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Long- versus medium-run identification in fractionally integrated VAR models

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Abstract. We state that long-run restrictions that identify structural shocks in VAR models with unit roots lose their original interpretation if the fractional integration order of the affected variable is below one. For such fractionally integrated models we consider a medium-run approach that employs restrictions on variance contributions over finite horizons. We show for alternative identification schemes that letting the horizon tend to infinity is equivalent to imposing the restriction of Blanchard and Quah (1989) introduced for the unit-root case.

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1 Introduction

Correct specification of integration orders is essential for valid inference in structural vector autoregressive (SVAR[1]) models, in particular, if identification of the structural shocks is related to their long-run effects. Therefore, the literature considered fractional time series models where the orders of integration may take on real (instead of integer) values and are estimated along with the other model parameters.

Recent results suggest that macroeconomic variables such as GDP or inflation may have integration orders smaller than one; see, e.g., Caporale and Gil-Alana (2013) or Gil-Alana (2011). This means that in multivariate fractional integration models, no shock could have an infinitely long-living effect on these variables, regardless of structural restrictions.

As a remedy we suggest the use of medium-run constraints for identification, which was considered in standard SVARs by Uhlig (2004) and Francis et al. (2013) as an alternative to long-run restrictions. We propose several approaches which constrain the variance contribution of selected shocks over a prespecified range of periods. For these finite-horizon criteria we show that by letting the number of periods tend to infinity they become formally identical to the computationally straightforward Blanchard and Quah (1989) condition. We thus provide an economic interpretation of the latter and justify its use in a fractional context.

2 Fractional SVARs and identification

2.1 The model

In order to avoid the restriction of integer integration orders for structural VAR analysis, fractionally integrated vector autoregressive (FIVAR) models have been used; see, e.g., Caporale and Gil-Alana (2011), Gil-Alana and Moreno (2009) or Lovcha (2009). Tschernig et al. (2013) introduced additional flexibility for the short-run dynamics by a fractional lag operator. The subsequent analysis will be based on the popular FIVAR model, noting that the results straightforwardly carry over to the more flexible model of Tschernig et al. (2013) as well. We assume that the bivariate time series $\boldsymbol{x}_t = (x_{1t},\ x_{2t})'$ is generated by

$$\boldsymbol{A}(L)\boldsymbol{\Delta}(L;\boldsymbol{d})\boldsymbol{x}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t, \quad t = 1, 2, \ldots, \tag{1}$$

where L is the lag or backshift operator ($L\boldsymbol{x}_t = \boldsymbol{x}_{t-1}$) and $\boldsymbol{\Delta}(L;\boldsymbol{d}) := \mathrm{diag}(\Delta^{d_1}, \Delta^{d_2})$ holds the fractional difference operators $\Delta^{d_j} = (1-L)^{d_j}$ of real orders d_1 and d_2 (see, e.g., Baillie, 1996). Starting values are set to zero, i.e., $\boldsymbol{x}_t = \boldsymbol{0}$ for $t < 1$ although this assumption can be relaxed along the lines of Johansen (2008).

[1]Abbreviations used in the text: (S)VAR: (Structural) vector autoregression, FIVAR: Fractionally integrated vector autoregression, LRR: Long-run restriction, LRRS: Long-run restricted shock, LRUS: Long-run unrestricted shock

Analogously to standard non-cointegrated VAR models with unit roots, the fractionally differenced series $(\Delta^{d_1} x_{1,t}\ \ \Delta^{d_2} x_{2,t})'$ follows a stable VAR model with all roots of $\boldsymbol{A}(L) = \boldsymbol{I} - \boldsymbol{A}_1 L - \ldots - \boldsymbol{A}_p L^p$ outside the unit circle. In our structural setup, $\boldsymbol{\varepsilon}_t \sim IID(\boldsymbol{0}; \boldsymbol{I})$ is the vector of economic shocks and the impact matrix $\boldsymbol{B}$ holds their contemporaneous effects.

2.2 Long-run and finite-horizon identification schemes

Identification restrictions are needed to uniquely recover the elements of the impact matrix $\boldsymbol{B}$ from the reduced-form error covariance matrix $\boldsymbol{\Omega} = \mathrm{Var}(\boldsymbol{u}_t) = \boldsymbol{B}\boldsymbol{B}'$, where $\boldsymbol{u}_t = \boldsymbol{B}\boldsymbol{\varepsilon}_t$ is the reduced-form disturbance term. To this end, Blanchard and Quah (1989) introduced the concept of long-run restrictions which exclude an infinitely long-lasting impact of selected shocks on a specific variable. In a setup with $d_1 = d_2 = 1$, denote $\boldsymbol{\Xi}(z) := \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j z^j = \boldsymbol{A}(z)^{-1}\boldsymbol{B}$. The effect of a shock in $\boldsymbol{\varepsilon}_t$ on $\boldsymbol{x}_{t+h}$ in the distant future, $h \to \infty$, is given by $\boldsymbol{\Xi}(1)$. Identification of $\boldsymbol{B}$ can be obtained by constraining the permanent effect of, say, $\varepsilon_{2,t}$ on the first variable $x_{1,t}$ using the long-run restriction (LRR)

$$\boldsymbol{\Xi}(1) = \boldsymbol{A}(1)^{-1}\boldsymbol{B} = \begin{pmatrix} \xi_{11}(1) & 0 \\ \xi_{21}(1) & \xi_{22}(1) \end{pmatrix}. \tag{2}$$

We keep this ordering of shocks and refer to $\varepsilon_{1,t}$ as the long-run unrestricted shock (LRUS), while $\varepsilon_{2,t}$ is called the long-run restricted shock (LRRS). Below we will show that in fractional models the LRR (2) loses its original interpretation but features the meaning of a medium-run restriction in the limiting case.

Let $\boldsymbol{x}_t = \boldsymbol{\Theta}_0\boldsymbol{\varepsilon}_t + \boldsymbol{\Theta}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Theta}_2\boldsymbol{\varepsilon}_{t-2} + \ldots$ denote the vector moving average representation of model (1), and denote by $\theta_{ij,h}$ the ijth element of $\boldsymbol{\Theta}_h$, i.e. the impulse responses of the ith variable to the jth structural shock at horizon h.[2] Shocks to $x_{1,t}$ can have ever lasting effects on future realizations of this variable only if $d_1 \geq 1$. Formally, as established by Kokoszka and Taqqu (1995), the impulse responses generally evolve at a rate of order h^{d_1-1} and thus converge to zero at a hyperbolic rate if $d_1 < 1$. The impact of *any* shock to $x_{1,t}$ vanishes with an increasing horizon so that no long-run effect in the terminology of Blanchard and Quah (1989) exists. The economic interpretation of LRR (2) is no longer obvious in this context.

To clarify this interpretation we apply an approach focussing on specified finite horizons and show how it can approximate the long-term behavior. To quantify the influence of the structural shocks for a given horizon, note that the forecast error of $x_{i,t+h}$, $i \in \{1,2\}$, based on known coefficients and information up to period t is given by $\sum_{s=0}^{h-1}\sum_{j=1}^{2}\theta_{ij,s}\varepsilon_{j,t+h-s}$. Consider the forecast error variance

$$\mathrm{Var}_t(x_{i,t+h}) = \sum_{s=0}^{h-1}\left(\theta_{i1,s}^2 + \theta_{i2,s}^2\right) = \sum_{s=0}^{h-1}\theta_{i1,s}^2 + \sum_{j=0}^{h-1}\theta_{i2,s}^2, \quad i = 1,2, \tag{3}$$

[2] Chung (2001) discusses computation of impulse responses and their properties in the vector ARFIMA model while Do et al. (2013) introduce conceptually different generalized impulse responses in our FIVAR setup.

which can be decomposed into one variance component due to the LRUS, $\varepsilon_{1,t}$, and one due to the LRRS, $\varepsilon_{2,t}$. Thus, the share of the h-step forecast variance of variable i due to $\varepsilon_{j,t}$ is given by

$$\omega_{ij,h} = \frac{\sum_{s=0}^{h-1} \theta_{ij,s}^2}{\mathrm{Var}_t(x_{i,t+h})}. \tag{4}$$

In order to require a small impact of the LRRS on the behavior of the first variable h periods ahead, we consider three identification schemes which draw on restricting these variance shares or a variant thereof. We first choose an identification procedure that directly minimizes the forecast error variance share of the LRRS, i.e. FIN1

$$\min_{\boldsymbol{B}} \omega_{12,h} \qquad s.t. \quad \boldsymbol{B}\boldsymbol{B}' = \boldsymbol{\Omega}. \tag{5}$$

Since minimizing the contribution of the restricted shock amounts to maximizing the share of the unrestricted one, in our bivariate model this is identical to the constraint brought forward by Francis et al. (2013).

While economic theory hardly gives any guidance regarding an appropriate value of h, one may instead have an interval of horizons in mind which will be considered relevant. Then it would be reasonable to focus on a range $h \in [l; u]$, over which the LRRS should have minimal impact. Using the average forecast error variance contribution (4) for identification yields FIN2

$$\min_{\boldsymbol{B}} \frac{1}{u-l+1} \sum_{h=l}^{u} \omega_{12,h} \qquad s.t. \quad \boldsymbol{B}\boldsymbol{B}' = \boldsymbol{\Omega}. \tag{6}$$

If a shock $\varepsilon_{2,t}$ has a large effect on $x_{1,t}$ over the first few periods, this is also reflected by the longer-term forecast error variance since short-horizon impulse responses enter FIN1 (5) and FIN2 (6) through the sum in (3). The interpretation of the LRRS as having a restricted effect over longer horizons may suffer from this property. In order to avoid this problem we modify FIN1 and obtain FIN3

$$\min_{\boldsymbol{B}} \frac{\sum_{i=l}^{h} \theta_{12,i}^2}{\mathrm{Var}_t(x_{1,t+h})} \qquad s.t. \quad \boldsymbol{B}\boldsymbol{B}' = \boldsymbol{\Omega}, \tag{7}$$

where now the variance share of exclusively the successive $h-l$ shocks, $\varepsilon_{2,t+1}, \ldots, \varepsilon_{2,t+h-l}$, contributing to a $x_{1,t+h}$ is minimized. The restriction proposed by Uhlig (2004) is obtained as a special case by setting $l = h$ for FIN3. The computation of all three finite horizon restrictions is described in Appendix A.

3 Relation between long-run and finite-horizon restrictions

3.1 The long and the medium run in fractional models

Without the typical interpretation, but still referred to as the LRR in the following, restriction (2) can be likewise imposed in the fractional model. Regardless of $\boldsymbol{B}$, the "long-run covariance matrix" $\boldsymbol{A}(1)^{-1}\boldsymbol{\Omega}[\boldsymbol{A}(1)']^{-1}$ is a function of reduced form parameters only. Using the Cholesky decomposition, this matrix can be uniquely written as $\boldsymbol{\Xi}(1)\boldsymbol{\Xi}(1)'$ due to the triangularity imposed on $\boldsymbol{\Xi}(1)$. The impact matrix $\boldsymbol{B}$ is then straightforwardly calculated as $\boldsymbol{A}(1)\boldsymbol{\Xi}(1)$ which is exactly the same as in standard integrated VAR models.

The effect of the LRR in the fractional setting is not obvious: Tschernig et al. (2013) show that after technically imposing (2), one may write the first variable as

$$x_{1,t} = \underbrace{[\Delta^{-d_1}\xi_{11}(1) + \Delta^{1-d_1}\xi^*_{11}(L)]\,[\varepsilon_{1,t}I(t \geq 1)]}_{\theta_{11,0}\varepsilon_{1,t}+\theta_{11,1}\varepsilon_{1,t-1}+\ldots+\theta_{11,t-1}\varepsilon_{1,1}} \quad + \quad \underbrace{\Delta^{1-d_1}\xi^*_{12}(L)\,[\varepsilon_{2,t}I(t \geq 1)]}_{\theta_{12,0}\varepsilon_{2,t}+\theta_{12,1}\varepsilon_{2,t-1}+\ldots+\theta_{12,t-1}\varepsilon_{2,1}}\;, \tag{8}$$

where the $\xi^*_{1j}(L)$, $j = 1, 2$, are polynomials generating I(0) processes. This implies that $\theta_{11,h}$ and $\theta_{12,h}$ evolve at different rates for large horizons. While $\theta_{11,h} \sim h^{d_1-1}$ is typical for impulse responses of I(d_1) processes (see Chung, 2001), we observe a reduced rate $\theta_{12,h} \sim h^{d_1-2}$ for the responses to the LRRS.

Hence, the LRR (2) and the finite-horizon restrictions yield qualitatively different results with respect to the rate of decay of the impulse responses: Only the LRR (2) affects the memory property of the LRRS-driven component of $x_{1,t}$ (the second term in (8)). In contrast, since $\xi_{12}(1) \neq 0$ for the finite-horizon conditions FIN1 (5), FIN2 (6) and FIN3 (7), these restrictions do not lead to differing rates of $\theta_{11,h}$ and $\theta_{12,h}$.

However, the rate at which the impulse responses $\theta_{12,h}$ evolve is also the fundamental determinant of variance shares over longer horizons. Consequently, for the case $d_1 > 0.5$, we establish that identification by the restrictions FIN1 (5) and FIN3 (7) yields the same result as LRR (2) if $h \to \infty$, while FIN2 (6) is equivalent for $u \to \infty$.

To see the asymptotic equivalence between the LRR and the finite horizon restrictions, let a scalar $\beta \in [-1; 1]$ index the identification scheme (see (A.1) in the Appendix) so that $\xi_{12}(1; \beta)$ explicitly depends on β. Letting the horizon h tend to infinity, it is shown in Appendix B that the objective function of FIN1 (5) fulfills

$$\omega_{12,h} \longrightarrow \xi_{12}(1;\beta)^2 \left[\Gamma(d_1)^2(2d_1-1)\lim_{h\to\infty} h^{1-2d_1}\,\mathrm{Var}_t(x_{1,t+h})\right]^{-1} \quad \text{as } h \to \infty. \tag{9}$$

The latter expression is minimized by taking β as to satisfy the LRR (2) because then $\xi_{12}(1;\beta) = 0$ while the other terms are strictly positive. The same limit results for FIN2 and FIN3, since the difference between their objective functions and $\omega_{12,h}$ is negligible for large horizons u and h, respectively.

3.2 Practical implications

We assess the practical relevance of the previous result and the proximity of different identifying conditions for various values of u and h for a stylized process with reduced form

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & -0.5\\ 0 & 0.5\end{pmatrix}L\right]\begin{pmatrix}\Delta^{0.7} & 0\\ 0 & \Delta^{1.7}\end{pmatrix}\boldsymbol{x}_t = \boldsymbol{u}_t, \quad \boldsymbol{\Omega} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{10}$$

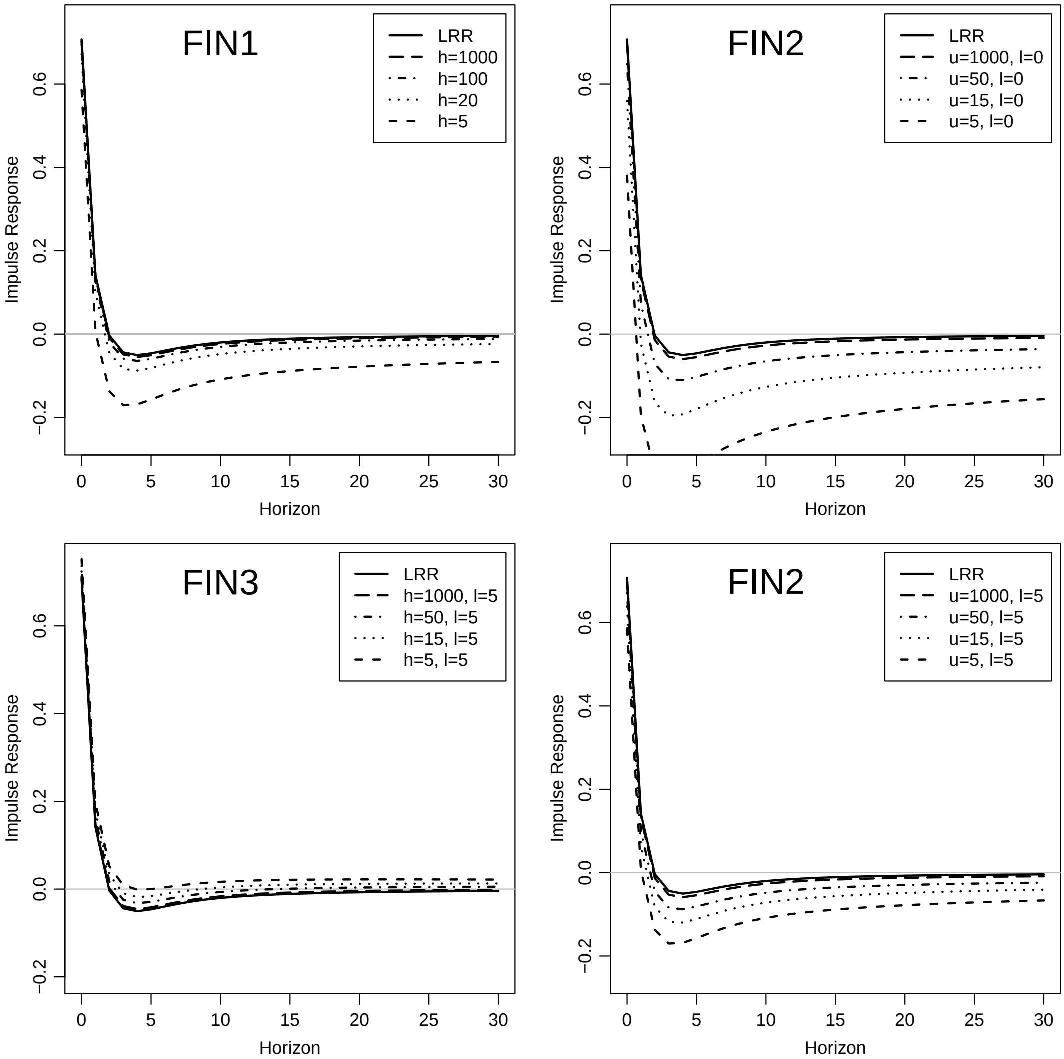

Figure 1: Impulse response function of the first variable to the second shock (LRRS) for the FIVAR process (10) and different identification restrictions imposed.

The integration orders are chosen according to the application to GDP and prices in Tschernig et al. (2013). The upper left panel of Figure 1 shows how the impulse response function of the first variable to the second shock (LRRS) depends on h for the finite-horizon restriction FIN1 (5). With increasing horizon h, the FIN1-based impulse responses approach those resulting from the LRR. For $h = 100$ and larger the differences are negligible. Likewise, the impulse responses of shocks identified by FIN2 (6) and FIN3 (7), shown in the other three panels, also reveal that with growing u both restrictions yield results closer to the LRR (2), especially for the latter restriction and with larger values of l. While here the impulse responses become negative before they converge to zero, this behaviour could be avoided in the more flexible model of Tschernig et al. (2013), for which our results also hold.

In practice, the restriction LRR is thus justified and applicable not only with the long run defined by infinitely long horizons. If identification is meant to focus on the medium run but with rather large horizons (in the present example, say, 50 upwards), the LRR can be taken as a good and technically straightforward approximation.

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A Computation of the impact matrix

For the computation of the impact matrix $\boldsymbol{B}$ satisfying the finite-horizon restrictions FIN1 (5), FIN2 (6) and FIN3 (7), in the given setup any identification scheme (up to sign differences) can be indexed by a single number $\beta \in [-1;1]$. Here we use

$$\boldsymbol{B} = \boldsymbol{P}\boldsymbol{D} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \sqrt{1-\beta^2} & \beta \\ -\beta & \sqrt{1-\beta^2} \end{pmatrix}, \tag{A.1}$$

where $\boldsymbol{P}$ is lower triangular and satisfies $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{\Omega}$, while $\boldsymbol{D}$ is orthonormal by construction.

Note that $\mathrm{Var}_t\,(x_{i,t+h})$, $i = 1, 2$, is independent of β and consider the variance component in (3) that is attributed to shock j. Denoting the reduced form MA coefficient matrices by $\boldsymbol{\Phi}_h = \boldsymbol{\Theta}_h \boldsymbol{B}^{-1}$ and the unit basis vector with 1 in the ith row by $\boldsymbol{e}_i$, the impulse response of variable i to shock j at horizon h is given by

$$\theta_{ij,h} = \boldsymbol{e}_i'\boldsymbol{\Phi}_h\boldsymbol{P}\boldsymbol{D}\boldsymbol{e}_j = \boldsymbol{e}_i'\boldsymbol{\Phi}_h\boldsymbol{P}\boldsymbol{d}_j, \quad \text{where } \boldsymbol{d}_1 = \begin{pmatrix} \sqrt{1-\beta^2} \\ -\beta \end{pmatrix}, \quad \boldsymbol{d}_2 = \begin{pmatrix} \beta \\ \sqrt{1-\beta^2} \end{pmatrix}.$$

Hence, the h-step forecast variance shares are obtained by

$$\omega_{ij,h} = \frac{\sum_{s=0}^{h-1} \boldsymbol{d}_j'\boldsymbol{P}'\boldsymbol{\Phi}_s'\boldsymbol{e}_i\boldsymbol{e}_i'\boldsymbol{\Phi}_s\boldsymbol{P}\boldsymbol{d}_j}{\mathrm{Var}_t(x_{i,t+h})} = \boldsymbol{d}_j'\boldsymbol{V}_{ih}\boldsymbol{d}_j. \tag{A.2}$$

Since $\boldsymbol{V}_{ih}$ is positive semidefinite and symmetric by construction, it can be represented as $\boldsymbol{V}_{ih} = \lambda_1\boldsymbol{v}_1\boldsymbol{v}_1' + \lambda_2\boldsymbol{v}_2\boldsymbol{v}_2'$ (Lütkepohl, 1996, Section 9.13.3, Result (2)), where $\lambda_1 \geq \lambda_2$ denote the nonnegative eigenvalues and $\boldsymbol{v}_1, \boldsymbol{v}_2$ the corresponding orthonormal eigenvectors, respectively. It follows that $\omega_{ij,h} = \lambda_1(\boldsymbol{d}_j'\boldsymbol{v}_1)^2 + \lambda_2(\boldsymbol{d}_j'\boldsymbol{v}_2)^2$ with $0 \leq (\boldsymbol{d}_j'\boldsymbol{v}_1)^2, (\boldsymbol{d}_j'\boldsymbol{v}_2)^2 \leq 1$, where the latter property is due to orthonormality of $\boldsymbol{d}_j$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Since both eigenvalues are nonnegative, the minimum of $\omega_{12,h}$ is obtained and thus restriction FIN1 (5) fulfilled for $\boldsymbol{d}_1 = \boldsymbol{v}_1$ and $\boldsymbol{d}_2 = \boldsymbol{v}_2$. The impact matrix is computed as $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{D}$. Analogously, FIN2 can be imposed by replacing $\boldsymbol{V}_{ih}$ by the average $\bar{\boldsymbol{V}}_{i,lu} = \frac{1}{u-l+1}\sum_{s=l}^{u}\boldsymbol{V}_{is}$, and FIN3 by using $\tilde{\boldsymbol{V}}_{i,lh} = \sum_{s=l}^{h-1}\boldsymbol{P}'\boldsymbol{\Phi}_s'\boldsymbol{e}_i\boldsymbol{e}_i'\boldsymbol{\Phi}_s\boldsymbol{P}$ instead.

B Proof of limit results

To obtain the result (9) note that from Chung (2001, Corollary 2) it follows that the squared impulse responses satisfy

$$\theta_{12,j}^2(\beta) = \frac{\xi_{12}(1;\beta)^2}{\Gamma(d_1)^2} j^{2d_1-2} + o(j^{2d_1-2}).$$

Further, Schotman et al. (2008, eq. A20) state that $\mathrm{Var}_t(x_{1,t+h}) = Ch^{2d_1-1} + o(h^{2d_1-1})$ with $C > 0$, which does not depend on β.

Define

$$X_h(\beta) := h^{1-2d_1} \sum_{j=1}^{h} \theta_{12,j}^2(\beta), \qquad Y_h := h^{1-2d_1} \mathrm{Var}_t(x_{1,t+h})$$

and likewise $Y_\infty := \lim_{h\to\infty} Y_h > 0$. By Schotman et al. (2008, eq. A19) $\sum_{i=1}^{k} i^a k^{-(a+1)} \overset{k\to\infty}{\longrightarrow} (a+1)^{-1}$ and therefore

$$\lim_{h\to\infty} X_h(\beta) = \frac{\xi_{12}(1;\beta)^2}{\Gamma(d_1)^2} \frac{1}{2d_1 - 1}. \tag{B.1}$$

The result (9) follows from $\omega_{12,h} = X_h(\beta)/Y_h$.

The same limit holds for FIN3 (7) with fixed l and $h \to \infty$ because

$$h^{1-2d_1} \sum_{j=1}^{h} \theta_{12,j}^2(\beta) - h^{1-2d_1} \sum_{j=l}^{h} \theta_{12,j}^2(\beta) = h^{1-2d_1} \sum_{j=1}^{l-1} \theta_{12,j}^2(\beta) = O(h^{1-2d_1}) = o(1).$$

The objective of FIN2 (6) is $(u-l+1)^{-1} \sum_{h=l}^{u} \omega_{12,h} = u^{-1} \sum_{h=1}^{u} \omega_{12,h} + o(1)$. Using Davidson (1994, Theorem 2.26), it satisfies

$$\lim_{u\to\infty} \frac{1}{u} \sum_{h=1}^{u} \omega_{12,h} = \lim_{h\to\infty} \omega_{12,h}$$

as was claimed in the text.