Noise Traders
in
Financial Markets

DISSERTATION

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1 Introduction

The present work focuses on the role of noise traders in financial markets. First and foremost, this task requires to define the term noise traders. There is a broad variety of definitions for noise traders in the literature (see Dow & Gorton (2006)). Subsequently, we adopt the two definitions of noise traders by Kyle (1985) and De Long et al. (1990). In chapter 2 we characterize noise traders as agents who trade randomly, whereas we describe them as agents who choose their demand dependent on past price changes in chapter 3.

Before taking a closer look at the behavior of noise traders, we discuss the role of noise in financial markets. Following Grossman (1976), noise is needed to solve the no trade argument put forward by Migrom & Stokey (1982). By this argument a market agent, even though he has additional private information, cannot profit from it. An agent with superior information is interested in buying an asset if the value of the asset, conditional on his information, exceeds its price. Thus, the price which is offered by the buyer is smaller than or equal to the value of the asset. The seller, who does not have additional information on the asset, knows that the buyer would not demand the asset if the price exceeded its value. Hence, no one trades since the seller is not worse off if he keeps the asset.

A formal framework for that issue is a Rational Expectations Equilibrium (REE) model. Agents in an REE model fully understand the model economy and the market mechanisms. If an informed agent uses his information for trading, it has an influence on the price. Therefore also uninformed agents learn from that information since they can deduce informed agents' information from the price. The price that forms in equilibrium reflects all information in the market. If information is costly, then no one has an incentive to gather information since the price already incorporates the "best" available information (see Grossman (1976)). One way to give informed agents the possibility to benefit from their additional information is the implementation of noise, for example via a stochastic asset supply (brought forward in Grossman (1976)). The price in this setup is not fully, but only partially revealing the information of informed agents. This idea of a noisy asset supply is also used in models by Grossman & Stiglitz (1980) and Hellwig (1980) which we adopt in the first section of this dissertation. The random supply, however, will be interpreted as the outcome of noise traders. Following Kyle (1985), it is assumed that these noise traders trade randomly and independently from the price. We will examine how noise traders perform in the market (via expected final wealth) and how different model parameters affect their performance. The definition of noise traders adopted here was used by Gloston & Milgrom (1985) as well, who explain the random trading behavior of noise traders by exogenous shocks like job losses or promotions. This definition of noise traders is worth a discussion which is outlined in section 2.5.

A result of chapter 2 is that noise traders' expected total wealth is decreasing. If noise traders performance is on average "bad" it gives rise to the question why they should survive in the market. One explanation is that arbitrage opportunities are limited. Reasons for that could be a limited time horizon (Dow & Gorton (1994)) or professional market agents (Shleifer & Vishny (1997) and Arnold (2009)). Nonetheless, section 2 does not focus on that issue since we consider a two-period model and not a long time horizon.

Since the learning process of market agents and the price formation in a REE model occur simultaneously, strategic behavior is absent. We will analyze an example for strategic behavior in the second part of this work by extending the model of De Long et al. (1990) (henceforth: DSSW) who introduce positive feedback traders. These are market agents that determine their demand by observing past price changes. If the price increased in the past, they demand the asset, if it decreased, they sell it. There may be different reasons for that behavior: it may come for example "from extrapolative expectations about prices, or trend chasing" (Shleifer, 2000, p.155). A rational agent can exploit this behavior: if he receives a positive signal about the future payoff, he knows that the price will increase tomorrow. This price increase will lead to a future positive demand of positive feedback traders. Anticipating this positive demand, rational agents will drive the prices even to a higher level and then sell the asset tomorrow. The presence of rational agents therefore has a rather destabilizing than stabilizing effect on the price. In section 3 we present an extension of the DSSW model by applying another time structure and adding a second signal.

2 Noise traders and information

The aim of this section is to analyze the performance of noise traders by considering their expected final wealth. We will see that the amount of expected wealth that noise traders lose is gained by rational agents. Hence, there is a transfer of expected wealth from noise traders to rational agents. Interestingly, this transfer decreases as the fraction of informed agents in the market increases.

For the remainder of this chapter we follow Kyle (1985) and describe noise traders as agents who randomly purchase or sell assets no matter what the price of the asset is. Their behavior can be explained by exogenous shocks (see Gloston & Milgrom (1985)). Since we do not assume a specific utility function for noise traders, our measure of how good or how bad they perform is the expected value of their final wealth. The only assumption we make is that a higher expected wealth is better for them.

In section 2.1 we present our model following Grossman & Stiglitz (1980) and Hellwig (1980). After that, we are in the position to give an expression for the expected transfer of wealth from noise traders to rational agents (section 2.2). We also examine the effect of different parameters on the expected final wealth. In section 2.3 we take a look at the expected utility of rational agents. The results there are in line with Grossman & Stiglitz (1980). Another distribution of the initial wealth will be discussed in section 2.4.

2.1 Model setup

The model we use is a modified version of the partially revealing REE model used by Grossman & Stiglitz (1980) with the notation of Hellwig (1980). There is a continuum of agents on the interval $[0,1]$. Following Grossman & Stiglitz (1980) and Hellwig (1980), agents can allocate their initial wealth to one riskless and one risky asset, where the former pays one unit and the latter X units of a single consumption good. Let p denote the price of the risky asset and take the riskfree asset to be numeraire. The initial endowment of agent i consists of a constant part w_{0i} and a given amount $\bar{Z}$ of the risky asset. The final wealth of agent i holding z_i units of the risky assets is given by

$$w_{1i} = w_{0i} + p\bar{Z} + z_i(X - p). \tag{2.1}$$

Following Grossman & Stiglitz (1980), we assume that agents maximize their expected utility $E[U]$, with

$$U = -e^{-\rho w_{1i}},$$ [1]

where ρ is a parameter for absolute risk aversion.

2.1.1 Asset supply

According to both papers cited above, we assume that the supply is risky. We explain this random supply by noise traders. Given a fixed amount of $2\bar{Z}$ assets where one half $(= \bar{Z})$ is held by the agents we introduced above (RAT) and the other half $(= \bar{Z})$ is possessed by agents we call noise traders (NT) (upper panel of figure 2.1). As already stated in the beginning of this chapter, noise traders are hit by exogenous shocks and therefore randomly[2] buy or sell the asset. The total supply of noise traders is given by

[1] The choice of this specific utility function (CARA) used by Grossman and Stiglitz (1980) is consistent with assumption A1 in (Hellwig, 1980, p.479).

[2] On the one hand side there are noise traders that demand the asset. On the other hand side some noise traders sell it. We are just interested in the overall reaction of noise traders. Also note, that noise traders are not interested in the price of the asset. They just sell or purchase the asset.

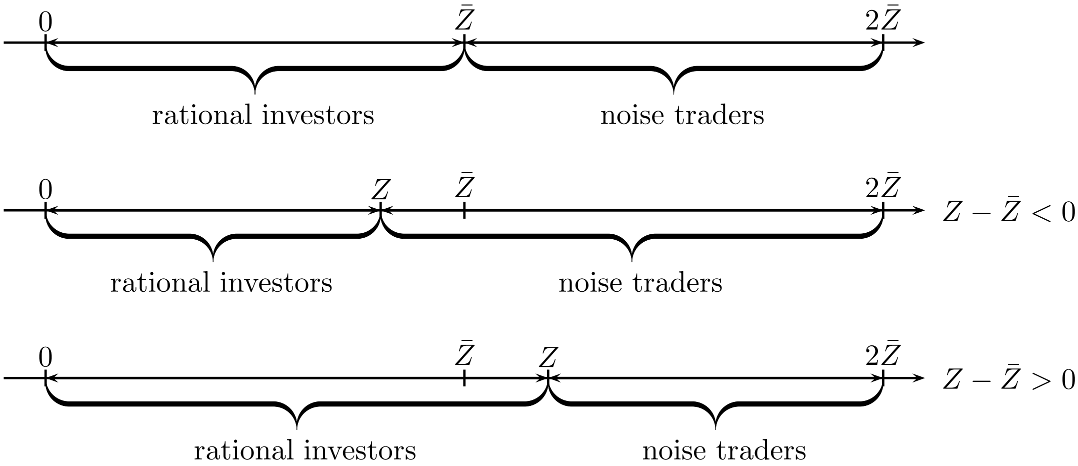


Figure 2.1: Initial (top) and final (middle and bottom) asset allocation for two different realizations of Z

$Z - \bar{Z} \sim N(0, \Delta^2)$. Hence, on average the amount of assets supplied or demanded by noise traders is zero. In the case of a positive realization of $Z - \bar{Z}$ (see lower panel of figure 2.1) noise traders sell part of their assets, whereas if $Z - \bar{Z}$ is negative (see middle panel of figure 2.1) they purchase assets. Therefore, the total (stochastic) supply of assets for rational agents is given by

$$\underbrace{\bar{Z}}_{\text{assets held by RAT}} + \underbrace{(Z - \bar{Z})}_{\text{total supply of NT}} = Z \sim N(\bar{Z}, \Delta^2).$$ [3]

Recall that we refrain from using a specific utility function for noise traders and instead consider their final wealth

$$\begin{aligned} w_{1,NT} &= \underbrace{\big(\bar{Z} - (Z - \bar{Z})\big)}_{\text{assets held by NT at date 1}} X + \underbrace{\big(Z - \bar{Z}\big)}_{\text{assets NT sold to/bought from RAT}} p \\ &= X\bar{Z} - (Z - \bar{Z})(X - p), \end{aligned}$$

[3] By assuming that $\bar{Z}$ is "large enough" we can neglect the cases where the total supply is negative. For example if $\bar{Z} = 3\Delta$, the probability that the realization of Z is non-negative is 99.865%

which depends on the realizations of the payoff X and the supply Z. Hence, expected wealth is

$$\begin{aligned} E[w_{1,NT}] &= E[X\bar{Z} - (Z-\bar{Z})(X-p)] \\ &= \bar{X}\bar{Z} - E[(Z-\bar{Z})(X-p)] \\ &= \bar{X}\bar{Z} - E[(Z-\bar{Z})]E[(X-p)] - Cov[Z-\bar{Z}, X-p] \\ &= \bar{X}\bar{Z} - (\bar{Z}-\bar{Z})E[(X-p)] - Cov[Z-\bar{Z}, X-p] \\ &= \bar{X}\bar{Z} - Cov[Z-\bar{Z}, X-p]. \end{aligned}$$

Note, that the expected wealth $E[w_{1,NT}]$ is decreasing in the covariance between $Z-\bar{Z}$ and $X-p$.[4] If the covariance is not equal to zero there is a transfer of expected wealth from noise traders to rational agents. To verify this, we consider the expected total final wealth of rational agents. Recall that the final wealth of a rational agent is

$$w_{1i} = w_{0i} + p\bar{Z} + z_i(X-p).$$

The total final wealth of all rational agents in equilibrium is therefore

$$w_{1,RAT} = w_0 + p\bar{Z} + Z(X-p),$$

where $w_0 = \int_0^1 w_{0i}\,\mathrm{d}i$ and $\int_0^1 z_i\,\mathrm{d}i = Z$.[5] Hence, the expected total final wealth of rational agents is

$$\begin{aligned} E[w_{1,RAT}] &= E[w_0 + \bar{Z}p + Z(X-p)] \\ &= E[w_0 + \bar{Z}X + (Z-\bar{Z})(X-p)] \\ &= w_0 + \bar{Z}\bar{X} + E[(Z-\bar{Z})(X-p)] \\ &= w_0 + \bar{Z}\bar{X} + E[Z-\bar{Z}]E[X-p] + Cov[Z-\bar{Z}, X-p] \\ &= w_0 + \bar{Z}\bar{X} + Cov[Z-\bar{Z}, X-p]. \end{aligned}$$

So, in contrast to noise traders' expected wealth, the expected wealth of rational agents is increasing in the covariance between $Z-\bar{Z}$ and $X-p$. Note, that the total expected

[4]This is also true for alternative partitions of the $2\bar{Z}$ assets among NT and RAT (see section 2.4).

[5]This is the market clearing condition we will specify later.

wealth of both groups is

$$\begin{aligned} E[w_{1,NT}] + E[w_{1,RAT}] &= \bar{X}\bar{Z} - Cov[Z - \bar{Z}, X - p] + w_0 + \bar{Z}\bar{X} + Cov[Z - \bar{Z}, X - p] \\ &= w_0 + 2\bar{Z}\bar{X} \end{aligned}$$

which is the constant initial wealth of rational agents w_0 plus the total of assets $2\bar{Z}$ multiplied by the expected value $\bar{X}$. This means that the amount of expected wealth which noise traders lose is gained by rational agents and vice versa. The following table gives an overview over the initial and final holdings and the corresponding wealth.

	rational speculators	noise traders	total
initial holdings	$\bar{Z}$	$\bar{Z}$	$2\bar{Z}$
initial wealth	$w_0 + p\bar{Z}$	$p\bar{Z}$	$w_0 + 2p\bar{Z}$
final holdings	$\bar{Z} + (Z - \bar{Z}) = Z$	$\bar{Z} - (Z - \bar{Z}) = 2\bar{Z} - Z$	$2\bar{Z}$
final wealth	$w_0 + X\bar{Z} + (X - p)(Z - \bar{Z})$	$X\bar{Z} - (X - p)(Z - \bar{Z})$	$w_0 + 2X\bar{Z}$

In order to specify the covariance (and therefore the transfer of wealth) more precisely, we will now continue with the presentation of the model.

2.1.2 Asset demand

We subdivide rational agents into two groups: a fraction of $1 - \tau$ type-A agents and a fraction of τ type-B agents. We will use the superscripts A and B to indicate that the parameter refers to the corresponding agent.[6] Both types of agents use information to rationally form expectations of the future payoff of the risky asset. Heterogeneity among the two groups of agents is induced by the amount of information they use. Specifically, we assume that type-A agents are only able to learn from the price, whereas type-B agents additionally have the ability to process[7] a private signal $y_i = X + \epsilon_i$,

[6]Type-A agents correspond to the uninformed agents in Grossman & Stiglitz (1980), type-B agents to informed agents.

[7]We refrain from introducing costs for gathering information as in Grossman & Stiglitz (1980). In our context some agents have the ability to process information, whereas others have not.

where $\epsilon_i \sim N(0, \sigma_\epsilon^2)\ \forall i$. Therefore, the individual demand of type-A and type-B agents is[8]

$$z_i^A = \frac{E[X|p] - p}{\rho Var[X|p]} \tag{2.2}$$

and

$$z_i^B = \frac{E[X|p, y_i] - p}{\rho Var[X|p, y_i]}. \tag{2.3}$$

2.1.3 Price

Before taking a closer look at the expectations of the two types of agents, we have to consider the price. For given information, agents derive their expectation of X based on p (type A) and p and y_i (type B), respectively. As outlined in Hellwig (1980), "expectations formation and market clearing cannot be treated separately" (Hellwig, 1980, p.480). When agents form their expectations, they consider the price of the asset. The expectations influence the market clearing condition and therefore the price. Thus, expectations formation and market clearing must be considered simultaneously. This is a fixed-point problem that has a linear solution. Following Hellwig (1980), we first assume that the price is linear in X and Z, i.e.

$$p = \pi_0 + \pi_1 X - \gamma Z, \tag{2.4}$$

where π_0, π_1 and γ are constant. Using this assumption, we will see later that the price is in fact a linear combination of X and Z. Anticipating the form of the price function, we can now take a closer look at the expectations on X of the two different types of agents.

2.1.4 Expectation formation

Agents of type A

As stated above, type-A agents form their expectations conditional only on the price.

[8] For a derivation see for example Grossman (1976) or Grossman & Stiglitz (1980).

The variance-covariance-matrix of the normal vector

$$\begin{pmatrix} X \\ p \end{pmatrix}$$

with mean

$$\begin{pmatrix} \bar{X} \\ \pi_0 + \pi_1 \bar{X} - \gamma \bar{Z} \end{pmatrix}$$

is given by

$$V^A = \begin{pmatrix} \sigma_X^2 & \pi_1 \sigma_X^2 \\ \pi_1 \sigma_X^2 & \pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2 \end{pmatrix}.$$

Therefore, following Raiffa & Schlaifer (1961), the conditional distribution of X given p is also normal with mean

$$E[X|p] = \alpha_{0i}^A + \alpha_{2i}^A p \tag{2.5}$$

and variance $Var[X|p] = \beta_i^A$.[9] Since, by assumption, all agents of type A are identical the subscript i can be omitted. Specifically,

$$\begin{aligned} E[X|p] &= \bar{X} + \frac{\pi_1 \sigma_X^2}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2}(p - \pi_0 - \pi_1 \bar{X} + \gamma \bar{Z}) \\ &= \frac{\pi_1^2 \sigma_X^2 \bar{X} + \gamma^2 \Delta^2 \bar{X} - \pi_0 \pi_1 \sigma_X^2 - \pi_1^2 \sigma_X^2 \bar{X} + \pi_1 \gamma \sigma_X^2 \bar{Z}}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} + \frac{\pi_1 \sigma_X^2}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} p \\ &= \frac{\gamma^2 \Delta^2 \bar{X} - \pi_0 \pi_1 \sigma_X^2 + \pi_1 \gamma \sigma_X^2 \bar{Z}}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} + \frac{\pi_1 \sigma_X^2}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} p \end{aligned}$$

[9]Let $(X, S) \sim N(\mu, \Sigma)$ be a $n-$dimensional random variable with

$$\mu = \begin{pmatrix} \mu_X \\ \mu_S \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \Sigma_{X,X} & \Sigma_{X,S} \\ \Sigma_{S,X} & \Sigma_{S,S} \end{pmatrix}.$$

Then

$$(X|S = s) \sim N\left(\mu_X + \Sigma_{X,S}\Sigma_{S,S}^{-1}(s - \mu_S), \Sigma_{X,X} - \Sigma_{X,S}\Sigma_{S,S}^{-1}\Sigma_{S,X}\right). \tag{2.6}$$

(Raiffa & Schlaifer, 1961, p.250).

and

$$\begin{aligned} Var[X|p] &= \sigma_X^2 - \frac{\pi_1^2 \sigma_X^4}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} \\ &= \frac{\pi_1^2 \sigma_X^4 + \gamma^2 \Delta^2 \sigma_X^2 - \pi_1^2 \sigma_X^4}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} \\ &= \frac{\gamma^2 \Delta^2 \sigma_X^2}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} \\ &= \beta^A. \end{aligned} \tag{2.7}$$

Therefore,

$$\alpha_0^A = \frac{\gamma^2 \Delta^2 \bar{X} - \pi_0 \pi_1 \sigma_X^2 + \pi_1 \gamma \sigma_X^2 \bar{Z}}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2} \tag{2.8}$$

and

$$\alpha_2^A = \frac{\pi_1 \sigma_X^2}{\pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2}. \tag{2.9}$$

Agents of type B

In contrast to type-A agents, agents of type B form their expectation of the payoff dependent not only on the price p but also conditional on a private signal y_i. Hence, the conditional expectation given p and y_i is

$$E[X|y_i, p] = \alpha_{0i}^B + \alpha_{1i}^B y_i + \alpha_{2i}^B p \tag{2.10}$$

with variance $Var[X|y_i, p] = \beta_i^B$. Since we assume that the private signal has the same variance for all agents of type B, the subscript i at α_{0i}^B, α_{1i}^B, α_{2i}^B, and β_i^B can be omitted. The variance-covariance matrix of the normal vector

$$\begin{pmatrix} X \\ y_i \\ p \end{pmatrix}$$

with mean

$$\begin{pmatrix} \bar{X} \\ \bar{X} \\ \pi_0 + \pi_1 \bar{X} - \gamma \bar{Z} \end{pmatrix}$$

is

$$V^B = \begin{pmatrix} \sigma_X^2 & \sigma_X^2 & \pi_1 \sigma_X^2 \\ \sigma_X^2 & \sigma_X^2 + \sigma_\epsilon^2 & \pi_1 \sigma_X^2 \\ \pi_1 \sigma_X^2 & \pi_1 \sigma_X^2 & \pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2 \end{pmatrix}.$$

Following again Raiffa & Schlaifer (1961), $E[X|y_i,p]$ and $Var[X|y_i,p]$ are given by[10]

$$
\begin{aligned}
E[X|y_i,p] &= \bar{X} + \begin{pmatrix} \sigma_X^2 & \pi_1\sigma_X^2 \end{pmatrix} \begin{pmatrix} \sigma_X^2+\sigma_\epsilon^2 & \pi_1\sigma_X^2 \\ \pi_1\sigma_X^2 & \pi_1^2\sigma_X^2+\gamma^2\Delta^2 \end{pmatrix}^{-1} \\
& \quad \begin{pmatrix} y_i-\bar{X} \\ p-\pi_0-\pi_1\bar{X}+\gamma\bar{Z} \end{pmatrix} \\
&= \bar{X} + \frac{1}{(\sigma_X^2+\sigma_\epsilon^2)(\pi_1^2\sigma_X^2+\gamma^2\Delta^2)-\pi_1^2\sigma_X^4} \\
& \quad \begin{pmatrix} \sigma_X^2 & \pi_1\sigma_X^2 \end{pmatrix} \begin{pmatrix} \pi_1^2\sigma_X^2+\gamma^2\Delta^2 & -\pi_1\sigma_X^2 \\ -\pi_1\sigma_X^2 & \sigma_X^2+\sigma_\epsilon^2 \end{pmatrix} \begin{pmatrix} y_i-\bar{X} \\ p-\pi_0-\pi_1\bar{X}+\gamma\bar{Z} \end{pmatrix} \\
&= \bar{X} + \frac{1}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} \\
& \quad \begin{pmatrix} \sigma_X^2(\pi_1^2\sigma_X^2+\gamma^2\Delta^2)-\pi_1^2\sigma_X^4 \\ -\pi_1\sigma_X^4+\pi_1\sigma_X^2(\sigma_X^2+\sigma_\epsilon^2) \end{pmatrix}^t \begin{pmatrix} y_i-\bar{X} \\ p-\pi_0-\pi_1\bar{X}+\gamma\bar{Z} \end{pmatrix} \\
&= \bar{X} - \frac{\gamma^2\Delta^2\sigma_X^2\bar{X}+\pi_1\sigma_X^2\sigma_\epsilon^2(\pi_0+\pi_1\bar{X}-\gamma\bar{Z})}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} \\
& \quad + \frac{\gamma^2\Delta^2\sigma_X^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} y_i \\
& \quad + \frac{\pi_1\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} p
\end{aligned}
$$

and

$$
\begin{aligned}
Var[X|y_i,p] &= \sigma_X^2 - \begin{pmatrix} \sigma_X^2 & \pi_1\sigma_X^2 \end{pmatrix} \begin{pmatrix} \sigma_X^2+\sigma_\epsilon^2 & \pi_1\sigma_X^2 \\ \pi_1\sigma_X^2 & \pi_1^2\sigma_X^2+\gamma^2\Delta^2 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_X^2 \\ \pi_1\sigma_X^2 \end{pmatrix} \\
&= \sigma_X^2 - \begin{pmatrix} \frac{\gamma^2\Delta^2\sigma_X^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} & \frac{\pi_1\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} \end{pmatrix} \begin{pmatrix} \sigma_X^2 \\ \pi_1\sigma_X^2 \end{pmatrix} \\
&= \frac{\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} \\
&= \beta^B. \qquad (2.11)
\end{aligned}
$$

[10] $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Therefore,

$$\begin{aligned}\alpha_0^B &= \bar{X} - \frac{\gamma^2\Delta^2\sigma_X^2\bar{X} + \pi_1\sigma_X^2\sigma_\epsilon^2(\pi_0 + \pi_1\bar{X} - \gamma\bar{Z})}{\sigma_\epsilon^2\gamma^2\Delta^2 + \sigma_X^2(\pi_1^2\sigma_\epsilon^2 + \gamma^2\Delta^2)} \\ &= \frac{\gamma^2\Delta^2\sigma_\epsilon^2\bar{X} - \pi_0\pi_1\sigma_X^2\sigma_\epsilon^2 + \gamma\pi_1\sigma_X^2\sigma_\epsilon^2\bar{Z}}{\sigma_\epsilon^2\gamma^2\Delta^2 + \sigma_X^2(\pi_1^2\sigma_\epsilon^2 + \gamma^2\Delta^2)}, \end{aligned} \tag{2.12}$$

$$\alpha_1^B = \frac{\gamma^2\Delta^2\sigma_X^2}{\sigma_\epsilon^2\gamma^2\Delta^2 + \sigma_X^2(\pi_1^2\sigma_\epsilon^2 + \gamma^2\Delta^2)} \tag{2.13}$$

and

$$\alpha_2^B = \frac{\pi_1\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2 + \sigma_X^2(\pi_1^2\sigma_\epsilon^2 + \gamma^2\Delta^2)}. \tag{2.14}$$

2.1.5 Market clearing

Given the demand of both types of agents z_i^A and z_i^B and the supply Z of the asset, market clearing requires

$$Z = \int_0^\tau z_i^B \,\mathrm{d}i + \int_\tau^1 z_i^A \,\mathrm{d}i.$$

Note, that since agents of type A all have the same information about the future payoff (i.e. the price) $\int_\tau^1 z_i^A \,\mathrm{d}i = (1-\tau)z_i^A$.
Using (2.2), (2.3), (2.5), and (2.10) the market clearing condition becomes

$$\begin{aligned} Z &= (1-\tau)\frac{E[X|p] - p}{\rho\beta^A} + \int_0^\tau \frac{E[X|y_i, p] - p}{\rho\beta^B} \\ &= (1-\tau)\frac{\alpha_0^A + (\alpha_2^A - 1)p}{\rho\beta^A} + \tau\frac{\alpha_0^B + \alpha_1^B X + (\alpha_2^B - 1)p}{\rho\beta^B} \\ &= \frac{(1-\tau)\alpha_0^A\beta^B + \tau\alpha_0^B\beta^A + \left((1-\tau)(\alpha_2^A - 1)\beta^B + \tau(\alpha_2^B - 1)\beta^A\right)p + \tau\alpha_1^B\beta^A X}{\rho\beta^A\beta^B}. \end{aligned}$$

Solving for p, we get

$$\begin{aligned} p = &-\frac{(1-\tau)\alpha_0^A\beta^B + \tau\alpha_0^B\beta^A}{(1-\tau)(\alpha_2^A - 1)\beta^B + \tau(\alpha_2^B - 1)\beta^A} \\ &-\frac{\tau\alpha_1^B\beta^A}{(1-\tau)(\alpha_2^A - 1)\beta^B + \tau(\alpha_2^B - 1)\beta^A}X \\ &+\frac{\rho\beta^A\beta^B}{(1-\tau)(\alpha_2^A - 1)\beta^B + \tau(\alpha_2^B - 1)\beta^A}Z. \end{aligned}$$

Hence,

$$\pi_0 = -\frac{(1-\tau)\alpha_0^A\beta^B + \tau\alpha_0^B\beta^A}{(1-\tau)(\alpha_2^A-1)\beta^B + \tau(\alpha_2^B-1)\beta^A},$$

$$\pi_1 = -\frac{\tau\alpha_1^B\beta^A}{(1-\tau)(\alpha_2^A-1)\beta^B + \tau(\alpha_2^B-1)\beta^A}$$

and

$$\gamma = -\frac{\rho\beta^A\beta^B}{(1-\tau)(\alpha_2^A-1)\beta^B + \tau(\alpha_2^B-1)\beta^A}.$$

It can be seen that a linear price in X and Z actually solves our problem.

Plugging (2.7), (2.8), (2.9), (2.11), (2.12), (2.13), and (2.14) in the three preceding equations we get[11]

$$\begin{aligned}\pi_0 &= \frac{(1-\tau)\alpha_0^A\beta^B + \tau\alpha_0^B\beta^A}{(1-\tau)(1-\alpha_2^A)\beta^B + \tau(1-\alpha_2^B)\beta^A} \\ &= \frac{\sigma_\epsilon^2\left(\gamma^2\Delta^2\bar{X} + \pi_1\sigma_X^2\left(\gamma\bar{Z}-\pi_0\right)\right)}{\gamma^2\Delta^2\sigma_\epsilon^2 + \sigma_X^2\left(\gamma^2\Delta^2\tau + (\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)},\end{aligned} \tag{2.15}$$

$$\begin{aligned}\pi_1 &= \frac{\tau\alpha_1^B\beta^A}{(1-\tau)(1-\alpha_2^A)\beta^B + \tau(1-\alpha_2^B)\beta^A} \\ &= \frac{\gamma^2\Delta^2\tau\sigma_X^2}{\gamma^2\Delta^2\sigma_\epsilon^2 + \sigma_X^2\left(\gamma^2\Delta^2\tau + (\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)}\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\gamma &= \frac{\rho\beta^A\beta^B}{(1-\tau)(1-\alpha_2^A)\beta^B + \tau(1-\alpha_2^B)\beta^A} \\ &= \frac{\gamma^2\Delta^2\rho\sigma_X^2\sigma_\epsilon^2}{\gamma^2\Delta^2\sigma_\epsilon^2 + \sigma_X^2\left(\gamma^2\Delta^2\tau + (\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)}.\end{aligned} \tag{2.17}$$

Solving (2.15) for π_0 leads to

$$\pi_0 = \frac{\gamma\sigma_\epsilon^2\left(\gamma\Delta^2\bar{X} + \pi_1\bar{Z}\sigma_X^2\right)}{\gamma^2\Delta^2\sigma_\epsilon^2 + \sigma_X^2\left(\gamma^2\Delta^2\tau + \pi_1^2\sigma_\epsilon^2\right)}. \tag{2.18}$$

We have to solve equations (2.16) and (2.17) simultaneously because both contain γ and π_1. Since the right hand sides in both equations have the same denominator we

[11] These calculations are very extensive so we handle them in appendix 2.A.

get[12]

$$\begin{aligned} \frac{\gamma^2\Delta^2\rho\sigma_X^2\sigma_\epsilon^2}{\gamma} &= \frac{\gamma^2\Delta^2\tau\sigma_X^2}{\pi_1} \\ \Leftrightarrow \gamma^2\Delta^2\rho\sigma_X^2\sigma_\epsilon^2\pi_1 &= \gamma^2\Delta^2\rho\sigma_X^2\sigma_\epsilon^2\gamma \\ \Leftrightarrow \gamma &= \frac{\rho\sigma_\epsilon^2\pi_1}{\tau}. \end{aligned} \tag{2.19}$$

Plugging (2.19) in (2.16) and solving for π_1 (see appendix 2.A) we get

$$\pi_1 = \frac{\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}. \tag{2.20}$$

Using this together with (2.19) we obtain

$$\begin{aligned} \gamma &= \frac{\rho\sigma_\epsilon^2}{\tau}\frac{\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)} \\ &= \frac{\rho\sigma_X^2\sigma_\epsilon^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}. \end{aligned} \tag{2.21}$$

Finally, we can plug π_1 and γ in equation (2.18) yielding

$$\pi_0 = \frac{\rho\sigma_\epsilon^2\left(\Delta^2\rho\bar{X}\sigma_\epsilon^2+\tau\bar{Z}\sigma_X^2\right)}{\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}. \tag{2.22}$$

2.2 The transfer of wealth from noise traders to rational investors

In section 2.1.1 we have shown that the expected value of the final wealth is

$$E[w_{1,RAT}] = w_0 + \bar{Z}\bar{X} + Cov[Z-\bar{Z}, X-p]$$

for rational agents and

$$E[w_{1,NT}] = \bar{Z}\bar{X} - Cov[Z-\bar{Z}, X-p]$$

for noise traders. Hence, the transfer of expected wealth from noise traders to rational agents is exactly the covariance between $Z-\bar{Z}$ and $X-p$. Since we have specified the

[12]Note, that $\pi_0 = \gamma = 0$ also solves (2.19). However, this does not solve equations (2.16) and (2.17) since the right hand side of both equations is not defined for $\pi_1 = \gamma = 0$.

price function and its parameters, we are now in the position to calculate the covariance

$$\begin{aligned} Cov[Z-\bar{Z},X-p] &= Cov[Z,X-\pi_0-\pi_1X+\gamma Z] \\ &= Cov[Z,\gamma Z] \\ &= \gamma\Delta^2 \\ &= \frac{\rho\sigma_X^2\sigma_\epsilon^2\Delta^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}, \end{aligned}$$

since $\bar{Z}$ is constant and X and Z are uncorrelated. This covariance is in fact always positive since $\rho > 0$, $\tau \in [0,1]$, and all variances are positive. Therefore, noise traders lose expected wealth and this is actually transferred to rational speculators. To understand that, we compare the final wealth for realizations $Z > \bar{Z}$ and $Z < \bar{Z}$ and compare it to the case when the realization of Z is equal to $\bar{Z}$. Recall that the price is

$$p = \pi_0 + \pi_1 X - \gamma Z.$$

Since $\gamma > 0$, the price increases as Z decreases and vice versa. Consider that we have realizations Z and X. If $Z > \bar{Z}$, noise traders overall sell assets (actually $Z - \bar{Z} > 0$ assets). This increased supply to rational agents decreases (compared to $Z = \bar{Z}$) the price via γ. Thus the rational agents buy a positive amount $Z - \bar{Z}$ for a "lower" price. If in contrast $Z < \bar{Z}$, noise traders demand the asset (exactly $\bar{Z} - Z$ assets) and the reduced supply to rational agents increases the price via γ (again compared to the realization $Z = \bar{Z}$). Thus, rational agents sell the asset at a "higher" price. In both cases rational agents are better off than in the case when the realization of Z is $\bar{Z}$. This explains why rational agents' expected wealth (less w_0) is always larger than noise traders' expected wealth.

In the following we examine the effect of the different parameters on the variance. We therefore differentiate $\gamma\Delta^2$ with respect to the corresponding parameter.

The influence of σ_X^2 on the transfer of wealth

First, we examine the effect of the variance of the risky asset on the transfer of wealth.

Considering

$$\begin{aligned}
\frac{\partial(\gamma\Delta^2)}{\partial\sigma_X^2} &= \frac{\rho\sigma_\epsilon^2\Delta^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} \\
&\quad -\frac{\rho\sigma_X^2\sigma_\epsilon^2\Delta^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\tau(\Delta^2\rho^2\sigma_\epsilon^2+\tau)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} \\
&= \frac{\rho^3\Delta^4\sigma_\epsilon^6\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} > 0,
\end{aligned}$$

we see that the transfer of wealth is increasing in the variance of X. We have seen in the price function that the future payoff X of the asset has an impact on the price. An increase in σ_X^2 decreases the price since the price incorporates more risk. Hence, the price decreases such that noise traders are paid worse for the assets they sell.

The influence of Δ^2 on the transfer of wealth

Next, we study the effect of an increased variance in the supply on the expected wealth of noise traders. The derivative of $\gamma\Delta^2$ with respect to Δ^2 is

$$\begin{aligned}
\frac{\partial\gamma\Delta^2}{\partial\Delta^2} &= \frac{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)\left(2\Delta^2\rho^3\sigma_X^2\sigma_\epsilon^4+\tau\rho\sigma_X^2\sigma_\epsilon^2\Delta^2\right)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} \\
&\quad -\frac{\left(\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\rho^2\sigma_\epsilon^2\right)\rho\sigma_X^2\sigma_\epsilon^2\Delta^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} \\
&= \frac{\Delta^4\rho^5\sigma_X^2\sigma_\epsilon^8+\rho\tau\sigma_X^4\sigma_\epsilon^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)^2}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} > 0.
\end{aligned}$$

Here happens the same as in the case when the variance of the payoff increases. An increase in Δ^2 increases the risk that is incorporated in the price. This decreases the price and therefore noise traders sell their assets at a cheaper price. So their expected wealth decreases in Δ^2.

The influence of τ on the transfer of wealth

The last issue to analyze is the effect of τ on the expected wealth of noise traders. Therefore, we consider

$$\begin{aligned}
\frac{\partial\gamma\Delta^2}{\partial\tau} &= \frac{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)\rho\sigma_X^2\sigma_\epsilon^2\Delta^2}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} \\
&\quad -\frac{\left(\tau\rho^2\sigma_X^2\Delta^2\sigma_\epsilon^2+2\tau\sigma_X^2\right)\rho\sigma_X^2\sigma_\epsilon^2\Delta^2\left(\Delta^2\rho^2\sigma\epsilon^2+\tau\right)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2} \\
&= -\frac{\Delta^2\rho\sigma_X^2\sigma_\epsilon^2\left(\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)^2-\Delta^2\rho^2\sigma_\epsilon^4\right)}{\left(\Delta^2\rho^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)\right)^2}.
\end{aligned}$$

This derivative is negative if and only if

$$\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)^2-\Delta^2\rho^2\sigma_\epsilon^4>0 \Leftrightarrow \sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)^2>\Delta^2\rho^2\sigma_\epsilon^4.$$

The function

$$\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)^2-\Delta^2\rho^2\sigma_\epsilon^4$$

is a upward open parabola in τ which is monotonically increasing for $\tau > 0$ since the apex $(-\Delta^2\rho^2\sigma_\epsilon^2, -\Delta^2\rho^2\sigma_\epsilon^4)$ is in the third quadrant. Therefore, it is sufficient to show that the function is positive for $\tau = 0$. In that case we have

$$\sigma_X^2\Delta^4\rho^4\sigma_\epsilon^4-\Delta^2\rho^2\sigma_\epsilon^4>0 \Leftrightarrow \sigma_X^2\Delta^2\rho^2>1.$$

If we assume that the latter inequality holds,[13] the transfer of wealth decreases as τ increases. So the higher the fraction of informed investors (= investors that learn from the price and a private signal (type-B)) in the market, the lower is the transfer of wealth from noise traders to rational agents. At first sight one might think that the other way round is true: the more informed rational agents, the better the opportunities for them to exploit noise traders. The reason why that is not the case is the following: if the number of type-B agents increases, the price incorporates more information and less risk. This decline of risk increases the price. Therefore noise traders get a higher price for their assets.

2.3 Expected utility

For completeness, we also consider the expected utility of rational agents similar as in Grossman & Stiglitz (1980). As stated in the model presentation, agents are maximizing their expected utility. Recall, that

$$U=-e^{-\rho w_{1i}}=-e^{-\rho(w_{0i}+\bar{Z}p+z_i(X-p))}.$$

Since z_i and $X-p$ are normally distributed, the exponent of U contains a product of two normal distributed random variables. Following Brunnermeier (2001) we compute expected utility by using the following lemma:

[13] This assumption is not too unrealistic since we use a risk aversion parameter ρ larger than one.

Lemma 1. *Let $w \sim N(0,\Sigma)$ be a multinomial random variable with positive definite variance-covariance-matrix. Then*

$$E[e^{w^tAw+b^tw+d}] = |I-2\Sigma A|^{\frac{1}{2}} e^{\frac{1}{2}(I-2\Sigma A)^{-1}\Sigma b+d},$$

where I is the identity matrix, A is a symmetric matrix, b a vector, d a scalar, and $|\,.\,|$ the determinant (Brunnermeier, 2001, p.64).

The expected utilities of both types of agents have to be derived separately because the amount of shares held by agents of type A and type B are different.

Agents of type A

In order to apply lemma 1, w_{1i} has to be brought in the appropriate form:

$$\begin{aligned}
w_{1i} &= w_{0i} + z_i^A(X-p) + \bar{Z}p \\
&= w_{0i} + (z_i^A - \bar{z}_i^A)(X-p) + \bar{z}_i^A(X-p) + \bar{Z}(p-\bar{p}) + \bar{Z}\bar{p} \\
&= w_{0i} + (z_i^A - \bar{z}_i^A)(X-p-(\bar{X}-\bar{p})) + (z_i^A - \bar{z}_i^A)(\bar{X}-\bar{p}) \\
&\quad + \bar{z}_i^A(X-p-(\bar{X}-\bar{p})) + \bar{z}_i^A(\bar{X}-\bar{p}) + \bar{Z}(p-\bar{p}) + \bar{Z}\bar{p} \\
&= \begin{pmatrix} z_i^A - \bar{z}_i^A & X-p-(\bar{X}-\bar{p}) & p-\bar{p} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_i^A - \bar{z}_i^A \\ X-p-(\bar{X}-\bar{p}) \\ p-\bar{p} \end{pmatrix} \\
&\quad + \begin{pmatrix} \bar{X}-\bar{p} & \bar{z}_i^A & \bar{Z} \end{pmatrix} \begin{pmatrix} z_i^A - \bar{z}_i^A \\ X-p-(\bar{X}-\bar{p}) \\ p-\bar{p} \end{pmatrix} + w_{0i} + \bar{z}_i^A(\bar{X}-\bar{p}) + \bar{Z}\bar{p} \qquad (2.23)
\end{aligned}$$

where $\bar{z}_i^A$, $\bar{X}$, and $\bar{p}$ are the means of z_i^A, X, and p. Using

$$A = \begin{pmatrix} 0 & -\frac{\rho}{2} & 0 \\ -\frac{\rho}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} -\rho(\bar{X}-\bar{p}) \\ -\rho\bar{z}_i^A \\ -\rho\bar{Z} \end{pmatrix},$$

$$d = -\rho \left(w_{0i} + \bar{z}_i^A(\bar{X} - \bar{p}) + \bar{Z}\bar{p} \right),$$

and

$$\Sigma = \left(\begin{array}{ccc} \sigma^2_{z_i^A} & Cov[z_i^A, X-p] & Cov[z_i^A, p] \\ Cov[z_i^A, X-p] & \sigma^2_{X-p} & Cov[X-p, p] \\ Cov[z_i^A, p] & Cov[X-p, p] & \sigma^2_p \end{array} \right)$$

we are able to calculate the expected utility.[14] The final thing to determine are the entries in the matrix Σ which are

$$\begin{aligned}
\sigma^2_{z_i^A} &= Var\left[\frac{\alpha_0^A + (\alpha_2^A - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^A} \right] \\
&= \left(\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A} \right)^2 \sigma_X^2 + \left(\frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A} \right)^2 \Delta^2, \\
Cov[z_i^A, X-p] &= Cov\left[\frac{\alpha_0^A + (\alpha_2^A - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^A}, X - \pi_0 - \pi_1 X + \gamma Z \right] \\
&= \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}(1 - \pi_1)\sigma_X^2 - \frac{(\alpha_2^A - 1)\gamma^2}{\rho\beta^A}\Delta^2, \\
Cov[z_i^A, p] &= Cov\left[\frac{\alpha_0^A + (\alpha_2^A - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^A}, \pi_0 + \pi_1 X - \gamma Z \right] \\
&= \frac{(\alpha_2^A - 1)\pi_1^2}{\rho\beta^A}\sigma_X^2 + \frac{(\alpha_2^A - 1)\gamma^2}{\rho\beta^A}\Delta^2, \\
\sigma^2_{X-p} &= Var[X - \pi_0 - \pi_1 X + \gamma Z] \\
&= (1 - \pi_1)^2 \sigma_X^2 + \gamma^2 \Delta^2, \\
Cov[X-p, p] &= Cov[X - \pi_0 - \pi_1 X + \gamma Z, \pi_0 + \pi_1 X - \gamma Z] \\
&= (1 - \pi_1)\pi_1 \sigma_X^2 - \gamma^2 \Delta^2
\end{aligned}$$

and

$$\begin{aligned}
\sigma_p^2 &= Var[\pi_0 + \pi_1 X - \gamma Z] \\
&= \pi_1^2 \sigma_X^2 + \gamma^2 \Delta^2.
\end{aligned}$$

For the ease of exposition we plot (see figure 2.2)[15] the expected utility for the example[16] $\rho = 5$, $\sigma_X = 1$, $\bar{X} = 5$, $\bar{Z} = 5$, $\Delta = 1$, $\sigma_\epsilon = \frac{1}{10}$, and $w_0 = 1$ over the fraction of type-B

[14]Note, that another decomposition is also possible. To check that the results are correct we used another decomposition with the variable vector $\left(\begin{array}{cc} X - \bar{X} & Z - \bar{Z} \end{array} \right)^t$ (see appendix 2.B).

[15]Figures 2.2, 2.3, and 2.4 were produced using Wolfram Mathematica. The source code can be found in appendix 2.C.

[16]We choose $\bar{X}$ and $\bar{Z}$ large enough (five times the standard deviation) so that the probability of a

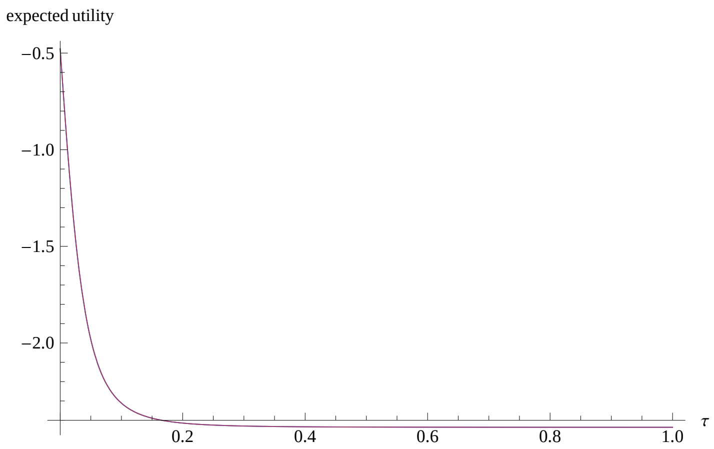


Figure 2.2: Expected utility for type-A agents

agents ($\tau \in (0,1]$).[17]

Agents of type B

By rearranging w_{1i}[18] as in (2.23), we get

$$A = \begin{pmatrix} 0 & -\frac{\rho}{2} & 0 \\ -\frac{\rho}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} -\rho(\bar{X} - \bar{p}) \\ -\rho\bar{z}_i^B \\ -\rho\bar{Z} \end{pmatrix},$$

$$d = -\rho\left(w_{0i} + \bar{z}_i^B(\bar{X} - \bar{p}) + \bar{Z}\bar{p}\right)$$

negative realization of X and Z is negligible (2.86652×10^{-5} percent).

[17]The expected utility drops very steeply at the beginning for small values of τ. The reason for that is that the variance of the signal is very small. If σ_ϵ^2 increases, the expected utility function becomes flatter since the price does not become informative "so fast". This is in line with the results in Grossman & Stiglitz (1980) since "[a]n increase in the quality of information [...] increases the informativeness of the price system" (Grossman & Stiglitz, 1980, p.399).

[18]We also checked the results with another decomposition where the normal vector was $\begin{pmatrix} X - \bar{X} & Z - \bar{Z} & \epsilon_i \end{pmatrix}^t$ (see appendix 2.B).

and

$$\Sigma = \begin{pmatrix} \sigma^2_{z_i^B} & Cov[z_i^B, X-p] & Cov[z_i^B, p] \\ Cov[z_i^B, X-p] & \sigma^2_{X-p} & Cov[X-p,p] \\ Cov[z_i^B, p] & Cov[X-p,p] & \sigma^2_p \end{pmatrix}.$$

The entries in the matrix Σ are

$$\begin{aligned}
\sigma^2_{z_i^B} &= Var\left[\frac{\alpha_0^B + \alpha_1^B(X+\epsilon_i) + (\alpha_2^B - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^B}\right] \\
&= \left(\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\right)^2 \sigma_X^2 + \left(\frac{(\alpha_2^B - 1)\gamma}{\rho\beta^B}\right)^2 \Delta^2 + \left(\frac{\alpha_1^B}{\rho\beta^B}\right)^2 \sigma_\epsilon^2, \\
Cov[z_i^B, X-p] &= Cov\left[\frac{\alpha_0^B + \alpha_1^B(X+\epsilon_i) + (\alpha_2^B - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^B},\right. \\
&\qquad \left. X - \pi_0 - \pi_1 X + \gamma Z\right] \\
&= \frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}(1-\pi_1)\sigma_X^2 - \frac{(\alpha_2^B - 1)\gamma}{\rho\beta^B}\gamma\Delta^2, \\
Cov[z_i^B, p] &= Cov\left[\frac{\alpha_0^B + \alpha_1^B(X+\epsilon_i) + (\alpha_2^B - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^B},\right. \\
&\qquad \left. \pi_0 + \pi_1 X - \gamma Z\right] \\
&= \frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\pi_1\sigma_X^2 + \frac{(\alpha_2^B - 1)\gamma}{\rho\beta^B}\gamma\Delta^2, \\
\sigma^2_{X-p} &= Var[X - \pi_0 - \pi_1 X + \gamma Z] \\
&= (1-\pi_1)^2\sigma_X^2 + \gamma^2\Delta^2, \\
Cov[X-p, p] &= Cov[X - \pi_0 - \pi_1 X + \gamma Z, \pi_0 + \pi_1 X - \gamma Z] \\
&= (1-\pi_1)\pi_1\sigma_X^2 - \gamma^2\Delta^2,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_p^2 &= Var[\pi_0 + \pi_1 X - \gamma Z] \\
&= \pi_1^2\sigma_X^2 + \gamma^2\Delta^2.
\end{aligned}$$

Again, the expected utility is plotted as a function of τ using the same parameters as in the previous paragraph (figure 2.3).

Comparison of type-A and type-B agents

When we draw both expected utility functions in a $E[U]-\tau-$diagram (figure 2.4), we see that agents who are able to process additional information (type B) are always

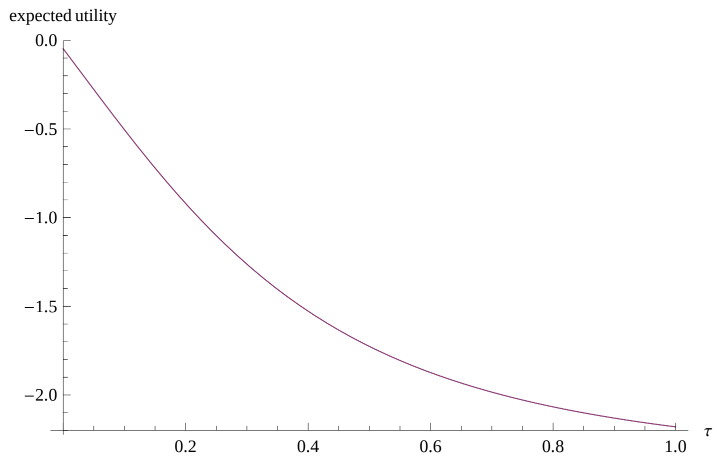


Figure 2.3: Expected utility for type-B agents

better off compared to agents who form their expectations using only the price (type A). This is different to the model of Grossman & Stiglitz (1980). The reason is that we do not consider costs of information for type-B agents. We assume that some agents have the ability to process information whereas others have not. If we introduced costs of information, the upper curve would be shifted downwards and, hence, there would exist a point where the expected utilities of both groups are identical.

Although rational agents' expected wealth is increasing in τ (see sections 2.1.1 and 2.2), their expected utility is decreasing in our example for both groups of rational agents. As τ increases, the number of agents that additionally learn from a private signal is increasing. If there are more agents who can process information then the price contains more information and rather reflects the future value X of the asset. Therefore, the price becomes more appropriate and the exploitation of noise traders decreases (which is in line with our results in section 2.2).

2.4 Other distributions of the $2\bar{Z}$ assets

All results stated above where made on the assumption that in the beginning both noise traders and rational agents hold $\bar{Z}$ assets. In this section we will show that other

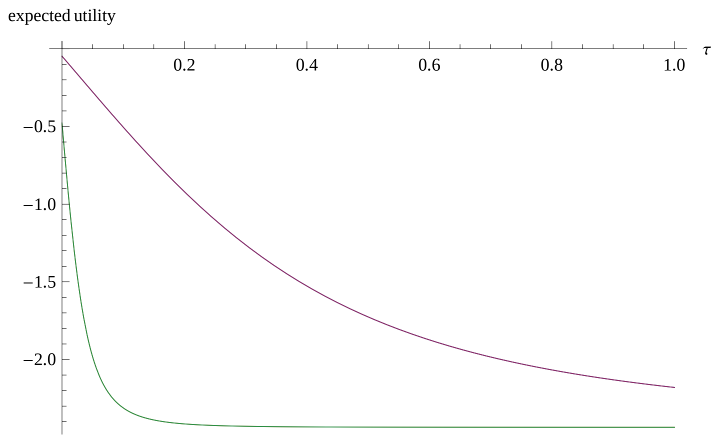


Figure 2.4: Expected utility for type-A (lower line) and type-B (upper line) agents

initial distributions of the $2\bar{Z}$ have an impact on the expected wealth but do not change our results qualitatively.

Let us assume that rational agents' initial endowment of the risky asset is $\theta 2\bar{Z}$ and that noise traders hold $(1-\theta)2\bar{Z}$ where $\theta \in [0,1]$. So initial wealth of these two groups is $w_0 + 2\theta\bar{Z}p$ and $2(1-\theta)\bar{Z}p$, respectively.

Noise traders are again hit by exogenous shocks and therefore demand or sell the asset. Before they are hit by the shock they hold $2(1-\theta)\bar{Z}$ whereas they possess $2\bar{Z} - Z \sim N(\bar{Z}, \Delta^2)$ assets later. The difference is

$$\underbrace{2\bar{Z} - Z}_{\text{final holdings (NT)}} \quad - \quad \underbrace{2(1-\theta)\bar{Z}}_{\text{initial holdings (NT)}} \quad = 2\theta\bar{Z} - Z,$$

which is the net supply/demand of noise traders. If $2\theta\bar{Z} > Z$, noise traders demand the asset, if $2\theta\bar{Z} < Z$ they sell it. Note, that if $\theta = \frac{1}{2}$ we are in the situation from section 2.1.1 and the expected transfer of assets is equal to zero. Since rational agents hold $2\theta\bar{Z}$ assets in the beginning, their total supply is

$$\underbrace{2\theta\bar{Z}}_{\text{initial holdings (RAT)}} \quad - \quad \underbrace{\left(2\theta\bar{Z} - Z\right)}_{\text{transfer from/to NT}} \quad = Z.$$

An initial (upper panel) and a final (middle and lower panel) asset allocation for two different realizations of Z are depicted in figure 2.5. Rational agents' wealth is

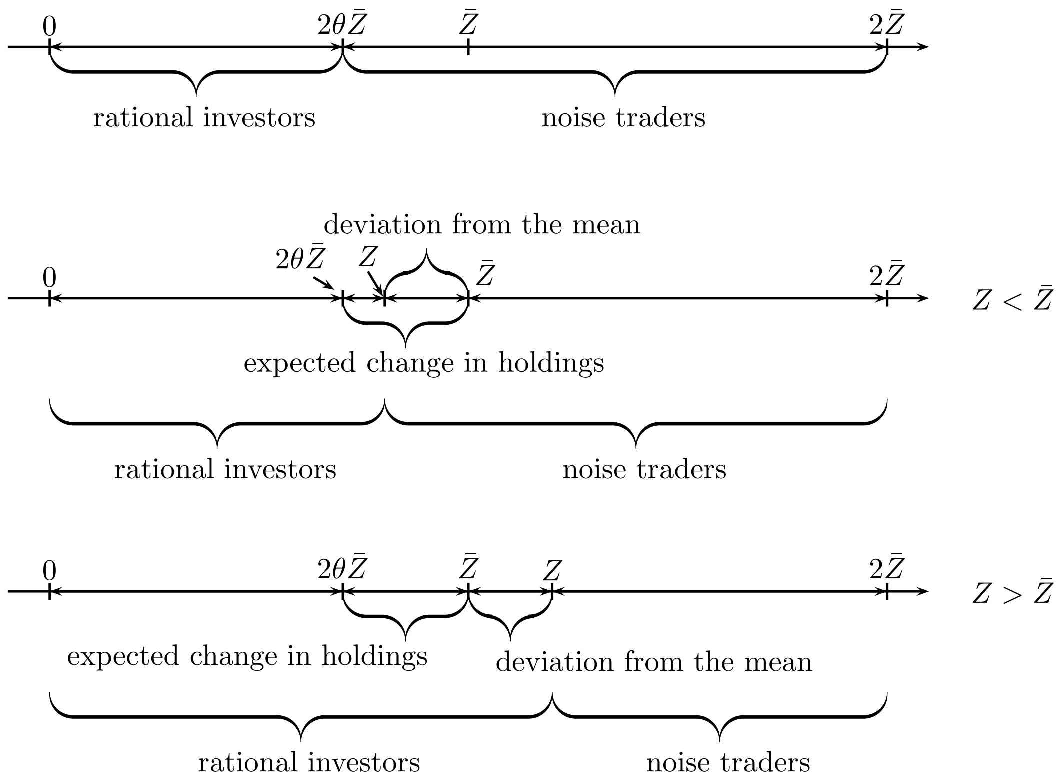


Figure 2.5: Initial (top) and final (middle and bottom) asset allocation for different realizations of Z.

$$\begin{aligned} w_{1,RAT} &= w_0 + 2\theta\bar{Z}X - (2\theta\bar{Z} - Z)(X - p) \\ &= w_0 + 2\theta\bar{Z}X - (2\theta\bar{Z} - Z + \bar{Z} - \bar{Z})(X - p) \\ &= w_0 + 2\theta\bar{Z}X + (Z - \bar{Z})(X - p) + (\bar{Z} - 2\theta\bar{Z})(X - p). \end{aligned}$$

The first part $w_0 + 2\theta\bar{Z}X$ is the initial wealth of the rational agents. The fourth summand $(\bar{Z} - 2\theta\bar{Z})(X - p)$ is the expected change in the holdings of rational agents and the third summand $(Z - \bar{Z})(X - p)$ is the deviation from the expected holding $\bar{Z}$. Recall, that noise traders' final holdings are

$$\underbrace{2(1-\theta)\bar{Z}}_{\text{initial holdings (NT)}} + \underbrace{2\theta\bar{Z} - Z}_{\text{transfer from/to RAT}}$$

and their final wealth is

$$\begin{aligned} w_{1,NT} &= 2(1-\theta)\bar{Z}X + (2\theta\bar{Z} - Z)(X-p) \\ &= 2(1-\theta)\bar{Z}X + (2\theta\bar{Z} - Z + \bar{Z} + \bar{Z} - 2\bar{Z})(X-p) \\ &= 2(1-\theta)\bar{Z}X - (Z-\bar{Z})(X-p) - (2(1-\theta)\bar{Z} - \bar{Z})(X-p). \end{aligned}$$

The first summand $2(1-\theta)\bar{Z}X$ is the value of the initial asset holdings in the end. The third summand is the expected change of asset holdings multiplied by its worth and the second summand is the value of the deviation from the expected asset holding $\bar{Z}$. Summing up the wealth of the two groups we get the total final wealth

$$\begin{aligned} w_1^t &= w_{1,RAT} + w_{1,NT} \\ &= w_0 + 2\theta X\bar{Z} + (Z-\bar{Z})(X-p) + (\bar{Z} - 2\theta\bar{Z})(X-p) \\ &\quad +2(1-\theta)X\bar{Z} - (Z-\bar{Z})(X-p) - (2(1-\theta)\bar{Z} - \bar{Z})(X-p) \\ &= w_0 + 2\theta X\bar{Z} + 2(1-\theta)X\bar{Z} \\ &= w_0 + 2X\bar{Z} \end{aligned}$$

which is just the total amount of assets $2\bar{Z}$ times the payoff X plus rational speculators' constant part of the initial wealth w_0. The following table gives an overview over the initial and final number of assets and the initial and final wealth of the two groups of agents:[19]

	rational speculators	noise traders	total
initial number of assets	$2\theta\bar{Z}$	$2(1-\theta)\bar{Z}$	$2\bar{Z}$
initial wealth	$w_0 + 2\theta\bar{Z}p$	$2(1-\theta)\bar{Z}p$	$w_0 + 2\bar{Z}p$
final number of assets	$2\theta\bar{Z} - (2\theta\bar{Z} - Z) = Z$	$2(1-\theta)\bar{Z} + (2\theta\bar{Z} - Z)$ $= 2\bar{Z} - Z$	$2\bar{Z}$
final wealth	$w_0 + 2\theta X\bar{Z}$ $+(Z-\bar{Z})(X-p)$ $+(\bar{Z} - 2\theta\bar{Z})(X-p)$	$2(1-\theta)X\bar{Z} - (Z-\bar{Z})(X-p)$ $-(2(1-\theta)\bar{Z} - \bar{Z})(X-p)$	$w_0 + 2X\bar{Z}$

[19]For $\theta = \frac{1}{2}$ this table coincides with the table in section 2.1.1.

As in section 2.1.1, we consider the expected final wealth of the two groups of agents:

$$\begin{aligned}
E[w_{1,RAT}] &= E[w_0 + 2\theta X\bar{Z} + (Z-\bar{Z})(X-p) + (\bar{Z} - 2\theta\bar{Z})(X-p)] \\
&= w_0 + 2\theta\bar{X}\bar{Z} + E[(Z-\bar{Z})(X-p)] + E[(\bar{Z} - 2\theta\bar{Z})(X-p)] \\
&= w_0 + 2\theta\bar{X}\bar{Z} + E[Z-\bar{Z}]E[X-p] + Cov[Z-\bar{Z}, X-p] \\
&\quad +(\bar{Z} - 2\theta\bar{Z})E[X-p] \\
&= w_0 + 2\theta\bar{X}\bar{Z} + Cov[Z-\bar{Z}, X-p] + (\bar{Z} - 2\theta\bar{Z})(\bar{X} - \bar{p})
\end{aligned}$$

and

$$\begin{aligned}
E[w_{1,NT}] &= E[2(1-\theta)X\bar{Z} - (Z-\bar{Z})(X-p) - (2(1-\theta)\bar{Z} - \bar{Z})(X-p)] \\
&= 2(1-\theta)\bar{X}\bar{Z} - E[(Z-\bar{Z})(X-p)] - (2(1-\theta)\bar{Z} - \bar{Z})E[X-p] \\
&= 2(1-\theta)\bar{X}\bar{Z} - E[(Z-\bar{Z})]E[(X-p)] - Cov[Z-\bar{Z}, X-p] \\
&\quad -(2(1-\theta)\bar{Z} - \bar{Z})(\bar{X} - \bar{p}) \\
&= 2(1-\theta)\bar{X}\bar{Z} - Cov[Z-\bar{Z}, X-p] - (2(1-\theta)\bar{Z} - \bar{Z})(\bar{X} - \bar{p}).
\end{aligned}$$

The expected wealth of rational speculators consists of their initial wealth at date 1 $w_0 + 2\theta\bar{X}\bar{Z}$, the wealth change that comes from the expected change in asset holdings and the covariance between the deviation of Z from $\bar{Z}$ and the excess return $X - p$. Note again, that if $\theta = \frac{1}{2}$ we have the special case from above since $\bar{Z} - 2\theta\bar{Z}$ and $2(1-\theta)\bar{Z} - \bar{Z}$ cancel out.

2.5 Summary

In this chapter we presented the REE model by Grossman & Stiglitz (1980). We interpreted the stochastic supply as noise traders and showed that their expected wealth is transferred to rational market agents. This transfer decreases as the fraction of informed agents in the market increases. If there are more informed agents in the market, the price becomes more informative. Therefore it incorporates less risk and noise traders trade the asset to a more appropriate price.

Although we introduced noise traders, we were not too specific about their identity. Following Gloston & Milgrom (1985) we explained their trading behavior via exogenous events like job losses, job promotions or marriage and childbirth. This interpretation is worth discussing. A point of criticism is mentioned by Dow & Gorton (2006). They state that the probability of having to sell an asset is not equal to the probability of buying an asset. This idea is intuitive since when someone needs money he has to sell assets, whereas if someone has earned money it is not necessary to purchase assets. Therefore, assuming that noise traders' expected change in holdings $Z - \bar{Z}$ is equal to zero might be disputable. Using another decomposition of the $2\bar{Z}$ assets (with $(1-\theta)2\bar{Z} > \bar{Z} \Leftrightarrow \theta < \frac{1}{2}$) solves this problem since in that case noise traders on average sell a portion of their asset holdings.
Before considering strategic behavior of rational market agents to exploit noise traders in the next chapter we present the specific calculations that were skipped so far. Since the calculations are extensive, a mathematica source code is presented in the end where we compared our calculations with the ones calculated by mathematica.

2.A Calculations

In this section we present the calculations from section 2.1.5 and prove equations (2.15), (2.16) and (2.17) before we show that (2.20) and (2.22) hold. Since the denominators of (2.15), (2.16) and (2.17) are all equal we first simplify that term.

$$
\begin{aligned}
& (1-\tau)\left(1-\alpha_2^A\right)\beta^B+\tau\left(1-\alpha_2^B\right)\beta^A \\
= \; & (1-\tau)\left(1-\frac{\pi_1\sigma_X^2}{\pi_1^2\sigma_X^2+\gamma^2\Delta^2}\right)\frac{\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} \\
& +\tau\left(1-\frac{\pi_1\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)}\right)\frac{\gamma^2\Delta^2\sigma_X^2}{\pi_1^2\sigma_X^2+\gamma^2\Delta^2} \\
= \; & \frac{(1-\tau)\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2-\pi_1\sigma_X^2\right)\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)} \\
& +\frac{\tau\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\pi_1^2\sigma_\epsilon^2+\sigma_X^2\gamma^2\Delta^2-\pi_1\sigma_X^2\sigma_\epsilon^2\right)\gamma^2\Delta^2\sigma_X^2}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)} \\
= \; & \frac{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2-\pi_1\sigma_X^2\right)\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2+\tau\sigma_X^4\Delta^4\gamma^4}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)} \\
= \; & \frac{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)\right)}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}.
\end{aligned}
$$

The numerators in equations (2.15), (2.16) and (2.17) are

$$
\begin{aligned}
(1-\tau)\alpha_0^A\beta^B+\tau\alpha_0^B\beta^A \;=\; & (1-\tau)\frac{\left(\gamma^2\Delta^2\bar{X}-\pi_0\pi_1\sigma_X^2+\pi_1\gamma\sigma_X^2\bar{Z}\right)\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)\right)} \\
& +\tau\frac{\left(\gamma^2\Delta^2\sigma_\epsilon^2\bar{X}-\pi_0\pi_1\sigma_X^2\sigma_\epsilon^2+\gamma\pi_1\sigma_X^2\sigma_\epsilon^2\bar{Z}\right)\gamma^2\Delta^2\sigma_X^2}{\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)\right)\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)} \\
= \; & \frac{\left(\gamma^2\Delta^2\bar{X}-\pi_0\pi_1\sigma_X^2+\pi_1\gamma\sigma_X^2\bar{Z}\right)\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)\right)},
\end{aligned}
$$

$$
\begin{aligned}
\tau\alpha_1^B\beta^A \;=\; & \tau\frac{\gamma^2\Delta^2\sigma_X^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)}\frac{\gamma^2\Delta^2\sigma_X^2}{\pi_1^2\sigma_X^2+\gamma^2\Delta^2} \\
= \; & \frac{\tau\Delta^4\gamma^4\sigma_X^4}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)\right)},
\end{aligned}
$$

and

$$
\begin{aligned}
\rho\beta^A\beta^B \;=\; & \rho\frac{\gamma^2\Delta^2\sigma_X^2}{\pi_1^2\sigma_X^2+\gamma^2\Delta^2}\frac{\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)} \\
= \; & \frac{\rho\sigma_\epsilon^2\gamma^4\Delta^4\sigma_X^4}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2)\right)},
\end{aligned}
$$

respectively. So we have

$$\begin{aligned}
\pi_0 &= \frac{\frac{\left(\gamma^2\Delta^2\bar{X}-\pi_0\pi_1\sigma_X^2+\pi_1\gamma\sigma_X^2\bar{Z}\right)\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}}{\frac{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\pi_1\sigma_\epsilon^2\right)\right)}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}} \\
&= \frac{\left(\gamma^2\Delta^2\bar{X}-\pi_0\pi_1\sigma_X^2+\pi_1\gamma\sigma_X^2\bar{Z}\right)\gamma^2\Delta^2\sigma_X^2\sigma_\epsilon^2}{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)\right)} \\
&= \frac{\left(\gamma^2\Delta^2\bar{X}-\pi_0\pi_1\sigma_X^2+\pi_1\gamma\sigma_X^2\bar{Z}\right)\sigma_\epsilon^2}{\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)},
\end{aligned}$$

$$\begin{aligned}
\pi_1 &= \frac{\frac{\tau\Delta^4\gamma^4\sigma_X^4}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}}{\frac{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\pi_1\sigma_\epsilon^2\right)\right)}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}} \\
&= \frac{\tau\Delta^4\gamma^4\sigma_X^4}{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)\right)} \\
&= \frac{\tau\Delta^2\gamma^2\sigma_X^2}{\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)},
\end{aligned}$$

and

$$\begin{aligned}
\gamma &= \frac{\frac{\rho\sigma_\epsilon^2\gamma^4\Delta^4\sigma_X^4}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}}{\frac{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\pi_1\sigma_\epsilon^2\right)\right)}{\left(\pi_1^2\sigma_X^2+\gamma^2\Delta^2\right)\left(\sigma_\epsilon^2\gamma^2\Delta^2+\sigma_X^2\left(\pi_1^2\sigma_\epsilon^2+\gamma^2\Delta^2\right)\right)}} \\
&= \frac{\rho\sigma_\epsilon^2\gamma^4\Delta^4\sigma_X^4}{\gamma^2\Delta^2\sigma_X^2\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)\right)} \\
&= \frac{\rho\sigma_\epsilon^2\gamma^2\Delta^2\sigma_X^2}{\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)}.
\end{aligned}$$

Rearranging (2.15) yields (2.18) since

$$\begin{aligned}
\Leftrightarrow \quad \pi_0\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)\right) &= \left(\gamma^2\Delta^2\bar{X}-\pi_0\pi_1\sigma_X^2+\pi_1\gamma\sigma_X^2\bar{Z}\right)\sigma_\epsilon^2 \\
\Leftrightarrow \quad \pi_0\left(\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+\pi_1^2\sigma_\epsilon^2\right)\right) &= \left(\gamma^2\Delta^2\bar{X}+\pi_1\gamma\sigma_X^2\bar{Z}\right)\sigma_\epsilon^2 \\
\Leftrightarrow \quad \pi_0 &= \frac{\gamma\sigma_\epsilon^2\left(\gamma\Delta^2\bar{X}+\pi_1\sigma_X^2\bar{Z}\right)}{\gamma^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\gamma^2\Delta^2\tau+\pi_1^2\sigma_\epsilon^2\right)}.
\end{aligned}$$

Now we come to equation (2.20). Plugging (2.19) in (2.16), we get

$$\pi_1 = \frac{\rho\sigma_\epsilon^2\left(\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\right)^2\Delta^2\sigma_X^2}{\left(\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\right)^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\left(\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\right)^2\Delta^2\tau+(\pi_1-1)\,\pi_1\sigma_\epsilon^2\right)}.$$

Rearranging yields[20]

$$
\begin{aligned}
\tfrac{1}{\tau^2}\rho^2\sigma_\epsilon^4\Delta^2\tau\sigma_X^2\pi_1^2 \quad &= \frac{1}{\tau^2}\rho^2\sigma_\epsilon^6\Delta^2\pi_1^3+\frac{1}{\tau^2}\rho^2\sigma_\epsilon^2\sigma_X^2\Delta^2\tau\pi_1^3+\sigma_X^2\sigma_\epsilon^2\pi_1^2-\sigma_X^2\sigma_\epsilon^2\pi_1^2\\
\Leftrightarrow \quad \rho^2\sigma_\epsilon^4\Delta^2\tau\sigma_X^2+\sigma_X^2\sigma_\epsilon^2\tau^2 \quad &= \left(\rho^2\sigma_\epsilon^6\Delta^2+\rho^2\sigma_\epsilon^2\sigma_X^2\Delta^2\tau+\tau^2\sigma_X^2\sigma_\epsilon^2\right)\pi_1\\
\Leftrightarrow \quad \pi_1 \quad &= \frac{\tau\sigma_X^2\sigma_\epsilon^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\sigma_\epsilon^2\left(\rho^2\Delta^2\sigma_\epsilon^4+\tau\sigma_X^2(\Delta^2\rho^2\sigma_\epsilon^2+\tau)\right)}\\
&= \frac{\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}{\rho^2\Delta^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\Delta^2\rho^2\sigma_\epsilon^2+\tau\right)}.
\end{aligned}
$$

Finally we show that equation (2.22) holds. Plugging (2.19) in (2.15), we have[21]

$$
\begin{aligned}
\pi_0 \quad &= \quad \frac{\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\sigma_\epsilon^2\left(\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\Delta^2\bar{X}+\pi_1\bar{Z}\sigma_X^2\right)}{\left(\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\right)^2\Delta^2\sigma_\epsilon^2+\sigma_X^2\left(\left(\frac{\rho\sigma_\epsilon^2\pi_1}{\tau}\right)^2\Delta^2\tau+\pi_1^2\sigma_\epsilon^2\right)}\\
&= \quad \frac{\frac{\pi_1^2}{\tau^2}\rho\sigma_\epsilon^4\left(\rho\sigma_\epsilon^2\Delta^2\bar{X}+\tau\bar{Z}\sigma_X^2\right)}{\frac{\pi_1^2}{\tau^2}\left(\rho\Delta^2\sigma_\epsilon^6+\sigma_X^2\left(\rho\sigma_\epsilon^4\Delta^2\tau+\sigma_\epsilon^2\tau^2\right)\right)}\\
&= \quad \frac{\rho\sigma_\epsilon^2\left(\rho\sigma_\epsilon^2\Delta^2\bar{X}+\tau\bar{Z}\sigma_X^2\right)}{\rho\Delta^2\sigma_\epsilon^4+\tau\sigma_X^2\left(\rho\sigma_\epsilon^2\Delta^2+\tau\right)}.
\end{aligned}
$$

2.B Alternative decompositions of the final wealth

We mentioned in footnote 14 and 18 that there are also other decompositions of the final wealth which lead to the same expected utility. To make sure that there is no mistake in the mathematica file, it contains another decomposition of the final wealth. It is shown in appendix 2.C that both decompositions lead to the same expected utility. These other decompositions are presented in this section. We will see that the matrix A, the vector b and the scalar d from lemma 1 will be much more complicated than before but the variance-covariance-matrix will be pretty simple.

Type-A agents

As mentioned in footnote 14, we consider the vector $\begin{pmatrix} X-\bar{X} & Z-\bar{Z}\end{pmatrix}^t$ of random variables. The initial wealth was $w_{1i}=w_{0i}+z_i^A(X-p)+\bar{Z}p$. Using equations (2.2),

[20] Recall from footnote 12 that $\pi_1 \neq 0$.

[21] Assuming again that $\pi_0 \neq 0 \neq \gamma$.

(2.4), and (2.5) we get

$$\begin{aligned}
w_{1i} &= w_{0i} + \frac{\alpha_0^A + (\alpha_2^A - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^A}(X - \pi_0 - \pi_1 X + \gamma Z) \\
&\quad + \bar{Z}(\pi_0 + \pi_1 X - \gamma Z) \\
&= w_{0i} - \frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}\pi_0 + \bar{Z}\pi_0 + \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}(1 - \pi_1)X^2 \\
&\quad - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\gamma Z^2 + \left(\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}(1 - \pi_1)\right) XZ \\
&\quad + \left(\frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}(1 - \pi_1) - \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\pi_0 + \bar{Z}\pi_1\right) X \\
&\quad + \left(\frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\pi_0 - \frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}\gamma - \bar{Z}\gamma\right) Z.
\end{aligned}$$

Using that

$$XZ = X(Z - \bar{Z}) + X\bar{Z} = (X - \bar{X})(Z - \bar{Z}) + \bar{X}(Z - \bar{Z}) + (X - \bar{X})\bar{Z} + \bar{X}\bar{Z}, \quad (2.24)$$

we get

$$\begin{aligned}
w_{1i} &= \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}(1 - \pi_1)(X - \bar{X})^2 - \left(\frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\gamma\right)(Z - \bar{Z})^2 \\
&\quad + \left(\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}(1 - \pi_1)\right)(X - \bar{X})(Z - \bar{Z}) \\
&\quad + \left(2\bar{X}\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}(1 - \pi_1) + \left(\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}(1 - \pi_1)\right)\bar{Z}\right. \\
&\quad + \left.\left(\frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}(1 - \pi_1) - \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\pi_0 + \bar{Z}\pi_1\right)\right)(X - \bar{X}) \\
&\quad + \left(-2\bar{Z}\frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\gamma + \left(\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}(1 - \pi_1)\right)\bar{X}\right. \\
&\quad + \left.\left(\frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\pi_0 - \frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}\gamma - \bar{Z}\gamma\right)\right)(Z - \bar{Z}) \\
&\quad + w_{0i} - \frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}\pi_0 + \bar{Z}\pi_0 + \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}(1 - \pi_1)\bar{X}^2 \\
&\quad + \left(\frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}(1 - \pi_1)\right)\bar{X}\bar{Z} - \frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\gamma\bar{Z}^2 \\
&\quad + \left(\frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}(1 - \pi_1) - \frac{(\alpha_2^A - 1)\pi_1}{\rho\beta^A}\pi_0 + \bar{Z}\pi_1\right)\bar{X} \\
&\quad + \left(\frac{(\alpha_2^A - 1)\gamma}{\rho\beta^A}\pi_0 - \frac{\alpha_0^A + (\alpha_2^A - 1)\pi_0}{\rho\beta^A}\gamma - \bar{Z}\gamma\right)\bar{Z}.
\end{aligned}$$

So

$$A = \begin{pmatrix} -\rho\frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}(1-\pi_1) & -\frac{\rho}{2}\left(\frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}(1-\pi_1)\right) \\ -\frac{\rho}{2}\left(\frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}(1-\pi_1)\right) & \rho\left(\frac{(\alpha_2^A-1)\gamma^2}{\rho\beta^A}\right) \end{pmatrix},$$

$$b = \begin{pmatrix} -\rho\left(2\bar{X}\frac{(\alpha_2^A-1)\pi_1(1-\pi_1)}{\rho\beta^A} + \left(\frac{(\alpha_2^A-1)\pi_1\gamma}{\rho\beta^A} - \frac{(\alpha_2^A-1)\gamma(1-\pi_1)}{\rho\beta^A}\right)\bar{Z}\right. \\ \left. + \left(\frac{\left(\alpha_0^A+(\alpha_2^A-1)\pi_0\right)(1-\pi_1)}{\rho\beta^A} - \frac{(\alpha_2^A-1)\pi_1\pi_0}{\rho\beta^A} + \bar{Z}\pi_1\right)\right) \\ \\ -\rho\left(\left(\frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}(1-\pi_1)\right)\bar{X}\right. \\ \left. + \left(\frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}\pi_0 - \frac{\alpha_0^A+(\alpha_2^A-1)\pi_0}{\rho\beta^A}\gamma - \bar{Z}\gamma\right) - 2\bar{Z}\frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}\gamma\right) \end{pmatrix}$$

and

$$\begin{aligned} d &= w_{0i} - \frac{\alpha_0^A + (\alpha_2^A-1)\pi_0}{\rho\beta^A}\pi_0 + \bar{Z}\pi_0 + \frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}(1-\pi_1)\bar{X}^2 \\ &+ \left(\frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}\gamma - \frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}(1-\pi_1)\right)\bar{X}\bar{Z} - \frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}\gamma\bar{Z}^2 \\ &+ \left(\frac{\alpha_0^A + (\alpha_2^A-1)\pi_0}{\rho\beta^A}(1-\pi_1) - \frac{(\alpha_2^A-1)\pi_1}{\rho\beta^A}\pi_0 + \bar{Z}\pi_1\right)\bar{X} \\ &+ \left(\frac{(\alpha_2^A-1)\gamma}{\rho\beta^A}\pi_0 - \frac{\alpha_0^A + (\alpha_2^A-1)\pi_0}{\rho\beta^A}\gamma - \bar{Z}\gamma\right)\bar{Z}. \end{aligned}$$

The variance-covariance-matrix in this case is

$$\Sigma = \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \Delta^2 \end{pmatrix}.$$

Type-B agents

Recall footnote 18 where we mentioned the vector $\begin{pmatrix} X-\bar{X} & Z-\bar{Z} & \epsilon_i \end{pmatrix}^t$ of random variables. Plugging equations (2.3), (2.4), and (2.10) final wealth of a type-B agent becomes

$$
\begin{aligned}
w_{1i} &= w_{0i} + \bar{Z}p + z_i^B(X - p) \\
&= w_{0i} + \bar{Z}(\pi_0 + \pi_1 X - \gamma Z) \\
&\quad + \frac{\alpha_0^B + \alpha_1^B(X + \epsilon_i) + (\alpha_2^B - 1)(\pi_0 + \pi_1 X - \gamma Z)}{\rho\beta^B}(X - \pi_0 - \pi_1 X + \gamma Z) \\
&= w_{0i} + \bar{Z}\pi_0 - \frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}\pi_0 + \frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}(1 - \pi_1)X^2 \\
&\quad - \frac{(\alpha_2^B - 1)\gamma^2}{\rho\beta^B}Z^2 + \left(\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B - 1)\gamma}{\rho\beta^B}(1 - \pi_1)\right)XZ \\
&\quad + \frac{\alpha_1^B(1 - \pi_1)}{\rho\beta^B}X\epsilon_i + \frac{\alpha_1^B\gamma}{\rho\beta^B}Z\epsilon_i - \frac{\alpha_1^B}{\rho\beta^B}\pi_0\epsilon_i \\
&\quad + \left(\frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}(1 - \pi_1) - \frac{\alpha_1^B}{\rho\beta^B}\pi_0 + \bar{Z}\pi_1\right)X \\
&\quad + \left(\frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}\gamma + \frac{(\alpha_2^B - 1)\gamma}{\rho\beta^B}\pi_0 - \bar{Z}\gamma\right)Z.
\end{aligned}
$$

Using again equation (2.24), we have

$$
\begin{aligned}
w_{1i} &= \frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}(1-\pi_1)(X-\bar{X})^2 - \frac{(\alpha_2^B-1)\gamma^2}{\rho\beta^B}(Z-\bar{Z})^2 \\
&+ \left(\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)(X-\bar{X})(Z-\bar{Z}) \\
&+ \frac{\alpha_1^B(1-\pi_1)}{\rho\beta^B}(X-\bar{X})\epsilon_i + \frac{\alpha_1^B\gamma}{\rho\beta^B}(Z-\bar{Z})\epsilon_i \\
&+ \left(\left(\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)\bar{Z}\right. \\
&+ 2\bar{X}\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}(1-\pi_1) \\
&+ \left.\left(\frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}(1-\pi_1) - \frac{\alpha_1^B}{\rho\beta^B}\pi_0 + \bar{Z}\pi_1\right)\right)(X-\bar{X}) \\
&+ \left(\left(\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)\bar{X} - 2\bar{Z}\frac{(\alpha_2^B-1)\gamma^2}{\rho\beta^B}\right. \\
&+ \left.\left(\frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}\gamma + \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}\pi_0 - \bar{Z}\gamma\right)\right)(Z-\bar{Z}) \\
&+ \left(\frac{\alpha_1^B(1-\pi_1)}{\rho\beta^B}\bar{X} + \frac{\alpha_1^B\gamma}{\rho\beta^B}\bar{Z} - \frac{\alpha_1^B}{\rho\beta^B}\pi_0\right)\epsilon_i \\
&+ w_{0i} + \bar{Z}\pi_0 - \frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}\pi_0 + \frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}(1-\pi_1)\bar{X}^2 \\
&- \frac{(\alpha_2^B-1)\gamma^2}{\rho\beta^B}\bar{Z}^2 \\
&+ \left(\frac{\alpha_1^B + (\alpha_2^B - 1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)\bar{X}\bar{Z} \\
&+ \left(\frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}(1-\pi_1) - \frac{\alpha_1^B}{\rho\beta^B}\pi_0 + \bar{Z}\pi_1\right)\bar{X} \\
&+ \left(\frac{\alpha_0^B + (\alpha_2^B - 1)\pi_0}{\rho\beta^B}\gamma + \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}\pi_0 - \bar{Z}\gamma\right)\bar{Z}.
\end{aligned}
$$

So

$$
A = \begin{pmatrix}
-\frac{\alpha_1^B+(\alpha_2^B-1)\pi_1}{\beta^B}(1-\pi_1) & -\frac{\left(\alpha_1^B+(\alpha_2^B-1)\pi_1\right)\gamma-(\alpha_2^B-1)\gamma(1-\pi_1)}{2\beta^B} & -\frac{\alpha_1^B(1-\pi_1)}{2\beta^B} \\
-\frac{\left(\alpha_1^B+(\alpha_2^B-1)\pi_1\right)\gamma-(\alpha_2^B-1)\gamma(1-\pi_1)}{2\beta^B} & \frac{(\alpha_2^B-1)\gamma^2}{\beta^B} & -\frac{\alpha_1^B\gamma}{2\beta^B} \\
-\frac{\alpha_1^B(1-\pi_1)}{2\beta^B} & -\frac{\alpha_1^B\gamma}{2\beta^B} & 0
\end{pmatrix},
$$

$$b = \begin{pmatrix} -\rho\left(\left(\frac{\alpha_1^B+(\alpha_2^B-1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)\bar{Z} + 2\bar{X}\frac{\alpha_1^B+(\alpha_2^B-1)\pi_1}{\rho\beta^B}(1-\pi_1) + \right. \\ \left. \left(\frac{\alpha_0^B+(\alpha_2^B-1)\pi_0}{\rho\beta^B}(1-\pi_1) - \frac{\alpha_1^B}{\rho\beta^B}\pi_0 + \bar{Z}\pi_1\right)\right) \\ \\ -\rho\left(\left(\frac{\alpha_1^B+(\alpha_2^B-1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)\bar{X} - 2\bar{Z}\frac{(\alpha_2^B-1)\gamma^2}{\rho\beta^B} + \right. \\ \left. \left(\frac{\alpha_0^B+(\alpha_2^B-1)\pi_0}{\rho\beta^B}\gamma + \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}\pi_0 - \bar{Z}\gamma\right)\right) \\ \\ -\rho\left(\frac{\alpha_1^B(1-\pi_1)}{\rho\beta^B}\bar{X} + \frac{\alpha_1^B\gamma}{\rho\beta^B}\bar{Z} - \frac{\alpha_1^B}{\rho\beta^B}\pi_0\right) \end{pmatrix}$$

and

$$\begin{aligned} d &= w_{0i} + \bar{Z}\pi_0 - \frac{\alpha_0^B + (\alpha_2^B-1)\pi_0}{\rho\beta^B}\pi_0 \\ &+ \frac{\alpha_1^B + (\alpha_2^B-1)\pi_1}{\rho\beta^B}(1-\pi_1)\bar{X}^2 - \frac{(\alpha_2^B-1)\gamma^2}{\rho\beta^B}\bar{Z}^2 \\ &+ \left(\frac{\alpha_1^B + (\alpha_2^B-1)\pi_1}{\rho\beta^B}\gamma - \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}(1-\pi_1)\right)\bar{X}\bar{Z} \\ &+ \left(\frac{\alpha_0^B + (\alpha_2^B-1)\pi_0}{\rho\beta^B}(1-\pi_1) - \frac{\alpha_1^B}{\rho\beta^B}\pi_0 + \bar{Z}\pi_1\right)\bar{X} \\ &+ \left(\frac{\alpha_0^B + (\alpha_2^B-1)\pi_0}{\rho\beta^B}\gamma + \frac{(\alpha_2^B-1)\gamma}{\rho\beta^B}\pi_0 - \bar{Z}\gamma\right)\bar{Z}. \end{aligned}$$

The variance-covariance matrix is again very simple:

$$\Sigma = \begin{pmatrix} \sigma_X^2 & 0 & 0 \\ 0 & \Delta^2 & 0 \\ 0 & 0 & \sigma_\epsilon^2 \end{pmatrix}.$$

2.C Mathematica source code

On the next pages is the commented (expressions in (* *) are comments) mathematica source code where the expectation formation, the market clearing, the comparative statics and the expected utility functions are computed.

In[1]:= $\texttt{solPrice} = \{p \to \pi_0 + \pi_1 X - \gamma Z\};$ (*substitution rule for the price*)

In[2]:= $\texttt{para} = \left\{\rho \to 5,\ \sigma_X \to 1,\ \overline{X} \to 5,\ \overline{Z} \to 5,\ \Delta \to 1,\ \sigma_\epsilon \to \frac{1}{10},\ w_0 \to 1\right\};$ (*set of parameters*)

Type A

In[3]:= $\texttt{VA} = \left\{\left\{\sigma_X{}^2,\ \pi_1\,\sigma_X{}^2\right\},\ \left\{\pi_1\,\sigma_X{}^2,\ \pi_1{}^2\,\sigma_X{}^2 + \gamma^2\,\Delta^2\right\}\right\};$
(*variance-covariance-matrix of the vector $(X\ p)^t$*)

In[4]:= MatrixForm[VA]

Out[4]//MatrixForm=

$$\begin{pmatrix} \sigma_X^2 & \pi_1\,\sigma_X^2 \\ \pi_1\,\sigma_X^2 & \gamma^2\,\Delta^2 + \pi_1^2\,\sigma_X^2 \end{pmatrix}$$

In[5]:= $\texttt{Alpha0A} = \left\{\alpha_{A0} \texttt{->} \overline{X} - \frac{\texttt{VA[[1, 2]]}}{\texttt{VA[[2, 2]]}}\left(\pi_0 + \pi_1\,\overline{X} - \gamma\,\overline{Z}\right)\right\}$ (*α_0 for agents of type A*)

Out[5]= $\left\{\alpha_{A0} \to \overline{X} - \frac{\pi_1\left(-\gamma\,\overline{Z} + \pi_0 + \overline{X}\,\pi_1\right)\sigma_X^2}{\gamma^2\,\Delta^2 + \pi_1^2\,\sigma_X^2}\right\}$

In[6]:= $\texttt{Alpha2A} = \left\{\alpha_{A2} \texttt{->} \frac{\texttt{VA[[1, 2]]}}{\texttt{VA[[2, 2]]}}\right\}$ (*α_2 for agents of type A*)

Out[6]= $\left\{\alpha_{A2} \to \frac{\pi_1\,\sigma_X^2}{\gamma^2\,\Delta^2 + \pi_1^2\,\sigma_X^2}\right\}$

In[7]:= $\texttt{betaA} = \left\{\beta_A \to \texttt{VA[[1, 1]]} - \frac{\texttt{VA[[1, 2]]}}{\texttt{VA[[2, 2]]}}\,\texttt{VA[[1, 2]]}\right\}$ // Simplify
(*conditional variance of the payoff X given the price p of the type A agents*)

Out[7]= $\left\{\beta_A \to \frac{\gamma^2\,\Delta^2\,\sigma_X^2}{\gamma^2\,\Delta^2 + \pi_1^2\,\sigma_X^2}\right\}$

In[8]:= $\texttt{shareA} = \frac{\alpha_{A0} + (\alpha_{A2} - 1)\,p}{\rho\,\beta_A};$ (*number of shares an agent of type A holds*)

Type B

In[9]:= $\texttt{VB} = \left\{\left\{\sigma_X{}^2,\ \sigma_X{}^2,\ \pi_1\,\sigma_X{}^2\right\},\ \left\{\sigma_X{}^2,\ \sigma_X{}^2 + \sigma_\epsilon{}^2,\ \pi_1\,\sigma_X{}^2\right\},\ \left\{\pi_1\,\sigma_X{}^2,\ \pi_1\,\sigma_X{}^2,\ \pi_1{}^2\,\sigma_X{}^2 + \gamma^2\,\Delta^2\right\}\right\};$
(*variance-covariance-matrix of the vector $(X\ y_i\ p)^t$*)

In[10]:= MatrixForm[VB]

Out[10]//MatrixForm=

$$\begin{pmatrix} \sigma_X^2 & \sigma_X^2 & \pi_1\,\sigma_X^2 \\ \sigma_X^2 & \sigma_X^2 + \sigma_\epsilon^2 & \pi_1\,\sigma_X^2 \\ \pi_1\,\sigma_X^2 & \pi_1\,\sigma_X^2 & \gamma^2\,\Delta^2 + \pi_1^2\,\sigma_X^2 \end{pmatrix}$$

In[11]:= $\texttt{Alpha0B} = \left\{\alpha_{B0} \texttt{->} \overline{X} - \texttt{VB[[1, 2 ;; 3]].Inverse[VB[[2 ;; 3, 2 ;; 3]]]}.\left\{\overline{X},\ \pi_0 + \pi_1\,\overline{X} - \gamma\,\overline{Z}\right\}\right\}$ //
Simplify (*α_0 for agents of type B*)

Out[11]= $\left\{\alpha_{B0} \to \left(\left(\gamma^2\,\Delta^2\,\overline{X} + \left(\gamma\,\overline{Z} - \pi_0\right)\pi_1\,\sigma_X^2\right)\sigma_\epsilon^2\right) \Big/ \left(\gamma^2\,\Delta^2\,\sigma_\epsilon^2 + \sigma_X^2\left(\gamma^2\,\Delta^2 + \pi_1^2\,\sigma_\epsilon^2\right)\right)\right\}$

In[12]:= $\texttt{Alpha1B} = \{\alpha_{B1} \texttt{->} \texttt{VB[[1, 2 ;; 3]].Inverse[VB[[2 ;; 3, 2 ;; 3]]][[1]]}\}$ //
Simplify (*α_1 for agents of type B*)

Out[12]= $\left\{\alpha_{B1} \to \frac{\gamma^2\,\Delta^2\,\sigma_X^2}{\gamma^2\,\Delta^2\,\sigma_\epsilon^2 + \sigma_X^2\left(\gamma^2\,\Delta^2 + \pi_1^2\,\sigma_\epsilon^2\right)}\right\}$

In[13]:=
```
Alpha2B = {α_B2 -> VB[[1, 2 ;; 3]].Inverse[VB[[2 ;; 3, 2 ;; 3]]][[2]]} //
  Simplify (*α_2 for agents of type B*)
```

Out[13]= $\left\{\alpha_{B2} \to \frac{\pi_1 \sigma_X^2 \sigma_\epsilon^2}{\gamma^2 \Delta^2 \sigma_\epsilon^2 + \sigma_X^2 \left(\gamma^2 \Delta^2 + \pi_1^2 \sigma_\epsilon^2\right)}\right\}$

In[14]:=
```
betaB = {β_B → σ_X^2 - VB[[1, 2 ;; 3]].Inverse[VB[[2 ;; 3, 2 ;; 3]]].VB[[1, 2 ;; 3]]} //
  Simplify (*conditional variance of the payoff X given
   the private signal y_i and the price p of the type B agents*)
```

Out[14]= $\left\{\beta_B \to \frac{\gamma^2 \Delta^2 \sigma_X^2 \sigma_\epsilon^2}{\gamma^2 \Delta^2 \sigma_\epsilon^2 + \sigma_X^2 \left(\gamma^2 \Delta^2 + \pi_1^2 \sigma_\epsilon^2\right)}\right\}$

In[15]:= shareB = $\frac{1}{\rho\, \beta_B}$ $(\alpha_{B0} + \alpha_{B1} (X + \epsilon_i) + (\alpha_{B2} - 1)\, p)$;(*number of shares an agent of type B holds*)

In[16]:= aggshareB = $\frac{\alpha_{B0} + \alpha_{B1} X + (\alpha_{B2} - 1)\, p}{\rho\, \beta_B}$;
(*number of shares all agents of type B hold together*)

Market Clearing

In[17]:=
```
Marketclearing = τ aggshareB + (1 - τ) shareA
   (*market clearing condition with τ type-B agents and (1-τ) type-A agents*)
```

Out[17]= $\frac{(1-\tau)\,(\alpha_{A0} + p\,(-1 + \alpha_{A2}))}{\rho\, \beta_A} + \frac{\tau\,(\alpha_{B0} + X\,\alpha_{B1} + p\,(-1 + \alpha_{B2}))}{\rho\, \beta_B}$

In[18]:=
```
Solve[Marketclearing == Z, p][[1]] (*solving the market clearing condition for p*)
```

Out[18]= $\{p \to (-\tau\,\alpha_{B0}\,\beta_A - X\,\tau\,\alpha_{B1}\,\beta_A - \alpha_{A0}\,\beta_B + \tau\,\alpha_{A0}\,\beta_B + Z\,\rho\,\beta_A\,\beta_B) \,/\, (-\tau\,\beta_A + \tau\,\alpha_{B2}\,\beta_A - \beta_B + \tau\,\beta_B + \alpha_{A2}\,\beta_B - \tau\,\alpha_{A2}\,\beta_B)\}$

In[19]:=
```
Collect[(-τ α_B0 β_A - X τ α_B1 β_A - α_A0 β_B + τ α_A0 β_B + Z ρ β_A β_B) /
   (-τ β_A + τ α_B2 β_A - β_B + τ β_B + α_A2 β_B - τ α_A2 β_B), {X, Z}]
 (*solving the market clearing price for X and Z to get the
  coefficients in the price function p = π_0+π_1 X - γ Z*)
```

Out[19]= $-(X\,\tau\,\alpha_{B1}\,\beta_A) \,/\, (-\tau\,\beta_A + \tau\,\alpha_{B2}\,\beta_A - \beta_B + \tau\,\beta_B + \alpha_{A2}\,\beta_B - \tau\,\alpha_{A2}\,\beta_B) + (Z\,\rho\,\beta_A\,\beta_B) \,/\, (-\tau\,\beta_A + \tau\,\alpha_{B2}\,\beta_A - \beta_B + \tau\,\beta_B + \alpha_{A2}\,\beta_B - \tau\,\alpha_{A2}\,\beta_B) + (-\tau\,\alpha_{B0}\,\beta_A - \alpha_{A0}\,\beta_B + \tau\,\alpha_{A0}\,\beta_B) \,/\, (-\tau\,\beta_A + \tau\,\alpha_{B2}\,\beta_A - \beta_B + \tau\,\beta_B + \alpha_{A2}\,\beta_B - \tau\,\alpha_{A2}\,\beta_B)$

In[20]:=
```
Pi0 =
  (-τ α_B0 β_A - α_A0 β_B + τ α_A0 β_B) / (-τ β_A + τ α_B2 β_A - β_B + τ β_B + α_A2 β_B - τ α_A2 β_B) //. {Alpha0A[[1]],
     Alpha2A[[1]], Alpha0B[[1]], Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} //
  Simplify (*defining π_0 and plugging in the values of the α's and β's*)
```

Out[20]= $\left(\left(\gamma^2 \Delta^2 \overline{X} + \left(\gamma\,\overline{Z} - \pi_0\right) \pi_1 \sigma_X^2\right) \sigma_\epsilon^2\right) \Big/ \left(\gamma^2 \Delta^2 \sigma_\epsilon^2 + \sigma_X^2 \left(\gamma^2 \Delta^2 \tau + (-1 + \pi_1)\, \pi_1 \sigma_\epsilon^2\right)\right)$

In[21]:=
```
Pi1 = - (τ α_B1 β_A) / (-τ β_A + τ α_B2 β_A - β_B + τ β_B + α_A2 β_B - τ α_A2 β_B) //. {Alpha0A[[1]], Alpha2A[[1]],
     Alpha0B[[1]], Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} //
  Simplify (*defining π_1 and plugging in the values of the α's and β's*)
```

Out[21]= $\left(\gamma^2 \Delta^2 \tau\, \sigma_X^2\right) \Big/ \left(\gamma^2 \Delta^2 \sigma_\epsilon^2 + \sigma_X^2 \left(\gamma^2 \Delta^2 \tau + (-1 + \pi_1)\, \pi_1 \sigma_\epsilon^2\right)\right)$

In[22]:=
```
Gam = - ρ β_A β_B / (-τ β_A + τ α_B2 β_A - β_B + τ β_B + α_A2 β_B - τ α_A2 β_B) //. {Alpha0A[[1]], Alpha2A[[1]],
     Alpha0B[[1]], Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} //
  Simplify (*defining γ and plugging in the values of the α's and β's*)
```

Out[22]= $\left(\gamma^2 \Delta^2 \rho\, \sigma_X^2 \sigma_\epsilon^2\right) \Big/ \left(\gamma^2 \Delta^2 \sigma_\epsilon^2 + \sigma_X^2 \left(\gamma^2 \Delta^2 \tau + (-1 + \pi_1)\, \pi_1 \sigma_\epsilon^2\right)\right)$

In[23]:= `Pi0sol1 = Solve[Pi0 == π0, π0][[1]] // Simplify (*solving the upper π0 function for π0*)`

Out[23]= $\left\{\pi_0 \to \left(\gamma \left(\gamma \Delta^2 \overline{X} + \overline{Z} \pi_1 \sigma_X^2\right) \sigma_\epsilon^2\right) / \left(\gamma^2 \Delta^2 \sigma_\epsilon^2 + \sigma_X^2 \left(\gamma^2 \Delta^2 \tau + \pi_1^2 \sigma_\epsilon^2\right)\right)\right\}$

In[24]:= `Pi1Gammasol = Solve[{Pi1 == π1, Gam == γ}, {π1, γ}][[1]] // Simplify`
`(*solving the upper π1 and γ function for π1 and γ simultaneously*)`

Out[24]= $\left\{\pi_1 \to \frac{\tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}{\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}, \gamma \to \frac{\rho \sigma_X^2 \sigma_\epsilon^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}{\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}\right\}$

In[25]:= `Pi0sol = Pi0sol1 //. Pi1Gammasol // Simplify (*plugging π1 and γ in π0*)`

Out[25]= $\left\{\pi_0 \to \frac{\rho \sigma_\epsilon^2 \left(\tau \overline{Z} \sigma_X^2 + \Delta^2 \rho \overline{X} \sigma_\epsilon^2\right)}{\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}\right\}$

In[26]:= `Pi1sol = {Pi1Gammasol[[1]]} (*defining π1*)`

Out[26]= $\left\{\pi_1 \to \frac{\tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}{\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}\right\}$

In[27]:= `Gammasol = {Pi1Gammasol[[2]]} (*defining γ*)`

Out[27]= $\left\{\gamma \to \frac{\rho \sigma_X^2 \sigma_\epsilon^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}{\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}\right\}$

Comparative Statics on $\gamma \Delta^2$

In[28]:= `γ Δ^2 //. Gammasol //. Δ^2 → x (*replacing Δ^2 for the derivative*)`

Out[28]= $\frac{x \rho \sigma_X^2 \sigma_\epsilon^2 \left(\tau + x \rho^2 \sigma_\epsilon^2\right)}{x \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + x \rho^2 \sigma_\epsilon^2\right)}$

In[29]:= `D[%, x] // Simplify (*deriving with respect to x = Δ^2*)`

Out[29]= $\left(x^2 \rho^5 \sigma_X^2 \sigma_\epsilon^8 + \rho \tau \sigma_X^4 \sigma_\epsilon^2 \left(\tau + x \rho^2 \sigma_\epsilon^2\right)^2\right) / \left(x \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + x \rho^2 \sigma_\epsilon^2\right)\right)^2$

In[30]:= `% //. x → Δ^2 (*resubstituting Δ^2*)`

Out[30]= $\left(\Delta^4 \rho^5 \sigma_X^2 \sigma_\epsilon^8 + \rho \tau \sigma_X^4 \sigma_\epsilon^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)^2\right) / \left(\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)\right)^2$

In[31]:= `D[γ Δ^2 //. Gammasol //. σX^2 → x, x] // Simplify`
`(*deriving γΔ^2 with respect to σX^2 after substitution*)`

Out[31]= $\frac{\Delta^4 \rho^3 \sigma_\epsilon^6 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)}{\left(x \tau^2 + x \Delta^2 \rho^2 \tau \sigma_\epsilon^2 + \Delta^2 \rho^2 \sigma_\epsilon^4\right)^2}$

In[32]:= `% //. x → σX^2(*resubstituting σX^2*)`

Out[32]= $\left(\Delta^4 \rho^3 \sigma_\epsilon^6 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)\right) / \left(\tau^2 \sigma_X^2 + \Delta^2 \rho^2 \tau \sigma_X^2 \sigma_\epsilon^2 + \Delta^2 \rho^2 \sigma_\epsilon^4\right)^2$

In[33]:= `D[γ Δ^2 //. Gammasol, τ] // Simplify (*deriving γΔ^2 with respect to τ*)`

Out[33]= $-\left(\Delta^2 \rho \sigma_X^2 \sigma_\epsilon^2 \left(-\Delta^2 \rho^2 \sigma_\epsilon^4 + \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)^2\right)\right) / \left(\Delta^2 \rho^2 \sigma_\epsilon^4 + \tau \sigma_X^2 \left(\tau + \Delta^2 \rho^2 \sigma_\epsilon^2\right)\right)^2$

Expected Utility of agents of type A

- **variant 1**

In[34]:= `testA1 = -ρ (w0 + p Z̄ + zi (X - p)); (*expression in the exponent of the utility function*)`

In[35]:=
$$\texttt{vecwA1} = \{z_i - ZA,\ X - p - \overline{Xmp},\ p - \overline{p}\};\ \text{(*vector for the decomposition*)}$$

In[36]:=
$$\texttt{matAA1} = \left\{\left\{0, \frac{-\rho}{2}, 0\right\}, \left\{\frac{-\rho}{2}, 0, 0\right\}, \{0, 0, 0\}\right\};\ \text{(*matrix A*)}$$

In[37]:=
$$\texttt{vecbA1} = \{-\rho\,\overline{Xmp},\ -\rho\,ZA,\ -\rho\,\overline{Z}\};\ \text{(*vector b*)}$$

In[38]:=
$$\texttt{skadA1} = -\rho\left(w_0 + \overline{p}\,\overline{Z} + ZA\,(\overline{Xmp})\right);\ \text{(*scalar d*)}$$

In[39]:=
```
testA1 - vecwA1.matAA1.vecwA1 - vecwA1.vecbA1 - skadA1 //
 Simplify (*test, whether the decomposition was correct*)
```

Out[39]= 0

In[40]:=
```
I3 = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}; (*identity matrix*)
```

In[41]:=
$$\Sigma A1 = \Bigg\{\Bigg\{\left(\frac{(\alpha_{A2}-1)\,\pi_1}{\rho\,\beta_A}\right)^2 \sigma_X^2 + \left(\frac{(\alpha_{A2}-1)\,\gamma}{\rho\,\beta_A}\right)^2 \Delta^2,$$
$$\left(\frac{(\alpha_{A2}-1)\,\pi_1}{\rho\,\beta_A}\right)(1-\pi_1)\,\sigma_X^2 - \left(\frac{(\alpha_{A2}-1)\,\gamma}{\rho\,\beta_A}\right)\gamma\,\Delta^2,\ \left(\frac{(\alpha_{A2}-1)\,\pi_1}{\rho\,\beta_A}\right)\pi_1\,\sigma_X^2 + \left(\frac{(\alpha_{A2}-1)\,\gamma}{\rho\,\beta_A}\right)\gamma\,\Delta^2\Bigg\},$$
$$\Bigg\{\left(\frac{(\alpha_{A2}-1)\,\pi_1}{\rho\,\beta_A}\right)(1-\pi_1)\,\sigma_X^2 - \left(\frac{(\alpha_{A2}-1)\,\gamma}{\rho\,\beta_A}\right)\gamma\,\Delta^2,\ (1-\pi_1)^2\,\sigma_X^2 + \gamma^2\,\Delta^2,\ (1-\pi_1)\,\pi_1\,\sigma_X^2 - \gamma^2\,\Delta^2\Bigg\},$$
$$\Bigg\{\left(\frac{(\alpha_{A2}-1)\,\pi_1}{\rho\,\beta_A}\right)\pi_1\,\sigma_X^2 + \left(\frac{(\alpha_{A2}-1)\,\gamma}{\rho\,\beta_A}\right)\gamma\,\Delta^2,\ (1-\pi_1)\,\pi_1\,\sigma_X^2 - \gamma^2\,\Delta^2,\ \pi_1^2\,\sigma_X^2 + \gamma^2\,\Delta^2\Bigg\}\Bigg\};$$

(*variance-covariance-matrix*)

In[42]:=
$$\texttt{EUA1} = \frac{-1}{\rho}\,\frac{1}{\sqrt{\texttt{Det[I3 - 2 }\Sigma\texttt{A1.matAA1]}}}$$
$$\texttt{Exp}\left[\frac{1}{2}\ \texttt{vecbA1.Inverse[I3 - 2 }\Sigma\texttt{A1.matAA1].}\Sigma\texttt{A1.vecbA1 + skadA1}\right]\ \texttt{// Simplify;}$$

(*computation of the expected utility*)

In[43]:=
$$\texttt{EUA11} = \texttt{EUA1} \mathbin{//.} \Big\{ZA \to \frac{1}{\rho\,\beta_A}\left(\alpha_{A0} + \left(-\overline{Z}\,\gamma + \pi_0 + \overline{X}\,\pi_1\right)(-1+\alpha_{A2})\right),$$
$$\overline{Xmp} \to (1-\pi_1)\,\overline{X} - \pi_0 + \gamma\,\overline{Z},\ \overline{p} \to \pi_0 + \pi_1\,\overline{X} - \gamma\,\overline{Z}\Big\}\ \texttt{// Simplify;}$$

(*substitutiong the means of z_i^A, X-p and p*)

In[44]:=
```
EUA12 = EUA11 //. {Alpha0A[[1]], Alpha2A[[1]], Alpha0B[[1]],
     Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} // Simplify;
```

(*plugging the α's and the β's in the expected utility function*)

In[45]:=
```
EUA13 = EUA12 //. {Pi0sol[[1]], Pi1sol[[1]], Gammasol[[1]]} // Simplify;
```

(*plugging π_0, π_1 and γ in the expected utility *)

In[46]:=
```
EUA14 = EUA13 //. para;
```

(*plugging the parameter example in the expected utility function*)

- **variant 2**

In[47]:=
$$\texttt{testA2} = -\rho\left(w_0 + p\,\overline{Z} + z_i\,(X-p)\right)\ \texttt{//. \{}z_i \texttt{ -> shareA\} //.}$$
```
   solPrice (*expression in the exponent of the utility function*)
```

Out[47]=
$$-\rho\left(\overline{Z}\,(-Z\,\gamma + \pi_0 + X\,\pi_1) + w_0 + \frac{1}{\rho\,\beta_A}\,(X + Z\,\gamma - \pi_0 - X\,\pi_1)\,(\alpha_{A0} + (-Z\,\gamma + \pi_0 + X\,\pi_1)\,(-1+\alpha_{A2}))\right)$$

In[48]:= `Collect[testA2, {X, Z}] (*collecting all expressions with X and Z*)`

Out[48]=
$$X\left(-Z\rho\left(-\frac{\gamma(-1+\alpha_{A2})}{\rho\,\beta_A}+\frac{2\gamma\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\rho\left(\overline{Z}\pi_1+\frac{\alpha_{A0}}{\rho\,\beta_A}-\frac{\pi_1\alpha_{A0}}{\rho\,\beta_A}+\frac{\pi_0(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{2\pi_0\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)\right)-Z\rho\left(-\gamma\overline{Z}+\frac{\gamma\alpha_{A0}}{\rho\,\beta_A}+\frac{2\gamma\pi_0(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\rho\left(\overline{Z}\pi_0+W_0-\frac{\pi_0\alpha_{A0}}{\rho\,\beta_A}-\frac{\pi_0^2(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-X^2\rho\left(\frac{\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{\pi_1^2(-1+\alpha_{A2})}{\rho\,\beta_A}\right)+\frac{Z^2\gamma^2(-1+\alpha_{A2})}{\beta_A}$$

In[49]:= `vecwA2 =` $\{X-\overline{X},\ Z-\overline{Z}\}$`; (*vector for the decomposition*)`

In[50]:=
$$\text{matAA2}=\left\{\left\{-\rho\left(\frac{\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{\pi_1^2(-1+\alpha_{A2})}{\rho\,\beta_A}\right),\ \frac{-\rho}{2}\left(-\frac{\gamma(-1+\alpha_{A2})}{\rho\,\beta_A}+\frac{2\gamma\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)\right\},\left\{\frac{-\rho}{2}\left(-\frac{\gamma(-1+\alpha_{A2})}{\rho\,\beta_A}+\frac{2\gamma\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right),\ \frac{\gamma^2(-1+\alpha_{A2})}{\beta_A}\right\}\right\};$$
`(*matrix A*)`

In[51]:=
$$\text{vecbA2}=\left\{-\rho\left(\overline{Z}\pi_1+\frac{\alpha_{A0}}{\rho\,\beta_A}-\frac{\pi_1\alpha_{A0}}{\rho\,\beta_A}+\frac{\pi_0(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{2\pi_0\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\overline{Z}\rho\left(-\frac{\gamma(-1+\alpha_{A2})}{\rho\,\beta_A}+\frac{2\gamma\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-2\overline{X}\rho\left(\frac{\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{\pi_1^2(-1+\alpha_{A2})}{\rho\,\beta_A}\right),\ -\rho\left(-\gamma\overline{Z}+\frac{\gamma\alpha_{A0}}{\rho\,\beta_A}+\frac{2\gamma\pi_0(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\overline{X}\rho\left(-\frac{\gamma(-1+\alpha_{A2})}{\rho\,\beta_A}+\frac{2\gamma\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)+\frac{2\overline{Z}\gamma^2(-1+\alpha_{A2})}{\beta_A}\right\};$$
`(*vector b*)`

In[52]:=
$$\text{skadA2}=\overline{X}\left(-\overline{Z}\rho\left(-\frac{\gamma(-1+\alpha_{A2})}{\rho\,\beta_A}+\frac{2\gamma\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\rho\left(\overline{Z}\pi_1+\frac{\alpha_{A0}}{\rho\,\beta_A}-\frac{\pi_1\alpha_{A0}}{\rho\,\beta_A}+\frac{\pi_0(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{2\pi_0\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}\right)\right)-\overline{Z}\rho\left(-\gamma\overline{Z}+\frac{\gamma\alpha_{A0}}{\rho\,\beta_A}+\frac{2\gamma\pi_0(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\rho\left(\overline{Z}\pi_0+W_0-\frac{\pi_0\alpha_{A0}}{\rho\,\beta_A}-\frac{\pi_0^2(-1+\alpha_{A2})}{\rho\,\beta_A}\right)-\overline{X}^2\rho\left(\frac{\pi_1(-1+\alpha_{A2})}{\rho\,\beta_A}-\frac{\pi_1^2(-1+\alpha_{A2})}{\rho\,\beta_A}\right)+\frac{\overline{Z}^2\gamma^2(-1+\alpha_{A2})}{\beta_A};$$
`(*scalar d*)`

In[53]:= `testA2 - vecwA2.matAA2.vecwA2 - vecwA2.vecbA2 - skadA2 // Simplify (*test, whether the decomposition was correct*)`

Out[53]= 0

In[54]:= `I2 = {{1, 0}, {0, 1}}; (*identity matrix*)`

In[55]:= $\Sigma A2 = \{\{\sigma_X^2, 0\}, \{0, \Delta^2\}\}$`; (*variance-covariance-matrix*)`

In[56]:=
$$\text{EUA2}=\frac{-1}{\rho}\ \frac{1}{\sqrt{\text{Det}[\text{I2}-2\,\Sigma\text{A2.matAA2}]}}\ \text{Exp}\left[\frac{1}{2}\ \text{vecbA2.Inverse}[\text{I2}-2\,\Sigma\text{A2.matAA2}].\Sigma\text{A2.vecbA2}+\text{skadA2}\right]\ //\ \text{Simplify};$$
`(*computation of the expected utility*)`

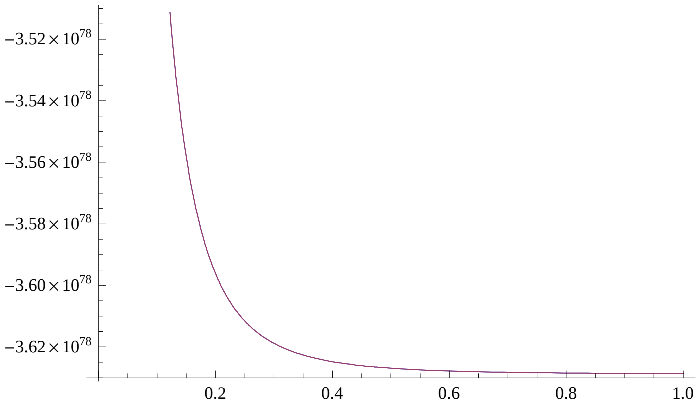
In[57]:= `EUA21 = EUA2 //. {Alpha0A[[1]], Alpha2A[[1]], Alpha0B[[1]], Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} // Simplify;`
`(*plugging the` α`'s and the` β`'s in the expected utility function*)`

In[58]:=
```
EUA22 = EUA21 //. {Pi0sol[[1]], Pi1sol[[1]], Gammasol[[1]]} // Simplify;
(*plugging π0, π1 and γ in the expected utility *)
```

In[59]:=
```
EUA23 = EUA22 //. para;
(*plugging the parameter example in the expected utility function*)
```

In[60]:=
```
Plot[{EUA14, EUA23}, {τ, 0, 1}](*mutual plot of variant 1 and variant
 2 to check whether both decompositions lead to the same result*)
```

Out[60]=



Expected Utility of agents of type B

- **variant 1**

In[61]:= $\texttt{testB1} = -\rho\left(w_0 + p\,\overline{Z} + z_i\,(X - p)\right)$; (*expression in the exponent of the utility function*)

In[62]:= $\texttt{vecwB1} = \left\{z_i - ZB,\ X - p - \overline{Xmp},\ p - \overline{p}\right\}$; (*vector for the decomposition*)

In[63]:= $\texttt{matAB1} = \left\{\left\{0, \frac{-\rho}{2}, 0\right\}, \left\{\frac{-\rho}{2}, 0, 0\right\}, \{0, 0, 0\}\right\}$; (*matrix A*)

In[64]:= $\texttt{vecbB1} = \left\{-\rho\,\overline{Xmp},\ -\rho\,ZB,\ -\rho\,\overline{Z}\right\}$; (*vector b*)

In[65]:= $\texttt{skadB1} = -\rho\left(w_0 + \overline{p}\,\overline{Z} + ZB\,(\overline{Xmp})\right)$; (*scalar d*)

In[66]:=
```
testB1 - vecwB1.matAB1.vecwB1 - vecwB1.vecbB1 - skadB1 //
 Simplify(*test, whether the decomposition was correct*)
```

Out[66]= 0

In[67]:=

$$\Sigma B1 = \left\{\left\{\left(\frac{\alpha_{B1}}{\rho\,\beta_B}\right)^2 \sigma_\epsilon^2 + \left(\frac{(\alpha_{B2}-1)\,\pi_1 + \alpha_{B1}}{\rho\,\beta_B}\right)^2 \sigma_X^2 + \left(\frac{(\alpha_{B2}-1)\,\gamma}{\rho\,\beta_B}\right)^2 \Delta^2,\ \left(\frac{(\alpha_{B2}-1)\,\pi_1 + \alpha_{B1}}{\rho\,\beta_B}\right)(1-\pi_1)\,\sigma_X^2 - \left(\frac{(\alpha_{B2}-1)\,\gamma}{\rho\,\beta_B}\right)\gamma\,\Delta^2,\ \left(\frac{(\alpha_{B2}-1)\,\pi_1 + \alpha_{B1}}{\rho\,\beta_B}\right)\pi_1\,\sigma_X^2 + \left(\frac{(\alpha_{B2}-1)\,\gamma}{\rho\,\beta_B}\right)\gamma\,\Delta^2\right\},$$

$$\left\{\left(\frac{(\alpha_{B2}-1)\,\pi_1 + \alpha_{B1}}{\rho\,\beta_B}\right)(1-\pi_1)\,\sigma_X^2 - \left(\frac{(\alpha_{B2}-1)\,\gamma}{\rho\,\beta_B}\right)\gamma\,\Delta^2,\ (1-\pi_1)^2\,\sigma_X^2 + \gamma^2\,\Delta^2,\ (1-\pi_1)\,\pi_1\,\sigma_X^2 - \gamma^2\,\Delta^2\right\},$$

$$\left\{\left(\frac{(\alpha_{B2}-1)\,\pi_1 + \alpha_{B1}}{\rho\,\beta_B}\right)\pi_1\,\sigma_X^2 + \left(\frac{(\alpha_{B2}-1)\,\gamma}{\rho\,\beta_B}\right)\gamma\,\Delta^2,\ (1-\pi_1)\,\pi_1\,\sigma_X^2 - \gamma^2\,\Delta^2,\ \pi_1^2\,\sigma_X^2 + \gamma^2\,\Delta^2\right\}\right\};$$

(*variance-covariance-matrix*)

In[68]:=

$$\texttt{EUB1} = \frac{-1}{\rho}\,\frac{1}{\sqrt{\texttt{Det[I3 - 2 ΣB1.matAB1]}}}\ \texttt{Exp}\left[\frac{1}{2}\,\texttt{vecbB1.Inverse[I3 - 2 ΣB1.matAB1].ΣB1.vecbB1 + skadB1}\right]\ \texttt{// Simplify;}$$

(*computation of the expected utility*)

In[69]:= EUB11 = EUB1 //. $\left\{ZB \to \frac{1}{\rho\,\beta_B}\left(\alpha_{B0} + \overline{X}\,\alpha_{B1} + \left(-\overline{Z}\,\gamma + \pi_0 + \overline{X}\,\pi_1\right)(-1 + \alpha_{B2})\right),\ \overline{Xmp} \to (1 - \pi_1)\,\overline{X} - \pi_0 + \gamma\,\overline{Z},\ \overline{p} \to \pi_0 + \pi_1\,\overline{X} - \gamma\,\overline{Z}\right\}$ // Simplify;
(*substitutiong the means of z_i^B, X-p and p*)

In[70]:= EUB12 = EUB11 //. {Alpha0A[[1]], Alpha2A[[1]], Alpha0B[[1]], Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} // Simplify;
(*plugging the α's and the β's in the expected utility function*)

In[71]:= EUB13 = EUB12 //. {Pi0sol[[1]], Pi1sol[[1]], Gammasol[[1]]} // Simplify;
(*plugging π_0, π_1 and γ in the expected utility *)

In[72]:= EUB14 = EUB13 //. para // Simplify;
(*plugging the parameter example in the expected utility function*)

- **variant 2**

In[73]:= testB2 = $-\rho\left(w_0 + p\,\overline{Z} + z_i\,(X - p)\right)$ //. $z_i \to$ shareB //. solPrice (*expression in the exponent of the utility function*)

Out[73]= $$-\rho\left(\overline{Z}\,(-Z\,\gamma + \pi_0 + X\,\pi_1) + w_0 + \frac{1}{\rho\,\beta_B}(X + Z\,\gamma - \pi_0 - X\,\pi_1)\,\left(\alpha_{B0} + (-Z\,\gamma + \pi_0 + X\,\pi_1)\,(-1 + \alpha_{B2}) + \alpha_{B1}\,(X + \epsilon_i)\right)\right)$$

In[74]:= Collect[%, {X, Z, ϵ_i}] (*collecting all expressions with X, Z and ϵ_i*)

Out[74]= $$-\rho\left(\overline{Z}\,\pi_0 + w_0 - \frac{\pi_0\,\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_0^2\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - X^2\,\rho\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B} - \frac{\pi_1^2\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) + \frac{Z^2\,\gamma^2\,(-1 + \alpha_{B2})}{\beta_B} + \frac{\pi_0\,\alpha_{B1}\,\epsilon_i}{\beta_B} + X\left(-Z\,\rho\left(\frac{\gamma\,\alpha_{B1}}{\rho\,\beta_B} - \frac{\gamma\,(-1 + \alpha_{B2})}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - \rho\left(\overline{Z}\,\pi_1 + \frac{\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_0\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_0\,(-1 + \alpha_{B2})}{\rho\,\beta_B} - \frac{2\,\pi_0\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - \rho\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B}\right)\epsilon_i\right) + Z\left(-\rho\left(-\gamma\,\overline{Z} + \frac{\gamma\,\alpha_{B0}}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_0\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - \frac{\gamma\,\alpha_{B1}\,\epsilon_i}{\beta_B}\right)$$

In[75]:= vecwB2 = $\{X - \overline{X},\ Z - \overline{Z},\ \epsilon_i\}$; (*vector for the decomposition*)

In[76]:= matAB2 = $$\left\{\left\{-\rho\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B} - \frac{\pi_1^2\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right),\ \frac{-\rho}{2}\left(\frac{\gamma\,\alpha_{B1}}{\rho\,\beta_B} - \frac{\gamma\,(-1 + \alpha_{B2})}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right),\ \frac{-\rho}{2}\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B}\right)\right\},\right.$$
$$\left\{\frac{-\rho}{2}\left(\frac{\gamma\,\alpha_{B1}}{\rho\,\beta_B} - \frac{\gamma\,(-1 + \alpha_{B2})}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right),\ \frac{\gamma^2\,(-1 + \alpha_{B2})}{\beta_B},\ \frac{-1}{2}\,\frac{\gamma\,\alpha_{B1}}{\beta_B}\right\},$$
$$\left.\left\{\frac{-\rho}{2}\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B}\right),\ \frac{-1}{2}\,\frac{\gamma\,\alpha_{B1}}{\beta_B},\ 0\right\}\right\};$$ (*matrix A*)

In[77]:= vecbB2 = $$\left\{-2\,\overline{X}\,\rho\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B} - \frac{\pi_1^2\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - \overline{Z}\,\rho\left(\frac{\gamma\,\alpha_{B1}}{\rho\,\beta_B} - \frac{\gamma\,(-1 + \alpha_{B2})}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - \rho\left(\overline{Z}\,\pi_1 + \frac{\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_0\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_0\,(-1 + \alpha_{B2})}{\rho\,\beta_B} - \frac{2\,\pi_0\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right),\right.$$
$$\frac{2\,\overline{Z}\,\gamma^2\,(-1 + \alpha_{B2})}{\beta_B} - \overline{X}\,\rho\left(\frac{\gamma\,\alpha_{B1}}{\rho\,\beta_B} - \frac{\gamma\,(-1 + \alpha_{B2})}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_1\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right) - \rho\left(-\gamma\,\overline{Z} + \frac{\gamma\,\alpha_{B0}}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_0\,(-1 + \alpha_{B2})}{\rho\,\beta_B}\right),$$
$$\left.\frac{\pi_0\,\alpha_{B1}}{\beta_B} - \rho\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B}\right)\overline{X} - \frac{\gamma\,\alpha_{B1}}{\beta_B}\,\overline{Z}\right\};$$ (*vector b*)

In[78]:= skadB2 = $-\rho\left(\overline{Z}\,\pi_0 + w_0 - \frac{\pi_0\,\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_0^2\,(-1+\alpha_{B2})}{\rho\,\beta_B}\right) - \overline{X}^2\,\rho\left(\frac{\alpha_{B1}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_1\,(-1+\alpha_{B2})}{\rho\,\beta_B} - \frac{\pi_1^2\,(-1+\alpha_{B2})}{\rho\,\beta_B}\right) + \frac{\overline{Z}^2\,\gamma^2\,(-1+\alpha_{B2})}{\beta_B} + \overline{X}\left(-\overline{Z}\,\rho\left(\frac{\gamma\,\alpha_{B1}}{\rho\,\beta_B} - \frac{\gamma\,(-1+\alpha_{B2})}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_1\,(-1+\alpha_{B2})}{\rho\,\beta_B}\right) - \rho\left(\overline{Z}\,\pi_1 + \frac{\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_1\,\alpha_{B0}}{\rho\,\beta_B} - \frac{\pi_0\,\alpha_{B1}}{\rho\,\beta_B} + \frac{\pi_0\,(-1+\alpha_{B2})}{\rho\,\beta_B} - \frac{2\,\pi_0\,\pi_1\,(-1+\alpha_{B2})}{\rho\,\beta_B}\right)\right) + \overline{Z}\left(-\rho\left(-\gamma\,\overline{Z} + \frac{\gamma\,\alpha_{B0}}{\rho\,\beta_B} + \frac{2\,\gamma\,\pi_0\,(-1+\alpha_{B2})}{\rho\,\beta_B}\right)\right)$; (*scalar d*)

In[79]:= testB2 - vecwB2.matAB2.vecwB2 - vecwB2.vecbB2 - skadB2 // Simplify(*test, whether the decomposition was correct*)

Out[79]= 0

In[80]:= ΣB2 = $\{\{\sigma_X{}^2, 0, 0\}, \{0, \Delta^2, 0\}, \{0, 0, \sigma_\epsilon{}^2\}\}$; (*variance-covariance-matrix*)

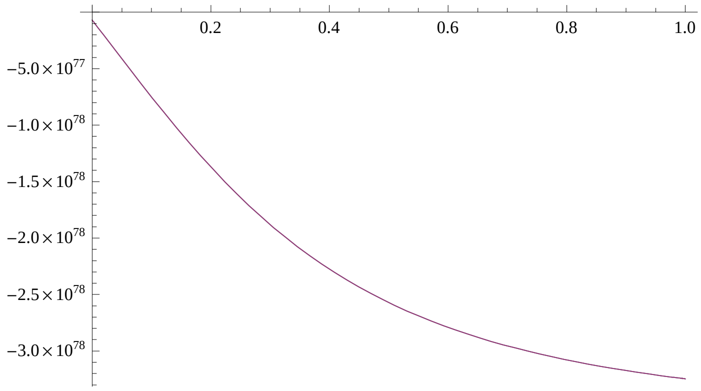
In[81]:= EUB2 = $\frac{-1}{\rho}\,\frac{1}{\sqrt{\text{Det[I3 - 2 ΣB2.matAB2]}}}$ Exp[$\frac{1}{2}$ vecbB2.Inverse[I3 - 2 ΣB2.matAB2].ΣB2.vecbB2 + skadB2] // Simplify;
(*computation of the expected utility*)

In[82]:= EUB21 = EUB2 //. {Alpha0A[[1]], Alpha2A[[1]], Alpha0B[[1]], Alpha1B[[1]], Alpha2B[[1]], betaA[[1]], betaB [[1]]} // Simplify;
(*plugging the α's and the β's in the expected utility function*)

In[83]:= EUB22 = EUB21 //. {Pi0sol[[1]], Pi1sol[[1]], Gammasol[[1]]} // Simplify;
(*plugging π_0, π_1 and γ in the expected utility *)

In[84]:= EUB23 = EUB22 //. para // Simplify;
(*plugging the parameter example in the expected utility function*)

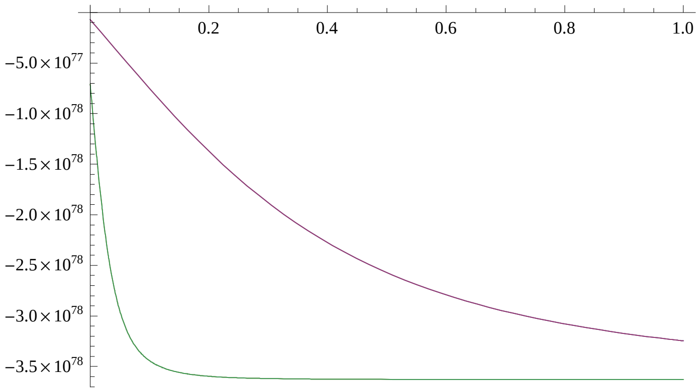
In[85]:= Plot[{EUB14, EUB23}, {τ, 0, 1}](*mutual plot of variant 1 and variant 2 to check whether both decompositions lead to the same result*)

Out[85]=


Mutual Plot

In[86]:=
```
Plot[{EUB14, EUB23, EUA14, EUA23}, {τ, 0, 1}]
(*mutual plot of type-A and type-B agents' expected utility*)
```

Out[86]=



3 Positive Feedback Traders

This section presents an extended version of Arnold & Brunner (2012). As mentioned in the introduction, the noise traders we consider here are positive feedback traders. Recall that these are agents whose demand depends on past price changes. At first sight one might think that the presence of rational speculators in markets with positive feedback traders eliminates arbitrage opportunities and therefore stabilizes prices. De Long et al. (1990) present a model with positive feedback traders where the presence of utility maximizing rational speculators rather destabilizes asset prices. The reason why they destabilize the prices is because they anticipate the behavior of noise traders and therefore drive prices to a higher or lower level. When arbitrageurs receive a positive signal, they demand the asset. This demand results in a price increase. Rational speculators anticipate that positive feedback traders will have a positive demand tomorrow so they drive the price to an even higher level. When positive feedback traders demand the asset tomorrow, rational speculators sell their holdings. This effect disappears in the absence of rational speculators.

We consider a generalized version of the DSSW model with a noiseless signal. We have additional trading dates and a second (also noiseless) informative signal. This allows us to check how the results in DSSW depend on the timing (how many trading dates; when does information arrive).

In section 3.1 we present the model setup. Section 3.2 derives the solution of the rational speculators' maximization problem. Section 3.3 describes the case with one signal and the different results that come from different time setups. In section 3.4 we consider the case with two signals and examine the different price reactions.

3.1 Model

Following De Long et al. (1990), we have three groups of agents in our model: positive feedback traders (measure one), rational speculators (measure μ, where $0 \leq \mu \leq 1$) and passive investors (measure $1-\mu$). To distinguish variables referring to positive feedback traders, rational speculators and passive investors, we use the superscripts f,r and e, respectively.

There are $T+2$ (where $T \geq 2$) dates, starting with period 0 and ending with period $T+1$.

In the model we have one consumption good, which agents consume at $T+1$ and two assets, where one is risky and one is safe. The safe asset is in perfect elastic supply and has zero net return. The supply of the risky asset is $S \geq 0$, which is exogenously given. The return on the risky asset at $T+1$ is

$$v + \Phi + \theta,$$

where $v \geq 0$ is certain and Φ and θ are random variables. The first random variable Φ is the sum of two i.i.d. shocks ϕ' and ϕ''. Both shocks have mean zero, a finite variance and a symmetric density. Since we also want to analyze the case where investors receive only one signal, we allow for $\sigma^2_{\phi'} = 0$. In contrast to DSSW we only concentrate on the case with noiseless signals, which means that rational speculators learn about the realizations of ϕ' and ϕ'' at $t^{r'}$ and $t^{r''}$, respectively.[1] We assume $1 \leq t^{r'} < t^{r''} \leq T$. Passive investors observe the realizations of ϕ' and ϕ'' at $t^{e'}$ and $t^{e''}$. Since we do not want that passive investors know the realization of ϕ' and ϕ'' before rational speculators do, we assume that $t^{r'} < t^{e'}$ and $t^{r''} < t^{e''} \leq T$. The second random variable θ also has mean zero and a finite variance.[2] θ is realized in the last period and none of the agents

[1] DSSW consider the two cases where the signal is noiseless and where it has some noise. Rational speculators receive a noisy signal about the realization of Φ. After they have received this signal they can exclude certain realizations of Φ but they do not know exactly which state will be realized. Using a noisy signal may be an objective for future research.

[2] In their paper, DSSW assume that θ is normal. They need this assumption to transform their CARA utility function into a mean-variance utility function. We drop this assumption and assume that rational speculators maximize a mean-variance utility function.

receives a signal about it before $T+1$. This ensures that the price in the last period has an uncertain component and therefore rational agents have a finite demand.

The price of the asset at date $t = 0, 1, \ldots, T{+}1$ is p_t and the demand of the positive feedback traders, the rational speculators and the passive investors at date $t = 0, 1, \ldots, T$ are denoted D_t^f, D_t^r and D_t^e, respectively. The demand of the positive feedback traders consists of two parts. We assume that if the price is constant their demand is equal to the supply of the risky asset S. After price changes in the past, they choose their demand depending on whether the price has risen (higher demand) or fallen (lower demand). Current price changes have no impact on the demand of positive feedback traders.

$$D_t^f = \begin{cases} S & t = 0, 1, \\ S + \sum_{l=1}^{t-1} \beta_l \Delta p_{t-l} & t = 2, \ldots, T, \end{cases} \tag{3.1}$$

where $\Delta p_t = p_t - p_{t-1}$ and $\beta_l \geq 0$ for $l = 1, 2, \ldots, T-1$.

The demand of the passive investors is driven by the difference of the current price and their expectations of the fundamental value. In other words, if the asset is too cheap from their point of view they buy the asset, if it is too expensive they sell it. Note that they only use their signals to form their expectation so

$$D_t^e = \begin{cases} \alpha(v - p_t) & 0 \leq t < t^{e'} \\ \alpha(v + \phi' - p_t) & t^{e'} \leq t < t^{e''} \\ \alpha(v + \phi' + \phi'' - p_t) & t^{e''} \leq t \leq T, \end{cases} \tag{3.2}$$

where α is a constant. Since we want that the demand functions of the passive investors and the rational speculators are the same at $T+1$, we set $\alpha = \frac{1}{2\gamma\sigma_\theta^2}$.[3]

The third group are the rational speculators. Their preferences are given by the mean-variance-utility function $\mu_W - \gamma\sigma_W^2$, where $\gamma > 0$ is a risk aversion coefficient and μ_W and σ_W^2 are the mean and the variance of the final wealth.

Definition 1. *Prices p_t $(t = 0, 1, \ldots, T)$ and demands D_t^f $(t = 0, 1, \ldots, T)$, D_t^r, and D_t^e $(t = 0, 1, \ldots, T)$ are an* equilibrium *if*

[3] We will see in section 3.2 that passive investors' demand then coincides with rational speculators' demand at date T.

- D_t^f *satisfies (3.1),*
- D_t^e *satisfies (3.2),*
- $D_t^r \quad (t = 0, 1, \ldots, T)$ *is the time-consistent solution of the mean-variance utility maximization problem, given current information, and*
- *the market for the risky asset clears at each date so*

$$D_t^f + \mu D_t^r + (1-\mu) D_t^e = S, \qquad t = 0, 1, \ldots, T. \tag{3.3}$$

We consider equilibria where the asset prices are linear functions of the realized shocks with intercept v for $\mu > 0$:

$$p_t = \begin{cases} v & 0 \leq t < t^{r'} \\ v + (1+\nu')\phi' & t^{r'} \leq t < t^{r''} \\ v + (1+\lambda')\phi' + (1+\lambda'')\phi'' & t^{r''} \leq t < T. \end{cases} \tag{3.4}$$

The parameters ν', λ' and λ'' will be specified later. Considering linear price functions makes it easier for us to solve the rational speculators' utility maximization problem in section 3.2 and therefore characterizing the equilibrium. However, we will show later that the equilibrium is unique (section 3.4.7). Using the parameters ν', λ' and λ'' we can define what it means for the price to under- and to overreact.

Definition 2. *If* $\nu' > 0$ $(\nu' < 0)$ *at* $t^{r'}$ *or* $\lambda' > 0$ $(\lambda' < 0)$ *at* $t^{r''}$*, then the price* overreacts (underreacts) *to the date-*$t^{r'}$ *signal. It* overreacts (underreacts) *to the date-*$t^{r''}$ *signal if* $\lambda'' > 0$ $(\lambda'' < 0)$.

Note, that the price reaction to contemporary shocks is not predictable, however the price change from $t^{r''} - 1$ to $t^{r''}$

$$p_{t^{r''}} - p_{t^{r''}-1} = (\lambda' - \nu')\phi' + (1+\lambda')\phi'' \tag{3.5}$$

contains a predictable component $(\lambda' - \nu')\phi'$ for rational speculators who observe ϕ' in $t^{r'}$.

3.2 Investment behavior

As mentioned above, rational speculators choose their demand by maximizing their mean-variance utility function. Let W_t^r denote the rational speculators' wealth and C_t^r denote their investment in the safe asset in period $t = 0, \ldots, T$. Therefore their date-t wealth is $W_t^r = C_t^r + p_t D_{t-1}^r$. So rational speculators' final wealth

$$W_{T+1}^r = C_{t-1}^r - \sum_{\tau=t}^{T} p_\tau \left(D_\tau^r - D_{\tau-1}^r\right) + (v + \Phi + \theta) D_T^r, \quad t = 1, \ldots, T-1$$

is the current holding of the safe asset plus the gains from trading the risky asset between t and T and the return from clearing their position at date T. Rearranging and using $W_t^r = C_t^r + p_t D_{t-1}^r$ yields

$$\begin{aligned} W_{T+1}^r &= C_{t-1}^r - \sum_{\tau=t}^{T} p_\tau \left(D_\tau^r - D_{\tau-1}^r\right) + (v + \Phi + \theta) D_T^r \\ &= C_{t-1}^r + p_t D_{t-1}^r + \sum_{t'=t}^{T-1} \left(p_{t'+1} - p_{t'}\right) D_{t'}^r + \left(v + \phi' + \phi'' + \theta - p_T\right) D_T^r \\ &= W_t^r + \sum_{t'=t}^{T-1} \left(p_{t'+1} - p_{t'}\right) D_{t'}^r + \left(v + \phi' + \phi'' + \theta - p_T\right) D_T^r, \end{aligned} \tag{3.6}$$

for $t' = 0, 1, \ldots, T-1$. We solve this maximization problem recursively.

Proposition 1. *The demand of rational speculators at date T is*

$$D_T^r = \frac{v + \phi' + \phi'' - p_T}{2\gamma\sigma_\theta^2} = \alpha(v + \phi' + \phi'' - p_T). \tag{3.7}$$

Proof. The final wealth in the last period can be written as

$$W_{T+1}^r = W_T^r + (v + \phi' + \phi'' + \theta - p_T) D_T^r.$$

Since the realizations of ϕ' and ϕ'' are already known at T, the mean is

$$E_T^r W_{T+1}^r = W_T^r + (v + \phi' + \phi'' - p_T) D_T^r$$

and the variance is

$$\sigma_{W_{T+1|T}}^2 = E_T^r (W_{T+1} - E_T^r W_{T+1}^r)^2 = \sigma_\theta^2 (D_T^r)^2.$$

So the mean-variance utility function of the rational speculators is

$$U = W_T^r + (v + \phi' + \phi'' - p_T)D_T^r - \gamma\sigma_\theta^2 (D_T^r)^2.$$

The first order condition for the maximization problem is

$$\frac{\partial U}{\partial D_T^r} = (v + \phi' + \phi'' - p_T) - 2\gamma\sigma_\theta^2 D_T^r = 0.$$

Solving for D_T^r yields

$$D_T^r = \frac{v + \phi' + \phi'' - p_T}{2\gamma\sigma_\theta^2} = \alpha(v + \phi' + \phi'' - p_T).$$

□

Since we defined $\alpha = \frac{1}{2\gamma\sigma_\theta^2}$ before, the utility functions of the rational speculators and the passive investors in the last period are the same. Note, that the date-T demand of rational speculators is bounded since we have fundamental risk at $T+1$ via θ.
Before solving for the demand at other dates, we recall (3.4). Since the prices are the constant when no new information arrives, the final wealth (3.6) does not depend on the demand at these dates. Therefore we only solve for $D_{t^{r'}-1}^r$ and $D_{t^{r''}-1}^r$.

Proposition 2. *The demand functions of a rational speculator at $t^{r'} - 1$ and $t^{r''} - 1$ are given by*

$$D_{t^{r'}-1}^r = \frac{v - p_{t^{r'}-1}}{2\gamma(1+\nu')^2\sigma_{\phi'}^2} \tag{3.8}$$

if $\sigma_{\phi'}^2 > 0$, and,

$$D_{t^{r''}-1}^r = \frac{v + (1+\lambda')\phi' - p_{t^{r''}-1}}{2\gamma(1+\lambda'')^2\sigma_{\phi''}^2} - \frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}. \tag{3.9}$$

Proof. Using (3.4) for $t = T$, (3.6) for $t = t^{r''} - 1$ and (3.7), we get

$$W_{T+1}^r = W_{t^{r''}-1}^r + (p_{t^{r''}} - p_{t^{r''}-1})D_{t^{r''}-1}^r + \alpha(\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'').$$

Since θ is independent, the expected value of the rational speculators given the information in $t^{r''} - 1$ is

$$E_{t^{r''}-1}^r W_{T+1}^r = W_{t^{r''}-1}^r + (E_{t^{r''}-1}^r p_{t^{r''}} - p_{t^{r''}-1})D_{t^{r''}-1}^r + \alpha(\lambda'^2\phi'^2 + \lambda''^2\sigma_{\phi''}^2).$$

The conditional variance $\sigma^2_{W_{T+1}|t^{r''}-1} = E^r_{t^{r''}-1}\left[\left(W_{T+1} - E^r_{t^{r''}-1}W_{T+1}\right)^2\right]$ is

$$\begin{aligned}
\sigma^2_{W_{T+1}|t^{r''}-1} &= E^r_{t^{r''}-1}\Big[\Big((p_{t^{r''}} - E^r_{t^{r''}-1}p_{t^{r''}})D^r_{t^{r''}-1} \\
&\quad +\alpha((\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'') - (\lambda'^2\phi'^2 + \lambda''^2\sigma^2_{\phi''}))\Big)^2\Big] \\
&= E^r_{t^{r''}-1}\left[(p_{t^{r''}} - E^r_{t^{r''}-1}p_{t^{r''}})^2 D^{r2}_{t^{r''}-1}\right] \\
&\quad +2\alpha E^r_{t^{r''}-1}\Big[(p_{t^{r''}} - E^r_{t^{r''}-1}p_{t^{r''}}) \\
&\qquad \left((\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'') - (\lambda'^2\phi'^2 + \lambda''^2\sigma^2_{\phi''})\right)\Big] D^r_{t^{r''}-1} \\
&\quad +E^r_{t^{r''}-1}\left[\left((\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'') - (\lambda'^2\phi'^2 + \lambda''^2\sigma^2_{\phi''})\right)^2\right]
\end{aligned}$$

The first summand is $(1+\lambda'')^2\sigma^2_{\phi''}D^{r2}_{t^{r''}-1}$. Expanding the second summand, we get

$$\begin{aligned}
& 2\alpha E^r_{t^{r''}-1}\left[(1+\lambda'')\phi'' D^r_{t^{r''}-1}(\lambda''^2\phi''^2 + 2\lambda'\lambda''\phi'\phi'' - \theta\lambda'\phi' - \theta\lambda''\phi'' - \lambda''^2\sigma^2_{\phi''})\right] \\
= {} & 2\alpha D^r_{t^{r''}-1}\Big[E^r_{t^{r''}-1}\left((1+\lambda'')\lambda''^2\phi''^3\right) + E^r_{t^{r''}-1}\left((1+\lambda'')\phi''^2 2\lambda'\phi'\lambda''\right) \\
& - E^r_{t^{r''}-1}\left((1+\lambda'')\phi''\theta\lambda'\phi'\right) - E^r_{t^{r''}-1}\left((1+\lambda'')\theta\lambda''\phi''^2\right) \\
& - E^r_{t^{r''}-1}\left((1+\lambda'')\phi''\lambda''\sigma^2_{\phi''}\right)\Big] \\
= {} & 4\alpha(1+\lambda'')\lambda'\lambda''\phi'\sigma^2_{\phi''}D^r_{t^{r''}-1}.
\end{aligned}$$

All other expectations cancel out since ϕ'' has mean zero ($E[\phi''] = 0$) and is symmetric ($E[\phi''^3] = 0$) and θ has also mean zero and is not correlated with ϕ'' ($E[\theta\phi''] = 0$). The third summand of the conditional variance does not depend on $D^r_{t^{r''}-1}$, so when we derive the optimal demand, it cancels out.

The mean-variance utility function without constants is

$$(E^r_{t^{r''}-1}p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} - \gamma\left[(1+\lambda'')^2\sigma^2_{\phi''}D^{r2}_{t^{r''}-1} + 4\alpha(1+\lambda'')\lambda'\lambda''\phi'\sigma^2_{\phi''}D^r_{t^{r''}-1}\right].$$

The first order condition for maximization at $t^{r''}+1$ is

$$(E^r_{t^{r''}-1}p_{t^{r''}} - p_{t^{r''}-1}) - \gamma\left[[(1+\lambda'')^2\sigma^2_{\phi''}2D^r_{t^{r''}-1} + 4\alpha(1+\lambda'')\lambda'\lambda''\phi'\sigma^2_{t^{r''}-1}\right] = 0,$$

and solving for $D^r_{t^{r''}-1}$ yields (3.9).

We solve for $D^r_{t^{r'}-1}$ analogously. Consider final wealth

$$W^r_{T+1} = W^r_{t^{r'}-1} + (p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1} + (p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} + (v + \phi' + \phi'' + \theta - p_T)D^r_T.$$

From (3.4) we have

$$p_{t^{r''}} - p_{t^{r''}-1} = (\lambda' - \nu')\phi' + (1+\lambda'')\phi'',$$

and using (3.9) we get

$$W^r_{T+1} = W^r_{t^{r'}-1} + (p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1} + \Gamma(\phi', \phi'', \theta)$$

where we define for simplicity

$$\begin{aligned}\Gamma(\phi', \phi'', \theta) &= [(\lambda' - \nu')\phi' + (1+\lambda'')\phi''] \left[\frac{\lambda' - \nu'}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''}{1+\lambda''}\right]\phi' \\ &\quad +\alpha(\lambda'\phi' + \lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'').\end{aligned}$$

The expectation of the final wealth in $t^{r'} - 1$ is

$$E^r_{t^{r'}-1}W^r_{T+1} = W^r_{t^{r'}-1} + (E^r_{t^{r'}-1}p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1} + E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)$$

and the conditional variance is

$$\begin{aligned}\sigma^2_{W_{T+1}|t^{r'}-1} &= E^r_{t^{r'}-1}\left[\left(W^r_{T+1} - E^r_{t^{r'}-1}W^r_{T+1}\right)^2\right] \\ &= E^r_{t^{r'}-1}\Big[\Big(W^r_{t^{r'}-1} + (p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1} + \Gamma(\phi', \phi'', \theta) \\ &\quad - W^r_{t^{r'}-1} - (E^r_{t^{r'}-1}p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1} - E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)\Big)^2\Big] \\ &= E^r_{t^{r'}-1}\left[\left(p_{t^{r'}} - E^r_{t^{r'}-1}p_{t^{r'}}\right)^2 D^{r2}_{t^{r'}-1}\right] \\ &\quad +2E^r_{t^{r'}-1}\left[\left(p_{t^{r'}} - E^r_{t^{r'}-1}p_{t^{r'}}\right) D^r_{t^{r'}-1}\left(\Gamma(\phi', \phi'', \theta) - E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)\right)\right] \\ &\quad +E^r_{t^{r'}-1}\left[\left(\Gamma(\phi', \phi'', \theta) - E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)\right)^2\right] \\ &= (1+\nu')^2\sigma^2_{\phi'}D^{r2}_{t^{r'}-1} + E^r_{t^{r'}-1}\left[\left(\Gamma(\phi', \phi'', \theta) - E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)\right)^2\right] \\ &\quad +2E^r_{t^{r'}-1}\left[(1+\nu')\phi'\left(\Gamma(\phi', \phi'', \theta) - E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)\right)\right] D^r_{t^{r'}-1}.\end{aligned}$$

Since ϕ' has mean zero, $E^t_{t^{r'}-1}\left[\phi' E^t_{t^{r'}-1}\Gamma(\phi', \phi'', \theta)\right] = 0$, so the third summand becomes

$$2(1+\nu')E^r_{t^{r'}-1}\left[\Gamma(\phi', \phi'', \theta)\phi'\right] D^r_{t^{r'}-1}.$$

Substituting for $\Gamma(\phi', \phi'', \theta)$, we get

$$2(1+\nu')D^r_{t^{r'}-1}\left[\frac{\lambda'-\nu'}{1+\lambda''}\left(\frac{\lambda'-\nu'}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - 2\alpha\lambda'\lambda''\right)E^r_{t^{r'}-1}(\phi'^3)\right.$$
$$+\left(\frac{\lambda'-\nu'}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - 2\alpha\lambda'\lambda''\right)E^r_{t^{r'}-1}(\phi'^2\phi'') + \alpha\lambda'^2 E^r_{t^{r'}-1}(\phi'^3)$$
$$\left.+\,2\alpha\lambda'\lambda'' E^r_{t^{r'}-1}(\phi'^2\phi'') + \alpha\lambda''^2 E^r_{t^{r'}-1}(\phi'\phi''^2) - \alpha\lambda' E^r_{t^{r'}-1}(\phi'^2\theta) - \alpha\lambda'' E^r_{t^{r'}-1}(\phi'\phi''\theta)\right].$$

All expected values are zero because of symmetry and independence. Therefore the mean-variance utility function (without constants) is

$$(v - p_{t^{r'}-1})D^r_{t^{r'}-1} - \gamma(1+\nu')^2\sigma^2_{\phi'}D^{r2}_{t^{r'}-1}.$$

Taking the first order condition

$$(v - p_{t^{r'}-1}) - 2\gamma(1+\nu')^2\sigma^2_{\phi'}D^r_{t^{r'}-1} = 0$$

and solving for $D^r_{t^{r'}-1}$ yields (3.8). □

Equation (3.8) and the first summand of equation (3.9) are the expected price changes at $t^{r'}$ and $t^{r''}$ respectively, divided by 2γ times the price variance. The second summand of equation (3.9) comes from the fact, that there is positive correlation between the returns on investment at dates $t^{r''}-1$ and T. The returns at these dates are

$$\begin{aligned}(p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} &= \left(v + (1+\lambda')\phi' + (1+\lambda'')\phi'' - v - (1+\nu')\phi'\right)D^r_{t^{r''}-1}\\ &= \left((\lambda'-\nu')\phi' + (1+\lambda'')\phi''\right)D^r_{t^{r''}-1}\end{aligned}$$

and

$$\begin{aligned}(v+\phi'+\phi''+\theta-p_T)D^r_T &= \alpha(v+\phi'+\phi''-v-(1+\lambda')\phi'-(1+\lambda'')\phi'')\\ &\quad (v+\phi'+\phi''-v-(1+\lambda')\phi'-(1+\lambda'')\phi''+\theta)\\ &= \alpha(\lambda'\phi'+\lambda''\phi''-\theta)(\lambda'\phi'+\lambda''\phi'')\\ &= \alpha\left((\lambda'\phi')^2 + (\lambda''\phi'')^2 + 2\lambda'\lambda''\phi'\phi'' - \theta\lambda'\phi' - \theta\lambda''\phi''\right),\end{aligned}$$

respectively. Let us assume that $\lambda' > 0$, $\lambda'' > 0$, $\phi' > 0$ and $\phi'' > -\frac{\lambda'\phi'}{\lambda''}$.[4] Then an increase in ϕ'' increases both, returns $\left((\lambda'-\nu')\phi' + (1+\lambda'')\phi''\right)D^r_{t^{r''}-1}$ and $\alpha(\lambda'\phi' +$

[4]If this holds, the expected payoff $\lambda'\phi' + \lambda''\phi''$ is positive.

$\lambda''\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'')$. To hedge against this covariance effect, rational speculators reduce their demand even though ϕ' and therefore the predictable part of (3.5) is positive. This may happen when the variance of the second signal becomes too large (see section 3.4).

We can also show analytically that the second term represents the covariance between the returns at $t^{r''}$ and T. Let $r(t)$ denote the return at date t. Let us consider again the mean-variance utility function $\mu_W - \gamma\sigma_W^2$. Since we consider the maximization problem at $t^{r''} - 1$, the only stochastic components in W_{T+1} are the returns at $t^{r''} - 1$ and T. Therefore we can rewrite the mean-variance utility function as

$$\begin{aligned}\mu_W - \gamma\sigma_W^2 &= \mu_W - \gamma\sigma^2_{r(t^{r''}-1)+r(T)} \\ &= \mu_W - \gamma\left(\sigma^2_{r(t^{r''}-1)} + 2\sigma_{r(t^{r''}-1),r(T)} + \sigma^2_{r(T)}\right),\end{aligned}$$

where $\sigma_{r(t^{r''}-1),r(T)}$ is the covariance of the returns on the investment at $t^{r''} - 1$ and T.[5] Note, that $\sigma^2_{r(T)}$ cancels out, when we derive with respect to $D^r_{t^{r''}-1}$. Now, let us calculate the date-$t^{r''} - 1$ covariance $\sigma_{r(t^{r''}-1),r(T)}$:[6]

$$\begin{aligned}\sigma_{r(t^{r''}-1),r(T)} &= \operatorname{Cov}\Big(\left[(\lambda' - \nu')\phi' + (1+\lambda'')\phi''\right] D^r_{t^{r''}-1}, \\ &\qquad \alpha\left((\lambda'\phi')^2 + (\lambda''\phi'')^2 + 2\lambda'\lambda''\phi'\phi'' - \theta\lambda'\phi' - \theta\lambda''\phi''\right)\Big) \\ &= (1+\lambda'')\alpha\lambda''^2 \operatorname{Cov}(\phi'', \phi''^2) D^r_{t^{r''}-1} \\ &\quad + (1+\lambda'')2\alpha\lambda'\lambda''\phi' \operatorname{Cov}(\phi'', \phi'') D^r_{t^{r''}-1} \\ &\quad - (1+\lambda'')\alpha\lambda'\phi' \operatorname{Cov}(\phi'', \theta) D^r_{t^{r''}-1} - (1+\lambda'')\alpha\lambda'' \operatorname{Cov}(\phi'', \theta\phi'') D^r_{t^{r''}-1} \\ &= 2\alpha\lambda'\lambda''(1+\lambda'')\phi'\sigma^2_{\phi''} D^r_{t^{r''}-1},\end{aligned}$$

since λ', λ'' and ϕ' are known at $t^{r''} - 1$, θ is independent and $\operatorname{Cov}(\phi'', \phi''^2) = E(\phi''^3) - E(\phi'')E(\phi''^2) = 0$. Since we derived the mean-variance utility function with respect to the demand, $D^r_{t^{r''}-1}$ cancels out in the covariance and when solving for $D^r_{t^{r''}-1}$ we divide through $2\gamma(1+\lambda'')\sigma^2_{\phi''}$. This yields

$$-\gamma\frac{\partial 2\sigma_{r(t^{r''}-1),r(T)}}{\partial D^r_{t^{r''}-1}} \frac{1}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} = -\frac{4\gamma\alpha(1+\lambda'')\lambda'\lambda''\phi'\sigma^1_{\phi''}}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} = -\frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''},$$

[5] Let X, Y be two random variables. Then $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + 2\operatorname{Cov}(X,Y) + \operatorname{Var}(Y)$.

[6] Let X, Y, Z be random variables and $a, b, c \in \mathbb{R}$. Then Cov(aX,bY+cZ) $= ab\operatorname{Cov}(X,Y) + ac\operatorname{Cov}(X,Z)$.

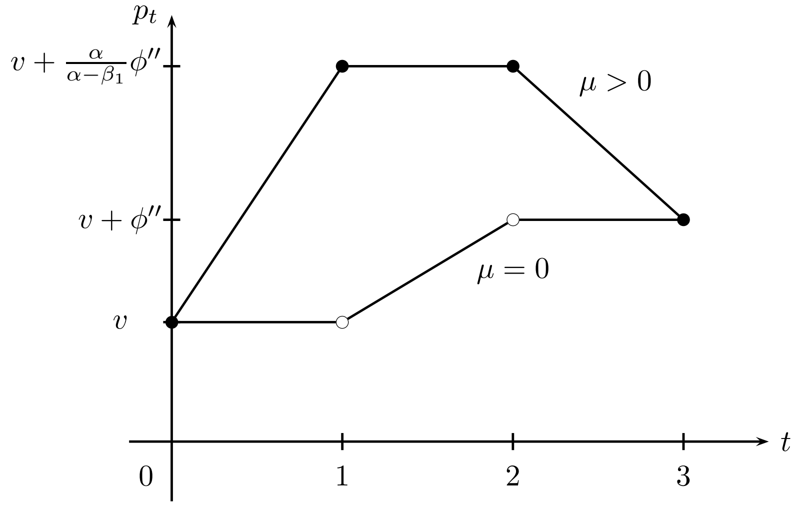


Figure 3.1: Destabilizing speculation

which coincides with the second term on the right-hand side in equation (3.9).
In the case of one signal, the second term in (3.9) cancels out and we have again the expected price change over 2γ times the variance.

3.3 Case with one signal

Now, we want to study if and to what extent the time structure of the model has an impact on the asset price. Before considering the model with two signals we first turn to the case with one signal. We therefore set $\sigma^2_{\phi'} = 0$ so that $\phi' = 0$.

3.3.1 Destabilizing rational speculation (the DSSW Model)

We show, that the DSSW model is a special case of our setup. As in DSSW, we consider the cases where $\mu = 0$ and $\mu > 0$ (absence and presence of rational speculators). A difference to DSSW is that we allow for $v \geq 0, S \geq 0$ and θ non-normal.[7]

[7] As mentioned in Section 3.1, rational speculators maximize mean-variance utility $\mu_W - \gamma\sigma^2_W$. If θ were normal, one could interpret this as the representation of a CARA utility function. However,

Proposition 3. *Let $T = 2$, $t^{r''} = 1$ and $t^{e''} = 2$. Then the equilibrium prices are*

$$p_0 = v,\ p_1 = p_2 = v + \left(1 + \frac{\beta_1}{\alpha - \beta_1}\right)\phi'' \tag{3.10}$$

for $\mu > 0$ and

$$p_0 = p_1 = v, \quad p_2 = v + \phi'' \tag{3.11}$$

for $\mu = 0$.

Proof. First we consider the case $\mu > 0$. From (3.4) we know that $p_0 = v$ and $p_1 = p_2$. The demands in period 2 are $D_2^e = S + \beta_1(p_1 - p_0)$ and $D_2^r = D_2^f = \alpha(v + \phi'' - p_2)$. Then the market clearing condition is

$$\begin{aligned} & S && = S + \beta_1(p_1 - p_0) + \alpha(v + \phi'' - p_2) \\ \Leftrightarrow\quad & 0 && = \beta_1 p_2 - \beta_1 v + \alpha(v + \phi'' - p_2) \\ \Leftrightarrow\quad & (\alpha - \beta_1)p_2 && = (\alpha - \beta_1)v + \alpha\phi'' \\ \Leftrightarrow\quad & p_2 && = v + \frac{\alpha}{\alpha - \beta_1}\phi'' = v + \left(\frac{\beta_1}{\alpha - \beta_1}\right)\phi''. \end{aligned}$$

In period 0 $D_0^f = S$, $D_0^e = \alpha(v - p_0)$ and $D_0^f = \frac{v - p_0}{2\gamma(1+\lambda'')^2\sigma_{\phi''}^2}$. The market clearing condition in this case is

$$S = S + (1 - \mu)\alpha(v - p_0) + \mu\frac{v - p_0}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2}$$

so $p_0 = v$ is an equilibrium price. Note, that in contrast to DSSW, we do not treat period 0 as a "reference period" where "there is no trading" (De Long et al., 1990, p.387) but as a period where all market agents have the opportunity to trade.

For $\mu = 0$, $D_t^e = \alpha(v - p_t)$ and $D_t^f = S$ for $t = 0, 1$. Therefore the market clears if $\alpha(v - p_t) = 0$ so $p_0 = p_1 = v$. This yields $D_2^f = S$ and together with $D_2^e = \alpha(v + \phi'' - p_2)$ and the market clearing condition $S = S + \alpha(v + \phi'' - p_2)$ we have $p_2 = v + \phi''$. □

Remark 1. *Let $\underline{\phi}''$ and $\underline{\theta}$ denote lower bounds for the realizations of ϕ'' and θ. By assuming that*

$$\underline{\phi}'' + \underline{\theta} \geq -v \Leftrightarrow v + \underline{\phi}'' + \underline{\theta} \geq 0$$

we interpret it just as the representation of their preferences. This allows us to set up a non-negativity condition for the prices (see remark 1).

and

$$\underline{\phi}'' \geq -\left(\frac{\alpha-\beta_1}{\alpha}\right) v \Leftrightarrow v + \left(1+\frac{\beta_1}{\alpha-\beta_1}\right)\underline{\phi}'' \geq 0$$

we make sure that prices do not become negative with certainty. Note that we have to exclude normality of θ to do this.

As we can see in the upper proposition and in figure 3.1, the presence of arbitrageurs increases the reaction of the signal on the price (for a positive realization of ϕ''). Rational speculators demand the asset in period 1 since $D_1^r = -D_1^e = -\alpha(v - p_1) = \left(1+\frac{\beta_1}{\alpha-\beta_1}\right)\phi''$. This demand increases the price at date 1 so positive feedback traders have a positive demand at date 2. Since rational speculators anticipate this positive demand they push the price to an even higher level because they know that there will be someone who buys the asset from them.
In the absence of rational speculators ($\mu = 0$), passive investors receive the signal at date 2 and have therefore a positive demand that increases the price. Since this is the last trading period, positive feedback traders have no further opportunities to react on that price change. This changes when we add another trading period. Next, we will see that a small change in the time structure leads to a different result.

3.3.2 Stabilizing rational speculation

Now we consider the same setup as above with one slight difference: we have another trading period at the end. As in DSSW, we assume that positive feedback traders' demand in period t only depends on the price change from $t-2$ to $t-1$, i.e. $\beta_k = 0\ \forall\, k = 2, 3, \ldots$. Since we still consider the case with one signal we have $\sigma^2_{\phi'} = 0$ so $\phi' = 0$.

Proposition 4. *Let $T = 3$, $t^{r''} = 1$ and $t^{e''} = 2$. The equilibrium asset prices are*

$$p_0 = v \text{ and } p_t = v + \phi'', \quad t = 1, 2, 3, \tag{3.12}$$

for $\mu > 0$ and

$$p_0 = p_1 = v, \quad p_2 = v + \phi'', \quad p_3 = v + \left(1+\frac{\beta_1}{\alpha}\right)\phi'' \tag{3.13}$$

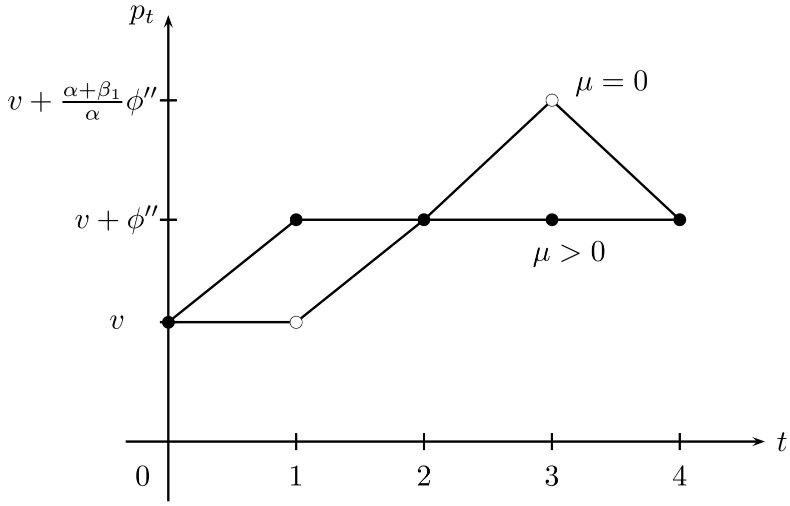


Figure 3.2: Stabilizing speculation

for $\mu = 0$.

Proof. For $\mu > 0$ with $t^{r''} = 1$ we have $p_1 = p_2 = p_3$ from (3.4). The demand functions in the last period are $D_3^e = S$ and $D_3^r = D_3^f = \alpha(v + \phi'' - p_3)$. Rearranging the market clearing condition in period 3 and using $p_1 = p_2 = p_3$ yields

$$\begin{aligned} & S && = S + \alpha(v + \phi'' - p_3) \\ \Leftrightarrow \quad & 0 && = v + \phi'' - p_3 \\ \Leftrightarrow \quad & p_3 && = v + \phi''. \end{aligned}$$

The market clearing condition in period 0 is

$$\begin{aligned} S &= S + (1-\mu)\alpha(v - p_0) + \mu\frac{v - p_0}{2\gamma(1+\nu')^2\sigma_{\phi'}^2} \\ 0 &= \left[(1-\mu)\alpha + \mu\frac{1}{2\gamma(1+\nu')^2\sigma_{\phi'}^2}\right](v - p_0) \end{aligned}$$

so $p_0 = v$. This is again consistent with (3.4).

For $\mu = 0$, the demands of positive feedback traders in period 0 and 1 are S. The demand of passive investors in these periods are $D_0^e = \alpha(v - p_0)$ and $D_1^e = \alpha(v - p_1)$, respectively. The market clearing condition with these demands yields $p_0 = p_1 = v$. Therefore $D_2^f = S$. Passive investors receive the signal in period $2\,(= t^{e''})$ so their demand is $D_2^e = \alpha(v + \phi'' - p_2)$. Plugging the demand functions in the market clearing condition we get

$$\begin{aligned} S &= S + \alpha(v + \phi'' - p_2) \\ \Leftrightarrow \quad 0 &= (v + \phi'' - p_2) \\ \Leftrightarrow \quad p_2 &= v + \phi''. \end{aligned}$$

Since the passive investors do not get any new signal in the last period, their demand function is again $D_3^e = \alpha(v + \phi'' - p_3)$. Since the period-1 and period-2 prices are not identical, the demand function of the positive feedback traders is $D_3^f = S + \beta_1(p_2 - p_1) = \beta_1\phi''$. Plugging this in the market clearing condition, we get

$$\begin{aligned} S &= S + \beta_1\phi'' + \alpha(v + \phi'' - p_3) \\ \Leftrightarrow \quad 0 &= \frac{\beta_1}{\alpha}\phi'' + v + \phi'' - p_3 \\ \Leftrightarrow \quad p_3 &= v + \left(1 + \frac{\beta_1}{\alpha}\right)\phi''. \end{aligned}$$

□

This proposition shows that rational speculation may also be stabilizing. Figure 3.2 depicts the two price paths in the presence (filled dots) and in the absence (non-filled dots) of rational speculators with a positive realization of ϕ''.

If rational speculators are present in the market ($\mu > 0$), they receive the signal at date 1 and therefore demand the asset so the price increases. Positive feedback traders have a positive demand in period 2 due to this price rise. Since the price is constant when no new information arrives (i.e. at dates 2 and 3; otherwise rational speculators' demand would be $\pm\infty$), positive feedback traders' date-2 demand is absorbed by rational speculators. Therefore positive feedback traders' demand at date 3 is equal to zero and the price stays at $v + \phi''$.[8]

[8]If we allowed for $\beta_2 \neq 0$, the situation would change since positive feedback traders' date-3 demand would not be equal to S. We discuss this case in subsection 3.3.4.

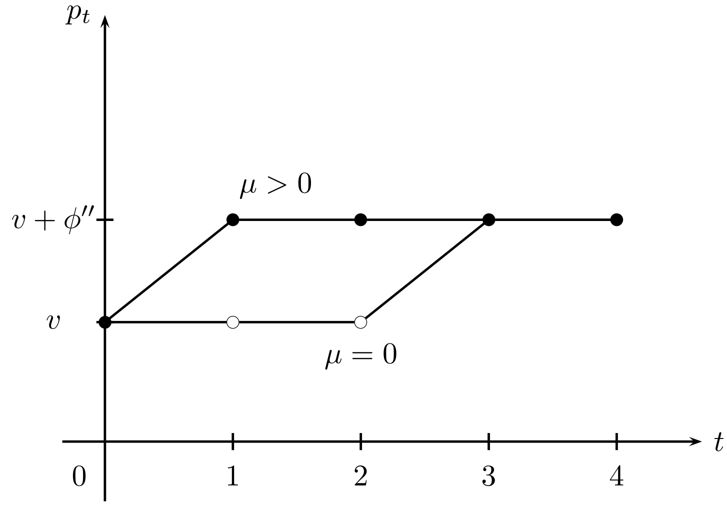


Figure 3.3: Neutral speculation

This changes in the absence of rational speculators ($\mu = 0$). Then, passive investors receive the signal at date 2, so they demand the asset and the price rises. In contrast to the situation with $T = 2$, positive feedback traders now have time to react to the price change from date 1 to date 2. This price change yields a positive demand of the positive feedback traders at date 3 so the price exceeds $v + \phi''$ since passive investors satisfy positive feedback traders' demand.

3.3.3 Neutral rational speculation

We have seen so far that rational speculation can either be destabilizing (as in DSSW) or stabilizing. In this subsection we show that neutral speculation (there is overreaction to the signal neither in the case when rational speculators are present nor when they are absent) is also possible. The only difference to the proposition above is the date when passive investors receive their signal.

Proposition 5. *Let $T = 3$, $t^{r''} = 1$, $t^{e''} = 3$ and $\beta_2 = 0$. The equilibrium asset prices are $p_0 = v$ and*

$$p_t = v + \phi'', \quad t = 1, 2, 3,$$

for $\mu > 0$ *and*

$$p_0 = p_1 = p_2 = v, \quad p_3 = v + \phi''$$

for $\mu = 0$.

Proof. For $\mu > 0$ the proof is identical to the proof of (3.12). Since in the case with $\mu = 0$ no one receives a signal before date 3, the demand functions are $D_t^e = \alpha(v - p_t)$ for $t = 0, 1, 2$. With $D_0^f = S$ we get $S = S + \alpha(v - p_0) \Leftrightarrow p_0 = v$. For the same reason $p_1 = v$. Since $p_0 = p_1 = v$, $D_2^f = S$ so $p_2 = v$. At date 3, $D_3^e = \alpha(v + \phi'' - p_3)$ and $D_3^f = S + \beta_1(p_2 - p_1) = S$ so the market clearing condition $S = S + \alpha(v + \phi'' - p_3)$ yields $p_3 = v + \phi''$. □

As mentioned above, there is no overreaction to the signal passive investors on the one hand side and rational speculators on the other hand side receive. The development of the price is depicted in figure 3.3 (for $\phi'' > 0$). The reason why there is no overreaction if $\mu > 0$ is the same as in the stabilizing case: from (3.4) we know that price is constant except when rational speculators receive their signal. Since this price change is too far in the past, it has no impact on the demand of the positive feedback traders ($\beta_2 = 0$). In the absence of rational speculators ($\mu = 0$), the price change occurs in the last period so positive feedback traders do not react on it since it is too recent. So in this case there is also no overreaction and we have neutral speculation.

3.3.4 The general case

We have seen so far that small changes in the time setup have a large impact on the behavior of equilibrium prices. The only difference between the DSSW model and the model with stabilizing speculation is that we added another trading period in the end. Giving the signal to passive investors one period later led to the case with neutral speculation. The three different results special cases of the generalization in this section. We first analyze when and how prices overreact in the presence of rational speculators before we consider the cases when they are absent.

Proposition 6. *The equilibrium asset prices are $p_t = v$ for $t = 0, 1, \ldots, t^{r''} - 1$ and*

$$p_t = v + \left(1 + \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)\phi'', \quad t = t^{r''}, \ldots, T, \tag{3.14}$$

for $\mu > 0$.

Proof. Let $\mu > 0$. From (3.4), we know that $p_t = v$ for $t = 0, 1, \ldots, t^{r''} - 1$ and $p_{t^{r''}} = \ldots = p_T$. From (3.7) we have $D_T^r = \alpha(v + \phi'' - p_T)$. Since passive investors receive their signal before T ($t^{e''} \leq T$), their demand function is also $D_T^e = \alpha(v + \phi'' - p_T)$. Since there is only one price change from period $t^{r''} - 1$ to $t^{r''}$, positive feedback traders' demand is $D_T^f = S + \beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1})$. Thus the market clearing condition in T is

$$S = S + \beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1}) + \alpha(v + \phi'' - p_T).$$

Rearranging this we get

$$\begin{aligned}
0 \quad &= \beta_{T-t^{r''}}(p_T - v) + \alpha(v + \phi'' - p_T) \\
\Leftrightarrow \quad (\alpha - \beta_{T-t^{r''}})p_T &= (\alpha - \beta_{T-t^{r''}})v + \alpha\phi'' \\
\Leftrightarrow \quad p_T &= v + \frac{\alpha}{\alpha - \beta_{T-t^{r''}}}\phi'' = v + \left(1 + \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)\phi'',
\end{aligned}$$

which proves (3.14).
Note that $p_t = v$ for $t = 0, 1, \ldots, t^{r''} - 1$ is actually an equilibrium:

$$D_t^e = \alpha(v - p_t) = \alpha(v - v) = 0$$

and

$$D_t^f = \frac{v - p_{t^{r''}-1}}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2} = \frac{v - v}{2\gamma(1 + \lambda'')^2\sigma_{\phi''}^2} = 0$$

so the market clears for $t = 0, 1, \ldots, t^{r''} - 1$. □

We see from (3.14) that in the case when rational investors are present ($\mu > 0$), the extent of overreaction depends only on one feedback parameter $\beta_{T-t^{r''}}$. If this parameter is positive there is overreaction (for a positive realization of ϕ''). In the DSSW model, $\beta_{T-t^{r''}} = \beta_{2-1} = \beta_1 > 0$ so prices overshoot. If $\beta_{T-t^{r''}} = 0$ the price goes to $v + \phi''$ and stays there. In the latter two models with $T = 3$ and $t^{r''} = 1$ (stabilizing and neutral speculation) we have $\beta_{T-t^{r''}} = \beta_{3-1} = \beta_2 = 0$ (by assumption) so there is no overreaction.

When we assume that β_l is a decreasing sequence (that means price changes further back in the past have a weaker impact on the current demand of positive feedback traders), the degree of overreaction is decreasing when the time span $T - t^{r''}$ between the arrival of the signal and the final trading date is increasing. In other words: The earlier the information arrives (holding the number of trading dates constant), the smaller is the extent of overshooting. Or: The more trading dates we have (holding the date when the signal arrives fixed), the smaller is the degree of overreaction. Therefore, the model is better used to explain price overshooting over short than in long time horizons.

Next we consider two setups where rational speculators are absent ($\mu = 0$). Although both cases only differ in the values of β_l they lead to very different results.

Proposition 7. *For $\mu = 0$, the asset prices obey $p_t = v$ for $t = 0, 1, \ldots, t^{e''} - 1$ and*

$$p_t = v + \left[\sum_{\tau=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha} \right)^{\tau} \right] \phi'', \quad t = t^{e''}, \ldots, T, \tag{3.15}$$

if $\beta_l = \beta \quad \forall l$ is constant.

Proof. For $\mu = 0$ and $\beta_l = \beta \quad \forall l$ the market clearing condition before the arrival of the signal in $t^{e''}$ (so at dates $t = 0, 1, \ldots, t^{e''} - 1$) is just $S = S + \alpha(v - p_t)$. Therefore $p_t = v$ for $t = 0, 1, \ldots, t^{e''} - 1$.

Now we prove (3.15). Consider the market clearing condition at $t = t^{e''}, \ldots, T$

$$\begin{aligned} S &= S + \beta(p_{t-1} - p_{t-2}) + \ldots + \beta(p_{t^{e''}} - p_{t^{e''}-1}) + \alpha(v + \phi'' - p_t) \\ \Leftrightarrow \quad 0 &= \beta p_{t-1} - \beta v + \alpha(v + \phi'' - p_t) \\ \Leftrightarrow \quad 0 &= \frac{\beta}{\alpha} p_{t-1} - \frac{\beta}{\alpha} v + v + \phi'' - p_t \\ \Leftrightarrow \quad p_t &= \frac{\alpha - \beta}{\alpha} v + \phi'' + \frac{\beta}{\alpha} p_{t-1}. \end{aligned}$$

To solve this differential equation, we first have to eliminate the constant. Let $p_t =$

$p_{t-1} = \bar{p}$. Then

$$\begin{aligned}
& \bar{p} && = \frac{\alpha-\beta}{\alpha}v + \phi'' + \frac{\beta}{\alpha}\bar{p} \\
\Leftrightarrow\ & \bar{p}\left(\tfrac{\alpha-\beta}{\beta}\right) && = \frac{\alpha-\beta}{\alpha}v + \phi'' \\
\Leftrightarrow\ & \bar{p} && = v + \frac{\alpha}{\alpha-\beta}\phi''.
\end{aligned}$$

Next we define $\tilde{p}_t = p_t - \bar{p} \Leftrightarrow p_t = \tilde{p}_t + \bar{p}$. Using this in the upper differential equation, we get

$$\begin{aligned}
& \tilde{p}_t + v + \tfrac{\alpha}{\alpha-\beta}\phi'' && = \frac{\alpha-\beta}{\alpha}v + \phi'' + \frac{\beta}{\alpha}\tilde{p}_{t-1} + \frac{\beta}{\alpha}\left(v + \frac{\alpha}{\alpha-\beta}\phi''\right) \\
\Leftrightarrow\ & \tilde{p}_t && = \frac{\beta}{\alpha}\tilde{p}_{t-1}.
\end{aligned}$$

This is a homogenous linear differential equation of degree one, so its solution is

$$\tilde{p}_t = \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \tilde{p}_{t^{e''}}.$$

Substituting $\tilde{p}_t = p_t - \bar{p}$ and using that $p_{t^{e''}} = v + \phi''$ and footnote 9 yields

$$\begin{aligned}
& p_t - v - \tfrac{\alpha}{\alpha-\beta}\phi'' && = \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \left(p_{t^{e''}} - v - \frac{\alpha}{\alpha-\beta}\phi''\right) \\
\Leftrightarrow\ & p_t && = v + \frac{\alpha}{\alpha-\beta}\phi'' + \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \left(v + \phi'' - v - \frac{\alpha}{\alpha-\beta}\phi''\right) \\
\Leftrightarrow\ & p_t && = v + \frac{\alpha}{\alpha-\beta}\phi'' + \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \left(-\frac{\beta}{\alpha-\beta}\right)\phi'' \\
\Leftrightarrow\ & p_t && = v + \frac{\alpha}{\alpha-\beta}\phi'' + \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}+1} \left(-\frac{\alpha}{\alpha-\beta}\right)\phi'' \\
\Leftrightarrow\ & p_t && = v + \frac{\alpha}{\alpha-\beta}\phi'' \left[1 - \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}+1}\right] \\
\Leftrightarrow\ & p_t && = v + \frac{1 - \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}+1}}{1 - \frac{\beta}{\alpha}}\phi'' \\
\Leftrightarrow\ & p_t && = v + \left[\sum_{\tau=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{\tau}\right]\phi'',
\end{aligned}$$

which proves (3.15). □

We see that a change in the time setup has again an impact on the price. The more trading periods we have (keeping the date when the information arrives fixed), the higher is the final price. On the other hand, the earlier the signal arrives (keeping the number of trading dates constant), the higher is the price at the final date. This is in contrast with the results from proposition 6 where it was the other way around. Considering the infinite geometric series we get

$$p_t < v + \left[\sum_{\tau=0}^{\infty}\left(\frac{\beta}{\alpha}\right)^{\tau}\right]\phi'' = v + \left(1 + \frac{\beta}{\alpha-\beta}\right)\phi''$$

for $\phi'' > 0$.[9] That means that the price is always increasing and moves to the price level we have in the presence of rational speculators (although it never reaches it). Another conclusion we draw from proposition 7 is that there is no overreaction in the absence of rational speculators when the signal arrives at date T.

Corollary 1. *Let $\mu = 0$ and $\beta \neq 0$. If $t^{e''} = T$ then*

$$p_t = v \text{ for } t = 0, \ldots, T-1 \text{ and } p_T = v + \phi''.$$

Proof. Before the signal arrives, the demand functions of the positive feedback traders and the passive investors are $D_t^f = S$ and $D_t^e = \alpha(v - p_t)$, respectively. Therefore the market clearing condition is

$$S = S + \alpha(v - p_t)$$

[9]Since $\beta < \alpha$, $\frac{\beta}{\alpha} < 1$. By definition

$$\sum_{\tau=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\tau} = 1+\left(\frac{\beta}{\alpha}\right)+\left(\frac{\beta}{\alpha}\right)^{2}+\ldots+\left(\frac{\beta}{\alpha}\right)^{k} \text{ and } \left(\frac{\beta}{\alpha}\right)\sum_{\tau=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\tau} = \left(\frac{\beta}{\alpha}\right)+\left(\frac{\beta}{\alpha}\right)^{2}+\ldots+\left(\frac{\beta}{\alpha}\right)^{k+1}.$$

Subtracting these two and rearranging leads to

$$\sum_{\tau=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\tau} - \left(\frac{\beta}{\alpha}\right)\sum_{\tau=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\tau} = 1 - \left(\frac{\beta}{\alpha}\right)^{k+1} \Leftrightarrow \sum_{\tau=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\tau} = \frac{1-\left(\frac{\beta}{\alpha}\right)^{k+1}}{1-\left(\frac{\beta}{\alpha}\right)}.$$

Since $\lim\limits_{k\to\infty}\dfrac{1-\left(\frac{\beta}{\alpha}\right)^{k+1}}{1-\left(\frac{\beta}{\alpha}\right)} = \dfrac{1}{1-\left(\frac{\beta}{\alpha}\right)}$ we have $\displaystyle\sum_{\tau=0}^{\infty}\left(\frac{\beta}{\alpha}\right)^{\tau} = \frac{1}{1-\left(\frac{\beta}{\alpha}\right)} = \frac{\alpha}{\alpha-\beta} = 1 + \frac{\beta}{\alpha-\beta}$.

which yields $p_t = v$ for $t = 0, \ldots, T-1$. When passive investors learn about ϕ'' in period T, their demand function becomes $D^e_T = \alpha(v + \phi'' - p_t)$. Since there has been no price change in the past, positive feedback traders' demand is again $D^f_T = S$. Plugging this in the market clearing condition $S = S + \alpha(v + \phi'' - p_T)$ yields $p_T = v + \phi''$. $\square$

In this setup, positive feedback traders have "no time" to react on the price change that results from new information passive investors receive. Positive feedback traders want to buy the asset but do not have the possibility to do so. That is the reason why there is no overreaction in the DSSW model and in the case with neutral speculation since passive investors learn about ϕ'' in T.

Changing the values of β_l leads to another interesting result:

Proposition 8. *For $\mu = 0$, the asset prices obey $p_t = v$ for $t = 0, 1, \ldots, t^{e''} - 1$ and*

$$p_t = v + \phi'' + \frac{\beta_1}{\alpha}(p_{t-1} - p_{t-2}), \quad t = t^{e''}, \ldots, T, \tag{3.16}$$

if $\beta_l = 0$ for $l \geq 2$.

Proof. For $\mu = 0$ and $0 = \beta_2 = \beta_3 = \ldots$ the market clearing in each period t is

$$\begin{aligned} 0 &= \beta_1(p_{t-1} - p_{t-2}) + \alpha(v + \phi'' - p_t) \\ \Leftrightarrow \quad p_t &= v + \phi'' + \frac{\beta_1}{\alpha}(p_{t-1} - p_{t-2}) \end{aligned}$$

which proves (3.16). $\square$

Corollary 2. *The price response to the signal in the setup of proposition 8 is characterized by damped fluctuations.*

Proof. From proposition 8 we have the difference equation

$$p_t = v + \phi'' + \frac{\beta_1}{\alpha}(p_{t-1} - p_{t-2}).$$

To find a solution, we first eliminate the constant. We set $p_t = p_{t-1} = p_{t-2} = \bar{p}$. Then

$$\begin{aligned} \bar{p} &= v + \phi'' + \frac{\beta_1}{\alpha}(\bar{p} - \bar{p}) \\ &= v + \phi''. \end{aligned}$$

Now we define $p_t = \tilde{p}_t + \bar{p} = \tilde{p}_t + v + \phi''$. Plugging this into (3.16) yields a homogeneous difference equation of degree 2:

$$\begin{aligned}
\tilde{p}_t + v + \phi'' &= v + \phi'' + \frac{\beta_1}{\alpha}(\tilde{p}_{t-1} + v + \phi'' - \tilde{p}_{t-2} - v - \phi'') \\
&= v + \phi'' + \frac{\beta_1}{\alpha}(\tilde{p}_{t-1} - \tilde{p}_{t-2}) \\
\Leftrightarrow \tilde{p}_t &= \frac{\beta_1}{\alpha}(\tilde{p}_{t-1} - \tilde{p}_{t-2}). \qquad (3.17)
\end{aligned}$$

We try a solution of the form $\tilde{p}_t = A\xi^t$ for some A and some ξ. Plugging this into (3.17) yields

$$\begin{aligned}
A\xi^t &= \frac{\beta_1}{\alpha}A\xi^{t-1} - \frac{\beta_1}{\alpha}A\xi^{t-2} \\
\Leftrightarrow \quad 0 &= \xi^2 - \frac{\beta_1}{\alpha}\xi + \frac{\beta_1}{\alpha}.
\end{aligned}$$

Solving for this quadratic function for ξ we obtain

$$\xi_{1/2} = \frac{\frac{\beta_1}{\alpha} \pm \sqrt{\frac{\beta_1^2}{\alpha^2} - 4\frac{\beta_1}{\alpha}}}{2}.$$

ξ_1 and ξ_2 are complex numbers if and only if the expression under the square root is negative. This follows from our assumption $0 \leq \beta_1 < \alpha$ since

$$\begin{aligned}
\beta_1 &< \alpha \\
\Rightarrow \beta_1 &< 4\alpha \\
\frac{\beta_1}{\alpha} &< 4 \\
\frac{\beta_1^2}{\alpha^2} - 4\frac{\beta_1}{\alpha} &< 0.
\end{aligned}$$

Therefore

$$\xi_{1/2} = \frac{\beta_1}{2\alpha} \pm i\frac{\sqrt{4\frac{\beta_1}{\alpha} - \frac{\beta_1^2}{\alpha^2}}}{2},$$

where $i = \sqrt{-1}$. The solution of (3.17) is of the form

$$\tilde{p}_t = A_1\xi_1^t + A_2\xi_2^t.$$

It converges to zero for all values of A_1 and A_2 if $\lim_{t\to\infty} \xi_1^t = 0$ and $\lim_{t\to\infty} \xi_2^t = 0$. To calculate $\xi_{1/2}^t$, rewrite $\xi_{1/2} = a_{1/2} + b_{1/2}i$ in polar coordinates $r_{1/2}(\cos(\varphi_{1/2}) + i\sin(\varphi_{1/2}))$ where

$r_{1/2} = \sqrt{a^2+b^2}$. Since $\xi_{1/2}^t = r_{1/2}^t(\cos(t\varphi_{1/2}) + i\sin(t\varphi_{1/2}))$ it is sufficient to show that $\lim_{t\to\infty} r_{1/2}^t = 0$. Since $r_{1/2}$ is real, this is true if $|r_{1/2}| < 1$.

$$\begin{aligned}
r_{1/2} &= \sqrt{\frac{\beta_1^2}{4\alpha^2} + \frac{4\frac{\beta_1}{\alpha} - \frac{\beta_1^2}{\alpha^2}}{4}} \\
&= \sqrt{\frac{\beta_1^2}{4\alpha^2} + \frac{4\frac{\alpha\beta_1}{\alpha^2} - \frac{\beta_1^2}{\alpha^2}}{4}} \\
&= \sqrt{\frac{\beta_1^2 + 4\alpha\beta_1 - \beta_1^2}{4\alpha^2}} \\
&= \sqrt{\frac{4\alpha\beta_1}{4\alpha^2}} \\
&= \sqrt{\frac{\beta_1}{\alpha}}
\end{aligned}$$

Since we assumed $\beta_1 < \alpha$,

$$\begin{aligned}
&\Rightarrow\ r_{1/2} < 1 \\
&\Rightarrow\ \lim_{t\to\infty} r_{1/2}^t = 0 \\
&\Rightarrow\ \lim_{t\to\infty} \xi_1^t = 0 \text{ and } \lim_{t\to\infty} \xi_2^t = 0 \\
&\Rightarrow\ \lim_{t\to\infty} \tilde{p}_t = 0.
\end{aligned}$$

With $\lim_{t\to\infty} \tilde{p}_t = 0$ and $p_t = \tilde{p}_t + \bar{p}$ the price converges to $v + \phi''$. □

This price behavior can be explained as follows: passive investors demand a positive quantity of the asset when they receive the signal, so the price rises (for $\phi'' > 0$) until $p_{t^{e''}} = v + \phi''$. Positive feedback traders' demand at $t^{e''} + 1$ is positive due to this price increase and therefore the price increases even further. Passive investors have a negative demand (since $p_{t^{e''}+1} > v + \phi''$) so they satisfy positive feedback traders' demand. However there is a date t, where D_t^e exceeds D_t^f. At that point, the price falls and both, passive investors and rational speculators sell the asset until its price falls below $v + \phi''$. When the price is below $v + \phi''$, only positive feedback traders sell the asset (due to the price decrease before) and passive investors buy the asset (since it is too cheap now). Therefore the price decrease weakens until, at some point, the price goes up again. When it exceeds $v + \phi''$ the cycle starts again. Note, that this only works when we assume $\alpha > \beta$. If β exceeded α, the price would go to infinity.

3.3.5 The effect of measure in the presence of rational speculators

As we have seen in proposition 6 and as remarked by DSSW, the "path of prices in the case of a noiseless signal is discontinuous: $\mu = 0$ and $0 < \mu << 1$ are not nearly equivalent" (De Long et al., 1990, p.388). This is true since after the signal arrived the price is determined by the date$-T$ market clearing conditions which coincide for rational speculators and passive investors in the case with one signal. We will see in section 3.4 that this changes when we add a second signal.

A point one could think about is whether the sizes of the three groups of investors have an impact on the preceding results. In this section, we show that it has (under certain assumptions) no impact on the direction but on the extent of the effect. Therefore we define the measures η_1, $\eta_2\mu$ and $\eta_2(1-\mu)$ for positive feedback traders, rational speculators and passive investors, respectively. With this choice of η_1 and η_2, we follow DSSW "since changes in μ keep the risk-bearing capacity of the economy constant" (De Long et al., 1990, p.386), because the total of passive investors and rational speculators stays constant.

Proposition 9. *Let η_1, $\eta_2\mu$ and $\eta_2(1-\mu)$ be the measures of positive feedback traders, rational speculators and passive investors, respectively. Then the equilibrium asset prices are $p_t = v$ for $t = 0, 1, \ldots, t^{r''} - 1$ and*

$$p_t = v + \left(1 + \frac{\eta_1 \beta_{T-t^{r''}}}{\eta_2 \alpha - \eta_1 \beta_{T-t^{r''}}}\right) \phi'', \quad t = t^{r''}, \ldots, T \tag{3.18}$$

for $\mu > 0$.

Proof. Since we want positive feedback traders to absorb the supply of the asset if there is no price change we set $S = 0$ for the remainder of this section.[10] Then the market clearing condition becomes

$$0 = \eta_1 D_t^f + \eta_2 \mu D_t^r + \eta_2 (1-\mu) D_t^e.$$

[10] We could also modify the supply to $\eta_1 S$ or the positive feedback traders' demand function by replacing S by $\frac{S}{\eta_1}$.

We have $p_0 = p_1 = \ldots = p_{t^{r''}-1}$ and $p_{t^{r''}} = \ldots = p_T$ from equation (3.4). Then the period$-T$ demand function of positive feedback traders is $D_T^f = \beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1})$. Both passive investors' and rational speculators' demand function is $D_T^e = D_T^r = \alpha(v + \phi'' - p_T)$. Plugging this into the market clearing condition yields (3.18), since

$$\begin{aligned}
0 \quad & = \eta_1\beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1}) + \eta_2\alpha(v + \phi'' - p_T) \\
& = \eta_1\beta_{T-t^{r''}}(p_T - v) + \eta_2\alpha(v + \phi'' - p_T) \\
\Leftrightarrow \quad (\eta_2\alpha - \eta_1\beta_{T-t^{r''}})p_T & = (\eta_2\alpha - \eta_1\beta_{T-t^{r''}})v + \eta_2\alpha\phi'' \\
\Leftrightarrow \quad p_T & = v + \frac{\eta_2\alpha}{\eta_2\alpha - \eta_1\beta_{T-t^{r''}}}\phi'' = v + \left(1 + \frac{\eta_1\beta_{T-t^{r''}}}{\eta_2\alpha - \eta_1\beta_{T-t^{r''}}}\right)\phi''.
\end{aligned}$$

□

In order to have overreaction (for a positive realization of ϕ'') it is necessary that $\eta_2\alpha - \eta_1\beta_{T-t^{r''}} > 0 \Leftrightarrow \frac{\eta_2}{\eta_1} > \frac{\beta_{T-t^{r''}}}{\alpha}$. If this inequality is violated, there is either no equilibrium price ($\eta_2\alpha - \eta_1\beta_{T-t^{r''}} = 0$) or underreaction ($\eta_2\alpha - \eta_1\beta_{T-t^{r''}} < 0$).
To understand why we need this assumption we define the total market demand

$$md_t = D_t^f + D_t^e + D_t^p.$$

Note, that the market clears if $md_t = 0$. Considering the market demand at date T and using (3.4) we have[11]

$$\begin{aligned}
md_T &= \eta_1\beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1}) + \eta_2\alpha(v + \phi'' - p_T) \\
&= \eta_1\beta_{T-t^{r''}}(p_T - v) + \eta_2\alpha(v + \phi'' - p_T) \\
&= (\eta_2\alpha - \eta_1\beta_{T-t^{r''}})v + \alpha\phi'' - (\eta_2\alpha - \eta_1\beta_{T-t^{r''}})p_T.
\end{aligned}$$

So the market demand is a linear function of p_T with slope $-(\eta_2\alpha - \eta_1\beta_{T-t^{r''}})$. For $\eta_2\alpha > \eta_1\beta_{T-t^{r''}}$ the intercept of the market demand is greater than zero and the slope is negative so there exists a positive equilibrium price (see left panel of figure 3.4). As η_2 decreases or η_1 increases, the slope becomes smaller and the intercept approaches $\alpha\phi''$ so the price increases (middle panel of figure 3.4). This can be shown analytically

[11] For the demand functions see proof of proposition 9.

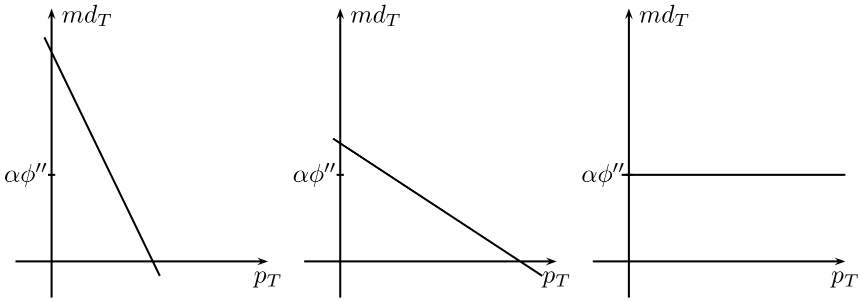


Figure 3.4: The market demand as $\eta_2\alpha > \eta_1\beta_1$ (left panel), as η_2 decreases (or η_1 increases or both)(middle panel) and as $\eta_2\alpha = \eta_1\beta_{T-t^{r''}}$ (right panel).

by deriving the degree of overreaction with respect to η_1 and η_2 since

$$\begin{aligned}
\frac{\partial}{\partial\eta_1}\frac{\eta_1\beta_{T-t^{r''}}}{\eta_2\alpha-\eta_1\beta_{T-t^{r''}}} &= \frac{(\eta_2\alpha-\eta_1\beta_{T-t^{r''}})\beta_{T-t^{r''}}-\eta_1\beta_{T-t^{r''}}(-\beta_{T-t^{r''}})}{(\eta_2\alpha-\eta_1\beta_{T-t^{r''}})^2} \\
&= \frac{\eta_2\alpha\beta_{T-t^{r''}}-\eta_1\beta^2_{T-t^{r''}}+\eta_1\beta^2_{T-t^{r''}}}{(\eta_2\alpha-\eta_1\beta_{T-t^{r''}})^2} \\
&= \frac{\eta_2\alpha\beta_{T-t^{r''}}}{(\eta_2\alpha-\eta_1\beta_{T-t^{r''}})^2} > 0
\end{aligned}$$

and

$$\frac{\partial}{\partial\eta_2}\frac{\eta_1\beta_{T-t^{r''}}}{\eta_2\alpha-\eta_1\beta_{T-t^{r''}}} = \frac{-\eta_1\alpha\beta_{T-t^{r''}}}{(\eta_2\alpha-\eta_1\beta_{T-t^{r''}})^2} < 0.$$

When $\eta_2\alpha = \eta_1\beta_{T-t^{r''}}$ then the market demand is constant ($md_T = \alpha\phi''$) and the market never clears (right panel of figure 3.4). If we increased η_1 (decrease η_2) further this would yield negative prices which we do not consider.

One interpretation is to consider $\eta_2\alpha$ and $\eta_1\beta_l$ as "trading power" of the market agents. Positive feedback traders' demand at date T is

$$\eta_1 D^f_T = \eta_1\beta_{T-t^{r''}}(p_{t^{r''}} - p_{t^{r''}-1}) = \eta_1\beta_{T-t^{r''}}(p_T - v).$$

It is driven by two components: the price change ($p_T - v$) and the reaction to it ($\beta_{T-t^{r''}}$) multiplied by the measure of the group (η_1). At date T, passive investors' and rational speculators' demand function is

$$\eta_1(\mu D^r_T + (1-\mu)D^e_T) = \eta_2\alpha(v + \phi'' - p_T).$$

Again, this demand is driven by two components: the price change ($v+\phi''-p_T$) and the reaction to it (α) multiplied by the size of the group (η_2). Note that if $v \leq p_T \leq v+\phi''$ then both, $p_T - v$ and $v + \phi'' - p_T$ are positive so all date$-T$ demand functions are greater than zero (for η_1, η_2, α and $\beta_{T-t^{r''}}$ positive) and the market never clears. So a price between v and $v + \phi''$ is never realized.

Let us assume for the remainder of this subsection that $\phi'' \geq 0$.

If $\eta_2\alpha > \eta_1\beta_{T-t^{r''}}$, then rational speculators and passive investors can satisfy positive feedback traders' demand if the price change that affects positive feedback traders' demand is (in absolute value but with opposite sign) larger than the price change that drives rational speculators' and passive investors' demand (i.e. $v+\phi''-p_T < 0 < p_T-v$). This is true if $p_T > v + \phi''$, so we have overreaction.

If $\eta_2\alpha = \eta_1\beta_{T-t^{r''}}$, the reaction to the price changes is the same for both groups (positive feedback traders on the one hand and passive and rational investors on the other hand). Since the market has to clear, both price changes also have to be equal (with opposite sign). This is only possible if $\phi'' = 0$ ($\Leftrightarrow p_T - v = v + \phi'' - p_T$). Then the slope and the intercept of the market demand are both equal to zero so every price satisfies the market clearing condition.

If $\eta_2\alpha < \eta_1\beta_{T-t^{r''}}$, positive feedback traders react stronger to price changes than rational speculators and passive investors do. If we had $p_T > v + \phi''$, then positive feedback traders' demand would exceed passive and rational investors' demand and there would be no equilibrium price. To have market clearing at T, the price change that affects positive feedback traders must be smaller (and of opposite sign) than the price change of passive and rational investors (i.e. $p_T - v < 0 < v + \phi'' - p_T$). This is the case if $p_T < v$. Then, rational speculators short the asset even though they receive a positive signal and buy it in the last period whereas positive feedback traders sell it. Therefore we have underreaction in this case. It might happen that prices get negative but we exclude these cases.

Note, that if we set $\eta_1 = \eta_2$, then both cancel out in equation (3.18) and we have the same overreaction as in the case when the measures are 1, μ and $1 - \mu$.

3.3.6 The effect of measure in the absence of rational speculators

The measure also may have an impact on the results in the absence of rational speculators. Let us therefore take the setup from above (measure of positive feedback traders is η_1, measure of passive investors is η_2) and set $\mu = 0$. We apply this setup to the two propositions 7 and 8.

Corollary 3. *Let η_1 be the measure of positive feedback traders and η_2 be the measure of passive investors. The behavior of the price p_t in propositions 7 and 8 is the same for $t = 0, 1, \ldots, t^{e''} - 1$ and does not change qualitatively if*

$$\frac{\eta_1}{\eta_2} < \frac{\alpha}{\beta} \text{ and } \frac{\eta_1}{\eta_2} < \frac{\alpha}{\beta_1}, \tag{3.19}$$

respectively.

Proof. The price behavior before the signal arrives is the same as in both proposition 7 and 8. The demand functions are $D_t^f = S$ and $D_t^e = \alpha(v - p_t)$ for $t = 0, 1, \ldots, t^{e''} - 1$. As in proof of proposition 9, we set $S = 0$ so the market clearing condition is

$$0 = \eta_1 D_t^f + \eta_2 D_t^e = \eta_2 \alpha (p_t - v)$$

and we have $p_t = v$.

For $t \geq t^{e''}$ we first consider the situation in proposition 7. The market clears if

$$\begin{aligned}
& 0 && = \eta_1 \left(\beta(p_{t-1} - p_{t-2}) + \ldots + \beta(p_{t^{e''}} - p_{t^{e''}-1}) \right) + \eta_2 \alpha (v + \phi'' - p_t) \\
\Leftrightarrow \quad & 0 && = \eta_1 \beta p_t - \eta_1 \beta v + \eta_2 \alpha (v + \phi'' - p_t) \\
\Leftrightarrow \quad & 0 && = \frac{\eta_1 \beta}{\eta_2 \alpha} p_{t-1} - \frac{\eta_1 \beta}{\eta_2 \alpha} v + v + \phi'' - p_t \\
\Leftrightarrow \quad & p_t && = \frac{\eta_2 \alpha - \eta_1 \beta}{\eta_2 \alpha} v + \phi'' + \frac{\eta_1 \beta}{\eta_2 \alpha} p_{t-1}.
\end{aligned}$$

The proof of proposition 7 applies here if

$$\frac{\eta_1 \beta}{\eta_2 \alpha} < 1 \Leftrightarrow \frac{\eta_1}{\eta_2} < \frac{\alpha}{\beta}.$$

Now we turn to the situation in proposition 8. The market clearing condition is

$$\begin{aligned}
& 0 && = \eta_1 \beta_1 (p_{t-1} - p_{t-2}) + \eta_2 \alpha (v + \phi'' - p_t) \\
\Leftrightarrow \quad & p_t && = v + \phi'' + \frac{\eta_1 \beta_1}{\eta_2 \alpha} (p_{t-1} - p_{t-2}).
\end{aligned}$$

As in the proof proposition 8 we assume

$$\frac{\eta_1\beta_1}{\eta_2\alpha} < 1 \Leftrightarrow \frac{\eta_1}{\eta_2} < \frac{\alpha}{\beta_1}.$$

□

This proposition tells us essentially the same as proposition 9. If the measure of positive feedback traders is not too large (compared to the measure of passive investors), then the price functions are almost the same as with measure one.

Remark 2. *If equation (3.19) is violated in the situation of proposition 7, then prices go to infinity.*

Proof. Applying the proof from proposition 7 and replacing $\frac{\beta}{\alpha}$ by $\frac{\eta_1\beta}{\eta_2\alpha}$ we get the homogenous linear differential equation

$$\tilde{p}_t = \frac{\eta_1\beta}{\eta_1\alpha}\tilde{p}_{t-1}.$$

The solution to this differential equation is

$$\tilde{p}_t = \left(\frac{\eta_1\beta}{\eta_1\alpha}\right)^{t-t^{e''}} \tilde{p}_{t^{e''}},$$

so $\tilde{p}_t$ and therefore p_t go to infinity as t increases if $\frac{\eta_1\beta}{\eta_2\alpha} > 1$. □

If $\eta_1\beta > \eta_2\alpha$ ($\eta_1\beta_1 > \eta_2\alpha$), then positive feedback traders have more "trading power" than passive investors. Their demand after the initial price change is always larger than the absolute value of passive investors' demand so the price goes to ∞.
Note that if $\eta_1 = \eta_2$, then equation (3.19) becomes $\alpha > \beta$ and $\alpha > \beta_1$, respectively and we are again in the situation of proposition 7 and 8.

3.3.7 Fundamental value and bubbles

We have seen in this section that slight changes in the time setup and different assumptions on the values of β_l can change the price response to the signal dramatically.

But in which cases is there a bubble?

In the original model with destabilizing speculation, DSSW say that "informed rational speculators can push prices away from fundamentals" (De Long et al., 1990, p.388). They are talking in this context about a "positive-feedback bubble" (De Long et al., 1990, p.392) which they distinguish from a rational bubble. But to have a positive-feedback bubble one must know what the fundamental value is. Following the definition of the bubble in DSSW, one might think that the fundamental value is the equilibrium price in the absence of rational speculators. So a bubble is the difference between the equilibrium price in the presence ($\mu > 0$) and in the absence ($\mu = 0$) of rational speculators. The bubble in their model would therefore be

$$p_{1,\mu>0} - p_{1,\mu=0} = v + (1 + \frac{\beta_1}{\alpha - \beta_1})\phi'' - v = \frac{\alpha}{\alpha - \beta_1}$$

at date 1 and

$$p_{2,\mu>0} - p_{2,\mu=0} = v + (1 + \frac{\beta_1}{\alpha - \beta_1})\phi'' - v - \phi'' = \frac{\beta_1}{\alpha - \beta_1}$$

at date 2. When we consider the case with stabilizing speculation, we have a bubble with

$$p_{1,\mu>0} - p_{1,\mu=0} = v + \phi'' - v = \phi''$$

at date 1 and

$$p_{3,\mu>0} - p_{3,\mu=0} = v + \phi'' - v - \left(1 + \frac{\beta_1}{\alpha}\right)\phi'' = -\frac{\beta_1}{\alpha}\phi''$$

at date 3. So in that case we first have a positive and then a negative bubble after a *positive* signal was realized. This raises doubts to the upper definition of a bubble. The definition of the fundamental value is also worth discussing. Consider $\beta_l = \beta$ for all l. In the presence of rational speculators the price immediately rises to

$$p_{t,\mu>0} = v + \left(1 + \frac{\beta}{\alpha - \beta}\right)\phi''$$

(see proposition 6). In the absence of rational speculators in this setup the price is

$$p_{t,\mu=0} = v + \left[\sum_{\tau=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{\tau}\right]\phi''$$

(see proposition 7) and for a large $t - t^{e''}$ it converges to $v + \left(1 + \frac{\beta}{\beta-\alpha}\right)\phi''$. So in the presence of rational speculators the price jumps to the higher level whereas in the absence it slowly converges to it. One could therefore think that the fundamental value is $v + \left(1 + \frac{\beta}{\beta-\alpha}\right)\phi''$. This is in line with Loewenstein & Willard (2006). They argue that the consumption risk which is caused by noise traders is fundamental and should therefore be reflected in the price. When rational speculators invest in the asset they have to be compensated for the risk they carry. Let us consider D_2^r in the DSSW model. Solving for the period$-T$ price yields

$$p_T = v + \phi'' - \frac{D_T^r}{\alpha}.$$

The price therefore contains a premium or a discount when rational speculators' demand is not equal to zero.

Next, we consider rational bubbles. Let us think of the fundamental value as the discounted sum of future payoffs. When we have another equilibrium price, then we define the difference between this price and the fundamental value as a rational bubble. To have such a bubble, we need two equilibrium prices. By showing that the equilibrium price is unique, we can exclude these kind of bubbles. We will show in section 3.4.7 that the equilibrium price we consider is actually unique.

The final sort of a bubble we want to consider is a "speculative bubble". An agent does not necessary buy an asset to hold it to maturity but rather sells it earlier for a higher price (De Long et al., 1990, p.380). Define a buy-and-hold strategy that begins at date t as a marginal change in the holdings of the asset from date t to T ($dD_t^r = dD_{t+1}^r = \ldots = dD_T^r \neq 0$). The fundamental value at date t is the price with which the buy-and-hold strategy does not affect rational speculators' utility. The speculative bubble is then the difference between the equilibrium price and the fundamental value.

Proposition 10. *A speculative bubble does not exist in the model with one signal.*

Proof. Let $U_t^r = E_t^r W_{T+1}^r - \gamma\sigma^{r2}_{W_{t+1}^r|t}$ denote rational speculators' utility function at date t. Deriving the utility with respect to dD_t^r (we understand this as the total derivative since $dD_t^r = dD_{T+1}^r = \ldots = dD_T^r$), we have

$$\frac{dU_t^r}{dD_t^r} = \frac{dE_t^r W_{T+1}^r}{dD_t^r} - \gamma \frac{d\sigma^{r2}_{W_{t+1}^r|t}}{dD_t^r}.$$

Let us first consider $\frac{dE_t^r W_{T+1}^r}{dD_t^r}$. The date-$t$ expectation of the final wealth (3.6) is

$$E_t^r W_{T+1}^r = W_t^r + E_t^r\left[(p_{t+1} - p_t)D_t^r\right] + \ldots + E_t^r\left[(p_T - p_{T-1})D_{T-1}^r\right] + E_t^r\left[(v + \Phi - p_T)D_T^r\right].$$

Taking the total derivative we have

$$\begin{aligned}
dE_t^r W_{T+1}^r &= E_t^r(p_{t+1} - p_t)dD_t^r + \ldots + E_t^r(p_T - p_{T-1})dD_{T-1}^r \\
&\quad + E_t^r(v + \Phi - p_T)dD_T^r \\
&= (-p_t + v + E_t^r\Phi)dD_t^r \\
\Leftrightarrow \frac{dE_t^r W_{T+1}^r}{dD_t^r} &= -p_t + v + E_t^r\Phi.
\end{aligned}$$

Plugging the derivative of U_t^r in that function and solving for p_t, we have

$$p_t = v + E_t^r\Phi - \gamma \frac{d\sigma^{r2}_{W_{t+1}^r|t}}{dD_t^r} - \frac{dU_t^r}{dD_t^r}. \tag{3.20}$$

If the buy-and-hold strategy is unprofitable (i.e. $\frac{dU_t^r}{dD_t^r} = 0$), then the equilibrium price equals the fundamental value and therefore a speculative bubble does not exist. So we have to show that $\frac{dU_t^r}{dD_t^r} = 0$. We therefore need to calculate $\frac{dE_t^r W_{T+1}^r}{dD_t^r}$ and $\frac{d\sigma^{r2}_{W_{t+1}^r|t}}{dD_t^r}$. Note that

$$\frac{d\sigma^{r2}_{W_{t+1}^r|t}}{dD_t^r} = 2E_t^r\left[(W_{T+1}^r - E_t^r W_{T+1}^r)\frac{d(W_{T+1}^r - E_t^r W_{T+1}^r)}{dD_t^r}\right]. \tag{3.21}$$

For $t = t^{r''}, t^{r''}+1, \ldots, T$ equation (3.6) becomes $W_{T+1}^r = W_t^r + (v + \phi'' + \theta - p_T)D_T^r$. The date$-t$ expectation of the final wealth is $E_t^r W_{T+1}^r = W_t^r + (v + \phi'' - p_T)D_T^r$ so

$$\frac{d(W_{T+1}^r - E_t^r W_{T+1}^r)}{dD_T^r} = \frac{d\theta D_t^r}{dD_t^r} = \theta.$$

Together with $\frac{dE_t^r W_{T+1}^r}{dD_t^r} = (v + \phi'' - p_T)$, we have

$$\begin{aligned}
\frac{dU_t^r}{dD_t^r} &= (v + \phi'' - p_T) - 2\gamma E_t^r\left[\theta D_T^r \theta\right] \\
&= (v + \phi'' - p_T) - 2\gamma E_t^r\left[\alpha(v + \phi'' - p_T)\theta^2\right] \\
&= (v + \phi'' - p_T)(1 - 2\gamma\alpha\sigma_\theta^2) \\
&= 0,
\end{aligned}$$

since $\alpha = \frac{1}{2\gamma\sigma_\theta^2}$.

For $t = 0, 1, \ldots, t^{r''} - 1$, final wealth is

$$W_{T+1}^r = W_t^r + (p_{t^{r''}} - p_{t^{r''}-1})D_{t^{r''}-1}^r + (v + \phi'' + \theta - p_T)D_T^r.$$

The date$-t$ expectation is

$$E_t^r W_{T+1}^r = W_t^r + E_t^r \left[(p_{t^{r''}} - p_{t^{r''}-1})D_{t^{r''}-1}^r\right] + E_t^r \left[(v + \phi'' + \theta - p_T)D_T^r\right]$$

and the total derivative is

$$dE_t^r W_{T+1}^r = E_t^r \left(p_{t^{r''}} - p_{t^{r''}-1}\right) dD_{t^{r''}-1}^r + E_t^r \left(v + \phi'' + \theta - p_T\right) dD_T^r.$$

Together with (3.4) and $dD_t^r = dD_{T+1}^r = \ldots = dD_T^r$, we have

$$\begin{aligned} \frac{dE_t^r W_{T+1}^r}{dD_t^r} &= (E_t^r p_T - v) + (v + E_t^r \phi'' + E_t^r \theta - E_t^r p_T) \\ &= 0. \end{aligned}$$

The total derivative of the difference between final wealth and its expectation is

$$\begin{aligned} d(W_{T+1}^r - E_t^r W_{T+1}^r) &= (p_{t^{r''}} - p_{t^{r''}-1})dD_{t^{r''}-1}^r + (v + \phi'' + \theta - p_T)D_T^r \\ &\quad - E_t^r \left[p_{r''} - p_{t^{r''}-1}\right] dD_{t^{r''}-1}^r - E_t^r \left[v + \phi'' + \theta - p_T\right] dD_T^r \end{aligned}$$

so

$$\begin{aligned} \frac{d(W_{T+1}^r - E_t^r W_{T+1}^r)}{dD_t^r} &= p_{t^{r''}} - p_{t^{r''}-1} + (v + \phi'' + \theta - p_T) \\ &\quad - E_t^r \left[p_{t^{r''}} - p_{t^{r''}-1}\right] - E_t^r \left[v + \phi'' + \theta - p_T\right] \\ &= -v + (v + \phi'' + \theta) - (v - v) - (v - v) \\ &= \phi'' + \theta. \end{aligned}$$

Since $\phi' = 0$ and $p_{t^{r''}-1} = v$ we have $D_{t^{r''}-1}^r = 0$ (see equation (3.9)). Therefore the derivative of the variance is

$$\begin{aligned} \frac{d\sigma_{T+1|t}^2}{dD_t^r} &= 2E_t^r \left[\left((v + \phi'' + \theta - v - (1 + \lambda'')\phi''\right)\alpha(v + \phi'' - v - (1 + \lambda'')\phi'') \right. \\ &\quad \left. - E_t^r \left[(v + \phi'' + \theta - v - (1 + \lambda'')\phi'')\alpha(v + \phi'' - v - (1 + \lambda'')\phi'')\right]\right) (\phi'' + \theta)\Big] \\ &= 2E_t^r \left[\left(-\alpha\lambda''\phi''(\theta - \lambda''\phi'') - E_t^r \left[-\alpha\theta\lambda''\phi'' + \alpha\lambda''^2\phi''^2\right]\right) (\phi'' + \theta)\right] \\ &= 2E_t^r \left[\left(-\alpha\theta\lambda''\phi'' + \alpha\lambda''^2\phi''^2 - \alpha\lambda''^2\sigma_{\phi''}^2\right) (\phi'' + \theta)\right] \\ &= 0, \end{aligned}$$

since θ is has mean zero and is independent ($E_t^r[\theta\phi''] = E_t^r[\theta\phi''^2] = 0$) and ϕ'' has also mean zero and is symmetric ($E_t^r[\phi''^3] = 0$). Therefore $\frac{dU_t^r}{dD_t^r} = 0$, which proves the proposition. $\square$

Although a speculative bubble does not exist in the case with one signal, we will see in section 3.4.8 that such a speculative bubble may exist in the case with two signals (between the arrival of both signals).

3.4 Case with two signals

After the discussion of the model with one signal, we now turn to the case with two signals. In this section we assume that $\sigma^2_{\phi'} > 0$ so we have two signals. We first derive a general formula for the price reaction and then specify the different results in certain time setups. After that we show that the equilibrium is unique and that a speculative bubble may exist.

3.4.1 The general case

Other than in the case with one signal, we first show the general case and then discuss several special time setups that lead to different price pathes.

Proposition 11. *Let $\sigma^2_{\phi''} > 0$ and $D = \mathbb{1}_{t^{e'} < t^{r''}}$. For $\mu > 0$, the equilibrium asset prices satisfy (3.4) with*

$$\nu' = \frac{\beta_{T-t^{r'}} - (\alpha - \beta_{T-t^{r''}})\lambda'}{\beta_{T-t^{r''}} - \beta_{T-t^{r'}}} \tag{3.22}$$

$$= \frac{2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\left[(1-\mu)\alpha(D-1) + \beta_{(t^{r''}-1)-t^{r'}}\right]}{\mu + \left[(1-\mu)\alpha - \beta_{(t^{r''}-1)-t^{r'}}\right]2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} + \frac{\mu\left[1 - 4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right]\lambda'}{\mu + \left[(1-\mu)\alpha - \beta_{(t^{r''}-1)-t^{r'}}\right]2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} \tag{3.23}$$

and

$$\lambda'' = \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}. \tag{3.24}$$

Proof. To prove this proposition we plug the demand functions of the three types of agents into the market clearing condition and solve for the prices p_t. We look at the market clearing conditions at dates before information arrives ($t^{r'} - 1$, $t^{r''} - 1$ and T). For simplicity we set $S = 0$.

From equations (3.1), (3.2) and (3.9), the market clears at $t^{r'} - 1$ if

$$\begin{aligned} 0 &= D^f_{t^{r'}-1} + \mu D^r_{t^{r'}-1} + (1-\mu) D^e_{t^{r'}-1} \\ &= \mu \frac{v - p_{t^{r'}-1}}{2\gamma(1+\nu')^2\sigma^2_{\phi'}} + (1-\mu)\alpha(v - p_{t^{r'}-1}) \\ &= \left(\frac{\mu}{2\gamma(1+\nu')^2\sigma^2_{\phi'}} + (1-\mu)\alpha \right)(v - p_{t^{r'}-1}). \end{aligned}$$

Therefore $p_{t^{r'}-1} = v$ since $\frac{\mu}{2\gamma(1+\nu')^2\sigma^2_{\phi'}} + (1-\mu)\alpha > 0$
We get the demand functions at $t^{r''} - 1$ again from equations (3.1), (3.2) and (3.9). However, we have to consider different cases. If rational speculators receive both signals in sequent periods (i.e. $t^{r'} = t^{r''} - 1$) then positive feedback traders have no time to react to the first price change. Their demand is equal to zero in this case. If there is at least one trading period between the arrival of the two signals (i.e. $t^{r'} < t^{r''} - 1$) then positive feedback traders' demand is

$$D^f_{t^{r''}-1} = \beta_{t^{r''}-1-t^{r'}}(p_{t^{r'}} - p_{t^{r'}-1}). \tag{3.25}$$

By defining $\beta_0 = 0$, equation (3.25) also holds for $t^{r'} = t^{r''} - 1$. Passive investors' demand depends on weather they have already received their signal at $t^{r''} - 1$. If this is the case (i.e. $t^{e'} \le t^{r''} - 1$) then $D^e_{t^{r''}-1} = \alpha(v + \phi' - p_{t^{r''}-1})$. If not (i.e. $t^{e'} > t^{r''} - 1$) then $D^e_{t^{r''}-1} = \alpha(v - p_{t^{r''}-1})$. We have defined

$$D = \mathbb{1}_{t^{e'} < t^{r''}} = \begin{cases} 0 & \text{if } t^{e'} > t^{r''} - 1 \\ 1 & \text{if } t^{e'} \le t^{r''} - 1 \end{cases}$$

and therefore combine these two cases in

$$D^e_{t^{r''}-1} = \alpha(v + D\phi' - p_{t^{r''}-1}).$$

With (3.4) and (3.9), the market clearing condition is

$$\begin{aligned}
0 &= \beta_{t^{r''}-1-t^{r'}}(p_{t^{r'}} - p_{t^{r'}-1}) + (1-\mu)\alpha(v + D\phi' - p_{t^{r''}-1}) \\
&\quad +\mu\left[\frac{v + (1+\lambda')\phi' - p_{t^{r''}-1}}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}\right] \\
&= -\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}} + \frac{\mu}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right] p_{t^{r''}-1} \\
&\quad + \left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}} + \frac{\mu}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right] v \\
&\quad + \left[(1-\mu)\alpha D + \frac{\mu(1+\lambda') - 4\alpha\gamma\mu\lambda'\lambda''(1+\lambda'')\sigma^2_{\phi''}}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right]\phi' \\
&= -\left[\frac{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right] p_{t^{r''}-1} \\
&\quad + \left[\frac{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right] v \\
&\quad + \left[\frac{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}D(1-\mu)\alpha + (\mu - 4\alpha\gamma\mu\lambda''(1+\lambda'')\sigma^2_{\phi''})\lambda'}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right]\phi'.
\end{aligned}$$

Therefore

$$\begin{aligned}
p_{t^{r''}-1} &= v + \left[\frac{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}\right]\phi' \\
&\quad + \left[\frac{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}D(1-\mu)\alpha - 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}\right]\phi' \\
&\quad + \left[\frac{\left(\mu - 4\alpha\gamma\mu\lambda''(1+\lambda'')\sigma^2_{\phi''}\right)\lambda'}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}\right]\phi' \\
&= v + \left[1 + \frac{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha(D-1) + \beta_{t^{r''}-1-t^{r'}}\right]}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}\right. \\
&\quad \left. + \frac{\left(\mu - 4\alpha\gamma\mu\lambda''(1+\lambda'')\sigma^2_{\phi''}\right)\lambda'}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}\right]\phi''
\end{aligned}$$

so

$$\begin{aligned}
\nu' = &\frac{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha(D-1) + \beta_{t^{r''}-1-t^{r'}}\right]}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]} \\
&+ \frac{\mu - 4\alpha\gamma\mu\lambda''(1+\lambda'')\sigma^2_{\phi''}}{\mu + 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left[(1-\mu)\alpha - \beta_{t^{r''}-1-t^{r'}}\right]}\lambda'
\end{aligned}$$

which proves (3.23).

The demand functions in period T are given by the equations (3.1), (3.2) and (3.7). Plugging these into the market clearing condition and using (3.4) we get

$$\begin{aligned}
0 &= \beta_{T-t^{r'}}(p_{t^{r'}} - p_{t^{r'}-1}) + \beta_{T-t^{r''}}(p_{t^{r''}} - p_{r''-1}) + \alpha(v + \phi' + \phi'' - p_T) \\
&= \beta_{T-t^{r'}}(1+\nu')\phi' + \beta_{T-t^{r''}}(p_T - v - (1+\nu')\phi') + \alpha(v + \phi' + \phi'' - p_T) \\
&= -(\alpha - \beta_{T-t^{r''}})p_T + (\alpha - \beta_{T-t^{r''}})v + (\alpha - \beta_{T-t^{r''}})\phi' \\
&\quad + \beta_{T-t^{r'}}(1+\nu')\phi' - \beta_{T-t^{r''}}\nu'\phi' + \alpha\phi'' \\
&= -(\alpha - \beta_{T-t^{r''}})p_T + (\alpha - \beta_{T-t^{r''}})v + (\alpha - \beta_{T-t^{r''}})\phi' + (\alpha - \beta_{T-t^{r''}})\phi'' \\
&\quad + \beta_{T-t^{r'}}\phi' + (\beta_{T-t^{r'}} - \beta_{T-t^{r''}})\nu'\phi' + \beta_{T-t^{r''}}\phi''.
\end{aligned}$$

Solving for the price yields to

$$p_T = v + \left(1 + \frac{\beta_{T-t^{r'}} + (\beta_{T-t^{r'}} - \beta_{T-t^{r''}})\nu'}{\alpha - \beta_{T-t^{r''}}}\right)\phi' + \left(1 + \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)\phi''$$

So

$$\lambda' = \frac{\beta_{T-t^{r'}} + (\beta_{T-t^{r'}} - \beta_{T-t^{r''}})\nu'}{\alpha - \beta_{T-t^{r''}}}$$

and

$$\lambda'' = \frac{\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}$$

which proves (3.24). To get (3.22), we solve the upper equation for ν'.

$$\begin{aligned}
& & \lambda' &= \frac{\beta_{T-t^{r'}} + (\beta_{T-t^{r'}} - \beta_{T-t^{r''}})\nu'}{\alpha - \beta_{T-t^{r''}}} \\
&\Leftrightarrow & (\alpha - \beta_{T-t^{r''}})\lambda' &= \beta_{T-t^{r'}} + (\beta_{T-t^{r'}} - \beta_{T-t^{r''}})\nu' \\
&\Leftrightarrow & (\beta_{T-t^{r''}} - \beta_{T-t^{r'}})\nu' &= \beta_{T-t^{r'}} - (\alpha - \beta_{T-t^{r''}})\lambda' \\
&\Leftrightarrow & \nu' &= \frac{\beta_{T-t^{r'}} - (\alpha - \beta_{T-t^{r''}})\lambda'}{\beta_{T-t^{r''}} - \beta_{T-t^{r'}}}
\end{aligned}$$

This proves (3.22). □

Before considering special time setups we can draw some general conclusions from proposition 11.

We see that the response to the second signal (3.24) is essentially the same as in the general case with one signal. When we assume again that β_l is a decreasing sequence then the response to the second signal is again smaller, the earlier the second signal arrives (or the more trading periods there are).

The reaction to the first signal is given by the two equations (3.22) and (3.23). Since the relation between λ' and ν' is linear and since we have two equations, we can solve for both parameters. Note, that when we maintain the assumptions that $\beta_l > \beta_k$ for $l < k$ and $\alpha > \beta_l \, \forall l$, then equation (3.22) tells us that λ' and ν' are inversely related. An increase in ν' leads to a decrease in λ' and vice versa. In other words, the stronger the reaction to the first signal when rational speculators receive it, the weaker the reaction to it when the second signal arrives. In the extreme case, when one of the two parameters is negative, then the other is positive. So it might be the case that we have underreaction to the first signal in the beginning (i.e. $\nu' < 0$) and overreaction after the second signal arrives ($\lambda' > 0$). The opposite case is of course also possible (first overreaction ($\nu' > 0$), then underreaction ($\lambda' < 0$)).

We will see in the following subsections that the equilibrium price can be obtained by finding the intersection of the two lines given by equations (3.22) and (3.23). So it may happen for certain parameter conditions that an equilibrium price does not exist. This happens when the lines are parallel or, mathematically speaking, when their slopes are the same, that is if

$$\frac{-(\alpha - \beta_{T-t^{r''}})}{\beta_{T-t^{r''}} - \beta_{T-t^{r'}}} = \frac{\mu \left[1 - 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}\right]}{\mu + 2\gamma(1+\lambda'')^2 \left[(1-\mu)\alpha - \beta_{t^{r''}-t^{r'}-1}\right] \sigma^2_{\phi''}}. \tag{3.26}$$

Another point worth mentioning is that the variance of the first signal $\sigma^2_{\phi'}$ does not affect the prices. This is due to the fact that both rational speculators and passive investors have zero demand (since $p_t = v$ for $t = 0, 1, \ldots, t^{r'} - 1$, $D^r_{t^{r'}-1} = 0$) before the arrival of of the first signal.

In the next three subsections we consider different time setups that lead to different results. In the proof of proposition 11 we considered different demand functions of passive investors and positive feedback traders which depended on the time structure of the model. We allowed D to be either zero or one and $\beta_{t^{t''}-1-t^{r'}}$ to be equal to zero or not equal to zero. Since we have two cases for D and two cases for $\beta_{t^{t''}-1-t^{r'}}$ we have four cases overall:

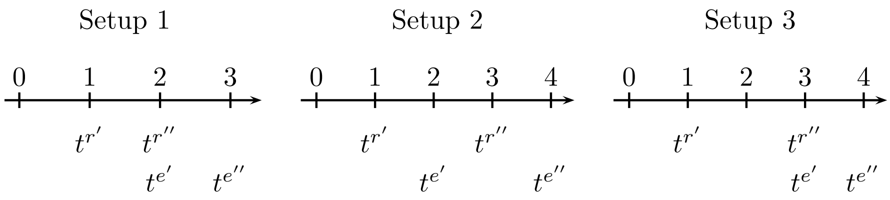


Figure 3.5: The three different time setups.

- $t^{e'} < t^{r''}$ and $t^{r''} - 1 - t^{r'} = 0$ ($\Rightarrow D = 1, \beta_{(t^{r''}-1)-t^{r'}} = 0$)
- $t^{e'} < t^{r''}$ and $t^{r''} - 1 - t^{r'} > 0$ ($\Rightarrow D = 1, \beta_{(t^{r''}-1)-t^{r'}} \geq 0$)
- $t^{e'} \geq t^{r''}$ and $t^{r''} - 1 - t^{r'} = 0$ ($\Rightarrow D = 0, \beta_{(t^{r''}-1)-t^{r'}} = 0$)
- $t^{e'} \geq t^{r''}$ and $t^{r''} - 1 - t^{r'} > 0$ ($\Rightarrow D = 0, \beta_{(t^{r''}-1)-t^{r'}} \geq 0$)

The first case is by assumption not possible. Let $t^{e'} < t^{r''}$ and $t^{r''} - 1 - t^{r'} = 0 \Leftrightarrow t^{r''} = t^{r'} + 1$. Then

$$t^{e'} < t^{r''} = t^{r'} + 1.$$

But this means that passive investors receive their first signal at least at the same date or even before rational speculators learn about the first signal. This contradicts our model assumption $t^{r'} < t^{e'}$ so only three cases are left.

These cases are discussed in the next three subsections. For simplicity we consider specific dates instead of using $t^{r'}$, $t^{r''}$, $t^{e'}$, $t^{e''}$ and T. The time structure of these three setups is depicted in figure 3.5.

3.4.2 Setup 1

Corollary 4. *Let $T = 3$, $t^{r'} = 1$, $t^{r''} = 2$, $t^{e'} = 2$ and $t^{e''} = 3$. Then equations (3.22) and (3.23) become*

$$\nu' = \frac{\beta_2 - (\alpha - \beta_1)\lambda'}{\beta_1 - \beta_2} = \frac{-2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''} + \mu\left[1 - 4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right]\lambda'}{\mu + 2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}}. \tag{3.27}$$

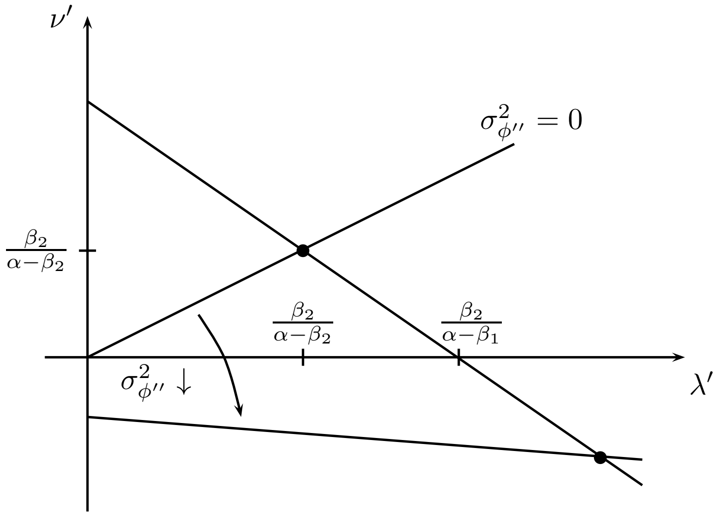


Figure 3.6: Determination of λ' and ν'

Proof. Since $t^{e'} = t^{r''} = 2 \Rightarrow D = 0$ and since $t^{r'} = t^{r''} - 1 = 1 \Rightarrow \beta_{1-1} = \beta_0 = 0$. Then (3.27) follows immediately from proposition 11. □

We solve for λ' and ν' graphically. We therefore draw the two lines

$$\nu' = \frac{\beta_2 - (\alpha - \beta_1)\lambda'}{\beta_1 - \beta_2}$$

and

$$\nu' = \frac{-2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''} + \mu\left[1 - 4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right]\lambda'}{\mu + 2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}}$$

from equation (3.27) in a λ'-ν'−diagram. The intersection of the two functions gives us the equilibrium values of λ' and ν'. Both functions are linear in λ' which means that they are both lines. We first consider

$$\nu' = \frac{\beta_2 - (\alpha - \beta_1)\lambda'}{\beta_1 - \beta_2}. \tag{3.28}$$

If we assume that earlier news have a weaker impact on positive feedback traders' demand (i.e. $\beta_1 > \beta_2$), then (3.28) is a decreasing function with positive intercept

$\frac{\beta_2}{\beta_1-\beta_2}$. It intersects the λ'−axis at $\frac{\beta_2}{\alpha-\beta_1}$ since

$$\begin{aligned} 0 &= \frac{\beta_2-(\alpha-\beta_1)\lambda'}{\beta_1-\beta_2} \\ \Leftrightarrow \quad (\alpha-\beta_2)\lambda' &= \beta_2 \\ \Leftrightarrow \quad \lambda' &= \frac{\beta_2}{\alpha-\beta_1}. \end{aligned}$$

To draw

$$\nu' = \frac{-2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''} + \mu\left[1-4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right]\lambda'}{\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}}, \tag{3.29}$$

we first consider the case $\sigma^2_{\phi''}=0$. Then (3.29) becomes $\nu' = \frac{0}{\mu}+\frac{\mu}{\mu}\lambda' = \lambda'$ so it is the line through the origin with slope 1. This line intersects (3.28) at $\frac{\beta_2}{\alpha-\beta_2}$ since

$$\begin{aligned} \lambda' &= \frac{\beta_2-(\alpha-\beta_1)\lambda'}{\beta_1-\beta_2} \\ \Leftrightarrow \quad (\beta_1-\beta_2)\lambda' &= \beta_2-(\alpha-\beta_1)\lambda' \\ \Leftrightarrow \quad (\alpha-\beta_2)\lambda' &= \beta_2 \\ \Leftrightarrow \quad \lambda' &= \frac{\beta_2}{\alpha-\beta_2}. \end{aligned}$$

To see how an increase in $\sigma^2_{\phi''}$ affects λ' and ν', we derive the intercept and the slope with respect to $\sigma^2_{\phi''}$:

$$\begin{aligned} & \frac{\partial}{\partial\sigma^2_{\phi''}}\frac{-2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}}{\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}} \\ = \quad & \frac{(\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''})(-2\alpha\gamma(1-\mu)(1+\lambda'')^2)}{\left[\mu+(1-\mu)\alpha 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\right]^2} \\ & -\frac{2\alpha\gamma(1-\mu)(1+\lambda'')^2(-2\alpha\gamma(1-\mu)(1+\lambda'')^2)\sigma^2_{\phi''}}{\left[\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}\right]^2} \\ = \quad & \frac{-2\alpha\gamma\mu(1-\mu)(1+\lambda'')^2}{\left[\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}\right]^2} < 0 \end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\sigma^2_{\phi''}}\frac{\mu\left(1-4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}\right)}{\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}} \\
=\;& \frac{(\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''})(-4\alpha\gamma\lambda''\mu(1+\lambda''))}{\left[\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}\right]^2} \\
&-\frac{\left(\mu-4\alpha\gamma\lambda''\mu(1+\lambda'')\sigma^2_{\phi''}\right)2\alpha\gamma(1-\mu)(1+\lambda'')^2}{\left[\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}\right]^2} \\
=\;& \frac{-4\alpha\gamma\lambda''\mu^2(1+\lambda'')-2\alpha\gamma\mu(1-\mu)(1+\lambda'')^2}{\left[\mu+2\alpha\gamma(1-\mu)(1+\lambda'')^2\sigma^2_{\phi''}\right]^2}<0
\end{aligned}$$

since $0<\mu<1$, $\alpha>0$, $\gamma>0$ and $\lambda''>0$. Therefore, as $\sigma^2_{\phi''}$ rises, the intercept and the slope decrease so $\nu'<\frac{\beta_2}{\alpha-\beta_2}<\lambda'$. We can show that the intercept converges to -1 and the slope to $-\frac{2\mu\lambda''}{(1-\mu)(1+\lambda'')}$ as $\sigma^2_{\phi''}\to\infty$. Since the denominator and the numerator of (3.29) are linear functions in $\sigma^2_{\phi''}$, they both go to $\pm\infty$ as $\sigma^2_{\phi''}\to\infty$ and we can apply l'Hospital's rule[12]. Therefore

$$\lim_{\sigma^2_{\phi''}\to\infty}\frac{-2(1-\mu)\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2}{\mu+2(1-\mu)\alpha\gamma(1+\lambda'')^2\sigma^2_{\phi''}}=\lim_{\sigma^2_{\phi''}\to\infty}\frac{-2(1-\mu)\alpha\gamma(1+\lambda'')^2}{2(1-\mu)\alpha\gamma(1+\lambda'')^2}=-1 \tag{3.30}$$

and

$$\lim_{\sigma^2_{\phi''}\to\infty}\frac{\mu\left(1-4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right)}{\mu+(1-\mu)\alpha2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}=\lim_{\sigma^2_{\phi''}\to\infty}\frac{-4\alpha\lambda''\gamma\mu(1+\lambda'')}{(1-\mu)\alpha2\gamma(1+\lambda'')^2}=-\frac{2\mu\lambda''}{(1-\mu)(1+\lambda'')}. \tag{3.31}$$

So for $\sigma^2_{\phi''}$ large enough $\nu'<-1$. That means there there may be underreaction to a positive signal. If the realizations of ϕ' and ϕ'' are both positive then two possible price paths are depicted in figure 3.7. The solid line represents the case where the variance of the second signal is close to zero. In this case, the intersection of the two lines represented by (3.28) and (3.29) is in the first quadrant in the $\lambda'-\nu'-$diagram. Since $\sigma^2_{\phi''}>0$ they intersect below the 45°-line (see figure 3.6). Therefore $\lambda'>\nu'$. We have relatively little overreaction to the first signal after it arrived but a much stronger reaction to it when the second signal arrives. Note that there exists $\sigma^2_{\phi''}>0$ for which $\nu'=0$ so we have neither under- nor overreaction after the first signal arrived. As

[12]Let $f(x)=\frac{m(x)}{n(x)}$. As $x\to a$, then $m(x)$ and $n(x)$ go both to 0 or $\pm\infty$. Then $\lim\limits_{x\to a}\frac{m(x)}{n(x)}=\lim\limits_{x\to a}\frac{m'(x)}{n'(x)}$ (Chiang & Wainwright, 2005, p.399).

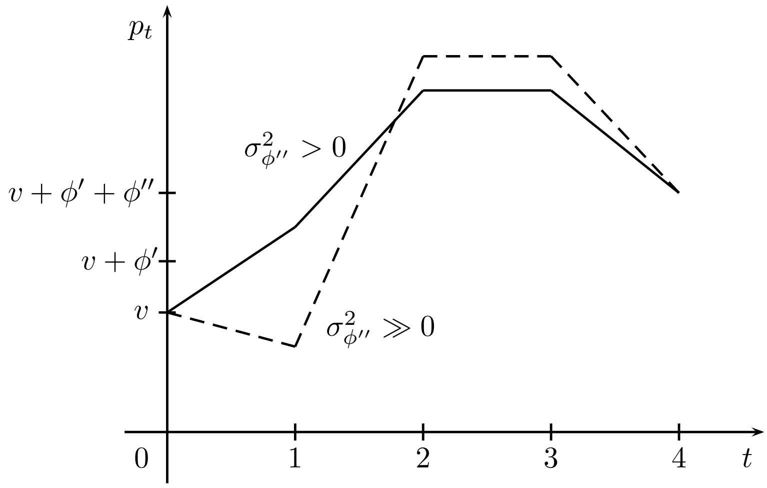


Figure 3.7: Possible prices reactions

$\sigma^2_{\phi''}$ increases, the intersection of (3.28) and (3.29) moves to the fourth quadrant in the λ'-$\nu'-$coordinate system (see figure 3.6). Then $\nu' < 0 < \lambda'$. That means there is underreaction to the first signal when it is realized but overreaction when the second signal arrives. The dashed line in figure 3.7 illustrates the price path in this case.

Another point worth mentioning is that it might be the case that there exists no equilibrium. This is true when the slopes of equations (3.28) and (3.29) are equal. To see that this might happen, we consider the limit of the slope of (3.29) which is $-\frac{2\mu\lambda''}{(1-\mu)(1+\lambda'')} = -\frac{2\mu\beta_1}{(1-\mu)\alpha}$ as $\sigma^2_{\phi''} \to \infty$. As long as

$$\begin{aligned} -\frac{2\mu\beta_1}{(1-\mu)\alpha} &> -\frac{\alpha-\beta_1}{\beta_1-\beta_2} \\ \Leftrightarrow -\frac{2\mu}{1-\mu} &> -\frac{\alpha-\beta_1}{\beta_1-\beta_2}\frac{\alpha}{\beta_1}, \end{aligned}$$

the slope of equation (3.28) is steeper than the slope of equation (3.29). The left-hand side of this inequality is equal to zero for $\mu = 0$ and as μ increases, $\frac{\mu}{1-\mu}$ increases and goes to ∞ as μ goes to 1. So the left-hand side of the upper inequality lies between $-\infty$ and zero for $\mu \in [0,1]$. Since $\frac{\alpha-\beta_1}{\beta_1-\beta_2}\frac{\alpha}{\beta_1} > 0$ (we assumed $\alpha > \beta_1$ and $\beta_1 > \beta_2$), there are values of μ such that

$$-\frac{2\mu}{1-\mu} = -\frac{\alpha-\beta_1}{\beta_1-\beta_2}\frac{\alpha}{\beta_1}$$

and

$$-\frac{2\mu}{1-\mu} < -\frac{\alpha-\beta_1}{\beta_1-\beta_2}\frac{\alpha}{\beta_1}.$$

In the case of equality, the slopes of (3.28) and (3.29) are equal so both lines are parallel. Since they have no intersection point, there exists no equilibrium (they cannot be identical since the intercept of (3.28) is positive ($\frac{\beta_2}{\beta_1-\beta_2}$) and the intercept of (3.29) is negative (in the interval $(-1,0]$)). If the inequality holds, then the line given by (3.29) is steeper than the line given by (3.28). Since the intercept of (3.28) is greater than zero ($\frac{\beta_2}{\beta_1-\beta_2}$) and the intercept of (3.29) is in $(-1,0]$, the lines intersect in the second quadrant of the $\lambda'-\nu'-$diagram. Therefore $\lambda' < 0 < \nu'$. So we have overreaction to the first signal when it arrives, but underreaction when the second signal arrives.

But why is it that an increase in the variance of the second signal ($\sigma^2_{\phi''}$) has an influence on the degree and even the direction of under- and overreaction of the reaction to the first signal? The reason is the covariance effect mentioned in section 3.2. Recall that

$$D^r_{t^{r''}-1} = \frac{v+(1+\lambda')\phi' - p_{t^{r''}-1}}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}.$$

We have shown that the covariance between the returns on investment at date $t^{r''}-1$ and T is given by the second summand. As $\sigma^2_{\phi''}$ increases, the first part $\left(\frac{v+(1+\lambda')\phi'-p_{t^{r''}-1}}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}\right)$ of the demand decreases[13] whereas the second term $\left(\frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}\right)$ increases. Therefore the demand decreases and can even becomes negative. So if the variance of the second signal is large, rational speculators are willing to hedge against the impact of ϕ'' at date T.

3.4.3 Setup 2

Corollary 5. *Let $T=4$, $t^{r'}=1$, $t^{r''}=3$, $t^{e'}=2$ and $t^{e''}=4$. Then equations (3.22) and (3.23) become*

$$\nu' = \frac{\beta_3-(\alpha-\beta_1)\lambda'}{\beta_1-\beta_3} \tag{3.32}$$

[13] Note that an increase in $\sigma^2_{\phi''}$ also increases λ'. But since λ' (and therefore the numerator) is bounded (this comes from equations (3.30) and (3.31)), there exists $\sigma^2_{\phi''}$ above which every increase in the variance decreases the first term of the demand.

and

$$\nu' = \frac{2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\beta_1 + \mu\left(1 - 4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right)\lambda'}{\mu + \left[(1-\mu)\alpha - \beta_1\right]2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}. \tag{3.33}$$

Proof. Since $t^{e'} = 2$ and $t^{r''} = 3 \Rightarrow D = 1$ and since $t^{r'} = 1$ and $t^{r''} = 3 \Rightarrow \beta_{3-1-1} = \beta_1$. Then (3.33) follows immediately from proposition 11. □

The determination of λ' and ν' is a little bit more complicated in this setup. This is because, in contrast to the setup in the previous section, it may happen that the denominator becomes zero or negative. We therefore have to distinguish different parameter combinations that lead to very different results. Instead of starting with $\sigma^2_{\phi''} = 0$ and increasing it, we assume that is positive and constant and only vary the measure μ of rational speculators.

Let us begin with decomposing the linear function in λ' (3.33) in its intercept

$$\frac{2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\beta_1}{\mu + \left[(1-\mu)\alpha - \beta_1\right]2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} \tag{3.34}$$

and its slope

$$\frac{\mu - 4\alpha\gamma\mu\lambda''(1+\lambda'')\sigma^2_{\phi''}}{\mu + \left[(1-\mu)\alpha - \beta_1\right]2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}. \tag{3.35}$$

We first examine the intercept a little closer. The numerator is constant in μ, so in our case μ affects only the denominator of the intercept. The denominator is a linear function in μ

$$\mu + 2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\left[(1-\mu)\alpha - \beta_1\right] = \mu\left(1 - 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2\right) + (\alpha - \beta_1)2\gamma\sigma^2_{\phi''}(1+\lambda'')^2$$

and greater than zero for $\mu = 0$ since $\alpha > \beta_1$, $\gamma > 0$, $\sigma^2_{\phi''} > 0$ and $\lambda'' > 0$. It is increasing (function I in the left panel of figure 3.8) if

$$1 > 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$$

and decreasing (functions II and III in the left panel of figure 3.8) if

$$1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2.$$

We have to distinguish two different cases when it is decreasing. It may be positive for $\mu = 1$ (and therefore positive for all $\mu \in (0, 1]$; function II in the left panel of

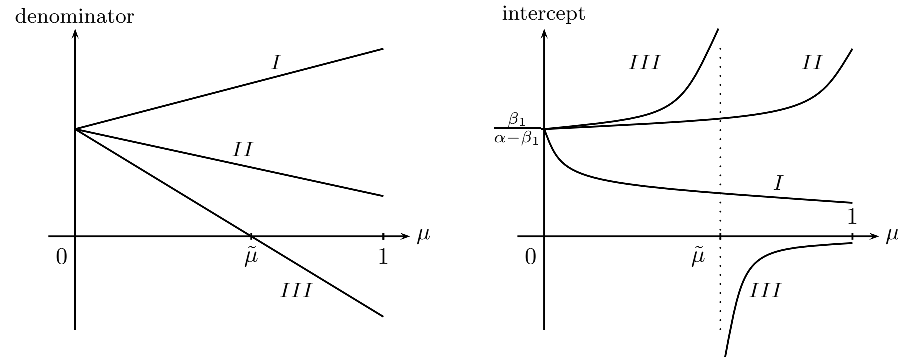


Figure 3.8: Shape of the denominator (left panel) and of (3.34) (right panel).

figure 3.8) or negative for $\mu = 1$ (in that case $\exists\, \tilde{\mu} \in (0, 1]$ such that the denominator is equal to zero; function III in the left panel of figure 3.8). To see that, we calculate the denominator for $\mu = 1$ and get $1 - 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2$. If

$$1 > 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2,$$

the denominator is positive for $\mu = 1$, if

$$1 < 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2,$$

it is negative. Since $\alpha > \beta_1$, we have

$$2\alpha\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 > 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2,$$

so there are three different cases to consider:

I $2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 < 1$

II $2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 < 1 < 2\alpha\gamma\sigma^2_{\phi''}(1 + \lambda'')^2$

III $1 < 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1 + \lambda'')^2$.

Therefore we have three different shapes of the intercept as a function of μ depending on the parameter constellation. In case I, the denominator is positive and increasing in μ so the intercept is positive, bounded and decreasing in μ (function I in the right panel of figure 3.8). If the denominator is positive and decreasing (case II), the intercept is also positive, bounded but increasing (function II in the right panel of figure 3.8). In the third case (III), the intercept is always increasing[14] but it is positive (and goes to ∞) for $\mu < \tilde{\mu}$ and negative (comes from $-\infty$) for $\tilde{\mu} < \mu$ (function III in the right panel of figure 3.8).

For the determination of λ' and ν' we have to check whether the intercept of (3.33) for $\mu = 0$ is smaller or larger than the intercept of (3.32). The intercept of (3.33) for $\mu = 0$ is

$$\frac{2\beta_1\gamma(1+\lambda'')^2\sigma^2_{\phi''}}{(\alpha-\beta_1)2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} = \frac{\beta_1}{\alpha-\beta_1}.$$

Then

$$\begin{aligned}
\frac{\beta_1}{\alpha-\beta_1} &\lesseqgtr \frac{\beta_3}{\beta_1-\beta_3} \\
\Leftrightarrow \beta_1(\beta_1-\beta_3) &\lesseqgtr \beta_3(\alpha-\beta_1) \\
\Leftrightarrow \beta_1^2-\beta_1\beta_3 &\lesseqgtr \alpha\beta_3-\beta_1\beta_3 \\
\Leftrightarrow \beta_1^2 &\lesseqgtr \alpha\beta_3,
\end{aligned}$$

so the intercept of (3.33) with $\mu = 0$ is smaller than the intercept of (3.32) if $\beta_1^2 < \alpha\beta_3$ and larger if $\beta_1^2 > \alpha\beta_3$.

Next, we consider the slope

$$\frac{\mu - 4\alpha\gamma\mu\lambda''(1+\lambda'')\sigma^2_{\phi''}}{\mu + [(1-\mu)\alpha-\beta_1]\,2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}$$

of equation (3.33). The denominator coincides with the denominator of the intercept so the the results from above apply here.[15] We therefore still have to examine the nu-

[14]This can be validated by taking the derivative of the intercept with respect to μ

$$\frac{\partial(3.34)}{\partial\mu} = -\frac{2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\beta_1(1-2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2)}{\left(\mu+[(1-\mu)\alpha-\beta_1]\,2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\right)^2} > 0.$$

[15]Recall the three cases

I $2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1$

II $2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$

III $1 < 2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$.

The behavior of the denominator in these cases is as depicted in the left panel of figure 3.8.

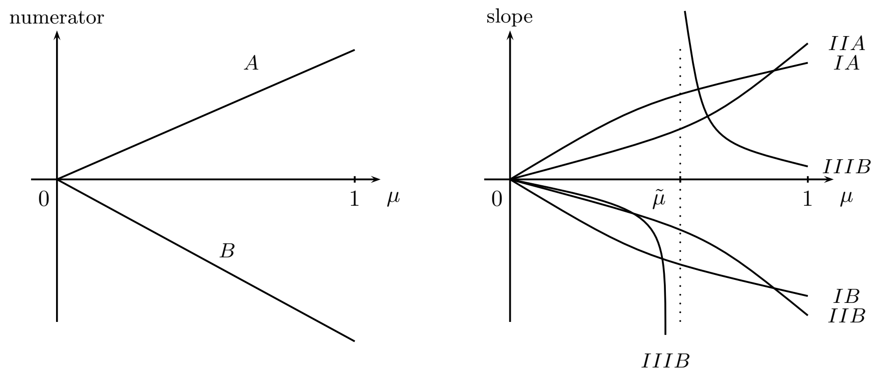


Figure 3.9: Shape of the numerator of the slope (left panel) and of (3.35) (right panel).

merator to characterize the behavior of the slope. Like the denominator, the numerator is a linear function in μ. For $\mu = 0$ it is equal to zero and for $\mu = 1$ we have

$$1 - 4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''}$$

which is also the slope of the numerator. We therefore have to consider two cases. The numerator is positive and increasing (case A in the left panel of figure 3.9) if

$$1 > 4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''}$$

and negative and decreasing (case B in the left panel of figure 3.9) if

$$1 < 4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''}.$$

Note, that

$$\alpha\lambda'' = \alpha\frac{\beta_1}{\alpha - \beta_1} = \beta_1\frac{\alpha}{\alpha - \beta_1} = \beta_1(1 + \lambda''),$$

so

$$4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''} = 4\beta_1\gamma(1 + \lambda'')^2\sigma^2_{\phi''} > 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2.$$

This yields five different cases:[16]

IA $2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1 + \lambda'')^2 < 1$ and $4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''} < 1$

[16] In case III, we have $1 < 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2$. Since $4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''} > 2\beta_1\gamma\sigma^2_{\phi''}(1 + \lambda'')^2$ we have $1 < 4\alpha\gamma\lambda''(1 + \lambda'')\sigma^2_{\phi''}$ so case III together with case A is not possible.

IB $2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1$ and $1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$

IIA $2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ and $4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''} < 1$

IIB $2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ and $1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$

IIIB $1 < 2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ and $1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$.

In the following we will characterize these cases. The slope is a rational function of the form

$$\frac{a\mu}{b+c\mu}, \tag{3.36}$$

where $a = 1-4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$, b is positive and constant and $c = 1-2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$. Therefore a is positive (negative) if $1 > 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$ ($1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$) and c is positive (negative) if $1 > 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ ($1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$). The first derivative of the rational function is

$$\left(\frac{a\mu}{b+c\mu}\right)' = \frac{(b+c\mu)a - a\mu c}{(b+c\mu)^2} = \frac{ab}{(b+c\mu)^2}$$

and the second derivative is

$$\left(\frac{a\mu}{b+c\mu}\right)'' = \frac{-ab2c(b+c\mu)}{(b+c\mu)^4} = \frac{-2abc}{(b+c\mu)^3}.$$

Note that all cases have in common that the numerator is zero for $\mu = 0$ so all slope-functions go through the origin.

Case IA

In this case we have $2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1$ and $4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''} < 1$. Therefore a and c in (3.36) are both positive. Hence, the first derivative is also positive and the second derivative is negative (since the denominator is positive) so the slope is increasing and concave. The function is depicted in the the right panel of figure 3.9.

So overall, the intercept decreases and the slope increases.

Case IB

As in case IA, c is again positive but a is negative since $1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$. The first derivative is therefore negative and the second derivative is positive (since the denominator is positive). The resulting convex and decreasing function is also depicted in the the right panel of figure 3.9.

As in case IA, the intercept decreases in this case, whereas the slope decreases.

Case IIA

We now consider the case where $2\beta_1\gamma\sigma^2_{\phi''}(1+\lambda'')^2 < 1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ and $4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''} < 1$ so a is now positive and c is negative. The slope is therefore increasing and convex (since the denominator is positive; see right panel of figure 3.9).

In this case both the slope and the intercept increase as μ increases. This may lead to mean reversion which we will discuss after the consideration of all cases.

Case IIB

If we have $1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$ instead of $4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''} < 1$, then both a and c are negative so the slope is a decreasing concave (since the denominator is positive) function (again right panel in figure 3.9).

Now, the intercept is increasing as above, but the slope decreases as μ increases. This may lead to a overreaction in the beginning and to a underreaction afterwards. We will discuss this example subsequently.

Case IIIB

As in case IIB, we have $1 < 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ and $1 < 4\alpha\gamma\lambda''(1+\lambda'')\sigma^2_{\phi''}$ so a and c are again both negative. Therefore the slope is again decreasing. But since the denominator of the slope is positive for $\mu < \tilde{\mu}$ and negative for $\tilde{\mu} < \mu$, the slope is concave for $\mu < \tilde{\mu}$ and convex for $\tilde{\mu} < \mu$. For $\mu \to \tilde{\mu}^{\pm}$, the slope goes to $\mp\infty$. As all other cases, this is also depicted in the right panel of figure 3.9.

So overall, the intercept increases as μ increases but is positive as long as $\mu < \bar{\mu}$ and negative otherwise. In this parameter constellation almost all price reactions are possible for different values of μ.

Note, that it is very important for the results whether the intercept of (3.32) lies above or below the intercept of (3.33) for $\mu = 0$.

We will now pick out two examples of price reactions. We have seen in the discussion of setup 1 that it might happen that the reaction to the first signal is smaller when it arrives than when the second signal arrives ($0 < \nu' < \lambda'$). For a large variance we even had underreaction of the first signal when it arrived and a strong overreaction later ($\nu' < 0 < \lambda'$). But other price reactions are also possible. We will now discuss the

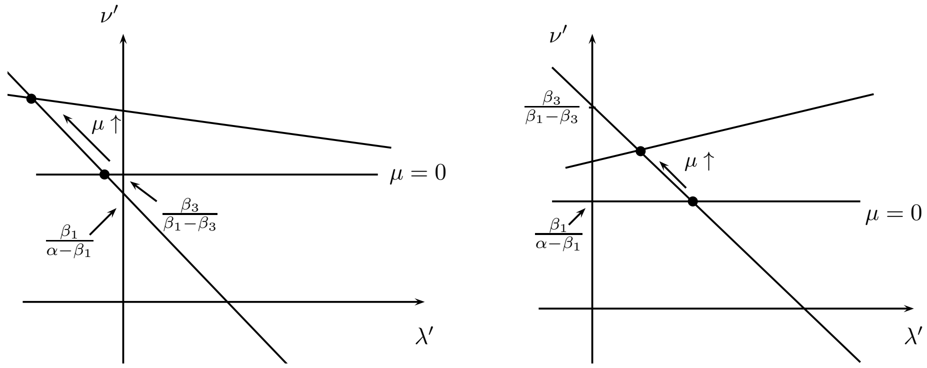


Figure 3.10: Determination of ν' and λ' in the case IIB with $\alpha\beta_3 < \beta_1^2$ (left panel) and in the case IIA with $\beta_1^2 < \alpha\beta_3$ (right panel).

remaining two possible price reactions ($\lambda' < 0 < \nu'$ and $0 < \lambda' < \nu'$).

Let us consider case IIB, where the intercept of equation (3.32) is smaller than the intercept of equation (3.33) so $\alpha\beta_3 < \beta_1^2$. For $\mu = 0$, the line given by equation (3.33) has slope zero and lies above the intercept of the ν'-axis and the line given by equation (3.32). We have seen above that if μ increases, the intercept of (3.33) increases and its slope decreases. Therefore the intersection of (3.32) and (3.33) lies in the second quadrant of the coordinate system so $\lambda' < 0 < \nu'$. This situation is depicted in the left panel of figure 3.10. If rational speculators receive a positive signal in that case ($\phi' > 0$), the price increases to a higher level before it falls very strongly when the second signal arrives. A possible price path is depicted by the dashed line in figure 3.11.

The next case we take a closer look at is case IIA where $\beta_1^2 < \alpha\beta_3$. In that case, the line given by equation (3.33) lies below the intersection of (3.32) with the ν'-axis for $\mu = 0$. As μ increases both, the slope and the intercept of (3.33) increase and for small values of μ, both lines intersect in the first quadrant of the coordinate system (see right panel of figure 3.10). In that case $0 < \lambda' < \nu'$, so we have mean reversion. There is a strong overreaction to the first signal that weakens (but stays positive) when the second signal arrives. A possible price path is depicted by the solid line in figure 3.11.

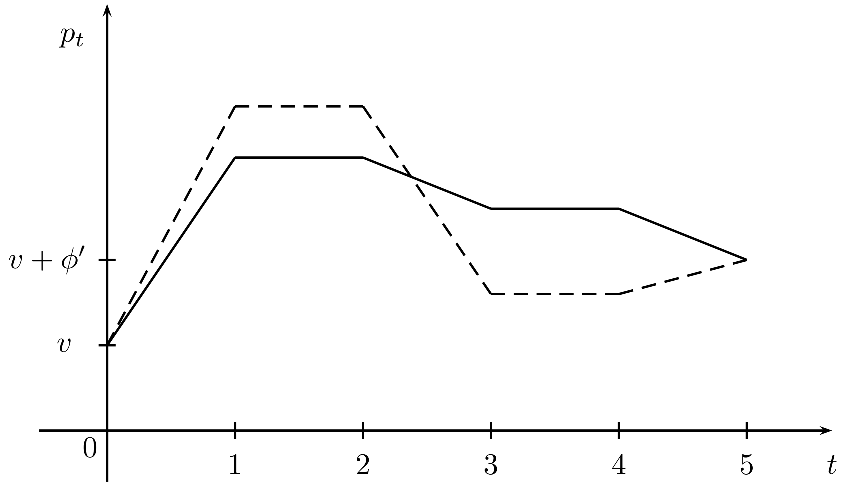


Figure 3.11: Possible price pathes for $\lambda' < 0 < \nu'$ (dashed line) and $0 < \lambda' < \nu'$ (solid line) if $\phi'' = 0$.

3.4.4 Setup 3

For completeness, we also consider the third time setup.

Corollary 6. *Let $T = 4$, $t^{r'} = 1$, $t^{r''} = 3$, $t^{e'} = 3$ and $t^{e''} = 4$. Then equations (3.22) and (3.23) become*

$$\nu' = \frac{\beta_3 - (\alpha - \beta_1)\lambda'}{\beta_1 - \beta_3} = \frac{-2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\left((1-\mu)\alpha - \beta_1\right) + \mu\left(1 - 4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right)\lambda'}{\mu + \left((1-\mu)\alpha - \beta_1\right)2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}. \tag{3.37}$$

Proof. Since $t^{e'} = t^{r''} = 3 \Rightarrow D = 0$ and since $t^{r'} = 1$ and $t^{r''} = 3 \Rightarrow \beta_{3-1-1} = \beta_1$. Then (3.37) follows immediately from proposition 11. $\square$

We decompose the right hand side of (3.37) again in its intercept

$$\frac{-2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\left((1-\mu)\alpha - \beta_1\right)}{\mu + \left((1-\mu)\alpha - \beta_1\right)2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} \tag{3.38}$$

and its slope

$$\frac{\mu\left(1 - 4\alpha\lambda''\gamma(1+\lambda'')\sigma^2_{\phi''}\right)\lambda'}{\mu + \left((1-\mu)\alpha - \beta_1\right)2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}.$$

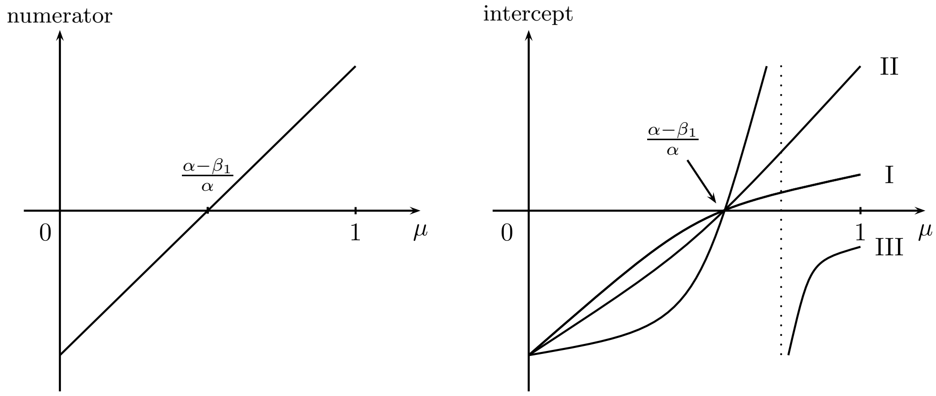


Figure 3.12: Shape of the numerator of the intercept (left panel) and the intercept (right panel).

The slope is equal to the slope in setup 2, so the cases discussed there apply here. The only difference to section 3.4.3 is that the numerator of the intercept differs. It is not constant as before but a linear function in μ with intercept $-2\gamma\sigma^2_{\phi''}(1+\lambda'')^2(\alpha-\beta_1)$ and slope $2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\alpha\mu$. For $\mu = 0$, the numerator of (3.38) is

$$-2\gamma\sigma^2_{\phi''}(1+\lambda'')^2(\alpha-\beta_1) < 0,$$

since $\alpha > \beta_1$ and

$$2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\beta_1 > 0$$

for $\mu = 1$. The numerator is therefore a linear increasing function that is equal to zero if

$$\begin{aligned}
& -2\gamma\sigma^2_{\phi''}(1+\lambda'')^2\left[(1-\mu)\alpha-\beta_1\right] &= 0 \\
\Leftrightarrow \quad & (1-\mu)\alpha-\beta_1 &= 0 \\
\Leftrightarrow \quad & \alpha-\beta_1 &= \alpha\mu \\
\Leftrightarrow \quad & \mu &= \frac{\alpha-\beta_1}{\alpha}.
\end{aligned}$$

The numerator is depicted in the left panel of figure 3.12. The slope is therefore a rational function of the form $\frac{a\mu+b}{c\mu+d}$, where $a = 2\gamma\sigma^2_{\phi''}\alpha(1+\lambda'')^2 > 0$, $b = -2\gamma\sigma^2_{\phi''}(1+$

$\lambda'')^2(\alpha-\beta_1) < 0$, $c = 1 - 2\alpha\gamma\sigma^2_{\phi''}(1+\lambda'')^2$ and $d = (\alpha-\beta_1)2\gamma\sigma^2_{\phi''}(1+\lambda'')^2 > 0$. Note, that $d = -b$ and $a = 1 - c$. The first and second derivatives are

$$\begin{aligned}\left(\frac{a\mu+b}{c\mu+d}\right)' &= \frac{(c\mu+d)a-(a\mu+b)c}{(c\mu+d)^2}\\ &= \frac{da-bc}{(c\mu+d)^2}\\ &= \frac{d}{(c\mu+d)^2} > 0\end{aligned}$$

and

$$\begin{aligned}\left(\frac{d}{(c\mu+d)^2}\right)' &= \frac{(c\mu+d)^2\,0-2d(c\mu+d)c}{(c\mu+d)^4}\\ &= \frac{-2dc}{(c\mu+d)^3}.\end{aligned}$$

So the intercept of the second equality in (3.37) is always increasing. Its curvature however depends on the parameters. Recall the three cases I, II and III from section 3.4.3 when we discussed the numerator of equation (3.34). In case I, $c > 0$. In case II, $c < 0$ but $c\mu + d > 0$ for $\mu \in (0, 1]$ and in case III we also had $c < 0$ but $c\mu + d > 0$ for $\mu \in (0, \bar{\mu})$ and $c\mu + d < 0$ for $\mu \in (\bar{\mu}, 1]$. Therefore the intercept is concave in case I and case III for $\mu > \bar{\mu}$ and convex in case II and case III for $\mu < \bar{\mu}$. For $\mu = 0$, the intercept is always negative since its numerator is then negative and the denominator positive. The shape of the intercept in the three cases is depicted in the right panel of figure 3.12.

Combining the intercept and the slope from section 3.4.3 yields the following cases:

Cases IA and IIA

In this case we have $\nu' < 0 < \lambda'$ for small values of μ. As μ increases, the intercept becomes positive and we first have $0 < \nu' < \lambda'$ and then $0 < \lambda' < \nu'$. If the intercept of the second equation of (3.37) exceeds the intercept of the first equation of (3.37) we have $\lambda' < 0 < \nu'$. The difference between the cases IA and IIA is the growth rate of the intercept and the slope and therefore the values for large μ.

Cases IB and IIB

For small values of μ, the intersection of the two lines given by equation (3.37) lies again in the fourth quadrant so $\nu' < 0 < \lambda'$. As μ increases, the slope decreases and the intercept increases. Therefore it is possible that we have $0 < \nu' < \lambda'$ or $0 < \lambda' < \nu'$, but for large slopes also $\lambda' < 0 < \nu'$. If the intercept of the second equation of (3.37) exceeds the intercept of the first equation of (3.37) we have $\lambda' < 0 < \nu'$ (if the slope of the second equality of equation (3.37) is smaller than the slope of the first equality). Again, the difference between IB and IIB lies in the growth rate of the slope and the intercept.

Case IIIB

In this case, both the slope and the intercept go to $\pm\infty$ and come from $\pm\infty$ as μ increases. Therefore all kinds of price reactions are possible in this parameter constellation ($\nu' < 0 < \lambda'$, $0 < \nu' < \lambda'$, $0 < \lambda' < \nu'$ and $\lambda' < 0 < \nu'$).

3.4.5 The possibility of non-existence of an equilibrium

In this section we discuss why it is possible that there exists a value of μ for which no equilibrium price exists. Recall therefore from section 3.3.5 where we discussed the effect of measure in the case with one signal. There, we defined the date$-t$ market demand as

$$md_t = D_t^f + (1-\mu)D_t^e + \mu D_t^r.$$

Now we turn to the case with two signals. The definition of the market demand also applies here. The situation after the arrival of the second signal is essentially the same as in section 3.3.5. The response to the first signal however may not exist. So we take a closer look at the solution for the price in period $t^{r''}-1$. With equation (3.4) and the demand functions (3.1), (3.2) and (3.9) the market demand at $t^{r''}-1$ is

$$\begin{aligned} md_{t^{r''}-1} &= \beta_{(t^{r''}-1)-t^{r'}}(p_{t^{r'}} - p_{t^{r'}-1}) + (1-\mu)\alpha(v - p_{t^{r''}-1}) \\ &\quad +\mu\left(\frac{v + (1+\lambda')\phi' - p_{t^{r''}-1}}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}\right) \\ &= -\Big((1-\mu)\alpha - \beta_{(t^{r''}-1)-t^{r'}} + \mu\alpha'\Big)\,p_{t^{r''}-1} + \Upsilon \end{aligned} \tag{3.39}$$

where Υ does not depend on $p_{t^{r''}-1}$. We remember that we defined α as 1 over 2γ times the variance of the price in the next period ($= \frac{1}{2\gamma\sigma_\theta^2}$). Defining $\alpha' = \frac{1}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}}$, the slope of $md_{t^{r''}-1}$ is

$$(1-\mu)\alpha - \beta_{(t^{r''}-1)-t^{r'}} + \alpha'\mu.$$

We assumed that $\alpha > \beta_l$ for all l. Assuming additionally that $\alpha' > \beta_l$ for all l, the slope is always greater zero since

$$(1-\mu)\alpha - \beta_l + \mu\alpha' > (1-\mu)\beta_l - \beta_l + \mu\beta_l = 0$$

for $0 < \mu < 1$. This changes if we allow for $\alpha' < \beta_l < \alpha$. Then it may happen that there exists a μ for which

$$(1-\mu)\alpha - \beta_l + \mu\alpha' = (\alpha - \beta_l) + \mu(\alpha' - \alpha) = 0$$

since $\alpha' - \alpha < 0$ and $\alpha - \beta_l > 0$. So the slope of md is equal to zero and there exists no equilibrium in this case. The interpretation is the same as in the case with one signal. If the "trading power" of passive investors ($(1-\mu)\alpha$) and rational speculators ($\mu\alpha'$) together is smaller than the "trading power" of positive feedback traders ($\beta_{t^{r''}-t^{r'}-1}$), we do not have an equilibrium.

3.4.6 Price behavior in the absence of rational speculators

In the preceding sections we have analyzed the price behavior in the presence of rational speculators. In this section we consider the model with two signals in the absence of rational speculators ($\mu = 0$). For simplicity, we consider again only the two special cases with $\beta_l = \beta = const.$ for all l and $\beta_2 = \beta_3 = \ldots = 0$.

Proposition 12. *For $\mu = 0$, the asset prices obey $p_t = v$ for $0, 1, \ldots, t^{e'} - 1$,*

$$p_t = v + \left[\sum_{\tau=0}^{t-t^{e'}} \left(\frac{\beta}{\alpha}\right)^{\tau}\right] \phi' \text{ for } t = t^{e'}, \ldots, t^{e''} - 1$$

and

$$p_t = v + \left[\sum_{\tau=0}^{t-t^{e'}} \left(\frac{\beta}{\alpha}\right)^{\tau}\right] \phi' + \left[\sum_{\varsigma=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{\varsigma}\right] \phi'' \text{ for } t = t^{e''}, \ldots, T, \tag{3.40}$$

if $\beta_l = \beta$ is constant.

Proof. The proof that $p_t = v$ for $0, 1, \ldots, t^{e'} - 1$ and

$$p_t = v + \left[\sum_{\tau=0}^{t-t^{e'}} \left(\frac{\beta}{\alpha} \right)^{\tau} \right] \phi' \text{ for } t = t^{e'}, \ldots, t^{e'} - 1,$$

is essentially the same as the proof of proposition 7 (replace ϕ'' by ϕ' and $t^{e''}$ by $t^{e'}$), so we only have to show (3.40). Consider the market clearing condition at date $t = t^{e''}, t^{e''} + 1, \ldots, T$

$$\begin{aligned}
& S &&= S + \beta(p_{t-1} - p_{t-2}) + \ldots + \beta(p_{t^{e'}} - p_{t^{e'}-1}) + \alpha(v + \phi' + \phi'' - p_t) \\
\Leftrightarrow \quad & 0 &&= \beta p_{t-1} - \beta v + \alpha(v + \phi' + \phi'' - p_t) \\
\Leftrightarrow \quad & 0 &&= \frac{\beta}{\alpha} p_{t-1} - \frac{\beta}{\alpha} v + v + \phi' + \phi'' - p_t \\
\Leftrightarrow \quad & p_t &&= \frac{\beta}{\alpha} p_{t-1} + \frac{\alpha - \beta}{\alpha} v + \phi' + \phi''.
\end{aligned}$$

To solve the differential equation, we first have to eliminate the constant. Set $p_t = p_{t-1} = \bar{p}$ so

$$\begin{aligned}
& \bar{p} &&= \frac{\alpha - \beta}{\alpha} v + \phi' + \phi'' + \frac{\beta}{\alpha} \bar{p} \\
\Leftrightarrow \quad & \tfrac{\alpha-\beta}{\alpha} \bar{p} &&= \frac{\alpha - \beta}{\alpha} v + \phi' + \phi'' \\
\Leftrightarrow \quad & \bar{p} &&= v + \frac{\alpha}{\alpha - \beta} \phi' + \frac{\alpha}{\alpha - \beta} \phi''.
\end{aligned}$$

Next, we decompose the price in the steady state and the deviation from it so $p_t = \tilde{p}_t + \bar{p}$. Then we get a differential equation without constant since

$$\begin{aligned}
\tilde{p}_t + v + \frac{\alpha}{\alpha - \beta} \phi' + \frac{\alpha}{\alpha - \beta} \phi'' &= \frac{\beta}{\alpha} \left(\tilde{p}_{t-1} + v + \frac{\alpha}{\alpha - \beta} \phi' + \frac{\alpha}{\alpha - \beta} \phi'' \right) \\
& \quad + \frac{\alpha - \beta}{\alpha} v + \phi' + \phi'' \\
\Leftrightarrow \tilde{p}_t &= \frac{\beta}{\alpha} \tilde{p}_{t-1}.
\end{aligned}$$

Its solution is

$$\tilde{p}_t = \left(\frac{\beta}{\alpha} \right)^{t-t^{e''}} \tilde{p}_{t^{e''}}.$$

Substituting $\tilde{p}_t = p_t - \bar{p}$ and using footnote 9 yields $p_t - \bar{p} = \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} (p_{t-1} - \bar{p}) \Leftrightarrow$

$$\begin{aligned}
p_t &= v + \frac{\alpha}{\alpha-\beta}\phi' + \frac{\alpha}{\alpha-\beta}\phi'' \\
&\quad + \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \left(v + \sum_{\tau=0}^{t^{e''}-t^{e'}} \left(\frac{\beta}{\alpha}\right)^{\tau} \phi' + \phi'' - v - \frac{\alpha}{\alpha-\beta}\phi' - \frac{\alpha}{\alpha-\beta}\phi'' \right) \\
&= v + \frac{\alpha}{\alpha-\beta}\phi' + \frac{\alpha}{\alpha-\beta}\phi'' \\
&\quad + \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \left(\frac{\alpha}{\alpha-\beta} \left(1 - \left(\frac{\beta}{\alpha}\right)^{t^{e''}-t^{e'}+1} \right) \phi' - \frac{\alpha}{\alpha-\beta}\phi' - \frac{\beta}{\alpha-\beta}\phi'' \right) \\
&= v + \frac{\alpha}{\alpha-\beta}\phi' + \frac{\alpha}{\alpha-\beta}\phi'' - \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{t^{e''}-t^{e'}+1} \phi' - \left(\frac{\beta}{\alpha}\right)^{t-t^{e''}+1} \frac{\alpha}{\alpha-\beta}\phi'' \\
&= v + \frac{1-\left(\frac{\beta}{\alpha}\right)^{t-t^{e'}+1}}{1-\frac{\beta}{\alpha}}\phi' + \frac{1-\left(\frac{\beta}{\alpha}\right)^{t-t^{e''}+1}}{1-\frac{\beta}{\alpha}}\phi'' \\
&= v + \left[\sum_{\tau=0}^{t-t^{e'}} \left(\frac{\beta}{\alpha}\right)^{\tau} \right] \phi' + \left[\sum_{\varsigma=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{\varsigma} \right] \phi''
\end{aligned}$$

which proves (3.40). □

The results here are essentially the same as in the setup with one signal. After the first signal arrives, the price increases monotonically to $v + (1 + \frac{\beta}{\alpha-\beta})\phi'$ for $t^{e'} \leq t \leq t^{e''} - 1$ (see discussion that follows proposition 7). As the price is expressed in (3.40) it can be decomposed into the two effects caused by ϕ' on the one hand side and ϕ'' on the other hand side. The effect of ϕ' on the price is described by $\left[\sum_{\tau=0}^{t-t^{e'}} \left(\frac{\beta}{\alpha}\right)^{\tau} \right] \phi'$, the effect of ϕ'' by $\left[\sum_{\varsigma=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{\varsigma} \right] \phi''$. If the realizations of ϕ' and ϕ'' are positive then both effects on p_t are positive (since $\beta, \alpha > 0$) so the price increases as t increases. After the arrival of the second signal the price moves towards

$$v + (1 + \frac{\beta}{\alpha-\beta})\phi' + (1 + \frac{\beta}{\alpha-\beta})\phi''$$

since the limit of (3.40) is (see footnote 9)

$$\lim_{t\to\infty} v + \left[\sum_{\tau=0}^{t-t^{e'}} \left(\frac{\beta}{\alpha}\right)^{\tau} \right] \phi' + \left[\sum_{\varsigma=0}^{t-t^{e''}} \left(\frac{\beta}{\alpha}\right)^{\varsigma} \right] \phi'' = v + \left(1 + \frac{\beta}{\alpha-\beta}\right)\phi' + \left(1 + \frac{\beta}{\alpha-\beta}\right)\phi''.$$

Proposition 13. *For $\mu = 0$, the asset prices obey $p_t = v$ for $t = 0, \ldots, t^{e'} - 1$,*

$$p_t = v + \phi' + \frac{\beta_1}{\alpha}\left(p_{t-1} - p_{t-2}\right) \text{ for } t = t^{e'}, \ldots, t^{e''} \tag{3.41}$$

and

$$p_t = v + \phi' + \phi'' + \frac{\beta_1}{\alpha}\left(p_{t-1} - p_{t-2}\right) \text{ for } t = t^{e''}, \ldots, T \tag{3.42}$$

if $\beta_l = 0$ for all $l \geq 2$. The price response in this case is characterized by damped fluctuations.

Proof. For $\mu = 0$ and $0 = \beta_2 = \beta_3 = \ldots$ the market clearing condition for dates between $t^{e'}$ and $t^{e''} - 1$ (after passive investors received the first signal but before they receive the second) is

$$\begin{aligned} 0 &= \beta_1\left(p_{t-1} - p_{t-2}\right) + \alpha(v + \phi' - p_t) \\ \Leftrightarrow p_t &= v + \phi' + \frac{\beta_1}{\alpha}\left(p_{t-1} - p_{t-2}\right). \end{aligned}$$

After passive investors receive the second signal (between $t^{e''}$ and T), the market clearing condition is

$$\begin{aligned} 0 &= \beta_1\left(p_{t-1} - p_{t-2}\right) + \alpha\left(v + \phi' + \phi'' - p_t\right) \\ \Leftrightarrow p_t &= v + \phi' + \phi'' + \frac{\beta_1}{\alpha}\left(p_{t-1} - p_{t-2}\right). \end{aligned}$$

This proves (3.41) and (3.42).

To see that these differential equations characterize damped fluctuations, we eliminate the constants (3.41) and (3.42). We set $\bar{p}' = p_t = p_{t-1} = p_{t-2}$ in (3.41) and $\bar{p}'' = p_t = p_{t-1} = p_{t-2}$ in (3.42). Then

$$\begin{aligned} \bar{p}' &= v + \phi + \frac{\beta_1}{\alpha}\left(\bar{p}' - \bar{p}'\right) \\ &= v + \phi' \end{aligned}$$

and

$$\begin{aligned} \bar{p}'' &= v + \phi' + \phi'' + \frac{\beta_1}{\alpha}\left(\bar{p}'' - \bar{p}''\right) \\ &= v + \phi' + \phi''. \end{aligned}$$

Replacing p_t in (3.41) by $\tilde{p}'_t + \bar{p}'$ and p_t in (3.42) by $\tilde{p}''_t + \bar{p}''$, we get

$$\begin{aligned}
\tilde{p}'_t + v + \phi' &= v + \phi' + \frac{\beta_1}{\alpha}\left(\tilde{p}'_{t-1} + v + \phi' - \tilde{p}''_{t-2} - v - \phi'\right) \\
&= v + \phi' + \frac{\beta_1}{\alpha}\left(\tilde{p}'_{t-1} - \tilde{p}''_{t-2}\right) \\
\Leftrightarrow \tilde{p}'_t &= \frac{\beta_1}{\alpha}\left(\tilde{p}'_{t-1} - \tilde{p}''_{t-2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{p}''_t + v + \phi' + \phi'' &= v + \phi' + \phi'' + \frac{\beta_1}{\alpha}\left(\tilde{p}''_{t-1} + v + \phi' + \phi'' - \tilde{p}''_{t-2} - v - \phi' - \phi''\right) \\
&= v + \phi' + \phi''\frac{\beta_1}{\alpha}\left(\tilde{p}''_{t-1} - \tilde{p}''_{t-2}\right) \\
\Leftrightarrow \tilde{p}''_t &= \frac{\beta_1}{\alpha}\left(\tilde{p}''_{t-1} - \tilde{p}''_{t-2}\right).
\end{aligned}$$

Now we have two homogeneous linear differential equations of degree two so the rest of the proof is identical to the proof in corollary 2. □

The price reaction is again essentially the same as in the case with one signal. The price is equal to v before the first signal arrives. Since passive investors demand a positive quantity of the asset after they learn about the signal, the price increases to $v+\phi'$. This price change leads to a positive demand two periods after the signal arrived so the price overshoots. The further the price moves away from $v+\phi'$, the larger is the negative demand of the passive investors in absolute value. There is a point where their negative demand exceeds positive feedback traders' demand. At that date the price falls. This price decrease leads to a negative demand of positive feedback traders. Since passive investors may also have a negative demand in this case, the price falls below $v+\phi'$. Now passive investors' demand becomes positive since the asset is too "cheap" from their point of view. This weakens the decrease of the price so positive feedback traders' negative demand decreases. At some point, passive investors' demand exceeds positive feedback traders' demand so the price increases and the cycle starts again. Note that prices only move to $v+\phi'$ and $v+\phi'+\phi''$, respectively because $\frac{\beta_1}{\alpha} < 1$.

3.4.7 Uniqueness

We have seen in the previous sections that prices of the form given by (3.4) are an equilibrium. However, there might in principle exist another price sequence that leads to an equilibrium and does not satisfy (3.4). We show in this section that the equilibrium formed by the considered prices is unique.
To show uniqueness we use the concept of an information partition that can be found for example in Magill & Quinzii (1998). Let there be a well-defined probability space and a (finite) state space Ω.

Definition 3. *A* partition *of Ω is a collection of mutually disjoint subsets of Ω whose union is Ω. A partition $\mathbb{F}_t$ is said to be* finer *than $\mathbb{F}_{t-1}$ if for $\sigma \in \mathbb{F}_t$ and $\sigma' \in \mathbb{F}_{t-1}$, either $\sigma \subset \sigma'$ or $\sigma \cap \sigma' = \emptyset$.*

We can describe the unfolding of information by a sequence of partitions of Ω,

$$\mathbb{F} = (\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_T)$$

where $\mathbb{F}_0 = \Omega$, $\mathbb{F}_t$ is a finer partition than $\mathbb{F}_{t-1}$ and $\mathbb{F}_T = \{\{s\}\}_{s\in\Omega}$. When information increases, then certain states may be excluded so a partition becomes finer since $\sigma \subset \sigma'$. In period 0, an agent has no information about which state will be realized. As time unfolds, his information increases and he has full information in period T. This unfolding of information can be illustrated graphically by an event tree. Figure 3.13 depicts an example with 6 states and 4 periods. The partitions in this example are $\mathbb{F}_0 = \{1, 2, 3, 4, 5, 6\}$, $\mathbb{F}_1 = \{\{1\}, \{2, 3, 4\}, \{5, 6\}\}$, $\mathbb{F}_2 = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6\}\}$ and $\mathbb{F}_3 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$. This figure also illustrates the concept of consistency that is given by our definition of a finer partition: We say that the revealed information is consistent if an agent cannot deduce information at date t about the realization of a state by observing past paths. When we take two states $s, s' \in \sigma_t$ from a subset $\sigma_t \in \mathbb{F}_t$, then both paths in the event tree have to be equal. Consider $\{2, 3\} \in \mathbb{F}_2$ in our example. An agent cannot decide whether state 2 or state 3 will be realized by looking at their price paths up to date 2. Both states 2 and 3 were in the same subsets

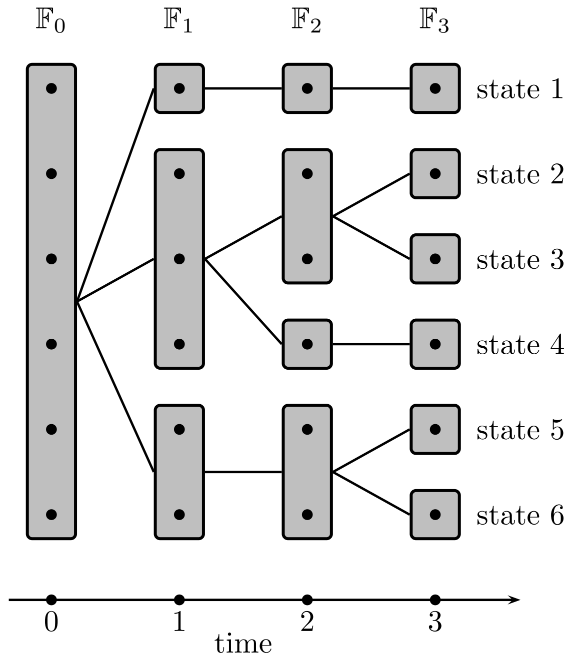


Figure 3.13: An example for the unfolding of information.

at date 0 ($\{1,2,3,4,5,6\} \in \mathbb{F}_0$) and at date 1 ($\{2,3,4\} \in \mathbb{F}_1$).
Now we come back to our model setup. Let $(\mathbb{F}_t)_{t=0,1,\ldots,T}$ be the sequence of information partitions that is getting finer as t increases. We denote the value of a random variable in state $s \in \Omega$ by adding the subscript s. When rational speculators learn about ϕ' at date $t^{r'}$, then $\phi'_s = \phi'_{s'}$ for $s, s' \in \sigma_t$ if $\sigma_t \subset \sigma_{t^{r'}}$ for $t^{r'} < t$. The same is true for ϕ''. In other words, if rational speculators receive the realization of a signal it stays constant over all possible states.

Proposition 14. *Let $(p_{t,s}, D^f_{t,s}, D^p_{t,s}, D^e_{t,s})_{t=0,1,\ldots,T,s\in\Omega}$ be an equilibrium. Then the equilibrium is generically unique.*

Proof. This follows with lemmas 2, 3 and 4 below. □

We show the uniqueness of the equilibrium by backward induction. First, by using the date$-T$ market clearing condition, we show that p_T is a function of the realizations of the two signals and the past prices. Since ϕ', ϕ'' and $p_0, \ldots, p_{T-1}$ are known to rational speculators at $T-1$, p_T is uniquely determined. Repeating this argument at date $t^{r''}-1$ ($t^{r'}-1$), where we show that $p_{t^{r''}-1}$ ($p_{t^{r'}-1}$) is a function of ϕ' and $p_0,\ldots,p_{t^{r''}-2}$ ($p_0,\ldots,p_{t^{r'}-2}$), proves that $p_{t^{r''}-1}$ ($p_{t^{r'}-1}$) is uniquely determined. That all prices stay constant when no information arrives follows from the absence of arbitrage opportunities.

Lemma 2.

$$p_{t,s} = p_{T,s} \text{ for } t = t^{r''}, t^{r''}+1, \ldots, T \text{ and all } s \in \Omega.$$

Proof. For each state s, the market clearing condition at date T is

$$0 = \sum_{l=1}^{T-1} \beta_l \Delta p_{T-l,s} + \alpha(v + \phi'_s + \phi''_s - p_{T,s}).$$

Solving for p_T leads to

$$p_{T,s} = v + \phi'_s + \phi''_s + \frac{1}{\alpha}\sum_{l=1}^{T-1} \beta_l \Delta p_{T-l}. \tag{3.43}$$

Let us assume that $t^{r''} < T$. Then rational speculators have already observed both signals ϕ'_s and ϕ''_s at $T-1$. That means that the realization of both signals is uniform in all states $s \in \sigma_{T-1} \in \mathbb{F}_{T-1}$. Consistency of information revelation implies that the prices $p_{t,s}$ are also identical in all states in σ_{T-1} for all $t = 0,1,\ldots,T-1$. So all variables on the right hand side of equation (3.43) are identical over all states $s \in \sigma_T \in \mathbb{F}_T$ with $\sigma_T \subset \sigma_{T-1}$. Therefore

$$p_{T,s} = p_{T,s'}$$

for $s \in \sigma_T \subset \sigma_{T-1}$ and $s' \in \sigma_T \subset \sigma_{T-1}$.

Next, we show that the price is constant for $t = t^{r''},\ldots,T$. Let us assume that there is a state $s \in \sigma_T$ for which $p_{T,s} \neq p_{T-1,s}$. Then the capital gains $(p_{T,s} - p_{T-1,s})D^r_{T-1,s}$ are increasing (if $p_{T,s} > p_{T-1,s}$) or decreasing (if $p_{T,s} < p_{T-1,s}$) in rational speculators' demand so there exists no solution to the optimization problem. Therefore $p_{T,s} = p_{T-1,s}$. By replacing $p_{T,s}$ by $p_{T-1,s}$ in equation (3.43) and using the argument from above we get $p_{T-1,s} = p_{T-2,s}$. Repeating this argument recursively we get

$$p_{t,s} = p_{T,s} \text{ for } t = t^{r''}, t^{r''}+1, \ldots, T \text{ and } s \in \Omega.$$

Note that the case $t^{r''} = T$ is trivial. □

Lemma 3.

$$p_{t,s} = p_{t^{r''}-1,s} \text{ for } t = t^{r'}, \ldots, t^{r''} - 1 \text{ and } s \in \Omega.$$

Proof. We consider the date $t^{r''}-1$ utility maximization problem of rational speculators. For simplicity we drop the subscript s when it is not relevant for the calculation. Rational speculators' final wealth is

$$W^r_{T+1} = W^r_{t^{r''}-1} + (p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} + \alpha(p_T - v - \phi' - \phi'' - \theta)(p_T - v - \phi' - \phi'').$$

Before calculating the mean and the variance of the final wealth to derive the optimal demand, we rewrite p_T using $p_t = p_T$ for $t = t^{r''}, t^{r''} - 1, \ldots, T$.

$$\begin{aligned}
p_T &= v + \phi' + \phi'' + \frac{1}{\alpha}\sum_{l=1}^{T-1} \beta_l \Delta p_{T-l} \\
&= v + \phi' + \phi'' + \frac{1}{\alpha}\sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} + \frac{\beta_{T-t^{r''}}}{\alpha}\left(p_{t^{r''}} - p_{t^{r''}-1}\right) \\
\Leftrightarrow p_T\left(1 - \frac{\beta_{T-t^{r''}}}{\alpha}\right) &= v + \phi' + \phi'' + \frac{1}{\alpha}\sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \frac{\beta_{T-t^{r''}}}{\alpha} p_{t^{r''}-1} \\
\Leftrightarrow p_T &= \frac{1}{\alpha - \beta_{T-t^{r''}}} \\
&\quad \left[\alpha(v + \phi' + \phi'') + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right].
\end{aligned}$$

The mean of $p_T = p_{T-1} = \ldots = p_{t^{r''}}$ at date $t^{r''} - 1$ is

$$E_{t^{r''}-1} p_{t^{r''}} = \frac{1}{\alpha - \beta_{T-t^{r''}}}\left[\alpha(v + \phi') + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right].$$

Then the expected final wealth is

$$\begin{aligned}
E^r_{t^{r''}-1}W^r_{T+1} &= \left(E^r_{t^{r''}-1}p_{t^{r''}} - p_{t^{r''}-1}\right) D^r_{t^{r''}-1} + C \\
&= \frac{1}{\alpha - \beta_{T-t^{r''}}} \left(\alpha(v+\phi') - (\alpha - \beta_{T-t^{r''}})p_{t^{r''}-1}\right. \\
&\quad \left. + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right) D^r_{t^{r''}-1} + C \\
&= \frac{1}{\alpha - \beta_{T-t^{r''}}} \left(\alpha(v + \phi' - p_{t^{r''}-1}) + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l}\right) D^r_{t^{r''}-1} + C,
\end{aligned}$$

where C is a constant that does not depend on $D^r_{t^{r''}-1}$ and therefore is not relevant for our optimization problem. Next, we determine the conditional variance of the final wealth

$$\begin{aligned}
\sigma^{r2}_{W_{T+1}|t^{r''}-1} &= E^r_{t^{r''}-1}\left(W^r_{T+1} - E^r_{t^{r''}-1}W^r_{T+1}\right)^2 \\
&= E^r_{t^{r''}-1}\Big(W^r_{t+1} + (p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} + \alpha p_T^2 - 2\alpha v p_T - 2\alpha\phi' p_T \\
&\quad -2\alpha\phi'' p_T + \alpha v^2 + 2\alpha\phi' v + 2\alpha\phi'' v + \alpha\phi'^2 + 2\alpha\phi'\phi'' + \alpha\phi''^2 - \alpha\theta p_T \\
&\quad +\alpha v\theta + \alpha\theta\phi' + \alpha\theta\phi'' \\
&\quad -W^r_{t+1} - (E^r_{t^{r''}-1}p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} - \alpha E^r_{t^{r''}-1}p_T^2 + 2\alpha v E^r_{t^{r''}-1}p_T \\
&\quad +2\alpha\phi' E^r_{t^{r''}-1}p_T - 2\alpha E^r_{t^{r''}-1}(\phi'' p_T) - \alpha v^2 - 2\alpha\phi' v - 2\alpha v E^r_{t^{r''}-1}\phi'' \\
&\quad -\alpha\phi'^2 - 2\alpha\phi' E^r_{t^{r''}-1}\phi'' - \alpha E^r_{t^{r''}-1}\phi''^2 + \alpha E^r_{t^{r''}-1}(\theta p_T) - \alpha v E^r_{t^{r''}-1}\theta \\
&\quad - \alpha\phi' E^r_{t^{r''}-1}\theta - \alpha E^r_{t^{r''}-1}(\theta\phi'')\Big)^2 \\
&= E^r_{t^{r''}-1}((p_{t^{r''}} - E^r_{t^{r''}-1}p_{t^{r''}})D^r_{t^{r''}-1} + \alpha(p_T^2 - E^r_{t^{r''}-1}p_T^2) \\
&\quad -2\alpha v(p_T - E^r_{t^{r''}-1}p_T) - 2\alpha\phi'(p_T - E^r_{t^{r''}-1}p_T) \\
&\quad -2\alpha(\phi'' p_T - E^r_{t^{r''}-1}(\phi'' p_T)) + 2\alpha v(\phi'' - E^r_{t^{r''}-1}\phi'') \\
&\quad +2\alpha\phi'(\phi'' - E^r_{t^{r''}-1}\phi'') + \alpha(\phi''^2 - E^r_{t^{r''}-1}\phi''^2) - \alpha(\theta p_T - E^r_{t^{r''}-1}(\theta p_T)) \\
&\quad +\alpha v(\theta - E^r_{t^{r''}-1}\theta) + \alpha\phi'(\theta - E^r_{t^{r''}-1}\theta)) + \alpha(\theta\phi'' - E^r_{t^{r''}-1}(\theta\phi'')))^2.
\end{aligned}$$

To simplify this expression, we calculate

$$\begin{aligned}
p_T - E^r_{t^{r''}-1}p_T &= \frac{1}{\alpha - \beta_{T-t^{r''}}}\left[\alpha(v+\phi'+\phi'') + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right] \\
&\quad - \frac{1}{\alpha - \beta_{T-t^{r''}}}\left[\alpha(v+\phi') + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right] \\
&= \frac{\alpha}{\alpha - \beta_{T-t^{r''}}}\phi''.
\end{aligned}$$

Plugging this into the conditional variance and together with $E^r_{t^{r''}-1}\phi'' = 0$, $E^r_{t^{r''}-1}\phi''^3 = 0$ (symmetry), $E^r_{t^{r''}-1}\theta = 0$, $Cov(\theta, p_T) = Cov(\theta, \phi'') = 0$ (independence of θ),

$$\begin{aligned}
E^r_{t^{r''}-1}(\phi'' p_T^2) &= E^r_{t^{r''}-1}\Bigg\{\left(\frac{1}{\alpha - \beta_{T-t^{r''}}}\right)^2 \\
&\quad \Bigg[\alpha^2\phi''^3 + 2\alpha\phi''^2(\alpha v + \alpha\phi' + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}}) \\
&\quad + (\alpha v + \alpha\phi' + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}})^2\Bigg]\Bigg\} \\
&= \left(\frac{1}{\alpha - \beta_{T-t^{r''}}}\right)^2 2\alpha\sigma^2_{\phi''} \\
&\quad \left(\alpha v + \alpha\phi' + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
E^r_{t^{r''}-1}(\phi''^2 p_T) &= \frac{1}{\alpha - \beta_{t-t^{r''}}}(\alpha v E^r_{t^{r''}-1}\phi''^2 + \alpha\phi' E^r_{t^{r''}-1}\phi''^2 + \alpha E^r_{t^{r''}-1}\phi''^3 \\
&\quad + E^r_{t^{r''}-1}\phi''^2 \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - E^r_{t^{r''}-1}\beta_{T-t^{r''}} p_{t^{r''}-1}) \\
&= \frac{1}{\alpha - \beta_{T-t^{r''}}}\sigma^2_{\phi''}\left(\alpha v + \alpha\phi' + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \beta_{T-t^{r''}} p_{t^{r''}-1}\right),
\end{aligned}$$

we have

$$
\begin{aligned}
\sigma^{r2}_{W_{T+1}|t^{r''}-1} \quad = \quad & E^r_{t^{r''}-1}\left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\phi''\right)^2\left(D^r_{t^{r''}-1}\right)^2 + 2\frac{\alpha}{\alpha-\beta_{t-t^{r''}}}D^r_{t^{r''}-1} \\
& E^r_{t^{r''}-1}\Big[\alpha\phi'' p_T^2 - 2\alpha\phi'' E^r_{t^{r''}-1}p_T^2 - 2\alpha v\left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\phi''\right)\phi'' \\
& -2\alpha\phi'\phi''\left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\phi''\right) - 2\alpha\phi''^2 p_T + 2\alpha\phi'' E^r_{t^{r''}-1}(p_T\phi'') \\
& +2\alpha v\phi''^2 + 2\alpha\phi'\phi''^2 + \alpha\phi''^3 - \alpha\phi'' E^r_{t^{r''}-1}(\phi''^2) - \alpha\phi''\theta p_T + \alpha v\phi''\theta \\
& +\alpha\phi'\phi''\theta + \alpha\phi''^2\theta\Big] + C'
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{r2}_{W_{T+1}|t^{r''}-1} \quad = \quad & \left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\right)^2\sigma^2_{\phi''}\left(D^r_{t^{r''}-1}\right)^2 + 2\frac{\alpha}{\alpha-\beta_{t-t^{r''}}}D^r_{t^{r''}-1} \\
& \left[\left(\frac{1}{\alpha-\beta_{T-t^{r''}}}\right)^2 2\alpha^2\sigma^2_{\phi''}\alpha v + \left(\frac{1}{\alpha-\beta_{T-t^{r''}}}\right)^2 2\alpha^2\sigma^2_{\phi''}\alpha\phi'\right. \\
& + \left(\frac{1}{\alpha-\beta_{T-t^{r''}}}\right)^2 2\alpha^2\sigma^2_{\phi''}\sum_{l=T-(t^{r''}-1)}^{T-1}\beta_l\Delta p_{T-l} \\
& - \left(\frac{1}{\alpha-\beta_{T-t^{r''}}}\right)^2 2\alpha^2\sigma^2_{\phi''}\beta_{T-t^{r''}}p_{t^{r''}-1} - 2\alpha v\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\sigma^2_{\phi''} \\
& -2\alpha\phi'\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\sigma^2_{\phi''} - 2\alpha\frac{1}{\alpha-\beta_{T-t^{r''}}}\sigma^2_{\phi''}\alpha v - 2\alpha\frac{1}{\alpha-\beta_{T-t^{r''}}}\sigma^2_{\phi''}\alpha\phi' \\
& -2\alpha\frac{1}{\alpha-\beta_{T-t^{r''}}}\sigma^2_{\phi''}\sum_{l=T-(t^{r''}-1)}^{T-1}\beta_l\Delta p_{T-l} \\
& \left. +2\alpha\frac{1}{\alpha-\beta_{T-t^{r''}}}\sigma^2_{\phi''}\beta_{T-t^{r''}}p_{t^{r''}-1} + 2\alpha v\sigma^2_{\phi''} + 2\alpha\phi'\sigma^2_{\phi''}\right] + C'
\end{aligned}
$$

$$\begin{aligned}
\sigma^{r2}_{W_{T+1}|t^{r''}-1} &= \left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\right)^2 \sigma^2_{\phi''}\left(D^r_{t^{r''}-1}\right)^2 + 4\frac{\alpha^2}{\alpha-\beta_{t-t^{r''}}} D^r_{t^{r''}-1}\sigma^2_{\phi''} \\
&\left[\frac{\alpha^2-2\alpha(\alpha-\beta_{T-t^{r''}})+\left(\alpha-\beta_{T-t^{r''}}\right)^2}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} v\right. \\
&\frac{\alpha^2-2\alpha(\alpha-\beta_{T-t^{r''}})+\left(\alpha-\beta_{T-t^{r''}}\right)^2}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} \phi'' \\
&+\frac{\alpha-(\alpha-\beta_{T-t^{r''}})}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} \\
&\left.-\frac{\alpha-(\alpha-\beta_{T-t^{r''}})}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} \beta_{T-t^{r''}} p_{t^{r''}-1}\right] + C'
\end{aligned}$$

$$\begin{aligned}
\sigma^{r2}_{W_{T+1}|t^{r''}-1} &= \left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\right)^2 \sigma^2_{\phi''}\left(D^r_{t^{r''}-1}\right)^2 + 4\frac{\alpha^2}{\alpha-\beta_{t-t^{r''}}} D^r_{t^{r''}-1}\sigma^2_{\phi''} \\
&\left[\frac{\alpha^2-2\alpha^2+2\alpha\beta_{T-t^{r''}}+\alpha^2-2\alpha\beta_{T-t^{r''}}+\beta^2_{T-t^{r''}}}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} v\right. \\
&\frac{\alpha^2-2\alpha^2+2\alpha\beta_{T-t^{r''}}+\alpha^2-2\alpha\beta_{T-t^{r''}}+\beta^2_{T-t^{r''}}}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} \phi'' \\
&\left.+\frac{\beta_{T-t^{r''}}}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l} - \frac{\beta_{T-t^{r''}}}{\left(\alpha-\beta_{T-t^{r''}}\right)^2} \beta_{T-t^{r''}} p_{t^{r''}-1}\right] \\
&+C'
\end{aligned}$$

$$\begin{aligned}
\sigma^{r2}_{W_{T+1}|t^{r''}-1} &= \left(\frac{\alpha}{\alpha-\beta_{T-t^{r''}}}\right)^2 \sigma^2_{\phi''}\left(D^r_{t^{r''}-1}\right)^2 + \frac{4\alpha^2\beta_{T-t^{r''}}}{\left(\alpha-\beta_{T-t^{r''}}\right)^3} \sigma^2_{\phi''} D^r_{t^{r''}-1} \\
&\left[\beta_{T-t^{r''}}(v+\phi'-p_{t^{r''}-1}) + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l}\right] + C'
\end{aligned}$$

where C' is again a constant that does not depend on $D^r_{t^{r''}-1}$. Setting the derivative of the mean-variance utility function with respect to $D^r_{t^{r''}-1}$ equal to zero yields

$$
\begin{aligned}
\frac{\partial}{\partial D^r_{t^{r''}-1}}(\mu_W - \gamma\sigma^2_W) &= \frac{1}{\alpha - \beta_{T-t^{r''}}}\left[\alpha(v + \phi' - p_{t^{r''}-1}) + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l}\right] \\
&\quad -2\gamma\left(\frac{\alpha}{\alpha - \beta_{T-t^{r''}}}\right)^2 \sigma^2_{\phi''} D^r_{t^{r''}-1} - \gamma\frac{4\alpha^2\beta_{T-t^{r''}}}{\left(\alpha - \beta_{T-t^{r''}}\right)^3}\sigma^2_{\phi''} \\
&\quad \left[\beta_{T-t^{r''}}(v + \phi' - p_{t^{r''}-1}) + \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l}\right] \\
&\overset{!}{=} 0.
\end{aligned}
$$

Solving for $D^r_{t^{r''}-1}$ yields

$$
\begin{aligned}
D^r_{t^{r''}-1} &= \left(\frac{\alpha - \beta_{T-t^{r''}}}{2\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)(v + \phi' - p_{t^{r''}-1}) \\
&\quad + \left(\frac{\alpha - \beta_{T-t^{r''}}}{2\gamma\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right) \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l}.
\end{aligned}
$$

So the market demand in period $t^{r''} - 1$ is

$$
\begin{aligned}
md_{t^{r''}-1} &= D^f_{t^{r''}-1} + (1-\mu)D^e_{t^{r''}-1} + \mu D^r_{t^{r''}-1} \\
&= \sum_{l=1}^{t^{r''}-2} \beta_l \Delta p_{t^{r''}-1-l,s} + (1-\mu)\alpha(v + D\phi'_s - p_{t^{r''}-1,s}) \\
&\quad +\mu\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)(v + \phi'_s - p_{t^{r''}-1,s}) \\
&\quad +\mu\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\gamma\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right) \sum_{l=T-(t^{r''}-1)}^{T-1} \beta_l \Delta p_{T-l,s}.
\end{aligned} \tag{3.44}
$$

The market clears if $md_{t^{r''}-1} = 0$. As we saw in section 3.4.5, there exists no equilibrium price if $\frac{\partial md_{t^{r''}-1}}{\partial p_{t^{r''}-1}} = 0$. We therefore assume that

$$
\begin{aligned}
\frac{\partial md_{t^{r''}-1}}{\partial p_{t^{r''}-1}} &= -(1-\mu)\alpha + \mu\left[-\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)\right. \\
&\quad \left. + \beta_{T-(t^{r''}-1)}\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)\right] \\
&\neq 0.
\end{aligned}
$$

Since $md(p_{t^{r''}-1,s})$ is a linear function of the price, the market clears at date $t^{r''}-1$ for a price level $p_{t^{r''}-1,s}$. To see that this price level is uniquely determined, we show that

$$p_{t^{r''}-1,s} = p_{t^{r''}-1,s'} \text{ for } s \in \sigma_{t^{r''}-1} \in \mathbb{F}_{t^{r''}-1} \text{ and } s' \in \sigma'_{t^{r''}-1} \in \mathbb{F}_{t^{r''}-1}.$$

Consider $\sigma_{t^{r''}-1}, \sigma'_{t^{r''}-1} \subset \sigma_{t^{r''}-2}$ with $\sigma_{t^{r''}-1}, \sigma'_{t^{r''}-1} \in \mathbb{F}_{t^{r''}-1}$ and $\sigma_{t^{r''}-2} \in \mathbb{F}_{t''-2}$. Since ϕ'_s is already realized at $t^{r''}-2 \geq t^{r'}$, it is uniform in every state $s \in \sigma_{t^{r''}-2}$. The consistency of information implies that all past prices are also uniform across the states in $\sigma_{t^{r''}-2}$. Therefore the date$-t^{r''}-1$ price is unique.
Now we show that the price is constant between $t^{r'}$ and $t^{r''}-1$. This follows from the absence of arbitrage opportunities. Let us assume that $p_{t^{r''}-1,s} \neq p_{t^{r''}-2,s}$. Then the capital gains $(p_{t^{r''}-1,s} - p_{t^{r''}-2,s})D^r_{t^{r''}-2,s}$ are monotonically increasing or decreasing in the rational speculators' demand. So there exists no solution for their maximization problem. Thus, $p_{t^{r''}-1,s} = p_{t^{r''}-2,s}$. Applying the arbitrage argument recursively to earlier dates, we have

$$p_{t,s} = p_{t^{r''}-1,s} \text{ for } t = t^{r'}, \ldots, t^{r''}-1 \text{ and } \quad s \in \Omega.$$

□

Now we can show that the prices in the case with one signal are unique.

Corollary 7. *The equilibrium is generically unique in the case with one signal.*

Proof. When we have only one signal, then $\phi'_s = 0$. Using $\phi'_s = 0$ and $p_{t,s} = p_{t^{r''}-1,s}$ for $t = 0, 1, \ldots, t^{r''}-1, s \in \Omega$ we can simplify equations (3.43) and (3.44) and get

$$\begin{aligned} p_{T,s} &= v + \phi'_s + \phi''_s + \frac{1}{\alpha}\sum_{l=1}^{T-1} \beta_l \Delta p_{T-l,s} \\ &= v + \phi''_s + \frac{1}{\alpha}\beta_{T-t^{r''}}\left(p_{t^{r''},s} - p_{0,s}\right) \\ \Leftrightarrow p_{T,s} &= \frac{\alpha(v+\phi''_s) + \beta_{T-t^{r''}} p_{0,s}}{\alpha - \beta_{T-t^{r''}}} \end{aligned}$$

and the market clearing condition

$$0 = \left[\mu\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right) + (1-\mu)\alpha\right](v - p_{0,s}).$$

By setting $p_{0,s} = v$, we get the same results as in proposition 6. To have uniqueness, we have to assume that term before $v - p_{0,s}$ is not equal to zero, i.e.

$$\mu \left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) + (1-\mu)\alpha \neq 0.$$

One can see that there is a parameter constellation for which any price $p_{0,s}$ together with $p_{T,s}$ is an equilibrium. $\square$

When we considered the different setups in the case with two signals we have seen that there is a parameter constellation such that there exists no solution for ν' and λ'. This is the case when the two linear functions given by equations (3.22) and (3.23) are parallel in the $\lambda' - \nu'-$diagram which is the case when their slopes are equal. So there exists no solution if

$$\frac{\alpha - \beta_{T-t^{r''}}}{\beta_{T-t^{r''}} - \beta_{T-t^{r'}}} = \frac{\mu \left[4\gamma\sigma^2_{\phi''}\beta_{T-t^{r''}}(1+\lambda'')^2 - 1\right]}{\mu + 2\gamma\sigma^2_{\phi''}(1+\lambda'')^2 \left[(1-\mu)\alpha - \beta_{t^{r''}-t^{r'}-1}\right]}. \tag{3.45}$$

We can show that this non-existence condition coincides with the condition we get from equation (3.44) for the existence of an equilibrium.

Corollary 8. *There is no equilibrium in equation (3.44) (for an intercept not equal to zero) if and only if (3.45) holds.*

Proof. Equation (3.44) tells us, that the market demand is a linear function of $p_{t^{r''}-1}$. The market clears if the market demand is zero, so if the line given by (3.44) intersects the $p_{t^{r''}-1}-$axis in a $p_{t^{r''}-1} - md_{t^{r''}-1}-$diagram. If the slope of $md_{t^{r''}-1}$ is not equal to zero, an equilibrium always exists. If its slope is equal to zero (and the intercept does not equal zero), the market never clears and an equilibrium does not exist.
We therefore have to show that when the slope of the market demand is equal to zero if and only if (3.45) holds. Recall equation (3.44). Using that $p_{t^{r'}} = \ldots = p_{t^{r''}-1}$ and

$p_{t^{r''}} = \ldots = p_T$, we have

$$\begin{aligned} md_{t^{r''}-1} &= \sum_{l=t^{r''}-t^{r'}}^{t^{r''}-2} \beta_l \Delta p_{(t^{r''}-1)-l} + \beta_{t^{r''}-t^{r'}-1}(p_{t^{r''}-1} - p_{t^{r'}-1}) \\ &+\mu \left[\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) \left(v + \phi' - p_{t^{r''}-1} \right) \right. \\ &+ \left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha^2\gamma\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) \\ &\left. \left(\sum_{l=T-(t^{r'}+1)}^{T-1} \beta_l \Delta p_{T-l} + \beta_{T-t^{r'}}(p_{t^{r''}-1} - p_{t^{r'}-1}) \right) \right] \\ &+(1-\mu)\alpha(v + D\phi' - p_{t^{r''}-1}). \end{aligned} \tag{3.46}$$

Deriving this with respect to $p_{t^{r''}-1}$, we get

$$\begin{aligned} \frac{\partial(3.46)}{\partial p_{t^{r''}-1}} &= \beta_{t^{r''}-t^{r'}-1} - (1-\mu)\alpha \\ &-\mu \left[\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) \right. \\ &\left. - \left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha^2\gamma\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) \beta_{T-t^{r'}} \right]. \end{aligned} \tag{3.47}$$

So the slope of the market demand is given by (3.47). Rearranging (3.45) and using $1 + \lambda'' = \frac{\alpha}{\alpha - \beta_{T-t^{r''}}}$ yields

$$\begin{aligned} 0 &= \left(\beta_{T-t^{r''}} - \beta_{T-t^{r'}}\right) 4\gamma\mu\beta_{T-t^{r''}} \left(1+\lambda''\right)^2 \sigma^2_{\phi''} - \left(\beta_{T-t^{r''}} - \beta_{T-t^{r'}}\right)\mu \\ &- \left(\alpha - \beta_{T-t^{r''}}\right)\mu - \left(\alpha - \beta_{T-t^{r''}}\right) 2\gamma(1+\lambda'')^2\sigma^2_{\phi''}\left(1-\mu\right)\alpha \\ &+ \left(\alpha - \beta_{T-t^{r''}}\right) 2\gamma\left(1+\lambda''\right)^2 \beta_{t^{r''}-t^{r'}-1}\sigma^2_{\phi''} \\ &= \mu \left[\frac{\left(\beta_{T-t^{r''}} - \beta_{T-t^{r'}}\right) 4\gamma\sigma^2_{\phi''}\beta_{T-t^{r''}}(1+\lambda'')^2 + \left(\beta_{T-t^{r'}} - \alpha\right)}{\left(\alpha - \beta_{T-t^{r''}}\right) 2\gamma\sigma^2_{\phi''}\left(1+\lambda''\right)^2} \right] \\ &+ \left(1-\mu\right)\alpha + \beta_{t^{r''}-t^{r'}-1} \\ &= \mu \left[\frac{\left(\beta_{T-t^{r''}} - \beta_{T-t^{r'}}\right) 2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} - \frac{\left(\alpha - \beta_{T-t^{r'}}\right)\left(\alpha - \beta_{T-t^{r''}}\right)}{2\gamma\alpha^2\sigma^2_{\phi''}} \right] \\ &+ \left(1-\mu\right)\alpha + \beta_{t^{r''}-t^{r'}-1} \\ &= \left(1-\mu\right)\alpha + \beta_{t^{r''}-t^{r'}-1} \\ &-\mu \left[\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\gamma\alpha\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) - \left(\frac{\alpha - \beta_{T-t^{r'}}}{2\gamma\alpha^2\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}} \right) \beta_{T-t^{r'}} \right]. \end{aligned}$$

This means that if (3.45) holds, (3.47) is equal to zero so there exists no equilibrium. Note that this proves both directions since all transformations are equivalent. □

The final thing we need to show for the proof of proposition 14 is

Lemma 4.

$$p_{t,s} = p_{t^{r'}-1,s} \text{ for } t = 0, \ldots, t^{r'} - 1 \text{ and } s \in \Omega.$$

Proof. To get the price, we have to solve the rational speculators' utility maximization problem at $t^{r'} - 1$. Before considering the final wealth we simplify a few expressions for convenience. Let $\tilde{p}_t = p_t - v$ and rewrite (3.46) as

$$\begin{aligned} 0 \;=\; & \sum_{l=t^{r''}-t^{r'}}^{t^{r''}-2} \beta_l \Delta\tilde{p}_{(t^{r''}-1)-l} + \beta_{t^{r''}-t^{r'}-1}\left(\tilde{p}_{t^{r''}-1} - \tilde{p}_{t^{r'}-1}\right) \\ & +\mu\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha\gamma\sigma^2_{\phi''}} - \frac{2\beta^2_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)(\phi' - \tilde{p}_{t^{r''}-1}) \\ & +\mu\left(\frac{\alpha - \beta_{T-t^{r''}}}{2\alpha^2\gamma\sigma^2_{\phi''}} - \frac{2\beta_{T-t^{r''}}}{\alpha - \beta_{T-t^{r''}}}\right)\left(\sum_{l=T-(t^{r'}-1)}^{T-1} \beta_l \Delta\tilde{p}_{T-l} + \beta_{T-t^{r'}}\left(\tilde{p}_{t^{r''}-1} - \tilde{p}_{t^{r'}-1}\right)\right) \\ & +(1-\mu)\alpha(D\phi' - \tilde{p}_{t^{r''}-1}), \end{aligned}$$

where we assume that the right-hand side is not constant in $\tilde{p}_{t^{r''}-1}$. Solving for $\tilde{p}_{t^{r''}-1}$ yields to a function that is linear in the first signal ϕ', the price differences $\Delta\tilde{p}_t$ and the price level at $t^{r'} - 1$. We therefore rewrite $\tilde{p}_{t^{r''}-1}$ as

$$\tilde{p}_{t^{r''}-1} = h(\phi', \Delta\tilde{p}_1, \ldots, \Delta\tilde{p}_{t^{r'}-1}, \tilde{p}_{t^{r'}-1}) \equiv a'_h\phi' + \sum_{t=1}^{t^{r'}-1} b_{h,t}\Delta\tilde{p}_t + c_h\tilde{p}_{t^{r'}-1},$$

where a'_h, $b_{h,t}\,(t = 1, 2, \ldots, t^{r'} - 1)$ and c_h are constants. The demand of rational speculators is also linear in the first signal ϕ', the past price differences $\Delta\tilde{p}_t$ and the price levels at $t^{r''} - 1$ and at $t^{r'} - 1$. Since we have seen that $\tilde{p}_{t^{r''}-1}$ is linear in ϕ', $\Delta\tilde{p}_t\,(t = 1, 2, \ldots, t^{r'} - 1)$ and $\tilde{p}_{t^{r'}-1}$ we rewrite the demand as

$$D^r_{t^{r''}-1} = g(\phi', \Delta\tilde{p}_1, \Delta\tilde{p}_{t^{r'}-1}, \tilde{p}_{t^{r'}-1}) \equiv a'_g\phi' + \sum_{t=1}^{t^{r'}-1} b_{g,t}\Delta\tilde{p}_t + c_g\tilde{p}_{t^{r'}-1},$$

where a'_g, $b_{g,t}\,(t=1,2,\ldots,t^{r'}-1)$ and c_g are again constants. Next we simplify $\tilde{p}_T$. We therefore consider equation (3.43) and use $p_T = \ldots = p_{t^{r''}}$ and $p_{t^{r''}-1} = \ldots = p_{t^{r'}}$. Then

$$\begin{aligned}
\tilde{p}_T &= \phi' + \phi'' + \sum_{l=T-t^{r'}+1}^{T-1} \beta_l \Delta\tilde{p}_{T-l} + \frac{\beta_{T-t^{r''}}}{\alpha}\left(\tilde{p}_T - \tilde{p}_{t^{r''}-1}\right) \\
&\quad + \frac{\beta_{T-t^{r'}}}{\alpha}\left(\tilde{p}_{t^{r''}-1} - \tilde{p}_{t^{r'}-1}\right) \\
\Leftrightarrow \tilde{p}_T\frac{\alpha-\beta_{T-t^{r''}}}{\alpha} &= \phi' + \phi'' + \sum_{l=T-t^{r'}+1}^{T-1} \beta_l \Delta\tilde{p}_{T-l} - \frac{\beta_{T-t^{r''}}}{\alpha}\tilde{p}_{t^{r''}-1} \\
&\quad + \frac{\beta_{T-t^{r'}}}{\alpha}\left(\tilde{p}_{t^{r''}-1} - \tilde{p}_{t^{r'}-1}\right) \\
\Leftrightarrow \tilde{p}_T &= \frac{1}{\alpha-\beta_{T-t^{r''}}}\Bigg[\alpha(\phi'+\phi'') + \sum_{l=T-t^{r'}+1}^{T-1} \beta_l \Delta\tilde{p}_{T-l} \\
&\quad -\beta_{T-t^{r''}}\tilde{p}_{t^{r''}-1} + \beta_{T-t^{r'}}\left(\tilde{p}_{t^{r''}-1} - \tilde{p}_{t^{r'}-1}\right)\Bigg],
\end{aligned}$$

so $\tilde{p}_T$ is linear in the two signals ϕ' and ϕ'', $\Delta\tilde{p}_t\,(t=1,2,\ldots,t^{r'}-1)$ and $\tilde{p}_{t^{r'}-1}$ (note that $\tilde{p}_{t^{r''}-1}$ are linear in ϕ', $\Delta\tilde{p}_t\,(t=1,2,\ldots,t^{r'}-1)$ and $\tilde{p}_{t^{r'}-1}$), so we have

$$\tilde{p}_T = f(\phi',\phi'',\Delta\tilde{p}_1,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}) \equiv a'_f\phi' + a''_f\phi'' + \sum_{t=1}^{t^{r'}-1} b_{f,t}\Delta\tilde{p}_t + c_f\tilde{p}_{t^{r'}-1},$$

with constants a'_f, a''_f, $b_{f,t}\,(t=1,2,\ldots,t^{r'}-1)$ and c_f. Since we have $p_T - v - \phi' - \phi'' = \tilde{p}_T - \phi' - \phi'$ in the final wealth we define

$$\begin{aligned}
\tilde{p}_T - \phi' - \phi' &= k(\phi,\phi'',\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}) \\
&\equiv a'_k\phi' + a''_k\phi'' + \sum_{t=1}^{t^{r'}-1} b_{f,t}\Delta\tilde{p}_t + c_f\tilde{p}_{t^{r'}-1} \\
&= (a'_f-1)\phi' + (a''_f-1)\phi'' + \sum_{t=1}^{t^{r'}-1} b_{f,t}\Delta\tilde{p}_t + c_f\tilde{p}_{t^{r'}-1}.
\end{aligned}$$

By setting both signals ϕ' and ϕ'' equal to zero, we define

$$\begin{aligned}
\tilde{h}\left(\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right) &= h\left(0,\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right), \\
\tilde{g}\left(\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right) &= g\left(0,\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right), \\
\tilde{f}\left(\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right) &= f\left(0,0,\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right) \text{ and} \\
\tilde{k}\left(\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right) &= k\left(0,0,\Delta\tilde{p}_1,\ldots,\Delta\tilde{p}_{t^{r'}-1},\tilde{p}_{t^{r'}-1}\right).
\end{aligned}$$

With the linear functions h, g, f and k we can rewrite the final wealth as

$$\begin{aligned} W_T &= W_{t^{r'}-1} + (\tilde{p}_{t^{r'}} - \tilde{p}_{t^{r'}-1})D^r_{t^{r'}-1} + (\tilde{p}_{t^{r''}} - \tilde{p}_{t^{r''}-1})D^r_{t^{r''}-1} \\ &\quad + \alpha(p_T - \phi' - \phi'' - \Theta)(p_T - \phi' - \phi'') \\ &= W_{t^{r'}-1} + (h - \tilde{p}_{t^{r'}-1})D^r_{t^{r'}-1} + (f-h)g + \alpha(k-\theta)k. \end{aligned}$$

The expected value of W_T at date $t^{r'}-1$ $E^r_{t^{r'}-1}W_T$ is

$$E^r_{t^{r'}-1}W_T = W_{t^{r'}-1} + (\tilde{h} - \tilde{p}_{t^{r'}-1})D^r_{t^{r'}-1} + E^r_{t^{r'}-1}\left[(f-h)g\right] + \alpha E^r_{t^{r'}-1}k^2,$$

since θ has mean zero and is independently distributed. The conditional variance $\sigma^2_{W_T|t^{r'}-1}$ is

$$\begin{aligned} \sigma^2_{W_T|t^{r'}-1} &= E^r_{t^{r'}-1}\left[\left(W_T - E^r_{t^{r'}-1}W_T\right)^2\right] \\ &= E^r_{t^{r'}-1}\left[\left(W_{t^{r'}-1} + \left(h - \tilde{p}_{t^{r'}-1}\right)D^r_{t^{r'}-1} + (f-h)g + \alpha(k-\theta)k\right.\right. \\ &\quad \left.\left. - W_{t^{r'}-1} - \left(\tilde{h} - \tilde{p}_{t^{r'}-1}\right)D^r_{t^{r'}-1} - E^r_{t^{r'}-1}\left[(f-h)g\right] - \alpha E^r_{t^{r'}-1}k^2\right)^2\right] \\ &= E^r_{t^{r'}-1}\left[\left(\left(h-\tilde{h}\right)D^r_{t^{r'}-1} + (f-h)g - E^r_{t^{r'}-1}\left[(f-h)g\right]\right.\right. \\ &\quad \left.\left. - \alpha\left(k^2 - k\theta - E^r_{t^{r'}-1}k^2\right)\right)^2\right] \\ &= E^r_{t^{r'}-1}\left[\left(h-\tilde{h}\right)^2\right]\left(D^r_{t^{r'}-1}\right)^2 + C''' + 2D^r_{t^{r'}-1} \cdot \\ &\quad E^r_{t^{r'}-1}\left[\left(h-\tilde{h}\right)\left((f-h)g - E^r_{t^{r'}-1}\left[(f-h)g\right] + \alpha(k^2 - \theta k - E^r_{t^{r'}-1}k^2)\right)\right] \end{aligned}$$

where C''' does not depend on $D^r_{t^{r'}-1}$ and is therefore irrelevant for mean-variance utility maximization. Using again the fact that θ has mean zero and is independently distributed and that $E^r_{t^{r'}-1}(h-\tilde{h}) = a'_h E^r_{t^{r'}-1}\phi' = 0$ we get

$$\begin{aligned} \sigma^2_{W_T|t^{r'}-1} &= E^r_{t^{r'}-1}\left[\left(h-\tilde{h}\right)^2\right]\left(D^r_{t^{r'}-1}\right)^2 + C''' \\ &\quad + 2D^r_{t^{r'}-1}\left(E^r_{t^{r'}-1}\left[(h-\tilde{h})(f-h)g\right] + E^r_{t^{r'}-1}\left[\alpha(h-\tilde{h})k^2\right]\right). \end{aligned}$$

Since $h - \tilde{h} = a'_h\phi'$,

$$E^r_{t^{r'}-1}\left(h-\tilde{h}\right)^2 = a'_h\sigma^2_{\phi'}.$$

To replace $E^r_{t^{r'}-1}\left[(h-\tilde{h})(f-h)g\right]$ and $E^r_{t^{r'}-1}\left[(h-\tilde{h})k^2\right]$ in the variance, we first calculate $(f-h)g$ and k^2:

$$\begin{aligned}(f-h)g &= (\tilde{f}+a'_f\phi'+a''_f\phi''-\tilde{h}-a'_h\phi')(\tilde{g}+a'_g\phi')\\ &= (\tilde{f}-\tilde{h})\tilde{g}+(\tilde{f}-\tilde{h})a'_g\phi'+a'_f\phi'\tilde{g}+a'_fa'_g(\phi')^2\\ &\quad +a''_f\phi''\tilde{g}+a''_f\phi''a'_g\phi'-a'_h\phi'\tilde{g}-a'_ha'_g(\phi')^2\end{aligned}$$

and

$$\begin{aligned}k^2 &= (a'_k\phi'+a''_k\phi''+\tilde{k})^2\\ &= (a'_k\phi')^2+(a''_k\phi'')^2+\tilde{k}^2+2a'_ka''_k\phi'\phi''+2a'_k\phi'\tilde{k}+2a''_k\phi''\tilde{k}.\end{aligned}$$

So

$$\begin{aligned}E^r_{t^{r'}-1}\left[(h-\tilde{h})(f-h)g\right] &= a'_hE^r_{t^{r'}-1}\left[\phi'(f-h)g\right]\\ &= a'_h\Big(E^r_{t^{r'}-1}\left[\phi'(\tilde{f}-\tilde{h})\tilde{g}\right]+E^r_{t^{r'}-1}\left[(\tilde{f}-\tilde{h})a'_g(\phi')^2\right]\\ &\quad +E^r_{t^{r'}-1}\left[a'_f\tilde{g}(\phi')^2\right]+E^r_{t^{r'}-1}\left[a'_fa'_g(\phi')^3\right]\\ &\quad +E^r_{t^{r'}-1}\left[a''_f\phi''\tilde{g}\phi'\right]+E^r_{t^{r'}-1}\left[a''_f\phi''a'_g(\phi')^2\right]\\ &\quad -E^r_{t^{r'}-1}\left[a'_h(\phi')^2\tilde{g}\right]-E^r_{t^{r'}-1}\left[a'_ha'_g(\phi')^3\right]\Big)\\ &= a'_h\left[(\tilde{f}-\tilde{h})a'_g+(a'_f-a'_h)\tilde{g}\right]\sigma^2_{\phi'}\end{aligned}$$

since $E^r_{t^{r'}-1}\phi' = E^r_{t^{r'}-1}(\phi'^3) = 0$ (mean zero and symmetry of the density of ϕ') and $E^r_{t^{r'}-1}\phi'' = E^r_{t^{r'}-1}(\phi'\phi'') = 0$ (mean zero and independence of ϕ' and ϕ''). Moreover,

$$\begin{aligned}E^r_{t^{r'}-1}\left[(h-\tilde{h})k^2\right] &= a'_h\Big(E^r_{t^{r'}-1}\left[(a'_k)^2(\phi')^3\right]+E^r_{t^{r'}-1}\left[(a''_k)^2\phi'\phi''^2\right]\\ &\quad +E^r_{t^{r'}-1}\left[\tilde{k}^2\phi'\right]+E^r_{t^{r'}-1}\left[2a'_ka''_k(\phi')^2\phi''\right]\\ &\quad +E^r_{t^{r'}-1}\left[2a'_k(\phi')^2\tilde{k}\right]+E^r_{t^{r'}-1}\left[2a''_k\phi''\tilde{k}\phi'\right]\Big)\\ &= 2a'_ha'_k\tilde{k}\sigma^2_{\phi'},\end{aligned}$$

because of the symmetry of ϕ' and the independence of ϕ' and ϕ''. So

$$\sigma^2_{W_T|t^{r'}-1} = a'_h\left[a'_h\left(D^r_{t^{r'}-1}\right)^2+2D^r_{t^{r'}-1}\Big((\tilde{f}-\tilde{h})a'_g+(a'_f-a'_h)\tilde{g}+2\alpha a'_k\tilde{k}\Big)\right]\sigma^2_{\phi'}+C'''',$$

where C'''' does not depend on $D^r_{t^{r'}-1}$. So the mean-variance utility function is

$$\begin{aligned}\mu_W - \gamma\sigma_W^2 &= (\tilde{h} - \tilde{p}_{t^{r'}-1})D^r_{t^{r'}-1} \\ &\quad -\gamma a'_h \left[a'_h \left(D^r_{t^{r'}-1}\right)^2 + 2D^r_{t^{r'}-1}\left((\tilde{f}-\tilde{h})a'_g + (a'_f - a'_h)\tilde{g} + 2\alpha a'_k\tilde{k}\right)\right]\sigma^2_{\phi'} \\ &\quad +C''''',\end{aligned}$$

where C''''' again does not depend on $D^r_{t^{r'}-1}$. Deriving with respect to $D^r_{t^{r'}-1}$, we get

$$\begin{aligned}\frac{\partial}{\partial D^r_{t^{r'}-1}}\mu_W - \gamma\sigma_W^2 &= (\tilde{h} - \tilde{p}_{t^{r'}-1}) - 2\gamma\left(a'_h\right)^2 D^r_{t^{r'}-1}\sigma^2_{\phi'} \\ &\quad -2a'_h\gamma\left((\tilde{f}-\tilde{h})a'_g + (a'_f - a'_h)\tilde{g} + 2\alpha a'_k\tilde{k}\right)\sigma^2_{\phi'} \\ &\stackrel{!}{=} 0.\end{aligned}$$

Solving for $D^r_{t^{r'}-1}$ yields to

$$\begin{aligned}D^r_{t^{r'}-1} &= \frac{\tilde{h} - \tilde{p}_{t^{r'}-1}}{2\gamma\left(a'_h\right)^2\sigma^2_{\phi'}} - \frac{1}{a'_h}\left((\tilde{f}-\tilde{h})a'_g + (a'_f - a'_h)\tilde{g} + 2a'_k\tilde{k}\right) \\ &\equiv \tilde{D}(\Delta\tilde{p}_1, \ldots, \Delta\tilde{p}_{t^{r'}-1}, \tilde{p}_{t^{r'}-1}),\end{aligned}$$

where $\tilde{D}$ is linear in $\Delta p_t\,(t = 1, 2, \ldots, t^{r'} - 1)$ and in $\tilde{p}_{t^{r'}-1}$. So the market clearing condition in period $t^{r'} - 1$ is

$$0 = \sum_{l=1}^{t^{r'}-2} \beta_l \Delta\tilde{p}_{t^{r'}-1-l} + \mu\tilde{D} - (1-\mu)\alpha\tilde{p}_{t^{r'}-1}.$$

With constants $b_{p,t}$ and c_p, we have

$$\tilde{p}_{t^{r'}-1} = p(\Delta\tilde{p}_1, \ldots, \Delta\tilde{p}_{t^{r'}-2}, \tilde{p}_{t^{r'}-2}) \equiv \sum_{t=1}^{t^{r'}-2} b_{p,t}\Delta\tilde{p}_t + c_p\tilde{p}_{t^{r'}-2}.$$

Next, we show that

$$\tilde{p}_{t,s} = \tilde{p}_{t^{r'}-1,s} \text{ for } t = 0, 1, \ldots, t^{r'} - 1 \text{ and } s \in \Omega.$$

The reasoning is the same as before. Let us assume that $p_{t^{r'}-1,s} \neq p_{t^{r'}-2,s}$ for $s \in \sigma_{t^{r'}-2}$. Then the capital gains $(p_{t^{r'}-1} - p_{t^{r'}-2})D^r_{t^{r'}-2}$ are monotonically increasing in $D^r_{t^{r'}-2}$. So a solution for their optimization problem does not exist. Therefore $\tilde{p}_{t^{r'}-1,s} =$

$\tilde{p}_{t^{r'}-2,s}$. Repeating this argument recursively, we get $\tilde{p}_{t,s} = \tilde{p}_{t^{r'}-1,s}$ for $t = 0, 1, \ldots, t^{r'} - 1$ and $s \in \Omega$. Using this, we get

$$\tilde{p}_{t^{r'}-1} = p\left(0, \ldots, 0, \tilde{p}_{t^{r'}-2}\right) = c_p \tilde{p}_{t^{r'}-2}.$$

So

$$\tilde{p}_{t,s} = 0,\ t = 0, 1, \ldots, t^{r'} - 1,\ s \in \Omega$$

is uniquely determined unless $c_p = 1$. □

3.4.8 Bubbles

We now want continue our discussion on bubbles from section 3.3.7. We have seen in the previous section that the equilibrium is unique so rational bubbles cannot exist. But how about the speculative bubbles we mentioned earlier? Recall that we defined a speculative bubble as the difference between the fundamental value and the equilibrium price, where the fundamental value was the price that made a buy-and-hold strategy unprofitable. Such a bubble did not exist in the case with one signal but, as we will see that it may exist in the case with two signals.

Proposition 15. *Given that fundamental value is the price that makes a buy-and-hold strategy unprofitable, the equilibrium price differs from the fundamental value by*

$$\frac{\lambda''(\lambda' - \nu')}{1 + \lambda''}\phi''$$

for dates $t = t^{r'}, \ldots, t^{r''} - 1$ (between the arrival of the first and the second signal). At all other dates, the equilibrium price coincides with the fundamental value.

Proof. The proof is almost in the same manner as the proof of proposition 10 (especially for dates $t = 0, 1, \ldots, t^{r'} - 1$ and $t = t^{r''}, t^{r''} + 1, \ldots, T$). Recall that in order to show that equilibrium price equals fundamental value, we have to show that $\frac{dU_t^r}{dD_t^r} = 0$ in equation (3.20):

$$p_t = v + E_t^r \Phi - \gamma \frac{d\sigma^{r2}_{W^r_{t+1}|t}}{dD_t^r} - \frac{dU_t^r}{dD_t^r}.$$

Final wealth for dates $t = t^{r''}, t^{r''}+1, \ldots, T$ is $W^r_{T+1} = W^r_t + (v + \phi' + \phi'' + \theta - p_T)D^r_T$ so

$$E^r_t W^r_{T+1} = W^r_t + (v + \phi' + \phi'' - p_T)D^r_T$$

and

$$\frac{d(W^r_t - E^r_t W^r_{T+1})}{dD^r_T} = \theta.$$

Together with (3.21), $\frac{dE^r_t W^r_{T+1}}{dD^r_t} = (v + \phi' + \phi'' - p_T)$ and $\alpha = \frac{1}{2\gamma\sigma^2_\theta}$, we have

$$\begin{aligned}
\frac{dU^r_t}{dD^r_t} &= \frac{dE^r_t W^r_{T+1}}{dD^r_t} - \gamma \frac{d\sigma^{r2}_{W^r_{t+1}|t}}{dD^r_t} \\
&= (v + \phi' + \phi'' - p_T) - 2\gamma E^r_t \left[\theta D^r_T \theta\right] \\
&= (v + \phi' + \phi'' - p_T) - 2\gamma E^r_t \left[\theta^2 \alpha (v + \phi' + \phi'' - p_T)\right] \\
&= (v + \phi' + \phi'' - p_T)(1 - 2\alpha\gamma\sigma^2_\theta) \\
&= 0.
\end{aligned}$$

Next we consider the time span between the arrival of the two signals when final wealth is

$$W^r_{T+1} = W^r_t + (p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} + (v + \phi' + \phi'' + \theta - p_T)D^r_T$$

for $t = t^{r'}, t^{r'}+1, \ldots, t^{r''}-1$. Its date$-t$ expectation is

$$E^r_t W^r_{T+1} = W^r_t + E^r_t \left[(p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1}\right] + E^r_t \left[(v + \phi' + \phi'' + \theta - p_T)D^r_T\right],$$

so together with (3.4) we have

$$\begin{aligned}
\frac{dE^r_t W^r_{T+1}}{dD^r_t} &= E^r_t \left(p_{t^{r''}} - p_{t^{r''}-1}\right) + E^r_t \left(v + \phi' + \phi'' + \theta - p_T\right) \\
&= -(1 + \nu')\phi' + \phi' \\
&= -\nu'\phi'
\end{aligned}$$

and

$$\begin{aligned}
\frac{d(W^r_{T+1} - E^r_t W^r_{T+1})}{dD^r_t} &= (p_{t^{r''}} - p_{t^{r''}-1}) + (v + \phi' + \phi'' + \theta - p_T) \\
&\quad - E^r_t \left(p_{t^{r''}} - p_{t^{r''}-1}\right) - E^r_t \left(v + \phi' + \phi'' + \theta - p_T\right) \\
&= \phi'' + \theta.
\end{aligned}$$

Using again (3.4) and (3.21), we get

$$\begin{aligned}
\frac{d\sigma^2_{W^r_{T+1}|t}}{dD^r_t} &= 2E^r_t\left[\left\{(p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} + (v+\phi'+\phi''+\theta)D^r_T\right.\right. \\
&\quad \left.\left. -E^r_t\left[(p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1}\right] - E^r_t\left[(v+\phi'+\phi''+\theta)D^r_T\right]\right\}(\phi''+\theta)\right] \\
&= 2E^r_t\left[\left\{((\lambda'-\nu')\phi' + (1+\lambda'')\phi'')\left(\frac{(\lambda'-\nu')\phi'}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}\right)\right.\right. \\
&\quad +\alpha(\lambda'\phi'+\lambda''\phi'')(\lambda'\phi'+\lambda''\phi''-\theta) - (\lambda'-\nu')\phi' \\
&\quad \left.\left.\left(\frac{(\lambda'-\nu')\phi'}{2\gamma(1+\lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''\phi'}{1+\lambda''}\right) - \alpha(\lambda'^2\phi'^2 + \lambda''^2\sigma^2_{\phi''})\right\}(\phi''+\theta)\right] \\
&= 2E^r_t\left[\left\{\frac{(\lambda'-\nu')\phi'\phi''}{2\gamma(1+\lambda'')\sigma^2_{\phi''}} - 2\alpha\lambda'\lambda''\phi'\phi'' + 2\alpha\lambda'\lambda''\phi'\phi''\right.\right. \\
&\quad \left.\left. -\alpha\lambda'\phi'\theta - \alpha\lambda''\phi''\theta + \alpha\lambda''^2\phi''^2 - \alpha\lambda''^2\sigma^2_{\phi''}\right\}(\phi''+\theta)\right] \\
&= 2\left[\frac{\lambda'-\nu'}{2\gamma(1+\lambda'')} - \alpha\lambda'\sigma^2_\theta\right]\phi',
\end{aligned}$$

since ϕ'' has mean zero, is symmetric and uncorrelated with θ which has also mean zero. Therefore,

$$\begin{aligned}
\frac{dU^r_t}{dD^r_t} &= \frac{dE^r_t W^r_{T+1}}{dD^r_t} - \gamma\frac{d\sigma^2_\theta}{dD^r_t} \\
&= -\nu'\phi' - 2\gamma\left[\frac{\lambda'-\nu'}{2\gamma(1+\lambda'')} - \alpha\lambda\sigma^2_\theta\right]\phi' \\
&= \left[-\nu' - \frac{\lambda'-\nu'}{1+\lambda''} + \lambda'\right]\phi' \\
&= \left[\frac{(\lambda'-\nu')(1+\lambda'')}{1+\lambda''} - \frac{\lambda'-\nu'}{1+\lambda''}\right]\phi' \\
&= \frac{\lambda''(\lambda'-\nu')}{1+\lambda''}\phi',
\end{aligned}$$

which is the price difference between the fundamental value and the equilibrium price. Finally we consider the dates before the arrival of the first signal. For $t = 0, 1, \ldots, t^{r'}-1$, final wealth is

$$W^r_{T+1} = W^r_t + (p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1} + (p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1} + (v+\phi'+\phi''+\theta-p_T)D^r_T.$$

Its date$-t$ expectation is

$$\begin{aligned}
E^r_t W^r_{T+1} &= W^r_t + E^r_t\left[(p_{t^{r'}} - p_{t^{r'}-1})D^r_{t^{r'}-1}\right] + E^r_t\left[(p_{t^{r''}} - p_{t^{r''}-1})D^r_{t^{r''}-1}\right] \\
&\quad +E^r_t\left[(v+\phi'+\phi''+\theta-p_T)D^r_T\right],
\end{aligned}$$

so

$$\begin{aligned}
\frac{dE_t^r W_{T+1}^r}{dD_t^r} &= E_t^r\left(p_{t^{r'}} - p_{t^{r'}-1}\right) + E_t^r\left(p_{t^{r''}} - p_{t^{r''}-1}\right) + E_t^r\left(v + \phi' + \phi'' + \theta - p_T\right) \\
&= -p_{t^{r'}-1} + v \\
&= 0,
\end{aligned}$$

and together with (3.4) we have

$$\begin{aligned}
\frac{d(W_{T+1}^r - E_t^r W_{T+1}^r)}{dD_t^r} &= \left(p_{t^{r'}} - p_{t^{r'}-1}\right) + \left(p_{t^{r''}} - p_{t^{r''}-1}\right) + \left(v + \phi' + \phi'' + \theta - p_T\right) \\
&\quad - E_t^r\left(p_{t^{r'}} - p_{t^{r'}-1}\right) - E_t^r\left(p_{t^{r''}} - p_{t^{r''}-1}\right) \\
&\quad - E_t^r\left(v + \phi' + \phi'' + \theta - p_T\right) \\
&= \phi' + \phi'' + \theta.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{dU_t^r}{dD_t^r} &= -2\gamma \frac{d\sigma^2_{W_{T+1}^r|t}}{dD_t^r} \\
&= -2\gamma E_t^r\Bigg[\bigg\{(p_{t^{r'}} - p_{t^{r'}-1})D_{t^{r'}-1}^r + (p_{t^{r''}} - p_{t^{r''}-1})D_{t^{r''}-1}^r \\
&\quad + (v + \phi' + \phi'' + \theta - p_T)D_T^r - E_t^r\left[(p_{t^{r'}} - p_{t^{r'}-1})D_{t^{r'}-1}^r\right] \\
&\quad - E_t^r\left[(p_{t^{r''}} - p_{t^{r''}-1})D_{t^{r''}-1}^r\right] - E_t^r\left[(v + \phi' + \phi'' + \theta - p_T)D_T^r\right]\bigg\}(\phi' + \phi'' + \theta)\Bigg] \\
&= -2\gamma E_t^r\Bigg[\Bigg\{((\lambda' - \nu')\phi' + (1 + \lambda'')\phi'')\left(\frac{(\lambda' - \nu')\phi'}{2\gamma(1 + \lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha\lambda'\lambda''\phi'}{1 + \lambda''}\right) \\
&\quad + \alpha(\lambda'\phi' + \lambda'\phi'' - \theta)(\lambda'\phi' + \lambda''\phi'') - \frac{(\lambda' - \nu')^2\sigma^2_{\phi'}}{2\gamma(1 + \lambda'')^2\sigma^2_{\phi''}} + \frac{2\alpha\lambda'\lambda''(\lambda' - \nu')\sigma^2_{\phi}}{1 + \lambda''} \\
&\quad - \alpha\left(\lambda'^2\sigma^2_{\phi'} + \lambda''^2\sigma^2_{\phi''}\right)\Bigg\}(\phi' + \phi'' + \theta)\Bigg] \\
&= -2\gamma E_t^r\Bigg[\Bigg\{\frac{(\lambda' - \nu')^2\phi'^2}{2\gamma(1 + \lambda'')^2\sigma^2_{\phi''}} - \frac{2\alpha(\lambda' - \nu')\lambda'\lambda''\phi'^2}{1 + \lambda''} + \frac{(\lambda' - \nu')\phi'\phi''}{2\gamma(1 + \lambda'')\sigma^2_{\phi''}} \\
&\quad - 2\alpha\lambda'\lambda''\phi'\phi'' + \alpha(\lambda'^2\phi'^2 + \lambda''^2\phi''^2 + 2\lambda'\lambda''\phi'\phi'' - \theta\lambda'\phi' - \theta\lambda''\phi'') \\
&\quad - \frac{(\lambda' - \nu')^2\sigma^2_{\phi'}}{2\gamma(1 + \lambda'')^2} + \frac{2\alpha\lambda'\lambda''(\lambda' - \nu')\sigma^2_{\phi}}{1 + \lambda''} - \alpha\left(\lambda'^2\sigma^2_{\phi'} + \lambda''^2\sigma^2_{\phi''}\right)\Bigg\}(\phi' + \phi'' + \theta)\Bigg] \\
&= 0,
\end{aligned}$$

since ϕ' and ϕ'' have mean zero, are uncorrelated and symmetric and since θ has mean zero and is uncorrelated with ϕ' and ϕ''. □

Proposition 15 shows that, unlike in the case with one signal (see proposition 10), a speculative bubble may exist in the case with two signals. The reason for the existence of the bubble is the time inconsistency of the optimal plans in the dynamic mean-variance setup (see Chen et al. (1971)). Before the arrival of the first and after the arrival of the second signal there is no scope for time inconsistency. Therefore, there is no bubble in the case with one signal. But in the case with two signals a buy-and-hold strategy might become profitable after the arrival of the first signal, so the asset price deviates from the fundamental value. This bubble is large for a positive realization of the first signal ($\phi' > 0$), if the predictable part of the price change $(\lambda' - \nu')\phi'$ is large. Note that Allen et al. (1993) state that it is a necessary for a finite bubble to exist that agents are short-sale constrained and there are information asymmetries. Since we do not have short-sale constraints or information asymmetries in our model, proposition 15 shows that the fact that these conditions are necessary depends on the time consistency of the optimal plans.

3.5 Results

We have seen in the preceding paragraphs that a different time setup and different assumptions on the values of β_l have a strong impact on the price response to the signals.

Adding just one additional trading period in the case with one signal led to a very different result than in DSSW. Instead of destabilizing speculation we had stabilizing or neutral speculation. Adding more trading dates led to the result that, in the presence of rational speculators, only one feedback parameter is of interest. Modifying the measure of the three groups of agents did not change the results qualitatively as long as certain parameter assumptions hold.

In the case with two signals, there was a wide range of price dynamics. The reactions to the first signal ranged from over- and then to underreaction ($\lambda' < 0 < \nu'$), over mean reversion ($0 < \lambda' < \nu'$), and an increasing reaction ($0 < \nu' < \lambda'$) to underreaction at the beginning and overreaction in the end ($\nu' < 0 < \lambda'$). Even after good news it may happen that rational speculators go short and the asset price falls. The reaction to the second signal was essentially the same as the reaction in the case with one signal. In contrast to the case with one signal rational speculators' measure μ was very important for the reaction to the first signal.

In our discussion of bubbles, we discussed several definitions of bubbles. By proving that the equilibrium is unique we showed that rational bubbles cannot exist. Considering speculative bubbles that come from a buy-and-hold strategy, we saw that such a bubble may exist only in the case with two signals.

A point for future research is to consider the model with more than two signals, although this might prove to be mathematically challenging. Another interesting question is, whether and how the results change when the signals become noisy.

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