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Abstract. The main aim of the present article is to prove that any globally hyperbolic spacetime M can be smoothly isometrically embedded in Lorentz-Minkowski $\mathbb{L}^N$, for some N, in the spirit of Nash's theorem. This will be a consequence of the following two results, with interest in its own right: (1) a Lorentzian manifold is isometrically embeddable in $\mathbb{L}^N$ if and only if it is a stably causal spacetime which admits a smooth time function τ with $|\nabla\tau| > 1$, and (2) any globally hyperbolic spacetime (M, g) admits a global orthogonal decomposition $M = \mathbb{R} \times S, g = -\beta dt^2 + g_t$ with bounded function β. The role of the so-called "folk problems on smoothability" is stressed.

Keywords: causality theory, globally hyperbolic, isometric embedding, conformal embedding

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1 Introduction

A celebrated theorem by J. Nash [15] states that any C^3 Riemannian manifold can be isometrically embedded in any open subset of some Euclidean space $\mathbb{R}^N$ for large N. Greene [9] showed, by means of a simple reasoning, that Nash's theorem can be extended to indefinite metrics, i.e., any semi-Riemannian manifold M can be smoothly isometrically embedded in any open subset of semi-Euclidean space $\mathbb{R}^N_s$ for N, s large enough. Moreover, he also reduced the Nash value for N (and accordingly for the index s, in the indefinite case), by working with the implicit function theorem by Schwartz [19]. Independently, Clarke [7] also showed the possibility to embed semi-Riemannian manifolds in $\mathbb{R}^N_s$ and, by using Kuiper's technique in [11], reduced Nash N for C^k isometric embeddings when $3 \leq k < \infty$.

Nevertheless, a new problem appears when a semi-Riemannian manifold of index s is going to be embedded in a semi-Euclidean space of the same index $\mathbb{R}^N_s$. We will focus in the simplest case $s = 1$, i.e., the isometric embedding of a Lorentzian manifold (M, g) in[1] $\mathbb{L}^N$. Such an embedding will not exist in general: recall, for example, the case when M admits a timelike closed curve –which contradicts the possibility of an embedding in $\mathbb{L}^N$. So, the first task is to characterize the class of isometrically embeddable spacetimes. Our first result (Section 3) is then:

Theorem 1.1 *Let (M, g) be a Lorentzian manifold. The following assertions are equivalent:*
(i) (M, g) admits a isometric embedding in $\mathbb{L}^N$ for some $N \in \mathbb{N}$.
(ii) (M, g) is a stably causal spacetime with a steep temporal function*, i.e., a smooth function τ such that $g(\nabla\tau, \nabla\tau) \leq -1$.*

This theorem is carried out by using some simple arguments, which essentially reduce the hardest problem to the Riemannian case. So, this result (and the subsequent ones on isometric embeddings) is obtained under the natural technical conditions: (a) (M, g) must be C^k with $3 \leq k \leq \infty$, and all the other elements will be as regular as permitted by k, and (b) the smallest value of N is $N = N_0(n) + 1$, where n is the dimension of M and $N_0(n)$ is the optimal bound in the Riemannian case (see [10] for a recent summary on this bound). We will not care on the local problem (a summary in Lorentzian signature can be found in [20]); recall also that, locally, any spacetime fulfills condition (ii). So, the main problem we will consider below, is the existence of a steep temporal function as stated in (ii).

It is known that any stably causal spacetime admits a *time* function, which can be smoothed into a *temporal* one τ (see Section 2 for definitions and background). Nevertheless, the condition of being *steep*, $|\nabla\tau| \geq 1$ cannot be fulfilled for all stably causal spacetimes. In fact, a simple counterexample, which works even in the causally simple case, is provided below (Example 3.3). Notice that causal simplicity is the level in the standard *causal hierarchy of spacetimes* immediately below global hyperbolicity. So, the natural question is to wonder if any globally hyperbolic spacetime admits a steep temporal function τ. The existence of embeddings in $\mathbb{L}^N$ for globally hyperbolic spacetimes was also studied by Clarke [7, Sect. 2]. Nevertheless, his result cannot be regarded as complete,

[1] Our motivation is purely geometrical, with natural conclusions for relativistic spacetimes, or other theories which use General Relativity. Nevertheless, a more fundamental application would appear for the suggestive hypothesis of *brane world*. In order to explain the weakness of gravitational force, this theory assumes that our whole Universe is isometrically embedded in some *bulk* space, see for example [12]. $\mathbb{L}^N$ (or some quotient manifold) would be a toy choice of bulk, but notice that both, Nash's result and our techniques, work for isometric embeddings in arbitrarily curved manifolds.

because it is affected by the so-called "folk problems" of smoothability of causally-constructed functions, as will be discussed in the Appendix.

Apart from the consequence of the embedding in $\mathbb{L}^N$, the existence of a steep temporal τ is relevant for the structure of globally hyperbolic spacetimes. Recently, any globally hyperbolic spacetime (M, g) has been proved to admit a *Cauchy orthogonal decomposition*

$$M = \mathbb{R} \times S, \qquad g = -\beta d\mathcal{T}^2 + g_{\mathcal{T}}, \tag{1.1}$$

where $\beta > 0$ is a function on M, $g_{\mathcal{T}}$ is a Riemannian metric on $S_{\mathcal{T}} := \{\mathcal{T}\} \times S$ smoothly varying with $\mathcal{T}$, and each slice $S_{\mathcal{T}}$ becomes a Cauchy hypersurface [4]. Moreover, further properties have been achieved [5]: any compact acausal spacelike submanifold with boundary can be extended to a (smooth) spacelike Cauchy hypersurface Σ, and any such Σ can be regarded as the a slice $\mathcal{T}$ =constant for a suitable Cauchy orthogonal decomposition (1.1). Apart from the obvious interest in the foundations of classical General Relativity, such results have applications in fields such as the wave equation or quantization, see for example [1, 17]. One of the authors suggested possible analytical advantages of a strengthened decomposition (1.1), where additional conditions on the elements $\beta, g_{\mathcal{T}}$ are imposed [14]. In particular, such a decomposition is called a *b-decomposition* if the function β is bounded; this property turns out equivalent to the existence of a steep temporal function (Lemma 3.5).

Our second result is then (Section 4):

Theorem 1.2 *Any globally hyperbolic spacetime admits a steep Cauchy temporal function $\mathcal{T}$ and, so, a Cauchy orthogonal b-decomposition.*

Remark 1.3 From the technical viewpoint, the decomposition (1.1) was carried out in [4] by proving the existence of a Cauchy temporal function; moreover, a simplified argument shows the existence of a temporal function in any stably causal spacetime ([4], see also the discussion in [18]). Our proof re-proves the existence of the Cauchy temporal function with different and somewhat simpler arguments, as well as a stronger conclusion. Nevertheless, we use some technical elements (remarkably, Proposition 4.2) which hold in the globally hyperbolic case, but not in the stably causal one.[2]

Finally, it is worth emphasizing the following consequence of previous two theorems, obtained by taking into account additionally that causality is a conformal invariant.

Corollary 1.4 *(1) Any globally hyperbolic spacetime can be isometrically embedded in some $\mathbb{L}^N$.*

(2) A Lorentzian manifold is a stably causal spacetime if and only if it admits a conformal embedding in some $\mathbb{L}^N$. In this case, a representative of its conformal class will have a finite-valued time-separation (Lorentzian distance) function.

After some preliminaries in the next section, the following two ones are devoted, respectively, to prove Theorems 1.1 and 1.2, as well as to discuss some of their consequences. Finally, in the Appendix, Clarke's technique for globally hyperbolic spacetimes is discussed, and new causal problems on smoothability, which may have their own interest, are suggested.

[2]Notice that only differentiability C^1 is needed for these results.

2 Preliminaries

In what follows, any semi-Riemannian manifold will be C^k, with $3 \leq k \leq \infty$ as in Nash's theorem, and will be assumed connected without loss of generality. Any geometric element on the manifold will be smooth if it has the higher order of differentiability allowed by k. For a immersion $i : M \to \bar{M}$ only injectivity of each $di_p, p \in M$ is required; the injectivity of i, as well as being a homeomorphism onto its image, are required additionally for i to be an embedding.

Our notation and conventions on causality will be standard as, for example, in [2] or [16]. Nevertheless, some terminology on the solution of the so-called "folk problems of smoothability" introduced in [3, 4] are also used here (see [13] for a review). In particular, a Lorentzian manifold (M, g) is a manifold M endowed with a metric tensor g of index one $(-, +, \ldots, +)$, a tangent vector $v \in T_pM$ in $p \in M$, is timelike (resp. spacelike; lightlike; causal) when $g(v, v) < 0$ (resp, $g(v, v) > 0$; $g(v, v) = 0$ but $v \neq 0$; v is timelike or lightlike); so, following [13], the vector 0 will be regarded as non-spacelike and non-causal – even though this is not by any means the unique convention in the literature. For any vector v, we write $|v| := \sqrt{|g(v, v)|}$. A spacetime is a time-orientable Lorentzian manifold, which will be assumed time-oriented (choosing any of its two time-orientations) when necessary; of course, the choice of the time-orientation for submanifolds conformally immersed in $\mathbb{L}^N$ will agree with the induced from the canonical time-orientation of $\mathbb{L}^N$. The associated *time-separation* or *Lorentzian distance* function will be denoted by d, $d(p, q) := \sup_{c \in \Omega(p,q)} l(c)$ where the supremum is taken over the space $\Omega(p, q)$ of future-directed causal C^1 curves from p to q parametrized over the unit interval (if this space is empty, d is defined equal to 0), and $l(c) := \int_0^1 |\dot{c}(t)|dt$ for such a curve. The following elements of causality must be taken into account (they are explained in detail in [13]).

- A *time function* t on a spacetime is a continuous function which increases strictly on any future-directed causal curve. It is well-known that, for a spacetime, the existence of such a function is equivalent to be *stably causal*. Recently [4], it has been proved that this is also equivalent to the existence of a *temporal function* τ, i.e., a smooth time function with everywhere past-directed timelike gradient $\nabla\tau$. Along the present paper, a temporal function will be called *steep* if $|\nabla\tau| \geq 1$; as we will see, not all stably causal spacetimes admit a steep temporal function.

- A spacetime is called globally hyperbolic if it is causal[3] and the intersections $J^+(p) \cap J^-(q)$ are compact for all $p, q \in M$. Globally hyperbolic spacetimes are the most relevant from both, the geometric and physical viewpoints, and lie at the top of the so-called *causal ladder* or *causal hierarchy* of spacetimes. In fact, the last steps of this hierarchy are: stable causality, causal continuity, causal simplicity and global hyperbolicity.

- A time or temporal function is called *Cauchy* if it is onto on $\mathbb{R}$ and all its level hypersurfaces are Cauchy hypersurfaces (i.e., topological hypersurfaces crossed exactly once by any inextensible timelike curve). A classical theorem by Geroch [8] asserts the equivalence between: (i) to be globally hyperbolic, (ii) to admit a Cauchy hypersurface, and (iii) to admit a Cauchy temporal function. Moreover, the results in [3, 4] also ensures the equivalence with: (iv) to

[3]The classical definition impose *strong causality* instead of causality, but that condition can be weakened, [6].

admit a (smooth) spacelike Cauchy hypersurface, and (v) to admit a Cauchy temporal function $\mathcal{T}$. As a consequence, the full spacetime admits a orthogonal Cauchy decomposition as in (1.1).

The following simple results are useful for the discussions below.

Proposition 2.1 *Let (M, g) be a spacetime.*

(1) If τ is a temporal function then there exists a conformal metric $g^* = \Omega g$, $\Omega > 0$, such that τ is steep.

(2) If $\mathcal{T}$ is a Cauchy temporal function and τ is a temporal function then $\mathcal{T} + \tau$ is a Cauchy temporal function. Moreover, $\mathcal{T} + \tau$ is steep if so is either τ or $\mathcal{T}$.

Proof. (1) As $\nabla^*\tau = \nabla\tau/\Omega$, choose any $\Omega \leq |\nabla\tau|$.

(2) $\mathcal{T} + \tau$ is temporal (and steep, if so is any of the two functions) because of the reversed triangle inequality. In order to check that its level hypersurfaces are Cauchy, consider any future-directed timelike curve $\gamma : (a_-, a_+) \to M$. It is enough to check that $\lim_{s\to\pm a}(\mathcal{T}+\tau)(\gamma(s)) = \pm\infty$. But this is obvious, because $\lim_{s\to\pm a}\mathcal{T}(\gamma(s)) = \pm\infty$ (as $\mathcal{T}$ is Cauchy) and $\tau(\gamma(s))$ is increasing. ∎

3 Characterization of isometrically embeddable Lorentzian manifolds

Proposition 3.1 *Let (M, g) be a Lorentzian manifold. If there exists a conformal immersion $i : M \to \mathbb{L}^N$ then (M, g) is a stably causal spacetime.*

Moreover, if i is a isometric immersion, then: (a) the natural time coordinate $t = x^0$ of $\mathbb{L}^N$ induces a steep temporal function on M, and (b) the time-separation d of (M, g) is finite-valued.

Proof. Notice that $x^0 \circ i$ is trivially smooth and also a time function (as x^0 increases on $i\circ\gamma$, where γ is any future-directed causal curve in M), which proves stable causality.

If i is isometric, then $|\nabla(x^0 \circ i)| \geq 1$ because, at each $p \in M$, $\nabla(x^0 \circ i)_p$ is the projection of $\nabla x^0_{i(p)}$ onto the tangent space $di(T_pM)$, and its orthogonal $di(T_pM)^\perp$ in $T_{i(p)}\mathbb{L}^N$ is spacelike. This proves (a), for (b) notice that the finiteness of d is an immediate consequence of the finiteness of the time-separation d_0 on $\mathbb{L}^N$ and the straightforward inequality $d(p, q) \leq d_0(i(p), i(q))$ for all $p, q \in M$. ∎

Remark 3.2 As a remarkable difference with the Riemannian case, Proposition 3.1 yields obstructions for the existence of both, conformal and isometric immersions in $\mathbb{L}^N$. In particular, non-stably causal spacetimes cannot be conformally immersed, and further conditions on the time-separation are required for the existence of an isometric immersion. In fact, it is easy to find even causally simple spacetimes splitted as in (1.1) (with levels of $\mathcal{T}$ non-Cauchy) which cannot be isometrically immersed in $\mathbb{L}^N$, as the following example shows.

Example 3.3 Let $M = \{(t, x) \in \mathbb{R}^2 : x > 0\}$, $g = (-dt^2 + dx^2)/x^2$. This is conformal to $\mathbb{R}\times\mathbb{R}^+ \subset \mathbb{L}^2$ and, thus, causally simple. It is easy to check that $d(p, q) = \infty$ for $p = (-2, 1), q = (2, 1)$ (any sequence of causal curves $\{\gamma_m\}_m$ connecting p and q whose images contain $\{(t, 1/m) : |t| < 1\}$ will have diverging lengths). Thus, (M, g) cannot be isometrically immersed in $\mathbb{L}^N$.

Nash's theorem will be essential for the proof of the following result.

Proposition 3.4 *If a spacetime (M, g) admits a steep temporal function τ then it can be isometrically embedded in $\mathbb{L}^N$ for some N.*

For the proof, recall first.

Lemma 3.5 *If a spacetime (M, g) admits a temporal function τ then the metric g admits a orthogonal decomposition*

$$g = -\beta d\tau^2 + \bar{g} \tag{3.1}$$

where $\beta = |\nabla\tau|^{-2}$ and $\bar{g}$ is a positive semi-definite metric on M with radical spanned by $\nabla\tau$.

In particular, if τ is steep then $\beta \leq 1$.

Proof. The orthogonal decomposition (3.1) follows by taking $\bar{g}$ as the trivial extension of $g|_{(\nabla\tau)^\perp}$ to all TM. To determine the value of β, recall that $d\tau(\nabla\tau) = g(\nabla\tau, \nabla\tau) = -\beta\,(d\tau(\nabla\tau))^2$ ■

Proof of Proposition 3.4. Consider the orthogonal decomposition in Lemma 3.5. Even though M, by topology change of the levelsets, does not need to split as a product $\mathbb{R} \times S$, we can rewrite (3.1) as

$$g = -\beta d\tau^2 + g_\tau, \tag{3.2}$$

where each g_{τ_0} is Riemannian metric on the slice $S_{\tau_0} = \tau^{-1}(\tau_0)$ varying smoothly with τ_0. Moreover, each $p \in M$ will be written as (τ, x) where $x \in S_{\tau(p)}$.

Now, consider the auxiliary Riemannian metric

$$g_R := (4 - \beta)d\tau^2 + g_\tau.$$

By Nash theorem, there exists an isometric embedding $i_{\text{nash}} : (M, g_R) \hookrightarrow \mathbb{R}^{N_0}$. Then, a simple computation shows that the required isometric embedding $i : (M, g) \hookrightarrow \mathbb{L}^{N_0+1}$ is just:

$$i(\tau, x) = (2\tau, i_{\text{nash}}(\tau, x)).$$

■

Remark 3.6 (1) From the proof, it is clear that the hypotheses on steepness can be weakened just by assuming that $\nabla\tau$ is lower bounded by some positive function $\epsilon(\tau) > 0$. In fact, this is equivalent to require $\beta(\tau, x) \leq A(\tau)^2 := 1/\epsilon(\tau)$, and the proof would work by taking $g_R := (4A(\tau)^2 - \beta)d\tau^2 + g_\tau$ and $i(\tau, x) = \left(2\int_0^\tau A(s)ds, i_{\text{nash}}(\tau, x)\right)$. Nevertheless, no more generality would be obtained in this case, because of the following two different arguments: (a) it is easy to check that, if this weaker condition holds, then a suitable composition $\hat{\tau} = f \circ \tau$ for some increasing function f on $\mathbb{R}$ would be steep and temporal, and (b) the existence of a steep temporal function would be ensured by taking the isometric embedding $i : M \hookrightarrow \mathbb{L}^N$ and restricting the natural coordinate $t = x^0$ as in Proposition 3.1.

(2) Notice that Proposition 3.1 yields a necessary condition for the existence of a isometric embedding and Proposition 3.4 a sufficient one. Both together prove trivially Theorem 1.1, as well as Corollary 5.1(2) (notice also Proposition 2.1(1)). Recall that, as a difference with Nash's theorem, Proposition 3.4 does *not* allow to prove that the spacetime is isometrically embedded in an *arbitrarily small* open subset, which cannot be expected now.

4 The Cauchy orthogonal b-decomposition of any globally hyperbolic spacetime

In order to obtain a steep Cauchy temporal function in a globally hyperbolic spacetime, Proposition 2.1(2) reduces the problem to find a steep temporal function (not necessarily Cauchy), as the existence of a Cauchy temporal function is ensured in [4]. Nevertheless, we will prove directly the existence of a steep Cauchy temporal function $\mathcal{T}$, proving Theorem 1.2 with independence of the results in [4] (recall Remark 1.3).

So, in what follows (M, g) will be a globally hyperbolic spacetime, and we will assume that t is a Cauchy time function as given by Geroch [8]. The following notation will be also used here. Regarding t,

$$T_a^b = t^{-1}([a, b]), \qquad S_a = t^{-1}(a).$$

For any $p \in M$, j_p is the function

$$q \mapsto j_p(q) = \exp(-1/d(p, q)^2).$$

For any $A, B \subset M$,

$$J(A, B) := J^+(A) \cap J^-(B)$$

in particular $J(p, S) := J^+(p) \cap J^-(S)$ for S any (Cauchy) hypersurface.

4.1 Some technical elements

In the next two propositions we will introduce a pair of technical tools for the proof. But, first, consider the following straightforward lemma, which will be claimed several times.

Lemma 4.1 *Let τ be a function such that $g(\nabla\tau, \nabla\tau) < 0$ in some open subset U and let $K \subset U$ compact. For any function f there exists a constant c such that $g(\nabla(f + c\tau), \nabla(f + c\tau)) < -1$ on K.*

Proof. Notice that at each x in the compact subset K the quadratic polynomial $g(\nabla(f(x) + c\tau(x)), \nabla(f(x) + c\tau(x)))$ becomes smaller than -1 for some large c. ■

The following "cone semi-time function" will be useful from a technical viewpoint.

Proposition 4.2 *Let S be a Cauchy hypersurface, $p \in J^-(S)$. For all neighborhood V of $J(p, S)$ there exists a smooth function $\tau \geq 0$ such that:*

(i) $\mathrm{supp}\,\tau \subset V$

(ii) $\tau > 1$ on $S \cap J^+(p)$.

(iii) $\nabla\tau$ is timelike and past-directed in $\mathrm{Int}(\mathrm{Supp}\ (\tau) \cap J^-(S))$.

(iv) $g(\nabla\tau, \nabla\tau) < -1$ on $J(p, S)$.

Proof. Let t be a Cauchy time function such that[4] $S = S_a := t^{-1}(a)$, and let $K \subset V$ be a compact subset such that $J(p, S_a) \subset \mathrm{Int}\ (K)$. This compactness yields some $\delta > 0$ such that:

[4] Along the proof, we will use this lemma only for Cauchy hypersurfaces which are slices of a prescribed time function. However, any Cauchy hypersurface can be written as such a slice for some Cauchy time function. In fact, it is easy to obtain a proof by taking into account that both, $I^+(S)$ and $I^-(S)$ are globally hyperbolic and, thus, admit a Cauchy time function –for details including the non-trivial case that S is smooth spacelike and t is also required to be temporal, see [5]).

for every $x \in K$ there exists a convex neighborhood $U_x \subset V$ with $\partial^+ U_x \subset J^+(S_{t(x)+2\delta})$, where $\partial^+ U_x := \partial U_x \cap J^+(x)$. Now, choose $a_0 < a_1 := t(p) < \ldots < a_n = a$ with $a_{i+1} - a_i < \delta/2$, and construct τ by induction on n as follows.

For $n = 1$, cover $J(p, S) = \{p\}$ with a set type $I^+(x) \cap U_x$ with $x \in K \cap T_{a_0}^{a_1}$ and consider the corresponding function j_x. For a suitable constant $c > 0$, the product cj_x satisfies both, (ii), (iii) and (iv). To obtain smoothability preserving (i), consider the open covering $\{I^-(S_{a+\delta}), I^+(S_{a+\delta/2})\}$ of M, and the first function $0 \leq \mu \leq 1$ of the associated partition of the unity (Supp $\mu \subset I^-(S_{a+\delta})$). The required function is just $\tau = c\mu j_x$.

Now, assume by induction that the result follows for any chain $a_0 < \ldots < a_{n-1}$. So, for any $k \leq n-1$, consider $J(p, S_{a_k})$ and choose a compact set $\hat{K} \subset \text{Int } K$ with $J(p, S) \subset \text{Int } \hat{K}$. Then, there exists a function $\hat{\tau}$ which satisfies condition (i) above for $V = \text{Int } \hat{K} \cap I^-(S_{a_{k+1}})$ and conditions (ii), (iii), (iv) for $S = S_{a_k}$. Now, cover $\hat{K} \cap T_{a_k}^{a_{k+1}}$ with a finite number of sets type $I^+(x^i) \cap U_{x^i}$ with $x^i \in K \cap T_{a_{k-1}}^{a_{k+1}}$, and consider the corresponding functions j_{x^i}.

For a suitable constant $c > 0$, the sum $\hat{\tau} + c \sum_i j_{x^i}$ satisfies (iii) for $S = S_{a_{k+1}}$. This is obvious in $J^-(S_{a_k})$ (for any $c > 0$), because of the convexity of timelike cones and the reversed triangle inequality. To realize that this can be also obtained in $T_{a_k}^{a_{k+1}}$, where $\nabla\tau$ may be non-timelike, notice that the support of $\nabla\hat{\tau}|_{T_{a_k}^{a_{k+1}}}$ is compact, and it is included in the interior of the support of $\sum_i j_{x^i}$, where the gradient of the sum is timelike; so, use Lemma 4.1. As $J^+(p, S_{a_{k+1}})$ is compact, conditions (ii), (iv) can be trivially obtained by choosing, if necessary, a bigger c.

Finally, smoothability (and (i)), can be obtained again by using the open covering $\{I^-(S_{a_{k+1}+\delta}), I^+(S_{a_{k+1}+\delta/2})\}$ of M, and the corresponding first function μ of the associated partition of the unity, i.e. $\tau = \mu(\hat{\tau} + c \sum_i j_{x^i})$. ■

In order to extend locally defined time functions to a global time one, one cannot use a partition of the unity (as stressed in previous proof, as $\nabla\tau$ is not always timelike when μ is non-constant). Instead, local time functions must be added directly and, then, coverings as the following ones will be useful.

Definition 4.3 *Let S be a Cauchy hypersurface. A* fat cone covering *of S is a sequence of points $p_i' \ll p_i, i \in \mathbb{N}$ such that both*[5]*, $\mathcal{C}' = \{I^+(p_i') : i \in \mathbb{N}\}$ and $\mathcal{C} = \{I^+(p_i) : i \in \mathbb{N}\}$ yield a locally finite covering of S.*

Proposition 4.4 *Any Cauchy hypersurface S admits a fat cone covering $p_i' \ll p_i, i \in \mathbb{N}$.*

Moreover, both $\mathcal{C}$ and $\mathcal{C}'$ yield also a finite subcovering of $J^+(S)$.

Proof. Let $\{K_j\}_j$ be a sequence of compact subsets of S satisfying $K_j \subset \text{Int } K_{j+1}$, $S = \cup_j K_j$. Each $K_j \backslash \text{ Int } K_{j-1}$ can be covered by a finite number of sets type $I^+(p_{jk}), k = 1 \ldots k_j$ such that $I^+(p_{jk}) \cap S \subset K_{j+1} \backslash K_{j-2}$. Moreover, by continuity of the set-valued function I^+, this last inclusion is fulfilled if each p_{jk} is replaced by some close $p_{jk}' \ll p_{jk}$, and the required pairs $p_i'(= p_{jk}')$, $p_i(= p_{jk})$, are obtained.

For the last assertion, take $q \in J^+(S)$ and any compact neighborhood $W \ni q$. As $J^-(W) \cap S$ is compact, it is intersected only by finitely many elements of $\mathcal{C}, \mathcal{C}'$, and the result follows. ■

[5]Strictly, we will need only the local finiteness of $\mathcal{C}'$.

4.2 Proof of the b-decomposition

Definition 4.5 *Let $p', p \in T^a_{a-1}$, $p' \ll p$. A* steep forward cone function (SFC) *for (a, p', p) is a smooth function $h^+_{a,p',p} : M \to [0, \infty)$ which satisfies the following:*

1. *$supp(h^+_{a,p',p}) \subset J^+(p', S_{a+2})$,*
2. *$h^+_{a,p',p} > 1$ on $S_{a+1} \cap J^+(p)$,*
3. *If $x \in J^-(S_{a+1})$ and $h^+_{a,p',p}(x) \neq 0$ then $\nabla h^+_{a,p',p}(x)$ is timelike and past-directed, and*
4. *$g(\nabla h^+_{a,p',p}, \nabla h^+_{a,p',p}) < -1$ on $J(p, S_{a+1})$.*

Now, Proposition 4.2 applied to $S = S_{a+1}, V = I^-(S_{a+2}) \cap I^+(p')$ yields directly:

Proposition 4.6 *For all (a, p', p) there exists a SFC.*

The existence of a fat cone covering (Proposition 4.4) allows to find a function h^a_+ which in some sense globalizes the properties of a SFC.

Lemma 4.7 *Choose $a \in \mathbb{R}$ and take any fat cone covering $\{p'_i \ll p_i | i \in \mathbb{N}\}$ for $S = S_a$. For every positive sequence $\{c_i \geq 1 | i \in \mathbb{N}\}$, the non-negative function $h^+_a := (|a| + 1) \sum_i c_i h^+_{a,p'_i,p_i}$ satisfies:*

1. *$supp(h^+_a) \subset J(S_{a-1}, S_{a+2})$,*
2. *$h^+_a > |a| + 1$ on[6] S_{a+1},*
3. *If $x \in J^-(S_{a+1})$ and $h^+_a(x) \neq 0$ then $\nabla h^+_a(x)$ is timelike and past-directed, and*
4. *$g(\nabla h^+_a, \nabla h^+_a) < -1$ on $J(S_a, S_{a+1})$.*

Proof. Obvious. ■

The gradient of h^+_a will be spacelike at some subset of $J(S_{a+1}, S_{a+2})$. So, in order to carry out the inductive process which proves Theorem 1.2, a strengthening of Lemma 4.7 will be needed.

Lemma 4.8 *Let $h^+_a \geq 0$ as in Lemma 4.7. Then there exists a function h^+_{a+1} which satisfies all the properties corresponding to Lemma 4.7 and additionally:*

$$g(\nabla(h^+_a + h^+_{a+1}), \nabla(h^+_a + h^+_{a+1})) < -1 \qquad \text{on } J(S_{a+1}, S_{a+2}) \tag{4.1}$$

(so, this inequality holds automatically on all $J(S_a, S_{a+2})$).

Proof. Take a fat cone covering $\{p'_i \ll p_i | i \in \mathbb{N}\}$ for $S = S_{a+1}$. Now, for each p_i consider a constant $c_i \geq 1$ such that $c_i h^+_{a+1,p'_i,p_i} + h^+_a$ satisfies inequality (4.1) on $J^+(p_i, S_{a+2})$ (see Lemma 4.1). The required function is then $h^+_{a+1} = (|a| + 2) \sum_i c_i h^+_{a+1,p'_i,p_i}$. ■

Now, we have the elements to complete our main proof.

[6] This condition is imposed in order to ensure that the finally obtained temporal function is Cauchy. It could be dropped if one looks only for a temporal function and, then, uses Proposition 2.1(2).

Proof of Theorem 1.2. Consider the function h_a^+ provided by Lemma 4.7 for $a = 0$, and apply inductively Lemma 4.8 for $a = n \in \mathbb{N}$. Then, we obtain a function $\mathcal{T}^+ = \sum_{n=0}^{\infty} \geq 0$ with nowhere spacelike gradient, which is a steep temporal function on $J^+(S_0)$ with support in $J^+(S_{-1})$. Analogously, one can obtain a function $\mathcal{T}^- \geq 0$ which is a steep temporal function with the reversed time orientation, on $J^-(S_0)$. So, $\mathcal{T} = \mathcal{T}^+ - \mathcal{T}^-$ is clearly a steep temporal function on all M.

Moreover, the levels hypersurfaces of $\mathcal{T}$ are Cauchy. In fact, consider any future-directed causal curve γ, and reparametrized it with the Cauchy time function t. Then,

$$\lim_{t\to\infty} \mathcal{T}(\gamma(t)) \left(= \lim_{n\in\mathbb{N}} \mathcal{T}^+(\gamma(n+1)) \geq \lim_{n\in\mathbb{N}} h_n^+(\gamma(n+1)) \right) = \infty, \quad \lim_{t\to-\infty} \mathcal{T}(\gamma(t)) = -\infty,$$

and γ crosses all the levels of $\mathcal{T}$, as required. ■

5 Appendix

Clarke [7] developed the following method in order to embed isometrically any manifold M endowed with a semi-Riemannian (or even degenerate) metric g in some semi-Euclidean space $\mathbb{R}^N_s$. First, he proved that, for some $p \geq 0$, there exists a function $f : M \to \mathbb{R}^p_p$ such that the (possibly degenerate) pull-back metric $g(f)$ on M induced from f satisfies $g_R = g - g(f) > 0$. So, the results for positive definite metrics are applicable to (M, g_R), and one can construct a Riemannian isometric embedding $f_R : M \to \mathbb{R}^{N_0}$ (f_R can be constructed from Nash result, but Clarke develops a technique based on Kuiper's [11], which works when the embedding is required C^k, with $k < \infty$, and allows to reduce Nash value for N_0). Then, the required embedding $i : M \to \mathbb{R}^N_p$ is obtained as a product $i(x) = (f_1(x), f_R(x))$ for $N = p + N_0$.

In Lorentzian signature, Clarke's optimal value for p is 2. Nevertheless, he claims that, if (M, g) is a globally hyperbolic spacetime, then one can take $p = 1$ [7, Lemma 8]. Our purpose in this Appendix is to analyze this question and show:

(A) the required condition $g - g(f) > 0$ on f is essentially equivalent to be a steep temporal function, and

(B) the success of the construction of f in [7, Lemma 8] depends on a new problem of smoothability, which may have interest in its own right.

In order to make clear these points, we will particularize the proof of [7, Lemma 8] to a very simple case, and will follow most of the notation there. As a previous remark, Clarke assumed that the existence of a temporal function τ had been already proved, as this question (one of the folk problems of smoothability) seemed true when his article was written. At any case, we will assume even that τ is Cauchy temporal, as we know now that such a τ exists. Then, consider a globally hyperbolic spacetime which can be written as

$$M = \mathbb{R}^2, \qquad g = -V^2 d\tau^2 + M^2 dy^2,$$

where (τ, y) are the natural coordinates of $\mathbb{R}^2$ and V, M are two positive functions on M. Easily, a function $f : M \to \mathbb{R}^1_1$ satisfies $g - g(f) > 0$ if and only if:

$$-V^2(\partial_y f)^2 + M^2(\partial_\tau f)^2 > V^2 M^2, \tag{5.1}$$

and this is trivially equivalent to $g(\nabla f, \nabla f) < -1$. This proves (A) in our particular example, and a deeper study shows that Clarke's requirements in [7, Lemma 8] are also equivalent in general (eventually, taking into account Remark 3.6(1)).

Now, consider any smooth function $\sigma \geq 0$ on M invariant through the flow of $\nabla\tau$ such that $\sigma^{-1}([0, s])$ is compact for all s, and let $Y = \sigma^{-1}([0, 1])$; in our simplified example, we can put $\sigma(\tau, y) = |y|^2$. Outside Y the two lightlike vector fields,

$$A_\pm = M\partial_\tau \pm V\partial_\sigma,$$

are well defined, and equation (5.1) can be also rewritten as

$$(A_+ f)(A_- f) > V^2 M^2. \tag{5.2}$$

So, the crux is to construct a function f which satisfies (5.2) outside Y, among other conditions. Clarke's proposal is the following. Let

$$H^\pm(t, s) = J^\pm(\tau^{-1}(0)) \cap J^\mp(\tau^{-1}(t) \cap \sigma^{-1}([0, s])).$$

Choosing certain volume element ω, function f is defined as:

$$f(x) = \int_{H^+(\tau(x),\sigma(x))} \omega \tag{5.3}$$

whenever $\tau(x) > \epsilon > 0$ and outside a neighborhood of[7] Y. Notice that $A_\pm$ are future directed, and A_+ points outwards the region $\sigma^{-1}([0, \sigma(x)])$ at each $x \in M\backslash Y$. So, if f is C^1, then one would have $A_+(f) > A_-(f) > 0$. Moreover, Clarke claims that (5.2) can be also achieved by choosing ω large enough (and eventually, redefining of τ).

At what extent can one assume that f is C^1 (or, at least, that it can be smoothed to a function which satisfies the required conditions)? For each measurable set $Z \subset M$, consider its ω-measure $\mu(Z) = \int_Z \omega$. In any causally continuous spacetime it is known that the functions $x \mapsto \mu(J^\pm(x))$ are continuous, if $\mu(M) < \infty$. Moreover, if M is globally hyperbolic and S is any topological Cauchy hypersurface, then $I^+(S)$ is a globally hyperbolic spacetime in its own right, and function $x \mapsto \mu(J(S, x)), x \in I^+(S)$, becomes continuous, even dropping the finiteness of μ. Nevertheless, neither functions $\mu(J^\pm(x))$ nor $\mu(J(S, x))$ are smooth in general (see figure). In Clarke's case, the fact that $S = \tau^{-1}(0)$ is not only smooth but spacelike, may help to smoothness. However, recall that the definition of f also uses function σ. Such a σ can be defined by taking some auxiliary complete Riemannian metric on S, and smoothing along the cut locus the squared distance function to a fixed point $y_0 \in S$. The behavior of f at the points $x \in M$ such that the boundary of $S \cap J^\mp(\tau^{-1}(x) \cap \sigma^{-1}([0, \sigma(x)])$ intersects the cut locus may complicate the situation.

Summing up, the smoothability of f becomes a non trivial problem, which may have interest not only to complete Clarke's proof but also in its own right. But, at any case, our solution to the embedding problem becomes a much more direct and self-contained way.

[7]For $\tau(x) < -\epsilon < 0$, function f is negative and defined dually in terms of H^-, for $\tau(x) = 0$, f is 0, and a more technical definition is given for f on a neighborhood of $Y \cup \tau^{-1}(0)$. However, this is not relevant for our discussion.

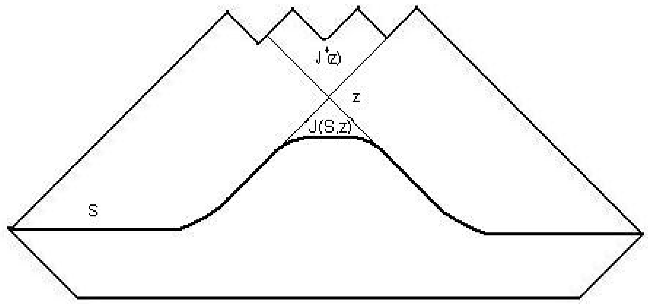

Figure 1: The depicted open subset of $\mathbb{L}^2$ is globally hyperbolic, and S a smooth Cauchy hypersurface. Functions $J^+(x)$ and $J(S,x)$ are not smooth at $z \in I^+(S)$.

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