
Challenges of capital allocation in one- and multi-period credit risk



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*Living at risk is jumping off the cliff and
building your wings on the way down.*

RAY BRADBURY

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1 Background and motivation

"Managing risk is not just about assessing and monitoring all the things that could go wrong. Rather it is about understanding all the things that need to go right for an organization to achieve its mission and objectives." (United Nations joint staff pension fund [2010])

One of the main tasks of financial institutions is risk transformation, i.e., the conversion of risky investments to lower risk, for example by diversification. Therefore, the quality of risk assessment is a major success factor and one of the most important competitive advantages for banks. In the current environment of sovereign crisis and new regulatory requirements, the optimal use of economic capital and a high quality of risk assessment techniques are especially crucial for banks to achieve their objectives. Hence, the importance of risk management is constantly increasing.

Risk in banking can be divided into three main types: credit risk, market risk and operational risk. This dissertation focuses on the first type. Credit risk concerns the loss of value of credit instruments due to a reduced ability of the counterparty to meet its obligations. The responsibility of credit risk management is to determine the solvency of the counterparty as well as the value of collaterals for each instrument. Furthermore, risk management departments have to assess the overall portfolio risk and its diversification benefits. This is measured through credit risk models. With such models, a bank can determine loss distributions of portfolios or single asset classes, and they form the foundation of economic capital calculation. This leads to a close link between the development and application of credit risk models and capital allocation, i.e., the calculation of the contribution of single assets or asset classes to the portfolio risk.

Economic capital calculation and allocation were pioneered by America's Bankers' Trust in the 1970s (Scott [2002]). They calculated risks and, based on that, the RORAC (return on risk adjusted capital) and charged for the adopted capital, especially on the trading floors. The aim of calculating risk-related returns was to give traders an incentive to

reduce gambling. Lending departments of many of the leading banks picked up the concept. Shocked after large loan losses due to the debt crisis of the 1980s, the major global banks felt they needed a better way of quantifying credit risks. At the same time the results of Black and Scholes [1973] and Merton [1974] made it possible to create new sophisticated models to measure credit risk. Structural credit risk models were developed and followed by reduced form models (Jarrow and Turnbull [1995]). One of the first adopters was JP Morgan. They came up with CreditMetrics, a nowadays heavily used model, and made it publicly available in 1997 (Crouhy et al. [2000]). They were followed by many of their peers, like KMV Corporation or Credit Suisse Financial Products, which released CreditRisk+, a reduced form model, also in 1997. At the end of the 1990s, a fair number of banks pursued internal credit-risk models. Ever since, these models have been developed further, their parametrization has been improved and they are applied to evaluate the risk of business units (see e.g., Hamerle and Rösch [2006]). At the beginning of the 21st century, first mechanisms and algorithms were introduced in order to enable credit portfolio management that surpasses simple steering by return and costs (Rockafellar and Uryasev [2000]).

However, these applications are still mostly restricted to evaluation purposes and do not directly influence top-management decisions. Furthermore, in most cases, capital is only allocated on a business unit level, not on a transaction level (Baer et al. [2011]), and most of the models have a one-year perspective and ignore long-term effects. This means that a lot of the potential advantages are not fully leveraged. The missing allocation of capital on a transaction level, for example, can lead to closing of transactions or loans that destroy economic value. And the backwards looking evaluation can lead to the hindsight of a wrong business decision, instead of influencing a business decision going forward. On the other hand, capital allocation and the calculation of risk-related returns are the prerequisites for risk-related incentives and decisions, which are vital to give the contrast to purely margin- or opportunity-driven decisions and in determining a bank's future strategy. In the recent past and the wake of the global financial crisis, economic capital has gained even more importance. The impact of not anticipated losses has been significant and caused increasing interest in risk-capital models. Due to the insights after the crisis, risk models experience a revival and the demand for practical relevant models increases, which can be integrated into the daily decision process and close the gap between theoretical models and real-life portfolios.

Literature provides several concepts of capital allocation (Stoughton and Zechner [2000], Tasche [2004a], Mausser and Rosen [2007]) and portfolio optimization algorithms (Rocka-

fellar and Uryasev [2000], Hallerbach [2004], Stoughton and Zechner [2007], Dorfleitner et al. [2012]). From a theoretical viewpoint, the tools and concepts for portfolio management are given in a one- and multi-period setting. Different side conditions or the issue of asymmetric information are covered. In most cases practical aspects and applicability on active portfolio steering are not the focus. Challenges that might arise from application of theoretical concepts to real-life scenarios are only briefly touched, such as the effects of granularity, heterogeneity or limited availability of input data. In this context, we want to reflect challenges of the application of risk models, gradient capital allocation and portfolio optimization, consider the limits of model applicability and create awareness of trade-offs.

This dissertation consists of three parts, each of which is an autonomous article. They are devoted to the challenges and limits of capital allocation on the basis of credit-risk models. We aim to address the impact of practical complexities like granularity or heterogeneity on portfolio optimization decisions and compare short-term and long-term optimization results.

Chapter 2 analyzes the conditions under which per-unit capital allocation with several homogeneous asset classes can be justified. Specifically, it considers the effect of portfolio size on allocated capital and portfolio optimization. It analyses the minimum number of assets that is necessary to justify the assumption that the loss distribution of an asset class is independent of the asset class size, so that per-unit risk exists. Gradient allocation is based on the derivative of risk with regard to the asset class size. A portfolio optimization approach based on gradient allocation implicitly assumes that risk scales linearly with the number of obligors. Therefore, the existence of a per-unit risk per obligor is the foundation of a successful RORAC-based portfolio optimization. We prove for a one- and two-factor model and give Monte Carlo evidence for other models, that for two or more homogeneous asset classes the loss distribution functions and their copula converge. This implies that in large subportfolios, a per-unit risk exists, and, multiplied by the number of assets, leads to the subportfolio risk. Hence, for all common credit risk models, portfolio optimization based on gradient allocation is justified as long as the single asset classes have a minimum number of obligors. The barrier of asset class size is dependent on a number of input parameters, such as probability of default, correlation or the chosen risk measure. We give a number of examples and sensitivities for this barrier. If the minimum asset class size is not achieved, per-unit risk capital allocation could lead to erroneous business decisions. In most cases, the risk of a new obligor in a small asset class is overestimated.

Chapter 3 considers moderately inhomogeneous asset classes or subportfolios. The existence of a per-unit risk, and thus the applicability of gradient capital allocation and portfolio optimization, requires a number of conditions. We define the minimum requirements for capital allocation on a subportfolio level. In practice the use of per-unit risk or exposure-weighted per-unit risk is not uncommon. As explained earlier, in most cases risk is only allocated on a business unit level and then broken down by exposure. Therefore, we aim at increasing the awareness of potential pitfalls of the use of per-unit risk for optimization algorithms and show the importance of sensitivity analysis and stress testing. We give evidence that per-unit risk is valid in a moderately inhomogeneous asset class. However, the higher the fluctuation of input parameters, the more important gets the size of the asset class, i.e., it needs more obligors to reach constant per-unit risk. Additionally, we show by simulation that deviant input parameters, like correlation or exposure, can influence the results significantly. As a consequence, increasing one asset class based on an optimization algorithm should sustain the specific asset class composition, i.e., the distribution of all parameters. A second consideration of this chapter is the treatment of a potential systematic under- or overestimation of risk in one asset class, e.g., by a wrong estimation of correlation. This can be tackled by stress testing and the definition and consideration of all relevant scenarios. We suggest two solutions. In the first option, each scenario is weighted by its assumed probability and the bank bases its decisions on expected values. However, if highly improbable stress scenarios are chosen, the probability of the event is extremely low and hard to measure. An alternative approach is to add constraints to the optimization algorithm, e.g., by setting limits for capital ratios or losses in the case of stress. The optimization algorithm then, as a side condition, has to exclude all portfolio compositions that would lead to a capital ratio underneath or a loss above the given barrier in stress. By doing that, one ensures a minimum amount of profitability under stress by waiving return in the base case scenario.

Chapter 4 considers the trade-off between short-term profitability and sustainability of business decisions. It analyses the effects of one-period risk measurement in comparison to multi-period risk measurement. This chapter defines the relevant loss processes, of which risk can be measured. We differentiate between loss and cumulative loss, and considers the effects of different assumptions, such as replacement of write-offs, different maturities or rating migration. This presetting is incorporated into the applied credit risk models. Based on the so-defined different types of loss processes, risk measures can be introduced. Value-at-risk and expected shortfall are expanded in different ways in a multi-period setting with

deviant results for the measured risk in absolute and relative terms. A new risk measure, expected shortfall as weighted capital requirement with discount rate, is defined. It displays the future capital requirement of a loss process as present value of cash flows from in- or decrease of capital requirements. The chapter shows that one-period capital allocation principles and portfolio optimization can be applied to a multi-period setting. Portfolio optimization decisions with a view on multi-period risk can be different from the one-period perspective. Hence, there is a trade-off between short-term and long-term capital needs. This leads to a number of practical challenges in interpretation, implementation and communication, such as the necessary IT and reporting structure.

Finally, we give a short summary of the results and refer to future research areas.

From a methodological view, our results are based on mathematical proof, analytical calculation and Monte-Carlo simulation. We use instruments from probability theory and apply these mathematical derived results to practical situations and challenges. Our objective is to identify and disclose challenges of the application of theoretically developed capital allocation in real life.

2 Justification of per-unit risk capital allocation in portfolio credit risk models

This research project is joint work with Gregor Dorfleitner. The paper has been submitted for publication to *The International Journal of Theoretical and Applied Finance*.

Abstract

Risk capital allocation is based on the assumption that the risk of a homogeneous portfolio is scaled up and down with the portfolio size. In this article we show that this assumption is true for large portfolios, but has to be revised for small ones. On basis of numerical examples we calculate the minimum portfolio size that is necessary to limit the error of gradient risk capital allocation and the resulting error in a portfolio optimization algorithm or pricing strategy. We show the dependency of this minimum portfolio size on different parameters like the probability of default and on the credit risk model that is used.

2.1 Introduction

Calculation and allocation of risk capital is one of the major tasks of risk management in banks. As consequence of the financial crisis, risk management departments continue to gather more influence on business decisions and risk capital gains importance. We consider risk capital and its allocation as one cornerstone of portfolio steering and analyze the conditions concerning homogeneity and asset class size under which per-unit capital allocation with several asset classes can be justified.

When talking about risk capital, one has to differentiate between regulatory capital and economic risk capital. Regulatory capital is necessary to fulfill regulatory requirements and is meant to ensure that the bank is able to meet all its obligations. Economic risk capital is calculated by using a more flexible internal model that does not underly regulatory rules and can therefore represent bank's specifics in a more accurate way. In this paper we restrict ourselves to economic risk capital with a focus on internal portfolio steering.

To reach risk-based decisions, it is necessary to allocate risk or respectively risk capital to the relevant asset classes or obligors. There are three options to determine risk contributions: stand-alone contribution, incremental contribution or marginal contribution (Mausser and Rosen [2007]). Stand-alone contribution calculates the risk of one asset class without considering the rest of the portfolio. Diversification effects are ignored. Incremental contribution is calculated by comparing the risk of the total portfolio with the risk of the portfolio without one asset class. Incremental risk then becomes the resulting delta. This approach is useful for portfolios consisting of few large deals. Marginal risk contribution is calculated through an allocation principle like gradient allocation, which is based on the derivative of the risk measure with respect to the number of obligors. Tasche [2004a] and Tasche [2008] demonstrate that the gradient allocation (also called Euler allocation) is a tool well-suited to measuring the risk of single asset classes or single obligors in portfolios with homogeneous asset classes. The axiomatic framework behind capital allocation principles is provided from a mathematical perspective in Kalkbrener [2005] and from another viewpoint by Tasche [2004a], Buch and Dorfleitner [2008], Merton and Perold [1993] or Stoughton and Zechner [2007]. Each allocation method is connected with a risk measure that can be chosen coherently as introduced in Artzner et al. [1999] and Acerbi [2002], e.g., expected shortfall. Nevertheless, the non-coherent risk measure value-at-risk (VaR) is used in many cases because it is common in practice. Various literary contributions focus on the application of capital allocation to credit portfolios with the target to develop an analytical formula for the risk contribution of one subportfolio, such as Kalkbrener et al.

[2004], Tasche [2004b] and Tasche [2009] based on the work of Gouriéroux et al. [2003]. Mausser and Rosen [2007] give an overview of calculation methods for risk contributions.

Based on the return and the risk contribution of an obligor, deduced from allocations principles, a large number of performance figures have been discussed over the last years and decades, the most well-known ones being RORAC (return on risk adjusted capital, also known or slightly differently defined as RAROC or RARORAC) and EVA[®] (Economic Value Added). Rockafellar and Uryasev [2000] as well as Buch et al. [2011] introduce algorithms that allow the calculation of the optimal amount of capital that should be invested to each subportfolio. The same approach is the foundation for the work of Krokhamal et al. [2001] and Hallerbach [2004], who add constraints to the optimization problem. Application of the algorithms leads to the optimal amount of businesses per asset class, optimal in a sense of the maximization of RORAC. The loss fluctuations of subportfolios are supposed to have a linear structure. The authors implicitly assume a specific loss distribution per obligor that is multiplied with the number of obligors in the subportfolio. Usually, the limit loss distribution of the asset class can serve for this purpose, which approximates the real loss distribution but ignores granularity in the asset classes.

This issue has been addressed in a one-asset class case by so-called granularity adjustments. They are mentioned the first time by Wilde [2001] and are mathematically extended by Gordy [2003], who additionally presents a formula for the specific case of CreditRisk+. Based on this work, Emmer and Tasche [2005] deduce a formula of an granularity adjustment in a structural one-factor model. This tool is very useful for small portfolios consisting of one homogeneous asset class. Gordy [2003] also presents a way to calculate or estimate granularity adjustments for heterogeneous portfolios within certain limits. Finally, Voropaev [2011] derives an elegant formula for granularity adjustment with VaR as risk measure.

In this paper we apply gradient-based capital allocation to loan portfolios and analyze the conditions under which this approach is justifiable. Credit portfolios are typically characterized by the individuality of the single deals or obligors. For each obligor default is a binary event. Hence, the loss distribution of the complete portfolio differs from the loss distribution per obligor whenever there is no perfect dependence. We will show that under a number of reasonable conditions, each asset class has a limit loss distribution, so that even in loan portfolios the incremental risk of an obligor can be approximated by the marginal risk for any asset class with a minimum number of obligors. We base the discussion on the results of McNeil et al. [2005] and Schoenbucher [2006], who prove that

limit-loss distributions exist for a number of credit risk models. We generalize the results for a setting with more than one asset classes and calculate the error of an application of gradient allocation on asset classes of finite size for several examples. Furthermore, we provide evidence that portfolio optimization based on gradient allocation is justifiable in both cases, when several asset classes are scaled up or down proportionally or non-proportionally.

The remainder of this paper is structured as follows: In Section 2.2 we motivate the discussion through an example, which shows how per-unit capital allocation can trigger wrong business decisions in an inadequate business environment, and we introduce the notation and the target for the following sections. In Section 2.3 we provide the mathematical background and show that portfolio optimization based on gradient capital allocation rules makes sense for large portfolios. To broaden the theoretical results we perform different simulations in Section 2.4. There, we give evidence that per-unit risk allocation is justifiable even for portfolios with less strict conditions in a way that we veer towards real world scenarios. In Section 2.5 we conclude with a discussion of our findings.

2.2 Motivation

2.2.1 Motivating example

In order to motivate the discussion we demonstrate the potential pitfalls of capital allocation models in small portfolios by presenting a short example. We show that capital allocation rules can lead to an erroneous calculation of the necessary risk capital whenever there is no perfect dependence of the single assets within each asset class.

We consider a Bernoulli mixture model, or more specifically a two-factor Poisson mixture model, for details see e.g., Crouhy et al. [2000]. As model assumptions, we choose two gamma distributed factors ($\sim \Gamma(0.4, 2.5)$). We assume that the portfolio consists of ten obligors in two subportfolios that consist of 5 obligors of identical exposure (EaD) equal to 1 each. In this model, the probability of default (PD) is random with an expected value of 3%. We choose the model parameter so that the correlation between each two obligors in one asset class is 3%. Furthermore, the two asset classes are assumed to be independent of each other, i.e., the correlation between the two asset classes is 0. This is achieved by choosing the factor loading of the second factor equal to 0 for the first asset class and the other way round. Loss given default (LGD) is beta-distributed ($\sim B(0.5, 0.5)$) with mean 0.5 for all obligors.

For these assumptions, we determine the loss distribution function and can deduce the risk of one new obligor based on three concepts:

1. Incremental risk,
2. even allocation based on the assumption of a homogeneous risk measure and a portfolio-size-independent loss distribution,
3. even allocation adjusted by granularity adjustment.

The loss distribution function for the original portfolio is denoted L_0 and for a portfolio with one extra obligor in the first asset class L_1 . Furthermore, the granularity adjustment for the portfolio, as defined in Gordy [2003], is denoted GA . Via Monte Carlo simulation, we evaluate the risk characteristics of the portfolio and receive as portfolio risks $Var_{0.995}(L_0) = 1.90$ and $Var_{0.995}(L_1) = 1.97$. Furthermore the granularity adjustment is $GA = 1.55$.

The different concepts lead to different risk evaluations. We can calculate the additional risk through adding one obligor as:

1. Incremental risk: $Var_{0.995}(L_1) - Var_{0.995}(L_0) = 1.97 - 1.90 = 0.07$,
2. even allocated risk per-unit: $Var_{0.995}(L_0)/10 = 0.19$,
3. adjusted allocated risk: $(Var_{0.995}(L_0) - GA)/10 = (1.90 - 1.55)/10 = 0.04$.

The assumption of a homogeneous risk measure overestimate the additional risk of a new obligor in a small portfolio significantly. The adjustment by an allocated share of granularity adjustment mitigates this effect, but has the tendency to overestimate the granularity effect for very small portfolios. Additionally, it considers the weighted averages of portfolio characteristics like correlation and ignores the specific parameters per asset class. Furthermore, in a situation with less symmetry regarding asset classes the allocation rules for the granularity adjustment are not obvious.

In summary, we outline that existing allocation methods do not capture the true incremental risk in this specific situation and no investment decision should be based on a standard algorithm in this case. There is an additional important conclusion: The per-unit risk in this case is not constant, i.e., the new obligor adds a lower risk to the portfolio than the existing obligors, even if it has the exact same characteristics. Under the assumption of a constant profit margin, the new obligor increases a performance indicator like RORAC, while an optimization algorithm based on gradient allocation would assume positive homogeneity of risk and wrongly lead to a constant RORAC and ultimately to an incorrect business decision.

2.2.2 Problem statement and notation

Assume a bank's credit portfolio consisting of n subportfolios or asset classes. We use these two expressions equivalently. In practice, one asset class can be defined by common characteristics of the obligors like the industry, the country or a specific range of ratings. An asset class $i \in \{1, \dots, n\}$ consists of $u_i \in \mathbb{N}$ obligors. Loss occurs when an obligor k_i ($k_i = 1, \dots, u_i$) defaults within a given time period. Typically, a period of one year is chosen. This event is described by the random variable $X_{i,k_i} \in \{0, 1\}$ for each obligor in asset class i , where $X_{i,k_i} = 1$ indicates default and $X_{i,k_i} = 0$ indicates no default. For obligor k_i we denote the exposure at default $EaD_{i,k_i} \in [0, 1]$ and loss given default $LGD_{i,k_i} \in [0, 1]$. The loss of the bank due to one obligor k_i is therefore given by $L_{i,k_i} = X_{i,k_i} \cdot EaD_{i,k_i} \cdot LGD_{i,k_i}$ and the loss of an asset class by $L_i := L_i(u_i) = \sum_{k=1}^{u_i} L_{i,k_i}$. The total loss of the portfolio then is calculated as follows:

$$L(u) = \sum_{i=1}^n L_i = \sum_{i=1}^n \sum_{k=1}^{u_i} X_{i,k_i} \cdot EaD_{i,k_i} \cdot LGD_{i,k_i}, \quad (2.1)$$

with $u = (u_1, \dots, u_n)$. If obligor k_i defaults, the bank suffers a loss L_{i,k_i} ; if the obligor does not default it gains a fixed return. Traditionally for credit risk only losses are considered. Given a risk measure ρ , the risk of the portfolio can be calculated as $\rho(L)$. Formally ρ is a mapping from the set of random variables to the positive real numbers. ρ can be chosen coherent (Artzner et al. [1999]). Furthermore, in the following we denote by $X_i := \frac{1}{u_i} \sum_{k=1}^{u_i} X_{i,k_i}$ the fraction of defaults in the asset class i . An asset class is differentiated from the other asset classes by a number of characteristics. As long as not stated differently we assume that within one asset class i all obligors have:

- the same (unconditional) probability of default $P(X_{i,k_i} = 1) = PD_i$,
- the same correlation $\text{corr}(X_{i,k_i}; X_{i,l_i}) = \varrho_i$ ($k_i, l_i = 1, \dots, u_i$) between each other,
- the same correlation $\text{corr}(X_{i,k_i}; X_{j,l_j}) = \varrho_{ij}$ ($k_i = 1, \dots, u_i, l_j = 1, \dots, u_j$) to obligors of another asset class j ,
- the same exposure at default $EaD_{i,k_i} = EaD_i \in [0, 1]$,
- the same distribution of loss given defaults $LGD_{i,k_i} = LGD_i \in [0, 1]$.

Section 2.2.1 will show that in this setting gradient allocation will not necessarily lead to identical risk for identical obligors within one asset class due to the missing linearity of losses. To apply gradient allocation the following condition is necessary: There exists a

random variable \tilde{X}_i , such that

$$\sum_{i=1}^n \sum_{k=1}^{u_i} L_{i,k_i} \sim \sum_{i=1}^n u_i \cdot \tilde{X}_i, \quad (2.2)$$

where \sim is equality in distribution or a close enough approximation. \tilde{X}_i represents the average fluctuation of losses in asset class i . The existence and form of \tilde{X}_i has to be determined. Under the assumption that EaD_i and LGD_i are fixed real numbers, one can set $EaD_i = LGD_i = 1$. We will assume this for the following sections as long as not stated otherwise. This changes condition (2.2) as follows:

$$\sum_{i=1}^n \sum_{k=1}^{u_i} X_{i,k_i} \sim \sum_{i=1}^n u_i \cdot \tilde{X}_i. \quad (2.3)$$

This condition can be decomposed for large portfolios into two steps: Let l_{i,u_i} be the distribution function of $L_i(u_i)$ for $i = 1, \dots, n$. Firstly, for any single asset class, proof has to be given that there is an \tilde{X}_i with distribution function \tilde{l}_i , for which

$$\textbf{Step 1: } \frac{1}{u_i} l_{i,u_i} \rightarrow \tilde{l}_i,$$

for $u_i \rightarrow \infty$ as a weak convergence on the space of univariate distribution functions.

Secondly, the dependency structure of the asset classes has to be considered, i.e., the convergence of the copula of the loss distribution functions of any pair of asset classes i, j with $i \neq j$ has to be proven.

$$\textbf{Step 2: } C_{i,j}^{u_i, u_j}(l_{i,u_i}, l_{j,u_j}) \rightarrow C_{i,j} \text{ pointwise,}$$

for all $u_i \rightarrow \infty$ and $u_j = q \cdot u_i$, q constant, where $C_{i,j}$ and $C_{i,j}^{u_i, u_j}$ are copulas. The convergence for any proportion follows if step two is true for all q . By putting these two steps together, one can use the following lemma.

Lemma 1. *Let $\{l_{i,u_i} : u_i \in \mathbb{Z}_+\}$ and $\{l_{j,u_j} : u_j = q \cdot u_i, q \text{ const}\}$ be two sequences of univariate distribution functions and let $\{C_{i,j}^{u_i, u_j} : u_i \in \mathbb{Z}_+, u_j = q \cdot u_i\}$ be a sequence of copulas; then, for every $u_i \in \mathbb{Z}_+$, a bivariate distribution function is defined through*

$$l_{i,j}^{u_i, u_j}(x, y) := C_{i,j}^{u_i, u_j}(l_{i,u_i}(x); l_{j,u_j}(y)).$$

If the sequences $\{l_{i,u_i}\}$ and $\{l_{j,u_j}\}$ converge to \tilde{l}_i and to \tilde{l}_j respectively in the weak convergence on the space of univariate distribution functions, and if the sequence of copulas $\{C_{i,j}^{u_i,u_j}\}$ converges to the copula $C_{i,j}$ pointwise in $[0,1]^2$, then the sequence $\{l_{i,j}^{u_i,u_j}\}$ converges in the weak topology of the space of bivariate distribution functions against $C_{i,j}(\tilde{l}_i(x); \tilde{l}_j(y))$.

A proof of this lemma can be found in Sempi [2004].

With this lemma, one can show by induction that the joint distribution function of the losses in the asset classes converges weakly. With this result, the convergence of the sum of losses can be concluded, or, alternatively, the convergence of the total loss.

Theorem 1. *Let l_u be the distribution function of total portfolio losses $L(u)$ with $u = (u_1, \dots, u_n)$. Assume the limit distribution function of losses \tilde{l}_i for each asset class i , $i = 1, \dots, n$, exists and is piecewise continuous. If the limit copula $C_{i,j}(\tilde{l}_i(x); \tilde{l}_j(y))$ of any pair of distribution functions exists and is piecewise continuous, the total loss distribution function l_u of the portfolio converges for $u_i \rightarrow \infty$ for any given proportion $u_1 : u_2 : \dots : u_n$ of asset class sizes and the limit per-unit risks per asset class exists.*

A proof of this theorem can be found in A.1.

Note that the assumption of piecewise continuity of losses is not a significant restriction in a real world loan portfolio.

Under the assumption that approximation (2.3) is valid, gradient allocation can be used to calculate the risk contribution of each asset class or obligor and truly measures the additional necessary risk capital of any additional obligor of that kind. We denote the risk contribution of an asset class as $\rho(L_i|L)$, so that $\sum_i \rho(L_i|L) = \rho(L(u))$. An application of gradient allocation according to Tasche [2008] then states that for the risk contribution of obligor k_i we have:

$$\rho^{p.u.}(X_{i,k_i}) = \frac{1}{u_i} \rho(L_i|L) = \frac{1}{u_i} \frac{\partial \rho(L(u))}{\partial u_i}(u_1, \dots, u_n). \quad (2.4)$$

According to the Euler Theorem, the sum of all per-unit risks then adds up to the total risk of the portfolio. Based upon the existence of a per-unit risk $\rho^{p.u.}(X_{i,k_i})$ all theoretical results that use gradient allocation can be applied. In particular, the following approximation can be used:

$$\rho\left(\sum_{i=1}^n \sum_{k=1}^{u_i} X_{i,k_i}\right) \simeq \rho\left(\sum_{i=1}^n u_i \cdot \tilde{X}_i\right). \quad (2.5)$$

2.3 Theoretical results

The example of Section 2.2.1 highlights that there are cases in which the assumption of constant per-unit risk leads to significant errors. This section will prove that under some restrictions this error is small enough to be ignored. The analysis is based on existing results of asymptotic loss distributions, that are put into the context of capital allocation and per-unit risk. We show that there exists a per-unit risk per obligor so that up- and downscaling of risk as it is used in portfolio optimization, based on risk capital allocation is justifiable, i.e., approximation (2.5) is valid.

2.3.1 Factor models - prerequisites

We start with analyzing factor models (also called static structural models, see McNeil et al. [2005]) in the next subsection and then extend this view to mixture models.

Following Rosen and Saunders [2010] or Dorfleitner et al. [2012], we identify each obligor with a so called creditworthiness index, which is an obligor specific random variable. In general, the creditworthiness index is based on the Merton model, which was originally formulated for asset values. In the context of portfolio credit risk modeling it is a hidden variable (see e.g., Crouhy et al. [2000]). Various alternatives to the Merton model and its use of Brownian motion have been discussed in literature, e.g., the Levy simple structural model (Baxter [2007]). The obligor defaults if its CWI_{i,k_i} falls below a given barrier S_i within a given time period (usually one year). Therefore, X_{i,k_i} is expressed as:

$$X_{i,k_i} = \mathbb{1}_{\{CWI_{i,k_i} < S_i\}}.$$

In the factor model, we use CWI_{i,k_i} as a weighted sum of systematic risk factors M_j , and an obligor-specific idiosyncratic factor E_{i,k_i} , which is independent of other idiosyncratic risk factors the systematic factors M_j . The vector of systematic factors is denoted as $\mathbf{M} = (M_j)_j$.

$$CWI_{i,k_i} = \sum_{j=1}^m \alpha_{i,j} M_j + \alpha_{i,E} E_{i,k_i}, \quad (2.6)$$

where M_j for $j = 1, \dots, m$ and E_{i,k_i} for $k_i = 1, \dots, u_i$ are standard normally distributed. Moreover, $\alpha_{i,E}$ is chosen in a way that CWI_{i,k_i} itself is standard normally distributed. To prevent the calculations from becoming too technical we will focus on a one-factor model, i.e., $m = 1$ and we write $M_1 = M$.

In this case it follows that $\text{corr}(CWI_{i,k_i}, CWI_{i,l_i}) = \alpha_{i,j}^2 = \alpha_i^2$.¹ This model is very similar to the CreditMetrics model of JP Morgan or the KMV model (see e.g., Crouhy et al. [2000]).²

2.3.2 Factor models - one asset class

In the case of one asset class we will omit the index i indicating the number of the asset class. As first step, we prove that there exists an \tilde{X} which satisfies $\sum_{k=1}^u X_k \sim u \cdot \tilde{X}$. and therefore for any homogeneous risk measure:

$$\rho\left(\sum_{k=1}^u X_k\right) = u \cdot \rho(\tilde{X}). \quad (2.7)$$

From now on we will refer to $\rho(\tilde{X})$ as per-unit risk of an obligor.

The probability of default of one given obligor k is conditional on the state of the factor $M = c$:

$$PD(c) = P[CWI_k < S | M = c] = \Phi\left(\frac{S - \alpha c}{\sqrt{1 - \alpha^2}}\right) \quad (2.8)$$

for all k . With this equation we conclude:

Theorem 2. *Assume we have a portfolio consisting of one asset class. Let S be the default threshold and α^2 the correlation between the obligors' CWIs. Then the loss distribution of $X := 1/u \sum_{k=1}^u X_k$ representing the default proportion of the complete portfolio based on a one-factor model as defined above converges against a limit distribution function \tilde{l} and*

$$\tilde{l}(x) = \Phi\left(\frac{1}{\alpha}\left(\sqrt{1 - \alpha^2} \Phi^{-1}(x) - S\right)\right), \quad x \in [0, 1]. \quad (2.9)$$

For a proof see Schoenbucher [2006].

With this loss distribution function, the risk (measured as a function only depending on \tilde{l}) converges against a limit $\rho(X) = \rho(\frac{1}{u} \sum_{k=1}^u X_k) \rightarrow \rho(\tilde{X})$. Thus, \tilde{X} can be defined by this limit and for any fixed u the total portfolio risk can be approximated by $u \cdot \rho(\tilde{X})$ for every homogeneous risk measure.

¹If we choose a one-factor model, we determine the correlation between two asset classes by choosing the correlation within the asset classes ($\text{corr}(CWI_{i,k_i}, CWI_{j,l_i}) = \alpha_i \cdot \alpha_j$). In a multi-factor model all correlation can be chosen individually.

²In CreditMetrics the probability of default is given by rating tables and rating transition matrices which we ignore for our discussion.

2.3.3 Factor models - more than one asset classes

More relevant for the question of capital allocation is the case of more asset classes. Thus, we consider portfolios of two asset classes. The results can be easily translated into more than two asset classes by induction. We again assume that each asset class is homogeneous as defined in Section 2.2, but the asset classes differ from one another. We still assume for simplicity that all assets have the same exposure at default and loss given default equals 1, but the probability of default and correlation can be different.

In a general setting with the notation introduced in Section 2.3.1 we use the following lemma.

Lemma 2. *Assume a portfolio of n asset classes. Let \mathbf{M} be a vector of systematic factors, $(c_j)_j \in \mathbb{R}^m$ a vector of constants and let X be the fraction of defaults in the portfolio (i.e., $0 \leq X \leq 1$).*

Under the assumption that $\frac{u_i}{\sum_{k=1}^n u_k}$ converges for all i , and conditional on $\mathbf{M} = (c_j)_j$ the convergence

$$X - \frac{1}{\sum_{i=1}^n u_i} \left(\sum_{i=1}^n \left(u_i PD_i((c_j)_j) \right) \right) \xrightarrow{a.s.} 0$$

holds, where $PD_i((c_j)_j) = P[CWI_i < S_i | \mathbf{M} = (c_j)_j]$.

This lemma is an extension of the law of large numbers and follows from the work of Lucas et al. [2001].

For two asset classes formula (2.6) implies:

$$\begin{aligned} CWI_{1,k_1} &= \alpha_1 M + \sqrt{1 - \alpha_1^2} E_{1,k_1} && \text{for all } k = 1, \dots, u_1 \text{ from asset class 1,} \\ CWI_{2,k_2} &= \alpha_2 M + \sqrt{1 - \alpha_2^2} E_{2,k_2} && \text{for all } l = 1, \dots, u_2 \text{ from asset class 2,} \\ \alpha_i^2 &= \text{corr}(CWI_{i,k_i}, CWI_{i,l_i}), && \text{for } i = 1, 2, \\ \alpha_1 \alpha_2 &= \text{corr}(CWI_{1,k_1}, CWI_{2,l_2}). \end{aligned}$$

We denote the probability of default of assets from the two asset classes PD_1 and PD_2 . We can now consider two cases: Case 1 assumes an asset class with a fixed number of obligors while the second asset class is scaled up. Case 2 considers a proportional upscaling of both asset classes.

Theorem 3. *Let X be the fraction of defaults in the portfolio (i.e., $0 \leq X \leq 1$). Then the following holds:*

1. *If we fix the number of obligors u_2 of the second asset class and only increase the number of obligors u_1 of asset class 1, we get*

$$P\left[|X - PD_1(c)| > \epsilon | M = c\right] \xrightarrow{a.s.} 0 \text{ as } u_1 \rightarrow \infty,$$

2. *If we increase the number of obligors of both asset classes simultaneously, whilst retaining a fixed proportion ($u_1 : u_2 = a : b$, with $a, b > 0$), we obtain*

$$P\left[\underbrace{\left|X - \frac{a}{a+b} PD_1(c)\right|}_{=:a'} - \underbrace{\frac{b}{a+b} PD_2(c)}_{=:b'} > \epsilon | M = c\right] \xrightarrow{a.s.} 0 \text{ as } u_1, u_2 \rightarrow \infty.$$

The proof of this theorem follows directly from Lemma 2 with $n = 2, m = 1$.

Based on this and the one asset class case of Schoenbucher [2006], in the following we generalize the results for limit loss distributions for more than one asset classes.

Theorem 4. *Assume we have a portfolio consisting of two asset classes or subportfolios. Let S_1 and S_2 be the default thresholds for the two subportfolios, and α_1^2 and α_2^2 the correlation within the obligors of the subportfolios. Then the loss distribution of the complete portfolio based on a one-factor model as defined before converges against a limit distribution function \tilde{l} , and \tilde{l} is given as follows:*

1. *For a fix number of obligors in the second subportfolio u_2 :*

$$\tilde{l}(x) = \Phi\left(\frac{1}{\alpha_1}\left(\sqrt{1 - \alpha_1^2} \Phi^{-1}(x) - S_1\right)\right), \quad x \in [0, 1].$$

2. *For fixed proportion between the number of obligors of the two subportfolios ($u_1 : u_2 = a : b$, with $a, b > 0$ and $a' = \frac{a}{a+b}, b' = \frac{b}{a+b}$):*

$$\begin{aligned} \tilde{l}(x) = & \int_{x'=s_1}^{s_2} \min \left[\Phi\left(\frac{1}{\alpha_1}\left(\sqrt{1 - \alpha_1^2} \Phi^{-1}\left(\frac{x - x'}{a'}\right) - S_1\right)\right); \right. \\ & \left. \Phi\left(\frac{1}{\alpha_2}\left(\sqrt{1 - \alpha_2^2} \Phi^{-1}\left(\frac{x'}{b'}\right) - S_2\right)\right) \right] dx' \end{aligned}$$

$$= \int_{x'=s_1}^{s_2} C^{FH} \left(\tilde{l}_1 \left(\frac{x-x'}{a'} \right), \tilde{l}_2 \left(\frac{x'}{b'} \right) \right) dx', \quad x \in [0, 1],$$

with $s_1 = \max(0; x - a')$, $s_2 = \min(x; b')$, C^{FH} Frechét-Hoeffding upper bound copula, \tilde{l}_i limit loss distribution of asset class i ($i = 1, 2$).

A proof of this theorem can be found in A.2.

Again, the loss distribution converges against a limit distribution. We can calculate the per-unit risk of one obligor in the two cases by

1. $u_2 =: c$ fix and $u_1 \gg u_2$.

$\sum_1^{u_2} X_{2,k_2}$ is bounded by a constant c , so $\rho(X) \leq \rho(\frac{1}{u_1+c} \sum X_{1,k_1} + \frac{c}{u_1+c})$. Hence, the second term in the brackets converges to zero if u_1 gets larger, so $\rho(\tilde{X})$ is an approximation for the average risk contribution for one obligor from the first asset class.

2. $u_1 : u_2 = q$ fix $\Rightarrow u_1 + u_2 = u_2 \cdot (q + 1)$, where $q \in \mathbb{Q}_+$ and $u_2 \rightarrow \infty$.

When define the risk of the limit loss distribution function as follows:

$$R_q := \lim_{\substack{u_1, u_2 \rightarrow \infty \\ u_1/u_2 = q}} \rho \left(\frac{1}{u_1 + u_2} \left(\sum_{k_1=1}^{u_1} X_{1,k_1} + \sum_{k_2=1}^{u_2} X_{2,k_2} \right) \right). \quad (2.10)$$

R_q now describes one "package" consisting of $\frac{q}{q+1}$ obligors of asset class 1 and $\frac{1}{q+1}$ obligors of asset class 2. To use this for portfolio optimization, one then has to split the risk of the package to the single obligors.

A more general way of modeling two asset classes is achieved through increasing the number of systematic factors. This approach has the advantage of a better presentation of concentration risks. In a two-factor model, the two asset classes are described as follows:

$$\begin{aligned} CWI_{1,k_1} &= \frac{1}{\sqrt{\alpha_{11}^2 + \alpha_{12}^2 + 1}} (\alpha_{11}M_1 + \alpha_{12}M_2 + E_{1,k_1}), \\ CWI_{2,k_2} &= \frac{1}{\sqrt{\alpha_{21}^2 + \alpha_{22}^2 + 1}} (\alpha_{21}M_1 + \alpha_{22}M_2 + E_{2,k_2}), \end{aligned}$$

where $\mathbf{M} = (M_1, M_2)$ is a two-dimensional random vector of systematic factors with

$\mathbf{M} \sim N_2(0, \Omega)$ normally distributed with a given covariance matrix Ω . M_1, M_2 and the idiosyncratic factors E_{1,k_1}, E_{2,k_2} are standard normally distributed. We choose the systematic factors orthogonally, i.e., independently without loss of generality. For the conditional probabilities of default in this case we obtain

$$\begin{aligned} PD_i(c_1, c_2) &= P[CWI_i < S_i | (M_1, M_2) = (c_1, c_2)] = \\ &= P\left[E_{1,k_1} < \sqrt{\alpha_{i1}^2 + \alpha_{i2}^2 + 1} S_i - \alpha_{i1}c_1 - \alpha_{i2}c_2\right] \\ &= \Phi\left(\sqrt{\alpha_{i1}^2 + \alpha_{i2}^2 + 1} S_i - \alpha_{i1}c_1 - \alpha_{i2}c_2\right). \end{aligned}$$

The loss distribution function can be calculated via two-dimensional integration over all values that can be realized by M_1 and M_2 . This is analytically complex. Due to the independence of the systematic factors M_1 and M_2 we obtain for every single asset class:

$$\tilde{l}_i(x) = \Phi_{0, \sqrt{\alpha_{i1}^2 + \alpha_{i2}^2}}\left(\Phi^{-1}(x) - \sqrt{\alpha_{i1}^2 + \alpha_{i2}^2} S_i\right), \quad x \in [0, 1]$$

where $\Phi_{\mu, \sigma}$ is the normal distribution with mean μ and standard deviation σ . In the general case we have to solve the following integral.

$$\tilde{l}(x) = \int_{\mathbb{R}^2} P[X \leq x | \mathbf{M} = (c_1, c_2)] f(c_1, c_2) dc_1 dc_2,$$

where $f : \mathbb{R}^2 \rightarrow [0, 1]$ denotes the density function of \mathbf{M} . From Lemma 2 we deduce the existence of a limit distribution function of the complete portfolio for any fix limit proportion of the two asset classes, i.e., for $\frac{u_i}{u_1 + u_2}$ converges for $i = 1, 2$. The limit distribution then only depends on the proportion of the asset classes, the probabilities of default and the factor loadings defined by the choice of $\alpha_{i,j}$ for $i, j = 1, 2$.

$$\tilde{l}(x) = \int_{\mathbb{R}^2} \mathbb{1}_{\{a'PD_1(c_1, c_2) + b'PD_2(c_1, c_2)\}} f(c_1, c_2) dc_1 dc_2.$$

We conclude that even in a more-factor threshold model, the limit of the loss distribution exists under a number of reasonable assumptions. This allows us to use gradient allocation and consequently portfolio optimization tools in this setting as well. Again, we have based the results on some restrictions, namely the assumption of homogeneous asset classes as well as the condition of a proportional up-scaling of the number of obligors in the asset classes.

2.3.4 Mixture models

So far we have discussed factor models for which defaults occur when the creditworthiness index (*CWI*) falls below a threshold. Mixture models are a more general class of models (McNeil et al. [2005]). In these models, the systematic factors still form a base for calculating the probability of default, but the precise mechanism of how default is calculated can be defined in various ways.

For an asset class i , let X_{i,k_i} , $k_i = 1, \dots, u_i$ be a random variable. In the case of a binomial random variable, the model is called Bernoulli mixture model. Then, the probability of default for obligor k_i is defined by

$$P[X_{i,k_i} = 1 | \mathbf{M} = (c_j)_j] = p_{i,k_i}(\mathbf{M}), \text{ with } j = 1, \dots, m,$$

and is a random variable itself. The distribution of p_{i,k_i} describes the approach in a closer way. The very common model CreditRisk+, which was proposed by Credit Suisse in 1997. It is a Poisson mixture model, and thus, p_{i,k_i} is Poisson distributed and it follows:

$$P[L_i = r | \mathbf{M} = (c_j)_j] = \exp\left(-\sum_{k_i=1}^{u_i} \lambda_{k_i}((c_j)_j)\right) \frac{\left(\sum_{k_i=1}^{u_i} \lambda_{k_i}((c_j)_j)\right)^r}{r!}. \quad (2.11)$$

In particular, CreditRisk+ is a one-factor model with $\lambda_{k_i}(M) = c_{k_i}M$, where $c_{k_i} > 0$ is a constant, and M is assumed to be $\Gamma(\alpha, \beta)$ -distributed. For further details see Crouhy et al. [2000] and McNeil et al. [2005]. Asset classes are differentiated by their distributions of default probabilities and the correlation within the asset class and to another asset class. We additionally release the definition of a homogeneous asset class by allowing different exposures per obligor. For a given obligor k_i ($k_i \in \{1, \dots, u_i\}$) the exposure at default EaD_{i,k_i} is deterministic with values in $(0, 1]$, and the loss given default LGD_{i,k_i} is a random variable with values in $(0, 1]$ that is independent of the default indicator X_{i,k_i} .

We focus on one asset class according to step one in Section 2.2.2 and omit index i . For the further discussion we make the following assumptions.

1. There are functions $l_u : \mathbb{R}^m \rightarrow [0, 1]$ such that conditional on \mathbf{M} , the losses $(L(u))_{u \in \mathbb{N}}$ form a sequence of independent random variables with mean $l_u((c_j)_j) = E[L(u) | \mathbf{M} = (c_j)_j]$.

2. There exists a function $\tilde{l} : \mathbb{R}^u \rightarrow \mathbb{R}$ such that

$$\lim_{u \rightarrow \infty} \frac{1}{u} E[L(u) | \mathbf{M} = (c_j)_j] = \tilde{l}((c_j)_j).$$

3. There is a constant $c < \infty$ such that $\sum_{k=1}^u (EaD_k/k)^2 < c$ for all u .

This means we demand independence of the obligors (or their losses) at a given state of economy. The second assumption states that the expected loss for a given state of economy converges, which means the essential composition of the asset class, in terms of PD , EaD and LGD , must converge to a fixed constant. Finally, the third assumption prevents the exposure from growing with the number of obligors approaching ∞ . Thus far we have obtained the result by giving each exposure a weight of $1/u$ for u obligors in the portfolio. The theorem shows that for every Bernoulli mixture model under a few basic assumptions the loss distribution converges against a limiting distribution. Once this becomes certain, the desired approximative equality (2.3) is valid for every risk measure.

Based on these assumptions, we can draw a conclusion for the limit loss distribution.

Theorem 5. *Let $u \in \mathbb{N}$ be the number of obligors in the portfolio. If the above assumptions 1.-3. hold, then*

$$\lim_{u \rightarrow \infty} \frac{1}{u} L(u) = \tilde{l}((c_j)_j), \quad P(\cdot | \mathbf{M} = (c_j)_j) - a.s.$$

A proof of this theorem can be found in Frey and McNeil [2003].

In the special case of a one-factor Bernoulli mixture model, we obtain a stronger result:

Theorem 6. *Let $\mathbf{M} = M$ be a one-dimensional random variable with distribution function G . Assume that the conditional asymptotic loss function $\tilde{l}(c)$ is strictly increasing and right continuous and that G is strictly increasing at $q_\eta(M)$, i.e., $G(q_\eta(M) + \delta) > \eta$ for every $\delta > 0$. Thus, if assumptions 1.-3. hold, then*

$$\lim_{u \rightarrow \infty} \frac{1}{u} q_\eta(L(u)) \rightarrow \tilde{l}(q_\eta(M)).$$

A proof of this theorem can be found in Frey and McNeil [2003]. This theorem proves that under the given conditions the tail of the limit loss distribution only depends on the tail of the factor M . Hence, for any quantile-based risk measure, there exists a limit per-unit risk.

At first glance the definition of a mixture model appears to be different from the threshold model we previously discussed. However, McNeil et al. [2005] prove that every multifactor threshold model can be equivalently described by a Bernoulli mixture model. With this equivalence, we can apply all results in this section to the setting we have considered so far in Section 2.3.2 and 2.3.3. Nevertheless, sections 2.3.2 and 2.3.3 provide additional information through the analytically calculated limit distribution functions. Furthermore, we can mathematically prove the convergence of the distribution function of the complete portfolio and hence the copula function (see Theorem 1).

2.3.5 Granularity adjustments

Given the convergence of a loss distribution, which can be assumed according to the previous sections for many sets of conditions, there still remains an error for finite asset classes. Due to the finite granularity, the asset class will keep undiversified idiosyncratic risk. The so-called granularity adjustment approximates the remaining idiosyncratic risk for a finite asset class and captures the error made with an accuracy of $o(1/n)$. The granularity adjustment is defined as the second order Taylor expansion of the difference between risk of loss distribution and risk of limit loss distribution.

In a one-asset class setting, this approach leads to acceptable approximations. Nevertheless, the application in a more-asset-classes setting does not lead to the desired results regarding portfolio optimization. Granularity adjustments in a two-asset-classes case is so far not covered in literature. One approach for risk capital calculation according to Gordy [2003] is to consider the two asset classes as one heterogeneous portfolio. Based on weighted averages of input parameters, Gordy [2003] shows a reasonable approximation for the portfolio granularity adjustment. The resulting value does not solve the issue of portfolio optimization targets for two reasons: Firstly, the result is highly dependent on a specific portfolio composition. Secondly, an allocation method of the granularity adjustment to the different asset classes does not exist so far. Therefore, the adjustment cannot be incorporated in the optimization algorithm. Developing an allocation principle comprising granularity adjustments might be an interesting topic for further research.

2.3.6 Summary of theoretical results

For factor models of the discussed form, we have seen that the loss distribution of the total portfolio converges if the number of obligors increases. The same holds true for Bernoulli mixture models under the condition of convergence of the copula of the loss distribution

functions of the asset classes. This result was based on some economically reasonable assumptions. This means that there always exists a limit loss distribution function \tilde{l} , which describes the losses of large portfolios. Based on the limit distribution function \tilde{l} for large portfolios the per-unit risk $\rho(\tilde{X})$ is constant, meaning it is independent of the portfolio size.

In the case of one asset class, the total portfolio risk can be calculated via $u \cdot \rho(\tilde{X})$. This implies that a portfolio consisting of a sum of u obligors can be represented as $\sum_{k=1}^u X_k \stackrel{d}{\sim} u \cdot \tilde{X}$. With this approximative equality capital allocation and portfolio optimization based on capital allocation are acceptable. In the case of two or more asset classes the limit loss distribution also exists as long as the asset classes are up-scaled in a fixed proportion. The risk calculated from it describes the risk of a package consisting of a specific proportion of obligors of the asset classes.

Summarizing, we have obtained several theoretical results. Firstly, in large portfolios we can allocate a per-unit risk to every obligor for factor and Bernoulli mixture models, which can be used to estimate the risk of a new obligor of the same characteristics. Secondly, the per-unit risk exists for any risk measure we choose. Thirdly, under the used assumptions portfolio optimization algorithms based on gradient allocation are justifiable. Nevertheless, the theoretical discussion opens up the following questions: Which error do we make per asset class in small portfolios? And how many obligors are necessary to limit this error? What happens if the portfolio is not perfectly homogeneous? What happens if we scale two or more asset classes up or down and the proportion is not fixed? The next section will deal with these questions based on Monte Carlo simulation.

2.4 Evidence from simulation

In this section we supplement the analytically derived results from the previous section through simulation. In particular, we investigate the questions left unanswered in the previous section. This includes the speed of convergence, the dependence on input variables and the effect of an increase of asset classes in a non-fixed proportion.

2.4.1 General model assumptions

To make all results comparable we fix some assumptions and input parameters for the simulations for all following sections. All assumptions hold as long as not stated otherwise. We analyze the dependency of the loss distribution function and the portfolio size and

determine how many obligors are necessary to gain a constant per-unit risk. In order to do this, we compare portfolios with identical characteristics but a different number of obligors. For this reason, we will indicate the number of obligors of the first asset in the two scenarios by u_i and u'_i ($i = 1, 2$).

- number of obligors
 - case 1: $u_i = 100$, $u'_i = 1,000$,
 - case 2: $u_i = 1,000$, $u'_i = 1,500$,
- exposure: $EaD_i = 1/u_i$ and accordingly $EaD'_i = 1/u'_i$ in the case of one asset class, and respectively $EaD_i = 1/(u_1 + u_2)$ and $EaD'_i = 1/(u'_1 + u_2)$ for two asset classes,
- loss given default: LGD_i beta-distributed random variable ($\sim B(0.5, 0.5)$).

As risk measure we choose the VaR. Note that all considerations in the theoretical discussion were made pertaining to the loss distribution function, and thus, any quantile-based, homogeneous risk measure could be chosen. Still, the setting is more flexible than the one we chose in the theoretical part because we allow random LGD s that were not part of the theoretical discussion for factor models. Furthermore, the setting allows us to analyze the influence of the asset class size on characteristics of the loss distribution. We compare the VaR or the quantile of loss distributions between portfolios of different sizes. We weight the exposures so that the calculated VaR corresponds to the per-unit risk of one obligor; see also equation (2.7).

Based on these assumptions, we simulate different scenarios, analyze the output graphically and draw conclusions for the per-unit risk.

2.4.2 One asset class

As above, we consider a single asset class first. Even if this case is not relevant for capital allocation, it can nevertheless produce results, like the dependency of minimal asset class size and the asset class parameters, that can be transferred to more asset classes. Thus, this section serves as preparation for the following sections.

In addition to the assumptions of Section 2.4.1 we choose the following parameters for a first discussion: (unconditional) $PD = 2\%$ and correlation $\varrho = 2.73\%$.

In the case of a structural one-factor model this translates into $\alpha^2 \approx 0.16$.³ There,

³To use a realistic input parameter we choose the correlation according to the Basel II formula for big corporations: $\alpha^2 = 0.12 \frac{1-e^{-50PD}}{1-e^{-50}} + 0.24(1 - \frac{1-e^{-50PD}}{1-e^{-50}})$; see e.g., Basel Committee on Banking Supervision [2011].

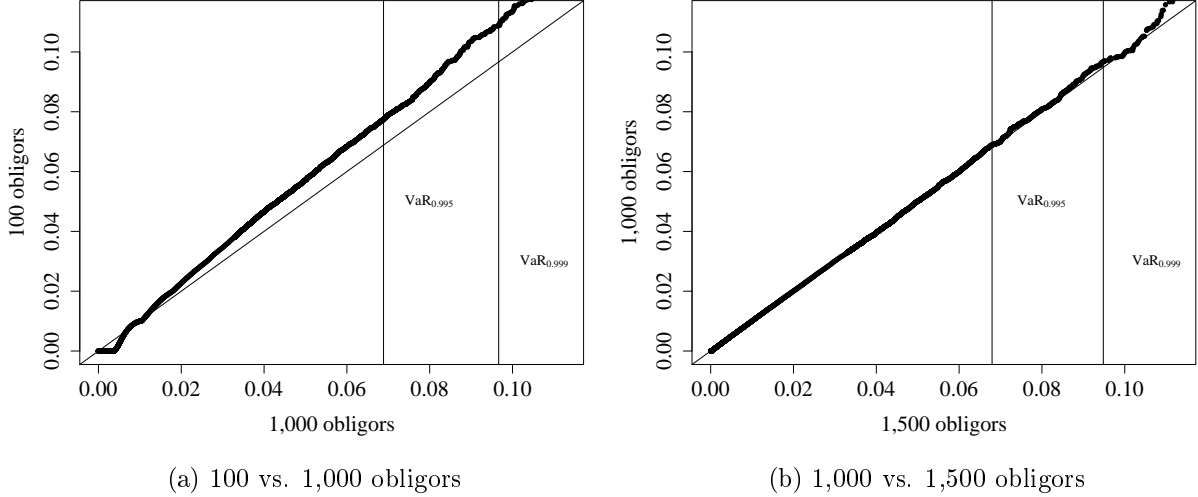


Figure 2.1: Comparison of Q-Q-plots for one asset class simulated with 100,000 model runs in a one-factor model as described in formula (2.6). Both figures describe asset classes with $PD = 2\%$ and $\varrho = 2.73\%$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class size on the x-axis.

the density function of losses for the smaller portfolio (100 obligors) differs significantly from the density function of the larger portfolio (1,000 obligors). The difference is due to the fatter tails in the loss density, meaning a higher per-unit risk. The per-unit risk in a portfolio consisting of 1,000 obligors or 1,500 obligors is almost identical, while it is clearly higher in a small portfolio. This conclusion can also be drawn from the Q-Q-plot of the two portfolios in Figure 2.1a, which is steeper than the bisecting line, while the plot in Figure 2.1b is almost identical with the bisecting line. At a confidence level of $\eta = 0.995$ the per-unit risk in the small portfolio is 0.078, while it is 0.069 in a portfolio of 1,000 obligors, which corresponds to a decrease by 12.4% for larger portfolios. Figure 2.1b shows that this effect disappears for large u . As mentioned in Section 2.3.5, this issue can be mitigated by the consideration of granularity adjustments in the one-asset-class case.

Since these results follow directly from the convergence of the distribution function, the same behavior can be expected from alternative risk measures such as expected shortfall, an example can be found in A.3.

In Figure 2.2 we fix the confidence level η for the VaR at 0.99 and look at the per-

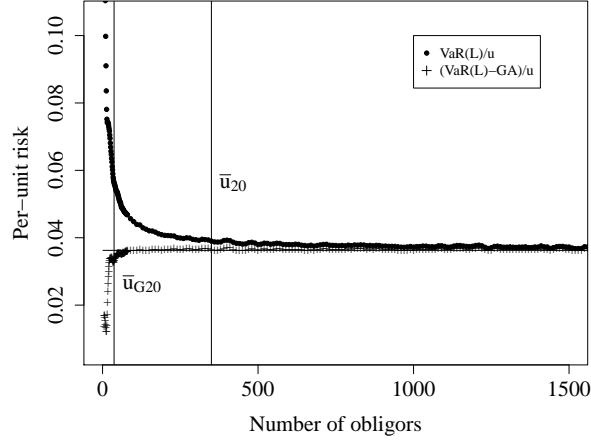


Figure 2.2: Per-unit risk vs. portfolio size simulated with 100,000 model runs for a portfolio size of 1 to 2,000 obligors ($PD = 1\%$, $\eta = 0.99$, correlation according to Basel II formula for big corporations). \bar{u}_{20} and \bar{u}_{G20} mark the number of necessary obligors to reach a maximum delta between per-unit risk and limit per-unit risk of $\epsilon = 20\text{bp}$ before and after granularity adjustment (GA).

unit risk depending on the number of obligors. The dots show per-unit risk without consideration of granularity adjustments, the crosses include granularity adjustments. In the first case, per-unit risk is larger for small portfolios but then converges. From the simulation result we calculate the minimum \bar{u}_ϵ for a given ϵ to obtain:

$$\frac{1}{\bar{u}_\epsilon} \rho \left(\sum_{k=1}^{\bar{u}_\epsilon} X_k \cdot LGD_k \right) - \rho(\tilde{X}) \leq \epsilon.$$

If we choose, for example, a maximum error of $\epsilon = 20 \text{ bp} = 0.002$, we obtain $\bar{u}_\epsilon = \bar{u}_{20} = 350^4$. This means that the per-unit risk is overrated by maximal 20 bp as long as the portfolio has a minimum size of 350 obligors. Taking into account granularity adjustments, convergence is reached significantly faster and the number of necessary obligors reduces to $\bar{u}_{G20} = 30$. As a comparison we choose $\epsilon = 10 \text{ bp}$ and obtain an obviously higher $\bar{u}_{10} = 534$ and $\bar{u}_{G10} = 43$.

As a next step we analyze the sensitivity of this result to the input parameters. Ta-

⁴The calculation is based on simulation for a one-factor model. The limit-per-unit risk is approximated by the per-unit risk in a portfolio of 2,000 obligors.

ble 2.1a shows the number of obligors necessary to gain a constant per-unit risk with a maximum error of $\epsilon = 20$ bp depending on the choice of η and PD . The simulation reveals that the number of necessary obligors depends on both parameters. We see that a higher confidence level demands a larger portfolio to gain a constant per-unit risk. This is due to the fact that the limit loss distribution function $\tilde{l}(x)$, given in equation (2.9), is approximated in a better way for a larger number of obligors. However, there is a high dependency on the PD of the obligors, too. If we look at $\tilde{l}(x)$ in equation (2.9), we see that the probability of defaults shifts the argument in the normal distribution function to the left. This means that a higher probability of defaults leads to a movement of the data points out of the tails. In this way, the observations match the mathematical derivation.

We conduct the same discussion for an alternative credit risk model, namely a Poisson mixture model as introduced in Section 2.3.4. As additional input parameters to Section 2.4.1 we choose shape and scale of common factor $\gamma_1 = 0.8^5$, $\gamma_2 = 1/\gamma_1$. These parameters guarantee that correlation corresponds to the correlation chosen for the factor model (see Section 2.4.2). The main result for a mixture model is similar to the one for a factor model. In small portfolios the per-unit risk is higher than in larger portfolios. However, if the portfolio size is high enough the per-unit risk converges to the same limits as in a factor model. The effect can again be reduced by consideration of granularity adjustments. The speed of convergence as well as the range of minimum obligors are in the same order of magnitude as for the one-factor model, as shown in Table 2.1b. There is one small difference in the results: The dependency of the number of necessary obligors on the chosen quantile is slightly higher in a mixture model than in a structural one-factor model.

2.4.3 More than one asset classes

Next, we discuss the most relevant scenario, namely the case of more than one asset classes and consider an example with two asset classes. From the previous sections we know that the marginal distributions, meaning the loss distributions of the single asset classes, converge for a large number of obligors. In this section we will give evidence of the convergence of the copula. With the existence of a limit copula we know that, additional to the limit loss distributions per asset class, there is a limit dependency structure for all

⁵This value is calculated, using $\varrho = \frac{1}{PD(1-PD)} \left(\int_{-\infty}^{\infty} \Phi^2\left(\frac{S-\alpha c}{\sqrt{1-\alpha^2}}\right) d\phi(c) - PD^2 \right) = \frac{1}{PD(1-PD)} \left(\binom{1+\gamma_1}{2} \left(\frac{2PD}{2PD+\gamma_1}\right)^2 \left(\frac{\gamma_1}{2PD+\gamma_1}\right)^{\gamma_1} - PD^2 \right)$

$\eta \backslash PD$	0.5%	1%	2%	5%	10%	0.5%	1%	2%	5%	10%
0.95	194	228	284	356	424	134	198	254	352	372
0.97	226	274	320	428	440	204	256	322	380	406
0.99	290	350	382	444	516	292	354	428	486	492
0.999	309	406	477	544	554	442	506	578	588	560

(a) One-factor model
(b) Mixture model

Table 2.1: Number of necessary obligors to achieve constant per-unit risk with a maximum error of 20 bp simulated with 100,000 model runs in a structural one-factor and a mixture model as described in equation (2.6) and (2.11) with VaR as risk measure.

combinations of asset class sizes. As explained in Section 2.2.2, this gives evidence of the existence of per-unit risks. For any error ϵ and any ratio of asset class sizes the per-unit risks can therefore be calculated. In the case of two asset classes this means: For every pair of large number of obligors in two asset classes (u_1, u_2) we can approximate the following equality with an error ϵ :

$$\rho \left(\sum_{k_1=1}^{u_1} X_{1,k_1} \cdot LGD_{1,k_1} + \sum_{k_2=1}^{u_2} X_{2,k_2} \cdot LGD_{2,k_2} \right) = \rho \left(u_1 \tilde{X}_1 + u_2 \tilde{X}_2 \right).$$

This result drawn from simulation is very powerful and more general than theoretical proof in Theorems 4 and 5.

We choose the input parameters for the model as follows:

- $PD_1 = 1\%$,
- $PD_2 = 2\%$,
- $\alpha_1^2 = 0.19$, so $\varrho_1 = 2.28\%^6$,
- $\alpha_2^2 = 0.16$, so $\varrho_2 = 2.73\%^6$,
- $\alpha_1 \alpha_2 = 0.12$, so $\varrho_{12} = 2.49\%$,
- LGD_1, LGD_2 beta-distributed random variables ($\sim B(0.5, 0.5)$).

The resulting loss distribution function and empirical copula for a one-factor model are shown in Figure 2.3.

In order to draw initial conclusions about the limit loss distribution function one can look at the plot of the contour lines of the copula based on a one-factor model, and additionally, on a two-factor model as well as a mixture model (with $\gamma_1 = 0.6$) as shown in

⁶To use a realistic input parameter we choose the correlation according to the Basel II formula for big corporations; see e.g., Basel Committee on Banking Supervision [2011].

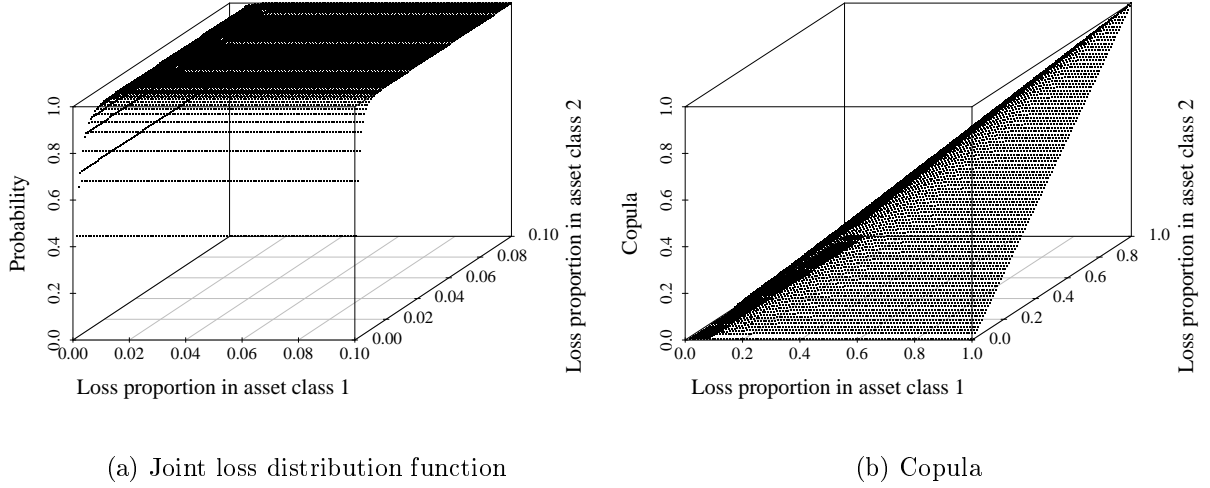


Figure 2.3: Simulated joint loss distribution function and empirical copula of loss variables of two asset classes for $u_1 = u_2 = 2,000$ with input parameters as introduced in Section 2.4.3. Simulated with 10,000 model runs for the loss distribution and 10,000 nodes.

Figure 2.4. The copulas in Figure 2.4a and 2.4c show similarity with the Frechét-Hoeffding upper bond. This result is in concordance with the theoretical result in Theorem 4 for the one-factor model.

When simulating the copula for alternative pairs of u_1 and $u_2 \in \{1, \dots, 2,000\}$ the average distance of the copula functions to the copulas in Figure 2.4 decreases. As an example, the results of this simulation are shown in Table 2.2 for a one-factor model, and respectively, for a two-factor model and mixture model. The convergence seems slow since errors smaller than 1% are only produced with approximately 1,500 obligors per asset class. However, the error is clearly smaller if we focus on the cases of the high number of defaults that are relevant for risk measurement. If we only consider the highest 10% of occurring default numbers per asset class, for example in the case of 100 obligors per asset class, the average error reduces from 0.1585 to 0.0114. Hence, through simulation, we provide evidence of the convergence of the copula function. Based on the convergence of the copula and respectively of the joint distribution function of the two asset classes, we can deduce the convergence of the per-unit risk, independently of the chosen risk measure.

To visualize the results we consider one specification of the model by choosing a specific proportion of asset class sizes according to case 2 of Theorem 4 in Section 2.3.3. We will

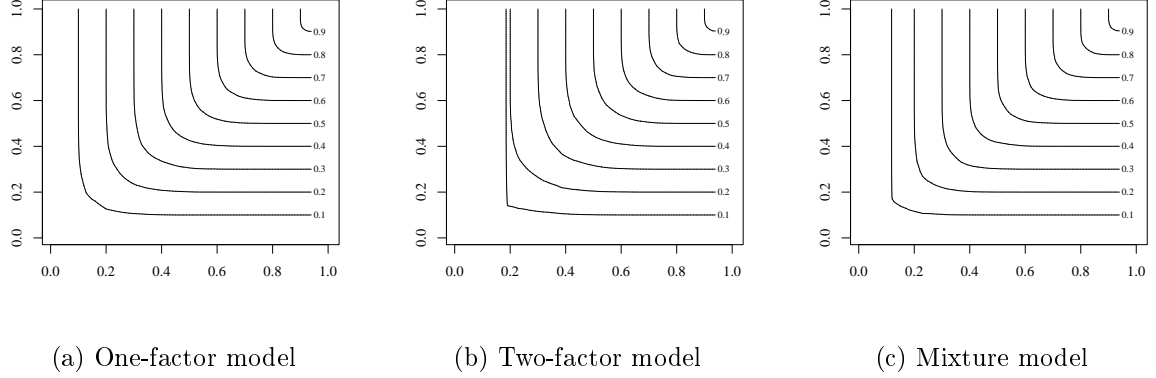


Figure 2.4: Contour lines of empirical copulas of loss variables for two asset classes for $u_1 = u_2 = 2,000$ with input parameters as introduced in Section 2.4.3. Simulated with 10,000 model runs for the loss distribution and 10,000 nodes.

$u_1 \backslash u_2$	10	100	500	1,000	1,500
One-factor model					
10	0.3168	0.2750	0.2497	0.2487	0.2485
100	0.2915	0.1585	0.0848	0.0694	0.0653
500	0.2861	0.1380	0.0354	0.0170	0.0116
1,000	0.2858	0.1327	0.0304	0.0115	0.0068
1,500	0.2857	0.1360	0.0289	0.0099	0.0053
Two-factor model					
10	0.3067	0.2847	0.2647	0.2595	0.2576
100	0.2902	0.2105	0.1337	0.1065	0.0978
500	0.2890	0.1839	0.0698	0.0342	0.0229
1,000	0.2836	0.1763	0.0618	0.0271	0.0111
1,500	0.2834	0.1715	0.0630	0.0255	0.0107
Mixture model					
10	0.3125	0.2721	0.2525	0.2452	0.2442
100	0.2881	0.1631	0.0916	0.0830	0.0785
500	0.3164	0.1302	0.0457	0.0276	0.0231
1,000	0.2766	0.1256	0.0327	0.0163	0.0097
1,500	0.2810	0.1246	0.0287	0.0113	0.0052

Table 2.2: Convergence of copula measured as average distance of the copula with $u_1 = u_2 = 2,000$ with input parameters as introduced in Section 2.4.3. Simulated with 10,000 model runs for each loss distribution and 10,000 nodes per copula.

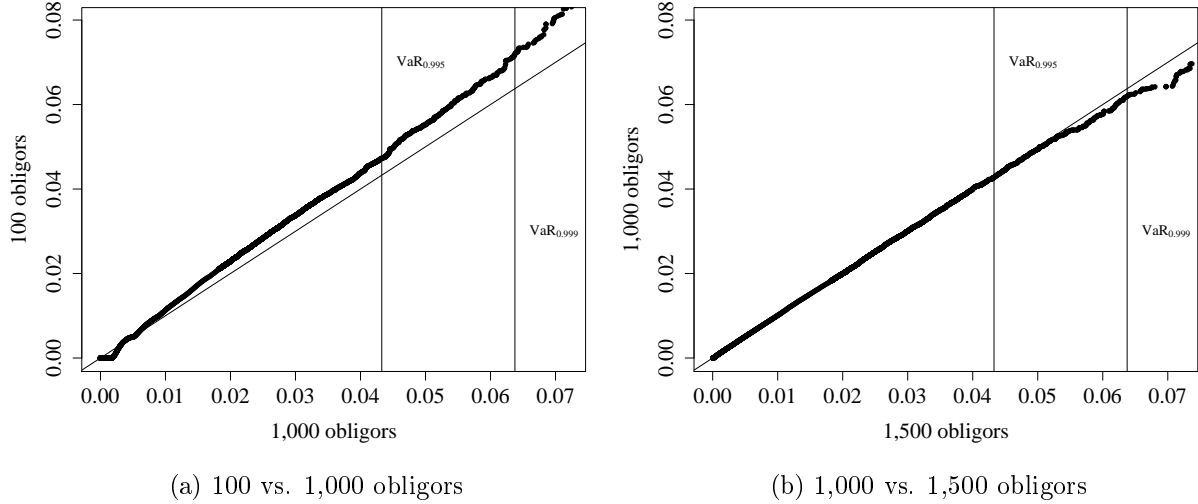


Figure 2.5: Comparison of Q-Q-plots for two asset classes with fix proportion of number of obligors (100, 1,000 or 1,500 each) simulated with 100,000 model runs in a one-factor model as described in formula (2.6). Both figures describe asset classes with $PD_1 = 1\%$, $PD_2 = 2\%$ and $\varrho_1 = 2.28\%$, $\varrho_2 = 2.73\%$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class sizes on the x-axis.

see how the per-unit risk changes with the number of obligors and also ascertain how many obligors are necessary to reach a constant per-unit risk with a maximum error of 20 bp. We use the following assumption: $u_1 : u_2 = 1$, i.e., both asset classes are the same size.

Looking at the Q-Q-plot we see that the line in Figure 2.5a has a higher slope than the bisecting line. This shows that the quantiles of small portfolios are higher, meaning that the VaR contribution per obligor is higher. In Figure 2.5b, the line almost equals the angle bisector, meaning that the per-unit risk in a 1,000 obligor portfolio is the same as that in a 1,500 obligor portfolio. In Figure 2.5a, the VaR line is slightly less steep than in the case of one asset class. This is due to the lower average probability of default of the portfolio. As seen before, a lower PD leads to a lower number of necessary obligors for a constant per-unit risk. The equivalent results can be drawn for expected shortfall as risk measure. For details see A.3.

Table 2.3 shows the minimum number of necessary packages to achieve a constant R_q and respectively a constant per-unit risk. In the case of $PD_1 = 1\%$ and $PD_2 = 2\%$ and with VaR at a confidence level of 95% as risk measure 236 obligors in total, i.e., 118 obligors per asset class, are necessary to achieve convergence of the per-unit risk. The

comparatively low number is due to the proportional up-scaling. If, on the other hand, the number of obligors of each asset class is changed individually, each asset class must be in a region of constant per-unit risk.

$\eta \backslash PD$	1%/2%	2%/5%	1%/10%	5%/10%
0.95	236	296	332	428
0.97	284	348	416	456
0.99	388	428	432	464
0.999	432	460	480	572

Table 2.3: Number of necessary obligors to achieve constant per-unit risk with a maximum error of 20 bp; two asset classes with a fix proportion of number of obligors, correlation according to Basel II formula for big corporations; simulated with 100,000 model runs per asset class size.

The results are in the same order of magnitude as the results we observed with only one asset class. One conspicuous feature needs to be pointed out though. Comparing the case $PD_1 = 2\%$, $PD_2 = 5\%$ with the case of $PD_1 = 1\%$, $PD_2 = 10\%$ yields a similar number of obligors even if the average PD differs. This is due to the fact that the number of obligors necessary does not increase linearly with the PD .

2.5 Conclusion

In this paper we show under which conditions it is justifiable to use the assumption of constant per-unit risk in portfolio credit risk models. This result is especially relevant in portfolio optimization or performance measurement. We study the asymptotic behavior of loss distributions in order to show that, irrespective of the risk measure we use, for a large homogeneous asset class the risk per obligor converges to a limit per-unit risk. We supplement this result through several simulations, showing the effect of the error being made by assuming constant per-unit risk to be limited, as long as each asset class has a minimum number of obligors. In the simulated examples, on average, this minimum portfolio size was approximately 400 obligors per asset class. Simulations show that the exact number is highly dependent on input parameters such as probability of default or the risk measure.

For a single homogeneous asset class, this effect can be reduced by the consideration of granularity adjustments. The adjustment of the total asset class risk by the remaining idiosyncratic risk, allows to give a better approximation of the incremental risk of a new obligor and reduces the necessary asset class size significantly. However, this result is hard

to transfer in a setting of more than one asset classes, and therefore not applicable for portfolio optimization decisions.

We prove for a one- and two-factor model and give Monte Carlo evidence for other models, that the copula of the loss distributions of two asset classes converges as well. By putting these results together, we can conclude that in all common credit risk models portfolio optimization based on gradient allocation is justified as long as the single asset classes are a minimum size. However, if this minimum size is not achieved, the assumption of constant per-unit risk could lead to false business decisions. In most cases, the risk of a new obligor in a small asset class might be overestimated. Notice that all results are based on the assumption of homogeneous asset classes that can be in- or decreased without changing the asset class characteristics. Furthermore, only one time period was considered.

Consideration of portfolios of certain inhomogeneity, in order to gain proximity to real world scenarios, can be found Dorfleitner and Pfister [2013], where the authors examine what happens when increasing one asset class leads to a change in the asset class characteristics. Furthermore, a number of additional constraints or stress scenarios are added in order to challenge a business decision that is based on the purely mathematical optimization algorithm.

3 Capital allocation and per-unit risk in inhomogeneous and stressed credit portfolios

This research project is joint work with Gregor Dorfleitner. It was presented at the HypoVeinsbank - Member of UniCredit in Munich in November 2011 as part of the "HVB-Stiftungsfond". The paper has been published in *The Journal of Fixed Income*.

Abstract

This paper considers the application of gradient allocation and portfolio optimization to credit portfolios. The assumption of linearity of risk that is implied by the use of gradient allocation is challenged in a setting of inhomogeneity and stress scenarios. In order to see the effects of mathematically derived portfolio optimization on real life examples, a number of insightful examples are considered. This challenges mathematical results and enables a financial interpretation of the latter. The results reveal that per-unit risk is not disturbed by moderate inhomogeneity in most cases, whereas a change in portfolio composition as well as stress can influence the portfolio optimization decision significantly. The importance of incorporation of sensitivity analysis and stress testing into reporting structures and optimization decisions is emphasized.

3.1 Introduction

As a result of the financial crisis in 2008 and 2009 capital requirements have become a central concern of banks and governments. Credit risk, being a main driver of capital requirements, consequently gained significance. Therefore, measuring credit risk in a conservative and stress resistant manner has gained in importance.

The target of credit risk measurement is to evaluate the performance of a bank's credit portfolio and to steer portfolio decisions through risk aspects. In order to do this, one has to allocate risk or risk capital respectively to each asset class. Risk capital allocation in banks has been extensively studied from a mathematical and economical perspective. The theoretical aspects and axiomatic frameworks behind capital allocation principles can be seen in Tasche [2004a] and from another viewpoint in Kalkbrener [2005]. The approach of Tasche [2004a] is also the basis of the work of Buch and Dorfleitner [2008], who analyze the coherence of allocation principles. Merton and Perold [1993] and Stoughton and Zechner [2007] explain the principles considering their practical aspects. An overview of different methodologies for capital allocation and their advantages is given by Koyluoglu and Stoker [2002]. Several techniques to apply capital allocation to credit portfolios can be found in literature. Their goal is to develop an analytical formula that calculates the risk contribution of one asset class. Several calculation methods for risk contributions based on gradient allocation have been introduced to credit portfolios. An overview of this is provided in Mausser and Rosen [2007]. Tasche [2009] uses kernel estimators to derive a formula for value-at-risk contributions in credit portfolios. An expected shortfall formula for risk contribution is calculated in Kalkbrener et al. [2004].

Based on the allocated capital, a number of algorithms have been introduced in order to optimize portfolios by ratios, such as return on risk adjusted capital (RORAC) or alternative efficiency measures. Rockafellar and Uryasev [2000] introduce an algorithm to optimize the portfolio composition. The authors assume that the return per obligor is independent of the number of obligors in the asset class. Based on this assumption they show that there exists a portfolio composition that minimizes the risk function measured with the conditional value-at-risk as risk measure. Similarly, Buch et al. [2011] introduce an algorithm based on the work of Tasche [2004a] and Tasche [2008] that allows banks to optimize the RORAC in such a way that it is taken into account that segment managers have superior knowledge of each of their asset classes, while the overall risk information is only available at the bank's headquarter. This algorithm is based upon gradient allocation and consequently on the assumption that risk of segments scales linearly with the number

of obligors. The linearity assumption in the work of Buch et al. [2011] demands a per-unit risk per asset class that is independent of the number of obligors. This means the total risk of an asset class can be calculated as the product of number of obligors and per-unit risk per obligor. Dorfleitner and Pfister [2012] show for factor models and Bernoulli mixture models that this assumption is justifiable under a number of conditions, namely perfect homogeneity in the subportfolios or asset classes and a number of obligors above a certain threshold. This barrier depends on the specific asset classes and the chosen risk model and measure.

Several authors have studied the topic of inhomogeneity as well as stress and their effects on credit losses. Wehrspohn [2003] calculates loss distributions for heterogeneous portfolios by clustering each asset class into a finite number of homogeneous "buckets". Hanson and Pesaran [2008] define heterogeneity by a fluctuation of default probabilities and show, based on simulation, that heterogeneity leads to an increase of expected and a decrease of unexpected losses. Grundke [2005] finds a definition for inhomogeneity by allowing variable input parameters such as probability of default. Additionally, he looks at the effect of stochastic interest rates and credit spreads. The same definition of heterogeneity is used in Rosen and Saunders [2009] with a focus on hedging strategies for systematic risk. Bonti et al. [2006] show the effects of stress in specific countries or industries on a loan portfolio by stressing one common factor in a portfolio model. Roesch and Scheule [2007] discuss which input parameters in a loan portfolio should be stressed focusing on retail loans. Current stress tests are given by the European Banking Authority [2011]. An example of the effects of stress on credit portfolios in Germany is presented in Mager and Schmieder [2008].

In this paper we give evidence that portfolio optimization based on risk driven indicators is possible even in moderately inhomogeneous or stressed asset classes. In this way, we close the gap between mathematical solutions for portfolio optimization based on homogeneous asset classes and the discussions of real-world portfolios with inhomogeneous parameters. We show that moderate inhomogeneity in most cases does not affect the existence of a constant per-unit risk but might influence the minimum number of obligors that is necessary to limit the error made by the application of an algorithm. On the other hand, a strong level of deviance in one of the input parameters can lead to a divergence in per-unit risk. The same is true for stress tests based on stressing systematic factors. Hence, we suggest that portfolio steering decisions have to be based on three fundamentals, these being the base case portfolio optimization, a sensitivity analysis and stress testing.

This article is structured as follows: In Section 3.2 we explain the essential concepts and principles that are used in the subsequent sections and introduce our notation. In Section 3.3 the definition of moderate inhomogeneity, deviance in input parameters and stress is given. Finally, in Section 3.4 we demonstrate the change in per-unit risk and in the pre-conditions of portfolio optimization in several examples of inhomogeneous and stressed asset classes compared to the homogeneous case. Furthermore we discuss the effects on portfolio decisions when we allow inhomogeneity and stress in an asset class. We conclude this article with a discussion and a brief outlook on the challenges a bank has to face in this setting.

3.2 Principles and notation

3.2.1 Notation

Consider a credit portfolio of n asset classes. Asset classes are typically defined in concordance with regulatory requirements and therefore given by industry, country or obligor size or kind of debt instrument. We restrict ourselves to the credit portfolio of a bank and asset classes solely exposed to credit risk. Each asset class i consists of a number of obligors u_i . The random variable $X_{i,k_i} \in \{0, 1\}$ is the binary random variable that indicates default ($X_{i,k_i} = 1$) or no default ($X_{i,k_i} = 0$) of obligor k_i in asset class i within a given time interval. The characteristics of an obligor $k_i \in \{1, \dots, u_i\}$, of asset class i are denoted as follows:

- (unconditional) probability of default PD_{i,k_i} ; if all obligors within the asset class have the same PD , it follows $PD_{i,k_i} = PD_i$,
- correlation between each other $\varrho_{i,k_i,j,k_j} = \text{corr}(X_{i,k_i}, X_{j,k_j})$ for $k_i \in 1, \dots, u_i$ and $k_j \in 1, \dots, u_j$,
- exposure at default EaD_{i,k_i} ,
- loss given defaults LGD_{i,k_i} .

All parameters refer to one time period. For the ease of notation, when talking about one asset class the index i will be omitted.

Based on a credit portfolio model the loss of the portfolio ($=: L$) is calculated by:

$$L(u) = \sum_{i=1}^n L_i = \sum_{i=1}^n \sum_{k_i=1}^{u_i} X_{i,k_i} \cdot EaD_{i,k_i} \cdot LGD_{i,k_i}, \quad (3.1)$$

where L_i is the loss in asset class i .

3.2.2 Credit portfolio model and risk measures

The following discussion focuses on a factor model like CreditMetrics for static portfolios, see e.g., McNeil et al. [2005]. A factor model is based upon the firm-value model introduced by Merton [1974]. Obligor k_i of asset class i defaults when the firm-value, represented by a continuous random variable, falls below a given threshold S_{i,k_i} . Following Rosen and Saunders [2010] and Dorfleitner et al. [2012] this random variable is called credit worthiness index CWI_{i,k_i} , which is defined for each obligor k_i as follows:

$$CWI_{i,k_i} = \sum_{j=1}^m \alpha_{i,j} M_j + \alpha_{i,E} E_{i,k_i}, \quad (3.2)$$

where M_j for $j = 1, \dots, m$ represents systematic risks driving credit events and E_{i,k_i} the idiosyncratic risk of obligor k_i . All M_j and E_{i,k_i} are standard normally distributed. Moreover, $\alpha_{i,E}$ is chosen in a way such that CWI_{i,k_i} itself is standard normally distributed. Obligor k_i defaults if $X_{i,k_i} = \mathbb{1}_{\{CWI_{i,k_i} < S_{i,k_i}\}}$ is 1. In the special case of a one-factor model ($j = 1$), α equals the square root of the correlation between the CWIs of two obligors within the asset class.

With this model we can calculate the loss distribution of any credit portfolio using formula (3.1). The risk of the portfolio can then be measured by risk measures like value-at-risk (VaR), expected shortfall (ES) or any spectral risk measure (Acerbi [2002]). We denote the risk measure by ρ and the confidence level of the risk measure η . Given the risk we can calculate the economic capital as the difference between risk and expected loss.

3.2.3 Allocation principle and portfolio optimization

To measure the risk contribution of one asset class or the per-unit risk of one obligor, an allocation principle such as gradient allocation can be used; see Tasche [2004a]. We denote the risk contribution of an asset class by $\rho(L_i|L)$, so that $\sum_i \rho(L_i|L) = \rho(L(u))$. With gradient allocation according to Tasche [2008] in granular homogeneous asset classes, the per-unit risk of one obligor for any homogeneous risk measure ρ is:

$$\rho^{p.u.}(X_{i,k_i}) = \frac{1}{u_i} \rho(L_i|L) = \frac{1}{u_i} \frac{\partial \rho(L(u))}{\partial u_i}(u_1, \dots, u_n). \quad (3.3)$$

The sum of per-unit risks ($u_i \cdot \rho^{p.u.}(X_{i,k_i})$) in one asset class then describes the risk contribution of the asset class. According to the Euler Theorem the sum of all risk contributions then adds up to the total risk of the portfolio.

Knowing the capital requirement of every asset class in the portfolio enables risk managers to steer portfolio decisions based on the value of return per allocated risk capital. To do so, several indicators have been defined in the literature to date. We will use the RORAC concept for the further discussion as introduced in Tasche [2008]

$$RORAC_i = \frac{E[g_i - L_i]}{\rho(L_i|L)},$$

where g_i describes the gain of asset class i if the loans are repaid regularly. It is a random variable which is independent on L , and $E[L_i]$ measures the expected loss and, accordingly, $E[g_i - L_i]$ the expected return of the asset class. RORAC can be used for ex ante performance measurement or as the target quantity for optimization.

3.2.4 Per-unit risk in homogeneous credit portfolios

Dorffleitner and Pfister [2012] show that under a number of reasonable conditions the assumption of a constant per-unit risk independent of the number of obligors is justified for large, homogeneous asset classes. As soon as a minimum number of obligors in one asset class is obtained, the difference between the actual incremental risk of one obligor and the calculated per-unit risk becomes limited.

In this setting, a homogeneous asset class is defined as follows.

Definition 1. *An asset class is called homogeneous if all obligors have the same probability of default, correlation to each other, correlation to obligors of a second asset class, exposure at default and distribution of loss given default on a given probability space.*

In the case of an infinitely granular homogeneous asset class, the loss distribution converges for one- and two-factor models as well as for Bernoulli mixture models as shown in McNeil et al. [2005] and in Dorffleitner and Pfister [2012]. Through this convergence, we can deduce the existence of a constant per-unit risk. Furthermore, the error made for homogeneous portfolios with finite granularity is smaller than a given $\epsilon > 0$ when the number of obligors exceeds a certain barrier. The per-unit risk of an homogeneous asset

class can be calculated by dividing the total risk of the asset class by the number of the obligors. Figure 3.1 shows the plot of per-unit risks depending on the number of obligors in the asset class. There the barrier for constant per-unit risk with a maximum error of $\epsilon = 20$ bp is $\bar{u}_{20} = 361$. One can also ascertain that the per-unit risk for smaller portfolios is dependent on the number of obligors and typically higher than the limit value.

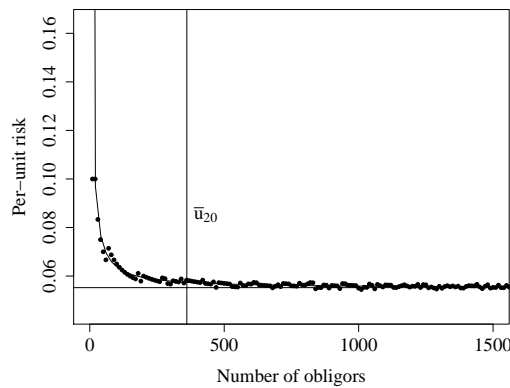


Figure 3.1: Per-unit risk vs. portfolio size simulated in a one-factor model with $PD = 1\%$, $\eta = 0.99$ and correlation according to Basel II formula for big corporations. \bar{u}_{20} marks the minimum portfolio size so that the delta to the limit per-unit risk is maximal 20 bp; calculated via non-linear regression ($y = a + b/x$) with limit per-unit risk equal to $a + b/2,000$.

Dorfleitner and Pfister [2012] show that the assumption of constant per-unit risk is justified in the case of more than one homogeneous asset class for one-factor models and give evidence, through the employment of Monte-Carlo simulation, for two-factor models and Bernoulli mixture models that the copula of the loss distributions of two asset classes also converges. This result supplements the conclusion that the distribution function in a more asset class setting converges as soon as the marginal distributions of the single asset classes converge. With this knowledge, one can deduce the existence of per-unit risks for two or more asset classes.

The results allow us to conclude that portfolio optimization based on gradient allocation and consequently on constant per-unit risk can be conducted with a limited error in large inhomogeneous portfolios. However, it can lead to high errors in small portfolios. Since all portfolio optimization algorithms use the assumption of constant per-unit risk, they should only be employed for large, homogeneous portfolios. If they are applied to small portfolios, incorrect business decisions can be the consequence.

3.3 Per-unit risk in inhomogeneous and stressed credit portfolios

Real-world credit portfolios cannot fulfill the assumption of absolute homogeneity. Credit portfolios always have individual loans and obligors with individual characteristics like the probability of default. Additionally, some input parameters, like correlations, cannot be measured exactly. This leads to a certain level of inhomogeneity in every asset class. Furthermore, external and internal events can cause a spontaneous change in asset class characteristics. In this section we will define inhomogeneity and stress in order to measure the grade of deviation of an idealized model solution from reality.

3.3.1 Credit portfolios of moderate inhomogeneity

In a moderately inhomogeneous asset class as seen in reality, a number of parameters will take values within given ranges because of the following reasons:

- Obligor in an asset class do not have exactly the same probability of default, but the asset class gives a general indication of credit quality.
- Correlation between obligors is not measurable in its exactness and can change at any time.
- Exposure at default is hard to track and therefore EaD values are sometimes out-dated.

For these reasons we will broaden the definition of one asset class.

Definition 2. *An asset class is called moderately inhomogeneous if it has*

- *a fixed probability of default PD_{i_k} for each obligor i_k in a given interval $PD \in [PD_{min}; PD_{max}]$,*
- *stochastic correlation of credit worthiness index (CWI as defined in Section 3.2.2) to the CWI of other obligors within the same asset class that is uniformly distributed on a given interval $\alpha \in [\alpha_{min}; \alpha_{max}]$,*
- *stochastic exposure at default with a known distribution on a given interval $EaD \in [EaD_{min}; EaD_{max}]$.*

By this definition, inhomogeneity is allowed within the asset class but the input parameters are bounded by the limits of the given intervals. The first assumption shows that obligors in one portfolio might not have the exact same PD , which can follow from rating

down- or upgrades over time or may be caused by the limited number of subportfolios or asset classes that does not allow one to create one asset class per PD value. The second assumption shows the limited measurability of correlations. The third assumption shows that the exposure of an obligor might change within one time period, which is especially relevant for obligors with undrawn credit lines.

3.3.2 Credit portfolios with deviant input parameters

An asset class does not exhibit an identical distribution of input parameters over time. Input parameters of existing obligors might change at any time or new obligors might have different characteristics, so that the overall distribution of one and more parameters in the asset class changes. We still consider only one time period, but wish to study what happens if parameters are wrongly estimated.

We consider the following input parameters:

- probability of default,
- correlation,
- exposure (EaD and LGD).

A deviance in these parameters can be a result of business decisions. It might be generated by a reduction of creditworthiness of obligors over time through economic or political changes, e.g., increased taxes or a change in consumer behavior. Other input parameters like correlation between the obligors might change as a result of an acquisition of a company by another. Loss given default can increase, e.g., by a reduction of collateral values. We can simulate this effect by checking risk (and hence RORAC) sensitivity against the obligor's specific input variables. According to Roesch and Scheule [2007] this change in input parameters can be characterized as being stress as well. The constructed stress scenario describes the effect of worsened obligor characteristics. This can be used to assess the credit risk per asset class very conservatively. In this case, the deviance in input parameters will be chosen more extremely.

3.3.3 Stressed credit portfolios

Typically, a stress scenario describes a macro economic scenario that occurs with very low probability. The most prominent examples are given by the European Banking Authority (EBA, European Banking Authority [2011]). These scenarios are then translated into stressed input parameters, like PD or LGD , and stressed systematic factors.

Stress tests became more prominent after the crisis of 2008 and 2009 when new regulatory recommendations and rules had to be implemented. Stress can be defined in different ways. One method considers sensitivities to input parameters as explained above. Alternatively, stress can be caused by a change in market variables. This might result from an economic downturn of single industrial sectors or a single event or shock like the default of a sovereign state or a natural catastrophe. This kind of stress will have an effect on the systematic factors in a credit risk model while idiosyncratic factors are uninfluenced. We simulate a stress event by changing the distribution of the systematic factors in the credit risk model as introduced in Bonti et al. [2006].

When considering capital allocation and portfolio optimization, the fact of a moderately inhomogeneous asset class as well as deviance in input parameters and stress might have an effect on the per-unit risk calculation of an asset class. This might be true with respect to absolute values and the minimum number of obligors that is necessary to conduct portfolio optimization.

3.4 Monte Carlo evidence

In this section we give evidence for the existence of per-unit risk in inhomogeneous and stressed portfolios and demonstrate the limits of its validity. Inhomogeneity and stress in parameters and the effect on per-unit risk is analytically complex and thus best carried out via Monte Carlo simulation. All simulations are based on a factor model. In Section 3.4.1 and Section 3.4.2 the focus will be on one asset class. The results are transferable to a setting of more asset classes because in a one- or two-factor model the convergence of the marginal distributions leads to a convergence of the joint distribution functions. Evidence of this effect can be found in Dorfleitner and Pfister [2012].

3.4.1 Credit portfolios of moderate inhomogeneity

Firstly, we consider portfolios which do not fulfill the preconditions of homogeneity but instead the broader definition of moderate inhomogeneity as introduced above. We consider several examples.

First we set a base case asset class as reference, which is homogeneous. Example one describes an asset class with obligors that are all similar in their default structure. It is assumed that the large exposures are monitored and tracked frequently due to the required

tracking of large exposure restrictions, while other exposure values might be outdated and therefore only their distribution is known. This example describes a specialized portfolio, e.g., the corporate loan portfolio of a bank. Example two describes an asset class of higher heterogeneity within the default probabilities. All obligors are assumed to be of equal size in terms of exposure at default. This example might describe a retail portfolio. Example three shows the effects of changes of the uncertain parameter correlation. The asset class is assumed to be homogeneous for all parameters apart from correlation. Correlation changes within a given range.

For all examples we fix the following suppositions:

- risk measure: value-at-risk at a confidence level $\eta \in \{0.95, 0.97, 0.99, 0.999\}$,
- loss given default: LGD random variable $\in [0, 1]$ uniformly distributed (*u.d.*),
- credit risk model: one-factor threshold model.

In Table 3.1 we see the model assumptions for the single examples. The intervals are chosen in a way that the expected values μ of input parameters are equal in all examples. The abbreviation *u.d.* indicates uniform distribution. If no distribution information is listed in Table 3.1, the distribution is a step distribution with the given expected values μ . All correlation entries are rounded.

As base case we choose a homogeneous asset class with identical PD , correlation and exposure for all obligors. The values from this example are used as expected values for the three examples of moderate inhomogeneity. Inhomogeneity 1 describes a portfolio where all input parameters are within given ranges. The distribution of the parameters is known as well as the upper and lower bound. In order to isolate the effects we additionally consider Inhomogeneity 2 where only the PD is given within a range while all other parameters are fixed. We choose a broader range in which most obligors have a PD between 0% and 2% and few obligors have a PD up to 10%. Inhomogeneity 3 follows the same approach for the correlation of CWIs.

For the different examples, the loss distribution $L(u)$ of each case is simulated. The value-at-risk of this distribution $VaR_\eta(L(u))$ is the total risk of the asset class. We can interpret $\frac{1}{u}VaR_\eta(L(u))$ as per-unit risk as it is done when gradient allocation is used. A comparison of the results for per-unit risk of the base case with the examples shows whether gradient allocation leads to erroneous results. Additionally we consider the number of

3 Capital allocation and per-unit risk in inhomogeneous and stressed credit portfolios

		PD	corr. of CWIs α^2	EaD
Base Case	a	1%	0.19	1
	b	2%	0.16	1
Inhomogeneity 1	a	[0.5%, 1.5%] <i>u.d.</i>	[0.18, 0.21] <i>u.d.</i>	[0.5, 1.45] ⁷ <i>u.d.</i>
	b	[1.5%, 2.5%] <i>u.d.</i>	[0.15, 0.18] <i>u.d.</i>	[0.5, 1.45] ⁷ <i>u.d.</i>
Inhomogeneity 2	a	[0%, 10%], $\mu = 1\%$	0.19	1
	b	[0%, 10%], $\mu = 2\%$	0.16	1
Inhomogeneity 3	a	1%	[0, 0.25], $\mu = 0.19$	1
	b	2%	[0, 0.25], $\mu = 0.16$	1

Table 3.1: Input parameters for a one-factor model as introduced in Section 3.2.2 for four examples. All entries for correlation are rounded and are calculated with the Basel II formula for big corporations.

obligors that is necessary to ensure that the incremental risk of one obligor is approximated by the per-unit risk with an error $\epsilon < 20$ bp.

Comparing the per-unit risks in the base case of a homogeneous asset class with examples Inhomogeneity 1 and 3 reveals that changes in exposure and correlation result in no deviation of per-unit risk as long as average values are the same as shown in Table 3.2. If on the other hand the interval of the chosen PD s widens, as it does in example two, per-unit risk decreases. This is in line with the results of Hanson and Pesaran [2008] and can be explained by an induced decrease in the correlation of the CWIs in the asset class with higher variance of the default probabilities. A demonstrative example of this effect can be found in Hanson and Pesaran [2008].

	Base case		Inhomogeneity 1		Inhomogeneity 2		Inhomogeneity 3	
η \diagup Avg. PD	1%	2%	1%	2%	1%	2%	1%	2%
0.95	0.019	0.032	0.019	0.032	0.017	0.030	0.019	0.032
0.97	0.023	0.039	0.024	0.039	0.021	0.036	0.023	0.039
0.99	0.037	0.056	0.037	0.056	0.031	0.050	0.037	0.056
0.999	0.070	0.095	0.070	0.096	0.056	0.081	0.071	0.096

Table 3.2: Per-unit risk for homogeneous vs. moderately inhomogeneous asset classes as defined in Table 3.1. As risk measure VaR_η is chosen. All values are simulated via Monte-Carlo simulation with 100,000 model runs.

⁷For 90% of obligors, $EaD = 1.5$ for 10% of obligors.

A comparison of the minimum number of obligors which is necessary to ensure that incremental and per-unit risks coincide provides some interesting results. The number is clearly higher if concentration risk is increased in the portfolio as is the case of Inhomogeneity 1. The results can be seen in Table 3.3. Particularly in the case of high confidence levels, the minimum number of obligors is very high. This is a result of the concentration risk of the largest 10% of obligors which increases the per-unit risk. Therefore, the loss distribution converges more slowly. Inhomogeneity in PD and correlation, on the other hand, reduce the minimum number of obligors. The barrier is slightly lower, on the average 7%, than in the base case.

η	Avg. PD	Base case		Inhomogeneity 1		Inhomogeneity 2		Inhomogeneity 3	
		1%	2%	1%	2%	1%	2%	1%	2%
0.95		194	240	233	263	192	219	184	224
0.97		232	284	273	305	226	254	215	261
0.99		302	370	346	397	296	326	278	332
0.999		413	507	531	658	428	459	389	427

Table 3.3: Numbers of necessary obligors to gain constant per-unit risk for homogeneous vs. moderately inhomogeneous asset classes as defined in Table 3.1 with VaR_η as risk measure; simulated via Monte-Carlo simulation with 100,000 model runs and 200 nodes from $u = 10$ to $u = 2,000$.

The cause of this effect can be found in the dependency of the minimum number of obligors on the probability of default. The number of obligors does not increase linearly with the PD but in a convex curve as shown in Table 3.2a. In this case, the base case with variable PD is plotted. Since we chose an expected PD of 1% or respectively 2% most of the obligors have default probabilities within a range where the curve is highly convex and the number of necessary obligors is low. The same is true for the dependency on correlation as shown in Figure 3.2b. Due to the choice of interval, a large share of obligors has a CWI correlation to the other obligors above 0.2 where the minimum number of obligors is relatively low.

Based on this discussion we can conclude that gradient capital allocation and, consequently, a portfolio optimization algorithm can be used for asset classes of moderate inhomogeneity and the optimal portfolio size will be the same as under the assumption of a homogeneous asset class, as long as the overall portfolio composition does not change in terms of the distribution of all parameters when the portfolio size is either de- or increased

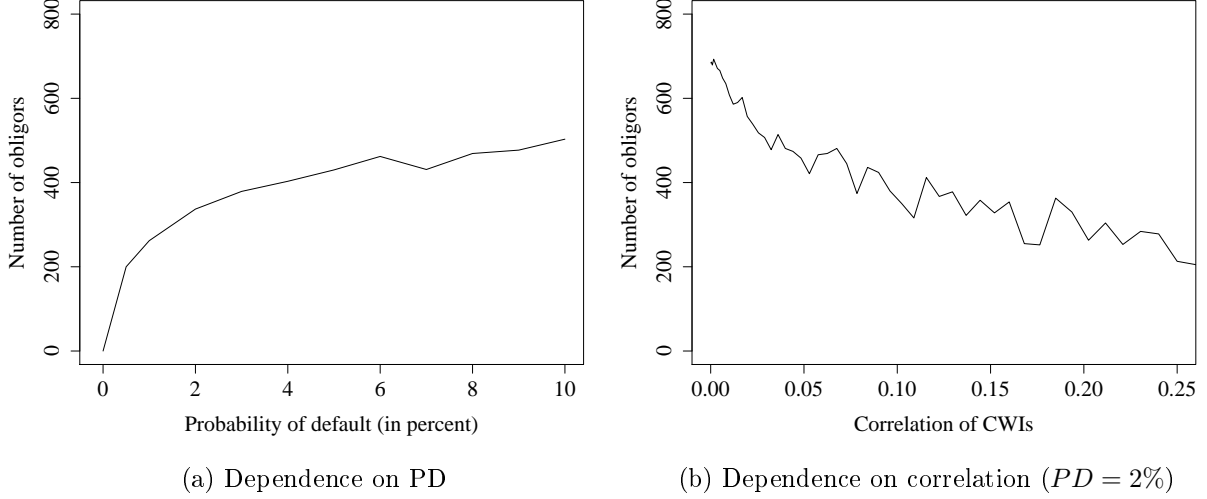


Figure 3.2: Minimum number of obligors for varying PD and CWI correlation (homogeneous asset class, $\eta = 0.95$); correlations are chosen according to the Basel II formula for big corporations: $\alpha^2 = 0.12 \frac{1-e^{-50PD}}{1-e^{-50}} + 0.24(1 - \frac{1-e^{-50PD}}{1-e^{-50}})$. See e.g., the Basel Committee on Banking Supervision [2011]; simulated via Monte-Carlo simulation with 100,000 model runs per PD and correlation value.

respectively. This means per-unit risk exists and is constant in a moderately inhomogeneous setting and can therefore be calculated with the average of the input parameters. If the PD or correlation variability is high, this calculation will give a conservative estimate of the minimum number of obligors in the asset class as long as most of the obligors have relatively low PD or high correlation. If a high fluctuation of PD is simulated, the calculation of per-unit risk has to be adjusted.

3.4.2 Credit portfolios with deviance in the input parameters

In this section the description of an inhomogeneous portfolio is expanded according to Section 3.3.2. Deviant input parameters due to an unmeasured change of parameters in time or due to misjudgment are considered.

As a base case we consider a homogeneous asset class of 1,000 obligors with the following parameters

- number of obligors: $u = 1,000$,
- probability of default: $PD = 2\%$ for the original asset class,

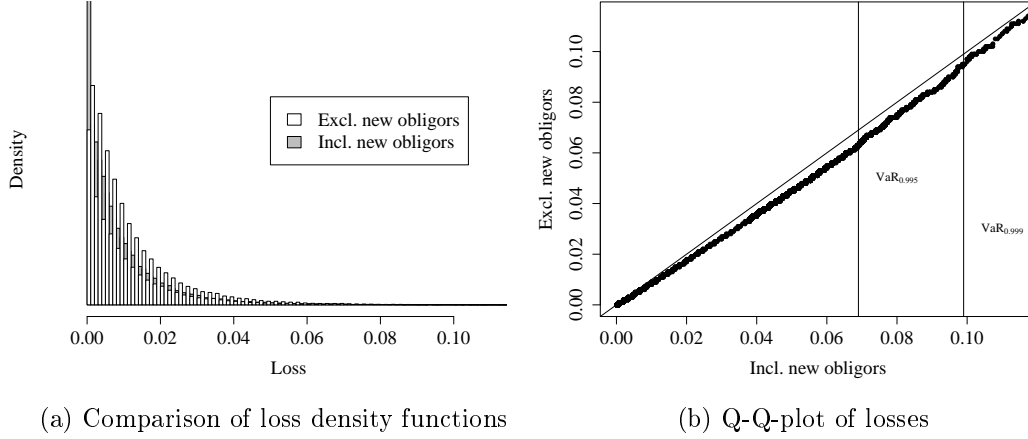


Figure 3.3: Portfolio loss per obligor $\frac{1}{u}L(u)$ represented by its density function and quantile (VaR) before and after adding 500 obligors with increased $PD = 3\%$; the distribution is modeled in a one-factor model with input parameters as defined in Section 3.2.2 and simulated via Monte-Carlo simulation with 100,000 model runs. The vertical lines indicate $VaR_{0.995}$ and $VaR_{0.999}$ for the larger portfolio on the x-axis.

- exposure at default: $EaD = 1/u$,
- loss given default: LGD random variable equal 0% or 100% with probability 0.5,
- correlation between the CWIs of the obligors $\alpha^2 = 0.16$.

We calculate the per-unit risks of this asset class via a one-factor credit risk model.

Deviant probability of default

The first scenario is defined by adding obligors of a significantly different probability of default to the asset class. This effect can, for example, be caused by a change in industry or business strategy, e.g., increasing the asset class is possible because the internal guideline for issuing loans has changed, or simply due to the misjudgment of the real PD . We assume that 500 new obligors are added to the asset class:

- probability of default: $PD' = 3\%$,
- exposure at default: $EaD' = 1/u'$ with $u' = u + 500 = 1,500$.

Based on the results for homogeneous asset classes which are summarized in Section 3.2.4 one would expect to find a portfolio of 1,000 obligors already in an area of constant

per-unit risk. However, due to the change in one input parameter, the per-unit risk no longer exists.

The density function of losses for the asset class including the new obligors with higher PD has fatter tails and respectively a higher VaR for high confidence levels as shown in Figure 3.3. The effect can be seen at any confidence level but is more significant for high confidence levels, which are the most relevant for risk measurement. On average, the per-unit risk increases by 0.005 due to the new obligors for $\eta \in [0.95, 0.999]$. The change in per-unit risk slightly increases with the confidence level; for example the per-unit value-at-risk at a confidence level of 95% is 0.033 for the homogeneous portfolio with a PD of 2% while it is 0.037 in the new portfolio with a share of obligors with higher PD . At a confidence level of 99.5% the per-unit risk jumps from 0.069 to 0.075 when adding the new obligors.

Deviant correlation

Next we consider the case that new obligors have a significantly different correlation to each other and to the rest of the obligors in the asset class. This effect can be caused by a set of new obligors that have business connections with each other. It is also possible for correlations to change over time. They may also have been underestimated in an existing asset class. We assume that 500 new obligors are added to the asset class with a correlation between their CWIs of 20%. This leads to the following scenario:

- correlation between CWIs of new obligors $\alpha'^2 = 0.20$,
- resulting correlation between CWIs of existing and new obligors $\alpha \cdot \alpha' = 0.18$.
- exposure at default: $EaD' = 1/u'$ with $u' = 1,500$.

The density function of losses for an asset class including the new obligors with higher correlation has a higher VaR for high confidence levels as plotted in Figure 3.4b. Up to a confidence level of 70%, the per-unit value-at-risk in the original asset class is slightly higher what we see in Figure 3.4b. However, this case is not relevant for risk measurement where only high confidence levels are used. For confidence levels above 70%, the new obligors increase the per-unit risk. Additionally, the delta between the two per-unit value-at-risks increases with the confidence level. As an example we look at a confidence level of $\eta = 95\%$ and compare it with $\eta = 99.5\%$. The per-unit risk in the first case is 0.033 for the homogeneous asset class with a correlation of 0.16, but it is 0.035 in the new asset

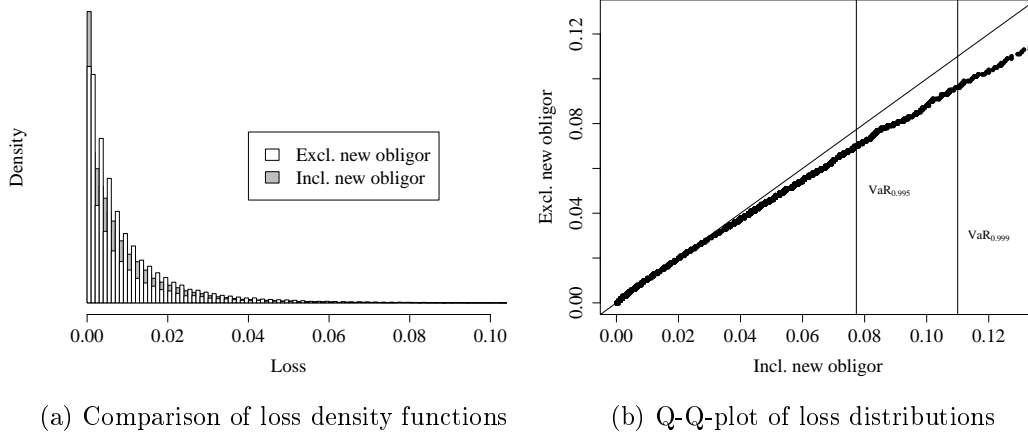


Figure 3.4: Portfolio loss per obligor $\frac{1}{u}L(u)$ represented by its density function and quantile (VaR) before and after adding 500 obligors with increased correlation $\alpha'^2 = 0.20$ to the rest of the asset class; the distribution is modeled in a one-factor model with input parameters as defined in Section 3.4.2 and simulated via Monte-Carlo simulation with 100,000 model runs. The vertical lines indicate $\text{VaR}_{0.995}$ and $\text{VaR}_{0.999}$ for the larger portfolio on the x-axis.

class with a share of obligors with higher correlation to the rest of the asset class. For $\eta = 99.5\%$ the two per-unit risks are 0.070 and 0.077.

Deviant exposures

Another important input variable to be considered is the exposure, i.e., exposure at default and loss given default. In order to simulate the effect of concentration in one asset class, one obligor with the following characteristics is added:

- exposure at default: $EaD' = 0.1$, i.e., the obligor has 10% of the EaD of the complete original asset class,
- $PD' = 2\%$,
- $\alpha'^2 = 0.16$,
- loss given default: $LGD' = 100\%$.

The exposure, measured as $EaD' \cdot LGD' = 0.1$, of the new obligor is therefore 20% of the total expected exposure of the existing portfolio ($= E[1,000 \cdot EaD \cdot LGD] = 0.5$).

The density function of losses shown in Figure 3.5a splits into two cases based on the binary event of default or no default of the new obligor. As a result, the Q-Q-plot

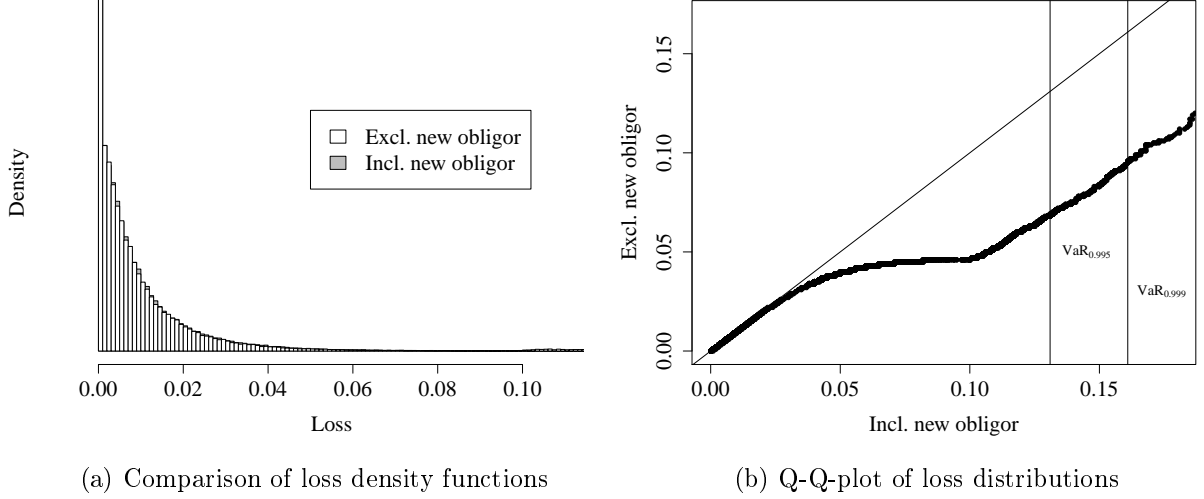


Figure 3.5: Portfolio loss per obligor $\frac{1}{u}L(u)$ represented by its density function and quantile (VaR) before and after adding one obligor with the larger exposure of 0.1; the distribution is modeled in a one-factor model with input parameters as defined in Section 3.4.2 and simulated via Monte-Carlo simulation with 100,000 model runs. The vertical lines indicate $\text{VaR}_{0.995}$ and $\text{VaR}_{0.999}$ for the larger portfolio on the x-axis.

in Figure 3.5b shows the new portfolio to bear a much higher risk by bending to the right. This result can be explained as a consequence of high concentration risk. The diversification benefits in the asset class are diminished by adding the large exposure. For capital allocation, this means that by adding one obligor with significantly higher exposure than the average of the existing asset class, we increase concentration risk and therefore total risk. The additional risk capital has to be higher than calculated via capital allocation rule to compensate for the concentration risk. In the example above, the total $\text{VaR}_{0.99}(L(u))$ increases from 57 to 116 by adding the new obligor, which means an increase by 103%, while pure linear upscaling suggests only an increase of the original VaR by 20% due to the increased expected exposure.

3.4.3 Stressed credit portfolios

In this section, per-unit risk of asset classes in a base scenario and a stress scenario are compared. To display the effect of stress on the output of the credit risk model, we assume that the stress scenario only has an effect on the systematic factors and consequently the credit worthiness of the obligors and the capital requirement.

In a challenging economical environment, the simplified assumption of standard normally distributions of systematic factors can be incorrect and lead to erroneous results. Stressed systematic factors are especially relevant in multi-factor models since the stress of one factor will have different effects on different asset classes. In this way, we can simulate stress in single industrial sectors or regional areas and identify concentration risk in specific sectors.

For the implementation of stress scenarios in a factor model, we follow Bonti et al. [2006]. We assume for a two-factor model that each systematic factor corresponds to an industry or a region. Each asset class has specific weights for each factor that represent the importance of the factor for the obligors in this asset class. We then simulate the loss distribution for the base case, in which all systematic factors are standard normally distributed. A second model run shows the stress scenario with one systematic factor displaying an alternative distribution function. Candidates for this distribution should, in practice, be derived from macroeconomic specialists with a view to the market and with the goal to implement macro-economic scenarios like a decrease in GDP growth as proposed by the European Banking Authority [2011]. According to Bonti et al. [2006], they should be chosen in a plausible, consistent, adapted and reportable way.

In the following analysis, we choose a two-factor-model for a portfolio consisting of two asset classes. For the two asset classes we chose the following input parameters:

- number of obligors: $u_1 = u_2 = 1,000$,
- probability of default: $PD_1 = 1\%$, $PD_2 = 2\%$,
- first systematic weight: $\alpha_{11} = 0.4$, $\alpha_{21} = 0.1$,
- second systematic weight: $\alpha_{12} = 0.1$, $\alpha_{22} = 0.4$,
- exposure: $EaD = 1/(u_1 + u_2)$,
- loss given default: LGD random variable equal 50% or 100% with probability 0.5,
- risk measure: $VaR_{0.99}$.

Stress is caused by a change in the distribution of the first systematic factor as described in Bonti et al. [2006]. We consider two scenarios. The first scenario describes a shift of the systematic factor to the left, the second scenario assumes a Gamma distribution of the first systematic factor. These distributions are simplified assumptions that have to be verified or adjusted by empirical results based on historical data.

In contrast to the examples of Section 3.4.2, stress of a systematic factor has an influence on all asset classes. Therefore, the severity of the stress effect per asset class has to be

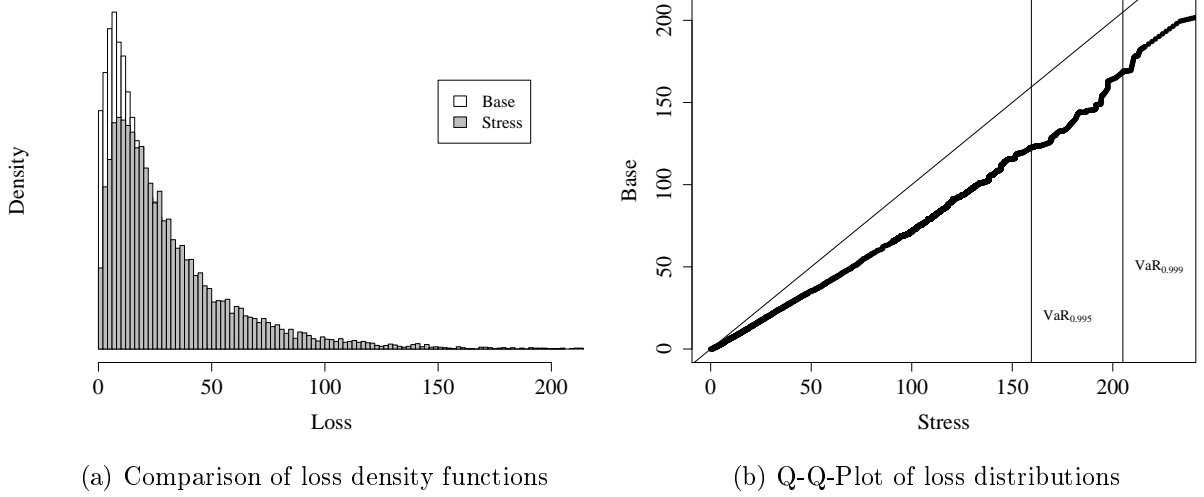


Figure 3.6: Portfolio loss per obligor $\frac{1}{u}L(u)$ represented by its density function and quantile (VaR) before and after stress; the distribution is modeled in a two-factor model with input parameters as defined in Section 3.4.3 and simulated via Monte-Carlo simulation with 100,000 model runs. The vertical lines indicate $\text{VaR}_{0.995}$ and $\text{VaR}_{0.999}$ for the stressed portfolio on the x-axis.

analyzed and the resulting portfolio optimization advice has to be considered. In scenario one, the systematic factor is now normally distributed with expected value $\mu_{M_1} = -0.2$ and variance $\sigma_{M_1} = 1$. The results are plotted in Figure 3.6. As one might expect, the portfolio shows a relevant increase of risk under stress.

The special interest lies on the contribution of each asset class to this effect. The per-unit risk of one obligor from each asset class is calculated with gradient allocation before and after stress according to formula (3.3). We used numerical differentiation to receive the results. As shown in Table 3.4 for both asset classes, the per-unit risk increases in the stress scenario, but the effect is more significant in the first asset class. Based upon the assumption of a fixed return per obligor that is not influenced by stress, the portfolio optimization advice changes in the different scenarios. Assuming a (marginal) RORAC of 10.1% in the first and 10.0% in the second asset class in the base case the portfolio optimization advice is to increase the size of the first asset class, whereas in the first stress scenario the asset class size should be decreased. A calculation of the same example with the second stress scenario reveals the same result. This stress scenario is defined by a $\Gamma(2, 2)$ -distributed (where $\Gamma(\gamma_1, \gamma_2)$ is the Gamma distribution with shape γ_1 and scale γ_2)

first systematic factor M_1 that is shifted to the left by subtracting $4.2 = 1.1 \cdot E[\Gamma(2, 2)]$. Again different optimization advices follow in the stress versus the base scenario.

	Asset class 1			Asset class 2		
	Base Case	Stress 1	Stress 2	Base Case	Stress 1	Stress 2
$\rho^{p.u.}$	0.0219	0.0305	0.0328	0.0615	0.0827	0.0829
total risk	21.9	30.5	32.8	61.5	82.7	82.9
expected loss	7.5	9.1	9.1	15	15.7	15.7
capital requ.	14.4	21.4	23.7	46.5	67.1	67.3
return	1.5	1.5	1.5	4.7	4.7	4.7
RORAC	10.1%	6.8%	6.1%	10.0%	6.9%	6.9%
Advice	Increase	Decrease	Decrease	Decrease	Increase	Increase

Table 3.4: RORAC calculation in base and stress scenarios for two asset classes modeled in a two-factor model. The stress scenarios are chosen as given in Section 3.4.3

We conclude that the stress scenario can lead to a recommendation for the portfolio composition that differs from the original one. A portfolio decision has to include a set of stress scenarios to ensure that the effect of economic stress on the bank is limited.

3.5 Conclusion and managerial implications

This paper applies gradient allocation to a number of examples with a focus on credit risk management and portfolio optimization. In practice, the use of per-unit risk or exposure-weighted per-unit risk is not uncommon. Therefore, we aim at increasing the awareness of potential pitfalls the use of per-unit risk for optimization algorithms in order to show the importance of sensitivity analysis regarding inhomogeneity and stress testing of risk models.

For the practical perspective it is crucial that an algorithm withstands inhomogeneity. We give evidence that the concept of per-unit risk is not destroyed by moderate inhomogeneity. The higher the fluctuation of input parameters, the more important it is to ensure a minimum number of obligors in the asset class, so that per-unit risk remains constant. Additionally, we show by simulation that deviant input parameters (like correlation or exposure) as they will occur in real-world portfolios can influence the results significantly. As a consequence, increasing one asset class based on an optimization algorithm should sustain the specific asset class composition, i.e., the distribution of all parameters. Another important result is that in stress scenarios, the changing loss distribution can lead to

a different portfolio optimization advice. As discussed below, for every portfolio decision different scenarios should be taken into account.

Based on these results, one has to think about potential solutions to reduce erroneous decisions. In the case of inhomogeneity of an asset class, two approaches can be pursued: First of all, one asset class will include obligors of limited ranges of PD s, correlations and exposures. Then the per-unit risk has to be adjusted accordingly for all obligors in the asset class. The average per-unit risk will be allocated to each obligor and revised when the portfolio composition changes. The advantage of this approach is that the amount of data needed for portfolio optimization is limited. Another option would be to allocate different values of per-unit risk to the different types of obligors or defining new asset classes. This approach represents the risk structure in a better way. However, the calculation of single risks for a real-world portfolio with a lot of slightly different input parameters in this case is very complex. This should only be the solution in severe cases, in order to keep monitoring, reporting and any IT processes manageable.

For stress scenarios there are two general options: In the first option, each scenario is weighted by its assumed probability and the bank bases its decisions on expected values. However, when highly improbable stress scenarios are chosen, so that the probability of the event is extremely low and hard to measure, it does not lead to the desired result. An alternative approach is to add constraints to the optimization algorithm, which the bank uses, e.g., by setting limits for losses or capital ratios in the case of stress. This means the bank defines a minimum capital ratio that has to be reached even in the case of a stress scenario. The optimization algorithm then has to exclude all portfolio compositions that would lead to a capital ratio underneath the barrier in stress as a side condition. By doing that, one ensures a minimum amount of profitability under stress by waiving return in the base case scenario.

The ideas presented in this paper provide an overview on practical challenges of the application of gradient capital allocation. The list of chosen examples give evidence that mathematically derived results should be financially interpreted before their execution. This raises some questions that should be subject to further research.

- From a mathematical perspective, the side conditions resulting from stress testing have to be incorporated into optimization tools.
- From a practical perspective, a process to incorporate stress tests into portfolio optimization decisions that prevents portfolio compositions which lead to a capital quote below a certain barrier in the case of stress has to be developed.

3 Capital allocation and per-unit risk in inhomogeneous and stressed credit portfolios

- For a sustainable decision, a multi-period view has to be considered. It is crucial to analyze the effects of a change in input parameters over a longer time frame.

4 Capital allocation in credit portfolios in a multi-period setting

The paper has been submitted for publication to the *Review of Managerial Science*.

Abstract

The article gives an overview on existing techniques of multi-period risk measurement and capital allocation, and applies these techniques to credit portfolios with a focus on practical aspects. The effects of the choice of considered loss process concerning the handling of write-offs and matured assets or rating migration are displayed, and the impact on portfolio optimization decisions is discussed. We point out the trade-off between short-term and long-term profitability and allude to the practical challenges of an application of multi-period risk measurement.

4.1 Introduction

The financial crisis from 2008 to 2010 gave indication that there exists a bias for short-term profit maximization in banks that can contravene the target of sustainable profitability. There are several ways to address this issue: by a new risk culture, new incentive systems or adjusted risk modeling. One specific way and focus of this article is to change credit risk assessment techniques. We show that optimization based on one-period risk measurement can reduce long-term profitability, and that this effect can be mitigated by choosing multi-period risk measures.

Credit risk typically is assessed in a one-period view. This approach is expanded in a regulatory environment by maturity adjustments. However, there also exists a broad discussion on dynamic credit risk models of discrete or continuous type. In this article we focus on discrete multi-period models. An overview of these models can be found in Bluhm et al. [2002], Bielecki and Rutkowski [2002], Duffie and Singleton [2003], Schoenbucher [2001], McNeil et al. [2005] or Hull and White [2008]. In general, credit risk models segment in two types: structural models, which base on the Merton model for firm values with (time-dependent) risk factors (e.g., Hamerle et al. [2007]), and reduced form models, where default time is triggered by an intensity function (Jarrow and Turnbull [1995]). The latter are more popular in practice because they require less detailed firm specific information (Jarrow and Protter [2004]) and are, therefore, the focus of our work. There are several classes of reduced-form risk models. The simplest class is the one of Conditionally Independent Defaults. More sophisticated models include correlation of default events over time, e.g., Copula Models or models with Interacting Intensities (Schoenbucher and Schubert [2003]; Laurent and Gregory [2003] and Frey and Backhaus [2004]). Based on these risk models, loss distributions and risk can be determined.

Two streams of literature deal with multi-period risk measurement, one considers risk as real number, the other as random process. The first option is usually discussed with a focus on market risk, where multi-period Value at Risk forecasts or respectively volatility forecasts as modification of the GARCH forecast are the key issues (e.g., Kleindorfer and Li [2005] and Kinatader and Wagner [2010]). Credit risk measurement, on the other hand, is based on the loss distribution that results from a credit risk model. The measure of risk or economic capital requirement can be transferred from the one-period setting (Artzner et al. [2003]; Artzner et al. [2007]; Frittelli and Scandolo [2006]; Cherny [2007] and Cherny [2009]). Alternatively, the conditional risk per time step can be considered, so that risk is a time-dependent random process (Pflug [2006]; Riedel [2004]; Roorda et al. [2005] and

Cheridito and Kupper [2010]). The advantage of risk as a real number is the immediate applicability for capital allocation and portfolio optimization, whereas a risk process is useful for forecasting purposes. This article focuses on the first kind of risk and its application areas.

Capital allocation in a dynamic setting can be transferred from the one-period setting. Allocation principles, like gradient allocation as introduced in Tasche [2004a], can be used to determine the marginal capital amount of one subportfolio, asset class or credit instrument (Desmedt et al. [2004]; Cherny [2009]; Assa [2009] and Buch et al. [2011]). Depending on the chosen risk measure, the allocated capital can be a real number or a process.

The allocated capital requirement and the return of the instrument or subportfolio form the decision drivers in portfolio optimization models. Models in a one-period setting have been introduced by Li and Ng [2000] as a simple mean-variance optimization approach or by Rockafellar and Uryasev [2000] and based on this by Pflug [2006] in a more complex setting. Furthermore, Stoughton and Zechner [2000] focus on incentive systems and the role of learning in portfolio optimization decisions and Hallerbach [2004] considers optimization techniques with side conditions. Finally, Tasche [2004a] and Buch et al. [2011] analyze RORAC (return on risk-adjusted capital) optimization based on gradient allocation, the latter with a focus on asymmetric information.

This paper applies the introduced methods of risk measurement and capital allocation to the specific case of a credit portfolio in a multi-period setting. We discuss which definitions of loss can be used to calculate risk and introduce a new risk measure named weighted capital requirement. We synthesize the existing streams of literature, discuss the application in practical terms and consider the effects on portfolio management decisions and the challenges for portfolio managers. We conclude with the main result that a portfolio optimization decision is dependent on the choice of considered time frame, especially when rating migration is considered.

The paper is structured as follows: Section 4.2 introduces the notation. Section 4.3 considers the types of loss processes where risk can be measured, and introduces the credit risk models which are used to determine the associated distribution. Based on these processes, we discuss in Section 4.4 which kinds of risk measures are appropriate in a multi-period credit portfolio and analyze consequences of the choice of risk measure. Capital allocation via the introduced risk measures is explained in Section 4.5. Finally, in Section 4.6, all considerations of the previous sections are consolidated in an example, and we explain via this example the difference in portfolio optimization decisions that are based on one-period

versus multi-period risk measurement and capital allocation. Section 4.7 concludes the article with a summary of the results and practical challenges of multi-period risk assessment and gives an outlook on further research topics.

4.2 Notation and objective

We consider a portfolio structured in N subportfolios, called asset classes. An asset class is a set of obligors in a credit portfolio. Each obligor is identified with its default indicator variable $Y_{n,i_n}^t \in \{0, 1\}$, a random variable that describes default of obligor i_n in asset class n in a given time period t by $Y_{n,i_n}^t = 1$. We consider $T \in \mathbb{N}$ time periods. Let $u^t = (u_1^t, \dots, u_N^t)$ be the deterministic vector of asset class sizes in terms of number of obligors in time period $t \in \{1, \dots, T\}$ and denote $u_n = \max_{t=1, \dots, T} u_n^t$. Furthermore, let $u = \sum_{n=1}^N u_n$ be the upper bound of number of obligors in the portfolio.

One asset class is defined by common characteristics. As characteristics we consider the following variables:

- The maturity of obligor $i_n \in \{1, \dots, u_n\}$ in asset class n is denoted $m_{n,i_n} \in \{1, \dots, T\}$. The maximum maturity in asset class n is denoted m_n , and hence $m_{n,i_n} \leq m_n \leq T$.
- The unconditional probability of default of each obligor in asset class n in time period t is called PD_n^t . We set $PD_n^t = 0$ for $m_{n,i_n} < t \leq T$ and, for asset classes with inhomogeneous PD , we introduce PD_{n,i_n}^t as PD of obligor i_n in asset class n .
- The conditional probability of default of one obligor in asset class n is $PD_n^t | \mathcal{F}_{t-1}$.
- The correlation between the default events of an obligor in asset class m to an obligor in asset class n is denoted $\varrho_{m,n} = \varrho_{m,n}^t(Y_{m,i_m}^t, Y_{n,i_n}^t)$. Correlation is assumed to be the same for all obligors in one asset class and to be constant over time.

$x_n^t = \frac{1}{u_n^t} \sum_{i_n=1}^{u_n^t} Y_{n,i_n}^t \in [0, 1]$ is a random variable and indicates the fraction of defaults in asset class n in time period t . Based on this definition, we introduce the following types of default vectors or bundlings of elements of the type x_n^t :

- $X_n^t = (x_n^1, \dots, x_n^t) \in [0, 1]^t$ random vector of fraction of defaults in asset class n up to time period t
- $X_n = X_n^T = (x_n^1, \dots, x_n^T) \in [0, 1]^T$ random vector of fraction of defaults in asset class n up to the end of the considered time frame
- $X^t = (x_1^t, \dots, x_N^t)' \in [0, 1]^N$ fraction of defaults per asset class in time period $t \in \{1, \dots, T\}$, where x' is the transposed of x

We assume that exposure at default (*EaD*) and loss given default (*LGD*) are equal to 1 for all obligors. Hence, the portfolio loss at time t is given by $l^t = u^t \cdot X^t$, and the cumulative loss from period 1 to t by $L^t = \sum_{i=1}^t l^i$, and accordingly for asset class n : $l_n^t = u_n^t \cdot x_n^t$ and $L_n^t = \sum_{i=1}^t l_n^i$.

Let (Ω, P, \mathcal{F}) be a probability space where $\mathcal{F} = (\mathcal{F}_t)_{1 \leq t \leq T}$ is a filtration, such that \mathcal{F}_t represents all information given at time t and $X_n^t \in \mathcal{F}_t$ (i.e., X_n^t is \mathcal{F}_t -measurable for each n). $\mathcal{L}_\infty = \mathcal{L}_\infty(\Omega, P, \mathcal{F}) = \{Z : \Omega \rightarrow \mathbb{R} | Z \in \mathcal{F}, \|Z\|_{\mathcal{L}_\infty} < \infty\}$ is the space of all (bounded) credit instruments. Ω' denotes a new sample space defined via $\Omega' = \Omega \times \{1, \dots, T\}$. We set P' as probability measure on Ω' , defined by $P'(\bigcup_{1 \leq t \leq T} \{E^t\} \times \{t\}) = \sum_{t=1}^T w_t P(E^t)$, $\sum_{t=1}^T w_t = 1$ for $E^t \subset \Omega$; see Artzner et al. [2007]. In particular, we introduce for $(w_1, \dots, w_s, \dots, w_T) = (0, \dots, 1, \dots, 0)$ $P^s(\bigcup_{1 \leq t \leq T} \{E^t\} \times \{t\}) = P(E^s)$. In this way a random process on Ω is transformed to a random variable on Ω' .

One task of a risk manager is to decide in which of the so-defined N subportfolios the bank should invest. The target typically is to maximize return per risk, where, in our case, return is fix and given. To determine risk, three important steps are necessary: Define the parameter or process of which risk is measured, define the measure that is used and define the way in which the risk is allocated to the subportfolios. These three steps and the effects on the optimization decision form the objective of the subsequent sections.

4.3 Credit loss processes

4.3.1 Characteristics of credit loss processes

In order to discuss risk of credit portfolios, the definition of the characteristics of the analyzed loss process is crucial. In a multi-period setting, the first decision to make is whether losses $(l^t)_t$ or cumulative losses $(L^t)_t$ should be considered. In the next step we fix the characteristics of the loss process. We will base our analysis and discussion on the basic assumptions of zero growth. This means no additional credit instruments are added to the portfolio. Given this assumption, there are four dimensions that have to be considered when talking about credit loss processes: replacement of write-offs, replacement of matured assets, maturities and rating migration. Specifically, we analyze four types of credit processes as introduced below and summarized in Table 4.1.

	Type 1	Type 2	Type 3	Type 4
Replacement of write-offs	no	yes	no	no
Replacement of matured assets	no	no	no	no
Different maturities	no	no	yes	no
Rating migration	no	no	no	yes

Table 4.1: Characteristics of the four types of credit loss processes analyzed in this article

- Type 1: In each asset class, we assume no replacement of write-offs, identical maturities of all obligors, and no rating migration:
 $u_n^s \geq u_n^t$ for all $s \leq t \leq m_n$,
 $m_{n,i_n} = m_n = T$ for all $i_n \in \{1, \dots, u_n^1\}$ and
 $PD_n^t = PD_n$ for all $t \in \{1, \dots, T\}$.
- Type 2: We assume replacement of write-offs at the beginning of each period, identical maturities and no rating migration:
 $u_n^s = u_n^t$ for all $s \leq t \leq m_n$,
 $m_{n,i_n} = m_n = T$ for all $i_n \in \{1, \dots, u_n^1\}$, and
 $PD_n^t = PD_n$ for all $t \in \{1, \dots, T\}$.
- Type 3: We assume no replacement of write-offs or matured assets, different maturities for each obligor and no rating migration. We do not consider replacement of matured assets because the process would not be distinct from Type 1:
 $u_n^s \geq u_n^t$ for all $s \leq t \leq m_n$,
 $m_{n,i_n} \leq m_n = T$ for all $i_n \in \{1, \dots, u_n^1\}$ and
 $PD_n^t = PD_n$ for all $t \in \{1, \dots, m_{n,i_n}\}$, for each obligor i_n ; $PD_n^t = 0$ for $t > m_{n,i_n}$.
- Type 4: We assume no replacement of write-offs and identical maturities, but allow rating migration in each time period:
 $u_n^s \geq u_n^t$ for all $s \leq t \leq m_n$,
 $m_{n,i_n} = m_n = T$ for all $i_n \in \{1, \dots, u_n^1\}$ and
there exists an $s \neq t$ for at least one obligor i_n with $PD_{n,i_n}^s \neq PD_{n,i_n}^t$ for $s, t \in \{1, \dots, T\}$. We denote $PD_n^t = \frac{1}{u_n^t} \sum_{i=1}^{u_n^t} PD_{n,i_n}^t$ the average probability of default of asset class n in time period t . Typically, the probability of default changes with the rating, e.g., as given in the *S&P* rating migration matrix; see Table 4.2.

	AAA	AA	A	BBB	BB	B	CCC
AAA	90.23	8.99	0.56	0.05	0.08	0.03	0.05
AA	0.58	90.00	8.65	0.56	0.06	0.08	0.02
A	0.04	2.00	91.59	5.71	0.40	0.17	0.02
BBB	0.01	0.13	3.89	90.71	4.18	0.68	0.16
BB	0.02	0.04	0.18	5.82	84.23	7.98	0.83
B	0.00	0.05	0.15	0.25	6.35	83.69	5.04
CCC	0.00	0.00	0.21	0.32	0.97	17.01	54.66

Table 4.2: One-year migration matrix (in %) of average global corporate transition rates based on *S&P* data (1981-2011) excluding unrated corporates; rows indicate initial rating, columns indicate rating after one year

4.3.2 Simple credit risk trees

We illustrate the processes introduced above using probability trees. This allows us to make certain concepts, like filtrations, more tangible and to define and illustrate several terms in the latter discussion of risk measurement. We consider an exemplary portfolio consisting of two independent obligors, i.e., $N = 2, u_1 = u_2 = 1, T = 3$. The resulting tree structures for the four types of credit processes are shown in Figure 4.1.

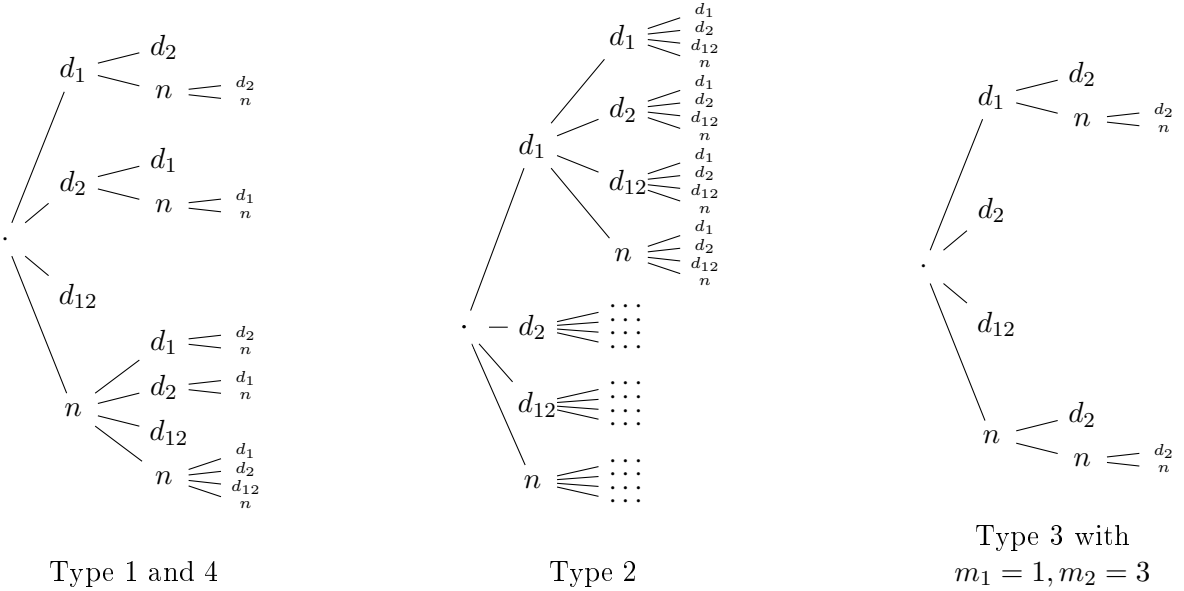


Figure 4.1: Credit loss trees for two obligors for the four credit loss processes given in Section 4.3.1, using the notation from Section 4.3.2

There d_n indicates default of the obligor in asset class $n \in 1, 2$, d_{12} indicates default of both obligors and n indicates no default at the given time period. This means $d_1 \hat{=} (x_1^1 =$

1, $x_2^1 = 0$) and so on. Tree Types 1 and 4 only differ in their distribution of default probabilities.

The filtration $\mathcal{F} = (\mathcal{F}_t)_t$ is given by the available information at any given time t . In the case of trees, the filtration corresponds to the partitions of the space Ω that represents the available information at time $T = 3$. For tree Type 1, this means

$$\Omega = \mathcal{F}_3 = \{[d_1d_2], [d_1nd_2], [d_1nn], [d_2d_1], [d_2nd_1], [d_2nn], [d_{12}], [nd_1d_2], [nd_1n], [nd_2d_1], [nd_2n], [nd_{12}], [nnd_1], [nnd_2], [nnd_{12}], [nnn]\}.$$

$$\mathcal{F}_0 = \{\emptyset\},$$

$$\mathcal{F}_1 = \{[d_1], [d_2], [d_{12}], [n]\},$$

$$\mathcal{F}_2 = \{[d_1d_2], [d_1n], [d_2d_1], [d_2n], [d_{12}], [nd_1], [nd_2], [nd_{12}], [nn]\}.$$

Hence, \mathcal{F}_t is a refinement of \mathcal{F}_{t-1} .

4.3.3 Multi-period credit risk models

In order to transfer results from the simple tree structure of the previous section to a more realistic setting, default and loss processes of a more complex portfolio have to be considered. This is typically achieved by credit risk models. We give a short introduction of the models used in the following sections, namely Conditionally Independent Defaults and Copula Models.

Hazard rates

According to Duffie and Singleton [2003] or McNeil et al. [2005], an intensity or hazard rate $h_n(t)$, or respectively h_n^t in a discrete setting, of asset class n describes the chance of default of obligor $i_n \in \{1, \dots, u_n\}$ at time t given survival up to time t . The cumulative hazard rate $H_n(t) = \int_0^t h_n(u)du$ or $H_n^t = \sum_{u=1}^t h_n^u$ is defined accordingly. $\tau_{i_n} = H_n^{-1}(E_{i_n})$ is called stopping time with E_{i_n} standard exponentially distributed. Default of obligor i_n up to time t occurs if $\tau_{i_n} < t$, i.e., if $E_{i_n} < H_n(t)$. Furthermore, the so-called survival function $S_n(t) = 1 - P(\tau_{i_n} \leq t) = \exp(-\int_0^t h_n(u)du)$ describes the probability that obligor i_n does not default before time t .

The hazard rate can be chosen in three different ways: constant, deterministic time-varying or stochastic. Examples are:

- $h_n(t) = c \in \mathbb{R}$ for all t , which describes a constant PD over time,
- $h_n(t) = c_t$, $c_t \in \mathbb{R}$, which corresponds to a rating migration in discrete time steps,
- $h_n(t) = \alpha_t + \sum_{j=1}^m \alpha_{n,j} M_{t,j} + E_{t,i_n}$ with $M_{t,j}, E_{t,i_n}$ CIR-square-root diffusions and $\alpha_t, \alpha_{n,j} \in \mathbb{R}^+$. For further details see, e.g., Duffie and Garleanu [2003].

As prerequisite for our further work we define:

Definition 3. Let $(\Omega, P, (\mathcal{F}_t)_{t=1, \dots, T})$ be given. If for $t_i \in \{1, \dots, T\}$ for all $i = 1, \dots, u$ the following three assumptions hold:

1. $H(t)$ hazard rate process is strictly increasing,
2. For all $t_i > 0 : P(\tau_i \geq t_i | \mathcal{F}_T) = P(\tau_i \geq t_i | \mathcal{F}_{t_i})$,
3. $P(\tau_1 \leq t_1, \dots, \tau_u \leq t_u | \mathcal{F}_T) = \prod_{i=1}^u P(\tau_i \leq t_i | \mathcal{F}_T)$,

then τ_i is called doubly stochastic conditionally independent random time.

In the subsequent sections, we apply two models to deduce multi-period loss functions. Here, we consider only one asset class, so the index n will be skipped.

Conditionally Independent Defaults Model

The simplest way of stepping from the default of one obligor to loss probabilities of one portfolio is by the assumption of conditional independence. Conditional independence for a given point in time t then means:

$$P(L^t = x) = E[P(L^t = x | \mathcal{F}_T)] = E \left[\prod_{\#j=x} P(\tau_j \leq t | \mathcal{F}_T) \cdot \prod_{\#j=u-x} P(\tau_j > t | \mathcal{F}_T) \right],$$

where $\#j = x$ is the set of all index vectors (j_1, \dots, j_t) with $\sum_{i=1}^t j_i = x$.

Copula Model

The second option applied is Copula Models, where a dependence structure between default times is introduced. Copula Models can be defined for deterministic as well as stochastic hazard rates. We will focus on the deterministic case.

Random times τ_1, \dots, τ_u follow a Copula Model with u -dimensional survival copula C , if there is an u -dimensional random vector $U \sim C$, independent of \mathcal{F}_T , such that

$$\tau_i = \inf\{t \geq 0 : \exp(-H(t)) \leq U_i\}, \quad 1 \leq i \leq u.$$

For deterministic hazard rates $h(t)$ and corresponding survival function $S(t)$ in a Copula Model the default probability for $t_i \in \{1, \dots, T\}$ is given by

$$P(\tau_1 > t_1, \dots, \tau_u > t_u) = C(S(t_1), \dots, S(t_u)).$$

A frequently employed copula is the one-factor Gaussian Copula, which corresponds to a one-factor structural model in the one-period case, as shown in McNeil et al. [2005].

4.3.4 Link of credit risk model and process type

Loss process types, as introduced in Section 4.3.1, are defined by two components of the applied credit risk model: the hazard rate and the vector of asset class sizes. Besides that, all input parameters define the asset class characteristics, but not the process type. The mapping of process types, hazard rate and asset class size can be seen in Table 4.3. For the readability of the table, we define two parameters: $D_n^t = \sum_{i_n=1}^{u_n} \mathbb{1}_{\{Y_{n,i_n}^t=1\}}$ and $M_n^t = \sum_{i_n=1}^{u_n} \left(\mathbb{1}_{\{Y_{n,i_n}^t=0\}} \cdot \mathbb{1}_{\{m_{n,i_n}=t\}} \right)$.

	Type 1	Type 2	Type 3	Type 4
Hazard rate h_n^t	PD_n	PD_n	PD_n	PD_n^t
Cumulative hazard rate H_n^t	$t \cdot PD_n$	$t \cdot PD_n$	$t \cdot PD_n$	$\sum_{i=1}^t PD_n^i$
Number of obligors u_n^t	$u_n^{t-1} - D_n^t$	u_n^{t-1}	$u_n^{t-1} - D_n^t - M_n^t$	$u_n^{t-1} - D_n^t$

Table 4.3: Risk model parameters of asset class n in time period t for credit loss process types as defined in Section 4.3.1

Be aware that the application of a credit risk model, like Conditional Independent Defaults or Copula Model, already accounts for the reduction of asset class size due to defaulted obligors. Hence, for technical implementation, one does not have to subtract D_n^t for process types one, three and four but has to add the number of replaced defaulted obligors for process type two.

Inserting the input parameters according to Table 4.3 in a credit risk model leads to the correspondent cumulative loss distribution. To deduce the loss distribution per period, one has to consider the difference of cumulative losses in period t and $t - 1$.

4.4 Multi-period risk measurement

The next essential step to come to a portfolio optimization decision is the definition of the risk measure, so that the risk of the portfolio can be determined. The so-defined portfolio risk is part of the target parameter of portfolio optimization.

4.4.1 Basic concepts

In a static model, a risk measure is a map of a random variable (RV) into the real numbers. This can be generalized in a dynamic setting by measuring risk of random processes (RP) instead of random variables. There are two concepts for dynamic risk measurement: by a real-valued number or a random process.

The more relevant type of risk measurement for capital allocation is risk measurement by real numbers. This concept allows to describe today's riskiness of one credit instrument with given maturity of one or more time periods. Risk as random process on the other hand describes ongoing riskiness of a credit instrument either with focus on its final value or on the risk trajectory.

For risk measurement by real numbers, there are two representations of risk measures that are presented in Frittelli and Scandolo [2006] and Artzner et al. [2007], the first being:

Definition 4. *Let \mathcal{L} be a vector space of random vectors, $\mathcal{C} \subset \mathcal{L}_\Sigma^\infty = \{Z \in \mathcal{L}^\infty \mid \sum_{i=1}^T Z_i \in \mathbb{R}\}$ and $\pi : \mathcal{C} \rightarrow \mathbb{R}$. Then any map $\rho : \mathcal{L} \rightarrow \mathbb{R}$ is called risk measure or capital requirement if*

$$\rho(X) = \rho_{\mathcal{A}, \mathcal{C}, \pi}(X) = \inf\{\pi(Y) \in \mathbb{R} \mid Y \in \mathcal{C}, X + Y \in \mathcal{A}\}, X \in \mathcal{L}$$

for some set $\mathcal{A} \subset \mathcal{L}$, provided it is a finite value.

According to Frittelli and Scandolo [2006], from a practical viewpoint, \mathcal{A} represents the fixed set of acceptable positions, \mathcal{C} represents the positions achievable by means of permitted hedging strategy and π describes the initial cost or treatment of capital over time, i.e., the choice of π determines among other things whether freed cash from one year can be reinvested in the following years. Finally, \mathcal{L} represents the vector space of considered loss processes. Overall, the definition states that risk is measured as the minimum amount of capital that has to be invested in order to make the portfolio acceptable. Therefore, \mathcal{A} is also called acceptability set. The definition of \mathcal{A} as a convex cone can be found in Artzner et al. [2007] but more general acceptance sets can also be considered.

Frittelli and Scandolo [2006] introduce two specific risk measures that will be considered in the subsequent sections:

Definition 5. Let $\mathcal{L}_\Sigma^\infty$ be a vector space of random vectors as defined above and $\mathcal{C} \subset \mathcal{L}_\Sigma^\infty$. Then for some set $\mathcal{A} \subset \mathcal{L}^\infty$:

1. A risk measure ρ is called *simple capital requirement* for $T = 1$ and $\pi(Y) = Y$.
2. A risk measure ρ is called *standard capital requirement* for $\pi(Y) = \sum_{t=1}^T Y^t$, provided it is a finite value.

The second equivalent representation for convex risk measures is given in Artzner et al. [2007]:

Proposition 1. If ρ fulfills the Fatou property, there is a closed convex set \mathcal{P}' of probabilities on (Ω', \mathcal{F}') absolutely continuous with respect to P' , such that:

$$\rho(X) = -\inf_{Q' \in \mathcal{P}'} E_{Q'}[X] = -\inf \sum_{t=1}^T w_t E_P[f_t X^t; f = (f_t)_t \in \mathcal{D}],$$

for a random vector X with values in a sample space Ω , where \mathcal{D} is a set of density functions of probability measures $Q' \in \mathcal{P}'$ with respect to P , called *determining system*, and $w_t > 0$ with $\sum_{t=1}^T w_t E_P[f_t] = 1$. $f_t : \Omega \rightarrow \mathbb{R}$ is a \mathcal{F}^t -measurable and non-negative function on Ω for all t .

One can define a risk process $\rho(X) = (\rho^t(X))_{t=1, \dots, T}$ for a random vector X as time-dependent random process, e.g., as the negative of a utility process as it is done in Cherny [2009]. Risk processes are less relevant in this setting because they cannot be used for today's capital allocation purposes. Different concepts of risk processes can be found in Frittelli and Gianin [2004], Artzner et al. [2007], Pflug [2006] or Cheridito and Kupper [2010]. The main application areas are risk forecasting and planning processes. Coherence of a risk measure as defined by Artzner et al. [1999] can be extended to a risk process in the multi-period setting by definition of dynamic consistency; see, e.g., Riedel [2004] or Pflug [2006].

In the following we consider the two most common risk measures, VaR (Value at Risk) and ES (Expected Shortfall), for a random process $\tilde{L} = (\tilde{L}^t)_t$, which will later be identified with the chosen loss process $(l^t)_t$ or $(L^t)_t$. Both measures can be expanded in a multi-period

setting, as demonstrated in the following examples. For latter use we define

$$\begin{aligned}\mathcal{C}_t &= \{Y | Y \text{ is } \mathcal{F}^t\text{-measurable}\}, \\ \mathcal{A}_\alpha^t &= \{Z \in \mathcal{L}^\infty | P(Z < 0) \leq \alpha\} \text{ for } \alpha \in (0, 1) \text{ and } Z \in \mathcal{F}^t, \text{ and} \\ \tilde{\mathcal{A}}_\alpha^t &= \{Z \in \mathcal{L}^\infty | E(Z \cdot \mathbb{1}_{A^t}) \geq 0, \forall A^t \in \mathcal{F}^t \text{ s.t. } P(A^t) > \alpha\}.\end{aligned}$$

1. One-period view: simple capital requirement; see, e.g., Frittelli and Scandolo [2006]. This definition coincides with the common definition of VaR as quantile of the loss distribution function and ES as expected loss given that the loss exceeds a certain barrier.

$$\begin{aligned}\rho_{\mathcal{A}_\alpha^1, \mathbb{R}}(\tilde{L}) &= VaR_\alpha(\tilde{L}) = \inf\{y \in \mathbb{R} | P(\tilde{L} + y < 0) \leq \alpha\} \\ \rho_{\tilde{\mathcal{A}}_\alpha^1, \mathbb{R}}(\tilde{L}) &= ES_\alpha(\tilde{L}) = \sup\{-E(\tilde{L}|A) | A \in \mathcal{F}, P(A) > \alpha\}\end{aligned}$$

2. More-than-one periods: product-type standard capital requirement; based on the product-type acceptance sets given in Frittelli and Scandolo [2006].

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_\alpha^1 \times \mathcal{A}_\alpha^2 \times \dots \times \mathcal{A}_\alpha^T, \\ \tilde{\mathcal{A}} &= \tilde{\mathcal{A}}_\alpha^1 \times \tilde{\mathcal{A}}_\alpha^2 \times \dots \times \tilde{\mathcal{A}}_\alpha^T, \\ \text{then } \rho_{\mathcal{A}, \mathcal{C}_0}(\tilde{L}) &= \sum_{t=1}^T VaR_\alpha(\tilde{L}^t), \\ \text{and } \rho_{\tilde{\mathcal{A}}, \mathcal{C}_0}(\tilde{L}) &= \sum_{t=1}^T ES_\alpha(\tilde{L}^t).\end{aligned}$$

3. More-than-one periods: product-type capital requirement with focus on final values; based on the product-type acceptance sets given in Frittelli and Scandolo [2006]. This approach only accounts for loss at the end of maturity. The difference to the one-period setting is that asset class characteristics, like PD , can change over time.

$$\begin{aligned}\mathcal{A} &= \mathcal{L}^\infty \times \mathcal{L}^\infty \times \dots \times \mathcal{L}^\infty \times \mathcal{A}_\alpha^T, \\ \tilde{\mathcal{A}} &= \mathcal{L}^\infty \times \mathcal{L}^\infty \times \dots \times \mathcal{L}^\infty \times \tilde{\mathcal{A}}_\alpha^T, \\ \text{then } \rho_{\mathcal{A}, \mathcal{C}_0}(\tilde{L}) &= VaR_\alpha(\tilde{L}^T), \\ \text{and } \rho_{\tilde{\mathcal{A}}, \mathcal{C}_0}(\tilde{L}) &= ES_\alpha(\tilde{L}^T).\end{aligned}$$

In illiquid markets where interference of risk managers is not possible, the focus is on final values. However, the concept of capital requirements with focus on final values ignores an increase of capital requirements by rating downgrades for $t < T$ as well as the timing of default.

4. More-than-one periods: product-type weighted capital requirement; based on the cumulative-stopping risk given in Assa [2009]. We use a discrete version of cumulative-stopping risk. In the easiest form, this risk measure describes the arithmetic mean of the risk in future time periods. By changing the weights, this approach is generalized in a way that it is able to account for influence factors like time value of money.

$$\begin{aligned}
 \mathcal{A} &= \mathcal{A}_\alpha^1 \times \mathcal{A}_\alpha^2 \times \dots \times \mathcal{A}_\alpha^T, \\
 \tilde{\mathcal{A}} &= \tilde{\mathcal{A}}_\alpha^1 \times \tilde{\mathcal{A}}_\alpha^2 \times \dots \times \tilde{\mathcal{A}}_\alpha^T, \\
 \pi(Y) &= \sum_{t=1}^T w_t Y^t \text{ with } \sum_{t=1}^T w_t = 1, \\
 \text{then } \rho_{\mathcal{A}, \mathcal{C}_0}(\tilde{L}) &= \sum_{t=1}^T w_t VaR_\alpha(\tilde{L}^t), \\
 \text{and } \rho_{\tilde{\mathcal{A}}, \mathcal{C}_0}(\tilde{L}) &= \sum_{t=1}^T w_t ES_\alpha(\tilde{L}^t).
 \end{aligned}$$

5. More-than-one periods: product-type discounted capital requirement. The identification of risk with capital requirement in a multi-period setting translates into the present value of the discounted future cash flows triggered by in- or decrease of capital requirements per period. Expected Shortfall in this sense can be described as follows:

$$\begin{aligned}
 \rho_\alpha(\tilde{L}) &= ES_\alpha(\tilde{L}^1) + \frac{1}{1+r} \left(ES_\alpha(\tilde{L}^2) - ES_\alpha(\tilde{L}^1) \right) + \dots + \\
 &\quad + \frac{1}{(1+r)^{T-1}} \left(ES_\alpha(\tilde{L}^T) - ES_\alpha(\tilde{L}^{T-1}) \right) - \frac{1}{(1+r)^T} ES_\alpha(\tilde{L}^T) \\
 &= \sum_{t=1}^T \left(\frac{r}{(1+r)^t} ES_\alpha(\tilde{L}^t) \right),
 \end{aligned}$$

where r is the discount rate. $ES_\alpha(\tilde{L}^t) - ES_\alpha(\tilde{L}^{t-1})$ describes the change of capital requirements in period t that occurs due to rating migration or maturing assets. In

the first period the full capital requirement $ES_\alpha(\tilde{L}^1)$ has to be raised. At the end of the last period the remaining capital $ES_\alpha(\tilde{L}^T)$ is freed, if we assume that all remaining assets mature. In this manner, only opportunity costs of capital per period are taken into account. This implies that unexpected losses over the complete time frame are 0, i.e., loss approaches expected loss. Therefore, this definition should only be used for large T .

6. More than one periods: product-type weighted capital requirement with discount rate. A potential approach of considering opportunity costs without ignoring unexpected loss is a combination of Example 3 with Example 5. In Example 3 we ignored opportunity costs and timing of default events, while in Example 5 we only focus on opportunity costs. We can define the total risk as sum of opportunities up to time $T - 1$ and discounted final-value risk at time T :

$$\rho_\alpha(\tilde{L}) = \sum_{t=1}^{T-1} \left(\frac{r}{(1+r)^t} ES_\alpha(\tilde{L}^t) \right) + \frac{1}{(1+r)^{T-1}} ES_\alpha(\tilde{L}^T)$$

In this sense, the combination of discounted and final-value focus risk measurement is a weighted capital requirement with weights $w_t = \frac{r}{(1+r)^t}$ for $t = 1, \dots, T - 1$ and $w_T = \frac{1}{(1+r)^{T-1}}$.

As an alternative, we set $ES_\alpha(\tilde{L}^t) = 0$ for $t > T$ and can then interpret the discounted capital requirement as weighted capital requirement (Example 4) with $w_t = \frac{r}{(1+r)^t}$ for $r \in [0, 1)$. It follows for $T \rightarrow \infty$ that $\lim_{T \rightarrow \infty} \sum_{t=1}^T w_t = 1$.

7. More-than-one periods: utility-based standard capital requirement; based on the utility-based acceptance sets given in Frittelli and Scandolo [2006].

$$\begin{aligned} \mathcal{A} &= \{Z \in \mathcal{L}^\infty | N(Z) > N(Z^*)\}, \text{ with } N \text{ utility functional, i.e., } N : \mathcal{L} \rightarrow \mathbb{R} \\ &\text{is concave and strictly increasing with } N(0) = 0, \text{ and } Z^* \text{ reference process,} \\ &\text{e.g., } N^t(Z) = E(Z^t \cdot \mathbb{1}_{A^t} | \mathcal{F}^{t-1}), \quad \forall A^t \in \mathcal{F}^{t-1}, P(A^t) > \alpha, \text{ and } Z^* = \mathbf{0}, \\ \text{then } \rho(\tilde{L}) &= \rho_{\mathcal{A}, \mathcal{C}_0}(\tilde{L}^1, \tilde{L}^2, \dots, \tilde{L}^T) = \sum_{t=1}^T \sup_{A^t} \{-E(\tilde{L}^t | A^t)\} = \sum_{t=1}^T ES(\tilde{L}^t), \\ \text{and } \rho(\tilde{L}) &= \rho_{\mathcal{A}, \mathcal{L}_\Sigma^\infty}(\tilde{L}^1, \tilde{L}^2) = \inf_{Y \in \mathcal{A}_\alpha^1} \left\{ \sup_{A^2} \left(-E[\tilde{L}^1 + \tilde{L}^2 - Y | A^2] \right) \right\}. \end{aligned}$$

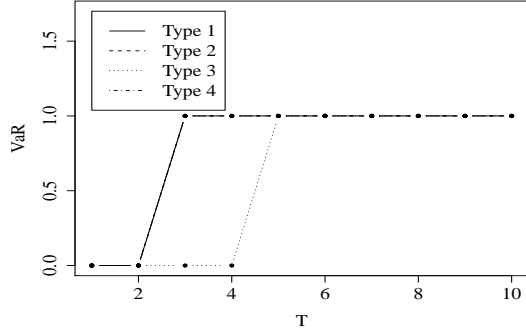
4.4.2 Application on credit loss trees

In order to visualize the effects of alternative loss processes as well as risk measures on portfolio risk, we apply risk measurement, as introduced above, to the simple example of a credit loss tree (see Section 4.3.2), where determination of risk is analytically solvable. We choose the example of a product-type capital requirement with focus on final values at the end of period T . As credit process, we consider cumulative losses of a credit portfolio of independent obligors. Therefore, we fix the following suppositions:

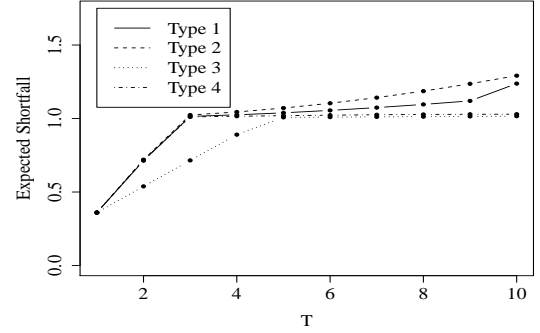
- The portfolio consists of $u = 2$ obligors.
- As risk measure, we choose multi-period VaR and ES with focus on final values as introduced in Example 3 in the previous section.
- The confidence level of the risk measure is $\alpha = 0.95$.
- The initial rating of both obligors is BB, which corresponds to $PD_1^1 = PD_2^1 = 0.9\%$.
- We consider one to ten periods, i.e., $T = 1, \dots, 10$.
- The correlation between the loss indicators of both obligors is $\varrho_{1,2} = 0$.
- We analyze the effects of rating migration. One obligor improves, the other worsens the PD by 0.1% per time period.

Based on the given data, we calculate the risk of the portfolio in a model of Conditionally Independent Defaults for one to ten periods. As loss processes, we consider cumulative loss processes of Types 1 to 4. Figure 4.2 shows the resulting risk in the four cases as a function of considered time frame. The analysis reveals that both, VaR and ES with focus on final values, grow with T for cumulative losses, as shown in Figure 4.2. The increase is high in the first two or three periods, but gets smaller for more time periods due to the high discreteness of the example. VaR is a step function of time with few informative value. Furthermore, the calculation shows that risk is highest if defaulted assets are replaced. Obviously, risk is decreased by reduced maturity of assets. We want to point out that Types 1 to 3 lead to almost identical ES values up to time period 4, but onwards the gap widens.

The two main results are that risk increases with the considered time frame T . And secondly, the chosen process type has an increasing influence on portfolio risk, the more time periods are considered.



(a) VaR with focus on final values for cumulative losses L^T in a simple tree



(b) ES with focus on final values for cumulative losses L^T in a simple tree

Figure 4.2: Risk as final-values-focused capital requirement of cumulative loss distribution for four types of credit loss trees as introduced in Section 4.3.1

4.4.3 Application on credit risk models

After these definitions and examples, we are now in the position to determine the risk of a credit portfolio. We therefore consider a portfolio with 100 obligors. As credit processes, loss as well as cumulative loss are considered. We determine standard capital requirements as well as risk with focus on final values and weighted capital requirements as introduced in Section 4.4. Therefore, we fix the following assumptions:

- The portfolio consists of $u = 100$ obligors.
- As risk measure, we choose multi-period VaR and ES (with focus on final values, as standard and weighted capital requirement with discount rate).
- The confidence level of the risk measure is $\alpha = 0.95$.
- The initial rating of the obligors is BB, which corresponds to $PD^1 = 0.9\%$.
- We consider one to ten periods, i.e., $T = 1, \dots, 10$.
- As discount rate for weighted capital requirements, we set $r = 0.1$.
- We analyze the effects of rating migration according to the *S&P* transition matrix (Table 4.2) with rating BB for $t = 1$. The resulting conditional average portfolio PD for the 10 considered time periods is:
 $PD^1 = 0.9\%, PD^2 = 1.54\%, PD^3 = 2.03\%, PD^4 = 2.47\%, PD^5 = 3.17\%, PD^6 = 3.44\%, PD^7 = 3.66\%, PD^8 = 3.84\%, PD^9 = 3.98\%, PD^{10} = 4.09\%$

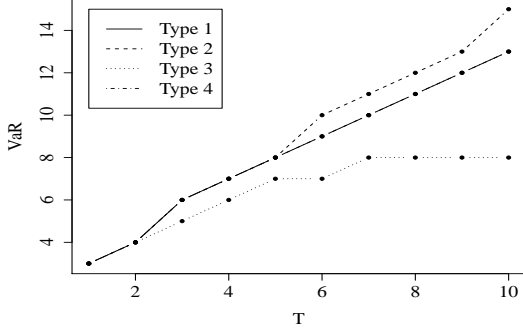
- We choose the model of Conditionally Independent Defaults and the Copula Model with Gauss-Copula with covariance matrix Σ as risk models, where

$$\Sigma = \begin{pmatrix} 1 & 0.17 & \cdots & 0.17 & 0.1 & \cdots & 0.1 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \\ 0.17 & & \cdots & 1 & 0.1 & & \cdots & 0.1 \\ 0.1 & \cdots & & 0.1 & 1 & 0.14 & \cdots & 0.14 \\ \vdots & \ddots & & \vdots & \vdots & \ddots & & \\ 0.1 & & \cdots & 0.1 & 0.14 & & \cdots & 1 \end{pmatrix},$$

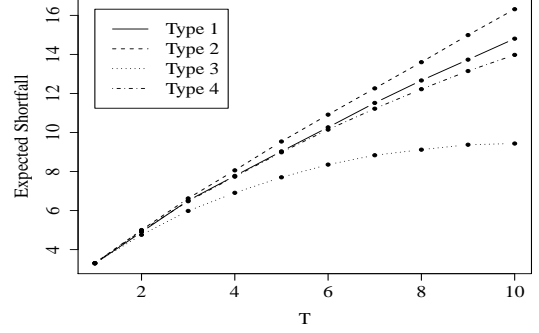
which is assumed to be constant over time.

Based on these assumptions, we determine via Monte Carlo simulation the loss or cumulative loss function for different time frames $T \in \{1, \dots, 10\}$ and calculate the portfolio risk according to the definitions given in Section 4.4.1 in Examples 2, 3 and 4. The results are shown on the following pages. This allows us to analyze the effects of the chosen process, credit risk model, risk measure and time frame on portfolio risk.

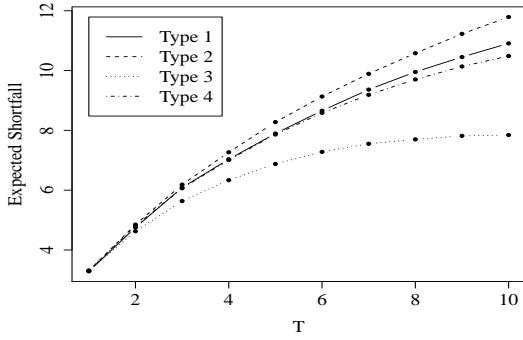
Risk increases with time in all considered cases, as one can see in Figures 4.3 and 4.4. Hence, assets with high maturity lead to higher risk values in general. The results are in line for VaR and ES as shown in Figures 4.3a and 4.3b. The risk of cumulative losses with focus on final values and of losses as standard capital requirements increases nearly linearly with T . In a Copula model (Figure 4.4), risk is higher than in a model of Conditionally Independent Defaults due to the correlation of default events. Especially, a loss process of Type 2 leads to clearly higher risk for large T . This effect can be explained through higher default rates, which have, additionally to the direct effect on risk, the secondary effect of a higher number of replaced assets for the following periods. Besides this, the results for Conditionally Independent Defaults and Copula Models are comparable. If we include rating migration according to the *S&P* transition matrix, we see a stronger risk increase with time in Figures 4.3e and 4.3f, which is caused by a worsening of the average portfolio *PD*. It is mentionable that high initial ratings lead to an above-average risk increase due to the very low risk in a one-period setting for a confidence level of 95%. For higher confidence levels, this effect reverses to the opposite. If we compare, for example, the one-period risk as weighted capital requirement with discount rate r of an AA-rated asset ($ES_{95\%}^1 = 0.4$) with its ten-period risk ($ES_{95\%}^{10} = 2.9$), the long-term risk is 7.1 times higher. For a BB-rated company, it is only 6.9 times higher. However, for $\alpha = 99\%$ the



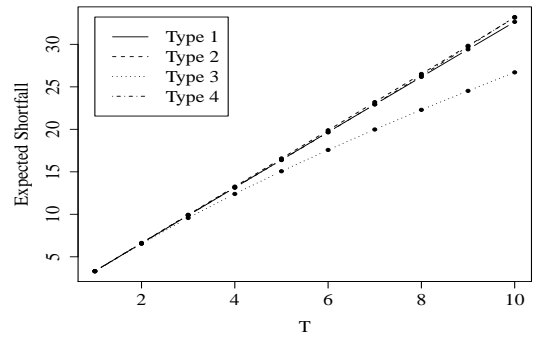
(a) VaR of cumulative losses L^T with focus on final values



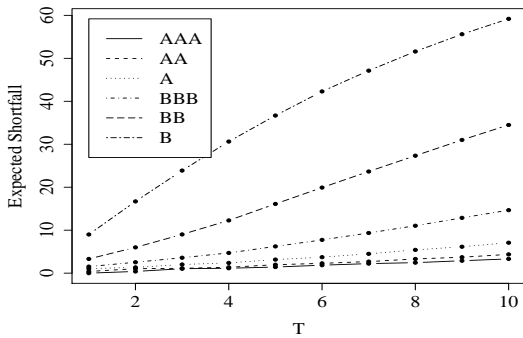
(b) ES of cumulative losses L^T with focus on final values



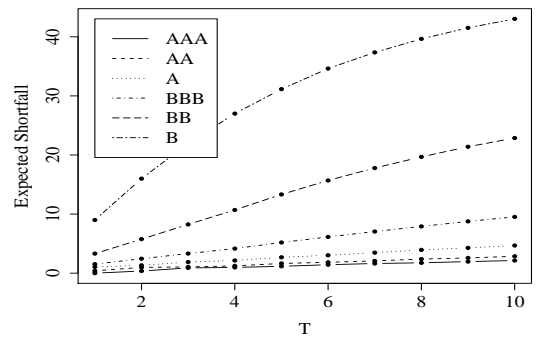
(c) ES of cumulative losses L^T as weighted capital requirement



(d) ES of loss l^T as standard capital requirement

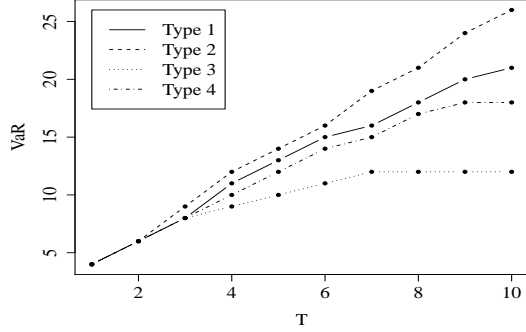


(e) ES of cumulative loss L^T with focus on final values for different initial ratings

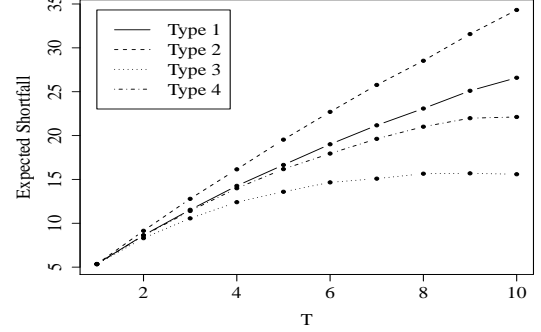


(f) ES of cumulative loss L^T as weighted capital requirement for different initial ratings

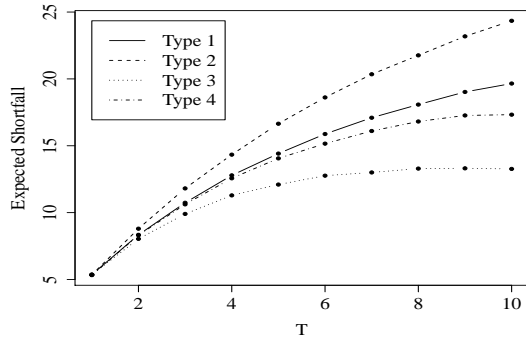
Figure 4.3: Risk as standard, weighted (with discount rate $r = 10\%$) or final-values-focused capital requirement of loss distribution for different types of credit loss processes as introduced in Section 4.4.3, simulated in a risk model of Conditionally Independent Defaults with 100,000 model runs



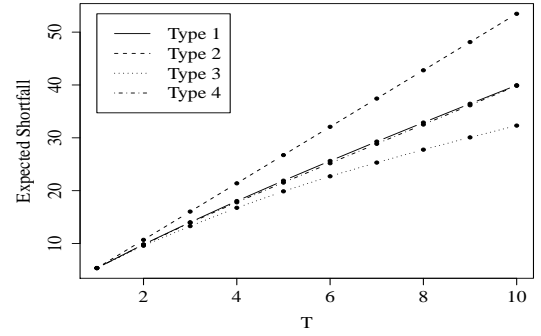
(a) VaR of cumulative losses L^T with focus on final values



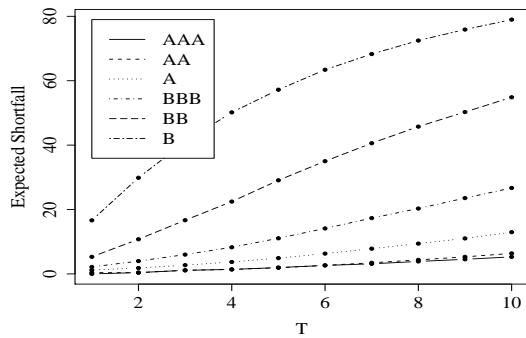
(b) ES of cumulative losses L^T with focus on final values



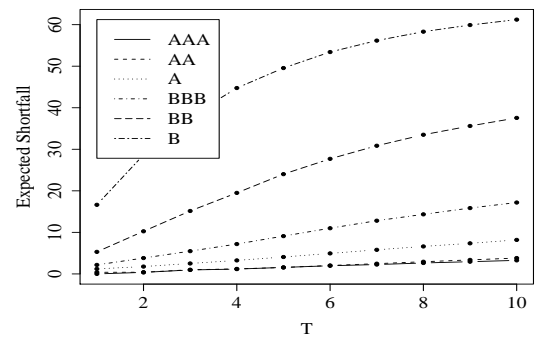
(c) ES of cumulative losses L^T as weighted capital requirement



(d) ES of loss l^T as standard capital requirement



(e) ES of cumulative loss L^T with focus on final values for different initial ratings



(f) ES of cumulative loss L^T as weighted capital requirement for different initial ratings

Figure 4.4: Risk as standard, weighted (with discount rate $r = 10\%$) or final-values-focused capital requirement of loss distribution for different types of credit loss processes as introduced in Section 4.4.3, simulated in a Copula Model with 100,000 model runs

factor for an AA-rated asset is 3.6 while it is 5.9 for the BB-rated asset. This means, short term risk measurement can over- or underrate the risk of an asset, depending on credit quality and chosen risk measure or quantile. The results are also shown in Figure 4.5. Furthermore, this result reveals that maturity effects decrease when the quantile is increased. This is in line with the work of Kalkbrener and Overbeck [2002].

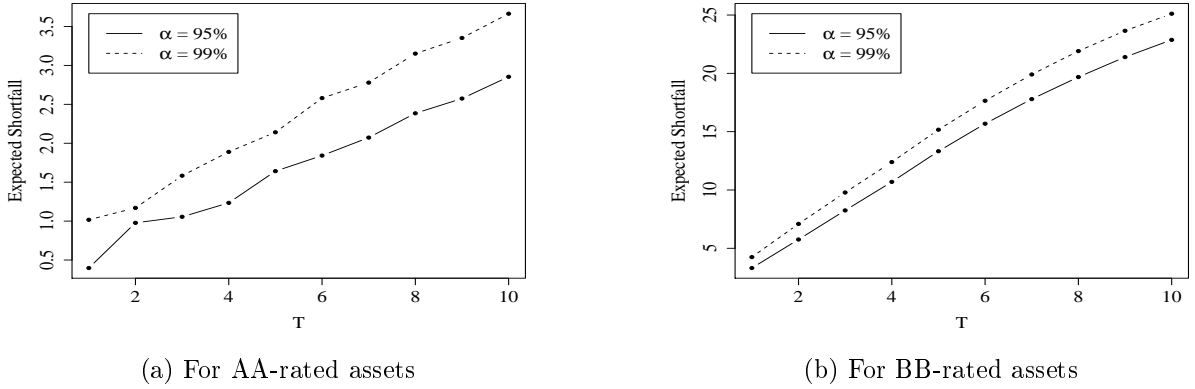


Figure 4.5: ES of cumulative loss L^T as weighted capital requirement with confidence levels $\alpha = 95\%$, and $\alpha = 99\%$; simulated in a model of Conditionally Independent Defaults with 100,000 model runs

Our analysis in this section shows that multi-period risk measurement bases on a number of different potential loss processes and risk measures and leads to significantly different results. Hence, it is a challenge to choose the most relevant process and risk measure.

Relevance of a risk measure depends on the purpose of risk measurement. Therefore, three dimensions should be considered: Comparability with historical data, relevance of timing of default events and cost of capital. If results have to be comparable with historical data, risk managers in many cases will be forced to use VaR, because VaR is the most commonly used risk measure. Otherwise, ES has the advantage of coherence and contains more information about highly improbable scenarios. The question of timing of default leads to a decision between a risk measure with focus on final values, where only the outcome at maturity counts, and a standard or weighted capital requirement, where each period matters. Differences might be triggered by rating migration or options of interference. As Figures 4.3 and 4.4 show, high maturities lead to higher risk for a final-value focused risk measure than for a weighted risk measure with discount rate. The reason for this effect is that the weight for high cumulative losses at the end of the considered time frame is one for a final-value focused risk measure, while it is lower for weighted risk measures. If

economic capital is a limiting factor and a worsening of ratings in early periods, therefore, is critical, the risk measure should consider more than the final value. Also, if portfolio managers have the chance to react on a change in portfolio characteristics, the risk measure should reflect these changes. Finally, the cost of capital differentiates between standard and weighted capital requirements. Standard capital requirement does not differentiate between a capital requirement in early versus late periods. Therefore, it should only be used if the discount rate is low, whereas weighted capital requirement reflects time value of capital cost.

4.5 Multi-period capital allocation

When the risk or economic capital of the complete portfolio is determined, the next step for portfolio valuation and optimization is allocation of risk to the subportfolios. This can be done through allocation of real-valued capital requirements or by defining a risk allocation process. In the case of real-valued capital allocation, the definition from the one-period setting can be transferred. Let A^ρ be an allocation principle, so that $\sum_{n=1}^N A_n^\rho = \rho(\tilde{L})$. In particular, gradient allocation is given for any differentiable risk measure ρ by

$$A_m^\rho = \lim_{h \rightarrow 0} \frac{\rho(\sum_{n \neq m} \tilde{L}_n + h \tilde{L}_m) - \rho(\sum_{n \neq m} \tilde{L}_n)}{h}.$$

As example we consider ES as standard capital requirement and $\tilde{L} = (\tilde{L}^t)_t = (l^t)_t$ as loss process: $\rho((l^t)_t) = \sum_{t=1}^T ES(l^t)$, then

$$\begin{aligned} \rho(l_m) &= \lim_{h \rightarrow 0} \frac{\sum_{t=1}^T ES(\sum_{n \neq m} l_n^t + h l_m^t) - \sum_{t=1}^T ES(\sum_{n \neq m} l_n^t)}{h} \\ &= \sum_{t=1}^T \lim_{h \rightarrow 0} \frac{ES(\sum_{n \neq m} l_n^t + h l_m^t) - ES(\sum_{n \neq m} l_n^t)}{h}. \end{aligned}$$

The resulting allocated capital of subportfolio m equals the sum over all periods of allocated capital per time period.

If this allocation principle is applied for portfolio management purposes, it implicitly assumes that each subportfolio is homogeneous or moderately heterogeneous; see Dorfleitner and Pfister [2012], Dorfleitner and Pfister [2013]. For small or inhomogeneous subportfolios, alternatives, like incremental risk measurement, should be used.

If we consider a capital requirement process $\rho = (\rho^t)_t$, it has to fulfill the three conditions of normalization, monotonicity and the translation property as defined in Cheridito and Kupper [2010]. According to Cherny [2009] a utility allocation can be defined for this kind of risk process. If we choose gradient allocation to calculate the utility contribution or respectively the risk contribution of one asset class at time t , we obtain:

$$\rho^t(\tilde{L}_m) = \lim_{h \rightarrow 0} \frac{\rho^t(\sum_{n \neq m} \tilde{L}_n + h \tilde{L}_m) - \rho^t(\sum_{n \neq m} \tilde{L}_n)}{h}$$

ρ^t can be interpreted as risk of the portfolio at time t given all future information up to time $t-1$. Desmedt et al. [2004] for example defines ρ^t as follows: Let $R^t(\tilde{L}) = E[\sum \tilde{L}^t | \mathcal{F}^t]$, then $\rho^t = \bar{\rho}(\sum \tilde{L}^t - R^t | \mathcal{F}^t)$, where $\bar{\rho}$ is a one-period risk measure.

The target of our work is to use capital allocation for portfolio optimization. Therefore, we focus on the first case of allocation of real-valued capital requirements. Our analysis examines the dependence of allocated capital and considered time frame. We introduce an example in order to examine the effects of the chosen risk measure and time frame on allocated capital. We consider a portfolio consisting of two asset classes with 100 obligors each ($u_1^1 = u_2^1 = 100$). The two asset classes are independent. We will consider the allocated capital as weighted capital requirement (ES_α with $\alpha = 0.95$, discount rate $r = 0.1$) for $T = 1, T = 5$ and $T = 10$. The first asset class is fixed and has an initial rating of AA. Asset class 2 at time period $t = 1$ has an average asset class rating of AAA in the first case, AA in the second case, ... or B in the last case, according to the *S&P* rating definition. The rating, and hence the *PD*, of both asset classes will migrate according to the modified *S&P* transition matrix given in Table 4.2. We compare the absolute risk of the second asset class for all cases, based on the cumulative loss process (Type 4), as well as the relative proportion of allocated risk as fraction of the total capital requirement of the portfolio. The results are given in Table 4.4.

Rating migration has a significant influence. As shown in Table 4.4 for a confidence level of $\alpha = 95\%$, the relative share of capital of higher initial rating goes down for ES as weighted capital requirement. The same calculation for higher confidence levels leads to the opposite result, i.e., worse-rated credit instruments need an even higher share of required capital when two or more periods are considered. This result gives a first indication that the chosen time frame has an influence on portfolio management decisions.

	$T = 1$		$T = 5$		$T = 10$	
	abs	rel	abs	rel	abs	rel
AAA	0.0	0.0%	1.2	41.4%	2.1	42.8%
AA	0.4	50.0%	1.6	50.0%	2.9	50.0%
A	1.1	72.9%	2.7	62.7%	4.7	61.8%
BBB	1.5	79.5%	5.2	76.5%	9.5	76.7%
BB	3.3	89.4%	13.3	89.0%	22.9	88.9%
B	9.0	95.8%	31.1	95.0%	43.0	93.8%

Table 4.4: Absolute and relative risk of the second asset class (rating of first asset class: AA) for different initial ratings as weighted capital requirement with $\alpha = 95\%$ and discount rate $r = 10\%$; modeled in a model of Conditionally Independent Defaults with 100,000 simulation runs

4.6 Effects on portfolio optimization

In order to discuss the effects of multi-period risk measurement on portfolio optimization decisions, one has to define a target parameter. In this section, we will use RORAC. The definition of RORAC in a multi-period setting is dependent on the chosen loss process, risk measure and allocation principle. Hence, there are a lot of different options to calculate RORAC. We want to analyze if the chosen definition has an impact on the portfolio management decision. In this section, we use the following two alternative RORAC definitions, which match the two risk measures analyzed in Section 4.4.3:

$$RORAC = \frac{\text{Cumulative return}}{\text{ES with focus on final values} - \text{Expected cumulative loss}} \quad (4.1)$$

or

$$RORAC = \frac{\text{Present value (PV) of cumulative return}}{\text{ES as weighted capital requirement} - \text{PV of expected cum. loss}} \quad (4.2)$$

In the one-period setting the two formulas coincide and meet the classic definition.

We revisit the example of the previous section in order to analyze the effects of choice of risk measure and RORAC definition on a portfolio optimization decision. Two asset classes with different initial ratings are given. Assume each asset class consists of 100 obligors at time $t = 1$ and each non-defaulted obligor leads to a return of 0.0006 in the first and 0.002 in the second asset class. If we focus on the case where the first asset class had a initial rating of AA and the second asset class BB, we can determine the RORAC per asset class.

We calculate the average PD with the rating transition matrix given in Table 4.2.

	Asset class 1		
	one-period	final values	weighted
Time periods T	1	10	10
PD_1^T	0.02%	0.29%	0.29%
Cum. return	0.06	0.60	0.40
$ES_{95\%}$	0.40	4.37	2.85
$ES_{99\%}$	1.02	5.41	3.65
Expected cum. loss	0.02	1.42	0.80
RORAC (95%)	15.91%	20.28%	19.75%
RORAC (99%)	6.02%	14.98%	14.20%

	Asset class 2		
	one-period	final values	weighted
Time periods T	1	10	10
PD_2^T	0.90%	4.09%	4.09%
Cum. return	0.20	1.81	1.25
$ES_{95\%}$	3.31	34.50	22.87
$ES_{99\%}$	4.25	37.25	25.12
Expected cum. loss	0.90	25.62	15.94
RORAC (95%)	8.30%	20.39%	18.03%
RORAC (99%)	5.98%	15.57%	13.61%

Table 4.5: RORAC per asset class for two asset classes with different initial rating (AA and BB) for $T = 1$ and $T = 10$ with $\alpha = 95\%$ and $\alpha = 99\%$ in a model of Conditionally Independent Defaults; RORAC is calculated according to formulas (4.1) and (4.2)

The ES is determined via simulation of the cumulative loss distribution with a model of Conditionally Independent Defaults as introduced in Section 4.3.3. We use the definition of capital requirement with focus on final values and weighted capital requirements from Section 4.4 with a discount rate of 10%. The expected cumulative loss is deduced from the simulated loss distribution. Finally, we calculate the expected cumulative return by multiplying the return per deal with the expected number of deals per period. In order to receive the present value used for the second case, we discount the yearly return with a discount rate of 10%. If we follow the basic concept of an optimization algorithm as introduced, e.g., in Rockafellar and Uryasev [2000], we have to invest in the asset class with the higher RORAC. Using the one-period ES, this leads to an increase of Asset class 1 for both confidence levels. However, if a ten-period ES with focus on final values is used, the RORAC is higher in Asset class 2, as shown in Table 4.5. The results also demonstrate that the RORAC varies considerably with the chosen risk measure and time frame.

This example illustrates that the portfolio optimization decision is significantly influenced by the chosen risk measure and time frame. This leads to a necessity to define a clear optimization target and to trade short-term profitability against sustainability.

4.7 Conclusion and practical aspects

In order to apply multi-period credit risk measurement, capital allocation and portfolio optimization to credit portfolios, a number of practical aspects have to be considered. First of all, it is crucial to define the relevant process of which risk shall be measured. It has to be differentiated between loss and cumulative loss, and one has to be aware of the effects of different assumptions, such as replacement of write-offs, replacement of matured assets or rating migration. We defined these assumptions, and showed how this presetting has to be incorporated in an applied credit risk model.

Based on the so-defined different types of loss processes, risk measures can be introduced. VaR and ES can be expanded in different ways in a multi-period setting with deviant results in absolute terms. We introduced ES as weighted capital requirement with and without discount rate as risk measure in order to display the future capital requirement of a loss process as present value of cash flows.

In order to achieve a risk-return-based portfolio management decision, the resulting portfolio risk has to be allocated to asset classes. One-period capital allocation principles and portfolio optimization can be applied to a multi-period setting. We proofed based on an example that portfolio optimization decisions with a view on multi-period risk can

be different from the one-period view. Hence, there is a trade-off between short-term and long-term capital needs. This means, if multi-period risk measurement and portfolio optimization are applied, risk management departments face a number of different practical issues and challenges in three areas: interpretation, implementation and communication.

In the first area, the main issue is that the new assessment technique leads to a number of alternative risk numbers depending on the chosen time frame, loss process and risk measure. It is crucial to interpret each number correctly and to choose the most relevant one for the decision process. Furthermore, the multi-period risk measure will differ from the (maturity-adjusted) regulatory capital requirement; see Kalkbrener and Overbeck [2002]. This deviance has to be interpreted as well, and a consideration and weighting of sustainability and long-term risk reduction versus short-term capital needs is required.

Implementation is closely linked to the interpretation result. Systems and IT infrastructure have to provide the option to consider all different types of relevant risk measures. Also, the reporting structure has to exhibit the different types of risks and processes, and every affected employee has to be trained to read the new numbers.

Finally, the multi-period setting leads to a higher complexity in communication between risk modeling experts and management or externals. While the rather simple concept of VaR can be communicated to non-specialists, the rather complex time-dependent risk concept that leads to a number of different outcomes per credit instrument might lead to confusion. Overall, the barriers of a more sustainable understanding of risk measurement should not be underestimated, but can be overcome.

All these challenges of application are interesting food for further thoughts. Furthermore, our results are based on models of Conditionally Independent Defaults and Copula Models with time-independent copula. An indication that default risk dependencies change over time, based on the example of the subprime crisis, can be found in Grundke [2010]. It is subject to further research to transfer the results to alternative models or parameters, such as time-varying correlation or respectively copula.

5 Summary and future research

This dissertation analyzes the applicability and challenges of gradient capital allocation with a focus on credit risk and its influence on a RORAC-based portfolio optimization decision. There are different angle of views that can be persued: The first is the effect of the subportfolio size on an optimization advice. The second analyzed question is the impact of inhomogeneity and stress on capital allocation and portfolio optimization, and finally, the third part considers the influence of chosen time frame, with a focus on one-period versus multi-period risk measures and allocation techniques.

Regarding the first consideration, we found out that allocated capital can only be used for optimization purposes if each subportfolio has a minimum size, in terms of number of assets or obligors. This minimum size depends on the probability of default and the correlation of the subportfolio and on the chosen risk measure. Moderate inhomogeneity in the subportfolio does not influence the results, as long as the subportfolio characteristics, like average default probability, are independent of the subportfolio size. On the other hand, systematic under- or overestimation of parameters as well as stress scenarios can have a significant influence on the optimization advice. Therefore, it is crucial to base each decision on three fundamentals: the RORAC-optimization algorithm result in the base case, a sensitivity analysis and a scenario-based stress test. Scenarios can be included in the decision process via constraints or side conditions of the algorithm. Furthermore, it is necessary to explicitly define the target parameter of portfolio optimization. This concerns especially the considered time frame. A multi-period RORAC can lead to a different portfolio management decision than a one-period RORAC. Therefore, the risk measure and the time horizon have to be determined accordingly to the current target of the bank. Short-term capital constraints have to be traded against long-term profitability.

In the risk management department, these results can influence the business process in different ways. The size of the asset classes has to be tracked and small asset classes have to be considered separately from the rest of the portfolio. Furthermore, regular sensitivity checks and stress tests should be implemented. Finally, the data base and modeling systems

should be expanded in a way, that they have the capability to measure multi-period risk. Of course, along with these technical implementations, trainings, incentives and reporting structures have to be adjusted.

If a financial institution fulfills the technical and personal requirements, active portfolio steering and risk-based decisions are possible. Nevertheless, the mathematical result still bases on model assumptions and is subject to model risk. We only showed some potential pitfalls of the application of optimization algorithms. Other potential model errors cannot be neglected. Thus, each decision has to be challenged by personal experience of risk managers and top management. The model serves as supplement and aid, it will never make the decision.

The work opens room for further research. From a mathematical perspective, the results can be implemented into an optimization algorithm. This includes especially stress testing. For this purpose, scenarios have to be defined from a macroeconomic perspective. Then, the resulting side conditions have to be included into the algorithm.

Another connected research area is business organization and processes. The impact of our results on organizational structure, incentive systems, processes and reporting can be analyzed.

A Appendix to Chapter 2

A.1 Proof of Theorem 1

Proof. From Lemma 1 follows for any pair of asset classes that the limit of the joint distribution function $\{l_{i,j}^{u_i,u_j}\}$ exists. If the marginal distributions \tilde{l}_i and the copula functions are piecewise continuous, it follows that the joint distribution function as composition of piecewise continuous functions is also piecewise continuous and bounded by $f(x) \equiv 1$. It follows that the integral of the function exists and consequently the loss distribution function of the two asset classes i and j . With induction, the existence of the total loss function of the complete portfolio can be concluded.

The per-unit risk can be calculated via gradient allocation; see approximation (2.3).

□

A.2 Proof of Theorem 4

Proof. The first claim follows directly from Theorem 2. If only the number of obligors in the first asset class is increased, the share of obligors in the second asset class converges to zero. The term for the second subportfolio converges to 0, because X describes the fraction of defaults and with the first asset class increasing, the share of the second asset class becomes smaller. To prove the second claim we calculate:

$$\begin{aligned}
 \tilde{l}(x) &= P[X \leq x] = \int_{-\infty}^{\infty} P[X \leq x | M = c] \phi(c) dc \\
 &= \int_{-\infty}^{\infty} \mathbb{1}_{\{a'PD_1(c) + b'PD_2(c) \leq x\}} \phi(c) dc \\
 &= \int_{-\infty}^{\infty} \left(\int_{x'=0}^x \left(\mathbb{1}_{\{PD_1(c) \leq \frac{x-x'}{a'}\}} \cdot \mathbb{1}_{\{PD_2(c) \leq \frac{x'}{b'}\}} \right) dx' \right) \phi(c) dc \\
 &= \int_{x'=0}^x \int_{-y}^{\infty} \phi(c) dc \, dx',
 \end{aligned}$$

where

$$0 \leq \frac{x - x'}{a'} \leq 1, \quad 0 \leq \frac{x'}{b'} \leq 1, \quad \text{i.e., } x - a' \leq x' \leq b';$$

$$y = \min \left(\frac{1}{\alpha_1} \left(\sqrt{1 - \alpha_1^2} \Phi^{-1} \left(\frac{x - x'}{a'} \right) - S_1 \right); \frac{1}{\alpha_2} \left(\sqrt{1 - \alpha_2^2} \Phi^{-1} \left(\frac{x'}{b'} \right) - S_2 \right) \right).$$

For the second line we used part 2 of Theorem 3.

By using that Φ is the antiderivative of ϕ , we obtain the formula in the theorem. □

A.3 Simulation results for expected shortfall as risk measure

The effects described in Section 2.4.2 for VaR as risk measure are very similar to the results of an accordant simulation for expected shortfall. Again, in a portfolio consisting of one asset class of 100 obligors the per-unit risk is higher than in a portfolio of 1,000 obligors, but then remains constant for an even higher number of obligors. To give an example, in both cases, the difference between per-unit risk in a granular portfolio and a diversified portfolio is 14% for a confidence level of $\eta = 99.5\%$, as can be seen in Figure A.1.

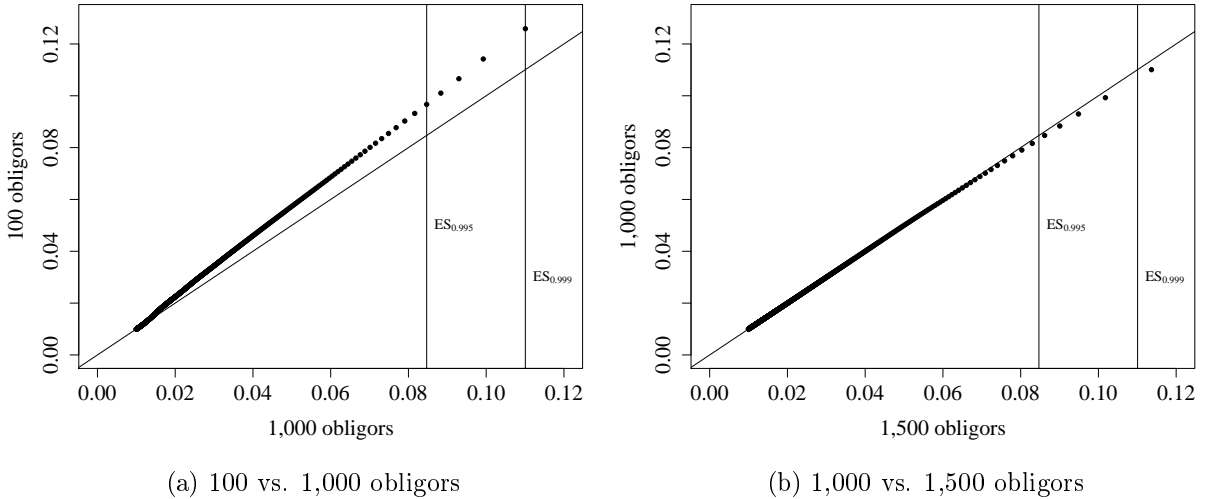


Figure A.1: Comparison of expected shortfall contribution for one asset class simulated with 100,000 model runs in a one-factor model as described in formula (2.6). Both figures describe asset classes with $PD = 2\%$ and $\varrho = 2.73\%$. The vertical lines mark the VaR with $\eta = 0.995$ and $\eta = 0.999$ for the larger asset class size on the x-axis.

A Appendix to Chapter 2

$\eta \backslash PD$	0.5%	1%	2%	5%	10%	1%/2%	2%/5%	1%/10%	5%/10%
0.95	264	285	336	426	487	318	392	438	462
0.97	299	321	373	471	537	355	434	478	508
0.99	372	390	444	563	632	430	518	553	596
0.999	502	533	591	742	783	593	694	710	719

(a) One asset class
(b) Two asset classes

Table A.1: Number of necessary obligors to achieve constant per-unit risk with a maximum error of 20 bp simulated with 100,000 model runs in a structural one-factor with ES as risk measure.

If we examine the dependency on the input parameters PD and η for expected shortfall, we obtain the results displayed in Table A.1a for one asset class and Table A.1b for two asset classes. As expected, the general results are similar to the results we calculated for VaR. The dependency on input parameters is nearly identical. However, the number of necessary obligors is higher than for VaR. This can be explained by the higher sensitivity towards concentration risks (see Bonti et al. [2006]).

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