The conjecture of Birch and Swinnerton-Dyer for Jacobians of constant curves over higher dimensional bases over finite fields

Dissertation zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
an der Fakultät für Mathematik der Universität Regensburg

vorgelegt von

Timo Keller

aus Clausen

2013

Promotionsgesuch eingereicht am: 5. Juli 2013
Die Arbeit wurde angeleitet von: Uwe Jannsen

Vorsitzender: Prof. Dr. Felix Finster

1. Gutachter: Prof. Dr. Uwe Jannsen
2. Gutachter: Prof. Dr. Walter Gubler

weiterer Prüfer: Prof. Dr. Günter Tamme

Contents

1 Preface

Diophantine equations, i. e. polynomial equations

$$f(X_1, \dots, X_n) = 0, \quad f \in \mathbf{Z}[X_1, \dots, X_n]$$

with f a polynomial with integer coefficients, and their integer or rational solutions constitute a central subject of number theory. Hilbert asked in his famous speech on the International Congress of Mathematicians 1900 if there is an algorithm which decides if a given diophantine equation is solvable in the rationals. This question was in general answered negatively by Matiyasevich in 1970.

One can restrict the question to certain classes of diophantine equations, for example to diophantine equations in two variables. Geometrically seen, these are curves and can again be separated by their genus, which depends on their degree (and their singularities). For curves of genus 0 there is the theorem of Hasse-Minkowski from which one can derive an algorithm deciding if there are rational solutions. By a theorem of Faltings, curves of genus > 1 have only finitely many rational solutions (but the theorem does not lead to an algorithm). It remain the curves of genus 1. If such a curve is smooth and possesses a rational point, it is called an *elliptic curve* (the rational point belongs to the datum). One special feature of these curves is that one has a natural law of an Abelian group on $E(\mathbf{Q})$, the set of rational solutions of the Weierstraß equation together with a point 0 at infinity.

Algebraic number theory is about global fields: On the one hand number fields, i. e. finite extensions of $\mathbf{Q}$, on the other hand function fields, i. e. finite extensions of $\mathbf{F}_q(T)$. Their theories fertilise each other, but the function field side is commonly easier, e. g. since there are no archimedean places and since it is more geometrical. For example, for global function fields the holomorphic continuation and the functional equation of the L-series of an elliptic curve follows from the existence of a Weil cohomology theory, whereas over $\mathbf{Q}$ it is only known since the proof of modularity of elliptic curves. In the function field case, one has the possibility to pass to the algebraic closure of the finite ground field; in the number field case, a replacement for this is Iwasawa theory.

In the following, let K be a function field and A/K be an Abelian variety, a higher dimensional generalisation of elliptic curves. By the *theorem of Mordell-Weil*, the group $A(K)$ is finitely generated and therefore (non-canonically) isomorphic to $\operatorname{Tor} A(K) \oplus \mathbf{Z}^r$ with the finite torsion subgroup $\operatorname{Tor} A(K)$. It turns out that the torsion subgroup is easily computed, so it remains to determine the *rank* $r \in \mathbf{N}$ (if r is known, one can calculate generators of $A(K)$). Now, the **Birch-Swinnerton-Dyer conjecture** provides information about r: One defines the *L-function* of the Abelian variety as

$$L(A/K, s) = \prod_{v \text{ place}} L_v(A/K, q^{-s})^{-1}$$

with the Euler factors $L_v(A/K, T) \in \mathbf{Z}[T]$ given by certain polynomials depending on the number of rational points of the reduction of A at the places v. The L-function converges and can be continued to the whole complex plane, where it satisfies a functional equation relating $L(A/K, s)$ with $L(A/K, 2 - s)$.

The *weak* Birch-Swinnerton-Dyer conjecture states that the rank r is equal to the vanishing order of the L-series at $s = 1$:

$$\operatorname*{ord}_{s=1} L(A/K, s) = \operatorname{rk} A(K)$$

The full Birch-Swinnerton-Dyer conjecture further describes the leading Taylor coefficient at $s = 1$ in terms of global data of E ("L-series encode local-global principles"). This is due to John Tate in [Tat66b]. Let $X/\mathbf{F}_q$ be the smooth projective geometrically connected model of K and let $\mathscr{A}/X$ be the Néron model of A/K. For the leading Taylor coefficient at $s = 1$, one should have the formula

$$\lim_{s\to 1} \frac{L(A/K,s)}{(s-1)^r} = \frac{|\text{Ш}(A/K)|\, R \prod_v c_v}{|\mathrm{Tor}\, A(K)| \cdot |\mathrm{Tor}\, A^\vee(K)|}$$

Here,

$$R = |\det \hat{h}(\cdot,\cdot)|$$

is the *regulator* of the *canonical height pairing* $\hat{h} : A(K) \times A^\vee(K) \to \mathbf{R}$. The factors

$$c_v = \left|\mathscr{A}(K_v)/\mathscr{A}^0(K_v)\right|$$

are the *Tamagawa numbers* ($c_v = 1$ if A has good reduction at v, which is the case for almost all v), and finally

$$\text{Ш}(A/K) = \ker\left(\mathrm{H}^1(K,A) \to \prod_{v \text{ place}} \mathrm{H}^1(K_v, A)\right)$$

the *Tate-Shafarevich group*, which is conjecturally finite. It classifies locally trivial A-torsors. A famous quote of John Tate [Tat74], p. 198 (for the conjecture over $\mathbf{Q}$) is:

"This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined[1] *to the order of a group* Ш *which is not known to be finite!"*

From the finiteness of the Tate-Shafarevich group as well as from the equality $\mathrm{rk}\, A(K) = \mathrm{ord}_{s=1} L(A/K,s)$ one would get algorithms for computing the Mordell-Weil group $A(K)$.

In his PhD thesis [Mil68], Milne proved the finiteness of the Tate-Shafarevich group and the Birch-Swinnerton-Dyer conjecture for *constant* Abelian varieties, i. e. those coming from base change from the finite constant field. Work of Peter Schneider [Sch82b] and Werner Bauer [Bau92], which was completed in the article [KT03] of Kazuya Kato and Fabien Trihan in 2003, proved that already the finiteness of one ℓ-primary component (ℓ prime, $\ell = \mathrm{char}\, K$ allowed) of the Tate-Shafarevich group of an Abelian variety over a global function field implies the Birch-Swinnerton-Dyer conjecture. For these investigations, one considers the unique connected smooth projective curve C with function field K, as well as the Néron model $\mathscr{A}/C$ of A/K.

The obvious generalisation of the weak Birch-Swinnerton-Dyer conjecture to *higher dimensional* function fields $K = \mathbf{F}_q(X)$, namely that

$$\operatorname*{ord}_{s=1} L(A/K,s) = \mathrm{rk}\, A(K), \tag{1.0.1}$$

was already formulated by Tate in [Tat65], p. 104, albeit for the vanishing order at $s = \dim X$, which is equivalent to (1.0.1) by the functional equation. For the leading coefficient, there is no conjecture up to now. Note that the vanishing order depends only on the generic fibre.

[1] by now proven for elliptic curves over $\mathbf{Q}$

The main results. The aim of this paper is to formulate and prove an analogue of the conjecture of Birch and Swinnerton-Dyer for certain Abelian schemes over higher dimensional bases over finite fields.

Given an Abelian variety A over the generic point of a base scheme X, one would like to spread it out over the whole of X as an Abelian scheme. It turns out that this is not always possible, e. g. over the integers $\operatorname{Spec}\mathbf{Z}$, there is no non-trivial Abelian scheme at all. But if one drops the condition that the spread out scheme is proper, there is such a model, called the Néron model, satisfying a universal property, called the Néron mapping property, if $\dim X = 1$: For an Abelian scheme $\mathscr{A}/X$, there is an isomorphism

$$\mathscr{A} \xrightarrow{\sim} g_*g^*\mathscr{A} \tag{1.0.2}$$

on the smooth site of X. Here $g : \{\eta\} \hookrightarrow X$ is the inclusion of the generic point. Intuitively, this means that smooth morphisms to the generic fibre can be spread out to the whole of $\mathscr{A}/X$.

In our situation where $\dim X \in \mathbf{N}$ is arbitrary, we prove that if there is an Abelian scheme $\mathscr{A}/X$, then it satisfies a weakened version of the universal property alluded to above, called the weak Néron mapping property, in the sense that the isomorphism (1.0.2) only holds on the étale site.

Theorem 1 (The weak Néron model)**.** *Let X be a regular Noetherian, integral, separated scheme with $g : \{\eta\} \hookrightarrow X$ the inclusion of the generic point. Let $\mathscr{A}/X$ be an Abelian scheme. Then*

$$\mathscr{A} \xrightarrow{\sim} g_*g^*\mathscr{A}$$

as étale sheaves on X.

In the following, let $k = \mathbf{F}_q$ be a finite field with $q = p^n$ elements, $\ell \neq p$ a prime, X/k a smooth projective geometrically connected variety, $\mathscr{C}/X$ a smooth projective relative curve admitting a section and B/k an Abelian variety.

One important invariant of an Abelian scheme that turns up in the conjecture of Birch and Swinnerton-Dyer is the Tate-Shafarevich group. This group classifies (everywhere) locally trivial A-torsors.

Theorem 2 (The Tate-Shafarevich group)**.** *Define the Tate-Shafarevich group of an Abelian scheme $\mathscr{A}/X$ by*

$$Ш(\mathscr{A}/X) := \mathrm{H}^1(X, \mathscr{A}).$$

Denote the quotient field of the strict Henselisation of $\mathscr{O}_{X,x}$ by K_x^{nr}, the inclusion of the generic point by $j : \{\eta\} \hookrightarrow X$ and $j_x : \operatorname{Spec}(K_x^{nr}) \hookrightarrow \operatorname{Spec}(\mathscr{O}_{X,x}^{sh}) \hookrightarrow X$. Then we have

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\left(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x\in X} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\right).$$

One can replace the product over all points by

(a) the closed points

$$\mathrm{H}^1(X, \mathscr{A}) = \ker\left(\mathrm{H}^1(K, j^*\mathscr{A}) \to \bigoplus_{x\in |X|} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\right)$$

*or (b) the codimension-*1 *points if one disregards the p-torsion* ($p = \mathrm{char}\, k$) *(for* $\dim X \leq 2$, *this also holds for the p-torsion), and if one considers the following situation:* X/k *is smooth projective and* $\mathscr{C}/X$ *is a smooth projective relative curve admitting a section, and* $\mathscr{A} = \mathbf{Pic}^0_{\mathscr{C}/X}$*:*

$$\mathrm{H}^1(X, \mathbf{Pic}^0_{\mathscr{C}/X}) = \ker\left(\mathrm{H}^1(K, j^*\mathbf{Pic}^0_{\mathscr{C}/X}) \to \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K_x^{nr}, j_x^*\mathbf{Pic}^0_{\mathscr{C}/X})\right),$$

One can also replace K_x^{nr} *by the quotient field of the completion* $\hat{\mathscr{O}}^{sh}_{X,x}$ *in the case of* $x \in X^{(1)}$*.*

Next, we partially generalise the relation between the Birch-Swinnerton-Dyer conjecture and the Artin-Tate conjecture to a higher dimensional basis. The classical **Artin-Tate conjecture** for a surface S over k is about the Brauer group of S and its invariants like the (rank of the) Néron-Severi group $\mathrm{NS}(S)$ and its intersection pairing.

Conjecture 1 (Artin-Tate conjecture)**.** *Let* S/k *be a smooth projective geometrically connected surface. Let* $P_i(S/k, T)$ *be the characteristic polynomial of the geometric Frobenius on* $\mathrm{H}^i(\bar{S}, \mathbf{Q}_\ell)$*, where the Frobenius acts via functoriality on the second factor of* $\bar{S} = S \times_k \bar{k}$*. Then the Brauer group* $\mathrm{Br}(S)$ *is finite and*

$$P_2(S, q^{-s}) \sim \frac{|\mathrm{Br}(S)|\, |\det(D_i.D_j)|}{q^{\alpha(S)}\, |\mathrm{Tor}\, \mathrm{NS}(S)|^2}(1 - q^{1-s})^{\rho(S)} \quad \textit{for } s \to 1,$$

where

$$\alpha(S) = \chi(S, \mathscr{O}_S) - 1 + \dim \mathbf{Pic}^0(S),$$

$\mathrm{NS}(S)$ *is the Néron-Severi group of* S*,* $\rho(S) = \mathrm{rk}\, \mathrm{NS}(S)$ *and* $(D_i)_{1 \leq i \leq \rho(S)}$ *is a base for* $\mathrm{NS}(S)$ *mod torsion. The symbol* $(D_i.D_j)$ *denotes the total intersection multiplicity of* D_i *and* D_j*.*

Conjecture (d) in [Tat66b], p. 306–13, concerns a surface S which is a relative curve $\mathscr{C}/X$ over a curve X over a finite field. It states the equivalence of the Birch-Swinnerton-Dyer conjecture for the Jacobian of the generic fibre of $\mathscr{C}/X$ and the Artin-Tate conjecture for $\mathscr{C}$: The rank of the Mordell-Weil group should be related to the rank of the Néron-Severi group of the surface, the order of the Tate-Shafarevich group to the order of the Brauer group of $\mathscr{C}$, and the height pairing to the intersection pairing on $\mathrm{NS}(\mathscr{C})$. For the equivalence of the full Birch-Swinnerton-Dyer and the Artin-Tate conjecture *for the base a curve*, see Gordon's PhD thesis [Gor79].

Let X be smooth projective over k of arbitrary dimension. We prove the following partial generalisation, concerning the finiteness part of the conjecture.

Theorem 3 (The Artin-Tate and the Birch-Swinnerton-Dyer conjecture)**.** *Let* $\mathscr{C}/X$ *be a smooth projective relative curve over a regular variety* X/k*. The finiteness of the (*ℓ*-torsion of the) Brauer group of* $\mathscr{C}$ *is equivalent to the finiteness of the (*ℓ*-torsion of the) Brauer group of the base* X *and the finiteness of the (*ℓ*-torsion of the) Tate-Shafarevich group of* $\mathbf{Pic}_{\mathscr{C}/X}$*: One has an exact sequence*

$$0 \to K_2 \to \mathrm{Br}(X) \xrightarrow{\pi^*} \mathrm{Br}(\mathscr{C}) \to Ш(\mathbf{Pic}_{\mathscr{C}/X}/X) \to K_3 \to 0,$$

where the groups K_i are annihilated by δ, the index of the generic fibre C/K, e.g. $\delta = 1$ if $\mathscr{C}/X$ has a section, and their prime-to-p part finite, and $K_i = 0$ if π has a section. Here, $Ш(\mathbf{Pic}_{\mathscr{C}/X}/X)$ sits in a short exact sequence

$$0 \to \mathbf{Z}/d \to Ш(\mathbf{Pic}^0_{\mathscr{C}/X}/X) \to Ш(\mathbf{Pic}_{\mathscr{C}/X}/X) \to 0,$$

where $d \mid \delta$.

Now we come to our first statement on the conjecture of Birch and Swinnerton-Dyer for Abelian schemes $\mathscr{A}/X$ over higher dimensional schemes $X/\mathbf{F}_q$. The proof we give is a generalisation of the methods of Peter Schneider [Sch82a] and [Sch82b] which assume $\dim X = 1$. The theorem relates the vanishing order of a certain L-function at $s = 1$ to the Mordell-Weil rank, and the leading Taylor coefficient at $s = 1$ to a product of regulators of certain cohomological pairings, the order of the ℓ-primary component of the Tate-Shafarevich group and the torsion subgroup, for each single prime $\ell \neq p$. A main problem was to find the "correct" definition of the L-function: One has to throw out the factors coming from dimension > 1 since these cause additional cohomological terms in the special L-value which cannot be identified in terms of geometric invariants of the Abelian scheme.

Conjecture 2 (The conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher-dimensional bases, cohomological version)**.** *Set $\bar{X} = X \times_k \bar{k}$. Define the L-function of the Abelian scheme $\mathscr{A}/X$ by*

$$L(\mathscr{A}/X, s) = \frac{P_1(\mathscr{A}/X, q^{-s})}{P_0(\mathscr{A}/X, q^{-s})}$$

where

$$P_i(\mathscr{A}/X, t) = \det(1 - \mathrm{Frob}_q^{-1}\, t \mid \mathrm{H}^i(\bar{X}, \mathrm{R}^1\pi_*\mathbf{Q}_\ell)).$$

Here $\pi : \mathscr{A} \to X$ is the structure morphism. Let ρ be the vanishing order of $L(\mathscr{A}/X, s)$ at $s = 1$ and define the leading coefficient $c = L^*(\mathscr{A}/X, 1)$ of $L(\mathscr{A}/X, s)$ at $s = 1$ by

$$L(\mathscr{A}/X, s) \sim c \cdot (\log q)^\rho (s-1)^\rho \quad \textit{for } s \to 1.$$

Define pairings on cohomology groups modulo torsion

$$\langle \cdot, \cdot \rangle_\ell : \mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} \to \mathrm{H}^{2d}(X, \mathbf{Z}_\ell(d)) \overset{\mathrm{pr}_1^*}{\to} \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell,$$
$$(\cdot, \cdot)_\ell : \mathrm{H}^2(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} \to \mathrm{H}^{2d+1}(X, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell.$$

Then the pairings are non-degenerate, $Ш(\mathscr{A}/X)[\ell^\infty]$ is finite, and one has the equality for the leading Taylor coefficient

$$|c|_\ell^{-1} = \left| \frac{\det \langle \cdot, \cdot \rangle_\ell}{\det(\cdot, \cdot)_\ell} \right|_\ell^{-1} \cdot \frac{|Ш(\mathscr{A}/X)[\ell^\infty]|}{|\mathrm{Tor}\, \mathscr{A}(X)[\ell^\infty]| \cdot \left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|}$$

where $\mathscr{A}(X) = A(K)$ with A the generic fibre of $\mathscr{A}/X$ and $K = k(X)$ the function field of X.

Our conditional result is:

Theorem 4 (The conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher-dimensional bases, cohomological version)**.** *In the situation of Conjecture 2, the following statements are equivalent:*

(a) $\rho = \mathrm{rk}_{\mathbf{Z}}\, \mathscr{A}(X)$

(b) $\langle\cdot,\cdot\rangle_\ell$ *and* $(\cdot,\cdot)_\ell$ *are non-degenerate and* $|\text{Ш}(\mathscr{A}/X)[\ell^\infty]| < \infty$

(c) Conjecture 2 holds.

In the case of a constant Abelian scheme, i. e. where $\mathscr{A} = B \times_k X$ for an Abelian variety B/k, we can improve this result by replacing the cohomological height pairing with geometric pairing which is given by an integral trace pairing.

Theorem 5 (The height pairing)**.** *Let* X/k *be a smooth projective geometrically connected variety with Albanese* A *such that* $\mathbf{Pic}_{X/k}$ *is reduced. Denote the constant Abelian scheme* $B \times_k X/X$ *by* $\mathscr{A}/X$*. Then the trace pairing*

$$\mathrm{Hom}_k(A,B) \times \mathrm{Hom}_k(B,A) \stackrel{\circ}{\to} \mathrm{End}(A) \stackrel{\mathrm{Tr}}{\to} \mathbf{Z}$$

tensored with $\mathbf{Z}_\ell$ *equals the cohomological pairing*

$$\langle\cdot,\cdot\rangle_\ell : \mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} \to \mathrm{H}^{2d}(X, \mathbf{Z}_\ell(d)) \stackrel{\mathrm{pr}_1^*}{\to} \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell.$$

If X/k *is a curve, this equals the following height pairing*

$$\gamma(\alpha) : X \stackrel{\varphi}{\to} A \stackrel{\alpha}{\to} B,$$
$$\gamma'(\beta) : X \stackrel{\varphi}{\to} A \stackrel{c}{\to} A^\vee \stackrel{\beta^\vee}{\to} B^\vee,$$
$$(\gamma(\alpha), \gamma'(\beta))_{ht} = \deg_X(-(\alpha\varphi, \beta^\vee c\varphi)^* \mathscr{P}_B),$$

where $\varphi : X \to A$ *is the Abel-Jacobi map associated to a rational point of* X *and* $c : A \xrightarrow{\sim} A$ *the canonical principal polarisation associated to the theta divisor, and this is equal to the usual Néron-Tate canonical height pairing.*

Finally, building upon work of Milne [Mil68], we give a proof of the conjecture of Birch and Swinnerton-Dyer for constant Abelian schemes over certain higher dimensional bases which works for all primes, including the characteristic, at once. Again, one important step was to find the "correct" definition of the L-function of a constant Abelian scheme. As in the first theorem on the Birch-Swinnerton-Dyer conjecture, one has to throw out the factors coming from dimension > 1 of the base X.

Theorem 6 (The conjecture of Birch and Swinnerton-Dyer for constant Abelian schemes over higher-dimensional bases, version with the height pairing)**.** *Assume* $\bar{X} = X \times_k \bar{k}$ *satisfies*

(a) the Néron-Severi group of $\bar{X}$ *is torsion-free;*

(b) the dimension of $\mathrm{H}^1_{\mathrm{Zar}}(\bar{X}, \mathscr{O}_{\bar{X}})$ *as a vector space over* $\bar{k}$ *equals the dimension of the Albanese of* $\bar{X}/\bar{k}$*.*

Let $\mathscr{A} = B \times_k X$*. Define the* L*-function of the constant Abelian scheme* $\mathscr{A}/X$ *as the* L*-function of the motive*

$$h^1(B) \otimes (h^0(X) \oplus h^1(X)) = h^1(B) \oplus (h^1(B) \otimes h^1(X)),$$

namely

$$L(B \times_k X/X, s) = \frac{L(h^1(B) \otimes h^1(X), s)}{L(h^1(B), s)}$$

with $L(h^i(X) \otimes h^1(B), t) = \det(1 - \mathrm{Frob}_q^{-1} t \mid \mathrm{H}^i(\bar{X}, \mathbf{Q}_\ell) \otimes \mathrm{H}^1(\bar{B}, \mathbf{Q}_\ell))$. *Let* $d = \dim B$ *and* $g = \dim \mathrm{Alb}(X)$ *and* $R_{\log}(B)$ *the determinant of the above height pairing multiplied with* $\log q$. *Then one has* $\mathscr{A}(X) = A(K)$ *with* A *the generic fibre of* $\mathscr{A}$ *and* $K = k(X)$ *the function field, and the following holds:*

1. *The Tate-Shafarevich group* $Ш(\mathscr{A}/X)$ *is finite.*
2. *The vanishing order equals the Mordell-Weil rank:* $\operatorname*{ord}_{s=1} L(\mathscr{A}/X, s) = \mathrm{rk}\, A(K)$.
3. *There is the equality for the leading coefficient*

$$L^*(\mathscr{A}/X, 1) = q^{(g-1)d} \frac{|Ш(\mathscr{A}/X)|\, R_{\log}(B)}{|\mathrm{Tor}\, A(K))|}.$$

For the question when (a) and (b) hold, see Theorem 4.2.10, Remark 4.2.11, Example 4.2.12 and Example 4.2.25.

For a constant Abelian variety $\mathscr{A} = B \times_k X/X$, the two definitions of the L-function in Theorem 4 and Theorem 6 agree. For a motivation for the definitions of the L-functions, see Remark 4.2.31 below.

Combining the two results on the conjecture of Birch and Swinnerton-Dyer, one can identify the remaining two expressions in Theorem 4 under the assumptions of Theorem 6: One has

$$|\det(\cdot,\cdot)_\ell|_\ell^{-1} = 1,$$
$$\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right| = 1,$$

$\langle\cdot,\cdot\rangle_\ell$ and $(\cdot,\cdot)_\ell$ are non-degenerate and $Ш(\mathscr{A}/X)[\ell^\infty]$ is finite.

Structure of the thesis. In section 2, we collect some basic and technical facts.

Section 3.1 is concerned with the proof of a cohomological vanishing theorem $\mathrm{R}^q\pi_*\mathbf{G}_m = 0$ for $q > 1$. In section 3.2, we introduce the notion of a weak Néron model and prove that an Abelian scheme is a weak Néron model of its generic fibre, see Theorem 1. Section 3.3 contains the definition of and theorems on the Tate-Shafarevich group in the higher dimensional basis case, see Theorem 2. In section 3.4, we give an alternative proof of a statement in the previous section. In the final subsection 3.5, we relate the Tate-Shafarevich and the Brauer group, see Theorem 3.

Section 4 is about the Birch-Swinnerton-Dyer conjecture: In section 4.1, we define the L-function of an Abelian scheme and state a cohomological form of a Birch-Swinnerton-Dyer conjecture and give a criterion under which conditions this theorem holds, see Theorem 4. We specialise to constant Abelian schemes in section 4.2. The height pairing, Theorem 5, is treated in section 4.2.1, and the Birch-Swinnerton-Dyer conjecture for constant Abelian schemes, Theorem 6, in section 4.2.2.

Danksagung. Ich danke: meinem Doktorvater Uwe Jannsen für jegliche Unterstützung, die er mir hat zuteilwerden lassen; für viele hilfreiche Gespräche und Hinweise Brian Conrad, Patrick Forré, Walter Gubler, Armin Holschbach, Peter Jossen, Moritz Kerz, Klaus Künnemann, Niko Naumann, Maximilian Niklas, Tobias Sitte, Johannes Sprang, Jakob Stix, Georg Tamme und, von mathoverflow, Angelo, anon, Martin Bright, Kestutis Cesnavicius, Torsten Ekedahl, Laurent Moret-Bailly, ulrich und xuhan; für das Probelesen Patrick Forré, Peter Jossen und Niko Naumann; für die angenehme Arbeitsatmosphäre allen Mitgliedern der Fakultät für Mathematik Regensburg; für die Unterstützung zu Schulzeiten Gunter Malle und Arno Speicher; für hilfreiche Ratschläge Jürgen Braun; der Studienstiftung des deutschen Volkes für die finanzielle und ideelle Förderung; schließlich meiner Familie dafür, dass ich mich zuhause immer wohlfühlen konnte.

2 Preliminaries

Notation. Let A be an Abelian group. Let $\operatorname{Tor} A$ be the torsion subgroup of A, $A_{\mathrm{Tors}} = A/\operatorname{Tor} A$. Let $\operatorname{Div} A$ be the maximal divisible subgroup of A and $A_{\mathrm{Div}} = A/\mathrm{Div}A$. Denote the cokernel of $A \xrightarrow{n} A$ by A/n and its kernel by $A[n]$, and the p-primary subgroup $\varinjlim_n A[p^n]$ by $A[p^\infty]$.

Canonical isomorphisms are often denoted by "=".

If not stated otherwise, all cohomology groups are taken with respect to the étale topology.

We denote Pontryagin duality, duals of R-modules or ℓ-adic sheaves and Abelian schemes by $(-)^\vee$. It should be clear from the context which one is meant.

The Henselisation of a (local) ring A is denoted by A^h and the strict Henselisation by A^{sh}.

The ℓ-adic valuation $|\cdot|_\ell$ is taken to be normalised by $|\ell|_\ell = \ell^{-1}$.

2.1 Algebra

Lemma 2.1.1. *Given a spectral sequence $E_2^{p,q} \Rightarrow E^n$, one has an exact sequence*

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to \ker(E^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}.$$

Proof. See [NSW00], p. 81, (2.1.3) Proposition. □

We have the following properties and notions for Abelian groups.

Lemma 2.1.2. *Let $f : A \to B$, $g : B \to C$ be homomorphisms. Then*

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$$

is exact.

Proof. Apply the snake lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker}(f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle gf} & & \downarrow{\scriptstyle g} & & \downarrow & & \\ 0 & \longrightarrow & C & \xrightarrow{\mathrm{id}} & C & \longrightarrow & 0. & & \end{array}$$

□

Lemma 2.1.3. *The tensor product of an Abelian torsion group A with a divisible Abelian group B is trivial.*

Proof. Take an elementary tensor $a \otimes b$. There is an $n > 0$ such that $na = 0$, so, by divisibility of B there is an b' such that $nb' = b$, so $a \otimes b = a \otimes nb' = na \otimes b' = 0 \otimes b' = 0$. □

Lemma 2.1.4. *Let A be an Abelian ℓ-torsion group such that $A[\ell]$ is finite. Then A is a cofinitely generated $\mathbf{Z}_\ell$-module.*

Proof. Equip A with the discrete topology. Applying Pontryagin duality to $0 \to A[\ell] \to A \xrightarrow{\ell} A$ gives us that $A^\vee/\ell$ is finite, hence by [NSW00], p. 179, (3.9.1) Proposition ($A^\vee$ being profinite as a dual of a discrete torsion group), $A^\vee$ is a finitely generated $\mathbf{Z}_\ell$-module, hence A a cofinitely generated $\mathbf{Z}_\ell$-module. □

Definition 2.1.5. *Let A be an Abelian group and ℓ a prime number. Then the* **ℓ-adic Tate module** *$T_\ell A$ of A is the projective limit*

$$T_\ell A = \varprojlim \left(\dots \xrightarrow{\ell} A[\ell^{n+1}] \xrightarrow{\ell} A[\ell^n] \xrightarrow{\ell} \dots \xrightarrow{\ell} A[\ell] \to 0 \right).$$

The **rationalised ℓ-adic Tate module** *is defined as $V_\ell A = T_\ell A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$.*

Lemma 2.1.6. *One has $T_\ell A = \mathrm{Hom}(\mathbf{Q}_\ell/\mathbf{Z}_\ell, A)$.*

Proof. One has $\mathrm{Hom}(\mathbf{Q}_\ell/\mathbf{Z}_\ell, A) = \mathrm{Hom}(\varinjlim_n \frac{1}{\ell^n}\mathbf{Z}/\mathbf{Z}, A) = \varprojlim_n \mathrm{Hom}(\frac{1}{\ell^n}\mathbf{Z}/\mathbf{Z}, A) = \varprojlim_n A[\ell^n]$. □

Lemma 2.1.7. *Let A be a finite Abelian group. Then $T_\ell A$ is trivial.*

Proof. There is an n_0 such that $A[\ell^n]$ is stationary for $n \geq n_0$, i. e. $\ell^{n_0} A = 0$, so there is no non-zero infinite sequence $(\dots, a_n, a_{n-1}, \dots, a_0)$ with $\ell a_{i+1} = a_i$ since no non-zero element of A is infinitely ℓ-divisible. □

Lemma 2.1.8. *Let A be a non-finite cofinitely generated $\mathbf{Z}_\ell$-module. Then $T_\ell A$ is a non-trivial $\mathbf{Z}_\ell$-module.*

Proof. Since A is a cofinitely generated $\mathbf{Z}_\ell$-module, $A \cong B \oplus (\mathbf{Q}_\ell/\mathbf{Z}_\ell)^n$ with B finite, so $T_\ell A \cong \mathbf{Z}_\ell^n$. As A is not finite, $n > 0$. □

Lemma 2.1.9. *Let A be an Abelian group and $T_\ell A = \varprojlim_n A[\ell^n]$ its ℓ-adic Tate module. Then $T_\ell A$ is torsion free.*

Proof. Let $a = (\dots, a_m, \dots, a_1, a_0) \in T_\ell A$ with $\ell^n a = 0$. Then there is a $n_0 \in \mathbf{N}$ minimal such that $a_{n_0} \neq 0$. Denote the order of a_{n_0} by ℓ^m, $m > 0$. If there is an $i > 0$ such that $\mathrm{ord}(a_{n_0+i}) < \ell^{m+i}$, then $0 = \ell^{m+i-1} a_{n_0+i} = \ell^{m+i-2} a_{n_0+i-1} = \dots = \ell^{m-1} a_{n_0}$, contradiction. Hence for $i \gg 0$, we have $\ell^{n+1} \mid \mathrm{ord}(a_{n_0+i}) \mid \mathrm{ord}(a)$, contradiction to $\ell^n a = 0$. □

Remark 2.1.10. Note that, in contrast, for an ℓ-adic sheaf $\mathscr{F}_n$, $\varprojlim_n \mathrm{H}^i(X, \mathscr{F}_n)$ need *not* be torsion-free.

Definition 2.1.11. *For a profinite group G, a G-module M is* **discrete** *iff*

$$M = \varinjlim_U M^U$$

for U running through the open normal subgroups of G.

Lemma 2.1.12. *Let G be a profinite group and M a discrete G-module. Then $\mathrm{H}^q(G, M)$ is torsion for $q > 0$.*

In particular, Galois cohomology is torsion in positive degrees.

Proof. We have an isomorphism

$$\varinjlim_U \mathrm{H}^q(G/U, M^U) \xrightarrow{\sim} \mathrm{H}^q(G, M),$$

the limit taken over the open normal subgroups U of G. As $\mathrm{H}^q(G/U, M^U)$ is torsion for $q > 0$ because it is killed by $|G/U| < \infty$, the result follows. □

Definition 2.1.13. *A homomorphism of* $\mathbf{Z}_\ell$*-modules* $f : A \to B$ *is called a* **quasi-isomorphism** *if* $\ker(f)$ *and* $\operatorname{coker}(f)$ *are finite. In this case, define*

$$q(f) = \left| \frac{|\operatorname{coker}(f)|}{|\ker(f)|} \right|_\ell .$$

Lemma 2.1.14. *In the situation of the previous definition, one has:*

1. *Assume* A, B *are finitely generated Abelian groups of the same rank with bases* $(a_i)_{i=1}^n$ *and* $(b_i)_{i=1}^n$ *and* $f(a_i) = \sum_j z_{ij} b_j$ *modulo torsion. Then* f *is a quasi-isomorphism iff* $\det(z_{ij}) \neq 0$*. In this case,*
$$q(f) = \left| \det(z_{ij}) \cdot \frac{|\operatorname{Tor} B|}{|\operatorname{Tor} A|} \right|_\ell .$$
2. *Assume given* $f : A \to B$ *and* $g : B \to C$*. If two of* f, g, gf *are quasi-isomorphism, so is the third, and* $q(gf) = q(g) \cdot q(f)$*.*
3. *For the Pontrjagin dual* $f^\vee : B^\vee \to A^\vee$, f *is a quasi-isomorphism iff* $f^\vee$ *is, and then* $q(f) \cdot q(f^\vee) = 1$*.*
4. *Suppose* ϑ *is an endomorphism of a finitely generated* $\mathbf{Z}_\ell$*-module* A*. Let* f *be the homomorphism* $\ker(\vartheta) \to \operatorname{coker}(\vartheta)$ *induced by the identity. Then* f *is a quasi-isomorphism iff* $\det(T - \vartheta_{\mathbf{Q}}) = T^\rho R(T)$ *with* $\rho = \operatorname{rk}_{\mathbf{Z}_\ell}(\ker(\vartheta))$ *and* $R(0) \neq 0$*. In this case,* $q(f) = |R(0)|_\ell$*.*

Proof. See [Tat66b], p. 306-19–306-20, Lemma z.1–z.4. □

2.2 Geometry

Definition 2.2.1. *A* **projective morphism** $X \to Y$ *is a morphism that factors as a closed immersion into a (possibly twisted) projective bundle* $X \hookrightarrow \mathbf{P}(\mathscr{E}) \to Y$*.*

Lemma 2.2.2. *Let* $f : X \to Y$ *be a smooth projective morphism of locally Noetherian schemes. Then the following are equivalent:*

1. *One has* $f_* \mathscr{O}_X = \mathscr{O}_Y$ *(Zariski sheaves).*
2. *The fibres of* f *are geometrically connected.*

If these hold, $\mathbf{G}_{m,Y} \xrightarrow{\sim} f_* \mathbf{G}_{m,X}$ *as Zariski, étale or fppf sheaves.*

Proof. Since f is smooth, having geometrically integral fibres is equivalent to having geometrically connected fibres. Hence:

$1 \implies 2$: See [Liu06], p. 200 f., Theorem 5.3.15/17.

$2 \implies 1$: See [Liu06], p. 208, Exercise 5.3.12.

If 2 holds, the last statement follows since the fibres of a base change of f are also geometrically connected if the fibres of f are so. □

Lemma 2.2.3. *For a morphism of schemes* $f : X \to Y$*, the edge maps* $\mathrm{H}^p(Y, f_*\mathscr{F}) \to \mathrm{H}^p(X, \mathscr{F})$ *in the Leray spectral sequence are equal to* f^**.*

Proof. Choose an injective resolution $\mathscr{F} \to J^\bullet$ and an injective resolution $f_*\mathscr{F} \to I^\bullet$. Apply the exact functor f^* to the latter to obtain $f^*f_*\mathscr{F} \to f^*I^\bullet$, and we have the adjunction composed with the first injective resolution $f^*f_*\mathscr{F} \to \mathscr{F} \to J^\bullet$. Since $f^*I^\bullet$ is exact and $J^\bullet$ is injective, one gets a map $f^*I^\bullet \to J^\bullet$, and an adjoint map $I^\bullet \to f_*J^\bullet$. Taking global sections $\mathrm{H}^0(Y, I^\bullet) \to \mathrm{H}^0(Y, f_*J^\bullet)$ and cohomology yields the edge map $\mathrm{H}^p(Y, f_*\mathscr{F}) \to \mathrm{H}^p(X, \mathscr{F})$. □

Now we construct a Leray spectral sequence for étale cohomology with supports.

Theorem 2.2.4. *If $i : Z \hookrightarrow Y$ is a closed immersion and $\pi : X \to Y$ is a morphism,*

$$\begin{array}{ccc} Z' & \overset{i'}{\hookrightarrow} & X \\ \downarrow & & \downarrow \pi \\ Z & \overset{i}{\hookrightarrow} & Y \end{array}$$

there is a E_2-spectral sequence for étale sheaves $\mathscr{F}$

$$\mathrm{H}^p_Z(Y, \mathrm{R}^q\pi_*\mathscr{F}) \Rightarrow \mathrm{H}^{p+q}_{Z'}(X, \mathscr{F}),$$

where $i' : Z' \hookrightarrow X$ is the fibre product $\mathrm{pr}_2 : Z \times_Y X \hookrightarrow X$.

Proof. This is the Grothendieck spectral sequence for the composition of functors generalising the Leray spectral sequence [Mil80], p, 89, Theorem III.1.18 (a)

$$\begin{aligned} F :&\mathscr{F} \mapsto \pi_*\mathscr{F} \\ G :&\mathscr{F} \mapsto \mathrm{H}^0_Z(Y, \mathscr{F}), \end{aligned}$$

since

$$\begin{aligned} (GF)(\mathscr{F}) &= \mathrm{H}^0_Z(Y, \pi_*\mathscr{F}) \\ &= \ker((\pi_*\mathscr{F})(Y) \to (\pi_*\mathscr{F})(Y \setminus Z)) \\ &= \ker(\mathscr{F}(X) \to \mathscr{F}(\pi^{-1}(Y \setminus Z))) \\ &= \ker(\mathscr{F}(X) \to \mathscr{F}(X \setminus Z')) \\ &= \mathrm{H}^0_{Z'}(X, \mathscr{F}). \end{aligned}$$

We have to check if $\pi_*(-)$ maps injectives to $\mathrm{H}^0_Z(Y, -)$-acyclics. Then [Mil80], p. 309, Theorem B.1 establishes the existence of the spectral sequence.

Injective sheaves $\mathscr{I}$ are flabby (defined in [Mil80], p. 87, Example III.1.9 (c)) and π_* maps flabby sheaves to flabby sheaves ([Mil80], p. 89, Lemma III.1.19). Therefore, it follows from the long exact localisation sequence [Mil80], p. 92, Proposition III.1.25

$$\begin{aligned} 0 &\to \mathrm{H}^0_Z(Y, \pi_*\mathscr{I}) \to \mathrm{H}^0(Y, \pi_*\mathscr{I}) \to \mathrm{H}^0(Y \setminus Z, \pi_*\mathscr{I}) \\ &\to \mathrm{H}^1_Z(Y, \pi_*\mathscr{I}) \to \mathrm{H}^1(Y, \pi_*\mathscr{I}) \to \mathrm{H}^1(Y \setminus Z, \pi_*\mathscr{I}) \\ &\to \mathrm{H}^2_Z(Y, \pi_*\mathscr{I}) \to \mathrm{H}^2(Y, \pi_*\mathscr{I}) \to \mathrm{H}^2(Y \setminus Z, \pi_*\mathscr{I}) \to \dots \end{aligned}$$

and $\mathrm{H}^p(Y, \pi_*\mathscr{I}) = 0 = \mathrm{H}^p(Y \setminus Z, \pi_*\mathscr{I})$ for $p > 0$ that $\mathrm{H}^q_Z(Y, \pi_*\mathscr{I}) = 0$ for $q > 1$. For $\mathrm{H}^1_Z(Y, \pi_*\mathscr{I}) = 0$, it remains to show that $\mathrm{H}^0(Y, \pi_*\mathscr{I}) \to \mathrm{H}^0(Y \setminus Z, \pi_*\mathscr{I})$ is surjective. For this, setting $j : U = X \setminus Z' \hookrightarrow X$, apply $\mathrm{Hom}(-, \mathscr{I})$ to the exact sequence $0 \to j_!\mathscr{O}_U \to \mathscr{O}_X$ $(U = X \setminus Z')$ and get

$$\mathscr{I}(X) = \mathrm{Hom}(\mathscr{O}_X, \mathscr{I}) \twoheadrightarrow \mathrm{Hom}(j_!\mathscr{O}_U, \mathscr{I}) = \mathrm{Hom}(\mathscr{O}_U, \mathscr{I}|_U) = \mathscr{I}(U),$$

the arrow being surjective since $\mathscr{I}$ is injective. □

Lemma 2.2.5. *Let I be a filtered category and $(i \mapsto X_i)$ a contravariant functor from I to schemes over X. Assume that all schemes are quasi-compact and that the transition maps $X_i \leftarrow X_j$ are affine. Let $X_\infty = \varprojlim X_i$, and, for a sheaf $\mathscr{F}$ on $X_{\text{ét}}$, let $\mathscr{F}_i$ and $\mathscr{F}_\infty$ be its inverse images on X_i and X_∞ respectively. Then*

$$\varinjlim \mathrm{H}^p((X_i)_{\text{ét}}, \mathscr{F}_i) \xrightarrow{\sim} \mathrm{H}^p((X_\infty)_{\text{ét}}, \mathscr{F}_\infty).$$

Assume the $X_i \subseteq X$ are open, the transition morphisms are affine and all schemes are quasi-compact. Let $Z \hookrightarrow X$ be a closed subscheme. Then

$$\varinjlim \mathrm{H}^p_{Z \cap X_i}((X_i)_{\text{ét}}, \mathscr{F}_i) \xrightarrow{\sim} \mathrm{H}^p_{Z \cap X_\infty}((X_\infty)_{\text{ét}}, \mathscr{F}_\infty).$$

Proof. See [Mil80], p. 88, Lemma III.1.16 for the first statement. The second one follows from the first, the long exact localisation sequence (note that the morphisms $(X \setminus Z) \cap X_i \leftarrow (X \setminus Z) \cap X_j$ are affine as well since they are base changes of affine morphisms) and the five lemma. □

Now we construct a Mayer-Vietoris sequence for cohomology with supports.

Theorem 2.2.6. *Let Y_1 and Y_1 be closed subschemes of X and $\mathscr{F}$ a sheaf on X. Then there is a long exact sequence of cohomology with supports*

$$\ldots \to \mathrm{H}^i_{Y_1 \cap Y_2}(X, \mathscr{F}) \to \mathrm{H}^i_{Y_1}(X, \mathscr{F}) \oplus \mathrm{H}^i_{Y_2}(X, \mathscr{F}) \to \mathrm{H}^i_{Y_1 \cup Y_2}(X, \mathscr{F}) \to \ldots$$

Proof. Let $\mathscr{I}$ be an injective sheaf on X. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{Y_1 \cap Y_2}(X, \mathscr{I}) & \longrightarrow & \Gamma_{Y_1}(X, \mathscr{I}) \oplus \Gamma_{Y_2}(X, \mathscr{I}) & \longrightarrow & \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, \mathscr{I}) & \longrightarrow & \Gamma(X, \mathscr{I}) \oplus \Gamma(X, \mathscr{I}) & \longrightarrow & \Gamma(X, \mathscr{I}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X \setminus (Y_1 \cap Y_2), \mathscr{I}) & \longrightarrow & \Gamma(X \setminus Y_1, \mathscr{I}) \oplus \Gamma(X \setminus Y_2, \mathscr{I}) & \longrightarrow & \Gamma(X \setminus (Y_1 \cup Y_2), \mathscr{I}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The maps are induced by the restrictions, the two maps into the direct sum have the opposite sign and the map out of the direct sum is induced by the summation.

Since $\mathscr{I}$ is injective, by the same argument as in the proof of Theorem 2.2.4 the columns are exact. The second row is trivially exact and the third row is exact since $\mathscr{I}$ is a sheaf and $\mathscr{I}$ is injective. Hence by the snake lemma, the first row is exact.

Applying this to an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$, we get an exact sequence of complexes

$$0 \to \Gamma_{Y_1 \cap Y_2}(X, \mathscr{I}^\bullet) \to \Gamma_{Y_1}(X, \mathscr{I}^\bullet) \oplus \Gamma_{Y_2}(X, \mathscr{I}^\bullet) \to \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}^\bullet) \to 0$$

and from this the long exact sequence in the usual way. □

Lemma 2.2.7. *Let $f : X \to S$ be a morphism of schemes. Then f is locally of finite presentation iff*

$$\mathrm{Mor}_S(\lim_{i\in I} T_i, X) = \varinjlim_{i\in I} \mathrm{Mor}_S(T_i, X)$$

for any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine.

Proof. See [EGAIV$_3$], p. 52, Proposition 8.14.2. □

Theorem 2.2.8 (Lang-Steinberg)**.** *Let X_0/k be a scheme such that $X_0 \times_k \bar{k}$ is an Abelian variety. Then X_0 has a k-rational point.*

Proof. See [Mum70], p. 205, Theorem 3. □

Theorem 2.2.9 (Zariski-Nagata purity)**.** *Let X be a locally notherian regular scheme, $U \hookrightarrow X$ open with closed complement Z of codimension ≥ 2. Then the functor $X' \mapsto X' \times_X U$ of the category of étale coverings of X to the category of étale coverings of U is an equivalence of categories.*

Proof. See [SGA1], Exp. X, Corollaire 3.3. □

3 The Brauer and the Tate-Shafarevich group

3.1 Higher direct images and the Brauer group

All cohomology groups are with respect to the étale topology unless stated otherwise.

Lemma 3.1.1. *Let X be a scheme and ℓ a prime invertible on X. Then there are exact sequences*

$$0 \to \mathrm{H}^{i-1}(X, \mathbf{G}_m) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell/\mathbf{Z}_\ell \to \mathrm{H}^i(X, \mu_{\ell^\infty}) \to \mathrm{H}^i(X, \mathbf{G}_m)[\ell^\infty] \to 0$$

for each $i \geq 1$.

Proof. This follows from the long exact sequence induced by the Kummer sequence (which is exact by the invertibility of ℓ)

$$1 \to \mu_{\ell^n} \to \mathbf{G}_m \overset{\ell^n}{\to} \mathbf{G}_m \to 1$$

and passage to the colimit. □

Definition 3.1.2. *A* **variety** *over a field k is a separated scheme of finite type over k.*

Recall the definition [Mil80], IV.2, p. 140 ff. of the **Brauer group** $\mathrm{Br}(X)$ of a scheme X as the group of equivalence classes of Azumaya algebras on X.

Definition 3.1.3. $\mathrm{Br}'(X) := \mathrm{Tor}\,\mathrm{H}^2(X, \mathbf{G}_m)$ *is called the* **cohomological Brauer group***.*

Theorem 3.1.4. $\mathrm{Br}'(X) = \mathrm{H}^2(X, \mathbf{G}_m)$ *if X is a regular integral quasi-compact scheme.*

Proof. See [Mil80], p. 106 f., Example 2.22: We have an injection $\mathrm{H}^2(X, \mathbf{G}_m) \hookrightarrow \mathrm{H}^2(K, \mathbf{G}_m)$ and the latter is torsion as Galois cohomology by Lemma 2.1.12. □

Theorem 3.1.5. *There is an injection* $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$*, where* $\mathrm{Br}(X)$ *is the Brauer group of X.*

Proof. See [Mil80], p. 142, Theorem 2.5. □

Theorem 3.1.6. *Let X be a scheme endowed with an ample invertible sheaf. Then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. See [dJ]. □

Corollary 3.1.7. *Let X/k be a smooth projective geometrically connected variety. Then* $\mathrm{Br}(X) = \mathrm{Br}'(X) = \mathrm{H}^2(X, \mathbf{G}_m)$.

Proof. The first equality follows from Theorem 3.1.6 since X/k is projective, and the second equality follows from Theorem 3.1.4. □

Theorem 3.1.8. *Let X be a smooth projective geometrically connected variety over a finite field $k = \mathbf{F}_q$, $q = p^n$.*

(a) $\mathrm{H}^i(X, \mathbf{G}_m)$ *is torsion for $i \neq 1$, finite for $i \neq 1, 2, 3$ and $= 0$ for $i > 2\dim(X) + 1$.*

(b) For $\ell \neq p$ and $i = 2, 3$, one has $\mathrm{H}^i(X, \mathbf{G}_m)[\ell^\infty] = (\mathbf{Q}_\ell/\mathbf{Z}_\ell)^{\rho_{i,\ell}} \oplus C_{i,\ell}$*, where $C_{i,\ell}$ is finite and $= 0$ for all but finitely many ℓ, and $\rho_{i,\ell}$ a non-negative integer.*

Proof. See [Lic83], p. 180, Proposition 2.1 a)–c), f). □

Corollary 3.1.9. *Let X be a smooth projective geometrically connected variety over a finite field $k = \mathbf{F}_q$, $q = p^n$. Let $\ell \neq p$ be prime. Then one has*

$$\mathrm{H}^i(X, \mu_{\ell^\infty}) \xrightarrow{\sim} \mathrm{H}^i(X, \mathbf{G}_m)[\ell^\infty]$$

for $i \neq 2$.

Proof. This follows from Lemma 3.1.1 and Theorem 3.1.8 by Lemma 2.1.3 since $\mathrm{H}^{i-1}(X, \mathbf{G}_m) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell/\mathbf{Z}_\ell$ is the tensor product of a torsion group (for $i \neq 2$) with a divisible group (Lemma 2.1.3). □

The following is a generalisation of [Gro68], pp. 98–104, Théorème (3.1) from the case of X/Y with $\dim X = 2$, $\dim Y = 1$ to X/Y with relative dimension 1. One can remove the assumption $\dim X = 1$ if one uses Artin's approximation theorem [Art69], p. 26, Theorem (1.10) instead of Greenberg's theorem on p. 104, l. 4 and l. −2, and replaces "proper" by "projective" and does some other minor modifications; also note that in our situation the Brauer group coincides with the cohomological Brauer group by Theorem 3.1.8 and Theorem 3.1.6. For the convenience of the reader, we reproduce the full proof of Theorem 3.1.10 and Theorem 3.1.16 here.

Theorem 3.1.10. *Let $f : \mathscr{C} \to X$ be a smooth projective morphism with fibres of dimension ≤ 1, $\mathscr{C}$ and X regular and X the spectrum of a Henselisation of a variety at a prime ideal with closed point x, and $\mathscr{C}_0 \hookrightarrow \mathscr{C}$ the subscheme $f^{-1}(x)$. Then the canonical homomorphism*

$$\mathrm{H}^2(\mathscr{C}, \mathbf{G}_m) \to \mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_m)$$

induced by the closed immersion $\mathscr{C}_0 \hookrightarrow \mathscr{C}$ is bijective.

Proof. Let $X = \mathrm{Spec}(A)$, $X_n = \mathrm{Spec}(A/\mathfrak{m}^{n+1})$, $\mathscr{C}_n = \mathscr{C} \times_X X_n$.

Note that for $\mathscr{C}$ and X, Br, Br′ and $\mathrm{H}^2(-, \mathbf{G}_m)$ are equal since there is an ample sheaf (Theorem 3.1.6) and by regularity (Theorem 3.1.4).

There are exact sequences for every n

$$0 \to \mathscr{F} \to \mathbf{G}_{m,\mathscr{C}_{n+1}} \to \mathbf{G}_{m,\mathscr{C}_n} \to 1 \tag{3.1.1}$$

with $\mathscr{F}$ a coherent sheaf on $\mathscr{C}_0$: Zariski-locally on the source, $\mathscr{C} \to X$ is of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and hence $\mathscr{C}_n \to X_n$ of the form $\mathrm{Spec}(B/\mathfrak{m}^{n+1}) \to \mathrm{Spec}(A/\mathfrak{m}^{n+1})$. There is an exact sequence

$$1 \to (1 + \mathfrak{m}^n/\mathfrak{m}^{n+1}) \to (B/\mathfrak{m}^{n+1})^\times \to (B/\mathfrak{m}^n)^\times \to 1.$$

The latter map is surjective since $\mathfrak{m}^n/\mathfrak{m}^{n+1} \subset B/\mathfrak{m}^{n+1}$ is nilpotent (*deformation of units:* Let $f : B \to A$ be a surjective ring homomorphism with nilpotent kernel. If $f(b)$ is a unit, so is b: this is because a unit plus a nilpotent element is a unit: Let $f(b)c = 1_A$. Then there is a $\bar{c} \in B$ such that $b\bar{c} - 1_B \in \ker(f)$, so $b\bar{c}$ is a unit, so b is invertible in B). By the logarithm, $(1+\mathfrak{m}^n/\mathfrak{m}^{n+1}) \xrightarrow{\sim} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a coherent sheaf on $\mathrm{Spec}(B)$. The sequences for a Zariski-covering of $\mathscr{C}_0$ glue to an exact sequence of sheaves on $\mathscr{C}_0$ (3.1.1), equivalently, on $\mathscr{C}_n$ for any n since these have the same underlying topological space.

Therefore, the associated long exact sequence to (3.1.1) yields

$$\mathrm{H}^2(\mathscr{C}_0, \mathscr{F}) \to \mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_{m,\mathscr{C}_{n+1}}) \to \mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_{m,\mathscr{C}_n}) \to \mathrm{H}^3(\mathscr{C}_0, \mathscr{F}).$$

Now, $\mathrm{H}^p_{\text{ét}}(\mathscr{C}_0, \mathscr{F}) = \mathrm{H}^p_{\text{Zar}}(\mathscr{C}_0, \mathscr{F})$ since $\mathscr{F}$ is coherent by [SGA4.2], VII 4.3. Thus, since $\dim \mathscr{C}_0 \leq 1$, $\mathrm{H}^2(\mathscr{C}_0, \mathscr{F}) = \mathrm{H}^3(\mathscr{C}_0, \mathscr{F}) = 0$. Thus we get an isomorphism

$$\mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_{m,\mathscr{C}_{n+1}}) \xrightarrow{\sim} \mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_{m,\mathscr{C}_n})$$

Next note that $\mathscr{C}_0 \hookrightarrow \mathscr{C}_n$ is a closed immersion defined by a nilpotent ideal sheaf, so there is an equivalence of categories of étale $\mathscr{C}_0$-sheaves and étale $\mathscr{C}_n$-sheaves by [Mil80], p. 30, Theorem I.3.23, so we get

$$\mathrm{H}^2(\mathscr{C}_{n+1}, \mathbf{G}_m) \xrightarrow{\sim} \mathrm{H}^2(\mathscr{C}_n, \mathbf{G}_m).$$

Taking torsion, it follows that $\mathrm{Br}'(\mathscr{C}_{n+1}) \xrightarrow{\sim} \mathrm{Br}'(\mathscr{C}_n)$, and then Theorem 3.1.6 yields that the $\mathrm{Br}(\mathscr{C}_{n+1}) \to \mathrm{Br}(\mathscr{C}_n)$ are isomorphisms (in fact, injectivity suffices for the following). Therefore the injectivity of $\mathrm{Br}(\mathscr{C}) \to \mathrm{Br}(\mathscr{C}_0)$ follows from the

Lemma 3.1.11. *Let $f : \mathscr{C} \to X$ be a projective smooth morphism with X the spectrum of a Henselisation of a variety at a regular prime ideal. Suppose the transition maps of $(\mathrm{Pic}(\mathscr{C}_n))_{n \in \mathbf{N}}$ are surjective (in fact, the Mittag-Leffler condition would suffice). Then the canonical homomorphism*

$$\mathrm{Br}(\mathscr{C}) \to \varprojlim_{n \in \mathbf{N}} \mathrm{Br}(\mathscr{C}_n)$$

is injective.

One can apply Lemma 3.1.11 in our situation since the transition maps $\mathrm{Pic}(\mathscr{C}_{n+1}) \to \mathrm{Pic}(\mathscr{C}_n)$ are surjective by

Theorem 3.1.12. *Let A be a Henselian local ring, $S = \mathrm{Spec}(A)$ with closed point s_0, $f : X \to S$ separated and of finite presentation, and $X_0 := f^{-1}(s_0)$ of dimension ≤ 1. Then for every closed subscheme X_0' of X with the same underlying space as X_0 and of finite presentation over S, the canonical homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0')$ is surjective.*

Proof. See [EGAIV$_4$], p. 288, Corollaire (21.9.12). □

Proof of Lemma 3.1.11. Let $\underline{A}$ be an Azumaya algebra over $\mathscr{C}$ which lies in the kernel of the map in this lemma, i. e. such that for every $n \in \mathbf{N}$ there is an isomorphism

$$u_n : \underline{A}_n \cong \underline{\mathrm{End}}(\underline{V}_n) \tag{3.1.2}$$

with $\underline{V}_n$ a locally free $\mathscr{O}_{\mathscr{C}_n}$-module. Such a $\underline{V}_n$ is uniquely determined by $\underline{A}_n$ modulo tensoring with an invertible sheaf $\underline{L}_n$:

Lemma 3.1.13. *Let X be a quasi-compact scheme, quasi-projective over an affine scheme. Assume $\underline{A} \in \mathrm{H}^1(X, \mathrm{PGL}_n)$ is an Azumaya algebra trivialised by $\underline{A} \cong \underline{\mathrm{End}}(\underline{V})$ with $\underline{V} \in \mathrm{H}^1(X, \mathrm{GL}_n)$ a locally free sheaf of rank n. Then every other such $\underline{V}'$ differs from $\underline{V}$ by tensoring with an invertible sheaf.*

Proof. Consider for $n \in \mathbf{N}$ the central extension of étale sheaves on X (see [Mil80], p. 146)

$$1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

By [Mil80], p. 143, Step 3, this induces a long exact sequence in (Čech) cohomology of pointed sets

$$\mathrm{Pic}(X) = \check{\mathrm{H}}^1(X, \mathbf{G}_m) \xrightarrow{g} \check{\mathrm{H}}^1(X, \mathrm{GL}_n) \xrightarrow{h} \check{\mathrm{H}}^1(X, \mathrm{PGL}_n) \xrightarrow{f} \check{\mathrm{H}}^2(X, \mathbf{G}_m).$$

Note that by assumption and [Mil80], p. 104, Theorem III.2.17, $\check{\mathrm{H}}^1(X, \mathbf{G}_m) = \mathrm{H}^1(X, \mathbf{G}_m) = \mathrm{Pic}(X)$ and $\check{\mathrm{H}}^2(X, \mathbf{G}_m) = \mathrm{H}^2(X, \mathbf{G}_m)$. Further, $\mathrm{Br}(X) = \mathrm{Br}'(X)$ since a scheme quasi-compact and quasi-projective over an affine scheme has an ample line bundle ([Liu06], p. 171, Corollary 5.1.36), so Theorem 3.1.6 applies and $\mathrm{Br}'(X) \hookrightarrow \mathrm{H}^2(X, \mathbf{G}_m)$. Since $\underline{A}$ is an Azumaya algebra, $f(\underline{A}) = [\underline{A}] \in \mathrm{Br}(X) \hookrightarrow \mathrm{H}^2(X, \mathbf{G}_m)$. Therefore f factors through $\mathrm{Br}(X) \hookrightarrow \mathrm{H}^2(X, \mathbf{G}_m)$.

Assume the Azumaya algebra $\underline{A} \in \mathrm{H}^1(X, \mathrm{PGL}_n)$ lies in the kernel of f, i.e. there is a $\underline{V}$ such that $\underline{A} \cong \underline{\mathrm{End}}(\underline{V})$. Then it comes from $\underline{V} \in \mathrm{H}^1(X, \mathrm{GL}_n)$ by [Mil80], p. 143, Step 2 (h is the morphism $\underline{V} \mapsto \underline{\mathrm{End}}(\underline{V})$). So, since $\mathbf{G}_m$ is central in GL_n, by the analogue of [Ser02], p. 54, Proposition 42 for étale Čech cohomology, if $\underline{V}' \in \mathrm{H}^1(X, \mathrm{GL}_n)$ also satisfies $\underline{A} \cong \underline{\mathrm{End}}(\underline{V}')$, they differ by an invertible sheaf. □

Because of surjectivity of the transition maps of $(\mathrm{Pic}(\mathscr{C}_n))_{n \in \mathbf{N}}$, one can choose the $\underline{V}_n$, u_n such that the $\underline{V}_n$ and u_n form a projective system:

$$\underline{V}_n = \underline{V}_{n+1} \otimes_{\mathscr{O}_{\mathscr{C}_{n+1}}} \mathscr{O}_{\mathscr{C}_n} \tag{3.1.3}$$

and the isomorphisms (3.1.2) also form a projective system: Construct the $\underline{V}_n$, u_n inductively. Take $\underline{V}_0$ such that

$$\underline{A} \otimes_{\mathscr{O}_{\mathscr{C}}} \mathscr{O}_{\mathscr{C}_0} \cong \underline{A}_0 \cong \underline{\mathrm{End}}(\underline{V}_0).$$

One has

$$\underline{A}_n = \underline{A} \otimes_{\mathscr{O}_{\mathscr{C}}} \mathscr{O}_{\mathscr{C}_n}$$

and by Lemma 3.1.13, there is an invertible sheaf $\mathscr{L}_n \in \mathrm{Pic}(\mathscr{C}_n)$ such that

$$\underline{V}_{n+1} \otimes_{\mathscr{O}_{\mathscr{C}_{n+1}}} \mathscr{O}_{\mathscr{C}_n} \xrightarrow{\sim} \underline{V}_n \otimes_{\mathscr{O}_{\mathscr{C}_n}} \mathscr{L}_n.$$

By assumption, there is an invertible sheaf $\mathscr{L}_{n+1} \in \mathrm{Pic}(\mathscr{C}_{n+1})$ such that $\mathscr{L}_{n+1} \otimes_{\mathscr{O}_{\mathscr{C}_{n+1}}} \mathscr{O}_{\mathscr{C}_n} \cong \mathscr{L}_n$, so redefine $\underline{V}_{n+1}$ as $\underline{V}_{n+1} \otimes_{\mathscr{O}_{\mathscr{C}_{n+1}}} \mathscr{L}_{n+1}^{-1}$. Then (3.1.3) is satisfied.

Let $\hat{X}$ be the completion of X, and denote by $\hat{\mathscr{C}}, \hat{\underline{A}}, \ldots$ the base change of $\mathscr{C}, \underline{A}, \ldots$ by $\hat{X} \to X$.

Recall that an adic Noetherian ring A with defining ideal $\mathscr{I}$ is a Noetherian ring with a basis of neighbourhoods of zero of the form $\mathscr{I}^n$, $n > 0$ such that A is complete and Hausdorff in this topology. For such a ring A, there is the formal spectrum $\mathrm{Spf}(A)$ with underlying space $\mathrm{Spec}(A/\mathscr{I})$.

Theorem 3.1.14. *Let A be an adic Noetherian ring, $Y = \mathrm{Spec}(A)$ with $\mathscr{I}$ a defining ideal, $Y' = V(\mathscr{I})$, $f : X \to Y$ a separated morphism of finite type, $X' = f^{-1}(Y')$. Let $\hat{Y} = Y_{/Y'} = \mathrm{Spf}(A)$, $\hat{X} = X_{/X'}$ the completions of Y and X along Y' and X', $\hat{f} : \hat{X} \to \hat{Y}$ the extension of f to the completions. Then the functor $\mathscr{F} \rightsquigarrow \mathscr{F}_{/X'} = \hat{\mathscr{F}}$ is an equivalence of categories of coherent $\mathscr{O}_X$-modules with proper support on $\mathrm{Spec}(A)$ to the category of coherent $\mathscr{O}_{\hat{X}}$-modules with proper support on $\mathrm{Spf}(A)$.*

Proof. See [EGAIII$_1$], p. 150, Théorème (5.1.4). □

According to Theorem 3.1.14, to give a projective system $(\underline{V}_n, u_n)_{n\in\mathbf{N}}$ on $(\mathscr{C}_n)_{n\in\mathbf{N}}$ as in (3.1.2) and (3.1.3) is equivalent to giving a locally free module $\hat{\underline{V}}$ on $\hat{\mathscr{C}}$ and an isomorphism

$$\hat{u} : \hat{\underline{A}} \xrightarrow{\sim} \underline{\mathrm{End}}(\hat{\underline{V}}). \tag{3.1.4}$$

If $X = \hat{X}$, we are done: $\underline{A} = \hat{\underline{A}}$ is trivial.

In the general case, one has to pay attention to the fact that one does not know if with the preceding construction $\hat{\underline{V}}$ comes from a locally free module $\underline{V}$ on $\mathscr{C}$. However, there is a locally free module $\underline{\mathscr{E}}$ on $\mathscr{C}$ such that there exists an epimorphism

$$\hat{\underline{\mathscr{E}}} \to \hat{\underline{V}}.$$

Indeed, choosing a projective immersion for $\hat{\mathscr{C}}$ (by projectivity of $\hat{\mathscr{C}}/\operatorname{Spec}(\hat{A})$) with an ample invertible sheaf $\mathscr{O}_{\hat{\mathscr{C}}}(1)$, it suffices to take a direct sum of sheaves of the form $\mathscr{O}_{\hat{\mathscr{C}}}(-N)$, $N \gg 0$. Now, for $N \gg 0$, there is an epimorphism $\mathscr{O}_{\hat{\mathscr{C}}}^{\oplus k} \twoheadrightarrow \hat{\underline{V}}(N)$ for a suitable $k \in \mathbf{N}$, so twisting with $\mathscr{O}_{\hat{\mathscr{C}}}(-N)$ gives

$$\mathscr{O}_{\hat{\mathscr{C}}}(-N)^{\oplus k} \twoheadrightarrow \hat{\underline{V}}$$

("there are enough vector bundles"). Set $\underline{\mathscr{E}} = \mathscr{O}_{\mathscr{C}}(-N)^{\oplus k}$.

Now consider, for schemes X' over X, the contravariant functor $F : (\mathrm{Sch}/X)^\circ \to (\mathrm{Set})$ given by $F(X') =$ the set of pairs $(\underline{V}', \varphi')$, where $\underline{V}'$ is a quotient of a locally free module $\underline{\mathscr{E}}' = \underline{\mathscr{E}} \otimes_X X'$ and $\varphi' : \underline{A}' = \underline{A} \otimes_X X' \xrightarrow{\sim} \underline{\mathrm{End}}(\underline{V}')$.

Since f is projective and flat, by [SGA4.3], p. 133 f., Lemme XIII 1.3, one sees that the functor F is representable by a scheme, also denoted F, locally of finite type over X, hence locally of finite presentation since our schemes are Noetherian (what matters is the functor being locally of finite presentation, not its representability). By assumption of Lemma 3.1.11 and (3.1.4), $(\hat{\underline{V}}, \hat{u})$ is an element from $F(\hat{X})$. By Artin approximation (the following Theorem 3.1.15), $F(\hat{X}) \neq \emptyset$ implies $F(X) \neq \emptyset$:

Theorem 3.1.15 (Artin approximation)**.** *Let R be a field, and let A be the Henselisation of an R-algebra of finite type at a prime ideal. Let $\mathfrak{m}$ be a proper ideal of A. Let F be a functor $(A - \mathbf{Alg}) \to (\mathbf{Set})$ which is locally of finite presentation. Denote by $\hat{A}$ the $\mathfrak{m}$-adic completion of A. Then $F(\hat{A}) \neq \emptyset$ implies $F(A) \neq \emptyset$: Given $\bar{\xi} \in F(\hat{A})$, for every integer c, there is a $\xi \in F(A)$ such that*

$$\xi \equiv \bar{\xi} \pmod{\mathfrak{m}^c}.$$

Proof. See [Art69], p. 26, Theorem (1.10) resp. Theorem (1.12). □

This proves that $\underline{A}$ is isomorphic to an algebra of the form $\underline{\mathrm{End}}(\underline{V})$ with $\underline{V}$ locally free over $\mathscr{C}$, so it is trivial as an element of $\mathrm{Br}(\mathscr{C})$. □

The surjectivity in Theorem 3.1.10 is shown analogously: Take an element of $\mathrm{Br}(\mathscr{C}_0)$, represented by an Azumaya algebra $\underline{A}_0$. As $\mathrm{Br}(\mathscr{C}_n) \xrightarrow{\sim} \mathrm{Br}(\mathscr{C}_0)$, see above, there is a compatible system of Azumaya algebras $\underline{A}_n$ on $\mathscr{C}_n$. Therefore, as above, there is an Azumaya algebra $\hat{\underline{A}}$ on $\hat{\mathscr{C}}$. Choose a locally free module $\underline{\mathscr{E}}$ on $\mathscr{C}$ such that there is an epimorphism $\hat{\underline{\mathscr{E}}} \twoheadrightarrow \hat{\underline{A}}$ and consider the functor $F : (\mathrm{Sch}/X)^\circ \to (\mathrm{Set})$ defined by $F(X') =$ the set of pairs $(\underline{B}', p')$ where $\underline{B}'$ is a locally free module of $\underline{\mathscr{E}} \otimes_X X'$ and p' a multiplication law on $\underline{B}'$ which makes it into an

Azumaya algebra. Then F is representable by a scheme locally of finite type over X *(loc. cit.)*, and the point in $F(x)$ (recall that x is the closed point of X) defined by $\underline{A}_0$ gives us a point in $F(\hat{X})$, so by Artin approximation Theorem 3.1.15 it comes from a point in $F(X)$, which proves surjectivity. $\square$

As a corollary of Theorem 3.1.10, we get

Theorem 3.1.16. *Let $\pi : \mathscr{C} \to X$ be projective and smooth with $\mathscr{C}$ and X regular, all fibres of dimension 1 and X be a variety. Then*

$$\mathrm{R}^q\pi_*\mathbf{G}_m = 0 \quad \textit{for } q > 1.$$

Proof. By [SGA4.2], VIII 5.2, resp. [Mil80], p. 88, III.1.15 one can assume X strictly local and we must prove $\mathrm{H}^i(\mathscr{C}, \mathbf{G}_m) = 0$ for $i > 1$. By the proper base change theorem [Mil80], p. 224, Corollary VI.2.7, one has for torsion sheaves $\mathscr{F}$ on $\mathscr{C}$ with restriction $\mathscr{F}_0$ to the closed fibre $\mathscr{C}_0$ restriction isomorphisms

$$\mathrm{H}^i(\mathscr{C}, \mathscr{F}) \to \mathrm{H}^i(\mathscr{C}_0, \mathscr{F}_0).$$

Since $\dim \mathscr{C}_0 = 1$, the latter term vanishes for $i > 2$ and for $i > 1$ if $\mathscr{F}$ is p-torsion, where p is the residue field characteristic. Therefore

$$\mathrm{cd}(\mathscr{C}) \leq 2, \text{ and } \mathrm{cd}_p(\mathscr{C}) \leq 1.$$

The relation $\mathrm{H}^i(\mathscr{C}, \mathbf{G}_m) = 0$ for $i > 2$ follows from the fact that these groups are torsion by Theorem 3.1.8 (a) and from

Lemma 3.1.17. *Let X be a scheme, ℓ a prime number and n an integer such that $\mathrm{cd}_\ell(X) \leq n$. Then $\mathrm{H}^i(X, \mathbf{G}_m)[\ell^\infty] = 0$ if $i > n+1$, resp. $i > n$ if ℓ is invertible on X.*

Proof. If ℓ is invertible on X, the Kummer sequence

$$1 \to \mu_{\ell^r} \to \mathbf{G}_m \xrightarrow{\ell^r} \mathbf{G}_m \to 1$$

induces a long exact sequence in cohomology, part of which is

$$0 = \mathrm{H}^{i+1}(X, \mu_{\ell^r}) \to \mathrm{H}^{i+1}(X, \mathbf{G}_m) \xrightarrow{\ell^r} \mathrm{H}^{i+1}(X, \mathbf{G}_m) \to \mathrm{H}^{i+2}(X, \mu_{\ell^r}) = 0,$$

for $i > n$, i. e. multiplication by ℓ^r induces an isomorphism on $\mathrm{H}^{i+1}(X, \mathbf{G}_m)$ for $i > n$. If $c \in \mathrm{H}^{i+1}(X, \mathbf{G}_m)[\ell^r] \subseteq \mathrm{H}^{i+1}(X, \mathbf{G}_m)[\ell^\infty]$, then $0 = \ell^r c$, therefore $c = 0$ by the injectivity of ℓ^r, so $\mathrm{H}^i(X, \mathbf{G}_m)[\ell^\infty] = 0$ for $i > n$.

If $\ell = p$, one has an exact sequence

$$1 \to \ker(\ell^r) \to \mathbf{G}_m \xrightarrow{\ell^r} \mathbf{G}_m \to \mathrm{coker}(\ell^r) \to 1$$

of étale sheaves which splits up into

$$\begin{array}{ccccc} 1 \to \ker(\ell^r) \to & \mathbf{G}_m & \xrightarrow{\qquad\ell^r\qquad} & \mathbf{G}_m & \to \mathrm{coker}(\ell^r) \to 1 \\ & \searrow & & \nearrow & \\ & & \mathrm{im}(\ell^r) & & \end{array}$$

By the same argument as in the case ℓ invertible on X, one finds $\mathrm{H}^i(X, \mathbf{G}_m) \xrightarrow{\sim} \mathrm{H}^i(X, \mathrm{im}(\ell^r))$ for $i > n$, and, since $\ell^r \operatorname{coker}(\ell^r) = 0$, $\mathrm{H}^i(X, \mathrm{im}(\ell^r)) \xrightarrow{\sim} \mathrm{H}^i(X, \mathbf{G}_m)$ for $i > n+1$. So, altogether

$$\ell^r : \mathrm{H}^i(X, \mathbf{G}_m) \xrightarrow{\sim} \mathrm{H}^i(X, \mathrm{im}(\ell^r)) \xrightarrow{\sim} \mathrm{H}^i(X, \mathbf{G}_m)$$

is injective for $i > n+1$, and therefore $\mathrm{H}^i(X, \mathbf{G}_m)[\ell^\infty] = 0$ for $i > n+1$. □

It remains to treat the case $i = 2$, i. e. to prove

$$\mathrm{H}^2(\mathscr{C}, \mathbf{G}_m) = 0.$$

If ℓ is invertible on X, then

$$\mathrm{H}^2(\mathscr{C}, \mathbf{G}_m)[\ell^\infty] = 0$$

follows as in the case $i > 2$. From the Kummer sequence Lemma 3.1.1 and the inclusion $\mathscr{C}_0 \hookrightarrow \mathscr{C}$, one gets a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(\mathscr{C}) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell & \longrightarrow & \mathrm{H}^2(\mathscr{C}, \mu_{\ell^\infty}) & \longrightarrow & \mathrm{H}^2(\mathscr{C}, \mathbf{G}_m)[\ell^\infty] & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}(\mathscr{C}_0) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell & \longrightarrow & \mathrm{H}^2(\mathscr{C}_0, \mu_{\ell^\infty}) & \longrightarrow & \mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_m)[\ell^\infty] & \longrightarrow & 0, \end{array}$$

and the middle vertical arrow is bijective by proper base change [Mil80], p. 224, Corollary VI.2.7, and the first vertical arrow is surjective by Theorem 3.1.12 and the right exactness of the tensor product. Hence, by the five lemma, the right vertical morphism ist bijective.

Lemma 3.1.18. *Let C/K be a projective regular curve over a separably closed field. Then* $\mathrm{Br}(C) = \mathrm{Br}'(C) = \mathrm{H}^2(C, \mathbf{G}_m) = 0$.

Proof. One has $\mathrm{Br}(C) = \mathrm{Br}'(C) = 0$ since C is a proper curve over a separably closed field by [Gro68], p. 132, Corollaire (5.8). Moreover, Theorem 3.1.4 implies $\mathrm{Br}'(C) = \mathrm{H}^2(C, \mathbf{G}_m)$. □

Thus the diagram gives us $\mathrm{H}^2(\mathscr{C}, \mathbf{G}_m)[\ell^\infty] = 0$

For $\ell = p$, one uses Theorem 3.1.10, which gives us

$$\mathrm{H}^2(\mathscr{C}, \mathbf{G}_m) \xrightarrow{\sim} \mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_m),$$

and $\mathrm{H}^2(\mathscr{C}_0, \mathbf{G}_m) = 0$ by Lemma 3.1.18. □

Remark 3.1.19. Note that the difficult Theorem 3.1.10 is only needed for the p-torsion in Theorem 3.1.16.

We now draw some consequences from Theorem 3.1.16:

Corollary 3.1.20. *In the situation of Theorem 3.1.16, assume we have locally Noetherian separated schemes with geometrically reduced and connected fibres. Then one has the long exact sequence*

$$\begin{aligned} 0 \to &\mathrm{H}^1(X, \mathbf{G}_m) \xrightarrow{\pi^*} \mathrm{H}^1(\mathscr{C}, \mathbf{G}_m) \to \mathrm{H}^0(X, \mathrm{R}^1\pi_*\mathbf{G}_m) \to \\ &\mathrm{H}^2(X, \mathbf{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathscr{C}, \mathbf{G}_m) \to \mathrm{H}^1(X, \mathrm{R}^1\pi_*\mathbf{G}_m) \to \\ &\mathrm{H}^3(X, \mathbf{G}_m) \xrightarrow{\pi^*} \mathrm{H}^3(\mathscr{C}, \mathbf{G}_m) \to \mathrm{H}^2(X, \mathrm{R}^1\pi_*\mathbf{G}_m) \to \\ &\mathrm{H}^4(X, \mathbf{G}_m) \xrightarrow{\pi^*} \mathrm{H}^4(\mathscr{C}, \mathbf{G}_m) \to \mathrm{H}^3(X, \mathrm{R}^1\pi_*\mathbf{G}_m) \to \dots \end{aligned}$$

Proof. This follows from the Leray spectral sequence and Theorem 3.1.16 combined with [Wei97], p. 124, Exercise 5.2.2 (spectral sequence with two rows; here for a cohomological spectral sequence). One has $\pi_*\mathbf{G}_m = \mathbf{G}_m$ by Lemma 2.2.2. The edge maps $\mathrm{H}^p(X, \mathbf{G}_m) = \mathrm{H}^p(X, \pi_*\mathbf{G}_m) \to \mathrm{H}^p(\mathscr{C}, \mathbf{G}_m)$ are identified as π^* by Lemma 2.2.3. □

Lemma 3.1.21. *Let $\pi : \mathscr{C} \to X$ be a proper relative curve with X integral and let D be an irreducible Weil divisor in the generic fibre C of π. Then $\pi_{\bar{D}} : \bar{D} \to X$ is a finite morphism.*

Proof. Take D an irreducible Weil divisor in C, i. e. a closed point of C. Then $\bar{D}/X$ is finite of degree $\deg(D)$: $\bar{D}/X$ is of finite type, generically finite and dominant (since D lies over the generic point of X) and $\bar{D}$ (as a closure of an irreducible set) and X are irreducible, $\mathscr{C}$ and X are integral, hence by [Har83], p. 91, Exercise II.3.7 there is a dense open subset $U \subseteq X$ such that $\bar{D}|_U \to U$ is finite. Since $\pi|_{\bar{D}} : \bar{D} \to X$ is proper (since it factors as a composition of a closed immersion and a proper morphism $\bar{D} \hookrightarrow \mathscr{C} \xrightarrow{\pi} X$), it suffices to show that it is quasi-finite. If there is an $x \in X$ such that $\pi|_{\bar{D}}^{-1}(x)$ is not finite, we must have $\pi|_{\bar{D}}^{-1}(x) = \mathscr{C}_x$, i. e. the whole curve as a fibre, since the fibres of π are irreducible. But then $\bar{D}$ would have more than one irreducible component, a contradiction. □

Corollary 3.1.22. *If in the situation of Corollary 3.1.20, π has a section $s : X \to \mathscr{C}$, one has split short exact sequences*

$$0 \longrightarrow \mathrm{H}^i(X, \mathbf{G}_m) \underset{\pi^*}{\overset{s^*}{\rightleftarrows}} \mathrm{H}^i(\mathscr{C}, \mathbf{G}_m) \longrightarrow \mathrm{H}^{i-1}(X, \mathrm{R}^1\pi_*\mathbf{G}_m) \longrightarrow 0$$

for $i \geq 1$.

In the general case, denote by C/K the generic fibre of $\mathscr{C}/X$, and assume that for every Weil divisor D in C, $\bar{D} \subseteq \mathscr{C}$ has everywhere the same dimension as X with no embedded components, and denote by δ the greatest common divisor of the degrees of Weil divisors on C/K, i. e. the index of C/K. Then one has an exact sequence

$$0 \to K_2 \to \mathrm{H}^2(X, \mathbf{G}_m) \xrightarrow{\pi^*} \mathrm{H}^2(\mathscr{C}, \mathbf{G}_m) \to \mathrm{H}^1(X, \mathrm{R}^1\pi_*\mathbf{G}_m) \to K_3 \to 0,$$

where $K_i = \ker(\mathrm{H}^i(X, \mathbf{G}_m) \xrightarrow{\pi^*} \mathrm{H}^i(\mathscr{C}, \mathbf{G}_m))$ are Abelian groups annihilated by δ whose prime-to-p torsion is finite.

Proof. The first assertion is obvious from the previous Corollary 3.1.20 and the existence of a section.

For the second claim, take an irreducible Weil divisor D in C. Then $\pi_{\bar{D}} : \bar{D} \to X$ is a finite morphism by Lemma 3.1.21. We have the commutative diagram

$$\begin{array}{ccc} \bar{D} & \xhookrightarrow{i} & \mathscr{C} \\ & \searrow^{\pi|_{\bar{D}}} & \downarrow \pi \\ & & X. \end{array}$$

By the Leray spectral sequence, we have that $\mathrm{H}^i(\bar{D}, \mathbf{G}_m) = \mathrm{H}^i(X, \pi|_{\bar{D},*}\mathbf{G}_m)$ as $\pi|_{\bar{D}}$ is finite, hence exact for the étale topology, see [Mil80], p. 72, Corollary II.3.6. If $\pi|_{\bar{D}}$ is also flat, by finite locally freeness we have a norm map $\pi|_{\bar{D},*}\mathbf{G}_m \to \mathbf{G}_m$ whose composite with the inclusion $\mathbf{G}_m \to \pi|_{\bar{D},*}\mathbf{G}_m$ is the δ-th power map. If not, there is still a norm map since f is flat in codimension 1 since $\bar{D}$ has everywhere the same dimension as X with no embedded components, so one can take the norm there, which then will land in $\mathbf{G}_m$ as X is normal.

We have $\pi|_{\bar{D}*} \circ i^* \circ \pi^* = \pi|_{\bar{D}*} \circ \pi|_{\bar{D}}^* = \deg(D)$, so $\ker(\pi^* : \mathrm{H}^*(X, \mathbf{G}_m) \to \mathrm{H}^*(\mathscr{C}, \mathbf{G}_m))$ is annihilated by $\deg(D)$ for all D, hence by the index δ. Now the finiteness of the prime-to-p part of the K_i follows from Theorem 3.1.8. □

In the following, assume π is smooth (automatically projective since $\mathscr{C}, X$ are projective over $\mathbf{F}_q$), with all geometric fibres integral and of dimension 1, and that it has a section $s : X \to \mathscr{C}$. Assume further that $\pi_*\mathscr{O}_\mathscr{C} = \mathscr{O}_X$ holds universally and π is cohomologically flat in dimension 0, e. g. if π is a flat proper morphism of locally Noetherian separated schemes with geometrically connected fibres (Lemma 2.2.2).

We recall some definitions from [FGI+05], p. 252, Definition 9.2.2.

Definition 3.1.23. *The* **relative Picard functor** $\mathrm{Pic}_{X/S}$ *on the category of (locally Noetherian) S-schemes is defined by*

$$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X \times_S T)/\,\mathrm{pr}_2^*\,\mathrm{Pic}(T).$$

Its associated sheaves in the Zariski, étale and fppf topology are denoted by

$$\mathrm{Pic}_{X/S,\mathrm{Zar}}, \quad \mathrm{Pic}_{X/S,\text{ét}}, \quad \mathrm{Pic}_{X/S,\mathrm{fppf}}\,.$$

Now we come to the representability of the relative Picard functor by a group scheme, whose connected component of unity is an Abelian scheme.

Theorem 3.1.24. $\mathrm{Pic}_{\mathscr{C}/X}$ *is represented by a separated smooth X-scheme $\mathbf{Pic}_{\mathscr{C}/X}$ locally of finite type.* $\mathrm{Pic}^0_{\mathscr{C}/X}$ *is represented by an Abelian X-scheme $\mathbf{Pic}^0_{\mathscr{C}/X}$. For every T/X,*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(\mathscr{C} \times_X T) \to \mathbf{Pic}_{\mathscr{C}/X}(T) \to 0$$

is exact.

Proof. Since π has a section $s : X \to \mathscr{C}$ and $\pi_*\mathscr{O}_\mathscr{C} = \mathscr{O}_X$ holds universally, by [FGI+05], p. 253, Theorem 9.2.5

$$\mathrm{Pic}_{\mathscr{C}/X} \xrightarrow{\sim} \mathrm{Pic}_{\mathscr{C}/X,\mathrm{Zar}} \xrightarrow{\sim} \mathrm{Pic}_{\mathscr{C}/X,\text{ét}} \xrightarrow{\sim} \mathrm{Pic}_{\mathscr{C}/X,\mathrm{fppf}}\,.$$

Since π is projective and flat with geometrically integral fibres, by [FGI+05] p. 263, Theorem 9.4.8, $\mathbf{Pic}_{\mathscr{C}/X}$ exists, is separated and locally of finite type over X and represents $\mathrm{Pic}_{\mathscr{C}/X,\text{ét}}$. Since X is Noetherian and $\mathscr{C}/X$ projective, by *loc. cit.*, $\mathbf{Pic}_{\mathscr{C}/X}$ is a disjoint union of open subschemes, each an increasing union of open quasi-projective X-schemes.

By [BLR90], p. 259 f., Proposition 4, $\mathbf{Pic}^0_{\mathscr{C}/X}/X$ is an Abelian scheme (this uses that $\mathscr{C}/X$ is a relative *curve* or an Abelian scheme).

The last assertion follows from [BLR90], p. 204, Proposition 4. □

Theorem 3.1.25. *Let $\pi : X \to S$ be a smooth proper morphism. Then $\mathrm{R}^1\pi_*\mathbf{G}_m = \mathbf{Pic}_{X/S}$, the higher direct image taken with respect to the fppf or étale topology.*

Proof. See [BLR90], p. 202 f. □

Theorem 3.1.26. *Let $\mathscr{A}/X$ be an Abelian scheme over a locally Noetherian, integral, geometrically unibranch (e. g., normal) base X. Then $\mathscr{A}/X$ is projective.*

Proof. See [Ray70], p. 161, Théorème XI 1.4. □

3.2 The weak Néron model

Theorem 3.2.1 (The weak Néron model)**.** *Let S be a regular, Noetherian, integral, separated scheme with $g : \{\eta\} \hookrightarrow S$ the inclusion of the generic point. Let X/S be a smooth projective variety with geometrically integral fibres that admits a section such that its Picard functor is representable (e. g., X/S a smooth projective curve admitting a section or an Abelian scheme). Then*

$$\mathbf{Pic}_{X/S} \xrightarrow{\sim} g_*g^*\mathbf{Pic}_{X/S}$$

as étale sheaves on S. Let $\mathscr{A}/X$ be an Abelian scheme. Then

$$\mathscr{A} \xrightarrow{\sim} g_*g^*\mathscr{A}$$

as étale sheaves on X.

We call this the **weak Néron mapping property** and $\mathscr{A}$ a **weak Néron model** of its generic fibre, while a Néron model of A/K in the usual sense is a model $\mathscr{A}/S$ which satisfies the Néron mapping property $\mathscr{A} \xrightarrow{\sim} g_*g^*\mathscr{A}$ for the *smooth* topology.

The main idea for injectivity is to use the separatedness of our schemes, and the main idea for surjectivity is that Weil divisors spread out and that the Picard group equals the Weil divisor class group by regularity.

Proof. Let $f : \mathbf{Pic}_{X/S} \to g_*g^*\mathbf{Pic}_{X/S}$ be the natural map of étale sheaves induced by adjointness. Let $\bar{s} \to S$ be a geometric point. We have to show that $(\operatorname{coker}(f))_{\bar{s}} = 0 = (\ker(f))_{\bar{s}}$.

Taking stalks and using Lemma 2.2.7, we get the following commutative diagram:

$$\begin{array}{ccccc}
\mathbf{Pic}_{X/S}(\mathscr{O}^{sh}_{S,s}) & \xrightarrow{f} & \mathbf{Pic}_{X/S}(\operatorname{Quot}(\mathscr{O}^{sh}_{S,s})) & \twoheadrightarrow & (\operatorname{coker}(f))_{\bar{s}} \\
\uparrow & & \uparrow & & \\
\operatorname{Pic}(X \times_S \mathscr{O}^{sh}_{S,s}) & \longrightarrow & \operatorname{Pic}(X \times_S \operatorname{Quot}(\mathscr{O}^{sh}_{S,s})) & & \\
\uparrow & & \uparrow & & \\
\operatorname{Div}(X \times_S \mathscr{O}^{sh}_{S,s}) & \longrightarrow & \operatorname{Div}(X \times_S \operatorname{Quot}(\mathscr{O}^{sh}_{S,s})) & &
\end{array}$$

Note that $\mathscr{O}^{sh}_{S,s}$ is a domain since it is regular as a strict Henselisation of a regular local ring by [Fu11], p. 111, Proposition 2.8.18. By Lemma 2.2.7, one has $(\mathbf{Pic}_{X/S})_{\bar{s}} = \mathbf{Pic}_{X/S}(\mathscr{O}^{sh}_{S,s})$ and $(g_*g^*\mathbf{Pic}_{X/S})_{\bar{s}} = \mathbf{Pic}_{X/S}(\mathscr{O}^{sh}_{S,s} \otimes_{\mathscr{O}_{S,s}} K(S))$. But for a regular local ring A, $A^{sh} \otimes_A \operatorname{Quot}(A) = \operatorname{Quot}(A^{sh})$:

Lemma 3.2.2. *Let A be a regular local ring. Then*

$$A^h \otimes_A \operatorname{Quot}(A) = \operatorname{Quot}(A^h),$$
$$A^{sh} \otimes_A \operatorname{Quot}(A) = \operatorname{Quot}(A^{sh}).$$

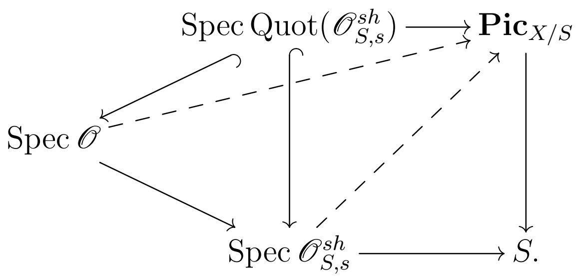
Proof. By [Mil80], p. 38, Remark 4.11, A^{sh} is the localisation at a maximal ideal lying over the maximal ideal $\mathfrak{m}$ of A of the integral closure of A in $(\operatorname{Quot}(A)^{\mathrm{sep}})^I$ with $I \subseteq \operatorname{Gal}(\operatorname{Quot}(A)^{\mathrm{sep}}/\operatorname{Quot}(A))$ the inertia subgroup. Pick an element $a \in A^{sh}$. It is a root of a monic polynomial $f(T) \in A[T]$, which we can assume to be monic irreducible since A is a regular local ring and hence factorial by [Mat86], p 163, Theorem 20.3, and hence $A[T]$ is factorial by [Mat86], p. 168, Exercise 20.2. Since A is factorial and $f \in A[T]$ is irreducible, by the lemma of Gauß [Bos03], p. 64, Korollar 6, f is also irreducible over $\operatorname{Quot}(A)$. Hence $A[a] \otimes_A \operatorname{Quot}(A) = \operatorname{Quot}(A)[T]/(f(T))$ is a field, and $A^{sh} \otimes_A \operatorname{Quot}(A)$ is a directed colimit of fields since tensor products commute with colimits, and hence itself a field, namely $\operatorname{Quot}(A^{sh})$. Now note that localisation does not change the quotient field.

The proof for A^h is the same: Just replace the inertia group by the decomposition group. □

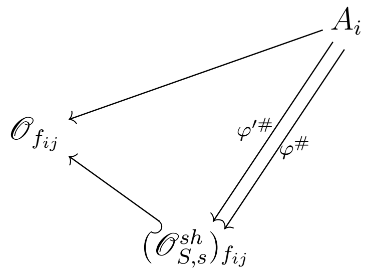
By [Har83], p. 145, Corollary II.6.16, one has a surjection from Div to Pic since S is Noetherian, integral, separated and locally factorial. By Theorem 3.1.24, the upper vertical arrows are surjective (under the assumption that X/S has a section)

But here, the lower horizontal map is surjective: A preimage under $\iota : X \times_S \operatorname{Quot}(\mathscr{O}_{S,s}^{sh}) \to X \times_S \mathscr{O}_{S,s}^{sh}$ of $D \in \operatorname{Div}(X \times_S \operatorname{Quot}(\mathscr{O}_{S,s}^{sh}))$ is $\bar{D} \in \operatorname{Div}(X \times_S \mathscr{O}_{S,s}^{sh})$, the closure taken in $X \times_S \mathscr{O}_{S,s}^{sh}$. In fact, note that D is closed in $X \times_S \operatorname{Quot}(\mathscr{O}_{S,s}^{sh})$ since it is a divisor; the closure of an irreducible subset is irreducible again, and the codimension is also 1 since the codimension is the dimension of the local ring at the generic point η_D of D, and the local ring of η_D in $X \times_S \operatorname{Quot}(\mathscr{O}_{S,s}^{sh})$ is the same as the local ring of η_D in $X \times_S \mathscr{O}_{S,s}^{sh}$ as it is the colimit of the global sections taken for all open neighbourhoods of η_D. Hence $(\operatorname{coker}(f))_{\bar{s}} = 0$.

For $(\ker(f))_{\bar{s}} = 0$, consider the diagram



We want to show that a lift $\operatorname{Spec} \mathscr{O}_{S,s}^{sh} \to \mathbf{Pic}_{X/S}$ of $\operatorname{Spec} \operatorname{Quot}(\mathscr{O}_{S,s}^{sh}) \to \mathbf{Pic}_{X/S}$ is unique. As $\mathbf{Pic}_{X/S}/S$ is separated, this is true for all valuation rings $\mathscr{O} \subset \operatorname{Quot}(\mathscr{O}_{S,s}^{sh})$ by the valuative criterion of separatedness [EGAII], p. 142, Proposition (7.2.3). But by [Mat86], p. 72, Theorem 10.2, every local ring $(\mathscr{O}_{S,s}^{sh})_{\mathfrak{p}}$ is dominated by a valuation ring $\mathscr{O}$ of $\operatorname{Quot}(\mathscr{O}_{S,s}^{sh})$. It follows from the valuative criterion for separatedness that the lift is topologically unique. Assume φ, φ' are two lifts. Now cover $\mathbf{Pic}_{X/S}/S$ by open affines $U_i = \operatorname{Spec} A_i$ and their preimages $\varphi^{-1}(U_i) = \varphi'^{-1}(U_i)$ by standard open affines $\{D(f_{ij})\}_j$.



It follows that $\varphi = \varphi'$.

Lemma 3.2.3. *Let S be a locally Noetherian scheme, $f : X \to S$ be a proper flat morphism and $\mathscr{E}$ a locally free sheaf on X. Then the Euler characteristic*

$$\chi_{\mathscr{E}}(s) = \sum_{i \geq 0} (-1)^i \dim \mathrm{H}^i(X_s, \mathscr{E}_s)$$

is locally constant on S.

Proof. See [EGAIII$_2$], p. 76 f., Théorème (7.9.4). □

For the statement about $\mathbf{Pic}^0_{X/S}$ for curves: Restricting the isomorphism to $\mathbf{Pic}^0$ gives a well-defined homomorphism since a line bundle being of degree 0 can be checked on geometric fibres by [Con], p. 3, Proposition 4.1 and [Con], p. 1, Theorem 2.2. Injectivity is immediate. Surjectivity follows since if $\mathscr{L}$ is a preimage in $\mathbf{Pic}$ of a line bundle lying in $\mathbf{Pic}^0$, it must already lie in $\mathbf{Pic}^0$ since, again, being of degree 0 can be checked fibrewise.

Now we prove the last statement of the theorem not only for curves, but also for Abelian schemes, so one can deduce the statement for Abelian varieties by noting that $\mathscr{A} = (\mathscr{A}^\vee)^\vee = \mathbf{Pic}^0_{\mathscr{A}^\vee/S}$.

We want to show that

$$\mathbf{Pic}^0_{X/S}(S') \to \mathbf{Pic}^0_{X/S}(S'_\eta) \tag{3.2.1}$$

is bijective for any étale S-scheme S'.[2]

First note that such an S' is regular, so its connected components are integral, and S'_η is the disjoint union of the generic points of the connected components of S', so we can replace $X \to S$ with the restrictions of the base change $X' \to S'$ over each connected component of S' separately to reduce to checking for $S' = S$.

For any section $S \to \mathbf{Pic}_{X/S}$, since S is connected and $\mathbf{Pic}^\tau_{X/S}$ is open and closed in $\mathbf{Pic}_{X/S}$ by [SGA6], p. 647 f., exp. XIII, Théorème 4.7, the preimage of $\mathbf{Pic}^\tau_{X/S}$ under the section is open and closed in S, hence empty or S. Thus, if even a single point of S is carried into $\mathbf{Pic}^\tau_{X/S}$ under the section, then the whole of S is. More generally, when using S'-valued points of $\mathbf{Pic}_{X/S}$ for any étale S-scheme S', such a point lands in $\mathbf{Pic}^\tau_{X/S}$ if and only if some point in each connected component of S' does, such as the generic point of each connected component S'_η.

This proves the statement in (3.2.1) for $\mathbf{Pic}^0_{X/S}$ replaced by $\mathbf{Pic}^\tau_{X/S}$, and hence for $\mathbf{Pic}^0_{X/S}$ in cases where it coincides with $\mathbf{Pic}^\tau_{X/S}$ (i. e., when the geometric fibers have component group for $\mathbf{Pic}_{X/S}$—i. e., Néron-Severi group—that is torsion-free, e. g. for Abelian schemes or curves, see Example 4.2.25 below). □

3.3 The Tate-Shafarevich group

Proposition 3.3.1. *Let X be integral and $\mathscr{A}/X$ be an Abelian scheme. Then $\mathrm{H}^i(X, \mathscr{A})$ is torsion for $i > 0$.*

Proof. Consider the Leray spectral sequence for the inclusion $g : \{\eta\} \hookrightarrow X$ of the generic point

$$\mathrm{H}^p(X, \mathrm{R}^q g_* g^* \mathscr{A}) \Rightarrow \mathrm{H}^{p+q}(\eta, g^* \mathscr{A}).$$

Calculation modulo the Serre subcategory of torsion sheaves, and exploiting the fact that Galois cohomology groups are torsion in dimension > 0 by Lemma 2.1.12, and therefore also the higher direct images $\mathrm{R}^q g_* g^* \mathscr{A}$ are torsion sheaves by [Mil80], p. 88, Proposition III.1.13 (the higher direct images are the sheaf associated with the presheaf "cohomology of the preimages"), the spectral sequence degenerates giving

$$\mathrm{H}^p(X, g_* g^* \mathscr{A}) = E_2^{p,0} = E^p = \mathrm{H}^p(\eta, g^* \mathscr{A}) = 0$$

[2]The following argument has been communicated to me by Brian Conrad.

for $p > 0$. Because of the weak Néron mapping property we have $\mathrm{H}^p(X, \mathscr{A}) \xrightarrow{\sim} \mathrm{H}^p(X, g_* g^* \mathscr{A})$, which finishes the proof. □

This gives a necessary condition for the existence of a weak Néron model:

Remark 3.3.2. In [MB], Moret-Bailly constructs, using [Ray70], XIII, A-torsors of infinite order over the affine line with two points identified, and even over normal two-dimensional schemes.

Definition 3.3.3. *Define the* **Tate-Shafarevich group** *of an Abelian scheme* $\mathscr{A}/X$ *by*

$$Ш(\mathscr{A}/X) = \mathrm{H}^1(X, \mathscr{A}).$$

Theorem 3.3.4. *Let* $\mathscr{A}/X$ *be an Abelian scheme. For* $x \in X$*, denote the quotient field of the strict Henselisation of* $\mathscr{O}_{X,x}$ *by* K_x^{nr}*, the inclusion of the generic point by* $j : \{\eta\} \hookrightarrow X$ *and let* $j_x : \mathrm{Spec}(K_x^{nr}) \hookrightarrow \mathrm{Spec}(\mathscr{O}_{X,x}^{sh}) \hookrightarrow X$ *be the composition. Then we have*

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\left(\mathrm{H}^1(K, j^*\mathscr{A}) \to \prod_{x \in X} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\right).$$

Proof. By Lemma 2.1.1, the Leray spectral sequence $\mathrm{H}^p(X, \mathrm{R}^q j_*(j^*\mathscr{A}))) \Rightarrow \mathrm{H}^{p+q}(K, j^*\mathscr{A})$ yields the exactness of $0 \to E_2^{1,0} \to E^1 \to E_2^{0,1}$, i. e.

$$\mathrm{H}^1(X, j_* j^*\mathscr{A}) = \ker\left(\mathrm{H}^1(K, j^*\mathscr{A}) \to \mathrm{H}^0(X, \mathrm{R}^1 j_*(j^*\mathscr{A}))\right).$$

Since

$$\mathrm{H}^0(X, \mathrm{R}^1 j_*(j^*\mathscr{A})) \to \prod_{x \in X} \mathrm{R}^1 j_*(j^*\mathscr{A})_{\bar{x}}$$

is injective ([Mil80], p. 60, Proposition II.2.10: If a section of an étale sheaf is non-zero, there is a geometric point for which the stalk of the section is non-zero) and

$$\mathrm{R}^1 j_*(j^*\mathscr{A})_{\bar{x}} = \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})$$

by Lemma 3.2.2, the theorem follows from Lemma 2.1.2 and the weak Néron mapping property $\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \mathrm{H}^1(X, j_* j^*\mathscr{A})$. □

Theorem 3.3.5 (The Tate-Shafarevich group)**.** *In the situation of Theorem 3.3.4, one can replace the product over all points by*

*(a) the codimension-*1 *points if one disregards the* p*-torsion (*$p = \mathrm{char}\, k$*) (for* $\dim X \leq 2$*, this also holds for the* p*-torsion), assuming* X/k *smooth projective and let* $\mathscr{A}/X$ *be an Abelian scheme such that the vanishing theorem Lemma 3.3.11 below is satisfied:*

$$\mathrm{H}^1(X, \mathscr{A}) = \ker\left(\mathrm{H}^1(K, j^*\mathscr{A}) \to \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\right),$$

or (b) the closed points

$$\mathrm{H}^1(X, \mathscr{A}) = \ker\left(\mathrm{H}^1(K, j^*\mathscr{A}) \to \bigoplus_{x \in |X|} \mathrm{H}^1(K_x^{nr}, j_x^*\mathscr{A})\right)$$

One can also replace K_x^{nr} *by the quotient field of the completion* $\hat{\mathscr{O}}_{X,x}^{sh}$ *in the case of* $x \in X^{(1)}$*.*

Proof. We first establish some vanishing results for étale cohomology with supports.

Lemma 3.3.6. *Let X/k be a smooth variety and $\pi : \mathscr{C} \to X$ a smooth proper relative curve which admits a section $s : X \to \mathscr{C}$. Let $Z \hookrightarrow X$ be a reduced closed subscheme of codimension ≥ 2. Assume $\dim X \leq 2$. Then*

$$\mathrm{H}^i_Z(X, \mathbf{Pic}_{\mathscr{C}/X}) = 0 \quad \text{for } i \leq 2.$$

For $\dim X > 2$, this holds at least up to p-torsion.

We use the following lemmata.

Lemma 3.3.7. *In the situation of the previous lemma, one has*

$$\mathrm{H}^i_Z(X, \mathbf{G}_m) = 0 \quad \text{for } i \leq 2,$$

and for $i = 3$ at least away from p.

Proof of Lemma 3.3.7. See [Gro68], p. 133 ff.: Using the local-to-global spectral sequence ([Gro68], p. 133, (6.2))

$$E_2^{p,q} = \mathrm{H}^p(X, \mathscr{H}^q_Z(\mathbf{G}_m)) \Rightarrow \mathrm{H}^{p+q}_Z(X, \mathbf{G}_m)$$

and [Gro68], p. 133–135

$$\begin{aligned} \mathscr{H}^0_Z(\mathbf{G}_m) &= 0 \quad && \text{[Gro68], p. 133, (6.3)} \\ \mathscr{H}^1_Z(\mathbf{G}_m) &= 0 \quad && \text{[Gro68], p. 133, (6.4) since the codimension of } Z \text{ in } X \text{ is} \neq 1 \\ \mathscr{H}^2_Z(\mathbf{G}_m) &= 0 \quad && \text{[Gro68], p. 134, (6.5)} \\ \mathscr{H}^3_Z(\mathbf{G}_m)^{(p')} &= 0 \quad && \text{[Gro68], p. 134 f., Thm. (6.1),} \end{aligned}$$

(even $\mathscr{H}^3_Z(\mathbf{G}_m) = 0$ for $\dim X = 2$; if not, we have to calculate modulo suitable Serre subcategories in the following), we have $E_2^{p,q} = 0$ for $q \leq 3$, and hence the result $E^n = 0$ for $0 \leq n \leq 3$ follows from the exact sequences

$$0 \to E_2^{n,0} \to E^n \to E_2^{0,n}$$

for $n = 1, 2, 3$. □

Proof of Lemma 3.3.6.

Lemma 3.3.8. *If $f : X \to Y$ is a flat morphism of locally Noetherian schemes and $Z \hookrightarrow Y$ is a closed immersion of codimension $\geq c$, then also the base change $Z' := Z \times_Y X \hookrightarrow X$ is a closed immersion of codimension $\geq c$.*

Proof. Without loss of generality, let $Z' = \overline{\{z'\}}$ be irreducible. Since all involved schemes are locally Noetherian and f is flat, by [GW10], p. 464, Corollary 14.95 ($f : X \to Y$, $y = f(x)$, then the codimension of $\overline{\{x\}}$ is $\geq$ the codimension of $\overline{\{f(x)\}}$), we have $\operatorname{codim}_X Z' \geq \operatorname{codim}_Y \overline{f(z')}$. But $\overline{f(z')} \subseteq \overline{Z} = Z$, so $\operatorname{codim}_Y \overline{f(z')} \geq \operatorname{codim}_Y Z \geq c$, hence the result. □

By Theorem 3.1.16 and the Leray spectral sequence with supports Theorem 2.2.4, we get a long exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2 \to E_2^{1,1} \to E^{3,0} \to E^3 \to E_2^{2,1} \to E^{4,0} \to E^4 \qquad (3.3.1)$$

by [Wei97], p. 124, Exercise 5.2.2 (spectral sequence with two rows; here for a cohomological spectral sequence). But $E_2^{i,0} = \mathrm{H}^i_Z(X, \mathbf{G}_m) = 0$ for $i \leq 3$ by Lemma 3.3.7 (for $i = 3$ at least away from p). Therefore the long exact sequence (3.3.1) yields isomorphisms

$$E^i \to E_2^{i-1,1}$$

for $i \leq 2$, but $E^i = \mathrm{H}^i_{Z'}(\mathscr{C}, \mathbf{G}_m) = 0$ for $i \leq 3$, again by Lemma 3.3.7 and Lemma 3.3.8, hence $\mathrm{H}^i_Z(X, \mathrm{R}^1\pi_*\mathbf{G}_m) = E_2^{i-1,1} = 0$ for $i \leq 2$ from (3.3.1). For the vanishing of $E_2^{2,1}$ note that

$$0 = E^3 \to E_2^{2,1} \to E^{4,0} \to E^4$$

is exact by (3.3.1), but by Lemma 2.2.3 the latter map is $\pi^* : \mathrm{H}^4_Z(X, \mathbf{G}_m) \hookrightarrow \mathrm{H}^4_{Z'}(\mathscr{C}, \mathbf{G}_m)$, which is injective as π admits a section. □

Now for the proof of Theorem 3.3.5, at least for prime-to-p torsion:

First note that the map $\mathrm{H}^1(X, \mathscr{A}) \to \ker(\ldots)$ in Theorem 3.3.5 is well-defined since if $x \in \mathrm{H}^1(X, \mathscr{A})$ is restricted to $\mathrm{H}^1(K, \mathscr{A})$ via $g : \{\eta\} \hookrightarrow X$, its pullback to $K_x^{nr} = \mathrm{Spec}(\mathrm{Quot}(\mathscr{O}_{X,x}^{sh}))$ factors as

$$\mathrm{H}^1(X, \mathscr{A}) \to \mathrm{H}^1(\mathrm{Spec}(\mathscr{O}_{X,x}^{sh}), \mathscr{A}) \to \mathrm{H}^1(\mathrm{Spec}(\mathrm{Quot}(\mathscr{O}_{X,x}^{sh})), \mathscr{A}),$$

but the étale site of $\mathrm{Spec}(\mathscr{O}_{X,x}^{sh})$ is trivial:

Lemma 3.3.9. *Let $S = \mathrm{Spec}(R)$ be the spectrum of a strictly Henselian local ring. Then the global sections functor on the étale site for Abelian sheaves is exact, so all cohomology groups in positive dimension vanish.*

Proof. See [Tam94], p. 124, Lemma (6.2.3). □

Lemma 3.3.10. *Let X be a normal scheme and $\mathscr{C}/X$ a smooth proper relative curve. Then there is an exact sequence*

$$0 \to \mathbf{Pic}^0_{\mathscr{C}/X} \to \mathbf{Pic}_{\mathscr{C}/X} \to \mathbf{Z} \to 0.$$

Proof. Without loss of generality, assume X connected (as all schemes are of finite type over a field, so there are only finitely many connected components all of which are open: Every connected component is closed, and they are finite in number, so they are open). Let $g = 1 - \chi_{\mathscr{O}_{\mathscr{C}}}$ be the genus of $\mathscr{C}/X$ (well-defined because of Lemma 3.2.3). Consider

$$\deg : \mathbf{Pic}_{\mathscr{C}/X} \to \mathbf{Z},$$
$$\mathrm{Pic}(\mathscr{C} \times_X Y)/\,\mathrm{pr}_2^* \mathrm{Pic}(Y) \ni \mathscr{L} \mapsto \chi_{\mathscr{L}}(y) - (1-g), \quad y \in Y;$$

this is a well-defined morphism of Abelian sheaves on the small étale site of X because of Lemma 3.2.3.

Now the statement follows from [Con], p. 3 f., Proposition 4.1 and Theorem 4.4. □

Lemma 3.3.11. *Let X/k be a smooth variety and $\mathscr{C}/X$ a smooth proper relative curve. Assume* $\dim X \leq 2$. *Let $Z \hookrightarrow X$ be a reduced closed subscheme of codimension ≥ 2. Then*

$$\mathrm{H}^i_Z(X, \mathbf{Pic}^0_{\mathscr{C}/X}) = 0 \quad \text{for } i \leq 2.$$

If $\dim X > 2$, *this holds at least up to p-torsion.*

Proof. Taking the long exact sequence associated to the short exact sequence of Lemma 3.3.10 with respect to $\mathrm{H}^i_Z(X, -)$, by Lemma 3.3.6, it suffices to show that $\mathrm{H}^i_Z(X, \mathbf{Z}) = 0$ for $i = 0, 1, 2$.

For this, consider the long exact sequence

$$\ldots \to \mathrm{H}^i_Z(X, \mathbf{Z}) \to \mathrm{H}^i(X, \mathbf{Z}) \to \mathrm{H}^i(X \setminus Z, \mathbf{Z}) \to \ldots$$

It suffices to show that $\mathrm{H}^i(X, \mathbf{Z}) \to \mathrm{H}^i(X \setminus Z, \mathbf{Z})$ is an isomorphism for $i = 0, 1$ and an injection for $i = 2$.

Lemma 3.3.12. *Let X be geometrically unibranch (e. g. normal) and $i : U \hookrightarrow X$ dominant. Then for any constant sheaf A, one has $A \xrightarrow{\sim} i_* i^* A$ as étale sheaves.*

Proof. See [SGA4.3], p. 25 f., IX Lemme 2.14.1. □

Lemma 3.3.13. *Let X be a connected normal Noetherian scheme with generic point η. Then* $\mathrm{H}^p(X, \mathbf{Q}) = 0$ *for all $p > 0$ and* $\mathrm{H}^1(X, \mathbf{Z}) = 0$.

Proof. Denote the inclusion of the generic point by $g : \{\eta\} \hookrightarrow X$. Since X is connected normal, by Lemma 3.3.12

$$A \xrightarrow{\sim} g_* g^* A.$$

As $\mathrm{R}^q g_*(g^* \mathbf{Q}) = 0$ for $q > 0$ (since Galois cohomology is torsion and $\mathbf{Q}$ is uniquely divisible; then use [Mil80], p. 88, Proposition III.1.13), the Leray spectral sequence

$$\mathrm{H}^p(X, \mathrm{R}^q g_* g^* \mathbf{Q}) \Rightarrow \mathrm{H}^{p+q}(\eta, \mathbf{Q})$$

degenerates to $\mathrm{H}^p(X, \mathbf{Q}) = \mathrm{H}^p(\eta, \mathbf{Q})$, which is trivial as, again, Galois cohomology is torsion and $\mathbf{Q}$ is uniquely divisible.

Similarly, as $\mathrm{R}^1 g_*(g^* \mathbf{Z}) = 0$ (since $\mathbf{Z}$ carries the trivial Galois action and homomorphisms from profinite groups to discrete groups have finite image, and $\mathbf{Z}$ has no nontrivial finite subgroups; again, use [Mil80], p. 88, Proposition 1.13), the Leray spectral sequence gives $\mathrm{H}^1(X, \mathbf{Z}) = \mathrm{H}^1(\eta, \mathbf{Z}) = 0$. □

Remark 3.3.14. If X is not normal, $\mathrm{H}^i(X, \mathbf{Z}) \neq 0$ in general:[3] Consider $X = \mathrm{Spec}(k[X, Y]/(Y^2 - (X^3 + X^2)))$, the normalisation morphism $\pi : \mathbf{A}^1_k \to X$ and the inclusion $i : \{x\} \hookrightarrow X$ of $x = (0, 0)$. There is a short exact sequence of étale sheaves on X

$$0 \to \mathbf{Z}_X \to \pi_* \mathbf{Z}_{\mathbf{A}^1_k} \to i_* \mathbf{Z}_{\{x\}} \to 0.$$

Taking the long exact cohomology sequence yields $\mathrm{H}^1(X, \mathbf{Z}_X) = \mathbf{Z}$.

[3] http://mathoverflow.net/questions/84414/etale-cohomology-with-coefficients-in-the-integers

For $i = 0$, this map is $\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}$.

For $i = 1$, both groups are equal to 0 by Lemma 3.3.13.

For $i = 2$, this map is $\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2(X \setminus Z, \mathbf{Z})$. Consider the long exact sequence associated to

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

Proposition 3.3.15. *Let X be a connected Noetherian scheme and $\bar{x}$ a geometric point. Then*

$$\mathrm{H}^1(X, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}_{cont}(\pi_1^{\text{ét}}(X, \bar{x}), \mathbf{Q}/\mathbf{Z})$$

and

$$\mathrm{H}^1(X, \mathbf{Z}_\ell) = \mathrm{Hom}_{cont}(\pi_1^{\text{ét}}(X, \bar{x}), \mathbf{Z}_\ell).$$

Proof. This follows from [Fu11], p. 245, Proposition 5.7.20 (for a connected Noetherian scheme X and a *finite* group G, one has $\mathrm{H}^1(X, G) = \mathrm{Hom}_{cont}(\pi_1^{\text{ét}}(X, \bar{x}), G)$) by passing to the colimit over $G_n = \frac{1}{n}\mathbf{Z}/\mathbf{Z}$, or passing to the limit over $G_n = \mathbf{Z}/\ell^n$, respectively. □

In the following, we omit the base point. Now, X being normal because smooth over a field, $\mathrm{H}^2(X, \mathbf{Z}) = \mathrm{H}^1(X, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}_{cont}(\pi_1^{\text{ét}}(X), \mathbf{Q}/\mathbf{Z})$. (For the first equality, use the long exact sequence associated to $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ and $\mathrm{H}^p(X, \mathbf{Q}) = 0$ for $p > 0$ by Lemma 3.3.13. For the second equality use Proposition 3.3.15.) Since $\pi_1^{\text{ét}}(X \setminus Z) \to \pi_1^{\text{ét}}(X)$ is an isomorphism because Z is of codimension ≥ 2 (by Theorem 2.2.9), $\mathrm{H}^2(X, \mathbf{Z}) \to \mathrm{H}^2(X \setminus Z, \mathbf{Z})$ is an isomorphism:

$$\begin{array}{ccc} \mathrm{H}^2(X, \mathbf{Z}) & \xrightarrow{\cong} & \mathrm{Hom}_{cont}(\pi_1^{\text{ét}}(X), \mathbf{Q}/\mathbf{Z}) \\ \downarrow & & \downarrow \cong \\ \mathrm{H}^2(X \setminus Z, \mathbf{Z}) & \xrightarrow{\cong} & \mathrm{Hom}_{cont}(\pi_1^{\text{ét}}(X \setminus Z), \mathbf{Q}/\mathbf{Z}) \end{array}$$

□

Lemma 3.3.16. *We have*

$$\varinjlim_U \mathrm{H}^p(U, \mathscr{F}|_U) \xrightarrow{\sim} \mathrm{H}^p(K, \mathscr{F}_\eta),$$

the colimit with respect to the restriction maps, where U runs through the non-empty standard affine open subschemes $D(f_i)$ of an non-empty affine open subscheme $\mathrm{Spec}(A) \subseteq X$.

Proof. Corollary to Lemma 2.2.5. □

Let $\emptyset \neq U \hookrightarrow X$ be open with reduced closed complement $Y \hookrightarrow X$. Then one has the long exact localisation sequence [Mil80], p. 92, Proposition III.1.25

$$0 \to \mathrm{H}^0_Y(X, \mathscr{A}) \to \mathrm{H}^0(X, \mathscr{A}) \to \mathrm{H}^0(U, \mathscr{A}) \to \mathrm{H}^1_Y(X, \mathscr{A}) \to \mathrm{H}^1(X, \mathscr{A}) \to \mathrm{H}^1(U, \mathscr{A}) \to \dots.$$

Because of the injectivity of $\mathrm{H}^0(X, \mathscr{A}) \hookrightarrow \mathrm{H}^0(U, \mathscr{A})$ (If two sections of $\mathscr{A}/X$ coincide on U open dense, they agree on X since $\mathscr{A}$ is separated and X reduced), the exactness of the sequence yields $\mathrm{H}^0_Y(X, \mathscr{A}) = 0$, and hence a short exact sequence

$$0 \to \mathrm{H}^1_Y(X, \mathscr{A})/\mathscr{A}(U) \to \mathrm{H}^1(X, \mathscr{A}) \to \ker\big(\mathrm{H}^1(U, \mathscr{A}) \to \mathrm{H}^2_Y(X, \mathscr{A})\big) \to 0. \tag{3.3.2}$$

Lemma 3.3.17 (Excision of codimension ≥ 2 subschemes)**.** *One can excise subschemes $Z \hookrightarrow Y$ of codimension ≥ 2 in X:*

$$\ker\left(\mathrm{H}^1(U,\mathscr{A}) \to \mathrm{H}^2_Y(X,\mathscr{A})\right) = \ker\left(\mathrm{H}^1(U,\mathscr{A}) \to \mathrm{H}^2_{Y\setminus Z}(X\setminus Z,\mathscr{A}|_{X\setminus Z})\right). \tag{3.3.3}$$

Proof. From the long exact localisation sequence for cohomology with supports [Mil80], p. 92, Remark III.1.26, and from Lemma 3.3.11 one gets the injectivity

$$0 \to \mathrm{H}^2_Y(X,\mathscr{A}) \hookrightarrow \mathrm{H}^2_{Y\setminus Z}(X\setminus Z,\mathscr{A}|_{X\setminus Z}), \tag{3.3.4}$$

hence by Lemma 2.1.2 the claim. □

Lemma 3.3.18. *Let $Y \hookrightarrow X$ be a closed subscheme with open complement $U = X\setminus Y$ and with all of its irreducible components of codimension 1 in X. Denote the finitely many irreducible components of Y by $(Y_i)_{i=1}^n$. Then*

$$\ker\left(\mathrm{H}^1(U,\mathscr{A}) \to \mathrm{H}^2_Y(X,\mathscr{A})\right) = \ker\left(\mathrm{H}^1(U,\mathscr{A}) \to \bigoplus_{i=1}^n \mathrm{H}^2_{Y_i\setminus Z_i}(X\setminus Z_i,\mathscr{A})\right) \tag{3.3.5}$$

with certain closed subschemes $Z_i \hookrightarrow Y_i$.

Proof. Excise the intersections $Y_i \cap Y_j$ for $i \neq j$ (they are of codimension ≥ 2 in X since our schemes are catenary as they are varieties by [Liu06], p. 338, Corollary 8.2.16 and

$$2 = 1 + 1 \leq \operatorname{codim}(Y_i \cap Y_j \hookrightarrow Y) + \operatorname{codim}(Y \hookrightarrow X) = \operatorname{codim}(Y_i \cap Y_j \hookrightarrow X)).$$

Now, by a repeated application of the Mayer-Vietoris sequence with supports Theorem 2.2.6, one gets the claim. □

Lemma 3.3.19. *One has*

$$\ker\left(\mathrm{H}^1(U,\mathscr{A}) \to \mathrm{H}^2_Y(X,\mathscr{A})\right) = \ker\left(\mathrm{H}^1(U,\mathscr{A}) \to \bigoplus_{i=1}^n \mathrm{H}^2_{\{x_i\}}(X\setminus \tilde{Z}_i,\mathscr{A})\right) \tag{3.3.6}$$

with the $x_i \in X \setminus \tilde{Z}_i$ closed points and generic points of the Y_i and $\tilde{Z}_i \hookrightarrow X$ certain subschemes.

Proof. This follows basically by excising (using (3.3.3)) everything except the generic points of the Y_i in (3.3.5). The only technical difficulty is that for applying Lemma 2.2.5 one has to make sure that the transition maps are affine.

Fix an irreducible component Y_i of Y and call it Y with x its generic point. We have to construct an injection

$$\mathrm{H}^2_{Y\setminus Z}(X\setminus Z,\mathscr{A}) \hookrightarrow \mathrm{H}^2_{\{x\}}(X\setminus \tilde{Z},\mathscr{A}). \tag{3.3.7}$$

Lemma 3.3.20. *Let X be a regular variety over a finite field and $x \in X^{(1)}$. Then there is an open affine subscheme* $\operatorname{Spec} A = X_0 \subseteq X$ *containing x such that A is a unique factorisation domain.*

Proof. The class group $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ (using that X is regular) is finitely generated since $\mathrm{NS}(X)$ is so by [Mil80], p. 215, Theorem IV.3.25 and $\mathrm{Pic}^0(X)$ is the group of rational points of the Picard scheme and the ground field is finite. Excise the finite set of generators of the Picard group using [Har83], p. 133, Proposition II.6.5. (If $\overline{\{x\}}$ is one of the generators, replace it by a moving lemma.) Then the rest of the variety has trivial class group. Now take an open affine subset of the remaining scheme containing x. It is normal and has trivial class group, so by [Har83], p. 131, Proposition II.6.2 it is a unique factorisation domain. □

Using Lemma 3.3.20, choose an affine open $X_0 := \operatorname{Spec} A \subset X \setminus Z$ containing the point x such that A is a unique factorisation domain. One has $\mathrm{H}^q_Y(X, \mathscr{A}) \hookrightarrow \mathrm{H}^q_{Y\cap X_0}(X_0, \mathscr{A})$: $X \setminus X_0$ is a closed subscheme $V \hookrightarrow X$ such that $V \cap Y \hookrightarrow Y$ is of codimension ≥ 1 and hence (since varieties are catenary by [Liu06], p. 338, Corollary 2.16) $V \cap Y \hookrightarrow X$ of codimension ≥ 2, so we conclude by excision (3.3.3). We construct a sequence $(X_i)_{i=0}^\infty$ of standard affine open subsets $X_i = D(f_i) \subset X_0$, $X_{i+1} \subset X_i$, all of them containing the point x such that

$$\mathrm{H}^q_{Y\cap X_i}(X_i, \mathscr{A}) \hookrightarrow \mathrm{H}^q_{Y\cap X_{i+1}}(X_{i+1}, \mathscr{A}), \quad \text{and thus by (3.3.3)}$$
$$\ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \mathrm{H}^2_Y(X, \mathscr{A})\right) = \ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \mathrm{H}^2_{Y\cap X_i}(X_i, \mathscr{A})\right),$$

and such that $\varprojlim_j X_i \cap Y = \{x\}$. Then (3.3.7) follows from Lemma 2.2.5.

Construction of the $(X_i)_{i=0}^\infty$. Since X is countable, one can choose an enumeration $(Z_i)_{i=1}^\infty$ of the closed integral subschemes of codimension 1 of X_0 not equal to $X_0 \cap Y$.

Given $X_i = D(f_i)$, take $f \in \operatorname{Spec} A_{f_i}$ such that $V(f) = Z_{i+1}$. (By the converse of Krull's Hauptidealsatz [Eis95], p. 233, Corollary 10.5, since Z_{i+1} is of codimension 1 in X_0, there is an $f \in A$ such that $Z_{i+1} \subseteq V(f)$ is minimal. Since X_0 is a unique factorisation domain, one can assume f prime, hence $Z_{i+1} = V(f)$ by codimension reasons.) Then $V(f) \cap (X^{(1)} \cap Y) = \emptyset$ and $V(f) \cap Y \subsetneq Y$ has codimension ≥ 1 in Y, hence (again, varieties being catenary) $V(f) \cap Y$ has codimension ≥ 2 in X. Therefore one can apply (3.3.4) to yield an injection

$$\mathrm{H}^2_{Y\cap X_i}(X_i, \mathscr{A}) \hookrightarrow \mathrm{H}^2_{(Y\cap X_i)\setminus V(f)}(X_i \setminus (Y \cap V(f)), \mathscr{A}).$$

Now, by excision ([Mil80], p. 92, Proposition III.1.27) of $V(f)$, one has

$$\mathrm{H}^2_{(Y\cap X_i)\setminus V(f)}(X_i \setminus (Y \cap V(f)), \mathscr{A}) \xrightarrow{\sim} \mathrm{H}^2_{(Y\cap X_i)\setminus V(f)}(X_i \setminus V(f), \mathscr{A}).$$

Set $X_{i+1} = X_i \setminus V(f) = D(f_i f)$, $f_{i+1} = f_i f$.

Apply this to the direct summands in (3.3.5). □

Conclusion of the proof I, surjectivity. Applying excision in the form of [Mil80], p. 93, Corollary III.1.28 to (3.3.6) (using that the $x_i \in X \setminus \tilde{Z}_i$ are closed points), one gets

$$\ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \mathrm{H}^2_Y(X, \mathscr{A})\right) = \ker\left(\mathrm{H}^1(U, \mathscr{A}) \overset{(r_i)}{\to} \bigoplus_{i=1}^n \mathrm{H}^2_{\{x_i\}}(X^h_{x_i}, \mathscr{A})\right).$$

Now (r_i) factors as

$$r_i : \mathrm{H}^1(U, \mathscr{A}) \overset{j^*_{x_i}}{\to} \mathrm{H}^1(K^h_{x_i}, \mathscr{A}) \overset{\delta_i}{\to} \mathrm{H}^2_{\{x_i\}}(X^h_{x_i}, \mathscr{A}),$$

where the latter map is the boundary map of the localisation sequence associated to the discrete valuation ring $\mathscr{O}^h_{X,x_i}$ (X is normal as it is smooth over a field, x_i is a codimension-1 point, and the Henselisation of a normal ring is normal again by [Fu11], p. 106, Proposition 2.8.10, and normal rings are (R_1))

$$X^h_{x_i} = \mathrm{Spec}(\mathrm{Quot}(\mathscr{O}^h_{X,x_i})) \cup \{x_i\}$$

(x_i is the closed point [Henselisation preserves residue fields], and $\mathrm{Spec}(K^h_{x_i})$ is the generic point of $X^h_{x_i}$). One has $\mathrm{H}^1(X^h_{x_i}, \mathscr{A}) = \mathrm{H}^1(\kappa(x_i), \mathscr{A})$ and the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(X^h_{x_i}, \mathscr{A}) \overset{\mathrm{inf}}{\to} \mathrm{H}^1(K^h_x, \mathscr{A}) \overset{\mathrm{res}}{\to} \mathrm{H}^1(K^{nr}_x, \mathscr{A}).$$

Since $\mathscr{O}^{sh}_{X,x_i}$ is a discrete valuation ring (as above; the strict Henselisation of a normal ring is normal again by [Fu11], p. 111, Proposition 2.8.18), the valuative criterion of properness [EGAII], p. 144 f., Théorème (7.3.8) gives us $\mathscr{A}(K^{nr}_{x_i}) = \mathscr{A}(X^{sh}_{x_i})$. Hence one can write $j_i^* = \mathrm{inf}$ as the inflation

$$\mathrm{H}^1(\kappa(x_i), \mathscr{A}(X^{sh}_{x_i})) \hookrightarrow \mathrm{H}^1(K^h_{x_i}, \mathscr{A}),$$

so the cokernel of $j^*_{x_i}$ injects into $\mathrm{H}^1(K^{nr}_{x_i}, \mathscr{A})$ and in $\mathrm{H}^2_{\{x_i\}}(X^h_{x_i}, \mathscr{A})$, so we get

$$\ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \bigoplus_{i=1}^n \mathrm{coker}(j_{x_i})\right) \xrightarrow{\sim} \ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \bigoplus_{i=1}^n \mathrm{H}^2_{\{x_i\}}(X^h_{x_i}, \mathscr{A})\right)$$
$$\ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \bigoplus_{i=1}^n \mathrm{coker}(j_{x_i})\right) \xrightarrow{\sim} \ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \bigoplus_{i=1}^n \mathrm{H}^1(K^{nr}_{x_i}, \mathscr{A})\right)$$

and hence

$$\ker\left(\mathrm{H}^1(U, \mathscr{A}) \to \mathrm{H}^2_Y(X, \mathscr{A})\right) = \ker\left(\mathrm{H}^1(U, \mathscr{A}) \overset{(r_i)}{\to} \bigoplus_{i=1}^n \mathrm{H}^1(K^{nr}_{x_i}, \mathscr{A})\right).$$

Conclusion of the proof II, surjectivity. Taking the limit over all Y (choose an enumeration $(Y_i)_{i=1}^\infty$ of the integral closed subschemes of codimension 1 of X [X is countable]) yields by Lemma 3.3.16 and the fact that (**Ab**) satisfies (AB5)

$$\ker\left(\mathrm{H}^1(K, \mathscr{A}) \to \varinjlim_{Y} \mathrm{H}^2_Y(X, \mathscr{A})\right) = \ker\left(\mathrm{H}^1(K, \mathscr{A}) \to \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K^{nr}_x, \mathscr{A})\right).$$

Now (3.3.2) gives us

$$0 \to \varinjlim \mathrm{H}^1_Z(X, \mathscr{A})/\mathscr{A}(U) \to \mathrm{H}^1(X, \mathscr{A}) \to \ker\left(\mathrm{H}^1(K, \mathscr{A}) \to \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K^{nr}_x, \mathscr{A})\right) \to 0.$$

Injectivity. But the latter, surjective, map factors as (with the isomorphism from Theorem 3.3.4)

$$\mathrm{H}^1(X, \mathscr{A}) \xrightarrow{\sim} \ker\left(\mathrm{H}^1(K, \mathscr{A}) \to \prod_{x \in X} \mathrm{H}^1(K_x^{nr}, \mathscr{A})\right)$$
$$\hookrightarrow \ker\left(\mathrm{H}^1(K, \mathscr{A}) \to \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(K_x^{nr}, \mathscr{A})\right),$$

so the claim follows.

Now one has to check that the isomorphism is in fact the one induced by the natural maps. We skip this. □

Alternative proof of Theorem 3.3.5. The injectivity: For $g : \{\eta\} \hookrightarrow X$ the inclusion of the generic point, consider the Leray spectral sequence

$$\mathrm{H}^p(X, \mathrm{R}^q g_* g^* \mathscr{A}) \Rightarrow \mathrm{H}^{p+q}(K, g^* \mathscr{A}).$$

Because of $\mathscr{A} \xrightarrow{\sim} g_* g^* \mathscr{A}$ as étale sheaves (the weak Néron mapping property for $\mathscr{A} = \mathbf{Pic}^0_{\mathscr{C}/X}$), the low term exact sequence yields the exactness of

$$0 \to \mathrm{H}^1(U, \mathscr{A}) \hookrightarrow \mathrm{H}^1(K, g^* \mathscr{A}) \to \mathrm{H}^0(U, \mathrm{R}^1 g_* g^* \mathscr{A}) \tag{3.3.8}$$

for all $\emptyset \neq U \hookrightarrow X$ open. Therefore, the map in question is injective. Since Galois cohomology is torsion in positive degrees, $\mathrm{H}^1(U, \mathscr{A})$ is also torsion.

The surjectivity (we only give a very rough sketch since this is not needed): Now let $\alpha \in \ker(\mathrm{H}^1(K, \mathscr{A}) \to \oplus_{x \in X^{(1)}} \mathrm{H}^1(K_x^{nr}, \mathscr{A}))$. Then for a suitable $U \hookrightarrow X$ containing all codimension-1-points of X, α maps to 0 in $\mathrm{H}^0(U, \mathrm{R}^1 g_* g^* \mathscr{A})$ in (3.3.8). (We claim this here without proof.) Therefore it comes from an $\tilde{\alpha} \in \mathrm{H}^1(U, \mathscr{A})$.

So take $\tilde{\alpha} \in \mathrm{H}^1_{\text{ét}}(U, \mathscr{A})$. Since $\mathscr{A}/X$ is a smooth, projective (by Theorem 3.1.26), commutative group scheme, by [Mil80], p. 114, Theorem III.3.9, the canonical map $\mathrm{H}^1_{\text{ét}}(U, \mathscr{A}) \to \mathrm{H}^1_{\text{fl}}(U, \mathscr{A})$ is an isomorphism. Since $\mathrm{H}^2_{\text{ét}}(U, \mathscr{A})$ is torsion, there is an $n > 1$ such that $n\tilde{\alpha} = 0$. By the long exact cohomology sequence associated to the exact sequence

$$0 \to \mathscr{A}[n] \to \mathscr{A} \xrightarrow{n} \mathscr{A} \to 0$$

of flat sheaves, $\tilde{\alpha}$ comes from an element of

$$\mathrm{H}^1_{\text{fl}}(U, \mathscr{A}[n]) \twoheadrightarrow \mathrm{H}^1_{\text{fl}}(U, \mathscr{A})[n].$$

Since $\mathscr{A}[n]/X$ is finite ($n > 1$), hence affine, by [Mil80], p. 121, Theorem III.4.3, a sheaf on X_{fl} that is an $\mathscr{A}[n]$-torsor is representable.

Now, the $\mathscr{A}[n]/U$-torsor $\tilde{\alpha}$ spreads out to a $\mathscr{A}[n]/X$-torsor $\bar{\tilde{\alpha}}$: Since X is regular, this follows from [MB85b], p. 490, Lemme 2, which is stated there without proof. Hence α comes from $\mathrm{H}^1_{\text{ét}}(X, \mathscr{A})$, so the map in question is surjective. □

Remark 3.3.21. For generalising the previous Theorem 3.3.5 from Jacobians to arbitrary Abelian schemes, it would be desirable to have a generalisation of the crucial Lemma 3.3.11, which depends on Theorem 3.1.16. It seems that with the different approach of the second proof this could be circumvented.

Closed points suffice in Theorem 3.3.5. In the proof of Theorem 3.3.4, even

$$\mathrm{H}^0(X, \mathrm{R}^1 j_*(j^* \mathscr{A})) \to \prod_{x \in |X|} \mathrm{R}^1 j_*(j^* \mathscr{A})_{\bar{x}}$$

is injective by [Mil80], p. 65, Remark II.2.17 (b): If a section of an étale sheaf is non-zero, there is a *closed* geometric point for which the stalk of the section is non-zero. This is because X/k is a variety and hence Jacobson. □

One can replace the strict Henselisation $\mathscr{O}_{X,x}^{sh}$ of $\mathscr{O}_{X,x}$ by its completion $\hat{\mathscr{O}}_{X,x}^{sh}$ (respectively by their quotient fields) in the case of $x \in X^{(1)}$: (The following is a generalisation of [Mil68], p. 99 to higher dimensions, replacing Greenberg's approximation theorem by Artin approximation.)

Lemma 3.3.22. *Let $(A, \mathfrak{m})$ be a discrete valuation ring of a variety. Let Z be a smooth proper scheme of finite type over A^{sh}. The following are equivalent:*

1. *Z has a point over* $\mathrm{Quot}(A^{sh})$.
2. *Z has a point over A^{sh}.*
3. *Z has points over $A^{sh}/\mathfrak{m}^n A^{sh} = \hat{A}^{sh}/\mathfrak{m}^n \hat{A}^{sh}$ for all $n \gg 0$.*
4. *Z has a point over $\hat{A}^{sh}$.*
5. *Z has a point over* $\mathrm{Quot}(\hat{A}^{sh})$.

In 3, the equality $A^{sh}/\mathfrak{m}^n A^{sh} = \hat{A}^{sh}/\mathfrak{m}^n \hat{A}^{sh}$ holds by [Eis95], p. 183, Theorem 7.1.

Proof. One has 1 $\iff$ 2 and 4 $\iff$ 5 by the valuative criterion for properness [EGAII], p. 144 f., Théorème (7.3.8), note that we are in codimension 1.

2 $\implies$ 3 and 4 $\implies$ 3 are trivial.

3 $\implies$ 4 is trivial since smooth implies formally smooth [EGAIV$_4$], Définition (17.1.1).

4 $\implies$ 2 follows from Artin approximation, see Theorem 3.1.15, applied to the functor $T \mapsto Z(T)$ locally of finite presentation. □

3.4 Appendix to the proof of Lemma 3.3.6

We give here an alternative, more geometric than cohomological proof which works only for $i \leq 1$. We will only sketch it since we do not need this: Let $\mathscr{A}/X$ be an Abelian scheme over X/k smooth. We have the long exact localisation sequence

$$\ldots \to \mathrm{H}^i_Z(X, \mathscr{A}) \to \mathrm{H}^i(X, \mathscr{A}) \to \mathrm{H}^i(X \setminus Z, \mathscr{A}) \to \mathrm{H}^{i+1}_Z(X, \mathscr{A}) \to \ldots,$$

so for $i = 0$, the claim $\mathrm{H}^i_Z(X, \mathscr{A}) = 0$ is equivalent to the injectivity of

$$\mathrm{H}^0(X, \mathscr{A}) \to \mathrm{H}^0(X \setminus Z, \mathscr{A}),$$

which is clear from [Har83], p. 105, Exercise II.4.2 since $\mathscr{A}/X$ is separated, X is reduced and $X \setminus Z \hookrightarrow X$ is dense.

For $i = 1$ the claim $\mathrm{H}^i_Z(X, \mathscr{A}) = 0$ is equivalent to

$$\mathrm{H}^0(X, \mathscr{A}) \to \mathrm{H}^0(X \setminus Z, \mathscr{A})$$

being surjective and

$$\mathrm{H}^1(X, \mathscr{A}) \to \mathrm{H}^1(X \setminus Z, \mathscr{A})$$

being injective. The surjectivity of $\mathrm{H}^0(X, \mathscr{A}) \to \mathrm{H}^0(X \setminus Z, \mathscr{A})$ follows e. g. from

Theorem 3.4.1. *Let S be a normal Noetherian base scheme, and let $u : Z \to G$ be an S-rational map from a smooth S-scheme Z to a smooth and separated S-group scheme G. Then, if u is defined in codimension ≤ 1, it is defined everywhere.*

Proof. See [BLR90], p. 109, Theorem 1. □

For the injectivity of $\mathrm{H}^1(X, \mathscr{A}) \to \mathrm{H}^1(X \setminus Z, \mathscr{A})$: If a principal homogeneous space P/X for $\mathscr{A}/X$ is trivial over $X \setminus Z$, then it is trivial over X. This is true because X is smooth. The trivialisation over $X \setminus Z$ gives a rational map from X to the principal homogenous space and any such map (with X a regular scheme) extends to a morphism. The reason for this is that Abelian varieties do not contain rational curves by [Mil86], p. 107, Corollary 3.8, but all positive dimensional fibres of a birational proper morphism $X' \to X$ are covered by rational curves:

Theorem 3.4.2. *Let X, Y be normal, excellent schemes and $f : Y \to X$ a birational morphism of finite type. Let $E \subset Y$ be the* exceptional set

$$E(f) = \{y \in Y \mid f \textit{ is not a local isomorphism at } y\}$$

of f. Assume X is regular. Then E has ruled components over X.

Proof. See [Kol95], p. 286, Theorem 1.2. □

For the surjectivity of $\mathrm{H}^1(X, \mathscr{A}) \to \mathrm{H}^1(X \setminus Z, \mathscr{A})$: This means that any principal homogeneous space $P/(X \setminus Z)$ extends to a principal homogeneous space $\bar{P}/X$. If $P \in \mathrm{H}^1(X \setminus Z, \mathscr{A})$ has finite order prime to the characteristic, it comes from an element of $\mathrm{H}^1(X \setminus Z, \mathscr{A}[n])$ for some n prime to the characteristic. By [Mil80], p. 123, Corollary III.4.7, we have $PHS(\mathscr{A}[n]/X) \xrightarrow{\sim} \mathrm{H}^1(X_{\mathrm{fl}}, \mathscr{A}[n])$ (Čech cohomology) since $\mathscr{A}[n]/X$ is affine. Since $\mathscr{A}[n]/X$ is smooth because n is invertible on X, [Mil80], p. 123, Remark III.4.8 (a) shows that we can take étale cohomology as well, and by [Mil80], p. 101, Corollary III.2.10, one can take derived functor cohomology instead of Čech cohomology. By Zariski-Nagata purity Theorem 2.2.9, one can extend this to a $\bar{P}/X$, for which we have to show that it represents an element of $\mathrm{H}^1(X, \mathscr{A}[n])$, i. e. that it is a $\mathscr{A}[n]$-torsor, and then we get the desired element of $\mathrm{H}^1(X, \mathscr{A})$.

Now we need to show that if $P/(X \setminus Z)$ is an $\mathscr{A}|_{X \setminus Z}[n]$-torsor and $\bar{P}$ an extension of P to a finite étale covering of X, then $\bar{P}/X$ is also an $\mathscr{A}[n]$-torsor. For this, we use the following

Theorem 3.4.3. *Let S be a connected scheme, $G \to S$ a finite flat group scheme, and $X \to S$ a scheme over S equipped with a left action $\rho : G \times_S X \to X$. These data define a G-torsor over S if and only if there exists a finite locally free surjective morphism $Y \to S$ such that $X \times_S Y \to Y$ is isomorphic, as a Y-scheme with $G \times_S Y$-action, to $G \times_S Y$ acting on itself by left translations.*

Proof. See [Sza09], p. 171, Lemma 5.3.13. □

That $P/(X \setminus Z)$ is an $\mathscr{A}|_{X\setminus Z}[n]$-torsor amounts to saying that there is an operation

$$\mathscr{A}|_{X\setminus Z}[n] \times_{X\setminus Z} P \to P$$

as in the previous Theorem 3.4.3. Since this is étale locally isomorphic to the canonical action

$$\mathscr{A}|_{X\setminus Z}[n] \times_{X\setminus Z} \mathscr{A}|_{X\setminus Z}[n] \xrightarrow{\mu} \mathscr{A}|_{X\setminus Z}[n]$$

which is finite étale, by faithfully flat descent the operation defines an étale covering, so extends by Zariski-Nagata purity uniquely to an étale covering $H \to X$, which by uniqueness has to be isomorphic to $\mathscr{A}[n] \times_X \bar{P} \to \bar{P}$. Now one has to check the condition in Theorem 3.4.3.

For the injectivity of $\mathrm{H}^2(X, \mathscr{A}) \to \mathrm{H}^2(X \setminus Z, \mathscr{A})$, I do not have a geometric proof.

3.5 Relation of the Brauer and Tate-Shafarevich group

From the above, one gets the "finiteness part" of the relation of the (generalised) Artin-Tate and the Birch-Swinnerton-Dyer conjecture:

Theorem 3.5.1 (The Artin-Tate and the Birch-Swinnerton-Dyer conjecture)**.** *Let $\pi : \mathscr{C} \to X$ be projective and smooth with $\mathscr{C}$ and X regular, all fibres of dimension 1 and X be a variety. Then one has an exact sequence*

$$0 \to K_2 \to \mathrm{Br}(X) \xrightarrow{\pi^*} \mathrm{Br}(\mathscr{C}) \to Ш(\mathbf{Pic}_{\mathscr{C}/X}/X) \to K_3 \to 0$$

in which the groups K_i annihilated by δ, the index of the generic fibre C/K, e. g. $\delta = 1$ if $\mathscr{C}/X$ has a section, and their prime-to-p parts are finite, and $K_i = 0$ if π has a section. Here, $Ш(\mathbf{Pic}_{\mathscr{C}/X}/X)$ sits in a short exact sequence

$$0 \to \mathbf{Z}/d \to Ш(\mathbf{Pic}^0_{\mathscr{C}/X}/X) \to Ш(\mathbf{Pic}_{\mathscr{C}/X}/X) \to 0,$$

where $d \mid \delta$.

Hence the finiteness of the (ℓ-torsion of the) Brauer group of $\mathscr{C}$ is equivalent to the finiteness of the (ℓ-torsion of the) Brauer group of the base X and the finiteness of the (ℓ-torsion of the) Tate-Shafarevich group of $\mathbf{Pic}_{\mathscr{C}/X}$.

Proof. Combining Corollary 3.1.22 (here the theory of the Picard functor and the exactness of the sequence still works [we claim this here without proof] if we do not have a section, but étale locally a section; and the latter is the case since a smooth morphism factors locally into an étale morphism followed by an affine projection) with Theorem 3.1.25 and Theorem 3.3.5 yields the exact sequence

$$0 \to K_2 \to \mathrm{Br}(X) \xrightarrow{\pi^*} \mathrm{Br}(\mathscr{C}) \to \mathrm{H}^1(X, \mathbf{Pic}_{\mathscr{C}/X}) \to K_3 \to 0$$

with the prime-to-p part of the K_i finite, and $K_i = 0$ if π has a section.

Now the long exact sequence associated to the short exact sequence in Lemma 3.3.10 yields the exact sequence

$$\mathrm{H}^0(X, \mathbf{Pic}_{\mathscr{C}/X}) \to \mathrm{H}^0(X, \mathbf{Z}) \to \mathrm{H}^1(X, \mathbf{Pic}^0_{\mathscr{C}/X}) \to \mathrm{H}^1(X, \mathbf{Pic}_{\mathscr{C}/X}) \to \mathrm{H}^1(X, \mathbf{Z}) = 0,$$

and $\mathrm{H}^1(X, \mathbf{Z}) = 0$ using Lemma 3.3.13. Now, choose a Weil divisor D on the generic fibre C/K of $\mathscr{C}/X$ with degree δ the index of C/K. By Lemma 3.1.21 and [Con], p. 3, Proposition 4.1, $\bar{D}$ is a Weil divisor on $\mathscr{C}$ of degree δ, and its image under $\mathrm{H}^0(X, \mathbf{Pic}_{\mathscr{C}/X}) \to \mathrm{H}^0(X, \mathbf{Z}) = \mathbf{Z}$ (here we use that X is connected) is δ. Hence $\mathrm{coker}(\mathrm{H}^0(X, \mathbf{Pic}_{\mathscr{C}/X}) \to \mathrm{H}^0(X, \mathbf{Z}) = \mathbf{Z})$ is a quotient of $\mathbf{Z}/\delta$. □

4 The special L-value in cohomological terms

4.1 The L-function

Let $k = \mathbf{F}_q$ be a finite field with $q = p^n$ elements and let $\ell \neq p$ be a prime. For a variety X/k denote by $\bar{X}$ its base change to a separable closure $\bar{k}$ of k.

Denote by Frob_q the arithmetic Frobenius, by Γ the absolute Galois group of our finite base field $k = \mathbf{F}_q$, by n the dimension of the base scheme X and by $\bar{X}$ the base change $X \times_k \bar{k}$.

Let X/k be a smooth projective geometrically connected variety.

Definition 4.1.1. *Let $\pi : \mathscr{A} \to X$ be an Abelian scheme of (relative) dimension d. Then its* ***L*-function** *is defined as*

$$L(\mathscr{A}/X, s) = \prod_{x \in |X|} \det\left(1 - q^{-s\deg(x)} \mathrm{Frob}_x^{-1} \mid (\mathrm{R}^1\pi_*\mathbf{Q}_\ell)_{\bar{x}}\right)^{-1}.$$

Here, Frob_x^{-1} *is the geometric Frobenius and* $(\mathrm{R}^1\pi_*\mathbf{Q}_\ell)_{\bar{x}} = \mathrm{H}^1(\overline{\mathscr{A}}_x, \mathbf{Q}_\ell)$ *by proper base change.*

Proposition 4.1.2. *One has*

$$L(\mathscr{A}/X, s) = \prod_{i=0}^{2d} \det\left(1 - q^{-s} \mathrm{Frob}_q^{-1} \mid \mathrm{H}^i(\bar{X}, \mathrm{R}^1\pi_*\mathbf{Q}_\ell)\right)^{(-1)^{i+1}},$$

where the Frobenius acts via functoriality on the second factor of $\bar{X} = X \times_k \bar{k}$.

Proof. This follows from the Lefschetz trace formula [KW01], p. 7, Theorem 1.1. □

Definition 4.1.3. *Let*

$$P_i(\mathscr{A}/X, s) = \det\left(1 - q^{-s} \mathrm{Frob}_q^{-1} \mid \mathrm{H}^i(\bar{X}, \mathrm{R}^1\pi_*\mathbf{Q}_\ell)\right).$$

and define the **relative *L*-function** *of an Abelian scheme $\mathscr{A}/X$ by*

$$L(\mathscr{A}/X, s) = \frac{P_1(\mathscr{A}/X, q^{-s})}{P_0(\mathscr{A}/X, q^{-s})}.$$

For our purposes, it is better to consider the following L-function:

Definition 4.1.4. *Let*

$$L_i(\mathscr{A}/X, s) = \det(1 - q^{-s} \mathrm{Frob}_q^{-1} \mid \mathrm{H}^i(\bar{X}, V_\ell\mathscr{A})).$$

Remark 4.1.5. Note that

$$\begin{aligned} P_i(\mathscr{A}/X, s) &= \det(1 - q^{-s} \mathrm{Frob}_q^{-1} \mid \mathrm{H}^i(\bar{X}, \mathrm{R}^1\pi_*\mathbf{Q}_\ell)) \\ &= \det(1 - q^{-s} \mathrm{Frob}_q^{-1} \mid \mathrm{H}^i(\bar{X}, V_\ell(\mathscr{A}^\vee))(-1)) \quad \text{by (4.1.1) below} \\ &= \det(1 - q^{-s} q \, \mathrm{Frob}_q^{-1} \mid \mathrm{H}^i(\bar{X}, V_\ell\mathscr{A})) \quad \text{by Lemma 4.2.21 below} \\ &= L_i(\mathscr{A}/X, s - 1), \end{aligned}$$

so the vanishing order of $P_i(\mathscr{A}/X, s)$ at $s = 1$ is equal to the vanishing order of $L_i(\mathscr{A}/X, s)$ at $s = 0$, and the respective leading coefficients agree.

Definition 4.1.6. *A $\mathbf{Q}_\ell[\Gamma]$-module is said to be* **pure of weight n** *if all eigenvalues α of the geometric Frobenius automorphism are algebraic integers which have absolute value $q^{n/2}$ under all embeddings $\iota : \mathbf{Q}(\alpha) \hookrightarrow \mathbf{C}$.*

We often use the yoga of weights:

Theorem 4.1.7. *Let $f : X \to Y$ be a smooth proper morphism of schemes of finite type over $\mathbf{F}_q$ and $\mathscr{F}$ a smooth sheaf pure of weight n. Then $\mathrm{R}^i f_* \mathscr{F}$ is a smooth sheaf pure of weight $n+i$ for any i.*

Proof. Apply Poincaré duality to [Del80], p. 138, Théorème 1. □

The following is a generalisation of [Sch82a], p. 134–138 and [Sch82b], p. 496–498.
For the definition of an ℓ-adic sheaf see [FK88], p. 122, Definition 12.6.

Lemma 4.1.8. *Let $(G_n)_{n\in\mathbf{N}}$ be a Barsotti-Tate group consisting of finite* étale *group schemes. Then it is an ℓ-adic sheaf.*

Proof. By [Tat67], p. 161, (2)

$$0 \to \ker[\ell] \to G_{n+1} \overset{[\ell]}{\to} G_n \to 0$$

is exact. But $\ker[\ell] = \ell^n G_{n+1}$. Furthermore, $G_n = 0$ for $n < 0$ and $\ell^{n+1} G_n = 0$ by [Tat67], p. 161, (ii). Finally, the G_n are constructible since they are finite *étale* group schemes. □

Corollary 4.1.9. *For ℓ invertible on X, $T_\ell \mathscr{A} = (\mathscr{A}[\ell^n])_{n\in\mathbf{N}}$ is an ℓ-adic sheaf.*

Proof. This follows from Lemma 4.1.8 since ℓ is invertible on X, so $\mathscr{A}[\ell^n]/X$ is finite étale. □

Theorem 4.1.10. *Let K be an arbitrary field and A/K be an Abelian variety. Then we have an isomorphism of (ℓ-adic discrete) G_K-modules, equivalently, by [Mil80], p. 53, Theorem II.1.9, of (ℓ-adic) étale sheaves on* $\operatorname{Spec} K$,

$$T_\ell(A) = \mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell)^\vee.$$

Proof. Consider the Kummer sequence

$$1 \to \mu_{\ell^n} \to \mathbf{G}_m \overset{\ell^n}{\to} \mathbf{G}_m \to 1$$

on $\bar{A}$. Taking étale cohomology, one gets an exact sequence of G_K-modules

$$0 \to \mathbf{G}_m(\bar{A})/\ell^n \to \mathrm{H}^1(\bar{A}, \mu_{\ell^n}) \to \mathrm{H}^1(\bar{A}, \mathbf{G}_m)[\ell^n] \to 0.$$

Since $\Gamma(\bar{A}, \mathscr{O}_{\bar{A}}) = \bar{K}$ is separably closed and $\ell \neq \operatorname{char} K$, $\mathbf{G}_m(\bar{A})$ is ℓ-divisible (one can extract ℓ-th roots), and hence

$$\mathrm{H}^1(\bar{A}, \mu_{\ell^n}) \xrightarrow{\sim} \mathrm{H}^1(\bar{A}, \mathbf{G}_m)[\ell^n] = \operatorname{Pic}(\bar{A})[\ell^n] = \operatorname{Pic}^0(\bar{A})[\ell^n],$$

the latter equality since $\operatorname{NS}(\bar{A})$ is torsion-free by [Mum70], p. 178, Corollary 2. Taking Tate modules $\varprojlim_n$ yields

$$\mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell(1)) \xrightarrow{\sim} T_\ell \operatorname{Pic}^0(\bar{A}), \tag{4.1.1}$$

so (the first equality coming from the perfect Weil pairing (4.2.4))

$$\mathrm{Hom}(T_\ell A, \mathbf{Z}_\ell(1)) = T_\ell(A^\vee) = \mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell(1)),$$

so

$$(T_\ell A)^\vee = \mathrm{Hom}(T_\ell A, \mathbf{Z}_\ell) = \mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell),$$

so

$$T_\ell A = \mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell)^\vee.$$

Alternatively, $\pi_1(A, 0) = \prod_\ell T_\ell(A)$ by [Mum70], p. 171, and $\mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell) = \mathrm{Hom}(\pi_1(A, 0), \mathbf{Z}_\ell)$ by Proposition 3.3.15. □

Remark 4.1.11. Note that both $T_\ell(-)$ and $\mathrm{H}^1(-, \mathbf{Z}_\ell)^\vee$ are covariant functors.

Lemma 4.1.12. *Isogenous Abelian varieties over a finite field k have the same number of points.*

Proof. Let $f : A \to B$ be an k-isogeny. Take Galois invariants of

$$0 \to \ker f(\bar{k}) \to A(\bar{k}) \to B(\bar{k}) \to 0$$

and using Lang-Steinberg Theorem 2.2.8 in the form $\mathrm{H}^1(k, A) = 0 = \mathrm{H}^1(k, B)$ and the Herbrand quotient $h(\ker f(\bar{k})) = 1$ (since $\ker f(\bar{k})$ is finite) yields $|A(k)| = |B(k)|$.

Alternatively, use that $A(\mathbf{F}_{q^n}) = \ker(1 - \mathrm{Frob}_q^n)$ and $f(1 - \mathrm{Frob}_q^n) = (1 - \mathrm{Frob}_q^n)f$ and $\deg f > 0$ is finite, and take degrees. □

Lemma 4.1.13. *Let $(\mathscr{F}_n)_{n \in \mathbf{N}}$, $\mathscr{F} = \varprojlim_n \mathscr{F}_n$ be an ℓ-adic sheaf. For every i, there is a short exact sequence*

$$0 \to \mathrm{H}^{i-1}(\bar{X}, \mathscr{F})_\Gamma \to \mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^i(\bar{X}, \mathscr{F})^\Gamma \to 0. \tag{4.1.2}$$

The following argument is a generalisation of [Mil88], p. 78, Lemma 3.4.

Proof. Since Γ has cohomological dimension 1, we get from the Hochschild-Serre spectral sequence for $\bar{X}/X$ [Mil80], p. 106, Remark III.2.21 (b) short exact sequences for every n and i

$$0 \to \mathrm{H}^{i-1}(\bar{X}, \mathscr{F}_n)_\Gamma \to \mathrm{H}^i(X, \mathscr{F}_n) \to \mathrm{H}^i(\bar{X}, \mathscr{F}_n)^\Gamma \to 0.$$

Since all involved groups are finite (because the two outer groups are finite by [Mil80], p. 224, Corollary VI.2.8 since $\bar{X}/\bar{k}$ is proper and $\mathscr{F}_n$ is constructible by definition of an ℓ-adic sheaf), the system satisfies the Mittag-Leffler condition, so taking the projective limit yields an exact sequence

$$0 \to \varprojlim_n (\mathrm{H}^{i-1}(\bar{X}, \mathscr{F}_n)_\Gamma) \to \mathrm{H}^i(X, \mathscr{F}) \to \varprojlim_n (\mathrm{H}^i(\bar{X}, \mathscr{F}_n)^\Gamma) \to 0.$$

Write $M[n]$ for $\mathrm{H}^i(\bar{X}, \mathscr{F}_n)$. Breaking the exact sequence

$$0 \to M[n]^\Gamma \to M[n] \stackrel{\mathrm{Frob}-1}{\to} M[n] \to M[n]_\Gamma \to 0$$

into two short exact sequences and applying $\varprojlim_n$, one obtains, setting $Q[n] = (\mathrm{Frob} - 1)M[n]$, two exact sequences

$$0 \to \varprojlim_n (M[n]^\Gamma) \to \varprojlim_n (M[n]) \overset{\mathrm{Frob}-1}{\to} \varprojlim_n (Q_n) \to \varprojlim_n{}^1 (M[n]^\Gamma) \tag{4.1.3}$$

$$0 \to \varprojlim_n (Q[n]) \to \varprojlim_n (M[n]) \to \varprojlim_n (M[n]_\Gamma) \to \varprojlim_n{}^1 (Q[n]) \tag{4.1.4}$$

Since the $M[n]$, and hence the $M[n]^\Gamma$ are finite (argument as above), they form a Mittag-Leffler system, and hence one gets from (4.1.3) an exact sequence

$$0 \to \varprojlim_n (M[n]^\Gamma) \to \varprojlim_n (M[n]) \overset{\mathrm{Frob}-1}{\to} \varprojlim_n (Q_n) \to 0.$$

Similarly, the $Q[n] \subseteq M[n]$ are finite, and hence

$$0 \to \varprojlim_n (Q[n]) \to \varprojlim_n (M[n]) \to \varprojlim_n (M[n]_\Gamma) \to 0$$

is exact from (4.1.4). Combining the above two short exact sequences, one gets the exactness of

$$0 \to \varprojlim_n (M[n]^\Gamma) \to \varprojlim_n (M[n]) \overset{\mathrm{Frob}-1}{\to} \varprojlim_n (M[n]) \to \varprojlim_n (M[n]_\Gamma) \to 0,$$

which shows that for all i

$$\varprojlim_n (\mathrm{H}^i(\bar{X}, \mathscr{F}_n)^\Gamma) = \varprojlim_n (M[n]^\Gamma) = \ker(\varprojlim_n (M[n]) \overset{\mathrm{Frob}-1}{\to} \varprojlim_n (M[n])) = \mathrm{H}^i(\bar{X}, \mathscr{F})^\Gamma$$

$$\varprojlim_n (\mathrm{H}^i(\bar{X}, \mathscr{F}_n)_\Gamma) = \varprojlim_n (M[n]_\Gamma) = \mathrm{coker}(\varprojlim_n (M[n]) \overset{\mathrm{Frob}-1}{\to} \varprojlim_n (M[n])) = \mathrm{H}^i(\bar{X}, \mathscr{F})_\Gamma,$$

which is what we wanted. □

This implies

$$\mathrm{H}^{2d}(\bar{X}, T_\ell \mathscr{A})_\Gamma \xrightarrow{\sim} \mathrm{H}^{2d+1}(X, T_\ell \mathscr{A}) \tag{4.1.5}$$

since $\mathrm{H}^{2d+1}(\bar{X}, T_\ell \mathscr{A}) = 0$ for dimension reasons. Because of $\mathrm{H}^i(\bar{X}, T_\ell \mathscr{A}) = 0$ for $i > 2d$, it follows that $\mathrm{H}^i(X, T_\ell \mathscr{A}) = 0$ for $i > 2d + 1$. Furthermore, one has

$$\mathbf{Z}_\ell = (\mathbf{Z}_\ell)_\Gamma = \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d))_\Gamma \xrightarrow{\sim} \mathrm{H}^{2d+1}(X, \mathbf{Z}_\ell(d)),$$

the second equality by Poincaré duality [Mil80], p. 276, Theorem VI.11.1 (a).

Lemma 4.1.14. *Let* Frob *be a topological generator of* Γ *and* M *be a finitely generated* $\mathbf{Z}_\ell$*-module with continuous* Γ*-action. Then the following are equivalent:*

1. $\det(1 - \mathrm{Frob} \mid M \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell) \neq 0$.
2. $\mathrm{H}^0(\Gamma, M) = M^\Gamma$ *is finite.*
3. $\mathrm{H}^1(\Gamma, M)$ *is finite.*

If one of these holds, we have $\mathrm{H}^1(\Gamma, M) = M_\Gamma$ *and*

$$|\det(1 - \mathrm{Frob} \mid M \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell)|_\ell = \frac{|\mathrm{H}^0(\Gamma, M)|}{|\mathrm{H}^1(\Gamma, M)|} = \frac{|M^\Gamma|}{|M_\Gamma|} = \frac{|\ker(1 - \mathrm{Frob})|}{|\mathrm{coker}\,(1 - \mathrm{Frob})|} = q(1 - \mathrm{Frob}).$$

Proof. See [BN78], p. 42, Lemma (3.2). If M is torsion, $\mathrm{H}^1(\Gamma, M) = M_\Gamma$ by [NSW00], p. 69, (1.6.13) Proposition (i). For the last equality, see Lemma 2.1.14. □

Since $T_\ell \mathscr{A} = \mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell)^\vee$ (see Theorem 4.1.10) has weight -1, $\mathrm{H}^i(\bar{X}, T_\ell \mathscr{A})$ has weight $i - 1$. So the conditions of Lemma 4.1.14 are fulfilled for the Γ-module $M = \mathrm{H}^i(\bar{X}, T_\ell \mathscr{A})$ and $i \neq 1$. Therefore infinite groups in the short exact sequences in (4.1.2) can only occur in the following two sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})_\Gamma & \xrightarrow{\beta} & \mathrm{H}^2(X, T_\ell\mathscr{A}) & \longrightarrow & \mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma & \longrightarrow & 0 \\ & & & \overset{f}{\nwarrow} & & & & & \\ 0 & \longrightarrow & \mathrm{H}^0(\bar{X}, T_\ell\mathscr{A})_\Gamma & \longrightarrow & \mathrm{H}^1(X, T_\ell\mathscr{A}) & \xrightarrow{\alpha} & \mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})^\Gamma & \longrightarrow & 0 \end{array} \tag{4.1.6}$$

Here, f is induced by the identity on $\mathrm{H}^1(\bar{X}, T_\ell \mathscr{A})$. Since $\mathrm{H}^2(\bar{X}, T_\ell \mathscr{A})^\Gamma$ and $\mathrm{H}^0(\bar{X}, T_\ell \mathscr{A})_\Gamma$ are finite (having weight $2 - 1 \neq 0$ and $0 - 1 \neq 0$), α and β are quasi-isomorphisms, i. e. they have finite kernel and cokernel.

Recall that

$$L_1(\mathscr{A}/X, t) = \det(1 - \mathrm{Frob}_q^{-1} t \mid \mathrm{H}^1(\bar{X}, V_\ell \mathscr{A})).$$

Define $\tilde{L}_1(\mathscr{A}/X, t)$ and ρ by

$$\begin{aligned} L_1(\mathscr{A}/X, t) &= (t - 1)^\rho \cdot \tilde{L}_1(\mathscr{A}/X, t), \\ \rho &= \operatorname*{ord}_{t=1} L_1(\mathscr{A}/X, t) \in \mathbf{N}. \end{aligned}$$

By writing Frob_q^{-1} in Jordan normal form, one sees that ρ is equal to

$$\dim_{\mathbf{Q}_\ell} \bigcup_{n \geq 1} \ker(1 - \mathrm{Frob}_q^{-1})^n \geq \dim_{\mathbf{Q}_\ell} \ker(1 - \mathrm{Frob}_q^{-1}) = \dim_{\mathbf{Q}_\ell} \mathrm{H}^1(\bar{X}, V_\ell \mathscr{A})^\Gamma,$$

i. e.

$$\rho \geq \dim_{\mathbf{Q}_\ell} \mathrm{H}^1(\bar{X}, V_\ell \mathscr{A})^\Gamma,$$

and that equality holds iff the operation of the Frobenius on $\mathrm{H}^1(\bar{X}, V_\ell \mathscr{A})$ is semi-simple at 1, i. e.

$$\dim_{\mathbf{Q}_\ell} \bigcup_{n \geq 1} \ker(1 - \mathrm{Frob}_q^{-1})^n = \dim_{\mathbf{Q}_\ell} \ker(1 - \mathrm{Frob}_q^{-1}),$$

i. e. the generalised eigenspace at 1 equals the eigenspace, which is equivalent to $f_{\mathbf{Q}_\ell}$ in (4.1.6) being an isomorphism, i. e. f being a quasi-isomorphism. From (4.1.6), since $\mathrm{H}^0(\bar{X}, T_\ell \mathscr{A})_\Gamma$ is finite, one sees that

$$\dim_{\mathbf{Q}_\ell} \mathrm{H}^1(\bar{X}, V_\ell \mathscr{A})^\Gamma = \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(\bar{X}, T_\ell \mathscr{A})^\Gamma = \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell \mathscr{A}).$$

Corollary 4.1.15. *If $i \neq 1$, one has*

$$|L_i(\mathscr{A}/X, q^{-1})|_\ell = \frac{\left|\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|}{\left|\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A})_\Gamma\right|}.$$

Proof. This follows from Lemma 4.1.14 1 if $i \neq 1$ since $\mathrm{H}^i(\bar{X}, V_\ell\mathscr{A})$ has weight $i-1$. □

Lemma 4.1.16. $\rho = \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A})$ *iff f is a quasi-isomorphism. In this case,*

$$|\tilde{L}_1(\mathscr{A}/X, q^{-1})|_\ell^{-1} = q(f) = \frac{|\operatorname{coker} f|}{|\ker f|} \quad \textit{and}$$
$$|\tilde{L}_1(\mathscr{A}/X, q^{-1})|_\ell^{-1} = q((\beta f\alpha)_{\mathrm{Tors}}) \cdot \frac{\left|\mathrm{H}^0(\bar{X}, T_\ell\mathscr{A})_\Gamma\right|}{\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|} \cdot \frac{|\operatorname{Tor} \mathrm{H}^2(X, T_\ell\mathscr{A})|}{|\operatorname{Tor} \mathrm{H}^1(X, T_\ell\mathscr{A})|}.$$

The idea is that for infinite cohomology groups $\mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})$, one should insert a regulator term $q(f)$ or $q((\beta f\alpha)_{\mathrm{Tors}})$.

Proof. From the discussion above, the first statement follows. Assuming this, one has

$$\begin{aligned}
|\tilde{L}_1(\mathscr{A}/X, q^{-1})| &= \frac{|(\mathrm{Frob}_q - 1)\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A})^\Gamma|}{|(\mathrm{Frob}_q - 1)\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A})_\Gamma|} \\
&= \frac{|(\mathrm{Frob}_q - 1)\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A})^\Gamma|}{|(\mathrm{Frob}_q - 1)\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A}) : (\mathrm{Frob}_q - 1)^2\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A})|} \\
&= \frac{|\ker f|}{|\operatorname{coker} f|} = q(f)^{-1}.
\end{aligned}$$

For the second equation,

$$\begin{aligned}
q(f) &= \frac{q(\operatorname{Tor}(\beta f))}{q(\beta)} \cdot q((\beta f)_{\mathrm{Tors}}) \quad \text{by Lemma 2.1.14 2} \\
&= \frac{1}{\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|} \cdot \frac{|\operatorname{Tor} \mathrm{H}^2(X, T_\ell\mathscr{A})_\Gamma|}{\left|\operatorname{Tor} \mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|} \cdot q((\beta f)_{\mathrm{Tors}}) \\
&= q((\beta f\alpha)_{\mathrm{Tors}}) \cdot \frac{\left|\mathrm{H}^0(\bar{X}, T_\ell\mathscr{A})_\Gamma\right|}{\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|} \cdot \frac{|\operatorname{Tor} \mathrm{H}^2(X, T_\ell\mathscr{A})|}{|\operatorname{Tor} \mathrm{H}^1(X, T_\ell\mathscr{A})|} \quad \text{since } \operatorname{coker}(\alpha) = 0.
\end{aligned}$$ □

Lemma 4.1.17. *Let $\ell \neq p$ be invertible on X. Then the sequence*

$$0 \to \mathscr{A}(X) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \to \mathrm{H}^1(X, \mathscr{A}[\ell^\infty]) \to \mathrm{H}^1(X, \mathscr{A})[\ell^\infty] \to 0$$

is exact.

Proof. The short exact sequence of étale sheaves

$$0 \to \mathscr{A}[\ell^n] \to \mathscr{A} \to \mathscr{A} \to 0$$

induces

$$0 \to \mathscr{A}(X)/\ell^n \to \mathrm{H}^1(X, \mathscr{A}[\ell^n]) \to \mathrm{H}^1(X, \mathscr{A})[\ell^n] \to 0. \tag{4.1.7}$$

Passing to the colimit $\varinjlim_n$ yields the result. □

Lemma 4.1.18. *Let ℓ be invertible on X. Then the $\mathbf{Z}_\ell$-corank of $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$ is finite.*

Proof. From (4.1.7), one sees that $\mathrm{H}^1(X,\mathscr{A})[\ell]$ is finite as it is a quotient of $\mathrm{H}^1(X,\mathscr{A}[\ell])$ and $\mathscr{A}[\ell]/X$ is constructible. Hence $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$ is cofinitely generated by Lemma 2.1.4. □

Lemma 4.1.19. *Let X/k be proper and ℓ be invertible on X. There is a long exact sequence*

$$\ldots \to \mathrm{H}^i(X,T_\ell\mathscr{A}(n)) \to \mathrm{H}^i(X,T_\ell\mathscr{A}(n))\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell \to \mathrm{H}^i(X,\mathscr{A}[\ell^\infty](n)) \to \ldots$$

which induces isomorphisms

$$\mathrm{H}^{i-1}(X,\mathscr{A}[\ell^\infty](n))_{\mathrm{Div}} \to \operatorname{Tor}\mathrm{H}^i(X,T_\ell\mathscr{A}(n)).$$

Proof. Consider for $m,m'\in\mathbf{N}$ the short exact sequence of étale sheaves

$$0\to\mathscr{A}[m]\hookrightarrow\mathscr{A}[mm']\xrightarrow{\cdot m}\mathscr{A}[m']\to 0.$$

Twisting with $\mathbf{Z}_\ell(n)$ gives the exact sequence

$$0\to\mathscr{A}[m](n)\hookrightarrow\mathscr{A}[mm'](n)\xrightarrow{\cdot m}\mathscr{A}[m'](n)\to 0.$$

Setting $m=\ell^\mu, m'=\ell^\nu$, the associated long exact sequence is

$$\ldots\to\mathrm{H}^i(X,\mathscr{A}[\ell^\mu](n))\to\mathrm{H}^i(X,\mathscr{A}[\ell^{\mu+\nu}](n))\to\mathrm{H}^i(X,\mathscr{A}[\ell^\nu](n))\to\ldots.$$

Passing to the projective limit $\varprojlim_\mu$ and then to the inductive limit $\varinjlim_\nu$ yields the desired long exact sequence since all involved cohomology groups are finite by [Mil80], p. 224, Corollary VI.2.8 since X/k is proper and our sheaves are constructible.

For the second statement, consider the exact sequence

$$\mathrm{H}^{i-1}(X,T_\ell\mathscr{A}(n))\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell\xrightarrow{f}\mathrm{H}^{i-1}(X,\mathscr{A}[\ell^\infty](n))\xrightarrow{d}\mathrm{H}^i(X,T_\ell\mathscr{A}(n))\xrightarrow{g}\mathrm{H}^i(X,T_\ell\mathscr{A}(n))\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell.$$

Since $\mathrm{H}^i(X,T_\ell\mathscr{A}(n))$ is a finitely generated $\mathbf{Z}_\ell$-module (since $(\mathscr{A}[\ell^n])_{n\in\mathbf{N}}$ is an ℓ-adic sheaf) and g is induced by the identity, we have $\ker g=\operatorname{Tor}\mathrm{H}^i(X,T_\ell\mathscr{A}(n))$. Since $\mathrm{H}^{i-1}(X,T_\ell\mathscr{A}(n))$ is a finitely generated $\mathbf{Z}_\ell$-module and $\mathrm{H}^{i-1}(X,\mathscr{A}[\ell^\infty](n))$ is a cofinitely generated ℓ-torsion module, we have $\operatorname{im} f=\operatorname{Div}\mathrm{H}^{i-1}(X,\mathscr{A}[\ell^\infty](n))$. The claim follows from the exactness of the sequence. □

Lemma 4.1.20. *Assume ℓ is invertible on X. Then one has the following identities for the étale cohomology groups of X:*

$$\mathrm{H}^i(X,T_\ell\mathscr{A})=0\quad\textit{for } i\neq 1,2,\ldots,2d+1 \tag{4.1.8}$$
$$\operatorname{Tor}\mathrm{H}^1(X,T_\ell\mathscr{A})=\mathrm{H}^0(X,\mathscr{A}[\ell^\infty])_{\mathrm{Div}}=\operatorname{Tor}\mathrm{H}^0(X,\mathscr{A})[\ell^\infty] \tag{4.1.9}$$
$$\operatorname{Tor}\mathrm{H}^2(X,T_\ell\mathscr{A})=\mathrm{H}^1(X,\mathscr{A}[\ell^\infty])_{\mathrm{Div}} \tag{4.1.10}$$
$$\mathrm{H}^1(X,\mathscr{A}[\ell^\infty])_{\mathrm{Div}}=\text{Ш}(\mathscr{A}/X)[\ell^\infty]\quad\textit{if } \text{Ш}(\mathscr{A}/X)[\ell^\infty] \textit{ is finite} \tag{4.1.11}$$

Proof. (4.1.8): For $i>2d+1$ this follows from (4.1.2), and it holds for $i=0$ since $\mathrm{H}^0(X,\mathscr{A}[\ell^n])\subseteq\operatorname{Tor}\mathscr{A}(X)$ is finite (since $\mathscr{A}(X)$ is a finitely generated Abelian group by the Mordell-Weil theorem) hence its Tate-module is trivial by Lemma 2.1.7.

(4.1.9) and (4.1.10) From Lemma 4.1.19, we get

$$\left|\operatorname{Tor}\mathrm{H}^i(X, T_\ell\mathscr{A})\right| = \left|\mathrm{H}^{i-1}(X, \mathscr{A}[\ell^\infty])_{\mathrm{Div}}\right|$$

The desired equalities follow by plugging in $i = 1, 2$.

Further, one has $\mathrm{H}^0(X, \mathscr{A}[\ell^\infty])_{\mathrm{Div}} = \operatorname{Tor}\mathrm{H}^0(X, \mathscr{A})[\ell^\infty]$ in (4.1.9) because $\mathrm{H}^0(X, \mathscr{A}[\ell^\infty])$ is cofinitely generated by the Mordell-Weil theorem.

Finally, (4.1.11) holds since by Lemma 4.1.17, $\mathrm{H}^1(X, \mathscr{A}[\ell^\infty])_{\mathrm{Div}} = \mathrm{H}^1(X, \mathscr{A})[\ell^\infty]$ if the latter is finite, and this equals $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$. □

Now we have two pairings

$$\langle\cdot,\cdot\rangle_\ell : \mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} \to \mathrm{H}^{2d}(X, \mathbf{Z}_\ell(d)) \stackrel{\mathrm{pr}_1^*}{\to} \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell, \tag{4.1.12}$$

$$(\cdot,\cdot)_\ell : \mathrm{H}^2(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} \to \mathrm{H}^{2d+1}(X, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell. \tag{4.1.13}$$

Lemma 4.1.21. *The regulator term $q((\beta f\alpha)_{\mathrm{Tors}})$ is defined iff f is a quasi-isomorphism and then equals*

$$\left|\frac{\det\langle\cdot,\cdot\rangle_\ell}{\det(\cdot,\cdot)_\ell}\right|_\ell^{-1}$$

where both pairings are non-degenerate.

Proof. Using $\mathrm{H}^{2d+1}(X, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell$ and $\mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell$, there is a commutative diagram of pairings

$$\begin{array}{ccccc}
\mathrm{H}^2(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times & & \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} & \stackrel{\cup}{\to} & \mathbf{Z}_\ell \\
\uparrow \beta & & \downarrow \cong & & \| \\
(\mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})_\Gamma)_{\mathrm{Tors}} \times & & (\mathrm{H}^{2d-1}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))^\Gamma)_{\mathrm{Tors}} & \stackrel{\cup}{\to} & \mathbf{Z}_\ell \\
\uparrow f & & \| & & \| \\
(\mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})^\Gamma)_{\mathrm{Tors}} \times & & (\mathrm{H}^{2d-1}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))^\Gamma)_{\mathrm{Tors}} & \stackrel{\cup}{\to} & \mathbf{Z}_\ell \\
\uparrow \alpha\, \cong & & \cong \uparrow & & \| \\
\mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times & & \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} & \stackrel{\cup}{\to} & \mathbf{Z}_\ell
\end{array}$$

By Poincaré duality (using that $T_\ell\mathscr{A}$ is a smooth sheaf since the $\mathscr{A}[\ell^n]$ are étale), the pairing in the second line is non-degenerate, hence the pairing in the first line is too (β is a quasi-isomorphism). (The upper right and the lower right arrows are isomorphisms since their kernel is $(\mathrm{H}^{2d-2}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} = 0$ for weight reasons: $(2d-2)+(-1)-2(d-1) = -1 \neq 0$; the lower left arrow is an isomorphism since α is a quasi-isomorphism.) Hence if f is a quasi-isomorphism, all pairings in the diagram are non-degenerate, and then the claimed equality for the regulator $q((\beta f\alpha)_{\mathrm{Tors}}) = |\operatorname{coker}(\beta f\alpha)_{\mathrm{Tors}}|$ follows. □

Lemma 4.1.22. *Let ℓ be invertible on X. Then one has a short exact sequence*

$$0 \to \mathscr{A}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \stackrel{\delta}{\to} \mathrm{H}^1(X, T_\ell\mathscr{A}) \to \varprojlim_n(\mathrm{H}^1(X, \mathscr{A})[\ell^n]) \to 0.$$

Proof. Since ℓ is invertible on X, the short exact sequence of étale sheaves

$$0 \to \mathscr{A}[\ell^n] \to \mathscr{A} \xrightarrow{\ell^n} \mathscr{A} \to 0$$

induces a short exact sequence

$$0 \to \mathscr{A}(X)/\ell^n \xrightarrow{\delta} \mathrm{H}^1(X, \mathscr{A}[\ell^n]) \to \mathrm{H}^1(X, \mathscr{A})[\ell^n] \to 0$$

in cohomology, and passing to the limit $\varprojlim_n$ gives us the desired short exact sequence. □

Lemma 4.1.23. *One has $\rho \geq \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A})$. Then $1 \iff 2 \iff 3$ and $4 \iff 5$ in the following; further $3 \iff 4$ assuming $\rho = \mathrm{rk}_{\mathbf{Z}} \mathscr{A}(X)$.*

1. *$\langle\cdot,\cdot\rangle_\ell$ and $(\cdot,\cdot)_\ell$ are non-degenerate.*
2. *f is a quasi-isomorphism.*
3. *Equality holds $\rho = \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A})$.*
4. *The canonical injection $\mathscr{A}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\delta} \mathrm{H}^1(X, T_\ell\mathscr{A})$ is surjective.*
5. *The ℓ-primary part of the Tate-Shafarevich group $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$ is finite.*

Furthermore, the following are equivalent:

(a) $\rho = \mathrm{rk}_{\mathbf{Z}} \mathscr{A}(X)$

(b) $\langle\cdot,\cdot\rangle_\ell$ e non-degenerate and the ℓ-primary part of the Tate-Shafarevich group $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$ is finite.

Proof. $1 \iff 2$: See Lemma 4.1.21. $2 \iff 3$: This is Lemma 4.1.16. $3 \iff 4$: One has $\rho = \mathrm{rk}_{\mathbf{Z}} \mathscr{A}(X) = \mathrm{rk}_{\mathbf{Z}_\ell}(\mathscr{A}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell)$ and by Lemma 4.1.22 $\mathrm{rk}_{\mathbf{Z}_\ell}(\mathscr{A}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell) \leq \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A})$, so this is an equality iff δ in 3. is onto. $4 \iff 5$: By Lemma 2.1.7 and Lemma 2.1.8 $\varprojlim_n(\mathrm{H}^1(X, \mathscr{A})[\ell^n]) = T_\ell(\mathrm{H}^1(X, \mathscr{A}))$ is trivial iff $\mathrm{H}^1(X, \mathscr{A})[\ell^\infty] = \text{Ш}(\mathscr{A}/X)[\ell^\infty]$ is finite since $\text{Ш}(\mathscr{A}/X)[\ell^\infty]$ is a cofinitely generated $\mathbf{Z}_\ell$-module by Lemma 4.1.18.

(a) $\Longrightarrow$ (b): Since δ in 4 is injective, one has $\mathrm{rk}_{\mathbf{Z}} \mathscr{A}(X) \leq \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A}) \leq \rho$. Therefore, $\rho = \mathrm{rk}_{\mathbf{Z}} \mathscr{A}(X)$ implies equality, and 3 and 4 follow, so 1–5 hold. (b) $\Longrightarrow$ (a): (b) is equivalent to 1–5, so from 4 one gets $\mathscr{A}(X) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\sim} \mathrm{H}^1(X, T_\ell\mathscr{A})$, but by 3, $\rho = \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A}) = \mathrm{rk}_{\mathbf{Z}} \mathscr{A}(X)$. □

Define c by

$$\begin{aligned} L(\mathscr{A}/X, s) &\sim c \cdot (1 - q^{1-s})^\rho \\ &\sim c \cdot (\log q)^\rho (s-1)^\rho \quad \text{for } s \to 1, \end{aligned}$$

see Remark 4.2.37. Note that $c \in \mathbf{Q}$ since $L(\mathscr{A}/X, s)$ is a rational function with $\mathbf{Q}$-coefficients in q^{-s}, and $c \neq 0$ since ρ is the vanishing order of the L-function at $s = 1$ by definition of ρ and the Riemann hypothesis.

Corollary 4.1.24. *If $\rho = \mathrm{rk}_{\mathbf{Z}_\ell} \mathrm{H}^1(X, T_\ell\mathscr{A})$, then*

$$|c|_\ell^{-1} = q((\beta f \alpha)_{\mathrm{Tors}}) \cdot \frac{|\mathrm{Tor}\, \mathrm{H}^2(X, T_\ell\mathscr{A})|}{|\mathrm{Tor}\, \mathrm{H}^1(X, T_\ell\mathscr{A})| \cdot \left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right|}.$$

Proof. Using Lemma 4.1.16 for $\tilde{L}_1(\mathscr{A}/X,t)$ and Corollary 4.1.15 for $L_0(\mathscr{A}/X,t)$, one gets

$$\begin{aligned}|c|_\ell^{-1} &= q((\beta f\alpha)_{\mathrm{Tors}}) \cdot \frac{\left|\mathrm{H}^0(\bar{X},T_\ell\mathscr{A})_\Gamma\right|}{\left|\mathrm{H}^2(\bar{X},T_\ell\mathscr{A})^\Gamma\right|} \cdot \frac{|\mathrm{Tor}\,\mathrm{H}^2(X,T_\ell\mathscr{A})|}{|\mathrm{Tor}\,\mathrm{H}^1(X,T_\ell\mathscr{A})|} \cdot \frac{\left|\mathrm{H}^0(\bar{X},T_\ell\mathscr{A})^\Gamma\right|}{\left|\mathrm{H}^0(\bar{X},T_\ell\mathscr{A})_\Gamma\right|} \\ &= q((\beta f\alpha)_{\mathrm{Tors}}) \cdot \frac{1}{\left|\mathrm{H}^2(\bar{X},T_\ell\mathscr{A})^\Gamma\right|} \cdot \frac{|\mathrm{Tor}\,\mathrm{H}^2(X,T_\ell\mathscr{A})|}{|\mathrm{Tor}\,\mathrm{H}^1(X,T_\ell\mathscr{A})|} \cdot \frac{\left|\mathrm{H}^0(\bar{X},T_\ell\mathscr{A})^\Gamma\right|}{1}.\end{aligned}$$

For $0 = \mathrm{H}^0(X,T_\ell\mathscr{A}) \xrightarrow{\sim} \mathrm{H}^0(\bar{X},T_\ell\mathscr{A})^\Gamma$ use (4.1.2) with $i=0$ and (4.1.8). □

Theorem 4.1.25 (The conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher-dimensional bases, cohomological version)**.** *One has $\rho \geq \mathrm{rk}_{\mathbf{Z}_\ell}\mathrm{H}^1(X,T_\ell\mathscr{A})$ and the following are equivalent:*

(a) $\rho = \mathrm{rk}_{\mathbf{Z}}\mathscr{A}(X)$

(b) $\langle\cdot,\cdot\rangle_\ell$ and $(\cdot,\cdot)_\ell$ are non-degenerate and $|Ш(\mathscr{A}/X)[\ell^\infty]| < \infty$.

If these hold, we have

$$|c|_\ell^{-1} = \left|\frac{\det\langle\cdot,\cdot\rangle_\ell}{\det(\cdot,\cdot)_\ell}\right|_\ell^{-1} \cdot \frac{|Ш(\mathscr{A}/X)[\ell^\infty]|}{|\mathrm{Tor}\,\mathscr{A}(X)[\ell^\infty]| \cdot \left|\mathrm{H}^2(\bar{X},T_\ell\mathscr{A})^\Gamma\right|}.$$

Proof. The first statement is Lemma 4.1.23. Now identify the terms in Corollary 4.1.24 using Lemma 4.1.20 (cohomology groups) and Lemma 4.1.21 (regulator). □

Remark 4.1.26. For the vanishing of $\mathrm{H}^2(\bar{X},T_\ell\mathscr{A})^\Gamma$ see Remark 4.2.41 below.

4.2 The case of a constant Abelian scheme

In this section, we specialise to the case where $\mathscr{A} = A \times_k X$ is a *constant* Abelian variety. Let $\Gamma = G_k$ be the absolute Galois group of the finite ground field $k = \mathbf{F}_q$, $q = p^n$.

Lemma 4.2.1. *1. There is an isomorphism $\mathscr{A}[m] \xrightarrow{\sim} A[m]\times_k X$ of finite flat group schemes resp. of constructible sheaves (for $\mathrm{char}\,k \nmid m$) on X.*

2. There is an isomorphism $T_\ell\mathscr{A} = (T_\ell A)\times_k X$ of ℓ-adic sheaves on X for $\ell \neq \mathrm{char}\,k$.

3. There is an isomorphism of Abelian groups

$$\mathscr{A}(X) = \mathrm{Mor}_X(X,\mathscr{A}) \xrightarrow{\sim} \mathrm{Mor}_k(X,A), (f : X \to \mathscr{A}) \mapsto \mathrm{pr}_1 \circ f,$$

and under this isomorphism $\mathrm{Tor}\,\mathscr{A}(X)$ corresponds to the subset of constant morphisms

$$\mathrm{Tor}\,\mathscr{A}(X) \xrightarrow{\sim} \{f : X \to A \mid f(X) = \{a\}\} = \mathrm{Hom}_k(k,A) = A(k).$$

Proof. 1. Consider the fibre product diagram

$$\begin{array}{ccc} A[m] & \longrightarrow & k \\ \downarrow & & \downarrow{\scriptstyle 0} \\ A & \xrightarrow{[m]} & A \end{array}$$

and apply $- \times_k X$.

2. This follows from 1.

3. The inverse is given by $(f : X \to A) \mapsto ((f, \mathrm{id}_X) : X \to A \times_k X = \mathscr{A})$.

   For the second statement: If $f : X \to A$ is constant a, (f, id_X) has finite order $\operatorname{ord} a$ in $A(k)$ since k and thus $A(k)$ is finite. Conversely, if $f : X \to \mathscr{A}$ has finite order n, the image of $\mathrm{pr}_1 \circ f$ lies in the discrete set of n-torsion points (since $\mathrm{pr}_1 : A \times_k X \to A$ is a morphism of group schemes), so is constant because X is connected. □

Example 4.2.2. The rank of the Mordell-Weil group of a constant Abelian variety over a projective space has rank 0, since there are no non-constant k-morphisms $\mathbf{P}^n_k \to A$, see [Mil86], p. 107, Corollary 3.9.

Corollary 4.2.3. *Assume X has a k-rational point x_0. Then there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(k) & \longrightarrow & \mathscr{A}(X) & \longrightarrow & \mathrm{Hom}_k(\mathrm{Alb}_{X/k}, A) & \longrightarrow & 0 \\ & & \downarrow \cong & & \| & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Tor}\,\mathscr{A}(X) & \longrightarrow & \mathscr{A}(X) & \longrightarrow & \mathscr{A}(X)_{\mathrm{Tors}} & \longrightarrow & 0, \end{array}$$

and, if k is a finite *field,*

$$\mathrm{rk}\,\mathscr{A}(X) = r(f_A, f_{\mathrm{Alb}_{X/k}}),$$

where $r(f_A, f_B)$ for A and B Abelian varieties over a finite field is defined in [Tat66a], p. 138.

Proof. The lower row is trivially exact. By the universal property of the Albanese (use that X has a k-rational point x_0), one has $\{f \in \mathrm{Mor}_k(X, A) \mid f(x_0) = 0\} = \mathrm{Hom}_k(\mathrm{Alb}_{X/k}, A)$. Thus the upper row is exact. The left hand vertical arrow is an isomorphism because of Lemma 4.2.1 3. Now the five lemma implies that the right hand vertical arrow is an isomorphism since it is a well-defined homomorphism: Precompose $f : \mathrm{Alb}_{X/k} \to A$ with the Abel-Jacobi map $\varphi : X \to \mathrm{Alb}_{X/k}$ associated to x_0.

The equality for the rank follows from [Tat66a], p. 139, Theorem 1 (a). □

4.2.1 Heights

Lemma 4.2.4. *Let M and N be torsion-free finitely generated $\mathbf{Z}_\ell$-modules, resp. continuous $\mathbf{Z}_\ell[\Gamma]$-modules. Then one has*

$$M \otimes_{\mathbf{Z}_\ell} N = \mathrm{Hom}_{\mathbf{Z}_\ell-\mathbf{Mod}}(M^\vee, N), \tag{4.2.1}$$

$$(M \otimes_{\mathbf{Z}_\ell} N)^\Gamma = \mathrm{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}(M^\vee, N), \tag{4.2.2}$$

where $(-)^\vee$ denotes the $\mathbf{Z}_\ell$-dual.

Proof. This is clear. □

Lemma 4.2.5. *The Weil pairing induces a perfect pairing of torsion-free finitely generated $\mathbf{Z}_\ell[\Gamma]$-modules.*

$$T_\ell\mathscr{A} \times T_\ell(\mathscr{A}^\vee) \to \mathbf{Z}_\ell(1) \tag{4.2.3}$$

$$\mathrm{Hom}(T_\ell\mathscr{A}, \mathbf{Z}_\ell) = T_\ell(\mathscr{A}^\vee)(-1) \tag{4.2.4}$$

Proof. See [Mum70], p. 186. □

Remark 4.2.6. By [Mil80], p. 53, Theorem II.1.9, a $\mathbf{Z}_\ell[\Gamma]$-module is a $\mathbf{Z}_\ell$-sheaf on (the étale site of) $k = \mathbf{F}_q$.

Lemma 4.2.7. *Let $\mathscr{A} = A \times_k X$ be a constant Abelian scheme. Then one has $\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A}) = \mathrm{H}^i(\bar{X}, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} T_\ell A$ as ℓ-adic sheaves on the étale site of k.*

Proof. By the projection formula for $\pi : \bar{X} \to \bar{k}$, one has

$$\begin{aligned}\mathrm{H}^i(\bar{X}, T_\ell\mathscr{A}) &= \mathrm{R}^i\pi_*(T_\ell\mathscr{A})\\ &= \mathrm{R}^i\pi_*(\mathbf{Z}_\ell \otimes_{\mathbf{Z}_\ell} \pi^* T_\ell A)\\ &= \mathrm{R}^i\pi_*(\mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} T_\ell A\\ &= \mathrm{H}^i(\bar{X}, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} T_\ell A\end{aligned}$$

as ℓ-adic sheaves on the étale site of k. □

Theorem 4.2.8 (The height pairing)**.** *Let X/k be a smooth projective geometrically connected variety with Albanese A such that $\mathbf{Pic}_{X/k}$ is reduced. Denote the constant Abelian scheme $B \times_k X/X$ by $\mathscr{A}/X$. Then the trace pairing*

$$\mathrm{Hom}_k(A, B) \times \mathrm{Hom}_k(B, A) \xrightarrow{\circ} \mathrm{End}(A) \xrightarrow{\mathrm{Tr}} \mathbf{Z}$$

tensored with $\mathbf{Z}_\ell$ equals the cohomological pairing

$$\langle \cdot, \cdot \rangle_\ell : \mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} \to \mathrm{H}^{2d}(X, \mathbf{Z}_\ell(d)) \xrightarrow{\mathrm{pr}_1^*} \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) = \mathbf{Z}_\ell.$$

Proof. We have

$$\begin{aligned}\mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} &= \mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})^\Gamma \quad \text{by (4.1.6)}\\ &= \left(\mathrm{H}^1(\bar{X}, \mathbf{Z}_\ell(1)) \otimes_{\mathbf{Z}_\ell} (T_\ell A)(-1)\right)^\Gamma \quad \text{by Lemma 4.2.7}\\ &= (T_\ell \operatorname{Pic}(X) \otimes_{\mathbf{Z}_\ell} (T_\ell A)(-1))^\Gamma \quad \text{by the Kummer sequence}\\ &= \mathrm{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}\left(((T_\ell A)(-1))^\vee, T_\ell \operatorname{Pic}(X)\right) \quad \text{by (4.2.2)}\\ &= \mathrm{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}(\mathrm{Hom}(T_\ell(A^\vee), \mathbf{Z}_\ell)^\vee, T_\ell \operatorname{Pic}(X)) \quad \text{by (4.2.4)}\\ &= \mathrm{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}(T_\ell(A^\vee), T_\ell \operatorname{Pic}(X))\\ &= \mathrm{Hom}_k(A^\vee, \operatorname{Pic}(X)) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \quad \text{by the Tate conjecture [Tat66a].}\end{aligned}$$

Note that $\mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})^\Gamma$ is torsion-free since $\mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})$ is so, and this holds because of the Künneth formula and since $\mathrm{H}^1(\bar{X}, \mathbf{Z}_\ell(1)) = T_\ell \operatorname{Pic}(X)$ is torsion-free because of Lemma 2.1.9. Therefore, in (4.1.6) above, $\ker \alpha = \mathrm{H}^0(\bar{X}, T_\ell\mathscr{A})_\Gamma$ is the whole torsion subgroup of $\mathrm{H}^1(X, T_\ell\mathscr{A})$.

Remark 4.2.9. Note that $T_\ell A = T_\ell \bar{A}$ as $\mathbf{Z}_\ell[\Gamma]$-modules.

$T_\ell\mathscr{A}$ has weight -1 and $T_\ell(\mathscr{A}^\vee)(d-1)$ has weight $-1 - 2(d-1) = -2d + 1$ and we have

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^{2d-1}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))_\Gamma & \xrightarrow{\beta} & \mathrm{H}^{2d}(X, T_\ell(\mathscr{A}^\vee)(d-1)) & \to & \mathrm{H}^{2d}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))^\Gamma & \to & 0 \\ & & & \nwarrow^{f} & & & & & \\ 0 & \to & \mathrm{H}^{2d-2}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))_\Gamma & \to & \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1)) & \xrightarrow{\alpha} & \mathrm{H}^{2d-1}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))^\Gamma & \to & 0, \end{array} \tag{4.2.5}$$

where only the four groups connected by f, α and β can be infinite.

The perfect Poincaré duality pairing

$$\mathrm{H}^1(\bar{X}, \mathbf{Z}_\ell(1)) \times \mathrm{H}^{2d-1}(\bar{X}, \mathbf{Z}_\ell(d-1)) \to \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) \xrightarrow{\sim} \mathbf{Z}_\ell \tag{4.2.6}$$

identifies $\mathrm{H}^{2d-1}(\bar{X}, \mathbf{Z}_\ell(d-1))$ with $T_\ell \operatorname{Pic}(X)^\vee$.

Theorem 4.2.10. *Let X/k be a smooth projective geometrically connected variety with a k-rational point. Then the reduced Picard variety $(\mathbf{Pic}^0_{X/k})_{\mathrm{red}}$ is dual to $\mathrm{Alb}(X)$ and $\mathbf{Pic}^0_{X/k}$ is reduced iff $\dim \mathbf{Pic}^0_{X/k} = \dim_k \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{O}_X)$.*

Proof. By [Moc12], Proposition A.6 (i) or [FGI+05], p. 289 f., Remark 9.5.25, $(\mathbf{Pic}^0_{X/k})_{\mathrm{red}}$ is dual to $\mathrm{Alb}(X)$. By [FGI+05], p. 283, Corollary 9.5.13, the Picard variety is reduced (and then smooth and an Abelian scheme) iff equality holds in $\dim \mathbf{Pic}^0_{X/k} \leq \dim_k \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{O}_X)$. □

Remark 4.2.11. The integer $\alpha(X) := \dim_k \mathrm{H}^1_{\mathrm{Zar}}(X, \mathscr{O}_X) - \dim \mathbf{Pic}^0_{X/k}$ is called the **defect of smoothness**.

Example 4.2.12. One has e. g. $\alpha(X) = 0$ if (b) from Theorem 4.2.23 below is satisfied. This holds true for X an Abelian variety or a curve.

$$\begin{aligned}
\mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} &= \mathrm{H}^{2d-1}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))^\Gamma \quad \text{by (4.1.6)} \\
&= \left(\mathrm{H}^{2d-1}(\bar{X}, \mathbf{Z}_\ell(d-1)) \otimes_{\mathbf{Z}_\ell} T_\ell(A^\vee)\right)^\Gamma \quad \text{by Lemma 4.2.7} \\
&= (T_\ell \operatorname{Pic}(X)^\vee \otimes_{\mathbf{Z}_\ell} T_\ell(A^\vee))^\Gamma \quad \text{by (4.2.6)} \\
&= \operatorname{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}(T_\ell \operatorname{Pic}(X), T_\ell(A^\vee)) \quad \text{by (4.2.2)} \\
&= \operatorname{Hom}_k(\operatorname{Pic}(X), A^\vee) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \quad \text{by the Tate conjecture [Tat66a].}
\end{aligned}$$

Example 4.2.13. In particular, if the characteristic polynomials of the Frobenius on $\operatorname{Pic}(X)$ and $A^\vee$ are coprime, $\operatorname{Hom}_k(A^\vee, \operatorname{Pic}(X)) = 0 = \operatorname{Hom}_k(\operatorname{Pic}(X), A^\vee)$ and the discriminants of the parings $\langle \cdot, \cdot \rangle_\ell$ and $(\cdot, \cdot)_\ell$ are equal to 1.

In the general case, we have to investigate if the following diagram commutes:

Lemma 4.2.14. *Let $f : A \to B$ an homomorphism of an Abelian variety A and $e_A : T_\ell A \times T_\ell A^\vee \to \mathbf{Z}_\ell(1)$ and $e_B : T_\ell B \times T_\ell B^\vee \to \mathbf{Z}_\ell(1)$ be the perfect Weil pairings. Then*

$$e_B(f(a), b) = e_A(a, f^\vee(b))$$

for all $a \in T_\ell A$ and $b \in T_\ell B^\vee$.

Proof. See [Mum70], p. 186, (I). □

Corollary 4.2.15. *Let $f : A \to A$ an endomorphism of an Abelian variety A. Then*

$$\operatorname{Tr}_{T_\ell(A)}(f) = \operatorname{Tr}_{T_\ell(A^\vee)}(f^\vee).$$

Proof. The relation $e_A(f(x), y) = e_A(x, f^\vee(y))$ means that $f^\vee$ is the transpose of f with respect to the bilinear form $V_\ell(A) \times V_\ell(A^\vee) \to \mathbf{Q}_\ell(1)$ (the Weil pairing). Choosing orthonormal bases, the matrix of $f^\vee$ is the transpose of the matrix of f, so they have the same trace. □

$$
\begin{array}{ccc}
\mathrm{H}^1(X, T_\ell\mathscr{A})_{\mathrm{Tors}} \times \mathrm{H}^{2d-1}(X, T_\ell(\mathscr{A}^\vee)(d-1))_{\mathrm{Tors}} & \xrightarrow{\cup} \mathrm{H}^{2d}(X, \mathbf{Z}_\ell(d)) \to \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) \to & \mathbf{Z}_\ell \\
\Big\downarrow \mathrm{pr}_1^* \quad (1) & & \Big\| \\
\mathrm{H}^1(\bar{X}, T_\ell\mathscr{A})^\Gamma \times \mathrm{H}^{2d-1}(\bar{X}, T_\ell(\mathscr{A}^\vee)(d-1))^\Gamma & \xrightarrow{\cup} \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) \to & \mathbf{Z}_\ell \\
\cong \Big\uparrow \quad (2) & & \Big\| \\
\Big(\mathrm{H}^1(\bar{X}, \mathbf{Z}_\ell(1)) \otimes_{\mathbf{Z}_\ell} (T_\ell A)(-1)\Big)^\Gamma \times \Big(\mathrm{H}^{2d-1}(\bar{X}, \mathbf{Z}_\ell(d-1)) \otimes_{\mathbf{Z}_\ell} T_\ell(A^\vee)\Big)^\Gamma & \to \mathrm{H}^{2d}(\bar{X}, \mathbf{Z}_\ell(d)) \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell \to & \mathbf{Z}_\ell \\
\Big\downarrow \cong \quad (3) & & \Big\| \\
\Big(T_\ell \operatorname{Pic}(X) \otimes_{\mathbf{Z}_\ell} (T_\ell A)(-1)\Big)^\Gamma \times \Big(T_\ell \operatorname{Pic}(X)^\vee \otimes_{\mathbf{Z}_\ell} T_\ell(A^\vee)\Big)^\Gamma & \to \operatorname{End}(T_\ell \operatorname{Pic}(X)) \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell \xrightarrow{\mathrm{Tr}} & \mathbf{Z}_\ell \\
\Big\downarrow \cong \quad (4) & & \Big\| \\
\operatorname{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}\big(\underbrace{((T_\ell A)(-1))^\vee}_{T_\ell(A^\vee)}, T_\ell \operatorname{Pic}(X)\big) \times \operatorname{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}\big(T_\ell \operatorname{Pic}(X), T_\ell(A^\vee)\big) & \to \operatorname{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}\big(\underbrace{((T_\ell A)(-1))^\vee}_{T_\ell(A^\vee)}, T_\ell(A^\vee)\big) \to & \mathbf{Z}_\ell \\
\Big\downarrow \cong \quad (5) & & \Big\| \\
\operatorname{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}\big(T_\ell(A^\vee), T_\ell \operatorname{Pic}(X)\big) \times \operatorname{Hom}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}\big(T_\ell \operatorname{Pic}(X), T_\ell(A^\vee)\big) & \xrightarrow{\circ} \operatorname{End}_{\mathbf{Z}_\ell[\Gamma]-\mathbf{Mod}}(T_\ell A^\vee) \xrightarrow{\mathrm{Tr}} & \mathbf{Z}_\ell \\
\cong \Big\uparrow \quad (6) & & \Big\| \\
\operatorname{Hom}_k(A^\vee, \operatorname{Pic}(X)) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \times \operatorname{Hom}_k(\operatorname{Pic}(X), A^\vee) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell & \xrightarrow{\circ} \operatorname{End}_k(A^\vee) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\mathrm{Tr}} & \mathbf{Z}_\ell \\
\cong \Big\uparrow \sigma \quad (7) & & \Big\| \\
\operatorname{Hom}_k(\operatorname{Pic}(X), A^\vee) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \times \operatorname{Hom}_k(A^\vee, \operatorname{Pic}(X)) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell & \xrightarrow{\circ} \operatorname{End}_k(\operatorname{Pic}(X)^\vee) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\mathrm{Tr}} & \mathbf{Z}_\ell \\
\cong \Big\uparrow (-)^\vee \quad (8) & & \Big\| \\
\operatorname{Hom}_k(\operatorname{Alb}(X), A) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \times \operatorname{Hom}_k(A, \operatorname{Alb}(X)) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell & \xrightarrow{\circ} \operatorname{End}_k(\operatorname{Alb}(X)) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\mathrm{Tr}} & \mathbf{Z}_\ell
\end{array}
$$

(1) commutes since $\cup$-product commutes with restrictions.

(2) commutes because of the associativity of the $\cup$-product.

(3) commutes since, in general, one has a commutative diagram of finitely generated free modules over a ring R

$$\begin{array}{ccc} A \times B & \xrightarrow{\langle\cdot,\cdot\rangle} & R \\ \downarrow{\scriptstyle\cong} & & \| \\ C \times C^\vee & \longrightarrow & R \end{array}$$

identifying B with the dual of $C \cong A$ with a perfect pairing $\langle \cdot, \cdot \rangle$ and the canonical pairing $C \times C^\vee \to K$: Choose a basis (a_i) of A and the dual basis (b_i) of B; these are mapped to the bases (c_i) and (c_i') of C and $C^\vee$. Then, under the top horizontal map, $\langle a_i, b_j \rangle = \delta_{ij}$ with the Kronecker symbol δ_{ij}, and under the bottom horizontal map $(c_i, c_j') = \delta_{ij}$.

(4) commutes since, in general, one has a commutative diagram of finitely generated free modules over a ring R

$$\begin{array}{ccccc} (M \otimes_R N^\vee) \times (M^\vee \otimes_R N) & \longrightarrow & \mathrm{End}_{R-\mathbf{Mod}}(M) \otimes_R \mathrm{End}_{R-\mathbf{Mod}}(N) & \xrightarrow{\mathrm{Tr}_M \otimes_R \mathrm{Tr}_N} & R \\ \downarrow{\scriptstyle\cong} & & & & \| \\ \mathrm{Hom}_{R-\mathbf{Mod}}(N, M) \times \mathrm{Hom}_{R-\mathbf{Mod}}(M, N) & \xrightarrow{\circ} & \mathrm{End}_{R-\mathbf{Mod}}(N) & \xrightarrow{\mathrm{Tr}_N} & R \end{array}$$

For proving this, choose bases (a_i) of M and (b_i) of N and their dual bases (a_i') of $M^\vee$ and (b_i') of $N^\vee$. The element $(a_i \otimes b_j', a_k' \otimes b_l)$ of $(M \otimes_K N^\vee) \times (M^\vee \otimes_K N)$ is sent by the upper horizontal arrows to $\delta_{ik}\delta jl$, and by the left vertical arrow to $(b_m \mapsto b_j'(b_m)a_i, a_n \mapsto a_k'(a_n)b_l)$. The latter element is mapped by the lower left horizontal arrow to $b_m \mapsto a_k'(b_j'(b_m)a_i)b_l$ and the trace of this endomorphism is $\delta_{jl}\delta_{ki}$. Therefore, the diagram commutes.

(5) commutes because of precomposing with the isomorphism $(T_\ell\mathscr{A}(-1))^\vee \xrightarrow{\sim} T_\ell(\mathscr{A}^\vee)$ coming from the perfect Weil pairing.

(6) commutes because of [Lan58], p. 186 f., Theorem 3.

(7) commutes because of $\mathrm{Tr}(\alpha\beta) = \mathrm{Tr}(\beta\alpha)$, see [Lan58], p. 187, Corollary 1.

(8) commutes because of Corollary 4.2.15 and since $\mathrm{Pic}(X)$ is dual to $\mathrm{Alb}(X)$ by Theorem 4.2.10 since $\mathbf{Pic}_{X/k}$ is reduced. □

Remark 4.2.16. One has $T_\ell(A^\vee) \neq T_\ell(A)^\vee$ as ℓ-adic sheaves on the étale site of k since their weights are different; however, the Weil pairing gives us under a *non-canonical* isomorphism $\mathbf{Z}_\ell(1) \cong \mathbf{Z}_\ell$ a non-canonical isomorphism $T_\ell(A^\vee) = \mathrm{Hom}(T_\ell(A), \mathbf{Z}_\ell(1)) \cong T_\ell(A)^\vee$, which does not respect the weights. In fact, one has $T_\ell(A^\vee) = T_\ell(A)^\vee(1)$.

The case of a curve as a basis. Let X/k be a smooth projective geometrically connected *curve* with function field $K = K(X)$, base point $x_0 \in X(k)$, Albanese A and Abel-Jacobi map $\varphi : X \to A$ and canonical principal polarisation $c : A \xrightarrow{\sim} A^\vee$, and B/k be an Abelian variety. Let $\alpha \in \mathrm{Hom}_k(A, B)$ and $\beta \in \mathrm{Hom}_k(B, A)$ be homomorphisms.

Let

$$\langle \alpha, \beta \rangle = \mathrm{Tr}(\beta \circ \alpha : A \to A) \in \mathbf{Z}$$

be the trace pairing, the trace being taken as an endomorphism of A as in [Lan58]. By [Lan58], p. 186 f., Theorem 3, this equals the trace taken as an endomorphism of the Tate module $T_\ell A$ or

$\mathrm{H}^1(\bar{A}, \mathbf{Z}_\ell)$ (they are dual to each other, so for the trace, it does not matter which one we are taking, see Theorem 4.1.10).

We now show that our trace pairing is equivalent to the usual Néron-Tate height pairing on curves and is thus a sensible generalisation in the case of a higher dimensional base.

Theorem 4.2.17 (The trace and the height pairing for curves)**.** *Let X/k be a smooth projective geometrically connected* curve *with Albanese A. Then the trace pairing*

$$\mathrm{Hom}_k(A, B) \times \mathrm{Hom}_k(B, A) \xrightarrow{\circ} \mathrm{End}(A) \xrightarrow{\mathrm{Tr}} \mathbf{Z}$$

equals the following height pairing

$$\begin{aligned} \gamma(\alpha) &: X \xrightarrow{\varphi} A \xrightarrow{\alpha} B, \\ \gamma'(\beta) &: X \xrightarrow{\varphi} A \xrightarrow{c} A^\vee \xrightarrow{\beta^\vee} B^\vee, \\ (\gamma(\alpha), \gamma'(\beta))_{ht} &= \deg_X(-(\alpha\varphi, \beta^\vee c\varphi)^* \mathscr{P}_B), \end{aligned}$$

where $\varphi : X \to A$ is the Abel-Jacobi map associated to a rational point of X and $c : A \xrightarrow{\sim} A$ the canonical principal polarisation associated to the theta divisor, and this is equivalent to the usual Néron-Tate canonical height pairing.

Proof. By [Mil68], p. 100, we have

$$\langle \alpha, \beta \rangle = \deg_X((\mathrm{id}_X, \beta\alpha\varphi)^* \delta_1),$$

where $\delta_1 \in \mathrm{Pic}(X \times_k A)$ is a divisorial correspondence such that

$$(\mathrm{id}_X, \varphi)^* \delta_1 = \Delta_X - \{x_0\} \times X - X \times \{x_0\},$$

which we define to be Δ^* (Δ^*_X is the idempotent cutting out $h(X) - h^0(X) - h^2(X) = h^1(X)$ for X a smooth projective geometrically connected curve).

Proposition 4.2.18. *Let X, Y be Abelian varieties over k and $f \in \mathrm{Hom}_k(X, Y)$. Then*

$$(f \times \mathrm{id}_{Y^\vee})^* \mathscr{P}_Y \cong (\mathrm{id}_X \times f^\vee)^* \mathscr{P}_X$$

in $\mathrm{Pic}(X \times_k Y^\vee)$.

Proof. By the universal property of the Poincaré bundle $\mathscr{P}_X$ applied to $(f \times \mathrm{id}_{\mathscr{P}_y})^* \mathscr{P}_Y$, there exists a unique map $\hat{f} : X^\vee \to Y^\vee$ such that

$$(f \times \mathrm{id}_{Y^\vee})^* \mathscr{P}_Y \cong (\mathrm{id}_X \times \hat{f})^* \mathscr{P}_X. \tag{4.2.7}$$

It remains to show that $\hat{f} = f^\vee$.

Let T/k be a variety and $\mathscr{L} \in \mathrm{Pic}^0_0(Y \times_k T)$ arbitrary. By the universal property of the Poincaré bundle $\mathscr{P}_Y$, there exists $g : T \to Y^\vee$ such that $\mathscr{L} = (\mathrm{id}_Y \times g)^* \mathscr{P}_Y$. We want to show

$\hat{f}_* : Y^\vee(T) \to X^\vee(T), g \mapsto \hat{f}g$ equals $f^\vee : \mathrm{Pic}^0_0(Y \times_k T) \to \mathrm{Pic}^0_0(X \times_k T), \mathscr{L} \mapsto f^*\mathscr{L}$. Now we have

$$\begin{aligned} f^\vee(\mathscr{L}) &= (f \times \mathrm{id}_T)^*\mathscr{L} \\ &= (f \times \mathrm{id}_T)^*(\mathrm{id}_Y \times g)^*\mathscr{P}_Y \\ &= (f \times g)^*\mathscr{P}_Y \\ &= (\mathrm{id}_X \times g)^*(f \times \mathrm{id}_{Y^\vee})^*\mathscr{P}_Y \\ &= (\mathrm{id}_X \times g)^*(\mathrm{id}_X \times \hat{f})^*\mathscr{P}_X \quad \text{by (4.2.7)} \\ &= (\mathrm{id}_X \times \hat{f}g)^*\mathscr{P}_X \\ &= \hat{f}_*(\mathscr{L}) \end{aligned}$$

□

There is the following property of the Theta divisor Θ of the Jacobian A of C on A (which is defined in [BG06], p. 272, Remark 8.10.8) and let $\Theta^- = [-1]^*\Theta$ with ϑ and ϑ^- denoting the respecting divisor class).

Proposition 4.2.19. *Let* $c_A = m^*\vartheta^- - \mathrm{pr}_1^*\vartheta^- - \mathrm{pr}_2^*\vartheta^- \in \mathrm{Pic}(A \times_k A)$. *Then*

$$(\varphi, \mathrm{id}_A)^*c_A = -\delta_1 \tag{4.2.8}$$

and

$$(\mathrm{id}_A, \varphi_{\vartheta^-})^*\mathscr{P}_A = c_A. \tag{4.2.9}$$

Proof. See [BG06], p. 279, Propositions 8.10.19–20. □

The Theta divisor induces the canonical principal polarisation $\varphi_\vartheta = c : A \xrightarrow{\sim} A^\vee$ Therefore

$$\begin{aligned} (\gamma(\alpha), \gamma'(\beta)) &= (\alpha\varphi, \beta^\vee c\varphi)^*\mathscr{P}_B \quad \text{by definition} \\ &= (\alpha\varphi \times c\varphi)^*(\mathrm{id}_X, \beta^\vee)^*\mathscr{P}_B \\ &= (\alpha\varphi \times c\varphi)^*(\beta, \mathrm{id}_{A^\vee})^*\mathscr{P}_A \quad \text{by Proposition 4.2.18} \\ &= (\beta\alpha\varphi, c\varphi)^*\mathscr{P}_A \\ &= (\beta\alpha\varphi, \varphi_\vartheta\varphi)^*\mathscr{P}_A \\ &= (\beta\alpha\varphi, \varphi)^*(\mathrm{id}_A \times \varphi_\vartheta)^*\mathscr{P}_A \\ &= (\beta\alpha\varphi, \varphi)^*c_A \quad \text{by (4.2.9)} \\ &= (\varphi, \beta\alpha\varphi)^*c_A \quad \text{by symmetry} \\ &= -(\mathrm{id}_X, \beta\alpha\varphi)^*\delta_1 \quad \text{by (4.2.8)} \end{aligned}$$

Summing up, one has

$$\begin{aligned} (\gamma(\alpha), \gamma'(\beta))_{ht} &= \deg_X(-(\mathrm{id}_X, \beta\alpha\varphi)^*\delta_1) \\ &= -\langle \alpha, \beta \rangle . \end{aligned}$$

By [MB85a], p. 72, Théorème 5.4, this pairing equals the Néron-Tate canonical height pairing. □

Remark 4.2.20. Note that our height coming from the trace pairing is normalised since it is bilinear.

4.2.2 The special L-value

Lemma 4.2.21. *Let $f : A \to B$ be an isogeny of Abelian varieties over a field k and $\ell \neq \operatorname{char} k$. Then f induces an Galois equivariant isomorphism $V_\ell A \xrightarrow{\sim} V_\ell B$ of rationalised Tate modules.*

Proof. There is the exact sequence of étale sheaves over k

$$0 \to \ker(f) \to A \to B \to 0.$$

This induces an exact sequence of G_k-modules

$$0 \to \ker(f)(\bar{k}) \to A(\bar{k}) \to B(\bar{k}) \to 0.$$

Since for an abelian group M, one has $T_\ell M = \operatorname{Hom}(\mathbf{Q}_\ell/\mathbf{Z}_\ell, M)$, applying $\operatorname{Hom}(\mathbf{Q}_\ell/\mathbf{Z}_\ell, -)$ to the above exact sequence yields (writing, by abuse of notation, $T_\ell A$ for $T_\ell A(\bar{k})$)

$$0 \to T_\ell \ker(f)(\bar{k}) \to T_\ell A \to T_\ell B \to \operatorname{Ext}^1(\mathbf{Q}_\ell/\mathbf{Z}_\ell, \ker(f)(\bar{k})).$$

Since $\ker(f)$ is a finite group scheme, we have $T_\ell \ker(f)(\bar{k}) = 0$ by Lemma 2.1.7

$$0 \to T_\ell A \to T_\ell B \to \operatorname{Ext}^1(\mathbf{Q}_\ell/\mathbf{Z}_\ell, \ker(f)(\bar{k})).$$

Since $T_\ell A$ and $T_\ell B$ have the same rank as f is an isogeny (or since $\operatorname{Ext}^1(\mathbf{Q}_\ell/\mathbf{Z}_\ell, \ker(f)(\bar{k}))$ is finite), tensoring with $\mathbf{Q}_\ell$ yields the desired isomorphism. □

Lemma 4.2.22. *Let $k = \mathbf{F}_q$ be a finite field and A/k be an Abelian variety of dimension g. Denote the eigenvalues of the Frobenius Frob_q on $V_\ell A$ by $(\alpha_i)_{i=1}^{2g}$. Then $\alpha_i \mapsto q/\alpha_i$ is a bijection.*

Proof. The Weil pairing induces a perfect Galois equivariant pairing

$$V_\ell A \times V_\ell A^\vee \to \mathbf{Q}_\ell(1),$$

and, choosing a polarisation $f : A \to A^V$, by Lemma 4.2.21, we also have by precomposing a perfect Galois equivariant pairing

$$\langle \cdot, \cdot \rangle : V_\ell A \times V_\ell A \to \mathbf{Q}_\ell(1).$$

Now let v_i be an eigenvector of Frob_q on $V_\ell A$ with eigenvalue α_i. Then, since the pairing $\langle \cdot, \cdot \rangle$ is perfect, there is an eigenvector v_j of Frob_q on $V_\ell A$ such that $\langle v_i, v_j \rangle = x \neq 0$ (otherwise, we would have $\langle v_i, v_j \rangle = 0$ for all eigenvectors v_j, but there is a basis of eigenvectors on the Tate module since the Frobenius acts semi-simply). Now, since the pairing is Galois equivariant, $qx = \operatorname{Frob}_q(x) = \operatorname{Frob}_q \langle v_i, v_j \rangle = \langle \operatorname{Frob}_q v_i, \operatorname{Frob}_q v_j \rangle = \langle \alpha_i v_i, \alpha_j v_j \rangle = \alpha_i \alpha_j \langle v_i, v_j \rangle = \alpha_i \alpha_j x$. Since $x \neq 0$, the statement follows. □

Theorem 4.2.23. *Let $k = \mathbf{F}_q$, $q = p^n$ be a finite field and X/k a smooth projective and geometrically connected variety and assume $\bar{X} = X \times_k \bar{k}$ satisfies*

(a) the Néron-Severi group of $\bar{X}$ is torsion-free and

(b) the dimension of $\mathrm{H}^1_{\mathrm{Zar}}(\bar{X}, \mathscr{O}_{\bar{X}})$ as a vector space over $\bar{k}$ equals the dimension of the Albanese of $\bar{X}/\bar{k}$.

If B/k is an Abelian variety, then $\mathrm{H}^1(X,B)$ *is finite and its order satisfies the relation*

$$q^{gd}\prod_{a_i\neq b_j}\left(1-\frac{a_i}{b_j}\right)=\left|\mathrm{H}^1(X,B)\right|\left|\det\langle\alpha_i,\beta_j\rangle\right|,$$

where A/k is the Albanese of X/k, g and d are the dimensions of A and B, respectively, $(a_i)_{i=1}^{2g}$ and $(b_j)_{j=1}^{2d}$ are the roots of the characteristic polynomials of the Frobenius of A/k and B/k, $(\alpha_i)_{i=1}^{r}$ and $(\beta_i)_{i=1}^{r}$ are bases for $\mathrm{Hom}_k(A,B)$ *and* $\mathrm{Hom}_k(B,A)$, *and $\langle\alpha_i,\beta_j\rangle$ is the trace of the endomorphism $\beta_j\alpha_i$ of A.*

Remark 4.2.24. Note that the $\mathrm{Hom}_k(A,B)$ and $\mathrm{Hom}_k(B,A)$ are free $\mathbf{Z}$-modules of the same rank $r=r(f_A,f_B)\leq 4gd$ by [Tat66a], p. 139, Theorem 1 (a), with f_A and f_B the characteristic polynomials of the Frobenius of A/k and B/k. (Another argument for them having the same rank is that the category of Abelian varieties up to isogeny is semi-simple.) Furthermore, $\mathrm{H}^1(X,B)=\mathrm{H}^1(X,B\times_k X)=Ш(B\times_k X/X)$ since for $U\to X$, one has $B(U)=(B\times_k X)(U)$ by the universal property of the fibre product.

Proof. See [Mil68], p. 98, Theorem 2. □

Example 4.2.25. (a) and (b) are satisfied for $X=A$ an Abelian variety or a curve: (a) because of [Mum70], p. 178, Corollary 2, and (b) since $A^\vee=\mathbf{Pic}^0_{A/k}$ is an Abelian variety, in particular smooth and reduced. See also Theorem 4.2.10, Remark 4.2.11 and Example 4.2.12.

Combining the above, we get

Corollary 4.2.26. *In the situation of Theorem 4.2.23, one has*

$$q^{gd}\prod_{a_i\neq b_j}\left(1-\frac{a_i}{b_j}\right)=\left|Ш(B\times_k X/X)\right|R(B).$$

Definition 4.2.27. *Define the* ***L-function of $B\times_k X/X$*** *as the L-function of the motive*

$$h^1(B)\otimes(h^0(X)\oplus h^1(X))=h^1(B)\oplus(h^1(B)\otimes h^1(X)),$$

namely

$$L(B\times_k X/X,s)=\frac{L(h^1(B)\otimes h^1(X),s)}{L(h^1(B),s)}.$$

Here, the Künneth projectors are algebraic by [Jan92], p. 451, Remarks 2).

Theorem 4.2.28. *The two L-functions Definition 4.1.3 and Definition 4.2.27 agree for constant Abelian schemes.*

Proof. One has $V_\ell B=\mathrm{H}^1(\bar{B},\mathbf{Q}_\ell)^\vee$ by Theorem 4.1.10, $(V_\ell B)^\vee=(V_\ell B^\vee)(-1)$ by Remark 4.2.16, $V_\ell(B)\cong V_\ell(B^\vee)$ by Lemma 4.2.21 and the existence of a polarisation, $\mathrm{H}^i(\bar{X},V_\ell\mathscr{A})=\mathrm{H}^i(\bar{X},\mathbf{Q}_\ell)\otimes V_\ell B$ by Lemma 4.2.7 and $V_\ell\mathscr{A}=(V_\ell B)\times_k X$ by Lemma 4.2.1 since $\mathscr{A}/X$ is constant. Using this, one gets

$$\begin{aligned}L(h^i(X)\otimes h^1(B),t)&=\det(1-\mathrm{Frob}_q^{-1}t\mid \mathrm{H}^i(\bar{X},\mathbf{Q}_\ell)\otimes\mathrm{H}^1(\bar{B},\mathbf{Q}_\ell))\\&=\det(1-\mathrm{Frob}_q^{-1}t\mid \mathrm{H}^i(\bar{X},\mathbf{Q}_\ell)\otimes V_\ell(B^\vee)(-1))\\&=\det(1-\mathrm{Frob}_q^{-1}t\mid \mathrm{H}^i(\bar{X},\mathbf{Q}_\ell)\otimes V_\ell(B)(-1))\\&=\det(1-\mathrm{Frob}_q^{-1}t\mid \mathrm{H}^i(\bar{X},(V_\ell B)\times_k X)(-1))\\&=\det(1-\mathrm{Frob}_q^{-1}q^{-1}t\mid \mathrm{H}^i(\bar{X},V_\ell\mathscr{A}))\\&=L_i(\mathscr{A}/X,q^{-1}t),\end{aligned}$$

for $i = 0, 1$. Now conclude using $h^1(B) = h^0(X) \otimes h^1(B)$ since X is connected. □

Remark 4.2.29. Note that

$$\operatorname*{ord}_{t=1} L(\mathscr{A}/X, t) = \operatorname*{ord}_{s=1} L(\mathscr{A}/X, q^{-1}q^s).$$

Remark 4.2.30. Note that this L-function does not satisfy a functional equation coming from Poincaré duality.

Remark 4.2.31. Now let us explain how we came up with this definition of the L-function.

I omit the characteristic Polynomials $L_i(\mathscr{A}/X, t)$ in higher dimensions $i > 1$ since otherwise cardinalities of cohomology groups would turn up in the special L-value for which I have no interpretation as in the case $i = 0$ and the cardinality of the ℓ-torsion of the Mordell-Weil group or in the case $i = 1$ and the cardinality of the ℓ-torsion of the Tate-Shafarevich group. In the case of a curve C as a basis, my definition is the same as the classical definition of the L-function up to an $L_2(t)$-factor. This factor contributes basically only a factor $|\operatorname{Tor}\mathscr{A}^\vee(X)[\ell^\infty]|$ in the denominator. In the classical curve case $\dim X = 1$, the L-function can also be represented as a product over all closed points $x \in |X|$ of Euler factors.

I came up with my definition of the L-function for a higher dimensional basis via the constant Abelian scheme case: The main contribution of the motive of an Abelian variety is h^1 since the motive of an Abelian scheme is an exterior algebra over h^1. Thus it suggests itself to take $h^1(B)$ as a piece for the motive of the L-function. In the classical case is the motive of $B \times_k C/C$ for a curve C/k just $h^1(B) \otimes h^*(C)$. Here, $h^*(C)$ decomposes as $h^0(C) \oplus h^1(C) \oplus h^2(C)$. The summand $h^1(B) \otimes h^2(C)$ contributes only the cardinality of $\operatorname{Tor}\mathscr{A}^\vee(X)$, which is well-understood. For a higher dimensional basis X one has to omit the higher terms $h^i(X)$ for $i > 1$ since otherwise factors in the special L-value would turn up which I cannot interpret.

We expand

$$\begin{aligned} L(B \times_k X/X, s) &= \frac{L(h^1(B) \otimes h^1(X), s)}{L(h^1(B), s)} \\ &= \frac{\prod_{j=1}^{2d} \prod_{i=1}^{2g} (1 - a_i b_j q^{-s})}{\prod_{j=1}^{2d} (1 - b_j q^{-s})}. \end{aligned}$$

By Lemma 4.2.22, one has for the numerator

$$\prod_{j=1}^{2d} \prod_{i=1}^{2g} (1 - a_i b_j q^{-s}) = \prod_{j=1}^{2d} \prod_{i=1}^{2g} \Big(1 - \frac{a_i}{b_j} q^{1-s}\Big), \tag{4.2.10}$$

and the denominator has no zeros at $s = 1$ by the Riemann hypothesis. Therefore

$$\operatorname*{ord}_{s=1} L(B \times_k X/X, s) = r(f_A, f_b)$$

is equal to the number $r(f_A, f_B)$ of pairs (i, j) such that $a_i = b_j$, which equals by [Tat66a], p. 139, Theorem 1 (a) the rank r of $(B \times_k X)(X)$:

$$\begin{aligned} r(f_A, f_B) &= \operatorname{rk} \operatorname{Hom}_k(A, B) \\ &= \operatorname{rk} \operatorname{Hom}_k(X, B) \quad \text{by the universal property of the Albanese} \\ &= \operatorname{rk} \operatorname{Hom}_X(X, B \times_k X), \end{aligned}$$

see Corollary 4.2.3.

Lemma 4.2.32. *The denominator evaluated at $s = 1$ equals*

$$\prod_{j=1}^{2d}(1 - b_j q^{-1}) = \frac{|\mathrm{Tor}\,(B \times_k X)(X)|}{q^d}.$$

Proof.

$$\begin{aligned}
\prod_{j=1}^{2d}(1 - b_j q^{-1}) &= \prod_{j=1}^{2d}\Big(1 - \frac{1}{b_j}\Big) && \text{by Lemma 4.2.22}\\
&= \prod_{j=1}^{2d}\frac{b_j - 1}{b_j}\\
&= \prod_{j=1}^{2d}\frac{1 - b_j}{b_j} && \text{since } 2d \text{ is even}\\
&= \frac{\deg(\mathrm{id} - \mathrm{Frob}_q)}{q^d} && \text{by Lemma 4.2.22 and [Lan58], p. 186 f., Theorem 3}\\
&= \frac{|B(\mathbf{F}_q)|}{q^d} && \text{since } \mathrm{id} - \mathrm{Frob}_q \text{ is separable}\\
&= \frac{|\mathrm{Tor}\,(B \times_k X)(X)|}{q^d} && \text{by Lemma 4.2.13.}
\end{aligned}$$

□

Remark 4.2.33. Note that, if X/k is a smooth *curve*, $(B \times_k X)(X) = B(K)$ with $K = k(X)$ the function field of X by the valuative criterion since X/k is a smooth curve and B/k is proper. For general X, setting $\mathscr{A} = B \times_k X$ and $K = k(X)$ the function field, $(B \times_k X)(X) = \mathscr{A}(X) = A(K)$ also holds true because of the weak Néron mapping property.

Remark 4.2.34. One has $|\mathrm{Tor}\,(B \times_k X)(X)| = |B(k)| = |B^\vee(k)| = |\mathrm{Tor}\,(B \times_k X)^\vee(X)|$ by Lemma 4.2.13 and Lemma 4.1.12.

Definition 4.2.35. *Define the* **regulator** $R_{\log}(B)$ *by* $\det(\langle\cdot,\cdot\rangle \cdot \log q) =: R_{\log}(B)$.

Remark 4.2.36. We have $R_{\log}(B) = (\log q)^r \cdot R(B)$ with $R(B) = \det(\langle\cdot,\cdot\rangle)$.

Remark 4.2.37. We have

$$\begin{aligned}
1 - q^{1-s} &= 1 - \exp(-(s-1)\log q)\\
&= (\log q)(s-1) + O\big((s-1)^2\big) \quad \text{for } s \to 1
\end{aligned}$$

using the Taylor expansion of exp.

Putting everything together,

$$\begin{aligned}
\lim_{s\to 1}\frac{L(\mathscr{A}/X, s)}{(s-1)^r} &= \frac{q^d(\log q)^r}{|\mathrm{Tor}\,\mathscr{A}(X)|}\prod_{a_i \neq b_j}\Big(1 - \frac{a_i}{b_j}\Big) && \text{by Lemma 4.2.32 and (4.2.10)}\\
&= \frac{q^{(g-1)d}(\log q)^r}{|\mathrm{Tor}\,\mathscr{A}(X)|}\,|\text{Ш}(\mathscr{A}/X)|\,R(B) && \text{by Corollary 4.2.26.}
\end{aligned}$$

We conclude with our main theorem.

Theorem 4.2.38 (The conjecture of Birch and Swinnerton-Dyer for constant Abelian schemes over higher-dimensional bases, version with the height pairing)**.** *In the situation of Theorem 4.2.23, one has:*

1. *The Tate-Shafarevich group $Ш(\mathscr{A}/X)$ is finite.*
2. *The vanishing order equals the Mordell-Weil rank: $\operatorname{ord}_{s=1} L(\mathscr{A}/X, s) = \operatorname{rk} A(K)$.*
3. *There is the equality for the leading Taylor coefficient*
$$L^*(\mathscr{A}/X, 1) = q^{(g-1)d} \frac{|Ш(\mathscr{A}/X)|\, R_{\log}(B)}{|\operatorname{Tor} A(K)|}.$$

Remark 4.2.39. Note that the factor $q^{(g-1)d}$ also appears in [Bau92], p. 286, Theorem 4.6 (ii).

Combining Theorem 4.1.25 and Theorem 4.2.38 and using Theorem 4.2.28, one can identify the remaining two expressions in Theorem 4.1.25:

Corollary 4.2.40. *Assuming (a) and (b) of Theorem 4.2.23, in Theorem 4.1.25 resp. Lemma 4.1.23, all equalities hold and one has*
$$|\det(\cdot,\cdot)_\ell|_\ell^{-1} \cdot \left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right| = 1.$$
Since both factors are positive integers, it follows that
$$|\det(\cdot,\cdot)_\ell|_\ell^{-1} = 1,$$
$$\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right| = 1.$$

Remark 4.2.41. In fact, $\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right| = 1$ (under the assumption (a) above that $\mathrm{NS}(\bar{X})$ is torsion-free) can also be seen directly: The long exact sequence associated to the short exact sequence of étale sheaves on $\bar{X}$
$$1 \to \mu_{\ell^n} \to \mathbf{G}_m \overset{\ell^n}{\to} \mathbf{G}_m \to 1$$
yields the exactness of
$$0 \to \mathrm{H}^1(\bar{X}, \mathbf{G}_m)/\ell^n \to \mathrm{H}^2(\bar{X}, \mu_{\ell^n}) \to \mathrm{H}^2(\bar{X}, \mathbf{G}_m)[\ell^n] \to 0.$$
Combining with the exactness of
$$0 \to \mathrm{Pic}^0(\bar{X}) \to \mathrm{Pic}(\bar{X}) \to \mathrm{NS}(\bar{X}) \to 0$$
and the divisibility of $\mathrm{Pic}^0(\bar{X})$ (hence $\mathrm{H}^1(\bar{X}, \mathbf{G}_m)/\ell^n = \mathrm{NS}(\bar{X})/\ell^n$) and passage to the inverse limit $\varprojlim_n$ gives us
$$0 \to \mathrm{NS}(\bar{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \to \mathrm{H}^2(\bar{X}, \mathbf{Z}_\ell(1)) \to T_\ell \mathrm{H}^2(\bar{X}, \mathbf{G}_m) \to 0$$
since the $\mathrm{NS}(\bar{X})/\ell^n$ are finite by [Mil80], p. 215, Theorem V.3.25, so they satisfy the Mittag-Leffler condition. As $\mathrm{NS}(\bar{X})$ is torsion-free (by assumption (a) above) and $T_\ell\mathrm{H}^2(\bar{X}, \mathbf{G}_m)$ too (by Lemma 2.1.9), it follows that $\mathrm{H}^2(\bar{X}, \mathbf{Z}_\ell(1))$ is torsion-free, so also
$$\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma = (\mathrm{H}^2(\bar{X}, \mathbf{Z}_\ell(1)) \otimes_{\mathbf{Z}_\ell} T_\ell\mathscr{A}(-1))^\Gamma,$$
so
$$\left|\mathrm{H}^2(\bar{X}, T_\ell\mathscr{A})^\Gamma\right| = 1$$
since this group is finite (having weight $2 - 1 = 1 \neq 0$) and torsion-free (as a subgroup of a tensor product of torsion-free groups).

References

[Art69] Artin, Michael: *Algebraic approximation of structures over complete local rings.* In: Publ. Math. IHÉS, **36** (1969), 23–58.

[Bau92] Bauer, Werner: *On the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic $p > 0$.* In: Invent. Math., **108**(2) (1992), 263–287.

[BG06] Bombieri, Enrico and Gubler, Walter: *Heights in Diophantine geometry.* New Mathematical Monographs **4**. Cambridge: Cambridge University Press. xvi, 652 p. 2006.

[BLR90] Bosch, Siegfried; Lütkebohmert, Werner and Raynaud, Michel: *Néron models.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, **21**. Berlin etc.: Springer-Verlag. x, 325 p. 1990.

[BN78] Bayer, Pilar and Neukirch, Jürgen: *On values of zeta functions and ℓ-adic Euler characteristics.* In: Invent. Math., **50** (1978), 35–64.

[Bos03] Bosch, Siegfried: *Algebra.* fünfte Auflage, Springer-Lehrbuch. Berlin: Springer 2003.

[Con] Conrad, Brian: *Math 248B. Picard functors for curves*. URL `http://math.stanford.edu/~conrad/248BPage/handouts/pic.pdf`.

[Del80] Deligne, Pierre: *La conjecture de Weil : II.* In: Publ. Math. IHÉS, **52**(2) (1980), 137–252.

[dJ] de Jong, Aise Johan: *A result of Gabber.* URL `www.math.columbia.edu/~dejong/papers/2-gabber.pdf`.

[Eis95] Eisenbud, David: *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics. **150**. Berlin: Springer-Verlag. xvi, 785 p. 1995.

[FGI+05] Fantechi, Barbara; Göttsche, Lothar; Illusie, Luc et al.: *Fundamental algebraic geometry: Grothendieck's FGA explained.* Mathematical Surveys and Monographs **123**. Providence, RI: American Mathematical Society (AMS). x, 339 p. 2005.

[FK88] Freitag, Eberhard and Kiehl, Reinhardt: *Etale cohomology and the Weil conjecture*. Springer-Verlag 1988.

[Fu11] Fu, Lei: *Étale cohomology theory.* Nankai Tracts in Mathematics **13**. Hackensack, NJ: World Scientific. ix, 611 p. 2011.

[Gor79] Gordon, William J.: *Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer.* In: Compos. Math., **38** (1979), 163–199.

[Gro68] ——— *Le groupe de Brauer. III : Exemples et complements.* Dix Exposes Cohomologie Schemas, Advanced Studies Pure Math. **3**, 88–188. 1968.

[GW10] Görtz, Ulrich and Wedhorn, Torsten: *Algebraic geometry I. Schemes. With examples and exercises.* Advanced Lectures in Mathematics. Wiesbaden: Vieweg+Teubner. vii, 615 p. 2010.

[Har83] Hartshorne, Robin: *Algebraic geometry. Corr. 3rd printing.* Graduate Texts in Mathematics, **52**. New York-Heidelberg-Berlin: Springer-Verlag. XVI, 496 p. 1983.

[Jan92] Jannsen, Uwe: *Motives, numerical equivalence, and semi-simplicity.* In: Invent. Math., **107**(3) (1992), 447–452.

[Kol95] Kollár, János: *Rational curves on algebraic varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. **32**. Berlin: Springer-Verlag. viii, 320 p. 1995.

[KT03] Kato, Kazuya and Trihan, Fabien: *On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$.* In: Invent. Math., **153**(3) (2003), 537–592.

[KW01] Kiehl, Reinhardt and Weissauer, Rainer: *Weil conjectures, perverse sheaves and l-adic Fourier transform.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. **42**. Berlin: Springer. xii, 375 p. 2001.

[Lan58] Lang, Serge: *Abelian varieties.* Interscience Tracts in Pure and Applied Mathematics, **7**. New York-London: Interscience Publishers. XII, 256 p. 1958.

[Lic83] Lichtenbaum, Stephen: *Zeta-functions of varieties over finite fields at $s = 1$.* Arithmetic and geometry, Pap. dedic. I. R. Shafarevich, Vol. I: Arithmetic, Prog. Math. **35**, 173–194. 1983.

[Liu06] Liu, Qing: *Algebraic geometry and arithmetic curves. Transl. by Reinie Erné.* Oxford Graduate Texts in Mathematics **6**. Oxford: Oxford University Press. xv, 577 p. 2006.

[Mat86] Matsumura, Hideyuki: *Commutative ring theory*. Cambridge Studies in Advanced Mathematics, **8**, Cambridge University Press 1986.

[MB] Moret-Bailly (mathoverflow.net/users/7666), Laurent: *étale cohomology with values in an abelian scheme is torsion?* MathOverflow. `http://mathoverflow.net/questions/82082` (version: 2011-11-28), URL `http://mathoverflow.net/questions/82082`.

[MB85a] Moret-Bailly, Laurent: *Métriques permises.* Szpiro, Lucien (ed.), Séminaire sur les pinceaux arithmétiques: La conjecture de Mordell, Paris, 1983–84, Astérisque **127**, 29–87. 1985.

[MB85b] ——— *Un théorème de pureté pour les familles de courbes lisses.* In: C. R. Acad. Sci., Paris, Sér. I, **300** (1985), 489–492.

[Mil68] Milne, James S.: *The Tate-Šafarevič Group of a Constant Abelian Variety.* In: Invent. Math., **6** (1968), 91–105.

[Mil80] ——— *Étale cohomology.* Princeton Mathematical Series. **33**. Princeton, New Jersey: Princeton University Press. XIII, 323 p. 1980.

[Mil86] ——— *Abelian varieties.* Arithmetic geometry, Pap. Conf., Storrs/Conn. 1984, 103–150 (1986). 1986.

[Mil88] ——— *Motivic cohomology and values of zeta functions.* In: Compos. Math., **68**(1) (1988), 59–102.

[Moc12] Mochizuki, Shinichi: *Topics in absolute anabelian geometry I: Generalities.* In: J. Math. Sci., Tokyo, **19**(2) (2012), 139–242.

[Mum70] Mumford, David: *Abelian varieties*. Oxford University Press. viii, 242 p. 1970.

[NSW00] Neukirch, Jürgen; Schmidt, Alexander and Wingberg, Kay: *Cohomology of Number Fields*. Grundlehren der mathematischen Wissenschaften, Vol. **323**, Springer-Verlag 2000.

[Ray70] Raynaud, Michel: *Faisceaux amples sur les schemas en groupes et les espaces homogenes.* Lecture Notes in Mathematics. **119**. Berlin-Heidelberg-New York: Springer-Verlag. III, 219 p. 1970.

[Sch82a] Schneider, Peter: *On the values of the zeta function of a variety over a finite field.* In: Compos. Math., **46** (1982), 133–143.

[Sch82b] ——— *Zur Vermutung von Birch und Swinnerton-Dyer über globalen Funktionenkörpern.* In: Math. Ann., **260** (1982), 495–510.

[Ser02] Serre, Jean-Pierre: *Galois cohomology. Transl. from the French by Patrick Ion. 2nd printing.* Berlin: Springer 2002.

[Sza09] Szamuely, Tamás: *Galois groups and fundamental groups.* Cambridge Studies in Advanced Mathematics **117**. Cambridge: Cambridge University Press. ix, 270 p. 2009.

[Tam94] Tamme, Günter: *Introduction to étale cohomology. Translated by Manfred Kolster.* Universitext. Berlin: Springer-Verlag. ix, 186 p. 1994.

[Tat65] Tate, John T.: *Algebraic cycles and poles of zeta functions.* Arithmetical algebraic Geom., Proc. Conf. Purdue Univ. 1963, 93–110. 1965.

[Tat66a] ——— *Endomorphisms of Abelian varieties over finite fields.* In: Invent. Math., **2** (1966), 134–144.

[Tat66b] ——— *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog.* Dix Exposés Cohomologie Schémas, Advanced Studies Pure Math. **3**, 189–214 (1968); Sém. Bourbaki 1965/66, Exp. No. 306, 415–440. 1966.

[Tat67] ——— *p-divisible groups.* Proc. Conf. local Fields, NUFFIC Summer School Driebergen 1966, 158–183 (1967). 1967.

[Tat74] ——— *The arithmetic of elliptic curves.* In: Invent. Math., **23** (1974), 179–206.

[Wei97] Weibel, Charles A.: *An introduction to homological algebra*. Cambridge studies in advanced mathematics **38**, Cambridge university press 1997.

[EGAII] ——— *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : II. Étude globale élémentaire de quelques classes de morphismes.* In: Publ. Math. IHÉS, **8** (1961), 5–222.

[EGAIII_1] Grothendieck, Alexandre and Dieudonné, Jean: *Eléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie.* In: Publ. Math. IHÉS, **11** (1961), 5–167.

[EGAIII_2] ——— *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Seconde partie.* In: Publ. Math. IHÉS, **17** (1963), 5–91.

[EGAIV_3] ——— *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schémas et des morphismes de schémas, Troisième partie.* In: Publ. Math. IHÉS, **28** (1966), 5–255.

[EGAIV_4] ——— *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie.* In: Publ. Math. IHÉS, **32** (1967), 5–361.

[SGA1] Grothendieck, Alexandre: *Séminaire de Géométrie Algébrique du Bois Marie – 1960–61 – Revêtements étales et groupe fondamental – (SGA 1).* (Lecture notes in mathematics **224**), Berlin; New York: Springer-Verlag. xxii+447 1971.

[SGA4.2] Artin, Michael; Grothendieck, Alexandre and Verdier, Jean-Louis: *Séminaire de Géométrie Algébrique du Bois Marie – 1963–64 – Théorie des topos et cohomologie étale des schémas – (SGA 4) – vol. 2*. Lecture notes in mathematics (in French) **270**, Berlin; New York: Springer-Verlag. iv+418 1972.

[SGA4.3] ——— *Séminaire de Géométrie Algébrique du Bois Marie – 1963–64 – Théorie des topos et cohomologie étale des schémas – (SGA 4) – vol. 3*. Lecture notes in mathematics (in French) **305**, Berlin; New York: Springer-Verlag. vi+640 1972.

[SGA6] Berthelot, Pierre; Grothendieck, Alexandre and Illusie, Luc: *Séminaire de Géométrie Algébrique du Bois Marie – 1966–67 – Théorie des intersections et théorème de Riemann-Roch – (SGA 6).* Lecture notes in mathematics (in French) **225**, Berlin; New York: Springer-Verlag. xii+700 1971.