$\int_{\Omega_n} \operatorname{dist}^q\left(Du_n, SO(2)\right)\, d\lambda_2 \to 0 \text{ for } n \to \infty.$$

Using the first result we get, that the sequence $\{Du_n\}_{n\in\mathbb{N}}$ is bounded in $L^q\left(\widehat{\Omega}, \mathbb{R}^{2\times 2}\right)$. Since $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^q\left(\widehat{\Omega}; \mathbb{R}^2\right)$ by assumption we get, that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,q}\left(\widehat{\Omega}, \mathbb{R}^2\right)$. Thus it exists a subsequence, denoted again by $\{u_n\}_{n\in\mathbb{N}}$, and an $u \in L^q\left(\widehat{\Omega}; \mathbb{R}^2\right)$ with

$$u_n \rightharpoonup u \text{ in } L^q\left(\widehat{\Omega}, \mathbb{R}^2\right) \text{ and } Du_n \rightharpoonup Du \text{ in } L^q\left(\widehat{\Omega}, \mathbb{R}^{2\times 2}\right),$$

for $n \to \infty$. This implies

$$Du_n^+ \rightharpoonup Du^+ \text{ in } L^q\left(\widehat{\Omega}, \mathbb{R}^{2\times 2}\right) \text{ and } Du_n^- \rightharpoonup Du^- \text{ in } L^q\left(\widehat{\Omega}, \mathbb{R}^{2\times 2}\right).$$

Since $q \geq 2$ these weak convergences are also valid in $L^2\left(\widehat{\Omega}, \mathbb{R}^2\right)$ and $L^2\left(\widehat{\Omega}, \mathbb{R}^{2\times 2}\right)$. Consider the function

$$f : \mathbb{R}^{2\times 2} \to \mathbb{R}\,, F \mapsto \|F\|^2 - 2\det(F),$$

then $f(F) = 2\left\|F^-\right\|^2 \geq 0$ for the anti-conformal part F^- of F and thus $f(F) = 0$ if and only if $F \in Con(2)$. The weak lower semi-continuity of the L^2-Norm, and Equation (6.2) gives us

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \int_{\widehat{\Omega}} 2\operatorname{dist}^2\left(Du_n, SO(2)\right)\, d\lambda_2 \geq \limsup_{n\to\infty} \left(\int_{\widehat{\Omega}} 2\left\|Du_n^-\right\|^2\, d\lambda_2\right) \\ &\geq \int_{\widehat{\Omega}} 2\left\|Du^-\right\|^2 d\lambda_2 = \int_{\widehat{\Omega}} f(Du)\, d\lambda_2 \geq 0. \end{aligned} \tag{6.8}$$

Thus we have $f(Du(x)) = 0$ for a.e. $x \in \widehat{\Omega}$, which implies $Du(x) \in Con(2)$ for a.e. $x \in \widehat{\Omega}$. Next, we get by the Equation (6.4) in the proof of Lemma 6.4, that

$$\begin{aligned} \int_{\widehat{\Omega}} \left|\det\left(Du_n\right) - 1\right|\, d\lambda_2 \;\leq\; & \frac{1}{2}\int_{\widehat{\Omega}} \operatorname{dist}^2\left(Du_n\left(\mathbb{1} - \gamma\vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right)\, d\lambda_2 \\ & +2\int_{\widehat{\Omega}} \operatorname{dist}\left(Du_n\left(\mathbb{1} - \gamma\vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right)\, d\lambda_2, \end{aligned}$$

which implies $\det\left(Du_n\right) \to 1$ in $L^1\left(\widehat{\Omega}\right)$. In the following we assume, that $\widehat{\Omega}$ is connected and has Lipschitz boundary. Next, we want to show, that $\det\left(Du_n\right) \rightharpoonup \det\left(Du\right)$ in $L^{\frac{q}{2}}\left(\widehat{\Omega}\right)$ for $q \geq 2$. In the case $q > 2$ this is an immediate consequence of Lemma 3.3. If

$q = 2$ we show this by using Lemma 3.51 for $A_n = Du_n^+$, $B_n = Du_n^-$ and $A = Du^+ = Du$. By Equation (6.8) we get $Du_n^- \to 0$ in $L^2\left(\widehat{\Omega};\mathbb{R}^{2\times 2}\right)$ and thus also in $L^1\left(\widehat{\Omega};\mathbb{R}^{2\times 2}\right)$. Since $\det(F) = \det(F^+) + \det(F^-)$ for each $F \in \mathbb{R}^{2\times 2}$ and $\det(G) = -\frac{1}{2}\|G\|^2$ for each $G \in \mathit{Anticon}(2)$, we get by Lemma 6.4 (iv), that

$$\int_E \left|\det\left(Du_n^+\right)\right| \, d\lambda_2 \leq \int_E \left|\det\left(Du_n\right)\right| \, d\lambda_2 + \int_E \left|\det\left(Du_n^-\right)\right| \, d\lambda_2$$
$$\int_E C\left(\operatorname{dist}^2\left(Du_n\left(\mathbb{1} - \gamma\vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) + 1\right) \, d\lambda_2 + \int_E \left\|Du_n^-\right\| \, d\lambda_2 \stackrel{n\to\infty}{\longrightarrow} C\left|E\right|,$$

for all measurable sets $E \subseteq \widehat{\Omega}$, which implies the equi-integrability of the sequence $\{\det(Du_n^+)\}_{n\in\mathbb{N}}$. Thus we can apply Lemma 3.51 and get $\det(Du_n^+) \rightharpoonup \det(Du)$ in $L^1\left(\widehat{\Omega}\right)$. Finally $\det(Du_n^-) = -\frac{1}{2}\left\|Du_n^-\right\|^2 \to 0$ in $L^1\left(\widehat{\Omega}\right)$ implies $\det(Du_n) \rightharpoonup \det(Du)$ in $L^1\left(\widehat{\Omega}\right)$ for $q = 2$.

The uniqueness of weak limits implies $\det(Du(x)) = 1$ for a.e. $x \in \widehat{\Omega}$. Since we have shown above, that $Du(x) \in \mathit{Con}(2)$ for a.e. $x \in \widehat{\Omega}$ we get $Du(x) \in SO(2)$ for a.e. $x \in \widehat{\Omega}$. Finally Lemma 3.5 implies that Du is constant on $\widehat{\Omega}$ as asserted, since $\widehat{\Omega}$ is connected.

□

6.3 Motivation for the simplification

Next, we give a short motivation for the interchange of the variational part $V_x(\gamma,\Omega)$ by $V_y\left(\chi_{\{|\gamma|\leq\mu\}},\Omega\right)$, where $\mu > 0$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of simple laminates, with slip-strains $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n(x)$ depends on $\nabla u_n(x)$ only, in particular $\gamma_n(x)$ takes only two different values. If $\int_\Omega \operatorname{dist}^q\left(\nabla u_n\left(\mathbb{1} - \gamma_n\vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 \stackrel{n\to\infty}{\longrightarrow} 0$ and $\int_\Omega |\gamma_n|^p \, d\lambda_2 \stackrel{n\to\infty}{\longrightarrow} 0$, namely $\{(u_n,\gamma_n)\}_{n\in\mathbb{N}}$ is a sequence which shows Theorem 4.2, then both variational parts do not differ much.

Definition 6.8. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set. For a matrix $F^* \in \mathcal{M}$ we define the set of simple laminates, whose gradients differ from F^* by a rank-one matrix through*

$$\Lambda_{F^*} := \left\{l : \Omega \to \mathbb{R}^2 \;:\; \exists\vec{v} \in \mathbb{R}^2,\, \vec{w} \in \mathbb{S}^1,\, \lambda \in [0,1],\, \forall x \in \Omega,\, l(x) = F^*x + \chi_\lambda\left(x\cdot\vec{w}\right)\vec{v}\right\},$$

where $\chi_\lambda : \mathbb{R} \to \mathbb{R}$ is the one-periodic, Lipschitz-continuous function defined through $\chi_\lambda(0) = 0 = \chi_\lambda(1)$, $\chi_\lambda'(t) = \lambda$ for $t \in (0, 1-\lambda)$ and $\chi_\lambda'(t) = \lambda - 1$ for $t \in (1-\lambda, 1)$. For a laminate $l \in \Lambda_{F^*}$ we define

$$BV_l(\Omega) := \{\widetilde{\gamma} \in BV(\Omega) \;:\; \exists\gamma \in \widetilde{\gamma},\; \gamma(x) = \gamma(y) \Leftrightarrow \nabla l(x) = \nabla l(y) \text{ for a.e. } x, y \in \Omega\}.$$

Lemma 6.9. *Let $\Omega = B\left(0, \sqrt{2}\right) \subseteq \mathbb{R}^2$, $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $\alpha > 2$. Let Λ_{F^*} and $BV_l(\Omega)$ be defined as in Definition 6.8, where we choose $F^* = R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(2)} \backslash SO(2)$, with $R \in SO(2)$ and $\gamma_0 \neq 0$. Let $(u_n, \widetilde{\gamma}_n) \in \Lambda_{F^*} \times \left(BV_{u_n}(\Omega) \cap L^{\max\{p,q\}}(\Omega)\right)$ for $n \in \mathbb{N}$ such that*

$$\int_\Omega \operatorname{dist}^q \left(\nabla u_n \left(\mathbb{1} - \widetilde{\gamma}_n \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0 \tag{6.9}$$

$$\text{and} \qquad \int_\Omega |\widetilde{\gamma}_n|^p \, d\lambda_2 \leq 4 \left(\frac{|\gamma_0|}{\alpha}\right)^p \text{ for all } n \in \mathbb{N}. \tag{6.10}$$

Then we have for each $1 > \eta > 0$ that there exists an $N = N(\alpha, \eta) \in \mathbb{N}$ such that

$$V_x(\gamma_n, \Omega) \leq (1+\eta)\left(\frac{\alpha - 1}{\alpha - 2} + \eta\right) V_y\left(\chi_{\left\{|\gamma_n| \leq \frac{|\gamma_0|}{\alpha}\right\}}, \Omega\right)$$

$$\text{and} \qquad V_x(\gamma_n, \Omega) \geq (1-\eta)\left(\frac{\alpha - 3}{\alpha - 2} - \eta\right) V_y\left(\chi_{\left\{|\gamma_n| \leq \frac{|\gamma_0|}{\alpha}\right\}}, \Omega\right)$$

for all $n \geq N$. For the following statements we assume additionally $p > 1$ or

$$\int_\Omega |\widetilde{\gamma}_n|^p \, d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0$$

then we have that

$$V_x(\widetilde{\gamma}_n, \Omega) - V_y\left(\chi_{\left\{|\widetilde{\gamma}_n| \leq \frac{|\gamma_0|}{\alpha}\right\}}, \Omega\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Let $a_n, b_n, \mu_n \in \mathbb{R}$, $\lambda_n \in \left[0, \frac{1}{2}\right]$, $\gamma_{1,n}, \gamma_{2,n} \in \mathbb{R}$ such that

$$\widetilde{\gamma}_n(x) = \gamma_{1,n} \text{ if } \nabla u_n(x) \in R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} + \lambda_n R \begin{pmatrix} a_n & b_n \\ \mu_n a_n & \mu_n b_n \end{pmatrix}$$

$$\text{and} \qquad \widetilde{\gamma}_n(x) = \gamma_{2,n} \text{ if } \nabla u_n(x) \in R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} + (\lambda_n - 1) R \begin{pmatrix} a_n & b_n \\ \mu_n a_n & \mu_n b_n \end{pmatrix}$$

for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$. Then we have $\lambda_n + |\mu_n| \overset{n\to\infty}{\longrightarrow} 0$, $|a_n|, |b_n|, |\gamma_{2,n}| \overset{n\to\infty}{\longrightarrow} \infty$ and $\{\gamma_{1,n}\}_{n\in\mathbb{N}}$ is a bounded sequence. Especially we have

(i) $\lambda_n |a_n|^q \overset{n\to\infty}{\longrightarrow} 0$ *and if $p > 1$ we get* $\liminf\limits_{n\to\infty} \lambda_n |a_n|^{\frac{p}{p-1}} > 0$,

(ii) $\frac{|b_n|}{|a_n|^q} \overset{n\to\infty}{\longrightarrow} \infty$ *and if $p > 1$ we get* $\limsup\limits_{n\to\infty} \frac{|b_n|}{|a_n|^{\frac{p}{p-1}}} < \infty$,

(iii) $\frac{|\gamma_{2,n}|}{|a_n|^{q-1}} \overset{n\to\infty}{\longrightarrow} \infty$ *and if $p > 1$ we get* $\limsup\limits_{n\to\infty} \frac{|b_n|}{|a_n|^{\frac{p}{p-1}}} < \infty$,

(iv) $\liminf_{n\to\infty} \lambda_n |b_n| > 0$ *and* $\limsup_{n\to\infty} \lambda_n |b_n| < \infty$,

(v) $\frac{|a_n||\gamma_{2,n}|}{|b_n|} \overset{n\to\infty}{\longrightarrow} 1$.

Proof:
W.l.o.g. we can choose $F^* = \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix}$ with $\gamma_0 \in \mathbb{R} \setminus \{0\}$. Let $u_n : \Omega \to \mathbb{R}^2$ be defined by $u_n(x) = F^* x + \chi_{\lambda_n}(x \cdot \vec{w}_n)\,\vec{v}_n$ with $\vec{w}_n \in \mathbb{S}^1$, $\vec{v}_n \in \mathbb{R}^2$ and $\lambda_n \in [0,1]$. Since we get, by definition of χ_λ, that $\chi_{\lambda_n}(x \cdot \vec{w}_n) = \chi_{1-\lambda_n}(x \cdot (-\vec{w}_n))$ for all $x \in \Omega$ it suffices to investigate the case $\lambda_n \in \left[0, \frac{1}{2}\right]$. Define $\Omega_{1,n} := \{x \in \Omega \,:\, x \cdot \vec{w}_n - \lfloor x \cdot \vec{w}_n \rfloor \in (0, 1-\lambda_n)\}$, $\Omega_{2,n} := \{x \in \Omega \,:\, x \cdot \vec{w}_n - \lfloor x \cdot \vec{w}_n \rfloor \in (1-\lambda_n, 1)\}$, where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \,:\, n \leq x\}$ denotes the floor function of $x \in \mathbb{R}$. By definition of u_n we have

$$\nabla u_n(x) = \begin{cases} F^* + \lambda_n \vec{v}_n \otimes \vec{w}_n & \text{for} \quad x \in \Omega_{1,n} \\ F^* + (\lambda_n - 1)\,\vec{v}_n \otimes \vec{w}_n & \text{for} \quad x \in \Omega_{2,n} \end{cases}.$$

Let $\gamma_n \in \widetilde{\gamma}_n$ be a representative such that γ_n is equal to $\gamma_{1,n} \in \mathbb{R}$ on $\Omega_{1,n}$ and it is equal to $\gamma_{2,n} \in \mathbb{R}$ on $\Omega_{2,n}$, where we have used $\widetilde{\gamma}_n \in BV_{u_n}(\Omega)$. Using the shorter notation

$$\vec{v}_n \otimes \vec{w}_n = \begin{pmatrix} (v_n)_1 (w_n)_1 & (v_n)_1 (w_n)_2 \\ (v_n)_2 (w_n)_1 & (v_n)_2 (w_n)_2 \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ \mu_n a_n & \mu_n b_n \end{pmatrix},$$

with $a_n, b_n, \mu_n \in \mathbb{R}$, thereby we can assume, that there exists an $N \in \mathbb{N}$ such that such that $(v_n)_1 \neq 0$ for all $n \geq N$. Because if not, then there would exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $(v_{n_k})_1 = 0 = a_{n_k}$ for all $k \in \mathbb{N}$. This gives a contradiction to the Equations (6.17) and (6.18), since $\alpha > 1$. Thereby one can derive these equations by using first the old notation. In the following we choose $n \geq N$. We get

$$\nabla u_n(x)\,(\mathbb{1} - \gamma_n(x)\vec{e}_1 \otimes \vec{e}_2) = \begin{cases} X_1 & \text{for} \quad x \in \Omega_{1,n} \\ X_2 & \text{for} \quad x \in \Omega_{2,n} \end{cases},$$

where

$$X_1 := \begin{pmatrix} 1 & \gamma_0 - \gamma_{1,n} \\ 0 & 1 \end{pmatrix} + \lambda_n \begin{pmatrix} a_n & b_n - \gamma_{1,n} a_n \\ \mu_n a_n & \mu_n (b_n - \gamma_{1,n} a_n) \end{pmatrix}$$

$$\text{and} \qquad X_2 := \begin{pmatrix} 1 & \gamma_0 - \gamma_{2,n} \\ 0 & 1 \end{pmatrix} + (\lambda_n - 1) \begin{pmatrix} a_n & b_n - \gamma_{2,n} a_n \\ \mu_n a_n & \mu_n (b_n - \gamma_{2,n} a_n) \end{pmatrix}.$$

Then we get using the Assumption (6.9) that

$$|\Omega_{1,n}| \operatorname{dist}^q (X_1, SO(2)) + |\Omega_{2,n}| \operatorname{dist}^q (X_2, SO(2)) \overset{n\to\infty}{\longrightarrow} 0. \qquad (6.11)$$

Since $\Omega = B\left(0, \sqrt{2}\right) \subseteq \mathbb{R}^2$, there can be inscribed a square of side-length two, whose rectangular sides are given by the vectors $\vec{w}_n$ and $\vec{w}_n^\perp$, refer to Figure 6.1. Thus we have,

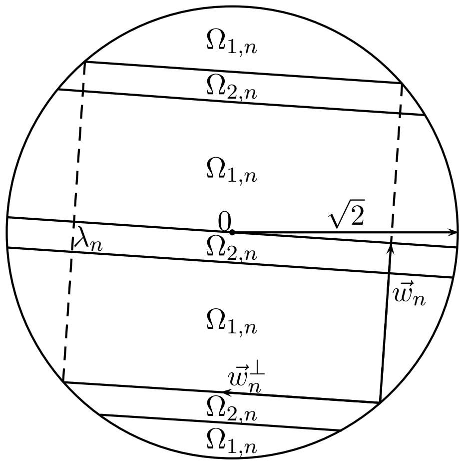

Figure 6.1:

since χ_λ is one-periodic for all $\lambda \in [0,1]$, that $|\Omega_{1,n}| \geq 4(1-\lambda_n)$ and $|\Omega_{2,n}| \geq 4\lambda_n$. The anti-conformal part of X_1 is given by

$$X_1^- = \frac{1}{2}\begin{pmatrix} \lambda_n(a_n - \mu_n b_n + \mu_n\gamma_{1,n}a_n) & \lambda_n(\mu_n a_n + b_n - \gamma_{1,n}a_n) + \gamma_0 - \gamma_{1,n} \\ \lambda_n(\mu_n a_n + b_n - \gamma_{1,n}a_n) + \gamma_0 - \gamma_{1,n} & -\lambda_n(a_n - \mu_n b_n + \mu_n\gamma_{1,n}a_n) \end{pmatrix}$$

and thus we have by Remark 6.3, that

$$\begin{aligned} &\operatorname{dist}^q(X_1, SO(2)) \geq \left\|X_1^-\right\|^q \qquad (6.12)\\ &= \left[\frac{1}{2}\lambda_n^2(a_n - \mu_n b_n + \mu_n\gamma_{1,n}a_n)^2 + \frac{1}{2}(\lambda_n(\mu_n a_n + b_n - \gamma_{1,n}a_n) + \gamma_0 - \gamma_{1,n})^2\right]^{\frac{q}{2}}\\ &\geq \frac{1}{2^{\max\{\frac{q}{2},1\}}}\left(\lambda_n^q|a_n - \mu_n b_n + \mu_n\gamma_{1,n}a_n|^q + |\lambda_n(\mu_n a_n + b_n - \gamma_{1,n}a_n) + \gamma_0 - \gamma_{1,n}|^q\right). \end{aligned}$$

The same calculations can be made with the matrix X_2, one only has to replace $\gamma_{1,n}$ with $\gamma_{2,n}$ and λ_n with $(\lambda_n - 1)$. Using the Equations (6.11) and (6.12) and the statement in between we get

$$(1-\lambda_n)\lambda_n^q|a_n - \mu_n b_n + \mu_n\gamma_{1,n}a_n|^q \overset{n\to\infty}{\longrightarrow} 0, \qquad (6.13)$$
$$(1-\lambda_n)|\lambda_n(\mu_n a_n + b_n - \gamma_{1,n}a_n) + \gamma_0 - \gamma_{1,n}|^q \overset{n\to\infty}{\longrightarrow} 0, \qquad (6.14)$$
$$\lambda_n(1-\lambda_n)^q|a_n - \mu_n b_n + \mu_n\gamma_{2,n}a_n|^q \overset{n\to\infty}{\longrightarrow} 0, \qquad (6.15)$$
$$\lambda_n|(\lambda_n - 1)(\mu_n a_n + b_n - \gamma_{2,n}a_n) + \gamma_0 - \gamma_{2,n}|^q \overset{n\to\infty}{\longrightarrow} 0 \qquad (6.16)$$

and finally the Assumption (6.10) implies by convexity

$$|(1-\lambda_n)\gamma_{1,n} + \lambda_n\gamma_{2,n}|^p \leq (1-\lambda_n)|\gamma_{1,n}|^p + \lambda_n|\gamma_{2,n}|^p \leq \left(\frac{|\gamma_0|}{\alpha}\right)^p \qquad (6.17)$$

for all $n \in \mathbb{N}$. By summing (6.14) and (6.16) using the convexity of $x \mapsto |x|^q$ we get

$$(1-\lambda_n)\,\lambda_n\,(\gamma_{2,n}-\gamma_{1,n})\,a_n + \gamma_0 - ((1-\lambda_n)\,\gamma_{1,n} + \lambda_n\gamma_{2,n}) \overset{n\to\infty}{\longrightarrow} 0. \tag{6.18}$$

Summing (6.13) and (6.15), using again the convexity of $x \mapsto |x|^q$ we get

$$(1-\lambda_n)\lambda_n\,(\gamma_{2,n}-\gamma_{1,n})\,\mu_n a_n \overset{n\to\infty}{\longrightarrow} 0. \tag{6.19}$$

Next, we show that $\mu_n \overset{n\to\infty}{\longrightarrow} 0$. Assume not, then there exists a $\rho > 0$ and a subsequence, denoted again with $\{\mu_n\}_{n\in\mathbb{N}}$, such that $|\mu_n| > \rho$ for all $n \in \mathbb{N}$, then (6.18) combined with (6.19) implies $\gamma_0 - ((1-\lambda_n)\,\gamma_{1,n} + \lambda_n\gamma_{2,n}) \overset{n\to\infty}{\longrightarrow} 0$, which is a contradiction to (6.17), since $\alpha > 1$. Thus we can assume, that $\mu_n \overset{n\to\infty}{\longrightarrow} 0$.
Next, we show that $\lambda_n \overset{n\to\infty}{\longrightarrow} 0$. Assume not, then there exists a $\rho > 0$ and a subsequence, denoted again with $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\frac{1}{2} \geq \lambda_n > \rho$ for all $n \in \mathbb{N}$. Then we get by (6.17), that $\{\gamma_{1,n}\}_{n\in\mathbb{N}}$ and $\{\gamma_{2,n}\}_{n\in\mathbb{N}}$ are bounded sequences. Assume now, that there exists a subsequence denoted again with $\{b_n\}_{n\in\mathbb{N}}$, so that $|b_n| \overset{n\to\infty}{\longrightarrow} \infty$, in particular there exists an $N \in \mathbb{N}$ so that $|b_n| \geq 1$ for all $n \geq N$. Then we get by $\frac{1}{|b_n|^q}$ times Equation (6.13), that $\frac{a_n}{b_n}(1+\mu_n\gamma_{1,n}) - \mu_n \overset{n\to\infty}{\longrightarrow} 0$, which implies that $\frac{a_n}{b_n} \overset{n\to\infty}{\longrightarrow} 0$. Additionally we get by $\frac{1}{|b_n|^q}$ times Equation (6.14), that $\lambda_n\left(\mu_n\frac{a_n}{b_n} + 1 - \gamma_{1,n}\frac{a_n}{b_n}\right) + \frac{\gamma_0-\gamma_{1,n}}{b_n} \overset{n\to\infty}{\longrightarrow} 0$, which gives a contradiction and we have that $\{b_n\}_{n\in\mathbb{N}}$ is bounded. Since (6.13) implies $a_n(1+\mu_n\gamma_{1,n}) - \mu_n b_n \overset{n\to\infty}{\longrightarrow} 0$ we get, that $a_n \overset{n\to\infty}{\longrightarrow} 0$. Thus we have again by (6.18), that $\gamma_0 - ((1-\lambda_n)\,\gamma_{1,n} + \lambda_n\gamma_{2,n}) \overset{n\to\infty}{\longrightarrow} 0$, which is a contradiction to (6.17). Thus we can assume that $\lambda_n \overset{n\to\infty}{\longrightarrow} 0$. This implies by (6.17), that $\{\gamma_{1,n}\}_{n\in\mathbb{N}}$ is a bounded sequence.
Next, we want to show that $\lambda_n a_n \overset{n\to\infty}{\longrightarrow} 0$. Assume not, then there exists a $\rho > 0$ and a subsequence denoted again with $\{\lambda_n a_n\}_{n\in\mathbb{N}}$, such that $|\lambda_n a_n| > \rho$ and $a_n \neq 0$ for all $n \in \mathbb{N}$. Then also $\frac{1}{|\lambda_n a_n|^q}$ times the Equations (6.13) and (6.14), respectively, converges to zero for $n \to \infty$. Thus we have $(1-\lambda_n)\left|1 - \mu_n\frac{b_n}{a_n} + \mu_n\gamma_{1,n}\right|^q \overset{n\to\infty}{\longrightarrow} 0$ and $(1-\lambda_n)\left|\mu_n + \frac{b_n}{a_n} - \gamma_{1,n} + \frac{\gamma_0-\gamma_{1,n}}{\lambda_n a_n}\right|^q \overset{n\to\infty}{\longrightarrow} 0$, which implies that $\left\{\mu_n\frac{b_n}{a_n}\right\}_{n\in\mathbb{N}}$ converges to one and $\left\{\frac{b_n}{a_n}\right\}_{n\in\mathbb{N}}$ is bounded. This gives the desired contradiction since $\mu_n \overset{n\to\infty}{\longrightarrow} 0$. Thus we have $\lambda_n a_n \overset{n\to\infty}{\longrightarrow} 0$ and by (6.18) and (6.17) we get, that $|\gamma_{2,n}| \overset{n\to\infty}{\longrightarrow} \infty$.
Using the above results the Equation (6.14) leads to

$$\lambda_n b_n + \gamma_0 - \gamma_{1,n} \overset{n\to\infty}{\longrightarrow} 0. \tag{6.20}$$

Since $\{\gamma_{1,n}\}_{n\in\mathbb{N}}$ is bounded and $|\gamma_{2,n}| \overset{n\to\infty}{\longrightarrow} \infty$, there exists an $N_1 \in \mathbb{N}$, such that $|\gamma_{1,n}| \leq |\gamma_{2,n}|$ and $\frac{|\gamma_0|}{\alpha} < |\gamma_{2,n}|$ for all $n \geq N_1$. Thus we have, using (6.17), that $|\gamma_{1,n}| \leq (1-\lambda_n)\,|\gamma_{1,n}| + \lambda_n\,|\gamma_{2,n}| \leq \frac{|\gamma_0|}{\alpha}$ for all $n \geq N_1$. Inserting this in (6.20) we get, that there exists an $\widetilde{N}_1 = \widetilde{N}_1(\alpha) \geq N_1$, so that $\left(1+\frac{2}{\alpha}\right)|\gamma_0| \geq |\lambda_n b_n| \geq \left(1-\frac{2}{\alpha}\right)|\gamma_0|$ for all $n \geq \widetilde{N}_1 \geq N_1$, which implies assertion (iv), $|b_n| \overset{n\to\infty}{\longrightarrow} \infty$ and $\frac{a_n}{b_n} = \frac{\lambda_n a_n}{\lambda_n b_n} \overset{n\to\infty}{\longrightarrow} 0$. Since $\left\{\frac{1}{\lambda_n |b_n|}\right\}_{n\geq \widetilde{N}_1}$ and thus $\left\{\frac{1}{\lambda_n |b_n|^q}\right\}_{n\geq \widetilde{N}_1}$ is bounded we get that $\frac{1}{\lambda_n |b_n|^q}$ times (6.16) converges to zero for $n \to \infty$, i.e.,

$$(\lambda_n - 1)\left(\mu_n \frac{a_n}{b_n} + 1 - \frac{\gamma_{2,n} a_n}{b_n}\right) + \frac{\gamma_0 - \gamma_{2,n}}{b_n} \overset{n\to\infty}{\longrightarrow} 0.$$

This leads to

$$\frac{\gamma_{2,n}\left(a_n - 1\right)}{b_n} \overset{n\to\infty}{\longrightarrow} 1, \tag{6.21}$$

since $\left|\frac{\gamma_{2,n}}{b_n}\right| = \frac{\lambda_n |\gamma_{2,n}|}{\lambda_n |b_n|} \leq \frac{\frac{1}{\alpha}|\gamma_0|}{\left(1-\frac{2}{\alpha}\right)|\gamma_0|} = \frac{1}{\alpha-2}$ for all $n \geq \widetilde{N}_1$. Next, we want to compare the variational part $V_x\left(\gamma_n, \Omega\right)$ with $V_y\left(\chi_{\left\{|\gamma_n| \leq \frac{|\gamma_0|}{\alpha}\right\}}, \Omega\right)$. In both cases we have six jump-lines, at least for all $n \in \mathbb{N}$ big enough such that $\lambda_n \leq \sqrt{2} - 1$, refer to Figure 6.1. The length of the two jump-lines near the origin can be bounded from above by the diameter $2\sqrt{2}$ of Ω and the sum of the length of the other four jump-lines can be bounded from above by $4 \cdot 2$, which is due to the convexity of the region $\Omega = B\left(0, \sqrt{2}\right)$. Thus the total jump-length can be bounded from above by $8 + 2\sqrt{2}$. Let $\zeta_1 > 0$, since $\lambda_n \overset{n\to\infty}{\longrightarrow} 0$ there exists an $N_2 = N_2(\alpha, \zeta_1) \geq \widetilde{N}_1 \in \mathbb{N}$, such that the total length can be bounded from below by $8 + 2\sqrt{2} - \zeta_1$. In order to compute both variational parts, we need the horizontal and the vertical latitude of the jump-lines and thus we need the modulus of the slope of the jump-lines. With respect to the first coordinate this is given by $|\frac{a_n}{b_n}|$. Therefore the extension of the total jump region in the first coordinate is given by the sum of all jump-lines times $\frac{|b_n|}{\sqrt{a_n^2 + b_n^2}}$ and the extension of the total jump region in the second coordinate is given by the sum of all jump-lines times $\frac{|a_n|}{\sqrt{a_n^2+b_n^2}}$. Since we have shown above, that $|\gamma_{1,n}| \leq \frac{|\gamma_0|}{\alpha} < |\gamma_{2,n}|$ for all $n \geq N_1$ we get

$$\left(8 + 2\sqrt{2} - \zeta_1\right) \cdot \frac{|b_n|}{\sqrt{a_n^2 + b_n^2}} \leq V_y\left(\chi_{\left\{|\gamma_n| \leq \frac{|\gamma_0|}{\alpha}\right\}}, \Omega\right) \leq \left(8 + 2\sqrt{2}\right) \cdot \frac{|b_n|}{\sqrt{a_n^2 + b_n^2}} \tag{6.22}$$

for all $n \geq N_2$ and, since the jump-height of γ_n is given by $|\gamma_{2,n} - \gamma_{1,n}|$, we have

$$\left(8 + 2\sqrt{2} - \zeta_1\right) \cdot \frac{|a_n|\,|\gamma_{2,n} - \gamma_{1,n}|}{\sqrt{a_n^2 + b_n^2}} \leq V_x\left(\gamma_n, \Omega\right) \leq \left(8 + 2\sqrt{2}\right) \cdot \frac{|a_n|\,|\gamma_{2,n} - \gamma_{1,n}|}{\sqrt{a_n^2 + b_n^2}} \tag{6.23}$$

for all $n \geq N_2$. Next, Equation (6.21) and $\frac{a_n}{b_n} \overset{n\to\infty}{\longrightarrow} 0$ implies $\frac{|\gamma_{2,n}||a_n - 1|}{\sqrt{b_n^2 + a_n^2}} \overset{n\to\infty}{\longrightarrow} 1$. Using this and $\left|\frac{\gamma_{2,n}}{b_n}\right| \leq \frac{1}{\alpha-2}$ for all $n \geq \widetilde{N}_1$, which was shown after Equation (6.21) we get, that for $\zeta_2 > 0$ there exists an $N_3 = N_3\left(\alpha, \zeta_1, \zeta_2\right) \geq N_2 \in \mathbb{N}$, such that

$$\frac{|a_n|\,|\gamma_{2,n} - \gamma_{1,n}|}{\sqrt{a_n^2 + b_n^2}} \leq \frac{|a_n - 1|\,|\gamma_{2,n}| + |\gamma_{2,n}| + |a_n|\,|\gamma_{1,n}|}{\sqrt{a_n^2 + b_n^2}} \leq 1 + \frac{1}{\alpha - 2} + \zeta_2 \tag{6.24}$$

and

$$\frac{|a_n|\,|\gamma_{2,n}-\gamma_{1,n}|}{\sqrt{a_n^2+b_n^2}} \geq \frac{|a_n-1|\,|\gamma_{2,n}|-|\gamma_{2,n}|-|a_n|\,|\gamma_{1,n}|}{\sqrt{a_n^2+b_n^2}} \geq 1-\frac{1}{\alpha-2}-\zeta_2 \tag{6.25}$$

for all $n \geq N_3$. Since $\frac{|b_n|}{\sqrt{a_n^2+b_n^2}} \overset{n\to\infty}{\longrightarrow} 1$, there exists for given $\zeta_3 > 0$ a natural number $N_4 = N_4(\alpha,\zeta_1,\zeta_2,\zeta_3) \geq N_3 \in \mathbb{N}$, such that $1-\zeta_3 \leq \frac{|b_n|}{\sqrt{a_n^2+b_n^2}} \leq 1+\zeta_3$ for all $n \geq N_4$. Thus we can conclude, using the Equations (6.22), (6.23) and (6.24), that

$$\begin{aligned} V_x(\gamma_n,\Omega) &\leq \left(8+2\sqrt{2}\right)\left(1+\frac{1}{\alpha-2}+\zeta_2\right) \\ &= \left(8+2\sqrt{2}-\zeta_1\right)\frac{1-\zeta_3}{1-\zeta_3}\frac{8+2\sqrt{2}}{8+2\sqrt{2}-\zeta_1}\left(\frac{\alpha-1}{\alpha-2}+\zeta_2\right) \\ &\leq V_y\left(\chi_{\left\{|\gamma_n|\leq\frac{\gamma_0}{\alpha}\right\}},\Omega\right)\frac{8+2\sqrt{2}}{(1-\zeta_3)\left(8+2\sqrt{2}-\zeta_1\right)}\left(\frac{\alpha-1}{\alpha-2}+\zeta_2\right) \end{aligned}$$

for all $n \geq N_4$. Thus we have for each $1 > \eta > 0$ there exists an $N_5 = N_5(\alpha,\eta) \in \mathbb{N}$, such that

$$V_x(\gamma_n,\Omega) \leq (1+\eta)\left(\frac{\alpha-1}{\alpha-2}+\eta\right)V_y\left(\chi_{\left\{|\gamma_n|\leq\frac{|\gamma_0|}{\alpha}\right\}},\Omega\right)$$

and analogously, that

$$V_x(\gamma_n,\Omega) \geq (1-\eta)\left(\frac{\alpha-3}{\alpha-2}-\eta\right)V_y\left(\chi_{\left\{|\gamma_n|\leq\frac{|\gamma_0|}{\alpha}\right\}},\Omega\right)$$

for all $n \geq N_5$ as asserted. Assume in the following, that $\int_\Omega |\widetilde{\gamma}_n|^p \; d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0$ or $p > 1$, in the first case we get for n big enough, that $\lambda_n\,|\gamma_{2,n}| \leq \lambda_n\,|\gamma_{2,n}|^p \leq \int_\Omega |\widetilde{\gamma}_n|^p \; d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0$ and in the second case we have by (6.17), that $\lambda_n\,|\gamma_{2,n}|^p \leq \left(\frac{|\gamma_0|}{\alpha}\right)^p \leq |\gamma_0|^p < \infty$, which implies since $p > 1$ and $|\gamma_{2,n}| \overset{n\to\infty}{\longrightarrow} \infty$, that $\lambda_n\,|\gamma_{2,n}| \overset{n\to\infty}{\longrightarrow} 0$. In both cases we get $\left|\frac{\gamma_{2,n}}{b_n}\right| = \frac{\lambda_n|\gamma_{2,n}|}{\lambda_n|b_n|} \overset{n\to\infty}{\longrightarrow} 0$, which implies, using (6.21), that $\frac{|\gamma_{2,n}a_n|}{|b_n|} \overset{n\to\infty}{\longrightarrow} 1$, i.e., assertion (v) is fulfilled and we get $|a_n| \overset{n\to\infty}{\longrightarrow} \infty$. Using the same calculations as above we get

$$V_x(\gamma_n,\Omega) - V_y\left(\chi_{\left\{|\gamma_n|\leq\frac{|\gamma_0|}{\alpha}\right\}},\Omega\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Next, $|a_n| \overset{n\to\infty}{\longrightarrow} \infty$ implies, that there exists an $N_6 \geq N_5 \in \mathbb{N}$, so that

$$\begin{aligned} \operatorname{dist}^q(X_2, SO(2)) &\geq \;\; |\|X_2\vec{e}_1\|-1|^q = \left|\sqrt{(1+(\lambda_n-1)\,a_n)^2+((\lambda_n-1)\,\mu_n a_n)^2}-1\right|^q \\ &\geq \;\; ||1+(\lambda_n-1)\,a_n|-1|^q \geq \left(\frac{|a_n|}{2}\right)^q, \end{aligned}$$

for all $n \geq N_6$. Using Equation (6.11) we get, that $\lambda_n |a_n|^q \overset{n\to\infty}{\longrightarrow} 0$. Next, (6.18) implies $\lambda_n a_n \gamma_{2,n} + \gamma_0 - \gamma_{1,n} \overset{n\to\infty}{\longrightarrow} 0$ and since $|\gamma_{1,n}| \leq \frac{|\gamma_0|}{\alpha}$ for all $n \geq N_1$ we get there exists an $\widetilde{N}_6 \geq N_6 \in \mathbb{N}$, so that $\left(1+\frac{2}{\alpha}\right)|\gamma_0| \geq \lambda_n |a_n| |\gamma_{2,n}| \geq \left(1-\frac{2}{\alpha}\right)|\gamma_0|$ for all $n \geq \widetilde{N}_6$. Next, since (6.17) implies $\lambda_n |\gamma_{2,n}|^p \leq \left(\frac{|\gamma_0|}{\alpha}\right)^p$ for all $n \in \mathbb{N}$ we get

$$\lambda_n |a_n|^{\frac{p}{p-1}} = \frac{(\lambda_n |a_n| |\gamma_{2,n}|)^{\frac{p}{p-1}}}{(\lambda_n |\gamma_{2,n}|^p)^{\frac{1}{p-1}}} \geq \alpha^{\frac{p}{p-1}} \left(1-\frac{2}{\alpha}\right)^{\frac{p}{p-1}}$$

for all $n \geq \widetilde{N}_6$, which shows assertion (i). Next, we have $\frac{|b_n|}{|a_n|^q} = \frac{\lambda_n |b_n|}{\lambda_n |a_n|^q} \overset{n\to\infty}{\longrightarrow} \infty$ and

$$\frac{|b_n|}{|a_n|^{\frac{p}{p-1}}} = \frac{\lambda_n |b_n|}{\lambda_n |a_n|^{\frac{p}{p-1}}} \leq \frac{1}{\alpha^{\frac{p}{p-1}}} \left(1-\frac{2}{\alpha}\right)^{\frac{p}{1-p}} \left(1+\frac{2}{\alpha}\right)|\gamma_0| \tag{6.26}$$

for all $n \geq \widetilde{N}_6$, which shows (ii). Next, we get (iii), since $\frac{|\gamma_{2,n}|}{|a_n|^{q-1}} = \frac{|\gamma_{2,n}||a_n|}{|b_n|} \frac{|b_n|}{|a_n|^q} \overset{n\to\infty}{\longrightarrow} \infty$ and

$$\frac{|\gamma_{2,n}|}{|a_n|^{\frac{1}{p-1}}} = \frac{\lambda_n |\gamma_{2,n}| |a_n|}{\lambda_n |a_n|^{\frac{p}{p-1}}} \leq \frac{1}{\alpha^{\frac{p}{p-1}}} \left(1-\frac{2}{\alpha}\right)^{\frac{p}{1-p}} \left(1+\frac{2}{\alpha}\right)|\gamma_0|.$$

Thus we have proven all statements.

□

Remark 6.10. *Consider the notations of Lemma 6.9, where we assume again* $p > 1$ *or* $\int_\Omega |\widetilde{\gamma}_n| \, d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0$. *If we write* $t = t(n) = |a_n|$ *and assume, that* λ_n, $|b_n|$, $|\gamma_{2,n}|$ *are, up to a constant, given by a power of* $|a_n|$, *i.e.,* $|\lambda_n| \sim t^{\eta_1}$, $|b_n| \sim t^{\eta_2}$, $|\gamma_{2,n}| \sim t^{\eta_3}$ *with* $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$. *Then Lemma 6.9* (iv) *implies* $\eta_2 = -\eta_1$ *and* (v) *implies* $\eta_3 = \eta_2 - 1$ *finally* (ii) *implies* $\eta_2 > q$ *and if* $p > 1$, *that* $\eta_2 \leq \frac{p}{p-1}$. *If one has* $p > 1$ *and* $\int_\Omega |\widetilde{\gamma}_n| \, d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0$ *we get by Equation (6.26), that* $\frac{|b_n|}{|a_n|^{\frac{p}{p-1}}} \overset{n\to\infty}{\longrightarrow} 0$ *and thus* $\eta_2 < \frac{p}{p-1}$. *Let* $\eta := \eta_2$ *and assume, that* $\int_\Omega |\widetilde{\gamma}_n| \, d\lambda_2 \overset{n\to\infty}{\longrightarrow} 0$, *which implies* $\gamma_{1,n} \overset{n\to\infty}{\longrightarrow} 0$ *by Equation (6.17) and by Equation (6.20), that* $\lambda_n b_n + \gamma_0 \overset{n\to\infty}{\longrightarrow} 0$. *Then we have* $\nabla u_n(x) \in R\{F_1, F_2\}$ *for all* $x \in \Omega$, *where* $R \in \mathrm{SO}(2)$ *as in Lemma 6.9 and*

$$F_1 := \begin{pmatrix} 1+\lambda_n a_n & \gamma_0 + \lambda_n b_n \\ \lambda_n \mu_n a_n & 1+\lambda_n \mu_n b_n \end{pmatrix} \overset{n\to\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$F_2 := \begin{pmatrix} 1+(\lambda_n-1)\,a_n & \gamma_0 + (\lambda_n-1)\,b_n \\ (\lambda_n-1)\,\mu_n a_n & 1+(\lambda_n-1)\,\mu_n b_n \end{pmatrix} \in \begin{pmatrix} \pm t & \pm t^\eta \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ o\,(t) & o\,(t^\eta) \end{pmatrix}.$$

Comparing this with the matrices $F_t(0)$ *and* $F_t(1)$, *defined in the Proof of Lemma 5.1, we recognize, that* $F_t(0)$ *and* $F_t(1)$ *have the same properties as* F_1 *and* F_2, *respectively. This means, that there is no fundamentally different simple laminate construction which shows Theorem 4.2 than the one given by* $F_t(0)$ *and* $F_t(1)$ *used in its proof.*

6.4 Lower bound for a simplified model

Until now we are not able to close the gap in the upper bound $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, shown in Section 5.3, and the lower bound δ. Considering now the simplified energy $\widetilde{I}_{\varepsilon,\delta}(u,\gamma)$, defined in Section 5.3 in Equation (5.22), then Theorem 6.15 shows that this energy has a lower bound, which scales as $\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$ for small δ. Next, we derive some statements used in the proof of Theorem 6.15.

Lemma 6.11. *Let $N \in \mathbb{N}$, $\omega \subseteq \mathbb{R}$ be open and connected, $A \subseteq \omega$ be a set which is equivalent to $\mathcal{I} := \bigcup_{l=1}^{N} I_l$ except for a $\mathcal{L}^1$-null set, where $I_l = [a_l, b_l]$, $a_l < b_l \in \mathbb{R}$ and $I_l \cap I_k = \emptyset$ for each $l \neq k \in \{1,\dots,N\}$, and fulfills $|A| < |\omega|$. Let $g : \omega \to \mathbb{R}$ be a function, which is monotone increasing on each I_l, $l \in \{1,\dots,N\}$ and $g|_{\mathcal{I}}$ be Lipschitz continuous. If it exists a constant $K > 0$ such that $M := |\{x \in A \,:\, g'(x) \geq K,\ g'(x) \text{ exists}\}| > 0$, then we get*

$$\int_A |g(x)|\ dx \cdot V(\chi_A, \omega) \geq K\frac{M^2}{4} > 0.$$

Proof:
Define $B_l := I_l \cap \{x \in A \,:\, g'(x) \geq K,\ g'(x) \text{ exists}\}$ and $M_l := |B_l|$ for $l \in \{1,\dots,N\}$. Using the definition of A, in particular $|\omega| > |A| > 0$, we get

$$V(\chi_A, \omega) = V(\chi_{\mathcal{I}}, \overline{\omega}) \geq \max\{2(N-1), 1\} \geq N.$$

Since g is monotone increasing on I_l and $g' \geq K$ on B_l, for each $l \in \{1,\dots,N\}$, one gets

$$\begin{aligned}\int_A |g(x)|\ dx &\geq \sum_{l=1}^{N} \int_{B_l} |g(x)|\ dx \geq \sum_{l=1}^{N} \int_0^{M_l} K\left|x - \frac{M_l}{2}\right|\ dx \\ &= \frac{K}{4}\sum_{l=1}^{N} M_l^2 \geq \frac{K}{4N}\left(\sum_{l=1}^{N} M_l\right)^2 = K\frac{M^2}{4N},\end{aligned}$$

where we used the convexity of $x \mapsto x^2$ in the last inequality. Finally we obtain

$$\int_A |g(x)|\ dx \cdot V(\chi_A, \omega) \geq K\frac{M^2}{4N} \cdot N = K\frac{M^2}{4}$$

as asserted.

□

Lemma 6.12. *Let $q \geq 1$, $N \in \mathbb{N}$, $\omega \subseteq \mathbb{R}$ be open and connected set, $A \subseteq \omega$ be a set which is equivalent to $\mathcal{I} := \bigcup_{l=1}^{N} I_l$ except for an $\mathcal{L}^1$- null set, where $I_l = [a_l, b_l] \subseteq \mathbb{R}$, $a_l < b_l$ and $I_l \cap I_k = \emptyset$ for each $l \neq k \in \{1, \ldots, N\}$, and fulfills $|A| < |\omega|$. Let $f : \overline{\omega} \to \mathbb{R}$ be Lipschitz-continuous. Assume there exist constants $K_1 > 0$ and $K_2 > 0$ with $K_2 < \left(\frac{K_1}{2}\right)^q \frac{|A|}{2^{\max\{2,q\}}}$ such that $\int_A |f'(x) - K_1|^q \; dx < K_2$, then we get*

$$\left| \left\{ x \in A \; : \; \left| f'(x) - K_1 \right| < \frac{K_1}{2}, \; f'(x) \; \textit{exists} \right\} \right| > \frac{3}{4} |A|$$

and

$$\int_\omega |f| \; dx \cdot V(\chi_A, \omega) > \frac{|A|^2 K_1}{128}.$$

Proof:
Define $A_{K_1} := \left\{ x \in A \; : \; |f'(x) - K_1| < \frac{K_1}{2}, \; f'(x) \text{ exists} \right\}$. Assume $|A_{K_1}| \leq \frac{3}{4} |A|$, then we would obtain by

$$K_2 > \int_A \left| f'(x) - K_1 \right|^q \; dx \geq \int_{A - A_{K_1}} \left| f'(x) - K_1 \right|^q \; dx \geq \left(\frac{K_1}{2} \right)^q \frac{1}{4} |A| > K_2$$

a contradiction to the assumption. This proves the first assertion. The second assertion will be shown with help of the previous Lemma. Our aim is to define a function g, which fulfills the requirements in Lemma 6.11 and additionally $\int_\omega |f(x)| \; dx \geq \int_A |g(x)| \; dx$. This is done by eliminating the oscillations from f, see Figure 6.2. The crucial point is, we can show that $\frac{1}{|A|} \int_{\{f'<0\} \cap A} f'(x) \; dx$ is bounded from below by $-\left(\frac{K_2}{|A|}\right)^{\frac{1}{q}}$. For small enough K_2 this suggests, that the oscillations can be eliminated by a subset E of A_{K_1}, with $|E| < |A_{K_1}|$. This will indicate, that one can define g in such way, that g is equal to f in the remaining part $A_{K_1} - E$, which has a positive measure. Thus we will get $g'(x) \geq \frac{K_1}{2}$ for all $x \in A_{K_1} - E$ and we can apply Lemma 6.11 for $K = \frac{K_1}{2}$. Let $A_+ := \{x \in A \; : \; f'(x) \geq 0, \; f'(x) \text{ exists}\}$ and $A_- := \{x \in A \; : \; f'(x) < 0, \; f'(x) \text{ exists}\}$, then we get, using Jensen's inequality,

$$\begin{aligned} \frac{1}{|A|} \int_{A_-} f'(x) \; dx &= -\frac{1}{|A|} \int_{A_-} \left| f'(x) \right| \; dx \geq -\frac{1}{|A|} \int_A \left| f'(x) - K_1 \right| \; dx \\ &\geq -\frac{1}{|A|^{\frac{1}{q}}} \left(\int_A \left| f'(x) - K_1 \right|^q \; dx \right)^{\frac{1}{q}} > -\left(\frac{K_2}{|A|} \right)^{\frac{1}{q}}, \qquad (6.27) \end{aligned}$$

since $q \geq 1$. By assumption we have $K_2 < \left(\frac{K_1}{2}\right)^q \frac{|A|}{2^q}$ and thus we get $\frac{|A|^{1-\frac{1}{q}} K_2^{\frac{1}{q}}}{\frac{K_1}{2}} < \frac{|A|}{2}$.

For an arbitrary measurable set $B \subseteq A_{K_1}$ with $|B| > \frac{|A|^{1-\frac{1}{q}} K_2^{\frac{1}{q}}}{\frac{K_1}{2}}$ we calculate

$$\frac{1}{|A|} \int_B f'(x)\, dx \geq \frac{|B|}{|A|} \frac{K_1}{2} > \left(\frac{K_2}{|A|} \right)^{\frac{1}{q}},$$

and combining this with Equation (6.27) implies

$$\frac{1}{|A|} \int_{B \cup A_-} f'(x)\, dx > \left(\frac{K_2}{|A|} \right)^{\frac{1}{q}} - \left(\frac{K_2}{|A|} \right)^{\frac{1}{q}} = 0. \tag{6.28}$$

Define now the function $g : \overline{\omega} \to \mathbb{R}$ by

$$g(x) := \begin{cases} \inf\limits_{y \geq x,\, y \in I_l} \{|f(y)|\} & \text{if} \quad f(x) \geq 0 \\ -\inf\limits_{y \leq x,\, y \in I_l} \{|f(y)|\} & \text{if} \quad f(x) < 0 \end{cases}, \text{ for } x \in I_l,\ l \in \{1, \ldots, N\}$$

and $g(x) := 0$ for $x \in \overline{\omega} \setminus \mathcal{I}$, see Figure 6.2. Then we get that g is monotone increasing on the sets $\{x \in I_l \ : \ f(x) \geq 0\}$ and $\{x \in I_l \ : \ f(x) < 0\}$. Since f is continuous and $f(x) = 0$ implies $g(x) = 0$ we get, that g is monotone increasing and continuous on I_l for each $l \in \{1, \ldots, N\}$. Define for $l \in \{1, \ldots, N\}$ the sets $U_l := (a_l, b_l)$, then one can

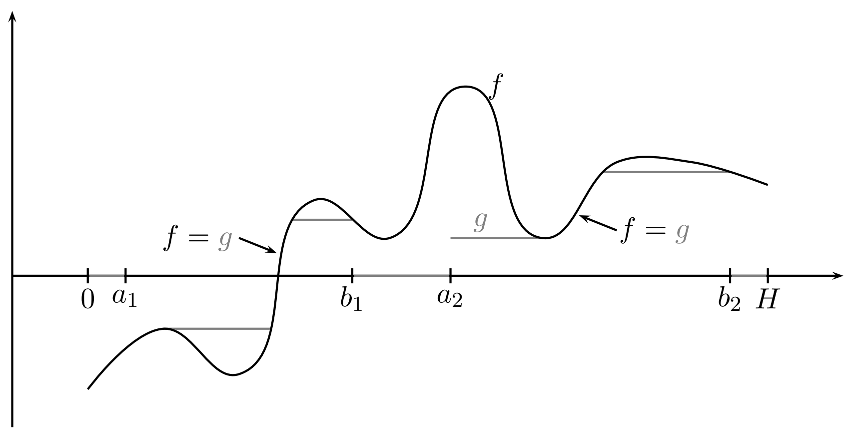

Figure 6.2: $N = 2,\ \omega = (0, H)$

show the following three sub-assertions.

(*i*) Let $x \in U_l$, $l \in \{1, \ldots, N\}$, with $f(x) \neq g(x)$, then it exists a $\delta > 0$, so that $g|_{(x-\delta, x+\delta)}$ is constant.

(*ii*) We get the inequality

$$\int_{\{x \in U_l \,:\, g'(x)=0,\, g'(x) \text{ exists}\}} f'(x)\, dx \leq 0.$$

(*iii*) Let $x \in U_l$, $l \in \{1, \ldots, N\}$, with $f(x) = g(x)$, $f'(x)$ and $g'(x)$ exist. Assume that $g'(x) \neq 0$ or $f(x) \neq 0$, then we get $f'(x) = g'(x)$.

Since $g|_{I_l}$ is continuous for each $l \in \{1, \ldots, N\}$, we get as a corollary of (*i*), that g is Lipschitz continuous on $\mathcal{I}$, since this is true for f. Thus (*ii*) is well-defined, since $\{x \in U_l \,:\, g'(x) = 0,\ g'(x) \text{ exists}\}$ is measurable.
Proof of (*i*): Let $x \in U_l$, since $f(x) = 0$ implies $g(x) = 0$, we can assume that $f(x) \neq 0$. Assume $f(x) > 0$ with $f(x) \neq g(x)$ and thus $f(x) > g(x)$ by definition of g. The case $f(x) < 0$ goes analogously. Choose now $x_{\text{inf}} \in I_l$, with $x_{\text{inf}} \geq x$, such that we have $f(x_{\text{inf}}) = \inf_{y \geq x,\, y \in I_l} |f(y)| = g(x) \geq 0$, then we get $x_{\text{inf}} > x$, since $f(x) > g(x) = f(x_{\text{inf}})$. Thus we have for all $z \in [x, x_{\text{inf}}) \subseteq U_l$, that

$$f(x_{\text{inf}}) \geq g(x_{\text{inf}}) \geq g(z) = \inf_{y \geq z,\, y \in I_l} |f(y)| \geq \inf_{y \geq x,\, y \in I_l} |f(y)| = f(x_{\text{inf}}). \tag{6.29}$$

Since f is continuous, there exists an $\varepsilon > 0$, so that $f(v) > g(x)$ for all $v \in [x - \varepsilon, x]$ and $x - \varepsilon \in U_l$. Let $v \in [x - \varepsilon, x]$, then we get $g(v) = \inf_{y \geq v,\, y \in I_l} |f(y)| = \inf_{y \geq x,\, y \in I_l} |f(y)| = g(x)$. Combining this with Equation (6.29), we get that g is constant in $(x - \varepsilon, x_{\text{inf}})$. Choose $\delta = \min\{\varepsilon, x_{\text{inf}} - x\}$, then we get assertion (*i*).
We continue with the proof of (*iii*): Let $x \in U_l$ and $l \in \{1, \ldots, N\}$, with $f(x) = g(x)$, so that $f'(x), g'(x)$ exist and assume that $g'(x) \neq 0$ or $f(x) \neq 0$. Assume $f(x) > 0$, then we get

$$\begin{aligned} f'(x) &= \lim_{h \to 0,\, h > 0} \frac{f(x-h) - f(x)}{-h} \leq \lim_{h \to 0,\, h > 0} \frac{g(x-h) - g(x)}{-h} = g'(x), \\ g'(x) &= \lim_{h \to 0,\, h > 0} \frac{g(x+h) - g(x)}{h} \leq \lim_{h \to 0,\, h > 0} \frac{f(x+h) - f(x)}{h} = f'(x), \end{aligned}$$

since $f(z) \geq g(z)$ for all $z \in U_l$ with $f(z) \geq 0$. Similarly we get $f'(x) = g'(x)$ in the case $f(x) < 0$. Consider now the case $f(x) = g(x) = 0$ and thus $g'(x) \neq 0$. If there is a sequence $\{h_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ with $h_n \neq 0$ for all $n \in \mathbb{N}$ and $h_n \overset{n \to \infty}{\longrightarrow} 0$, so that $g(x + h_n) = f(x + h_n)$, then we immediately get, that $g'(x) = f'(x)$. Assume now that such a sequence does not exist, which implies, that there exists a $\rho > 0$, so that $f(z) \neq g(z)$ for all $z \in (x - \rho, x + \rho) \setminus \{x\}$ and additionally $(x - \rho, x + \rho) \subseteq U_l$. Using

sub-assertion (i) we get, that g is constant on $(x-\rho, x)$ and on $(x, x+\rho)$. Finally the continuity of g implies $g'(x) = 0$, which is the desired contradiction. Thus we get (iii).
Proof of (ii): Since f is continuous the set $\{x \in I_l \ : \ f(x) = 0\}$ is closed. Denote now, if it exists, the smallest zero point of $f|_{I_l}$ by $y_{\min} \in I_l$ and denote with $y_{\max} \in I_l$ the biggest zero point of f, where we admit $y_{\min} = y_{\max}$. Define $Y := [y_{\min}, y_{\max}]$ in the case $y_{\min}$ exists and $Y := \emptyset$ otherwise. Thus we get $f(x) \neq 0$ if $x \in I_l - Y$. Define $Z := \{x \in U_l \ : \ f(x) \neq g(x)\}$, then Z is open, since f and g are continuous, and we get using (i) that $Z \subseteq X := \{x \in U_l \ : \ g'(x) = 0, \ g'(x) \text{ exists}\}$. Using (iii) we get that

$$\left(X \cap \left\{x \in U_l \ : \ f'(x) \text{ exists}\right\}\right) \setminus (Z \cup Y) \subseteq \widehat{X} := \left\{x \in U_l \ : \ f'(x) = 0, \ f'(x) \text{ exists}\right\}.$$

This implies

$$\int\limits_X f'(x) \, d\lambda_2 = \int\limits_{X \setminus (Z \cup Y)} f'(x) \, d\lambda_2 + \int\limits_Y f'(x) \, d\lambda_2 + \int\limits_Z f'(x) \, d\lambda_2 = \int\limits_Z f'(x) \, d\lambda_2,$$

where the second integral is zero since $f\left(y_{\min}\right) = 0 = f\left(y_{\max}\right)$ or $Y = \emptyset$. Since Z is open it has countable many open connected components with positive measure. Let $(a,b) \subseteq \mathbb{R}$ one of these connected components, then it suffices to show $\int\limits_{(a,b)} f'(x) \, dx \leq 0$. Using (i), then the definition of (a,b) implies $a = a_l$ or $f(a) = g(a)$. Analogously we have for b, that $b = b_l$ or $f(b) = g(b)$.
Consider now the case $a = a_l$ and $f(a) \neq g(a)$, then we get $f(a_l) > 0$, since $f(a_l) = 0$ implies $g(a_l) = 0$ and $f(a_l) < 0$ implies $f(a_l) = -\inf\limits_{y \leq a_l, \, y \in I_l} |f(y)| = g(a_l)$. Then we get $f(a_l) > g(a_l) \geq 0$ by definition of g. Assume first $f(b) > g(b)$, then we get $f(b) > 0$, since we have $g\left(b\right) = g\left(a_l\right) \geq 0$ by definition of b. Since $b = b_l$ by definition of (a,b), we get by $g\left(b_l\right) = \inf\limits_{y \geq b_l} \left|f\left(b_l\right)\right| = f\left(b_l\right)$ a contradiction to $f(b) > g(b)$. Thus we can assume $f(b) \leq g(b)$ and since g is constant on (a_l, b) we get $f(b) \leq g(a_l)$. This implies

$$\int\limits_{(a,b)} f'(x) \, dx = f(b) - f(a_l) \leq g(a_l) - f(a_l) < 0,$$

in the case $a = a_l$ and $f(a) \neq g(a)$. Analogously we get $\int\limits_{(a,b)} f'(x) \, dx < 0$ in the case $b = b_l$ and $f\left(b\right) \neq g\left(b\right)$. In the case $f(a) = g(a)$ and $f(b) = g(b)$ we get, that

$$\int\limits_{(a,b)} f'(x) \, dx = f(b) - f(a) = g(b) - g(a) = 0,$$

since g is constant on (a,b). This shows assertion (ii).
Furthermore, we get by construction, that $\int_\omega |f(x)| \ dx \geq \int_A |g(x)| \ dx$. Define the set

$E := A_{K_1} \cap G$, where $G := \{x \in A : g'(x) = 0, \, g'(x) \text{ exists}\}$, then the crucial point is now, that g fulfills $\left|\{x \in A : g'(x) \geq \frac{K_1}{2}, \, g'(x) \text{ exists}\}\right| \geq |A_{K_1} \setminus E| > \frac{1}{4}|A| > 0$, which will be shown later. Let $H := \left\{x \in \bigcup_{l=1}^{N} U_l : g'(x) \text{ exists}\right\}$, then we can show for an $x \in A_- \cap H$, that $g'(x) = 0$. Assume not, then we get by sub-assertion (i), that $f(x) = g(x)$, which implies using sub-assertion (iii), that $g'(x) = f'(x) < 0$ and thus gives a contradiction to $g'(x) \geq 0$. Therefore we have $A_- \cap H \subseteq G$ and conclude, that G is equal to $(A_-) \cup (A_+ \cap G)$ except for an $\mathcal{L}^1$-null set and thus

$$\int_{A_- \cup E} f'(x) \, dx \leq \int_{A_- \cup (A_+ \cap G)} f'(x) \, dx = \int_G f'(x) \, dx \leq 0,$$

where we used sub-assertion (ii) in the last inequality. Then, due to Equation (6.28) and a formula stated after Equation (6.27), we get $|E| \leq \frac{|A|^{1-\frac{1}{q}} K_2^{\frac{1}{q}}}{\frac{K_1}{2}} < \frac{|A|}{2}$ and thus $|A_{K_1} - E| > \frac{1}{4}|A|$, because of $|A_{K_1}| > \frac{3}{4}|A|$. Since f and $g|_{\mathcal{I}}$ are Lipschitz continuous we get $|\{x \in A : f'(x) \text{ exists}\}| = |\{x \in A : g'(x) \text{ exists}\}|$. Thus we have

$$\left|\left\{x \in A : \left|g'(x) - K_1\right| < \frac{K_1}{2}, \, g'(x) \text{ exists}\right\}\right| = |A_{K_1} - E|,$$

because we get for $x \in A \cap \bigcup_{l=1}^{N} U_l$ for which $f'(x)$ and $g'(x)$ exists, that $g'(x) \neq 0$ implies by (i) that $f(x) = g(x)$ and thus, using (iii), that $f'(x) = g'(x)$. Summarized $g : \overline{\omega} \to \mathbb{R}$ is a function, which is Lipschitz continuous on $\mathcal{I}$, monotone increasing on each I_l for $l \in \{1, \ldots, N\}$ and fulfills

$$M := \left|\left\{x \in A : g'(x) \geq \frac{K_1}{2}, \, g'(x) \text{ exists}\right\}\right| \geq |A_{K_1} - E| > \frac{1}{4}|A| > 0.$$

Thus we can apply Lemma 6.11 for $K := \frac{K_1}{2} > 0$ and the $M > 0$ defined above and conclude

$$\int_\omega |f(x)| \, dx \cdot V(\chi_A, \omega) \geq \int_A |g(x)| \, dx \cdot V(\chi_A, \omega) \geq K \frac{M^2}{4} > \frac{|A|^2 K_1}{128}$$

as asserted.

□

Before we can prove the lower bound for $\widetilde{I}_{\varepsilon,\delta}$ we will show it for the simpler energy $\widehat{I} = \widehat{I}_{\varepsilon,\delta;F^*}$, defined in Section 5.3 in Equation (5.23), given by

$$\widehat{I}(u, \gamma) = \fint_\Omega \frac{1}{\varepsilon} \|\nabla u (\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2) - R\|^q + |\gamma|^p \, d\lambda_2 + \frac{\delta}{|\Omega|} V_y\left(\chi_{\{\gamma=0\}}, \Omega\right),$$

for $F^* = R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(2)}$, with $R \in SO(2)$, $\gamma_0 \in \mathbb{R}$.

Theorem 6.13. *Let* $q \geq 1$, $F^* = R\begin{pmatrix}1 & \gamma_0\\ 0 & 1\end{pmatrix} \in \mathcal{M}^{(2)}$, $R \in SO(2)$, $\mu > 0$, $\gamma_0 \neq 0$ *and* $\Omega = (0,L)\times(0,H)$, $L, H > 0$. *Next, we define*

$$X_\mu = X_{\mu;p,q} := \left\{\gamma \in BV(\Omega)\cap L^{\max\{p,q\}}(\Omega)\ : \exists\, \overline{\gamma}\in\gamma,\ \forall x\in\Omega : |\overline{\gamma}(x)| \in \{0\}\cup[\mu,\infty)\right\}.$$

Let $C_B = C_B(\Omega, F^*, q) > 0$ *be a constant, independent of* ε, δ, p, *given by Corollary 5.13, such that*

$$\inf_{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2),\ \gamma\in X_\mu} \widehat{I}\,(u,\gamma) \leq C_B\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}},$$

for all $0 < \varepsilon, \delta \leq 1$, *then there exists a constant* $C = C\,(\Omega, C_B, F^*, q) > 0$ *independent of* $\varepsilon, \delta, \mu, p$ *and a* $0 < \rho < 1$, *such that*

$$\inf_{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2),\ \gamma\in X_\mu} \widehat{I}\,(u,\gamma) \geq \min\left\{C\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}, \frac{\mu^p}{8}\right\},$$

for all $0 < \varepsilon, \delta < \rho$.

Proof:
W.l.o.g. we can choose $F^* = \begin{pmatrix}1 & \gamma_0\\ 0 & 1\end{pmatrix}$, with $\gamma_0 > 0$, and $R = \mathbb{1}$, since the case $\gamma_0 < 0$ goes analogously. Choose $\rho \in (0,1)$ such that $24\cdot 2^{\max\{q+2,2q\}} C_B\,(\varepsilon\delta)^{\frac{q}{q+1}} < \gamma_0^q$, for all $0 < \varepsilon, \delta < \rho$. For $u = \begin{pmatrix}u_1\\ u_2\end{pmatrix} \in W^{1,\infty}_{F^*}\left(\Omega;\mathbb{R}^2\right)$ and $\gamma \in BV(\Omega)\cap L^{\max\{p,q\}}(\Omega)$ we get

$$\int_\Omega \|\nabla u\,(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2) - \mathbb{1}\|^q\ d\lambda_2 \geq \int_\Omega |\partial_x u_1 - 1|^q\ d\lambda_2 = \int_\Omega |\partial_x v|^q\ d\lambda_2,$$

where $v \in W^{1,\infty}(\Omega)$ is defined by $v\begin{pmatrix}x\\ y\end{pmatrix} = u_1\begin{pmatrix}x\\ y\end{pmatrix} - x - \gamma_0 y$. Then we get $v = 0$ on $\partial\Omega$ and we can conclude with help of Jensen's inequality, for each $l \in (0,L)$, that

$$\begin{aligned}\left(\int_\Omega |\partial_x v|^q\ d\lambda_2\right)^{\frac{1}{q}} &= |\Omega|^{\frac{1}{q}}\left(\frac{1}{|\Omega|}\int_\Omega |\partial_x v|^q\ d\lambda_2\right)^{\frac{1}{q}} \geq |\Omega|^{\frac{1}{q}-1}\int_\Omega |\partial_x v|\ d\lambda_2 \qquad (6.30)\\ &\geq \frac{1}{|\Omega|^{1-\frac{1}{q}}}\int_0^H\left(\int_0^l |\partial_x v|\ dx\right)dy \geq \frac{1}{|\Omega|^{1-\frac{1}{q}}}\int_0^H\left|\int_0^l \partial_x v\ dx\right|dy = \frac{1}{|\Omega|^{1-\frac{1}{q}}}\int_0^H |v_l(y)|\ dy,\end{aligned}$$

where $v_l : (0,H) \to \mathbb{R}$ is defined by $v_l(y) := v(l,y)$. Compound we get

$$\left(\widehat{I}\,(u,\gamma)\right)^{\frac{1}{q}} \geq \left(\frac{1}{|\Omega|}\int_\Omega \frac{1}{\varepsilon}|\partial_x v|^q\ d\lambda_2\right)^{\frac{1}{q}} \geq \frac{1}{\varepsilon^{\frac{1}{q}}\,|\Omega|}\int_0^H |v_l(y)|\ dy. \qquad (6.31)$$

Assume that $\left\{\left(u_n = \begin{pmatrix} u_{n,1} \\ u_{n,2} \end{pmatrix}, \gamma_n\right)\right\}_{n\in\mathbb{N}} \subseteq W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2) \times X_\mu$ is a minimal sequence of $\widehat{I}$, i.e., it exists a sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ with $\kappa_n \overset{n\to\infty}{\longrightarrow} 0$ and an $N \in \mathbb{N}$ big enough such that

$$\widehat{I}(u_n, \gamma_n) \leq \inf_{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2),\, \gamma\in X_\mu} \widehat{I}(u,\gamma) + \kappa_n \leq C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}} + \kappa_n \leq 2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}, \tag{6.32}$$

for all $n \geq N$. Define now $v_n \in W^{1,\infty}(\Omega)$ by $v_n\begin{pmatrix} x \\ y \end{pmatrix} = u_{n,1}\begin{pmatrix} x \\ y \end{pmatrix} - x - \gamma_0 y$ and for $l \in (0,L)$ we define $v_{n,l} \in W^{1,\infty}((0,H))$ by $v_{n,l}(y) := v_n(l,y)$. Then $v_n = 0$ on $\partial\Omega$ for all $n \in \mathbb{N}$. Choose a representant $\widetilde{\gamma}_n$ of γ_n for each $n \in \mathbb{N}$. Next, we define $\widetilde{\gamma}_{n,l} : (0,H) \to \mathbb{R}$ by $\widetilde{\gamma}_{n,l}(y) := \widetilde{\gamma}_n(l,y)$ for each $l \in (0,L)$, then we get, by Lemma 3.28 and Lemma 3.29, that $\widetilde{\gamma}_{n,l} \in BV(\omega)$ for almost every $l \in (0,L)$. Our goal is now to apply Lemma 6.12 for the Lipschitz continuous function $f = -v_{n,l} : (0,H) \to \mathbb{R}$, the sets $\omega = (0,H)$, $A = A_{n,l} := \{\widetilde{\gamma}_{n,l} = 0\} \subseteq \omega$ and the constants $K_1 = \gamma_0 > 0$, $K_2 = 12HC_B(\varepsilon\delta)^{\frac{q}{q+1}}$. Since we have chosen ρ and thus ε and δ, so that $24 \cdot 2^{\max\{2+q,2q\}} C_B (\varepsilon\delta)^{\frac{q}{q+1}} < \gamma_0^q$, we get $K_2 < \left(\frac{\gamma_0}{2}\right)^q \frac{\frac{H}{2}}{2^{max\{2,q\}}}$, which is smaller than $\left(\frac{K_1}{2}\right)^q \frac{|A|}{2^{\max\{2,q\}}}$ if $|A| > \frac{H}{2}$. Therefore we have to show, that there exists an $l \in (0,L)$ such that

$$H > |A| = |A_{n,l}| = |\{\widetilde{\gamma}_{n,l} = 0\}| > \frac{H}{2} > 0$$

and

$$\int_A |f' - K_1|^q \, dx = \int_A |v'_{n,l}(x) + \gamma_0|^q \, dx < K_2.$$

In the case $\gamma_0 < 0$, we define $K_1 = -\gamma_0 > 0$ and $f = v_{n,l}$. The assumption $|A| > 0$ implies, that we can use Corollary 3.35 and get a natural number $N \in \mathbb{N}$, so that A is equivalent to $\mathcal{I} := \bigcup_{l=1}^{N} I_l$ except for a $\mathcal{L}^1$-null set, where $I_l = [a_l, b_l]$, $a_l < b_l \in \mathbb{R}$ and $I_l \cap I_k = \emptyset$ for each $l \neq k \in \{1,\ldots,N\}$. Another reason why we have to add the assumption $|A_{n,l}| > \frac{H}{2}$ is that the lower bound in Lemma 6.12, i.e., $\frac{|A_{n,l}|^2 K_1}{128}$, must not converge to 0 for $n \to \infty$. Additionally the above l has to be chosen in such way that $\frac{L}{4} V\left(\chi_{A_{n,l}}, \omega\right) \leq V_y\left(\chi_{\{\widetilde{\gamma}_n = 0\}}, \Omega\right)$. We show these statements by proving the following four sub-assertions:

(i) For $S_1 := \left\{l \in (0,L) : |A_{n,l}| > \frac{H}{2}\right\}$ we get $|S_1| > \frac{3}{4}L$ or an immediate proof of the Theorem.

(ii) For $S_2 := \{l \in (0,L) : |A_{n,l}| < H\}$ we have $|S_2| > \frac{3}{4}L$.

(iii) For $S_3 := \left\{l \in (0,L) : \int_{A_{n,l}} \left|v'_{n,l}(y) + \gamma_0\right|^q dy < K_2\right\}$ we derive $|S_3| > \frac{3}{4}L$.

(*iv*) For $S_4 := \left\{ l \in (0,L) \ : \ V\left(\chi_{A_{n,l}}, \omega\right) \leq \frac{4}{L} V_y\left(\chi_{\{\widetilde{\gamma}_n = 0\}}, \Omega\right) \right\}$ we get $|S_4| > \frac{3}{4}L$.

This implies, that $|S_1 \cap S_2 \cap S_3 \cap S_4| > 0$, if S_i are measurable for $i \in \{1, \ldots, 4\}$, which implies the existence of the desired l. Using Fubini one can show, that S_i, $i \in \{1, \ldots, 4\}$ are measurable. More exactly, since $\gamma_n \in BV(\Omega)$ the set $\{x \in \Omega \ : \ \widetilde{\gamma}_n(x) = 0\}$ is measurable and the characteristic function $\chi_{\{x \in \Omega \, : \, \widetilde{\gamma}_n(x) = 0\}} : \Omega \to \{0,1\}$ is integrable. Fubini gives $l \mapsto \int_0^H \chi_{\{\widetilde{\gamma}_n = 0\}}(l,y) \ dy = |\{\widetilde{\gamma}_{n,l} = 0\}|$ is measurable and thus also S_1 and S_2. Since $\int_{\{\widetilde{\gamma}_n = 0\}} |\partial_y v_n(x,y) + \gamma_0|^q \ d\lambda_2 \leq \varepsilon |\Omega| \widehat{I}(u_n, \gamma_n) < \infty$ we get similarly that S_3 is measurable and $\int_{(0,L)} \left|V\left(\chi_{A_{n,l}}, \omega\right)\right| \ dl = V_y\left(\chi_{\{\widetilde{\gamma}_n = 0\}}, \Omega\right) \leq \frac{|\Omega|}{\delta} \widehat{I}(u_n, \gamma_n) < \infty$ implies that S_4 is measurable.

(*i*): Assume $|S_1| \leq \frac{3}{4}L$, then we get $\left|\left\{ l \in (0,L) \ : \ |A_{n,l}| \leq \frac{H}{2} \right\}\right| \geq \frac{1}{4}L$ and thus

$$\frac{1}{|\Omega|} \int_\Omega |\gamma_n|^p \ d\lambda_2 \geq \frac{1}{|\Omega|} \int\limits_{(0,L) \setminus S_1} \left(\int\limits_{(0,H) \setminus A_{n,l}} |\gamma_n|^p \ dy \right) dx \geq \frac{LH\mu^p}{8|\Omega|},$$

which proves the Theorem.

(*ii*): Assume $|S_2| \leq \frac{3}{4}L$, then we get $|\{l \in (0,L) \ : \ |A_{n,l}| = H\}| \geq \frac{1}{4}L$. For the set $Q := ((0,L) \setminus S_2) \times (0,H)$ we get, with the help of Jensen's inequality,

$$\begin{aligned}
2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}} |\Omega| &\geq \widehat{I}(u_n, \gamma_n) |\Omega| \geq \int_\Omega \frac{1}{\varepsilon} |\partial_y u_{n,1} - \gamma \partial_x u_{n,1}|^q \ d\lambda_2 \geq \int_Q \frac{1}{\varepsilon} |\partial_y u_{n,1}|^q \ d\lambda_2 \\
&\geq \frac{1}{\varepsilon} |Q|^{1-q} \left(\int_Q |\partial_y u_{n,1}| \ d\lambda_2 \right)^q \geq \frac{1}{\varepsilon} |Q|^{1-q} \left(\int_{(0,L) \setminus S_2} \left| \int_0^H \partial_y u_{n,1} \ dy \right| dx \right)^q \\
&= \frac{1}{\varepsilon} |Q|^{1-q} \left(\int_{(0,L) \setminus S_2} \gamma_0 H \ dx \right)^q = \frac{1}{\varepsilon} \gamma_0^q |Q| \geq \frac{1}{\varepsilon} \gamma_0^q \frac{1}{4} |\Omega|,
\end{aligned}$$

which is a contradiction to $8C_B (\varepsilon\delta)^{\frac{q}{q+1}} < \gamma_0^q$, for all $0 < \varepsilon, \delta < \rho$.

(*iii*): Assume $|S_3| \leq \frac{3}{4}L$, then $\left|\left\{ l \in (0,L) \ : \ \int_{A_{n,l}} \left| v'_{n,l}(y) + \gamma_0 \right|^q \ dy \geq K_2 \right\}\right| \geq \frac{1}{4}L$ and thus we get

$$\begin{aligned}
2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}} &\geq \frac{1}{|\Omega|} \int_{\{\widetilde{\gamma}_n = 0\}} \frac{1}{\varepsilon} |\partial_y u_{n,1}|^q \ d\lambda_2 = \frac{1}{\varepsilon} \frac{1}{|\Omega|} \int_{\{\widetilde{\gamma}_n = 0\}} |\partial_y v_n + \gamma_0|^q \ d\lambda_2 \\
&= \frac{1}{\varepsilon} \frac{1}{|\Omega|} \int_{(0,L)} \left(\int_{A_{n,l}} \left| v'_{n,l}(y) + \gamma_0 \right|^q \ dy \right) dl \geq \frac{1}{\varepsilon} \frac{1}{|\Omega|} \frac{1}{4} L K_2 = 3C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}},
\end{aligned}$$

which gives the desired contradiction and shows (iii).
(iv): Assume $|S_4| \leq \frac{3}{4}L$ then we have

$$\left|\left\{l \in (0,L)\ :\ V\left(\chi_{A_{n,l}},\omega\right) > \frac{4}{L}V_y\left(\chi_{\{\gamma_n=0\}},\Omega\right)\right\}\right| \geq \frac{1}{4}L$$

and we get by

$$V_y\left(\chi_{\{\gamma_n=0\}},\Omega\right) = \int\limits_0^L V\left(\chi_{A_{n,l}},\omega\right)\ dl > \frac{1}{4}L\frac{4}{L}V_y\left(\chi_{\{\gamma_n=0\}},\Omega\right),$$

the desired contradiction.
Thus we have proven the sub-assertions $(i)-(iv)$, which gives an $l \in (0,L)$ so that we have $H > |A| = |A_{n,l}| > \frac{H}{2} > 0$, $\int_{A_{n,l}} |f'(x)-\gamma_0|^q\ dx < K_2$ and additionally $V_y\left(\chi_{\{\widetilde{\gamma}_n=0\}},\Omega\right) \geq \frac{L}{4}V\left(\chi_{A_{n,l}},\omega\right)$. Using Lemma 6.12 we get

$$\int\limits_0^H |v_n(l,y)|\ dy\cdot V\left(\chi_A,(0,H)\right) = \int_\omega |f|\ dy\cdot V\left(\chi_A,\omega\right) \geq \frac{|A|^2 K_1}{128} > \frac{H^2K_1}{2^9}.$$

This implies, using Equation (6.31) and Equation (6.32), that

$$\begin{aligned}\left(2C_B\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}\right)^{\frac{1}{q}} V\left(\chi_A,\omega\right) &\geq \left(\widehat{I}\left(u_n,\gamma_n\right)\right)^{\frac{1}{q}} V\left(\chi_A,\omega\right)\\ &\geq \frac{1}{\varepsilon^{\frac{1}{q}}}\frac{1}{|\Omega|}\int\limits_0^H |v_{n,l}(y)|\ dy\cdot V\left(\chi_A,\omega\right) \geq \frac{H^2K_1}{2^9\,|\Omega|\,\varepsilon^{\frac{1}{q}}},\end{aligned}$$

and thus we get $V\left(\chi_A,\omega\right) \geq \frac{H^2K_1}{2^9|\Omega|(2C_B)^{\frac{1}{q}}}\frac{1}{(\varepsilon\delta)^{\frac{1}{q+1}}}$. Finally we can conclude

$$\inf_{\substack{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2)\\ \gamma\in X_\mu}} \widehat{I}(u,\gamma)+\kappa_n \geq \widehat{I}\left(u_n,\gamma_n\right) \geq \frac{\delta}{|\Omega|}V_y\left(\chi_{\{\widetilde{\gamma}_n=0\}},\Omega\right) \geq \frac{\delta}{4H}V\left(\chi_A,\omega\right) \geq C\frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}},$$

where $C = C\left(\Omega,F^*,C_B,q\right) = \frac{\gamma_0}{2^{11}L(2C_B)^{\frac{1}{q}}} > 0$ is independent of ε,δ,p,n, as asserted.

□

Next, we prove the same lower bound for the simplified energy $\widetilde{I} = \widetilde{I}_{\varepsilon,\delta}$, which was defined in Section 5.3 in Equation (5.22) and is given by

$$\widetilde{I}\left(u,\gamma\right) = \int_\Omega \frac{1}{\varepsilon}\operatorname{dist}^q\left(\nabla u\left(\mathbb{1}-\gamma\vec{e}_1\otimes\vec{e}_2\right),SO(2)\right)+|\gamma|^p\ d\lambda_2 + \frac{\delta}{|\Omega|}V_y\left(\chi_{\{\gamma=0\}},\Omega\right).$$

First of all, we give some lower bounds of this energy, which are used in the subsequent Theorem.

Lemma 6.14. *Let* $q \geq 1$, $F^* = R \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(2)}$, $R \in SO(2)$, *then we get for each* $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ *with* $u = F^*$ *on* $\partial\Omega$ *and each* $\gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)$, *that*

$$2^q \widetilde{I}(u,\gamma) \geq \frac{1}{\varepsilon}\left(\frac{1}{|\Omega|}\int_\Omega |\partial_x u_1 - 1| \; d\lambda_2\right)^q$$

and

$$\widetilde{I}(u,\gamma) + \frac{1}{\varepsilon}\left(2^{\frac{q}{2}}\left(\varepsilon \widetilde{I}(u,\gamma)\right)^{\frac{1}{2}}\right)^{\min\left\{\frac{2}{q},1\right\}} \geq \frac{1}{2^{q-1}\varepsilon\,|\Omega|}\int_\Omega |\partial_y u_1 - \gamma \partial_x u_1|^q \; d\lambda_2,$$

for all $\varepsilon, \delta > 0$.

Proof:
W.l.o.g. we can choose $F^* = \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix}$ and $R = \mathbb{1}$. By definition of $\widetilde{I}$ we have

$$\begin{aligned} |\Omega| \widetilde{I}_{\varepsilon,\delta}(u,\gamma) &\geq \int_\Omega \frac{1}{\varepsilon} \inf_{\vec{a}\in\mathbb{S}^1}\left[\|\nabla u \vec{e}_1 - \vec{a}\|^2 + \left\|\nabla u \vec{e}_2 - \gamma \nabla u \vec{e}_1 - \vec{a}^\perp\right\|^2\right]^{\frac{q}{2}} d\lambda_2 \qquad (6.33) \\ &= \int_\Omega \frac{1}{\varepsilon}\left[\|\nabla u \vec{e}_1 - \vec{a}_{\inf}\|^2 + \left\|\nabla u \vec{e}_2 - \gamma \nabla u \vec{e}_1 - \vec{a}_{\inf}^\perp\right\|^2\right]^{\frac{q}{2}} d\lambda_2 \\ &\geq \int_\Omega \frac{1}{\varepsilon}\left[(\partial_x u_1 - a_{\inf,1})^2 + (\partial_y u_1 - \gamma \partial_x u_1 + a_{\inf,2})^2\right]^{\frac{q}{2}} d\lambda_2, \end{aligned}$$

where we have defined $\vec{a}_{\inf} = \begin{pmatrix} a_{\inf,1} \\ a_{\inf,2} \end{pmatrix}$ as the pointwise infimum, if it is unique and as $\vec{e}_1$ otherwise for all $x \in \Omega$ for which $\nabla u(x)$ exists and zero else. This is done as in the proof of Lemma 6.6, where we have shown, that the function $\vec{a}_{\inf} : \Omega \to \mathbb{S}^1 \cup \{0\}$ is measurable, which implies that $a_{\inf,1} : \Omega \to [-1,1]$ is measurable. Now we want to show the first assertion. We get with the help of Equation (6.1) and Jensen's inequality, that

$$\begin{aligned} |\Omega| \widetilde{I}_{\varepsilon,\delta}(u,\gamma) &\geq \int_\Omega \frac{1}{\varepsilon}|\partial_x u_1 - a_{\inf,1}|^q \; d\lambda_2 \geq \frac{|\Omega|^{1-q}}{\varepsilon}\left(\int_\Omega |\partial_x u_1 - a_{\inf,1}| \; d\lambda_2\right)^q \\ &\geq \frac{|\Omega|^{1-q}}{\varepsilon}\left(\left|\int_\Omega |\partial_x u_1 - 1| \; d\lambda_2 - \int_\Omega |1 - a_{\inf,1}| \; d\lambda_2\right|\right)^q \\ &\geq \frac{|\Omega|^{1-q}}{\varepsilon}\left[\frac{1}{2^{q-1}}\left(\int_\Omega |\partial_x u_1 - 1| \; d\lambda_2\right)^q - \left|\int_\Omega 1 - a_{\inf,1} \; d\lambda_2\right|^q\right], \end{aligned}$$

since $1 - a_{\inf,1}(x) \geq 0$ for every $x \in \Omega$. Next, Lemma 3.3 gives $\int_\Omega 1 \, d\lambda_2 = \int_\Omega \partial_x u_1 \, d\lambda_2$, which implies

$$\begin{aligned}
|\Omega| \widetilde{I}_{\varepsilon,\delta}(u,\gamma) &\geq \frac{|\Omega|^{1-q}}{\varepsilon} \left[\frac{1}{2^{q-1}} \left(\int_\Omega |\partial_x u_1 - 1| \, d\lambda_2 \right)^q - \left(\int_\Omega |\partial_x u_1 - a_{\inf,1}| \, d\lambda_2 \right)^q \right] \\
&\geq \frac{|\Omega|^{1-q}}{\varepsilon} \left[\frac{1}{2^{q-1}} \left(\int_\Omega |\partial_x u_1 - 1| \, d\lambda_2 \right)^q - |\Omega|^{q-1} \int_\Omega |\partial_x u_1 - a_{\inf,1}|^q \, d\lambda_2 \right] \\
&\geq \frac{|\Omega|^{1-q}}{\varepsilon 2^{q-1}} \left(\int_\Omega |\partial_x u_1 - 1| \, d\lambda_2 \right)^q - |\Omega| \, \widetilde{I}_{\varepsilon,\delta}(u,\gamma).
\end{aligned}$$

Thus we get the first part of the lemma.
Next, we get, using Equation (6.1) and Equation (6.33), that

$$\begin{aligned}
\varepsilon \, |\Omega| \, \widetilde{I}_{\varepsilon,\delta}(u,\gamma) &\geq \int_\Omega |\partial_y u_1 - \gamma \partial_x u_1 + a_{\inf,2}|^q \, d\lambda_2 \\
&\geq \frac{1}{2^{q-1}} \int_\Omega |\partial_y u_1 - \gamma \partial_x u_1|^q \, d\lambda_2 - \int_\Omega |a_{\inf,2}|^q \, d\lambda_2. \qquad (6.34)
\end{aligned}$$

Investigate first the case $1 \leq q \leq 2$. In this case we can derive by Jensen's inequality for concave functions and Lemma 3.3 that,

$$\begin{aligned}
&\frac{1}{|\Omega|} \int_\Omega |a_{\inf,2}|^q \, d\lambda_2 = \frac{1}{|\Omega|} \int_\Omega \left(1 - a_{\inf,1}^2\right)^{\frac{q}{2}} \, d\lambda_2 \leq \frac{1}{|\Omega|^{\frac{q}{2}}} \left(\int_\Omega \left(1 - a_{\inf,1}^2\right) \, d\lambda_2 \right)^{\frac{q}{2}} \\
&\leq \left(\frac{2}{|\Omega|} \right)^{\frac{q}{2}} \left(\int_\Omega (1 - a_{\inf,1}) \, d\lambda_2 \right)^{\frac{q}{2}} \leq \left(\frac{2}{|\Omega|} \right)^{\frac{q}{2}} \left(\int_\Omega |\partial_x u_1 - a_{\inf,1}| \, d\lambda_2 \right)^{\frac{q}{2}} \\
&\leq \left(\frac{2}{|\Omega|} \right)^{\frac{q}{2}} \left(|\Omega|^{q-1} \int_\Omega |\partial_x u_1 - a_{\inf,1}|^q \, d\lambda_2 \right)^{\frac{1}{2}} \leq 2^{\frac{q}{2}} \left(\varepsilon \widetilde{I}_{\varepsilon,\delta}(u,\gamma) \right)^{\frac{1}{2}}, \qquad (6.35)
\end{aligned}$$

where we have used $\left(1 - a_{\inf,1}^2\right) = (1 + a_{\inf,1})(1 - a_{\inf,1}) \leq 2(1 - a_{\inf,1})$ since $a_{\inf,1} \leq 1$.
If $q > 2$ we have $\int_\Omega \left(1 - a_{\inf,1}^2\right)^{\frac{q}{2}} \, d\lambda_2 \leq \int_\Omega \left(1 - a_{\inf,1}^2\right) \, d\lambda_2$, since $1 - a_{\inf,1}^2(x) \in [0,1]$ for every $x \in \Omega$, and we get as above, that

$$\frac{1}{|\Omega|} \int_\Omega \left(1 - a_{\inf,1}^2\right)^{\frac{q}{2}} \, d\lambda_2 \leq \left(2^{\frac{q}{2}} \left(\varepsilon \widetilde{I}_{\varepsilon,\delta}(u,\gamma) \right)^{\frac{1}{2}} \right)^{\frac{2}{q}} = 2 \left(\varepsilon \widetilde{I}_{\varepsilon,\delta}(u,\gamma) \right)^{\frac{1}{q}}. \qquad (6.36)$$

Combining the Equations (6.34), (6.35) and (6.36), we get

$$\varepsilon \widetilde{I}_{\varepsilon,\delta}(u,\gamma) \geq \frac{1}{2^{q-1} |\Omega|} \int_\Omega |\partial_y u_1 - \gamma \partial_x u_1|^q \, d\lambda_2 - \left(2^{\frac{q}{2}} \left(\varepsilon \widetilde{I}(u,\gamma) \right)^{\frac{1}{2}} \right)^{\min\left\{\frac{2}{q},1\right\}},$$

which gives the desired second assertion.

□

Theorem 6.15. *Let* $q \geq 1$*,* $F^* = R\begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}^{(2)}$*, with* $R \in SO(2)$*,* $\mu > 0$ *and* $\Omega = (0, L) \times (0, H)$ *,* $L, H > 0$*. We define*

$$X_\mu = X_{\mu;p,q} := \left\{ \gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega) \ : \exists\, \overline{\gamma} \in \gamma,\ \forall x \in \Omega : |\overline{\gamma}(x)| \in \{0\} \cup [\mu, \infty) \right\}.$$

Let $C_B = C_B(\Omega, F^*, q) > 0$ *be a constant, independent of* ε, δ, p*, given by Corollary 5.12, such that*

$$\inf_{u \in W^{1,\infty}(\Omega;\mathbb{R}^2),\ \gamma \in X_\mu} \widetilde{I}\,(u, \gamma) \leq C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}},$$

for all $0 < \varepsilon, \delta \leq 1$*. Then there exists a constant* $C = C\,(\Omega, C_B, F^*, q) > 0$ *independent of* ε, δ, μ *and a* $0 < \rho < 1$*, such that*

$$\inf_{u \in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2),\ \gamma \in X_\mu} \widetilde{I}\,(u, \gamma) \geq \min\left\{ C \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}, \frac{\mu^p}{8} \right\},$$

for all $0 < \varepsilon, \delta < \rho$*.*

Proof:
The proof is similar to the proof of Theorem 6.13, some details can be found there. W.l.o.g. we can choose $F^* = \begin{pmatrix} 1 & \gamma_0 \\ 0 & 1 \end{pmatrix}$ with $\gamma_0 > 0$ and $R = \mathbb{1}$, where the case $\gamma_0 < 0$ goes analogously. Choose now $\rho \in (0, 1)$ such that we get $2^{\max\{2q+5, 3q+3\}} \widetilde{C} \varepsilon < \gamma_0^q$, for all $0 < \varepsilon, \delta < \rho$, where $\widetilde{C} := 2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}} + \frac{1}{\varepsilon} \left(2^{\frac{q}{2}} \left(2C_B\,(\varepsilon\delta)^{\frac{q}{q+1}} \right)^{\frac{1}{2}} \right)^{\min\left\{\frac{2}{q}, 1\right\}} > 0$. The $\widetilde{C}$ was defined in such way, that we get by Lemma 6.14,

$$\widetilde{C} \geq \frac{1}{\varepsilon 2^{q-1} |\Omega|} \int_\Omega |\partial_y u_{n,1} - \gamma \partial_x u_{n,1}|^q \ d\lambda_2 \tag{6.37}$$

if $I(u, \gamma) \leq 2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}$, which is fulfilled for example for minimizing sequences. For $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in W^{1,\infty}_{F^*}\left(\Omega; \mathbb{R}^2\right)$ and $\gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)$ we get by Lemma 6.14, that

$$2^q \widetilde{I}\,(u, \gamma) \geq \frac{1}{\varepsilon} \left(\frac{1}{|\Omega|} \int_\Omega |\partial_x u_1 - 1| \ d\lambda_2 \right)^q.$$

Next, we define $v \in W^{1,\infty}(\Omega)$ by $v\begin{pmatrix} x \\ y \end{pmatrix} = u_1 \begin{pmatrix} x \\ y \end{pmatrix} - x - \gamma_0 y$. Then we get $v = 0$ on $\partial\Omega$ and we get as in Equation (6.30) in the proof of Theorem 6.13, for each $l \in (0, L)$, that

$$2\left(\varepsilon \widetilde{I}\,(u, \gamma)\right)^{\frac{1}{q}} \geq \frac{1}{|\Omega|} \int_\Omega |\partial_x u_1 - 1| \ d\lambda_2 \geq \frac{1}{|\Omega|} \int_0^H |v_l(y)| \ dy, \tag{6.38}$$

where $v_l : (0,H) \to \mathbb{R}$ is defined by $v_l(y) := v(l,y)$.
Assume that $\left\{ \left(u_n = \begin{pmatrix} u_{n,1} \\ u_{n,2} \end{pmatrix}, \gamma_n \right) \right\}_{n\in\mathbb{N}} \subseteq W^{1,\infty}_{F^*}\left(\Omega;\mathbb{R}^2\right) \times X_\mu$ is a minimal sequence, i.e., it exists a sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ with $\kappa_n \overset{n\to\infty}{\longrightarrow} 0$ and a $N \in \mathbb{N}$ such that

$$\widetilde{I}\left(u_n,\gamma_n\right) \le \inf_{u\in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2),\ \gamma\in X_\mu} \widetilde{I}\left(u,\gamma\right) + \kappa_n \le 2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}} \tag{6.39}$$

for all $n \ge N$. Define now $v_n \in W^{1,\infty}(\Omega)$ by $v_n \begin{pmatrix} x \\ y \end{pmatrix} = u_{n,1}\begin{pmatrix} x \\ y \end{pmatrix} - x - \gamma_0 y$ and for $l \in (0,L)$ we define $v_{n,l} \in W^{1,\infty}\left((0,H)\right)$ by $v_{n,l}(y) := v_n(l,y)$. Then $v_n = 0$ on $\partial\Omega$ for all $n \in \mathbb{N}$. Choose a representant $\widetilde{\gamma}_n$ of γ_n for each $n \in \mathbb{N}$ and define the function $\widetilde{\gamma}_{n,l} : \omega \to \mathbb{R}$ by $\widetilde{\gamma}_{n,l}(y) := \widetilde{\gamma}_n(l,y)$ for each $l \in (0,L)$. As in the proof of Theorem 6.13, we want to apply Lemma 6.12 for the Lipschitz continuous function $f = -v_{n,l} : (0,H) \to \mathbb{R}$, the sets $\omega = (0,H)$ and $A = A_{n,l} := \{\widetilde{\gamma}_{n,l} = 0\}$ and for the constants $K_1 = \gamma_0 > 0$ and $K_2 = 2^{q+2}H\varepsilon\widetilde{C} > 0$. Then we have $K_2 < \left(\frac{\gamma_0}{2}\right)^q \frac{\frac{H}{2}}{2^{max\{2,q\}}} < \left(\frac{K_1}{2}\right)^q \frac{|A|}{2^{max\{2,q\}}}$ if $|A| > \frac{H}{2}$, for a suitable $l \in (0,L)$. As in the proof of Theorem 6.13, we need to show the sub-assertions $(i) - (iv)$, which are repeated in the following:

- (i) For $S_1 := \left\{ l \in (0,L) \ :\ |A_{n,l}| > \frac{H}{2} \right\}$ we get $|S_1| > \frac{3}{4}L$ or an immediate prove of the Theorem.
- (ii) For $S_2 := \{ l \in (0,L) \ :\ |A_{n,l}| < H \}$ we have $|S_2| > \frac{3}{4}L$.
- (iii) For $S_3 := \left\{ l \in (0,L) \ :\ \int_{A_{n,l}} \left| v'_{n,l}(x) + \gamma_0 \right|^q \, dy < K_2 \right\}$ we derive $|S_3| > \frac{3}{4}L$.
- (iv) For $S_4 := \left\{ l \in (0,L) \ :\ V\left(\chi_{A_{n,l}},\omega\right) \le \frac{4}{L} V_y\left(\chi_{\{\gamma_n=0\}},\Omega\right) \right\}$ we get $|S_4| > \frac{3}{4}L$.

Review, that we have already shown in Theorem 6.13, that S_i are measurable for each $i \in \{1,\ldots,4\}$ and $|S_1 \cap S_2 \cap S_3 \cap S_4| > 0$.
The proofs of (i) and (iv) are the same as in the proof of Theorem 6.13.
(ii): Assume $|S_2| \le \frac{3}{4}L$, then we get, using Equation (6.37), that

$$2^{q-1}\left|\Omega\right|\widetilde{C} \ge \int_\Omega \frac{1}{\varepsilon}\left|\partial_y u_{n,1} - \gamma\partial_x u_{n,1}\right|^q \, d\lambda_2 \ge \frac{1}{\varepsilon}\gamma_0^q \frac{1}{4}\left|\Omega\right|,$$

where the last inequality follows as in the proof of sub-assertion (ii) in Theorem 6.13. This is a contradiction to $2^{q+1}\widetilde{C}\varepsilon < \gamma_0^q$.
(iii): Assume $|S_3| \le \frac{3}{4}L$, then we get again with help of Equation (6.37), that

$$2^{q-1}\widetilde{C} \ge \frac{1}{|\Omega|}\int_{\{\widetilde{\gamma}_n=0\}} \frac{1}{\varepsilon}\left|\partial_y u_{n,1}\right|^q \, d\lambda_2 \ge \frac{1}{\varepsilon}\frac{1}{|\Omega|}\frac{1}{4}LK_2 = 2^q\widetilde{C},$$

where the last inequality follows as in the proof of sub-assertion (iii) in Theorem 6.13. This is the desired contradiction and shows (iii).
Thus we have proven the sub-assertions $(i)-(iv)$, which give an $l \in (0, L)$ such that we have $H > |A| = |A_{n,l}| > \frac{H}{2} > 0$, $\int_A |f'(x) - K_1|^q \ dx < K_2$ and additionally $V_y \left(\chi_{\{\gamma_n=0\}}, \Omega\right) \geq \frac{L}{4} V \left(\chi_{A_{n,l}}, \omega\right)$. Using Lemma 6.12 we get

$$\int\limits_0^H |v_n(l,y)| \ dy \cdot V\left(\chi_A, (0,H)\right) = \int_\omega |f| \ dy \cdot V\left(\chi_A, \omega\right) \geq \frac{|A|^2 K_1}{128} > \frac{H^2 K_1}{2^9}.$$

This implies, using Equation (6.38) and Equation (6.39), that

$$\begin{aligned} \left(2C_B \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}}\right)^{\frac{1}{q}} V\left(\chi_A, \omega\right) &\geq \left(\widetilde{I}\left(u_n, \gamma_n\right)\right)^{\frac{1}{q}} V\left(\chi_A, \omega\right) \\ &\geq \frac{1}{2\varepsilon^{\frac{1}{q}}} \frac{1}{|\Omega|} \int\limits_0^H |v_{n,l}(y)| \ dy \cdot V\left(\chi_A, \omega\right) \geq \frac{H^2 K_1}{2^{10} |\Omega| \varepsilon^{\frac{1}{q}}}, \end{aligned}$$

and thus we get $V\left(\chi_A, \omega\right) \geq \frac{H^2 K_1}{2^{10} |\Omega| (2C_B)^{\frac{1}{q}}} \frac{1}{(\varepsilon\delta)^{\frac{1}{q+1}}}$. Finally we can conclude

$$\inf_{\substack{u \in W^{1,\infty}_{F^*}(\Omega;\mathbb{R}^2) \\ \gamma \in X_\mu}} \widetilde{I}(u,\gamma) + \kappa_n \geq \widetilde{I}\left(u_n, \gamma_n\right) \geq \frac{\delta}{|\Omega|} V_y\left(\chi_{\{\widetilde{\gamma}_n=0\}}, \Omega\right) \geq \frac{\delta}{4H} V\left(\chi_A, \omega\right) \geq C \frac{\delta^{\frac{q}{q+1}}}{\varepsilon^{\frac{1}{q+1}}},$$

where $C = C\left(\Omega, F^*, C_B, q\right) = \frac{\gamma_0}{2^{12} L (2C_B)^{\frac{1}{q}}} > 0$ is independent of ε, δ, μ. This finalizes the proof.

□

7 Γ-convergence

In this chapter we show results for $p, q \geq 2$, which implies $\frac{1}{p} + \frac{1}{q} \leq 1$ and includes the case of linear hardening, i.e. $p = 2$. In the thesis of Carolin Kreisbeck [42, Theorem 7.28], see also Theorem 4.3, it was shown, that the energy functional $E_{cond,\varepsilon}$ converges in the sense of Γ-convergence to the functional E_{rigid} for $\varepsilon \to 0$, see the Equations (4.2) and (4.3) for the definition of $E_{cond,\varepsilon}$ and E_{rigid}. An interesting question is now, if the energy $E_{\varepsilon,\delta}$, defined in Equation (4.12), converges in the sense of Γ-convergence to the functional $E = \frac{1}{|\Omega|} E_{rigid}$ for $\varepsilon, \delta \to 0$. One expects a positive result if one chooses $\delta = \delta(\varepsilon)$ small in comparison to ε, e.g. $\delta = \varepsilon^\kappa$ with κ big enough. Unfortunately we are only able to show this, if we restrict ourselves to more regular functions, namely $u \in C^{1,\alpha}\left(\Omega; \mathbb{R}^2\right)$ with $\alpha \in (0, 1]$. In this section we choose $B^\infty(0, \rho)$ as subset of $\mathbb{R}^2$ and define for fixed $F \in \mathbb{R}^{2\times 2}$ the function $u_0 : \Omega \to \mathbb{R}^2$ by $u_0(x) = Fx$ for $x \in \Omega$. Finally, we define $\mathcal{X} = \mathcal{X}_F := W_F^{1,1}\left(\Omega; \mathbb{R}^2\right) \cup \left(L^1\left(\Omega; \mathbb{R}^2\right) \setminus W^{1,1}\left(\Omega; \mathbb{R}^2\right)\right)$.

Theorem 7.1. *Let $\Omega = B^\infty(0, L) \subseteq \mathbb{R}^2$, $F \in \mathbb{R}^{2\times 2}$, $p = q = 2$, $\alpha \in (0, 1]$ and $\delta = \varepsilon^\kappa$ with $\kappa > 1 + \max\left\{1, \frac{1}{2\alpha}\right\}$. Let $E_{\varepsilon,\delta}$ and E be the functionals defined in the Equations (4.12) and (4.13). Then, $E_{\varepsilon,\varepsilon^\kappa}$ converges to E in the sense of pointwise Γ-convergence on $Z := \left\{u \in C^{1,\alpha}\left(\Omega; \mathbb{R}^2\right) : u = F \text{ on } \partial\Omega\right\}$ as ε tends to zero, for the metric space $(X, d) = \left(\mathcal{X}, \|. - .\|_{L^1(\Omega;\mathbb{R}^2)}\right)$. This means,*

$$E[u] = \Gamma - \lim_{\varepsilon\to 0} E_{\varepsilon,\varepsilon^\kappa}[u] \text{ for each } u \in Z.$$

Furthermore the lower bound inequality (i) in Definition 3.36 is fulfilled for all functions $u \in L^1\left(\Omega; \mathbb{R}^2\right)$ and one has the following compactness result:

If $\{u_\varepsilon\}_{\varepsilon>0} \subseteq W^{1,1}\left(\Omega; \mathbb{R}^2\right)$ is a sequence of functions with bounded energy, i.e., it exists a $B > 0$, so that it holds for all $\varepsilon > 0$, that $E_{\varepsilon,\varepsilon^\kappa}[u_\varepsilon] \leq B < \infty$, and we have $u_\varepsilon - u_0 \in W_0^{1,1}\left(\Omega; \mathbb{R}^2\right)$ for all $\varepsilon > 0$. Then, there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$, with $\varepsilon_k \overset{k\to\infty}{\longrightarrow} 0$, and a function $u \in W^{1,2}\left(\Omega; \mathbb{R}^2\right)$ such that $u_{\varepsilon_k} \to u$ in $L^1\left(\Omega; \mathbb{R}^2\right)$ as $k \to \infty$, with $u - u_0 \in W_0^{1,1}\left(\Omega; \mathbb{R}^2\right)$ and $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere in Ω.

For simplicity we split the proof into the natural three parts, the lower bound, the compactness and the recovery sequence. The lower bound and compactness result follows immediately from the corresponding ones in [42, Theorem 7.18].

7.1 Lower bound and compactness

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain, $p = q = 2$, then the following was shown in [42, Theorem 7.18, Remark 7.25], for the case $\delta = 0$:

If $\{u_\varepsilon\}_{\varepsilon>0} \subseteq W^{1,1}\left(\Omega;\mathbb{R}^2\right)$ is a sequence of bounded energy, i.e., for all $\varepsilon > 0$ it holds $E_{\varepsilon,0}\left[u_\varepsilon\right] \leq B < \infty$ and we have $u_\varepsilon - u_0 \in W_0^{1,1}\left(\Omega;\mathbb{R}^2\right)$ for all $\varepsilon > 0$, then there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$, with $\varepsilon_k \overset{k\to\infty}{\longrightarrow} 0$, and a function $u \in W^{1,2}\left(\Omega;\mathbb{R}^2\right)$ such that $\liminf_{\varepsilon\to 0} E_{\varepsilon,0}\left[u_\varepsilon\right] = \liminf_{k\to\infty} E_{\varepsilon_k,0}\left[u_{\varepsilon_k}\right]$ and $u_{\varepsilon_k} \to u$ in $L^1\left(\Omega;\mathbb{R}^2\right)$ as $k \to \infty$, with $u - u_0 \in W_0^{1,1}\left(\Omega;\mathbb{R}^2\right)$ and $\nabla u \in \mathcal{N}^{(2)}$ almost everywhere in Ω. Furthermore, one has the lower bound inequality

$$\liminf_{\varepsilon\to 0} E_{\varepsilon,0}\left[u_\varepsilon\right] = \liminf_{k\to\infty} E_{\varepsilon_k,0}\left[u_{\varepsilon_k}\right] \geq E[u] = \frac{1}{|\Omega|}\int_\Omega W^{qc}_{rigid,2}\left(\nabla u\right)\, d\lambda_2.$$

Since $E_{\varepsilon,\delta} \geq E_{\varepsilon,0}$, the same is true for all $\delta > 0$ if one replaces $E_{\varepsilon,0}$ by $E_{\varepsilon,\delta}$ and one obtains the desired compactness result. Let $\{x_\varepsilon\}_{\varepsilon>0} \subseteq \mathcal{X}$ be a sequence with $\liminf_{\varepsilon\to 0} f_\varepsilon\left(x_\varepsilon\right) < \infty$, then there exists a $B > 0$ and a subsequence $\{x_{\varepsilon_n}\}_{n\in\mathbb{N}} \subseteq W_F^{1,1}\left(\Omega;\mathbb{R}^2\right)$, such that $E_{\varepsilon,\varepsilon^\kappa}\left[u_\varepsilon\right] \leq B$ and we get the demanded lower bound. Next, we want to show the existence of a recovery sequence for each $u \in Z$.

7.2 Recovery sequence

Remark 7.2. *Let $\rho > 0$, $0 < \alpha \leq 1$ and $v \in C^{1,\alpha}\left(\overline{B^\infty(0,\rho)}\right)$, then we have*

$$\frac{1}{\left|B^\infty\left(0,\rho\right)\right|}\int_{B^\infty(0,\rho)} \left|\nabla v(x) - \nabla v(0)\right|^2\, d\lambda_2 \leq \pi\,\|v\|^2_{C^{1,\alpha}\left(\overline{B^\infty(0,\rho)}\right)}\,\rho^{2\alpha}.$$

Proof: Let $x \in B^\infty\left(0,\rho\right) \setminus \{0\}$, then we get $\frac{\left|\partial_{x_i}v(x)-\partial_{x_i}v(0)\right|}{|x-0|^\alpha} \leq \|\partial_{x_i}v\|_{C^{0,\alpha}\left(\overline{B^\infty(0,\rho)}\right)}$, for $i \in \{1,2\}$, and thus we have

$$\left|\nabla v(x) - \nabla v(0)\right|^2 = \sum_{i=1}^{2}\left|\partial_{x_i}v(x) - \partial_{x_i}v(0)\right|^2 \leq \|v\|^2_{C^{1,\alpha}\left(\overline{B^\infty(0,\rho)}\right)}\,|x|^{2\alpha}.$$

Since

$$\int_{B^\infty(0,\rho)} |x|^{2\alpha}\, d\lambda_2 \leq \int_0^{2\pi}\left(\int_0^{\sqrt{2}\rho} r^{2\alpha}\cdot r\, dr\right) d\varphi = \frac{2\pi}{2\alpha+2}2^{\alpha+1}\rho^{2\alpha+2}$$

and $\alpha \in (0,1]$ we deduce the assertion.

□

Lemma 7.3. *Let $\varepsilon, \delta > 0$, $\rho \geq \varepsilon\delta$, $p \geq 2$ and $q \geq 1$. Suppose $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $u_0(y) = Fy$ with $F \in \mathcal{N}^{(2)}$. Then there exist $\gamma \in BV\left(B^\infty(0,\rho)\right) \cap L^{\max\{p,q\}}\left(B^\infty(0,\rho)\right)$ and $z \in W^{1,\infty}\left(B^\infty(0,\rho); \mathbb{R}^2\right)$, such that*

(i) $z = u_0$ *on* $\partial B^\infty(0,\rho)$,

(ii) $\int_{B^\infty(0,\rho)} |\gamma|^p \, d\lambda_2 \leq |B^\infty(0,\rho)| \, W^{qc}_{rigid,p}(F)$,

(iii) $\int_{\Omega_\rho} \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla z \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 + \delta V_x\left(\gamma, \Omega_\rho\right) \leq C \, |F\vec{e}_2|^q \, \rho^2 \sqrt{\frac{\delta}{\rho\varepsilon}}$,

(iv) $\|z - u_0\|_{L^\infty(B^\infty(0,\rho);\mathbb{R}^2)} \leq \sqrt{\rho\delta\varepsilon} \leq \rho$,

where $\Omega_\rho := B^\infty(0,\rho)$ and $C = C(q) > 0$ is independent of z, ρ, ε, δ, F, p.

Proof:
If $F \in \mathcal{M}^{(2)}$ then define $z = u_0$, which implies (i) and (iv), and define γ by the unique representation $F = R\left(\mathbb{1} + \gamma \vec{e}_1 \otimes \vec{e}_2\right)$ with $R \in SO(2)$. Then, γ satisfies (ii) in view of $|\gamma|^p = \left(|\gamma|^2\right)^{\frac{p}{2}} = \left(|F\vec{e}_2|^2 - 1\right)^{\frac{p}{2}} = W^{qc}_{rigid,p}(F)$. Since γ is constant on $B^\infty(0,\rho)$ and $\nabla z \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right) = R\left(\mathbb{1} + \gamma \vec{e}_1 \otimes \vec{e}_2\right)\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right) = R \in SO(2)$, we get

$$\int_{B^\infty(0,\rho)} \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla z \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 + \delta V_x\left(\gamma, B^\infty(0,\rho)\right) = 0.$$

Therefore (iii) is fulfilled.
From now on let $F \in \mathcal{N}^{(2)} - \mathcal{M}^{(2)}$. Following Lemma 4.1 there exist $F_0,\ F_1 \in \mathcal{M}^{(2)}$, with $F_0\vec{e}_2 = F_1\vec{e}_2$, so that $F = \lambda F_0 + (1-\lambda)F_1$ for a $\lambda \in (0,1)$. Let $\vec{a} \in \mathbb{R}^2$ such that $F_0 - F_1 = \vec{a} \otimes \vec{e}_1$, which implies $|\vec{a}| \leq 2$. Define on $B^\infty(0,\rho)$ the simple laminate $v_1 \in W^{1,\infty}\left(B^\infty(0,\rho); \mathbb{R}^2\right)$ by

$$v_1(\nu) = F(\nu) + h\chi_\lambda\left(\frac{\vec{e}_1 \cdot \nu}{h}\right)\vec{a},$$

with period $0 < h \leq \rho$ to be chosen later, where $\chi_\lambda : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous, one-periodic function, with $\chi_\lambda(0) = \chi_\lambda(1) = 0$ and $\chi'_\lambda(t) = \begin{cases} 1-\lambda \text{ if } t \in (0,\lambda) \\ -\lambda \text{ if } t \in (\lambda,1) \end{cases}$. Then we get for $\tau \in \mathbb{R}^2$ with $\frac{\vec{e}_1 \cdot \tau}{h} - \left\lfloor \frac{\vec{e}_1 \cdot \tau}{h} \right\rfloor \in (0,\lambda)$, that

$$\nabla v_1(\tau) = F + (1-\lambda)\vec{a} \otimes \vec{e}_1 = \lambda F_0 + (1-\lambda)F_1 + (1-\lambda)(F_0 - F_1) = F_0$$

and analogously $\nabla v_1(\tau) = F_1$ if $\frac{\vec{e}_1 \cdot \tau}{h} - \left\lfloor \frac{\vec{e}_1 \cdot \tau}{h} \right\rfloor \in (\lambda,1)$.
In order to fulfill the boundary-condition $v_1 = u_0$ on the lines $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ y = \pm\rho\right\}$

we cut the laminate by using affine interpolation between the values of v_1 in the three points $X_{k,u} := (kh, \rho)$, $Y_{k,u} := ((k+1)h, \rho)$ and $Z_{k,u} := (kh + \lambda h, \rho - h)$ for the top and $X_{k,l} := (kh, -\rho)$, $Y_{k,l} := ((k+1)h, -\rho)$ and $Z_{k,l} := (kh + \lambda h, -\rho + h)$ for the bottom, for each $k \in \mathbb{Z}$, for which $X_{k,u}, Y_{k,u}, Z_{k,u} \in \overline{B^\infty(0,\rho)}$, see Figure 7.1. This leads to a function $v_2 \in W^{1,\infty}\left(B^\infty(0,\rho); \mathbb{R}^2\right)$ with $\nabla v_2(x) \in \{F_0, F_1, G_l, G_u\}$ for a.e. $x \in B^\infty(0,\rho)$, where G_l and G_u are independent of h and k. To achieve this we can assume w.l.o.g. $k = 0$. Then we have $v_2(h, -\rho) - v_2(0, -\rho) = v_1(h, -\rho) - v_1(0, -\rho) = F \begin{pmatrix} h \\ 0 \end{pmatrix}$ and

$$\begin{aligned} v_2(\lambda h, -\rho + h) - v_2(0, -\rho) &= v_1(\lambda h, -\rho + h) - v_1(0, -\rho) \\ = F \begin{pmatrix} \lambda h \\ h \end{pmatrix} + h \chi_\lambda \left(\frac{\lambda h}{h} \right) \vec{a} &= h \left(F \begin{pmatrix} \lambda \\ 1 \end{pmatrix} + (1-\lambda)\lambda \vec{a} \right). \end{aligned}$$

Thus we get

$$G_l := \nabla v_2|_{D\left(X_{0,l}, Y_{0,l}, Z_{0,l}\right)} = F\vec{e}_1 \otimes \vec{e}_1 + (F\vec{e}_2 + (1-\lambda)\lambda\vec{a}) \otimes \vec{e}_2$$

and analogously

$$G_u := \nabla v_2|_{D(X_{0,u}, Y_{0,u}, Z_{0,u})} = F\vec{e}_1 \otimes \vec{e}_1 + (F\vec{e}_2 - (1-\lambda)\lambda\vec{a}) \otimes \vec{e}_2,$$

where $D(x,y,z) := \{\lambda_1 x + \lambda_2 y + (1 - \lambda_1 - \lambda_2)\, z : \lambda_1 \in (0,1), \lambda_2 \in (0, 1-\lambda_1)\}$ denotes the open triangle with vertices $x, y, z \in \mathbb{R}^2$. In order to fulfill the boundary condition on the lines $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ x = \pm\rho \right\}$ we define the function

$$\begin{aligned} z : B^\infty(0,\rho) &\to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{cases} v_2 \begin{pmatrix} x \\ y \end{pmatrix} & \text{if} \quad |x| \leq \left\lfloor \frac{\rho}{h} \right\rfloor \cdot h \\ F \begin{pmatrix} x \\ y \end{pmatrix} & \text{otherwise} \end{cases}, \end{aligned}$$

where $\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$, refer to Figure 7.1. Next, we define the auxiliary set $\Omega_h := \left\{ |y| \leq \rho - h,\ |x| \leq \left\lfloor \frac{\rho}{h} \right\rfloor \cdot h \right\} \cap \left\{ x \in \mathbb{R}^2 \ : \ \nabla v_2(x) \text{ exists} \right\}$, then we get by construction, that $\nabla z(\tau) \in \{F_0, F_1\} \subseteq \mathcal{M}^{(2)}$ for all $\tau \in \Omega_h$. Thus we can define the function $\gamma : B^\infty(0,\rho) \to \mathbb{R}$ for an $\tau \in \Omega_h$ by the unique representation $\nabla z(\tau) = R(\tau)\,(\mathbb{1} + \gamma \vec{e}_1 \otimes \vec{e}_2)$ with $R(\tau) \in SO(2)$ and otherwise we set $\gamma(\tau) = 0$. This definition of γ ensures $\nabla z(\tau)\,(\mathbb{1} - \gamma(\tau)\vec{e}_1 \otimes \vec{e}_2) \in SO(2)$ for a.e. $\tau \in \Omega_h$ and thus we get

$$\int_{\Omega_h} \operatorname{dist}^q \left(\nabla z \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right), SO(2) \right) \, d\lambda_2 = 0. \tag{7.1}$$

Furthermore we have for $\tau \in \Omega_h$, that

$$|\gamma(\tau)| = \left(|\nabla z(\tau)\vec{e}_2|^2 - 1 \right)^{\frac{1}{2}} = \left(|F\vec{e}_2|^2 - 1 \right)^{\frac{1}{2}} = \left(W^{qc}_{rigid,p}(F) \right)^{\frac{1}{p}}. \tag{7.2}$$

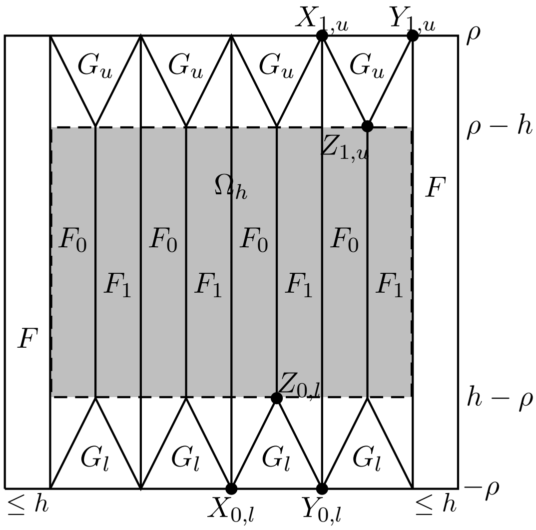

Figure 7.1: $\nabla z,\ \lambda = 0.5$

This definition ensures, that $z \in W^{1,\infty}\left(B^\infty(0,\rho);\mathbb{R}^2\right)$, with $z = u_0$ on $\partial B^\infty(0,\rho)$, and γ is a piecewise constant function in $BV\left(B^\infty(0,\rho)\right) \cap L^{\max\{p,q\}}\left(B^\infty(0,\rho)\right)$, which jumps between the values 0, $\left(W^{qc}_{rigid,p}\left(F\right)\right)^{\frac{1}{p}}$ and $-\left(W^{qc}_{rigid,p}\left(F\right)\right)^{\frac{1}{p}}$. Using Equation (7.2), we get

$$\int_{B^\infty(0,\rho)} |\gamma|^p \ d\lambda_2 = \int_{\Omega_h} W^{qc}_{rigid,p}(F)\ d\lambda_2 \leq \left|B^\infty\left(0,\rho\right)\right| W^{qc}_{rigid,p}(F),$$

which is (ii). Since $F_i \in \mathcal{M}^{(2)}$ for $i \in \{0,1\}$, we get $|F_i\vec{e}_1| = 1$, $\det\left(F_i\right) = JF_i\vec{e}_1 \cdot F_i\vec{e}_2 = 1$ and thus we get, using $F_0\vec{e}_2 = F\vec{e}_2 = F_1\vec{e}_2$, that

$$|F_i\vec{e}_2 - JF_i\vec{e}_1|^2 = |F_i\vec{e}_2|^2 - 2F_i\vec{e}_2 \cdot JF_i\vec{e}_1 + |JF_i\vec{e}_1|^2 = |F\vec{e}_2|^2 - 1, \tag{7.3}$$

for $i \in \{0,1\}$. Therefore we have

$$\operatorname{dist}^2\left(F_i, SO(2)\right) \leq \|F_i - \left(F_i\vec{e}_1 \otimes \vec{e}_1 + JF_i\vec{e}_1 \otimes \vec{e}_2\right)\|^2 = |F_i\vec{e}_2 - JF_i\vec{e}_1|^2 = |F\vec{e}_2|^2 - 1,$$

for $i \in \{0,1\}$. Using Equation (7.3) and $F_0\vec{e}_2 = F\vec{e}_2 = F_1\vec{e}_2$, we get

$$\begin{aligned}\operatorname{dist}^2\left(F, SO(2)\right) &= \inf_{\vec{a}\in\mathbb{S}^1}\left\{|F\vec{e}_1 - \vec{a}|^2 + \left|F\vec{e}_2 - \vec{a}^\perp\right|^2\right\} \\ &\leq \min_{i\in\{0,1\}}\left\{|F\vec{e}_1 - F_i\vec{e}_1|^2\right\} + |F\vec{e}_2 - JF_0\vec{e}_1|^2 \leq 1 + |F\vec{e}_2|^2 - 1,\end{aligned}$$

since $F\vec{e}_1$ lies on the line connecting $F_0\vec{e}_1$ and $F_1\vec{e}_1$ and $|F_0\vec{e}_1 - F_1\vec{e}_1| \leq 2$. Next, we get

$$\begin{aligned}\operatorname{dist}^2(G_l, SO(2)) &= \inf_{A\in SO(2)} \|F + (1-\lambda)\lambda\vec{a}\otimes\vec{e}_2 - A\|^2 \\ &\leq 2\left(\operatorname{dist}^2(F, SO(2)) + \left(\frac{|\vec{a}|}{4}\right)^2\right) \leq 3\,|F\vec{e}_2|^2\,,\end{aligned}$$

since we have for $F \in \mathcal{N}^{(2)}$, that $|F\vec{e}_2|^2 \geq 1$, and $|\vec{a}| \leq 2$ implies $2\left(\frac{|\vec{a}|}{4}\right)^2 \leq \frac{1}{2} < |F\vec{e}_2|^2$. Analogously we get $\operatorname{dist}^2(G_u, SO(2)) \leq 3\,|F\vec{e}_2|^2$. Combining the above inequalities, we get $\left\|\operatorname{dist}^2(\nabla z(.), SO(2))\right\|_{L^\infty(B^\infty(0,\rho))} \leq 3\,|F\vec{e}_2|^2$. Using this and Equation (7.1), we get

$$\begin{aligned}&\int_{B^\infty(0,\rho)} \operatorname{dist}^q(\nabla z\,(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2)\,, SO(2))\; d\lambda_2 = \int_{B^\infty(0,\rho)\setminus\Omega_h} \operatorname{dist}^q(\nabla z, SO(2))\; d\lambda_2 \\ &\leq 4\cdot h\cdot 2\rho\cdot\max\left\{\operatorname{dist}^q(A, SO(2))\;:\; A\in\{F, F_0, F_1, G_l, G_u\}\right\} \leq 8h\rho\cdot 3^{\frac{q}{2}}\,|F\vec{e}_2|^q\,.\end{aligned}$$

Next, we want to compute the variational part $V_x(\gamma, B^\infty(0,\rho))$. The jump length, i.e., the extension in y-direction of one jump-line, is calculated as $2\rho - 2h \leq 2\rho$, refer to Figure 7.1. The number of jump-lines is less than or equal to $4\left\lfloor\frac{\rho}{h}\right\rfloor + 1$ and the jump height has the upper bound $2\,|\gamma| = 2\left(W^{qc}_{rigid,p}(F)\right)^{\frac{1}{p}}$, see Equation (7.2). Thus we get,

$$V_x(\gamma, B^\infty(0,\rho)) \leq 2\rho\cdot\left(4\left\lfloor\frac{\rho}{h}\right\rfloor + 1\right)\cdot 2\left(W^{qc}_{rigid,p}(F)\right)^{\frac{1}{p}} \leq 20\left(W^{qc}_{rigid,p}(F)\right)^{\frac{1}{p}}\frac{\rho^2}{h}, \quad (7.4)$$

since $h \leq \rho$. This computation also regards jumps at the boundary $\{|x| = \pm\rho\}$, namely for the case $\frac{\rho}{h} \in \mathbb{N}$ unless in this case you must not add them. But this allows us to ignore the case $\frac{\rho}{h} \in \mathbb{N}$ in the following two lemmata. Since $\det(F) = 1$ and $|F\vec{e}_1| < 1$ we have $|F\vec{e}_2| \geq 1$ and thus we get $W^{qc}_{rigid,p}(F)^{\frac{1}{p}} = \left(|F\vec{e}_2|^2 - 1\right)^{\frac{1}{2}} \leq |F\vec{e}_2| \leq |F\vec{e}_2|^q$. Combining this with Equation (7.4), we deduce for $h = \sqrt{\rho\varepsilon\delta} \leq \rho \Leftrightarrow \varepsilon\delta \leq \rho$, that

$$\begin{aligned}&\int_{B^\infty(0,\rho)} \frac{1}{\varepsilon}\operatorname{dist}^q(\nabla z\,(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2)\,, SO(2))\; d\lambda_2 + \delta V_x(\gamma, B^\infty(0,\rho)) \\ &\qquad\leq C\,|F\vec{e}_2|^q\left(\frac{h\rho}{\varepsilon} + \frac{\delta\rho^2}{h}\right) = 2C\,|F\vec{e}_2|^q\,\rho^2\sqrt{\frac{\delta}{\rho\varepsilon}},\end{aligned}$$

with $C = C(q) > 0$ independent of z, ρ, ε, δ, F and p. Thus we have assertion (iii) and finally we get (iv) by

$$\|z - u_0\|_{L^\infty(B^\infty(0,\rho);\mathbb{R}^2)} \leq \lambda\,(1-\lambda)\,h\,|\vec{a}| \leq h = \sqrt{\rho\delta\varepsilon} \leq \rho.$$

This completes the proof.

□

In the following we restrict ourselves to $p = q = 2$. Presumably one can show the following for more general p and q. This was done for the model without self-energy in [42, Section 7.4]. The technics used therein could be useful to generalize our result.

Lemma 7.4. *Let $\varepsilon, \delta > 0$, $\rho \geq \varepsilon\delta$, $x_0 \in \mathbb{R}^2$ and $u \in W^{1,\infty}\left(B^\infty(x_0,\rho);\mathbb{R}^2\right)$, $F \in \mathcal{N}^{(2)}$. Define*

$$\eta := \frac{1}{|B^\infty(x_0,\rho)|}\int_{B^\infty(x_0,\rho)} |\nabla u - F|^2 \, d\lambda_2.$$

Then it exists $\gamma \in BV(B^\infty(x_0,\rho))$ and $w \in W^{1,\infty}\left(B^\infty(x_0,\rho);\mathbb{R}^2\right)$, such that

(i) $w = u$ *on* $\partial B^\infty(x_0,\rho)$,

(ii)

$$\int_{B^\infty(x_0,\rho)} |\gamma|^2 \, d\lambda_2 \leq \int_{\Omega_\rho} W^{qc}_{rigid,2}(\nabla u)\, d\lambda_2 + (\eta + \sqrt{\eta})\,|\Omega_\rho| + \sqrt{\eta}\int_{\Omega_\rho} |\nabla u \vec{e}_2|^2 \, d\lambda_2,$$

(iii)

$$\int_{\Omega_\rho} \frac{1}{\varepsilon}\operatorname{dist}^2\left(\nabla w\left(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right)\, d\lambda_2 + \delta V_x(\gamma,\Omega_\rho)$$
$$\leq \frac{\eta}{\varepsilon}\left(W^{qc}_{rigid,2}(F) + 2\right)|\Omega_\rho| + C\max\left\{\|F\|^2, \|\nabla u\|^2_{L^\infty\left(\overline{\Omega_\rho};\mathbb{R}^2\right)}\right\}\rho^2\sqrt{\frac{\delta}{\varepsilon\rho}},$$

(iv) $\|w - u\|_{L^\infty(B^\infty(x_0,\rho);\mathbb{R}^2)} \leq \rho$,

where $\Omega_\rho := B^\infty(x_0,\rho)$ and $C > 0$ is independent of u, F, ε, ρ, δ.

Proof:
After translation it suffices to show the statement for the origin $x_0 = 0$. Define now $\gamma \in BV(B^\infty(0,\rho))$ and $z, u_0 \in W^{1,\infty}\left(B^\infty(0,\rho);\mathbb{R}^2\right)$, Ω_h with $h \leq \rho$ as in the proof of Lemma 7.3, for the given $F \in \mathcal{N}^{(2)}$. Then we can define

$$w(\tau) := u(\tau) + z(\tau) - u_0(\tau) \text{ for } \tau \in B^\infty(0,\rho),$$

and get immediately *(i)*, i.e., $w = u$ on $\partial B^\infty(x_0,\rho)$. Using Equation (7.2) of Lemma 7.3 and $|F\vec{e}_2| \geq 1$, we obtain *(ii)* by

$$\int_{B^\infty(0,\rho)} |\gamma|^2 \, d\lambda_2 = \int_{\Omega_h} |\gamma|^2 \, d\lambda_2 \leq \int_{B^\infty(0,\rho)} \left(|F\vec{e}_2|^2 - 1\right) d\lambda_2$$
$$= \int_{B^\infty(0,\rho)} \left(|\nabla u\vec{e}_2|^2 - 1\right) d\lambda_2 + \int_{B^\infty(0,\rho)} \left(|F\vec{e}_2|^2 - |\nabla u\vec{e}_2|^2\right) d\lambda_2$$
$$\leq \int_{B^\infty(0,\rho)} W^{qc}_{rigid,2}(\nabla u)\, d\lambda_2 + (\eta + \sqrt{\eta})\,|B^\infty(0,\rho)| + \sqrt{\eta}\int_{B^\infty(0,\rho)} |\nabla u\vec{e}_2|^2 \, d\lambda_2.$$

Thereby, we used for the last inequality the definition of η and the algebraic estimate, $< a+b, a+b > - < b,b > = < a,a > +2 < a,b > \leq \left(1+\frac{1}{\sqrt{\eta}}\right)|a|^2 + \sqrt{\eta}\,|b|^2$ for $a,b \in \mathbb{R}^2$, which implies

$$
\begin{aligned}
&\int_{B^\infty(0,\rho)} |F\vec{e}_2|^2 - |\nabla u\vec{e}_2|^2 \ d\lambda_2 \\
&\leq \left(1+\frac{1}{\sqrt{\eta}}\right)\int_{B^\infty(0,\rho)} |F\vec{e}_2 - \nabla u\vec{e}_2|^2 \ d\lambda_2 + \sqrt{\eta}\int_{B^\infty(0,\rho)} |\nabla u\vec{e}_2|^2 \ d\lambda_2 \\
&\leq (\eta+\sqrt{\eta})\,|B^\infty(0,\rho)| + \sqrt{\eta}\int_{B^\infty(0,\rho)} |\nabla u\vec{e}_2|^2 \ d\lambda_2.
\end{aligned}
$$

We get using Equation (7.4) of Lemma 7.3, that

$$
\delta V_x\left(\gamma, B^\infty(0,\rho)\right) \leq 20\left(|F\vec{e}_2|^2 - 1\right)^{\frac{1}{2}} \frac{\rho^2}{h}\delta \leq 20\,|F\vec{e}_2|^2\,\frac{\rho^2}{h}\delta, \tag{7.5}
$$

since $|F\vec{e}_2| \geq 1$. Finally, we get by Equation (7.2) and $\nabla z(\tau)\left(\mathbb{1} - \gamma(\tau)\vec{e}_1\otimes\vec{e}_2\right) \in SO(2)$ for a.e. $\tau \in \Omega_h$, that

$$
\begin{aligned}
&\int_{B^\infty(0,\rho)} \frac{1}{\varepsilon}\,\mathrm{dist}^2\left(\nabla w\left(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2\right), SO(2)\right) \ d\lambda_2 \\
&\leq \int_{\Omega_h} \frac{1}{\varepsilon}\left\|\nabla w\left(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2\right) - \nabla z\left(\mathbb{1} - \gamma\vec{e}_1\otimes\vec{e}_2\right)\right\|^2 \ d\lambda_2 \\
&+ \int_{B^\infty(0,\rho)\setminus\Omega_h} \frac{1}{\varepsilon}\,\mathrm{dist}^2\left(\nabla w, SO(2)\right) \ d\lambda_2 \qquad (7.6)\\
&\leq \frac{1}{\varepsilon}\left(W^{qc}_{rigid,2}(F) + 2\right)\int_{\Omega_h} \|\nabla w - \nabla z\|^2 \ d\lambda_2 \\
&+ \frac{1}{\varepsilon}\,|B^\infty(0,\rho) - \Omega_h| \sup_{x\in B^\infty(0,\rho)}\left\{\mathrm{dist}^2\left(\nabla w(x), SO(2)\right)\right\}.
\end{aligned}
$$

Since we have shown in Lemma 7.3 that $\left\|\mathrm{dist}^2\left(\nabla z(.), SO(2)\right)\right\|_{L^\infty(B^\infty(0,\rho))} \leq 3\,|F\vec{e}_2|^2$ we conclude

$$
\begin{aligned}
&\mathrm{dist}^2\left(\nabla w(x), SO(2)\right) = \inf_{A\in SO(2)} \|\nabla u(x) + \nabla z(x) - F - A\|^2 \qquad (7.7)\\
&\leq 3\left(\|\nabla u(x)\|^2 + \|F\|^2 + \mathrm{dist}^2\left(\nabla z(x), SO(2)\right)\right) \leq 12\left(\|\nabla u(x)\|^2 + \|F\|^2\right),
\end{aligned}
$$

for almost every $x \in B^\infty(0,\rho)$. Since $\nabla w(x) - \nabla z(x) = \nabla u(x) - F$, for almost every $x \in B^\infty(0,\rho)$, by definition of w, we get

$$
\int_{\Omega_h} \|\nabla w - \nabla z\|^2 \ d\lambda_2 \leq |B^\infty(0,\rho)|\,\eta. \tag{7.8}
$$

Using $|B^\infty(0,\rho) \setminus \Omega_h| \le 4 \cdot h \cdot 2\rho = 8h\rho$ and the Equations (7.5)-(7.8), we get (iii), i.e.,

$$\begin{aligned}
&\int_{B^\infty(0,\rho)} \frac{1}{\varepsilon} \operatorname{dist}^2\left(\nabla w\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 + \delta V_x\left(\gamma, B^\infty(0,\rho)\right) \\
&\le \frac{\eta}{\varepsilon}\left(W^{qc}_{rigid,2}(F) + 2\right)|\Omega_\rho| + C \max\left\{\|F\|^2, \|\nabla u\|^2_{L^\infty(B^\infty(0,\rho);\mathbb{R}^2)}\right\}\left(\frac{h\rho}{\varepsilon} + \frac{\rho^2\delta}{h}\right) \\
&\le \frac{\eta}{\varepsilon}\left(W^{qc}_{rigid,2}(F) + 2\right)|\Omega_\rho| + C \max\left\{\|F\|^2, \|\nabla u\|^2_{L^\infty\left(\overline{B^\infty(0,\rho)};\mathbb{R}^2\right)}\right\}\rho^2\sqrt{\frac{\delta}{\varepsilon\rho}},
\end{aligned}$$

for $h = \sqrt{\rho\varepsilon\delta} \le \rho$, where $C > 0$ is independent of u, F, ρ, ε, δ. Finally we have $\|w - u\|_{L^\infty(B^\infty(0,\rho);\mathbb{R}^2)} = \|z - u_0\|_{L^\infty(B^\infty(0,\rho);\mathbb{R}^2)} \le \rho$, which establishes (iv) and finalizes the proof.

□

Lemma 7.5. *Let $L > 0$, $\varepsilon, \alpha \in (0,1]$, $\delta = \varepsilon^{1+\beta+\max\left\{1,\frac{1}{2\alpha}\right\}}$, where $\beta > 0$. Choose $u \in C^{1,\alpha}\left(\overline{B^\infty(0,L)};\mathbb{R}^2\right)$ with $\nabla u(x) \in \mathcal{N}^{(2)}$ for every $x \in B^\infty(0,L)$ and consider the elastic energy $W_e(F) = \operatorname{dist}^2(F, SO(2))$ for $F \in \mathbb{R}^{2\times 2}$. Then there are functions $\gamma \in BV(B^\infty(0,L))$ and $w \in W^{1,\infty}\left(B^\infty(0,L);\mathbb{R}^2\right)$ such that*

(i) $w = u$ *on* $\partial B^\infty(0,L)$,

(ii) $I(w,\gamma) \le \fint_{B^\infty(0,L)} W^{qc}_{rigid,2}(\nabla u)\, d\lambda_2 + C \max\left\{\|u\|^4_{C^{1,\alpha}\left(\overline{B^\infty(0,L)};\mathbb{R}^2\right)}, 1\right\}\varepsilon^{\frac{\alpha\beta}{4}}$,

(iii) $\|w - u\|_{L^\infty(B^\infty(0,L);\mathbb{R}^2)} \le \varepsilon$,

where $C = C(L) > 0$ is independent of u and ε.

Proof:
Let $\rho \ge \varepsilon\delta > 0$ to be chosen later and define $h = \frac{\sqrt{\rho\varepsilon\delta}}{2} \le \frac{\rho}{2}$. Consider now the lattice $\Gamma = \rho\left(\mathbb{Z} + \frac{1}{2}\right) \times \rho\left(\mathbb{Z} + \frac{1}{2}\right)$ and the intersection set $\Delta := \Gamma \cap B^\infty\left(0, L - \frac{\rho}{2}\right)$. Applying Lemma 7.4 for each $\tau \in \Delta$ with radius $\frac{\rho}{2}$ and $F = F_\tau = \nabla u(\tau) \in \mathcal{N}^{(2)}$ gives functions $w_\tau \in W^{1,\infty}\left(B^\infty\left(\tau, \frac{\rho}{2}\right);\mathbb{R}^2\right)$ and $\gamma_\tau \in BV\left(B^\infty\left(\tau, \frac{\rho}{2}\right)\right)$ with the properties presented there. Then we get $B^\infty\left(\tau_1, \frac{\rho}{2}\right) \cap B^\infty\left(\tau_2, \frac{\rho}{2}\right) = \emptyset$ for $\tau_1, \tau_2 \in \Delta$ with $\tau_1 \ne \tau_2$ and $B^\infty(0, L-\rho) \subseteq \bigcup_{\tau \in \Delta} \overline{B^\infty\left(\tau, \frac{\rho}{2}\right)} \subseteq B^\infty(0, L-\rho)$ by definition of Δ. Compose these function in order to obtain functions $w \in W^{1,\infty}\left(B^\infty(0,L);\mathbb{R}^2\right)$ and $\gamma \in BV(B^\infty(0,L))$. They are defined by

$$w(x) = \begin{cases} w_\tau(x) & \text{if } x \in B^\infty\left(\tau, \frac{\rho}{2}\right) \\ u(x) & \text{otherwise} \end{cases}$$

and

$$\gamma(x) = \begin{cases} \gamma_\tau(x) & \text{if } x \in B^\infty\left(\tau, \frac{\rho}{2}\right) \\ 0 & \text{otherwise} \end{cases},$$

thereby the Lipschitz-continuity of w is ensured because of $w_\tau = u$ on $\partial B^\infty(x_0,\rho)$ for each $\tau \in \Delta$. Furthermore we have $w = u$ on $\partial B^\infty(0,L)$, which gives assertion (i). Let $\|u\|_{1,\alpha} := \|u\|_{C^{1,\alpha}\left(\overline{B^\infty(0,L)};\mathbb{R}^2\right)}$ and $\|u\|_1 := \|u\|_{C^1\left(\overline{B^\infty(0,L)};\mathbb{R}^2\right)}$. Using Remark 7.2 we have $\eta_\tau = \frac{1}{\left|B^\infty\left(\tau,\frac{\rho}{2}\right)\right|}\int_{B^\infty\left(\tau,\frac{\rho}{2}\right)} |\nabla u(x) - \nabla u(\tau)|^2 \ d\lambda_2 \leq C\|u\|^2_{C^{1,\alpha}\left(\overline{B^\infty\left(\tau,\frac{\rho}{2}\right)};\mathbb{R}^2\right)}\rho^{2\alpha}$ and thus $\eta_\tau \leq C\|u\|^2_{1,\alpha}\rho^{2\alpha}$. Thus we get using the statement (ii) of Lemma 7.4, that

$$
\begin{aligned}
&\int_{B^\infty(0,L)} |\gamma|^2 \ d\lambda_2 = \sum_{\tau\in\Delta}\int_{B^\infty\left(\tau,\frac{\rho}{2}\right)} |\gamma_\tau|^2 \ d\lambda_2 \\
&\leq \sum_{\tau\in\Delta}\left[\int_{B^\infty\left(\tau,\frac{\rho}{2}\right)} W^{qc}_{rigid,2}(\nabla u) \ d\lambda_2 + (\eta_\tau + \sqrt{\eta_\tau})\left|B^\infty\left(\tau,\frac{\rho}{2}\right)\right|\right. \\
&\qquad \left. + \sqrt{\eta_\tau}\int_{B^\infty\left(\tau,\frac{\rho}{2}\right)} |\nabla u\vec{e}_2|^2 \ d\lambda_2\right] \\
&\leq \int_{B^\infty(0,L)} W^{qc}_{rigid,2}(\nabla u) \ d\lambda_2 + C\|u\|_{1,\alpha}\left(\|u\|_{1,\alpha}\rho^{2\alpha} + \rho^\alpha + \rho^\alpha\|u\|_1^2\right)|B^\infty(0,L)|.
\end{aligned} \tag{7.9}
$$

The case $\frac{\rho}{2h} \in \mathbb{N}$ prohibits to write $V_x(\gamma, B^\infty(0,L))$ as sum over all $V_x\left(\gamma_s, B^\infty\left(s,\frac{\rho}{2}\right)\right)$, because the jumps at the boundary $\left\{\zeta \in \partial B^\infty\left(\tau,\frac{\rho}{2}\right) \ : \ |\zeta\vec{e}_1 - \tau\vec{e}_1| = \frac{\rho}{2}\right\}$ are not considered. Since the additional jumps at the boundary, in the case $\frac{\rho}{2h} \in \mathbb{N}$, are accounted for in Equation (7.4) of Lemma 7.3, we get

$$
V_x(\gamma, B^\infty(0,L)) \leq C\sum_{\tau\in\Delta}|F_\tau\vec{e}_2|^2\frac{\rho^2}{h} \leq C\|u\|_1^2\sum_{\tau\in\Delta}\frac{\rho^2}{\sqrt{\rho\varepsilon\delta}}, \tag{7.10}
$$

where we used $h = \frac{\sqrt{\rho\varepsilon\delta}}{2} \leq \frac{\rho}{2}$. Next, we get

$$
\left|B^\infty(0,L)\setminus\bigcup_{\tau\in\Delta}B^\infty\left(\tau,\frac{\rho}{2}\right)\right| \leq |B^\infty(0,L)\setminus B^\infty(0,L-\rho)| \leq 8\rho L \tag{7.11}
$$

and for $x \in B^\infty(0,L)$, that

$$
\operatorname{dist}^2(\nabla u(x), SO(2)) \leq \|\nabla u(x) - \mathbb{1}\|^2 \leq 4\left(\|u\|_1^2 + 1\right). \tag{7.12}
$$

Using the Equations (7.10)-(7.12) and (*iii*) of Lemma 7.4, we can conclude

$$\begin{aligned}
&\int_{B^\infty(0,L)} \frac{1}{\varepsilon} \operatorname{dist}^2\left(\nabla w\left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 + \delta V_x\left(\gamma, B^\infty(0,L)\right) \\
&= \sum_{\tau\in\Delta} \int_{B^\infty\left(\tau,\frac{\rho}{2}\right)} \frac{1}{\varepsilon} \operatorname{dist}^2\left(\nabla w_\tau\left(\mathbb{1} - \gamma_\tau \vec{e}_1 \otimes \vec{e}_2\right), SO(2)\right) \, d\lambda_2 \\
&\qquad + \delta V_x\left(\gamma, B^\infty(0,L)\right) + \int_{B^\infty(0,L)\setminus\bigcup_{\tau\in\Delta} B^\infty\left(\tau,\frac{\rho}{2}\right)} \frac{1}{\varepsilon} \operatorname{dist}^2\left(\nabla u, SO(2)\right) \, d\lambda_2 \\
&\le \sum_{\tau\in\Delta} \left[\frac{\eta_\tau}{\varepsilon}\left(W^{qc}_{rigid,2}\left(\nabla u(\tau)\right) + 2\right)\left|B^\infty\left(\tau, \frac{\rho}{2}\right)\right| + C\left\|u\right\|_1^2 \rho^2 \sqrt{\frac{\delta}{\varepsilon\rho}}\right] + C\left(\left\|u\right\|_1^2 + 1\right)\frac{\rho L}{\varepsilon} \\
&\le C\left|B^\infty(0,L)\right|\left[\left\|u\right\|_{1,\alpha}^2\left(\left\|u\right\|_1^2 + 1\right)\frac{\rho^{2\alpha}}{\varepsilon} + \left\|u\right\|_1^2\sqrt{\frac{\delta}{\varepsilon\rho}}\right] + C\left(\left\|u\right\|_1^2 + 1\right)\frac{\rho L}{\varepsilon},
\end{aligned}$$

since $W^{qc}_{rigid,2}\left(\nabla u(\tau)\right) \le \left\|u\right\|_1^2$ and we have shown above, that $\eta_\tau \le C\left\|u\right\|_{1,\alpha}^2 \rho^{2\alpha}$. Using this formula and Equation (7.9), then we get

$$\begin{aligned}
I(w,\gamma) &\le \frac{1}{\left|B^\infty(0,L)\right|}\int_{B^\infty(0,L)} W^{qc}_{rigid,2}\left(\nabla u\right) \, d\lambda_2 + C\left(\left\|u\right\|_{1,\alpha}^2 \rho^{2\alpha} + \left\|u\right\|_{1,\alpha}\rho^\alpha\right) \\
&+ C\left\|u\right\|_{1,\alpha}\rho^\alpha\left\|u\right\|_1^2 + C\left[\left\|u\right\|_{1,\alpha}^2\left(\left\|u\right\|_1^2 + 1\right)\frac{\rho^{2\alpha}}{\varepsilon} + \left\|u\right\|_1^2\sqrt{\frac{\delta}{\varepsilon\rho}} + \frac{1}{L}\left(\left\|u\right\|_1^2 + 1\right)\frac{\rho}{\varepsilon}\right].
\end{aligned}$$

If we choose $\rho = \varepsilon^{\frac{\beta}{2}+\max\left\{1,\frac{1}{2\alpha}\right\}}$, then we get $\varepsilon \ge \rho \ge \varepsilon\delta$, since $\delta = \varepsilon^{1+\beta+\max\left\{1,\frac{1}{2\alpha}\right\}}$ and $\varepsilon \le 1$. Since we have $\rho^\alpha = \varepsilon^{\frac{\alpha\beta}{2}+\max\left\{1,\frac{1}{2\alpha}\right\}\alpha} \le \varepsilon^{\frac{\alpha\beta}{2}}$ and $\frac{\rho^{2\alpha}}{\varepsilon} = \varepsilon^{\alpha\beta+\max\left\{1,\frac{1}{2\alpha}\right\}2\alpha} \le \varepsilon^{\alpha\beta}$ we get, using $\rho^{2\alpha} \le \frac{\rho^{2\alpha}}{\varepsilon}$ and $\frac{\rho}{\varepsilon} \le \varepsilon^{\frac{\beta}{2}}$, that

$$\max\left\{\rho^{2\alpha}, \rho^\alpha, \frac{\rho^{2\alpha}}{\varepsilon}, \sqrt{\frac{\delta}{\varepsilon\rho}}, \frac{\rho}{\varepsilon}\right\} \le \max\left\{\varepsilon^{\frac{\alpha\beta}{2}}, \varepsilon^{\alpha\beta}, \varepsilon^{\frac{\beta}{4}}, \varepsilon^{\frac{\beta}{2}}\right\} \le \varepsilon^{\frac{\alpha\beta}{4}}$$

and can conclude (*ii*), i.e.,

$$I(w,\gamma) \le \frac{1}{\left|B^\infty(0,L)\right|}\int_{B^\infty(0,L)} W^{qc}_{rigid,2}\left(\nabla u\right) \, d\lambda_2 + C\max\left\{\left\|u\right\|_{1,\alpha}^4, 1\right\}\varepsilon^{\frac{\alpha\beta}{4}},$$

whereupon $C > 0$ is independent of u and ε, but depends on L. Finally we have

$$\left\|w - u\right\|_{L^\infty(B^\infty(0,L);\mathbb{R}^2)} = \sup_{\tau\in\Delta}\left\|w_\tau - u\right\|_{L^\infty\left(B^\infty\left(\tau,\frac{\rho}{2}\right);\mathbb{R}^2\right)} \le \frac{\rho}{2} \le \varepsilon,$$

which finalizes the proof.

□

Now we can show the existence of a recovery sequence and the proof of Theorem 7.1 is completed.

Proof of Theorem 7.1 (Recovery sequence):
Since $\delta = \varepsilon^{\kappa}$ with $\kappa > 1+\max\left\{1, \frac{1}{2\alpha}\right\}$ there exists a $\beta > 0$ such that $\delta = \varepsilon^{1+\beta+\max\left\{1,\frac{1}{2\alpha}\right\}}$. Let $u \in Z$ and w.l.o.g. $\nabla u(x) \in \mathcal{N}^{(2)}$ for every $x \in B^{\infty}(0, L)$. Then we get by Lemma 7.5 sequences $\{u_\varepsilon\}_{\varepsilon>0} \subseteq W^{1,\infty}\left(B^{\infty}(0,L);\mathbb{R}^2\right)$, with $u_\varepsilon = u$ on $\partial B^{\infty}(0,L)$, and $\{\gamma_\varepsilon\}_{\varepsilon>0} \subseteq BV\left(B^{\infty}(0,L)\right)$, such that $u_\varepsilon \to u \in L^{\infty}\left(B^{\infty}(0,L);\mathbb{R}^2\right)$ for $\varepsilon \to 0$, which implies, that $u_\varepsilon \to u$ in $L^1\left(B^{\infty}(0,L);\mathbb{R}^2\right)$ for $\varepsilon \to 0$, and

$$E_{\varepsilon,\varepsilon^{\kappa}}\left[u_\varepsilon\right] \leq I_{\varepsilon,\varepsilon^{\kappa}}\left(u_\varepsilon,\gamma_\varepsilon\right) \leq \frac{1}{\left|B^{\infty}(0,L)\right|}\int_{B^{\infty}(0,L)} W^{qc}_{rigid,2}\left(\nabla u\right)\, d\lambda_2 + C\varepsilon^{\frac{\alpha\beta}{4}},$$

where $C = C(u, L) > 0$ is independent of ε. Together, with the lower bound we get that

$$E[u] = \lim_{\varepsilon\to 0} E_{\varepsilon,\varepsilon^{\kappa}}\left[u_\varepsilon\right].$$

□

Corollary 7.6. *Let $\Omega = B^{\infty}(0,L)$, with $L > 0$, $p = q = 2$, $\delta = \varepsilon^{\kappa}$, with $\kappa > 1$. Let $F \in \mathcal{N}^{(2)}$ and $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$, $u_0(x) = Fx$, then we have*

$$E\left[u_0\right] = \inf_{u\in Z} E[u] = \inf_{u\in\mathcal{X}} E[u],$$

the function u_0 is the only minimum of $E : \mathcal{X} \to \overline{\mathbb{R}}$ and we get

$$E\left[u_0\right] = \lim_{\varepsilon\to 0}\inf_{x\in X} E_{\varepsilon,\varepsilon^{\kappa}}(x).$$

Let $\{\overline{u}_\varepsilon\}_{\varepsilon>0}$ be a sequence of minimizers of $E_{\varepsilon,\varepsilon^{\kappa}}$, then each cluster point of $\{\overline{u}_\varepsilon\}_{\varepsilon>0}$ is a minimum of E and thus equal to u_0.

Proof: We can show, that u_0 is the only minimum of E by

$$\begin{aligned}\inf_{u\in\mathcal{X}} E[u] &= \inf_{\varphi\in W^{1,1}_0(\Omega;\mathbb{R}^2)} E\left[u_0+\varphi\right] = \inf_{\substack{\varphi\in W^{1,1}_0(\Omega;\mathbb{R}^2)\\ F+\nabla\varphi\in\mathcal{N}^{(2)}\ a.e.}} \fint_{\Omega}\left(\left|\left(F+\nabla\varphi\right)\vec{e}_2\right|^2 - 1\right)\, d\lambda_2 \\ &= \inf_{\substack{\varphi\in W^{1,1}_0(\Omega;\mathbb{R}^2)\\ F+\nabla\varphi\in\mathcal{N}^{(2)}\ a.e.}} \fint_{\Omega}\left(\left|F\vec{e}_2\right|^2 + \left|\nabla\varphi\vec{e}_2\right|^2 - 1\right)\, d\lambda_2 = \fint_{\Omega}\left(\left|F\vec{e}_2\right|^2 - 1\right)\, d\lambda_2 = E\left[u_0\right],\end{aligned}$$

where we used Lemma 3.3, which implies $\int_{\Omega} F\vec{e}_2\cdot\nabla\varphi\vec{e}_2\, d\lambda_2 = 0$ for $\varphi \in W^{1,1}_0\left(\Omega;\mathbb{R}^2\right)$. The other assertions are an immediate consequence of the statements (M) and (LE) in Remark 3.37. Therefore it suffices to find a recovery sequence for u_0 and $\delta = \varepsilon^{\kappa}$, with $\kappa > 1$, which is possible because of Lemma 7.3.

□

8 Outlook

There are many interesting questions, which motivate a further research into this topic. On the one hand one may investigate the unsolved problems in this thesis. For the model without hardening, i.e., for $p = 1$, we do not know, if the energy I has the same lower scaling relation as the simplified energy $\widetilde{I}$. A negative answer would be very curious, but might reveal a fascinating non-trivial microstructure. In the case of linear hardening, i.e., $p = 2$, we do only know the pointwise Γ-limit of $E_{\varepsilon,\delta}$ for $C^{1,\alpha}$-functions. To obtain the full Γ-limit one presumably needs to approximate a function $u \in W^{1,1}\left(\Omega;\mathbb{R}^2\right)$, with $\nabla u(x) \in \mathcal{N}^{(2)}$ for a.e. $x \in \Omega$ by a piecewise affine function v, whose gradients are close to $\mathcal{N}^{(2)}$ and $W^{qc}_{rigid,2}\left(\nabla u(x)\right)$ is close to $W^{qc}_{rigid,2}\left(\nabla v(x)\right)$ for a.e. $x \in \Omega$ and if $\delta = \delta(\varepsilon)$ is small in comparison to ε. Thereby one of the difficulties is the volume constraint in the definition of $\mathcal{N}^{(2)}$, namely $\mathcal{N}^{(2)} \subseteq \left\{F \in \mathbb{R}^{2\times 2} \;:\; \det(F) = 1\right\}$.

On the other hand the model, we spend attention to, consist of many simplification. We restrict ourselves to a single-slip model, where in fact the different crystalline structures have many slip systems, e.g. a fcc-crystal has twelve slip systems. The case of two orthogonal slip-systems was already investigated in [4, 25], for the first time step. But for the case of two arbitrary slip-systems and without self-energy we only manage to compute the rank-one convex envelope partially, see Section 4.3.

The model we consider here is valid for single-crystals only. In order to obtain a more realistic model one has to extend this to polycrystals. Namely one needs to include grain boundary effects.

Besides the parameter δ is comparable to the modulus of the Burgers vector and thus to the lattice parameter. Then one might assume that the slip variable γ is bounded by $\frac{C}{\delta}$, where C is a constant depending on the diameter of Ω. Then the energy part $\int_\Omega |\gamma|^p \; d\lambda_2$ depends on δ and presumably the parameter p appears in the scaling relations.

In this thesis we consider the two-dimensional model only and one might investigate the three-dimensional case.

9 Appendix

9.1 A. Calculations

Assertion 1:

The dislocation tensors defined by Acharya and Bassani [2], Bilby, Bullough and Smith [13], Fox [34], Noll [56] and its corresponding transformed version in the deformed, lattice and reference configuration are given by the following spreadsheet.

vector\area	deformed	lattice	reference
local B-vector	$F_{el}\,\mathrm{curl}\left(F_{el}^{-1}\right)$ (Bilby et al.)	$\det(F_{el})\,F_{el}\,\mathrm{curl}\left(F_{el}^{-1}\right)F_{el}^{-T} = \frac{1}{\det(F_{pl})}FF_{pl}^{-1}\mathrm{curl}\,(F_{pl})\,F_{pl}^{T}$	$FF_{pl}^{-1}\mathrm{curl}\,(F_{pl})$
true B-vector	$\mathrm{curl}\left(F_{el}^{-1}\right)$ (Acharya et al.)	$\det(F_{el})\,\mathrm{curl}\left(F_{el}^{-1}\right)F_{el}^{-T} = \frac{1}{\det(F_{pl})}\mathrm{curl}\,(F_{pl})\,F_{pl}^{T}$ (Noll)	$\mathrm{curl}\,(F_{pl})$
reference B-vector	$F^{-1}F_{el}\,\mathrm{curl}\left(F_{el}^{-1}\right)$	$\det(F_{el})\,F^{-1}F_{el}\,\mathrm{curl}\left(F_{el}^{-1}\right)F_{el}^{-T} = \frac{1}{\det(F_{pl})}F_{pl}^{-1}\mathrm{curl}\,(F_{pl})\,F_{pl}^{T}$	$F_{pl}^{-1}\mathrm{curl}\,(F_{pl})$ (Fox)

Table 9.1: Different dislocation tensors

In the rows the different notions of the Burgers vector are marked, where the reference Burgers vector is equal to the local Burgers vector transformed with the inverse deformation gradient F^{-1}. In the columns the area is marked in which they are measured.

Proof: For the true Burgers vector, the calculation can be found in Subsection 2.4.3. The formulas for the local and reference Burgers vector can be simply obtained by a pre-multiplication of the corresponding true Burgers vector with F_{el} and F_{pl}^{-1}, respectively.

Assertion 2:

Let $F_{pl} = \mathbb{1} + \gamma \vec{e}_1 \otimes \vec{e}_2 \in \mathbb{R}^{3\times 3}$ and $\gamma : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function which is independent of the third variable. Then the dislocation tensors defined in [2, 13, 34, 56] transformed to the reference configuration are identical to the corresponding ones in the lattice configuration. Furthermore the three tensors defined by Acharya and Bassani [2], Fox [34] and Noll [56] transformed to the lattice configuration are equal to

$$\partial_1 \gamma \left(\vec{e}_1 \otimes \vec{e}_3 \right)$$

and by pre-multiplication with the deformation gradient F we obtain the tensor defined by Bilby, Bullough and Smith [13] transformed to the lattice configuration, namely

$$\partial_1 \gamma \left(F\vec{e}_1 \otimes \vec{e}_3 \right).$$

Proof: Using the previous assertion we see that the local, true and reference Burgers vector, measured per unit area in the lattice or reference configuration, depending on $\text{curl}\,(F_{pl}) = \partial_1 \gamma \left(\vec{e}_1 \otimes \vec{e}_3 \right)$. Furthermore we have

$$F_{pl}^{-1} \text{curl}\,(F_{pl}) = \text{curl}\,(F_{pl}) = \text{curl}\,(F_{pl})\, F_{pl}^T.$$

Finally, using $\det(F_{pl}) = 1$ we get the statement. □

Assertion 3:

Let $A = F_{pl} \in \mathbb{R}^{n\times n}$ with $n = 2$ or $n = 3$. The tensor $G(A) = \left(u_{pl}^{-1}\right)^* d\alpha_{pl}$ can be identified with

$$\widehat{G}_2\,(A) := \frac{1}{\det(A)} \begin{pmatrix} \partial_1 A_{12} - \partial_2 A_{11} \\ \partial_1 A_{22} - \partial_2 A_{21} \end{pmatrix} \text{ resp. } \widehat{G}_3\,(A) := \frac{1}{\det(A)} \text{Curl}\,(A) A^T$$

and for a matrix $A = A\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2\times 2}$ one can identify $\widehat{G}_2(A)$ with $\widehat{G}_3\,(\text{diag}\,(A,1))$.

Proof:

Let $x_i : \Omega \subseteq \mathbb{R}^n \to \mathbb{R},\ x \mapsto pr_i(x)$ be the i-th component of the standard chart.

Let $n = 2$ then we have $\alpha_{pl}(x)[v] = \sum_{i=1}^2 A\vec{e}_i dx_i(x)[v]$ and thus we get

$$d\alpha_{pl}[v,w] = \left(\partial_1 A\vec{e}_2 - \partial_2 A\vec{e}_1\right) dx_1 \wedge dx_2[v,w] = \left(\partial_1 A\vec{e}_2 - \partial_2 A\vec{e}_1\right) \det(v,w).$$

Finally we get

$$\left(u_{pl}^{-1}\right)^* d\alpha_{pl}(x)[v,w] = d\alpha_{pl}(x)[A^{-1}v, A^{-1}w] = \frac{1}{\det(A)} \left(\partial_1 A\vec{e}_2 - \partial_2 A\vec{e}_1\right) \det(v,w).$$

Let $n = 3$ then we have $\alpha_{pl}(x)[v] = \sum_{i=1}^3 A\vec{e}_i dx_i(x)[v]$ and thus

$$\begin{aligned} d\alpha_{pl}[v,w] \;=\; & \left(\partial_1 A\vec{e}_2 - \partial_2 A\vec{e}_1\right) \underbrace{dx_1 \wedge dx_2[v,w]}_{=(v\times w)_3} + \left(-\partial_1 A\vec{e}_3 + \partial_3 A\vec{e}_1\right) \underbrace{dx_3 \wedge dx_1[v,w]}_{=(v\times w)_2} \\ & + \left(\partial_2 A\vec{e}_3 - \partial_3 A\vec{e}_2\right) \underbrace{dx_2 \wedge dx_3[v,w]}_{=(v\times w)_1} = \text{Curl}\,(A)\,(v\times w)\,. \end{aligned}$$

Therefore we have $\left(u_{pl}^{-1}\right)^{*} d\alpha(x)[v,w] = d\alpha(x)[A^{-1}v, A^{-1}w] = \mathrm{Curl}\,(A)\left(A^{-1}v \times A^{-1}w\right)$. For $A = (a_{i,j})_{ij} \in \mathbb{R}^{3\times3}$ we define $\widehat{A}_{ij} \in \mathbb{R}^{2\times2}$ by deleting the i-th row and j-th column of A. Then the cofactor matrix of A is defined by $(\mathrm{cof}\,(A))_{ij} = (-1)^{i+j}\det\left(\widehat{A}_{ij}\right)$ or equivalently $\mathrm{cof}\,(A) = \det(A)A^{-T}$. One can compute

$$(\mathrm{cof}\,(A))_{ij} = a_{i+1,j+1}a_{i+2,j+2} - a_{i+2,j+1}a_{i+1,j+2} = (A\vec{e}_{j+1} \times A\vec{e}_{j+2})_i\,,$$

where we identify the indices four and five with one and two, respectively. Thus we have

$$\mathrm{cof}\,(A) = \begin{pmatrix} | & | & | \\ A\vec{e}_2 \times A\vec{e}_3 & A\vec{e}_3 \times A\vec{e}_1 & A\vec{e}_1 \times A\vec{e}_2 \\ | & | & | \end{pmatrix}.$$

The identity $A = \left(A^{-1}\right)^{-1} = \frac{1}{\det(A^{-1})} cof\left(A^{-1}\right)^T$ finally leads to

$$\begin{aligned} A^{-1}\vec{e}_1 \times A^{-1}\vec{e}_2 &= \frac{1}{\det(A)}A^T\vec{e}_3, \\ A^{-1}\vec{e}_2 \times A^{-1}\vec{e}_3 &= \frac{1}{\det(A)}A^T\vec{e}_1 \\ \text{and } A^{-1}\vec{e}_3 \times A^{-1}\vec{e}_1 &= \frac{1}{\det(A)}A^T\vec{e}_2. \end{aligned}$$

These relations give the desired identification of $G(A)$ with $\widehat{G}_3\,(A)$. Finally we have for $A = A\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2\times2}$, that $\widehat{G}_3(\mathrm{diag}\,(A,1)) = \begin{pmatrix} \widehat{G}_2(A) \\ 0 \end{pmatrix} \otimes \vec{e}_3$. □

Assertion 4:

The tensors $\widehat{G}_2\,(A)$ and $\widehat{G}_3\,(A)$ are invariant under compatible local changes in the reference configuration.

Proof:

Because of the identification of $\widehat{G}_2\,(A)$ and $\widehat{G}_3(\mathrm{diag}\,(A,1))$, for $A = A\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2\times2}$, it suffices to proof the three dimensional case. Consider two reference configurations Ω and $\widetilde{\Omega}$. Let $f : \widetilde{\Omega} \subseteq \mathbb{R}^3 \to \Omega \subseteq \mathbb{R}^3$ be a smooth, bijective map, then the corresponding plastic deformation at point $x \in \widetilde{\Omega}$ is $A(f(x))\nabla f(x)$ and we have to show, that $\widehat{G}_3\,(A \circ f\nabla f)\,(x) = \widehat{G}_3\,(A)\,(f(x))$ for all $x \in \widetilde{\Omega}$. This is equivalent to

$$\frac{1}{\det(\nabla f)}\mathrm{Curl}\,\left(A \circ f\nabla f\right)(\nabla f)^T = \mathrm{Curl}\,(A) \circ f \text{ on } \widetilde{\Omega}.$$

For the i-th row and j-th column this means

$$\epsilon_{skl}\partial_k\left((A \circ f)_{ir}\,\partial_l f_r\right)\partial_s f_j = \epsilon_{jkl}\partial_k A_{il} \circ f \det(\nabla f)\,.$$

The right hand side is equal to $(\partial_{j+1}A_{i,j+2}\circ f - \partial_{j+2}A_{i,j+1}\circ f)\det(\nabla f)$. Using product and chain rule we get $\epsilon_{skl}\partial_k\left((A\circ f)_{ir}\partial_l f_r\right)\partial_s f_j = \epsilon_{skl}\nabla A_{ir}\circ f\partial_k f\partial_l f_r\partial_s f_j$ for the left hand side, where we have used $\epsilon_{skl}\partial_k\partial_l f = 0$. If we compute each summand we get for $r = j$ that

$$\epsilon_{skl}\nabla A_{ij}\circ f\partial_k f\partial_l f_j\partial_s f_j = 0,$$

since $\epsilon_{skl} = -\epsilon_{lks}$, for $r = j+1$ that

$$\begin{aligned}&\epsilon_{skl}\nabla A_{i,j+1}\circ f\partial_k f\partial_l f_{j+1}\partial_s f_j = \epsilon_{skl}\partial_m A_{i,j+1}\circ f\partial_k f_m\partial_l f_{j+1}\partial_s f_j\\&= -\epsilon_{slk}\partial_{j+2}A_{i,j+1}\circ f\partial_k f_{j+2}\partial_l f_{j+1}\partial_s f_j = -\partial_{j+2}A_{i,j+1}\circ f\det(\nabla f),\end{aligned}$$

and analogously for $r = j+2$ that

$$\epsilon_{skl}\nabla A_{i,j+2}\circ f\partial_k f\partial_l f_{j+2}\partial_s f_j = \epsilon_{skl}\partial_{j+1}A_{i,j+2}\circ f\partial_k f_{j+1}\partial_l f_{j+2}\partial_s f_j = \partial_{j+1}A_{i,j+2}\circ f\det(\nabla f).$$

Summarized we get

$$\frac{1}{\det(\nabla f)}\mathrm{Curl}\,(A\circ f\nabla f)\,(\nabla f)^T = \mathrm{Curl}\,(A)\circ f \text{ on } \widetilde{\Omega}$$

as desired. □

Assertion 5:

The tensor $\widehat{G}_{el,3}\left(F_{el}^{-1}\right) = \det(F_{el})\,\mathrm{curl}\,\left(F_{el}^{-1}\right)\left(F_{el}^{-1}\right)^T$ is invariant under compatible changes in the deformed configuration.
Proof:
Let $\widehat{\Omega},\Omega\subseteq\mathbb{R}^3$ and $g:\widehat{\Omega}\subseteq\mathbb{R}^3\to u(\Omega)\subseteq\mathbb{R}^3$ be a smooth, bijective map, then the corresponding inverse elastic deformation at point $x\in\widehat{\Omega}$ is $F_{el}^{-1}(g(x))\nabla g(x)$ and we have to show that $\widehat{G}_{el,3}\left(F_{el}^{-1}\circ g\nabla g\right)(x) = \widehat{G}_{el,3}\left(F_{el}^{-1}\right)(g(x))$ for all $x\in\widehat{\Omega}$. We have analogously as in Assertion 4 that

$$\mathrm{curl}\,\left(F_{el}^{-1}\circ g\nabla g\right)(\nabla g)^T = \det(\nabla g)\,\mathrm{curl}\,\left(F_{el}^{-1}\right)\circ g \text{ on } \widehat{\Omega},$$

which implies

$$\begin{aligned}\widehat{G}_{el,3}\left(F_{el}^{-1}\circ g\nabla g\right) &= \det\left(F_{el}\circ g\,(\nabla g)^{-1}\right)\mathrm{curl}\,\left(F_{el}^{-1}\circ g\nabla g\right)(\nabla g)^T F_{el}^{-T}\circ g\\&= \det(F_{el}\circ g)\,\mathrm{curl}\,\left(F_{el}^{-1}\right)\circ gF_{el}^{-T}\circ g = \widehat{G}_{el,3}\left(F_{el}^{-1}\right)\circ g \text{ on } \widehat{\Omega},\end{aligned}$$

as asserted. □

Picture:

Notation:

$$A = F_t(1),\ B = F_t(0),\ C_\pm = B \pm \frac{m_0\widetilde{m}}{2^{k(t)}\theta^{k(t)}N}\vec{e}_1\otimes\vec{a}_t^2,\ D = \frac{m_0(1-\lambda_t)H_t}{2^{k(t)}\theta^{k(t)}2NL_t}\vec{e}_1\otimes\vec{a}_t^2$$

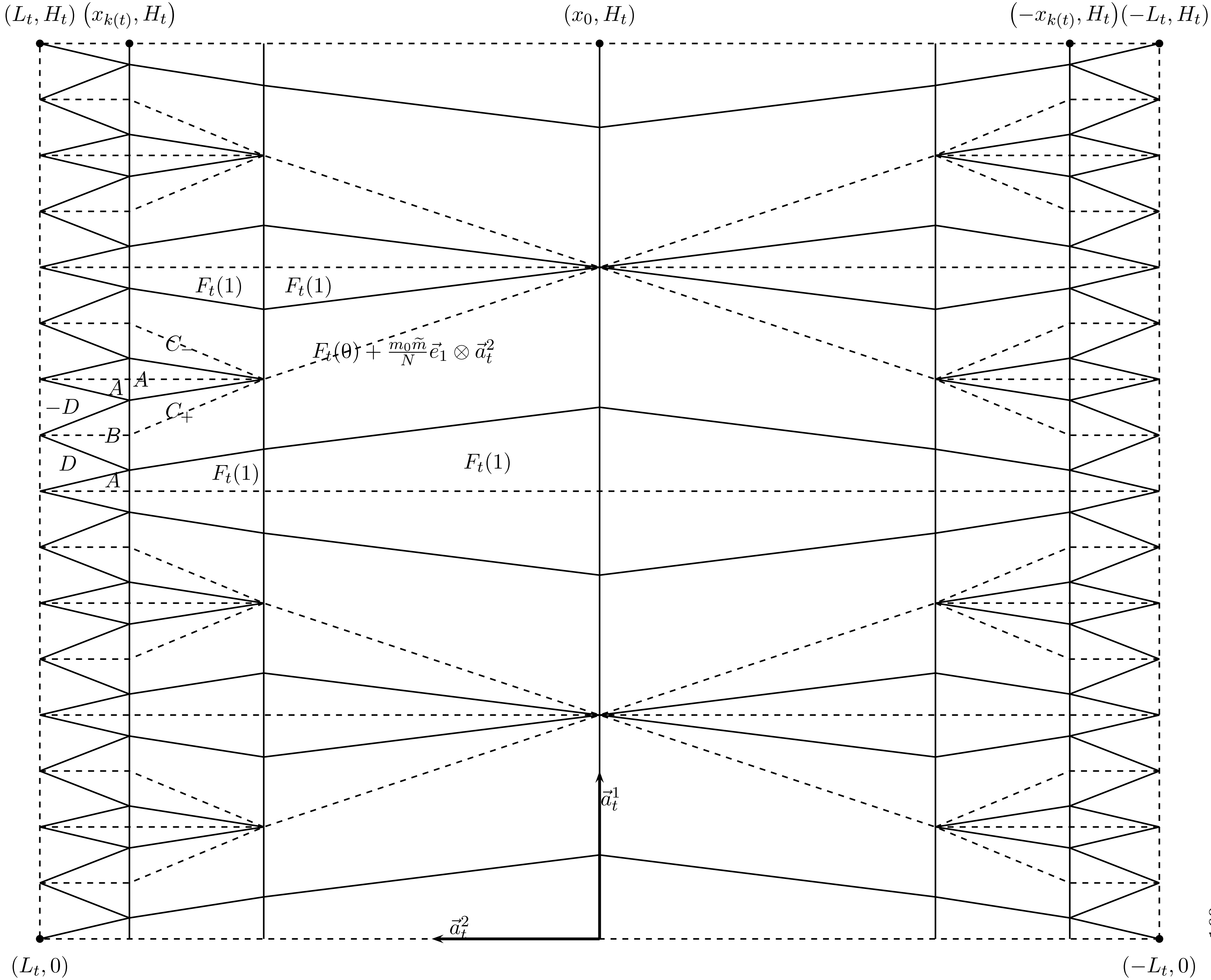

Figure 9.1: Picture 1: $\theta = 0.4$, $\lambda_t = \frac{3}{8}$, $k(t) = \lfloor -\frac{1}{2\theta} \ln(t) \rfloor = 2$

9.2 B. Notation and conventions

We collect some notations used in this thesis. We assume that one is the smallest natural number. Here and in the following index of notation we use $m, n, k \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$. The variable C is reserved for a generic constant, which can change its value in a chain of inequalities and only depends on nonrelevant parameters. Furthermore it can be multiple defined in one proof, i.e., the parameters, which are nonrelevant can change.
The sign $\approx$, denotes, that the relation is only approximately true and the sign $\sim$ means, that the relation is only approximately true and additionally a generic constant is hidden in the relation.
If it is not defined in another way, we denote the i-th component of a vector $\vec{a} \in \mathbb{R}^n$ by a_i for $i = 1, \ldots n$ and we define the modulus of $\vec{a} \in \mathbb{R}^n$ by $\|\vec{a}\| = |\vec{a}| := \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.
For a matrix $F \in \mathbb{R}^{m \times n}$ we denote the entry in the i-th row and j-th column commonly by F_{ij} and the norm of F is defined by $\|F\| := \sqrt{\sum_{i,j=1}^{n} F_{ij}^2}$.
For a vector $\vec{a} \in \mathbb{R}^2$, we denote the vector rotated by $\frac{\pi}{2}$ counter-clockwise with $\vec{a}^\perp := J\vec{a}$, where $J := \vec{e}_2 \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_2 \in \mathbb{R}^{2\times 2}$.
For sets $V, W \subseteq \mathbb{R}^n$ we say V is compactly contained in W, in formulas $V \subset\subset W$, if $V \subseteq \overline{V} \subseteq W$ and $\overline{V}$ is compact. For a function space $X(\Omega) \subseteq \{f : \Omega \to \mathbb{R}\}$, where $\Omega \subseteq \mathbb{R}^n$, we write $X(\Omega; \mathbb{R}^m) := [X(\Omega)]^m$ and $X_{loc}(\Omega) := \{f : \Omega \to \mathbb{R} \ : \ f \in X(U) \text{ for all } U \subset\subset \Omega\}$.
For a map $u \in W^{1,\infty}\left(\Omega; \mathbb{R}^2\right)$ and $F \in \mathbb{R}^{2\times 2}$, we sometimes use the unpurified abbreviation $\{\nabla u = F\}$ instead of $\{x \in \Omega \ : \nabla u(x) \text{ exists }, \nabla u(x) = F\}$, furthermore we use the shorter notation $u = F$ on $\partial\Omega$, instead of $u(x) = Fx$ for all $x \in \partial\Omega$.
The Lebesgue spaces $L^p(\Omega)$ and its corresponding norms $\|.\|_{L^p(\Omega)}$ are defined as in the Appendix D of [32]. We sometimes write $\|.\|_\infty$ instead of $\|.\|_{L^\infty}$.
The Hölder spaces, denoted by $C^{k,\alpha}(\Omega)$, with $k \in \mathbb{N}$ and $0 < \alpha \leq 1$ and its corresponding norms $\|.\|_{C^{k,\alpha}}$ are defined as in [32, Chapter 5].
The Sobolev spaces, denoted by $W^{k,p}(\Omega)$, with $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ and its corresponding norms $\|.\|_{k,p}$ are defined as in [32, Chapter 5].
We denote the space $W_F^{1,\infty}(\Omega; \mathbb{R}^m) = \left\{u \in W^{1,\infty}(\Omega; \mathbb{R}^n) \ : \ u - F \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)\right\}$.
Continuous functions $f : \Omega \to \mathbb{R}^m$ are implicitly defined on the closure of Ω, i.e. on $\overline{\Omega}$.
A collection of the notations used in this thesis is stated in the following.

9.3 C. Index of Notation

Numbers, sets and matrix operations

$\mathbb{N},\ \mathbb{Z},\ \mathbb{R},\ \overline{\mathbb{R}},\ \mathbb{C}$	natural, integer, real, extended real, complex numbers, $\overline{\mathbb{R}} = [-\infty, \infty]$
$\mathbb{R}^n$	n-dimensional Euclidean vector space
$\overline{\Omega},\ \mathring{\Omega},\ \partial\Omega$	closure, inner set and boundary of $\Omega \subseteq \mathbb{R}^n$
$B_\rho(x)$	ball around $x \in \mathbb{R}^n,\ B_\rho(x) := \{y \in \mathbb{R}^n\ :\ \lvert y-x\rvert < \rho\}\,,\ \rho > 0$
$B_\rho^\infty(x)$	$B_\rho^\infty(x) := \left\{y \in \mathbb{R}^n\ :\ \max\limits_{i\in\{1,\ldots,n\}} \lvert y_i - x_i\rvert < \rho\right\},\ \rho > 0,\ x \in \mathbb{R}^n$
$B_\rho(A)$	$B_\rho(A) := \{y \in \mathbb{R}^n\ :\ \exists x \in A\ :\ \lvert y-x\rvert < \rho\}\,,$ for $A \subseteq \mathbb{R}^n$ and $\rho > 0$
$S(x,y)$	$S(x,y) := \{\lambda x + (1-\lambda)y\ :\ \lambda \in (0,1)\}$ for $x, y \in \mathbb{R}^n$
$D(x,y,z)$	open triangle with vertices $x_1, x_2, x_3 \in \mathbb{R}^n$, $D(x_1,x_2,x_3) := \left\{\sum\limits_{i=1}^{3} \lambda_i x_i : \lambda_i \in (0,1),\ \sum\limits_{i=1}^{3} \lambda_i = 1\right\}$
$\Omega_{\mathrm{dist},\varepsilon}$	$\Omega_{\mathrm{dist},\varepsilon} := \left\{x \in \Omega\ :\ \inf\limits_{y\in\partial\Omega} \lvert y-x\rvert < \varepsilon\right\}$
Ω_ν	orthogonal projection of $\Omega \subseteq \mathbb{R}^n$ on $\pi_\nu := \{x \in \mathbb{R}^n\ : < x, \nu >= 0\}$
Ω_y^ν	defined by $\Omega_y^\nu := \{t \in \mathbb{R}\ :\ y + t\nu \in \Omega\}$
$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$	standard basis in $\mathbb{R}^n$
$< \vec{a} >_{\mathbb{R}}$	1-dimensional linear space with basis $\vec{a} \in \mathbb{R}^n$
$T\mathbb{R}^n$	tangential space of $\mathbb{R}^n$, $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$
$\mathbb{S}^n$	n-dimensional sphere, $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1}\ :\ \lvert x\rvert = 1\}$
$\mathbb{R}^{m\times n}$	space of real $m \times n$-matrices, $F = (F_{ij})_{i=1\ldots m,\, j=1\ldots n} \in \mathbb{R}^{m\times n}$
$\mathbb{1}$	identity in $\mathbb{R}^{n\times n}$
J	counter-clockwise rotation by $\frac{\pi}{2}$, $J := \vec{e}_2 \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_2 \in \mathbb{R}^{2\times 2}$
$\vec{x}^\perp$	counter-clockwise rotation of a vector $\vec{x} \in \mathbb{R}^2$ by $\frac{\pi}{2}$, $\vec{x}^\perp := J\vec{x}$
F^{-1}	inverse of a invertible matrix $F \in \mathbb{R}^{n\times n}$
F^T	transpose of a matrix $F \in \mathbb{R}^{m\times n}$
$\mathrm{diag}\,(F,1)$	3×3-matrix, for $F \in \mathbb{R}^{2\times 2}, (\mathrm{diag}\,(F,1))_{ij} := F_{ij}$ if $i, j \le 2$, $(\mathrm{diag}\,(F,1))_{33} := 1,\ (\mathrm{diag}\,(F,1))_{ij} := 0$ otherwise
$\det(F)$	denotes the determinant of a matrix $F \in \mathbb{R}^{n\times n}$
$\mathrm{cof}\,(F)$	cofactor of $F \in \mathbb{R}^{n\times n}$, $\mathrm{cof}\,(F) := \det(F)F^{-T}$
$\mathrm{rank}\,(F)$	rank of a matrix $F \in \mathbb{R}^{m\times n}$
$M(F)$	vector of all minors, i.e., subdeterminants, of $F \in \mathbb{R}^{m\times n}$
$\tau(n,m)$	length of $M(F)$ for $F \in \mathbb{R}^{m\times n}$

$F : G$	scalar product of $F, G \in \mathbb{R}^{m\times n}$, i.e., $F : G := \sum_{i=1}^{m} \sum_{j=1}^{n} F_{ij} G_{ij}$
$\|F\|$	Euclidean norm of $F \in \mathbb{R}^{m\times n}$, $\|F\| := \sqrt{F : F}$
$Gl(n)$	group of invertible $n \times n$-matrices
$Sl(n)$	group of $n \times n$-matrices with determinant one
$SO(n)$	group of orthogonal $n \times n$-matrices with determinant one
$\mathcal{M}^{(2)}$	$\{F \in \mathbb{R}^{2\times 2} \; : \; \|F\vec{e}_1\| = 1, \; \det(F) = 1\}$
$\mathcal{N}^{(2)}$	$\{F \in \mathbb{R}^{2\times 2} \; : \; \|F\vec{e}_1\| \leq 1, \; \det(F) = 1\}$
$Con(2)$	set of conform matrices
$Anticon(2)$	set of anticonform matrices
$K^{c\ (pc,\ qc,\ rc)}$	convex, polyconvex, quasiconvex, rank-one convex hull of $K \subseteq \mathbb{R}^{m\times n}$
$K^{(i)}$	one step rank-one convexification of $K^{(i-1)}$, $K^{(0)} = K \subseteq \mathbb{R}^{m\times n}$, $K^{(i)} := \{\lambda A + (1-\lambda)B \; : \; A, B \in K^{(i-1)}, \text{rank } (B - A) \leq 1, \; \lambda \in (0,1)\}$
$K^{lc} = K^{(lc)}$	lamination convex hull of $K \subseteq \mathbb{R}^{m\times n}$, $K^{lc} = K^{(lc)} = \bigcup_{i=0}^{\infty} K^{(i)}$

Measures and integrals

$(X, \mathcal{E})$	measure space, i.e., $\mathcal{E}$ is a σ-algebra in X
$\mathcal{E}_\mu$	completion of $\mathcal{E}$ with respect to a (positive) measure $\mu : \mathcal{E} \to \mathbb{R}^n$, $(\mu : \mathcal{E} \to [0, \infty])$
$\mathcal{B}(X)$	Borel σ-algebra on X, X topological vector space
$\|\mu\|$	total variation of a measure $\mu : \mathcal{E} \to \mathbb{R}^n$
$\mu^+, \; \mu^-$	positive, negative part of a real valued measure $\mu : \mathcal{E} \to \mathbb{R}$
Du	distributional derivative of $u \in L^1(\Omega)$, if $u \in \mathrm{BV}(\Omega)$, then $Du = (D_1 u, \ldots, D_n u) : \mathcal{B}(\Omega) \to \mathbb{R}^n$ is a Radon measure
$D_\nu u$	distributional derivative of $u \in L^1(\Omega)$ along $\nu \in \mathbb{R}^n \setminus \{0\}$
$V(u, \Omega)$	variation of $u \in [L^1_{loc}(\Omega)]^m$
$pV(f, \omega)$	pointwise variation of $f : \omega \to \mathbb{R}$, $\omega \subseteq \mathbb{R}$ open
$eV(f, \omega)$	essential variation of $f : \omega \to \mathbb{R}$, $\omega \subseteq \mathbb{R}$ open
$V_\nu(u, \Omega)$	variation of $u \in [L^1_{loc}(\Omega)]^m$ along $\nu \in \mathbb{S}^{n-1}$
$V_x(u, \Omega), \; V_y(u, \Omega)$	shorter notations for $V_{\vec{e}_1}(u, \Omega)$, $V_{\vec{e}_2}(u, \Omega)$
$P(E, \Omega)$	perimeter of a $\mathcal{L}^n$-measurable set E in Ω
$\lambda_n, \mathcal{L}^n$	n-dimensional Lebesgue measure
$\mathcal{H}^n$	n-dimensional Hausdorff measure
$\|\Omega\|$	volume of a Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$, $\|\Omega\| = \lambda_n(\Omega)$

$\int_{\partial\Omega} f\,dS$	surface integral of $f:\partial\Omega\to\mathbb{R}$, e.g. $\int_{\partial\Omega} f dS = \int_{\partial\Omega} f\, d\mathcal{H}^{n-1}$
$\int_C f ds$	line integral of a function $f : C \to \mathbb{R}$, on a curve $C \subseteq \mathbb{R}^2$, e.g. $\int_C f ds = \int_C f\, d\mathcal{H}^1$
$\int_C \vec{F}\, d\vec{s}$	line integral of a vector field $\vec{F} : C \to \mathbb{R}^n$, $\int_C \vec{F}\, d\vec{s} = \int_C \vec{F}\cdot \nu\, ds$, where $\nu(x)$ denotes the tangential vector at a point $x \in C$
$⨍_\Omega f\, dx$	mean value of $f : \Omega \to \mathbb{R}$, $⨍_\Omega f\, dx = \frac{1}{\lvert\Omega\rvert}\int_\Omega f\, dx$

Function spaces

$L(V,W)$	set of linear and continuous maps $f : V \to W$, V, W $\mathbb{R}$-vector spaces
$C(\Omega)$	continuous functions on Ω, normed by $\|f\|_\infty := \sup\limits_{x\in\Omega} \lvert f(x)\rvert$
$C_c(\Omega)$	continuous functions with compact support in Ω
$C_0(\Omega)$	completion of $C_c(\Omega)$ with respect to $\|.\|_\infty$
$C^1(\Omega)$	differentiable functions on Ω
$C^1_c(\Omega)$	differentiable functions with compact support in Ω
$C^{k,\alpha}\left(\overline{\Omega}\right)$	Hölder space, $k \in \mathbb{N}$, $\alpha \in (0, 1]$
$C^\infty(\Omega)$	space of infinitely differentiable functions on Ω
$C^\infty_c(\Omega)$, $\mathcal{D}(\Omega)$	space of infinitely differentiable functions with compact support in Ω
$\mathcal{D}'(\Omega)$	space of distributions on Ω, dual space of $\mathcal{D}(\Omega)$
$L^p(\Omega)$	Lebesgue space on $\Omega, 1 \le p \le \infty$
$W^{k,p}(\Omega)$	Sobolev space on Ω, $1 \le p \le \infty$, $k \in \mathbb{N}$
$W^{k,p}_0(\Omega)$	closure of $C^\infty_c(\Omega)$ with respect to $\|.\|_{k,p}$
$W^{k,p}_F(\Omega)$	$\left\{u \in W^{k,p}(\Omega;\mathbb{R}^n) \,:\, u - F \in W^{k,p}_0(\Omega;\mathbb{R}^n)\right\}$, $\Omega \subseteq \mathbb{R}^n$, $F \in \mathbb{R}^{n\times n}$
$BV(\Omega)$	space of functions of bounded variation in Ω
$Y_\Gamma = Y_{\Gamma;p,q}$	$\{\gamma \in BV(\Omega) \cap L^\infty(\Omega) \,:\, \exists\, \overline{\gamma} : \Omega \to \mathbb{R} \,:\, \overline{\gamma} \in \gamma,\ \overline{\gamma}(\Omega) \subseteq \{0, \Gamma\}\}$
$X_\mu = X_{\mu;p,q}$	$\left\{\gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega) \,:\, \exists\, \overline{\gamma} \in \gamma, \forall x \in \Omega : \lvert\overline{\gamma}(x)\rvert \in \{0\} \cup [\mu, \infty)\right\}$
$Z = Z_{F,\alpha,\Omega}$	$\left\{u \in C^{1,\alpha}\left(\Omega;\mathbb{R}^2\right) \,:\, u = F \text{ on } \partial\Omega\right\}$

Differential operator

f'	derivative of a differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$
$\frac{\partial u}{\partial x_i} = \partial_i u$	partial derivative of $u : \Omega \to \mathbb{R}^m$ in x_i-direction, $i \in \{1, \dots n\}$
$\partial_x u,\ \partial_y u$	other notations for $\partial_1 u,\ \partial_2 u$
∇u	gradient of $u : \Omega \to \mathbb{R}^m$
	$(\nabla u)_{i,j} = \partial_j u_i,\ i = 1 \dots m,\ j = 1 \dots n$
$\nabla_x u$	gradient of $u : \Omega \times \left[0, \widehat{T}\right] \to \mathbb{R}^m$
	with respect to the space coordinates $x \in \Omega$
$\nabla \times \varphi$	rotation of $\varphi : \Omega \to \mathbb{R}^3,\ (\nabla \times \varphi)_i = \sum\limits_{j,k=1}^{3} \epsilon_{ijk} \nabla_j \varphi_k,\ i \in \{1, \dots 3\}$
$\mathrm{div}\,(f)$	divergence of a function $f : \Omega \to \mathbb{R},\ \mathrm{div}\,(f) = \sum\limits_{i=1}^{n} \partial_i f$
$\dot{\mathbf{P}}, \dot{\mathbf{p}}$	derivative of a function $\mathbf{P}\,(\mathbf{p}) : \Omega \times [0, \widehat{T}] \to \mathbb{R}^{n \times n}\,(\mathbb{R}^n)$
	with respect to the time variable $t \in [0, \widehat{T}]$
$\frac{\partial \psi}{\partial F},\ D_F \psi$	matrix of partial derivatives of $\psi : \mathbb{R}^{n \times n} \times \mathbb{R}^m \to \mathbb{R}, (F, x) \mapsto \psi(F, x)$
	with respect to F_{ij}, $\left(\frac{\partial \psi}{\partial F}\right)_{ij} = \frac{\partial \psi}{\partial F_{ij}}$
$\frac{\partial \psi}{\partial \mathbf{p}},\ D_{\mathbf{p}} \psi$	vector of partial derivatives of $\psi : \mathbb{R}^{n \times n} \times \mathbb{R}^m \to \mathbb{R}, (F, \mathbf{p}) \mapsto \psi(F, \mathbf{p})$
	with respect to $\mathbf{p}_i$, $\left(\frac{\partial \psi}{\partial \mathbf{p}}\right)_i = \frac{\partial \psi}{\partial \mathbf{p}_i}$
$\mathrm{Curl}\,(F)$	Curl of a map $F : \Omega \to \mathbb{R}^{3 \times 3},\ \mathrm{Curl}\,(F)_{ij} = \mathrm{curl}\,(F)_{ij} = \sum\limits_{k,l=1}^{3} \epsilon_{jkl} \partial_k F_{il}$

Special functions and operations

$\vec{a} \otimes \vec{b}$	tensor product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, $\left(\vec{a} \otimes \vec{b}\right)_{ij} = a_i b_j$
$\vec{a} \cdot \vec{b},\ < \vec{a}, \vec{b} >$	scalar product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^n,\ \vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$
$\vec{a} \times \vec{b}$	cross product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, $\left(\vec{a} \times \vec{b}\right)_i = \sum\limits_{j,k=1}^{3} \epsilon_{ijk} a_j b_k$
$\angle\left(\vec{a}, \vec{b}\right)$	angle between $\vec{a} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$, $\angle\left(\vec{a}, \vec{b}\right) = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\lVert\vec{a}\rVert \lVert\vec{b}\rVert}\right)$
$\alpha \wedge \beta$	wedge product of a k-form α, and an l-form β
$\mathcal{O}\,(f)$	Landau symbol of $f : \mathbb{R} \to (0, \infty)$, $a \in \overline{\mathbb{R}}$, $g \in \mathcal{O}(f) \Leftrightarrow \lim\limits_{x \to a} \left\lvert \frac{g(x)}{f(x)} \right\rvert < \infty$
$o\,(f)$	Landau symbol of $f : \mathbb{R} \to (0, \infty)$, $a \in \overline{\mathbb{R}}$, $g \in o(f) \Leftrightarrow \lim\limits_{x \to a} \left\lvert \frac{g(x)}{f(x)} \right\rvert = 0$
sign	signum function, $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$, $\mathrm{sign}\,(x) = -1$ for $x < 0$,
	$\mathrm{sign}\,(x) = 1$ for $x > 0$, $\mathrm{sign}\,(0) = 0$
pr_i	projection on the i-th component, $pr_i : \Omega \to \mathbb{R}$, $pr_i(x) = x_i$

$\lfloor . \rfloor$	floor function, $\lfloor x \rfloor := \max\{k \in \mathbb{Z} \ : \ k \leq x\}$
$\lceil . \rceil$	ceiling function, $\lceil x \rceil := \min\{k \in \mathbb{Z} \ : \ k \geq x\}$
$(.)_+$	positive part of $x \in \mathbb{R}$, $(x)_+ := \max\{x, 0\}$
$\mathrm{width}\,(\Omega)$	width of $\Omega \subseteq \mathbb{R}^2$, $\mathrm{width}\,(\Omega) := \sup\{\lvert < x - y, \vec{e}_1 > \rvert \ : \ x, y \in \Omega\}$
$\mathrm{dist}\,(x, \Omega)$	distance from $x \in \mathbb{R}^n$ to $\Omega \subseteq \mathbb{R}^n$, $\mathrm{dist}\,(x, \Omega) = \inf\limits_{y \in \Omega} \lvert y - x \rvert$
$\mathrm{dist}_\nu\,(x, \partial\Omega)$	distance from $x \in \Omega \subseteq \mathbb{R}^n$ to $\partial\Omega$ in ν-direction, $\nu \in \mathbb{S}^{n-1}$
	$\mathrm{dist}_\nu\,(x, \partial\Omega) := \inf\{\lambda \in [0, \infty) \ : \ x + \lambda\nu \notin \Omega \text{ or } x - \lambda\nu \notin \Omega\}$
$\mathrm{dist}(F, K)$	distance from $F \in \mathbb{R}^{m \times n}$ to $K \subseteq \mathbb{R}^{m \times n}$, $\mathrm{dist}(F, K) = \inf\limits_{A \in K} \lVert F - A \rVert$
ϵ_{jkl}	Levi-Civita-Symbol,
	$\epsilon_{123} = 1, \ \epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}, \ i, k, j \in \{1, 2, 3\}$
χ_Σ	characteristic function of a set $\Sigma \subseteq \mathbb{R}^n$,
	$\chi_\Sigma(x) = 1$ for $x \in \Sigma$, $\chi_\Sigma(x) = 0$ for $x \notin \Sigma$
δ_U	δ-distribution supported on $U \subseteq \Omega$, $\delta_U \in \mathcal{D}'(\Omega)$,
	$< \delta_U, \varphi > = \int_U \varphi \, dx$ for all $\varphi \in \mathcal{D}(\Omega)$
$f\rvert_{\partial\Omega}$	restriction of $f : \overline{\Omega} \to \mathbb{R}$ to $\partial\Omega$
T	trace operator, $T : W^{1,p}(\Omega) \to L^p\,(\partial\Omega)$
E	extension operator, $E : [BV(\Omega)]^m \to [BV(\mathbb{R}^n)]^m$
f^+	positive part of a function $f : \Omega \to \mathbb{R}$, $f^+(x) = \max\{f(x), 0\}$
f^-	negative part of a function $f : \Omega \to \mathbb{R}$, $f^-(x) = -\min\{f(x), 0\}$
$f^*\alpha$	pullback of a one-form $\alpha : \mathbb{R}^m \to L\,(\mathbb{R}^m, \mathbb{R}^n)$ under $f : \mathbb{R}^m \to \mathbb{R}^m$
$\lVert \Phi \rVert$	operator norm of $\Phi : [C_0(X)]^n \to \mathbb{R}$
$f^{c\ (pc,\ qc,\ rc)}$	convex, poly-, quasi-, rank-one convex envelope of $f : \mathbb{R}^{m \times n} \to (-\infty, \infty]$

Model parameter

n	space dimension, $n = 2, 3$
$\Omega \subseteq \mathbb{R}^n$	reference configuration
u	time-dependent deformation, $u : [0, \widehat{T}] \times \Omega \to \mathbb{R}^n,\ \widehat{T} > 0$
$F = \nabla_x u$	deformation gradient, $F = F_{el}F_{pl}$
F_{el}	elastic part of the deformation gradient
F_{pl}	plastic part of the deformation gradient
$\mathbf{p} \in \mathbb{R}^M$	hardening parameters, $M = 1$ for the single-slip model
$(\mathbf{P}, \mathbf{p})$	internal variables, $\mathbf{P} = F_{pl}^{-1}$
$\psi_{total} = \psi + \psi_{self}$	total free energy density
ψ_{self}	self-energy density
$\psi,\ \widehat{\psi} = \psi_{el} + \psi_h$	free energy density without self-energy
ψ_{el}	elastic energy density
ψ_h	hardening energy density
ψ_{red}	reduced energy density
$\mathbf{T}$	first Piola-Kirchhoff stress tensor, $\mathbf{T} = \frac{\partial \psi}{\partial F}$
$\mathbf{Q},\ \overline{\mathbf{Q}}$	conjugate plastic stresses, $\mathbf{Q} = -\frac{\partial \psi}{\partial \mathbf{P}},\ \overline{\mathbf{Q}} = \mathbf{P}^T Q$
$\mathbf{q}$	conjugate hardening forces, $\mathbf{q} = -\frac{\partial \psi}{\partial \mathbf{p}}$
$\mathbb{Q}$	set of admissible stresses
$\phi, \widehat{\phi}$	yield function
U	dissipation function,
	$U\left(S, s\right) := \sup\limits_{(\overline{\mathbf{Q}}, \mathbf{q},) \in \mathbb{Q}} \left\{ \overline{\mathbf{Q}} : S + \mathbf{q} \cdot s \right\},\ S \in \mathbb{R}^{n \times n}, s \in \mathbb{R}^M$
t^k	time steps, $k = 0 \ldots N$
$\left(u^k, \mathbf{P}^k, \mathbf{p}^k\right)$	state variables at time t^k
u_b	boundary function
l	external loading, $< l(t), u >= \int_\Omega f(t) u dx + \int_{\partial\Omega} g(t) u dS$,
	where$f$ and g are the applied body and surface forces
$s,\ s^j$	slip direction
$m,\ m^j$	slip plane normal
τ, τ^j	critical resolved shear stress
$\dot{\sigma}$	slip rate
γ	amount of slip along (s, m)

Energy formulas

E_{self}	self-energy, $E_{self} = \delta \int_\Omega \left\| \widehat{G}_n \left(F_{pl}(x) \right) \right\| d\lambda_n$
$W_{cond,\varepsilon}$	condensed energy density,
	$W_{cond,\varepsilon}(F) = \inf_{\gamma \in \mathbb{R}} \left\{ \frac{1}{\varepsilon} W_e \left(F \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right) \right) + \|\gamma\|^p \right\}$ for $F \in \mathbb{R}^{2\times 2}$
$E_{cond,\varepsilon}$	condensed energy, $E_{cond,\varepsilon}[u] = \int_\Omega W_{cond,\varepsilon} \left(\nabla u(x) \right) \, d\lambda_2$
$W_{rigid,p}$	rigid energy density, $W_{rigid,p}(F) = \|\gamma\|^p$ if $F \in \mathcal{M}^{(2)}$, $W_{rigid,p}(F) = \infty$ else
$W_{rigid,el}$	rigid elastic energy density,
	$W_{rigid,el}(F) = 0$ if $F \in SO(2)$, $W_{rigid,el}(F) = \infty$ otherwise
E_{rigid}	rigid energy $E_{rigid}[u] := \int_\Omega W^{qc}_{rigid,p} \left(\nabla u \right) \, d\lambda_2$
$J, \ J_\varepsilon$	total energy without self-energy part,
	$J(u,\gamma) = ⨍_\Omega \frac{1}{\varepsilon} \left\| \nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right) - \mathbb{1} \right\|^q + \|\gamma\|^p \, d\lambda_2$
$W_e, \ W_{el}$	elastic energy density, $W_{el} = \frac{1}{\varepsilon} W_e$, e.g. $W_e(F) = \operatorname{dist}^q \left(F, SO(2) \right)$
$E_{\varepsilon,\delta}$	total energy, $E_{\varepsilon,\delta}[u] = \inf\limits_{\gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)} I_{\varepsilon,\delta}(u,\gamma)$
$I_{\varepsilon,\delta}$	$I_{\varepsilon,\delta}(u,\gamma) = ⨍_\Omega \frac{1}{\varepsilon} W_e(\nabla u(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2)) + \|\gamma\|^p \, d\lambda_2 + \frac{\delta}{\|\Omega\|} V_x(\gamma, \Omega)$
$\widetilde{E}_{\varepsilon,\delta}$	simplified total energy, $\widetilde{E}_{\varepsilon,\delta}[u] = \inf\limits_{\gamma \in BV(\Omega) \cap L^{\max\{p,q\}}(\Omega)} \widetilde{I}_{\varepsilon,\delta}(u,\gamma)$
$\widetilde{I}_{\varepsilon,\delta}(u,\gamma)$	$⨍_\Omega \frac{1}{\varepsilon} \operatorname{dist}^q \left(\nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right), SO(2) \right) + \|\gamma\|^p \, d\lambda_2 + \frac{\delta}{\|\Omega\|} V_y \left(\chi_{\{\gamma=0\}}, \Omega \right)$
$\widehat{I}(u,\gamma)$	$⨍_\Omega \frac{1}{\varepsilon} \left\| \nabla u \left(\mathbb{1} - \gamma \vec{e}_1 \otimes \vec{e}_2 \right) - R \right\|^q + \|\gamma\|^p \, d\lambda_2 + \frac{\delta}{\|\Omega\|} V_y \left(\chi_{\{\gamma=0\}}, \Omega \right)$

Geometric dislocation tensor and core energy

$G = G(F_{pl})$	geometric dislocation tensor, vector-valued two form
G^{el}	dislocation tensor in the lattice configuration depending on F_{el}, $G^{el} = \det(F_{el})\,\mathrm{curl}\,\left(F_{el}^{-1}\right)\left(F_{el}^{-1}\right)^T$
G^{pl}	dislocation tensor in the lattice configuration depending on F_{pl}, $G^{pl} = \frac{1}{\det(F_{pl})}\mathrm{Curl}\,(F_{pl})\,F_{pl}^T$
$\widehat{G}_3(F_{pl})$	identification of G for dimension three, $\widehat{G}_3(F_{pl}) = G^{pl}$
$\widehat{G}_2(F_{pl})$	identification of G for dimension two, $\widehat{G}_2(F_{pl}) = \frac{1}{\det(F_{pl})}\begin{pmatrix}\partial_1(F_{pl})_{12} - \partial_2(F_{pl})_{11}\\ \partial_1(F_{pl})_{22} - \partial_2(F_{pl})_{21}\end{pmatrix} \in \mathbb{R}^2$
u_{pl}	mapping from the reference to the lattice configuration, $u_{pl}: T\mathbb{R}^n \to T\mathbb{R}^n,\ u_{pl}(x,v) = (x, F_{pl}(x)v)$
u_{el}	mapping from the lattice to the deformed configuration, $u_{el}: T\mathbb{R}^n \to T\mathbb{R}^n,\ u_{el}(x,v) = (u(x), F_{el}(x)v)$
du	deformation in the tangential space, $du: T\mathbb{R}^n \to T\mathbb{R}^n,\ du = u_{el} \circ u_{pl}$
$\widehat{\alpha}_{pl}$	$\mathbb{R}^n$-valued one-form, $\widehat{\alpha}_{pl}: T\mathbb{R}^n \to L(T\mathbb{R}^n, \mathbb{R}^n),\ \widehat{\alpha}(x,w)[(y,v)] = F_{pl}(x)v$
α_{pl}	$\mathbb{R}^n$-valued one-form, $\alpha_{pl}: \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n),\ \alpha(x)[v] = F_{pl}(x)v$
α_{el}	$\mathbb{R}^n$-valued one-form, $\alpha_{el}: \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n),\ \alpha(x)[v] = F_{el}^{-1}(x)v$
$\vec{b},\ \vec{b}_l,\ \vec{b}_r$	true, local and reference Burgers vector, $\vec{b}_r = F^{-1}\vec{b}_l$
$b^{pl}(\Gamma)$	Burgers vector for a closed curve $\Gamma = \partial S$, and a two dimensional, compact submanifold $S \subseteq \mathbb{R}^n$
r_0	core radius
μ	average shear modulus
$\mathcal{T}$	self-energy per unit length, $\mathcal{T} = C\mu\,\lvert b\rvert^2$
$\mathcal{L}$	dislocation length

Bibliography

[1] Acharya, A. A counterpoint to Cermelli and Gurtin's criteria for choosing the 'correct' geometric dislocation tensor in finite plasticity. In *IUTAM Symposium on theoretical, computational and modelling aspects of inelastic media* (2008), B. D. Reddy, Ed., vol. 11, pp. 99–105.

[2] Acharya, A., and Bassani, J. L. Lattice incompatibility and a gradient theory of crystal plasticity. *J. Mech. Phys. Solids 48* (2000), 1565–1595.

[3] Adams, J., Conti, S., DeSimone, A., and Dolzmann, G. Relaxation of some transversally isotropic energies and applications to smectic A elastomers. *Math. Models Methods Appl. Sci. 18*, 1 (2008), 1–20.

[4] Albin, N., Conti, S., and Dolzmann, G. Infinite-order laminates in a model in crystal plasticity. *Proc. Roy. Soc. Edinburgh, Sect. A Math. 139* (2009), 685–708.

[5] Alt, H. W. *Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung*, fifth ed. Springer-Verlag, Berlin, Heidelberg, 2006.

[6] Alt, W. *Nichtlineare Optimierung: Eine Einführung in Theorie, Verfahren und Anwendungen*, first ed. Vieweg-Verlag, 2002.

[7] Ambrosio, L., Fusco, N., and Pallara, D. *Functions of bounded variation and free discontinuity problems.* Oxford University Press, New York, 2000.

[8] Ariza, M. P., and Ortiz, M. Discrete crystal elasticity and discrete dislocations in crystals. *Arch. Rat. Mech. Anal. 178* (2005), 149–226.

[9] Aubry, S., and Ortiz, M. The mechanics of deformation-induced subgrain-dislocation structures in metallic crystals at large strains. *Proc. Roy. Soc. Lond. A 459* (2003), 3131–3158.

[10] Ball, J. M. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal. 63* (1977), 337–403.

[11] Bartels, S., Carstensen, C., Hackl, K., and Hoppe, U. Effective relaxation for microstructure simulations: algorithms and applications. *Comput. Methods Appl. Mech. Eng. 193* (2004), 5143–5175.

[12] Bilby, B. A. Continuous distributions of dislocations. In *Progress in solid mechanics*, vol. 1. North-Holland publishing Co., 1960, ch. VII, pp. 331–398.

[13] Bilby, B. A., Bullough, R., and Smith, E. Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc. Roy. Soc. London A 231* (1955), 263–273.

[14] Bilby, B. A., and Smith, E. Continuous distributions of dislocations. III. *Proc. Roy. Soc. London A 236* (1956), 481–505.

[15] Braides, A. *Γ-convergence for beginners*, vol. 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2002.

[16] Braides, A., and Defranceschi, A. *Homogenization of multiple integrals*, vol. 12 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 1998.

[17] Burgers, J. M. Some considerations of the field of stress connected with dislocations in a regular crystal lattice. *Proc. Kon. Nederl. Akad. Wetensch. 42* (1939), 293–325, 378–399.

[18] Carstensen, C., Conti, S., and Orlando, A. Mixed analytical–numerical relaxation in finite single-slip crystal plasticity. *Continuum Mech. Thermodyn. 20* (2008), 275–301.

[19] Carstensen, C., Hackl, K., and Mielke, A. Non-convex potentials and microstructures in finite-strain plasticity. *Proc. Roy. Soc. Lond. A 458* (2002), 299–317.

[20] Cermelli, P., and Gurtin, M. E. On the characterization of geometrically necessary dislocations in finite plasticity. *J. Mech. Phys. Solids 49* (2001), 1539–1568.

[21] Conti, S. Branched microstructures: scaling and asymptotic self-similarity. *Comm. Pure Appl. Math. 53* (2000), 1448–1474.

[22] Conti, S. Relaxation of single-slip single-crystal plasticity with linear hardening. In *Multiscale Materials Modeling* (Freiburg, 2006), P. Gumbsch, Ed., Fraunhofer IRB, pp. 30–35.

[23] Conti, S., Dolzmann, G., and Klust, C. Relaxation of a class of variational models in crystal plasticity. *Proc. Roy. Soc. Lond. A 465* (2009), 1735–1742.

[24] Conti, S., Dolzmann, G., and Kreisbeck, C. Geometrically nonlinear models in crystal plasticity and the limit of rigid elasticity. *Proc. Appl. Math. Mech. 10* (2010), 3–6.

[25] Conti, S., Dolzmann, G., and Kreisbeck, C. Relaxation of a model in finite plasticity with two slip systems. *Math. Models Methods Appl. Sci.* (2013). (in print).

[26] Conti, S., and Ortiz, M. Dislocation microstructures and the effective behavior of single crystals. *Arch. Rat. Mech. Anal. 176* (2005), 103–147.

[27] Conti, S., and Theil, F. Single-slip elastoplastic microstructures. *Arch. Rat. Mech. Anal. 178* (2005), 125–148.

[28] Dacorogna, B. *Direct methods in the calculus of variations*, second ed. Springer-Verlag, Berlin, Heidelberg, New York, 2008.

[29] Dal Maso, G. *An introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and their Applications. Birkhäuser, 1993.

[30] Dolzmann, G. Numerical computation of rank-one convex envelopes. *SIAM J. Numer. Anal. 36* (1999), 1621–1635.

[31] Eshelby, J. D. The continuum theory of lattice defects. In *Solid State Physics*, F. D. Seitz and F. D. Turnbull, Eds., vol. 3. Academic Press, New York, 1955, pp. 79–144.

[32] Evans, L. C. *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics.* American Mathematical Society, 1998.

[33] Fox, N. A continuum theory of dislocations for single crystals. *J. Inst. Math. Appl. 2* (1966), 285–298.

[34] Fox, N. On the continuum theories of dislocations and plasticity. *Quart. Journ. Mech. and Appl. Math. 21* (1968), 67–75.

[35] Frank, F. C. LXXXIII. Crystal dislocations.—Elementary concepts and definitions. *Phil. Mag. Ser. VII 42* (1951), 809–819.

[36] Friedel, J. *Dislocations*, first ed., vol. 3. Pergamon Press, Oxford, New York, 1964.

[37] Kinderlehrer, D. *Remarks about equilibrium configurations of crystals.* Oxford Univ. Press, 1988, pp. 217–242.

[38] Kohn, R. V., and Müller, S. Branching of twins near an austenite-twinned-martensite interface. *Philosophical Magazine A 66* (1992), 697–715.

[39] Kohn, R. V., and Müller, S. Surface energy and microstructure in coherent phase transitions. *Comm. Pure Appl. Math. 47* (1994), 405–435.

[40] Kondo, K. On the geometrical and physical foundations of the theory of yielding. *Proceedings of the Second Japan National Congress for Applied Mechanics 2* (1952), 41–47.

[41] Kondo, K. Non-Riemannian geometry of imperfect crystals from a macroscopic viewpoint. In *RAAG Memoirs of the Unifying Study of Basic Problems in engineering and Physical Science by Means of Geometry*, K. Kondo, Ed., vol. 1. Gakuyusty Bunken Fukin-Kay, Tokyo, 1955, pp. 458–469.

[42] Kreisbeck, C. *Analytical aspects of relaxation for single-slip models in finite crystal plasticity.* PhD thesis, University of Regensburg, 2010.

[43] Kröner, E. Kontinuumstheorie der Versetzungen und Eigenspannungen. *Erg. angew. Math. 5* (1958), 1–179.

[44] Kröner, E. Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. *Arch. Rat. Mech. Anal. 4* (1960), 273–334.

[45] Kuhlmann-Wilsdorf, D. Theory of plastic deformation:- properties of low energy dislocation structures. *Materials Science and Engineering: A 113* (1989), 1–4.

[46] Lee, E. H. Elastic-plastic deformation at finite strains. *J. Appl. Mech. 36* (1969), 1–6.

[47] Lorent, A. The two-well problem with surface energy. *Proc. Roy. Soc. Edinburgh, Sect. A Math. 136* (2006), 795–805.

[48] Miehe, C., Schotte, J., and Lambrecht, M. Homogenization of inelastic solid materials at finite strains based on incremental minimization principles. Application to the texture analysis of polycrystals. *J. Mech. Phys. Solids 50* (2002), 2123–2167.

[49] Mielke, A. Energetic formulation of multiplicative elasto-plasticity using dissipation distances. *Continuum Mech. Thermodyn. 15* (2003), 351–382.

[50] Mielke, A. A new approach to elasto-plasticity using energy and dissipation functionals. In *Applied Mathematics Entering the 21st Century: Invited Talks from the ICIAM 2003 Congress* (2004), J. Hill and R. Moore, Eds., SIAM, pp. 315–335.

[51] Mielke, A., and Müller, S. Lower semicontinuity and existence of minimizers in incremental finite-strain elastoplasticity. *ZAMM Z. Angew. Math. Mech. 86* (2006), 233–250.

[52] Müller, S. Variational models for microstructure and phase transitions. Lecture notes. 2, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leibzig, 1998.

[53] Müller, S., and Šverák, V. Attainment results for the two-well problem by convex integration. In *Geometric analysis and the calculus of variations*, J. Jost, Ed. International Press, 1996, pp. 239–251.

[54] Müller, S., and Šverák, V. Convex integration with constraints and applications to phase transitions and partial differential equations. *J. Eur. Math. Soc. 1* (1999), 393–422.

[55] Morrey, C. B. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math. 2* (1952), 25–53.

[56] Noll, W. Materially uniform simple bodies with inhomogeneities. *Arch. Rat. Mech. Anal. 27* (1967), 1–32.

[57] Nye, J. F. Some geometrical relations in dislocated crystals. *Acta Metall. 1* (1953), 153–162.

[58] Orowan, E. Zur Kristallplastizität I: Tieftemperaturplastizität und Beckersche Formel. *Z. Phys. 89* (1934), 605–613.

[59] Orowan, E. Zur Kristallplastizität III: Über den Mechanismus des Gleitvorganges. *Z. Phys. 89* (1934), 634–659.

[60] Ortiz, M., and Repetto, E. A. Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids 47* (1999), 397–462.

[61] Phillips, R. *Crystals, defects and microstructures: Modeling across scales.* Cambridge University Press, 2001.

[62] Polanyi, M. On a form of lattice distortion that may render a crystal plastic. *Z. Phys. 89* (1934), 660–664.

[63] Rice, J. Inelastic constitutive relations for solids: An internal-variable theory and its applications to metal plasticity. *J. Mech. Phys. Solids, 19* (1971), 433–455.

[64] Rudin, W. *Real and complex analysis.* McGraw-Hill, New York, 1966.

[65] Scardia, L., and Zeppieri, C. I. Line-tension model for plasticity as the Γ-limit of a nonlinear dislocation energy. *SIAM J. Math. Anal. 44* (2012), 2372–2400.

[66] Simo, J. A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition: Part I. Continuum formulation. *Comput. Meth. Appl. Mech. Eng. 66* (1988), 199–219.

[67] Simo, J. A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition. Part II: Computational aspects. *Comput. Meth. Appl. Mech. Engng. 68* (1988), 1–31.

[68] Svendsen, B. Continuum thermodynamic models for crystal plasticity including the effects of geometrically-necessary dislocations. *J. Mech. Phys. Solids 50* (2002), 1297 – 1329.

[69] Šverák, V. Rank-one convexity does not imply quasiconvexity. *Proc. Roy. Soc. Edinburgh, Sect. A Math. 120* (1992), 185–189.

[70] Taylor, G. I. The mechanism of plastic deformation of crystals. Part I.-Theoretical. *Proc. Roy. Soc. London A 145* (1934), 362–387.

[71] Teodosiu, C. *Elastic models of crystal defects*. Springer, Berlin, 1982.

[72] Weertman, J., and Weertman, J. R. *Elementary dislocation theory*. Macmillan, New York, 1964.