Universität Regensburg
Mathematik

Every conformal class contains a metric of bounded geometry

Olaf Müller and Marc Nardmann

Preprint Nr. 09/2013

EVERY CONFORMAL CLASS CONTAINS A METRIC OF BOUNDED GEOMETRY

OLAF MÜLLER AND MARC NARDMANN

Abstract. We show that on every manifold, every conformal class of semi-Riemannian metrics contains a metric g such that each kth-order covariant derivative of the Riemann tensor of g has bounded absolute value a_k. This result is new also in the Riemannian case, where one can arrange in addition that g is complete with injectivity and convexity radius ≥ 1. One can even make the radii rapidly increasing and the functions a_k rapidly decreasing at infinity. We prove generalizations to foliated manifolds, where curvature, second fundamental form and injectivity radius of the leaves can be controlled similarly. Moreover, we explain a general principle that can be used to obtain analogous results for Riemannian manifolds equipped with arbitrary other additional geometric structures instead of foliations.

1. Introduction. Statement of results

A classical result due to R. E. Greene [10] says that every manifold admits a Riemannian metric of bounded geometry. It is therefore natural to ask a more refined question: Which conformal classes of Riemannian metrics on a given manifold M contain metrics of bounded geometry? The question is of course trivial on compact manifolds, because every metric there has bounded geometry. The problem on open manifolds has been considered by Eichhorn–Fricke–Lang [8], who proved that certain quite special conformal classes on manifolds of suitable topology contain metrics of bounded geometry. In the present article, we will show that on every manifold, *each* conformal class of Riemannian metrics contains a metric of bounded geometry. We also state and prove generalizations to foliated Riemannian manifolds and to semi-Riemannian manifolds of arbitrary signature, but let us first discuss the plain Riemannian case.

1.1. **Conventions.** $0 \in \mathbb{N}$. Manifolds are pure-dimensional, second countable, without boundary, and real-analytic. (Recall that the real-analyticity assumption is no loss of generality: For $r \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, every C^r-atlas contains a real-analytic subatlas, and every two such subatlases are real-analytically diffeomorphic; cf. e.g. [15].) Semi-Riemannian metrics and foliations are C^∞. A manifold-with-boundary may have an empty boundary. A compact exhaustion of a manifold M is a sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of M with $\bigcup_{i\in\mathbb{N}} K_i = M$ such that each K_i is contained in the interior of K_{i+1}. A compact exhaustion $(K_i)_{i\in\mathbb{N}}$ is **smooth** iff all K_i are C^∞ codimension-0 submanifolds-with-boundary of M.

1.2. **Definition.** Let M be a manifold, let $k \in \mathbb{N}$, let $\varepsilon, \iota \in C^0(M, \mathbb{R}_{>0})$. A Riemannian metric g on M **has k-geometry bounded by** (ε, ι) iff

- $\left|\nabla^i \operatorname{Riem}_g\right|_g \leq \varepsilon$ holds for every $i \in \{0, \dots, k\}$; and
- for each $x \in M$, the injectivity radius $\operatorname{inj}_g(x) \in \,]0, \infty]$ of g at the point x is $\geq \iota(x)$.

Here $\nabla^i \operatorname{Riem}_g$ denotes the ith covariant derivative with respect to g of the Riemann tensor Riem_g. (It does not matter whether we consider Riem_g as a $(4,0)$-tensor or $(3,1)$-tensor; the resulting functions $\left|\nabla^i \operatorname{Riem}_g\right|_g \in C^0(M, \mathbb{R}_{\geq 0})$ are the same in both cases.)

Let $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ be a compact exhaustion of M, let $\mathcal{E} = (\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence in $C^0(M, \mathbb{R}_{>0})$. A Riemannian metric g on M **has $(\infty, \mathcal{K})$-geometry bounded by** $(\mathcal{E}, \iota)$ iff

- for every $i \in \mathbb{N}$, the inequality $\left|\nabla^i \operatorname{Riem}_g\right|_g \leq \varepsilon_i$ holds on $M \setminus K_i$;
- $\operatorname{inj}_g \geq \iota$.

According to standard terminology, a Riemannian metric g on M **has bounded geometry** iff there exist a sequence $\mathcal{E} = (\varepsilon_i)_{i\in\mathbb{N}}$ of positive *constants* and a *constant* $\iota \in \mathbb{R}_{>0}$ such that

- for every $i \in \mathbb{N}$, the inequality $\left|\nabla^i \mathrm{Riem}_g\right|_g \le \varepsilon_i$ holds on M; and
- $\mathrm{inj}_g \ge \iota$.

For the relation of our "*k-geometry*" terminology to notions involving derivatives of the metric coefficients with respect to normal coordinates, see [7].

1.3. **Fact.** *Let g be a Riemannian metric on a manifold M. The following statements are equivalent:*

(1) g has bounded geometry.

(2) There exist a compact exhaustion $\mathcal{K}$ of M, a sequence $\mathcal{E} = (\varepsilon_i)_{i\in\mathbb{N}}$ of positive constants, and a constant $\iota \in \mathbb{R}_{>0}$ such that g has $(\infty, \mathcal{K})$-geometry bounded by $(\mathcal{E}, \iota)$.

Proof. (1)⇒(2) follows immediately from the fact that every manifold admits a compact exhaustion. (2)⇒(1) follows from the fact that a function on M which is bounded on the complement of a compact set K_i is bounded on M. □

Now we can state our main result for Riemannian metrics:

1.4. **Theorem.** *Let $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ be a smooth compact exhaustion of a manifold M, let $\iota, u_0 \in C^0(M, \mathbb{R}_{>0})$, let $\mathcal{E}$ be a sequence in $C^0(M, \mathbb{R}_{>0})$, let g_0 be a Riemannian metric on M. Then there exists a real-analytic function $u\colon M \to \mathbb{R}$ with $u > u_0$ such that the metric $\mathrm{e}^{2u} g_0$ is complete and has $(\infty, \mathcal{K})$-geometry bounded by $(\mathcal{E}, \iota)$.*

The statement that the conformal class of g_0 contains a metric with $(\infty, \mathcal{K})$-geometry bounded by $(\mathcal{E}, \iota)$ becomes of course the stronger the more rapidly the elements of $\mathcal{E}$ decay at infinity and the more rapidly ι increases at infinity. The property $u > u_0$ can be used for instance to make volumes and diameters of given compact subsets of M as large as desired.

1.5. **Corollary.** *Let M be a manifold. Every conformal class of Riemannian metrics on M contains a metric of bounded geometry. Every conformal class of Riemannian metrics on M that contains a real-analytic metric contains a real-analytic metric of bounded geometry.*

Proof. We choose a smooth compact exhaustion $\mathcal{K}$ of M, a sequence $\mathcal{E}$ of positive constants, and a constant $\iota > 0$. We apply Theorem 1.4 to a metric g_0 — a real-analytic one if possible — in the given conformal class. The resulting $g = \mathrm{e}^{2u} g_0$ satisfies (2) from Fact 1.3 and thus has bounded geometry. □

Remark 1. Every manifold admits a real-analytic Riemannian metric by the Morrey–Grauert embedding theorem; cf. [15] and the references therein. But not every conformal class of Riemannian metrics contains a real-analytic one. For instance, on every nonempty manifold of dimension ≥ 4 one can easily construct a metric whose Weyl tensor is not real-analytic.

Remark 2. In the introduction to their article [8], Eichhorn–Fricke–Lang state in passing that it be easy to endow $\mathbb{R}^n$ with a metric which is not conformally equivalent to any metric of bounded geometry. Corollary 1.5 disproves that.

1.6. **Corollary.** *Let $k \in \mathbb{N}$, let g_0 be a Riemannian metric on a manifold M, let $\varepsilon, \iota, u_0 \in C^0(M, \mathbb{R}_{>0})$. Then there exists a real-analytic $u\colon M \to \mathbb{R}$ with $u > u_0$ such that $\mathrm{e}^{2u} g_0$ has k-geometry bounded by (ε, ι).*

Proof. We choose a smooth compact exhaustion $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ of M with $K_i = \emptyset$ for $i \le k$. We define $\mathcal{E}$ to be the sequence all of whose entries are ε. Theorem 1.4 applied to $\mathcal{K}, \mathcal{E}, \iota$ proves the claim. □

Remark. As stated in 1.1, we assume metrics to be C^∞ for simplicity. Regularity C^{k+2} would suffice for the Corollary 1.6 on k-bounded geometry, though, as interested readers will have no difficulty to check.

1.7. **Remark.** Since the standard definition of bounded geometry involves the injectivity radius, we have used it in the statements above. Replacing inj_g by the convexity radius conv_g in Definition 1.2 yields superficially stronger statements 1.4, 1.5, 1.6, though, because every Riemannian metric g satisfies $\mathrm{conv}_g \le \mathrm{inj}_g$ (and stronger inequalities hold for complete metrics). However, 1.3, 1.4, 1.5, 1.6 remain true with conv_g instead of inj_g, as we state explicitly in Theorem 1.8 and prove in Section 3.

The proof of Theorem 1.4 is similar to Greene's construction of metrics of bounded geometry [10] in several aspects: Like us, Greene uses a compact exhaustion $(K_i)_{i\in\mathbb{N}}$, thereby decomposing M into cylinders Z_i diffeomorphic to $\mathbb{R}\times\partial K_i$ and "topology-changing" regions U_i; and like us, he modifies a start metric g_0 only conformally, the conformal factor being constant on each U_i. The extreme simplification compared to our situation occurs on each of the sets Z_i, where Greene can choose g_0 to be a product metric, namely a very long cylinder, the length depending on the g_0-geometry of the neighboring regions U_i and U_{i+1}. The only information he needs is that the functions $\left|\nabla^i \mathrm{Riem}_g\right|_g$ and inj_g^{-1} become small when g is multiplied by a large constant, and that they depend continuously on g with respect to the fine C^∞-topology. As we are not free to choose g_0, we have to work considerably harder in the proof of 1.4, both with respect to $\left|\nabla^i \mathrm{Riem}_g\right|_g$ and with respect to inj_g.

One might ask whether Theorem 1.4 could be improved with respect to extensions of metrics. For instance, 1.4 says that for every $\varepsilon\in C^0(M,\mathbb{R}_{>0})$, every conformal class of Riemannian metrics on M contains a metric g with $\left|\mathrm{Riem}_g\right|_g<\varepsilon$. In Gromov's h-principle language [9, 11], this means that a certain (open second-order) partial differential relation for functions $M\to\mathbb{R}$ satisfies the h-principle. Whenever something like that happens, one should ask whether the relation satisfies even an *h-principle for extensions*. In our case, the question is this: Given a closed subset A of a manifold M and a function $\varepsilon\in C^0(M,\mathbb{R}_{>0})$, is the following statement true?

> "*Let g_0 be a Riemannian metric on M that fulfills $\left|\mathrm{Riem}_{g_0}\right|_{g_0}<\varepsilon$ on A. Then there exists a function $u\in C^\infty(M,\mathbb{R})$ with $u\big|_A=0$ such that $g:=\mathrm{e}^{2u}g_0$ fulfills $\left|\mathrm{Riem}_g\right|_g<\varepsilon$ on M.*"

One can ask analogous questions for weaker relations like $\mathrm{scal}_g>-n(n-1)\varepsilon$ instead of $\left|\mathrm{Riem}_g\right|_g<\varepsilon$, where $n=\dim M$. One can also weaken the statement:

> "*Let g_0 be a Riemannian metric on M that fulfills $\mathrm{Riem}_{g_0}=0$ on A. Then there exists a function $u\in C^\infty(M,\mathbb{R})$ with $u\big|_A=0$ such that $g:=\mathrm{e}^{2u}g_0$ fulfills $\mathrm{scal}_g>-\varepsilon$ on M.*"

Even this second statement is false: On every manifold M of dimension $n\geq 3$, for every compact codimension-0 submanifold-with-boundary $A\notin\{\emptyset,M\}$, for every $\varepsilon\in C^0(M,\mathbb{R}_{>0})$, and for every given Riemannian metric $\tilde{g}_0$ on M that satisfies $\mathrm{Riem}_{\tilde{g}_0}=0$ on A, there exists a counterexample g_0 to the statement which is equal to $\tilde{g}_0$ on A [14]. Hence the h-principle for extensions fails completely here.

It should be pointed out that this problem disappears at least for relations like $\left|\sec_g\right|<\mathrm{const}\in\mathbb{R}_{>0}$ when we drop the restriction to a given conformal class. Then the h-principle for extensions holds in the following form: When A is a closed subset of M such that no connected component of $M\setminus A$ is relatively compact in M, then for every Riemannian metric g_0 on M which satisfies the relation on A, there exists a (possibly not complete) metric g which satisfies the relation on M and is equal to g_0 on A. (This is a consequence of [9, Theorem 7.2.4]; cf. [12] for details and generalizations.)

Now that we have seen that Theorem 1.4 is the best result one can hope for in the "plain Riemannian" setting, let us discuss the announced generalizations to foliated manifolds and semi-Riemannian metrics. The core of our proof of Theorem 1.4 is the construction of solutions to certain ordinary differential inequalities. This core argument does not involve any geometry. In Section 2, we will axiomatize the general situation it applies to by introducing the notion of a *flatzoomer*: a functional that assigns to functions $u\in C^\infty(M,\mathbb{R})$ — which in our context describe conformal factors — functions $\Phi(u)\in C^0(M,\mathbb{R})$ that satisfy certain estimates. For instance, for $i\in\mathbb{N}$ and a Riemannian metric g on M, the functional $\Phi_i\colon u\mapsto\left|\nabla^i \mathrm{Riem}_{g[u]}\right|_{g[u]}$ with $g[u]:=\mathrm{e}^{2u}g$ is a flatzoomer. Leaving some subtleties of the injectivity radius aside, Theorem 1.4 is obtained as a special case of a result about sequences of flatzoomers (e.g. the sequence $(\Phi_i)_{i\in\mathbb{N}}$), namely Theorem 4.1 below. More generally, one can use this abstract result to prove the following theorem.

1.8. **Theorem.** *Let $\mathscr{F}$ be a foliation on a manifold M, let g_0,h_0 be semi-Riemannian metrics (not necessarily of the same signature) on M which induce (nondegenerate) semi-Riemannian metrics on (the leaves of) $\mathscr{F}$. Let $\mathscr{K}=(K_i)_{i\in\mathbb{N}}$ be a smooth compact exhaustion of M, let $\iota,u_0\in C^0(M,\mathbb{R}_{>0})$, let $\mathscr{E}=(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence in $C^0(M,\mathbb{R}_{>0})$. Then there exists a real-analytic function $u\colon M\to\mathbb{R}$ with $u>u_0$ such that the metrics $g:=\mathrm{e}^{2u}g_0$ and $h:=\mathrm{e}^{2u}h_0$ have the following properties:*

(i) For every $i \in \mathbb{N}$, $\left|\nabla^i \operatorname{Riem}_g\right|_h < \varepsilon_i$ holds on $M \setminus K_i$.
(ii) If g_0 is Riemannian, then g is complete with convexity radius $\operatorname{conv}_g > \iota$.
(iii) For every $i \in \mathbb{N}$, $\left|\nabla^i \operatorname{Riem}_{g_{\mathscr{F}}}\right|_{h_{\mathscr{F}}} < \varepsilon_i$ holds on $M \setminus K_i$.
(iv) If $(g_0)_{\mathscr{F}}$ is Riemannian, then for each $\mathscr{F}$-leaf L, g_L is complete with $\operatorname{conv}_{g_L} > \iota\big|_L$.
(v) For every $i \in \mathbb{N}$, $\left|\nabla^i II^{\mathscr{F}}_g\right|_h < \varepsilon_i$ holds on $M \setminus K_i$.

Here $\nabla^i II^{\mathscr{F}}_g$ denotes the ith covariant derivative with respect to g of the second fundamental form (equivalently: the Weingarten tensor) of $\mathscr{F}$ with respect to g (cf. 2.6 for details); for a semi-Riemannian metric η on M, $\eta_{\mathscr{F}}$ denotes the field of bilinear forms induced by η on (the leaves of) $\mathscr{F}$; for every leaf L of $\mathscr{F}$, g_L denotes the metric on L induced by g; $\nabla^i \operatorname{Riem}_{g_{\mathscr{F}}}$ is the tensor field on $\mathscr{F}$ which assigns to each $x \in M$ the value of $\nabla^i \operatorname{Riem}_{g_L}$ (defined as in 1.2) at x, where L is the leaf through x (cf. 2.5); and the absolute value $|T|_\eta$ of a tensor field T with respect to a semi-Riemannian metric η is defined to be the function $\left||T|^2_\eta\right|^{1/2}$ (cf. 2.1).

Remark 1. Theorem 1.4 is a special case of 1.8: By taking $g_0 = h_0$ to be Riemannian and $\mathscr{F}$ to be the codimension-0 foliation whose only leaf is M, the statements (i) and (iii) become equal, the statements (ii) and (iv) become equal, and (v) becomes trivial because $II^{\mathscr{F}}_g = 0$. In that situation, Theorem 1.4 is the conjunction of (i) and (ii) (equivalently: of (iii) and (iv)).

Remark 2. The Riemannianness assumptions in Theorem 1.8(ii),(iv) cannot be avoided in general, as we discuss briefly in Section 3 below. In particular, not every conformal class of Lorentzian metrics contains a geodesically complete one, even on closed manifolds.

Apart from Theorem 1.4, arguably the most interesting cases of Theorem 1.8 are those where g is Lorentzian and h is either equal to g or Riemannian. The first case might be more natural, but one often wants sharper estimates for instance of Riem_g than an indefinite metric $h = g$ can provide; cf. the Remark after 2.4 below. In the second case, one will typically consider not an arbitrary Riemannian metric h but one that is obtained from g by a Wick rotation around some g-timelike line subbundle of TM.

The information that the metric g we get from Theorem 1.8 lies in a given conformal class is particularly important when we consider metrics which are not Riemannian: then the causal structure of a metric (which is an invariant of the conformal class) plays a crucial role in almost all considerations. For example, if the given g_0 is a globally hyperbolic or stably causal Lorentzian metric, then the metric g provided by 1.8 has the same property.

Let us consider the case where $g_0 = h_0$ is Riemannian in Theorem 1.8. Even if one is not interested in having a solution metric g in each conformal class, the conformal class construction is probably the only chance to prove the theorem for an arbitrary foliation $\mathscr{F}$. Since such a foliation does usually not fit to the structure of *any* compact exhaustion $(K_i)_{i\in\mathbb{N}}$ of M (in the sense that the boundaries ∂K_i are not leaves of $\mathscr{F}$), a Greene-style construction would not work, for instance. The problem becomes even more severe when g_0 or h_0 is not Riemannian.

Our method of proof, in particular Theorem 4.1, should be regarded as a construction kit for all kinds of theorems in the spirit of 1.8. Instead of a foliation, such theorems might involve other geometric objects, e.g. bundles, almost complex structures or symplectic forms. Functions built from a metric g and such objects will often define flatzoomers via conformal change of g; cf. Remark 2.7. The flatzoomer condition is always easy to check for a given example. Whenever it holds, one gets a theorem of the form 1.8 saying that the considered function is small for some (complete) metric g in the desired conformal class.

Theorem 1.8 might also be useful in the context of conformally invariant field theories on curved backgrounds, because it allows to choose a background metric convenient for analytic considerations.

The article is organized as follows: Flatzoomers are introduced in Section 2. Injectivity and convexity radii are discussed in Section 3. Section 4 contains the proofs of our main results.

Acknowledgments. We are grateful to Nadine Große for pointing out reference [8], and we thank Stefan Suhr for remarks on the completeness of Lorentzian metrics.

2. Flatzoomers

In this section, we introduce the notions *flatzoomer* and *quasi-flatzoomer* and give several examples.

2.1. **Notation** ($g[u]$, $\nabla^i_g T$, $|T|^2_g$, $\mathbb{R}\mathrm{Poly}^d_m$)**.** Let (M,g) be a semi-Riemannian manifold. For $u \in C^0(M,\mathbb{R})$, we denote the semi-Riemannian metric $\mathrm{e}^{2u}g$ by $g[u]$. For $i \in \mathbb{N}$, the ith covariant derivative with respect to g of a C^∞ tensor field T on M is denoted by $\nabla^i_g T$. The function $|T|^2_g \in C^\infty(M,\mathbb{R})$ is the total contraction of $T \otimes T$ via g in corresponding tensor indices; it might be negative if g is not Riemannian. The function $|T|_g \in C^0(M,\mathbb{R}_{\geq 0})$ is defined to be $\big||T|^2_g\big|^{1/2}$. Riem_g denotes the Riemann tensor, viewed as a tensor field of type $(4,0)$. We adopt the Besse sign convention for Riem_g [2].

For $m, d \in \mathbb{N}$, $\mathbb{R}\mathrm{Poly}^d_m$ denotes the (finite-dimensional) $\mathbb{R}$-vector space of real polynomials of degree $\leq d$ in m variables, equipped with its unique Hilbert space topology.

Remark. Recall that $|T|^2_g$ does not change when we raise or lower indices of T via g. In particular, functions like $\big|\nabla^i_g \mathrm{Riem}_g\big|^2_g$ do not depend on whether we consider Riem_g as a $(4,0)$- or $(3,1)$-tensor field. However, when h is another semi-Riemannian metric on M, then in general $\big|\nabla^i_g \mathrm{Riem}_g\big|^2_h$ depends on this choice. Nevertheless, the difference is hardly relevant anywhere in this article.

2.2. **Definition.** Let M be a manifold. A functional $\Phi\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^0(M,\mathbb{R})$ is a **flatzoomer** iff for some — and hence every — Riemannian metric η on M, there exist $k, d \in \mathbb{N}$, $\alpha \in \mathbb{R}_{>0}$, $u_0 \in C^0(M,\mathbb{R}_{\geq 0})$ and a polynomial-valued map $P \in C^0\big(M, \mathbb{R}\mathrm{Poly}^d_{k+1}\big)$ such that

$$\big|\Phi(u)(x)\big| \leq \mathrm{e}^{-\alpha u(x)} P(x)\Big(u(x), \big|\nabla^1_\eta u\big|_\eta(x), \ldots, \big|\nabla^k_\eta u\big|_\eta(x)\Big)$$

holds for all $x \in M$ and all $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ with $u(x) > u_0(x)$.

Proof of "and hence every". This is essentially straightforward and similar to but simpler than the proof of Example 2.5 below. We omit the details. □

2.3. **Example** (covariant derivatives of the Riemann tensor)**.** Let (M,g) be a semi-Riemannian manifold, let $k \in \mathbb{N}$. Then $\Phi\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^\infty(M,\mathbb{R})$ defined by

$$\Phi(u) := \Big|\nabla^k_{g[u]} \mathrm{Riem}_{g[u]}\Big|^2_{g[u]}$$

is a flatzoomer. (This will be proved after Example 2.5 below.)

Remark. Examples of this form, where Φ results from a given function like $\big|\nabla^k_g \mathrm{Riem}_g\big|^2_g$ by varying g conformally, motivate the terminology "*flatzoomer*": As the "zoom factor" u becomes larger, $|\Phi(u)|$ becomes smaller (because $\mathrm{e}^{-\alpha u}$ tends to 0) locally uniformly, provided the derivatives of u are bounded in a suitable way described by P and η. For instance, the curvature of $g[u]$ becomes smaller in the sense that $\big|\big|\mathrm{Riem}_{g[u]}\big|^2_{g[u]}\big|$ tends to 0; i.e., $g[u]$ becomes flatter.

We can generalize Example 2.3:

2.4. **Example** (covariant derivatives of the Riemann tensor again)**.** Let g, h be semi-Riemannian metrics (not necessarily of the same signature) on a manifold M, let $k \in \mathbb{N}$. Then $\Phi\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^\infty(M,\mathbb{R})$ defined by

$$\Phi(u) := \Big|\nabla^k_{g[u]} \mathrm{Riem}_{g[u]}\Big|^2_{h[u]}$$

is a flatzoomer. (This will be proved after Example 2.5 below.)

Remark. Especially interesting is the case where g is Lorentzian and h is Riemannian. There are many situations, in particular in General Relativity, where one would like to have a Lorentzian metric g on a manifold M which makes a certain codimension-1 foliation $\mathscr{F}$ on M spacelike, such that the curvature of g is controlled in a stronger sense than the absolute value of $\big|\mathrm{Riem}_g\big|^2_g$ being small:

Typically, one wants to control certain components $\mathrm{Riem}_g(e_i, e_j, e_k, e_l)$ of Riem_g, where $(e_0, \dots, e_{n-1})$ is a local orthonormal frame such that $e_1, \dots, e_{n-1}$ are tangential to the spacelike foliation $\mathscr{F}$ (and thus e_0 is timelike). However, the terms $\mathrm{Riem}_g(e_i, e_j, e_k, e_l)^2$ occur with different signs in the sum $\left|\mathrm{Riem}_g\right|_g^2$. Thus the condition of $\left|\mathrm{Riem}_g\right|_g^2$ being small is too weak; one wants that $\left|\mathrm{Riem}_g\right|_h^2$ is small for some *Riemannian* metric h. (When $\mathscr{F}$ is already given, it is natural to take the h which one obtains from g by changing the sign in the direction orthogonal to $\mathscr{F}$. Example 2.4 works with an arbitrary h, though.)

Even more generally than Example 2.4 (in the sense that 2.4 results from considering the codimension-0 foliation whose only leaf is M), we can consider the curvature of the leaves of a foliation on M instead of the curvature of the whole manifold M:

2.5. **Example** (covariant derivatives of the Riemann tensor of a foliation)**.** Let g, h be semi-Riemannian metrics on a manifold M, let $k \in \mathbb{N}$. Let $\mathscr{F}$ be a foliation on M such that g and h induce (nondegenerate) semi-Riemannian metrics $g_{\mathscr{F}}$ resp. $h_{\mathscr{F}}$ on the leaves of $\mathscr{F}$. (The condition that a metric $\tilde{g}$ on M induces a semi-Riemannian metric on the leaves of $\mathscr{F}$ is satisfied for instance when $\tilde{g}$ is Riemannian; more generally, when $\mathscr{F}$ is $\tilde{g}$-spacelike or $\tilde{g}$-timelike.) Then $\Phi\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to C^\infty(M, \mathbb{R})$ defined by

$$\Phi(u) := \left|\nabla^k_{g_{\mathscr{F}}[u]} \mathrm{Riem}_{g_{\mathscr{F}}[u]}\right|^2_{h_{\mathscr{F}}[u]}$$

is a flatzoomer.

Remark. Note that the signature of $g_{\mathscr{F}}$ resp. $h_{\mathscr{F}}$ is automatically constant on each connected component of M, for continuity reasons.

Proof of 2.5, and thus of 2.3 and 2.4. We write $g' := g_{\mathscr{F}}$ and $h' := h_{\mathscr{F}}$. For $r \in \mathbb{N}$, let $\mathscr{T}_r(\mathscr{F}) \to M$ denote the $\mathbb{R}$-vector bundle of $(r,0)$-tensors on $\mathscr{F}$; thus the fiber over $x \in M$ consists of the r-multilinear forms on $T_x\mathscr{F}$.

For $u \in C^\infty(M, \mathbb{R})$, the $(4,0)$-Riemann curvature of $g'[u]$ is [2, Theorem 1.159b]

$$\mathrm{Riem}_{g'[u]} = \mathrm{e}^{2u}\Big(\mathrm{Riem}_{g'} - g' \owedge \Big(\mathrm{Hess}_{g'} u - \mathrm{d}u \otimes \mathrm{d}u + \tfrac{1}{2} \left|\mathrm{d}u\right|^2_{g'} g'\Big)\Big). \tag{1}$$

With the notation $\nabla^{\tilde{g}} \equiv \nabla^1_{\tilde{g}}$, we have [2, Theorem 1.159a] for all sections X in $T\mathscr{F} \to M$ and all $v \in T\mathscr{F}$:

$$\nabla^{g'[u]}_v X = \nabla^{g'}_v X + \mathrm{d}u(X)\, v + \mathrm{d}u(v)\, X - g'(v, X)\, \mathrm{grad}_{g'} u. \tag{2}$$

Let $k, m \in \mathbb{N}$. We consider the (finite-dimensional) $\mathbb{R}$-vector space $\mathrm{PC}^{g',\mathscr{F}}_{k,m}$ of base-preserving vector bundle morphisms $\mathscr{T}_{k+4+2m}(\mathscr{F}) \to \mathscr{T}_{k+4}(\mathscr{F})$ which is spanned by all morphisms of the form $\xi \circ \pi$, where $\pi\colon \mathscr{T}_{k+4+2m}(\mathscr{F}) \to \mathscr{T}_{k+4+2m}(\mathscr{F})$ is a permutation of tensor indices (the same permutation over each $x \in M$) and $\xi\colon \mathscr{T}_{k+4+2m}(\mathscr{F}) \to \mathscr{T}_{k+4}(\mathscr{F})$ contracts each of the first m pairs of indices via g'.

We claim that for every $k \in \mathbb{N}$, there exist a number $\mu_k \in \mathbb{N}$ and, for each $i \in \{1, \dots, \mu_k\}$,

- a number $a_{k,i} \in \mathbb{N}$ and a section $\omega_{k,i}$ in $\mathscr{T}_{a_{k,i}}(\mathscr{F}) \to M$,
- numbers $c_{k,i,1}, \dots, c_{k,i,k+2} \in \mathbb{N}$,
- a number $m_{k,i} \in \mathbb{N}$ with $a_{k,i} + \sum_{\nu=1}^{k+2} \nu c_{k,i,\nu} = k + 4 + 2m_{k,i}$,
- and a morphism $\psi_{k,i} \in \mathrm{PC}^{g',\mathscr{F}}_{k,m_{k,i}}$

such that the following equation holds for all $u \in C^\infty(M, \mathbb{R})$:

$$\nabla^k_{g'[u]} \mathrm{Riem}_{g'[u]} \;=\; \mathrm{e}^{2u} \sum_{i=1}^{\mu_k} \psi_{k,i}\Big(\omega_{k,i} \otimes \Big(\nabla^1_{g'} u\Big)^{\otimes c_{k,i,1}} \otimes \cdots \otimes \Big(\nabla^{k+2}_{g'} u\Big)^{\otimes c_{k,i,k+2}}\Big). \tag{3}$$

We prove this by induction over k. Equation (1) shows that in the case $k=0$, (3) holds with $\mu_0=4$,

$$\begin{aligned}
&a_{0,1}=4\,, && a_{0,2}=2\,, && a_{0,3}=2\,, && a_{0,4}=4\,,\\
&\omega_{0,1}=\mathrm{Riem}_{g'}\,, && \omega_{0,2}=g'\,, && \omega_{0,3}=g'\,, && \omega_{0,4}=g'\owedge g'\,,\\
&c_{0,1,1}=0\,, && c_{0,2,1}=0\,, && c_{0,3,1}=2\,, && c_{0,4,1}=2\,,\\
&c_{0,1,2}=0\,, && c_{0,2,2}=1\,, && c_{0,3,2}=0\,, && c_{0,4,2}=0\,,\\
&m_{0,1}=0\,, && m_{0,2}=0\,, && m_{0,3}=0\,, && m_{0,4}=1\,,
\end{aligned}$$

for suitable morphisms $\psi_{0,1},\psi_{0,2},\psi_{0,3}\in\mathrm{PC}^{g',\mathscr{F}}_{0,0}$ and $\psi_{0,4}\in\mathrm{PC}^{g',\mathscr{F}}_{0,1}$.

Now we assume that (3) holds for some $k\in\mathbb{N}$ and verify it for $k+1$. Since all elements of $\mathrm{PC}^{g',\mathscr{F}}_{k,*}$ are g'-parallel, we obtain (using $\nabla^1_{g'[u]}u=\mathrm{d}u=\nabla^1_{g'}u$ and the product and chain rules)

$$\begin{aligned}
&\nabla^{k+1}_{g'[u]}\mathrm{Riem}_{g'[u]}=\nabla_{g'[u]}\nabla^{k}_{g'[u]}\mathrm{Riem}_{g'[u]}\\
&=\mathrm{e}^{2u}\sum_{i=1}^{\mu_k}\nabla_{g'[u]}\left(\psi_{k,i}\left(\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)\right)\\
&\quad+2\mathrm{e}^{2u}\,\mathrm{d}u\otimes\sum_{i=1}^{\mu_k}\psi_{k,i}\left(\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)\\
&=\underbrace{\mathrm{e}^{2u}\sum_{i=1}^{\mu_k}\hat{\psi}_{k,i}\left(\nabla_{g'[u]}\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)}_{\mathrm{I}:=}\\
&\quad\underbrace{+\mathrm{e}^{2u}\sum_{i=1}^{\mu_k}\sum_{j=1}^{k+2}\psi_{k,i,j}\left(\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\nabla_{g'[u]}\left(\left(\nabla^j_{g'}u\right)^{\otimes c_{k,i,j}}\right)\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)}_{\mathrm{II}:=}\\
&\quad\underbrace{+\mathrm{e}^{2u}\sum_{i=1}^{\mu_k}\tilde{\psi}_{k,i}\left(\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}+1}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,2}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)}_{\mathrm{III}:=}
\end{aligned}$$

for suitable $\hat{\psi}_{k,i},\,\psi_{k,i,j},\,\tilde{\psi}_{k,i}\in\mathrm{PC}^{g',\mathscr{F}}_{k+1,m_{k,i}}$. Summand III has already the desired form of the right-hand side of (3). Now we consider I. Writing $V_u(v,X):=\mathrm{d}u(X)v+\mathrm{d}u(v)X-g'(v,X)\,\mathrm{grad}_{g'}u$ for $v,X\in T_xM$, we deduce from (2) (by applying the product rule twice):

$$\begin{aligned}
&\left(\nabla^{g'[u]}_v\omega_{k,i}\right)\left(v_1,\dots,v_{a_{k,i}}\right)=\left(\nabla^{g'}_v\omega_{k,i}\right)\left(v_1,\dots,v_{a_{k,i}}\right)-\sum_{l=1}^{a_{k,i}}\omega_{k,i}\left(v_1,\dots,v_{l-1},V_u(v,v_l),v_{l+1},\dots,v_{a_{k,i}}\right)\\
&=\left(\nabla^{g'}_v\omega_{k,i}\right)\left(v_1,\dots,v_{a_{k,i}}\right)-\sum_{l=1}^{a_{k,i}}\omega_{k,i}\left(v_1,\dots,v_{l-1},v,v_{l+1},\dots,v_{a_{k,i}}\right)\nabla^1_{g'}u(v_l)\\
&\quad-\sum_{l=1}^{a_{k,i}}\omega_{k,i}\left(v_1,\dots,v_{a_{k,i}}\right)\nabla^1_{g'}u(v)+\left\langle\sum_{l=1}^{a_{k,i}}\omega_{k,i}\left(v_1,\dots,v_{l-1},\llcorner\lrcorner,v_{l+1},\dots,v_{a_{k,i}}\right),\,g'(v,v_l)\nabla^1_{g'}u(\llcorner\lrcorner)\right\rangle_{g'}\;;
\end{aligned}$$

hence

$$\begin{aligned}
&\hat{\psi}_{k,i}\left(\nabla_{g'[u]}\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)\\
&=\hat{\psi}_{k,i}\left(\nabla_{g'}\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)\\
&\quad+\sum_{l=1}^{a_{k,i}}\varphi_{k,i,l}\left(\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}+1}\otimes\left(\nabla^2_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)\\
&\quad+\sum_{l=1}^{a_{k,i}}\chi_{k,i,l}\left(g'\otimes\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}+1}\otimes\left(\nabla^2_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\cdots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)
\end{aligned}$$

for some $\varphi_{i,k,l} \in \mathrm{PC}^{g',\mathscr{F}}_{k+1,m_{k,i}}$ and $\chi_{i,k,l} \in \mathrm{PC}^{g',\mathscr{F}}_{k+1,m_{k,i}+1}$. This shows that also summand I has the desired form. A similar formula holds for each summand of

$$\nabla_{g'[u]}\left(\left(\nabla^j_{g'}u\right)^{\otimes c_{k,i,j}}\right) = \sum_{\nu=1}^{c_{k,i,j}} \left(\nabla^j_{g'}u\right)^{\otimes \nu-1} \otimes \nabla_{g'[u]}\nabla^j_{g'}u \otimes \left(\nabla^j_{g'}u\right)^{\otimes c_{k,i,j}-\nu},$$

which takes care of term II. Thus $\nabla^{k+1}_{g'[u]}\mathrm{Riem}_{g'[u]}$ has the required form (3). This completes the proof of our claim involving (3).

Let $u \in C^\infty(M,\mathbb{R}_{\geq 0})$. To compute $\Phi(u)$ at a point $x \in M$, we choose an h'-orthonormal basis $(e_1,\dots,e_n)$ of $T_x\mathscr{F}$. Then $(e_1[u],\dots,e_n[u])$ defined by $e_i[u] := \mathrm{e}^{-u}e_i$ is an $h'[u]$-orthonormal basis of $T_x\mathscr{F}$. Let $\varepsilon_i := h_{\mathscr{F}}(e_i,e_i) \in \{-1,1\}$. Thus

$$\begin{aligned}\Phi(u) = \left|\nabla^k_{g'[u]}\mathrm{Riem}_{g'[u]}\right|^2_{h'[u]} &= \sum_{a\in\{1,\dots,n\}^{k+4}} \varepsilon_{a_1}\dots\varepsilon_{a_{k+4}}\left(\nabla^k_{g'[u]}\mathrm{Riem}_{g'[u]}\right)\left(e_{a_1}[u],\dots,e_{a_{k+4}}[u]\right)^2\\ &= \mathrm{e}^{-2(k+4)u}\sum_{a\in\{1,\dots,n\}^{k+4}} \varepsilon_{a_1}\dots\varepsilon_{a_{k+4}}\left(\nabla^k_{g'[u]}\mathrm{Riem}_{g'[u]}\right)\left(e_{a_1},\dots,e_{a_{k+4}}\right)^2.\end{aligned}$$

Let η be any Riemannian metric on M. For suitable $d \in \mathbb{N}$ and $P \in C^0(M,\mathbb{R}\mathrm{Poly}^d_{k+2})$ not depending on u, we obtain at every $x \in M$, using (3),

$$\begin{aligned}\left|\Phi(u)(x)\right| &= \mathrm{e}^{-2(k+4)u(x)}\left|\left|\nabla^k_{g'[u]}\mathrm{Riem}_{g'[u]}\right|^2_{h'}(x)\right|\\ &\leq \mathrm{e}^{-2(k+2)u(x)}\left|\left|\sum_{i=1}^{\mu_k}\psi_{k,i}\left(\omega_{k,i}\otimes\left(\nabla^1_{g'}u\right)^{\otimes c_{k,i,1}}\otimes\dots\otimes\left(\nabla^{k+2}_{g'}u\right)^{\otimes c_{k,i,k+2}}\right)\right|^2_{h'}(x)\right|\\ &\leq \mathrm{e}^{-2(k+2)u(x)}P(x)\left(\left|\nabla^1_\eta u\right|_\eta(x),\dots,\left|\nabla^{k+2}_\eta u\right|_\eta(x)\right).\end{aligned}$$

Hence Φ is a flatzoomer. □

2.6. **Example** (covariant derivatives of the second fundamental form of a foliation)**.** Let g,h be semi-Riemannian metrics on a manifold M, let $k \in \mathbb{N}$. Let $\mathscr{F}$ be a foliation on M such that g induces a semi-Riemannian metric $g_{\mathscr{F}}$ on the leaves of $\mathscr{F}$. Let $\mathrm{pr}_g\colon TM \to T\mathscr{F}$ denote the g-orthogonal projection onto $T\mathscr{F}$; then $\mathrm{pr}_g^\perp := \mathrm{id}_{TM} - \mathrm{pr}_g$ is pointwise the g-orthogonal projection from T_xM onto the g-orthogonal complement of $T_x\mathscr{F}$ in T_xM. We consider the second fundamental form $I\!I^{\mathscr{F}}_g$ of $\mathscr{F}$ in (M,g) as a field of trilinear forms on M; i.e., for all $x \in M$ and $v,w,z \in T_xM$, we let

$$I\!I^{\mathscr{F}}_g(v,w,z) := g\left(\nabla^g_{\mathrm{pr}_g(v)}\left(\mathrm{pr}_g\circ\hat{w}\right), \mathrm{pr}_g^\perp(z)\right),$$

where $\hat{w}$ is any vector field on M with $\hat{w}(x) = w$ (the choice does not matter). Thus $I\!I^{\mathscr{F}}_g$ projects the input vectors $v,w \in T_xM$ to $T_x\mathscr{F}$, evaluates the second fundamental form of the $\mathscr{F}$-leaf through x in these projections, and translates the resulting vector (which is normal to $T_x\mathscr{F}$) into a 1-form.

Then $\Phi\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^\infty(M,\mathbb{R})$ defined by

$$\Phi(u) := \left|\nabla^k_{g[u]} I\!I^{\mathscr{F}}_{g[u]}\right|^2_{h[u]}$$

is a flatzoomer.

Proof. Let $u \in C^\infty(M,\mathbb{R})$. Clearly $\mathrm{pr} := \mathrm{pr}_g = \mathrm{pr}_{g[u]}$ and $\mathrm{pr}^\perp := \mathrm{pr}_g^\perp = \mathrm{pr}_{g[u]}^\perp$. All $v \in TM$ and vector fields X on M satisfy [2, Theorem 1.159a]

$$\nabla^{g[u]}_v X = \nabla^g_v X + \mathrm{d}u(X)v + \mathrm{d}u(v)X - g(v,X)\,\mathrm{grad}_g u. \tag{4}$$

This yields for all $x \in M$ and $v,w,z \in T_xM$:

$$\begin{aligned}I\!I^{\mathscr{F}}_{g[u]}(v,w,z) &= g[u]\left(\nabla^{g[u]}_{\mathrm{pr}(v)}\left(\mathrm{pr}\circ\hat{w}\right), \mathrm{pr}^\perp(z)\right)\\ &= g[u]\left(\nabla^g_{\mathrm{pr}(v)}\left(\mathrm{pr}\circ\hat{w}\right) - g\left(\mathrm{pr}(v),\mathrm{pr}(w)\right)\mathrm{grad}_g u,\ \mathrm{pr}^\perp(z)\right)\\ &= \mathrm{e}^{2u}\, I\!I^{\mathscr{F}}_g(v,w,z) - \mathrm{e}^{2u} g\left(\mathrm{pr}(v),\mathrm{pr}(w)\right)\mathrm{d}u\left(\mathrm{pr}^\perp(z)\right).\end{aligned} \tag{5}$$

For $r \in \mathbb{N}$, we define Π_r^g to be the set of sections in $\mathrm{End}(TM)^{\otimes r} \to M$ which have the form $p_1 \otimes \cdots \otimes p_r$ with $p_1, \ldots, p_r \in \{\mathrm{pr}, \mathrm{pr}^\perp, \mathrm{id}_{TM}\}$. Using the notation $\mathscr{T}_r(TM)$ and $\mathrm{PC}_{k,m}^{g,TM}$ from the proof of Example 2.5, we claim that for every $k \in M$, there exist a number $\mu_k \in \mathbb{N}$ and, for each $i \in \{1, \ldots, \mu_k\}$,

- a number $a_{k,i} \in \mathbb{N}$ and a section $\omega_{k,i}$ in $\mathscr{T}_{a_{k,i}}(TM) \to M$,
- numbers $c_{k,i,1}, \ldots, c_{k,i,k+1} \in \mathbb{N}$,
- a number $m_{k,i} \in \mathbb{N}$ with $a_{k,i} + \sum_{\nu=1}^{k+1} \nu c_{k,i,\nu} = k + 3 + 2m_{k,i}$,
- a section $p_{k,i} \in \Pi_{k+3+2m_{k,i}}^g$ and a morphism $\psi_{k,i} \in \mathrm{PC}_{k,m_{k,i}}^{g,TM}$

such that the following equation holds for all $u \in C^\infty(M, \mathbb{R})$:

$$\nabla_{g[u]}^k \mathit{II}_{g[u]}^{\mathscr{F}} = \mathrm{e}^{2u} \sum_{i=1}^{\mu_k} \psi_{k,i}\left(\left(\omega_{k,i} \otimes \left(\nabla_g^1 u\right)^{\otimes c_{k,i,1}} \otimes \cdots \otimes \left(\nabla_g^{k+1} u\right)^{\otimes c_{i,k,k+1}}\right) \circ p_{k,i}\right).$$

This claim is proved by induction over k in a similar way as in the proof of Example 2.5, with (5) as induction start and (4) being applied in the induction step. We omit the details.

Let $u \in C^\infty(M, \mathbb{R}_{\geq 0})$. An estimate analogous to the end of the proof of Example 2.5 yields now

$$\forall x \in M\colon \left|\Phi(u)(x)\right| \leq \mathrm{e}^{-2(k+1)u(x)} P(x)\left(\left|\nabla_\eta^1 u\right|_\eta(x), \ldots, \left|\nabla_\eta^{k+1} u\right|_\eta(x)\right)$$

for any Riemannian metric η on M and suitable $d \in \mathbb{N}$ and $P \in C^0(\mathbb{R}\mathrm{Poly}_{k+1}^d)$ not depending on u. Hence Φ is a flatzoomer. □

2.7. **Remark.** When we replace $h[u]$ by h in the definitions of the respective maps Φ in the Examples 2.4, 2.5, 2.6, then these maps are no longer flatzoomers, as one can tell easily from the proofs above. In contrast, after replacing one or both of the symbols $g[u]$ by g (while keeping $h[u]$) in one of the definitions, the resulting map Φ is still a flatzoomer. Replacing $\nabla_{g[u]}$ by $\nabla_{h[u]}$ or an arbitrary fixed connection $\tilde{\nabla}$ does not affect the flatzoomer property either.

As mentioned in the Introduction, when additional geometric objects — e.g. an almost complex structure J or a symplectic form — are given on M, one can construct many other examples of flatzoomers. These will typically be total $h[u]$-contractions $\Phi(u)$ of some tensor field T_u built from the additional objects and $g[u]$ resp. $h[u]$; e.g., T_u may be the $h[u]$-covariant derivative of the Nijenhuis tensor N_J, which has up to a sign exactly one not a priori vanishing total contraction. After lowering all upper indices of T_u via $h[u]$, we may assume that T_u is a field of multilinear forms. This T_u will usually for some $c \in \mathbb{Z}$ have the form e^{cu} times a polynomial in u and its derivatives; e.g., $c = 4$ in Example 2.5 with $T_u = \nabla^k \mathrm{Riem}_{g[u]} \otimes \nabla^k \mathrm{Riem}_{g[u]}$ (cf. (1)), $c = 4$ in Example 2.6 with $T_u = \nabla^k \mathit{II}_{g[u]}^{\mathscr{F}} \otimes \nabla^k \mathit{II}_{g[u]}^{\mathscr{F}}$ (cf. (5)), and $c = 2$ in the Nijenhuis derivative example. If the multilinear form T_u has more than c slots — which is the case in all these examples —, then the functional Φ is a flatzoomer.

2.8. **Example.** Let M be a manifold, let $m \in \mathbb{N}$. For $i \in \{1, \ldots, m\}$, let $\Phi_i\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to C^0(M, \mathbb{R})$ be a flatzoomer. Assume that $Q \in C^0\big(M \times (\mathbb{R}_{\geq 0})^m, \mathbb{R}_{\geq 0}\big)$ is homogeneous-polynomially bounded in the sense that there exist $r \in \mathbb{R}_{>0}$ and $c \in C^0(M, \mathbb{R}_{\geq 0})$ with

$$\forall x \in M\colon \forall v_1, \ldots, v_m \in [0,1]\colon Q(x, v_1, \ldots, v_m) \leq c(x) \cdot \big(v_1 + \cdots + v_m\big)^r.$$

Then the functional $\Phi\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to C^0(M, \mathbb{R}_{\geq 0})$ defined by

$$\Phi(u)(x) := Q\Big(x, \left|\Phi_1(u)(x)\right|, \ldots, \left|\Phi_m(u)(x)\right|\Big)$$

is a flatzoomer; cf. the proof sketch below.

This applies in particular to the function Q given by $Q(x, v) = \sum_{i=1}^m |v_i|$. Thus $\Phi := \sum_{i=1}^m |\Phi_i|$ is a flatzoomer. In this way, finitely many flatzoomers can be controlled by a single flatzoomer: if $|\Phi(u)| \leq \varepsilon$ holds for some $u \in C^\infty(M, \mathbb{R}_{\geq 0})$ and $\varepsilon \in C^0(M, \mathbb{R}_{>0})$, then $|\Phi_i(u)| \leq \varepsilon$ for every $i \in \{1, \ldots, m\}$.

Another example is obtained by taking $m = 1$ and $Q(s) = |s|^{1/2}$. In the situation of Example 2.4, the map $\tilde{\Phi} := |\Phi|^{1/2}\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to C^0(M, \mathbb{R}_{\geq 0})$ given by

$$\tilde{\Phi}(u) = \left|\nabla_{g[u]}^k \mathrm{Riem}_{g[u]}\right|_{h[u]}$$

is thus a flatzoomer. This generalizes in an obvious way to Example 2.5.

Sketch of proof of the flatzoomer property. This is completely analogous to the proof of 2.11 below: in the proof there, just replace every term of the form $\sup\{\text{something}(y) \mid y \in K_{l+1} \setminus K_{l-2}\}$ by something(x); every "$u > u_?$ on $K_{l+1} \setminus K_{l-2}$" by "$u(x) > u_?(x)$"; and the last sentence by "Thus Φ is a flatzoomer.". ☐

In order to prove Theorem 1.4, we have to control not only the functions $\left|\nabla_g^i \operatorname{Riem}_g\right|_g^2$ but also the inverse $\operatorname{inj}_g^{-1} \in C^0(M, \mathbb{R}_{>0})$ of the injectivity radius. However, the functional $\Phi\colon u \mapsto \operatorname{inj}_{g[u]}^{-1}$ is not a flatzoomer, because $\Phi(u)(x)$ cannot be bounded just in terms of some k-jet $j_x^k u$ of u *at the point* x; one has to take the values of u on a whole neighborhood of x into account. The following more general definition covers such functionals.

For a manifold M and a set $I \subseteq \mathbb{R}$, let $\operatorname{Fct}(M, I)$ denote the set of all (not necessarily continuous) functions $M \to I$.

2.9. **Definition.** Let $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ be a compact exhaustion of a manifold M, let $K_{-2} := K_{-1} := \emptyset$. A functional $\Phi\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to \operatorname{Fct}(M, \mathbb{R})$ is a **quasi-flatzoomer for** $\mathcal{K}$ iff for some — and hence every — Riemannian metric η on M, there exist $k, d \in \mathbb{N}$, $\alpha \in \mathbb{R}_{>0}$, $u_0 \in C^0(M, \mathbb{R}_{\geq 0})$ and $P \in C^0\big(M, \mathbb{R}\mathrm{Poly}_{k+1}^d\big)$ such that

$$\left|\Phi(u)(x)\right| \leq \sup\left\{ e^{-\alpha u(y)} P(y)\left(u(y), \left|\nabla_\eta^1 u\right|_\eta(y), \dots, \left|\nabla_\eta^k u\right|_\eta(y)\right) \,\middle|\, y \in K_{i+1} \setminus K_{i-2} \right\}$$

holds for all $i \in \mathbb{N}$ and $x \in K_i \setminus K_{i-1}$ and $u \in C^\infty(M, \mathbb{R}_{\geq 0})$ which satisfy $u > u_0$ on $K_{i+1} \setminus K_{i-2}$.

Proof of "and hence every". This is analogous to the proof of 2.2. ☐

2.10. **Example.** Every flatzoomer $\Phi\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to C^0(M, \mathbb{R})$ is a quasi-flatzoomer for every compact exhaustion of M. ☐

2.11. **Example.** Let $\mathcal{K} = (K_l)_{l\in\mathbb{N}}$ be a compact exhaustion of a manifold M, let $m \in \mathbb{N}$. For $i \in \{1, \dots, m\}$, let $\Phi_i\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to \operatorname{Fct}(M, \mathbb{R})$ be a quasi-flatzoomer for $\mathcal{K}$. Assume $Q \in C^0\big(M \times (\mathbb{R}_{\geq 0})^m, \mathbb{R}_{\geq 0}\big)$ is homogeneous-polynomially bounded in the sense of Example 2.8. Then $\Phi\colon C^\infty(M, \mathbb{R}_{\geq 0}) \to \operatorname{Fct}(M, \mathbb{R}_{\geq 0})$ defined by

$$\Phi(u)(x) := Q\Big(x, \left|\Phi_1(u)(x)\right|, \dots, \left|\Phi_m(u)(x)\right|\Big)$$

is a quasi-flatzoomer for $\mathcal{K}$.

Proof. Let $K_{-2} := K_{-1} := \emptyset$, let η be a Riemannian metric on M. For each $i \in \{1, \dots, m\}$, there exist $k_i, d_i \in \mathbb{N}$, $\alpha_i \in \mathbb{R}_{>0}$ and $b_i, u_i \in C^0(M, \mathbb{R}_{\geq 0})$ such that

$$\left|\Phi_i(u)(x)\right| \leq \sup\left\{ e^{-\alpha_i u(y)}\, b_i(y) \cdot \left(1 + \sum_{j=0}^{k_i} \left|\nabla_\eta^j u\right|_\eta(y)\right)^{d_i} \,\middle|\, y \in K_{l+1} \setminus K_{l-2} \right\}$$

holds for all $l \in \mathbb{N}$ and $x \in K_l \setminus K_{l-1}$ and $u \in C^\infty(M, \mathbb{R}_{\geq 0})$ which satisfy $u > u_i$ on $K_{l+1} \setminus K_{l-2}$. We consider $k := \max\{k_1, \dots, k_m\}$, $d := \max\{d_1, \dots, d_m\}$, $\alpha := \min\{\alpha_1, \dots, \alpha_m\}$ and the pointwise maxima $u_0 := \max\{u_1, \dots, u_m\}$, $b := \max\{b_1, \dots, b_m\}$ in $C^0(M, \mathbb{R}_{\geq 0})$. For every $i \in \{1, \dots, m\}$,

$$\left|\Phi_i(u)(x)\right| \leq \sup\left\{ e^{-\alpha u(y)}\, b(y) \cdot \left(1 + \sum_{j=0}^{k} \left|\nabla_\eta^j u\right|_\eta(y)\right)^{d} \,\middle|\, y \in K_{l+1} \setminus K_{l-2} \right\}$$

holds for all $l \in \mathbb{N}$ and $x \in K_l \setminus K_{l-1}$ and $u \in C^\infty(M, \mathbb{R}_{\geq 0})$ which satisfy $u > u_0$ on $K_{l+1} \setminus K_{l-2}$. This implies for all $l \in \mathbb{N}$ and $x \in K_l \setminus K_{l-1}$ and $u > u_0$:

$$
\begin{aligned}
\left|\Phi(u)(x)\right| &= Q\Big(x, \left|\Phi_1(u)(x)\right|, \ldots, \left|\Phi_m(u)(x)\right|\Big) \leq c(x) \cdot \Big(\left|\Phi_1(u)(x)\right| + \cdots + \left|\Phi_m(u)(x)\right|\Big)^r \\
&\leq m^r c(x) \left(\sup \left\{ b(y)\, \mathrm{e}^{-\alpha u(y)} \left(1 + \sum_{j=0}^{k} \left|\nabla_\eta^j u\right|_\eta (y)\right)^d \,\middle|\, y \in K_{l+1} \setminus K_{l-2} \right\} \right)^r \\
&\leq \sup \left\{ \mathrm{e}^{-\alpha r u(y)}\, m^r c(y) b(y)^r \left(1 + \sum_{j=0}^{k} \left|\nabla_\eta^j u\right|_\eta (y)\right)^{dr} \,\middle|\, y \in K_{l+1} \setminus K_{l-2} \right\}.
\end{aligned}
$$

Thus Φ is a quasi-flatzoomer for $\mathscr{K}$. □

3. Lower bounds on Riemannian injectivity and convexity radii

In this section, we first prove that the inverse convexity radius (and thus also the inverse injectivity radius) of Riemannian metrics on a manifold M — or, more generally, on a foliation on M — is a quasi-flatzoomer Φ with respect to conformal factors. In Section 4, we will construct a function $u \in C^\infty(M, \mathbb{R}_{\geq 0})$ with $\Phi(u) \leq 1$. This yields $\mathrm{conv}_{g[u]} \geq 1$, so in particular the metric $g[u]$ is complete. At the end of the section, we explain why this construction cannot be generalized to arbitrary semi-Riemannian metrics.

The standard lower estimates of the injectivity radius of a Riemannian metric due to Heintze–Karcher [13, Corollary 2.3.2] and Cheeger–Gromov–Taylor [5, Theorem 4.7] do apparently not imply the desired quasi-flatzoomer property directly in our situation. But we can just as well argue in a more elementary way. We use a local version of Klingenberg's lemma which does not assume completeness:

3.1. **Klingenberg's lemma.** *Let (M, g) be a Riemannian manifold, let $x \in M$, let $\delta, \ell, r \in \mathbb{R}_{>0}$. Assume*

- *the ball $B_r^g(x) := \{z \in M \mid \mathrm{dist}_g(x, z) \leq r\}$ (which is closed in M) is compact;*
- *$\sec_g(\sigma) \leq \delta$ holds for every $y \in B_r^g(x)$ and every 2-plane $\sigma \subseteq T_y M$;*
- *every periodic geodesic in (M, g) which is contained in $B_r^g(x)$ has length $\geq \ell$.*

Then $\mathrm{inj}_g(x) \geq \min\left\{\frac{\pi}{\sqrt{\delta}}, \frac{\ell}{2}, r\right\}$.

As we do not know a reference where Klingenberg's lemma is stated in this local form, let us review the proof. We need the following two results.

3.2. **Lemma** (Morse–Schönberg [3, Theorem II.6.3])**.** *Let $\delta, r \in \mathbb{R}_{>0}$, let $\gamma \colon [0, r] \to M$ be a unit-speed geodesic in a Riemannian manifold (M, g) such that $\mathrm{Riem}_g\big(\gamma'(t), w, \gamma'(t), w\big) < \delta g(w, w)$ holds for all $t \in [0, r]$ and $w \in T_{\gamma(t)} M$. If $r < \frac{\pi}{\sqrt{\delta}}$, then there is no conjugate point of $\gamma(0)$ along γ.* □

Recall that the **conjugate radius** $\mathrm{conj}_g(x) \in \,]0, \infty]$ of a point x in a (possibly incomplete) Riemannian manifold (M, g) is the number $\inf\{\varrho(v) \mid v \in T_x M, g(v, v) = 1\}$, where $\varrho(v)$ is the supremum of all $a \in \mathbb{R}_{>0}$ such that the maximal g-geodesic γ with $\gamma'(0) = v$ is defined on $[0, a]$ and has no conjugate point of $\gamma(0) = x$ along $\gamma|_{[0,a]}$.

3.3. **Lemma.** *Let (M, g) be a Riemannian manifold, let $x \in M$, let $\ell, r \in \mathbb{R}_{>0}$. Assume that*

- *$B := B_r^g(x)$ is compact;*
- *every periodic geodesic in (M, g) which is contained in B has length $\geq \ell$.*

Then $\mathrm{inj}_g(x) \geq \min\{\mathrm{conj}_g(x), \ell/2, r\}$.

Sketch of proof. Assume $\mathrm{inj}_g(x) < r$. Because of the compactness of B, the tangent space ball $B_{x,r}^g(0) := \{v \in T_x M \mid |v|_g \leq r\}$ is contained in the domain of $\exp_x^g$. Since $\mathrm{inj}_g(x)$ is the supremum of all $s \in \mathbb{R}_{>0}$ for which $\exp_x^g$ is a smooth embedding on $B_{x,s}^g(0)$, we have $\mathrm{inj}_g(x) = \mathrm{conj}_g(x)$ (if $\exp_x^g$ has a critical point at the boundary of $B_{x,r}^g(0)$) or $\exp_x^g$ is not injective on $B_{x,r}^g(0)$. In the latter case, there is a point

$y \in \operatorname{interior}(B)$ for which x and y are connected by two distinct length-minimizing geodesics such that y is not conjugate to x along either. Now repeat the proof of the complete case [4, Lemma 5.6] verbatim to get a periodic geodesic of length $2\operatorname{inj}_g(x)$. □

Proof of 3.1. By the sectional curvature assumption of 3.1, the curvature assumption of 3.2 holds for every unit-speed geodesic $\gamma\colon [0,r] \to M$ starting in x (because for every $t \in [0,r]$, it holds for all $w \in T_{\gamma(t)}M$ with $g(\gamma'(t), w) = 0$ and thus for all $w \in T_{\gamma(t)}M$). Hence 3.2 implies $\operatorname{conj}_g(x) \geq \min\{\pi/\delta^{1/2}, r\}$. Lemma 3.3 yields $\operatorname{inj}_g(x) \geq \min\{\pi/\delta^{1/2}, \ell/2, r\}$. □

Theorem 1.8 says not only that certain injectivity radii are large, but even that convexity radii are large. (The convexity radius function on a Riemannian manifold is by definition always pointwise ≤ the injectivity radius function.) Therefore we recall J. H. C. Whitehead's lower bound [4, Theorem 5.14]:

3.4. **Lemma.** *Let (M,g) be a Riemannian manifold, let $x \in M$, let $\delta, \iota, r \in \mathbb{R}_{>0}$. Assume that*

- *the ball $B := B^g_r(x)$ is compact;*
- *$\sec_g(\sigma) \leq \delta$ holds for every $y \in B$ and every 2-plane $\sigma \subseteq T_yM$;*
- *$\operatorname{inj}_g(y) \geq \iota$ holds for every $y \in B$.*

Then $\operatorname{conv}_g(x) \geq \min\left\{\frac{\pi}{2\sqrt{\delta}}, \frac{\iota}{2}, r\right\}$.

3.5. **Corollary** (to 3.1 and 3.4)**.** *Let (M,g) be a Riemannian manifold, let $x \in M$, let $\delta, \ell, r \in \mathbb{R}_{>0}$. Assume*

- *the ball $B := B^g_r(x)$ is compact;*
- *$\sec_g(\sigma) \leq \delta$ holds for every $y \in B$ and every 2-plane $\sigma \subseteq T_yM$;*
- *every periodic geodesic in (M,g) which is contained in B has length $\geq \ell$.*

Then $\operatorname{conv}_g(x) \geq \frac{1}{2}\min\left\{\frac{\pi}{\sqrt{\delta}}, \frac{\ell}{2}, \frac{r}{2}\right\}$.

Proof. $B' := B^g_{r/2}(x) \subseteq B$ is compact. For every $y \in B'$, the ball $B_y := B^g_{r/2}(y)$ is contained in B. Thus B_y is compact, $\sec_g \leq \delta$ holds on B_y, and every periodic geodesic in B_y has length $\geq \ell$. Hence Klingenberg's lemma implies $\operatorname{inj}_g(y) \geq \iota := \min\{\pi/\sqrt{\delta}, \ell/2, r/2\}$. Lemma 3.4, with $r/2$ and B' in the roles of r and B, yields

$$\operatorname{conv}_g(x) \geq \tfrac{1}{2}\min\left\{\tfrac{\pi}{\sqrt{\delta}}, \iota, r\right\} = \tfrac{1}{2}\min\left\{\tfrac{\pi}{\sqrt{\delta}}, \tfrac{\ell}{2}, \tfrac{r}{2}\right\}.$$ □

Now we introduce the quantities that feature prominently in Theorem 3.8:

3.6. **Definition.** Let $\mathscr{F}$ be a foliation on a manifold M. Let g be a Riemannian metric on $\mathscr{F}$ (i.e., a smooth section in the bundle $\operatorname{Sym}^2_+ T^*\mathscr{F} \to M$, whose fiber over x consists of the positive definite symmetric bilinear forms on the tangent space $T_x\mathscr{F}$ of the $\mathscr{F}$-leaf that contains x). For each leaf L of $\mathscr{F}$, g_L denotes the Riemannian metric on L that is the restriction of g. We define $\operatorname{conv}^{\mathscr{F}}_g : M \to \,]0,\infty]$ to be the function whose restriction to each $\mathscr{F}$-leaf L is $\operatorname{conv}_{g_L} \in C^0(L,]0,\infty])$. Analogously, $\operatorname{inj}^{\mathscr{F}}_g : M \to \,]0,\infty]$ denotes the function whose restriction to each $\mathscr{F}$-leaf L is $\operatorname{inj}_{g_L} \in C^0(L,]0,\infty])$.

$1/\operatorname{conv}^{\mathscr{F}}_g$ and $1/\operatorname{inj}^{\mathscr{F}}_g$ are defined as functions $M \to [0,\infty[$ in an obvious way.

3.7. **Remark.** In the situation of the preceding definition, the functions $\operatorname{conv}^{\mathscr{F}}_g$ and $\operatorname{inj}^{\mathscr{F}}_g$ are in general not continuous. For example, take the foliation $\mathscr{F}$ on $M := (\mathbb{R}^n \times \mathbb{R}) \setminus \{(0_n, 0)\}$ whose leaves are the sets $L_0 := (\mathbb{R}^n \setminus \{0_n\}) \times \{0\}$ and $L_t := \mathbb{R}^n \times \{t\}$ with $t \in \mathbb{R} \setminus \{0\}$, and take g to be the metric on $\mathscr{F}$ whose restriction to each L_i is the euclidean metric there. At each point of L_0, $\operatorname{conv}^{\mathscr{F}}_g = \operatorname{inj}^{\mathscr{F}}_g$ is not continuous, because it is constant ∞ on $\bigcup_{t\in\mathbb{R}\setminus\{0\}} L_t$ but finite-valued on L_0.

This is the reason why we allowed in Definition 2.9 the $\Phi(u)$ to be arbitrary functions $M \to \mathbb{R}$ instead of continuous ones.

Now we are ready to prove the main result of this section.

3.8. **Theorem.** *Let $\mathscr{F}$ be a foliation on a manifold M, let g be a Riemannian metric on $\mathscr{F}$, let $\mathscr{K} = (K_i)_{i\in\mathbb{N}}$ be a compact exhaustion of M. Then $\Phi\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to \mathrm{Fct}(M,\mathbb{R}_{\geq 0})$ given by*

$$\Phi(u) := 1/\operatorname{conv}^{\mathscr{F}}_{g[u]}$$

is a quasi-flatzoomer for $\mathscr{K}$. The same holds with $\operatorname{inj}^{\mathscr{F}}_{g[u]}$ instead of $\operatorname{conv}^{\mathscr{F}}_{g[u]}$.

Proof. Let $\mathscr{A}$ be a foliation atlas for $\mathscr{F}$. We choose a (parametrized) locally finite cover $\mathscr{U} = (U_i)_{i\in\mathbb{N}}$ of M by open sets U_i each of which has compact closure contained in the domain of some $\mathscr{A}$-chart φ_i.

For $i \in \mathbb{N}$, φ_i induces for each leaf L coordinates on $U_i \cap L$. For any $u \in C^\infty(M,\mathbb{R})$, we can consider the Christoffel symbols ${}^{g[u]_L}\Gamma^c_{ab}$ of the metric $g[u]_L$ with respect to these coordinates. Since U_i has compact closure in $\mathrm{dom}(\varphi_i)$, there exists a constant $A_i \in \mathbb{R}_{>0}$ such that

$$\left|{}^{g[u]_L}\Gamma^c_{ab}\right| \leq A_i \left(1 + \left|\mathrm{d}u\right|_g\right)$$

holds pointwise on $U_i \cap L$ for every $\mathscr{F}$-leaf L and every $u \in C^\infty(M,\mathbb{R})$: for $h = g_L$, we have

$$\begin{aligned}
{}^{h[u]}\Gamma^c_{ab} &= \frac{1}{2}\sum_m h[u]^{cm}\big(\partial_a h[u]_{bm} + \partial_b h[u]_{am} - \partial_m h[u]_{ab}\big)\\
&= \frac{1}{2}\sum_m \frac{h^{cm}}{\mathrm{e}^{2u}}\Big(\mathrm{e}^{2u}\big(\partial_a h_{bm} + \partial_b h_{am} - \partial_m h_{ab}\big) + 2\mathrm{e}^{2u}\big(\partial_a u\, h_{bm} + \partial_b u\, h_{am} - \partial_m u\, h_{ab}\big)\Big).
\end{aligned}$$

Let $n := \dim\mathscr{F}$. For $i \in \mathbb{N}$, we denote the (leafwise) euclidean metric on $\mathscr{F}|_{\mathrm{dom}(\varphi_i)}$, obtained via φ_i-pullback, by eucl_i. There exists a constant $C_i \in \mathbb{R}_{>0}$ such that

$$C_i\,|v|_{\mathrm{eucl}_i} \geq |v|_g \geq C_i^{-1}\,|v|_{\mathrm{eucl}_i}$$

holds for every $\mathscr{F}$-leaf L and every $x \in U_i \cap L$ and every $v \in T_xL$. We define $H_i := 4n^2 A_i C_i^3 \in \mathbb{R}_{>0}$.

Since $\mathscr{U}$ is locally finite, there exists an $H \in C^0(M,\mathbb{R}_{>0})$ with $\forall x \in M\colon \forall i \in \mathbb{N}\colon \big(x \in U_i \Rightarrow H(x) \geq H_i\big)$.

The Examples 2.5 and 2.8 (with $Q(s) = \frac{2}{\pi}|s|^{1/4}$) tell us that $\Phi_0\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^0(M,\mathbb{R}_{\geq 0})$ given by

$$\Phi_0(u) := \frac{2}{\pi}\left|\mathrm{Riem}_{g_{\mathscr{F}}[u]}\right|^{1/2}_{g_{\mathscr{F}}[u]}$$

is a flatzoomer. Moreover, $\Phi_1\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^0(M,\mathbb{R}_{\geq 0})$ given by

$$\Phi_1(u) := \mathrm{e}^{-u} H \cdot \left(1 + \left|\mathrm{d}u\right|_g\right)$$

is obviously a flatzoomer.

Let $K_{-2} := K_{-1} := \emptyset$. There exists a (sufficiently large) function $u_1 \in C^0(M,\mathbb{R})$ such that for every $i \in \mathbb{N}$, for every leaf L and for every $x \in (K_i \setminus K_{i-1}) \cap L$, there is a $j \in \mathbb{N}$ with

$$B^{g[u_1]_L}_1(x) \subseteq U_j \cap (K_{i+1} \setminus K_{i-2})\,.$$

Trivially, also $\Phi_2\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to C^0(M,\mathbb{R}_{\geq 0})$ given by $\Phi_2(u) := 4\mathrm{e}^{-u}\mathrm{e}^{u_1}$ is a flatzoomer.

By Example 2.8, $\Psi := \Phi_0 + \Phi_1 + \Phi_2$ is a flatzoomer; i.e., there exist $k, d \in \mathbb{N}$, $\alpha \in \mathbb{R}_{>0}$, $u_0 \in C^0(M,\mathbb{R}_{\geq 0})$, $P \in C^0(M,\mathbb{R}\mathrm{Poly}^d_{k+1})$ and a Riemannian metric η on M such that

$$0 \;\leq\; \Psi(u)(x) \;\leq\; \mathrm{e}^{-\alpha u(x)} P(x)\left(u(x),\, \left|\nabla^1_\eta u\right|_\eta(x), \ldots, \left|\nabla^k_\eta u\right|_\eta(x)\right)$$

holds for all $x \in M$ and all $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ with $u(x) > u_0(x)$. Without loss of generality, we may assume that u_0 is $\geq$ than each of the analogous functions which appear in the flatzoomer conditions of Φ_0, Φ_1, Φ_2.

We claim that

$$1/\operatorname{inj}^{\mathscr{F}}_{g[u]}(x) \;\le\; 1/\operatorname{conv}^{\mathscr{F}}_{g[u]}(x) \;\le\; \sup\Big\{ \mathrm{e}^{-\alpha u(y)} P(y) \Big(u(y), \big|\nabla^1_\eta u\big|_\eta(y), \ldots, \big|\nabla^k_\eta u\big|_\eta(y) \Big) \;\Big|\; y \in K_{i+1} \setminus K_{i-2} \Big\}$$

holds for all $i \in \mathbb{N}$ and $x \in K_i \setminus K_{i-1}$ and $u \in C^\infty(M, \mathbb{R}_{\ge 0})$ which satisfy $u > u_0$ on $K_{i+1} \setminus K_{i-2}$. This claim implies by Definition 2.9 that the theorem is true.

In order to prove the claim, only the second "$\le$" has to be checked. By Corollary 3.5, it suffices to verify that for all $i \in \mathbb{N}$ and leaves L and $x \in (K_i \setminus K_{i-1}) \cap L$ and $u \in C^\infty(M, \mathbb{R}_{\ge 0})$ which satisfy $u > u_0$ on $K_{i+1} \setminus K_{i-2}$, there exists an $r \in \mathbb{R}_{>0}$ such that $B^{g[u]_L}_r(x)$ is compact and the following inequalities hold (where $\sup \emptyset := 0$):

$$\frac{2}{\pi} \Big| \max\Big\{ \sec_{g[u]_L}(\sigma) \;\Big|\; z \in B^{g[u]_L}_r(x),\, \sigma \in \mathrm{Gr}_2(T_z L) \Big\} \Big|^{1/2} \le \sup\Big\{ \Phi_0(u)(y) \;\Big|\; y \in K_{i+1} \setminus K_{i-2} \Big\}, \tag{6}$$

$$\sup\Big\{ 4/\operatorname{length}(\gamma) \;\Big|\; \gamma \subset B^{g[u]_L}_r(x) \text{ is a periodic } g[u]_L\text{-geodesic} \Big\} \le \sup\Big\{ \Phi_1(u)(y) \;\Big|\; y \in K_{i+1} \setminus K_{i-2} \Big\}, \tag{7}$$

$$\frac{4}{r} \le \sup\Big\{ \Phi_2(u)(y) \;\Big|\; y \in K_{i+1} \setminus K_{i-2} \Big\}. \tag{8}$$

We will show that $r := 1/\sup\{ \mathrm{e}^{u_1(y) - u(y)} \mid y \in K_{i+1} \setminus K_{i-2} \}$ has these properties. It satisfies (8) tautologically. Moreover, with $q := \inf\{ \mathrm{e}^{u(y) - u_1(y)} \mid y \in K_{i+1} \setminus K_{i-2} \}$ we obtain

$$B^{g[u]_L}_r(x) = B^{\exp(2u - 2u_1) g[u_1]_L}_r(x) \subseteq B^{g[u_1]_L}_{r/q}(x) = B^{g[u_1]_L}_1(x) \subseteq U_j \cap (K_{i+1} \setminus K_{i-2})$$

for some $j \in \mathbb{N}$. In particular, $B^{g[u]_L}_r(x)$ is contained in a compact set and thus compact.

Inequality (6) is true: For each $z \in B^{g[u]_L}_r(x)$ and each $\sigma \in \mathrm{Gr}_2(T_z L)$, we choose a $g[u]_L$-orthonormal basis (e_1, e_2) of σ. This yields $\big|\sec_{g[u]_L}(\sigma)\big| = \big|\mathrm{Riem}_{g[u]_L}(e_1, e_2, e_1, e_2)\big| \le \big|\mathrm{Riem}_{g[u]_L}\big|_{g[u]_L}$. Since z lies in $K_{i+1} \setminus K_{i-2}$, the definition of $\Phi_0(u)$ implies (6).

It remains to check (7). Let $\gamma\colon [0, \ell] \to B := B^{g[u]_L}_r(x)$ be a periodic $g[u]_L$-geodesic which is parametrized by $g[u]_L$-arclength. Because of periodicity, we may assume after a change of parametrization that γ satisfies $u(\gamma(0)) = \min_{s \in [0,\ell]} u(\gamma(s))$.

Since B is contained in $U_j \subseteq \operatorname{dom}(\varphi_j)$, the euclidean metric eucl_j is defined on B and we can regard B as a subset of the vector space $\mathbb{R}^n$. Again by periodicity of γ, there exists a $\tau \in [0, \ell]$ with $\langle \gamma'(\tau), \gamma'(0) \rangle_{\mathrm{eucl}_j} < 0$. In particular, $\big|\gamma'(0)\big|_{\mathrm{eucl}_j} \le \big|\gamma'(\tau) - \gamma'(0)\big|_{\mathrm{eucl}_j}$.

Denoting the components (with respect to the chosen coordinates) of a vector $v \in T_x L$ with $x \in B$ by $v_1, \ldots, v_n$, we have the following estimates:

$$C_j\, |v|_{\mathrm{eucl}_j} \ge |v|_g \ge C_j^{-1}\, |v|_{\mathrm{eucl}_j}\,, \qquad\qquad n^{1/2}\, |v|_{\mathrm{eucl}_j} \ge \sum_{a=1}^{n} |v_a|\,.$$

In particular,

$$\forall s \in [0, \ell]\colon\; n^{1/2} C_j \mathrm{e}^{-u(\gamma(s))} = n^{1/2} C_j \mathrm{e}^{-u(\gamma(s))} \big|\gamma'(s)\big|_{g[u]} = n^{1/2} C_j \big|\gamma'(s)\big|_g \ge \sum_{a=1}^{n} \big|\gamma'_a(s)\big|\,.$$

Using this and $\forall c\colon |\partial_c|_{\mathrm{eucl}_j} = 1$ and the $g[u]_L$-geodesic equation

$$\forall s \in [0, \ell]\colon\; \gamma''(s) = \sum_{c=1}^{n} \gamma''_c(s)\, \partial_c\big(\gamma(s)\big) = -\sum_{a,b,c=1}^{n} {}^{g[u]_L}\Gamma^c_{ab}\big(\gamma(s)\big)\, \gamma'_a(s) \gamma'_b(s)\, \partial_c\big(\gamma(s)\big)\,,$$

we obtain

$$\begin{aligned}
1 &= \left|\gamma'(0)\right|_{g[u]} = \mathrm{e}^{u(\gamma(0))}\left|\gamma'(0)\right|_g \\
&\le C_j \mathrm{e}^{u(\gamma(0))}\left|\gamma'(0)\right|_{\mathrm{eucl}_j} \\
&\le C_j \mathrm{e}^{u(\gamma(0))}\left|\gamma'(\tau)-\gamma'(0)\right|_{\mathrm{eucl}_j} = C_j \mathrm{e}^{u(\gamma(0))}\left|\int_0^\tau \gamma''(s)\,\mathrm{d}s\right|_{\mathrm{eucl}_j} \\
&\le C_j \mathrm{e}^{u(\gamma(0))}\sum_{a,b,c=1}^{n}\int_0^\tau \left|{}^{g[u]_L}\Gamma^c_{ab}\big(\gamma(s)\big)\right|\cdot\left|\gamma'_a(s)\right|\cdot\left|\gamma'_b(s)\right|\,\mathrm{d}s \\
&\le C_j \mathrm{e}^{u(\gamma(0))}\sum_{a,b,c=1}^{n}\int_0^\tau A_j\cdot\Big(1+\left|\mathrm{d}u\right|_g(\gamma(s))\Big)\cdot\left|\gamma'_a(s)\right|\cdot\left|\gamma'_b(s)\right|\,\mathrm{d}s \\
&= nA_jC_j\mathrm{e}^{u(\gamma(0))}\int_0^\tau\Big(1+\left|\mathrm{d}u\right|_g(\gamma(s))\Big)\cdot\Big(\sum_{a=1}^{n}\left|\gamma'_a(s)\right|\Big)^2\,\mathrm{d}s \\
&\le n^2A_jC_j^3\int_0^\tau\Big(1+\left|\mathrm{d}u\right|_g(\gamma(s))\Big)\cdot\mathrm{e}^{u(\gamma(0))}\,\mathrm{e}^{-2u(\gamma(s))}\,\mathrm{d}s \\
&\le n^2A_jC_j^3\int_0^\tau\Big(1+\left|\mathrm{d}u\right|_g(\gamma(s))\Big)\cdot\mathrm{e}^{-u(\gamma(s))}\,\mathrm{d}s \\
&\le \ell n^2A_jC_j^3\left\|\mathrm{e}^{-u}\Big(1+\left|\mathrm{d}u\right|_g\Big)\right\|_{C^0(U_j\cap(K_{i+1}\setminus K_{i-2}))},
\end{aligned}$$

and thus

$$\begin{aligned}
4/\ell &\le H_j\left\|\mathrm{e}^{-u}\Big(1+\left|\mathrm{d}u\right|_g\Big)\right\|_{C^0(U_j\cap(K_{i+1}\setminus K_{i-2}))} \\
&\le \left\|H\mathrm{e}^{-u}\Big(1+\left|\mathrm{d}u\right|_g\Big)\right\|_{C^0(K_{i+1}\setminus K_{i-2})} = \sup\Big\{\Phi_1(u)(y)\,\Big|\,y\in K_{i+1}\setminus K_{i-2}\Big\}.
\end{aligned}$$

Hence also (7) is true. This completes the proof. □

It remains to explain why the statements (ii) and (iv) of Theorem 1.8 cannot be generalized to arbitrary semi-Riemannian metrics. One problem is that not every conformal class of, say, Lorentzian metrics contains a complete metric. (Recall that since there is no Lorentzian analogue of the Hopf–Rinow theorem, the notion of *completeness* of Lorentzian metrics refers always to geodesic completeness.)

3.9. **Example.** Let $m\in\mathbb{N}$, let M be a manifold which contains an open subset U diffeomorphic to $\mathbb{R}\times\mathbb{S}^1\times\mathbb{R}^m$; we identify U and $\mathbb{R}\times\mathbb{S}^1\times\mathbb{R}^m$ by the diffeomorphism. Let g_0 be a Lorentzian metric on M which has in a neighborhood of the circle $L:=\{0\}\times\mathbb{S}^1\times\{0_m\}\subset M$ the form

$$(g_0)_{(x,y,z)} = \begin{pmatrix} 0 & 1 & & & \\ 1 & x & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{pmatrix},$$

where x and z are the standard coordinates on $\mathbb{R}$ resp. $\mathbb{R}^m$ and where $y\in\mathbb{S}^1$. Then the conformal class of g_0 contains no metric all of whose lightlike geodesics are complete: For every g in the conformal class of g_0, the maximal domain $I\subseteq\mathbb{R}$ of the (lightlike) g-geodesic $\gamma\in C^\infty(I,M)$ with $\gamma(0)=(0,0,0)$ and $\gamma'(0)=(0,1,0)$ is bounded from above. (Here we consider $0\in\mathbb{R}/\mathbb{Z}=\mathbb{S}^1$.) The image of γ is L.

Proof. Let $u\in C^\infty(M,\mathbb{R}_{>0})$, let $g=ug_0$. We compute γ in the universal covering of $\mathbb{R}\times\mathbb{S}^1\times\mathbb{R}^m$, where we can use the standard global coordinates $(x_1,\dots,x_{m+2})$ (with $x=x_1$, $y=x_2$). The components $\gamma_1,\dots,\gamma_{m+2}$ solve the geodesic equation

$$\forall k\in\{1,\dots,m+2\}:\ \forall t\in I:\ \gamma''_k(t) = -\sum_{i,j=1}^{m+2}\Gamma^k_{ij}\big(\gamma(t)\big)\gamma'_i(t)\gamma'_j(t),$$

where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{m+2} g^{kl}\left(\partial_i g_{jl}+\partial_j g_{il}-\partial_l g_{ij}\right)\in C^\infty(\mathbb{R}^{m+2},\mathbb{R})$$

are the Christoffel symbols of g.

For $k \neq 2$, all Γ^k_{22} vanish on $\tilde{L} := \{0\} \times \mathbb{R} \times \{0_m\} \subset \mathbb{R}^{m+2}$: On $\tilde{L}$, we have $\big(g^{kl}\big) = \frac{1}{u} \cdot \big(\big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big) \oplus \mathrm{diag}(1, \dots, 1)\big)$ and thus, for $\kappa \geq 3$:

$$\Gamma^1_{22} = \frac{1}{2} \sum_l g^{1l} \big(2\partial_2 g_{2l} - \partial_l g_{22}\big) = \frac{1}{2u} \big(2\partial_2 g_{22} - \partial_2 g_{22}\big) = \frac{1}{2u} \partial_2(xu) = 0\,,$$
$$\Gamma^\kappa_{22} = \frac{1}{2} \sum_l g^{\kappa l} \big(2\partial_2 g_{2l} - \partial_l g_{22}\big) = \frac{1}{u} \partial_2 g_{2\kappa} - \frac{1}{2u} \partial_\kappa g_{22} = -\frac{1}{2u} \partial_\kappa (xu) = 0\,.$$

Hence for all $y, r \in \mathbb{R}$, the g-geodesic equation has a local solution $\gamma_{y,r}$ with $\gamma_{y,r}(0) = (0, y, 0_m)$ and $\gamma'_{y,r}(0) = (0, r, 0_m)$ such that all components of $\gamma_{y,r}$ except the 2-component vanish identically. This implies that the image of the maximal geodesic γ with $\gamma(0) = (0, 0, 0_m)$ and $\gamma'(0) = (0, 1, 0_m)$ is $\tilde{L}$.

To determine γ, we thus have to calculate only γ_2. On $\tilde{L}$, we compute

$$\Gamma^2_{22} = \frac{1}{2} \sum_l g^{2l} \big(2\partial_2 g_{2l} - \partial_l g_{22}\big) = \frac{1}{2u} \big(2\partial_2 g_{21} - \partial_1 g_{22}\big) = \frac{2\partial_2 u - \partial_1(xu)}{2u} = \frac{2\partial_2 u - u}{2u} = \frac{\partial_2 u}{u} - \frac{1}{2}\,.$$

With $w \in C^\infty(\mathbb{R}, \mathbb{R})$ given by $w(y) := \ln\big(u(0, y, 0_m)\big)$, $\gamma_2 \in C^\infty(I, \mathbb{R})$ is the maximal solution of

$$\gamma''_2(t) = \left(\frac{1}{2} - w'\big(\gamma_2(t)\big)\right) \gamma'_2(t)^2\,, \qquad\qquad \gamma_2(0) = 0\,, \quad \gamma'_2(0) = 1\,.$$

Since w is the pullback of a function on $\mathbb{S}^1$ via the universal covering $\mathbb{R} \ni y \mapsto [y] \in \mathbb{R}/\mathbb{Z}$, it is 1-periodic. In particular, $w - w(0)$ is bounded from above by some $C \in \mathbb{R}$. We obtain

$$\forall\, t \in I\colon\;\; \big(\ln \circ \gamma'_2\big)'(t) = \frac{\gamma''_2(t)}{\gamma'_2(t)} = \frac{1}{2}\gamma'_2(t) - w'\big(\gamma_2(t)\big)\gamma'_2(t) = \left(\frac{1}{2}\gamma_2 - w \circ \gamma_2\right)'(t)$$

and thus

$$\forall\, t \in I\colon\;\; \ln\big(\gamma'_2(t)\big) = \int_0^t \big(\ln \circ \gamma'_2\big)'(s)\, \mathrm{d}s = \frac{1}{2}\gamma_2(t) - w\big(\gamma_2(t)\big) + w(0) \geq \frac{1}{2}\gamma_2(t) - C\,.$$

This implies $\forall\, t \in I\colon \gamma'_2(t) \geq \mathrm{e}^{-C} \mathrm{e}^{\gamma_2(t)/2}$, hence

$$\forall\, t \in I \cap \mathbb{R}_{\geq 0}\colon\;\; 1 > 1 - \mathrm{e}^{-\gamma_2(t)/2} = \frac{1}{2} \int_0^{\gamma_2(t)} \frac{1}{\mathrm{e}^{\xi/2}}\, \mathrm{d}\xi = \frac{1}{2} \int_0^t \frac{\gamma'_2(s)}{\mathrm{e}^{\gamma_2(s)/2}}\, \mathrm{d}s \geq \frac{t}{2\mathrm{e}^C}\,,$$

i.e., $t < 2\mathrm{e}^C$. This proves that the domain I of γ is bounded from above. □

3.10. **Remark.** Note that the manifold M in Example 3.9 can even be compact, e.g. $M = \mathbb{T}^n$ for some $n \geq 2$. It is well-known that some compact manifolds admit incomplete Lorentzian metrics (cf. e.g. [1]), but we are not aware of a previous proven example in the literature of a conformal class without complete metric. Besides, as far as we know, it is an open question whether each manifold which admits a Lorentzian metric admits a complete one. (We are grateful to Stefan Suhr for remarks on these points.) We will not discuss here to which extent the completeness problem can be avoided by imposing causality conditions on the conformal class in question.

3.11. **Remark.** There is no natural useful notion of *injectivity radius* of a Lorentzian manifold (M, g), but one can define such a radius via an auxiliary Riemannian metric h on M: the "size" of the domain of $\exp^g_x$ in each tangent space $T_x M$ can be measured in terms of h. This "mixed" injectivity radius has been studied by Chen–LeFloch [6] in the situation when h is obtained from g by a Wick rotation around some timelike subbundle of TM. Unfortunately, this notion has still serious defects for the purposes of the present article. Even after dropping the completeness claims, the statements (ii) and (iv) of Theorem 1.8 do not become true in general for "mixed" injectivity radii. We will not go into details.

4. Proof of the main results

We will obtain Theorem 1.8 as a corollary to the following result about sequences of quasi-flatzoomers:

4.1. **Theorem.** *Let $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ be a smooth compact exhaustion of a manifold M, let $(\Phi_i)_{i\in\mathbb{N}}$ be a sequence of quasi-flatzoomers for $\mathcal{K}$, let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence in $C^0(M,\mathbb{R}_{>0})$, let $w \in C^0(M,\mathbb{R})$. Then there exists a real-analytic $u\colon M \to \mathbb{R}$ with $u > w$ such that*

$$\forall i \in \mathbb{N}\colon\ \big|\Phi_i(u)\big| < \varepsilon_i \ \text{ holds on } M\setminus K_i\,. \tag{9}$$

We need some preparations for the rather technical proof of 4.1. Lemma 4.3 below is the analytic key to our argument; see Remark 4.4 for further explanation.

4.2. **Definition.** As usual, $\phi^{(i)}$ denotes the ith derivative of a function $\phi \in C^\infty(I,\mathbb{R})$ on some interval $I \subseteq \mathbb{R}$. For $r \in \mathbb{R}_{\geq 0}$, we define

$$\mathit{Climbers}(r) := \Big\{\phi \in C^\infty\big([0,1],[0,r]\big)\ \Big|\ \phi(0)=0,\ \phi(1)=r,\ \forall i \in \mathbb{N}_{\geq 1}\colon \phi^{(i)}(0) = 0 = \phi^{(i)}(1)\Big\}\,.$$

A sequence $(\phi_n)_{n\in\mathbb{N}}$ in $C^\infty([0,1],\mathbb{R})$ is an **alpinist** iff $\forall n \in \mathbb{N}\colon \phi_n \in \mathit{Climbers}(n)$.

Let $k\in\mathbb{N}$, let $a\in\mathbb{R}_{>0}$, let $\Theta = (\phi_n)_{n\in\mathbb{N}}$ be a sequence in $C^\infty([0,1],\mathbb{R})$. We define the set

$$G_{k,a}[\Theta] := \left\{\max_{t\in[0,1]} \mathrm{e}^{-a\phi_n(t)}\left(1+\sum_{j=0}^{k}\big|\phi_n^{(j)}(t)\big|\right)\ \middle|\ n\in\mathbb{N}\right\} \subset \mathbb{R}_{>0}\,.$$

4.3. **Lemma.** *Let $a \in \mathbb{R}_{>0}$, let $k\in\mathbb{N}$. There is an alpinist $\Theta_{k,a}$ such that the set $G_{k,a}[\Theta_{k,a}]$ is bounded.*

(Here “bounded” means bounded from above, not away from 0.)

Proof of Lemma 4.3. We let $c := a/k$ if $k \geq 1$, and $c := 19.26$ if $k = 0$. For $n\in\mathbb{N}$, we consider

$$q_n := 1-\mathrm{e}^{-nc} \in [0,1[\,.$$

We choose some $\xi \in \mathit{Climbers}(1)$ and define a sequence $\Theta_{k,a} = (\phi_n)_{n\in\mathbb{N}}$ in $C^\infty([0,1],\mathbb{R})$ by

$$\phi_n := -\frac{1}{c}\ln\big(1-q_n\xi\big)\,. \tag{10}$$

The ϕ_n are well-defined because ξ is $[0,1]$-valued and hence $1-\big(1-\mathrm{e}^{-nc}\big)\xi$ is $[\mathrm{e}^{-nc},1]$-valued. Since $\xi \in \mathit{Climbers}(1)$, we have $\phi_n(0) = 0$ and $\phi_n(1) = -\frac{1}{c}\ln\big(1-q_n\big) = n$ and $\forall i\in\mathbb{N}_{\geq 1}\colon \phi_n^{(i)}(0) = 0 = \phi_n^{(i)}(1)$. Thus each ϕ_n lies in $\mathit{Climbers}(n)$, i.e., $\Theta_{k,a}$ is an alpinist.

We claim that for every $i\in\mathbb{N}_{\geq 1}$, there exists a polynomial $P_i \in \mathbb{R}[X_0,\dots,X_i]$ with

$$\forall n\in\mathbb{N}\colon\ \phi_n^{(i)} = \frac{P_i\left(q_n\xi^{(0)},\dots,q_n\xi^{(i)}\right)}{\big(1-q_n\xi\big)^i}\,.$$

We prove this by induction over i. For $i = 1$, the first derivatives $\phi_n' = \frac{1}{c}\,q_n\xi'/\big(1-q_n\xi\big)$ have the claimed form. If the ith derivatives $\phi_n^{(i)}$ have the claimed form, then the $(i+1)$st derivatives

$$\begin{aligned}\phi_n^{(i+1)} &= \left(\frac{P_i\left(q_n\xi^{(0)},\dots,q_n\xi^{(i)}\right)}{\big(1-q_n\xi\big)^i}\right)'\\ &= \frac{\big(1-q_n\xi\big)\cdot\sum_{\nu=0}^{i}\frac{\partial P_i}{\partial X_\nu}\left(q_n\xi^{(0)},\dots,q_n\xi^{(i)}\right)q_n\xi^{(\nu+1)} - i\,q_n\xi'\cdot P_i\left(q_n\xi^{(0)},\dots,q_n\xi^{(i)}\right)}{\big(1-q_n\xi\big)^{i+1}}\end{aligned}$$

have the claimed form as well. This completes the proof of the claim.

Since $\forall n\in\mathbb{N}\colon |q_n| \leq 1$, there exists for each $i \in\mathbb{N}_{\geq 1}$ a constant $C_i\in\mathbb{R}_{>0}$ with

$$\forall n\in\mathbb{N}\colon\ \left\|P_i\left(q_n\xi^{(0)},\dots,q_n\xi^{(i)}\right)\right\|_{C^0([0,1],\mathbb{R})} \leq C_i\,.$$

The supremum $S := \sup\left\{(1+s)/\mathrm{e}^{as} \,\middle|\, s \in \mathbb{R}_{\geq 0}\right\}$ exists in $\mathbb{R}_{>0}$. We obtain for all $n \in \mathbb{N}$ and $t \in [0,1]$:

$$\begin{aligned}\frac{1+\sum_{i=0}^{k}\left|\phi_n^{(i)}(t)\right|}{\mathrm{e}^{a\phi_n(t)}} &\leq \frac{1+\left|\phi_n(t)\right|}{\mathrm{e}^{a\phi_n(t)}} + \sum_{i=1}^{k}\frac{\left|P_i\left(q_n\xi^{(0)}(t),\dots,q_n\xi^{(i)}(t)\right)\right|}{\left(1-q_n\xi(t)\right)^i\,\mathrm{e}^{a\phi_n(t)}}\\ &\leq S+\sum_{i=1}^{k}\frac{C_i}{\left(1-q_n\xi(t)\right)^i\,\mathrm{e}^{a\phi_n(t)}} = S+\sum_{i=1}^{k}\frac{C_i\cdot\left(1-q_n\xi(t)\right)^k}{\left(1-q_n\xi(t)\right)^i} \leq S+\sum_{i=1}^{k}C_i\cdot 1^{k-i}.\end{aligned}$$

Hence the set $G_{k,a}[\Theta_{k,a}]$ is bounded by $S+\sum_{i=1}^{k}C_i$. □

4.4. **Remark.** If you suspect that the proof of 4.3 — in particular definition (10) — is unnecessarily complicated, the following example might be instructive. Consider any $\phi \in \mathit{Climbers}(1)$ which is for some $\delta \in]0,1[$ equal to $t \mapsto \mathrm{e}^{-1/t}$ on $]0,\delta]$. The sequence $\Theta = (\phi_n)_{n\in\mathbb{N}}$ given by $\phi_n = n\phi$ is an alpinist, but $G_{1,1}[\Theta]$ is not bounded.

Proof of the claim made in Remark 4.4. There is an $n_0 \in \mathbb{N}_{\geq 1}$ with $1/\ln(n_0) \leq \delta$. For $n \in \mathbb{N}$ with $n \geq n_0$, we consider $t_n := 1/\ln(n) \in]0,\delta]$. We have $\phi(t_n) = \mathrm{e}^{-1/t_n} = 1/n$ and thus $\mathrm{e}^{-\phi_n(t_n)} = 1/\mathrm{e}$. Moreover, $\phi'(t_n) = \mathrm{e}^{-1/t_n}/t_n^2 = \ln(n)^2/n$. Hence $\mathrm{e}^{-\phi_n(t_n)}\left(1+\left|\phi_n(t_n)\right|+\left|\phi_n'(t_n)\right|\right) \geq \frac{n}{\mathrm{e}}\ln(n)^2/n$. Since this tends to ∞ as $n \to \infty$, the set $G_{1,1}[\Theta]$ is not bounded. □

Proof of Theorem 4.1. Let $K_{-2} := K_{-1} := \emptyset$. For every $i \in \mathbb{N}$, the boundary Σ_i of the smooth codimension-zero submanifold-with-boundary K_i has an interior collar neighborhood $A_i \subseteq K_i \setminus K_{i-1}$ which can be diffeomorphically identified with $[0,1]\times\Sigma_i$ such that Σ_i is identified with $\{1\}\times\Sigma_i$. Let $\rho_i\colon A_i \to [0,1]$ denote the projection to the first factor.

We fix a Riemannian metric η on M. For $i,k \in \mathbb{N}$, the chain and product rules yield a constant $L_{i,k} \in \mathbb{R}_{>0}$ such that for all $x \in A_i$ and $f \in C^\infty([0,1],\mathbb{R})$, we have

$$1+\sum_{j=0}^{k}\left|\nabla_\eta^j\left(f\circ\rho_i\right)\right|_\eta(x) \leq L_{i,k}\cdot\left(1+\sum_{j=0}^{k}\left|f^{(j)}\left(\rho_i(x)\right)\right|\right). \tag{11}$$

For each $i \in \mathbb{N}$, the quasi-flatzoomer property of Φ_i gives us $k_i, d_i \in \mathbb{N}_{\geq 1}$ and $\theta_i, w_i \in C^0(M,\mathbb{R}_{>0})$ and $a_i \in \mathbb{R}_{>0}$ such that

$$\left|\Phi_i(u)(x)\right| \;\leq\; \sup\left\{\mathrm{e}^{-a_i u(y)}\,\theta_i(y)\cdot\left(1+\sum_{j=0}^{k_i}\left|\nabla_\eta^j u\right|_\eta(y)\right)^{d_i}\,\middle|\, y \in K_{l+1}\setminus K_{l-2}\right\} \tag{12}$$

holds for all $l \in \mathbb{N}$ and $x \in K_l \setminus K_{l-1}$ and $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ which satisfy $u > w_i$ on $K_{l+1}\setminus K_{l-2}$.

For each i, we replace Φ_i by $u \mapsto |\Phi_i(u)|^{1/d_i}$, replace a_i by a_i/d_i, replace θ_i by θ_i^{1/d_i}, and replace ε_i by ε_i^{1/d_i}. After this, we may assume without loss of generality that (12) holds with $d_i = 1$.

There is a function $\hat{w} \in C^0(M,\mathbb{R}_{>0})$ with $\hat{w} > w$ such that for each $l \in \mathbb{N}$, $\hat{w} > \max\{w, w_0, \dots, w_{l-1}\}$ holds pointwise on $K_{l+1}\setminus K_{l-2}$.

For $i,l \in \mathbb{N}$, we define $\check{\varepsilon}_{i,l} \in \mathbb{R}_{>0}$ and $\hat{\Phi}_{i,l}\colon C^\infty(M,\mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned}\check{\varepsilon}_{i,l} &:= \inf\left\{\varepsilon_i(x)\,\middle|\, x \in K_l\setminus K_{l-1}\right\},\\ \hat{\Phi}_{i,l}(u) &:= \sup\left\{\mathrm{e}^{-a_i u(y)}\,\theta_i(y)\cdot\left(1+\sum_{j=0}^{k_i}\left|\nabla_\eta^j u\right|_\eta(y)\right)\,\middle|\, y \in K_{l+1}\setminus K_{l-2}\right\}.\end{aligned}$$

Inequality (12) implies that, for any $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ which satisfies $u \geq \hat{w}$ (and hence satisfies for every $l \in \mathbb{N}$ the inequality $u|_{K_{l+1}\setminus K_{l-2}} > \max\{w_0,\dots,w_{l-1}\}|_{K_{l+1}\setminus K_{l-2}}$), the statement (9) is true if

$$\forall i,l \in \mathbb{N}\colon \left(l \geq i+1 \;\Rightarrow\; \hat{\Phi}_{i,l}(u) < \check{\varepsilon}_{i,l}\right). \tag{13}$$

(If (13) holds, then for all $i,l \in \mathbb{N}$ with $l \geq i+1$, we have on $K_l\setminus K_{l-1}$: $|\Phi_i(u)| \leq \hat{\Phi}_{i,l}(u) < \check{\varepsilon}_{i,l} \leq \varepsilon_i$, because $u > w_i$ is fulfilled on $K_{l+1}\setminus K_{l-2}$. Thus for all $i \in \mathbb{N}$, $|\Phi_i(u)| < \varepsilon_i$ holds on $M\setminus K_i$; i.e., (9) is true.)

For $l \in \mathbb{N}$, we define $\tilde{\varepsilon}_l \in \mathbb{R}_{>0}$ and $\tilde{\Phi}_l \colon C^\infty(M,\mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}$ by

$$\tilde{\varepsilon}_l := \min\left\{\check{\varepsilon}_{i,l} \,\middle|\, i \in \{0,\dots,l-1\}\right\},$$
$$\tilde{\Phi}_l(u) := \max\left\{\hat{\Phi}_{i,l}(u) \,\middle|\, i \in \{0,\dots,l-1\}\right\}.$$

For any $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ which satisfies $u \geq \hat{w}$, the statement (9) is true if

$$\forall l \in \mathbb{N} \colon \tilde{\Phi}_l(u) < \tilde{\varepsilon}_l. \tag{14}$$

(This follows from (13), because (14) implies for all $i, l \in \mathbb{N}$ with $l \geq i+1$: $\hat{\Phi}_{i,l}(u) \leq \tilde{\Phi}_l(u) < \tilde{\varepsilon}_l \leq \check{\varepsilon}_{i,l}$.)

For $l \in \mathbb{N}$, we define $\alpha_l \in \mathbb{R}_{>0}$ and $\kappa_l \in \mathbb{N}$ and $\vartheta_l \in C^0(M,\mathbb{R}_{>0})$ by

$$\begin{aligned} \alpha_l &:= \min\left\{a_i \,\middle|\, i \in \{0,\dots,l-1\}\right\}, \\ \kappa_l &:= \max\left\{k_i \,\middle|\, i \in \{0,\dots,l-1\}\right\}, \\ \vartheta_l(x) &:= \max\left\{\theta_i(x) \,\middle|\, i \in \{0,\dots,l-1\}\right\}. \end{aligned} \tag{15}$$

This yields for all $l \in \mathbb{N}$ and $u \in C^\infty(M,\mathbb{R}_{\geq 0})$:

$$\tilde{\Phi}_l(u) \leq \sup\left\{ e^{-\alpha_l u(y)}\, \vartheta_l(y) \cdot \left(1 + \sum_{j=0}^{\kappa_l} \left|\nabla^j_\eta u\right|_\eta(y)\right) \,\middle|\, y \in K_{l+1} \setminus K_{l-2} \right\}.$$

For $l \in \mathbb{N}$, we define

$$\tilde{\lambda}_l := \frac{\tilde{\varepsilon}_l}{\sup\left\{\vartheta_l(y) \,\middle|\, y \in K_{l+1} \setminus K_{l-2}\right\}} \in \mathbb{R}_{>0}.$$

We choose a monotonically decreasing sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ with $\forall i \in \mathbb{N} \colon \lambda_i < \tilde{\lambda}_i$.

Due to (14), for any $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ which satisfies $u \geq \hat{w}$, the statement (9) is true if

$$\forall i \in \mathbb{N} \colon \sup\left\{ e^{-\alpha_i u(y)} \left(1 + \sum_{j=0}^{\kappa_i} \left|\nabla^j_\eta u\right|_\eta(y)\right) \,\middle|\, y \in K_{i+1} \setminus K_{i-2} \right\} \leq \lambda_i. \tag{16}$$

For each $i \in \mathbb{N}$, Lemma 4.3 yields an alpinist $\Theta_i = (\varphi_i[n])_{n\in\mathbb{N}}$ such that the set $G_{k_{i+1},\alpha_{i+1}}[\Theta_i]$ is bounded from above by some $C_i \in \mathbb{R}_{>0}$.

For $i \in \mathbb{N}$, we consider $L_i := L_{i,\kappa_{i+1}}$. We define a sequence $b = (b_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ recursively by

$$\begin{aligned} b_i := \min\Big\{ \beta \in \mathbb{N} \,\Big|\, & \max\left\{1+\beta,\, L_i\beta + C_i L_i\right\} \leq \lambda_{i+1} e^{\alpha_{i+1}\beta} \quad \wedge \\ & \forall x \in K_i \setminus K_{i-1} \colon \hat{w}(x) \leq \beta \quad \wedge \quad \forall j \in \{0,\dots,i-1\} \colon b_j \leq \beta \Big\}; \end{aligned} \tag{17}$$

this is well-defined because the set on the right-hand side is nonempty. By construction, b is monotonically increasing. Hence the numbers $c_i := b_{i+1} - b_i$ lie in $\mathbb{N}$.

We define a function $u \in C^\infty(M,\mathbb{R}_{\geq 0})$ by

$$u(x) := \begin{cases} b_i & \text{if } \exists i \in \mathbb{N} \colon x \in K_i \setminus (A_i \cup K_{i-1}) \\ b_i + \varphi_i[c_i](\rho_i(x)) & \text{if } \exists i \in \mathbb{N} \colon x \in A_i \end{cases}.$$

Obviously u is indeed a well-defined and nonnegative function. It is smooth because all jth derivatives with $j \geq 1$ of $\varphi_i[c_i]$ vanish at 0 and 1 and because $b_i + \varphi_i[c_i](1) = b_i + c_i = b_{i+1}$.

Moreover, we have $u \geq \hat{w} > w$, because $\forall i \in \mathbb{N} \colon \forall x \in K_i \setminus K_{i-1} \colon u(x) \geq b_i \geq \hat{w}(x) > w(x)$.

Let $i \in \mathbb{N}$, let $y \in K_{i+1} \setminus K_{i-2}$. Then for some $\mu \in \{-1,0,1\}$, we have either $y \in K_{i+\mu} \setminus (A_{i+\mu} \cup K_{i+\mu-1})$ or $y \in A_{i+\mu}$. If $y \in K_{i+\mu} \setminus (A_{i+\mu} \cup K_{i+\mu-1})$, then the definition (17) of $b_{i+\mu}$ implies

$$1 + \sum_{j=0}^{\kappa_i} \left|\nabla^j_\eta u\right|_\eta(y) = 1 + b_{i+\mu} \leq \lambda_{i+\mu+1} e^{\alpha_{i+\mu+1} b_{i+\mu}} \leq \lambda_i e^{\alpha_i u(y)}, \tag{18}$$

because $(\lambda_l)_{l\in\mathbb{N}}$ and $(\alpha_l)_{l\in\mathbb{N}}$ decrease monotonically and $u(y) = b_{i+\mu} \geq 0$.

If $y \in A_{i+\mu}$, we consider $t := \rho_{i+\mu}(y)$. Since $G_{\kappa_{i+\mu+1},\alpha_{i+\mu+1}}[\Theta_{i+\mu}]$ is bounded from above by $C_{i+\mu}$, Definition 4.2 yields

$$\mathrm{e}^{-\alpha_{i+\mu+1}\varphi_{i+\mu}[c_{i+\mu}](t)}\left(1+\sum_{j=0}^{\kappa_{i+\mu+1}}\left|\varphi_{i+\mu}[c_{i+\mu}]^{(j)}(t)\right|\right) \le \sup\left(G_{\kappa_{i+\mu+1},\alpha_{i+\mu+1}}[\Theta_{i+\mu}]\right) \le C_{i+\mu}\,. \tag{19}$$

By (15), the sequence $(\kappa_l)_{l\in\mathbb{N}}$ increases monotonically, whereas $(\lambda_l)_{l\in\mathbb{N}}$ and $(\alpha_l)_{l\in\mathbb{N}}$ decrease monotonically. Since $\mathrm{e}^{\alpha_{i+\mu+1}\varphi_{i+\mu}[c_{i+\mu}](t)} \ge 1$, we deduce from (17) and (19):

$$\begin{aligned}
\lambda_i\,\mathrm{e}^{\alpha_i u(y)} &= \lambda_i\,\mathrm{e}^{\alpha_i b_{i+\mu}}\,\mathrm{e}^{\alpha_i\varphi_{i+\mu}[c_{i+\mu}](t)}\\
&\ge \lambda_{i+\mu+1}\,\mathrm{e}^{\alpha_{i+\mu+1}b_{i+\mu}}\,\mathrm{e}^{\alpha_{i+\mu+1}\varphi_{i+\mu}[c_{i+\mu}](t)}\\
&\ge L_{i+\mu}b_{i+\mu}\mathrm{e}^{\alpha_{i+\mu+1}\varphi_{i+\mu}[c_{i+\mu}](t)} + C_{i+\mu}L_{i+\mu}\,\mathrm{e}^{\alpha_{i+\mu+1}\varphi_{i+\mu}[c_{i+\mu}](t)}\\
&\ge L_{i+\mu}b_{i+\mu} + \frac{C_{i+\mu}L_{i+\mu}}{C_{i+\mu}}\left(1+\sum_{j=0}^{\kappa_{i+\mu+1}}\left|\varphi_{i+\mu}[c_{i+\mu}]^{(j)}(t)\right|\right)\\
&= L_{i+\mu}\cdot\left(1+b_{i+\mu}+\sum_{j=0}^{\kappa_{i+\mu+1}}\left|\varphi_{i+\mu}[c_{i+\mu}]^{(j)}(t)\right|\right).
\end{aligned}$$

Applying (11) to the function $f\colon s\mapsto b_{i+\mu}+\varphi_{i+\mu}[c_{i+\mu}](s)$, we thus obtain

$$1+\sum_{j=0}^{\kappa_i}\left|\nabla_\eta^j u\right|_\eta(y) \;\le\; 1+\sum_{j=0}^{\kappa_{i+\mu+1}}\left|\nabla_\eta^j u\right|_\eta(y) \;\le\; L_{i+\mu}\cdot\left(1+b_{i+\mu}+\sum_{j=0}^{\kappa_{i+\mu+1}}\left|\varphi_{i+\mu}[c_{i+\mu}]^{(j)}(t)\right|\right) \;\le\; \lambda_i\,\mathrm{e}^{\alpha_i u(y)}. \tag{20}$$

The inequalities (18) and (20) imply that (16) is true for the function u with $u \ge \hat{w} > w$ we have constructed. This shows that there exists a function $u \in C^\infty(M,\mathbb{R})$ with $u > w$ such that for every $i \in \mathbb{N}$, the inequality $|\Phi_i(u)| < \varepsilon_i$ holds on $M\setminus K_i$.

There exists a neighborhood of u with respect to the fine (i.e. Whitney) C^∞-topology on $C^\infty(M,\mathbb{R})$ all of whose elements v satisfy $v > w$ and, for every $i\in\mathbb{N}$, $|\Phi_i(v)| < \varepsilon_i$ on $M\setminus K_i$. Since real-analytic functions are fine-C^∞-dense in $C^\infty(M,\mathbb{R})$ (cf. e.g. [15, Theorem A]), Theorem 4.1 is proved. □

Now we can prove our main result stated in the Introduction:

Proof of Theorem 1.8. For $i\in\mathbb{N}$, consider the maps $\Psi_i, \Psi_i^{\mathscr{F}}, \Upsilon_i\colon C^\infty(M,\mathbb{R}_{\ge 0}) \to C^0(M,\mathbb{R}_{\ge 0})$ defined by

$$\Psi_i(u) := \left|\nabla^i_{g_0[u]}\mathrm{Riem}_{g_0[u]}\right|_{h_0[u]}, \qquad \Upsilon_i(u) := \left|\nabla^i_{g_0[u]}\, I\!I^{\mathscr{F}}_{g_0[u]}\right|_{h_0[u]},$$
$$\Psi_i^{\mathscr{F}}(u) := \left|\nabla^i_{(g_0)_{\mathscr{F}}[u]}\mathrm{Riem}_{(g_0)_{\mathscr{F}}[u]}\right|_{(h_0)_{\mathscr{F}}[u]}.$$

As stated in Example 2.8, for every flatzoomer Ξ also $u\mapsto|\Xi(u)|^{1/2}$ is a flatzoomer. Thus the Examples 2.4, 2.5, 2.6 show that Ψ_i, $\Psi_i^{\mathscr{F}}$, Υ_i are flatzoomers. By Example 2.10, they are quasi-flatzoomers for $\mathscr{K}$. For $i\in\mathbb{N}$, we define $\Phi_i\colon C^\infty(M,\mathbb{R}_{\ge 0}) \to C^0(M,\mathbb{R}_{\ge 0})$ by $\Phi_i(u) := \Psi_i(u)+\Psi_i^{\mathscr{F}}(u)+\Upsilon_i(u)$. Example 2.11 (see also 2.8) tells us that Φ_i is a quasi-flatzoomer.

Theorem 4.1, applied to the sequence $(\Phi_i)_{i\in\mathbb{N}}$, yields a real-analytic function $u\colon M\to\mathbb{R}$ with $u > u_0$ such that for every $i\in\mathbb{N}$, the inequality $\Phi_i(u) < \varepsilon_i$ holds on $M\setminus K_i$. Thus the statements (i), (iii), (v) of Theorem 1.8 are true. If $(g_0)_{\mathscr{F}}$ (and thus also g_0) is not Riemannian, the proof of 1.8 is now complete.

Otherwise we define a smooth compact exhaustion $\mathscr{K}' = (K_i')_{i\in\mathbb{N}}$ by $K_0' := \emptyset$ and $\forall i \ge 1\colon K_i' := K_{i-1}$, define $(\varepsilon_i')_{i\in\mathbb{N}}$ by $\varepsilon_0' := \frac{1}{\iota+1}$ and $\forall i\ge 1\colon \varepsilon_i' := \varepsilon_{i-1}$, and define $\forall i \ge 1\colon \Phi_i' := \Phi_{i-1}$. If $(g_0)_{\mathscr{F}}$, but not g_0, is Riemannian, then we consider $\Phi_0'\colon u\mapsto 1/\mathrm{conv}^{\mathscr{F}}_{(g_0)_{\mathscr{F}}[u]}$, which is a quasi-flatzoomer due to Theorem 3.8. If g_0 is Riemannian, we consider $\Phi_0'\colon u\mapsto 1/\mathrm{conv}^{\mathscr{F}}_{(g_0)_{\mathscr{F}}[u]} + 1/\mathrm{conv}_{g[u]}$, which is a quasi-flatzoomer due to Theorem 3.8 (applied also to the foliation whose only leaf is M) and Example 2.11.

Now Theorem 4.1, applied to $\mathscr{K}'$ and $(\Phi_i')_{i\in\mathbb{N}}$ and $(\varepsilon_i')_{i\in\mathbb{N}}$, shows that all statements of Theorem 1.8 are true, because the convexity radii are by construction $\ge \iota+1 \ge 1$, which implies in particular completeness of the metrics. □

The other results stated in Section 1 follow from Theorem 1.8, as explained there.

We end this article by stating explicitly, for future use elsewhere, one result about ordinary differential inequalities that has essentially been derived during the proof of Theorem 4.1.

4.5. **Theorem.** *Let $(\varepsilon_i)_{i\in\mathbb{N}}$ and $(\alpha_i)_{i\in\mathbb{N}}$ be sequences in $\mathbb{R}_{>0}$, let $(m_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{N}$, let $(P_i)_{i\in\mathbb{N}}$ be a sequence such that each P_i is a real polynomial (whose degree may depend on i) in m_i+1 real variables. Let $w\in C^0([0,\infty[,\mathbb{R})$. Then there exists a number $\mu\in\mathbb{R}$ such that for every $u_0\in[\mu,\infty[$, there is a function $u\in C^\infty([0,\infty[,\mathbb{R})$ with the following properties:*

(i) $u(0)=u_0$.

(ii) For each $i\in\mathbb{N}$, u is constant on the interval $\left[i,i+\frac{1}{2}\right]$.

(iii) $u>w$.

(iv) $\forall i\in\mathbb{N}\colon \forall x\in[i,i+1]\colon P_i\left(u(x),u'(x),\dots,u^{(m_i)}(x)\right)<\varepsilon_i \mathrm{e}^{\alpha_i u(x)}$.

Remark 1. In particular, the ordinary differential inequality (iv) can be solved globally with initial values $u(0)$ and $\forall i\geq 1\colon u^{(i)}(0)=0$ whenever $u(0)$ is sufficiently large. In contrast, the results of [14] show that even in simple special cases, the inequality (iv) can*not* be solved with $\forall i\geq 1\colon u^{(i)}(0)=0$ for arbitrary initial values $u(0)$ that satisfy $P_0(u(0),0,\dots,0)<\varepsilon_0\mathrm{e}^{\alpha_0 u(0)}$ (the properties (ii), (iii) do not matter for this conclusion).

Remark 2. The polynomials P_i are assumed to have constant coefficients here, for simplicity. But since they may depend on the interval $[i,i+1]$, an inequality of the form

$$\forall x\in[0,\infty[\colon P(x)\left(u(x),u'(x),\dots,u^{(m)}(x)\right)<\varepsilon(x)\mathrm{e}^{\alpha(x)u(x)},$$

for a polynomial-valued function $P\in C^0\left([0,\infty[,\mathbb{R}\mathrm{Poly}^d_{m+1}\right)$ and functions $\varepsilon,\alpha\in C^0([0,\infty[,\mathbb{R}_{>0})$, can always be strengthened to an inequality of the form (iv) and can then be solved using the theorem.

Sketch of proof of Theorem 4.5. For $M:=[0,\infty[$, consider the smooth compact exhaustion $(K_i)_{i\in\mathbb{N}}$ with $K_i:=[0,i+1]$; this M is a manifold-with-boundary, but that does not cause any problem. The maps $\Phi_i\colon C^\infty(M,\mathbb{R}_{\geq 0})\to C^\infty(M,\mathbb{R})$ given by $\Phi_i(u)(x):=\mathrm{e}^{-\alpha_i u(x)}P_i\left(u(x),u'(x),\dots,u^{(m_i)}(x)\right)$ are obviously (quasi-)flatzoomers. Revisiting the proof of Theorem 4.1 for our given data $(\Phi_i)_{i\in\mathbb{N}}$, $(\varepsilon_i)_{i\in\mathbb{N}}$, w, we choose the interior collar neighborhoods $A_i=\left[i+\frac{1}{2},i+1\right]$. Clearly there exists a number $\mu\in\mathbb{R}$ such that for every $u_0\in[\mu,\infty[$, we can choose the sequence b with $b_0=u_0$. The constructed function $u\in C^\infty(M,\mathbb{R})$ then satisfies (i)–(iv). □

References

[1] J. K. Beem, P. E. Ehrlich, K. L. Easley: *Global Lorentzian geometry* (2nd ed.). Marcel Dekker Inc., New York, 1996.

[2] A. L. Besse: *Einstein manifolds.* Springer-Verlag, Berlin, 1987.

[3] I. Chavel: *Riemannian geometry. A modern introduction* (2nd ed.). Cambridge Univ. Press, Cambridge, 2006.

[4] J. Cheeger, D. G. Ebin: *Comparison theorems in Riemannian geometry.* North-Holland Publishing Co., Amsterdam, 1975.

[5] J. Cheeger, M. Gromov, M. Taylor: *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds.* J. Diff. Geom. **17** (1982), 15–53.

[6] B.-L. Chen, P. G. LeFloch: *Injectivity radius of Lorentzian manifolds.* Commun. Math. Phys. **278** (2008), 679–713.

[7] J. Eichhorn: *The boundedness of connection coefficients and their derivatives.* Math. Nachr. **152** (1991), 145–158.

[8] J. Eichhorn, J. Fricke, A. Lang: *Metrics conformally equivalent to bounded geometry.* In: J. Slovák (ed.): Proceedings of the 15th Winter School "Geometry and Physics", Rendiconti Circ. Mat. Palermo Ser. II, Suppl. **43** (1996), 103–110.

[9] Y. Eliashberg, N. Mishachev: *Introduction to the h-principle.* American Math. Society, Providence, RI, 2002.

[10] R. E. Greene: *Complete metrics of bounded curvature on noncompact manifolds.* Arch. Math. **31** (1978/79), 89–95.

[11] M. Gromov: *Partial differential relations.* Springer-Verlag, Berlin, 1986.

[12] N. Große, M. Nardmann: *h-Principles for curvature inequalities.* In preparation.

[13] E. Heintze, H. Karcher: *A general comparison theorem with applications to volume estimates for submanifolds,* Ann. Sci. École Norm. Sup. **11** (1978), 451–470.

[14] O. Müller, M. Nardmann: *ODE-type obstructions to extending prescribed scalar curvature metrics in given conformal classes.* In preparation.

[15] K. Shiga: *Some aspects of real-analytic manifolds and differentiable manifolds.* J. Math. Soc. Japan **16** (1964), 128–142. Erratum: J. Math. Soc. Japan **17** (1965), 216–217.

Fakultät für Mathematik, Universität Regensburg
E-mail address: olaf.mueller@mathematik.uni-regensburg.de

Fachbereich Mathematik, Universität Hamburg
E-mail address: marc.nardmann@math.uni-hamburg.de