Volume-Preserving Mean Curvature and Willmore Flows with Line Tension

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Abstract

We show the short-time existence and uniqueness of solutions for the motion of an evolving hypersurface in contact with a solid container driven by volume-preserving Mean Curvature Flow (MCF) and line tension effect on the boundary. Difficulties arise due to the non-local nature of the resulting second order, nonlinear PDE, which will be overcome by a perturbation result from semigroup theory. In addition, we prove the same result for the Willmore flow with line tension, which results in a nonlinear PDE of fourth order. For both flows we will use a Hanzawa transformation to write the flows as graphs over a fixed reference hypersurface. We finish the thesis with an application of the generalized principle of linearized stability to prove stability of spherical caps under the volume-preserving Mean Curvature Flow with line tension.

Contents

1 Introduction

During the last three decades there has been a growing interest towards the field of curvature flows and evolving hypersurfaces. In the late 1970s Kenneth A. Brakke in [Bra78] was the first to study surfaces driven by the geometric evolution law $V_\Gamma = H_\Gamma$, meaning that the motion of a point on the surface in normal direction V_Γ is equal to the mean curvature of the surface in that point. This is known as the Mean Curvature Flow (MCF) and with the additional condition of volume conservation, this flow is a simplified model for the motion of soap bubbles or liquid droplets. In the 1980s several results related to this flow were proved, for example by Gage and Hamilton [GH86], Grayson [Gra87] as well as Huisken [Hui86]. One year later it was also Huisken and in 1998 Escher and Simonett [ES98b], who provided noteworthy results concerning the generalization to the volume-preserving MCF.

Following the remarkable research progress concerning the MCF other curvature flows were brought in the focus of general mathematical interest. In the last decade of the 20th century and the first years of the 21st century the Willmore flow was studied by several mathematicians such as Simon [Sim93] and Rivière [Riv06]. Very productive researchers in terms of the Willmore flow were Kuwert and Schätzle, who proved several results elucidating various aspects of this flow (cf. [KS01], [KS02], [KS04], [KS12], [KS13] and [BK03]). In the Willmore flow the motion of a surface V_Γ is not only proportional to its mean curvature, but to the Laplace-Beltrami operator of the mean curvature plus some lower order curvature related terms. This flow is a simplified model for the evolution of biomembranes.

This thesis is devoted to 2-dimensional surfaces in $\mathbb{R}^3$ that are brought in contact with a fixed solid container and the rules that govern their movements. Modeling a drop of liquid or a soap bubble physics suggest that the air-liquid-interface or the soap layer, which both can be viewed as an evolving hypersurface, tends to minimize its area. If such a surface gets into contact with some fixed impermeable boundary layer the mass conservation law makes it necessary to demand a constant volume condition. The occurring contact angle is mainly determined by the material constants and thereby the wettability of the container. But in particular on small length scales a second effect is entering the scenery, namely the line tension (cf. Section 1 of [BLK06]). This effect penalizes long contact curves and forces the drop or bubble to roll off more from the boundary. All these phenomena will control the motion of such an evolving hypersurface, which is schematically illustrated in Figure 1.

During this motion it seems unnatural to prescribe the boundary curve or the contact angle, since an arbitrary drop or bubble, which is brought in contact with a solid container, will not instantly have a boundary curve or contact angle that is energetically minimal. Prescribing the contact curve or the contact angle would correspond to Dirichlet or Neumann boundary conditions, respectively. Instead of doing so, we will impose boundary

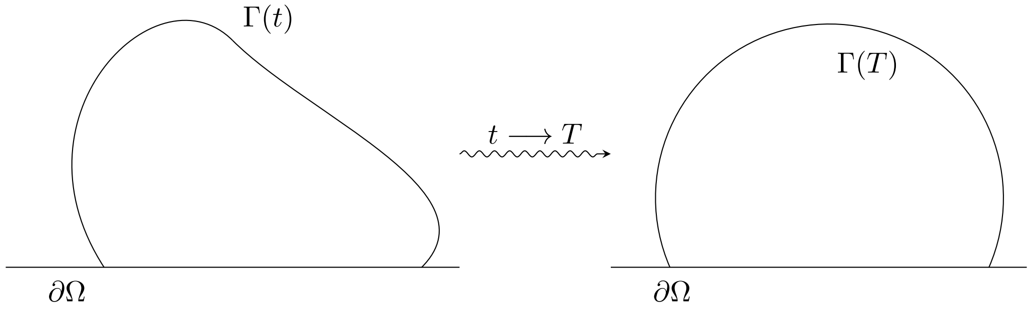

Figure 1: Evolving hypersurface $\Gamma(t)$ in contact with a container boundary $\partial\Omega$

conditions of relaxation type to allow the contact angle to change and the boundary curve to move. We will prove that for a sufficiently smooth initial droplet there is a small time interval in which we can guarantee that the initial droplet can evolve following the rules of this motion.

Biomembranes or lipid bilayers, such as the surface of a red blood cell, however, are related to the so called Helfrich energy (cf. [Can70] and [CV12]). The Willmore energy can be viewed as the simplest example of the Helfrich energy. Although the first considerations into this direction can be backtracked to Sophie Germain in 1831, the Willmore functional was dedicated to Thomas J. Willmore after his publication in 1965 (cf. [Wil65]). Instead of minimizing their areas these hypersurfaces try to minimize their bending energy. Including the wetting and line tension effect on the boundary we will again show wellposedness for short times.

The remaining part of this introductory section will be spent on a brief overview concerning the structure of this thesis.

In Section 2 we describe the general setting and introduce the notation that will be used. Moreover, we provide useful results that we will need in later sections, such as the variational formulas for area, volume and length. After presenting the preliminaries we will move on to investigate the volume-preserving MCF of an evolving hypersurface with line tension effects on the contact curve. We will motivate this flow by introducing the energy that will be minimized and deduce equations that have to hold for stationary surfaces. This motion will be governed by nonlinear PDEs of second order, which we will linearize around a fixed reference hypersurface in the final subsection.

Section 3 is devoted to the first goal of this thesis, this is to show the existence of solutions of the MCF for sufficiently short times. We will achieve this goal by first considering the short-time existence of solutions of the linearized flow and then apply a fixed point argument to prove the same statement for the original nonlinear flow. The non-local nature of the volume-preserving MCF will give rise to some technical difficulties, which will be overcome by utilizing semigroup theory and a perturbation argument.

After the intensive consideration of the MCF, we will study the Willmore flow in the same context in Section 4. The setting of the evolving hypersurface in contact with a static container remains the same as before, but now the rules governing the motion of the hypersurface will be given as the Willmore flow. Again we will include line tension effects and boundary conditions of relaxation type. The resulting nonlinear PDE will be in one way more difficult, since it will be of fourth order, and in another way easier, because it will not contain any non-local terms. At the end of the section we will consider the linearization of the PDE, where we will see that it is not even necessary to linearize each and every single term, which will cause the calculations to be much shorter than those for the MCF.

Following the same strategy as for the MCF we will prove the short-time existence of solutions for the motion driven by the Willmore flow in Section 5. The semigroup arguments will not be of interest here due to the purely local nature of the flow.

Finally, in Section 6 we will come back to the volume-preserving MCF with line tension. Yet, rather than proving short-time existence results we will consider the stability of spherical caps, which are the easiest stationary surfaces of the given flow. After some elementary relations that are useful to describe spherical caps, we will introduce the generalized principle of linearized stability which the stability analysis will rely on. We will use the middle part of this section to introduce the abstract setting concerning the involved operators and spaces. In the last part we indicate how the principle is applied in our given situation by checking the four assumptions that are needed to formulate our final stability result.

2 The volume-preserving MCF and its linearization

This section is devoted to the studies of the volume-preserving MCF with line tension effects on a boundary contact curve. We will first present the basic setting in which we will work. Followed by the introduction of the MCF in Subsection 2.2, this is only to set up the notation and provide elementary definitions and tools like the Transport Theorem. The last part of this section deals with the linearization of the MCF, which we will intensively study in Section 3.

2.1 Preliminaries

Here we introduce the setting which we want to consider throughout the whole thesis. We will not define each and every term that appears and instead only fix the notation that will be used in the present work. For a complete and extensive review of curvature terms and all the definitions related to evolving hypersurfaces we refer the reader to the book of Bär [Bär10] and Chapter 2 of [Dep10].

Let $\varnothing \neq \Omega \subseteq \mathbb{R}^3$ be an open, connected domain with smooth boundary $\partial\Omega$. Furthermore, let $\Gamma \subseteq \Omega$ be a connected, smooth hypersurface with boundary such that $\Gamma \cup \partial\Gamma$ is compact and $\varnothing \neq \partial\Gamma \subseteq \partial\Omega$. With $V \subseteq \Omega$ we want to denote the region between Γ and $\partial\Omega$ and D shall be defined as $D := \partial V \cap \partial\Omega$. In particular, we have $\partial D = \partial\Gamma$.

For a point $p \in \Gamma$ we denote the exterior normal to Γ in p by $n_\Gamma(p)$, where the term "exterior" should be understood with respect to V. Analogously, for the normal $n_{\partial\Omega}(p)$ for $p \in \partial\Omega$, which coincides with $n_D(p)$ if $p \in D \subseteq \partial\Omega$. Furthermore, for a point $p \in \partial\Gamma$ we want to denote by $n_{\partial\Gamma}(p)$ and $n_{\partial D}(p)$ the outer conormals to $\partial\Gamma$ and ∂D in p. In addition, we define the tangent vector to the curve $\partial\Gamma$ by $\vec{\tau}(p) := \frac{c'(t)}{|c'(t)|}$ and its curvature vector by $\vec{\varkappa}(p) := \frac{1}{|c'(t)|}\left(\frac{c'(t)}{|c'(t)|}\right)'$, where $c : (t-\varepsilon, t+\varepsilon) \longrightarrow \partial\Gamma$ is a parametrization of $\partial\Gamma$ around $p \in \partial\Gamma$ with $c(t) = p$. Moreover, we define for $p \in \partial\Gamma$ the angles $\alpha(p) := \sphericalangle(n_\Gamma(p), n_D(p))$, $\beta(p) := \sphericalangle(n_D(p), n_{\partial\Gamma}(p))$ and $\gamma(p) := \sphericalangle(n_{\partial D}(p), n_{\partial\Gamma}(p))$, for which we assume

$$0 < \alpha(p) < \pi \qquad \text{for all } p \in \partial\Gamma. \tag{2.1}$$

The whole situation is shown in Figure 2.

Remark 2.1: (i) By definition of the conormals and $\vec{\tau}$, we have two orthonormal bases of $\mathbb{R}^3$ in every point $p \in \partial\Gamma$, namely $\{\vec{\tau}(p), n_\Gamma(p), n_{\partial\Gamma}(p)\}$ and $\{\vec{\tau}(p), n_D(p), n_{\partial D}(p)\}$. W.l.o.g. we can assume the parametrization c from above to be oriented such that $(\vec{\tau}(p), n_{\partial\Gamma}(p), n_\Gamma(p))$ and $(\vec{\tau}(p), n_D(p), n_{\partial D}(p))$ form a right-handed coordinate system. Moreover, $\{\vec{\tau}(p), n_{\partial\Gamma}(p)\}$ and $\{\vec{\tau}(p), n_{\partial D}(p)\}$ are orthonormal bases of $T_p\Gamma$ and T_pD, respectively. We will use these bases frequently.

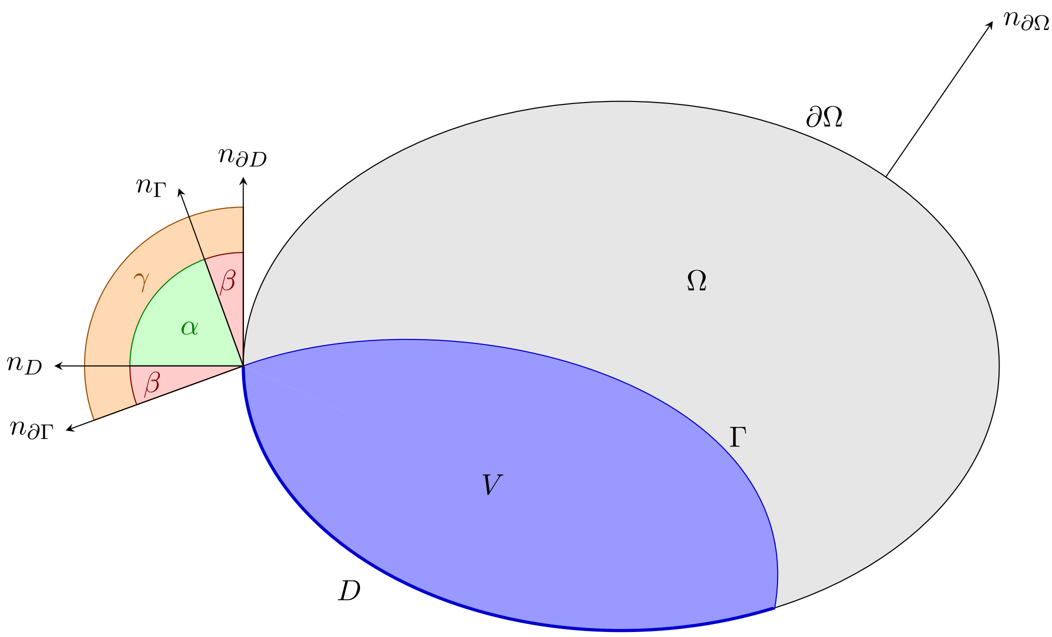

Figure 2: General situation and notation

(ii) On the one hand we obviously have $\alpha = \frac{\pi}{2} - \beta$, which shows

$$\cos(\beta) = \cos(-\beta) = \sin\left(\frac{\pi}{2} - \beta\right) = \sin(\alpha),$$
$$\sin(\beta) = -\sin(-\beta) = \cos\left(\frac{\pi}{2} - \beta\right) = \cos(\alpha)$$

and on the other hand $\frac{\pi}{2} + \beta = \gamma$, which gives

$$\cos(\gamma) = \cos\left(\frac{\pi}{2} + \beta\right) = -\sin(\beta) = -\cos(\alpha).$$

Since all vectors have unit length one gets the following angle relations

$$\begin{aligned} \langle n_\Gamma, n_D \rangle &= \cos(\alpha) \\ \langle n_D, n_{\partial\Gamma} \rangle &= \cos(\beta) = \sin(\alpha) \\ \langle n_{\partial D}, n_{\partial\Gamma} \rangle &= \cos(\gamma) = -\cos(\alpha), \end{aligned} \tag{2.2}$$

which will play an important role later on. $\square$

Before we can come to our main problem we have to present some variational results. To this end we need the so-called transport equation. We assume that we are given a fixed reference hypersurface $\Gamma^* \subseteq \Omega$ that is smooth up to the boundary $\partial\Gamma^* \subseteq \partial\Omega$ and an arbitrary function

$$\Phi : [0, T] \times \Gamma^* \longrightarrow \Omega : (t, q) \longmapsto \Phi(t, q),$$

with $\Phi(\partial\Gamma^*) \subseteq \partial\Omega$, which we have to specify later on. For fixed $t \in [0,T]$ we set

$$\Gamma(t) := \operatorname{Im}(\Phi(t,\bullet))$$

and obtain with $\Gamma := \bigcup_{t\in[0,T]} \{t\} \times \Gamma(t)$ an evolving hypersurface.

Definition 2.2: Let $\Gamma = (\Gamma(t))_{t\in I}$ be an evolving hypersurface in $\mathbb{R}^n$.
(i) For a fixed time $(t_0, p) \in \Gamma$, i.e. $p \in \Gamma(t_0)$, we define the normal velocity $V_\Gamma(t_0, p)$ of the evolving hypersurface Γ at (t_0, p) by choosing a curve

$$c : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}^n : \tau \longmapsto c(\tau)$$

with $c(\tau) \in \Gamma(\tau)$ and $c(t_0) = p$ and set

$$V_\Gamma(t_0, p) := n_\Gamma(t_0, p) \cdot \frac{d}{d\tau} c(\tau)\Big|_{\tau=t_0}.$$

(ii) For fixed $(t_0, p) \in \partial\Gamma$, i.e. $p \in \partial\Gamma(t_0)$, we define the normal boundary velocity $v_{\partial\Gamma}(t_0, p)$ at (t_0, p) by choosing a curve

$$c : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}^n : \tau \longmapsto c(\tau)$$

with $c(\tau) \in \partial\Gamma(\tau)$ and $c(t_0) = p$ and set

$$v_{\partial\Gamma}(t_0, p) := n_{\partial\Gamma}(t_0, p) \cdot \frac{d}{d\tau} c(\tau)\Big|_{\tau=t_0}.$$

(iii) For a fixed time $(t_0, p) \in \Gamma$, i.e. $p \in \Gamma(t_0)$, we define the normal time derivative $\partial_\Gamma^\circ f(t_0, p)$ of function $f : \Gamma \longrightarrow \mathbb{R}$ at (t_0, p) by choosing a curve

$$c : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}^n : \tau \longmapsto c(\tau)$$

with $c(\tau) \in \Gamma(\tau)$, $c(t_0) = p$ and

$$c'(\tau) = V_\Gamma(\tau, c(\tau)) n_\Gamma(\tau, c(\tau))$$

and define

$$\partial_\Gamma^\circ f(t_0, p) := \frac{d}{d\tau} f(\tau, c(\tau))\Big|_{\tau=t_0}. \qquad \square$$

It is clear that these definitions only make sense, when the two velocities and the normal time derivative are independent from the chosen curve c. We will not prove this in general and refer to Section 2.2 of [Dep10]. Yet, for the situation described before, where $\Gamma(t)$ is given as the graph of Φ over a reference hypersurface Γ^*, we can express the normal (boundary) velocity as given in the following lemma, which immediately shows its independence of the chosen curve.

Lemma 2.3: *(i) The normal velocity V_Γ at a point $(t,p) \in \Gamma$ with $p = \Phi(t,q)$ for some $q \in \Gamma^*$ is given by*

$$V_\Gamma(t,p) = n_\Gamma(t,p) \cdot \partial_t \Phi(t,q).$$

(ii) The normal boundary velocity $v_{\partial\Gamma}$ at a point $(t,p) \in \partial\Gamma$ with $p = \Phi(t,q)$ for some $q \in \partial\Gamma^*$ is given by

$$v_{\partial\Gamma}(t,p) = n_{\partial\Gamma}(t,p) \cdot \partial_t \Phi(t,q).$$

Proof: (i) Can be found in [Dep10] as Lemma 2.40.
(ii) First we choose a curve in the boundary of the fixed reference surface

$$\widehat{c} : (t-\varepsilon, t+\varepsilon) \longrightarrow \partial\Gamma^* : s \longmapsto \widehat{c}(s)$$

with $\widehat{c}(t) = q$. Remark that $\widehat{c}'(t)$ is linearly dependent to $\vec{\tau}^*(q)$. With this auxiliary curve we can define

$$c : (t-\varepsilon, t+\varepsilon) \longrightarrow \mathbb{R}^3 : s \longmapsto c(s) := \Phi(s, \widehat{c}(s)),$$

which is a curve as in the definition of the normal boundary velocity, since $c(s) \in \partial\Gamma(s)$ and $c(t) = \Phi(t, \widehat{c}(t)) = \Phi(t,q) = p$. Then we get

$$c'(s) = \partial_s \Phi(s, \widehat{c}(s)) + (\partial_{\widehat{c}(s)} \Phi(s, \widehat{c}(s)))(\widehat{c}'(s)),$$

which reads in $s = t$ as

$$c'(t) = \partial_t \Phi(t,q) + (\partial_q \Phi(t,q))(\widehat{c}'(t)),$$

where $(\partial_q \Phi(t,q))(\widehat{c}'(t))$ has only a contribution in $\vec{\tau}(t,p)$-direction. Multiplying this with $n_{\partial\Gamma}(t,p)$ gives

$$\begin{aligned} v_{\partial\Gamma}(t,p) = n_{\partial\Gamma}(t,p) \cdot \frac{d}{ds} c(s)\Big|_{s=t} &= n_{\partial\Gamma}(t,p) \cdot \partial_t \Phi(t,q) + \underbrace{n_{\partial\Gamma}(t,p) \cdot (\partial_q \Phi(t,q))(\widehat{c}'(t))}_{=0} \\ &= n_{\partial\Gamma}(t,p) \cdot \partial_t \Phi(t,q), \end{aligned}$$

where the second summand vanishes since $n_{\partial\Gamma}(t,p) \perp (\partial_q \Phi(t,q))(\widehat{c}'(t))$. ■

For more results concerning these velocities and the normal time derivative we refer once more to Section 2.2 of [Dep10]. Now we will introduce the basic tool for the calculations to follow, namely the Transport Theorem.

Theorem 2.4 (Transport Theorem): *For a smooth function $f : \Gamma \longrightarrow \mathbb{R}$ we obtain*

$$\frac{d}{dt} \int_{\Gamma(t)} f(t,p)\, d\mathcal{H}^2 = \int_{\Gamma(t)} \partial_\Gamma^\circ f(t,p) - f(t,p) V_\Gamma(t,p) H_\Gamma(t,p)\, d\mathcal{H}^2 + \int_{\partial\Gamma(t)} f(t,p) v_{\partial\Gamma}(t,p)\, d\mathcal{H}^1,$$

where $H_\Gamma(t,p)$ shall denote the mean curvature of Γ in (t,p).

Proof: A proof can be found in the appendix of [GW06]. ■

For later purposes we need the area and volume functional as well as the line energy and their variations.

Definition 2.5: (i) We define the area functional A of a 2-dimensional hypersurface Γ by

$$A(\Gamma) := \int_\Gamma 1\, d\mathcal{H}^2.$$

(ii) For a 2-dimensional hypersurface Γ as in Figure 2 and its enclosed domain V the volume functional Vol is defined as

$$\mathrm{Vol}(V) := \int_V 1 dx.$$

(iii) For the curve $\partial\Gamma$ the line energy is defined as

$$L(\partial\Gamma) := \int_{\partial\Gamma} 1\, d\mathcal{H}^1. \qquad \square$$

Now we specify the way how to vary Γ. We want to consider

$$\psi : \mathbb{R} \times \Gamma \longrightarrow \mathbb{R}^3 : (t,p) \longmapsto \psi(t,p) := p + t\zeta(p) \tag{2.3}$$

with a vector field

$$\zeta \in \mathcal{F}(\Gamma) := \{f \in C^\infty(\Gamma;\mathbb{R}^3) \mid f|_{\partial\Gamma} \cdot n_D = 0\}. \tag{2.4}$$

Basically ζ is the direction in which we want to vary. The condition (2.4) makes sure that on $\partial\Gamma$ there is only tangential movement to D. This guarantees that regarding Γ as the surface of a liquid droplet, we vary in a way such that the drop neither drains away nor detaches from $\partial\Omega$. Then we get a family of hypersurfaces by

$$\Gamma(t) := \mathrm{Im}(\psi(t,\bullet)).$$

In this special case one can express the normal velocity V_Γ and the normal boundary velocities $v_{\partial\Gamma}$ and $v_{\partial D}$ in terms of ζ due to Lemma 2.3. In fact one has

$$\begin{aligned} V_\Gamma(p) &= n_\Gamma(p) \cdot \zeta(p), \\ v_{\partial\Gamma}(p) &= n_{\partial\Gamma}(p) \cdot \zeta(p), \\ v_{\partial D}(p) &= n_{\partial D}(p) \cdot \zeta(p). \end{aligned} \tag{2.5}$$

Theorem 2.6: *As the first variation of area and volume we get*

$$\text{(i)} \qquad (\delta A(\Gamma))(\zeta) = \frac{d}{dt}\int_{\Gamma(t)} 1\, d\mathcal{H}^2 \bigg|_{t=0} = -\int_\Gamma H_\Gamma(p) V_\Gamma(p)\, d\mathcal{H}^2 + \int_{\partial\Gamma} v_{\partial\Gamma}(p)\, d\mathcal{H}^1$$
$$= -\int_\Gamma H_\Gamma(p)(n_\Gamma(p)\cdot\zeta(p))\, d\mathcal{H}^2 + \int_{\partial\Gamma} n_{\partial\Gamma}(p)\cdot\zeta(p)\, d\mathcal{H}^1.$$

$$\text{(ii)} \qquad (\delta\,\mathrm{Vol}(V))(\zeta) = \frac{d}{dt}\int_{V(t)} 1 dx \bigg|_{t=0} = \int_\Gamma V_\Gamma(p)\, d\mathcal{H}^2 = \int_\Gamma n_\Gamma(p)\cdot\zeta(p)\, d\mathcal{H}^2.$$

Proof: (i) Follows immediately from Theorem 2.4 by setting $f \equiv 1$ and (2.5).
(ii) First we observe

$$\begin{aligned}3\,\mathrm{Vol}(V(t)) &= \int_{V(t)} 3dx = \int_{V(t)} \mathrm{div}(\mathrm{Id}(x))dx \\ &= \underbrace{\int_{D(t)} p \cdot n_D(p)\, d\mathcal{H}^2}_{=:I_1} + \underbrace{\int_{\Gamma(t)} p \cdot n_\Gamma(t,p)\, d\mathcal{H}^2}_{=:I_2}\end{aligned}$$

by using Gauss' Theorem. A closer look at I_1, while keeping Lemma 2.38 from [Dep10] in mind, shows

$$\begin{aligned}\frac{d}{dt}I_1 &= \frac{d}{dt}\int_{D(t)} p \cdot n_D(p)\, d\mathcal{H}^2 \\ &= \int_{D(t)} \partial_D^\circ(p \cdot n_D(p)) - (p \cdot n_D(p))\underbrace{V_D(p)}_{=0} H_D(p)\, d\mathcal{H}^2 \\ &+ \int_{\partial D(t)} (p \cdot n_D(p)) v_{\partial D}(t,p)\, d\mathcal{H}^1 \\ &= \int_{D(t)} \underbrace{V_D(p)}_{=0}(n_D(p) \cdot n_D(p)) + p \cdot \underbrace{\partial_D^\circ n_D(p)}_{=0}\, d\mathcal{H}^2 + \int_{\partial\Gamma(t)} (p \cdot n_D(p)) v_{\partial D}(t,p)\, d\mathcal{H}^1 \\ &= \int_{\partial\Gamma(t)} (p \cdot n_D(p)) v_{\partial D}(t,p)\, d\mathcal{H}^1, \end{aligned} \tag{2.6}$$

where we used Theorem 2.4 and the fact that $V_D \equiv 0$ since D does not move in normal direction, which also gives $\partial_D^\circ n_D(p) = 0$.
An analogous calculation for I_2 in combination with Lemma 5.2 from [Dep10] gives

$$\begin{aligned}\frac{d}{dt}I_2 &= \frac{d}{dt}\int_{\Gamma(t)} p \cdot n_\Gamma(t,p)\, d\mathcal{H}^2 \\ &= \int_{\Gamma(t)} \partial_\Gamma^\circ(p \cdot n_\Gamma(t,p)) - (p \cdot n_\Gamma(t,p))V_\Gamma(t,p)H_\Gamma(t,p)\, d\mathcal{H}^2 \\ &+ \int_{\partial\Gamma(t)} (p \cdot n_\Gamma(t,p)) v_{\partial\Gamma}(t,p)\, d\mathcal{H}^1 \\ &= \int_{\Gamma(t)} V_\Gamma(t,p) \underbrace{(n_\Gamma(t,p) \cdot n_\Gamma(t,p))}_{=1} + p \cdot \underbrace{\partial_\Gamma^\circ n_\Gamma(t,p)}_{=-\nabla_\Gamma V_\Gamma(t,p)} - (p \cdot n_\Gamma(t,p))V_\Gamma(t,p)H_\Gamma(t,p)\, d\mathcal{H}^2 \\ &+ \int_{\partial\Gamma(t)} (p \cdot n_\Gamma(t,p)) v_{\partial\Gamma}(t,p)\, d\mathcal{H}^1 \\ &= \int_{\Gamma(t)} V_\Gamma(t,p) - p \cdot \nabla_\Gamma V_\Gamma(t,p) - (p \cdot n_\Gamma(t,p))V_\Gamma(t,p)H_\Gamma(t,p)\, d\mathcal{H}^2 \\ &+ \int_{\partial\Gamma(t)} (p \cdot n_\Gamma(t,p)) v_{\partial\Gamma}(t,p)\, d\mathcal{H}^1. \end{aligned} \tag{2.7}$$

Using Gauss' theorem on hypersurfaces (Theorem 2.29 in [Dep10]) with $f(t,p) = V_\Gamma(t,p)p$

we obtain

$$\begin{aligned}
-\int_{\Gamma(t)} p\cdot\nabla_\Gamma V_\Gamma(t,p)\,d\mathcal{H}^2 &= \int_{\Gamma(t)} \underbrace{\mathrm{div}_\Gamma(\mathrm{Id}(p))}_{=2} V_\Gamma(t,p)\,d\mathcal{H}^2 - \int_{\Gamma(t)} \mathrm{div}_\Gamma(V_\Gamma(t,p)p)\,d\mathcal{H}^2 \\
&= \int_{\Gamma(t)} 2V_\Gamma(t,p)\,d\mathcal{H}^2 + \int_{\Gamma(t)} (p\cdot n_\Gamma(t,p))H_\Gamma(t,p)V_\Gamma(t,p)\,d\mathcal{H}^2 \\
&\quad - \int_{\partial\Gamma(t)} (p\cdot n_{\partial\Gamma}(t,p))V_\Gamma(t,p)\,d\mathcal{H}^1.
\end{aligned}$$

Inserting this into equation (2.7) yields

$$\frac{d}{dt}I_2 = \int_{\Gamma(t)} 3V_\Gamma(t,p)\,d\mathcal{H}^2 + \int_{\partial\Gamma(t)} (p\cdot n_\Gamma(t,p))v_{\partial\Gamma}(t,p) - (p\cdot n_{\partial\Gamma}(t,p))V_\Gamma(t,p)\,d\mathcal{H}^1. \quad (2.8)$$

Combining equations (2.6) and (2.8) shows

$$\begin{aligned}
\frac{d}{dt}3\,\mathrm{Vol}(V(t)) &= \int\limits_{\partial\Gamma(t)} (p\cdot n_D(p))v_{\partial D}(t,p)\,d\mathcal{H}^1 + \int\limits_{\Gamma(t)} 3V_\Gamma(t,p)\,d\mathcal{H}^2 \\
&\quad + \int\limits_{\partial\Gamma(t)} (p\cdot n_\Gamma(t,p))v_{\partial\Gamma}(t,p) - (p\cdot n_{\partial\Gamma}(t,p))V_\Gamma(t,p)\,d\mathcal{H}^1 \\
&= \int\limits_{\Gamma(t)} 3V_\Gamma(t,p)\,d\mathcal{H}^2 \\
&\quad + \int\limits_{\partial\Gamma(t)} p\cdot(v_{\partial\Gamma}(t,p)n_\Gamma(t,p) - V_\Gamma(t,p)n_{\partial\Gamma}(t,p) + v_{\partial D}(t,p)n_D(p))\,d\mathcal{H}^1.
\end{aligned}$$

Finally, due to (2.5) this leads to

$$\frac{d}{dt}3\,\mathrm{Vol}(V(t))\Big|_{t=0} = 3\int_\Gamma n_\Gamma(p)\cdot\zeta(p)\,d\mathcal{H}^2 + \int_{\partial\Gamma} p\cdot W_1(p)\,d\mathcal{H}^1,$$

where $W_1(p) := (n_{\partial\Gamma}(p)\cdot\zeta(p))n_\Gamma(p) - (n_\Gamma(p)\cdot\zeta(p))n_{\partial\Gamma}(p) + (n_{\partial D}(p)\cdot\zeta(p))n_D(p)$. If we prove that $W_1\equiv 0$, we can divide the whole equation by 3 and finish the proof.
To show that $W_1\equiv 0$ we drop the argument p for convenience. With the help of (2.4) we get

$$\zeta = \langle\zeta, n_{\partial D}\rangle\, n_{\partial D} + \langle\zeta,\vec{\tau}\rangle\,\vec{\tau}.$$

Therefore, we obtain by Remark 2.1

$$\begin{aligned}
n_{\partial\Gamma}\cdot\zeta &= \langle\zeta,n_{\partial D}\rangle\langle n_{\partial\Gamma},n_{\partial D}\rangle = \langle\zeta,n_{\partial D}\rangle\cos(\gamma) = -\cos(\alpha)\langle\zeta,n_{\partial D}\rangle = -\langle\zeta,n_{\partial D}\rangle\langle n_\Gamma,n_D\rangle \\
n_\Gamma\cdot\zeta &= \langle\zeta,n_{\partial D}\rangle\langle n_\Gamma,n_{\partial D}\rangle = \cos(\beta)\langle\zeta,n_{\partial D}\rangle = \langle\zeta,n_{\partial D}\rangle\langle n_{\partial\Gamma},n_D\rangle \\
n_{\partial D}\cdot\zeta &= \langle\zeta,n_{\partial D}\rangle,
\end{aligned}$$

which gives

$$\begin{aligned}
W_1 &= (n_{\partial\Gamma}\cdot\zeta)n_\Gamma - (n_\Gamma\cdot\zeta)n_{\partial\Gamma} + (n_{\partial D}\cdot\zeta)n_D \\
&= -\langle\zeta,n_{\partial D}\rangle\langle n_\Gamma,n_D\rangle n_\Gamma - \langle\zeta,n_{\partial D}\rangle\langle n_{\partial\Gamma},n_D\rangle n_{\partial\Gamma} + \langle\zeta,n_{\partial D}\rangle n_D \\
&= \langle\zeta,n_{\partial D}\rangle(\underbrace{-\langle n_\Gamma,n_D\rangle n_\Gamma - \langle n_{\partial\Gamma},n_D\rangle n_{\partial\Gamma}}_{=-n_D} + n_D) = 0.
\end{aligned}$$

This proves the claim. ∎

Theorem 2.7: *The first variation of the line energy reads as*

$$(\delta L(\partial\Gamma))(\zeta) = \frac{d}{dt}\int_{\partial\Gamma(t)} 1\, d\mathcal{H}^1\bigg|_{t=0} = -\int_{\partial\Gamma} \vec{\varkappa}(p)\cdot\zeta(p)\, d\mathcal{H}^1.$$

Proof: For a parametrization

$$c(t,\bullet) : [0,1] \longrightarrow \partial\Gamma(t) : s \longmapsto c(t,s)$$

of $\partial\Gamma(t)$ with $c_s(t,s)\neq 0$ and $c(0,s)=p\in\partial\Gamma$ we get

$$\begin{aligned}
\frac{d}{dt}\int_{\partial\Gamma(t)} 1\, d\mathcal{H}^1\bigg|_{t=0} &= \frac{d}{dt}\int_0^1 |c_s(t,s)|ds\bigg|_{t=0} = \int_0^1 \frac{c_s(0,s)}{|c_s(0,s)|}\cdot\left(\frac{d}{dt}c_s(t,s)\bigg|_{t=0}\right)ds \\
&= \int_0^1 \frac{c_s(0,s)}{|c_s(0,s)|}\cdot(c_t(0,s))_s ds = -\int_0^1\left(\frac{c_s(0,s)}{|c_s(0,s)|}\right)_s\cdot c_t(0,s)ds \\
&= -\int_0^1\left(\frac{1}{|c_s(0,s)|}\left(\frac{c_s(0,s)}{|c_s(0,s)|}\right)_s\cdot\underbrace{c_t(0,s)}_{=\zeta(p)}\right)|c_s(0,s)|ds \\
&= -\int_{\partial\Gamma}\vec{\varkappa}(p)\cdot\zeta(p)\, d\mathcal{H}^1,
\end{aligned}$$

where the boundary terms of the partial integration vanish, because $\partial\Gamma(t)$ is a closed curve. ■

2.2 The Mean Curvature Flow

After these first basic facts we will now introduce the MCF as the most efficient way to decrease the area functional. This motivates the interest in the MCF, because it directly shows that each stationary surface will be a candidate for a local minimizer of this energy.

For this we want to consider the energy

$$\widetilde{E}(\Gamma) := \int_\Gamma 1\, d\mathcal{H}^2 - a\int_D 1\, d\mathcal{H}^2 + b\int_{\partial\Gamma} 1\, d\mathcal{H}^1 \tag{2.9}$$

for $a,b\in\mathbb{R}$ with $b\geq 0$. Under the additional constraint that the enclosed volume of V remains constant we search for conditions that a local minimizer of this energy has to satisfy.

Remark 2.8: This constraint of constant volume is physically reasonable if we look at V as a drop of liquid in contact with a nonporous membrane $\partial\Omega$. □

We want to bring this volume constraint into the energy with the help of a Lagrange multiplier. To do this we first have to show that such a Lagrange multiplier exists. To this end we vary our hypersurface Γ by

$$\psi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3 : (t,s,p) \longmapsto \psi(t,s,p) := p + t\zeta(p) + s\xi(p), \tag{2.10}$$

where we have to impose the condition

$$\zeta, \xi \in \widetilde{\mathcal{F}}(\mathbb{R}^3) := \{ f \in C^\infty(\mathbb{R}^3; \mathbb{R}^3) \mid f|_{\partial\Omega} \cdot n_{\partial\Omega} = 0 \} \tag{2.11}$$

as before to ensure that the drop neither drains away nor detaches from $\partial\Omega$. The new family of hypersurfaces is then given by $\Gamma(t,s) := \mathrm{Im}(\psi(t,s,\bullet))$ and it encloses regions $V(t,s) \subseteq \Omega$. Obviously we have $\Gamma(0,0) = \Gamma$ and $V(0,0) = V$. Moreover, we denote by

$$F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} : (t,s) \longmapsto F(t,s) := \mathrm{Vol}(V(t,s)) - \mathrm{Vol}(V) \tag{2.12}$$

the side constraint function. Here we have $F(0,0) = \mathrm{Vol}(V(0,0)) - \mathrm{Vol}(V) = 0$. We want to apply the implicit function theorem, for which we need the condition $\partial_s F(0,0) \neq 0$. To achieve this we fix an arbitrary function $\varphi \in C^\infty(\mathbb{R}^3; \mathbb{R})$ with $0 \leq \varphi(p) \leq 1$ and $\varnothing \neq \mathrm{supp}(\varphi) \Subset \Omega$ and $\mathrm{supp}(\varphi) \cap \Gamma \neq \varnothing$ and define $\xi(p) := \varphi(p) n_\Gamma(p)$. We see that the vector field $\xi \in \widetilde{\mathcal{F}}(\mathbb{R}^3)$ since $\xi|_{\partial\Omega} \equiv 0$ and especially we also have $\xi|_{\partial\Gamma} = \xi|_{\partial D} \equiv 0$. Then we obtain

$$\begin{aligned} \partial_s F(0,0) &= \int_\Gamma \xi(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2 = \int_{\mathrm{supp}(\varphi)\cap\Gamma} \varphi(p) \underbrace{(n_\Gamma(p) \cdot n_\Gamma(p))}_{=1}\, d\mathcal{H}^2 \\ &= \int_{\mathrm{supp}(\varphi)\cap\Gamma} \varphi(p)\, d\mathcal{H}^2 \neq 0. \end{aligned}$$

Therefore we know due to the implicit function theorem that there is an open interval $(-t_0, t_0)$ and a function $s(t)$ with $s(0) = 0$ and $F(t, s(t)) = 0$ for all $t \in (-t_0, t_0)$.
In particular, we can simplify (2.10) to

$$\widehat{\psi} : (-t_0, t_0) \times \partial V \longrightarrow \mathbb{R}^3 : (t,p) \longmapsto \widehat{\psi}(t,p) := p + t\zeta(p) + s(t)\xi(p)$$

and $\Gamma(t) := \widehat{\psi}(t,\Gamma) = \psi(t, s(t), \Gamma) = \Gamma(t, s(t))$. By construction all these hypersurfaces have constant volume during the variation. Moreover, by differentiating the equation $F(t, s(t)) = 0$ with respect to t we have $\partial_t F(t, s(t)) + \partial_s F(t, s(t)) s'(t) = 0$, which can be rearranged to

$$s'(t) = -\frac{\partial_t F(t, s(t))}{\partial_s F(t, s(t))},$$

where the denominator is not zero at least in a small neighborhood of $t = 0$. Utilizing the fact $s(0) = 0$, we get for $t = 0$

$$s'(0) = -\frac{\partial_t F(0, s(0))}{\partial_s F(0, s(0))} = -\frac{\partial_t F(0,0)}{\partial_s F(0,0)} = -\frac{\int\limits_\Gamma \zeta(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2}{\int\limits_\Gamma \xi(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2}.$$

Due to Lemma 2.3 the normal velocity in this case reads as $V_\Gamma(p) = (\zeta(p) + s'(0)\xi(p)) \cdot n_\Gamma(p)$

and for stationary solutions of our energy (2.9) with the volume constraint we see

$$
\begin{aligned}
0 &= \frac{d}{dt}\widetilde{E}(\Gamma(t))\Big|_{t=0} \\
&= -\int_\Gamma ((\zeta(p) + s'(0)\xi(p)) \cdot n_\Gamma(p)) H_\Gamma(p)\, d\mathcal{H}^2 + \int_{\partial\Gamma} (\zeta(p) + s'(0)\underbrace{\xi(p)}_{=0}) \cdot n_{\partial\Gamma}(p)\, d\mathcal{H}^1 \\
&\quad + a\int_D \underbrace{((\zeta(p) + s'(0)\xi(p)) \cdot n_D(p))}_{=0 \text{ because } \zeta,\xi\in\widetilde{\mathcal{F}}(\mathbb{R}^3)} H_D(p)\, d\mathcal{H}^2 - a\int_{\partial D} (\zeta(p) + s'(0)\underbrace{\xi(p)}_{=0}) \cdot n_{\partial D}(p)\, d\mathcal{H}^1 \\
&\quad - b\int_{\partial\Gamma} (\zeta(p) + s'(0)\underbrace{\xi(p)}_{=0}) \cdot \vec{\varkappa}(p)\, d\mathcal{H}^1 \\
&= -\int_\Gamma (\zeta(p) \cdot n_\Gamma(p)) H_\Gamma(p)\, d\mathcal{H}^2 + \int_{\partial\Gamma} \zeta(p) \cdot n_{\partial\Gamma}(p)\, d\mathcal{H}^1 - a\int_{\partial D} \zeta(p) \cdot n_{\partial D}(p)\, d\mathcal{H}^1 \\
&\quad - b\int_{\partial\Gamma} \zeta(p) \cdot \vec{\varkappa}(p)\, d\mathcal{H}^1 + \frac{\int\limits_\Gamma \zeta(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2}{\int\limits_\Gamma \xi(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2} \int_\Gamma (\xi(p) \cdot n_\Gamma(p)) H_\Gamma(p)\, d\mathcal{H}^2 \\
&= (\delta\widetilde{E}(\Gamma))(\zeta) + \underbrace{\frac{\int\limits_\Gamma (\xi(p) \cdot n_\Gamma(p)) H_\Gamma(p)\, d\mathcal{H}^2}{\int\limits_\Gamma \xi(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2}}_{=:-\lambda} \int_\Gamma \zeta(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2 \\
&= (\delta\widetilde{E}(\Gamma))(\zeta) - \lambda(\delta\,\mathrm{Vol}(V))(\zeta),
\end{aligned}
$$

where the Lagrange multiplier

$$
\lambda = -\frac{\int\limits_\Gamma (\xi(p) \cdot n_\Gamma(p)) H_\Gamma(p)\, d\mathcal{H}^2}{\int\limits_\Gamma \xi(p) \cdot n_\Gamma(p)\, d\mathcal{H}^2} = -\frac{\int\limits_\Gamma \varphi(p) H_\Gamma(p)\, d\mathcal{H}^2}{\int\limits_\Gamma \varphi(p)\, d\mathcal{H}^2} \tag{2.13}
$$

is constant with respect to the variation in ζ and hence the desired Lagrange multiplier, because

$$
(\delta\widetilde{E}(\Gamma))(\zeta) = \lambda(\delta\,\mathrm{Vol}(V))(\zeta).
$$

After showing the existence of an Lagrange multiplier λ we can combine the energy functional (2.9) and the volume constraint to obtain a new energy functional

$$
E(\Gamma) := \int_\Gamma 1\, d\mathcal{H}^2 - a\int_D 1\, d\mathcal{H}^2 + b\int_{\partial\Gamma} 1\, d\mathcal{H}^1 + \lambda\left(\int_V 1 dx - V_0\right) \tag{2.14}
$$

where $a, b, V_0 \in \mathbb{R}$ are given constants with $b \geq 0$ and $V_0 > 0$. We will vary our hypersurface by (2.3) with a vector field $\zeta \in \mathcal{F}(\Gamma)$ as defined in (2.4). For a more convenient notation we drop the argument "p" again and then the first variation of the energy functional (2.14) reads due to Theorem 2.6 and Theorem 2.7 as

$$
\begin{aligned}
(\delta E(\Gamma))(\zeta) = &-\int_\Gamma H_\Gamma(n_\Gamma \cdot \zeta)\, d\mathcal{H}^2 + \int_{\partial\Gamma} n_{\partial\Gamma} \cdot \zeta\, d\mathcal{H}^1 + a\int_D H_D \underbrace{(n_D \cdot \zeta)}_{=0}\, d\mathcal{H}^2 \\
&- a\int_{\partial D} n_{\partial D} \cdot \zeta\, d\mathcal{H}^1 - b\int_{\partial\Gamma} \vec{\varkappa} \cdot \zeta\, d\mathcal{H}^1 + \lambda\int_\Gamma n_\Gamma \cdot \zeta\, d\mathcal{H}^2 \\
= &\int_\Gamma (\lambda - H_\Gamma)(n_\Gamma \cdot \zeta)\, d\mathcal{H}^2 + \int_{\partial\Gamma} (n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) \cdot \zeta\, d\mathcal{H}^1
\end{aligned} \tag{2.15}
$$

for all $\zeta \in \mathcal{F}(\Gamma)$.

We want to find necessary conditions for a stationary solution Γ_{stat}, i.e. $(\delta E(\Gamma_{\text{stat}}))(\zeta) = 0$ for all $\zeta \in \mathcal{F}(\Gamma)$. For that we first choose $\zeta_1 \in \mathcal{F}(\Gamma)$ such that $\zeta_1|_{\partial\Gamma} \equiv 0$ and obtain

$$0 = (\delta E(\Gamma))(\zeta_1) = \int_\Gamma (\lambda - H_\Gamma)(n_\Gamma \cdot \zeta_1)\, d\mathcal{H}^2.$$

The fundamental lemma of calculus of variations shows $(\lambda - H_\Gamma)n_\Gamma = 0$ and hence

$$H_\Gamma = \lambda = const. \qquad \text{on } \Gamma. \tag{2.16}$$

Now we know that every stationary solution satisfies (2.16) and therefore (2.15) simplifies to

$$(\delta E(\Gamma_{\text{stat}}))(\zeta) = \int_{\partial\Gamma} (n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) \cdot \zeta\, d\mathcal{H}^1.$$

Unfortunately, we are not completely free in the choice of ζ, since we have to ensure $\zeta \in \mathcal{F}(\Gamma)$. But we can perform a trick in order to choose an arbitrary ζ again. The condition $\zeta|_{\partial\Gamma} \cdot n_D = 0$ means that $\zeta|_{\partial\Gamma}$ has no contribution in n_D-direction. This can be achieved for every $\zeta \in C^\infty(\Gamma; \mathbb{R}^3)$ via the orthogonal projection

$$\widehat{P} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : v \longmapsto \widehat{P}(v) := v - \langle v, n_D \rangle\, n_D. \tag{2.17}$$

Since every orthogonal projection is symmetric one can write

$$\left\langle n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}, \widehat{P}(\zeta) \right\rangle = \left\langle \widehat{P}(n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}), \zeta \right\rangle$$

and hence the second necessary condition can be written as

$$0 = \int\limits_{\partial\Gamma} (n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) \cdot \widehat{P}\left(\widetilde{\zeta}\right) d\mathcal{H}^1 = \int\limits_{\partial\Gamma} \widehat{P}(n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) \cdot \widetilde{\zeta}\, d\mathcal{H}^1$$

for all $\widetilde{\zeta} \in C^\infty(\Gamma; \mathbb{R}^3)$. Again via the fundamental lemma of calculus of variations we get

$$\widehat{P}(n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) = 0 \qquad \text{on } \partial\Gamma. \tag{2.18}$$

Now we want to rewrite this condition. Obviously, $\widehat{P}$ is linear and $\widehat{P}(n_{\partial D}) = n_{\partial D}$, because $\langle n_{\partial D}, n_D \rangle = 0$. Remember that we defined $\vec{\tau}(p) = \frac{c'(t)}{|c'(t)|}$ and $\vec{\varkappa}(p) = \frac{1}{|c'(t)|}\left(\frac{c'(t)}{|c'(t)|}\right)'$, where $c : (t-\varepsilon, t+\varepsilon) \longrightarrow \partial\Gamma$ is a parametrization around $p = c(t) \in \partial\Gamma$. Therefore we obtain

$$\begin{aligned}
\langle \vec{\tau}, \vec{\varkappa} \rangle &= \left\langle \frac{c'(t)}{|c'(t)|}, \frac{1}{|c'(t)|}\left(\frac{c'(t)}{|c'(t)|}\right)' \right\rangle = \frac{1}{|c'(t)|}\left\langle \frac{c'(t)}{|c'(t)|}, \left(\frac{c'(t)}{|c'(t)|}\right)' \right\rangle \\
&= \frac{1}{2|c'(t)|}\frac{d}{dt}\left\langle \frac{c'(t)}{|c'(t)|}, \frac{c'(t)}{|c'(t)|} \right\rangle = \frac{1}{2|c'(t)|}\frac{d}{dt}\langle \vec{\tau}, \vec{\tau} \rangle = \frac{1}{2|c'(t)|}\frac{d}{dt} 1 = 0,
\end{aligned}$$

which shows that $\vec{\varkappa} = \langle \vec{\varkappa}, n_D \rangle\, n_D + \langle \vec{\varkappa}, n_{\partial D} \rangle\, n_{\partial D}$ and hence $\widehat{P}(\vec{\varkappa}) = \langle \vec{\varkappa}, n_{\partial D} \rangle\, n_{\partial D}$. Combining these facts we can rewrite (2.18) as follows

$$0 = \widehat{P}(n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) = \widehat{P}(n_{\partial\Gamma}) - a n_{\partial D} - b\widehat{P}(\vec{\varkappa}) = \widehat{P}(n_{\partial\Gamma}) - (a + b\langle \vec{\varkappa}, n_{\partial D} \rangle) n_{\partial D}$$

and hence

$$\widehat{P}(n_{\partial\Gamma}) = (a + b\,\langle \vec{\varkappa}, n_{\partial D}\rangle)n_{\partial D}. \tag{2.19}$$

Multiplying (2.19) by $n_{\partial D}$, we obtain $\left\langle \widehat{P}(n_{\partial\Gamma}), n_{\partial D}\right\rangle = a + b\varkappa_{\partial D}$, where $\varkappa_{\partial D} := \langle \vec{\varkappa}, n_{\partial D}\rangle$ is the geodesic curvature of $\partial\Gamma$ in the hypersurface $\partial\Omega$. But we can still transform this equation some more by observing that

$$\begin{aligned} -\langle n_\Gamma, n_D\rangle &= -\cos(\alpha) = \cos(\gamma) = \langle n_{\partial\Gamma}, n_{\partial D}\rangle = \left\langle \widehat{P}(n_{\partial\Gamma}) + \langle n_{\partial\Gamma}, n_D\rangle\, n_D, n_{\partial D}\right\rangle \\ &= \left\langle \widehat{P}(n_{\partial\Gamma}), n_{\partial D}\right\rangle + \langle n_{\partial\Gamma}, n_D\rangle \underbrace{\langle n_D, n_{\partial D}\rangle}_{=0} = \left\langle \widehat{P}(n_{\partial\Gamma}), n_{\partial D}\right\rangle. \end{aligned}$$

Finally we derived the two necessary conditions for a stationary solution:

$$H_\Gamma = \lambda = const. \qquad \text{on } \Gamma \tag{2.20}$$

$$0 = a + b\varkappa_{\partial D} + \langle n_\Gamma, n_D\rangle \qquad \text{on } \partial\Gamma. \tag{2.21}$$

In the following we want to linearize these two equations. The condition (2.1), which we imposed at the beginning, is sufficient for the existence of a "curvilinear coordinate system" as invented by Vogel [Vog00]. We introduce this coordinate system now because with its help we can write an evolving hypersurface as a graph over a fixed reference surface Γ^*.

For $q \in \partial\Gamma^*$ and $w \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small there is a smooth function

$$\widetilde{t} : \partial\Gamma^* \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathbb{R} : (q, w) \longmapsto \widetilde{t}(q, w)$$

such that

$$q + wn_{\Gamma^*}(q) + \widetilde{t}(q, w)n_{\partial\Gamma^*}(q) \in \partial\Omega \qquad \forall\, w \in (-\varepsilon_0, \varepsilon_0).$$

Obviously, $\widetilde{t}(q, 0) = 0$ since $q + 0n_{\Gamma^*}(q)$ already lies in $\partial\Gamma^* \subseteq \partial\Omega$ without adding some multiple of $n_{\partial\Gamma^*}$. We can extend $\widetilde{t}$ smoothly to a function

$$t : \Gamma^* \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathbb{R} : (q, w) \longmapsto t(q, w)$$

such that $t(q, 0) = 0$ for all $q \in \Gamma^*$. Next we will use a special coordinate system

$$\Psi : \Gamma^* \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \Omega : (q, w) \longmapsto \Psi(q, w) := q + wn_{\Gamma^*}(q) + t(q, w)T(q), \tag{2.22}$$

where $T : \Gamma^* \longrightarrow \mathbb{R}^3$ is an arbitrary tangential vector field, that coincides with $n_{\partial\Gamma^*}$ on $\partial\Gamma^*$ and vanishes outside a small neighborhood of $\partial\Gamma^*$. By construction this curvilinear coordinate system satisfies $\Psi(q, 0) = q$ for all $q \in \Gamma^*$ and $\Psi(q, w) \in \partial\Omega$ for all $q \in \partial\Gamma^*$ and all $w \in (-\varepsilon_0, \varepsilon_0)$. Moreover, we can choose $\varepsilon_0 > 0$ small enough so that Ψ is a diffeomorphism onto its image. All technical details can be found in [Vog00].

Let $D^* \subseteq \partial\Omega$ be the analogous region for Γ^* as D is for Γ. Moreover, let $q \in \partial\Gamma^* = \partial D^*$ be fixed, $U \subseteq \mathbb{R}^3$ be an open neighborhood of q and assume that $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a smooth function describing $U \cap \partial\Omega$ as zero-level-set, i.e.

$$U \cap \partial\Omega = \{p \in \mathbb{R}^3 \mid F(p) = 0\}.$$

Then $\nabla F \perp T_q\partial\Omega$ and w.l.o.g. we assume $\frac{\nabla F}{\|\nabla F\|} = n_{D^*}$ on D^* - otherwise replace F by $-F$. By the choice of Ψ we obtain for all $q \in \partial\Gamma^*$

$$0 = F(\Psi(q,w)) = F(q + wn_{\Gamma^*}(q) + t(q,w)n_{\partial\Gamma^*}(q)) \qquad \forall\ w \in (-\varepsilon_0, \varepsilon_0).$$

Differentiating this equation with respect to w and setting $w = 0$ gives

$$\begin{aligned} 0 &= \nabla F(\Psi(q,0)) \cdot \partial_w\Psi(q,0) = \nabla F(q) \cdot (n_{\Gamma^*}(q) + t_w(q,0)n_{\partial\Gamma^*}(q)) \\ &= \langle \|\nabla F\|\, n_{D^*}(q), n_{\Gamma^*}(q)\rangle + t_w(q,0)\, \langle \|\nabla F\|\, n_{D^*}(q), n_{\partial\Gamma^*}(q)\rangle\,. \end{aligned}$$

Keeping the assumption (2.1) in mind one can rewrite this identity with the help of (2.2) to get

$$t_w(q,0) = -\frac{\|\nabla F\|\, \langle n_{D^*}(q), n_{\Gamma^*}(q)\rangle}{\|\nabla F\|\, \langle n_{D^*}(q), n_{\partial\Gamma^*}(q)\rangle} = -\frac{\cos(\alpha^*(q))}{\cos(\beta^*(q))} = -\frac{\cos(\alpha^*(q))}{\sin(\alpha^*(q))} = -\cot(\alpha^*(q)).$$

Hence we can write the vector $\partial_w\Psi(q,0)$ as

$$\partial_w\Psi(q,0) = n_{\Gamma^*}(q) - \cot(\alpha^*(q))n_{\partial\Gamma^*}(q) \qquad \forall\ q \in \partial\Gamma^*. \tag{2.23}$$

Utilizing (2.2) this leads to the following identities on the boundary $\partial\Gamma^*$, where we skip the argument q:

$$\begin{aligned} \langle \partial_w\Psi(0), \vec{\tau}^*\rangle &= \langle n_{\Gamma^*}, \vec{\tau}^*\rangle - \cot(\alpha^*)\, \langle n_{\partial\Gamma^*}, \vec{\tau}^*\rangle = 0 - \cot(\alpha^*)0 = 0, \\ \langle \partial_w\Psi(0), n_{D^*}\rangle &= \langle n_{\Gamma^*}, n_{D^*}\rangle - \cot(\alpha^*)\, \langle n_{\partial\Gamma^*}, n_{D^*}\rangle = \cos(\alpha^*) - \frac{\cos(\alpha^*)}{\sin(\alpha^*)}\sin(\alpha^*) = 0, \\ \langle \partial_w\Psi(0), n_{\partial D^*}\rangle &= \langle n_{\Gamma^*}, n_{\partial D^*}\rangle - \cot(\alpha^*)\, \langle n_{\partial\Gamma^*}, n_{\partial D^*}\rangle = \sin(\alpha^*) - \frac{\cos(\alpha^*)}{\sin(\alpha^*)}(-\cos(\alpha^*)) \\ &= \frac{\sin(\alpha^*)^2 + \cos(\alpha^*)^2}{\sin(\alpha^*)} = \frac{1}{\sin(\alpha^*)}. \end{aligned}$$

This shows that on the boundary $\partial\Gamma^*$ the vector $\partial_w\Psi(0)$ has the following coordinates with respect to the two orthonormal bases introduced in Remark 2.1:

$$\begin{aligned} \langle \partial_w\Psi(0), \vec{\tau}^*\rangle &= 0 & \langle \partial_w\Psi(0), \vec{\tau}^*\rangle &= 0 \\ \langle \partial_w\Psi(0), n_{\partial\Gamma^*}\rangle &= -\cot(\alpha^*) & \langle \partial_w\Psi(0), n_{D^*}\rangle &= 0 \\ \langle \partial_w\Psi(0), n_{\Gamma^*}\rangle &= 1 & \langle \partial_w\Psi(0), n_{\partial D^*}\rangle &= \tfrac{1}{\sin(\alpha^*)}. \end{aligned} \tag{2.24}$$

We note that the relation $\langle \partial_w\Psi(0), n_{\Gamma^*}\rangle = 1$ does also hold in Γ^* since

$$\langle \partial_w\Psi(0), n_{\Gamma^*}\rangle = \langle n_{\Gamma^*} + t_w(w)T, n_{\Gamma^*}\rangle = \underbrace{\langle n_{\Gamma^*}, n_{\Gamma^*}\rangle}_{=1} + t_w(w)\underbrace{\langle T, n_{\Gamma^*}\rangle}_{=0} = 1, \tag{2.25}$$

which is a fact used in Lemma 2.10 below.

As said before with the help of the curvilinear coordinate system Ψ we can write the evolving hypersurface as a family of graphs over the fixed hypersurface Γ^*. To this purpose we recall that we assumed our reference hypersurface Γ^* to be smooth up to the boundary $\partial\Omega$ and for $T \in (0, \infty)$ we choose a function

$$\varrho : [0, T] \times \Gamma^* \longrightarrow (-\varepsilon_0, \varepsilon_0) : (t, q) \longmapsto \varrho(t, q), \tag{2.26}$$

where we assume that ϱ is smooth enough such that all later terms are well-defined. We set $\Gamma_\varrho(t) := \mathrm{Im}(\Psi(\bullet, \varrho(t, \bullet)))$, which is why we will refer to ϱ as a "distance function" from Γ^* to $\Gamma_\varrho(t)$. This name is motivated by the fact that $\varrho(t, q)$ measures how far $q \in \Gamma^*$ has moved along the w-coordinate line at time t, which is also illustrated in Figure 3.

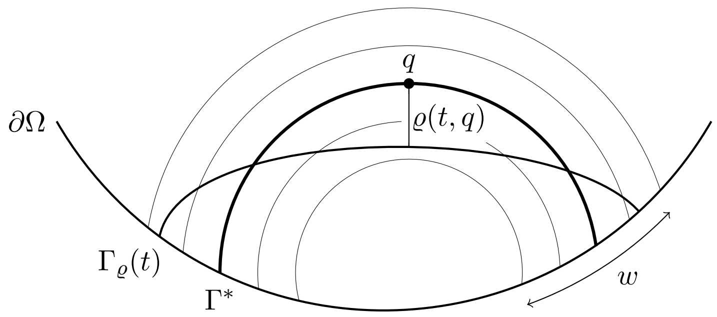

Figure 3: The distance function ϱ

Observe that by our construction of Ψ we have $\Gamma_0(t) = \Gamma^*$ for all $t \in [0, \infty)$. Define

$$\Phi : [0, \infty) \times \Gamma^* \longrightarrow \Omega : (t, q) \longmapsto \Phi(t, q) := \Psi(q, \varrho(t, q)) \tag{2.27}$$

and for fixed t we set $\Phi_t^\varrho(q) := \Phi(t, q)$, then $|\varrho(t, q)| < \varepsilon_0$ ensures that Φ_t^ϱ is a local diffeomorphism onto its image.

So far we set up all the necessary notation and relations to establish the flow that we want to consider. Inspired by the L_2-gradient flow of the energy (2.14) we demand the equation

$$V_\Gamma = \partial_t \Phi \cdot n_\Gamma = (-\nabla_{L_2} E) \cdot n_\Gamma = (H_\Gamma - \lambda)(n_\Gamma \cdot n_\Gamma) = H_\Gamma - \lambda \qquad \text{in } \Gamma,$$

where we used Lemma 2.3 to show the first equality. Additionally, there are several reasonable choices of boundary conditions. We will impose the boundary condition

$$v_{\partial D} = a + b\varkappa_{\partial D} + \langle n_\Gamma, n_D \rangle \qquad \text{on } \partial\Gamma \tag{2.28}$$

for all times $t \in [0, \infty)$. Hence we consider the flow

$$V_\Gamma(\Psi(q, \varrho(t, q))) = H_\Gamma(\Psi(q, \varrho(t, q))) - \overline{H}(\varrho(t)) \qquad \text{in } \Gamma^* \tag{2.29}$$

$$\begin{aligned} v_{\partial D}(\Psi(q, \varrho(t, q))) = a &+ b\varkappa_{\partial D}(\Psi(q, \varrho(t, q))) \\ &+ \langle n_\Gamma(\Psi(q, \varrho(t, q))), n_D(\Psi(q, \varrho(t, q))) \rangle \qquad \text{on } \partial\Gamma^*, \end{aligned} \tag{2.30}$$

where $\overline{H}(\varrho(t))$ is the mean value of the mean curvature, defined as

$$\overline{H}(\varrho(t)) := \fint_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}(t,p)\, d\mathcal{H}^2 := \frac{1}{\int\limits_{\Gamma_\varrho(t)} 1\, d\mathcal{H}^2} \int_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}(t,p)\, d\mathcal{H}^2.$$

One immediately sees that for $V_\Gamma = v_{\partial D} \equiv 0$ a solution of (2.29)-(2.30) is a hypersurface that satisfies the necessary conditions for a stationary solution (2.20)-(2.21).

Remark 2.9: (i) $\overline{H}(\varrho(t))$ is exactly the right choice for λ (cf. (2.13)) since

$$\begin{aligned}\frac{d}{dt}\operatorname{Vol}(\Gamma_\varrho(t)) &= \int_{\Gamma_\varrho(t)} V_{\Gamma_\varrho(t)}\, d\mathcal{H}^2 = \int_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)} - \overline{H}\, d\mathcal{H}^2 \\ &= \int_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}\, d\mathcal{H}^2 - \overline{H}\int_{\Gamma_\varrho(t)} 1\, d\mathcal{H}^2 \\ &= \int_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}\, d\mathcal{H}^2 - \int_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}\, d\mathcal{H}^2 = 0,\end{aligned}$$

which shows that (2.29)-(2.30) is volume preserving.
(ii) Observe that this term causes the flow (and also the resulting PDE later) to be non-local, since $\overline{H}$ contains information from the entire surface $\Gamma_\varrho(t)$. □

2.3 Linearization of the Mean Curvature Flow

In this subsection we want to linearize the volume-preserving MCF as given by the equations (2.29)-(2.30) around $\varrho \equiv 0$, which corresponds to a linearization around Γ^*. This result will be distributed over several lemmas.

Lemma 2.10: *For all $q \in \Gamma^*$ and all $t \in [0,\infty)$ we have*

$$\frac{d}{d\varepsilon} V_\Gamma(\Psi(q, \varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0} = \partial_t \varrho(t,q).$$

Proof: First we observe by Lemma 2.3

$$\begin{aligned}V_\Gamma(\Psi(q,\varrho(t,q))) &= n_\Gamma(\Psi(q,\varrho(t,q))) \cdot \frac{d}{dt}\Psi(q,\varrho(t,q)) \\ &= (n_\Gamma(\Psi(q,\varrho(t,q))) \cdot \partial_w \Psi(q,\varrho(t,q)))\partial_t\varrho(t,q) \end{aligned} \tag{2.31}$$

and with the help of the product rule this gives

$$\begin{aligned}\frac{d}{d\varepsilon} V_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}(n_\Gamma(\Psi(q,\varepsilon\varrho(t,q))) \cdot \partial_w\Psi(q,\varepsilon\varrho(t,q)))\partial_t\varepsilon\varrho(t,q)\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}(n_\Gamma(\Psi(q,\varepsilon\varrho(t,q))) \cdot \partial_w\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0} \underbrace{\partial_t\varepsilon\varrho(t,q)|_{\varepsilon=0}}_{=0} \\ &\quad + (n_\Gamma(\Psi(q,\varepsilon\varrho(t,q))) \cdot \partial_w\Psi(q,\varepsilon\varrho(t,q)))|_{\varepsilon=0}\,\partial_t\varrho(t,q) \\ &= (n_{\Gamma^*}(q) \cdot \partial_w\Psi(q,0))\partial_t\varrho(t,q) = \partial_t\varrho(t,q),\end{aligned}$$

where we used (2.25) in the last line. ■

Lemma 2.11: *For all $q \in \Gamma^*$ and all $t \in [0, \infty)$ we have*

$$\frac{d}{d\varepsilon} H_\Gamma(\Psi(q, \varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} = \Delta_{\Gamma^*}\varrho(t,q) + |\sigma^*|^2(q)\varrho(t,q) + \left(\nabla_{\Gamma^*} H_{\Gamma^*}(q) \cdot P\left(\partial_w \Psi(q,0)\right)\right)\varrho(t,q),$$

where Δ_{Γ^*} denotes the Laplace-Beltrami operator and ∇_{Γ^*} the surface gradient of Γ^*, $|\sigma^*|^2$ is defined as $|\sigma^*|^2 := (\varkappa_1^*)^2 + (\varkappa_2^*)^2$ with the principal curvatures $\varkappa_1^*, \varkappa_2^*$ of Γ^* and P denotes the projection onto the tangent space of Γ^* given by

$$P : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : v \longmapsto P(v) := v - \langle v, n_{\Gamma^*}\rangle\, n_{\Gamma^*}.$$

Proof: Exactly the same as in Lemma 3.5 of [Dep10]. The difference to the proof there is that we do not use $H_{\Gamma^*} \equiv const.$ in the last step, because we did not assume Γ^* to be a stationary solution of the energy (2.14). This generalizes the result and leaves us with the additional term $(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P(\partial_w \Psi(0)))\,\varrho(t)$. ■

Remark 2.12: For a boundary point $q \in \partial\Gamma^*$ we express the term $P(\partial_w \Psi(q,0))$ using (2.24) as follows

$$\begin{aligned} P(\partial_w\Psi(q,0)) &= \langle \partial_w\Psi(q,0), \vec{\tau}^*(q)\rangle\, \vec{\tau}^*(q) + \langle \partial_w\Psi(q,0), n_{\partial\Gamma^*}(q)\rangle\, n_{\partial\Gamma^*}(q) \\ &= -\cot(\alpha(q)) n_{\partial\Gamma^*}(q). \end{aligned}$$ □

Lemma 2.13: *For the linearization of the mean value of the mean curvature we get*

$$\begin{aligned} \frac{d}{d\varepsilon}\overline{H}(\varepsilon\varrho(t))\Big|_{\varepsilon=0} &= \fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2(q) - H_{\Gamma^*}(q)^2 + \overline{H}(\mathbb{O})H_{\Gamma^*}(q))\varrho(t,q)\, d\mathcal{H}^2 \\ &\quad - \frac{1}{\int_{\Gamma^*} 1\, d\mathcal{H}^2} \int_{\partial\Gamma^*} \Big(H_{\Gamma^*}(q) - \overline{H}(\mathbb{O})\Big) \cot(\alpha(q))\varrho(t,q)\, d\mathcal{H}^1, \end{aligned}$$

where $\mathbb{O}$ denotes the function $\varrho \equiv 0$.

Proof: Here we rename the surfaces $\Gamma_\varrho(t)$ in a way that makes the following calculations easier. We fix a time t and for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we set

$$\widetilde{\Gamma}(\varepsilon) := \mathrm{Im}(\Psi(\bullet, \varepsilon\varrho(t, \bullet))).$$

Obviously, these new hypersurfaces $\widetilde{\Gamma}$ are just renamed versions of the previous $\Gamma_\varrho(t)$, because $\widetilde{\Gamma}(\varepsilon) = \Gamma_{\varepsilon\varrho}(t)$, but now ε can be considered to be the time parameter of the evolution. We will write $\widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}$, $\widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}$, etc. for the terms of the evolving hypersurface $\left(\widetilde{\Gamma}(\varepsilon)\right)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$. In particular, an expression related to $\widetilde{\Gamma}$ evaluated for $\varepsilon = 0$ will result in the respective expression on Γ^*, because $\widetilde{\Gamma}(0) = \Gamma_0(t) = \Gamma^*$.
With this new notation (cf. Lemma 3.5 from [Dep10]) we write

$$\overline{H}(\varepsilon\varrho(t)) = \left(\int_{\widetilde{\Gamma}(\varepsilon)} 1\, d\mathcal{H}^2\right)^{-1} \left(\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon, p)\, d\mathcal{H}^2\right).$$

Then we get

$$
\begin{aligned}
\frac{d}{d\varepsilon}\overline{H}(\varepsilon\varrho(t))\bigg|_{\varepsilon=0} &= \left(\int_{\Gamma^*} 1\, d\mathcal{H}^2\right)^{-1} \left(\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2\bigg|_{\varepsilon=0}\right) \\
&\quad - \left(\int_{\Gamma^*} 1\, d\mathcal{H}^2\right)^{-2} \left(\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} 1\, d\mathcal{H}^2\bigg|_{\varepsilon=0}\right)\left(\int_{\Gamma^*} H_{\Gamma^*}(q)\, d\mathcal{H}^2\right) \\
&= \left(\int_{\Gamma^*} 1\, d\mathcal{H}^2\right)^{-1} \left(\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2\bigg|_{\varepsilon=0}\right) \\
&\quad - \left(\int_{\Gamma^*} 1\, d\mathcal{H}^2\right)^{-1} \left(\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} 1\, d\mathcal{H}^2\bigg|_{\varepsilon=0}\right)\underbrace{\left(⨍_{\Gamma^*} H_{\Gamma^*}(q)\, d\mathcal{H}^2\right)}_{=\overline{H}(\mathbb{O})}. \qquad (2.32)
\end{aligned}
$$

With the help of Theorem 2.4 the derivative of the area integral can be written as

$$
\begin{aligned}
\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} 1\, d\mathcal{H}^2\bigg|_{\varepsilon=0} &= -\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2 + \int_{\partial\widetilde{\Gamma}(\varepsilon)} \widetilde{v}_{\partial\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^1\bigg|_{\varepsilon=0} \\
&= -\int_{\widetilde{\Gamma}(0)} \widetilde{V}_{\widetilde{\Gamma}(0)}(0,q) H_{\Gamma^*}(q)\, d\mathcal{H}^2 + \int_{\partial\widetilde{\Gamma}(0)} \widetilde{v}_{\partial\widetilde{\Gamma}(0)}(0,q)\, d\mathcal{H}^1 \qquad (2.33)
\end{aligned}
$$

and we observe

$$
\begin{aligned}
\widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,\Psi(q,\varepsilon\varrho(t,q))) &= \widetilde{n}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,\Psi(q,\varepsilon\varrho(t,q)))\cdot\frac{d}{d\varepsilon}\Psi(q,\varepsilon\varrho(t,q)) \\
&= \left(\widetilde{n}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,\Psi(q,\varepsilon\varrho(t,q)))\cdot\partial_w\Psi(q,\varepsilon\varrho(t,q))\right)\varrho(t,q).
\end{aligned}
$$

Evaluated in $\varepsilon = 0$ we get using (2.24)

$$
\begin{aligned}
\widetilde{V}_{\widetilde{\Gamma}(0)}(0,q) = \widetilde{V}_{\widetilde{\Gamma}(0)}(0,\Psi(q,0)) &= (\widetilde{n}_{\widetilde{\Gamma}(0)}(0,\Psi(q,0))\cdot\partial_w\Psi(q,0))\varrho(t,q) \\
&= (n_{\Gamma^*}(q)\cdot\partial_w\Psi(q,0))\varrho(t,q) = \varrho(t,q). \qquad (2.34)
\end{aligned}
$$

Exactly the same calculations with $\widetilde{n}_{\partial\widetilde{\Gamma}}$ instead of $\widetilde{n}_{\widetilde{\Gamma}}$ show $\widetilde{v}_{\partial\widetilde{\Gamma}(0)}(0,q) = -\cot(\alpha(q))\varrho(t,q)$ for all $q \in \partial\Gamma^*$. Hence (2.33) can be rewritten as

$$
\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} 1\, d\mathcal{H}^2\bigg|_{\varepsilon=0} = -\int_{\Gamma^*} H_{\Gamma^*}(q)\varrho(t,q)\, d\mathcal{H}^2 - \int_{\partial\Gamma^*} \cot(\alpha(q))\varrho(t,q)\, d\mathcal{H}^1.
$$

With the help of Lemma 5.1 from [Dep10] the derivative of the curvature integral reads as

$$
\begin{aligned}
\frac{d}{d\varepsilon}\int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2\bigg|_{\varepsilon=0} &= \int_{\widetilde{\Gamma}(\varepsilon)} \partial^\circ \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p) - \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2 \\
&\quad + \int_{\partial\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\widetilde{v}_{\partial\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^1\bigg|_{\varepsilon=0} \\
&= \int_{\widetilde{\Gamma}(\varepsilon)} \Delta_{\widetilde{\Gamma}(\varepsilon)}\widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p) + |\widetilde{\sigma}|^2(\varepsilon,p)\widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2 \\
&\quad - \int_{\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)^2\widetilde{V}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\, d\mathcal{H}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\partial\widetilde{\Gamma}(\varepsilon)} \widetilde{H}_{\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\widetilde{v}_{\partial\widetilde{\Gamma}(\varepsilon)}(\varepsilon,p)\,d\mathcal{H}^1\Big|_{\varepsilon=0} \\
&= \int_{\Gamma^*} \Delta_{\Gamma^*}\varrho(t,q) + |\sigma^*|^2(q)\varrho(t,q) - H_{\Gamma^*}(q)^2\varrho(t,q)\,d\mathcal{H}^2 \\
&- \int_{\partial\Gamma^*} H_{\Gamma^*}(q)\cot(\alpha(q))\varrho(t,q)\,d\mathcal{H}^1.
\end{aligned}
$$

Using these variations we can continue with equation (2.32) as follows

$$
\begin{aligned}
\frac{d}{d\varepsilon}\overline{H}(\varepsilon\varrho(t))\Big|_{\varepsilon=0} &= \overline{H}(\mathbb{O})\left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\int_{\Gamma^*} H_{\Gamma^*}(q)\varrho(t,q)\,d\mathcal{H}^2 \\
&+ \overline{H}(\mathbb{O})\left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\int_{\partial\Gamma^*} \cot(\alpha(q))\varrho(t,q)\,d\mathcal{H}^1 \\
&+ \left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\int_{\Gamma^*} \Delta_{\Gamma^*}\varrho(t,q) + |\sigma^*|^2(q)\varrho(t,q) - H_{\Gamma^*}(q)^2\varrho(t,q)\,d\mathcal{H}^2 \\
&- \left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\int_{\partial\Gamma^*} H_{\Gamma^*}(q)\cot(\alpha(q))\varrho(t,q)\,d\mathcal{H}^1 \\
&= ⨍_{\Gamma^*} \overline{H}(\mathbb{O})H_{\Gamma^*}(q)\varrho(t,q)\,d\mathcal{H}^2 \\
&+ \left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\int_{\partial\Gamma^*} \overline{H}(\mathbb{O})\cot(\alpha(q))\varrho(t,q)\,d\mathcal{H}^1 \\
&+ ⨍_{\Gamma^*} \Delta_{\Gamma^*}\varrho(t,q) + |\sigma^*|^2(q)\varrho(t,q) - H_{\Gamma^*}(q)^2\varrho(t,q)\,d\mathcal{H}^2 \\
&- \left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\int_{\partial\Gamma^*} H_{\Gamma^*}(q)\cot(\alpha(q))\varrho(t,q)\,d\mathcal{H}^1 \\
&= ⨍_{\Gamma^*} \left(\Delta_{\Gamma^*} + |\sigma^*|^2(q) - H_{\Gamma^*}(q)^2 + \overline{H}(\mathbb{O})H_{\Gamma^*}(q)\right)\varrho(t,q)\,d\mathcal{H}^2 \\
&+ \left(\int_{\Gamma^*} 1\,d\mathcal{H}^2\right)^{-1}\left(\int_{\partial\Gamma^*} (\overline{H}(\mathbb{O}) - H_{\Gamma^*}(q))\cot(\alpha(q))\varrho(t,q)\,d\mathcal{H}^1\right). \qquad \blacksquare
\end{aligned}
$$

The Lemmas 2.10 - 2.13 show how the linearization of equation (2.29) looks like. We continue with the linearization of the equation (2.30).

Lemma 2.14: *For all $q \in \partial\Gamma^*$ and all $t \in [0,\infty)$ we have*

$$
\frac{d}{d\varepsilon}v_{\partial D}(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} = \frac{1}{\sin(\alpha^*(q))}\partial_t\varrho(t,q).
$$

Proof: First we observe by means of Lemma 2.3

$$
\begin{aligned}
v_{\partial D}(\Psi(q,\varrho(t,q))) &= n_{\partial D}(\Psi(q,\varrho(t,q)))\cdot\frac{d}{dt}\Psi(q,\varrho(t,q)) \\
&= (n_{\partial D}(\Psi(q,\varrho(t,q)))\cdot\partial_w\Psi(q,\varrho(t,q)))\partial_t\varrho(t,q) \qquad (2.35)
\end{aligned}
$$

and with the help of the product rule this leads to

$$
\begin{aligned}
\frac{d}{d\varepsilon}v_{\partial D}(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}(n_{\partial D}(\Psi(q,\varepsilon\varrho(t,q)))\cdot\partial_w\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0}\underbrace{\partial_t\varepsilon\varrho(t,q)|_{\varepsilon=0}}_{=0} \\
&+ (n_{\partial D}(\Psi(q,\varepsilon\varrho(t,q)))\cdot\partial_w\Psi(q,\varepsilon\varrho(t,q)))|_{\varepsilon=0}\,\partial_t\varrho(t,q) \\
&= (n_{\partial D^*}(q)\cdot\partial_w\Psi(q,0))\partial_t\varrho(t,q) = \frac{1}{\sin(\alpha^*(q))}\partial_t\varrho(t,q),
\end{aligned}
$$

where we used (2.24) in the last line. ■

Lemma 2.15: *For a vector $v(\varepsilon) = (v_1(\varepsilon), \ldots, v_n(\varepsilon)) \in \mathbb{R}^n$ depending on one parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $v \in C^1((-\varepsilon_0, \varepsilon_0))$ and $\|v(0)\| > 0$ one has*

$$\text{(i)} \qquad \frac{d}{d\varepsilon}\|v(\varepsilon)\|\Big|_{\varepsilon=0} = \frac{v(0)\cdot v'(0)}{\|v(0)\|}$$

$$\text{(ii)} \qquad \frac{d}{d\varepsilon}\frac{v(\varepsilon)}{\|v(\varepsilon)\|}\Big|_{\varepsilon=0} = P_{\frac{v(0)}{\|v(0)\|}}\left(\frac{v'(0)}{\|v(0)\|}\right),$$

where $P_v : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \longmapsto P_v(x) := x - \langle x, v\rangle\, v$ denotes the projection along v.

Proof: (i) An easy calculation shows

$$\begin{aligned}
\frac{d}{d\varepsilon}\|v(\varepsilon)\|\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\sqrt{v_1(\varepsilon)^2 + \ldots + v_n(\varepsilon)^2}\Big|_{\varepsilon=0} \\
&= \frac{1}{2\sqrt{v_1(\varepsilon)^2 + \ldots + v_n(\varepsilon)^2}} 2(v_1(\varepsilon)v_1'(\varepsilon) + \ldots + v_n(\varepsilon)v_n'(\varepsilon))\Big|_{\varepsilon=0} \\
&= \frac{v(\varepsilon)\cdot v'(\varepsilon)}{\|v(\varepsilon)\|}\Big|_{\varepsilon=0} = \frac{v(0)\cdot v'(0)}{\|v(0)\|}.
\end{aligned}$$

(ii) Using (i) we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon}\frac{v(\varepsilon)}{\|v(\varepsilon)\|}\Big|_{\varepsilon=0} &= \frac{v'(0)\,\|v(0)\| - v(0)\dfrac{v(0)\cdot v'(0)}{\|v(0)\|}}{\|v(0)\|^2} \\
&= \frac{v'(0)}{\|v(0)\|} - \left(\frac{v'(0)}{\|v(0)\|}\cdot\frac{v(0)}{\|v(0)\|}\right)\frac{v(0)}{\|v(0)\|} \\
&= P_{\frac{v(0)}{\|v(0)\|}}\left(\frac{v'(0)}{\|v(0)\|}\right).
\end{aligned}$$

■

Lemma 2.16: *Let $G \subseteq \overline{\mathbb{R}^2_+}$ be relative open (i.e. G is open in the subspace topology of $\overline{\mathbb{R}^2_+} \subseteq \mathbb{R}^2$) and let $F : \overline{G} \longrightarrow \Omega$ be the parametrization for a part of $\Gamma^* \cup \partial\Gamma^*$ with the properties*

$$\begin{aligned}
&\partial_1 F(x_0), \partial_2 F(x_0) \quad \textit{form an orthonormal basis of } \; T_{F(x_0)}\Gamma^* \\
&\partial_1 F(x_0) \times \partial_2 F(x_0) = n_{\Gamma^*}(F(x_0)) \qquad\qquad (2.36)\\
&\partial_1 F(x_0) = \vec{\tau}^*(F(x_0)) \quad \textit{and} \quad \partial_2 F(x_0) = n_{\partial\Gamma^*}(F(x_0)) \quad \textit{on} \;\; \partial\Gamma^*
\end{aligned}$$

for a fixed $x_0 \in G \cap \partial\mathbb{R}^2_+$. Then we have for all $F(x) = q \in \Gamma^*$

$$\text{(i)} \qquad \Psi(F(x), 0) = F(x) \qquad \textit{and} \qquad \partial_i\Psi(F(x), 0) = \partial_i F(x)$$

and for all $F(x) = q \in \partial\Gamma^*$

$$\text{(ii)} \qquad \partial_w\Psi(F(x), 0) = n_{\Gamma^*}(F(x)) - \cot(\alpha(F(x)))n_{\partial\Gamma^*}(F(x)),$$

$$\text{(iii)} \qquad \langle \partial_i\partial_w\Psi(F(x), 0), n_{\Gamma^*}(F(x))\rangle = \cot(\alpha(F(x)))\,\langle \partial_i n_{\Gamma^*}(F(x)), n_{\partial\Gamma^*}(F(x))\rangle$$

and for the fixed $F(x_0) = q_0 \in \partial\Gamma^*$

$$
\begin{aligned}
&\textit{(iv)} && (\partial_1\Psi \times \partial_2\Psi)(q_0,0) = n_{\Gamma^*}(q_0),\\
&\textit{(v)} && (\partial_w\Psi \times \partial_2\Psi)(q_0,0) = -\vec{\tau}^*(q_0),\\
&\textit{(vi)} && (\partial_1\Psi \times \partial_w\Psi)(q_0,0) = -n_{\partial\Gamma^*}(q_0) - \cot(\alpha(q_0))n_{\Gamma^*}(q_0),\\
&\textit{(vii)} && (\partial_1\Psi \times \partial_i\partial_w\Psi)(q_0,0) = \langle \partial_i\partial_w\Psi(q_0,0), n_{\partial\Gamma^*}(q_0)\rangle\, n_{\Gamma^*}(q_0)\\
&&& \qquad - \cot(\alpha(q_0))\,\langle \partial_i n_{\Gamma^*}(q_0), n_{\partial\Gamma^*}(q_0)\rangle\, n_{\partial\Gamma^*}(q_0),\\
&\textit{(viii)} && (\partial_i\partial_w\Psi \times \partial_2\Psi)(q_0,0) = \langle \partial_i\partial_w\Psi(q_0,0), \vec{\tau}^*(q_0)\rangle\, n_{\Gamma^*}(q_0)\\
&&& \qquad - \cot(\alpha(q_0))\,\langle \partial_i n_{\Gamma^*}(q_0), n_{\partial\Gamma^*}(q_0)\rangle\, \vec{\tau}^*(q_0).
\end{aligned}
$$

Proof: (i) The first equation is a property of the curvilinear coordinate system as we constructed it. The second claim we obtain by differentiation.
(ii) Equation (2.23) for $q = F(x) \in \partial\Gamma^*$.
(iii) Using (ii) and (2.25) we get

$$
\begin{aligned}
0 = \partial_i 1 &= \partial_i \langle \partial_w\Psi(F(x),0), n_{\Gamma^*}(F(x))\rangle\\
&= \langle \partial_i\partial_w\Psi(F(x),0), n_{\Gamma^*}(F(x))\rangle + \langle \partial_w\Psi(F(x),0), \partial_i n_{\Gamma^*}(F(x))\rangle .
\end{aligned}
$$

Since $\partial_i n_{\Gamma^*}$ lies in the $\vec{\tau}^*$-$n_{\partial\Gamma^*}$-plane we see

$$
\begin{aligned}
\langle \partial_i\partial_w\Psi(F(x),0), n_{\Gamma^*}(F(x))\rangle &= -\langle n_{\partial\Gamma^*}(F(x)), \partial_i n_{\Gamma^*}(F(x))\rangle \underbrace{\langle \partial_w\Psi(F(x),0), n_{\partial\Gamma^*}(F(x))\rangle}_{=-\cot(\alpha(F(x)))}\\
&\quad - \langle \vec{\tau}^*(F(x)), \partial_i n_{\Gamma^*}(F(x))\rangle \underbrace{\langle \partial_w\Psi(F(x),0), \vec{\tau}^*(F(x))\rangle}_{=0}\\
&= \cot(\alpha(F(x)))\,\langle n_{\partial\Gamma^*}(F(x)), \partial_i n_{\Gamma^*}(F(x))\rangle ,
\end{aligned}
$$

where we used (2.24).
(iv) Using (i) and (2.36) we obtain

$$
(\partial_1\Psi \times \partial_2\Psi)(F(x_0),0) = (\partial_1 F \times \partial_2 F)(x_0) = n_{\Gamma^*}(F(x_0)).
$$

(v) By (i), (ii) and (2.36) we have

$$
\begin{aligned}
(\partial_w\Psi \times \partial_2\Psi)(F(x_0),0) &= ((n_{\Gamma^*}\circ F) \times \partial_2 F)(x_0) - \cot(\alpha(F(x_0)))(\partial_2 F \times \partial_2 F)(x_0)\\
&= -\vec{\tau}^*(F(x_0)).
\end{aligned}
$$

(vi) Similar to (v), where we use (iv) in addition, we get

$$
\begin{aligned}
(\partial_1\Psi \times \partial_w\Psi)(F(x_0),0) &= (\partial_1 F \times (n_{\Gamma^*}\circ F))(x_0) - \cot(\alpha(F(x_0)))(\partial_1 F \times \partial_2 F)(x_0)\\
&= -n_{\partial\Gamma^*}(F(x_0)) - \cot(\alpha(F(x_0)))n_{\Gamma^*}(F(x_0)).
\end{aligned}
$$

(vii) Using the previous equations one observes

$$\begin{aligned}(\partial_1\Psi \times \partial_i\partial_w\Psi)(F(x_0),0) &= (\partial_1 F \times (\langle \partial_i\partial_w\Psi(F(x_0),0), \partial_1 F\rangle\, \partial_1 F \\ &\quad + \langle \partial_i\partial_w\Psi(F(x_0),0), \partial_2 F(x_0)\rangle\, \partial_2 F \\ &\quad + \langle \partial_i\partial_w\Psi(F(x_0),0), (n_{\Gamma^*}\circ F)\rangle\, (n_{\Gamma^*}\circ F)))(x_0) \\ &= 0 + \langle \partial_i\partial_w\Psi(F(x_0),0), \partial_2 F(x_0)\rangle\, (\partial_1 F \times \partial_2 F)(x_0) \\ &\quad + \langle \partial_i\partial_w\Psi(F(x_0),0), n_{\Gamma^*}(F(x_0))\rangle\, (\partial_1 F \times (n_{\Gamma^*}\circ F))(x_0) \\ &= \langle \partial_i\partial_w\Psi(F(x_0),0), n_{\partial\Gamma^*}(F(x_0))\rangle\, n_{\Gamma^*}(F(x_0)) \\ &\quad - \cot(\alpha(F(x_0)))\, \langle \partial_i n_{\Gamma^*}(F(x_0)), n_{\partial\Gamma^*}(F(x_0))\rangle\, n_{\partial\Gamma^*}(F(x_0)).\end{aligned}$$

(viii) Analogously to (vii) one can prove the final claim of the lemma

$$\begin{aligned}(\partial_i\partial_w\Psi \times \partial_2\Psi)(F(x_0),0) &= ((\langle \partial_i\partial_w\Psi(F(x_0),0), \partial_1 F\rangle\, \partial_1 F \\ &\quad + \langle \partial_i\partial_w\Psi(F(x_0),0), \partial_2 F(x_0)\rangle\, \partial_2 F \\ &\quad + \langle \partial_i\partial_w\Psi(F(x_0),0), (n_{\Gamma^*}\circ F)\rangle\, (n_{\Gamma^*}\circ F)) \times \partial_2 F)(x_0) \\ &= \langle \partial_i\partial_w\Psi(F(x_0),0), \partial_1 F(x_0)\rangle\, (\partial_1 F \times \partial_2 F)(x_0) + 0 \\ &\quad + \langle \partial_i\partial_w\Psi(F(x_0),0), n_{\Gamma^*}(F(x_0))\rangle\, ((n_{\Gamma^*}\circ F) \times \partial_2 F)(x_0) \\ &= \langle \partial_i\partial_w\Psi(F(x_0),0), \vec{\tau}^*(F(x_0))\rangle\, n_{\Gamma^*}(F(x_0)) \\ &\quad - \cot(\alpha(F(x_0)))\, \langle \partial_i n_{\Gamma^*}(F(x_0)), n_{\partial\Gamma^*}(F(x_0))\rangle\, \vec{\tau}^*(F(x_0)).\end{aligned}$$

This finishes the proof. ■

Lemma 2.17: *For the linearization of the angle condition we have*

$$\begin{aligned}&\frac{d}{d\varepsilon}\, \langle n_\Gamma(t,\Psi(q,\varepsilon\varrho(t,q))), n_D(\Psi(q,\varepsilon\varrho(t,q)))\rangle\Big|_{\varepsilon=0} = -\sin(\alpha(q))(\nabla_{\Gamma^*}\varrho(t,q)\cdot n_{\partial\Gamma^*}(q)) \\ &\quad + \cos(\alpha(q)) II_{\Gamma^*}(n_{\partial\Gamma^*}(q), n_{\partial\Gamma^*}(q))\varrho(t,q) - II_{D^*}(n_{\partial D^*}(q), n_{\partial D^*}(q))\varrho(t,q),\end{aligned}$$

where II_{Γ^*} and II_{D^*} are the second fundamental forms of Γ^* and D^* with respect to the normals n_{Γ^*} and n_{D^*}, respectively.

Proof: By the product rule we get

$$\begin{aligned}\frac{d}{d\varepsilon} n_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\cdot n_D(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} n_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0}\cdot n_{D^*}(q) \\ &\quad + n_{\Gamma^*}(q)\cdot \frac{d}{d\varepsilon} \underbrace{n_D(\Psi(q,\varepsilon\varrho(t,q)))}_{=:(3)}\Big|_{\varepsilon=0}\end{aligned} \tag{2.37}$$

and the normal can be written as

$$n_\Gamma(\Psi(q,\varepsilon\varrho(t,q))) = \frac{\partial_1(\Psi(q,\varepsilon\varrho(t,q)))\times\partial_2(\Psi(q,\varepsilon\varrho(t,q)))}{\|\partial_1(\Psi(q,\varepsilon\varrho(t,q)))\times\partial_2(\Psi(q,\varepsilon\varrho(t,q)))\|}.$$

For the vector

$$\begin{aligned}
v(\varepsilon) &:= \partial_1(\Psi(q,\varepsilon\varrho(t,q))) \times \partial_2(\Psi(q,\varepsilon\varrho(t,q))) \\
&= ((\partial_1\Psi)(q,\varepsilon\varrho(t,q)) + (\partial_w\Psi)(q,\varepsilon\varrho(t,q))\varepsilon\partial_1\varrho(t,q)) \\
&\times ((\partial_2\Psi)(q,\varepsilon\varrho(t,q)) + (\partial_w\Psi)(q,\varepsilon\varrho(t,q))\varepsilon\partial_2\varrho(t,q)) \\
&= ((\partial_1\Psi)(q,\varepsilon\varrho(t,q)) \times (\partial_2\Psi)(q,\varepsilon\varrho(t,q))) \\
&+ \varepsilon((\partial_1\Psi)(q,\varepsilon\varrho(t,q)) \times (\partial_w\Psi)(q,\varepsilon\varrho(t,q)))\partial_2\varrho(t,q) \\
&+ \varepsilon((\partial_w\Psi)(q,\varepsilon\varrho(t,q)) \times (\partial_2\Psi)(q,\varepsilon\varrho(t,q)))\partial_1\varrho(t,q) \\
&+ \varepsilon^2 \underbrace{((\partial_w\Psi)(q,\varepsilon\varrho(t,q)) \times (\partial_w\Psi)(q,\varepsilon\varrho(t,q)))}_{=0} \partial_1\varrho(t,q)\partial_2\varrho(t,q) \\
&= \underbrace{(\partial_1\Psi \times \partial_2\Psi)(q,\varepsilon\varrho(t,q))}_{=:(1)} \\
&+ \underbrace{(\partial_w\Psi \times \partial_2\Psi)(q,\varepsilon\varrho(t,q))\varepsilon\partial_1\varrho(t,q) + (\partial_1\Psi \times \partial_w\Psi)(q,\varepsilon\varrho(t,q))\varepsilon\partial_2\varrho(t,q)}_{=:(2)}
\end{aligned}$$

we get using Lemma 2.15(ii)

$$\frac{d}{d\varepsilon} n_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0} = P_{\frac{v(0)}{\|v(0)\|}}\left(\frac{v'(0)}{\|v(0)\|}\right) = P_{n_{\Gamma^*}(q)}(v'(0)), \tag{2.38}$$

because due to Lemma 2.16(iv) we have $v(0) = (\partial_1\Psi \times \partial_2\Psi)(q,0) = n_{\Gamma^*}(q)$. Inspired by [Dep10] we decompose the term $v'(0)$ as

$$\frac{d}{d\varepsilon} v(\varepsilon)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}(1)\bigg|_{\varepsilon=0} + \frac{d}{d\varepsilon}(2)\bigg|_{\varepsilon=0}. \tag{2.39}$$

Now we consider the terms (1), (2) and (3) separately. With Lemma 2.16(vii)+(viii) we first observe

$$\begin{aligned}
\frac{d}{d\varepsilon}(1)\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}(\partial_1\Psi \times \partial_2\Psi)(q,\varepsilon\varrho(t,q))\bigg|_{\varepsilon=0} \\
&= (\partial_1\partial_w\Psi(q,0) \times \partial_2\Psi(q,0))\varrho(t,q) + (\partial_1\Psi(q,0) \times \partial_2\partial_w\Psi(q,0))\varrho(t,q) \\
&= (\langle\partial_1\partial_w\Psi(q,0), \vec{\tau}^*(q)\rangle\, n_{\Gamma^*}(q) - \cot(\alpha(q))\,\langle\partial_1 n_{\Gamma^*}(q), n_{\partial\Gamma^*}(q)\rangle\,\vec{\tau}^*(q))\varrho(t,q) \\
&+ (\langle\partial_2\partial_w\Psi(q,0), n_{\partial\Gamma^*}(q)\rangle\, n_{\Gamma^*}(q) - \cot(\alpha(q))\,\langle\partial_2 n_{\Gamma^*}(q), n_{\partial\Gamma^*}(q)\rangle\, n_{\partial\Gamma^*}(q))\varrho(t,q) \\
&= (\langle\partial_1\partial_w\Psi(q,0), \vec{\tau}^*(q)\rangle + \langle\partial_2\partial_w\Psi(q,0), n_{\partial\Gamma^*}(q)\rangle)\varrho(t,q) n_{\Gamma^*}(q) \\
&- \cot(\alpha(q))(\langle\partial_1 n_{\Gamma^*}(q), n_{\partial\Gamma^*}(q)\rangle\,\vec{\tau}^*(q) + \langle\partial_2 n_{\Gamma^*}(q), n_{\partial\Gamma^*}(q)\rangle\, n_{\partial\Gamma^*}(q))\varrho(t,q) \\
&= \operatorname{div}_{\Gamma^*}(\partial_w\Psi(q,0))\varrho(t,q) n_{\Gamma^*}(q) \\
&+ \cot(\alpha(q))(II_{\Gamma^*}(\vec{\tau}^*(q), n_{\partial\Gamma^*}(q))\vec{\tau}^*(q) + II_{\Gamma^*}(n_{\partial\Gamma^*}(q), n_{\partial\Gamma^*}(q)) n_{\partial\Gamma^*}(q))\varrho(t,q).
\end{aligned}$$

Now consider the second term. With Lemma 2.16(v)+(vi) one can show

$$\begin{aligned}
\frac{d}{d\varepsilon}(2)\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\partial_1\varepsilon\varrho(t,q)(\partial_w\Psi\times\partial_2\Psi)(q,\varepsilon\varrho(t,q)) + \partial_2\varepsilon\varrho(t,q)(\partial_1\Psi\times\partial_w\Psi)(q,\varepsilon\varrho(t,q))\bigg|_{\varepsilon=0}\\
&= \partial_1\varrho(t,q)(\partial_w\Psi(q,0)\times\partial_2\Psi(q,0)) + 0 + \partial_2\varrho(t,q)(\partial_1\Psi(q,0)\times\partial_w\Psi(q,0)) + 0\\
&= -\partial_1\varrho(t,q)\vec{\tau}^*(q) - \partial_2\varrho(t,q)(n_{\partial\Gamma^*}(q) + \cot(\alpha(q))n_{\Gamma^*}(q))\\
&= -(\partial_1\varrho(t,q)\vec{\tau}^*(q) + \partial_2\varrho(t,q)n_{\partial\Gamma^*}(q)) - \partial_2\varrho(t,q)\cot(\alpha(q))n_{\Gamma^*}(q)\\
&= -\nabla_{\Gamma^*}\varrho(t,q) + \partial_1\varrho(t,q)\underbrace{\langle\partial_w\Psi(q,0),\vec{\tau}^*(q)\rangle}_{=0}n_{\Gamma^*}(q)\\
&\quad + \partial_2\varrho(t,q)\langle\partial_w\Psi(q,0),n_{\partial\Gamma^*}(q)\rangle n_{\Gamma^*}(q)\\
&= -\nabla_{\Gamma^*}\varrho(t,q) + \langle\partial_w\Psi(q,0),\partial_1\varrho(t,q)\vec{\tau}^*(q) + \partial_2\varrho(t,q)n_{\partial\Gamma^*}(q)\rangle n_{\Gamma^*}(q)\\
&= -\nabla_{\Gamma^*}\varrho(t,q) + \langle\partial_w\Psi(q,0),\nabla_{\Gamma^*}\varrho(t,q)\rangle n_{\Gamma^*}(q).
\end{aligned}$$

Inserting this into (2.39) we get

$$\begin{aligned}
\frac{d}{d\varepsilon}v(\varepsilon)\bigg|_{\varepsilon=0} &= \mathrm{div}_{\Gamma^*}(\partial_w\Psi(q,0))\varrho(t,q)n_{\Gamma^*}(q)\\
&\quad + \cot(\alpha(q))(II_{\Gamma^*}(\vec{\tau}^*(q),n_{\partial\Gamma^*}(q))\vec{\tau}^*(q) + II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))n_{\partial\Gamma^*}(q))\varrho(t,q)\\
&\quad - \nabla_{\Gamma^*}\varrho(t,q) + \langle\partial_w\Psi(q,0),\nabla_{\Gamma^*}\varrho(t,q)\rangle n_{\Gamma^*}(q)\\
&= -\nabla_{\Gamma^*}\varrho(t,q) + \mathrm{div}_{\Gamma^*}(\partial_w\Psi(q,0)\varrho(t,q))n_{\Gamma^*}(q)\\
&\quad + \cot(\alpha(q))(II_{\Gamma^*}(\vec{\tau}^*(q),n_{\partial\Gamma^*}(q))\vec{\tau}^*(q) + II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))n_{\partial\Gamma^*}(q))\varrho(t,q).
\end{aligned}$$

Due to (2.38) we have to project this vector along the normal n_{Γ^*} to obtain

$$\begin{aligned}
\frac{d}{d\varepsilon}n_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0} &= -\nabla_{\Gamma^*}\varrho(t,q) + \cot(\alpha(q))II_{\Gamma^*}(\vec{\tau}^*(q),n_{\partial\Gamma^*}(q))\vec{\tau}^*(q)\varrho(t,q)\\
&\quad + \cot(\alpha(q))II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))n_{\partial\Gamma^*}(q)\varrho(t,q).
\end{aligned}$$

Multiplied by $n_{D^*}(q)$ this reads due to Remark 2.1 as

$$\begin{aligned}
&\frac{d}{d\varepsilon}n_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0}\cdot n_{D^*}(q)\\
&= -(\nabla_{\Gamma^*}\varrho(t,q)\cdot n_{D^*}(q)) + \cot(\alpha(q))II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))\langle n_{\partial\Gamma^*}(q),n_{D^*}(q)\rangle\varrho(t,q)\\
&= -(\nabla_{\Gamma^*}\varrho(t,q)\cdot(\cos(\alpha(q))n_{\Gamma^*}(q) + \sin(\alpha(q))n_{\partial\Gamma^*}(q)))\\
&\quad + \cos(\alpha(q))II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))\varrho(t,q)\\
&= -\sin(\alpha(q))(\nabla_{\Gamma^*}\varrho(t,q)\cdot n_{\partial\Gamma^*}(q)) + \cos(\alpha(q))II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))\varrho(t,q)
\end{aligned}$$

and hence we have

$$\begin{aligned}
\frac{d}{d\varepsilon}n_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0}\cdot n_{D^*}(q) &= -\sin(\alpha(q))(\nabla_{\Gamma^*}\varrho(t,q)\cdot n_{\partial\Gamma^*}(q))\\
&\quad + \cos(\alpha(q))II_{\Gamma^*}(n_{\partial\Gamma^*}(q),n_{\partial\Gamma^*}(q))\varrho(t,q). \qquad (2.40)
\end{aligned}$$

Finally for part (3) we use the curve

$$c:[0,\varepsilon_0)\longrightarrow\Omega:\varepsilon\longmapsto c(\varepsilon):=\Psi(q,\varepsilon\varrho(t,q)).$$

Then we have $c(\varepsilon) \in \partial\Omega$ for all $\varepsilon \geq 0$, $c(0) = \Psi(q,0) = q$ and $c'(0) = \partial_w \Psi(q,0)\varrho(t,q)$. Since the vector $\partial_w \Psi(q,0) = \frac{1}{\sin(\alpha(q))} n_{\partial D^*}(q) \in T_q D^*$ and due to (2.24) we get

$$\begin{aligned}
\frac{d}{d\varepsilon}(3)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} n_D(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} \\
&= \partial_{\partial_w \Psi(q,0)\varrho(t,q)} n_{D^*}(q) = \partial_{\frac{1}{\sin(\alpha(q))} n_{\partial D^*}(q)} n_{D^*}(q)\varrho(t,q) \\
&= \frac{1}{\sin(\alpha(q))} \partial_{n_{\partial D^*}(q)} n_{D^*}(q)\varrho(t,q).
\end{aligned}$$

Therefore we obtain

$$n_{\Gamma^*}(q) \cdot \frac{d}{d\varepsilon}(3)\Big|_{\varepsilon=0} = \frac{1}{\sin(\alpha(q))} \left\langle n_{\Gamma^*}(q), \partial_{n_{\partial D^*}(q)} n_{D^*}(q) \right\rangle \varrho(t,q).$$

Writing $n_{\Gamma^*} = \sin(\alpha) n_{\partial D^*} + \cos(\alpha) n_{D^*}$ and considering the fact that $\partial_{n_{\partial D^*}(q)} n_{D^*}(q)$ lies in the $n_{\partial D^*}$-$\vec{\tau}^*$-plane shows

$$\begin{aligned}
\left\langle n_{\Gamma^*}, \partial_{n_{\partial D^*}} n_{D^*} \right\rangle &= \sin(\alpha) \left\langle n_{\partial D^*}, \partial_{n_{\partial D^*}} n_{D^*} \right\rangle + \cos(\alpha) \underbrace{\left\langle n_{D^*}, \partial_{n_{\partial D^*}} n_{D^*} \right\rangle}_{=0} \\
&= -\sin(\alpha) II_{D^*}(n_{\partial D^*}, n_{\partial D^*}).
\end{aligned}$$

In combination this leads to

$$n_{\Gamma^*}(q) \cdot \frac{d}{d\varepsilon}(3)\Big|_{\varepsilon=0} = -II_{D^*}(n_{\partial D^*}(q), n_{\partial D^*}(q))\varrho(t,q). \tag{2.41}$$

Coupling (2.40)-(2.41) as in (2.37) we finally arrive at

$$\begin{aligned}
&\frac{d}{d\varepsilon} \left\langle n_\Gamma(t, \Psi(q,\varepsilon\varrho(t,q))), n_D(\Psi(q,\varepsilon\varrho(t,q))) \right\rangle\Big|_{\varepsilon=0} = -\sin(\alpha(q))(\nabla_{\Gamma^*}\varrho(t,q) \cdot n_{\partial\Gamma^*}(q)) \\
&+ \cos(\alpha(q)) II_{\Gamma^*}(n_{\partial\Gamma^*}(q), n_{\partial\Gamma^*}(q))\varrho(t,q) - II_{D^*}(n_{\partial D^*}(q), n_{\partial D^*}(q))\varrho(t,q),
\end{aligned}$$

which is the desired statement. ∎

Lemma 2.18: *For the linearization of the geodesic curvature we have*

$$\begin{aligned}
\frac{d}{d\varepsilon} \varkappa_{\partial D}(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} &= \varrho_{\sigma\sigma}(t,q) + \varkappa_{D^*}(q) II_{D^*}(n_{\partial D^*}(q), n_{\partial D^*}(q))\varrho(t,q) \\
&\quad - \varkappa_{\partial D^*}(q) \left\langle \vec{\tau}^*(q), (n_{\partial D^*})_\sigma(q) \right\rangle \varrho(t,q) \\
&\quad - \left\langle n_{\partial D^*}(q), (n_{D^*})_\sigma(q) \right\rangle^2 \varrho(t,q),
\end{aligned}$$

where II_{D^*} is the second fundamental form of D^* with respect to the normal n_{D^*}, $\varkappa_{D^*}$ is the normal curvature of $\partial\Gamma^*$ in D^* defined as $\varkappa_{D^*}(q) := \langle \vec{\varkappa}^*(q), n_{D^*}(q) \rangle$ and σ denotes the arc-length-parameter of $\partial\Gamma^* = \partial D^*$.

<u>Proof</u>: First we denote by

$$c : [0,1] \longrightarrow \partial\Gamma^* : s \longmapsto c(s) \quad \text{with} \quad c(0) = c(1)$$

a parametrization of $\partial\Gamma^*$ and with

$$\begin{aligned}
&\widetilde{c} : [0, \varepsilon_0) \times [0,1] \longrightarrow \mathbb{R}^3 : (\varepsilon, s) \longmapsto \widetilde{c}(\varepsilon, s) \\
&\text{with} \quad \widetilde{c}(\varepsilon, 0) = \widetilde{c}(\varepsilon, 1) \quad \text{and} \quad \widetilde{c}(0, s) = c(s) \\
&\text{and} \quad \widetilde{c}_\varepsilon(0, s) = \varrho(t, c(s)) n_{\partial D^*}(c(s)) =: \zeta(s) \quad \forall\ s \in [0,1]
\end{aligned}$$

we denote a parametrization of $\partial\widetilde{\Gamma}(\varepsilon)$, where $\widetilde{\Gamma}(\varepsilon)$ is the evolving hypersurface from the proof of Lemma 2.13. If we choose the orientation of the parametrization $\widetilde{c}$ appropriately, we have the relation

$$\frac{\widetilde{c}_s(\varepsilon, s)}{\|\widetilde{c}_s(\varepsilon, s)\|} = \vec{\tau}(\varepsilon, c(s)) \qquad \forall\ \varepsilon \in [0, \varepsilon_0)\ \forall\ s \in [0,1]$$

and particularly

$$\frac{c_s(s)}{\|c_s(s)\|} = \vec{\tau}^*(c(s)) \qquad \forall\ s \in [0,1].$$

Therefore we have

$$\frac{1}{\|\widetilde{c}_s(\varepsilon, s)\|} \left(\frac{\widetilde{c}_s(\varepsilon, s)}{\|\widetilde{c}_s(\varepsilon, s)\|} \right)_s = \varkappa_{\partial \widetilde{D}(\varepsilon)}(\varepsilon, c(s)) n_{\partial \widetilde{D}(\varepsilon)}(\varepsilon, c(s)) + \varkappa_{\widetilde{D}(\varepsilon)}(\varepsilon, c(s)) n_{\widetilde{D}(\varepsilon)}(\varepsilon, c(s)).$$

The weak formulation of this reads as

$$\begin{aligned}
&-\int_{\partial\widetilde{\Gamma}(\varepsilon)} \frac{\widetilde{c}_s(\varepsilon, s(p))}{\|\widetilde{c}_s(\varepsilon, s(p))\|} \cdot \frac{\vec{\eta}_s(s(p))}{\|\widetilde{c}_s(\varepsilon, s(p))\|}\, d\mathcal{H}^1 \\
&\quad = \int_{\partial\widetilde{\Gamma}(\varepsilon)} \frac{1}{\|\widetilde{c}_s(\varepsilon, s(p))\|} \left(\frac{\widetilde{c}_s(\varepsilon, s(p))}{\|\widetilde{c}_s(\varepsilon, s(p))\|} \right)_s \cdot \vec{\eta}(s(p))\, d\mathcal{H}^1 \\
&\quad = \int_{\partial\widetilde{\Gamma}(\varepsilon)} \Big(\varkappa_{\partial \widetilde{D}(\varepsilon)}(\varepsilon, c(s(p))) n_{\partial \widetilde{D}(\varepsilon)}(\varepsilon, c(s(p))) \\
&\qquad\qquad + \varkappa_{\widetilde{D}(\varepsilon)}(\varepsilon, c(s(p))) n_{\widetilde{D}(\varepsilon)}(\varepsilon, c(s(p))) \Big) \cdot \vec{\eta}(s(p))\, d\mathcal{H}^1,
\end{aligned}$$

where $s(p)$ shall be given by $s(p) := (\widetilde{c}(\varepsilon, \bullet))^{-1}(p)$ for $p \in \partial\widetilde{\Gamma}(\varepsilon)$ and $\vec{\eta} : [0,1] \longrightarrow \mathbb{R}^3$ is an arbitrary vector field. Shortly we denote this by

$$0 = \int_{\partial\widetilde{\Gamma}(\varepsilon)} \frac{\widetilde{c}_s}{\|\widetilde{c}_s\|} \cdot \frac{\vec{\eta}_s}{\|\widetilde{c}_s\|}\, d\mathcal{H}^1 + \int_{\partial\widetilde{\Gamma}(\varepsilon)} \left(\varkappa_{\partial \widetilde{D}(\varepsilon)} n_{\partial \widetilde{D}(\varepsilon)} + \varkappa_{\widetilde{D}(\varepsilon)} n_{\widetilde{D}(\varepsilon)} \right) \cdot \vec{\eta}\, d\mathcal{H}^1. \tag{2.42}$$

Before we can differentiate the equation with respect to ε we have to do some auxiliary calculations:

1. Let $a(\varepsilon, s)$ be a quantity smoothly depending on ε and s. Using Lemma 2.15(i) we

have

$$
\begin{aligned}
&\frac{d}{d\varepsilon} \int_{\partial\widetilde{\Gamma}(\varepsilon)} a(\varepsilon, s(p))\, d\mathcal{H}^1 \Big|_{\varepsilon=0} \\
&\quad = \frac{d}{d\varepsilon} \int_I a(\varepsilon, s)\, \|\widetilde{c}_s(\varepsilon, s)\|\, ds \Big|_{\varepsilon=0} \\
&\quad = \int_I a_\varepsilon(\varepsilon, s)\, \|\widetilde{c}_s(\varepsilon, s)\|\, ds \Big|_{\varepsilon=0} + \int_I a(\varepsilon, s) \left(\frac{\widetilde{c}_s(\varepsilon, s) \cdot \widetilde{c}_{s\varepsilon}(\varepsilon, s)}{\|\widetilde{c}_s(\varepsilon, s)\|} \right) ds \Big|_{\varepsilon=0} \\
&\quad = \int_I a_\varepsilon(0, s)\, \|c_s(s)\|\, ds + \int_I a(0, s) \left(\frac{c_s(s)}{\|c_s(s)\|} \cdot \zeta_s(s) \right) ds \\
&\quad = \int_{\partial\Gamma^*} a_\varepsilon(0, s(q))\, d\mathcal{H}^1 + \int_{\partial\Gamma^*} a(0, s(q)) \left(\underbrace{\frac{c_s(s(q))}{\|c_s(s(q))\|}}_{=\vec{\tau}^*(c(s(q)))} \cdot \frac{\zeta_s(s(q))}{\|c_s(s(q))\|} \right) d\mathcal{H}^1,
\end{aligned}
$$

where $s(q)$ is the abbreviation for $s(q) := c^{-1}(q)$ with $q \in \partial\Gamma^*$.

2. If we denote with $\widetilde{P}$ the projection onto the $n_{\partial D^*}$-n_{D^*}-plane, which is also the $n_{\partial\Gamma^*}$-n_{Γ^*}-plane, we get by Lemma 2.15(ii)

$$
\frac{d}{d\varepsilon} \frac{\widetilde{c}_s(\varepsilon, s)}{\|\widetilde{c}_s(\varepsilon, s)\|} \Big|_{\varepsilon=0} = P_{\frac{c_s(s)}{\|c_s(s)\|}} \left(\frac{\widetilde{c}_{s\varepsilon}(0, s)}{\|c_s(s)\|} \right) = P_{\vec{\tau}^*(c(s))} \left(\frac{\zeta_s(s)}{\|c_s(s)\|} \right) = \widetilde{P} \left(\frac{\zeta_s(s)}{\|c_s(s)\|} \right).
$$

3. Using the second auxiliary calculation we obtain

$$
\begin{aligned}
&\frac{d}{d\varepsilon} \int_{\partial\widetilde{\Gamma}(\varepsilon)} \frac{\widetilde{c}_s(\varepsilon, s(p))}{\|\widetilde{c}_s(\varepsilon, s(p))\|} \cdot \frac{\vec{\eta}_s(s(p))}{\|\widetilde{c}_s(\varepsilon, s(p))\|}\, d\mathcal{H}^1 \Big|_{\varepsilon=0} \\
&\quad = \frac{d}{d\varepsilon} \int_I \frac{\widetilde{c}_s(\varepsilon, s)}{\|\widetilde{c}_s(\varepsilon, s)\|} \cdot \vec{\eta}_s(s) ds \Big|_{\varepsilon=0} = \int_I \widetilde{P} \left(\frac{\zeta_s(s)}{\|c_s(s)\|} \right) \cdot \vec{\eta}_s(s) ds \\
&\quad = \int_{\partial\Gamma^*} \widetilde{P} \left(\frac{\zeta_s(s(q))}{\|c_s(s(q))\|} \right) \cdot \frac{\vec{\eta}_s(s(q))}{\|c_s(s(q))\|}\, d\mathcal{H}^1.
\end{aligned}
$$

Using these auxiliary calculations we can differentiate the equation (2.42) with respect to ε and derive

$$
\begin{aligned}
0 = &\int_{\partial\Gamma^*} ((\varkappa_{\partial D^*})_\varepsilon n_{\partial D^*} + \varkappa_{\partial D^*} (n_{\partial D^*})_\varepsilon + (\varkappa_{D^*})_\varepsilon n_{D^*} + \varkappa_{D^*} (n_{D^*})_\varepsilon) \cdot \vec{\eta}\, d\mathcal{H}^1 \\
&+ \int_{\partial\Gamma^*} (\varkappa_{\partial D^*} n_{\partial D^*} + \varkappa_{D^*} n_{D^*}) \cdot \vec{\eta} \left(\vec{\tau}^* \cdot \frac{\zeta_s}{\|c_s\|} \right) d\mathcal{H}^1 + \int_{\partial\Gamma^*} \widetilde{P} \left(\frac{\zeta_s}{\|c_s\|} \right) \cdot \frac{\vec{\eta}_s}{\|c_s\|}\, d\mathcal{H}^1
\end{aligned}
$$

for $\varepsilon = 0$. Choosing $\vec{\eta}(s) := \xi(s) n_{\partial D^*}(c(s))$ with an arbitrary function $\xi : [0, 1] \longrightarrow \mathbb{R}$ we get

$$
\begin{aligned}
0 = &\int_{\partial\Gamma^*} (\varkappa_{\partial D^*})_\varepsilon \xi + \varkappa_{D^*} \xi ((n_{D^*})_\varepsilon \cdot n_{\partial D^*})\, d\mathcal{H}^1 \\
&+ \int_{\partial\Gamma^*} \varkappa_{\partial D^*} \xi \left(\vec{\tau}^* \cdot \frac{\zeta_s}{\|c_s\|} \right) d\mathcal{H}^1 + \int_{\partial\Gamma^*} \widetilde{P} \left(\frac{\zeta_s}{\|c_s\|} \right) \cdot \frac{(\xi n_{\partial D^*})_s}{\|c_s\|}\, d\mathcal{H}^1
\end{aligned}
$$

due to $(n_{\partial D^*})_\varepsilon \cdot n_{\partial D^*} = n_{D^*} \cdot n_{\partial D^*} = 0$ and $n_{\partial D^*} \cdot n_{\partial D^*} = 1$. Integration by parts yields the rewritten equation

$$\begin{aligned}\int_{\partial\Gamma^*} (\varkappa_{\partial D^*})_\varepsilon \xi \, d\mathcal{H}^1 = &\int_{\partial\Gamma^*} \left(\widetilde{P}\left(\frac{\zeta_s}{\|c_s\|}\right)\right)_s \cdot \frac{\xi n_{\partial D^*}}{\|c_s\|} \, d\mathcal{H}^1 \\ &- \int_{\partial\Gamma^*} \varkappa_{D^*}\xi((n_{D^*})_\varepsilon \cdot n_{\partial D^*}) \, d\mathcal{H}^1 \\ &- \int_{\partial\Gamma^*} \varkappa_{\partial D^*}\xi \left(\vec{\tau}^* \cdot \frac{\zeta_s}{\|c_s\|}\right) d\mathcal{H}^1. \end{aligned} \tag{2.43}$$

We will now investigate the first integrand step by step. Starting with

$$\zeta_s = \varrho_s n_{\partial D^*} + \varrho (n_{\partial D^*})_s \tag{2.44}$$

we project onto the $n_{\partial D^*}$-n_{D^*}-plane and obtain

$$\widetilde{P}\left(\frac{\zeta_s}{\|c_s\|}\right) = \frac{\varrho_s}{\|c_s\|} n_{\partial D^*} + \frac{1}{\|c_s\|}\varrho \left\langle (n_{\partial D^*})_s, n_{D^*}\right\rangle n_{D^*}.$$

Another differentiation with respect to s gives

$$\begin{aligned}\left(\widetilde{P}\left(\frac{\zeta_s}{\|c_s\|}\right)\right)_s = &\left(\frac{\varrho_s}{\|c_s\|}\right)_s n_{\partial D^*} + \frac{\varrho_s}{\|c_s\|}(n_{\partial D^*})_s + \left(\frac{1}{\|c_s\|}\varrho \left\langle (n_{\partial D^*})_s, n_{D^*}\right\rangle\right)_s n_{D^*} \\ &+ \frac{1}{\|c_s\|}\varrho \left\langle (n_{\partial D^*})_s, n_{D^*}\right\rangle (n_{D^*})_s\end{aligned}$$

and this finally leads to

$$\left(\widetilde{P}\left(\frac{\zeta_s}{\|c_s\|}\right)\right)_s \cdot \frac{\xi n_{\partial D^*}}{\|c_s\|} = \frac{1}{\|c_s\|}\left(\frac{\varrho_s}{\|c_s\|}\right)_s \xi + \frac{\varrho}{\|c_s\|^2}\left\langle (n_{\partial D^*})_s, n_{D^*}\right\rangle \left\langle (n_{D^*})_s, n_{\partial D^*}\right\rangle \xi.$$

Since $\left\langle (n_{\partial D^*})_s, n_{D^*}\right\rangle = \frac{d}{ds}\underbrace{\left\langle n_{\partial D^*}, n_{D^*}\right\rangle}_{=0} - \left\langle n_{\partial D^*}, (n_{D^*})_s\right\rangle = -\left\langle n_{\partial D^*}, (n_{D^*})_s\right\rangle$ we have

$$\left(\widetilde{P}\left(\frac{\zeta_s}{\|c_s\|}\right)\right)_s \cdot \frac{\xi n_{\partial D^*}}{\|c_s\|} = \varrho_{\sigma\sigma}\xi - \varrho \left\langle (n_{D^*})_\sigma, n_{\partial D^*}\right\rangle^2 \xi.$$

In addition, (2.44) shows that the third integrand in (2.43) reads as

$$\left(\vec{\tau}^* \cdot \frac{\zeta_s}{\|c_s\|}\right) = \varrho_\sigma \underbrace{(\vec{\tau}^* \cdot n_{\partial D^*})}_{=0} + \frac{\varrho}{\|c_s\|}\left\langle \vec{\tau}^*, (n_{\partial D^*})_s\right\rangle = \left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma\right\rangle \varrho.$$

Inserting these two facts into (2.43) we obtain

$$\begin{aligned}\int_{\partial\Gamma^*} (\varkappa_{\partial D^*})_\varepsilon \xi \, d\mathcal{H}^1 = &\int_{\partial\Gamma^*} \varrho_{\sigma\sigma}\xi - \varrho \left\langle (n_{D^*})_\sigma, n_{\partial D^*}\right\rangle^2 \xi \, d\mathcal{H}^1 \\ &- \int_{\partial\Gamma^*} \varkappa_{D^*} \left\langle (n_{D^*})_\varepsilon, n_{\partial D^*}\right\rangle \xi + \varkappa_{\partial D^*} \varrho \left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma\right\rangle \xi \, d\mathcal{H}^1.\end{aligned}$$

Since ξ was chosen arbitrarily we can again apply the fundamental lemma of calculus of variation to end up with

$$(\varkappa_{\partial D^*})_\varepsilon = \varrho_{\sigma\sigma} - \left\langle (n_{D^*})_\sigma, n_{\partial D^*}\right\rangle^2 \varrho - \varkappa_{D^*}\left\langle (n_{D^*})_\varepsilon, n_{\partial D^*}\right\rangle - \varkappa_{\partial D^*}\left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma\right\rangle \varrho.$$

We want to have an equation where ϱ is contained in every single term, therefore we rewrite the third term as

$$(n_{D^*})_\varepsilon = \partial_{\varrho n_{\partial D^*}} n_{D^*} = \varrho \partial_{n_{\partial D^*}} n_{D^*},$$

which leads to

$$\langle (n_{D^*})_\varepsilon, n_{\partial D^*} \rangle = \left\langle \partial_{n_{\partial D^*}} n_{D^*}, n_{\partial D^*} \right\rangle \varrho = -II_{D^*}(n_{\partial D^*}, n_{\partial D^*})\varrho.$$

Finally, we have the desired expression

$$(\varkappa_{\partial D^*})_\varepsilon = \varrho_{\sigma\sigma} - \langle (n_{D^*})_\sigma, n_{\partial D^*} \rangle^2 \varrho + \varkappa_{D^*} II_{D^*}(n_{\partial D^*}, n_{\partial D^*})\varrho - \varkappa_{\partial D^*} \langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \rangle \varrho. \quad \blacksquare$$

Combining the results from Lemma 2.10 to Lemma 2.18 the linearization of (2.29)-(2.30) is given by

$$\begin{aligned}
\partial_t \varrho(t) &= \Delta_{\Gamma^*} \varrho(t) + |\sigma^*|^2 \varrho(t) + \left(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P\left(\partial_w \Psi(0)\right)\right) \varrho(t) \\
&\quad - \fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2 - H^2_{\Gamma^*} + \overline{H}(\mathbb{O}) H_{\Gamma^*}) \varrho(t) \, d\mathcal{H}^2 \\
&\quad + \frac{1}{\int_{\Gamma^*} 1 \, d\mathcal{H}^2} \int_{\partial\Gamma^*} \left(H_{\Gamma^*} - \overline{H}(\mathbb{O})\right) \cot(\alpha) \varrho(t) \, d\mathcal{H}^1 && \text{in } [0,T] \times \Gamma^* && (2.45) \\
\partial_t \varrho(t) &= -\sin(\alpha)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} \varrho(t)) - \sin(\alpha) II_{D^*}(n_{\partial D^*}, n_{\partial D^*}) \varrho(t) \\
&\quad + \sin(\alpha)\cos(\alpha) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*}) \varrho(t) + b \sin(\alpha) \varrho_{\sigma\sigma}(t) \\
&\quad + b \sin(\alpha) \varkappa_{D^*} II_{D^*}(n_{\partial D^*}, n_{\partial D^*}) \varrho(t) - b \sin(\alpha) \varkappa_{\partial D^*} \langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \rangle \varrho(t) \\
&\quad - b \sin(\alpha) \langle n_{\partial D^*}, (n_{D^*})_\sigma \rangle^2 \varrho(t) && \text{on } [0,T] \times \partial\Gamma^* && (2.46) \\
\varrho(0) &= \varrho_0 && \text{in } \Gamma^*, && (2.47)
\end{aligned}$$

where we have dropped the argument q for a more convenient notation and introduced an initial condition.

This linearization will be the starting point for the short-time existence of solutions of the MCF in the section to follow.

3 Local existence of solutions of the volume-preserving MCF with line tension

In this section we will show that the flow (2.29)-(2.30) has a unique strong solution for short times. Throughout the whole section we assume $b > 0$. The case $b = 0$ is treated similarly, but will not be considered. Some difficulties will arise due to the non-local nature of the flow. Our goal will be achieved by first considering solutions of the linearized flow (2.45)-(2.47) and then apply a fixed point argument to transfer the short-time existence to the non-linear flow.

3.1 Short-time existence of solutions for the linearized volume-preserving Mean Curvature Flow

In a first step we want to show that for fixed $T > 0$ the flow

$$\partial_t \varrho(t) = \Delta_{\Gamma^*} \varrho(t) + |\sigma^*|^2 \varrho(t) + \left(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P\left(\partial_w \Psi(0)\right)\right) \varrho(t) \qquad \text{in } [0,T] \times \Gamma^* \tag{3.1}$$

$$\begin{aligned} \partial_t \varrho(t) = &- \sin(\alpha)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} \varrho(t)) - \sin(\alpha) II_{D^*}(n_{\partial D^*}, n_{\partial D^*}) \varrho(t) \\ &+ \sin(\alpha)\cos(\alpha) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*}) \varrho(t) + b \sin(\alpha) \varrho_{\sigma\sigma}(t) \\ &+ b \sin(\alpha) \varkappa_{D^*} II_{D^*}(n_{\partial D^*}, n_{\partial D^*}) \varrho(t) - b \sin(\alpha) \varkappa_{\partial D^*} \left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \right\rangle \varrho(t) \\ &- b \sin(\alpha) \left\langle n_{\partial D^*}, (n_{D^*})_\sigma \right\rangle^2 \varrho(t) \qquad \text{on } [0,T] \times \partial\Gamma^* \end{aligned} \tag{3.2}$$

$$\varrho(0) = \varrho_0 \qquad \text{in } \Gamma^*, \tag{3.3}$$

which is (2.45)-(2.47) without the non-local part, has a unique solution. To derive such a result we follow the work of [DPZ08].

Remark 3.1: In our first step we will not consider the non-local terms of (2.45), which are given by the operator

$$\begin{aligned} \mathcal{P}(\bullet) := &\fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2 - H_{\Gamma^*}^2 + \overline{H}(\mathbb{O}) H_{\Gamma^*}) \bullet \, d\mathcal{H}^2 \\ &- \frac{1}{\int\limits_{\Gamma^*} 1 \, d\mathcal{H}^2} \int_{\partial\Gamma^*} \Big(H_{\Gamma^*} - \overline{H}(\mathbb{O}) \Big) \cot(\alpha) \bullet \, d\mathcal{H}^1. \end{aligned}$$

Later we will show that $\mathcal{P}$ is only a lower order perturbation of the original differential operator and does not effect the result. $\square$

As a starting point we want to adopt the entire notation of [DPZ08] such that (3.1)-(3.3) is turned into a problem of the form

$$\begin{aligned}
\frac{d}{dt}u(t) + \mathcal{A}(t,q,D)u(t) &= f(t) && \text{in } J\times\Gamma^* \\
\frac{d}{dt}\rho(t) + \mathcal{B}_0(t,q,D)u(t) + \mathcal{C}_0(t,q,D_\partial)\rho(t) &= g_0(t) && \text{on } J\times\partial\Gamma^* \\
\mathcal{B}_1(t,q,D)u(t) + \mathcal{C}_1(t,q,D_\partial)\rho(t) &= g_1(t) && \text{on } J\times\partial\Gamma^* \\
u(0) &= u_0 && \text{in } \Gamma^* \\
\rho(0) &= \rho_0 && \text{on } \partial\Gamma^*,
\end{aligned}$$

which can be solved with the results of [DPZ08]. Although the authors only consider domains in $\mathbb{R}^n$ the results carry over to smooth manifolds. We would have to use a partition of unity and local coordinates several times, but for the sake of notation we skip these technicalities.

If we define $D := -i\nabla_{\Gamma^*}$ and $D_\partial := -i\partial_\sigma$ and drop the argument q, the operators and functions in our case read as

$$\begin{aligned}
\mathcal{A}(t,D) &:= -\Delta_{\Gamma^*} - |\sigma^*|^2 - (\nabla_{\Gamma^*}H_{\Gamma^*}\cdot P(\partial_w\Psi(0))) \\
\mathcal{B}_0(t,D) &:= \sin(\alpha)^2(n_{\partial\Gamma^*}\cdot\nabla_{\Gamma^*}) + \sin(\alpha)II_{D^*}(n_{\partial D^*},n_{\partial D^*}) \\
&\quad - \sin(\alpha)\cos(\alpha)II_{\Gamma^*}(n_{\partial\Gamma^*},n_{\partial\Gamma^*}) \\
\mathcal{C}_0(t,D_\partial) &:= -b\sin(\alpha)\partial_\sigma^2 - b\sin(\alpha)\varkappa_{D^*}II_{D^*}(n_{\partial D^*},n_{\partial D^*}) \\
&\quad + b\sin(\alpha)\varkappa_{\partial D^*}\left\langle\vec{\tau}^*,(n_{\partial D^*})_\sigma\right\rangle + b\sin(\alpha)\left\langle n_{\partial D^*},(n_{D^*})_\sigma\right\rangle^2 \\
\mathcal{B}_1(t,D) &:= 1 \\
\mathcal{C}_1(t,D_\partial) &:= -1 \\
u(t) &:= \varrho(t) \\
\rho(t) &:= \varrho(t)|_{\partial\Gamma^*}.
\end{aligned}$$

We note that the required condition "all $\mathcal{B}_j$ and at least one $\mathcal{C}_j$ are non-trivial" is satisfied. Moreover, in our case we have $E := F := \mathbb{R}$, which are obviously of type $\mathcal{HT}$ since the Hilbert-transform is continuous on $L_2(\mathbb{R};\mathbb{R})$. The interval we want to consider is $[0,T]$ denoted by J as in [DPZ08]. For a given $1<p<\infty$ the involved numbers are

$$\begin{aligned}
m &:= \frac{1}{2}\operatorname{ord}(\mathcal{A}) = 1, \\
m_0 &:= \operatorname{ord}(\mathcal{B}_0) = 1, & m_1 &:= \operatorname{ord}(\mathcal{B}_1) = 0, \\
k_0 &:= \operatorname{ord}(\mathcal{C}_0) = 2, & k_1 &:= \operatorname{ord}(\mathcal{C}_1) = 0, \\
\varkappa_0 &:= 1 - \frac{m_0}{2m} - \frac{1}{2mp} = \frac{1}{2} - \frac{1}{2p}, & \varkappa_1 &:= 1 - \frac{m_1}{2m} - \frac{1}{2mp} = 1 - \frac{1}{2p}, \\
l_0 &:= k_0 - m_0 + m_0 = 2, & l_1 &:= k_1 - m_1 + m_0 = 1, \\
l &:= \max\{l_0,l_1\} = 2.
\end{aligned}$$

Because of $l = 2 = 2m$ we have to consider the setting that is called "case 1" in [DPZ08]. In our situation the required function spaces simplify to

$$
\begin{aligned}
X &:= L_p(J; L_p(\Gamma^*; \mathbb{R})),\\
Z_u &:= W_p^1(J; L_p(\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^2(\Gamma^*; \mathbb{R})),\\
\pi Z_u &:= W_p^{2-\frac{2}{p}}(\Gamma^*; \mathbb{R}),\\
Y_0 &:= W_p^{\frac{1}{2}-\frac{1}{2p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
Y_1 &:= W_p^{1-\frac{1}{2p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
Z_\rho &:= W_p^{\frac{3}{2}-\frac{1}{2p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{3-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
\pi Z_\rho &:= W_p^{3-\frac{3}{p}}(\partial\Gamma^*; \mathbb{R}),\\
\pi_1 Z_\rho &:= W_p^{1-\frac{3}{p}}(\partial\Gamma^*; \mathbb{R}),
\end{aligned}
\tag{3.4}
$$

where we have to assure for the trace spaces that $\frac{2}{p} \notin \mathbb{N}$ and $\varkappa_0 > \frac{1}{p}$, i.e. $p > 3$. As the principle parts of the operators we obtain

$$
\begin{aligned}
\mathcal{A}^\sharp(t, q, -i\nabla_{\Gamma^*}) &= -\Delta_{\Gamma^*} = (-i\nabla_{\Gamma^*}) \cdot (-i\nabla_{\Gamma^*})\\
\mathcal{B}_0^\sharp(t, q, -i\nabla_{\Gamma^*}) &= i \sin(\alpha(q))^2 (n_{\partial\Gamma^*}(q) \cdot (-i\nabla_{\Gamma^*}))\\
\mathcal{C}_0^\sharp(t, q, -i\partial_\sigma) &= b \sin(\alpha(q))(-i\partial_\sigma)^2\\
\mathcal{B}_1^\sharp(t, q, -i\nabla_{\Gamma^*}) &= 1\\
\mathcal{C}_1^\sharp(t, q, -i\partial_\sigma) &= -1.
\end{aligned}
$$

To apply the theorems of [DPZ08] we have to check the respective assumptions. We remark that we can ignore the assumptions (LS_∞^-) and (LS_∞^+) due to the case $l = 2m$ and furthermore we can also ignore assumptions (SD), (SB) and (SC), since we have assumed all involved surfaces and operators to be smooth enough. Now we only have to revise the ellipticity assumption (E) and the Lopatinskii-Shapiro condition (LS).

To prove that condition (E) is satisfied we let $t \in J$, $q \in \Gamma^*$ and $\xi \in \mathbb{R}^2$ with $\|\xi\| = 1$. Then we see

$$
\begin{aligned}
\sigma(\mathcal{A}^\sharp(t, q, \xi)) &= \left\{\lambda \in \mathbb{C} \,\middle|\, \lambda - \mathcal{A}^\sharp(t, q, \xi) = 0\right\}\\
&= \left\{\lambda \in \mathbb{C} \,\middle|\, \lambda = \xi \cdot \xi = \|\xi\|^2 = 1\right\} = \{1\} \subseteq \mathbb{C}_+,
\end{aligned}
$$

where $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$.

In order to check condition (LS), the finite dimension of $E = F = \mathbb{R}$ allows us to prove the equivalent condition that the desired ODE given by

$$
\begin{aligned}
(\lambda + \mathcal{A}^\sharp(t, q, \widehat{\xi}, -i\partial_y))v(y) &= 0\\
\mathcal{B}_0^\sharp(t, q, \widehat{\xi}, -i\partial_y)v(0) + (\lambda + \mathcal{C}_0^\sharp(t, q, \widehat{\xi}))\sigma &= h_0\\
\mathcal{B}_1^\sharp(t, q, \widehat{\xi}, -i\partial_y)v(0) + \mathcal{C}_1^\sharp(t, q, \widehat{\xi})\sigma &= h_1
\end{aligned}
$$

has only the trivial solution in

$$C_0(\mathbb{R}_+;\mathbb{R})\times\mathbb{R} := \left\{ v : [0,\infty) \longrightarrow \mathbb{R} \,\middle|\, v \text{ is continuous and } \lim_{y\to\infty} v(y) = 0 \right\} \times \mathbb{R}$$

for $h_0 = h_1 = 0$, instead checking that there is a unique solution for arbitrary h_0 and h_1. Thus let $\widehat{\xi} \in \mathbb{R}$, $\lambda \in \overline{\mathbb{C}_+}$ with $|\widehat{\xi}| + |\lambda| \neq 0$ then we get

$$\begin{aligned}
\mathcal{A}^\sharp(t,q,\widehat{\xi},-i\partial_y) &= \widehat{\xi}^2 + (-i\partial_y)^2 = \widehat{\xi}^2 - \partial_y^2 \\
\mathcal{B}_0^\sharp(t,q,\widehat{\xi},-i\partial_y) &= i\sin(\alpha(q))^2 \left(n_{\partial\Gamma^*}(q)\cdot(\widehat{\xi},-i\partial_y)^T \right) \qquad & \mathcal{B}_1^\sharp(t,q,\widehat{\xi},-i\partial_y) = 1 \\
\mathcal{C}_0^\sharp(t,q,\widehat{\xi}) &= b\sin(\alpha(q))\widehat{\xi}^2 & \mathcal{C}_1^\sharp(t,q,\widehat{\xi}) = -1
\end{aligned}$$

and the ODE to be considered is

$$\begin{aligned}
&\text{(I)} \qquad && \lambda v(y) + \widehat{\xi}^2 v(y) - v''(y) = 0 \\
&\text{(II)} \qquad && i\sin(\alpha(q))^2 \left(n_{\partial\Gamma^*}(q)\cdot(\widehat{\xi}v(0),-iv'(0))^T \right) + \lambda\sigma + b\sin(\alpha(q))\widehat{\xi}^2\sigma = 0 \\
&\text{(III)} \qquad && v(0) - \sigma = 0.
\end{aligned}$$

Equation (I) immediately shows that $v(y) = c_1 e^{\mu y} + c_2 e^{-\mu y}$ with $\mu := \sqrt{\lambda + \widehat{\xi}^2} \neq 0$. Since μ and $-\mu$ appear in v we can w.l.o.g. choose for μ the complex square root that satisfies $\Re(\mu) > 0$. There is no chance that $\Re(\mu) = 0$, because this would mean that $\lambda + \widehat{\xi}^2 \in \mathbb{R}_-$, which is not possible due to the choice of λ and $\widehat{\xi}$. Since we require $v \in C_0(\mathbb{R}_+;\mathbb{R})$ one can see that $c_1 = 0$ due to

$$|e^{\mu y}| = |e^{\Re(\mu)y + i\Im(\mu)y}| = |\underbrace{e^{\Re(\mu)y}}_{>0}|\,\underbrace{|e^{i\Im(\mu)y}|}_{=1} = e^{\Re(\mu)y} \xrightarrow[y\to\infty]{} \begin{cases} \infty & \text{if } \Re(\mu) > 0 \\ 1 & \text{if } \Re(\mu) = 0 \\ 0 & \text{if } \Re(\mu) < 0 \end{cases}.$$

So far we know $v(y) = c_2 e^{-\mu y}$ and now (III) shows $c_2 = v(0) = \sigma$. As demanded in condition (LS) we identify the positive part of the last coordinate axis with the inner normal to $\partial\Gamma^*$, i.e. $-n_{\partial\Gamma^*} \mathrel{\widehat{=}} \binom{0}{1}$. With this identification (II) reads as

$$\begin{aligned}
0 &= i\sin(\alpha(q))^2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \widehat{\xi}v(0) \\ -iv'(0) \end{pmatrix} + \lambda\sigma + b\sin(\alpha(q))\widehat{\xi}^2\sigma \\
&= -\sin(\alpha(q))^2 v'(0) + (\lambda + b\sin(\alpha(q))\widehat{\xi}^2)\sigma.
\end{aligned}$$

This shows that we have

$$(\lambda + b\sin(\alpha(q))\widehat{\xi}^2)\sigma = \sin(\alpha(q))^2 v'(0) = -\mu\sigma\sin(\alpha(q))^2 e^{-\mu 0} = -\sigma\sin(\alpha(q))^2\sqrt{\lambda + \widehat{\xi}^2},$$

which is either the case for $\sigma = 0$ or if

$$\frac{\lambda + b\sin(\alpha(q))\widehat{\xi}^2}{\sin(\alpha(q))^2\sqrt{\lambda + \widehat{\xi}^2}} = -1. \tag{3.5}$$

In the case $\sigma = 0$ we would also have $v \equiv 0$, which would be the desired condition (LS). Therefore we only have to show that (3.5) is not possible. This is trivial, because $\lambda + b\sin(\alpha(q))\widehat{\xi}^2$ has non-negative real part since $b, \sin(\alpha(q)), \widehat{\xi}^2 \in \mathbb{R}_+$ and $\lambda \in \overline{\mathbb{C}_+}$. This can be written as $\arg(\lambda + b\sin(\alpha(q))\widehat{\xi}^2) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. In addition, we know $\lambda + \widehat{\xi}^2$ has positive real part, which is equivalent to

$$\arg\left(\sin(\alpha(q))^2\sqrt{\lambda + \widehat{\xi}^2}\right) = \arg\left(\sqrt{\lambda + \widehat{\xi}^2}\right) = \arg(\mu) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right).$$

Remember that we have chosen $\Re(\mu) > 0$, which causes $\arg(\mu) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ not to be an option. Together we get the contradiction

$$\pm\pi = \arg(-1) = \arg\left(\frac{\lambda + b\sin(\alpha(q))\widehat{\xi}^2}{\sin(\alpha(q))^2\sqrt{\lambda + \widehat{\xi}^2}}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right).$$

Upon having proved all assumptions from [DPZ08], we can state our first theorem.

Theorem 3.2: *Let $3 < p < \infty$, $J := [0, T]$ and the spaces be defined as in (3.4). Then the problem*

$$\frac{d}{dt}u(t) + \mathcal{A}(t, q, D)u(t) = f(t) \qquad \text{in } J \times \Gamma^* \tag{3.6}$$

$$\frac{d}{dt}\rho(t) + \mathcal{B}_0(t, q, D)u(t) + \mathcal{C}_0(t, q, D_\partial)\rho(t) = g_0(t) \qquad \text{on } J \times \partial\Gamma^* \tag{3.7}$$

$$\mathcal{B}_1(t, q, D)u(t) + \mathcal{C}_1(t, q, D_\partial)\rho(t) = g_1(t) \qquad \text{on } J \times \partial\Gamma^* \tag{3.8}$$

$$u(0) = u_0 \qquad \text{in } \Gamma^* \tag{3.9}$$

$$\rho(0) = \rho_0 \qquad \text{on } \partial\Gamma^* \tag{3.10}$$

has a unique solution $(u, \rho) \in Z_u \times Z_\rho$ if and only if

$$f \in X, \qquad u_0 \in \pi Z_u, \qquad \rho_0 \in \pi Z_\rho, \qquad g_0 \in Y_0, \qquad g_1 \in Y_1,$$
$$g_0(0) - \mathcal{B}_0(0)u_0 - \mathcal{C}_0(0)\rho_0 \in \pi_1 Z_\rho, \qquad \mathcal{B}_1(0)u_0 + \mathcal{C}_1(0)\rho_0 = g_1(0).$$

Proof: Follows from Theorem 2.1 in [DPZ08] adapted to our specific case. ■

From this theorem we deduce a corollary that gives us the existence and uniqueness of solutions of the flow (3.1)-(3.3) on each finite interval.

Corollary 3.3: *Let $3 < p < \infty$, $J := [0, T]$ and the spaces be defined as in (3.4). Then (3.1)-(3.3) has a unique solution $\varrho \in Z_u$ with $\varrho|_{\partial\Gamma^*} \in Z_\rho$ if and only if $\varrho_0 \in \pi Z_u$ and $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho$.*

Proof: Follows from Theorem 3.2 if we choose $f \equiv 0$, $g_0 \equiv 0$ and $g_1 \equiv 0$. Then we have exactly the right-hand sides of the flow (3.1)-(3.3) and can obviously drop the conditions

$f \in X$, $g_0 \in Y_0$, $g_1 \in Y_1$, because they are trivially satisfied. Also $\mathcal{B}_1(0)u_0 + \mathcal{C}_1(0)\rho_0 = g_1(0)$ is valid since $u_0|_{\partial\Gamma^*} = \varrho_0|_{\partial\Gamma^*} = \rho_0$. Finally, the condition

$$g_0(0) - \mathcal{B}_0(0, \bullet, -i\nabla_{\Gamma^*})\varrho_0 - \mathcal{C}_0(0, \bullet, -i\partial_\sigma)\varrho_0|_{\partial\Gamma^*} \in \pi_1 Z_\rho = W_p^{1-\frac{3}{p}}(\partial\Gamma^*; \mathbb{R})$$

can be ignored since on the one hand $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho = W_p^{3-\frac{3}{p}}(\partial\Gamma^*; \mathbb{R})$ and $\mathcal{C}_0$ is a second order differential operator and on the other hand $\varrho_0 \in \pi Z_u = W_p^{2-\frac{2}{p}}(\Gamma^*; \mathbb{R})$, $\mathcal{B}_0$ is of first order and the trace operator γ_0 maps from $W_p^{1-\frac{2}{p}}(\Gamma^*; \mathbb{R})$ to $W_p^{1-\frac{3}{p}}(\partial\Gamma^*; \mathbb{R})$ for $p > 3$. ■

Now we want to move on to the more important considerations about the non-local part $\mathcal{P}$, which we ignored in (3.1)-(3.3), but has to be considered for the flow (2.45)-(2.47). The basic ingredient will be a particular pertubation result of semigroup theory and the time-independence of the operators $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{C}_0$ and $\mathcal{C}_1$.

We remark that [DPZ08] also state some results in case that all operators are time-independent. Although in the above considerations we have written $\mathcal{A}(t, q, D)$ to fully adopt the notation we see that these operators are actually all time-independent. Hence we state another useful theorem from [DPZ08] for which we have to define the operator

$$A : \mathcal{D}(A) \longrightarrow \mathcal{W}(A) : \begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{A}(q, -i\nabla_{\Gamma^*}) & \mathbb{O} \\ \mathcal{B}_0(q, -i\nabla_{\Gamma^*}) & \mathcal{C}_0(q, -i\partial_\sigma) \end{pmatrix} \begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix},$$

where the domain and codomain are

$$\begin{aligned} \mathcal{D}(A) &:= \left\{ (\varrho, \widetilde{\varrho})^T \in W_p^2(\Gamma^*; \mathbb{R}) \times W_p^{3-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}) \,\middle|\, \varrho|_{\partial\Gamma^*} = \widetilde{\varrho} \right\} \\ \mathcal{W}(A) &:= L_p(\Gamma^*; \mathbb{R}) \times W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}). \end{aligned}$$

Remark 3.4: Note that the condition $\mathcal{B}_0(\bullet, -i\nabla_{\Gamma^*})\varrho + \mathcal{C}_0(\bullet, -i\partial_\sigma)\widetilde{\varrho} \in W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})$ from the original domain in [DPZ08] is automatically satisfied since this time on the one hand we have $\widetilde{\varrho} \in W_p^{3-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})$ and $\mathcal{C}_0$ is a second order differential operator and on the other hand $\varrho \in W_p^2(\Gamma^*; \mathbb{R})$, $\mathcal{B}_0$ is of first order and the trace operator γ_0 maps from $W_p^1(\Gamma^*; \mathbb{R})$ to $W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})$. □

Definition 3.5 (Maximal L_p-regularity): We say that a closed linear operator

$$A : \mathcal{D}(A) \subseteq X_1 \longrightarrow X_0$$

has maximal L_p-regularity on the interval $J \in \{[0, T], \mathbb{R}_+\}$ if for each $f \in L_p(J; X_0)$ there is a $u \in W_p^1(J; X_0) \cap L_p(J; \mathcal{D}(A))$ that satisfies

$$\begin{aligned} \frac{d}{dt}u(t) - Au(t) &= f(t) \quad \forall\, t \in J \\ u(0) &= 0. \end{aligned}$$

□

For this new operator A we get the following statement from [DPZ08].

Theorem 3.6: *Let $3 < p < \infty$. Then the operator $-A$ defined as above generates an analytic semigroup in $\mathcal{W}(A)$, which has the property of maximal L_p-regularity on each finite interval $J = [0,T]$. Moreover, there is $\omega \geq 0$ such that $-(A + \omega \,\mathrm{Id})$ has maximal L_p-regularity on the half-line $\mathbb{R}_+$.*

Proof: Adapt Theorem 2.2 from [DPZ08] to the given situation. ■

As a byproduct theorem we additionally get the following theorem, which is an improvement of Corollary 3.3 concerning the regularity of the involved spaces.

Theorem 3.7: *Let $3 < p < \infty$ and the spaces be defined as in (3.4). Then (3.1)-(3.3) has a unique solution $\varrho \in Z_u$ with*

$$\begin{aligned}\varrho|_{\partial\Gamma^*} \in W_p^1(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \cap W_p^{\frac{1}{2}-\frac{1}{2p}}(J; W_p^2(\partial\Gamma^*;\mathbb{R}))\\ \cap W_p^{1-\frac{1}{2p}}(J; L_p(\partial\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^{3-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))\end{aligned}$$

if and only if $\varrho_0 \in \pi Z_u$ and $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho$.
The same statement is true for $J = \mathbb{R}_+$ if ∂_t is replaced by $\partial_t + \omega \,\mathrm{Id}$ for some sufficiently large $\omega > 0$.

Proof: Follows from Corollary 2.3 from [DPZ08] adapted to the given situation. ■

Remark 3.8: For the same reason as in the proof of Corollary 3.3 we were allowed to erase the three conditions

$$\begin{aligned}&\mathcal{B}_0(\bullet, -i\nabla_{\Gamma^*})\varrho + \mathcal{C}_0(\bullet, -i\partial_\sigma)\varrho|_{\partial\Gamma^*} \in L_p(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))\\ &\mathcal{B}_0(\bullet, -i\nabla_{\Gamma^*})\varrho_0 + \mathcal{C}_0(\bullet, -i\partial_\sigma)\varrho_0|_{\partial\Gamma^*} \in \pi_1 Z_\rho\\ &\mathcal{B}_1(\bullet, -i\nabla_{\Gamma^*})\varrho_0 + \mathcal{C}_1(\bullet, -i\partial_\sigma)\varrho_0|_{\partial\Gamma^*} = g_1(0)\end{aligned}$$

from the original theorem in [DPZ08]. □

Now we use a perturbation argument for generators of analytic semigroups taken from [Paz83] to treat the non-local part $\mathcal{P}$. This is the essential ingredient needed to proof the existence of solutions for the flow (2.45)-(2.47).

Lemma 3.9: *Let A be the generator of an analytic semigroup on X. Let P be a closed linear operator satisfying $\mathcal{D}(P) \supseteq \mathcal{D}(A)$ and*

$$\|Px\|_X \leq \varepsilon \|Ax\|_X + M \|x\|_X \qquad \forall\ x \in \mathcal{D}(A). \tag{3.11}$$

Then there is a $\varepsilon_0 > 0$ such that, if $0 \leq \varepsilon \leq \varepsilon_0$, then $A + P$ is the generator of an analytic semigroup.

Proof: Can be found in [Paz83] on page 80. ■

In our case the perturbation operator P reads as follows

$$P : W_p^2(\Gamma^*;\mathbb{R}) \times W_p^{3-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}) \longrightarrow \mathbb{R} \times \{0\} : \begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix} \longmapsto \begin{pmatrix} P_1 & P_2 \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix},$$

where the operators P_1 and P_2 are defined as

$$\begin{aligned} P_1(\varrho) &:= - \fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2 - H_{\Gamma^*}^2 + \overline{H}(\mathbb{O}) H_{\Gamma^*}) \varrho \, d\mathcal{H}^2 \\ P_2(\widetilde{\varrho}) &:= \frac{1}{\int_{\Gamma^*} 1 \, d\mathcal{H}^2} \int_{\partial\Gamma^*} \Big(H_{\Gamma^*} - \overline{H}(\mathbb{O}) \Big) \cot(\alpha) \widetilde{\varrho} \, d\mathcal{H}^1. \end{aligned}$$

Due to the fact that Γ^* is bounded we can embed the space $\mathbb{R}$ into $L_p(\Gamma^*;\mathbb{R})$. Therefore, we can consider P as an operator

$$P : \mathcal{D}(P) \longrightarrow L_p(\Gamma^*;\mathbb{R}) \times W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})$$

with $\mathcal{D}(P) := W_p^2(\Gamma^*;\mathbb{R}) \times W_p^{3-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}) \supseteq \mathcal{D}(A)$ as required in Lemma 3.9. The argument $\mathbb{R} \hookrightarrow L_p(\Gamma^*;\mathbb{R})$ also shows that P is a closed linear operator. Now our goal is to prove that equation (3.11) is valid with arbitrarily small ε. Hence we would see $-A + P$ is also a generator of an analytic semigroup. The necessary steps to achieve this aim will be distributed to several lemmas. For a more convenient notation we define the spaces V and W to be

$$\begin{aligned} V &:= W_p^2(\Gamma^*;\mathbb{R}) \times W_p^{3-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}) \\ W &:= L_p(\Gamma^*;\mathbb{R}) \times W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}). \end{aligned}$$

Lemma 3.10: *For all $x \in \mathcal{D}(A)$ one has the estimate*

$$\|Px\|_W \leq c \, \|x\|_V^\theta \, \|x\|_W^{1-\theta} \tag{3.12}$$

for some $\theta \in (0,1)$.

Proof: First we see

$$\begin{aligned} \|P_1 \varrho\|_{L_p(\Gamma^*)} &= \left(\int_{\Gamma^*} |P_1 \varrho|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \\ &= A(\Gamma^*)^{\frac{1}{p}-1} \left| \int_{\Gamma^*} \Delta_{\Gamma^*} \varrho + |\sigma^*|^2 \varrho - H_{\Gamma^*}^2 \varrho + \overline{H}(\mathbb{O}) H_{\Gamma^*} \varrho \, d\mathcal{H}^2 \right|. \end{aligned}$$

Due to the compactness of $\Gamma^* \cup \partial\Gamma^*$ and the smoothness of Γ^* up to the boundary we have $|H_{\Gamma^*}| \leq c_1$, $|\sigma^*|^2 \leq c_2$ and $\overline{H}(\mathbb{O}) \leq c_3$. Hence we continue with the estimate from above

$$\begin{aligned} \|P_1 \varrho\|_{L_p(\Gamma^*)} &\leq A(\Gamma^*)^{\frac{1}{p}-1} \left(\left| \int_{\Gamma^*} \Delta_{\Gamma^*} \varrho \, d\mathcal{H}^2 \right| + c \int_{\Gamma^*} |\varrho| \, d\mathcal{H}^2 \right) \\ &= A(\Gamma^*)^{\frac{1}{p}-1} \left(\left| \int_{\partial\Gamma^*} n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} \varrho \, d\mathcal{H}^1 \right| + c \, \|\varrho\|_{L_1(\Gamma^*)} \right) \\ &\leq A(\Gamma^*)^{\frac{1}{p}-1} \left(\widehat{c} \, \|\nabla_{\Gamma^*} \varrho\|_{L_1(\partial\Gamma^*)} + c \, \|\varrho\|_{L_1(\Gamma^*)} \right), \end{aligned}$$

where we used Gauss' theorem on manifolds (cf. Remark 2.30(ii) of [Dep10]) in the second line. For every finite measure space (Ω, μ) one has $L_p(\Omega, \mu) \hookrightarrow L_1(\Omega, \mu)$ - in particular for $(\Omega, \mu) = (\Gamma^*,\, d\mathcal{H}^2)$ and $(\Omega, \mu) = (\partial\Gamma^*,\, d\mathcal{H}^1)$. Therefore, we get

$$\begin{aligned}\|P_1\varrho\|_{L_p(\Gamma^*)} &\leq A(\Gamma^*)^{\frac{1}{p}-1}\left(\widehat{c}\,\|\nabla_{\Gamma^*}\varrho\|_{L_1(\partial\Gamma^*)} + c\,\|\varrho\|_{L_1(\Gamma^*)}\right)\\ &\leq \widetilde{c}\left(\|\nabla_{\Gamma^*}\varrho\|_{L_p(\partial\Gamma^*)} + \|\varrho\|_{L_p(\Gamma^*)}\right).\end{aligned}$$

For every $\varepsilon > 0$ one has $W_p^\varepsilon(\Omega) \hookrightarrow L_p(\Omega)$ and thus we obtain

$$\|P_1\varrho\|_{L_p(\Gamma^*)} \leq \widetilde{c}\left(\|\nabla_{\Gamma^*}\varrho\|_{L_p(\partial\Gamma^*)} + \|\varrho\|_{L_p(\Gamma^*)}\right) \leq \widehat{c}\left(\|\nabla_{\Gamma^*}\varrho\|_{W_p^\varepsilon(\partial\Gamma^*)} + \|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)}\right).$$

Furthermore, the trace operator γ_0 is linear and bounded from $W_p^s(\Omega)$ to $W_p^{s-\frac{1}{p}}(\partial\Omega)$ for every $s > \frac{1}{p}$ and we have

$$\begin{aligned}\|P_1\varrho\|_{L_p(\Gamma^*)} &\leq \widehat{c}\left(\|\nabla_{\Gamma^*}\varrho\|_{W_p^\varepsilon(\partial\Gamma^*)} + \|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)}\right)\\ &\leq c'\left(\|\nabla_{\Gamma^*}\varrho\|_{W_p^{\frac{1}{p}+\varepsilon}(\Gamma^*)} + \|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)}\right)\\ &\leq c\,\|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)}\end{aligned}\tag{3.13}$$

Analogously for the operator P_2 we get the following

$$\begin{aligned}\|P_2\widetilde{\varrho}\|_{L_p(\Gamma^*)} &= \left(\int_{\Gamma^*}|P_2\widetilde{\varrho}|^p\; d\mathcal{H}^2\right)^{\frac{1}{p}} = A(\Gamma^*)^{\frac{1}{p}-1}\left|\int_{\partial\Gamma^*}\left(H_{\Gamma^*} - \overline{H}(\mathbb{O})\right)\cot(\alpha)\widetilde{\varrho}\,d\mathcal{H}^1\right|\\ &\leq c\left(\left|\int_{\partial\Gamma^*} H_{\Gamma^*}\cot(\alpha)\widetilde{\varrho}\,d\mathcal{H}^1\right| + \left|\int_{\partial\Gamma^*}\overline{H}(\mathbb{O})\cot(\alpha)\widetilde{\varrho}\,d\mathcal{H}^1\right|\right).\end{aligned}$$

Due to the compactness of $\partial\Gamma^*$ we see $|H_{\Gamma^*}| \leq c_1$, $\overline{H}(\mathbb{O}) \leq c_3$ and with assumption (2.1) we arrive at $|\cot(\alpha)| \leq c_4$. Hence we continue with

$$\|P_2\widetilde{\varrho}\|_{L_p(\Gamma^*)} \leq c\left(\left|\int_{\partial\Gamma^*} H_{\Gamma^*}\cot(\alpha)\widetilde{\varrho}\,d\mathcal{H}^1\right| + \left|\int_{\partial\Gamma^*}\overline{H}(\mathbb{O})\cot(\alpha)\widetilde{\varrho}\,d\mathcal{H}^1\right|\right) \leq \widehat{c}\,\|\widetilde{\varrho}\|_{L_1(\partial\Gamma^*)}.$$

For the same reasons mentioned above we get

$$\|P_2\widetilde{\varrho}\|_{L_p(\Gamma^*)} \leq \widehat{c}\,\|\widetilde{\varrho}\|_{L_1(\partial\Gamma^*)} \leq \widetilde{c}\,\|\widetilde{\varrho}\|_{L_p(\partial\Gamma^*)} \leq c'\,\|\widetilde{\varrho}\|_{W_p^{1+\varepsilon}(\partial\Gamma^*)}.$$

Since we only consider $x = \binom{\varrho}{\widetilde{\varrho}} \in \mathcal{D}(A)$ we know $\widetilde{\varrho} = \gamma_0\varrho$ and hence we gain

$$\|P_2\widetilde{\varrho}\|_{L_p(\Gamma^*)} \leq c'\,\|\widetilde{\varrho}\|_{W_p^{1+\varepsilon}(\partial\Gamma^*)} = c'\,\|\gamma_0\varrho\|_{W_p^{1+\varepsilon}(\partial\Gamma^*)} \leq c\,\|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)}.\tag{3.14}$$

Finally, we are in the position to have a look at the operator in focus. For P we obtain with (3.13) and (3.14) the following estimate

$$\begin{aligned}\|Px\|_W &\leq \|P_1\varrho + P_2\widetilde{\varrho}\|_{L_p(\Gamma^*)} + \|\mathbb{O}\varrho + \mathbb{O}\widetilde{\varrho}\|_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)}\\ &\leq \|P_1\varrho\|_{L_p(\Gamma^*)} + \|P_2\widetilde{\varrho}\|_{L_p(\Gamma^*)} \leq \widetilde{c}\,\|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)}.\end{aligned}$$

Example 2.12 from [Lun09] shows that $W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)$ is an interpolation space of exponent $\theta = \frac{1}{2}(1+\frac{1}{p}+\varepsilon) \in (0,1)$ with respect to $\left(L_p(\Gamma^*), W_p^2(\Gamma^*)\right)$, where we assume w.l.o.g. $\varepsilon < 1 - \frac{1}{p}$. This leads to

$$\begin{aligned}
\|Px\|_W &\leq \widetilde{c}\, \|\varrho\|_{W_p^{1+\frac{1}{p}+\varepsilon}(\Gamma^*)} \leq c\, \|\varrho\|^{\theta}_{W_p^2(\Gamma^*)} \|\varrho\|^{1-\theta}_{L_p(\Gamma^*)} \\
&\leq c \left(\|\varrho\|_{W_p^2(\Gamma^*)} + \|\widetilde{\varrho}\|_{W_p^{3-\frac{1}{p}}(\partial\Gamma^*)} \right)^{\theta} \left(\|\varrho\|_{L_p(\Gamma^*)} + \|\widetilde{\varrho}\|_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} \right)^{1-\theta} \\
&= c\, \|x\|_V^{\theta}\, \|x\|_W^{1-\theta}
\end{aligned}$$

and shows the desired result. ■

Lemma 3.11: *For all $g \in W$ there is some $\omega \geq 0$ such that for all $\lambda_0 > \omega$ the equation $\lambda_0 x + (A + 2\omega I)x = g$ has a solution $x \in \mathcal{D}(A)$. Moreover, we have $\|x\|_V \leq c(\lambda_0)\, \|g\|_W$.*

Proof: The maximal regularity stated in Theorem 3.6 shows there is some $\omega \geq 0$ such that for all $h \in L_p(\mathbb{R}_+; W)$ the equation

$$\begin{aligned}
u'(t) + (A + \omega I)u(t) &= h(t) && \text{in } \mathbb{R}_+ \times W \\
u(0) &= 0 && \text{in } W
\end{aligned} \tag{3.15}$$

has a unique solution $u \in W_p^1(\mathbb{R}_+; W) \cap L_p(\mathbb{R}_+; \mathcal{D}(A+\omega I))$. Let $g \in W$ and $\lambda_0 > \omega$ be arbitrary. We define

$$h(t) := 2(\lambda_0 - \omega)e^{-(\lambda_0-\omega)t} g \in L_p(\mathbb{R}_+; W)$$

and denote the unique solution of equation (3.15) by $u \in W_p^1(\mathbb{R}_+; W) \cap L_p(\mathbb{R}_+; \mathcal{D}(A))$, in which we used $\mathcal{D}(A + \omega I) = \mathcal{D}(A)$. Now we can define

$$x := \int_0^\infty e^{-(\lambda_0-\omega)t} u(t)dt \in \mathcal{D}(A).$$

As a result we see

$$\begin{aligned}
(\lambda_0 + (A + \omega I))x &= \int_0^\infty (\lambda_0 - \omega)e^{-(\lambda_0-\omega)t}u(t) + Ae^{-(\lambda_0-\omega)t}u(t)dt \\
&= \int_0^\infty -\left(\frac{d}{dt}e^{-(\lambda_0-\omega)t}\right) u(t) + e^{-(\lambda_0-\omega)t}Au(t)dt \\
&= -\left[e^{-(\lambda_0-\omega)t}u(t)\right]_0^\infty + \int_0^\infty e^{-(\lambda_0-\omega)t}\left(u'(t) + Au(t)\right)dt \\
&= 0 + \int_0^\infty e^{-(\lambda_0-\omega)t}\left(h(t) - \omega u(t)\right)dt \\
&= \int_0^\infty e^{-(\lambda_0-\omega)t}h(t)dt - \omega x \\
&= 2(\lambda_0 - \omega)\int_0^\infty e^{-2(\lambda_0-\omega)t}dt\ g - \omega x \\
&= \left[-e^{-2(\lambda_0-\omega)t}\right]_0^\infty g - \omega x = (-0+1)g - \omega x = g - \omega \operatorname{Id} x.
\end{aligned}$$

Therefore we know that $x \in \mathcal{D}(A)$ solves $(\lambda_0 + (A + 2\omega I))x = g$. For the norm estimate we have

$$\begin{aligned}\|x\|_V &\leq \int_0^\infty \underbrace{e^{-(\lambda_0-\omega)t}}_{\in L_{p'}(\mathbb{R}_+)} \underbrace{\|u(t)\|_V}_{\in L_p(\mathbb{R}_+)} dt \leq \left(\int_0^\infty \left(e^{-(\lambda_0-\omega)t}\right)^{p'} dt\right)^{\frac{1}{p'}} \left(\int_0^\infty \|u(t)\|_V^p \, dt\right)^{\frac{1}{p}} \\ &\leq \widehat{c} \, \|u\|_{L_p(\mathbb{R}_+;\mathcal{D}(A))} \leq \widetilde{c} \, \|h\|_{L_p(\mathbb{R}_+;W)} = 2(\lambda_0 - \omega)\widetilde{c} \left(\int_0^\infty \left(e^{-(\lambda_0-\omega)t}\right)^p dt\right)^{\frac{1}{p}} \|g\|_W \\ &= c \, \|g\|_W \, ,\end{aligned}$$

where the fourth estimate follows due to the maximal regularity of (3.15) and the closed graph theorem. This proves the desired statement. ■

Lemma 3.12: *For all $x \in \mathcal{D}(A)$ one has the estimate*

$$\|Px\|_W \leq c\left(\|Ax\|_W^\theta \, \|x\|_W^{1-\theta} + \|x\|_W\right) \tag{3.16}$$

for some $\theta \in (0, 1)$.

Proof: If we assign $\mu_0 := \lambda_0 + 2\omega \in \mathbb{R} \subseteq \mathbb{C}$ we get the solvability of $\mu_0 x + Ax = f$ in $\mathcal{D}(A)$ for every $f \in W$ and $\|x\|_V \leq c \, \|f\|_W$ from Lemma 3.11. This can be used to show

$$\|x\|_V \leq c \, \|f\|_W = c \, \|\mu_0 x + Ax\|_W \leq c|\mu_0| \, \|x\|_W + c \, \|Ax\|_W \leq \widetilde{c} \, (\|x\|_W + \|Ax\|_W) \, .$$

Using Lemma 3.10, we finally arrive at

$$\begin{aligned}\|Px\|_W &\leq c \, \|x\|_V^\theta \, \|x\|_W^{1-\theta} \leq c\widetilde{c}^\theta \, (\|x\|_W + \|Ax\|_W)^\theta \, \|x\|_W^{1-\theta} \\ &\leq \widehat{c}\left(\|x\|_W^\theta + \|Ax\|_W^\theta\right) \|x\|_W^{1-\theta} = \widehat{c} \, \|x\|_W + \widehat{c} \, \|Ax\|_W^\theta \, \|x\|_W^{1-\theta} \, ,\end{aligned}$$

where we used $(a + b)^\theta \leq (a^\theta + b^\theta)$. ■

Theorem 3.13: *Let $3 < p < \infty$. Then the operator $-A + P$ generates an analytic semigroup in $\mathcal{W}(A)$.*

Proof: We will use Lemma 3.9. Because of Theorem 3.6 we know that $-A$ generates an analytic semigroup. As stated immediately after the definition of P, the assumptions "$\mathcal{D}(P) \supseteq \mathcal{D}(A)$" and "$P$ closed" are satisfied and therefore only (3.11) remains to be proven. For $\theta \in (0, 1)$ as in Lemma 3.12 we define $p' := \frac{1}{\theta}$ and $q' := \frac{1}{1-\theta}$, which gives $1 < p', q' < \infty$ and $\frac{1}{p'} + \frac{1}{q'} = \theta + 1 - \theta = 1$. Young's inequality with epsilon leads to

$$\begin{aligned}\|Px\|_W &\leq c \, \|Ax\|_W^\theta \, \|x\|_W^{1-\theta} + c \, \|x\|_W \\ &\leq c\varepsilon \left(\|Ax\|_W^\theta\right)^{\frac{1}{\theta}} + c\left(\frac{\theta}{\varepsilon}\right)^{q-1} (1-\theta) \left(\|x\|_W^{1-\theta}\right)^{\frac{1}{1-\theta}} + c \, \|x\|_W \\ &= c\varepsilon \, \|Ax\|_W + M(\theta, \varepsilon) \, \|x\|_W\end{aligned}$$

in which we used Lemma 3.12 in the first inequality. Since $\varepsilon > 0$ can be chosen arbitrarily small, we get the desired statement (3.11) of Lemma 3.9. ■

Theorem 3.14: *Let $3 < p < \infty$. Then the flow (2.45)-(2.47) has a unique solution*

$$\varrho \in W_p^1([0,T]; L_p(\Gamma^*; \mathbb{R})) \cap L_p([0,T]; W_p^2(\Gamma^*; \mathbb{R}))$$

with boundary regularity

$$\varrho|_{\partial\Gamma^*} \in W_p^1([0,T]; W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})) \cap L_p([0,T]; W_p^{3-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}))$$

if $\varrho_0 \in W_p^2(\Gamma^*; \mathbb{R})$ and $\varrho_0|_{\partial\Gamma^*} \in W_p^{3-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})$.

Proof: Because of Theorem 3.13 we know that $-A + P$ generates an analytic semigroup. Assuming $x_0 \in \mathcal{D}(-A+P) = \mathcal{D}(A)$ we obtain due to Theorem 12.16 in [RR04] that the mild solution of the abstract Cauchy-problem

$$x'(t) - (-A+P)x(t) = 0 \tag{3.17}$$
$$x(0) = x_0, \tag{3.18}$$

which is given by $x(t) := e^{(-A+P)t}x_0$ is already a classical solution, i.e.

$$x \in C^1([0,T]; W) \cap C^0([0,T]; \mathcal{D}(A)).$$

Since

$$C^1([0,T]; W) \hookrightarrow W_p^1([0,T]; W) \quad \text{and} \quad C^0([0,T]; \mathcal{D}(A)) \hookrightarrow L_p([0,T]; \mathcal{D}(A))$$

we obtain the existence of a solution $x \in W_p^1([0,T]; W) \cap L_p([0,T]; \mathcal{D}(A))$. Conversely, by the same arguments as in the proof of Theorem 12.14 of [RR04] every strong solution is also a mild solution and hence we get uniqueness. As a solution

$$x \in W_p^1([0,T]; W) \cap L_p([0,T]; \mathcal{D}(A))$$

of (3.17)-(3.18) corresponds to a solution

$$\begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix} \in W_p^1([0,T]; W) \cap L_p([0,T]; \mathcal{D}(A))$$

of the problem

$$\begin{aligned} \frac{d}{dt}\varrho(t) + \mathcal{A}(q,D)\varrho(t) - \mathcal{P}\varrho(t) &= 0 && \text{in } [0,T] \times \Gamma^* \\ \frac{d}{dt}\widetilde{\varrho}(t) + \mathcal{B}_0(q,D)\varrho(t) + \mathcal{C}_0(q,D_\partial)\widetilde{\varrho}(t) &= 0 && \text{on } [0,T] \times \partial\Gamma^* \\ \mathcal{B}_1(q,D)\varrho(t) + \mathcal{C}_1(q,D_\partial)\widetilde{\varrho}(t) &= 0 && \text{on } [0,T] \times \partial\Gamma^* \\ \varrho(0) &= \varrho_0 && \text{in } \Gamma^* \\ \widetilde{\varrho}(0) &= \widetilde{\varrho}_0 && \text{on } \partial\Gamma^* \end{aligned}$$

the claim follows, because we can erase the third line if we replace $\widetilde{\varrho}$ by $\varrho|_{\partial\Gamma^*}$ due to $\begin{pmatrix} \varrho(t) \\ \widetilde{\varrho}(t) \end{pmatrix} \in \mathcal{D}(A)$ for all $t \in [0,T]$. Furthermore, we can delete the last condition, as we have imposed the condition $x_0 = \begin{pmatrix} \varrho_0 \\ \widetilde{\varrho}_0 \end{pmatrix} \in \mathcal{D}(A)$. ∎

3.2 Short-time existence of solutions for the volume-preserving Mean Curvature Flow

Now we want to prove the short-time existence of solutions of the non-linear flow

$$
\begin{aligned}
V_\Gamma(u(t)) &= H_\Gamma(u(t)) - \overline{H}(u(t)) && \text{in } [0,T]\times\Gamma^* && (3.19)\\
v_{\partial D}(\rho(t)) &= a + b\varkappa_{\partial D}(\rho(t)) + \langle n_\Gamma(u(t)), n_D(u(t))\rangle && \text{on } [0,T]\times\partial\Gamma^* && (3.20)\\
u(t) &= \rho(t) && \text{on } [0,T]\times\partial\Gamma^* && (3.21)\\
u(0) &= u_0 && \text{in } \Gamma^* && (3.22)\\
\rho(0) &= \rho_0 && \text{on } \partial\Gamma^*, && (3.23)
\end{aligned}
$$

where we have changed the notation to a more convenient one and have adopted the structure of the linearized PDE in Theorem 3.2. Inspired by [DGHSS10] we will use the contraction mapping principle to prove the desired short-time existence. We define the functions $\Phi := (u,\rho)$ and $\Phi_0 := (u_0,\rho_0)$, the spaces

$$
\begin{aligned}
\mathbb{E} &:= Z_u \times Z_\rho\\
\mathbb{F} &:= X \times Y_0 \times \{0\}\\
\mathbb{I} &:= \{(u_0,\rho_0)\in \pi Z_u \times \pi Z_\rho \mid u_0|_{\partial\Gamma^*} = \rho_0\}
\end{aligned}
$$

with their norms

$$
\begin{aligned}
\|\Phi\|_{\mathbb{E}} &:= \|u\|_{Z_u} + \|\rho\|_{Z_\rho}\\
\|f\|_{\mathbb{F}} &:= \left\|f^{(1)}\right\|_X + \left\|f^{(2)}\right\|_{Y_0}\\
\|\Phi_0\|_{\mathbb{I}} &:= \|u_0\|_{\pi Z_u} + \|\rho_0\|_{\pi Z_\rho}
\end{aligned}
$$

and the operator $L : \mathbb{E} \longrightarrow \mathbb{F}\times\mathbb{I}$ as the left-hand side of (3.6)-(3.10). Additionally, for the right hand side of the contraction mapping principle we define the non-linear operator $N : \mathbb{E} \longrightarrow \mathbb{F}$ as

$$
N(\Phi) := \begin{pmatrix} H_\Gamma(u) - \overline{H}(u) - V_\Gamma(u) + \frac{d}{dt}u + \mathcal{A}(q,D)u \\ a + b\varkappa_{\partial D}(\rho) + \langle n_\Gamma(u), n_D(u)\rangle - v_{\partial D}(\rho) + \frac{d}{dt}\rho + \mathcal{B}_0(q,D)u + \mathcal{C}_0(q,D_\partial)\rho \\ 0 \end{pmatrix}.
$$

In order to solve the equation $L\Phi = (N\Phi, \Phi_0)$ by the contraction mapping principle we need the following technical lemmas.

Lemma 3.15: *(i) Let $1 < p < \infty$ and $s,\alpha,\beta \geq 0$ and $r \geq 0$. Then for each $\sigma\in[0,1]$ one has*

$$
W_p^{s+\alpha}(J; W_p^r(\Omega;\mathbb{R})) \cap W_p^s(J; W_p^{r+\beta}(\Omega;\mathbb{R})) \hookrightarrow W_p^{s+\sigma\alpha}(J; W_p^{r+(1-\sigma)\beta}(\Omega;\mathbb{R})),
$$

where $\Omega \in \{\Gamma^*, \partial\Gamma^*\}$. Especially we get for $p > 3$

$$
\begin{aligned}
Z_u &\hookrightarrow W_p^{\sigma}(J; W_p^{2(1-\sigma)}(\Gamma^*;\mathbb{R}))\\
Z_\rho &\hookrightarrow W_p^{\sigma(\frac{3}{2}-\frac{1}{2p})}(J; W_p^{(1-\sigma)(3-\frac{1}{p})}(\partial\Gamma^*;\mathbb{R}))\\
Y_0 &\hookrightarrow W_p^{\sigma(\frac{1}{2}-\frac{1}{2p})}(J; W_p^{(1-\sigma)(1-\frac{1}{p})}(\partial\Gamma^*;\mathbb{R})).
\end{aligned}
$$

(ii) Let $1 < p < \infty$ and $s_1, s_2, r_1, r_2 \in \mathbb{R}$. Let $k \in \mathbb{N}_0$ be such that

$$\max\left\{0, k-1+\frac{1}{p}\right\} \leq s_1 < k + \frac{1}{p} < s_2 < k+1+\frac{1}{p}.$$

Then we have the following embedding

$$W_p^{s_2}(J; W_p^{r_2}(\Omega;\mathbb{R})) \cap W_p^{s_1}(J; W_p^{r_1}(\Omega;\mathbb{R})) \hookrightarrow BUC^k(J; W_p^{r_1-\gamma\left(k+\frac{1}{p}-s_1\right)}(\Omega;\mathbb{R})),$$

where $\gamma := \frac{r_1-r_2}{s_2-s_1}$ and $\Omega \in \{\Gamma^*, \partial\Gamma^*\}$. Especially we get for $p > 4$

$$Z_u \hookrightarrow BUC(J; W_p^{2-\frac{2}{p}}(\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; BUC^1(\Gamma^*;\mathbb{R})). \tag{3.24}$$

Proof: First we note that it is not necessary to distinguish between $W_p^s(J; W_p^r(\Gamma^*;\mathbb{R}))$ and $W_p^s(J; W_p^r(K;\mathbb{R}))$ for some open $K \subseteq \mathbb{R}^2$ with compact $\overline{K}$. Since we have assumed Γ^* to be smooth up to the boundary $\partial\Omega$ we have local C^∞-diffeomorphisms between subsets of Γ^* and K. To simplify the notation we assume to have one global C^∞-diffeomorphism Ψ between $\overline{K}$ and $\Gamma^* \cup \partial\Gamma^*$ and can define $\Phi := \Psi^{-1}$. Then for any $r \geq 0$ the pullback

$$\Phi^* : W_p^r(K) \longrightarrow W_p^r(\Gamma^*) : f \longmapsto \Phi^*(f) := f \circ \Phi$$

is an isomorphism between $W_p^r(K)$ and $W_p^r(\Gamma^*)$. For all $r = n \in \mathbb{N}$ this can be seen via induction with respect to n. The case $n = 0$ follows from

$$\begin{aligned}
\|\Phi^*(f)\|_{L_p(\Gamma^*)}^p &= \|f\circ\Phi\|_{L_p(\Gamma^*)}^p = \int_{\Gamma^*} |(f\circ\Phi)(q)|^p \, d\mathcal{H}^2 \\
&= \int_K |f(x)|^p \, \|\partial_{x_1}\Psi \times \partial_{x_2}\Psi\| \, dx \\
&\leq \underbrace{\left(\max_{x\in K} \|\partial_{x_1}\Psi \times \partial_{x_2}\Psi\|\right)}_{\leq\, c \quad (*)} \int_K |f(x)|^p dx \leq c\, \|f\|_{L_p(K)}^p \,,
\end{aligned}$$

where in $(*)$ we used $\Psi \in C^\infty(\overline{K}) \subseteq BUC^1(\overline{K})$ since $\overline{K}$ is compact. The converse estimate follows by exchanging to roles of Φ and Ψ. Due to a cumbersome notation in the case $n \geq 2$ we only show the case $n = 1$ using the statement for $n = 0$. The general case, which is proving the claim for $n = k+1$ using the case $n = k$, follows exactly in the same manner. For $n = 1$ we get

$$\begin{aligned}
\|\Phi^*(f)\|_{W_p^1(\Gamma^*)}^p &= \|\Phi^*(f)\|_{L_p(\Gamma^*)}^p + \sum_{|\alpha|=1} \|D^\alpha(f\circ\Phi)\|_{L_p(\Gamma^*)}^p \\
&\leq c\,\|f\|_{L_p(K)}^p + \sum_{|\alpha|=1} \|((\nabla f)\circ\Phi)\cdot D^\alpha\Phi\|_{L_p(\Gamma^*)}^p \\
&\leq c\,\|f\|_{L_p(K)}^p + \sum_{|\alpha|=1} \|(\nabla f)\circ\Phi\|_{L_p(\Gamma^*)}^p \underbrace{\|D^\alpha\Phi\|_{L_\infty(\Gamma^*)}^p}_{\leq\, \widehat{c}^p} \\
&= c\,\|f\|_{L_p(K)}^p + 2\widehat{c}^p \,\|(\nabla f)\circ\Phi\|_{L_p(\Gamma^*)}^p \\
&= c\,\|f\|_{L_p(K)}^p + 2\widehat{c}^p \,\|\Phi^*(\nabla f)\|_{L_p(\Gamma^*)}^p \\
&\leq c\,\|f\|_{L_p(K)}^p + \widetilde{c}\,\|\nabla f\|_{L_p(K)}^p \leq c\,\|f\|_{W_p^1(K)}^p \,,
\end{aligned}$$

where we used $\Phi \in C^\infty(\Gamma^* \cup \partial\Gamma^*) \subseteq BUC^1(\Gamma^* \cup \partial\Gamma^*)$, $|x \cdot y| \leq \|x\|\,\|y\|$ and Hölder's inequality in the third step. Now in the case $r \geq 0$ we can again prove the statement for $r = n + s$ with $n \in \mathbb{N}$ and $s \in (0,1)$ in the same manner as the case $r = 0 + s$ with $s \in (0,1)$ - only the notation becomes more involved. The case $r = s \in (0,1)$ can be seen as follows

$$
\begin{aligned}
\|\Phi^*(f)\|^p_{W^s_p(\Gamma^*)} &= c\left(\|\Phi^*(f)\|^p_{L_p(\Gamma^*)} + \int_{\Gamma^*}\int_{\Gamma^*} \frac{|(f\circ\Phi)(q) - (f\circ\Phi)(\widetilde{q})|^p}{\|q-\widetilde{q}\|^{sp+2}}\, d\mathcal{H}^2\, d\mathcal{H}^2\right) \\
&\leq \widetilde{c}\,\|f\|^p_{L_p(K)} \\
&\quad + c\int_{\Gamma^*}\int_{\Gamma^*} \frac{|f(\Phi(q)) - f(\Phi(\widetilde{q}))|^p}{\|\Phi(q)-\Phi(\widetilde{q})\|^{sp+2}} \underbrace{\left(\frac{\|\Phi(q)-\Phi(\widetilde{q})\|}{\|q-\widetilde{q}\|}\right)^{sp+2}}_{\leq\,\widehat{c}} d\mathcal{H}^2\, d\mathcal{H}^2 \\
&\leq c\left(\|f\|^p_{L_p(K)} + \int_{\Gamma^*}\int_{\Gamma^*} \frac{|f(\Phi(q)) - f(\Phi(\widetilde{q}))|^p}{\|\Phi(q)-\Phi(\widetilde{q})\|^{sp+2}}\, d\mathcal{H}^2\, d\mathcal{H}^2\right) \\
&\leq c\Big(\|f\|^p_{L_p(K)} \\
&\quad + \int_K\int_K \frac{|f(x)-f(\widetilde{x})|^p}{\|x-\widetilde{x}\|^{sp+2}} \underbrace{\|(\partial_{x_1}\Psi \times \partial_{x_2}\Psi)(x)\|\,\|(\partial_{x_1}\Psi \times \partial_{x_2}\Psi)(\widetilde{x})\|}_{\leq\,\widehat{c}\text{ as above}}\, dx\, d\widetilde{x}\Big) \\
&\leq \widetilde{c}\left(\|f\|^p_{L_p(K)} + \int_K\int_K \frac{|f(x)-f(\widetilde{x})|^p}{\|x-\widetilde{x}\|^{sp+2}} dx\, d\widetilde{x}\right) = c\,\|f\|^p_{W^s_p(K)}\,,
\end{aligned}
$$

where we used $\Phi \in C^\infty(\Gamma^* \cup \partial\Gamma^*) \subseteq BUC^1(\Gamma^* \cup \partial\Gamma^*)$ in the first and third inequality. Exchanging to roles of Φ and Ψ yields the converse inequalities. Due to the bijectivity of Φ^* we have the desired isomorphism.

(i) The main statement can be found as Lemma 4.3 in [DSS08]. The three additional embeddings are a direct consequence of the general embedding with $s = 0$, $\alpha = 1$, $r = 0$ and $\beta = 2$ or $s = 0$, $\alpha = \frac{3}{2} - \frac{1}{2p}$, $r = 0$ and $\beta = 3 - \frac{1}{p}$ or $s = 0$, $\alpha = \frac{1}{2} - \frac{1}{2p}$, $r = 0$ and $\beta = 1 - \frac{1}{p}$, respectively.

(ii) The main statement can be found as Lemma 4.4 in [DSS08]. The first part of the first additional embedding is obtained by the general embedding with $k = 0$, $s_1 = 0$, $s_2 = 1$, $r_1 = 2$ and $r_2 = 0$, whereas the second part follows from the usual Sobolev embeddings and our assumption $p > 4$. ■

Remark 3.16: The embedding (3.24) is only valid for $p > 4$ and will be crucial in the considerations to follow. This is the reason why we are forced to restrict the range of p from $p > 3$ in Theorem 3.2 and 3.14 to $p > 4$ in our final Theorem 3.22. □

Later in our most important technical lemma we will deal with quasi-linear differential operators. For this purpose it is helpful to know that the spaces containing the second highest derivatives are Banach algebras.

Since due to Lemma 3.15(i) with $\sigma = \frac{1}{2}$, we get $Z_u \hookrightarrow W_p^{\frac{1}{2}}(J; W^1_p(\Gamma^*;\mathbb{R}))$ and see

$$\nabla_{\Gamma^*} u \in W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W^1_p(\Gamma^*;\mathbb{R})).$$

Again because of Lemma 3.15(i) with $\sigma = \frac{2p-1}{3p-1}$, we get $Z_\rho \hookrightarrow W_p^{1-\frac{1}{2p}}(J; W_p^1(\partial\Gamma^*;\mathbb{R}))$ and arrive at

$$\partial_\sigma \rho \in W_p^{1-\frac{1}{2p}}(J; L_p(\partial\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})).$$

Therefore,

$$\begin{aligned}
\nabla^1 Z_u &:= W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R})) \\
\nabla^1 Z_\rho &:= W_p^{1-\frac{1}{2p}}(J; L_p(\partial\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))
\end{aligned}$$

are the spaces containing the first spacial derivatives of u and the first arc-length derivatives of ρ, respectively.

Lemma 3.17: *Let $4 < p < \infty$. Then each of the spaces*

$$\begin{aligned}
&\textit{(i)} && \nabla^1 Z_u = W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R})) \\
&\textit{(ii)} && \nabla^1 Z_\rho = W_p^{1-\frac{1}{2p}}(J; L_p(\partial\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))
\end{aligned}$$

is a Banach algebra up to a constant in the norm estimate of the product.

Proof: (i) First we use Lemma 3.15(ii) to obtain

$$\nabla^1 Z_u \hookrightarrow BUC(J; W_p^{1-\frac{2}{p}}(\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; BUC(\Gamma^*;\mathbb{R})),$$

where we have used $p > 4$ in the second embedding.
Now let $f, g \in \nabla^1 Z_u$ and define $(\delta_h f)(t) := f(t+h) - f(t)$ to obtain

$$\|fg\|_{W_p^s} \le c\,\|f\|_{L_\infty} \|g\|_{W_p^s} + \|g\|_{L_\infty} \|f\|_{W_p^s}$$

by $(\delta_h(fg))(t) = (\delta_h f)(t) g(t+h) + f(t)(\delta_h g)(t)$ and straight forward estimates. This fact will help us to show

$$\begin{aligned}
\|fg\|^p_{W_p^{\frac{1}{2}}(J;L_p(\Gamma^*))} &\le \|fg\|^p_{L_p(J;L_p(\Gamma^*))} + \int_0^T \int_0^{T-h} \frac{\|(\delta_h(fg))(t)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \\
&\le \|f\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \|g\|^p_{L_p(J;L_p(\Gamma^*))} \\
&+ \int_0^T \int_0^{T-h} \frac{\|(\delta_h f)(t) g(t+h)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \\
&+ \int_0^T \int_0^{T-h} \frac{\|f(t)(\delta_h g)(t)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \\
&\le c\,\|f\|^p_{\nabla^1 Z_u} \|g\|^p_{L_p(J;L_p(\Gamma^*))} \\
&+ \|f\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \int_0^T \int_0^{T-h} \frac{\|(\delta_h g)(t)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \\
&+ c\,\|f\|^p_{L_p(J;L_p(\Gamma^*))} \|g\|^p_{\nabla^1 Z_u}
\end{aligned}$$

$$
\begin{aligned}
&+ \|g\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \int_0^T \int_0^{T-h} \frac{\|(\delta_h f)(t)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \\
&\leq \widetilde{c} \|f\|^p_{\nabla^1 Z_u} \left(\|g\|^p_{L_p(J;L_p(\Gamma^*))} + \int_0^T \int_0^{T-h} \frac{\|(\delta_h g)(t)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \right) \\
&+ \widetilde{c} \|g\|^p_{\nabla^1 Z_u} \left(\|f\|^p_{L_p(J;L_p(\Gamma^*))} + \int_0^T \int_0^{T-h} \frac{\|(\delta_h f)(t)\|^p_{L_p(\Gamma^*)}}{|h|^{1+\frac{p}{2}}} dtdh \right) \\
&= \widehat{c} \left(\|f\|^p_{\nabla^1 Z_u} \|g\|^p_{W_p^{\frac{1}{2}}(J;L_p(\Gamma^*))} + \|f\|^p_{W_p^{\frac{1}{2}}(J;L_p(\Gamma^*))} \|g\|^p_{\nabla^1 Z_u} \right) \\
&\leq c \|f\|^p_{\nabla^1 Z_u} \|g\|^p_{\nabla^1 Z_u}
\end{aligned}
$$

and in addition

$$
\begin{aligned}
\|fg\|^p_{L_p(J;W_p^1(\Gamma^*))} &= \int_0^T \|(fg)(t)\|^p_{W_p^1(\Gamma^*)} dt \leq \int_0^T \|f(t)g(t)\|^p_{L_p(\Gamma^*)} + \|(\nabla_{\Gamma^*} fg)(t)\|^p_{L_p(\Gamma^*)} dt \\
&\leq \int_0^T \|f\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \|g(t)\|^p_{L_p(\Gamma^*)} dt \\
&+ \int_0^T \|\nabla_{\Gamma^*} f(t) g(t)\|^p_{L_p(\Gamma^*)} + \|f(t) \nabla_{\Gamma^*} g(t)\|^p_{L_p(\Gamma^*)} dt \\
&\leq \|f\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \int_0^T \|g(t)\|^p_{L_p(\Gamma^*)} + \|\nabla_{\Gamma^*} g(t)\|^p_{L_p(\Gamma^*)} dt \\
&+ \|g\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \int_0^T \|f(t)\|^p_{L_p(\Gamma^*)} + \|\nabla_{\Gamma^*} f(t)\|^p_{L_p(\Gamma^*)} dt \\
&\leq c \|f\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \|g\|^p_{L_p(J;W_p^1(\Gamma^*))} + c \|g\|^p_{L_\infty(J;L_\infty(\Gamma^*))} \|f\|^p_{L_p(J;W_p^1(\Gamma^*))} \\
&\leq \widetilde{c} \|f\|^p_{\nabla^1 Z_u} \|g\|^p_{\nabla^1 Z_u} .
\end{aligned}
$$

Combining both estimates one can see

$$
\|fg\|_{\nabla^1 Z_u} \leq \widehat{c} \|f\|_{\nabla^1 Z_u} \|g\|_{\nabla^1 Z_u} ,
$$

which proves the first claim.
(ii) Consider Lemma 3.15(ii) to obtain

$$
\nabla^1 Z_\rho \hookrightarrow BUC(J; W_p^{2-\frac{3}{p}}(\partial\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; BUC^1(\partial\Gamma^*;\mathbb{R})),
$$

where we used $p > 4$ in the second embedding. Now let $f, g \in \nabla^1 Z_\rho$ and obtain

$$
\begin{aligned}
\|fg\|^p_{W_p^{1-\frac{1}{2p}}(J;L_p(\partial\Gamma^*))} &\leq \|fg\|^p_{L_p(J;L_p(\partial\Gamma^*))} + \int_0^T \int_0^{T-h} \frac{\|(\delta_h(fg))(t)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+\left(1-\frac{1}{2p}\right)p}} dtdh \\
&\leq \|f\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \|g\|^p_{L_p(J;L_p(\partial\Gamma^*))} \\
&+ \int_0^T \int_0^{T-h} \frac{\|(\delta_h f)(t) g(t+h)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+p-\frac{1}{2}}} dtdh \\
&+ \int_0^T \int_0^{T-h} \frac{\|f(t)(\delta_h g)(t)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+p-\frac{1}{2}}} dtdh \\
&\leq c \|f\|^p_{\nabla^1 Z_\rho} \|g\|^p_{L_p(J;L_p(\partial\Gamma^*))}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|f\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \int_0^T \int_0^{T-h} \frac{\|(\delta_h g)(t)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+p-\frac{1}{2}}} dtdh \\
&+ c\,\|f\|^p_{L_p(J;L_p(\partial\Gamma^*))} \|g\|^p_{\nabla^1 Z_\rho} \\
&+ \|g\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \int_0^T \int_0^{T-h} \frac{\|(\delta_h f)(t)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+p-\frac{1}{2}}} dtdh \\
&\leq \widetilde{c}\,\|f\|^p_{\nabla^1 Z_\rho} \left(\|g\|^p_{L_p(J;L_p(\partial\Gamma^*))} + \int_0^T \int_0^{T-h} \frac{\|(\delta_h g)(t)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+p-\frac{1}{2}}} dtdh \right) \\
&+ \widetilde{c}\,\|g\|^p_{\nabla^1 Z_\rho} \left(\|f\|^p_{L_p(J;L_p(\partial\Gamma^*))} + \int_0^T \int_0^{T-h} \frac{\|(\delta_h f)(t)\|^p_{L_p(\partial\Gamma^*)}}{|h|^{1+p-\frac{1}{2}}} dtdh \right) \\
&= \widehat{c} \left(\|f\|^p_{\nabla^1 Z_\rho} \|g\|^p_{W_p^{1-\frac{1}{2p}}(J;L_p(\partial\Gamma^*))} + \|f\|^p_{W_p^{1-\frac{1}{2p}}(J;L_p(\partial\Gamma^*))} \|g\|^p_{\nabla^1 Z_\rho} \right) \\
&\leq c\,\|f\|^p_{\nabla^1 Z_\rho} \|g\|^p_{\nabla^1 Z_\rho}
\end{aligned}
$$

and in addition

$$
\begin{aligned}
\|fg\|^p_{L_p(J;W_p^{2-\frac{1}{p}}(\partial\Gamma^*))} &= \int_0^T \|(fg)(t)\|^p_{W_p^{2-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&\leq \int_0^T \|f(t)g(t)\|^p_{W_p^1(\partial\Gamma^*)} + \|(\partial_\sigma(fg))(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&\leq \underbrace{\|fg\|^p_{L_p(J;W_p^1(\partial\Gamma^*))}}_{\leq c\|f\|_{\nabla^1 Z_\rho}\|g\|_{\nabla^1 Z_\rho} \text{ as in (i)}} \\
&+ \int_0^T \|\partial_\sigma f(t) g(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} + \|f(t)\partial_\sigma g(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&\leq c\,\|f\|_{\nabla^1 Z_\rho} \|g\|_{\nabla^1 Z_\rho} + \|\partial_\sigma f\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \int_0^T \|g(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&+ \|g\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \int_0^T \|\partial_\sigma f(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&+ \|f\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \int_0^T \|\partial_\sigma g(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&+ \|\partial_\sigma g\|^p_{L_\infty(J;L_\infty(\partial\Gamma^*))} \int_0^T \|f(t)\|^p_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*)} dt \\
&\leq \widetilde{c} \Bigg(\|f\|_{\nabla^1 Z_\rho} \|g\|_{\nabla^1 Z_\rho} + \|f\|^p_{\nabla^1 Z_\rho} \|g\|^p_{L_p(J;W_p^{1-\frac{1}{p}}(\partial\Gamma^*))} \\
&+ \|f\|^p_{\nabla^1 Z_\rho} \|g\|^p_{L_p(W_p^{2-\frac{1}{p}}(\partial\Gamma^*))} \Bigg) \\
&\leq \widehat{c}\,\|f\|^p_{\nabla^1 Z_\rho} \|g\|^p_{\nabla^1 Z_\rho} .
\end{aligned}
$$

Combining these inequalities we come to the conclusion

$$
\|fg\|_{\nabla^1 Z_\rho} \leq \widetilde{c}\,\|f\|_{\nabla^1 Z_\rho} \|g\|_{\nabla^1 Z_\rho} ,
$$

which proves the second claim. ■

Lemma 3.18: *Let $J := [0,T]$ and $4 < p < \infty$ and $\mathbb{B}_r^{\mathbb{E}}(\mathbb{O}) := \{\Phi \in \mathbb{E} \,|\, \|\Phi\|_{\mathbb{E}} < r\}$. Then there exists an $r > 0$ such that $N(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O})) \subseteq \mathbb{F}$. Moreover, $N \in C^1(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O}); \mathbb{F})$ and $\|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E},\mathbb{F})} \leq cT^{\frac{1}{q}}$ for some $q > p$, where $DN : \mathbb{B}_r^{\mathbb{E}}(\mathbb{O}) \longrightarrow \mathcal{L}(\mathbb{E},\mathbb{F})$ denotes the Fréchet derivative of N.*

Proof: The linearization we calculated in Section 2.3 is indeed the Fréchet derivative as we will see later in this proof. Our first goal is to show

$$F(u) := H_\Gamma(u) - \overline{H}(u) - V_\Gamma(u) + \frac{d}{dt}u + \mathcal{A}(q,D)u \in X$$

$$G(u,\rho) := a + b\varkappa_{\partial D}(\rho) + \langle n_\Gamma(u), n_D(u)\rangle - v_{\partial D}(\rho) + \frac{d}{dt}\rho + \mathcal{B}_0(q,D)u + \mathcal{C}_0(q,D_\partial)\rho \in Y_0$$

for all $u \in \mathbb{B}^{Z_u}_{\frac{r}{\sqrt{2}}}(\mathbb{O})$ and $\rho \in \mathbb{B}^{Z_\rho}_{\frac{r}{\sqrt{2}}}(\mathbb{O})$. For $r > 0$ small enough all the terms appearing in F and G are well-defined and the linear parts of F and G can be omitted since

- $\mathcal{A}(q,D)u \in X$ due to $u \in Z_u \subseteq L_p(J; W_p^2(\Gamma^*;\mathbb{R}))$ and $\mathcal{A}$ is of second order in space.
- $\frac{d}{dt}u \in X$ due to $u \in Z_u \subseteq W_p^1(J; L_p(\Gamma^*;\mathbb{R}))$ and $\frac{d}{dt}$ is of first order in time.
- $\frac{d}{dt}\rho \in Y_0$ due to $\rho \in Z_\rho \subseteq W_p^{\frac{3}{2}-\frac{1}{2p}}(J; L_p(\partial\Gamma^*;\mathbb{R}))$ and $\frac{d}{dt}$ is of first order in time as well as $\rho \in Z_\rho \hookrightarrow W_p^1(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$ according to Lemma 3.15(i) with $\sigma = \frac{2p}{3p-1}$ and $\frac{d}{dt}$ is of first order in time.
- $\mathcal{B}_0(q,D)u \in Y_0$ due to $u \in Z_u \hookrightarrow W_p^{\frac{1}{2}}(J; W_p^1(\Gamma^*;\mathbb{R}))$ because of Lemma 3.15(i) with $\sigma = \frac{1}{2}$ and $\mathcal{B}_0$ is of first order in space. This leads to

  $$\mathcal{B}_0(q,D)u \in W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R}))$$

  and by (A.24) in [Gru95] the trace operator γ_0 maps as follows

  $$\gamma_0 : W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R})) \longrightarrow Y_0.$$

- $\mathcal{C}_0(q,D_\partial)\rho \in Y_0$ due to $\rho \in Z_\rho \subseteq L_p(J; W_p^{3-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$ and $\mathcal{C}_0$ is of second order in space as well as $\rho \in Z_\rho \hookrightarrow W_p^{\frac{1}{2}-\frac{1}{2p}}(J; W_p^2(\partial\Gamma^*;\mathbb{R}))$ due to Lemma 3.15(i) with $\sigma = \frac{p-1}{3p-1}$ and $\mathcal{C}_0$ is of second order in space.

Next we want to turn our attention to the two velocities in F and G. We first note

$$\sup_{t\in J}\sup_{q\in\Gamma^*} |n_\Gamma(u)\cdot\partial_w\Psi(u)| \leq \sup_{t\in J}\sup_{q\in\Gamma^*} \|n_\Gamma(u)\|\,\|\partial_w\Psi(u)\| = \sup_{t\in J}\sup_{q\in\Gamma^*} \|\partial_w\Psi(u)\| =: c < \infty$$

since J is compact, $\partial_w\Psi$ is continuous up to the boundary and because we have assumed $0 < \alpha < \pi$. Hence using (2.31) we obtain

$$\|V_\Gamma(u)\|_X \leq c\,\|\partial_t u(t,q)\|_{L_p(J;L_p(\Gamma^*;\mathbb{R}))} \leq c\,\|u(t,q)\|_{W_p^1(J;L_p(\Gamma^*;\mathbb{R}))} < \infty.$$

Analogously by means of (2.35) we get for the normal boundary velocity

$$\begin{aligned}
\|v_{\partial\Gamma}(\rho)\|_{W_p^{\frac{1}{2}-\frac{1}{2p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))} &\leq c\,\|\partial_t\rho(t,q)\|_{W_p^{\frac{1}{2}-\frac{1}{2p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))}\\
&\leq c\,\|\rho(t,q)\|_{W_p^{\frac{3}{2}-\frac{1}{2p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))} < \infty\\
\|v_{\partial\Gamma}(\rho)\|_{L_p(J;W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))} &\leq \widehat{c}\,\|\partial_t\rho(t,q)\|_{L_p(J;W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))} \leq \widehat{c}\,\|\rho(t,q)\|_{W_p^1(J;W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))}\\
&\leq \widetilde{c}\,\|\rho(t,q)\|_{Z_\rho} < \infty
\end{aligned}$$

due to the fact that $\rho \in Z_\rho \subseteq W_p^{\frac{3}{2}-\frac{1}{2p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))$ and $Z_\rho \hookrightarrow W_p^1(J;W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$ by Lemma 3.15(i). This proves $\|v_{\partial\Gamma}(\rho)\|_{Y_0} < \infty$.
Now we consider the angle term and the constant a. Since J, $\partial\Gamma^*$ and Γ^* are bounded, we can easily see that $\|c\|_{W_p^\alpha(J;W_p^\beta(\Gamma^*;\mathbb{R}))} < \infty$ and $\|c\|_{W_p^\alpha(J;W_p^\beta(\partial\Gamma^*;\mathbb{R}))} < \infty$ for each constant function c and all $\alpha,\beta \in [0,\infty)$. Therefore we can obviously estimate

$$\sup_{t\in J}\sup_{q\in\Gamma^*}|\langle n_\Gamma(u), n_D(u)\rangle| \leq \sup_{t\in J}\sup_{q\in\Gamma^*}\|n_\Gamma(u)\|\,\|n_D(u)\| = \sup_{t\in J}\sup_{q\in\Gamma^*} 1 = 1 < \infty$$

to conclude $\|\langle n_\Gamma(u), n_D(u)\rangle\|_{Y_0} \leq \|1\|_{Y_0} < \infty$ and $\|a\|_{Y_0} < \infty$.
Finally, we look at the remaining curvature terms. Due to Lemma 3.15(ii) we see that $|u(t,q)|$ and $|\nabla_{\Gamma^*}u(t,q)|$ remain bounded. This shows that for a maybe even smaller r the first fundamental form of all the hypersurfaces in the family $(\Gamma_\varrho(t))_{t\in J}$ is not degenerated. Because of the facts that on the one hand $H_\Gamma(u)$ depends linearly on the second spacial derivatives of u and on the other hand the coefficients, that involve only u and its first derivatives, are bounded as seen above, we get

$$\|H_\Gamma(u)\|_X \leq c\left(\left\|\nabla^2_{\Gamma^*}u\right\|_X + 1\right) \leq c\left(\|u\|_{L_p(J;W_p^2(\Gamma^*;\mathbb{R}))} + 1\right) \leq c\left(\|u\|_{Z_u} + 1\right) < \infty.$$

For the non-local mean integral we first have a look at the area. Surely, $\int\limits_{\Gamma(t)} 1\,d\mathcal{H}^2$ depends continuously on $t \in [0,T]$. Therefore, we obtain $0 < c \leq \int\limits_{\Gamma(t)} 1\,d\mathcal{H}^2 \leq C$. This leads to

$$\left|\overline{H}(u(t))\right| = \frac{1}{\int\limits_{\Gamma(t)} 1\,d\mathcal{H}^2}\left|\int_{\Gamma(t)} H_{\Gamma(t)}\,d\mathcal{H}^2\right| \leq \frac{1}{c}\left|\int_{\Gamma^*} H_\Gamma(u(t))J(u(t),\nabla_{\Gamma^*}u(t))\,d\mathcal{H}^2\right|,$$

where J is some determinant term that includes no second or higher order derivatives of u (cf. [Alt08]). Thus, the integrand depends affine linearly on the second space derivatives of u and the coefficients are bounded. The same argumentation as above finally leads to

$$\begin{aligned}
\left\|\overline{H}(u)\right\|_X &= \left(\int_J\int_{\Gamma^*}|\overline{H}(u)|^p\,d\mathcal{H}^2 dt\right)^{\frac{1}{p}} \leq C^{\frac{1}{p}}\left(\int_J|\overline{H}(u)|^p dt\right)^{\frac{1}{p}}\\
&\leq \frac{C^{\frac{1}{p}}}{c}\left(\int_J\left|\int_{\Gamma^*}H_\Gamma(u)J(u,\nabla_{\Gamma^*}u)\,d\mathcal{H}^2\right|^p dt\right)^{\frac{1}{p}}\\
&\leq \widehat{c}\,\|H_\Gamma(u)J(u,\nabla_{\Gamma^*}u)\|_{L_p(J;L_1(\Gamma^*;\mathbb{R}))}\\
&\leq \widehat{c}\,\|H_\Gamma(u)J(u,\nabla_{\Gamma^*}u)\|_X \leq c\left(\left\|\nabla^2_{\Gamma^*}u\right\|_X + 1\right) \leq c\left(\|u\|_{Z_u} + 1\right) < \infty.
\end{aligned}$$

For the geodesic curvature we observe $Z_\rho \hookrightarrow W_p^{\frac{1}{p}}(J; W_p^{3-\frac{3}{p}}(\partial\Gamma^*;\mathbb{R}))$ if we choose $\sigma = \frac{2}{3p-1}$ in Lemma 3.15(i). By usual Sobolev embeddings and the assumption $p > 4$ we get in addition

$$Z_\rho \hookrightarrow W_p^{\frac{1}{p}}(J; W_p^{3-\frac{3}{p}}(\partial\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; W_p^{3-\frac{3}{p}}(\partial\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; BUC^2(\partial\Gamma^*;\mathbb{R}))$$

and hence $|\rho(t,q)|$, $|\partial_\sigma\rho(t,q)|$ and $|\partial_\sigma^2\rho(t,q)|$ remain bounded. The continuous dependence of $\varkappa_{\partial D}(\rho)$ on ρ and its derivatives shows the boundedness and we get

$$\|\varkappa_{\partial D}(\rho)\|_{Y_0} \le \sup_{t\in J}\sup_{q\in\partial\Gamma^*} |\varkappa_{\partial D}(\rho)|\,\|1\|_{Y_0} \le c\,\|1\|_{Y_0} < \infty.$$

This shows $N(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O})) \subseteq \mathbb{F}$. What is left is $N \in C^1(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O});\mathbb{F})$.
To prove this we note that we have calculated in Section 2.3 the first variations of the parts of N. We will denote these variations by the prefix δ. Assuming the Lipschitz continuity of $\delta_\Phi N$ leads us to the Fréchet differentiability as follows

$$\begin{aligned}
&\|N(\Phi_2) - N(\Phi_1) - \delta_\Phi N(\Phi_1)(\Phi_2-\Phi_1)\|_{\mathbb{F}} \\
&\quad = \left\| \int_0^1 \delta_\Phi N(\Phi_1 + t(\Phi_2-\Phi_1))(\Phi_2-\Phi_1) - \delta_\Phi N(\Phi_1)(\Phi_2-\Phi_1)dt \right\|_{\mathbb{F}} \\
&\quad \le \left(\int_0^1 \|\delta_\Phi N(\Phi_1 + t(\Phi_2-\Phi_1)) - \delta_\Phi N(\Phi_1)\|_{\mathbb{F}}\, dt \right) \|\Phi_2-\Phi_1\|_{\mathbb{E}} \\
&\quad \le \left(\int_0^1 tL\,\|\Phi_2-\Phi_1\|_{\mathbb{E}}\, dt \right) \|\Phi_2-\Phi_1\|_{\mathbb{E}} \\
&\quad = L\,\|\Phi_2-\Phi_1\|_{\mathbb{E}}^2,
\end{aligned}$$

where we know that

$$N(\Phi_2) - N(\Phi_1) = \int_0^1 \delta_\Phi N(\Phi_1 + t(\Phi_2-\Phi_1))(\Phi_2-\Phi_1)dt$$

holds almost everywhere due to Section 2.3. Therefore for $N \in C^1(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O});\mathbb{F})$ and in addition $DN = \delta_\Phi N$. The Lipschitz continuity of $\delta_\Phi N$ remains to be proven. To simplify the formulas we look at each term in each component of N separately.
Starting with H_Γ we know that we can write

$$H_\Gamma(u) = \sum_{|\alpha|=2} a_\alpha(u, \nabla_{\Gamma^*}u)\partial^\alpha u + b(u, \nabla_{\Gamma^*}u)$$

with $a_\alpha, b \in C^3(U)$ and $U \subseteq \mathbb{R}\times\mathbb{R}^2$ a closed neighborhood of 0. Linearizing this we obtain

$$\begin{aligned}
(\delta_u H_\Gamma(u))(v) = \sum_{|\alpha|=2} &(\partial_1 a_\alpha(u, \nabla_{\Gamma^*}u)(\partial^\alpha u)v + \partial^\alpha u\,(\partial_2 a_\alpha(u,\nabla_{\Gamma^*}u)\cdot\nabla_{\Gamma^*}v) \\
&+ a_\alpha(u,\nabla_{\Gamma^*}u)\partial^\alpha v) + \partial_1 b(u,\nabla_{\Gamma^*}u)v + \partial_2 b(u,\nabla_{\Gamma^*}u)\cdot\nabla_{\Gamma^*}v.
\end{aligned}$$

Due to $a_\alpha, b \in C^3(U)$ the coefficients $a_\alpha, \partial_1 a_\alpha, \partial_2 a_\alpha, \partial_1 b$ and $\partial_2 b$ satisfy a Lipschitz condition on $\overline{B_r(0)} \subseteq \nabla^1 Z_u$, i.e.

$$\|\partial_1 a_\alpha(u,\nabla_{\Gamma^*}u) - \partial_1 a_\alpha(\widetilde{u},\nabla_{\Gamma^*}\widetilde{u})\|_{\nabla^1 Z_u} \le c\left(\|u-\widetilde{u}\|_{\nabla^1 Z_u} + \|\nabla_{\Gamma^*}u - \nabla_{\Gamma^*}\widetilde{u}\|_{\nabla^1 Z_u}\right)$$

for all $u, \widetilde{u} \in \overline{B_r(0)}$ and some $c > 0$. This Lipschitz condition can be seen via

$$\begin{aligned}
\|f(u) - f(\widetilde{u})\|_{\nabla^1 Z_u} &= \left\| \int_0^1 \frac{d}{dt} f(\widetilde{u} + t(u - \widetilde{u}))dt \right\|_{\nabla^1 Z_u} \\
&= \left\| \left(\int_0^1 f'(\widetilde{u} + t(u - \widetilde{u}))dt \right) (u - \widetilde{u}) \right\|_{\nabla^1 Z_u} \\
&= \underbrace{\left\| \int_0^1 f'(\widetilde{u} + t(u - \widetilde{u}))dt \right\|_{\nabla^1 Z_u}}_{\le c} \|u - \widetilde{u}\|_{\nabla^1 Z_u} \\
&\le c \|u - \widetilde{u}\|_{\nabla^1 Z_u},
\end{aligned}$$

where we have used Lemma 3.17. Because of $\|\bullet\|_{\nabla^1 Z_u} \le c \|\bullet\|_{Z_u}$, any two functions $u, \widetilde{u} \in \overline{B_r(0)} \subseteq Z_u$ are also in $\overline{B_{cr}(0)} \subseteq \nabla^1 Z_u$. Hence for $u, \widetilde{u} \in \overline{B_r(0)} \subseteq Z_u$ we get

$$\begin{aligned}
&\|\delta_u H_\Gamma(u) - \delta_u H_\Gamma(\widetilde{u})\|_{\mathcal{L}(Z_u, X)} \\
&\quad \le \sum_{|\alpha|=2} \|\partial_1 a_\alpha(u, \nabla_{\Gamma^*} u)\partial^\alpha u \operatorname{Id} - \partial_1 a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\partial^\alpha \widetilde{u} \operatorname{Id}\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|\partial^\alpha u \left(\partial_2 a_\alpha(u, \nabla_{\Gamma^*} u) \cdot \nabla_{\Gamma^*}\right) - \partial^\alpha \widetilde{u} \left(\partial_2 a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u}) \cdot \nabla_{\Gamma^*}\right)\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|a_\alpha(u, \nabla_{\Gamma^*} u)\partial^\alpha - a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\partial^\alpha\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|\partial_1 b(u, \nabla_{\Gamma^*} u) \operatorname{Id} - \partial_1 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u}) \operatorname{Id}\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|\partial_2 b(u, \nabla_{\Gamma^*} u) \cdot \nabla_{\Gamma^*} - \partial_2 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u}) \cdot \nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u, X)} \\
&\quad \le \sum_{|\alpha|=2} \left(\|\partial_1 a_\alpha(u, \nabla_{\Gamma^*} u)\partial^\alpha u - \partial_1 a_\alpha(u, \nabla_{\Gamma^*} u)\partial^\alpha \widetilde{u}\|_X\right. \\
&\quad + \left.\|\partial_1 a_\alpha(u, \nabla_{\Gamma^*} u)\partial^\alpha \widetilde{u} - \partial_1 a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\partial^\alpha \widetilde{u}\|_X\right) \|\operatorname{Id}\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \left(\|\partial^\alpha u \partial_2 a_\alpha(u, \nabla_{\Gamma^*} u) - \partial^\alpha \widetilde{u} \partial_2 a_\alpha(u, \nabla_{\Gamma^*} u)\|_X\right. \\
&\quad + \left.\|\partial^\alpha \widetilde{u} \partial_2 a_\alpha(u, \nabla_{\Gamma^*} u) - \partial^\alpha \widetilde{u} \partial_2 a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X\right) \|\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|a_\alpha(u, \nabla_{\Gamma^*} u) - a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X \|\partial^\alpha\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|\partial_1 b(u, \nabla_{\Gamma^*} u) - \partial_1 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X \|\operatorname{Id}\|_{\mathcal{L}(Z_u, X)} \\
&\quad + \|\partial_2 b(u, \nabla_{\Gamma^*} u) - \partial_2 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X \|\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u, X)}.
\end{aligned}$$

Since $\nabla^1 Z_u \hookrightarrow L_\infty(J; L_\infty(\Gamma^*; \mathbb{R}))$ we have $\|\bullet\|_{L_\infty(J; L_\infty(\Gamma^*; \mathbb{R}))} \le c \|\bullet\|_{\nabla^1 Z_u}$. Moreover, in the same manner as for the linear parts $\mathcal{A}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{C}_0$ and $\mathcal{C}_1$ we can prove $\|\operatorname{Id}\|_{\mathcal{L}(Z_u, X)} < \infty$, $\|\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u, X)} < \infty$ and $\|\partial^\alpha\|_{\mathcal{L}(Z_u, X)} < \infty$ for $|\alpha| = 2$, which enables us to continue the inequality above as follows

$$\begin{aligned}
\|\delta_u H_\Gamma(u) - \delta_u H_\Gamma(\widetilde{u})\|_{\mathcal{L}(Z_u, X)} \le c \sum_{|\alpha|=2} & \underbrace{\|\partial_1 a_\alpha(u, \nabla_{\Gamma^*} u)\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\le \widetilde{c} \|\partial_1 a_\alpha(u, \nabla_{\Gamma^*} u)\|_{\nabla^1 Z_u} \le c(r)} \underbrace{\|\partial^\alpha (u - \widetilde{u})\|_X}_{\le \|u - \widetilde{u}\|_{Z_u}} \\
& + \underbrace{\|\partial_1 a_\alpha(u, \nabla_{\Gamma^*} u) - \partial_1 a_\alpha(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\le \widetilde{c} \|\ldots\|_{\nabla^1 Z_u} \le \widehat{c} \left(\|u - \widetilde{u}\|_{\nabla^1 Z_u} + \|\nabla_{\Gamma^*} u - \nabla_{\Gamma^*} \widetilde{u}\|_{\nabla^1 Z_u} \right)} \underbrace{\|\partial^\alpha \widetilde{u}\|_X}_{\le r} \\
& + \underbrace{\|\partial_2 a_\alpha(u, \nabla_{\Gamma^*} u)\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\le \widetilde{c} \|\partial_2 a_\alpha(u, \nabla_{\Gamma^*} u)\|_{\nabla^1 Z_u} \le c(r)} \underbrace{\|\partial^\alpha (u - \widetilde{u})\|_X}_{\le \|u - \widetilde{u}\|_{Z_u}}
\end{aligned}$$

$$
\begin{aligned}
&+ \underbrace{\left\|\partial_2 a_\alpha(u, \nabla_{\Gamma^*} u) - \partial_2 a_\alpha(\widetilde{u}, \nabla_{\Gamma^*}\widetilde{u})\right\|_{L_\infty(J;L_\infty(\Gamma^*))}}_{\leq \widetilde{c}\|\dots\|_{\nabla^1 Z_u} \leq \widehat{c}\left(\|u-\widetilde{u}\|_{\nabla^1 Z_u} + \|\nabla_{\Gamma^*} u - \nabla_{\Gamma^*}\widetilde{u}\|_{\nabla^1 Z_u}\right)} \underbrace{\|\partial^\alpha \widetilde{u}\|_X}_{\leq r} \\
&+ 3c \left(\underbrace{\|u - \widetilde{u}\|_{\nabla^1 Z_u}}_{\leq \widetilde{c}\|u-\widetilde{u}\|_{Z_u}} + \underbrace{\|\nabla_{\Gamma^*} u - \nabla_{\Gamma^*}\widetilde{u}\|_{\nabla^1 Z_u}}_{\leq \widetilde{c}\|u-\widetilde{u}\|_{Z_u}} \right) \\
&\leq c(r) \|u - \widetilde{u}\|_{Z_u} .
\end{aligned}
$$

This shows the Lipschitz continuity of $\delta_u H_\Gamma : \overline{B_r(0)} \subseteq Z_u \longrightarrow \mathcal{L}(Z_u, X)$ and hence we see $H_\Gamma \in C^1(\overline{B_r(0)}, X)$.
Similar considerations can be made for $\varkappa_{\partial D}$. We can write

$$
\varkappa_{\partial D}(\rho) = a(\rho, \partial_\sigma \rho)\partial_\sigma^2 \rho + b(\rho, \partial_\sigma \rho)
$$

with $a, b \in C^4(U)$. Linearizing this we obtain

$$
\begin{aligned}
(\delta_\rho \varkappa_{\partial D}(\rho))(v) = {} & \partial_1 a(\rho, \partial_\sigma \rho)(\partial_\sigma^2 \rho) v + \partial_\sigma^2 \rho \left(\partial_2 a(\rho, \partial_\sigma \rho)\partial_\sigma v\right) + a(\rho, \partial_\sigma \rho)\partial_\sigma^2 v \\
& + \partial_1 b(\rho, \partial_\sigma \rho) v + \partial_2 b(\rho, \partial_\sigma \rho)\partial_\sigma v.
\end{aligned}
$$

As before $a, \partial_1 a, \partial_2 a, \partial_1 b$ and $\partial_2 b$ satisfy a Lipschitz condition on $\overline{B_r(0)} \subseteq \nabla^1 Z_\rho$ and due to $\|\bullet\|_{\nabla^1 Z_\rho} \leq c \|\bullet\|_{Z_\rho}$ any two functions $\rho, \widetilde{\rho} \in \overline{B_r(0)} \subseteq Z_\rho$ are also in $\overline{B_{cr}(0)} \subseteq \nabla^1 Z_\rho$. Hence for $\rho, \widetilde{\rho} \in \overline{B_r(0)} \subseteq Z_\rho$ we get

$$
\begin{aligned}
\|\delta_\rho \varkappa_{\partial D}(\rho) - \delta_\rho \varkappa_{\partial D}(\widetilde{\rho})\|_{\mathcal{L}(Z_\rho, Y_0)} \leq {} & \left\|\partial_1 a(\rho, \partial_\sigma \rho)\partial_\sigma^2 \rho \operatorname{Id} - \partial_1 a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\partial_\sigma^2 \widetilde{\rho} \operatorname{Id}\right\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \left\|\partial_\sigma^2 \rho \left(\partial_2 a(\rho, \partial_\sigma \rho)\partial_\sigma\right) - \partial_\sigma^2 \widetilde{\rho}\left(\partial_2 a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\partial_\sigma\right)\right\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \left\|a(\rho, \partial_\sigma \rho)\partial_\sigma^2 - a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\partial_\sigma^2\right\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \|\partial_1 b(\rho, \partial_\sigma \rho) \operatorname{Id} - \partial_1 b(\widetilde{\rho}, \partial_\sigma \widetilde{\rho}) \operatorname{Id}\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \|\partial_2 b(\rho, \partial_\sigma \rho)\partial_\sigma - \partial_2 b(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\partial_\sigma\|_{\mathcal{L}(Z_\rho, Y_0)} \\
\leq {} & \left(\left\|\partial_1 a(\rho, \partial_\sigma \rho)\partial_\sigma^2 \rho - \partial_1 a(\rho, \partial_\sigma \rho)\partial_\sigma^2 \widetilde{\rho}\right\|_{Y_0} \right. \\
& \left. + \left\|\partial_1 a(\rho, \partial_\sigma \rho)\partial_\sigma^2 \widetilde{\rho} - \partial_1 a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\partial_\sigma^2 \widetilde{\rho}\right\|_{Y_0} \right) \|\operatorname{Id}\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \left(\left\|\partial_\sigma^2 \rho \partial_2 a(\rho, \partial_\sigma \rho) - \partial_\sigma^2 \widetilde{\rho} \partial_2 a(\rho, \partial_\sigma \rho)\right\|_{Y_0} \right. \\
& \left. + \left\|\partial_\sigma^2 \widetilde{\rho} \partial_2 a(\rho, \partial_\sigma \rho) - \partial_\sigma^2 \widetilde{\rho} \partial_2 a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\right\|_{Y_0} \right) \|\partial_\sigma\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \|a(\rho, \partial_\sigma \rho) - a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\|_{Y_0} \left\|\partial_\sigma^2\right\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \|\partial_1 b(\rho, \partial_\sigma \rho) - \partial_1 b(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\|_{Y_0} \|\operatorname{Id}\|_{\mathcal{L}(Z_\rho, Y_0)} \\
& + \|\partial_2 b(\rho, \partial_\sigma \rho) - \partial_2 b(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\|_{Y_0} \|\partial_\sigma\|_{\mathcal{L}(Z_\rho, Y_0)} .
\end{aligned}
$$

Since $\nabla^1 Z_\rho \hookrightarrow L_\infty(J; L_\infty(\partial\Gamma^*; \mathbb{R}))$, we have $\|\bullet\|_{L_\infty(J;L_\infty(\partial\Gamma^*;\mathbb{R}))} \leq c \|\bullet\|_{\nabla^1 Z_\rho}$. Again we see $\|\mathrm{Id}\|_{\mathcal{L}(Z_\rho,Y_0)} < \infty$, $\|\partial_\sigma\|_{\mathcal{L}(Z_\rho,Y_0)} < \infty$ and $\|\partial_\sigma^2\|_{\mathcal{L}(Z_\rho,Y_0)} < \infty$, which brings us to

$$\begin{aligned}
\|\delta_\rho \varkappa_{\partial D}(\rho) - \delta_\rho \varkappa_{\partial D}(\widetilde{\rho})\|_{\mathcal{L}(Z_\rho,Y_0)} \leq\ & c \underbrace{\|\partial_1 a(\rho, \partial_\sigma \rho)\|_{\nabla^1 Z_\rho}}_{\leq c(r)} \underbrace{\left\|\partial_\sigma^2(\rho - \widetilde{\rho})\right\|_{Y_0}}_{\leq \|\rho-\widetilde{\rho}\|_{Z_\rho}} \\
& + c \underbrace{\|\partial_1 a(\rho, \partial_\sigma \rho) - \partial_1 a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\|_{\nabla^1 Z_\rho}}_{\leq \widetilde{c}\left(\|\rho-\widetilde{\rho}\|_{\nabla^1 Z_\rho} + \|\partial_\sigma\rho - \partial_\sigma\widetilde{\rho}\|_{\nabla^1 Z_\rho}\right)} \underbrace{\left\|\partial_\sigma^2 \widetilde{\rho}\right\|_{Y_0}}_{\leq r} \\
& + c \underbrace{\|\partial_2 a(\rho, \partial_\sigma \rho)\|_{\nabla^1 Z_\rho}}_{\leq c(r)} \underbrace{\left\|\partial_\sigma^2(\rho - \widetilde{\rho})\right\|_{Y_0}}_{\leq \|\rho-\widetilde{\rho}\|_{Z_\rho}} \\
& + c \underbrace{\|\partial_2 a(\rho, \partial_\sigma \rho) - \partial_2 a(\widetilde{\rho}, \partial_\sigma \widetilde{\rho})\|_{\nabla^1 Z_\rho}}_{\leq \widetilde{c}\left(\|\rho-\widetilde{\rho}\|_{\nabla^1 Z_\rho} + \|\partial_\sigma\rho - \partial_\sigma\widetilde{\rho}\|_{\nabla^1 Z_\rho}\right)} \underbrace{\left\|\partial_\sigma^2 \widetilde{\rho}\right\|_{Y_0}}_{\leq r} \\
& + 3c \left(\underbrace{\|\rho - \widetilde{\rho}\|_{\nabla^1 Z_\rho}}_{\leq \widetilde{c}\|\rho-\widetilde{\rho}\|_{Z_\rho}} + \underbrace{\|\partial_\sigma \rho - \partial_\sigma \widetilde{\rho}\|_{\nabla^1 Z_\rho}}_{\leq \widetilde{c}\|\rho-\widetilde{\rho}\|_{Z_\rho}} \right) \\
\leq\ & c(r)\, \|\rho - \widetilde{\rho}\|_{Z_\rho},
\end{aligned}$$

where we have used $\|fg\|_{Y_0} \leq \|f\|_{\nabla^1 Z_\rho} \|g\|_{Y_0}$, which is shown by same arguments as in the proof of Lemma 3.17. Again Lipschitz continuity of $\delta_\rho \varkappa_{\partial D} : \overline{B_r(0)} \subseteq Z_\rho \longrightarrow \mathcal{L}(Z_\rho, Y_0)$ is proven and we arrive at $\varkappa_{\partial D} \in C^1(\overline{B_r(0)}, Y_0)$.
Also the velocity V_Γ can be written as

$$V_\Gamma(u) = a(u, \nabla_{\Gamma^*} u)\partial_t u + b(u, \nabla_{\Gamma^*} u)$$

with $a, b \in C^3(U)$. Linearization gives

$$\begin{aligned}
(\delta_u V_\Gamma(u))(v) = \partial_1 a(u, \nabla_{\Gamma^*} u)(\partial_t u)v + \partial_t u\, (\partial_2 a(u, \nabla_{\Gamma^*} u) \cdot \nabla_{\Gamma^*} v) + a(u, \nabla_{\Gamma^*} u)\partial_t v \\
+ \partial_1 b(u, \nabla_{\Gamma^*} u)v + \partial_2 b(u, \nabla_{\Gamma^*} u) \cdot \nabla_{\Gamma^*} v.
\end{aligned}$$

For $u, \widetilde{u} \in \overline{B_r(0)} \subseteq Z_u$ we analogously come to the estimate

$$\begin{aligned}
&\|\delta_u V_\Gamma(u) - \delta_u V_\Gamma(\widetilde{u})\|_{\mathcal{L}(Z_u,X)} \\
&\quad \leq \|\partial_1 a(u, \nabla_{\Gamma^*} u)\partial_t u \,\mathrm{Id} - \partial_1 a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\partial_t \widetilde{u} \,\mathrm{Id}\|_{\mathcal{L}(Z_u,X)} \\
&\quad + \|\partial_t u\, (\partial_2 a(u, \nabla_{\Gamma^*} u) \cdot \nabla_{\Gamma^*}) - \partial_t \widetilde{u}\, (\partial_2 a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u}) \cdot \nabla_{\Gamma^*})\|_{\mathcal{L}(Z_u,X)} \\
&\quad + \|a(u, \nabla_{\Gamma^*} u)\partial_t - a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\partial_t\|_{\mathcal{L}(Z_u,X)} \\
&\quad + \|\partial_1 b(u, \nabla_{\Gamma^*} u) \,\mathrm{Id} - \partial_1 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u}) \,\mathrm{Id}\|_{\mathcal{L}(Z_u,X)} \\
&\quad + \|\partial_2 b(u, \nabla_{\Gamma^*} u) \cdot \nabla_{\Gamma^*} - \partial_2 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u}) \cdot \nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u,X)} \\
&\quad \leq (\|\partial_1 a(u, \nabla_{\Gamma^*} u)\partial_t u - \partial_1 a(u, \nabla_{\Gamma^*} u)\partial_t \widetilde{u}\|_X \\
&\quad + \|\partial_1 a(u, \nabla_{\Gamma^*} u)\partial_t \widetilde{u} - \partial_1 a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\partial_t \widetilde{u}\|_X)\, \|\mathrm{Id}\|_{\mathcal{L}(Z_u,X)}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\|\partial_t u \partial_2 a(u, \nabla_{\Gamma^*} u) - \partial_t \widetilde{u} \partial_2 a(u, \nabla_{\Gamma^*} u)\|_X\right. \\
&+ \left.\|\partial_t \widetilde{u} \partial_2 a(u, \nabla_{\Gamma^*} u) - \partial_t \widetilde{u} \partial_2 a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X\right) \|\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u, X)} \\
&+ \|a(u, \nabla_{\Gamma^*} u) - a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X \, \|\partial_t\|_{\mathcal{L}(Z_u, X)} \\
&+ \|\partial_1 b(u, \nabla_{\Gamma^*} u) - \partial_1 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X \, \|\mathrm{Id}\|_{\mathcal{L}(Z_u, X)} \\
&+ \|\partial_2 b(u, \nabla_{\Gamma^*} u) - \partial_2 b(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_X \, \|\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u, X)} \\
&\leq c \underbrace{\|\partial_1 a(u, \nabla_{\Gamma^*} u)\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\leq \widetilde{c} \|\partial_1 a(u, \nabla_{\Gamma^*} u)\|_{\nabla^1 Z_u} \leq c(r)} \underbrace{\|\partial_t (u - \widetilde{u})\|_X}_{\leq \|u - \widetilde{u}\|_{Z_u}} \\
&+ \underbrace{\|\partial_1 a(u, \nabla_{\Gamma^*} u) - \partial_1 a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\leq \widetilde{c} \|\ldots\|_{\nabla^1 Z_u} \leq \widetilde{c} \left(\|u - \widetilde{u}\|_{\nabla^1 Z_u} + \|\nabla_{\Gamma^*} u - \nabla_{\Gamma^*} \widetilde{u}\|_{\nabla^1 Z_u}\right)} \underbrace{\|\partial_t \widetilde{u}\|_X}_{\leq r} \\
&+ \underbrace{\|\partial_2 a(u, \nabla_{\Gamma^*} u)\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\leq \widetilde{c} \|\partial_2 a(u, \nabla_{\Gamma^*} u)\|_{\nabla^1 Z_u} \leq c(r)} \underbrace{\|\partial_t (u - \widetilde{u})\|_X}_{\leq \|u - \widetilde{u}\|_{Z_u}} \\
&+ \underbrace{\|\partial_2 a(u, \nabla_{\Gamma^*} u) - \partial_2 a(\widetilde{u}, \nabla_{\Gamma^*} \widetilde{u})\|_{L_\infty(J; L_\infty(\Gamma^*))}}_{\leq \widetilde{c} \|\ldots\|_{\nabla^1 Z_u} \leq \widetilde{c} \left(\|u - \widetilde{u}\|_{\nabla^1 Z_u} + \|\nabla_{\Gamma^*} u - \nabla_{\Gamma^*} \widetilde{u}\|_{\nabla^1 Z_u}\right)} \underbrace{\|\partial_t \widetilde{u}\|_X}_{\leq r} \\
&+ 3c \left(\underbrace{\|u - \widetilde{u}\|_{\nabla^1 Z_u}}_{\leq \widetilde{c} \|u - \widetilde{u}\|_{Z_u}} + \underbrace{\|\nabla_{\Gamma^*} u - \nabla_{\Gamma^*} \widetilde{u}\|_{\nabla^1 Z_u}}_{\leq \widetilde{c} \|u - \widetilde{u}\|_{Z_u}} \right) \\
&\leq c(r) \, \|u - \widetilde{u}\|_{Z_u} .
\end{aligned}
$$

Hence $\delta_u V_\Gamma : \overline{B_r(0)} \subseteq Z_u \longrightarrow \mathcal{L}(Z_u, X)$ is Lipschitz continuous as well, which shows the desired fact $V_\Gamma \in C^1(\overline{B_r(0)}, X)$.
The claim $v_{\partial D} \in C^1(\overline{B_r(0)}, Y_0)$ can be seen in the analogous way as for V_Γ replacing V_Γ by $v_{\partial D}$, u by ρ, δ_u by δ_ρ, ∇_{Γ^*} by ∂_σ, Z_u by Z_ρ and X by Y_0, respectively.
As seen before the integral mean of the mean curvature has the form

$$
\begin{aligned}
\overline{H}(u(t)) &= \frac{1}{\int\limits_{\Gamma(t)} 1 \, d\mathcal{H}^2} \int_{\Gamma(t)} H_{\Gamma(t)} \, d\mathcal{H}^2 \\
&= \frac{1}{\int\limits_{\Gamma^*} J(u(t), \nabla_{\Gamma^*} u(t)) \, d\mathcal{H}^2} \int_{\Gamma^*} H_\Gamma(u) J(u(t), \nabla_{\Gamma^*} u(t)) \, d\mathcal{H}^2,
\end{aligned}
$$

where J is some determinant term. Hence we can write

$$
\overline{H}(u) = \int_{\Gamma^*} \sum_{|\alpha|=2} a_\alpha(u, \nabla_{\Gamma^*} u) \partial^\alpha u + b(u, \nabla_{\Gamma^*} u) \, d\mathcal{H}^2
$$

with a_α and b similar to the considerations for H_Γ, simply including the terms $J(u, \nabla_{\Gamma^*} u)$

and $\left(\int\limits_{\Gamma^*} J(u(t),\nabla_{\Gamma^*}u(t))\,d\mathcal{H}^2\right)^{-1}$. With the same estimates as for H_Γ we obtain

$$\begin{aligned}\left\|\delta_u\overline{H}(u)-\delta_u\overline{H}(\widetilde{u})\right\|_{\mathcal{L}(Z_u,X)} &\leq \ldots \leq \int_{\Gamma^*} c(r)\,\|u-\widetilde{u}\|_{Z_u}\,d\mathcal{H}^2\\ &\leq \left(\int_{\Gamma^*} c(r)\,d\mathcal{H}^2\right)\|u-\widetilde{u}\|_{Z_u} \leq \widetilde{c}(r)\,\|u-\widetilde{u}\|_{Z_u}\end{aligned}$$

showing $\delta_u\overline{H}:\overline{B_r(0)}\subseteq Z_u\longrightarrow\mathcal{L}(Z_u,X)$ is Lipschitz continuous and $\overline{H}\in C^1(\overline{B_r(0)},X)$. Finally, the angle term is left to be considered. Here we can write

$$W(u):=\langle n_\Gamma(u),n_D(u)\rangle = b(u,\nabla_{\Gamma^*}u)$$

with $b\in C^3(U)$ implying the Lipschitz continuity of $\partial_1 b,\partial_2 b$ on $\overline{B_r(0)}\subseteq Z_u$. As before we estimate for $u,\widetilde{u}\in\overline{B_r(0)}\subseteq Z_u$

$$\begin{aligned}\|\delta_u W(u)-\delta_u W(\widetilde{u})\|_{\mathcal{L}(Z_u,\nabla^1 Z_u)} &\leq \|\partial_1 b(u,\nabla_{\Gamma^*}u)\operatorname{Id}-\partial_1 b(\widetilde{u},\nabla_{\Gamma^*}\widetilde{u})\operatorname{Id}\|_{\mathcal{L}(Z_u,\nabla^1 Z_u)}\\ &+\|\partial_2 b(u,\nabla_{\Gamma^*}u)\cdot\nabla_{\Gamma^*}-\partial_2 b(\widetilde{u},\nabla_{\Gamma^*}\widetilde{u})\cdot\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u,\nabla^1 Z_u)}\\ &\leq \|\partial_1 b(u,\nabla_{\Gamma^*}u)-\partial_1 b(\widetilde{u},\nabla_{\Gamma^*}\widetilde{u})\|_{\nabla^1 Z_u}\,\|\operatorname{Id}\|_{\mathcal{L}(Z_u,\nabla^1 Z_u)}\\ &+\|\partial_2 b(u,\nabla_{\Gamma^*}u)-\partial_2 b(\widetilde{u},\nabla_{\Gamma^*}\widetilde{u})\|_{\nabla^1 Z_u}\,\|\nabla_{\Gamma^*}\|_{\mathcal{L}(Z_u,\nabla^1 Z_u)}\\ &\leq 2c\left(\underbrace{\|u-\widetilde{u}\|_{\nabla^1 Z_u}}_{\leq\widetilde{c}\|u-\widetilde{u}\|_{Z_u}}+\underbrace{\|\nabla_{\Gamma^*}u-\nabla_{\Gamma^*}\widetilde{u}\|_{\nabla^1 Z_u}}_{\leq\widetilde{c}\|u-\widetilde{u}\|_{Z_u}}\right)\\ &\leq \widehat{c}\,\|u-\widetilde{u}\|_{Z_u}.\end{aligned}$$

Therefore, Lipschitz continuity of $\delta_u W:\overline{B_r(0)}\subseteq Z_u\longrightarrow\mathcal{L}(Z_u,\nabla^1 Z_u)$ is proven and we obtain $W\in C^1(\overline{B_r(0)},\nabla^1 Z_u)$. Taking equation (A.24) from [Gru95] into account we have for the trace operator

$$\gamma_0: W_p^{\frac{1}{2}}(J;L_p(\Gamma^*))\cap L_p(J;W_p^1(\Gamma^*))\longrightarrow W_p^{\frac{1}{2}-\frac{1}{2p}}(J;L_p(\partial\Gamma^*))\cap L_p(J;W_p^{1-\frac{1}{p}}(\partial\Gamma^*)),$$

which is equivalent to $\gamma_0:\nabla^1 Z_u\longrightarrow Y_0$ and proves $\gamma_0\circ W\in C^1(\overline{B_r(0)},Y_0)$.
All of these continuity statements show $N\in C^1(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O});\mathbb{F})$ if we choose the radius r appropriately.
Now we consider the remaining statement $\|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E},\mathbb{F})}\leq cT^{\frac{1}{q}}$. First we remark that without the $\overline{H}$-term in N we would obtain $DN[\mathbb{O}]=0$. Therefore, it suffices to consider $\left\|D\overline{H}[\mathbb{O}]\right\|_{\mathcal{L}(\mathbb{E},X)}$. We estimate with the help of Lemma 2.13

$$\left\|D\overline{H}[\mathbb{O}](u,\rho)\right\|_X\leq\left\|\frac{1}{\int\limits_{\Gamma^*}1\,d\mathcal{H}^2}\underbrace{\int\limits_{\Gamma^*}(\Delta_{\Gamma^*}+|\sigma^*|^2-H_{\Gamma^*}^2+\overline{H}(\mathbb{O})H_{\Gamma^*})u\,d\mathcal{H}^2}_{\text{independent of the space variable}}\right\|_X$$

$$
\begin{aligned}
&+ \left\| \frac{1}{\int_{\Gamma^*} 1 \, d\mathcal{H}^2} \underbrace{\int_{\partial\Gamma^*} (H_{\Gamma^*} - \overline{H}(\mathbb{O}) \cot(\alpha)\rho \, d\mathcal{H}^1}_{\text{independent of the space variable}} \right\|_X \\
&\leq A(\Gamma^*)^{\frac{1}{p}-1} \left\| \int_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2 - H_{\Gamma^*}^2 + \overline{H}(\mathbb{O}) H_{\Gamma^*}) u \, d\mathcal{H}^2 \right\|_{L_p(J)} \\
&+ A(\Gamma^*)^{\frac{1}{p}-1} \left\| \int_{\partial\Gamma^*} (H_{\Gamma^*} - \overline{H}(\mathbb{O})) \cot(\alpha)\rho \, d\mathcal{H}^1 \right\|_{L_p(J)} \\
&\leq c \left(\left\| \int_{\Gamma^*} \Delta_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_p(J)} + \left\| \int_{\Gamma^*} (\underbrace{|\sigma^*|^2 - H_{\Gamma^*}^2 + \overline{H}(\mathbb{O}) H_{\Gamma^*}}_{\text{bounded}}) u \, d\mathcal{H}^2 \right\|_{L_p(J)} \right. \\
&\left. + \left\| \int_{\partial\Gamma^*} \underbrace{(H_{\Gamma^*} - \overline{H}(\mathbb{O})) \cot(\alpha)}_{\text{bounded}} \rho \, d\mathcal{H}^1 \right\|_{L_p(J)} \right) \\
&\leq \widetilde{c} \left(\left\| \int_{\Gamma^*} \Delta_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_p(J)} + \left\| \int_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_p(J)} + \left\| \int_{\partial\Gamma^*} \rho \, d\mathcal{H}^1 \right\|_{L_p(J)} \right).
\end{aligned}
$$

Using Gauss' theorem for hypersurfaces (cf. Theorem 2.29 in [Dep10]) we can write the first integral as a lower order boundary integral. Choosing $q > p$ arbitrarily and $r > p$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and utilizing Hölder's inequality we can continue the estimate above as follows

$$
\begin{aligned}
\left\| D\overline{H}[\mathbb{O}](u,\rho) \right\|_X &\leq \widetilde{c} \left(\left\| \int_{\Gamma^*} \Delta_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_p(J)} + \left\| \int_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_p(J)} + \left\| \int_{\partial\Gamma^*} \rho \, d\mathcal{H}^1 \right\|_{L_p(J)} \right) \\
&= \widetilde{c} \left(\left\| \int_{\partial\Gamma^*} \nabla_{\Gamma^*} u \cdot n_{\partial\Gamma^*} \, d\mathcal{H}^1 \right\|_{L_p(J)} + \left\| \int_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_p(J)} \right. \\
&\left. + \left\| \int_{\partial\Gamma^*} \rho \, d\mathcal{H}^1 \right\|_{L_p(J)} \right) \\
&\leq \widehat{c} \left(\|1\|_{L_q(J)} \left\| \int_{\partial\Gamma^*} \nabla_{\Gamma^*} u \cdot n_{\partial\Gamma^*} \, d\mathcal{H}^1 \right\|_{L_r(J)} + \|1\|_{L_q(J)} \left\| \int_{\Gamma^*} u \, d\mathcal{H}^2 \right\|_{L_r(J)} \right. \\
&\left. + \|1\|_{L_q(J)} \left\| \int_{\partial\Gamma^*} \rho \, d\mathcal{H}^1 \right\|_{L_r(J)} \right) \\
&\leq cT^{\frac{1}{q}} \left(\|\nabla_{\Gamma^*} u\|_{L_r(J;L_1(\partial\Gamma^*))} + \|u\|_{L_r(J;L_1(\Gamma^*))} + \|\rho\|_{L_r(J;L_1(\partial\Gamma^*))} \right).
\end{aligned}
$$

As seen before we have $\nabla_{\Gamma^*} u \in \nabla^1 Z_u$ and by Lemma 3.15(i) with $\sigma = 1 - \frac{2}{p}$, the trace operator γ_0 and $r > p > 0$ we obtain the embeddings

$$
\begin{aligned}
\nabla^1 Z_u = W_p^{\frac{1}{2}}(J; L_p(\Gamma^*)) \cap L_p(J; W_p^1(\Gamma^*)) &\hookrightarrow W_p^{\frac{1}{2}-\frac{1}{p}}(J; W_p^{\frac{2}{p}}(\Gamma^*)) \\
&\hookrightarrow W_p^{\frac{1}{2}-\frac{1}{p}}(J; W_p^{\frac{1}{p}}(\partial\Gamma^*)) \hookrightarrow L_r(J; L_1(\partial\Gamma^*)),
\end{aligned}
$$

which shows $\|\nabla_{\Gamma^*} u\|_{L_r(J;L_1(\partial\Gamma^*))} \leq c\,\|\nabla_{\Gamma^*} u\|_{\nabla^1 Z_u} \leq \widehat{c}\,\|u\|_{Z_u}$. Without using the trace operator in the above estimate we can prove $\|u\|_{L_r(J;L_1(\Gamma^*))} \leq \|u\|_{\nabla^1 Z_u} \leq \widehat{c}\,\|u\|_{Z_u}$. Finally $\|\rho\|_{L_r(J;L_1(\partial\Gamma^*))} \leq c\,\|\rho\|_{Z_\rho}$, because of

$$Z_\rho = W_p^{\frac{3}{2}-\frac{1}{2p}}(J; L_p(\partial\Gamma^*)) \cap L_p(J; W_p^{3-\frac{1}{p}}(\Gamma^*)) \subseteq W_p^{\frac{3}{2}-\frac{1}{2p}}(J; L_p(\partial\Gamma^*)) \hookrightarrow L_r(J; L_1(\partial\Gamma^*)).$$

Using these three facts we see

$$\begin{aligned}\left\|D\overline{H}[\mathbb{O}](u,\rho)\right\|_X &\leq cT^{\frac{1}{q}}\left(\|\nabla_{\Gamma^*} u\|_{L_r(J;L_1(\partial\Gamma^*))} + \|u\|_{L_r(J;L_1(\Gamma^*))} + \|\rho\|_{L_r(J;L_1(\partial\Gamma^*))}\right)\\ &\leq \widehat{c}T^{\frac{1}{q}}\left(2\,\|u\|_{Z_u} + \|\rho\|_{Z_\rho}\right) \leq cT^{\frac{1}{q}}\,\|(u,\rho)\|_{\mathbb{E}}\end{aligned}$$

proving the desired estimate $\left\|D\overline{H}[\mathbb{O}]\right\|_{\mathcal{L}(\mathbb{E},X)} \leq cT^{\frac{1}{q}}$. ■

Remark 3.19: (i) An important fact for the following considerations is that L is an isomorphism. We do not need to consider the condition $g_0(0) - \mathcal{B}_0(0)u_0 - \mathcal{C}_0(0)\rho_0 \in \pi_1 Z_\rho$ in Theorem 3.2 since due to the same argumentation as in proof of Corollary 3.3 we see $-\mathcal{B}_0(0)u_0 - \mathcal{C}_0(0)\rho_0 \in \pi_1 Z_\rho$ and by (A.25) in [Gru95] we see that for $g_0 \in Y_0$ one has $g_0(0) \in W_p^{1-\frac{3}{p}}(\partial\Gamma^*;\mathbb{R}) = \pi_1 Z_\rho$. Moreover, the condition $\mathcal{B}_1(0)u_0 + \mathcal{C}_1(0)\rho_0 = g_1(0)$ can be dropped, because $g_1 \equiv 0$ and $(u_0,\rho_0) \in \mathbb{I}$. Due to Theorem 3.2 L is an isomorphism between $\mathbb{E}$ and $\mathbb{F} \times \mathbb{I}$.
(ii) Although we have not indicated this dependence so far, the spaces $\mathbb{E}$ and $\mathbb{F}$ actually depend on T and should have been better denoted by $\mathbb{E}_T$ and $\mathbb{F}_T$. The same is true for the operators L and N. The justification for this notational inexactness will be given in the following lemma. This will be the first and also last segment where we will use the exact notation to indicate the dependence on T. □

Lemma 3.20: *Let $T_0 > 0$ be fixed and $T \in (0,T_0)$ arbitrary.*
(i) There exists a bounded extension operator from $\mathbb{F}_T$ to $\mathbb{F}_{T_0}$ with norms uniformly bounded in $T \leq T_0$, i.e. for all $f \in \mathbb{F}_T$ there is a $\widetilde{f} \in \mathbb{F}_{T_0}$ with $\widetilde{f}\Big|_{[0,T]} = f$ and $\left\|\widetilde{f}\right\|_{\mathbb{F}_{T_0}} \leq c(T_0)\,\|f\|_{\mathbb{F}_T}$.
(ii) The operator norm of $L_T^{-1} : \mathbb{F}_T \times \mathbb{I} \longrightarrow \mathbb{E}_T$ is uniformly bounded in T.
(iii) There exists a bounded extension operator from $\mathbb{E}_T$ to $\mathbb{E}_{T_0}$, i.e. for all $\Phi \in \mathbb{E}_T$ there is a $\widetilde{\Phi} \in \mathbb{E}_{T_0}$ with $\widetilde{\Phi}\Big|_{[0,T]} = \Phi$ and $\left\|\widetilde{\Phi}\right\|_{\mathbb{E}_{T_0}} \leq c(T_0)\,\|\Phi\|_{\mathbb{E}_T}$.
(iv) The uniform estimate $\|DN_T[\Phi] - DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E}_T,\mathbb{F}_T)} \leq c(T_0)\,\|\Phi\|_{\mathbb{E}_T} < \infty$ holds for $\Phi \in B_r^{\mathbb{E}_T}(0)$.

<u>Proof</u>: (i) Let $(f_1, f_2, 0) \in \mathbb{F}_T$. To define the extension we solve

$$\begin{aligned}\frac{d}{dt}\widehat{g}(t) - \partial_\sigma^2\widehat{g}(t) &= 0 && \text{on } [0,T_0]\times\partial\Gamma^*\\ \widehat{g}(0) &= f_2(T) && \text{on } \partial\Gamma^*,\end{aligned}$$

where the trace in $t = T$ of a function $f_2 \in Y_0$ is an element of $\pi_1 Z_\rho$ (cf. (A.25) of [Gru95]). We obtain a unique

$$\widehat{g} \in Y_0^{T_0} := W_p^1([0,T_0]; W_p^{-1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \cap L_p([0,T_0]; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$$

with

$$\|\widehat{g}\|_{Y_0^{T_0}} \leq c(T_0)\left(\|0\| + \|f_2(T)\|_{\pi_1 Z_\rho}\right) = c(T_0)\,\|f_2(T)\|_{\pi_1 Z_\rho}\,.$$

We define the extension $(\widetilde{f}_1, \widetilde{f}_2, 0) \in \mathbb{F}_{T_0}$ by

$$(\widetilde{f}_1, \widetilde{f}_2, 0) := \begin{cases} (f_1, f_2, 0) & \text{for } t \in [0,T] \\ (0, \widehat{g}(t-T), 0) & \text{for } t \in (T, T_0] \end{cases}$$

and have the estimate

$$\begin{aligned} \left\|(\widetilde{f}_1, \widetilde{f}_2, 0)\right\|_{\mathbb{F}_{T_0}} &\leq \|(f_1, f_2, 0)\|_{\mathbb{F}_T} + \|(0, \widehat{g}, 0)\|_{\mathbb{F}_{T_0}} = \|(f_1, f_2, 0)\|_{\mathbb{F}_T} + \|\widehat{g}\|_{Y_0^{T_0}} \\ &\leq \|(f_1, f_2, 0)\|_{\mathbb{F}_T} + c(T_0)\underbrace{\|f_2(T)\|_{\pi_1 Z_\rho}}_{\leq c\|f_2\|_{Y_0^T}} \leq \widehat{c}(T_0)\,\|(f_1, f_2, 0)\|_{\mathbb{F}_T} \end{aligned}$$

where the uniform estimate $\|f_2(T)\|_{\pi_1 Z_\rho} \leq c\,\|f_2\|_{Y_0^T}$ follows from Theorem III.4.10.2 of [Ama95] and Lemma 7.2 of [Ama05].
(ii) We know that for $(f, \Phi_0) \in \mathbb{F}_T \times \mathbb{I}$ there is a unique solution $\Phi \in \mathbb{E}_T$ of $L_T\Phi = (f, \Phi_0)$. We use the extension $\widetilde{f} \in \mathbb{F}_{T_0}$ from (i) to obtain a unique solution $\widetilde{\Phi} \in \mathbb{E}_{T_0}$ such that $L_{T_0}\widetilde{\Phi} = (\widetilde{f}, \Phi_0)$. Comparing Φ and $\widetilde{\Phi}$ we see by the uniqueness that $\widetilde{\Phi}\big|_{[0,T]} = \Phi$ holds. Therefore we obtain the following estimate

$$\begin{aligned} \left\|L_T^{-1}(f, \Phi_0)\right\|_{\mathbb{E}_T} &= \|\Phi\|_{\mathbb{E}_T} \leq \left\|\widetilde{\Phi}\right\|_{\mathbb{E}_{T_0}} = \left\|L_{T_0}^{-1}(\widetilde{f}, \Phi_0)\right\|_{\mathbb{E}_{T_0}} \\ &\leq c(T_0)\left(\left\|\widetilde{f}\right\|_{\mathbb{F}_{T_0}} + \|\Phi_0\|_{\mathbb{I}}\right) \leq \widetilde{c}(T_0)\left(\|f\|_{\mathbb{F}_T} + \|\Phi_0\|_{\mathbb{I}}\right). \end{aligned}$$

This proves

$$\sup_{T \in (0, T_0]} \left\|L_T^{-1}\right\|_{\mathcal{L}(\mathbb{F}_T \times \mathbb{I}, \mathbb{E}_T)} \leq \widetilde{c}(T_0) < \infty,$$

where it is important to note that $\widetilde{c}(T_0)$ only depends on the fixed T_0 but not on T.
(iii) For $\Phi \in \mathbb{E}_T$ we define $(f, \Phi_0) := L_T\Phi \in \mathbb{F}_T \times \mathbb{I}$ and use the extension from (i) to obtain $(\widetilde{f}, \Phi_0) \in \mathbb{F}_{T_0} \times \mathbb{I}$. Solving $L_{T_0}\widetilde{\Phi} = (\widetilde{f}, \Phi_0)$ on $[0, T_0]$ leads to a unique $\widetilde{\Phi} \in \mathbb{E}_{T_0}$. Due to the uniqueness on $[0,T]$ we have $\widetilde{\Phi}\big|_{[0,T]} = \Phi$. Moreover, we use $\left\|\widetilde{f}\right\|_{\mathbb{F}_{T_0}} \leq c(T_0)\,\|f\|_{\mathbb{F}_T}$ from (i) to end up with

$$\begin{aligned} \left\|\widetilde{\Phi}\right\|_{\mathbb{E}_{T_0}} &\leq c(T_0)\left(\|f\|_{\mathbb{F}_{T_0}} + \|\Phi_0\|_{\mathbb{I}}\right) \leq \widehat{c}(T_0)\left(\|f\|_{\mathbb{F}_T} + \|\Phi_0\|_{\mathbb{I}}\right) \\ &= \widehat{c}(T_0)\,\|(f, \Phi_0)\|_{\mathbb{F}_T \times \mathbb{I}} = \widehat{c}(T_0)\,\|L_T\Phi\|_{\mathbb{F}_T \times \mathbb{I}} \leq \widehat{c}(T_0)\,\|L_T\|\,\|\Phi\|_{\mathbb{E}_T}\,, \end{aligned}$$

where $\|L_T\| \leq c$ follows from direct estimates, since the coefficients are uniformly bounded.
(iv) Via the extension from (iii) we see

$$N_T(\Phi) = N_T\left(\widetilde{\Phi}\big|_{[0,T]}\right) = N_{T_0}\left(\widetilde{\Phi}\right)\Big|_{[0,T]}$$

and hence $DN_T[\Phi](v) = DN_{T_0}[\widetilde{\Phi}](\widetilde{v})\Big|_{[0,T]}$. Therefore we get for the norms the following estimate

$$\begin{aligned}
\|DN_T[\Phi](v) - DN_T[\mathbb{O}](v)\|_{\mathbb{F}_T} &\leq \left\|DN_{T_0}[\widetilde{\Phi}](\widetilde{v}) - DN_{T_0}[\mathbb{O}](\widetilde{v})\right\|_{\mathbb{F}_{T_0}} \\
&\leq \left\|DN_{T_0}[\widetilde{\Phi}] - DN_{T_0}[\mathbb{O}]\right\|_{\mathcal{L}(\mathbb{E}_{T_0},\mathbb{F}_{T_0})} \|\widetilde{v}\|_{\mathbb{E}_{T_0}} \\
&\leq c(T_0) \left\|DN_{T_0}[\widetilde{\Phi}] - DN_{T_0}[\mathbb{O}]\right\|_{\mathcal{L}(\mathbb{E}_{T_0},\mathbb{F}_{T_0})} \|v\|_{\mathbb{E}_T} \\
&\leq c(T_0)\widetilde{c}(T_0) \left\|\widetilde{\Phi}\right\|_{\mathbb{E}_{T_0}} \|v\|_{\mathbb{E}_T} \\
&\leq \underbrace{c(T_0)^2\widetilde{c}(T_0)}_{=:\widehat{c}(T_0)} \|\Phi\|_{\mathbb{E}_T} \|v\|_{\mathbb{E}_T}.
\end{aligned}$$

This leads to $\|DN_T[\Phi] - DN_T[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E}_T,\mathbb{F}_T)} \leq \widehat{c}(T_0) \|\Phi\|_{\mathbb{E}_T} < \infty$. ■

The two recently proven lemmas are the main tools for the application of the contraction mapping principle.

Lemma 3.21: *Let $4 < p < \infty$ and $J := [0,T]$ where $T > 0$ must be chosen sufficiently small. Then there exists some $\varepsilon > 0$ such that for each $\Phi_0 \in \mathbb{I}$ with $\|\Phi_0\|_{\mathbb{I}} < \varepsilon$ there exists a unique solution $\Phi = (u,\rho) \in \mathbb{E}$ of the equation $L\Phi = (N(\Phi), \Phi_0)$.*

<u>Proof</u>: The equation $L\Phi = (N(\Phi), \Phi_0)$ is equivalent to the fixed point problem $K(\Phi) = \Phi$, where

$$K(\Phi) := L^{-1}(N(\Phi), \Phi_0) \qquad \forall\, \Phi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O}).$$

We set $\mathbb{X}_r := \left\{\Phi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O}) \mid \Phi(0) = \Phi_0\right\}$. By Lemma 3.20(iv) we can choose $r > 0$ independent of T such that

$$\sup_{\Psi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \leq \frac{1}{4 \sup\limits_{T\in[0,T_0]} \|L^{-1}\|} + \|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})}.$$

Then we see that for all $T \in [0, T_0]$ we have

$$\sup_{\Psi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \leq \frac{1}{4\|L^{-1}\|} + \|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})}.$$

Before stating the main estimate we have to look at $\|N(\mathbb{O})\|_{\mathbb{F}}$. Here we see

$$\begin{aligned}
\|N(\mathbb{O})\|_{\mathbb{F}} &= \left\|H_{\Gamma^*} - \overline{H}(\mathbb{O})\right\|_X + \|a + b\varkappa_{\partial D^*} + \langle n_{\Gamma^*}, n_{D^*}\rangle\|_{Y_0} \\
&= T^{\frac{1}{p}} \left\|H_{\Gamma^*} - \overline{H}(\mathbb{O})\right\|_{L_p(\Gamma^*;\mathbb{R})} + T^{\frac{1}{p}} \|a + b\varkappa_{\partial D^*} + \langle n_{\Gamma^*}, n_{D^*}\rangle\|_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})},
\end{aligned}$$

because all the terms $H_{\Gamma^*}, \overline{H}(\mathbb{O}), a, b\varkappa_{\partial D^*}$ and $\langle n_{\Gamma^*}, n_{D^*}\rangle$ are time-independent. Hence $\|N(\mathbb{O})\|_{\mathbb{F}} \xrightarrow[T\to 0]{} 0$. This fact and Lemma 3.18 show that for a sufficiently small time interval $[0,T]$ we get $\|N(\mathbb{O})\|_{\mathbb{F}} \leq \varepsilon$ and $\|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \leq \varepsilon$ for arbitrary small $\varepsilon > 0$. We use these facts in the estimate

$$\begin{aligned}
\|K(\Phi)\|_{\mathbb{E}} &\leq \left\|L^{-1}\right\| (\|N(\Phi)\|_{\mathbb{F}} + \|\Phi_0\|_{\mathbb{I}}) \leq \left\|L^{-1}\right\| (\|N(\Phi) - N(\mathbb{O})\|_{\mathbb{F}} + \|N(\mathbb{O})\|_{\mathbb{F}} + \|\Phi_0\|_{\mathbb{I}}) \\
&\leq \left\|L^{-1}\right\| \left(\sup_{\Psi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \|\Phi\|_{\mathbb{E}} + \|N(\mathbb{O})\|_{\mathbb{F}} + \|\Phi_0\|_{\mathbb{I}} \right) \\
&\leq \frac{1}{4} \|\Phi\|_{\mathbb{E}} + \left\|L^{-1}\right\| \|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \|\Phi\|_{\mathbb{E}} + \left\|L^{-1}\right\| \|N(\mathbb{O})\|_{\mathbb{F}} + \left\|L^{-1}\right\| \|\Phi_0\|_{\mathbb{I}} \\
&\leq \frac{r}{4} + \left\|L^{-1}\right\| r\varepsilon + 2\left\|L^{-1}\right\| \varepsilon
\end{aligned}$$

for every $\Phi \in \mathbb{X}_r$. By choosing

$$\varepsilon(r) \leq \min \left\{ \frac{1}{4 \sup\limits_{T \in [0,T_0]} \|L^{-1}\|}, \frac{r}{4 \sup\limits_{T \in [0,T_0]} \|L^{-1}\|} \right\}$$

independent of T, we get

$$\varepsilon = \varepsilon(r) \leq \min \left\{ \frac{1}{4\,\|L^{-1}\|}, \frac{r}{4\,\|L^{-1}\|} \right\} \qquad \forall\, T \in [0,T_0]$$

and hence $\|K(\Phi)\| \leq \frac{r}{4} + \frac{r}{4} + \frac{r}{2} = r$, i.e. $K(\mathbb{X}_r) \subseteq \mathbb{X}_r$. To see that K is contractive, we use Lemma 3.18 again and observe that for all $\Phi_1, \Phi_2 \in \mathbb{X}_r$ the following holds

$$\begin{aligned}
\|K(\Phi_1) - K(\Phi_2)\|_{\mathbb{E}} &\leq \left\|L^{-1}\right\| \|N(\Phi_1) - N(\Phi_2)\|_{\mathbb{F}} \\
&\leq \left\|L^{-1}\right\| \sup_{\Psi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \|\Phi_1 - \Phi_2\|_{\mathbb{E}} \\
&\leq \frac{1}{4} \|\Phi_1 - \Phi_2\|_{\mathbb{E}} + \left\|L^{-1}\right\| \|DN[\mathbb{O}]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \|\Phi_1 - \Phi_2\|_{\mathbb{E}} \\
&\leq \frac{1}{4} \|\Phi_1 - \Phi_2\|_{\mathbb{E}} + c \left\|L^{-1}\right\| T^{\frac{1}{q}} \|\Phi_1 - \Phi_2\|_{\mathbb{E}}\,.
\end{aligned}$$

Choosing T smaller than $\left(\frac{1}{4c\|L^{-1}\|}\right)^q$ we see $\|K(\Phi_1) - K(\Phi_2)\|_{\mathbb{E}} \leq \frac{1}{2} \|\Phi_1 - \Phi_2\|_{\mathbb{E}}$ and hence $K : \mathbb{X}_r \longrightarrow \mathbb{X}_r$ is a contraction and the assertion follows from the contraction mapping principle. ■

Transforming this statement into our original situation we can establish the following theorem.

Theorem 3.22: *Let $T > 0$ be sufficiently small and $4 < p < \infty$. Then there exists an $\varepsilon > 0$ such that for each $\varrho_0 \in \pi Z_u$ with $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho$ and $\|\varrho_0\|_{\pi Z_u} + \|\varrho_0|_{\partial\Gamma^*}\|_{\pi Z_\rho} < \varepsilon$ there exists a unique solution $\varrho \in Z_u$ with $\varrho|_{\partial\Gamma^*} \in Z_\rho$ of the system*

$$\begin{aligned}
V_\Gamma(\varrho(t)) &= H_\Gamma(\varrho(t)) - \overline{H}(\varrho(t)) && \textit{in } [0,T] \times \Gamma^* \\
v_{\partial D}(\varrho(t)) &= a + b\varkappa_{\partial D}(\varrho(t)) + \langle n_\Gamma(\varrho(t)), n_D(\varrho(t)) \rangle && \textit{on } [0,T] \times \partial\Gamma^* \\
\varrho(0) &= \varrho_0 && \textit{in } \Gamma^*.
\end{aligned}$$

Proof: Rewriting Lemma 3.21 in terms of ϱ instead of Φ immediately leads to the result.■

This theorem is the result of all the considerations of Section 2 and 3 and completes the first part of this thesis. In the next two sections we will examine the Willmore Flow with the same strategy to prove the analogous result.

4 The Willmore flow and its linearization

This section is devoted to the Willmore Flow with line tension effects on the boundary, where we will perform the analogous steps as in Section 2 for the MCF. The difference is that this time there is no need to take care of non-local terms, but instead we will have operators of higher order.

4.1 The Willmore Flow

As the MCF the Willmore Flow will also be introduced as the best way to minimize an energy functional, namely the Willmore energy. Here stationary surfaces are called Willmore surfaces and are possible local minimizers of the Willmore energy. Determining the shape of such a Willmore surface is in general very difficult, but will not play any role in our considerations.

Definition 4.1: The Willmore functional of a 2-dimensional hypersurface Γ is defined as

$$W(\Gamma) := \frac{1}{4}\int_\Gamma H_\Gamma^2 \, d\mathcal{H}^2. \qquad \square$$

Again our first step is to calculate the first variation of that energy, where we assume the way of varying to be the same as in (2.3)-(2.4).

Theorem 4.2: *As the first variation of the Willmore functional we get*

$$\begin{aligned}(\delta W(\Gamma))(\zeta) = &\frac{1}{2}\int_\Gamma \left(\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3\right)(n_\Gamma \cdot \zeta)\, d\mathcal{H}^2 \\ &+ \frac{1}{2}\int_{\partial\Gamma} \frac{1}{2}H_\Gamma^2 (n_{\partial\Gamma}\cdot\zeta) + H_\Gamma(\nabla_\Gamma(n_\Gamma\cdot\zeta)\cdot n_{\partial\Gamma}) - (\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma})(n_\Gamma\cdot\zeta)\, d\mathcal{H}^1.\end{aligned}$$

Proof: We apply again Theorem 2.4 and obtain

$$(\delta W(\Gamma))(\zeta) = \frac{d}{dt}\frac{1}{4}\int_{\Gamma(t)} H_\Gamma^2\, d\mathcal{H}^2\bigg|_{t=0} = \int_\Gamma \frac{1}{2}H_\Gamma \partial^\circ H_\Gamma - \frac{1}{4}H_\Gamma^3 V_\Gamma \, d\mathcal{H}^2 + \frac{1}{4}\int_{\partial\Gamma} H_\Gamma^2 v_{\partial\Gamma}\, d\mathcal{H}^1.$$

By Lemma 5.1 of [Dep10] we can write

$$(\delta W(\Gamma))(\zeta) = \frac{1}{2}\int_\Gamma H_\Gamma\left(\Delta_\Gamma V_\Gamma + V_\Gamma\sum_{i=1}^{2}\varkappa_i^2\right) - \frac{1}{2}H_\Gamma^3 V_\Gamma\, d\mathcal{H}^2 + \frac{1}{4}\int_{\partial\Gamma} H_\Gamma^2 v_{\partial\Gamma}\, d\mathcal{H}^1.$$

Applying partial integration twice yields

$$\begin{aligned}\int_\Gamma H_\Gamma \Delta_\Gamma V_\Gamma\, d\mathcal{H}^2 &= \int_{\partial\Gamma} H_\Gamma(\nabla_\Gamma V_\Gamma\cdot n_{\partial\Gamma})\, d\mathcal{H}^1 - \int_\Gamma \nabla_\Gamma H_\Gamma\cdot\nabla_\Gamma V_\Gamma\, d\mathcal{H}^2 \\ &= \int_{\partial\Gamma} H_\Gamma(\nabla_\Gamma V_\Gamma\cdot n_{\partial\Gamma})\, d\mathcal{H}^1 - \int_{\partial\Gamma}(\nabla_\Gamma H_\Gamma\cdot n_{\partial\Gamma})V_\Gamma\, d\mathcal{H}^1 \\ &\quad + \int_\Gamma (\Delta_\Gamma H_\Gamma)V_\Gamma\, d\mathcal{H}^2,\end{aligned}$$

which leads to

$$
\begin{aligned}
(\delta W(\Gamma))(\zeta) &= \frac{1}{2}\int_\Gamma (\Delta_\Gamma H_\Gamma)V_\Gamma + H_\Gamma V_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2} H_\Gamma^3 V_\Gamma \, d\mathcal{H}^2 \\
&\quad + \frac{1}{2}\int_{\partial\Gamma} \frac{1}{2} H_\Gamma^2 v_{\partial\Gamma} + H_\Gamma(\nabla_\Gamma V_\Gamma \cdot n_{\partial\Gamma}) - (\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma})V_\Gamma \, d\mathcal{H}^1.
\end{aligned}
$$

Hence the claim follows from the formulas (2.5). ■

Again our first aim is to find necessary conditions for stationary solutions of the Willmore functional with contact area functional and line tension, therefore we consider

$$
\mathrm{WE}(\Gamma) := \frac{1}{4}\int_\Gamma H^2 \, d\mathcal{H}^2 - a\int_D 1 \, d\mathcal{H}^2 + b\int_{\partial\Gamma} 1 \, d\mathcal{H}^1. \tag{4.1}
$$

Due to Theorems 2.6, 2.7 and 4.2 the first variation of the energy WE reads as

$$
\begin{aligned}
(\delta\,\mathrm{WE}(\Gamma))(\zeta) &= \frac{1}{2}\int_\Gamma \left(\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3\right)(n_\Gamma\cdot\zeta)\, d\mathcal{H}^2 \\
&\quad + \frac{1}{2}\int_{\partial\Gamma} \frac{1}{2}H_\Gamma^2 (n_{\partial\Gamma}\cdot\zeta) + H_\Gamma(\nabla_\Gamma(n_\Gamma\cdot\zeta)\cdot n_{\partial\Gamma}) - (\nabla_\Gamma H_\Gamma\cdot n_{\partial\Gamma})(n_\Gamma\cdot\zeta)\, d\mathcal{H}^1 \\
&\quad + a\int_D H_D \underbrace{(n_D\cdot\zeta)}_{=0}\, d\mathcal{H}^2 - a\int_{\partial D} n_{\partial D}\cdot\zeta \, d\mathcal{H}^1 - b\int_{\partial\Gamma} \vec{\varkappa}\cdot\zeta\, d\mathcal{H}^1 \\
&= \frac{1}{2}\int_\Gamma \left(\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3\right)(n_\Gamma\cdot\zeta)\, d\mathcal{H}^2 \\
&\quad + \frac{1}{2}\int_{\partial\Gamma} \frac{1}{2}H_\Gamma^2 (n_{\partial\Gamma}\cdot\zeta) + H_\Gamma(\nabla_\Gamma(n_\Gamma\cdot\zeta)\cdot n_{\partial\Gamma}) - (\nabla_\Gamma H_\Gamma\cdot n_{\partial\Gamma})(n_\Gamma\cdot\zeta)\, d\mathcal{H}^1 \\
&\quad - \int_{\partial\Gamma} a(n_{\partial D}\cdot\zeta) + b(\vec{\varkappa}\cdot\zeta)\, d\mathcal{H}^1.
\end{aligned} \tag{4.2}
$$

Before we derive necessary conditions for stationary solutions we should have a closer look at the second term in the boundary integral. The surface gradient ∇_Γ is defined via an orthonormal basis. Due to the fact that we are on $\partial\Gamma$ we use $\{\vec{\tau}, n_{\partial\Gamma}\}$ from Remark 2.1(i) to obtain

$$
\nabla_\Gamma(n_\Gamma\cdot\zeta) = \partial_{\vec{\tau}}(n_\Gamma\cdot\zeta)\vec{\tau} + \partial_{n_{\partial\Gamma}}(n_\Gamma\cdot\zeta)n_{\partial\Gamma}
$$

and multiplying this equation by $n_{\partial\Gamma}$ one arrives at

$$
\nabla_\Gamma(n_\Gamma\cdot\zeta)\cdot n_{\partial\Gamma} = \partial_{n_{\partial\Gamma}}(n_\Gamma\cdot\zeta) = (\partial_{n_{\partial\Gamma}} n_\Gamma)\cdot\zeta + n_\Gamma\cdot(\partial_{n_{\partial\Gamma}}\zeta)
$$

due to $\vec{\tau}\cdot n_{\partial\Gamma} = 0$ and $n_{\partial\Gamma}\cdot n_{\partial\Gamma} = 1$. It becomes apparent that (4.2) can be written as

$$
\begin{aligned}
(\delta\,\mathrm{WE}(\Gamma))(\zeta) &= \frac{1}{2}\int_\Gamma \left(\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3\right)(n_\Gamma\cdot\zeta)\, d\mathcal{H}^2 \\
&\quad + \frac{1}{2}\int_{\partial\Gamma} \frac{1}{2}H_\Gamma^2 (n_{\partial\Gamma}\cdot\zeta) + H_\Gamma\left((\partial_{n_{\partial\Gamma}} n_\Gamma)\cdot\zeta + n_\Gamma\cdot(\partial_{n_{\partial\Gamma}}\zeta)\right)\, d\mathcal{H}^1 \\
&\quad - \int_{\partial\Gamma} \frac{1}{2}(\nabla_\Gamma H_\Gamma\cdot n_{\partial\Gamma})(n_\Gamma\cdot\zeta) + a(n_{\partial D}\cdot\zeta) + b(\vec{\varkappa}\cdot\zeta)\, d\mathcal{H}^1.
\end{aligned} \tag{4.3}
$$

First we assume with the same $\mathcal{F}(\Gamma)$ as in (2.4) that $\zeta_1 \in \mathcal{F}(\Gamma)$ satisfies $\zeta_1|_{\partial\Gamma} \equiv 0$ and $\partial_{n_{\partial\Gamma}}\zeta_1|_{\partial\Gamma} = 0$ in order to see

$$0 = (\delta\,\mathrm{WE}(\Gamma))(\zeta_1) = \frac{1}{2}\int_\Gamma \left(\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3\right)(n_\Gamma \cdot \zeta_1)\, d\mathcal{H}^2,$$

which gives by means of the fundamental lemma of calculus of variations

$$\frac{1}{2}\left(\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3\right) n_\Gamma = 0.$$

Multiplied by $2n_\Gamma$ we get as first necessary condition

$$\Delta_\Gamma H_\Gamma + H_\Gamma \sum_{i=1}^{2} \varkappa_i^2 - \frac{1}{2}H_\Gamma^3 = 0 \qquad \text{on } \Gamma, \tag{4.4}$$

which is known as the Willmore equation.

Remark 4.3: Note the different version of the Willmore equation given by

$$\Delta_\Gamma H_\Gamma + \frac{1}{2}H_\Gamma\left(H_\Gamma^2 - 4K_\Gamma\right) = 0,$$

where K_Γ is the Gauss curvature of Γ. This is equivalent to (4.4) because

$$\begin{aligned}\frac{1}{2}\left(H_\Gamma^2 - 4K_\Gamma\right) &= \frac{1}{2}\left((\varkappa_1 + \varkappa_2)^2 - 4\varkappa_1\varkappa_2\right) = \frac{1}{2}\left(\varkappa_1^2 - 2\varkappa_1\varkappa_2 + \varkappa_2^2\right)\\ &= \frac{1}{2}\left(2\varkappa_1^2 + 2\varkappa_2^2 - (\varkappa_1+\varkappa_2)^2\right) = \varkappa_1^2 + \varkappa_2^2 - \frac{1}{2}H_\Gamma^2.\end{aligned}$$

□

Stationary solutions therefore satisfy (4.4) and hence (4.3) simplifies to

$$\begin{aligned}(\delta\,\mathrm{WE}(\Gamma_{\mathrm{stat}}))(\zeta) = \frac{1}{2}\int_{\partial\Gamma} &\frac{1}{2}H_\Gamma^2(n_{\partial\Gamma}\cdot\zeta) + H_\Gamma((\partial_{n_{\partial\Gamma}} n_\Gamma)\cdot\zeta + n_\Gamma\cdot(\partial_{n_{\partial\Gamma}}\zeta))\, d\mathcal{H}^1\\ &- \int_{\partial\Gamma} \frac{1}{2}(\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma})(n_\Gamma\cdot\zeta) + a(n_{\partial D}\cdot\zeta) + b(\vec{\varkappa}\cdot\zeta)\, d\mathcal{H}^1.\end{aligned}$$

Using this we see that for a stationary solution furthermore

$$\begin{aligned}0 = &\int_{\partial\Gamma} \frac{1}{2}H_\Gamma^2(n_{\partial\Gamma}\cdot\zeta) + H_\Gamma((\partial_{n_{\partial\Gamma}} n_\Gamma)\cdot\zeta + n_\Gamma\cdot(\partial_{n_{\partial\Gamma}}\zeta)) - (\nabla_\Gamma H_\Gamma\cdot n_{\partial\Gamma})(n_\Gamma\cdot\zeta)\, d\mathcal{H}^1\\ &- 2\int_{\partial\Gamma} a(n_{\partial D}\cdot\zeta) + b(\vec{\varkappa}\cdot\zeta)\, d\mathcal{H}^1\end{aligned}$$

has to hold for every $\zeta \in \mathcal{F}(\Gamma)$.

Now we want to construct a $\zeta_2 \in \mathcal{F}(\Gamma)$ with $\zeta_2|_{\partial\Gamma} \equiv 0$ and arbitrary $n_\Gamma \cdot \partial_{n_{\partial\Gamma}}\zeta_2|_{\partial\Gamma}$. Let $[0,1]\times[0,\varepsilon_0] =: D \subseteq \mathbb{R}^2$ and

$$\varphi : D \longrightarrow \Gamma : (s_1, s_2) \longmapsto \varphi(s_1, s_2)$$

be a parametrization of a neighborhood of $\partial\Gamma$ with $\varphi(\bullet, 0)$ one such of $\partial\Gamma$. Moreover, $\partial_{s_1}\varphi(s_1,s_2)\cdot\partial_{s_2}\varphi(s_1,s_2) = 0$ and $\partial_{s_1}\varphi(s_1,s_2) \neq 0 \neq \partial_{s_2}\varphi(s_1,s_2)$ shall hold. Then φ^{-1} is a chart of Γ and we have

$$\partial_{s_1}\varphi(s_1,0) = \alpha_1(s_1)\vec{\tau}(\varphi(s_1,0)) \quad \text{and} \quad \partial_{s_2}\varphi(s_1,0) = \alpha_2(s_1)n_{\partial\Gamma}(\varphi(s_1,0))$$

for two scalar functions $\alpha_1, \alpha_2 : [0,1] \longrightarrow \mathbb{R}$ that vanish nowhere. We define our desired function $\zeta_2 : \Gamma \longrightarrow \mathbb{R}^3$ as

$$\zeta_2(\varphi(s_1, s_2)) := s_2 g(s_1) n_\Gamma(\varphi(s_1, s_2)),$$

where $g : [0,1] \to \mathbb{R}$ is an arbitrary function. Then we obviously have $\zeta_2|_{\partial\Gamma} = 0$ since a point $p \in \partial\Gamma$ is obtained by $p = \varphi(s_1, 0)$ for an appropriate $s_1 \in [0,1]$. In addition, this shows $\zeta_2|_{\partial\Gamma} \cdot n_D = 0$ and hence $\zeta_2 \in \mathcal{F}(\Gamma)$. Moreover, we have

$$\begin{aligned}
(\partial_{n_{\partial\Gamma}} \zeta_2)(\varphi(s_1, 0)) &= \partial_{s_2}(\zeta_2 \circ \varphi)(s_1, s_2)\big|_{s_2=0} \, \alpha_2(s_1)^{-1} \\
&= \left(g(s_1) n_\Gamma(\varphi(s_1, s_2)) + s_2 g(s_1) \partial_{s_2}(n_\Gamma \circ \varphi)(s_1, s_2)\right)\big|_{s_2=0} \, \alpha_2(s_1)^{-1} \\
&= \left(g(s_1) n_\Gamma(\varphi(s_1, 0)) + 0\right) \alpha_2(s_1)^{-1} = \alpha_2(s_1)^{-1} g(s_1) n_\Gamma(\varphi(s_1, 0)).
\end{aligned}$$

Hence with g also $n_\Gamma \cdot \partial_{n_{\partial\Gamma}} \zeta_2|_{\partial\Gamma} = \alpha_2(s_1)^{-1} g(s_1)$ is arbitrary.
Now we use this particular $\zeta_2 \in \mathcal{F}(\Gamma)$ to get the next necessary condition for stationary solutions

$$0 = (\delta \, \mathrm{WE}(\Gamma_{\mathrm{stat}}))(\zeta_2) = \int_{\partial\Gamma} H_\Gamma (n_\Gamma \cdot (\partial_{n_{\partial\Gamma}} \zeta_2)) \, d\mathcal{H}^1$$

Again by the fundamental lemma we obtain the condition $H_\Gamma = 0$ on $\partial\Gamma$. Therefore, we can simplify (4.3) for stationary solutions even more to

$$(\delta \, \mathrm{WE}(\Gamma_{\mathrm{stat}}))(\zeta) = -\frac{1}{2} \int_{\partial\Gamma} (\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma})(n_\Gamma \cdot \zeta) + 2a(n_{\partial D} \cdot \zeta) + 2b(\vec{\varkappa} \cdot \zeta) \, d\mathcal{H}^1,$$

which shows that

$$0 = \int_{\partial\Gamma} ((\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) n_\Gamma + 2a n_{\partial D} + 2b\vec{\varkappa}) \cdot \zeta \, d\mathcal{H}^1$$

has to hold for all $\zeta \in \mathcal{F}(\Gamma)$. The same projection trick as in Section 2.2 and a final use of the fundamental lemma gives the vector identity

$$\widehat{P}\left((\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) n_\Gamma + 2a n_{\partial D} + 2b\vec{\varkappa}\right) = 0 \qquad \text{on } \partial\Gamma.$$

Again we can rewrite this equation to

$$\begin{aligned}
0 &= \widehat{P}\left((\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) n_\Gamma + 2a n_{\partial D} + 2b\vec{\varkappa}\right) \\
&= (\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) \widehat{P}(n_\Gamma) + 2a \widehat{P}(n_{\partial D}) + 2b \widehat{P}(\vec{\varkappa}) \\
&= (\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) \sin(\alpha) n_{\partial D} + 2a n_{\partial D} + 2b \varkappa_{\partial D} n_{\partial D}.
\end{aligned}$$

Multiplying by $\frac{1}{2} n_{\partial D}$ gives the scalar equation $\frac{1}{2}\sin(\alpha)(\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) + a + b\varkappa_{\partial D} = 0$ on $\partial\Gamma$.

So we have derived the necessary conditions for stationary solutions to be

$$\begin{aligned}
&\text{(a)} \quad \Delta_\Gamma H_\Gamma + \frac{1}{2} H_\Gamma \left(H_\Gamma^2 - 4K_\Gamma\right) = 0 && \text{in } \Gamma && (4.5) \\
&\text{(b)} \quad H_\Gamma = 0 && \text{on } \partial\Gamma && (4.6) \\
&\text{(c)} \quad \frac{1}{2}\sin(\alpha)(\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) + a + b\varkappa_{\partial D} = 0 && \text{on } \partial\Gamma, && (4.7)
\end{aligned}$$

where we used Remark 4.3.

With the same curvilinear coordinate system Ψ and distance function ϱ as in Section 2.2 we consider a flow that tends to minimize the energy (4.1), which reads as

$$V_\Gamma = \partial_t \Phi \cdot n_\Gamma = (-\nabla_{L_2} \mathrm{WE}) \cdot n_\Gamma = -\Delta_\Gamma H_\Gamma - \frac{1}{2} H_\Gamma \left(H_\Gamma^2 - 4K_\Gamma \right) \qquad \text{in } \Gamma.$$

Furthermore, we impose the boundary conditions

$$H_\Gamma = 0 \qquad \text{on } \partial\Gamma$$

and

$$v_{\partial D} = \frac{1}{2} \sin(\alpha)(\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}) + a + b\varkappa_{\partial D} \qquad \text{on } \partial\Gamma. \tag{4.8}$$

Hence the flow considered is given by

$$\begin{aligned} V_\Gamma(\Psi(q, \varrho(t,q))) &= -\Delta_\Gamma H_\Gamma(\Psi(q, \varrho(t,q))) - \frac{1}{2} H_\Gamma(\Psi(q, \varrho(t,q))) \\ &\quad \left(H_\Gamma(\Psi(q, \varrho(t,q)))^2 - 4K_\Gamma(\Psi(q, \varrho(t,q))) \right) && \text{in } \Gamma^* \quad (4.9) \\ H_\Gamma(\Psi(q, \varrho(t,q))) &= 0 && \text{on } \partial\Gamma^* \quad (4.10) \\ v_{\partial D}(\Psi(q, \varrho(t,q))) &= \frac{1}{2} \sin(\alpha(q))(\nabla_\Gamma H_\Gamma(\Psi(q, \varrho(t,q))) \cdot n_{\partial\Gamma}(\Psi(q, \varrho(t,q)))) \\ &\quad + a + b\varkappa_{\partial D}(\Psi(q, \varrho(t,q))) && \text{on } \partial\Gamma^*. \quad (4.11) \end{aligned}$$

4.2 Linearization of the Willmore Flow

We have already linearized some parts of this flow in Section 2.3 and will now only calculate the highest order derivatives of the remaining parts as we have seen in Section 3.1 that only these are important for short-time existence. Therefore, we will not compute the lower order terms exactly, only their structure matters.

Lemma 4.4: *The linearization of K_Γ contains only first and second order derivatives of ϱ, i.e.*

$$\frac{d}{d\varepsilon} K_\Gamma(\Psi(q, \varepsilon\varrho(t,q))) \bigg|_{\varepsilon=0} = \widetilde{K} \left(q, \varrho(t,q), \nabla_{\Gamma^*} \varrho(t,q), \nabla_{\Gamma^*}^2 \varrho(t,q) \right),$$

where $\widetilde{K}$ is a smooth function.

<u>Proof</u>: Using (5.9) of [Bär10] we see that the linearization of the Gauss curvature is given by

$$2\partial_\varepsilon K_\Gamma = \sum_{ijkl=1}^{2} g^{jk} g^{il} \left(\partial_i \partial_k \partial_\varepsilon g_{jl} - \partial_i \partial_l \partial_\varepsilon g_{jk} \right) - 2K_\Gamma \sum_{ij=1}^{2} g^{ij} \partial_\varepsilon g_{ij},$$

where $g_{ij} := \langle \partial_i P, \partial_j P \rangle$ are the entries of the first fundamental form and g^{ij} are the entries of its inverse. For an orthogonal parametrization P (i.e. $g_{ij} = 0 = g^{ij}$ for $i \neq j$ at one fixed point) this formula simplifies to

$$2\partial_\varepsilon K_\Gamma = \sum_{ij=1}^{2} g^{jj} g^{ii} \left(\partial_i \partial_j \partial_\varepsilon g_{ji} - \partial_i \partial_i \partial_\varepsilon g_{jj} \right) - 2K_\Gamma \sum_{i=1}^{2} g^{ii} \partial_\varepsilon g_{ii}.$$

Looking at the first sum only and using the permutability of the derivatives we see

$$\begin{aligned}
\sum_{ij=1}^{2} g^{jj} g^{ii} \left(\partial_i \partial_j \partial_\varepsilon g_{ji} - \partial_i \partial_i \partial_\varepsilon g_{jj} \right) &= \sum_{ij=1}^{2} g^{jj} g^{ii} \partial_\varepsilon \left(\partial_i \partial_j \langle \partial_j P, \partial_i P \rangle - \partial_i \partial_i \langle \partial_j P, \partial_j P \rangle \right) \\
&= \sum_{ij=1}^{2} g^{jj} g^{ii} \partial_\varepsilon \left(\langle \partial_i \partial_j \partial_j P, \partial_i P \rangle + \langle \partial_j \partial_j P, \partial_i \partial_i P \rangle \right. \\
&\quad + \langle \partial_i \partial_j P, \partial_j \partial_i P \rangle + \langle \partial_j P, \partial_i \partial_j \partial_i P \rangle \\
&\quad - \langle \partial_i \partial_i \partial_j P, \partial_j P \rangle - \langle \partial_i \partial_j P, \partial_i \partial_j P \rangle \\
&\quad \left. - \langle \partial_i \partial_j P, \partial_i \partial_j P \rangle - \langle \partial_j P, \partial_i \partial_i \partial_j P \rangle \right) \\
&= \sum_{ij=1}^{2} g^{jj} g^{ii} \partial_\varepsilon \left(\langle \partial_i \partial_j \partial_j P, \partial_i P \rangle + \langle \partial_j \partial_j P, \partial_i \partial_i P \rangle \right. \\
&\quad \left. - \langle \partial_i \partial_j P, \partial_i \partial_j P \rangle - \langle \partial_j P, \partial_i \partial_i \partial_j P \rangle \right).
\end{aligned}$$

The cases $i = j$ do not contribute to the sum, which leads to

$$\begin{aligned}
\sum_{ij=1}^{2} g^{jj} g^{ii} \left(\partial_i \partial_j \partial_\varepsilon g_{ji} - \partial_i \partial_i \partial_\varepsilon g_{jj} \right) &= g^{22} g^{11} \partial_\varepsilon \left(\langle \partial_1 \partial_2 \partial_2 P, \partial_1 P \rangle + \langle \partial_2 \partial_2 P, \partial_1 \partial_1 P \rangle \right. \\
&\quad \left. - \langle \partial_1 \partial_2 P, \partial_1 \partial_2 P \rangle - \langle \partial_2 P, \partial_1 \partial_1 \partial_2 P \rangle \right) \\
&\quad + g^{11} g^{22} \partial_\varepsilon \left(\langle \partial_2 \partial_1 \partial_1 P, \partial_2 P \rangle + \langle \partial_1 \partial_1 P, \partial_2 \partial_2 P \rangle \right. \\
&\quad \left. - \langle \partial_2 \partial_1 P, \partial_2 \partial_1 P \rangle - \langle \partial_1 P, \partial_2 \partial_2 \partial_1 P \rangle \right) \\
&= 2 g^{11} g^{22} \partial_\varepsilon \left(\langle \partial_1 \partial_1 P, \partial_2 \partial_2 P \rangle - \langle \partial_1 \partial_2 P, \partial_1 \partial_2 P \rangle \right).
\end{aligned}$$

Hence for K_Γ we get the expression

$$\begin{aligned}
\partial_\varepsilon K_\Gamma &= g^{11} g^{22} \partial_\varepsilon \left(\langle \partial_1 \partial_1 P, \partial_2 \partial_2 P \rangle - \langle \partial_1 \partial_2 P, \partial_1 \partial_2 P \rangle \right) \\
&\quad - K_\Gamma g^{11} \partial_\varepsilon \langle \partial_1 P, \partial_1 P \rangle - K_\Gamma g^{22} \partial_\varepsilon \langle \partial_2 P, \partial_2 P \rangle.
\end{aligned}$$

This proves that the highest order derivatives in the linearization of K_Γ are second order space derivatives of the parametrization P. If $F : G \subseteq \mathbb{R}^2 \longrightarrow \Omega : x \longmapsto F(x)$ denotes the parametrization of Γ^* then $\Gamma_{\varepsilon\varrho}(t)$ is parametrized over G via $P(x) := \Psi(F(x), \varepsilon\varrho(t, F(x)))$. Obviously, first and second order space derivatives of P result in derivatives $\nabla_{\Gamma^*} \varrho(t, q)$ and $\nabla^2_{\Gamma^*} \varrho(t, q)$. ■

For the next linearization of $\Delta_\Gamma H_\Gamma$ we need to indicate the dependence of the operator Δ_Γ on ϱ. Following the notation of [Dep10] we transform the surface gradient $\nabla_{\Gamma_\varrho(t)}$ and the

Laplace-Beltrami-Operator $\Delta_{\Gamma_\varrho(t)}$ onto the reference surface Γ^* using the pullback metric $g(\bullet,\bullet) := \langle d_q\Phi_t^\varrho(\bullet), d_q\Phi_t^\varrho(\bullet)\rangle_{\mathbb{R}^3}$, where Φ_t^ϱ is defined after (2.27). The operators then read as

$$\Delta_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}(\Psi(q,\varrho(t,q))) = \Delta_{\Gamma^*}^\varrho \widetilde{H}_\varrho(t,q) \tag{4.12}$$

$$\nabla_{\Gamma_\varrho(t)} H_{\Gamma_\varrho(t)}(\Psi(q,\varrho(t,q))) = d_q\Phi_t^\varrho \left(\nabla_{\Gamma^*}^\varrho \widetilde{H}_\varrho(t,q)\right), \tag{4.13}$$

where $\widetilde{H}_\varrho(t,q) := H_{\Gamma_\varrho(t)}(\Psi(q,\varrho(t,q)))$.

Lemma 4.5: *The linearization of $\Delta_\Gamma H_\Gamma$ has the form*

$$\frac{d}{d\varepsilon}\Delta_\Gamma H_\Gamma(\Psi(q,\varepsilon\varrho(t,q)))\Big|_{\varepsilon=0} = \Delta_{\Gamma^*}\Delta_{\Gamma^*}\varrho(t,q) + G_1\left(q,\varrho(t,q),\nabla_{\Gamma^*}\varrho(t,q),\nabla_{\Gamma^*}^2\varrho(t,q)\right),$$

where G_1 is a smooth function.

Proof: By the product rule we obtain

$$\frac{d}{d\varepsilon}\Delta_{\Gamma^*}^{\varepsilon\varrho}\widetilde{H}_{\varepsilon\varrho}(t,q)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Delta_{\Gamma^*}^{\varepsilon\varrho}\Big|_{\varepsilon=0}\widetilde{H}_0(t,q) + \Delta_{\Gamma^*}^0\frac{d}{d\varepsilon}\widetilde{H}_{\varepsilon\varrho}(t,q)\Big|_{\varepsilon=0}.$$

For $\varrho \equiv 0$ we obviously see $\Delta_{\Gamma^*}^0 = \Delta_{\Gamma^*}$ and $\widetilde{H}_0 = H_{\Gamma^*}$. In combination with Lemma 2.11 we have

$$\begin{aligned}\frac{d}{d\varepsilon}\Delta_{\Gamma^*}^{\varepsilon\varrho}\widetilde{H}_{\varepsilon\varrho}(t,q)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Delta_{\Gamma^*}^{\varepsilon\varrho}\Big|_{\varepsilon=0} H_{\Gamma^*}(q) + \Delta_{\Gamma^*}\Delta_{\Gamma^*}\varrho(t,q)\\ &+ \Delta_{\Gamma^*}\left(|\sigma^*|^2(q)\varrho(t,q) + (\nabla_{\Gamma^*}H_{\Gamma^*}(q)\cdot P(\partial_w\Psi(q,0)))\,\varrho(t,q)\right).\end{aligned}$$

What is left to show is that the linearization of the Laplace-Beltrami operator applied to H_{Γ^*} contains at most second order derivatives of ϱ.
To see this we use the well-known local coordinate representation of the Laplace-Beltrami operator (cf. (6.15) in [AE08]) to obtain

$$\begin{aligned}\Delta_{\Gamma^*} &= \frac{1}{\sqrt{\det(g)}}\sum_{jk=1}^2 \partial_j\left(g^{jk}\sqrt{\det(g)}\partial_k\right)\\ &= \sum_{jk=1}^2 g^{jk}\partial_j\partial_k + (\partial_j g^{jk})\partial_k + \frac{1}{\sqrt{\det(g)}}\partial_j\left(\sqrt{\det(g)}\right)g^{jk}\partial_k\\ &= \sum_{jk=1}^2 g^{jk}\partial_j\partial_k + (\partial_j g^{jk})\partial_k + \frac{1}{2}\sum_{il=1}^2 g^{jk}g^{il}\partial_j g_{il}\partial_k,\end{aligned}$$

where we have used

$$\frac{1}{\sqrt{\det(g)}}\partial_j\left(\sqrt{\det(g)}\right) = \frac{1}{2\det(g)}\partial_j(\det(g)) = \frac{1}{2}\,\mathrm{spur}(g^{-1}\partial_j g) = \frac{1}{2}\sum_{il=1}^2 g^{il}\partial_j g_{il}$$

(cf. page 247 of [Bär10] for the third equality). After renaming the indices several times and using

$$\partial_\varepsilon g^{jk} = -\sum_{il=1}^2 g^{ji}\left(\partial_\varepsilon g_{il}\right)g^{lk}$$

from Lemma 5.2.4 of [Bär10] this leads to

$$\Delta^{\varrho}_{\Gamma^*} = \sum_{jk=1}^{2} g^{jk}(\varrho)\left(\partial_j\partial_k - \sum_{i=1}^{2}\Gamma^i_{jk}(\varrho)\partial_i\right),$$

where Γ^k_{ij} are the Christoffel symbols. Using Lemma 5.2.4 from [Bär10] again we see

$$\partial_\varepsilon\Gamma^k_{ij} = \frac{1}{2}\sum_{m=1}^{2} g^{km}\left(\partial_i\partial_\varepsilon g_{jm} + \partial_j\partial_\varepsilon g_{im} - \partial_m\partial_\varepsilon g_{ij}\right) - \sum_{ml=1}^{2}\Gamma^m_{ij} g^{kl}\partial_\varepsilon g_{lm}.$$

Parameterizing $\Gamma_{\varepsilon\varrho}(t)$ over G via $P(x) := \Psi(F(x), \varepsilon\varrho(t, F(x)))$ as in proof of Lemma 4.4, we see that $\partial_\varepsilon g_{il}$ contains only first order derivatives of ϱ. Hence $\partial_\varepsilon g^{jk}$ and $\partial_\varepsilon\Gamma^k_{ij}$ contain at most first and second order space derivatives of ϱ, respectively. The desired expression

$$\frac{d}{d\varepsilon}\Delta^{\varepsilon\varrho}_{\Gamma^*}\bigg|_{\varepsilon=0} H_{\Gamma^*}(q) = \left(\sum_{jk=1}^{2}\partial_\varepsilon g^{jk}\left(\partial_j\partial_k - \sum_{i=1}^{2}\Gamma^i_{jk}\partial_i\right) - \sum_{ijk=1}^{2} g^{jk}\partial_\varepsilon\Gamma^i_{jk}\partial_i\right) H_{\Gamma^*}(q)$$

therefore includes no third or higher order derivatives of ϱ and the claim follows. ∎

Lemma 4.6: *The linearization of $\nabla_\Gamma H_\Gamma \cdot n_{\partial\Gamma}$ is of the form*

$$\begin{aligned}\frac{d}{d\varepsilon}\nabla_\Gamma H_\Gamma(\Psi(q,\varepsilon\varrho(t,q))) \cdot n_{\partial\Gamma}(\Psi(q,\varepsilon\varrho(t,q)))\bigg|_{\varepsilon=0} &= \nabla_{\Gamma^*}\Delta_{\Gamma^*}\varrho(t,q)\cdot n_{\partial\Gamma^*}(q)\\ &\quad + G_2\left(q, \varrho(t,q), \nabla_{\Gamma^*}\varrho(t,q)\right),\end{aligned}$$

where G_2 is a smooth function.

Proof: First we decompose the desired expression using $\nabla_{\Gamma_0(t)}H_{\Gamma_0(t)} = \nabla_{\Gamma^*}H_{\Gamma^*}$ and $n_{\partial\Gamma_0(t)} = n_{\partial\Gamma^*}$ as follows

$$\frac{d}{d\varepsilon}\nabla_{\Gamma_{\varepsilon\varrho}(t)}H_{\Gamma_{\varepsilon\varrho}(t)}\cdot n_{\partial\Gamma_{\varepsilon\varrho}(t)}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\nabla_{\Gamma_{\varepsilon\varrho}(t)}H_{\Gamma_{\varepsilon\varrho}(t)}\bigg|_{\varepsilon=0}\cdot n_{\partial\Gamma^*} + \nabla_{\Gamma^*}H_{\Gamma^*}\cdot\frac{d}{d\varepsilon}n_{\partial\Gamma_{\varepsilon\varrho}(t)}\bigg|_{\varepsilon=0}.$$

While the linearization of the conormal contains ϱ and its first derivatives, a closer look at the first term shows

$$\frac{d}{d\varepsilon}\nabla_{\Gamma_{\varepsilon\varrho}(t)}H_{\Gamma_{\varepsilon\varrho}(t)}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}d_q\Phi^{\varepsilon\varrho}_t\bigg|_{\varepsilon=0}\left(\nabla_{\Gamma^*}H_{\Gamma^*}\right) + \mathrm{Id}\left(\frac{d}{d\varepsilon}\nabla^{\varepsilon\varrho}_{\Gamma^*}\widetilde{H}_{\varepsilon\varrho}\bigg|_{\varepsilon=0}\right),$$

where we have used equation (4.13), $\nabla^0_{\Gamma^*} = \nabla_{\Gamma^*}$, $\widetilde{H}_0 = H_{\Gamma^*}$ and $d_q\Phi^0_t = \mathrm{Id}$. Here $\partial_\varepsilon d_q\Phi^{\varepsilon\varrho}_t$ contains ϱ and first order derivatives of ϱ, hence we only have to look at

$$\frac{d}{d\varepsilon}\nabla^{\varepsilon\varrho}_{\Gamma^*}\widetilde{H}_{\varepsilon\varrho}\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\nabla^{\varepsilon\varrho}_{\Gamma^*}\bigg|_{\varepsilon=0}H_{\Gamma^*} + \nabla_{\Gamma^*}\frac{d}{d\varepsilon}\widetilde{H}_{\varepsilon\varrho}\bigg|_{\varepsilon=0}.$$

Using the well-known (cf. page 280 of [AE06]) local coordinate representation of the surface gradient given by

$$\nabla^{\varrho}_{\Gamma^*} = \sum_{k=1}^{2} g^{ik}(\varrho)\partial_k$$

and once more Lemma 5.2.4 from [Bär10] we see that $\partial_\varepsilon \nabla^{\varepsilon\varrho}_{\Gamma^*}$ also includes no second or higher order derivatives of ϱ. Using Lemma 2.11 again we see

$$\nabla_{\Gamma^*} \left.\frac{d}{d\varepsilon} \widetilde{H}_{\varepsilon\varrho}\right|_{\varepsilon=0} = \nabla_{\Gamma^*} \Big(\Delta_{\Gamma^*} \varrho + |\sigma^*|^2 \varrho + \left(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P\left(\partial_w \Psi(q,0)\right)\right) \varrho \Big) .$$

This proves the desired statement. ■

Combining Lemmas 4.4 - 4.6 and the results of Section 2.3 we obtain the linearization of the flow (4.9)-(4.11) as

$$\begin{aligned}
\partial_t \varrho(t) &= -\Delta_{\Gamma^*} \Delta_{\Gamma^*} \varrho(t) + F_1(\varrho(t), \nabla_{\Gamma^*} \varrho(t), \nabla^2_{\Gamma^*} \varrho(t)) && \text{in } [0,T] \times \Gamma^* && (4.14)\\
0 &= \Delta_{\Gamma^*} \varrho(t) + |\sigma^*|^2 \varrho(t) + \left(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P\left(\partial_w \Psi(0)\right)\right) \varrho(t) && \text{on } [0,T] \times \partial\Gamma^* && (4.15)\\
\partial_t \varrho(t) &= \frac{1}{2} \sin(\alpha) (\nabla_{\Gamma^*} \Delta_{\Gamma^*} \varrho(t) \cdot n_{\partial\Gamma^*}) + b \sin(\alpha) \partial^2_\sigma \varrho(t) && &&\\
&\quad + F_2(\varrho(t), \nabla_{\Gamma^*} \varrho(t)) && \text{on } [0,T] \times \partial\Gamma^* && (4.16)\\
\varrho(0) &= \varrho_0 && \text{in } \Gamma^*, && (4.17)
\end{aligned}$$

where F_1 and F_2 are smooth functions and we have suppressed the argument q again.

In the following section we will use this linearization for the short-time existence of solutions of the Willmore flow.

5 Local existence of solutions of the Willmore flow with line tension

Now we will do the same considerations as in Section 3 to show that the flow (4.9)-(4.11) has a unique strong solution for short times. Again we assume w.l.o.g. $b > 0$ and follow the work of [DPZ08] by considering first the linearized Willmore flow from (4.14)-(4.17).

5.1 Short-time existence of solutions for the linearized Willmore Flow

First we adopt again all the involved notation from [DPZ08]. Skipping the argument q the operators and functions in our case read as

$$\begin{aligned}
\mathcal{A}(t,D) &:= \Delta_{\Gamma^*}\Delta_{\Gamma^*} + LOT \\
\mathcal{B}_0(t,D) &:= -\frac{1}{2}\sin(\alpha)(n_{\partial\Gamma^*}\cdot\nabla_{\Gamma^*}\Delta_{\Gamma^*}) + LOT \\
\mathcal{C}_0(t,D_\partial) &:= -b\sin(\alpha)\partial_\sigma^2 + LOT \\
\mathcal{B}_1(t,D) &:= \Delta_{\Gamma^*} + |\sigma^*|^2 + (\nabla_{\Gamma^*}H_{\Gamma^*}\cdot P\left(\partial_w\Psi(0)\right)) \\
\mathcal{C}_1(t,D_\partial) &:= 0 \\
\mathcal{B}_2(t,D) &:= 1 \\
\mathcal{C}_2(t,D_\partial) &:= -1 \\
u(t) &:= \varrho(t) \\
\rho(t) &:= \varrho(t)|_{\partial\Gamma^*},
\end{aligned}$$

where LOT stands for some unspecified lower order terms that will not play any role, because we only consider the principle parts of all operators later on.

We note again that the required condition “all $\mathcal{B}_j$ and at least one $\mathcal{C}_j$ are non-trivial” is satisfied. We still have $E := F := \mathbb{R}$ of type $\mathcal{HT}$ and the interval we want to consider is $[0,T]$ denoted by J. For a given $1 < p < \infty$ the involved numbers are

$$\begin{aligned}
&m := \frac{1}{2}\operatorname{ord}(\mathcal{A}) = 2, \\
&m_0 := \operatorname{ord}(\mathcal{B}_0) = 3, && m_1 := \operatorname{ord}(\mathcal{B}_1) = 2, && m_2 := \operatorname{ord}(\mathcal{B}_2) = 0, \\
&k_0 := \operatorname{ord}(\mathcal{C}_0) = 2, && k_1 := \operatorname{ord}(\mathcal{C}_1) = -\infty, && k_2 := \operatorname{ord}(\mathcal{C}_2) = 0, \\
&\varkappa_0 = \frac{1}{4} - \frac{1}{4p}, && \varkappa_1 = \frac{1}{2} - \frac{1}{4p}, && \varkappa_2 = 1 - \frac{1}{4p}, \\
&l_0 := k_0 - m_0 + m_0 = 2, && l_1 := k_1 - m_1 + m_0 = -\infty, && l_2 := k_2 - m_2 + m_0 = 3, \\
&l := \max\{l_0, l_1, l_2\} = 3,
\end{aligned}$$

where we define $\varkappa_i := 1 - \frac{m_i}{2m} - \frac{1}{2mp}$. Because of $l = 3 < 4 = 2m$ we have to consider the setting that is called “case 2” in [DPZ08]. Now the required function spaces simplify in our situation to

$$\begin{aligned}
X &:= L_p(J; L_p(\Gamma^*; \mathbb{R})),\\
Z_u &:= W_p^1(J; L_p(\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^4(\Gamma^*; \mathbb{R})),\\
\pi Z_u &:= W_p^{4-\frac{4}{p}}(\Gamma^*; \mathbb{R}),\\
Y_0 &:= W_p^{\frac{1}{4}-\frac{1}{4p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
Y_1 &:= W_p^{\frac{1}{2}-\frac{1}{4p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
Y_2 &:= W_p^{1-\frac{1}{4p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{4-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
Z_\rho &:= W_p^{\frac{5}{4}-\frac{1}{4p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap W_p^1(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{4-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})),\\
\pi Z_\rho &:= W_p^{4-\frac{4}{p}}(\partial\Gamma^*; \mathbb{R}),\\
\pi_1 Z_\rho &:= W_p^{1-\frac{5}{p}}(\partial\Gamma^*; \mathbb{R}),
\end{aligned} \tag{5.1}$$

where we have to assure for the π-spaces that $\frac{4}{p} \notin \mathbb{N}$ and $\varkappa_0 > \frac{1}{p}$, i.e. $p > 5$. As the principle parts of the operators we obtain

$$\begin{aligned}
\mathcal{A}^\sharp(t, q, -i\nabla_{\Gamma^*}) &= \Delta_{\Gamma^*}\Delta_{\Gamma^*} = ((-i\nabla_{\Gamma^*}) \cdot (-i\nabla_{\Gamma^*}))^2\\
\mathcal{B}_0^\sharp(t, q, -i\nabla_{\Gamma^*}) &= \frac{1}{2} i \sin(\alpha(q))(n_{\partial\Gamma^*}(q) \cdot (-i\nabla_{\Gamma^*})\,((-i\nabla_{\Gamma^*}) \cdot (-i\nabla_{\Gamma^*})))\\
\mathcal{C}_0^\sharp(t, q, -i\partial_\sigma) &= b \sin(\alpha(q))(-i\partial_\sigma)^2\\
\mathcal{B}_1^\sharp(t, q, -i\nabla_{\Gamma^*}) &= -\,((-i\nabla_{\Gamma^*}) \cdot (-i\nabla_{\Gamma^*}))\\
\mathcal{C}_1^\sharp(t, q, -i\partial_\sigma) &= 0\\
\mathcal{B}_2^\sharp(t, q, -i\nabla_{\Gamma^*}) &= 1\\
\mathcal{C}_2^\sharp(t, q, -i\partial_\sigma) &= -1.
\end{aligned}$$

Again we have to check some assumptions to apply the theorems of [DPZ08]. Due to the case $l < 2m$ this time we can only ignore the assumptions (LS_∞^+), (SD), (SB) and (SC), but have to check the assumptions (E), (LS) and henceforth (LS_∞^-).

For assumption (E) we let $t \in J$, $q \in \Gamma^*$ and $\xi \in \mathbb{R}^2$ with $\|\xi\| = 1$. Then we see

$$\begin{aligned}
\sigma(\mathcal{A}^\sharp(t, q, \xi)) &= \left\{\lambda \in \mathbb{C} \,\middle|\, \lambda - \mathcal{A}^\sharp(t, q, \xi) = 0\right\}\\
&= \left\{\lambda \in \mathbb{C} \,\middle|\, \lambda = (\xi \cdot \xi)^2 = \|\xi\|^4 = 1\right\} = \{1\} \subseteq \mathbb{C}_+.
\end{aligned}$$

For checking the condition (LS) the finite dimension of $E = F = \mathbb{R}$ makes it equivalent to prove that the desired ODE given by

$$\begin{aligned}
(\lambda + \mathcal{A}^\sharp(t, q, \widehat{\xi}, -i\partial_y))v(y) &= 0\\
\mathcal{B}_0^\sharp(t, q, \widehat{\xi}, -i\partial_y)v(0) + (\lambda + \mathcal{C}_0^\sharp(t, q, \widehat{\xi}))\sigma &= h_0\\
\mathcal{B}_1^\sharp(t, q, \widehat{\xi}, -i\partial_y)v(0) + \mathcal{C}_1^\sharp(t, q, \widehat{\xi})\sigma &= h_1\\
\mathcal{B}_2^\sharp(t, q, \widehat{\xi}, -i\partial_y)v(0) + \mathcal{C}_2^\sharp(t, q, \widehat{\xi})\sigma &= h_2
\end{aligned}$$

has only the trivial solution in $C_0(\mathbb{R}_+;\mathbb{R}) \times \mathbb{R}$ for $h_0 = h_1 = h_2 = 0$, instead checking that there is a unique solution for arbitrary h_0, h_1 and h_2. So we let $\widehat{\xi} \in \mathbb{R}$, $\lambda \in \overline{\mathbb{C}_+}$ with $|\widehat{\xi}| + |\lambda| \neq 0$. Then we get

$$
\begin{aligned}
\mathcal{A}^\sharp(t,q,\widehat{\xi},-i\partial_y) &= \left(\widehat{\xi}^2 - \partial_y^2\right)^2 = \widehat{\xi}^4 - 2\widehat{\xi}^2\partial_y^2 + \partial_y^4 \\
\mathcal{B}_0^\sharp(t,q,\widehat{\xi},-i\partial_y) &= \frac{1}{2} i \sin(\alpha(q)) \left(n_{\partial\Gamma^*}(q) \cdot (\widehat{\xi},-i\partial_y)^T(\widehat{\xi}^2 - \partial_y^2)\right) \\
\mathcal{C}_0^\sharp(t,q,\widehat{\xi}) &= b\sin(\alpha(q))\widehat{\xi}^2 \\
\mathcal{B}_1^\sharp(t,q,\widehat{\xi},-i\partial_y) &= -\widehat{\xi}^2 + \partial_y^2 \\
\mathcal{C}_1^\sharp(t,q,\widehat{\xi}) &= 0 \\
\mathcal{B}_2^\sharp(t,q,\widehat{\xi},-i\partial_y) &= 1 \\
\mathcal{C}_2^\sharp(t,q,\widehat{\xi}) &= -1
\end{aligned}
$$

and the ODE to be considered is

$$
\begin{aligned}
&\text{(I)} && \lambda v(y) + \widehat{\xi}^4 v(y) - 2\widehat{\xi}^2 v''(y) + v''''(y) = 0 \\
&\text{(II)} && -\frac{1}{2}\sin(\alpha(q))\left(\widehat{\xi}^2 v'(0) - v'''(0)\right) + \lambda\sigma + b\sin(\alpha(q))\widehat{\xi}^2\sigma = 0 \\
&\text{(III)} && -\widehat{\xi}^2 v(0) + v''(0) = 0 \\
&\text{(IV)} && v(0) - \sigma = 0,
\end{aligned}
$$

where we have identified $n_{\partial\Gamma^*} \widehat{=} \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$ as before. Now we will look at these equations step by step.

Via the ansatz $v(y) = e^{\mu y}$ equation (I) transforms into $\mu^4 - 2\widehat{\xi}^2\mu^2 + (\lambda + \widehat{\xi}^4) = 0$. Substituting $\varkappa = \mu^2$ we obtain $\varkappa_{1/2} = \widehat{\xi}^2 \pm \sqrt{-\lambda}$. This shows $\mu_{1/2} = \sqrt{\widehat{\xi}^2 \pm \sqrt{-\lambda}}$ and $\mu_{3/4} = -\sqrt{\widehat{\xi}^2 \pm \sqrt{-\lambda}} = -\mu_{1/2}$ for $\lambda \neq 0$, while in the case of $\lambda = 0$ one ends up with the two distinct double zeros $\mu_{1/2} = \pm|\widehat{\xi}|$. Hence the function v has the form

$$
v(y) = \begin{cases} c_1 e^{\mu_1 y} + c_2 e^{-\mu_1 y} + c_3 e^{\mu_2 y} + c_4 e^{-\mu_2 y} & \text{if} \quad \lambda \neq 0 \\ c_1 e^{|\widehat{\xi}| y} + c_2 e^{-|\widehat{\xi}| y} + c_3 y e^{|\widehat{\xi}| y} + c_4 y e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases}.
$$

We remark that we can again choose w.l.o.g. the roots that satisfy $\Re(\mu_i) > 0$ for $i \in \{1,2\}$ since μ_i and $-\mu_i$ both appear in v and there is still no possibility that $\Re(\mu_i) = 0$ since this would mean that $\widehat{\xi}^2 \pm \sqrt{-\lambda} \in \mathbb{R}_-$. But this is not possible due to the assumptions $\widehat{\xi} \in \mathbb{R}$, $\lambda \in \overline{\mathbb{C}_+}$ with $|\widehat{\xi}| + |\lambda| \neq 0$. Furthermore we require $v \in C_0(\mathbb{R}_+;\mathbb{R})$, which leads to $c_1 = c_3 = 0$ for the same reason as in Section 3.1.

So far we know that

$$
v(y) = \begin{cases} c_2 e^{-\mu_1 y} + c_4 e^{-\mu_2 y} & \text{if} \quad \lambda \neq 0 \\ c_2 e^{-|\widehat{\xi}| y} + c_4 y e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases}
$$

and now (IV) shows

$$\sigma = v(0) = \begin{cases} c_2 + c_4 & \text{if} \quad \lambda \neq 0 \\ c_2 & \text{if} \quad \lambda = 0 \end{cases},$$

which leads to

$$v(y) = \begin{cases} c_2 e^{-\mu_1 y} + (\sigma - c_2) e^{-\mu_2 y} & \text{if} \quad \lambda \neq 0 \\ \sigma e^{-|\widehat{\xi}| y} + c_4 y e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases}.$$

Now equation (III) given by $v''(0) = \widehat{\xi}^2 v(0)$ transforms into

$$\begin{cases} c_2 \mu_1^2 + (\sigma - c_2)\mu_2^2 = c_2 \widehat{\xi}^2 + (\sigma - c_2)\widehat{\xi}^2 & \text{if} \quad \lambda \neq 0 \\ \sigma \widehat{\xi}^2 - 2|\widehat{\xi}| c_4 = \sigma \widehat{\xi}^2 & \text{if} \quad \lambda = 0 \end{cases}.$$

Using $\mu_1^2 = \widehat{\xi}^2 + \sqrt{-\lambda}$ and $\mu_2^2 = \widehat{\xi}^2 - \sqrt{-\lambda}$ we see that these conditions transform into $c_2 = \frac{\sigma}{2}$ if $\lambda \neq 0$ and $c_4 = 0$ if $\lambda = 0$ (since $\widehat{\xi} = 0$ is not allowed for $\lambda = 0$). We therefore know so far

$$v(y) = \begin{cases} \frac{\sigma}{2} \left(e^{-\mu_1 y} + e^{-\mu_2 y} \right) & \text{if} \quad \lambda \neq 0 \\ \sigma e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases}.$$

Equation (II) is the only remaining and can be written as

$$\begin{cases} \frac{\sigma}{4} \sin(\alpha) \sqrt{-\lambda} (\mu_1 - \mu_2) = \lambda \sigma + b \sin(\alpha(q)) \widehat{\xi}^2 \sigma & \text{if} \quad \lambda \neq 0 \\ b \sin(\alpha) \widehat{\xi}^2 \sigma = 0 & \text{if} \quad \lambda = 0 \end{cases},$$

because $\widehat{\xi}^2 v'(0) - v'''(0) = \frac{\sigma}{2} \sqrt{-\lambda}(\mu_1 - \mu_2)$ as one can easily calculate. In the case $\lambda = 0$ we have $b, \sin(\alpha), \widehat{\xi}^2 \in \mathbb{R}_+$ which proves $\sigma = 0$ and hence $v \equiv 0$, which shows (LS) in this case. In the case $\lambda \neq 0$ we either have $\sigma = 0$ which would mean that (LS) is satisfied or

$$\underbrace{\frac{1}{4} \sin(\alpha) \sqrt{-\lambda} (\mu_1 - \mu_2)}_{=:L} = \underbrace{\lambda + b \sin(\alpha(q)) \widehat{\xi}^2}_{=:R}. \tag{5.2}$$

To prove condition (LS) completely we only have to show that (5.2) is not possible. First we remark that we can assume w.l.o.g. $\Im(\sqrt{-\lambda}) > 0$ since the other choice would change the roles of μ_1 and μ_2, but also causes $\sqrt{-\lambda}$ to bring an additional "$-$" into the left hand side. Therefore both sides are independent of the choice of $\sqrt{-\lambda}$. Now we distinguish two cases, that are visualized in Figures 4, 5 and 6.

Case 1 ($\Im(\lambda) \geq 0$): Due to our assumptions $\Im(\lambda) \geq 0$ and $\Im(\sqrt{-\lambda}) > 0$ we see that $\arg(\sqrt{-\lambda}) \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$. Denoting $c := \Re(\sqrt{-\lambda}) \leq 0$ and $d := \Im(\sqrt{-\lambda}) > 0$ we can write the

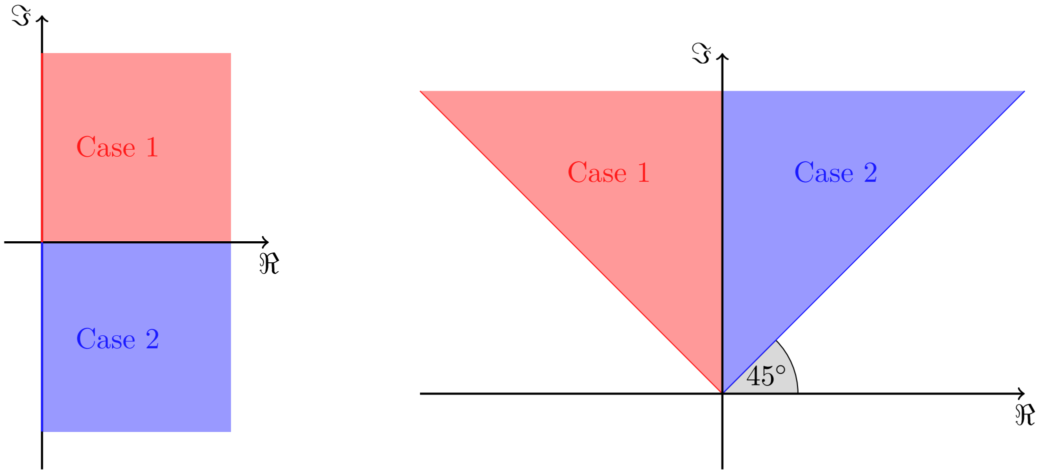

Figure 4: Location of λ and $\sqrt{-\lambda}$

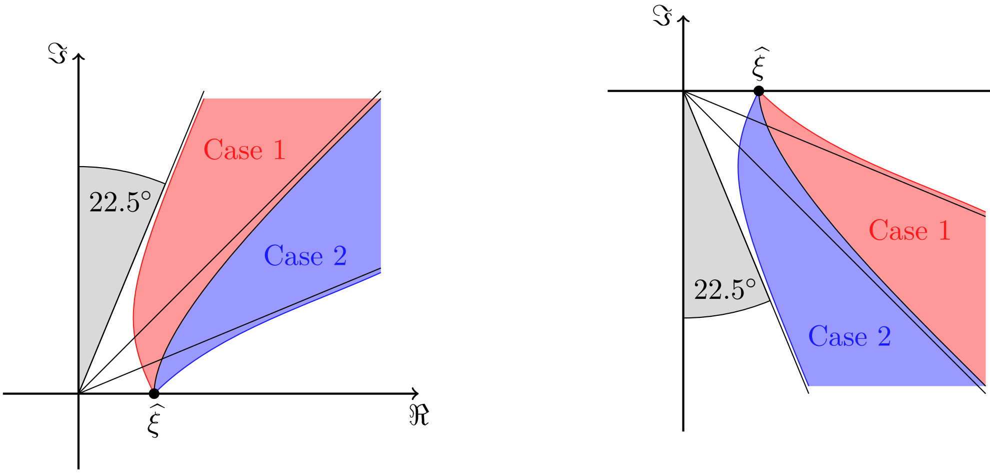

Figure 5: Location of μ_1 and μ_2

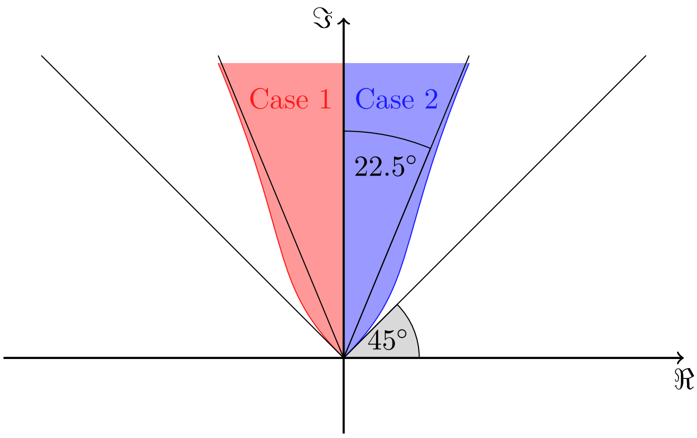

Figure 6: Location of $\mu_1 - \mu_2$

square roots μ_1 and μ_2 as follows

$$\mu_1 = \sqrt{\frac{\sqrt{(\widehat{\xi}^2+c)^2+d^2}+(\widehat{\xi}^2+c)}{2}} + i\underbrace{\operatorname{sgn}(\Im(\widehat{\xi}^2+\sqrt{-\lambda}))}_{=1}\sqrt{\frac{\sqrt{(\widehat{\xi}^2+c)^2+d^2}-(\widehat{\xi}^2+c)}{2}}$$
$$\mu_2 = \sqrt{\frac{\sqrt{(\widehat{\xi}^2-c)^2+d^2}+(\widehat{\xi}^2-c)}{2}} + i\underbrace{\operatorname{sgn}(\Im(\widehat{\xi}^2-\sqrt{-\lambda}))}_{=-1}\sqrt{\frac{\sqrt{(\widehat{\xi}^2-c)^2+d^2}-(\widehat{\xi}^2-c)}{2}}.$$

Due to $c \le 0$ we see that $\Re(\mu_1) \le \Re(\mu_2)$, which gives us $\Re(\mu_1-\mu_2) \le 0$ and the imaginary parts obviously satisfy $\Im(\mu_1) > 0$ and $\Im(\mu_2) < 0$, leading to $\Im(\mu_1-\mu_2) > 0$. This proves $\arg(\mu_1-\mu_2) \in \left[\frac{\pi}{2}, \pi\right)$. Hence we get

$$\arg(\sqrt{-\lambda}(\mu_1-\mu_2)) = \arg(\sqrt{-\lambda}) + \arg(\mu_1-\mu_2) \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] + \left[\frac{\pi}{2}, \pi\right) = \left[\pi, \frac{7\pi}{4}\right).$$

Therefore the whole left-hand side satisfies $\arg(L) \in \left[\pi, \frac{7\pi}{4}\right)$. The right-hand side obviously fulfills $\arg(R) \in \left[0, \frac{\pi}{2}\right]$ instead.

Case 2 ($\Im(\lambda) \le 0$): Because of the assumptions $\Im(\lambda) \le 0$ and $\Im(\sqrt{-\lambda}) > 0$ we see $\arg(\sqrt{-\lambda}) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. Denoting again $c := \Re(\sqrt{-\lambda}) \ge 0$ and $d := \Im(\sqrt{-\lambda}) > 0$ we can write the square roots μ_1 and μ_2 as in case 1 as

$$\mu_1 = \sqrt{\frac{\sqrt{(\widehat{\xi}^2+c)^2+d^2}+(\widehat{\xi}^2+c)}{2}} + i\underbrace{\operatorname{sgn}(\Im(\widehat{\xi}^2+\sqrt{-\lambda}))}_{=1}\sqrt{\frac{\sqrt{(\widehat{\xi}^2+c)^2+d^2}-(\widehat{\xi}^2+c)}{2}}$$
$$\mu_2 = \sqrt{\frac{\sqrt{(\widehat{\xi}^2-c)^2+d^2}+(\widehat{\xi}^2-c)}{2}} + i\underbrace{\operatorname{sgn}(\Im(\widehat{\xi}^2-\sqrt{-\lambda}))}_{=-1}\sqrt{\frac{\sqrt{(\widehat{\xi}^2-c)^2+d^2}-(\widehat{\xi}^2-c)}{2}}.$$

Due to $c \ge 0$ we get this time $\Re(\mu_1-\mu_2) \ge 0$ and as in case 1 we have $\Im(\mu_1-\mu_2) > 0$. This proves $\arg(\mu_1-\mu_2) \in \left(0, \frac{\pi}{2}\right]$. Hence we now obtain

$$\arg(\sqrt{-\lambda}(\mu_1-\mu_2)) = \arg(\sqrt{-\lambda}) + \arg(\mu_1-\mu_2) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] + \left(0, \frac{\pi}{2}\right] = \left(\frac{\pi}{4}, \pi\right].$$

Therefore we see $\arg(L) \in \left(\frac{\pi}{4}, \pi\right]$. The right-hand side obviously fulfills $\arg(R) \in \left[\frac{3\pi}{2}, 2\pi\right]$ instead.

In both cases we get $\arg(L) \neq \arg(R)$ which proves that (5.2) is not true in either case. This completes the proof of (LS).

Next we prove the validity of assumption (LS_∞^-). First we have to show that the system

$$\begin{aligned}(\lambda + \mathcal{A}^\sharp(t,q,\widehat{\xi},-i\partial_y))v(y) &= 0\\ \mathcal{B}_1^\sharp(t,q,\widehat{\xi},-i\partial_y)v(0) &= 0\\ \mathcal{B}_2^\sharp(t,q,\widehat{\xi},-i\partial_y)v(0) &= 0\end{aligned}$$

only has the trivial solution in $C_0(\mathbb{R}_+;\mathbb{R})$. So we let $\widehat{\xi} \in \mathbb{R}$, $\lambda \in \overline{\mathbb{C}_+}$ with $|\widehat{\xi}| + |\lambda| \neq 0$. Then the ODE to be considered is

$$\begin{aligned} &\text{(I)} \qquad \lambda v(y) + \widehat{\xi}^4 v(y) - 2\widehat{\xi}^2 v''(y) + v''''(y) = 0 \\ &\text{(II)} \qquad -\widehat{\xi}^2 v(0) + v''(0) = 0 \\ &\text{(III)} \qquad v(0) = 0. \end{aligned}$$

The same arguments as before show that the function v has the form

$$v(y) = \begin{cases} c_1 e^{\mu_1 y} + c_2 e^{-\mu_1 y} + c_3 e^{\mu_2 y} + c_4 e^{-\mu_2 y} & \text{if} \quad \lambda \neq 0 \\ c_1 e^{|\widehat{\xi}| y} + c_2 e^{-|\widehat{\xi}| y} + c_3 y e^{|\widehat{\xi}| y} + c_4 y e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases},$$

where we again choose w.l.o.g. $\Re(\mu_i) > 0$. The condition $v \in C_0(\mathbb{R}_+;\mathbb{R})$ leads to

$$v(y) = \begin{cases} c_2 e^{-\mu_1 y} + c_4 e^{-\mu_2 y} & \text{if} \quad \lambda \neq 0 \\ c_2 e^{-|\widehat{\xi}| y} + c_4 y e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases}$$

due to the same reason as before and now (III) shows that

$$0 = v(0) = \begin{cases} c_2 + c_4 & \text{if} \quad \lambda \neq 0 \\ c_2 & \text{if} \quad \lambda = 0 \end{cases},$$

which leads to

$$v(y) = \begin{cases} c_2 \left(e^{-\mu_1 y} - e^{-\mu_2 y}\right) & \text{if} \quad \lambda \neq 0 \\ c_4 y e^{-|\widehat{\xi}| y} & \text{if} \quad \lambda = 0 \end{cases}.$$

Now equation (II) given by $v''(0) = \widehat{\xi}^2 v(0)$ transforms into

$$\begin{cases} c_2 \mu_1^2 - c_2 \mu_2^2 = c_2 \widehat{\xi}^2 - c_2 \widehat{\xi}^2 = 0 & \text{if} \quad \lambda \neq 0 \\ -2|\widehat{\xi}| c_4 = 0 & \text{if} \quad \lambda = 0 \end{cases}.$$

Using $\mu_1^2 \neq \mu_2^2$ this gives $c_2 = 0$ if $\lambda \neq 0$ and $c_4 = 0$ if $\lambda = 0$. Hence we have shown $v \equiv 0$.

The second statement to prove in (LS_∞^-) is that for $|\widehat{\xi}| = 1$

$$\begin{aligned} \mathcal{A}^\sharp(t, q, \widehat{\xi}, -i\partial_y) v(y) &= 0 \\ \mathcal{B}_0^\sharp(t, q, \widehat{\xi}, -i\partial_y) v(0) + (\lambda + \mathcal{C}_0^\sharp(t, q, \widehat{\xi}))\sigma &= 0 \\ \mathcal{B}_1^\sharp(t, q, \widehat{\xi}, -i\partial_y) v(0) + \mathcal{C}_1^\sharp(t, q, \widehat{\xi})\sigma &= 0 \\ \mathcal{B}_2^\sharp(t, q, \widehat{\xi}, -i\partial_y) v(0) + \mathcal{C}_2^\sharp(t, q, \widehat{\xi})\sigma &= 0 \end{aligned}$$

only has the trivial solution in $C_0(\mathbb{R}_+;\mathbb{R}) \times \mathbb{R}$. So we let $\widehat{\xi} = \pm 1$, $\lambda \in \overline{\mathbb{C}_+}$ then the ODE to be considered is

$$\begin{aligned} &\text{(I)} \qquad v(y) - 2v''(y) + v''''(y) = 0 \\ &\text{(II)} \qquad -\frac{1}{2}\sin(\alpha(q))\left(v'(0) - v'''(0)\right) + \lambda\sigma + b\sin(\alpha(q))\sigma = 0 \\ &\text{(III)} \qquad -v(0) + v''(0) = 0 \\ &\text{(IV)} \qquad v(0) - \sigma = 0 \end{aligned}$$

due to $\widehat{\xi}^2 = 1$. Equation (I) transforms into $\mu^4 - 2\mu^2 + 1 = 0$, which has two double zeros $\mu_{1/2} = \pm 1$. Hence the function v has the form

$$v(y) = c_1 e^y + c_2 e^{-y} + c_3 y e^y + c_4 y e^{-y}.$$

Since $v \in C_0(\mathbb{R}_+;\mathbb{R})$ is required, we get $c_1 = c_3 = 0$ for the same reason as above. Equation (IV) shows that $\sigma = v(0) = c_2$, which leads to $v(y) = \sigma e^{-y} + c_4 y e^{-y}$. Now equation (III) reads as $\sigma - 2c_4 = \sigma$, hence $c_4 = 0$. We end up with $v(y) = \sigma e^{-y}$. Computing $v'(0) - v'''(0) = 0$ we see that (II) simplifies to $(\lambda + b\sin(\alpha(q)))\sigma = 0$. Since $\lambda \neq -b\sin(\alpha(q)) \in \mathbb{R}_-$ we get $\sigma = 0$ leading to $v \equiv 0$. Finally this completes (LS_∞^-).

Now we have proven all assumptions from [DPZ08] and can state the following theorem.

Theorem 5.1: *Let $5 < p < \infty$, $J := [0,T]$ and the spaces be defined as in (5.1). Then the problem*

$$\frac{d}{dt}u(t) + \mathcal{A}(t,q,D)u(t) = f(t) \qquad \text{in } J \times \Gamma^* \tag{5.3}$$

$$\frac{d}{dt}\rho(t) + \mathcal{B}_0(t,q,D)u(t) + \mathcal{C}_0(t,q,D_\partial)\rho(t) = g_0(t) \qquad \text{on } J \times \partial\Gamma^* \tag{5.4}$$

$$\mathcal{B}_1(t,q,D)u(t) + \mathcal{C}_1(t,q,D_\partial)\rho(t) = g_1(t) \qquad \text{on } J \times \partial\Gamma^* \tag{5.5}$$

$$\mathcal{B}_2(t,q,D)u(t) + \mathcal{C}_2(t,q,D_\partial)\rho(t) = g_2(t) \qquad \text{on } J \times \partial\Gamma^* \tag{5.6}$$

$$u(0) = u_0 \qquad \text{in } \Gamma^* \tag{5.7}$$

$$\rho(0) = \rho_0 \qquad \text{on } \partial\Gamma^* \tag{5.8}$$

has a unique solution $(u,\rho) \in Z_u \times Z_\rho$ if and only if

$$\begin{gathered} f \in X, \qquad u_0 \in \pi Z_u, \qquad \rho_0 \in \pi Z_\rho, \qquad g_0 \in Y_0, \\ g_1 \in Y_1, \qquad g_2 \in Y_2, \qquad g_0(0) - \mathcal{B}_0(0)u_0 - \mathcal{C}_0(0)\rho_0 \in \pi_1 Z_\rho, \\ \mathcal{B}_1(0)u_0 + \mathcal{C}_1(0)\rho_0 = g_1(0), \qquad \mathcal{B}_2(0)u_0 + \mathcal{C}_2(0)\rho_0 = g_2(0). \end{gathered}$$

Proof: Follows from Theorem 2.1 in [DPZ08] adapted to this specific case. ■

Corollary 5.2: *Let $5 < p < \infty$, $J := [0,T]$ and the spaces be defined as in (5.1). Then (4.14)-(4.17) has a unique solution $\varrho \in Z_u$ with $\varrho|_{\partial\Gamma^*} \in Z_\rho$ if and only if $\varrho_0 \in \pi Z_u$ and $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho$ and $\Delta_{\Gamma^*}\varrho_0 + |\sigma^*|^2\varrho_0 + (\nabla_{\Gamma^*}H_{\Gamma^*} \cdot P(\partial_w\Psi(0)))\,\varrho_0 = 0$.*

Proof: Follows from Theorem 5.1 if we choose $f \equiv 0$, $g_0 \equiv 0$, $g_1 \equiv 0$ and $g_2 \equiv 0$. Then we have exactly the right-hand sides of the flow (4.14)-(4.17) and can obviously drop the trivially satisfied conditions $f \in X$, $g_0 \in Y_0$, $g_1 \in Y_1$ and $g_2 \in Y_2$. Also $\mathcal{B}_2(0)u_0 + \mathcal{C}_2(0)\rho_0 = g_2(0)$ is valid since $u_0|_{\partial\Gamma^*} = \varrho_0|_{\partial\Gamma^*} = \rho_0$. Finally the condition

$$g_0(0) - \mathcal{B}_0(0,\bullet,-i\nabla_{\Gamma^*})\varrho_0 - \mathcal{C}_0(0,\bullet,-i\partial_\sigma)\varrho_0|_{\partial\Gamma^*} \in \pi_1 Z_\rho = W_p^{1-\frac{5}{p}}(\partial\Gamma^*;\mathbb{R})$$

can be ignored, because on the one hand $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho = W_p^{4-\frac{4}{p}}(\partial\Gamma^*;\mathbb{R})$, $\mathcal{C}_0$ is of second order and $W_p^{2-\frac{4}{p}}(\partial\Gamma^*;\mathbb{R}) \hookrightarrow W_p^{1-\frac{5}{p}}(\partial\Gamma^*;\mathbb{R})$ and on the other hand $\varrho_0 \in \pi Z_u = W_p^{4-\frac{4}{p}}(\Gamma^*;\mathbb{R})$,

$\mathcal{B}_0$ is of third order and the trace operator maps from $W_p^{1-\frac{4}{p}}(\Gamma^*;\mathbb{R})$ to $W_p^{1-\frac{5}{p}}(\partial\Gamma^*;\mathbb{R})$. Finally $\mathcal{B}_1(0)u_0 + \mathcal{C}_1(0)\rho_0 = g_1(0)$ simplifies to $\mathcal{B}_1(0)u_0 = 0$, where $u_0 = \varrho_0$ and

$$\mathcal{B}_1(0) = \Delta_{\Gamma^*} + |\sigma^*|^2 + \left(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P\left(\partial_w \Psi(0)\right)\right). \qquad \blacksquare$$

Since all the involved operators are time-independent we gain two byproduct theorems from [DPZ08]. The first is a semigroup formulation of the given problem and the second is an improvement of Corollary 5.2 in terms of the involved spaces.

Defining the operator

$$A : \mathcal{D}(A) \longrightarrow \mathcal{W}(A) : \begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix} \longmapsto \begin{pmatrix} \mathcal{A}(q, -i\nabla_{\Gamma^*}) & \mathbb{O} \\ \mathcal{B}_0(q, -i\nabla_{\Gamma^*}) & \mathcal{C}_0(q, -i\partial_\sigma) \end{pmatrix} \begin{pmatrix} \varrho \\ \widetilde{\varrho} \end{pmatrix},$$

where the domain and codomain are

$$\mathcal{D}(A) := \left\{ (\varrho, \widetilde{\varrho})^T \in W_p^4(\Gamma^*;\mathbb{R}) \times W_p^{4-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}) \,\middle|\, \varrho|_{\partial\Gamma^*} = \widetilde{\varrho}, \mathcal{B}_1(q, -i\nabla_{\Gamma^*})\varrho = 0 \right\}$$
$$\mathcal{W}(A) := L_p(\Gamma^*;\mathbb{R}) \times W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}).$$

Remark 5.3: Note that the condition $\mathcal{B}_0(\bullet, -i\nabla_{\Gamma^*})\varrho + \mathcal{C}_0(\bullet, -i\partial_\sigma)\widetilde{\varrho} \in W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})$ from the original domain in [DPZ08] is automatically satisfied, since $\widetilde{\varrho} \in W_p^{4-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})$, $\mathcal{C}_0$ is a second order differential operator and $W_p^{2-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}) \hookrightarrow W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})$ and furthermore $\varrho \in W_p^4(\Gamma^*;\mathbb{R})$, $\mathcal{B}_0$ is of third order and the trace operator maps from $W_p^1(\Gamma^*;\mathbb{R})$ to $W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})$. $\square$

Theorem 5.4: *Let $5 < p < \infty$. Then the operator $-A$ defined as above generates an analytic semigroup in $\mathcal{W}(A)$ which has the property of maximal L_p-regularity on each finite interval $J = [0, T]$. Moreover, there is $\omega \geq 0$ such that $-(A + \omega I)$ has maximal L_p-regularity on the half-line $J = \mathbb{R}_+$.*

Proof: Adapt Theorem 2.2 of [DPZ08] to the Willmore situation. $\blacksquare$

This leads to the improvement of Corollary 5.2, which reads as follows.

Theorem 5.5: *Let $5 < p < \infty$, $J := [0, T]$ and the spaces be defined as in (5.1). Then (4.14)-(4.17) has a unique solution $\varrho \in Z_u$ with*

$$\varrho|_{\partial\Gamma^*} \in W_p^1(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \cap W_p^{\frac{1}{4}-\frac{1}{4p}}(J; W_p^2(\partial\Gamma^*;\mathbb{R}))$$
$$\cap\, W_p^{1-\frac{1}{4p}}(J; L_p(\partial\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^{4-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$$

if and only if $\varrho_0 \in \pi Z_u$, $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho$ and $\mathcal{B}_1(q, -i\nabla_{\Gamma^*})\varrho_0 = 0$.
The same statement is true for $J = \mathbb{R}_+$ if ∂_t is replaced by $\partial_t + \omega I$ for some sufficiently large $\omega > 0$.

Proof: Follows from Corollary 2.3 of [DPZ08] in the Willmore situation. ■

Remark 5.6: For the same reason as in Remark 5.3 and the proof of Corollary 5.2 we were able to erase the three conditions

$$
\begin{aligned}
&\mathcal{B}_0(\bullet, -i\nabla_{\Gamma^*})\varrho + \mathcal{C}_0(\bullet, -i\partial_\sigma)\varrho|_{\partial\Gamma^*} \in L_p(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))\\
&\mathcal{B}_0(\bullet, -i\nabla_{\Gamma^*})\varrho_0 + \mathcal{C}_0(\bullet, -i\partial_\sigma)\varrho_0|_{\partial\Gamma^*} \in \pi_1 Z_\rho\\
&\mathcal{B}_2(\bullet, -i\nabla_{\Gamma^*})\varrho_0 + \mathcal{C}_2(\bullet, -i\partial_\sigma)\varrho_0|_{\partial\Gamma^*} = g_2(0)
\end{aligned}
$$

from the original theorem in [DPZ08]. □

5.2 Short-time existence of solutions for the Willmore Flow

Now we want to prove short-time existence of solutions of the non-linear flow

$$
\begin{aligned}
V_\Gamma(u(t)) &= -\Delta_\Gamma H_\Gamma(u(t)) - \frac{1}{2}H_\Gamma(u(t))\left(H_\Gamma(u(t))^2 - 4K_\Gamma(u(t))\right) && \text{in } J\times\Gamma^* && (5.9)\\
v_{\partial D}(\rho(t)) &= \frac{1}{2}\sin(\alpha)(\nabla_\Gamma H_\Gamma(u(t))\cdot n_{\partial\Gamma}(u(t))) + a + b\varkappa_{\partial D}(\rho(t)) && \text{on } J\times\partial\Gamma^* && (5.10)\\
0 &= H_\Gamma(u(t)) && \text{on } J\times\partial\Gamma^* && (5.11)\\
0 &= u(t) - \rho(t) && \text{on } J\times\partial\Gamma^* && (5.12)\\
u(0) &= u_0 && \text{in } \Gamma^* && (5.13)\\
\rho(0) &= \rho_0 && \text{on } \partial\Gamma^*, && (5.14)
\end{aligned}
$$

where we have changed the notation and adopted the structure of the linearized PDE in Theorem 3.2 again. To prove short-time existence we follow the same strategy as in Section 3.2. We define functions $\Phi := (u,\rho)$, $\Phi_0 := (u_0,\rho_0)$, spaces

$$
\begin{aligned}
\mathbb{E} &:= Z_u \times Z_\rho\\
\mathbb{F} &:= X \times Y_0 \times Y_1 \times \{0\}\\
\mathbb{I} &:= \{(u_0,\rho_0) \in \pi Z_u \times \pi Z_\rho \mid u_0|_{\partial\Gamma^*} = \rho_0\}
\end{aligned}
$$

with their norms

$$
\begin{aligned}
\|\Phi\|_{\mathbb{E}} &:= \|u\|_{Z_u} + \|\rho\|_{Z_\rho}\\
\|f\|_{\mathbb{F}} &:= \left\|f^{(1)}\right\|_X + \left\|f^{(2)}\right\|_{Y_0} + \left\|f^{(3)}\right\|_{Y_1}\\
\|\Phi_0\|_{\mathbb{I}} &:= \|u_0\|_{\pi Z_u} + \|\rho_0\|_{\pi Z_\rho}
\end{aligned}
$$

and the operator $L : \mathbb{E} \longrightarrow \mathbb{F}\times\mathbb{I}$ as the left-hand side of (5.3)-(5.8). For the right hand side of the contraction mapping principle we define the non-linear operator $N : \mathbb{E} \longrightarrow \mathbb{F}$ as

$$
N(\Phi) := \begin{pmatrix} F(u)\\ G(u,\rho)\\ H(u,\rho)\\ 0 \end{pmatrix},
$$

where

$$
\begin{aligned}
F(u) &:= -\Delta_\Gamma H_\Gamma(u) - \frac{1}{2} H_\Gamma(u)\left(H_\Gamma(u)^2 - 4K_\Gamma(u)\right) - V_\Gamma(u) + \frac{d}{dt}u + \mathcal{A}(q,D)u \\
G(u,\rho) &:= \frac{1}{2}\sin(\alpha)(\nabla_\Gamma H_\Gamma(u)\cdot n_{\partial\Gamma}(u)) + a + b\varkappa_{\partial D}(\rho) - v_{\partial D}(\rho) \\
&\quad + \frac{d}{dt}\rho + \mathcal{B}_0(q,D)u + \mathcal{C}_0(q,D_\partial)\rho \\
H(u,\rho) &:= H_\Gamma(u) + \mathcal{B}_1(q,D)u + \mathcal{C}_1(q,D)\rho.
\end{aligned}
$$

In order to solve the equation $L\Phi = (N\Phi, \Phi_0)$ by the contraction mapping principle we have to prove a technical lemma again.

Lemma 5.7: *(i) Let $5 < p < \infty$ and $\sigma \in [0,1]$. Then one gets*

$$
\begin{aligned}
Z_u &\hookrightarrow W_p^{\sigma}(J; W_p^{4(1-\sigma)}(\Gamma^*;\mathbb{R})) \\
Z_\rho &\hookrightarrow W_p^{\sigma(\frac{5}{4}-\frac{1}{4p})}(J; W_p^{(1-\sigma)(4-\frac{1}{p})}(\partial\Gamma^*;\mathbb{R})) \\
Y_0 &\hookrightarrow W_p^{\sigma(\frac{1}{4}-\frac{1}{4p})}(J; W_p^{(1-\sigma)(1-\frac{1}{p})}(\partial\Gamma^*;\mathbb{R})).
\end{aligned}
$$

(ii) Let $6 < p < \infty$. Then we have

$$
\begin{aligned}
Z_u &\hookrightarrow BUC(J; W_p^{4-\frac{4}{p}}(\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; BUC^3(\Gamma^*;\mathbb{R})) \qquad (5.15)\\
Z_\rho &\hookrightarrow BUC(J; W_p^{4-\frac{4}{p}}(\partial\Gamma^*;\mathbb{R})) \hookrightarrow BUC(J; BUC^3(\partial\Gamma^*;\mathbb{R})).
\end{aligned}
$$

<u>Proof</u>: As in the proof of Lemma 3.15 we do not distinguish between $W_p^\alpha(J; W_p^\beta(\Gamma^*;\mathbb{R}))$ and $W_p^\alpha(J; W_p^\beta(K;\mathbb{R}))$ for some open $K \subseteq \mathbb{R}^2$ with $\overline{K}$ compact.
(i) The three embeddings follow from Lemma 3.15(i) with $s = 0$, $\alpha = 1$, $r = 0$ and $\beta = 4$ or $s = 0$, $\alpha = \frac{5}{4} - \frac{1}{4p}$, $r = 0$ and $\beta = 4 - \frac{1}{p}$ or $s = 0$, $\alpha = \frac{1}{4} - \frac{1}{4p}$, $r = 0$ and $\beta = 1 - \frac{1}{p}$, respectively.
(ii) The first parts of the embeddings are obtained from Lemma 3.15(ii) with $k = 0$, $s_1 = 0$, $s_2 = 1$, $r_1 = 4$ and $r_2 = 0$ as well as $k = 0$, $s_1 = 0$, $s_2 = 1$, $r_1 = 4 - \frac{1}{p}$ and $r_2 = 1 - \frac{1}{p}$, whereas the second parts follow from the usual Sobolev embeddings and our assumption $p > 6$. ■

Remark 5.8: The embedding (5.15) is only valid for $p > 6$ and will be crucial in the considerations to follow. This is the reason why we are again forced to change the range of p from $p > 5$ in Theorem 5.1 to $p > 6$ in our final Theorem 5.14. □

Following the strategy of Section 3.2 we use Lemma 3.15(i) with $\sigma = \frac{1}{4}$ and $\sigma = \frac{1}{2}$ to get $Z_u \hookrightarrow W_p^{\frac{1}{4}}(J; W_p^3(\Gamma^*;\mathbb{R}))$ and $Z_u \hookrightarrow W_p^{\frac{1}{2}}(J; W_p^2(\Gamma^*;\mathbb{R}))$, respectively. This shows

$$
\begin{aligned}
\nabla^3_{\Gamma^*} u &\in W_p^{\frac{1}{4}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R})) \\
\nabla^2_{\Gamma^*} u &\in W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^2(\Gamma^*;\mathbb{R})).
\end{aligned}
$$

Next we get by the same lemma with $\sigma = \frac{2p-1}{4p-1}$ on the one hand

$$W_p^{\frac{5}{4}-\frac{1}{4p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{4-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})) \hookrightarrow W_p^{\tau}(J; W_p^2(\partial\Gamma^*; \mathbb{R})),$$

where $\tau := \frac{(5p-1)(2p-1)}{(4p-1)4p}$ and on the other hand

$$W_p^1(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{4-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})) \hookrightarrow W_p^{\frac{2}{3}-\frac{1}{3p}}(J; W_p^2(\partial\Gamma^*; \mathbb{R})),$$

where we used $\sigma = \frac{2}{3} - \frac{1}{3p}$. Since $\tau < \frac{2}{3} - \frac{1}{3p}$ we get

$$\partial_\sigma^2 \rho \in W_p^{\frac{2}{3}-\frac{1}{3p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R})).$$

Hence

$$\begin{aligned}
\nabla^3 Z_u &:= W_p^{\frac{1}{4}}(J; L_p(\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*; \mathbb{R})) \\
\nabla^2 Z_u &:= W_p^{\frac{1}{2}}(J; L_p(\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^2(\Gamma^*; \mathbb{R})) \\
\nabla^2 Z_\rho &:= W_p^{\frac{2}{3}-\frac{1}{3p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}))
\end{aligned}$$

are the spaces containing the third and second spacial derivatives of u and the second arc-length derivatives of ρ, respectively.

Lemma 5.9: *Let $6 < p < \infty$. Then the spaces*

$$\begin{aligned}
&\textit{(i)} && \nabla^3 Z_u = W_p^{\frac{1}{4}}(J; L_p(\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*; \mathbb{R})) \\
&\textit{(ii)} && \nabla^2 Z_u = W_p^{\frac{1}{2}}(J; L_p(\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^2(\Gamma^*; \mathbb{R})) \\
&\textit{(iii)} && \nabla^2 Z_\rho = W_p^{\frac{2}{3}-\frac{1}{3p}}(J; L_p(\partial\Gamma^*; \mathbb{R})) \cap L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}))
\end{aligned}$$

are Banach algebras up to a constant in the norm estimate of the product.

Proof: As we saw in Lemma 3.17 the crucial ingredient is the embedding into the space of bounded uniformly continuous functions. Therefore we will only prove these embeddings and the rest will follow in exactly the same manner as in Lemma 3.17.
(i) Lemma 3.15(ii) gives the embedding

$$\nabla^3 Z_u \hookrightarrow BUC(J; W_p^{1-\frac{4}{p}}(\Gamma^*; \mathbb{R})) \hookrightarrow BUC(J; BUC(\Gamma^*; \mathbb{R})),$$

where we used $p > 6$ in the second step.
(ii) Here Lemma 3.15(ii) gives the embedding

$$\nabla^2 Z_u \hookrightarrow BUC(J; W_p^{2-\frac{4}{p}}(\Gamma^*; \mathbb{R})) \hookrightarrow BUC(J; BUC^1(\Gamma^*; \mathbb{R})),$$

where again we have used $p > 6$ in the second step.
(iii) The same lemma as in (i) shows

$$\nabla^2 Z_\rho \hookrightarrow BUC(J; W_p^{4-\frac{4}{p}}(\partial\Gamma^*; \mathbb{R})) \hookrightarrow BUC(J; BUC^3(\partial\Gamma^*; \mathbb{R})),$$

where we used $p > 5$ in the second embedding. ∎

Lemma 5.10: *Let* $J := [0,T]$ *and* $6 < p < \infty$ *and* $\mathbb{B}_r^{\mathbb{E}}(\mathbb{O}) := \{\Phi \in \mathbb{E} \,|\, \|\Phi\|_{\mathbb{E}} < r\}$. *Then there exists an* $r > 0$ *such that* $N(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O})) \subseteq \mathbb{F}$. *Moreover,* $N \in C^1(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O}); \mathbb{F})$ *and* $DN[\mathbb{O}] = \mathbb{O}$, *where* DN *denotes the Fréchet derivative of* N.

Proof: The fact that $DN[\mathbb{O}] = \mathbb{O}$ is obvious due to the structure of the linearization in Section 4.2. The linearization we calculated in Section 4.2 is indeed the Fréchet derivative as we can see by the same argument as in the proof of Lemma 3.18. Only the Lipschitz continuity of $\delta_\Phi N$ is needed, which we will see at the end of the proof.
Our first goal is to show that $F(u) \in X$, $G(u,\rho) \in Y_0$ and $H(u) \in Y_1$ for all $u \in \mathbb{B}^{Z_u}_{\frac{r}{\sqrt{2}}}(\mathbb{O})$ and $\rho \in \mathbb{B}^{Z_\rho}_{\frac{r}{\sqrt{2}}}(\mathbb{O})$. For $r > 0$ small enough all the terms appearing in F, G and H are well-defined and the linear parts of F, G and H can be omitted since

- $\mathcal{A}(q,D)u \in X$ due to $u \in Z_u \subseteq L_p(J; W_p^4(\Gamma^*;\mathbb{R}))$ and $\mathcal{A}$ is of fourth order in space.
- $\frac{d}{dt}u \in X$ due to $u \in Z_u \subseteq W_p^1(J; L_p(\Gamma^*;\mathbb{R}))$ and $\frac{d}{dt}$ is of first order in time.
- $\frac{d}{dt}\rho \in Y_0$ due to $\rho \in Z_\rho \subseteq W_p^{\frac{5}{4}-\frac{1}{4p}}(J; L_p(\partial\Gamma^*;\mathbb{R}))$ and $\frac{d}{dt}$ is of first order in time as well as $\rho \in Z_\rho \subseteq W_p^1(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$ and $\frac{d}{dt}$ is of first order in time.
- $\mathcal{B}_0(q,D)u \in Y_0$ due to $u \in Z_u \hookrightarrow W_p^{\frac{1}{4}}(J; W_p^3(\Gamma^*;\mathbb{R}))$ because of Lemma 5.7(i) with $\sigma = \frac{1}{4}$ and $\mathcal{B}_0$ is of third order in space. This leads to
$$\mathcal{B}_0(q,D)u \in W_p^{\frac{1}{4}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R}))$$
and by (A.24) in [Gru95] the trace operator γ_0 maps as follows
$$\gamma_0 : W_p^{\frac{1}{4}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^1(\Gamma^*;\mathbb{R})) \longrightarrow Y_0.$$
- $\mathcal{C}_0(q,D_\partial)\rho \in Y_0$ due to $\rho \in Z_\rho \subseteq L_p(J; W_p^{4-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$ and $\mathcal{C}_0$ is of second order in space and
$$L_p(J; W_p^{2-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \hookrightarrow L_p(J; W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R}))$$
as well as $\rho \in Z_\rho \hookrightarrow W_p^{\frac{1}{4}-\frac{1}{4p}}(J; W_p^{4-\frac{4p}{5p-1}}(\partial\Gamma^*;\mathbb{R}))$ by Lemma 5.7(i) with $\sigma = \frac{p-1}{5p-1}$ and $\mathcal{C}_0$ is of second order in space and
$$W_p^{\frac{1}{4}-\frac{1}{4p}}(J; W_p^{2-\frac{4p}{5p-1}}(\partial\Gamma^*;\mathbb{R})) \hookrightarrow W_p^{\frac{1}{4}-\frac{1}{4p}}(J; L_p(\partial\Gamma^*;\mathbb{R}))$$
since $p > 6$.
- $\mathcal{B}_1(q,D)u \in Y_1$ due to $u \in Z_u \hookrightarrow W_p^{\frac{1}{2}}(J; W_p^2(\Gamma^*;\mathbb{R}))$ because of Lemma 5.7(i) with $\sigma = \frac{1}{2}$ and $\mathcal{B}_1$ is of second order in space. This leads to
$$\mathcal{B}_1(q,D)u \in W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^2(\Gamma^*;\mathbb{R}))$$
and by (A.24) in [Gru95] the trace operator γ_0 maps as follows
$$\gamma_0 : W_p^{\frac{1}{2}}(J; L_p(\Gamma^*;\mathbb{R})) \cap L_p(J; W_p^2(\Gamma^*;\mathbb{R})) \longrightarrow Y_1.$$

- $\mathcal{C}_1(q, D_\partial)\rho = 0 \in Y_1$ since $\mathcal{C}_1(q, D_\partial) = 0$.

Next we want to turn our attention to the two velocities in F and G. We first note that

$$\sup_{t\in J}\sup_{q\in\Gamma^*} |n_\Gamma(u)\cdot\partial_w\Psi(u)| \le \sup_{t\in J}\sup_{q\in\Gamma^*} \|n_\Gamma(u)\|\,\|\partial_w\Psi(u)\| = \sup_{t\in J}\sup_{q\in\Gamma^*} \|\partial_w\Psi(u)\| =: c < \infty,$$

since J is compact, $\partial_w\Psi$ is continuous up to the boundary $\partial\Gamma^*$ and we assumed $0 < \alpha < \pi$. Hence using (2.31) we obtain

$$\|V_\Gamma(u)\|_X \le c\,\|\partial_t u(t,q)\|_{L_p(J;L_p(\Gamma^*;\mathbb{R}))} \le c\,\|u(t,q)\|_{W_p^1(J;L_p(\Gamma^*;\mathbb{R}))} < \infty.$$

Analogously we get with (2.35) for the normal boundary velocity

$$\begin{aligned}
\|v_{\partial\Gamma}(\rho)\|_{W_p^{\frac14-\frac{1}{4p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))} &\le c\,\|\partial_t\rho(t,q)\|_{W_p^{\frac14-\frac{1}{4p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))}\\
&\le c\,\|\rho(t,q)\|_{W_p^{\frac54-\frac{1}{4p}}(J;L_p(\partial\Gamma^*;\mathbb{R}))} < \infty\\
\|v_{\partial\Gamma}(\rho)\|_{L_p(J;W_p^{1-\frac1p}(\partial\Gamma^*;\mathbb{R}))} &\le c\,\|\partial_t\rho(t,q)\|_{L_p(J;W_p^{1-\frac1p}(\partial\Gamma^*;\mathbb{R}))}\\
&\le c\,\|\rho(t,q)\|_{W_p^1(J;W_p^{1-\frac1p}(\partial\Gamma^*;\mathbb{R}))} < \infty
\end{aligned}$$

due to the fact that

$$\begin{aligned}
\rho \in Z_\rho \subseteq W_p^{\frac54-\frac{1}{4p}}(J;L_p(\partial\Gamma^*;\mathbb{R})) \cap L_p(J;W_p^{4-\frac1p}(\partial\Gamma^*;\mathbb{R})) &\hookrightarrow W_p^{\frac12-\frac{1}{2p}}(J;W_p^{1+\frac1p}(\partial\Gamma^*;\mathbb{R}))\\
&\hookrightarrow C^{\frac14-\frac{1}{4p}}(J;BUC^1(\partial\Gamma^*;\mathbb{R})).
\end{aligned}$$

This proves that $\|v_{\partial\Gamma}(\rho)\|_{Y_0} < \infty$.
Since J, $\partial\Gamma^*$ and Γ^* are bounded $\|c\|_{W_p^\alpha(J;W_p^\beta(\Gamma^*;\mathbb{R}))} < \infty$ and $\|c\|_{W_p^\alpha(J;W_p^\beta(\partial\Gamma^*;\mathbb{R}))} < \infty$ for every constant function c and every $\alpha,\beta \in [0,\infty)$. By Lemma 5.7(ii) we see that $|u(t,q)|$, $|\nabla_{\Gamma^*}u(t,q)|$, $|\nabla^2_{\Gamma^*}u(t,q)|$ and $|\nabla^3_{\Gamma^*}u(t,q)|$ stay bounded. This shows that for a maybe even smaller r the first fundamental form of all the hypersurfaces in the family $(\Gamma_\varrho(t))_{t\in J}$ is not degenerated. Because of the fact that $\Delta_\Gamma H_\Gamma(u)$ depends linearly on the fourth space derivatives of u and that the coefficients, involving only u and its first to third derivatives, are bounded, we get

$$\|\Delta_\Gamma H_\Gamma(u)\|_X \le c\left(\left\|\nabla^4_{\Gamma^*}u\right\|_X + 1\right) \le c\left(\|u\|_{L_p(J;W_p^4(\Gamma^*;\mathbb{R}))} + 1\right) \le c\left(\|u\|_{Z_u} + 1\right) < \infty.$$

Because of the fact that $|u(t,q)|$, $|\nabla_{\Gamma^*}u(t,q)|$ and $|\nabla^2_{\Gamma^*}u(t,q)|$ stay bounded and $H_\Gamma(u)$ and $K_\Gamma(u)$ depend continuously on u and its first and second space derivatives, we obtain

$$\begin{aligned}
\|H_\Gamma(u)\|_X &\le \sup_{t\in J}\sup_{q\in\Gamma^*}|H_\Gamma(u)|\,\|1\|_X \le c\,\|1\|_X < \infty\\
\left\|H_\Gamma(u)^2\right\|_X &\le \left(\sup_{t\in J}\sup_{q\in\Gamma^*}|H_\Gamma(u)|\right)^2 \|1\|_X \le c\,\|1\|_X < \infty\\
\|K_\Gamma(u)\|_X &\le \sup_{t\in J}\sup_{q\in\Gamma^*}|K_\Gamma(u)|\,\|1\|_X \le c\,\|1\|_X < \infty.
\end{aligned}$$

The same argument holds true for $\|H_\Gamma(u)\|_{Y_1}$ and $\|\nabla_\Gamma H_\Gamma(u)\|_{Y_0}$, because also $|\nabla^3_{\Gamma^*} u(t,q)|$ stays bounded.
For $\varkappa_{\partial D}(\rho)$ we observe by Lemma 5.7(ii) that $Z_\rho \hookrightarrow BUC(J; BUC^3(\partial\Gamma^*;\mathbb{R}))$ and hence $|\rho(t,q)|$, $|\partial_\sigma \rho(t,q)|$ and $|\partial^2_\sigma \rho(t,q)|$ stay bounded. In the same way as for $H_\Gamma(u)$, the continuous dependence of $\varkappa_{\partial D}(\rho)$ on ρ and its derivatives shows the boundedness and thus

$$\|\varkappa_{\partial D}(\rho)\|_{Y_0} \leq \sup_{t\in J}\sup_{q\in\partial\Gamma^*} |\varkappa_{\partial D}(\rho)|\,\|1\|_{Y_0} \leq c\,\|1\|_{Y_0} < \infty.$$

This shows $N(\mathbb{B}^{\mathbb{E}}_r(\mathbb{O})) \subseteq \mathbb{F}$. What is left is the fact that $N \in C^1(\mathbb{B}^{\mathbb{E}}_r(\mathbb{O});\mathbb{F})$, for which it is enough to show Lipschitz continuity of the several parts of N.
Again we start with the highest order terms included in $\Delta_\Gamma H_\Gamma$. We know that we can write

$$\Delta_\Gamma H_\Gamma(u) = \sum_{|\alpha|=4} a_\alpha(\underbrace{u, \nabla_{\Gamma^*} u, \nabla^2_{\Gamma^*} u, \nabla^3_{\Gamma^*} u}_{=:\mathbb{U}})\partial^\alpha u + b(\mathbb{U})$$

with $a_\alpha, b \in C^3(U)$ and $U \subseteq \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^{2\times 2}\times\mathbb{R}^{2\times 2\times 2}$ a closed neighborhood of 0. Linearizing this we obtain

$$\begin{aligned}(D_u\Delta_\Gamma H_\Gamma(u))(v) = \sum_{|\alpha|=4} \Big(&\partial^\alpha u \partial_1 a_\alpha(\mathbb{U})v + \partial^\alpha u\partial_2 a_\alpha(\mathbb{U})(\nabla_{\Gamma^*} v)\\ &+ \partial^\alpha u\partial_3 a_\alpha(\mathbb{U})(\nabla_{\Gamma^*} v)(\nabla_{\Gamma^*} v) + \partial^\alpha u\partial_4 a_\alpha(\mathbb{U})(\nabla_{\Gamma^*} v)(\nabla_{\Gamma^*} v)(\nabla_{\Gamma^*} v)\\ &+ a_\alpha(\mathbb{U})\partial^\alpha v\Big) + \partial_1 b(\mathbb{U})v + \partial_2 b(\mathbb{U})(\nabla_{\Gamma^*} v)\\ &+ \partial_3 b(\mathbb{U})(\nabla_{\Gamma^*} v)(\nabla_{\Gamma^*} v) + \partial_4 b(\mathbb{U})(\nabla_{\Gamma^*} v)(\nabla_{\Gamma^*} v)(\nabla_{\Gamma^*} v).\end{aligned}$$

Again by the smoothness of a and b the coefficients $\partial_i a_\alpha, \partial_i b$ with $i \in \{1,2,3,4\}$ and a_α satisfy a Lipschitz condition on $\overline{B_r(0)} \subseteq \nabla^3 Z_u$. Because of $\|\bullet\|_{\nabla^3 Z_u} \leq c\,\|\bullet\|_{Z_u}$, two functions $u, \widetilde{u} \in \overline{B_r(0)} \subseteq Z_u$ are also in $\overline{B_{cr}(0)} \subseteq \nabla^3 Z_u$. Analogously to H_Γ in Lemma 3.18 we use this time $\|\nabla^i_{\Gamma^*}\|_{\mathcal{L}(Z_u,X)} < \infty$ for $i \in \{0,1,2,3,4\}$ and the embedding $\nabla^3 Z_u \hookrightarrow X$ to prove for $u, \widetilde{u} \in \overline{B_r(0)} \subseteq Z_u$ the estimate

$$\begin{aligned}\Big\|D_u\Delta_\Gamma H_\Gamma(\mathbb{U}) - D_u\Delta_\Gamma H_\Gamma(\widetilde{\mathbb{U}})\Big\|_{\mathcal{L}(Z_u,X)}&\\ \leq \sum_{|\alpha|=4}\Big(&\|\partial_1 a_\alpha(\mathbb{U})\|_{L_\infty(J;L_\infty(\Gamma^*))}\|\partial^\alpha(u-\widetilde{u})\|_X\\ &+ \Big\|\partial_1 a_\alpha(\mathbb{U}) - \partial_1 a_\alpha(\widetilde{\mathbb{U}})\Big\|_{L_\infty(J;L_\infty(\Gamma^*))}\|\partial^\alpha \widetilde{u}\|_X\\ &+ \|\partial_2 a_\alpha(\mathbb{U})\|_{L_\infty(J;L_\infty(\Gamma^*))}\|\partial^\alpha(u-\widetilde{u})\|_X\\ &+ \Big\|\partial_2 a_\alpha(\mathbb{U}) - \partial_2 a_\alpha(\widetilde{\mathbb{U}})\Big\|_{L_\infty(J;L_\infty(\Gamma^*))}\|\partial^\alpha \widetilde{u}\|_X\\ &+ \|\partial_3 a_\alpha(\mathbb{U})\|_{L_\infty(J;L_\infty(\Gamma^*))}\|\partial^\alpha(u-\widetilde{u})\|_X\\ &+ \Big\|\partial_3 a_\alpha(\mathbb{U}) - \partial_3 a_\alpha(\widetilde{\mathbb{U}})\Big\|_{L_\infty(J;L_\infty(\Gamma^*))}\|\partial^\alpha \widetilde{u}\|_X\\ &+ \underbrace{\|\partial_4 a_\alpha(\mathbb{U})\|_{L_\infty(J;L_\infty(\Gamma^*))}}_{\leq \widetilde{c}\|\partial_i a_\alpha(\mathbb{U})\|_{\nabla^3 Z_u}\leq c(r)}\underbrace{\|\partial^\alpha(u-\widetilde{u})\|_X}_{\leq\|u-\widetilde{u}\|_{Z_u}}\end{aligned}$$

$$+\underbrace{\left\|\partial_4 a_\alpha(\mathbb{U})-\partial_4 a_\alpha(\widetilde{\mathbb{U}})\right\|_{L_\infty(J;L_\infty(\Gamma^*))}}_{\leq \widetilde{c}\|\ldots\|_{\nabla^3 Z_u}\leq \widehat{c}\sum_{i=0}^{3}\left\|\nabla^i_{\Gamma^*}u-\nabla^i_{\Gamma^*}\widetilde{u}\right\|_{\nabla^3 Z_u}}\underbrace{\|\partial^\alpha \widetilde{u}\|_X}_{\leq r}\Big)$$

$$+\,5c\underbrace{\sum_{i=0}^{3}\left\|\nabla^i_{\Gamma^*}u-\nabla^i_{\Gamma^*}\widetilde{u}\right\|_{\nabla^3 Z_u}}_{\leq 4\left\|u-\widetilde{u}\right\|_{Z_u}}\leq c(r)\left\|u-\widetilde{u}\right\|_{Z_u}.$$

This shows the Lipschitz continuity of $D_u\Delta_\Gamma H_\Gamma:\overline{B_r(0)}\subseteq Z_u\longrightarrow \mathcal{L}(Z_u,X)$ and hence we see $\Delta_\Gamma H_\Gamma\in C^1(\overline{B_r(0)},X)$.

The lower order terms given by

$$HK(u):=H_\Gamma(u)\Big(H_\Gamma(u)^2-4K_\Gamma(u)\Big)=b(u,\nabla_{\Gamma^*}u,\nabla^2_{\Gamma^*}u)$$

with $b\in C^3(U)$ can be treated exactly as the non-linear lower order terms in $\Delta_\Gamma H_\Gamma(u)$ leading to $HK\in C^1(\overline{B_r(0)},X)$.

Due to the higher regularity of Z_ρ the considerations for $\varkappa_{\partial D}$ this time are even easier than in Lemma 3.18. We can write

$$\varkappa_{\partial D}(\rho)=b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)$$

with $b\in C^5(U)$. Linearizing this we obtain

$$(D_\rho\varkappa_{\partial D}(\rho))(v)=\partial_1 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)v+\partial_2 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)\partial_\sigma v+\partial_3 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)\partial^2_\sigma v.$$

Here we can treat $\varkappa_{\partial D}$ as a non-linear operator, because the second order arc-length derivatives of ρ are still elements of the algebra $\nabla^2 Z_\rho$. Therefore

$$\begin{aligned}\|D_\rho\varkappa_{\partial D}(\rho)-D_\rho\varkappa_{\partial D}(\widetilde{\rho})\|_{\mathcal{L}(Z_\rho,Y_0)}&\leq\left\|\partial_1 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)\operatorname{Id}-\partial_1 b(\widetilde{\rho},\partial_\sigma\widetilde{\rho},\partial^2_\sigma\widetilde{\rho})\operatorname{Id}\right\|_{\mathcal{L}(Z_\rho,Y_0)}\\&+\left\|\partial_2 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)\partial_\sigma-\partial_2 b(\widetilde{\rho},\partial_\sigma\widetilde{\rho},\partial^2_\sigma\widetilde{\rho})\partial_\sigma\right\|_{\mathcal{L}(Z_\rho,Y_0)}\\&+\left\|\partial_3 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)\partial^2_\sigma-\partial_3 b(\widetilde{\rho},\partial_\sigma\widetilde{\rho},\partial^2_\sigma\widetilde{\rho})\partial^2_\sigma\right\|_{\mathcal{L}(Z_\rho,Y_0)}\\&\leq\left\|\partial_1 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)-\partial_1 b(\widetilde{\rho},\partial_\sigma\widetilde{\rho},\partial^2_\sigma\widetilde{\rho})\right\|_{Y_0}\|\operatorname{Id}\|_{\mathcal{L}(Z_\rho,Y_0)}\\&+\left\|\partial_2 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)-\partial_2 b(\widetilde{\rho},\partial_\sigma\widetilde{\rho},\partial^2_\sigma\widetilde{\rho})\right\|_{Y_0}\|\partial_\sigma\|_{\mathcal{L}(Z_\rho,Y_0)}\\&+\left\|\partial_3 b(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho)-\partial_3 b(\widetilde{\rho},\partial_\sigma\widetilde{\rho},\partial^2_\sigma\widetilde{\rho})\right\|_{Y_0}\left\|\partial^2_\sigma\right\|_{\mathcal{L}(Z_\rho,Y_0)}.\end{aligned}$$

Since $\nabla^2 Z_\rho\hookrightarrow Y_0$ as well as $\|\operatorname{Id}\|_{\mathcal{L}(Z_\rho,Y_0)}<\infty$, $\|\partial_\sigma\|_{\mathcal{L}(Z_\rho,Y_0)}<\infty$ and $\|\partial^2_\sigma\|_{\mathcal{L}(Z_\rho,Y_0)}<\infty$ we conclude using the smoothness of $\partial_i b$

$$\begin{aligned}&\|D_\rho\varkappa_{\partial D}(\rho)-D_\rho\varkappa_{\partial D}(\widetilde{\rho})\|_{\mathcal{L}(Z_\rho,Y_0)}\\&\qquad\leq 3c\Big(\underbrace{\|\rho-\widetilde{\rho}\|_{\nabla^2 Z_\rho}}_{\leq\left\|\rho-\widetilde{\rho}\right\|_{Z_\rho}}+\underbrace{\|\partial_\sigma\rho-\partial_\sigma\widetilde{\rho}\|_{\nabla^2 Z_\rho}}_{\leq\left\|\rho-\widetilde{\rho}\right\|_{Z_\rho}}+\underbrace{\left\|\partial^2_\sigma\rho-\partial^2_\sigma\widetilde{\rho}\right\|_{\nabla^2 Z_\rho}}_{\leq\left\|\rho-\widetilde{\rho}\right\|_{Z_\rho}}\Big)\\&\qquad\leq c\left\|\rho-\widetilde{\rho}\right\|_{Z_\rho}.\end{aligned}$$

This shows that $D_\rho \varkappa_{\partial D} : \overline{B_r(0)} \subseteq Z_\rho \longrightarrow \mathcal{L}(Z_\rho, Y_0)$ is Lipschitz continuous and thus $\varkappa_{\partial D} \in C^1(\overline{B_r(0)}, Y_0)$.
The calculations for V_Γ and $\varkappa_{\partial D}$ do not change significantly compared to Lemma 3.18 leading to $V_\Gamma \in C^1(\overline{B_r(0)}, X)$ and $v_{\partial D} \in C^1(\overline{B_r(0)}, Y_0)$.
From the first boundary condition the third order term remains unconsidered. Here we can write

$$\nabla H(u) := \nabla_\Gamma H_\Gamma(u) \cdot n_{\partial\Gamma}(u) = b(u, \nabla_{\Gamma^*} u, \nabla^2_{\Gamma^*} u, \nabla^3_{\Gamma^*} u)$$

with $b \in C^3(U)$ implying the Lipschitz continuity of $\partial_i b$ for all $i \in \{1, 2, 3, 4\}$. We estimate

$$\begin{aligned}
\|D_u \nabla H(u) - D_u \nabla H(\widetilde{u})\|_{\mathcal{L}(Z_u, \nabla^3 Z_u)} &\leq \left\| \partial_1 b(\mathbb{U}) \operatorname{Id} - \partial_1 b(\widetilde{\mathbb{U}}) \operatorname{Id} \right\|_{\mathcal{L}(Z_u, \nabla^3 Z_u)} \\
&+ \left\| \partial_2 b(\mathbb{U})(\nabla_{\Gamma^*}) - \partial_2 b(\widetilde{\mathbb{U}})(\nabla_{\Gamma^*}) \right\|_{\mathcal{L}(Z_u, \nabla^3 Z_u)} \\
&+ \left\| (\partial_3 b(\mathbb{U}) - \partial_3 b(\widetilde{\mathbb{U}}))(\nabla_{\Gamma^*})(\nabla_{\Gamma^*}) \right\|_{\mathcal{L}(Z_u, \nabla^3 Z_u)} \\
&+ \left\| (\partial_4 b(\mathbb{U}) - \partial_4 b(\widetilde{\mathbb{U}}))(\nabla_{\Gamma^*})(\nabla_{\Gamma^*})(\nabla_{\Gamma^*}) \right\|_{\mathcal{L}(Z_u, \nabla^3 Z_u)} \\
&\leq \sum_{i=0}^{3} \left\| \partial_i b(\mathbb{U}) - \partial_i b(\widetilde{\mathbb{U}}) \right\|_{\nabla^3 Z_u} \left\| \nabla^i_{\Gamma^*} \right\|_{\mathcal{L}(Z_u, \nabla^3 Z_u)} \\
&\leq 4c \sum_{i=0}^{3} \underbrace{\left\| \nabla^i_{\Gamma^*} u - \nabla^i_{\Gamma^*} \widetilde{u} \right\|_{\nabla^3 Z_u}}_{\leq \widetilde{c} \|u - \widetilde{u}\|_{Z_u}} \leq \widehat{c} \, \|u - \widetilde{u}\|_{Z_u}
\end{aligned}$$

proving $\nabla H \in C^1(\overline{B_r(0)}, \nabla^3 Z_u)$. Using equation (A.24) of [Gru95] gives $\gamma_0 : \nabla^3 Z_u \longrightarrow Y_0$ leading to $\gamma_0 \circ \nabla H \in C^1(\overline{B_r(0)}, Y_0)$.
The last thing to consider is the second boundary condition including the mean curvature H_Γ. Here we can write

$$H_\Gamma(u) = b(u, \nabla_{\Gamma^*} u, \nabla^2_{\Gamma^*} u)$$

with $b \in C^4(U)$ implying the Lipschitz continuity of $\partial_i b$ for all $i \in \{1, 2, 3\}$. We estimate

$$\begin{aligned}
\|D_u H(u) - D_u H(\widetilde{u})\|_{\mathcal{L}(Z_u, \nabla^2 Z_u)} &\leq \left\| \partial_1 b(\mathbb{U}) \operatorname{Id} - \partial_1 b(\widetilde{\mathbb{U}}) \operatorname{Id} \right\|_{\mathcal{L}(Z_u, \nabla^2 Z_u)} \\
&+ \left\| \partial_2 b(\mathbb{U})(\nabla_{\Gamma^*}) - \partial_2 b(\widetilde{\mathbb{U}})(\nabla_{\Gamma^*}) \right\|_{\mathcal{L}(Z_u, \nabla^2 Z_u)} \\
&+ \left\| \partial_3 b(\mathbb{U})(\nabla_{\Gamma^*})(\nabla_{\Gamma^*}) - \partial_3 b(\widetilde{\mathbb{U}})(\nabla_{\Gamma^*})(\nabla_{\Gamma^*}) \right\|_{\mathcal{L}(Z_u, \nabla^2 Z_u)} \\
&\leq \sum_{i=0}^{2} \left\| \partial_i b(\mathbb{U}) - \partial_i b(\widetilde{\mathbb{U}}) \right\|_{\nabla^2 Z_u} \left\| \nabla^i_{\Gamma^*} \right\|_{\mathcal{L}(Z_u, \nabla^2 Z_u)} \\
&\leq 3c \sum_{i=0}^{2} \underbrace{\left\| \nabla^i_{\Gamma^*} u - \nabla^i_{\Gamma^*} \widetilde{u} \right\|_{\nabla^2 Z_u}}_{\leq \widetilde{c} \|u - \widetilde{u}\|_{Z_u}} \leq \widehat{c} \, \|u - \widetilde{u}\|_{Z_u}
\end{aligned}$$

proving $H_\Gamma \in C^1(\overline{B_r(0)}, \nabla^2 Z_u)$. Taking into account equation (A.24) from [Gru95] once more γ_0 maps as follows $\gamma_0 : \nabla^2 Z_u \longrightarrow Y_1$ leading to $\gamma_0 \circ H_\Gamma \in C^1(\overline{B_r(0)}, Y_1)$.

All these continuity statements show that $N \in C^1(\mathbb{B}_r^{\mathbb{E}}(\mathbb{O});\mathbb{F})$, where we have chosen the radius r appropriately. ■

After proving this technical lemma we can again apply the contraction mapping principle in the following lemma.

Remark 5.11: Again it is important that L is an isomorphism. We do not need to consider $g_0(0)-\mathcal{B}_0(0)u_0-\mathcal{C}_0(0)\rho_0 \in \pi_1 Z_\rho$ from Theorem 5.1, since by the same argumentation as in proof of Corollary 5.2 we see $-\mathcal{B}_0(0)u_0-\mathcal{C}_0(0)\rho_0 \in \pi_1 Z_\rho$ and and by (A.25) in [Gru95] we see that for $g_0 \in Y_0$ one has $g_0(0) \in W_p^{1-\frac{5}{p}}(\partial\Gamma^*;\mathbb{R}) = \pi_1 Z_\rho$. Moreover, the condition $\mathcal{B}_2(0)u_0 + \mathcal{C}_2(0)\rho_0 = g_2(0)$ can be dropped, because $g_2 \equiv 0$ and $(u_0,\rho_0) \in \mathbb{I}$. Finally $\mathcal{B}_1(0)u_0 + \mathcal{C}_1(0)\rho_0 = g_1(0)$ reduces to $\mathcal{B}_1(0)u_0 = g_1(0)$. Due to Theorem 5.1 the operator L is an isomorphism between the spaces $\mathbb{E}$ and

$$\mathbb{F}_0 \times \mathbb{I} := \{(f, g_0, g_1, 0, u_0, \rho_0) \in \mathbb{F} \times \mathbb{I} \mid \mathcal{B}_1(0)u_0 = g_1(0)\} \times \mathbb{I}. \qquad \square$$

Lemma 5.12: *Let $T_0 > 0$ be fixed and $T \in (0, T_0)$ arbitrary.*
(i) There exists a bounded extension operator from $\mathbb{F}_T$ to $\mathbb{F}_{T_0}$, i.e. for all $f \in \mathbb{F}_T$ there is a $\widetilde{f} \in \mathbb{F}_{T_0}$ with $\widetilde{f}\big|_{[0,T]} = f$ and $\big\|\widetilde{f}\big\|_{\mathbb{F}_{T_0}} \leq c(T_0)\,\|f\|_{\mathbb{F}_T}$.
(ii) The operator norm of $L_T^{-1} : \mathbb{F}_T \times \mathbb{I} \longrightarrow \mathbb{E}_T$ is uniformly bounded in T.
(iii) There exists a bounded extension operator from $\mathbb{E}_T$ to $\mathbb{E}_{T_0}$, i.e. for all $\Phi \in \mathbb{E}_T$ there is a $\widetilde{\Phi} \in \mathbb{E}_{T_0}$ with $\widetilde{\Phi}\big|_{[0,T]} = \Phi$ and $\big\|\widetilde{\Phi}\big\|_{\mathbb{E}_{T_0}} \leq c(T_0)\,\|\Phi\|_{\mathbb{E}_T}$.
(iv) The uniform estimate $\|DN_T[\Phi]\|_{\mathcal{L}(\mathbb{E}_T,\mathbb{F}_T)} \leq c(T_0)\,\|\Phi\|_{\mathbb{E}_T} < \infty$ holds for $\Phi \in B_r^{\mathbb{E}_T}(0)$.

Proof: (i) Let $(f_1, f_2, f_3, 0) \in \mathbb{F}_T$. To define the extension we solve

$$\begin{aligned} \frac{d}{dt}\widehat{g}(t) - \partial_\sigma^4 \widehat{g}(t) &= 0 && \text{on } [0,T_0]\times\partial\Gamma^* \\ \widehat{g}(0) &= f_2(T) && \text{on } \partial\Gamma^*, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\widehat{h}(t) - \partial_\sigma^4 \widehat{h}(t) &= 0 && \text{on } [0,T_0]\times\partial\Gamma^* \\ \widehat{h}(0) &= f_3(T) && \text{on } \partial\Gamma^*, \end{aligned}$$

where the trace in $t = T$ of a function $f_2 \in Y_0$ is an element of $\pi_1 Z_\rho$ and for $f_3 \in Y_1$ it is an element of $\pi_2 Z_\rho := W_p^{2-\frac{5}{p}}(\partial\Gamma^*;\mathbb{R})$ (cf. (A.25) of [Gru95]). We obtain unique

$$\begin{aligned} \widehat{g} \in Y_0^{T_0} &:= W_p^1([0,T_0]; W_p^{-1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \cap L_p([0,T_0]; W_p^{3-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \\ \widehat{h} \in Y_1^{T_0} &:= W_p^1([0,T_0]; W_p^{-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \cap L_p([0,T_0]; W_p^{4-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})) \end{aligned}$$

with

$$\begin{aligned} \|\widehat{g}\|_{Y_0^{T_0}} &\leq c(T_0)\left(\|0\| + \|f_2(T)\|_{\pi_1 Z_\rho}\right) = c(T_0)\,\|f_2(T)\|_{\pi_1 Z_\rho} \\ \big\|\widehat{h}\big\|_{Y_1^{T_0}} &\leq c(T_0)\left(\|0\| + \|f_3(T)\|_{\pi_2 Z_\rho}\right) = \widetilde{c}(T_0)\,\|f_3(T)\|_{\pi_2 Z_\rho}. \end{aligned}$$

We define the extension $(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, 0) \in \mathbb{F}_{T_0}$ by

$$(\widetilde{f}_1, \widetilde{f}_2, 0) := \begin{cases} (f_1, f_2, f_3, 0) & \text{for } t \in [0, T] \\ (0, \widehat{g}(t - T), \widehat{h}(t - T), 0) & \text{for } t \in (T, T_0] \end{cases}$$

and have the estimate

$$\begin{aligned}
\left\| (\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, 0) \right\|_{\mathbb{F}_{T_0}} &\leq \|(f_1, f_2, f_3, 0)\|_{\mathbb{F}_T} + \left\| (0, \widehat{g}, \widehat{h}, 0) \right\|_{\mathbb{F}_{T_0}} \\
&= \|(f_1, f_2, f_3, 0)\|_{\mathbb{F}_T} + \|\widehat{g}\|_{Y_0^{T_0}} + \left\| \widehat{h} \right\|_{Y_1^{T_0}} \\
&\leq \|(f_1, f_2, f_3, 0)\|_{\mathbb{F}_T} + c(T_0) \underbrace{\|f_2(T)\|_{\pi_1 Z_\rho}}_{\leq c\|f_2\|_{Y_0^T}} + \widetilde{c}(T_0) \underbrace{\|f_3(T)\|_{\pi_2 Z_\rho}}_{\leq c\|f_3\|_{Y_1^T}} \\
&\leq \widehat{c}(T_0) \, \|(f_1, f_2, f_3, 0)\|_{\mathbb{F}_T}
\end{aligned}$$

where the uniform estimates $\|f_2(T)\|_{\pi_1 Z_\rho} \leq c \, \|f_2\|_{Y_0^T}$ and $\|f_3(T)\|_{\pi_1 Z_\rho} \leq c \, \|f_3\|_{Y_1^T}$ follow from Theorem III.4.10.2 of [Ama95] and Lemma 7.2 of [Ama05].
(ii) - (iv) Can be shown in the same way as in Lemma 3.20, where in (iv) we use $DN[\mathbb{O}] = 0$. ■

Lemma 5.13: *Let $6 < p < \infty$ and $J := [0, T]$ where $T > 0$ must be chosen sufficiently small. Then there exists some $\varepsilon > 0$ such that for each $\Phi_0 = (u_0, \rho_0) \in \mathbb{I}$ with $\|\Phi_0\|_{\mathbb{I}} < \varepsilon$ and $H_\Gamma(u_0) = 0$ there exists a unique solution $\Phi = (u, \rho) \in \mathbb{E}$ of $L\Phi = (N(\Phi), \Phi_0)$.*

Proof: We set $\mathbb{X}_r := \left\{ \Phi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O}) \mid \Phi(0) = \Phi_0 \right\}$. The equation $L\Phi = (N(\Phi), \Phi_0)$ is equivalent to the fixed point problem $K(\Phi) = \Phi$, where

$$K(\Phi) := L^{-1}(N(\Phi), \Phi_0) \qquad \forall \, \Phi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O}).$$

For the invertibility of L we have to make sure that $(N(\Phi), \Phi_0) \in \mathbb{F}_0$, which means $\mathcal{B}_1 u_0 = \mathcal{B}_1 \Phi_0^{(1)} = N^{(3)}(\Phi_0) = H_\Gamma(u_0) + \mathcal{B}_1 u_0 + \mathcal{C}_1 \rho_0 = H_\Gamma(u_0) + \mathcal{B}_1 u_0$. This is equivalent to the condition $H_\Gamma(u_0) = 0$.
Due to Lemma 5.12(iv) and Lemma 5.10 we can choose $r > 0$ independent of T such that

$$\sup_{\Psi \in \mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \leq \frac{1}{2 \sup\limits_{T \in [0, T_0]} \|L^{-1}\|}.$$

Before we can state the main estimate we have to take a look at $\|N(\mathbb{O})\|_{\mathbb{F}}$. Here we see

$$\begin{aligned}
\|N(\mathbb{O})\|_{\mathbb{F}} = &\left\| \Delta_{\Gamma^*} H_{\Gamma^*}(u) + \frac{1}{2} H_{\Gamma^*}(u) \left(H_{\Gamma^*}(u)^2 - 4K_{\Gamma^*}(u) \right) \right\|_X \\
&+ \left\| \frac{1}{2} \sin(\alpha)(\nabla_{\Gamma^*} H_{\Gamma^*}(u) \cdot n_{\partial\Gamma^*}(u)) + a + b\varkappa_{\partial D^*}(\rho) \right\|_{Y_0} + \|H_{\Gamma^*}(u)\|_{Y_1} \\
= &\, T^{\frac{1}{p}} \left\| \Delta_{\Gamma^*} H_{\Gamma^*}(u) + \frac{1}{2} H_{\Gamma^*}(u) \left(H_{\Gamma^*}(u)^2 - 4K_{\Gamma^*}(u) \right) \right\|_{L_p(\Gamma^*;\mathbb{R})} \\
&+ T^{\frac{1}{p}} \left\| \frac{1}{2} \sin(\alpha)(\nabla_{\Gamma^*} H_{\Gamma^*}(u) \cdot n_{\partial\Gamma^*}(u)) + a + b\varkappa_{\partial D^*}(\rho) \right\|_{W_p^{1-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})} \\
&+ T^{\frac{1}{p}} \, \|H_{\Gamma^*}(u)\|_{W_p^{2-\frac{1}{p}}(\partial\Gamma^*;\mathbb{R})},
\end{aligned}$$

because all the terms $\Delta_{\Gamma^*}H_{\Gamma^*}, H_{\Gamma^*}, K_{\Gamma^*}, a, \varkappa_{\partial D^*}$ and $n_{\partial\Gamma^*}$ are time-independent. Hence $\|N(\mathbb{O})\|_{\mathbb{F}} \xrightarrow[T\to 0]{} 0$ and for a sufficiently small time interval $[0,T]$ we get $\|N(\mathbb{O})\|_{\mathbb{F}} \leq \varepsilon$. We use this fact in the estimate

$$\begin{aligned}
\|K(\Phi)\|_{\mathbb{E}} &\leq \left\|L^{-1}\right\| (\|N(\Phi)\|_{\mathbb{F}} + \|\Phi_0\|_{\mathbb{I}}) \leq \left\|L^{-1}\right\| (\|N(\Phi) - N(\mathbb{O})\|_{\mathbb{F}} + \|N(\mathbb{O})\|_{\mathbb{F}} + \|\Phi_0\|_{\mathbb{I}}) \\
&\leq \left\|L^{-1}\right\| \left(\sup_{\Psi\in\mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \|\Phi\|_{\mathbb{E}} + \|N(\mathbb{O})\|_{\mathbb{F}} + \|\Phi_0\|_{\mathbb{I}} \right) \\
&\leq \frac{1}{2}\|\Phi\|_{\mathbb{E}} + \left\|L^{-1}\right\| \|N(\mathbb{O})\|_{\mathbb{F}} + \left\|L^{-1}\right\| \|\Phi_0\|_{\mathbb{I}} \\
&\leq \frac{r}{2} + 2\left\|L^{-1}\right\| \varepsilon
\end{aligned}$$

for every $\Phi \in \mathbb{X}_r$. By choosing

$$\varepsilon(r) \leq \frac{r}{4 \sup\limits_{T\in[0,T_0]} \|L^{-1}\|}$$

independent of T, we get $\|K(\Phi)\| \leq \frac{r}{2} + \frac{r}{2} = r$, i.e. $K(\mathbb{X}_r) \subseteq \mathbb{X}_r$. To see that K is contractive, we observe that for all $\Phi_1, \Phi_2 \in \mathbb{X}_r$ the following holds

$$\begin{aligned}
\|K(\Phi_1) - K(\Phi_2)\|_{\mathbb{E}} &\leq \left\|L^{-1}\right\| \|N(\Phi_1) - N(\Phi_2)\|_{\mathbb{F}} \\
&\leq \left\|L^{-1}\right\| \sup_{\Psi\in\mathbb{B}_r^{\mathbb{E}}(\mathbb{O})} \|DN[\Psi]\|_{\mathcal{L}(\mathbb{E};\mathbb{F})} \|\Phi_1 - \Phi_2\|_{\mathbb{E}} \\
&\leq \frac{1}{2}\|\Phi_1 - \Phi_2\|_{\mathbb{E}}.
\end{aligned}$$

Hence $K : \mathbb{X}_r \longrightarrow \mathbb{X}_r$ is a contraction and the assertion follows from the contraction mapping principle. ■

Transforming this statement to our original situation we end up with the following theorem.

Theorem 5.14: *Let $T > 0$ be sufficiently small and $6 < p < \infty$ then there exists an $\varepsilon > 0$ such that for each $\varrho_0 \in \pi Z_u$ with $\varrho_0|_{\partial\Gamma^*} \in \pi Z_\rho$, $\|\varrho_0\|_{\pi Z_u} + \|\varrho_0|_{\partial\Gamma^*}\|_{\pi Z_\rho} < \varepsilon$ and $H_\Gamma(\varrho_0) = 0$ on $\partial\Gamma^*$ there exists a unique solution $\varrho \in Z_u$ with $\varrho|_{\partial\Gamma^*} \in Z_\rho$ of the system*

$$\begin{aligned}
V_\Gamma(\varrho(t)) &= -\Delta_\Gamma H_\Gamma(\varrho(t)) - \frac{1}{2}H_\Gamma(\varrho(t))\left(H_\Gamma(\varrho(t))^2 - 4K_\Gamma(\varrho(t))\right) && \text{in } [0,T]\times\Gamma^* \\
v_{\partial D}(\varrho(t)) &= \frac{1}{2}\sin(\alpha)(\nabla_\Gamma H_\Gamma(\varrho(t)) \cdot n_{\partial\Gamma}(\varrho(t))) + a + b\varkappa_{\partial D}(\varrho(t)) && \text{on } [0,T]\times\partial\Gamma^* \\
0 &= H_\Gamma(\varrho(t)) && \text{on } [0,T]\times\partial\Gamma^* \\
\varrho(0) &= \varrho_0 && \text{in } \Gamma^*.
\end{aligned}$$

<u>Proof</u>: Rewriting Lemma 5.13 in terms of ϱ instead of Φ immediately gives the result. ■

This second short-time existence result finishes the sections concerning the Willmore flow and also the short-time existence considerations itself. In the last section of this thesis we will turn our attention to stability questions for the volume-preserving MCF.

6 Stability of spherical caps under the volume-preserving MCF

In this section we want to prove the stability of spherical caps on a flat surface, in the following only named SCs, under the volume-preserving MCF with line tension. Our first step in this direction will deal with the existence of stationary spherical caps - henceforth called SSCs - and the relations that have to be satisfied for an SC to be stationary.

6.1 Spherical Caps

Before we start with the existence of SSCs we need some relations of terms that describe spherical caps.

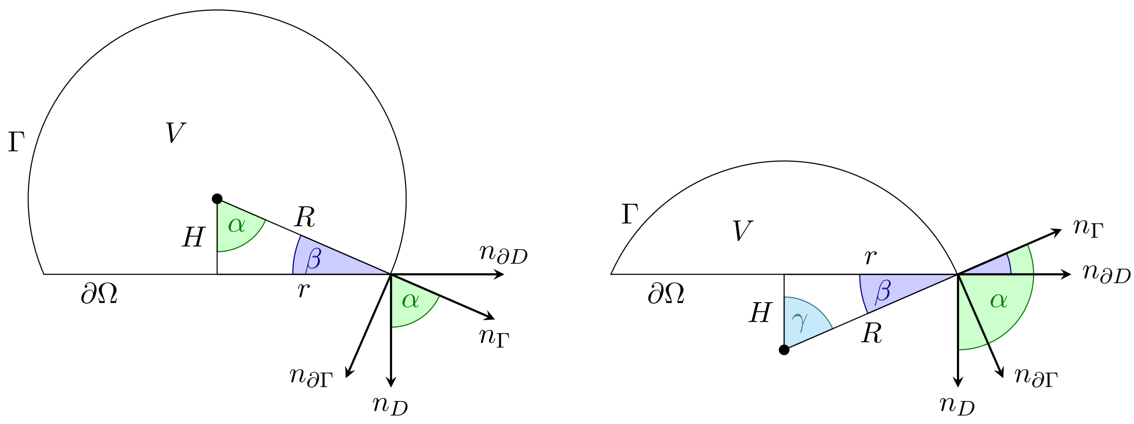

Figure 7: Spherical caps and the involved notation

Let Ω be the upper half space $\mathbb{R}^3_+ := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, then $\partial\Omega$ is the x-y-plane. The radius of the SC shall be denoted by R and the height of its center by H. Our convention will be that an SC whose center is above $\partial\Omega$ has a positive H and if the center is below the x-y-plane we declare H to be negative. The contact curve $\partial\Gamma = \partial D$ in this case is obviously an ordinary circle whose radius will be denoted by r and the angles shall be as in Figure 7. Remark that all angles - especially α - are constant in this situation.

Both cases and our sign convention for H lead to

$$H = r\cot(\alpha) \qquad \text{and} \qquad R = \frac{r}{\sin(\alpha)}. \tag{6.1}$$

By Remark 2.1(i) the triple $(\vec{\tau}, n_D, n_{\partial D})$ was supposed to be a right-handed orthonormal basis, hence we have to parametrize the contact circle clockwise looking down from the north pole. This causes the arc length derivative of $\vec{\tau}$, which is the curvature vector $\vec{\varkappa}$, to point inwards and away from $n_{\partial D}$. Therefore the geodesic curvature of the contact curve is negative, which means $\varkappa_{\partial D} = -\frac{1}{r}$.

Looking at the equations for stationary solutions we see that (2.20) is trivially satisfied for SCs. Hence an SSC only has to satisfy (2.21), which simplifies to

$$\cos(\alpha) = \frac{b}{r} - a \tag{6.2}$$

and additionally we will prescribe a given volume V_0.

To calculate the volume of an SC we use the well-known formula for the volume of solids of revolution. Our SC is a solid of revolution generated by rotating the function

$$f : [-R, H] \longrightarrow \mathbb{R} : x \longmapsto f(x) := \sqrt{R^2 - x^2}$$

around the z-axis, which also explains our sign convention for H. We can calculate

$$\begin{aligned}\mathrm{Vol}(V) &= \pi \int_{-R}^{H} f(x)^2 dx = \pi \int_{-R}^{H} R^2 - x^2 dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{H} \\ &= \pi \left(R^2 H - \frac{H^3}{3} + R^3 - \frac{R^3}{3} \right) = \frac{\pi}{3}(2R^3 + 3R^2 H - H^3),\end{aligned}$$

where V is the domain enclosed by Γ and D as in Figure 7. Making use of (6.1) we can express this in terms of α and r as follows

$$\mathrm{Vol}(V) = \frac{\pi}{3} r^3 \left(\frac{2 + 3\cos(\alpha) - \cos(\alpha)^3}{\sin(\alpha)^3} \right).$$

Now we know that an SSC fulfills (6.2) by which we can express its volume depending only on r as

$$V(r) := \mathrm{Vol}(V) = \frac{\pi}{3} r^3 \left(\frac{3\left(\frac{b}{r} - a\right) - \left(\frac{b}{r} - a\right)^3 + 2}{\sqrt{1 - \left(\frac{b}{r} - a\right)^2}^{\,3}} \right). \tag{6.3}$$

Next we answer the questions “Given any parameters $a \in \mathbb{R}$, $b > 0$ and $V_0 > 0$, is there an SSC?” and if not “For which values of a, b and V_0 are there SSCs?”.

Looking at equation (6.2) we immediately see that $-1 < \frac{b}{r} - a < 1$ has to hold. We can therefore distinguish the following cases:

1. Case ($a \leq -1$): Here we should have $a - 1 < \frac{b}{r} < a + 1 \leq 0$, which is not possible since $b > 0$ and $r > 0$.

2. Case ($-1 < a \leq 1$): Here the left inequality of $a - 1 < \frac{b}{r} < a + 1$ is always satisfied and we have to ensure $r \in \left(\frac{b}{a+1}, \infty\right)$.

3. Case ($a > 1$): Here both inequalities contribute to the restriction of r and we obtain $r \in \left(\frac{b}{a+1}, \frac{b}{a-1}\right)$.

This shows that there are definitely no SSCs if $a \leq -1$ and therefore from now on we restrict ourselves to $a > -1$ and $r \in I_r$ with

$$I_r := \begin{cases} \left(\frac{b}{a+1}, \infty\right) & \text{if } -1 < a \leq 1 \\ \left(\frac{b}{a+1}, \frac{b}{a-1}\right) & \text{if } a > 1 \end{cases}. \tag{6.4}$$

A close look on the function V from (6.3) shows that V has the following properties:

1. V is continuous on I_r.

2. $V(r) > 0$ for all $r \in I_r$, because of

$$\begin{aligned} V(r) &= \frac{\pi}{3} r^3 \left(\frac{2 + 3\cos(\alpha) - \cos(\alpha)^3}{\sin(\alpha)^3} \right) > r^3 \left(2 + 3\cos(\alpha) - \cos(\alpha)^3\right) \\ &> r^3 \left(2 + 3\cos(\pi) - \cos(\pi)^3\right) = 0. \end{aligned}$$

   Here we used that $2 + 3\cos(\alpha) - \cos(\alpha)^3$ is strictly decreasing for $\alpha \in (0, \pi)$.

3. For the limit in the left boundary point of I_r we obtain

$$\lim_{r \downarrow \frac{b}{a+1}} V(r) = \frac{\pi}{3} \left(\frac{b}{a+1}\right)^3 \lim_{\frac{b}{r} - a \uparrow 1} \left(\frac{3\left(\frac{b}{r} - a\right) - \left(\frac{b}{r} - a\right)^3 + 2}{\sqrt{1 - \left(\frac{b}{r} - a\right)^2}^3} \right) = \infty,$$

   because the numerator converges to 4 and the denominator to 0.

4. Consider again the case $-1 < a \leq 1$, where $r \longrightarrow \infty$ is allowed. Then we get

$$\begin{aligned} \lim_{r \to \infty} V(r) &= \lim_{r \to \infty} \frac{\pi}{3} r^3 \left(\frac{3\left(\frac{b}{r} - a\right) - \left(\frac{b}{r} - a\right)^3 + 2}{\sqrt{1 - \left(\frac{b}{r} - a\right)^2}^3} \right) \\ &= \frac{\pi}{3} \frac{-3a + a^3 + 2}{\sqrt{1 - a^2}^3} \lim_{r \to \infty} r^3 = \infty. \end{aligned}$$

5. Turning our attention to the case $a > 1$, where $r \longrightarrow \frac{b}{a-1}$ has to be considered, we

get with l'Hôpital's rule

$$\begin{aligned}
\lim_{r\uparrow\frac{b}{a-1}} V(r) &= \frac{\pi}{3}\left(\frac{b}{a-1}\right)^3 \lim_{r\uparrow\frac{b}{a-1}} \frac{3\left(\frac{b}{r}-a\right)-\left(\frac{b}{r}-a\right)^3+2}{\sqrt{1-\left(\frac{b}{r}-a\right)^2}^3} \\
&= \frac{\pi}{3}\left(\frac{b}{a-1}\right)^3 \lim_{r\uparrow\frac{b}{a-1}} \frac{-3\frac{b}{r^2}+3\left(\frac{b}{r}-a\right)^2\frac{b}{r^2}}{\frac{3}{2}\sqrt{1-\left(\frac{b}{r}-a\right)^2}(-2)(\frac{b}{r}-a)(-\frac{b}{r^2})} \\
&= \frac{\pi}{3}\left(\frac{b}{a-1}\right)^3 \lim_{r\uparrow\frac{b}{a-1}} \frac{\left(\frac{b}{r}-a\right)^2-1}{\sqrt{1-\left(\frac{b}{r}-a\right)^2}(\frac{b}{r}-a)} \\
&= -\frac{\pi}{3}\left(\frac{b}{a-1}\right)^3 \lim_{r\uparrow\frac{b}{a-1}} \frac{\left(\frac{b}{r}-a\right)^2-1}{\sqrt{1-\left(\frac{b}{r}-a\right)^2}} \\
&= \frac{\pi}{3}\left(\frac{b}{a-1}\right)^3 \lim_{r\uparrow\frac{b}{a-1}} \sqrt{1-\left(\frac{b}{r}-a\right)^2} = 0.
\end{aligned}$$

From properties 1 to 4 we see that in case of $-1 < a \leq 1$ there is some $c > 0$ such that $V(r) \geq c$ for all $r \in I_r$. So we can conclude that on the one hand there are no SSCs if $V_0 \in (0, c)$ and on the other hand there is at least one SSC for $V_0 \geq c$. The existence of SSCs can only be guaranteed for $a > 1$. Here properties 1 to 3 and 5 justify the intermediate value theorem which proves the existence of a solution of $V(r) = V_0$ for arbitrary $V_0 > 0$.

In the following considerations we assume

$$a > -1 \qquad \text{and} \qquad b > 0. \tag{6.5}$$

We will drop the given volume V_0 and answer the question "What are the possible angles α that an SSC can attain?". Note that $a \leq -1$ has to be excluded, since the stability analysis of an SSC would not make much sense if there is no SSC. As we already saw while answering the previous questions the feasible range for r is I_r from (6.4). SSCs satisfy $\cos(\alpha) = \frac{b}{r} - a$ and $\frac{b}{r} - a$ is obviously strictly decreasing in r. Thus

$$\cos(\alpha) = \frac{b}{r} - a \uparrow 1 \qquad \text{for} \quad r \downarrow \frac{b}{a+1}$$

and in case $a > 1$ we furthermore have

$$\cos(\alpha) = \frac{b}{r} - a \downarrow -1 \qquad \text{for} \quad r \uparrow \frac{b}{a-1},$$

which shows that all contact angles $\alpha \in (0, \pi)$ are possible.
Looking at the case $-1 < a \leq 1$ we obtain the limit

$$\cos(\alpha) = \frac{b}{r} - a \downarrow -a \qquad \text{for } r \longrightarrow \infty$$

and therefore only $\alpha \in (0, \arccos(-a))$ can appear as contact angle of an SSC. So we obtain the interval

$$I_\alpha := \begin{cases} (0, \arccos(-a)) & \text{if } -1 < a \leq 1 \\ (0, \pi) & \text{if } a > 1 \end{cases} \tag{6.6}$$

as the feasible range for α.

After we know which conditions have to hold for the contact angle α and the radius r, we can now start with the stability analysis of SCs.

6.2 The Generalized Principle of Linearized Stability

To prove the stability of SCs we assume that the reference hypersurface Γ^* from the previous chapters is now an SSC and we want to prove the stability of the zero-solution $\varrho \equiv 0$ for (2.29)-(2.30). To this end we will use the Generalized Principle of Linearized Stability (GPLS) as presented in [PSZ09] and start by setting up the abstract framework.

We begin by transforming the equations (2.29)-(2.30) into an abstract evolution equation of the form

$$\partial_t v(t) + A(v(t))v(t) = F(v(t)) \qquad t \in \mathbb{R}_+ \tag{6.7}$$

$$v(0) = v_0 \tag{6.8}$$

as given by (2.1) in [PSZ09]. As in (2.31) we can separate $\partial_t \varrho$ from V_Γ and transform (2.29) into

$$\begin{aligned} V_\Gamma(\varrho(t,q)) &= H_\Gamma(\varrho(t,q)) - \overline{H}(\varrho(t)) \\ (n_\Gamma(\varrho(t,q)) \cdot \partial_w \Psi(q, \varrho(t,q)))\partial_t \varrho(t,q) &= H_\Gamma(\varrho(t,q)) - \overline{H}(\varrho(t)) \\ \partial_t \varrho(t,q) &= \frac{H_\Gamma(\varrho(t,q)) - \overline{H}(\varrho(t))}{n_\Gamma(\varrho(t,q)) \cdot \partial_w \Psi(q, \varrho(t,q))}. \end{aligned}$$

Analogously we transform (2.30) using (2.35) into

$$\begin{aligned} v_{\partial D}(\varrho(t,q)) &= a + b\varkappa_{\partial D}(\varrho(t,q)) + \langle n_\Gamma(\varrho(t,q)), n_D(q, \varrho(t,q))\rangle \\ (n_{\partial D}(\varrho(t,q)) \cdot \partial_w \Psi(q, \varrho(t,q)))\partial_t \varrho(t,q) &= a + b\varkappa_{\partial D}(\varrho(t,q)) + \langle n_\Gamma(\varrho(t,q)), n_D(\varrho(t,q))\rangle \\ \partial_t \varrho(t,q) &= \frac{a + b\varkappa_{\partial D}(\varrho(t,q)) + \langle n_\Gamma(\varrho(t,q)), n_D(\varrho(t,q))\rangle}{n_{\partial D}(\varrho(t,q)) \cdot \partial_w \Psi(q, \varrho(t,q))}. \end{aligned}$$

Similar to the considerations in Section 3.1 we set for $4 < p < \infty$

$$\begin{aligned} X_1 &:= \mathcal{D}(A) := \left\{ (u, \rho) \in W_p^2(\Gamma^*; \mathbb{R}) \times W_p^{3-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}) \,\middle|\, u|_{\partial\Gamma^*} = \rho \right\} \\ X_0 &:= \mathcal{W}(A) := L_p(\Gamma^*; \mathbb{R}) \times W_p^{1-\frac{1}{p}}(\partial\Gamma^*; \mathbb{R}), \end{aligned}$$

where $X_1 \hookrightarrow X_0$ as demanded in [PSZ09] and we immediately see that

$$A_1(u,\rho)(u,\rho) := -\frac{H_\Gamma(u(t,q))}{n_\Gamma(u(t,q)) \cdot \partial_w \Psi(q, u(t,q))}$$
$$A_2(u,\rho)(u,\rho) := -\frac{a + b\varkappa_{\partial D}(\rho(t,q)) + \langle n_\Gamma(u(t,q)), n_D(u(t,q))\rangle}{n_{\partial D}(\rho(t,q)) \cdot \partial_w \Psi(q, \rho(t,q))}$$

are autonomous quasilinear operators. The nonlocal part $\overline{H}$ has to be included into F via

$$F_1(u,\rho) := -\frac{\overline{H}(u(t,q), \rho(t,q))}{n_\Gamma(u(t,q)) \cdot \partial_w \Psi(q, u(t,q))}$$
$$F_2(u,\rho) := 0.$$

If we define

$$v := (u,\rho) \qquad\qquad v_0 := (u_0, \rho_0)$$
$$A(v)v := \begin{pmatrix} A_1(v)v \\ A_2(v)v \end{pmatrix} \qquad\qquad F(v) := \begin{pmatrix} F_1(v) \\ F_2(v) \end{pmatrix}$$

we exactly have a problem like (6.7)-(6.8).

By interpolation results as in Theorem 4.3.1/1 and Definition 4.2.1/1 of [Tri78] we obtain

$$\Big(L_p(\Gamma^*), W_p^2(\Gamma^*)\Big)_{1-\frac{1}{p},p} = W_p^{2-\frac{2}{p}}(\Gamma^*)$$
$$\Big(W_p^{1-\frac{1}{p}}(\partial\Gamma^*), W_p^{3-\frac{1}{p}}(\partial\Gamma^*)\Big)_{1-\frac{1}{p},p} = W_p^{3-\frac{3}{p}}(\partial\Gamma^*).$$

Corollary 1.14 of [Lun09] shows that functions (u,ρ) belonging to $(X_0, X_1)_{1-\frac{1}{p},p}$ are traces at $t = 0$ of functions $v \in W_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1) \hookrightarrow BUC([0,\infty); (X_0, X_1)_{1-\frac{1}{p},p})$. This proves that the trace condition $u|_{\partial\Gamma^*} = \rho$ carries over from X_1 to the interpolation space and we have

$$X_\gamma := (X_0, X_1)_{1-\frac{1}{p},p} \subseteq \left\{ (u,\rho) \in W_p^{2-\frac{2}{p}}(\Gamma^*) \times W_p^{3-\frac{3}{p}}(\partial\Gamma^*) \,\middle|\, u|_{\partial\Gamma^*} = \rho \right\}.$$

For $V := B_\varepsilon(0) \subsetneq X_\gamma$ we can show $A \in C^1(V, \mathcal{L}(X_1, X_0))$ and $F \in C^1(V, X_0)$ by exactly the same arguments as in Lemmas 3.15 - 3.18, since the new terms $\frac{-1}{n_\Gamma(u)\cdot\partial_w\Psi(u)}$ and $\frac{-1}{n_{\partial D}(\rho)\cdot\partial_w\Psi(\rho)}$ only contain zero- and first-order derivatives of u and ρ, which are bounded for $(u,\rho) \in V$.

As already stated before we want to prove stability of SSCs, which means that we consider $v^* \equiv 0 \in \mathcal{E}$ parametrized over the SSC Γ^*, where $\mathcal{E}$ is the set of equilibria

$$\mathcal{E} := \{v \in V \cap X_1 \mid A(v)v = F(v)\} \subseteq V \cap X_1. \tag{6.9}$$

Clearly $\mathcal{E}$ is at least 2-dimensional since we can shift any stationary surface in x- and y-direction without changing the curvatures, surface area and contact angle. That we consider $v^* \equiv 0$ also explains why our notation differs slightly from that of [PSZ09]. In our special case there is no difference between what is called v and u in [PSZ09]. Hence we could use u for the same function as in Section 3.1.

The crucial assumption for applying the GPLS is "$A(v^*) = A(0)$ has the property of maximal L_p-regularity". First we remark that

$$\begin{aligned}(\delta A(v^*)v^*)(w) &= \frac{d}{d\varepsilon} A(v^* + \varepsilon w)(v^* + \varepsilon w)\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\Big(A(v^*) + \varepsilon A'(v^*)(w) + \mathcal{O}(\varepsilon^2)\Big)(v^* + \varepsilon w)\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} A(v^*)v^* + \varepsilon(A'(v^*)(w)v^* + A(v^*)w) + \mathcal{O}(\varepsilon^2)\Big|_{\varepsilon=0} \\ &= A'(v^*)(w)v^* + A(v^*)w + \mathcal{O}(\varepsilon)\big|_{\varepsilon=0} \\ &= A'(v^*)(w)v^* + A(v^*)w,\end{aligned}$$

which simplifies for $v^* \equiv 0$ to $(\delta A(0))(w) = A(0)w$. If we call

$$S(v) = S(u,\rho) = \begin{pmatrix} n_\Gamma(u)\cdot\partial_w\Psi(u) & 0 \\ 0 & n_{\partial D}(\rho)\cdot\partial_w\Psi(\rho)\end{pmatrix}$$

we have using (2.24)

$$S(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin(\alpha)}\end{pmatrix}.$$

Utilizing the fact that v^* parametrizes an SSC, i.e. $(A-F)(0) = 0$, we see

$$\begin{aligned}(\delta\mathbb{A})(0) &= \delta(S(A-F))(0) = (\delta S)(0)(A-F)(0) + S(0)(\delta(A-F))(0) \\ &= S(0)(\delta(A-F))(0) = S(0)(A(0) - F'(0)) = S(0)A_0,\end{aligned}$$

where $-\mathbb{A} := -S(A-F)$ is the operator of the right-hand side in (2.29)-(2.30) and $A_0 := A(0) - F'(0)$ is the operator from (2.5) of [PSZ09] adopted to our case $v^* \equiv 0$. Hence we see that we already calculated the linearization operator $A_0 = S^{-1}(0)(\delta\mathbb{A})(0)$ as the right-hand side of (2.45)-(2.46). In Section 3.1 we have shown in Theorem 3.6 that $A(0)$, which is A_0 without the non-local part $F'(0)$, has maximal L_p-regularity. This enables us to use Theorem 2.1 of [PSZ09] which in our situation reads as follows.

Theorem 6.1 (GPLS): *Let $4 < p < \infty$ and suppose that $v^* \equiv 0$ is normally stable, i.e.*
(a) near v^* the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1 of dimension $m \in \mathbb{N}$,
(b) the tangent space of $\mathcal{E}$ at v^* is given by $\mathcal{N}(A_0)$,
(c) 0 is a semi-simple eigenvalue of A_0, i.e. $\mathcal{N}(A_0) \oplus \mathcal{R}(A_0) = X_0$,
(d) $\sigma(A_0) \setminus \{0\} \subseteq \mathbb{C}_+$.
Then $v^* \equiv 0$ is stable in X_γ and there exists $\delta > 0$ such that the unique solution $v(t)$ of (6.7)-(6.8) with initial value $v_0 \in X_\gamma$ satisfying $\|v_0\|_{X_\gamma} < \delta$ exists on $\mathbb{R}^+$ and converges at an exponential rate in X_γ to some $v_\infty \in \mathcal{E}$.

6.3 Application

In the process of using the GPLS, it will be necessary to make use of a better suited parametrization of the SSC Γ^*. We will assume w.l.o.g. that the center of the SSC Γ^* lies on the z-axis and has height $H^* \in (-R^*, R^*)$ over or under the x-y-plane. The perfect fit for SCs are spherical coordinates shifted in z-direction by H^*, which will be introduced now.

Let a and b be given as in (6.5). Then we know by the previous considerations that for arbitrary $\alpha^* \in I_\alpha$ there is some $r^* \in I_r$ such that

$$\cos(\alpha^*) = \frac{b}{r^*} - a \qquad \text{and} \qquad \sin(\alpha^*) = \sqrt{1 - \left(\frac{b}{r^*} - a\right)^2}$$

as well as $R^* \in (0, \infty)$ and $H^* \in (-R^*, R^*)$ to satisfy

$$R^* := \frac{r^*}{\sin(\alpha^*)} \qquad \text{and} \qquad H^* := R^* \cos(\alpha^*).$$

Then the parametrization of Γ^* reads as

$$P(\varphi, \vartheta) := \begin{pmatrix} R^* \sin(\varphi)\sin(\vartheta) \\ R^* \cos(\varphi)\sin(\vartheta) \\ R^* \cos(\vartheta) + H^* \end{pmatrix} \qquad \text{with } \varphi \in [0, 2\pi] \text{ and } \vartheta \in [0, \pi - \alpha^*]. \tag{6.10}$$

The first and second derivatives of P are given by

$$P_\varphi(\varphi, \vartheta) = \begin{pmatrix} R^* \cos(\varphi)\sin(\vartheta) \\ -R^* \sin(\varphi)\sin(\vartheta) \\ 0 \end{pmatrix} \qquad P_\vartheta(\varphi, \vartheta) = \begin{pmatrix} R^* \sin(\varphi)\cos(\vartheta) \\ R^* \cos(\varphi)\cos(\vartheta) \\ -R^* \sin(\vartheta) \end{pmatrix}$$

$$P_{\varphi\varphi}(\varphi, \vartheta) = \begin{pmatrix} -R^* \sin(\varphi)\sin(\vartheta) \\ -R^* \cos(\varphi)\sin(\vartheta) \\ 0 \end{pmatrix} \qquad P_{\vartheta\vartheta}(\varphi, \vartheta) = \begin{pmatrix} -R^* \sin(\varphi)\sin(\vartheta) \\ -R^* \cos(\varphi)\sin(\vartheta) \\ -R^* \cos(\vartheta) \end{pmatrix}$$

$$P_{\varphi\vartheta}(\varphi, \vartheta) = P_{\vartheta\varphi}(\varphi, \vartheta) = \begin{pmatrix} R^* \cos(\varphi)\cos(\vartheta) \\ -R^* \sin(\varphi)\cos(\vartheta) \\ 0 \end{pmatrix}.$$

We can use this to calculate the special form of the following quantities in the case of Γ^*

being an SSC as follows

$$
\begin{aligned}
n_{\Gamma^*} &= \frac{P_\varphi \times P_\vartheta}{\|P_\varphi \times P_\vartheta\|} = \frac{1}{R^{*2}\sin(\vartheta)} \begin{pmatrix} R^{*2}\sin(\varphi)\sin(\vartheta)^2 \\ R^{*2}\cos(\varphi)\sin(\vartheta)^2 \\ R^{*2}\sin(\vartheta)\cos(\vartheta) \end{pmatrix} = \begin{pmatrix} \sin(\varphi)\sin(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix}, \\
G &= \begin{pmatrix} \langle P_\varphi, P_\varphi\rangle & \langle P_\vartheta, P_\varphi\rangle \\ \langle P_\varphi, P_\vartheta\rangle & \langle P_\vartheta, P_\vartheta\rangle \end{pmatrix} = \begin{pmatrix} R^{*2}\sin(\vartheta)^2 & 0 \\ 0 & R^{*2} \end{pmatrix}, \\
g &= \det(G) = R^{*4}\sin(\vartheta)^2, \qquad \sqrt{g} = R^{*2}\sin(\vartheta), \\
G^{-1} &= \begin{pmatrix} g^{11} & g^{21} \\ g^{12} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{R^{*2}\sin(\vartheta)^2} & 0 \\ 0 & \frac{1}{R^{*2}} \end{pmatrix}, \\
II_{\Gamma^*} &= \begin{pmatrix} \langle P_{\varphi\varphi}, n_{\Gamma^*}\rangle & \langle P_{\vartheta\varphi}, n_{\Gamma^*}\rangle \\ \langle P_{\varphi\vartheta}, n_{\Gamma^*}\rangle & \langle P_{\vartheta\vartheta}, n_{\Gamma^*}\rangle \end{pmatrix} = \begin{pmatrix} -R^*\sin(\vartheta)^2 & 0 \\ 0 & -R^* \end{pmatrix}, \\
W &= G^{-1} II_{\Gamma^*} = \begin{pmatrix} -\frac{1}{R^*} & 0 \\ 0 & -\frac{1}{R^*} \end{pmatrix}, \\
\varkappa_1^* &= \varkappa_2^* = -\frac{1}{R^*}, \qquad |\sigma^*|^2 = \frac{2}{R^{*2}}, \qquad H_{\Gamma^*} = -\frac{2}{R^*}, \qquad K_{\Gamma^*} = \frac{1}{R^{*2}}, \\
n_{\partial\Gamma^*} &= \left.\frac{P_\vartheta}{\|P_\vartheta\|}\right|_{\vartheta=\pi-\alpha^*} = \left.\begin{pmatrix} \sin(\varphi)\cos(\vartheta) \\ \cos(\varphi)\cos(\vartheta) \\ -\sin(\vartheta) \end{pmatrix}\right|_{\vartheta=\pi-\alpha^*} = \begin{pmatrix} -\sin(\varphi)\cos(\alpha^*) \\ -\cos(\varphi)\cos(\alpha^*) \\ -\sin(\alpha^*) \end{pmatrix}, \\
\vec{\tau}^* &= \left.\frac{P_\varphi}{\|P_\varphi\|}\right|_{\vartheta=\pi-\alpha^*} = \begin{pmatrix} \cos(\varphi) \\ -\sin(\varphi) \\ 0 \end{pmatrix}, \\
n_{D^*} &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \qquad n_{\partial D^*} = \frac{\vec{\tau}^* \times n_{D^*}}{\|\vec{\tau}^* \times n_{D^*}\|} = \begin{pmatrix} \sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}, \\
\vec{\varkappa}^* &= \left.\frac{\vec{\tau}^*_\varphi}{\|P_\varphi\|}\right|_{\vartheta=\pi-\alpha^*} = \left.\frac{1}{R^*\sin(\vartheta)} \begin{pmatrix} -\sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix}\right|_{\vartheta=\pi-\alpha^*} = \frac{1}{R^*\sin(\alpha^*)} \begin{pmatrix} -\sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix}, \\
\varkappa_{\partial D^*} &= \langle \vec{\varkappa}^*, n_{\partial D^*}\rangle = -\frac{1}{R^*\sin(\alpha^*)}, \\
\nabla_{\Gamma^*} &= g^{11}P_\varphi \partial_\varphi + g^{22}P_\vartheta\partial_\vartheta = \frac{1}{R^*\sin(\vartheta)} \begin{pmatrix} \cos(\varphi) \\ -\sin(\varphi) \\ 0 \end{pmatrix} \partial_\varphi + \frac{1}{R^*} \begin{pmatrix} \sin(\varphi)\cos(\vartheta) \\ \cos(\varphi)\cos(\vartheta) \\ -\sin(\vartheta) \end{pmatrix} \partial_\vartheta, \\
\Delta_{\Gamma^*} &= \frac{1}{\sqrt{g}}\left(\sqrt{g}g^{11}\partial_\varphi\right)_\varphi + \frac{1}{\sqrt{g}}\left(\sqrt{g}g^{22}\partial_\vartheta\right)_\vartheta = \frac{1}{R^{*2}\sin(\vartheta)}\left(\frac{1}{\sin(\vartheta)}\partial_{\varphi\varphi} + (\sin(\vartheta)\partial_\vartheta)_\vartheta\right) \\
&= \frac{1}{R^{*2}\sin(\vartheta)^2}\partial^2_{\varphi\varphi} + \frac{1}{R^{*2}}\partial^2_{\vartheta\vartheta} + \frac{1}{R^{*2}}\cot(\vartheta)\partial_\vartheta.
\end{aligned}
$$

Before we can check the assumptions of the GPLS it will be necessary to determine the nullspace of the operator A_0. The first step is to fit the equations (2.45)-(2.46) to the situation of Γ^* being an SSC with the above parametrization. Here we see that the first component of $-A_0\varrho$ simplifies to

$$\begin{aligned}
-(A_0\varrho)^{(1)} &= \Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho + \left(\underbrace{\nabla_{\Gamma^*} H_{\Gamma^*}}_{=0 \text{ as } H_{\Gamma^*} \text{ is constant}} \cdot P\left(\partial_w \Psi(0)\right) \right) \varrho \\
&\quad - \fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2 - H_{\Gamma^*}^2 + \underbrace{\overline{H}(\mathbb{O})}_{=H_{\Gamma^*}} H_{\Gamma^*}) \varrho \, d\mathcal{H}^2 \\
&\quad + \frac{1}{\int\limits_{\Gamma^*} 1 \, d\mathcal{H}^2} \int_{\partial\Gamma^*} \left(H_{\Gamma^*} - \underbrace{\overline{H}(\mathbb{O})}_{=H_{\Gamma^*}} \right) \cot(\alpha) \varrho \, d\mathcal{H}^1 \\
&= \Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho - \fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2) \varrho \, d\mathcal{H}^2. \qquad (6.11)
\end{aligned}$$

Searching for solutions of $0 = -(A_0\varrho)^{(1)}$ we immediately see that $\Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho = const.$ has to hold, since $\fint\limits_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2)\varrho \, d\mathcal{H}^2$ is constant. And vice versa if $\Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho$ is constant we get

$$\begin{aligned}
-(A_0\varrho)^{(1)} &= \Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho - \fint_{\Gamma^*} (\Delta_{\Gamma^*} + |\sigma^*|^2)\varrho \, d\mathcal{H}^2 \\
&= \Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho - (\Delta_{\Gamma^*} + |\sigma^*|^2)\varrho \fint_{\Gamma^*} 1 \, d\mathcal{H}^2 = 0.
\end{aligned}$$

Therefore it is equivalent to solve $c = \Delta_{\Gamma^*}\varrho + |\sigma^*|^2\varrho$ instead of $0 = -(A_0\varrho)^{(1)}$. Transforming the equation with respect to the parametrization from above we have to solve

$$c = \frac{1}{\sin(\vartheta)^2}\varrho_{\varphi\varphi} + \varrho_{\vartheta\vartheta} + \cot(\vartheta)\varrho_\vartheta + 2\varrho \qquad \text{in } (0, 2\pi) \times (0, \pi - \alpha^*), \qquad (6.12)$$

where the missing R^{*2} is included into the constant on the left side. For the boundary component we get

$$\begin{aligned}
-(A_0\varrho)^{(2)} &= -\sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho) - \sin(\alpha^*) \underbrace{II_{D^*}(n_{\partial D^*}, n_{\partial D^*})}_{=0 \text{ as } D \text{ is flat}} \varrho \\
&\quad + \sin(\alpha^*)\cos(\alpha^*) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*})\varrho \\
&\quad + b\sin(\alpha^*)\varrho_{\sigma\sigma} + b\sin(\alpha^*) \underbrace{\varkappa_{D^*}}_{=0} II_{D^*}(n_{\partial D^*}, n_{\partial D^*})\varrho \\
&\quad - b\sin(\alpha^*)\varkappa_{\partial D^*} \langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \rangle \varrho - b\sin(\alpha^*) \underbrace{\langle n_{\partial D^*}, (n_{D^*})_\sigma \rangle^2}_{=0 \text{ as } n_{D^*} \text{ is constant}} \varrho \\
&= -\sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho) + \sin(\alpha^*)\cos(\alpha^*) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*})\varrho \\
&\quad + b\sin(\alpha^*)\varrho_{\sigma\sigma} - b\sin(\alpha^*)\varkappa_{\partial D^*} \langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \rangle \varrho. \qquad (6.13)
\end{aligned}$$

Using the calculations above we have

$$
\begin{aligned}
n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} \varrho &= \frac{1}{R^*} \varrho_\vartheta \Big|_{\vartheta=\pi-\alpha^*} \\
II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*}) &= II_{\Gamma^*}\left(\frac{P_\vartheta}{\|P_\vartheta\|}\Big|_{\vartheta=\pi-\alpha^*}, \frac{P_\vartheta}{\|P_\vartheta\|}\Big|_{\vartheta=\pi-\alpha^*} \right) \\
&= \frac{1}{{R^*}^2} II_{\Gamma^*}(P_\vartheta, P_\vartheta) = -\frac{1}{R^*} \qquad (6.14) \\
\varrho_{\sigma\sigma} &= \frac{1}{\|P_\varphi\|} \partial_\varphi \left(\frac{\varrho_\varphi}{\|P_\varphi\|} \right) \Big|_{\vartheta=\pi-\alpha^*} = \frac{1}{{R^*}^2 \sin(\alpha^*)^2} \varrho_{\varphi\varphi} \\
\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \rangle &= \left\langle \vec{\tau}^*, \frac{(n_{\partial D^*})_\varphi}{\|P_\varphi\|} \right\rangle \Big|_{\vartheta=\pi-\alpha^*} \\
&= \frac{\langle \vec{\tau}^*, (n_{\partial D^*})_\varphi \rangle}{R^* \sin(\vartheta)} \Big|_{\vartheta=\pi-\alpha^*} = \frac{1}{R^* \sin(\alpha^*)} \qquad (6.15)
\end{aligned}
$$

and plugging this into the equation for $-(A_0\varrho)^{(2)}$ we end up with

$$
\begin{aligned}
-(A_0\varrho)^{(2)} &= -\sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} \varrho) + \sin(\alpha^*)\cos(\alpha^*) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*})\varrho \\
&\quad + b\sin(\alpha^*)\varrho_{\sigma\sigma} - b\sin(\alpha^*)\varkappa_{\partial D^*} \langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \rangle \varrho \\
&= \frac{\sin(\alpha^*)}{R^*} \left(-\sin(\alpha^*)\varrho_\vartheta - \cos(\alpha^*)\varrho + \frac{b}{R^* \sin(\alpha^*)^2} \varrho_{\varphi\varphi} + \frac{b}{R^* \sin(\alpha^*)^2} \varrho \right).
\end{aligned}
$$

We divide by $\frac{\sin(\alpha^*)}{R^*} \neq 0$ and obtain the first boundary condition for the nullspace to be

$$
0 = \frac{b}{R^* \sin(\alpha^*)^2} \varrho_{\varphi\varphi} + \frac{b}{R^* \sin(\alpha^*)^2} \varrho - \sin(\alpha^*)\varrho_\vartheta - \cos(\alpha^*)\varrho \Big|_{\vartheta=\pi-\alpha^*} \qquad \text{on } [0, 2\pi]. \qquad (6.16)
$$

Because we transformed A_0 into spherical coordinates $(\varphi, \vartheta) \in [0, 2\pi] \times [0, \pi - \alpha^*]$, we still have to impose four more boundary conditions. These represent the compatibility conditions on the "new" boundaries $\varphi = 0$, $\varphi = 2\pi$ and $\vartheta = 0$ that have not been present as we parametrized over Γ^*. The second and third boundary condition represent the periodicity in φ namely

$$
\begin{aligned}
0 &= \varrho|_{\varphi=0} - \varrho|_{\varphi=2\pi} && \text{on } [0, \pi - \alpha^*] \qquad (6.17) \\
0 &= \varrho_\varphi|_{\varphi=0} - \varrho_\varphi|_{\varphi=2\pi} && \text{on } [0, \pi - \alpha^*]. \qquad (6.18)
\end{aligned}
$$

The fourth boundary condition shall guarantee continuity in the "north pole" of the SSC. Here we demand

$$
const. = \varrho|_{\vartheta=0} \qquad \text{on } [0, 2\pi]. \qquad (6.19)
$$

Combining the equations (6.12) and (6.16)-(6.19) we have to solve the system

$$\begin{aligned}
c &= \frac{1}{\sin(\vartheta)^2}\varrho_{\varphi\varphi} + \varrho_{\vartheta\vartheta} + \cot(\vartheta)\varrho_\vartheta + 2\varrho && \text{in } (0,2\pi)\times(0,\pi-\alpha^*) && (6.20)\\
0 &= \frac{b}{R^*\sin(\alpha^*)^2}(\varrho_{\varphi\varphi}+\varrho) - \sin(\alpha^*)\varrho_\vartheta - \cos(\alpha^*)\varrho && \text{on } [0,2\pi]\times\{\pi-\alpha^*\} && (6.21)\\
0 &= \varrho|_{\varphi=0} - \varrho|_{\varphi=2\pi} && \text{on } [0,\pi-\alpha^*] && (6.22)\\
0 &= \varrho_\varphi|_{\varphi=0} - \varrho_\varphi|_{\varphi=2\pi} && \text{on } [0,\pi-\alpha^*] && (6.23)\\
const. &= \varrho|_{\vartheta=0} && \text{on } [0,2\pi] && (6.24)
\end{aligned}$$

to get all elements in the nullspace of A_0.

First we find a special solution of the inhomogeneous system by an educated guess. The function

$$\varrho^s(\varphi,\vartheta) := 1 + c_\alpha\cos(\vartheta) \tag{6.25}$$

with

$$c_\alpha := \frac{R^*\cos(\alpha^*)\sin(\alpha^*)^2 - b}{R^*\sin(\alpha^*)^2 - b\cos(\alpha^*)}$$

is what we are looking for. Obviously, this is only possible if $R^*\sin(\alpha^*)^2 \neq b\cos(\alpha^*)$. We claim that for $R^*\sin(\alpha^*)^2 = b\cos(\alpha^*)$ there exists no function that satisfies (6.20)-(6.24) with a $c \neq 0$ and will prove that fact later on in Lemma 6.7. It is an easy calculation to verify that ϱ^s satisfies the conditions (6.22)-(6.24) and we only check (6.20) and (6.21). For the interior equation we obtain

$$\begin{aligned}
\frac{1}{\sin(\vartheta)^2}\varrho^s_{\varphi\varphi} + \varrho^s_{\vartheta\vartheta} + \cot(\vartheta)\varrho^s_\vartheta + 2\varrho^s &= 0 - c_\alpha\cos(\vartheta) - c_\alpha\cot(\vartheta)\sin(\vartheta) + 2(1+c_\alpha\cos(\vartheta))\\
&= -2c_\alpha\cos(\vartheta) + 2 + 2c_\alpha\cos(\vartheta) = 2.
\end{aligned}$$

Now we will consider the equation (6.21). Differentiating ϱ we obtain

$$\begin{aligned}
\varrho^s_{\varphi\varphi}(\varphi,\pi-\alpha^*) &= 0\\
\varrho^s_\vartheta(\varphi,\pi-\alpha^*) &= -c_\alpha\sin(\vartheta)|_{\vartheta=\pi-\alpha^*} = -c_\alpha\sin(\alpha^*)\\
\varrho^s(\varphi,\pi-\alpha^*) &= 1 - c_\alpha\cos(\alpha^*),
\end{aligned}$$

which simplifies (6.21) to

$$\begin{aligned}
&\frac{b}{R^*\sin(\alpha^*)^2}(\varrho^s_{\varphi\varphi}+\varrho^s) - \sin(\alpha^*)\varrho^s_\vartheta - \cos(\alpha^*)\varrho^s\Big|_{\vartheta=\pi-\alpha^*}\\
&\quad= \frac{b}{R^*\sin(\alpha^*)^2}(0+1-c_\alpha\cos(\alpha^*)) + \sin(\alpha^*)c_\alpha\sin(\alpha^*) - \cos(\alpha^*)(1-c_\alpha\cos(\alpha^*))\\
&\quad= \frac{b}{R^*\sin(\alpha^*)^2}(1-c_\alpha\cos(\alpha^*)) + c_\alpha - \cos(\alpha^*)\\
&\quad= \frac{b - bc_\alpha\cos(\alpha^*) + c_\alpha R^*\sin(\alpha^*)^2 - R^*\cos(\alpha^*)\sin(\alpha^*)^2}{R^*\sin(\alpha^*)^2}\\
&\quad= \frac{b - R^*\cos(\alpha^*)\sin(\alpha^*)^2 + c_\alpha(R^*\sin(\alpha^*)^2 - b\cos(\alpha^*))}{R^*\sin(\alpha^*)^2} = 0.
\end{aligned}$$

A separation ansatz $\varrho(\varphi,\vartheta) = f(\varphi)g(\vartheta)$ is common practice to solve such a homogeneous system (6.20)-(6.24). But before we start with that, we want to justify this separation of variables following the ideas from Lecture 4 and 11 of [Sai07].

The operator $\Delta^B : X_1 \longrightarrow X_0$ is defined as

$$\Delta^B \varrho := \begin{pmatrix} -\Delta_{\Gamma^*}\varrho^{(1)} - |\sigma^*|^2 \varrho^{(1)} \\ \sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho^{(1)}) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*}\varrho^{(2)} - b\sin(\alpha^*)\varrho^{(2)}_{\sigma\sigma} - \frac{b}{R^{*2}\sin(\alpha^*)}\varrho^{(2)} \end{pmatrix}$$

and is symmetric with respect to the inner product defined by

$$\langle u,v\rangle_{\widetilde{L_2}} := \int_{\Gamma^*} f^{(1)} g^{(1)}\, d\mathcal{H}^2 + \int_{\partial\Gamma^*} \frac{1}{\sin(\alpha^*)^2} f^{(2)} g^{(2)}\, d\mathcal{H}^1$$

as one can see from

$$\begin{aligned} \left\langle \Delta^B u, v\right\rangle_{\widetilde{L_2}} &= \int_{\Gamma^*} \left(-\Delta_{\Gamma^*}u^{(1)} - |\sigma^*|^2 u^{(1)}\right) v^{(1)}\, d\mathcal{H}^2 + \int_{\partial\Gamma^*} (n_{\partial\Gamma^*}\cdot\nabla_{\Gamma^*}u^{(1)}) v^{(2)}\, d\mathcal{H}^1 \\ &\quad + \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*}u^{(2)} - \frac{b}{\sin(\alpha^*)}u^{(2)}_{\sigma\sigma} - \frac{b}{R^{*2}\sin(\alpha^*)^3}u^{(2)}\right) v^{(2)}\, d\mathcal{H}^1 \\ &= \int_{\Gamma^*} \nabla_{\Gamma^*}u^{(1)}\cdot\nabla_{\Gamma^*}v^{(1)} - |\sigma^*|^2 u^{(1)}v^{(1)}\, d\mathcal{H}^2 \\ &\quad + \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*} - \frac{b}{R^{*2}\sin(\alpha^*)^3}\right) u^{(2)}v^{(2)} - \frac{b}{\sin(\alpha^*)} u^{(2)}_{\sigma\sigma} v^{(2)}\, d\mathcal{H}^1 \\ &= \int_{\Gamma^*} \nabla_{\Gamma^*}u^{(1)}\cdot\nabla_{\Gamma^*}v^{(1)} - |\sigma^*|^2 u^{(1)}v^{(1)}\, d\mathcal{H}^2 \\ &\quad + \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*} - \frac{b}{R^{*2}\sin(\alpha^*)^3}\right) u^{(2)}v^{(2)} + \frac{b}{\sin(\alpha^*)} u^{(2)}_{\sigma} v^{(2)}_{\sigma}\, d\mathcal{H}^1, \end{aligned}$$

where we have used $\partial(\partial\Gamma^*) = \varnothing$ in the last step. Therefore all eigenvalues are real due to

$$\lambda \langle u,u\rangle_{\widetilde{L_2}} = \langle \lambda u,u\rangle_{\widetilde{L_2}} = \left\langle \Delta^B u,u\right\rangle_{\widetilde{L_2}} = \left\langle u,\Delta^B u\right\rangle_{\widetilde{L_2}} = \langle u,\lambda u\rangle_{\widetilde{L_2}} = \overline{\lambda}\langle u,u\rangle_{\widetilde{L_2}} \tag{6.26}$$

and the eigenfunctions corresponding to different eigenvalues are orthogonal, because of

$$\begin{aligned} (\lambda_1 - \lambda_2)\langle u,v\rangle_{\widetilde{L_2}} = \lambda_1\langle u,v\rangle_{\widetilde{L_2}} - \lambda_2\langle u,v\rangle_{\widetilde{L_2}} &= \langle \lambda_1 u,v\rangle_{\widetilde{L_2}} - \langle u,\lambda_2 v\rangle_{\widetilde{L_2}} \\ &= \left\langle \Delta^B u,v\right\rangle_{\widetilde{L_2}} - \left\langle u,\Delta^B v\right\rangle_{\widetilde{L_2}} = 0. \end{aligned}$$

Hence we have shown: All eigenvalues are real and all eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the $\widetilde{L_2}$-inner product.

Remark 6.2: This $\widetilde{L_2}$-inner product will also play an important role later on, while proving the solvability of (6.59)-(6.60). □

In (φ,ϑ)-coordinates Δ^B is given as

$$\Delta^B\varrho = \begin{pmatrix} -\frac{1}{R^{*2}\sin(\vartheta)^2}\varrho^{(1)}_{\varphi\varphi} - \frac{1}{R^{*2}}\varrho^{(1)}_{\vartheta\vartheta} - \frac{1}{R^{*2}}\cot(\vartheta)\varrho^{(1)}_{\vartheta} - \frac{2}{R^{*2}}\varrho^{(1)} \\ \frac{\sin(\alpha^*)^2}{R^*}\varrho^{(1)}_{\vartheta} + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*}\varrho^{(2)} - \frac{b}{R^{*2}\sin(\alpha^*)}(\varrho^{(2)}_{\varphi\varphi} + \varrho^{(2)}) \end{pmatrix},$$

where have to we impose the boundary conditions $\varrho|_{\varphi=0} = \varrho|_{\varphi=2\pi}$ and $\varrho_\varphi|_{\varphi=0} = \varrho_\varphi|_{\varphi=2\pi}$. We will decompose this operator into a part corresponding to differentiation with respect to φ and another part corresponding to differentiation with respect to ϑ. For a function $f : [0, 2\pi] \longrightarrow \mathbb{R}^2 : \varphi \longrightarrow (f^{(1)}(\varphi), f^{(2)}(\varphi))$ the φ-part shall be given as

$$\Delta_\varphi f := \begin{pmatrix} -f^{(1)}_{\varphi\varphi} \\ -f^{(2)}_{\varphi\varphi} \end{pmatrix}$$

with its boundary conditions $f(0) = f(2\pi)$ and $f_\varphi(0) = f_\varphi(2\pi)$. It is easy to see that the eigenvalues of this operator are k^2 for $k \in \mathbb{N}$. We use these eigenvalues of Δ_φ to define the ϑ-part of Δ^B as

$$\Delta^k_\vartheta g := \begin{pmatrix} -\frac{1}{R^{*2}} g^{(1)}_{\vartheta\vartheta} - \frac{1}{R^{*2}} \cot(\vartheta) g^{(1)}_\vartheta - \frac{1}{R^{*2}} \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right) g^{(1)} \\ \frac{\sin(\alpha^*)^2}{R^*} g^{(1)}_\vartheta(\pi - \alpha^*) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*} g^{(2)} - \frac{b(1-k^2)}{R^{*2}\sin(\alpha^*)} g^{(2)} \end{pmatrix}, \tag{6.27}$$

where $g^{(2)}$ is in $\mathbb{R}$ and $g^{(1)}$ is a function $g^{(1)} : [0, \pi - \alpha^*] \longrightarrow \mathbb{R}$ with $g^{(1)}(\pi - \alpha^*) = g^{(2)}$. Assume that we have an eigenpair (k^2, f_k) of Δ_φ and for this $k \in \mathbb{N}$ an eigenpair (μ_k, g_k) of Δ^k_ϑ. Then $(\mu_k, f_k g_k)$ is an eigenpair of Δ^B, since

$$\begin{aligned}
\Delta^B(f_k g_k) &= \begin{pmatrix} -\frac{1}{R^{*2}\sin(\vartheta)^2}(f^{(1)}_k)_{\varphi\varphi} g^{(1)}_k - \frac{1}{R^{*2}} f^{(1)}_k (g^{(1)}_k)_{\vartheta\vartheta} \\ -\frac{\cot(\vartheta)}{R^{*2}} f^{(1)}_k (g^{(1)}_k)_\vartheta - \frac{2}{R^{*2}} f^{(1)}_k g^{(1)}_k \\ \frac{\sin(\alpha^*)^2}{R^*} f^{(2)}_k (g^{(1)}_k)_\vartheta (\pi - \alpha^*) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*} f^{(2)}_k g^{(2)}_k \\ -\frac{b}{R^{*2}\sin(\alpha^*)}((f^{(2)}_k)_{\varphi\varphi} g^{(2)}_k + f^{(2)}_k g^{(2)}_k) \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{k^2}{R^{*2}\sin(\vartheta)^2} g^{(1)}_k - \frac{1}{R^{*2}}(g^{(1)}_k)_{\vartheta\vartheta} - \frac{1}{R^{*2}}\cot(\vartheta)(g^{(1)}_k)_\vartheta - \frac{2}{R^{*2}} g^{(1)}_k\right) f^{(1)}_k \\ \left(\frac{\sin(\alpha^*)^2}{R^*}(g^{(1)}_k)_\vartheta(\pi - \alpha^*) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*} g^{(2)}_k - \frac{b(1-k^2)}{R^{*2}\sin(\alpha^*)} g^{(2)}_k\right) f^{(2)}_k \end{pmatrix} \\
&= \begin{pmatrix} \mu_k f^{(1)}_k g^{(1)}_k \\ \mu_k f^{(2)}_k g^{(2)}_k \end{pmatrix} = \mu_k f_k g_k.
\end{aligned} \tag{6.28}$$

The next step in our separation ansatz justification is to show that there is an orthogonal basis of eigenfunctions of Δ^k_ϑ in a certain space. We define a weighted L_2- and W^1_2-space via

$$\begin{aligned}
\langle u, v\rangle_{\widehat{L_2}} &:= R^{*2}\int_0^{\pi-\alpha^*} u^{(1)} v^{(1)} \sin(\vartheta) d\vartheta + \frac{R^*}{\sin(\alpha^*)} u^{(2)} v^{(2)} \\
\widehat{L_2} &:= \left\{ f : [0, \pi - \alpha^*] \longrightarrow \mathbb{R}^2 \,\middle|\, \|f\|_{\widehat{L_2}} := \sqrt{\langle f, f\rangle_{\widehat{L_2}}} < \infty \right\} \\
\langle u, v\rangle_{\widehat{W^1_2}} &:= \int_0^{\pi-\alpha^*} u^{(1)}_\vartheta v^{(1)}_\vartheta \sin(\vartheta) d\vartheta + \int_0^{\pi-\alpha^*} \frac{k^2}{\sin(\vartheta)} u^{(1)} v^{(1)} d\vartheta + \langle u, v\rangle_{\widehat{L_2}} \\
\widehat{W^1_2} &:= \left\{ f \in \widehat{L_2} \,\middle|\, \partial_\vartheta f \in \widehat{L_2}, \|f\|_{\widehat{W^1_2}} := \sqrt{\langle f, f\rangle_{\widehat{W^1_2}}} < \infty, f^{(1)}(\pi - \alpha^*) = f^{(2)} \right\}
\end{aligned}$$

and a bilinear form $B : \widehat{W_2^1} \times \widehat{W_2^1} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} B(u,v) :=& \int_0^{\pi-\alpha^*} u_\vartheta^{(1)} v_\vartheta^{(1)} \sin(\vartheta) - \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right) u^{(1)} v^{(1)} \sin(\vartheta) d\vartheta \\ &+ \left(\cos(\alpha^*) - \frac{b(1-k^2)}{R^* \sin(\alpha^*)^2}\right) u^{(2)} v^{(2)}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \left\langle \Delta_\vartheta^k g, h \right\rangle_{\widehat{L_2}} =& \int_0^{\pi-\alpha^*} -g_{\vartheta\vartheta}^{(1)} h^{(1)} \sin(\vartheta) - \cos(\vartheta) g_\vartheta^{(1)} h^{(1)} - \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right) g^{(1)} h^{(1)} \sin(\vartheta) d\vartheta \\ &+ \sin(\alpha^*) g_\vartheta^{(1)}(\pi-\alpha^*) h^{(2)} + \left(\cos(\alpha^*) - \frac{b(1-k^2)}{R^* \sin(\alpha^*)^2}\right) g^{(2)} h^{(2)} \\ =& \int_0^{\pi-\alpha^*} -\left(\sin(\vartheta) g_\vartheta^{(1)}\right)_\vartheta h^{(1)} - \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right) g^{(1)} h^{(1)} \sin(\vartheta) d\vartheta \\ &+ \sin(\alpha^*) g_\vartheta^{(1)}(\pi-\alpha^*) h^{(2)} + \left(\cos(\alpha^*) - \frac{b(1-k^2)}{R^* \sin(\alpha^*)^2}\right) g^{(2)} h^{(2)} \\ =& -\left[\sin(\vartheta) g_\vartheta^{(1)} h^{(1)}\right]_0^{\pi-\alpha^*} + \int_0^{\pi-\alpha^*} \sin(\vartheta) g_\vartheta^{(1)} h_\vartheta^{(1)} d\vartheta \\ &- \int_0^{\pi-\alpha^*} \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right) g^{(1)} h^{(1)} \sin(\vartheta) d\vartheta + \sin(\alpha^*) g_\vartheta^{(1)}(\pi-\alpha^*) h^{(2)} \\ &+ \left(\cos(\alpha^*) - \frac{b(1-k^2)}{R^* \sin(\alpha^*)^2}\right) g^{(2)} h^{(2)} \\ =& \int_0^{\pi-\alpha^*} \sin(\vartheta) g_\vartheta^{(1)} h_\vartheta^{(1)} - \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right) g^{(1)} h^{(1)} \sin(\vartheta) d\vartheta \\ &+ \left(\cos(\alpha^*) - \frac{b(1-k^2)}{R^* \sin(\alpha^*)^2}\right) g^{(2)} h^{(2)} \\ =& B(g,h) \end{aligned} \tag{6.29}$$

for all $g, h \in \widehat{W_2^1}$. This bilinear form is bounded with respect to the norm defined on $\widehat{W_2^1}$. Moreover, the modified bilinear form

$$B_c : \widehat{W_2^1} \times \widehat{W_2^1} \longrightarrow \mathbb{R} : (u,v) \longmapsto B(u,v) + c \langle u, v \rangle_{\widehat{L_2}}$$

is also bounded and in addition positive definite for

$$c > \max\left\{ \frac{2}{R^{*2}}, \frac{b(1-k^2)}{R^{*2} \sin(\alpha^*)} - \frac{\cos(\alpha^*) \sin(\alpha^*)}{R^*} \right\} > 0.$$

Therefore B_c satisfies all assumptions for the Lemma of Lax-Milgram and there exists a bounded operator

$$\left(\Delta_\vartheta^k + c\,\mathrm{Id}\right)^{-1} : \widehat{L_2} \longrightarrow \widehat{W_2^1}$$

corresponding to a weak solution operator for $(\Delta_\vartheta^k + c\,\mathrm{Id})g = f$ with $f \in \widehat{L_2}$. Will show in Lemma 6.4 that regardless of our modified definition of the $\widehat{L_2}$- and $\widehat{W_2^1}$-space the compact embedding $\widehat{W_2^1} \hookrightarrow \widehat{L_2}$ holds true as usual. Therefore

$$\left(\Delta_\vartheta^k + c\,\mathrm{Id}\right)^{-1} : \widehat{L_2} \longrightarrow \widehat{W_2^1} \hookrightarrow \widehat{L_2}$$

is a compact operator. By the spectral theorem for compact operators we know that $\left(\Delta_\vartheta^k + c\,\mathrm{Id}\right)^{-1}$ has countably many eigenfunctions $(g_m^k)_{m\in\mathbb{N}}$, that form an orthonormal basis of $\widehat{L_2}$. The eigenfunctions are invariant under inversion and shifting, hence also the eigenfunctions of Δ_ϑ^k are an orthonormal basis of $\widehat{L_2}$ as well.

Remark 6.3: The spectral theorem for compact operators also states that the eigenvalues of $\left(\Delta_\vartheta^k + c\,\mathrm{Id}\right)^{-1}$ form a sequence converging to zero. In particular, the eigenvalues have no accumulation point other than 0. Therefore the eigenvalues of the uninverted operator have no accumulation point. The shift of the eigenvalues by c does not change this fact. Thus all eigenvalues of Δ_ϑ^k and with them also the eigenvalues of Δ^B are isolated. $\square$

It is well-known that also the eigenfunctions $(f_k)_{k\in\mathbb{N}}$ of Δ_φ, given by $\sin(k\varphi)$ and $\cos(k\varphi)$, form an orthogonal basis in $L_2([0,2\pi])$ (cf. for example Chapter V.4 in [Wer07]).
Now assume that there is an eigenfunction u of Δ^B corresponding to the eigenvalue λ that is not in the span of all functions that are in product form. Since we know that all eigenfunctions corresponding to different eigenvalues of Δ^B are orthogonal with respect to the $\widetilde{L_2}$-inner product and $f_k g_m^k$ is an eigenfunction of Δ^B we see that for arbitrary $k, m \in \mathbb{N}$ we would obtain

$$\begin{aligned}
0 = \left\langle u, f_k g_m^k \right\rangle_{\widetilde{L_2}} &= \int_0^{\pi-\alpha^*} \int_0^{2\pi} u^{(1)}(\varphi,\vartheta) f_k^{(1)}(\varphi) {g_m^k}^{(1)}(\vartheta) R^{*2} \sin(\vartheta) d\vartheta d\varphi \\
&+ \frac{1}{\sin(\alpha^*)^2} \int_0^{2\pi} u^{(2)}(\varphi) f_k^{(2)}(\varphi) {g_m^k}^{(2)} R^* \sin(\alpha^*) d\varphi \\
&= R^{*2} \int_0^{\pi-\alpha^*} \left(\int_0^{2\pi} u^{(1)}(\varphi,\vartheta) f_k^{(1)}(\varphi) d\varphi \right) {g_m^k}^{(1)}(\vartheta) \sin(\vartheta) d\vartheta \\
&+ \frac{R^*}{\sin(\alpha^*)} \left(\int_0^{2\pi} u^{(1)}(\varphi, \pi-\alpha^*) f_k^{(2)}(\varphi) d\varphi \right) {g_m^k}^{(2)} \\
&= \left\langle \int_0^{2\pi} u(\varphi,\vartheta) f_k(\varphi) d\varphi, g_m^k \right\rangle_{\widehat{L_2}} .
\end{aligned}$$

For each k the eigenfunctions $(g_m^k)_{m\in\mathbb{N}}$ are complete in $\widehat{L_2}$ and so we get

$$0 = \int_0^{2\pi} u(\varphi,\vartheta) f_k(\varphi) d\varphi$$

for all $k \in \mathbb{N}$ and almost every $\vartheta \in [0, \pi - \alpha^*]$. Since $(f_k)_{k\in\mathbb{N}}$ is complete in $L_2([0,2\pi])$ equipped with the usual L_2-inner product, we end up with $u(\varphi,\vartheta) = 0$ almost everywhere. Therefore we arrived at a contradiction to our assumption that u is an eigenfunction. This proves that all eigenfunctions are in the span of functions in product form and justifies the separation ansatz. The last missing ingredient is the proof of the compactness of the embedding $\widehat{W_2^1} \hookrightarrow \widehat{L_2}$, which we will present now.

Lemma 6.4: *The embedding* $\widehat{W_2^1} \hookrightarrow \widehat{L_2}$ *is compact.*

Proof: To this end let $(u_n)_{n\in\mathbb{N}} \subseteq \widehat{W_2^1}$ be bounded. Then we obtain for $t, s \in [0, \pi - \alpha^*]$

$$\begin{aligned}|u_n(t) - u_n(s)| &= \left|\int_s^t u_n'(x)dx\right| \leq \left|\int_s^t \sqrt{\sin(x)}u_n'(x)\frac{1}{\sqrt{\sin(x)}}dx\right| \\ &\leq \int_s^t \left|\sqrt{\sin(x)}u_n'(x)\right|\left|\frac{1}{\sqrt{\sin(x)}}\right| dx \\ &\leq \underbrace{\left(\int_s^t \sin(x)|u_n'(x)|^2 dx\right)^{\frac{1}{2}}}_{\leq c \text{ since } (u_n)_{n\in\mathbb{N}} \subseteq \widehat{W_2^1} \text{ in bounded}} \left(\int_s^t \frac{1}{\sin(x)}dx\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\sin(\pi - \alpha^*) > 0$ we can still find a linear function below $\sin(x)$ to continue the estimate as follows

$$|u_n(t) - u_n(s)| \leq \widehat{c}\left(\int_s^t \frac{1}{x}dx\right)^{\frac{1}{2}} = \widehat{c}\,(\ln(t) - \ln(s))^{\frac{1}{2}}. \tag{6.30}$$

The fact that the right-hand side is independent of n immediately shows that $(u_n)_{n\in\mathbb{N}}$ is equicontinuous on every compact interval $[a, \pi - \alpha^*] \subseteq (0, \pi - \alpha^*]$. Also on each such compact interval we have equivalence of the $\widehat{W_2^1}$- and W_2^1-norms due to $0 < c \leq \sin(\vartheta) \leq C$. Therefore the usual compact embedding $\widehat{W_2^1}([a, \pi - \alpha^*]) \hookrightarrow L_2([a, \pi - \alpha^*])$ holds. Here we define $\widehat{W_2^1}([a, \pi - \alpha^*])$ and $L_2([a, \pi - \alpha^*])$ in the same manner as $\widehat{W_2^1}$ and $\widehat{L_2}$ just the domain for the first component changes to $[a, \pi - \alpha^*]$ instead of $[0, \pi - \alpha^*]$. Hence the bounded sequence $(u_n)_{n\in\mathbb{N}}$ has a subsequence converging in $L_2([a, \pi - \alpha^*])$, which for simplicity shall be called $(u_n)_{n\in\mathbb{N}}$ again. Since L_2-convergence implies the pointwise convergence of a subsequence, we obtain a pointwise limit of $(u_n)_{n\in\mathbb{N}}$ on $[a, \pi - \alpha^*]$ for each $a > 0$.

Let $(a_i)_i \in \mathbb{N}$ be a monotone decreasing sequence converging to 0. Then we have for $i = 0$ a subsequence $(u_j^0)_{j\in\mathbb{N}}$ converging pointwise on $[a_0, \pi - \alpha^*]$. For $i = 1$ we can again extract a subsequence $(u_j^1)_{j\in\mathbb{N}} \subseteq (u_j^0)_{j\in\mathbb{N}}$ that converges on $[a_1, \pi - \alpha^*]$. Continuing this process, we find a diagonal sequence $(u_i^i)_{i\in\mathbb{N}}$, which converges pointwise on $(0, \pi - \alpha^*]$ to a function that we call u. For the sake of simplicity we name this subsequence $(u_n)_{n\in\mathbb{N}}$ again.

The estimate (6.30) also shows

$$|u_n(t)|^2 \leq c\left(|u_n(\pi - \alpha^*)|^2 + |\ln(t)|\right)$$

or by considering $u_n - u$ instead of u_n, we see

$$|u_n(t) - u(t)|^2 \leq c\left(|u_n(\pi - \alpha^*) - u(\pi - \alpha^*)|^2 + |\ln(t)|\right).$$

Therefore

$$\int_0^{\pi-\alpha^*} \sin(\vartheta)|u_n(\vartheta) - u(\vartheta)|^2 d\vartheta \leq \widetilde{c}\int_0^{\pi-\alpha^*} \sin(\vartheta)|\ln(\vartheta)|d\vartheta$$

and since $\lim\limits_{\vartheta\to 0} \sin(\vartheta)|\ln(\vartheta)| = 0$ we found a dominating function, which is still integrable. By dominated convergence theorem we get the $\widehat{L_2}$-convergence of $(u_n)_{n\in\mathbb{N}}$. This finally shows that the embedding $\widehat{W_2^1} \hookrightarrow \widehat{L_2}$ is compact. ■

After knowing that all solutions of the homogeneous system (6.20)-(6.24) will be in the span of functions with product structure $\varrho(\varphi,\vartheta) = f(\varphi)g(\vartheta)$, we can perform a separation ansatz to transform (6.20) with $c = 0$ into equations for f and g. Since we are only interested in non-trivial solutions for ϱ we can assume $f \not\equiv 0$ and $g \not\equiv 0$. We get

$$\begin{aligned} 0 &= \frac{1}{\sin(\vartheta)^2}\varrho_{\varphi\varphi} + \varrho_{\vartheta\vartheta} + \cot(\vartheta)\varrho_\vartheta + 2\varrho \\ &= \frac{1}{\sin(\vartheta)^2} f''g + fg'' + \cot(\vartheta)fg' + 2fg. \end{aligned} \tag{6.31}$$

This is equivalent to

$$-\frac{f''}{f} = \sin(\vartheta)^2\frac{g''}{g} + \sin(\vartheta)\cos(\vartheta)\frac{g'}{g} + 2\sin(\vartheta)^2,$$

where the left hand side is independent of ϑ and the right hand side is independent of φ. This justifies

$$-\frac{f''}{f} = \sin(\vartheta)^2\frac{g''}{g} + \sin(\vartheta)\cos(\vartheta)\frac{g'}{g} + 2\sin(\vartheta)^2 =: \lambda \in \mathbb{R} \tag{6.32}$$

This leads to the ODE $f'' + \lambda f = 0$ for f and a second ODE for g that we will examine later.

Remark 6.5: The fact that f or g could be zero in some points does not play any role for (6.32). For a fixed $\vartheta_0 \in [0, \pi - \alpha^*]$ with $g(\vartheta_0) \neq 0$ we definitely get the ODE $f'' + \lambda f = 0$ on the set $U := \{\varphi \in [0, 2\pi] | f(\varphi) \neq 0\}$. Assuming that $\varphi_0 \in U^c$ we see $f(\varphi_0) = 0$ and going back to (6.31) we get $0 = \frac{1}{\sin(\vartheta_0)^2} f''(\varphi_0)g(\vartheta_0)$. Since we assumed $g(\vartheta_0) \neq 0$, this leads to $f''(\varphi_0) = 0$ and therefore $f'' + \lambda f = 0$ is also valid for this φ_0. Interchanging the roles of f and g leads to the same result for g. $\square$

The equations (6.22) and (6.23) translate into boundary conditions for f namely

$$\begin{aligned} &(6.22) \quad \Leftrightarrow \quad 0 = f(0)g(\vartheta) - f(2\pi)g(\vartheta) \quad \forall\, \vartheta \in [0, \pi - \alpha^*] \quad \Leftrightarrow \quad f(0) = f(2\pi) \\ &(6.23) \quad \Leftrightarrow \quad 0 = f'(0)g(\vartheta) - f'(2\pi)g(\vartheta) \quad \forall\, \vartheta \in [0, \pi - \alpha^*] \quad \Leftrightarrow \quad f'(0) = f'(2\pi). \end{aligned}$$

The solution of $f'' + \lambda f = 0$ is obviously given by

$$f(\varphi) = \begin{cases} c_1 e^{\sqrt{-\lambda}\varphi} + c_2 e^{-\sqrt{-\lambda}\varphi} & \text{if } \lambda < 0 \\ c_1 + c_2\varphi & \text{if } \lambda = 0 \\ c_1\cos(\sqrt{\lambda}\varphi) + c_2\sin(\sqrt{\lambda}\varphi) & \text{if } \lambda > 0 \end{cases}.$$

If $\lambda < 0$ the boundary conditions $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$ show $c_1 = c_2 = 0$, which would give rise to the trivial solution $f \equiv 0$ that we wanted to ignore. If $\lambda = 0$ the boundary conditions only require $c_2 = 0$, which leaves $f \equiv c_1$ as a solution. And in the last case $\lambda > 0$ the boundary conditions simplify to

$$\begin{aligned} c_1 &= c_1\cos(2\pi\sqrt{\lambda}) + c_2\sin(2\pi\sqrt{\lambda}) \\ c_2\sqrt{\lambda} &= -c_1\sqrt{\lambda}\sin(2\pi\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(2\pi\sqrt{\lambda}). \end{aligned}$$

Dividing the second equation by $\sqrt{\lambda}$ and adding these two equations we can compare the coefficients and arrive at

$$\begin{aligned} 1 &= \cos(2\pi\sqrt{\lambda}) - \sin(2\pi\sqrt{\lambda}) \\ 1 &= \cos(2\pi\sqrt{\lambda}) + \sin(2\pi\sqrt{\lambda}). \end{aligned}$$

This shows $\cos(2\pi\sqrt{\lambda}) = 1$ and $\sin(2\pi\sqrt{\lambda}) = 0$, which is only possible if $\sqrt{\lambda} = k \in \mathbb{N}_+$. In this case we see $f(\varphi) = c_1\cos(k\varphi) + c_2\sin(k\varphi)$ with $k \in \mathbb{N}_+$. Also the solution in the case $\lambda = 0$ can be written in this form if we allow for $k = 0$. So we end up with the solutions

$$f_k(\varphi) = c_1\cos(k\varphi) + c_2\sin(k\varphi) \qquad \text{with } k \in \mathbb{N}. \tag{6.33}$$

We saw that we get non-trivial solutions only if $\lambda = k^2 \in \mathbb{N}$. Hence from (6.32) we get the following ODE for g

$$0 = g'' + \cot(\vartheta)g' + \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right)g. \tag{6.34}$$

So far we have not considered the boundary equations (6.21) and (6.24). Looking first at (6.24) we see

$$const. = \varrho|_{\vartheta=0} = f(\varphi)g(0),$$

which means that either $f(\varphi)$ is constant and $\lim\limits_{\vartheta\downarrow 0} g(\vartheta)$ exists or otherwise $g(0) = 0$. Since f is only constant if $k = 0$, we obtain the condition $g(0) = 0$ for all $k \geq 1$ and "$g(0)$ exists" for $k = 0$.

Last but not least (6.21) transforms into

$$0 = \frac{b(1-k^2)}{R^*\sin(\alpha^*)^2}g(\pi-\alpha^*) - \sin(\alpha^*)g'(\pi-\alpha^*) - \cos(\alpha^*)g(\pi-\alpha^*).$$

Hence (6.21) and (6.24) now read as

$$\begin{aligned} 0 &= g(0) && \text{if } k \geq 1 && (6.35) \\ \lim_{\vartheta\downarrow 0} g(\vartheta) &\text{ exists} && \text{if } k = 0 && (6.36) \\ 0 &= \left(\frac{b(1-k^2)}{R^*\sin(\alpha^*)^2} - \cos(\alpha^*)\right)g(\pi-\alpha^*) - \sin(\alpha^*)g'(\pi-\alpha^*) && \text{if } k \geq 0. && (6.37) \end{aligned}$$

For solving the system (6.34)-(6.37) we have to distinguish the cases $k = 0$, $k = 1$ and $k \geq 2$.

1. Case ($k = 0$): Here the general solution of (6.34) is

$$g_0(\vartheta) = c_1\cos(\vartheta) + c_2\left(\frac{1}{2}\cos(\vartheta)\ln\left(\frac{\cos(\vartheta)+1}{\cos(\vartheta)-1}\right) - 1\right)$$

as one can easily check by differentiation and (6.35) does not have to be considered. Due to (6.36) we must have $c_2 = 0$ and the solution reduces to $g_0(\vartheta) = c_1\cos(\vartheta)$. The equation (6.37) is then given by

$$\begin{aligned} 0 &= \left(\frac{b}{R^* \sin(\alpha^*)^2} - \cos(\alpha^*)\right)(-c_1\cos(\alpha^*)) - \sin(\alpha^*)(-c_1\sin(\alpha^*)) \\ &= c_1\left(1 - \frac{b\cos(\alpha^*)}{R^*\sin(\alpha^*)^2}\right). \end{aligned}$$

But this means that for $R^*\sin(\alpha^*)^2 \neq b\cos(\alpha^*)$ this equation is only satisfied for $c_1 = 0$ and we do not have any contributing functions from the case $k = 0$. If $R^*\sin(\alpha^*)^2 = b\cos(\alpha^*)$ one can choose any $c_1 \in \mathbb{R}$ and obtain $g_0(\vartheta) = c_1\cos(\vartheta)$ as the solution for $k = 0$. The significance of this special case will be clarified in Remark 6.8 below.

2. Case ($k = 1$): Again it is an easy but time-consuming calculation to check that now

$$g_1(\vartheta) = c_1\sin(\vartheta) + c_2\left(-\frac{1}{2}\sin(\vartheta)\ln\left(\frac{\cos(\vartheta)+1}{\cos(\vartheta)-1}\right) - \cot(\vartheta)\right)$$

is the general solution of (6.34). Due to $\frac{1}{2}\ln\left(\frac{\cos(\vartheta)+1}{\cos(\vartheta)-1}\right)' = \frac{-1}{\sin(\vartheta)}$ we get with l'Hôpital's rule

$$\lim_{\vartheta\downarrow 0}\frac{1}{2}\sin(\vartheta)\ln\left(\frac{\cos(\vartheta)+1}{\cos(\vartheta)-1}\right) = \lim_{\vartheta\downarrow 0}\frac{\frac{1}{2}\ln\left(\frac{\cos(\vartheta)+1}{\cos(\vartheta)-1}\right)}{\frac{1}{\sin(\vartheta)}} = \lim_{\vartheta\downarrow 0}\frac{\frac{1}{\sin(\vartheta)}}{\frac{\cos(\vartheta)}{\sin(\vartheta)^2}} = \lim_{\vartheta\downarrow 0}\tan(\vartheta) = 0$$

and thereafter

$$\begin{aligned} \lim_{\vartheta\downarrow 0} g_1(\vartheta) &= \lim_{\vartheta\downarrow 0}\left(-c_1\sin(\vartheta) + c_2\left(-\frac{1}{2}\sin(\vartheta)\ln\left(\frac{\cos(\vartheta)+1}{\cos(\vartheta)-1}\right) - \cot(\vartheta)\right)\right) \\ &= 0 + 0 - \lim_{\vartheta\downarrow 0} c_2\cot(\vartheta) = -c_2\lim_{\vartheta\downarrow 0}\cot(\vartheta). \end{aligned}$$

Hence the boundary condition (6.35) requires $c_2 = 0$. Hence the solution so far is $g_1(\vartheta) = c_1\sin(\vartheta)$. The boundary condition (6.36) does not have to be considered and (6.37) is now always valid, because

$$(0 - \cos(\alpha^*))(c_1\sin(\alpha^*)) - \sin(\alpha^*)(-c_1\cos(\alpha^*)) = 0.$$

This shows that $g_1(\vartheta) = c_1\sin(\vartheta)$ is the solution for $k = 1$.

3. Case ($k \geq 2$): Here we note the close relationship between the operator Δ_ϑ^k from (6.27) and the operator given by the right-hand sides of (6.34) and (6.37). We see that a solution of (6.34) and (6.37) would correspond to the eigenvalue 0 for the operator (6.27). Therefore it is enough to show that there is no eigenvalue 0 for $k \geq 2$ of Δ_ϑ^k. We assume that we would have an eigenfunction g of Δ_ϑ^k corresponding to the eigenvalue 0. Using

(6.29) we would obtain

$$
\begin{aligned}
0 &= \langle 0, g\rangle_{\widehat{L_2}} = \left\langle \Delta^k_\vartheta g, g\right\rangle_{\widehat{L_2}} = B(g,g) \\
&= \int_0^{\pi-\alpha^*} (g^{(1)}_\vartheta)^2 \sin(\vartheta) - \left(2 - \frac{k^2}{\sin(\vartheta)^2}\right)(g^{(1)})^2 \sin(\vartheta) d\vartheta \\
&+ \left(\cos(\alpha^*) - \frac{b(1-k^2)}{R^*\sin(\alpha^*)^2}\right)(g^{(2)})^2 \\
&= \int_0^{\pi-\alpha^*} (g^{(1)}_\vartheta)^2 \sin(\vartheta) + \underbrace{\left(\frac{k^2}{\sin(\vartheta)^2} - 2\right)}_{\geq \frac{4}{1}-2=2}(g^{(1)})^2 \sin(\vartheta) d\vartheta \\
&+ \underbrace{\left(\frac{b(k^2-1)}{R^*\sin(\alpha^*)^2} + \cos(\alpha^*)\right)}_{\geq \frac{3b}{R^*\sin(\alpha^*)^2}+\cos(\alpha^*)}(g^{(2)})^2 \\
&\geq \int_0^{\pi-\alpha^*} (g^{(1)}_\vartheta)^2 \sin(\vartheta) + 2(g^{(1)})^2\sin(\vartheta)d\vartheta + \left(\frac{3b}{R^*\sin(\alpha^*)^2} + \cos(\alpha^*)\right)(g^{(2)})^2. \qquad (6.38)
\end{aligned}
$$

For $b > C_{crit} := -\frac{1}{3}R^*\sin(\alpha^*)^2\cos(\alpha^*) = -\frac{1}{3}H^*\sin(\alpha^*)^2$ this is a contradiction, because the last term is strictly positive. Therefore we do not get any additional solutions from the cases $k \geq 2$.

Remark 6.6: (i) If $\cos(\alpha^*) \geq 0$, or equivalently $H^* \geq 0$, the critical constant C_{crit} is negative or zero and hence $b > C_{crit}$ is always satisfied. Therefore we have no nullspace elements for $k \geq 2$ in this case.
(ii) What we have done in the considerations for $k \geq 2$ above is actually much more valuable than it seems at the first glance. If we modify the calculations a little and assume that g is an eigenfunction of Δ^k_ϑ corresponding to an arbitrary eigenvalue λ. Then (6.38) reads as

$$\lambda \langle g, g\rangle_{\widehat{L_2}} = \left\langle \Delta^k_\vartheta g, g\right\rangle_{\widehat{L_2}} = B(g,g) > 0.$$

Yet, this shows that all eigenvalues μ_k of Δ^k_ϑ and due to (6.28) thereby also the eigenvalues of Δ^B are all positive for $k \geq 2$. □

Now we want to close the gap in our argument that occurred from ignoring the case $R^*\sin(\alpha^*)^2 = b\cos(\alpha^*)$ so far.

Lemma 6.7: *In the case $R^*\sin(\alpha^*)^2 = b\cos(\alpha^*)$ the system (6.20)-(6.24) has no solution if $c \neq 0$.*

Proof: We note that it suffices to consider $\cos(\alpha^*) > 0$, since $R^*\sin(\alpha^*)^2 = b\cos(\alpha^*)$ can not occur if $\cos(\alpha^*) \leq 0$. Moreover, we can ignore $b > C_{crit}$ in this case, since $C_{crit} = 0$.

Then we rewrite (6.20)-(6.24) for this particular situation and get

$$
\begin{aligned}
c &= \frac{1}{\sin(\vartheta)^2}\varrho_{\varphi\varphi} + \varrho_{\vartheta\vartheta} + \cot(\vartheta)\varrho_\vartheta + 2\varrho && \text{in } (0,2\pi)\times(0,\pi-\alpha^*) && (6.39)\\
0 &= \frac{1}{\cos(\alpha^*)}(\varrho_{\varphi\varphi} + \varrho) - \sin(\alpha^*)\varrho_\vartheta - \cos(\alpha^*)\varrho && \text{on } [0,2\pi]\times\{\pi-\alpha^*\} && (6.40)\\
0 &= \varrho|_{\varphi=0} - \varrho|_{\varphi=2\pi} && \text{on } [0,\pi-\alpha^*] && (6.41)\\
0 &= \varrho_\varphi|_{\varphi=0} - \varrho_\varphi|_{\varphi=2\pi} && \text{on } [0,\pi-\alpha^*] && (6.42)\\
const. &= \varrho|_{\vartheta=0} && \text{on } [0,2\pi] && (6.43)
\end{aligned}
$$

The ideas for this proof are taken from [Nar02]. The periodicity from (6.41)-(6.42) in φ justifies an ansatz of the form

$$\varrho(\varphi,\vartheta) = \sum_{m=-\infty}^{\infty} \widehat{\varrho}_m(\vartheta)e^{im\varphi}.$$

Using this in (6.39) we obtain

$$
\begin{aligned}
\sum_{m=-\infty}^{\infty} c\delta_{m0}e^{im\varphi} = c &= \left(\frac{1}{\sin(\vartheta)^2}\partial_{\varphi\varphi} + \partial_{\vartheta\vartheta} + \cot(\vartheta)\partial_\vartheta + 2\right)\varrho\\
&= \sum_{m=-\infty}^{\infty}\left(\widehat{\varrho}_m'' + \cot(\vartheta)\widehat{\varrho}_m' + \left(2 - \frac{m^2}{\sin(\vartheta)^2}\right)\widehat{\varrho}_m\right)e^{im\varphi},
\end{aligned}
$$

where δ_{ij} denotes the Kronecker delta. Interchanging the operator with the summation as well as the convergence of the sum is justified by the smoothness of ϱ on $[0,2\pi]\times[0,\pi-\alpha^*]$. The same ansatz in (6.40) and (6.43) gives

$$0 = \sum_{m=-\infty}^{\infty}\left(\left(\frac{1-m^2}{\cos(\alpha^*)} - \cos(\alpha^*)\right)\widehat{\varrho}_m - \sin(\alpha^*)\widehat{\varrho}_m'\right)e^{im\varphi}$$

and $const. = \sum_{m=-\infty}^{\infty}\widehat{\varrho}_m(0)e^{im\varphi}$, respectively. Since the Fourier series is unique we can equate the coefficients and this leads to the following two ODEs

$$
\begin{aligned}
c &= \widehat{\varrho}_0''(\vartheta) + \cot(\vartheta)\widehat{\varrho}_0'(\vartheta) + 2\widehat{\varrho}_0(\vartheta) && (6.44)\\
0 &= \sin(\alpha^*)\widehat{\varrho}_0(\pi-\alpha^*) - \cos(\alpha^*)\widehat{\varrho}_0'(\pi-\alpha^*) && (6.45)\\
&\lim_{\vartheta\downarrow 0}\widehat{\varrho}_0(\vartheta) \text{ exists} && (6.46)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \widehat{\varrho}_m''(\vartheta) + \cot(\vartheta)\widehat{\varrho}_m'(\vartheta) + \left(2 - \frac{m^2}{\sin(\vartheta)^2}\right)\widehat{\varrho}_m(\vartheta) && (6.47)\\
0 &= \left(\frac{1-m^2}{\cos(\alpha^*)} - \cos(\alpha^*)\right)\widehat{\varrho}_m(\pi-\alpha^*) - \sin(\alpha^*)\widehat{\varrho}_m'(\pi-\alpha^*) && (6.48)\\
0 &= \widehat{\varrho}_m(0) && (6.49)
\end{aligned}
$$

for $m \neq 0$. We start by investigating the second system. Assuming that we have a solution for it, we would get

$$\frac{m^2}{\sin(\vartheta)^2}\widehat{\varrho}_m = \frac{1}{\sin(\vartheta)}\left(\sin(\vartheta)\widehat{\varrho}_m'\right)' + 2\widehat{\varrho}_m.$$

Multiplying with $\sin(\vartheta)\widehat{\varrho}_m$ and integrating over $[0, \pi - \alpha^*]$ gives

$$\begin{aligned} m^2 \int_0^{\pi-\alpha^*} \frac{1}{\sin(\vartheta)}\widehat{\varrho}_m^2 d\vartheta &= \int_0^{\pi-\alpha^*} \left(\sin(\vartheta)\widehat{\varrho}_m'\right)' \widehat{\varrho}_m + 2\widehat{\varrho}_m^2 \sin(\vartheta) d\vartheta \\ &= \left[\sin(\vartheta)\widehat{\varrho}_m'\widehat{\varrho}_m\right]_0^{\pi-\alpha^*} - \int_0^{\pi-\alpha^*} \widehat{\varrho}_m'^2 \sin(\vartheta)d\vartheta + 2\int_0^{\pi-\alpha^*} \widehat{\varrho}_m^2 \sin(\vartheta)d\vartheta \\ &\leq \sin(\alpha^*)\widehat{\varrho}_m'(\pi-\alpha^*)\widehat{\varrho}_m(\pi-\alpha^*) + 2\int_0^{\pi-\alpha^*} \widehat{\varrho}_m^2 \sin(\vartheta)d\vartheta \\ &= \underbrace{\left(\frac{1-m^2}{\cos(\alpha^*)} - \cos(\alpha^*)\right)}_{<0} \widehat{\varrho}_m(\pi-\alpha^*)^2 + 2\int_0^{\pi-\alpha^*} \widehat{\varrho}_m^2 \sin(\vartheta)d\vartheta \\ &\leq 2\int_0^{\pi-\alpha^*} \widehat{\varrho}_m^2 \sin(\vartheta)d\vartheta. \end{aligned}$$

Yet, this leaves us with an upper bound for m^2, namely

$$m^2 \leq 2\frac{\int\limits_0^{\pi-\alpha^*} \widehat{\varrho}_m^2 \sin(\vartheta)d\vartheta}{\int\limits_0^{\pi-\alpha^*} \frac{1}{\sin(\vartheta)}\widehat{\varrho}_m^2 d\vartheta} \leq 2.$$

This shows that the system (6.47)-(6.49) only has to be considered for $m^2 = 1$. This reduces (6.47)-(6.49) to

$$0 = \widehat{\varrho}_1''(\vartheta) + \cot(\vartheta)\widehat{\varrho}_1'(\vartheta) + \left(2 - \frac{1}{\sin(\vartheta)^2}\right)\widehat{\varrho}_1(\vartheta) \tag{6.50}$$

$$0 = -\cos(\alpha^*)\widehat{\varrho}_1(\pi-\alpha^*) - \sin(\alpha^*)\widehat{\varrho}_1'(\pi-\alpha^*) \tag{6.51}$$

$$0 = \widehat{\varrho}_1(0). \tag{6.52}$$

The general solution of (6.50) is given by

$$\widehat{\varrho}_1(\vartheta) = -c_1\sin(\vartheta) + c_2\left(-\frac{1}{2}\sin(\vartheta)\ln\left(\frac{1+\cos(\vartheta)}{1-\cos(\vartheta)}\right) - \cot(\vartheta)\right).$$

For $\widehat{\varrho}_1$ to solve (6.52) we require $c_2 = 0$ and (6.51) is then immediately satisfied. Hence $\widehat{\varrho}_1(\vartheta) = -c_1\sin(\vartheta)$ is the complete solution of (6.50)-(6.52).
Now we consider the system (6.44)-(6.46). The general solution of (6.44) is given by

$$\widehat{\varrho}_0(\vartheta) = \frac{c}{2} + c_1\cos(\vartheta) + c_2\left(\frac{1}{2}\cos(\vartheta)\ln\left(\frac{1+\cos(\vartheta)}{1-\cos(\vartheta)}\right) - 1\right).$$

If $c_2 \neq 0$, the function would have a singularity in $\vartheta = 0$, which makes it necessary for (6.46) that $c_2 = 0$. Therefore we know that so far the solution $\widehat{\varrho}_0$ is of the form

$$\widehat{\varrho}_0(\vartheta) = \frac{c}{2} + c_1\cos(\vartheta).$$

The boundary condition (6.45) is only satisfied for $c = 0$ as one can see from

$$\begin{aligned} 0 &= \sin(\alpha^*)\widehat{\varrho}_0(\pi - \alpha^*) - \cos(\alpha^*)\widehat{\varrho}_0'(\pi - \alpha^*) \\ &= \sin(\alpha^*)\frac{c}{2} - c_1 \sin(\alpha^*)\cos(\alpha^*) + c_1 \cos(\alpha^*)\sin(\alpha^*) = \sin(\alpha^*)\frac{c}{2}. \end{aligned}$$

This is the contradiction that we are looking for. ■

Remark 6.8: (i) We continue the considerations from the previous proof one step further: Since $e^{i\varphi}$ and $e^{-i\varphi}$ can be transformed into $\sin(\varphi)$ and $\cos(\varphi)$ we end up with the solution

$$\varrho(\varphi, \vartheta) = c_1 \cos(\vartheta) + c_2 \cos(\varphi)\sin(\vartheta) + c_3 \sin(\varphi)\sin(\vartheta),$$

which is exactly what have obtained in the cases $k = 0$, $k = 1$ and $k \geq 2$ above.
(ii) Lemma 6.7 explains why we found for $R^* \sin(\alpha^*)^2 = b\cos(\alpha^*)$ an additional function while considering the case $k = 0$ above. This particular function compensates the missing special solution if $R^* \sin(\alpha^*)^2 = b\cos(\alpha^*)$, so that we always find three linearly independent functions in $\mathcal{N}(A_0)$ if we restrict ourselves to $b > C_{crit}$. □

If $b > C_{crit}$ then

$$\varrho(\varphi, \vartheta) = \begin{cases} c_1(1 + c_\alpha \cos(\vartheta)) \\ \quad + c_2 \cos(\varphi)\sin(\vartheta) + c_3 \sin(\varphi)\sin(\vartheta) & \text{if } R^* \sin(\alpha^*)^2 \neq b\cos(\alpha^*) \\ c_1 \cos(\vartheta) \\ \quad + c_2 \cos(\varphi)\sin(\vartheta) + c_3 \sin(\varphi)\sin(\vartheta) & \text{if } R^* \sin(\alpha^*)^2 = b\cos(\alpha^*) \end{cases}, \tag{6.53}$$

is the full solution to the inhomogeneous system (6.20)-(6.24).

Transforming (6.53) back to the usual x-y-z-coordinates, where z can be expressed by x and y as

$$z(x, y) = H^* \pm \sqrt{R^{*2} - x^2 - y^2},$$

one can see that the last two linearly independent summands that (6.53) consists of, are the expected shifts in x- and y-direction. In fact, using (6.10) we have

$$\sin(\varphi) = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \cos(\varphi) = \frac{y}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \sin(\vartheta) = \frac{1}{R^*}\sqrt{x^2 + y^2},$$

which shows

$$\widetilde{\varrho}_1(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \frac{1}{R^*} \sqrt{x^2 + y^2} = \frac{x}{R^*} \tag{6.54}$$

$$\widetilde{\varrho}_2(x, y) = \frac{y}{\sqrt{x^2 + y^2}} \frac{1}{R^*} \sqrt{x^2 + y^2} = \frac{y}{R^*}. \tag{6.55}$$

These $\widetilde{\varrho}_i$ with $i \in \{1, 2\}$ are obviously $C^\infty(\Gamma^*)$ and hence in $W_p^2(\Gamma^*; \mathbb{R})$. Furthermore, $\widetilde{\varrho}_1|_{\partial\Gamma^*} = \widetilde{\varrho}_1|_{z=0} = \frac{x}{R^*}$ and $\widetilde{\varrho}_2|_{\partial\Gamma^*} = \widetilde{\varrho}_2|_{z=0} = \frac{y}{R^*}$ are trivially in $W_p^{3-\frac{1}{p}}(\partial\Gamma^*)$.
The first linearly independent summand in (6.53) transforms using

$$\cos(\vartheta) = \frac{z(x, y) - H^*}{R^*}$$

into

$$\widetilde{\varrho}_0(x,y) = \begin{cases} \dfrac{R^* - c_\alpha H^*}{R^*} + c_\alpha \dfrac{z(x,y)}{R^*} & \text{if } R^* \sin(\alpha^*)^2 \neq b\cos(\alpha^*) \\ \dfrac{z(x,y)}{R^*} - \dfrac{H^*}{R^*} & \text{if } R^* \sin(\alpha^*)^2 = b\cos(\alpha^*) \end{cases}. \tag{6.56}$$

This is a combination of a radial expansion and a shift in z-direction, which is also in $C^\infty(\Gamma^*)$ and $\widetilde{\varrho}_2|_{\partial\Gamma^*} = \widetilde{\varrho}_2|_{z=0} = \frac{R^* - c_\alpha H^*}{R^*}$ is in $C^\infty(\partial\Gamma^*)$. Thus

$$v_i := \begin{pmatrix} \widetilde{\varrho}_i \\ \widetilde{\varrho}_i|_{\partial\Gamma^*} \end{pmatrix} \in V \cap X_1 \qquad \text{for each } i \in \{0,1,2\}. \tag{6.57}$$

Therefore we have $\mathcal{N}(A_0) = \operatorname{span}\{v_0, v_1, v_2\}$ and especially $\dim(\mathcal{N}(A_0)) = 3$ whenever $b > C_{crit}$.

Since the 3-dimensionality of $\mathcal{N}(A_0)$ will play a crucial role in all the considerations to follow, we assume from now on

$$b > C_{crit} = -\frac{1}{3} R^* \sin(\alpha^*)^2 \cos(\alpha^*) = -\frac{H^*}{3}\sin(\alpha^*)^2. \tag{6.58}$$

Now that we studied A_0 and its nullspace intensively, we still can not start checking the assumptions (a)-(d) from Theorem 6.1. For proving assumption (a) we first have to investigate the solvability of

$$-\Delta_{\Gamma^*} v^{(1)} - |\sigma^*|^2 v^{(1)} + \fint_{\Gamma^*} \Delta_{\Gamma^*} v^{(1)} + |\sigma^*|^2 v^{(1)}\, d\mathcal{H}^2 = f^{(1)} \qquad \text{in } \Gamma^* \tag{6.59}$$

$$\begin{aligned} \sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v^{(1)}) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*} v^{(1)} & \\ - b\sin(\alpha^*) v^{(2)}_{\sigma\sigma} - \frac{b}{{R^*}^2 \sin(\alpha^*)} v^{(2)} &= f^{(2)} \qquad \text{on } \partial\Gamma^* \end{aligned} \tag{6.60}$$

for a right-hand side $f = (f^{(1)}, f^{(2)})$.

First we will need the notion of a weak solution and later use semigroup arguments to show higher regularity of these solutions.

Definition 6.9 (Weak solution): We call

$$u = (u^{(1)}, u^{(2)}) \in H := \left\{ W_2^1(\Gamma^*) \times W_2^1(\partial\Gamma^*) \,\middle|\, u^{(1)}|_{\partial\Gamma^*} = u^{(2)}, \fint_{\Gamma^*} u^{(1)}\, d\mathcal{H}^2 = 0 \right\}$$

a weak solution of (6.59)-(6.60) for $f = (f^{(1)}, f^{(2)}) \in \widetilde{L_2}$ with

$$\widetilde{L_2} := \left\{ f \in L_2(\Gamma^*) \times L_2(\partial\Gamma^*) \,\middle|\, \fint_{\Gamma^*} f^{(1)}\, d\mathcal{H}^2 = 0 \right\}$$

$$\langle f, g\rangle_{\widetilde{L_2}} := \int_{\Gamma^*} f^{(1)} g^{(1)}\, d\mathcal{H}^2 + \int_{\partial\Gamma^*} \frac{1}{\sin(\alpha^*)^2} f^{(2)} g^{(2)}\, d\mathcal{H}^1$$

if we have

$$\begin{aligned}
&\int_{\Gamma^*} \nabla_{\Gamma^*} u^{(1)} \cdot \nabla_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 - \int_{\Gamma^*} |\sigma^*|^2 u^{(1)} v^{(1)} \, d\mathcal{H}^2 \\
&+ \int_{\partial\Gamma^*} \frac{b}{\sin(\alpha^*)} u_\sigma^{(2)} v_\sigma^{(2)} \, d\mathcal{H}^1 + \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*} - \frac{b}{R^* \sin(\alpha^*)^3} \right) u^{(2)} v^{(2)} \, d\mathcal{H}^1 \\
&\quad = \int_{\Gamma^*} f^{(1)} v^{(1)} \, d\mathcal{H}^2 + \int_{\partial\Gamma^*} \frac{1}{\sin(\alpha^*)^2} f^{(2)} v^{(2)} \, d\mathcal{H}^1
\end{aligned}$$

for all $v \in H$. □

This definition is motivated by the following calculation. If we assume that we would have a solution $u \in C^2$ of (6.59)-(6.60) then

$$\begin{aligned}
\int_{\Gamma^*} f^{(1)} v^{(1)} \, d\mathcal{H}^2 &= \int_{\Gamma^*} \left(-\Delta_{\Gamma^*} u^{(1)} - |\sigma^*|^2 u^{(1)} + \fint_{\Gamma^*} \Delta_{\Gamma^*} u^{(1)} + |\sigma^*|^2 u^{(1)} \, d\mathcal{H}^2 \right) v^{(1)} \, d\mathcal{H}^2 \\
&= - \int_{\Gamma^*} \left(\Delta_{\Gamma^*} u^{(1)} \right) v^{(1)} \, d\mathcal{H}^2 - \int_{\Gamma^*} |\sigma^*|^2 u^{(1)} v^{(1)} \, d\mathcal{H}^2 \\
&+ \left(\fint_{\Gamma^*} \Delta_{\Gamma^*} u^{(1)} + |\sigma^*|^2 u^{(1)} \, d\mathcal{H}^2 \right) \int_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 \\
&= \int_{\Gamma^*} \nabla_{\Gamma^*} u^{(1)} \cdot \nabla_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 - \int_{\Gamma^*} |\sigma^*|^2 u^{(1)} v^{(1)} \, d\mathcal{H}^2 \\
&- \int_{\partial\Gamma^*} \left(n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} u^{(1)} \right) v^{(1)} \, d\mathcal{H}^1 \\
&= \int_{\Gamma^*} \nabla_{\Gamma^*} u^{(1)} \cdot \nabla_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 - \int_{\Gamma^*} |\sigma^*|^2 u^{(1)} v^{(1)} \, d\mathcal{H}^2 \\
&- \int_{\partial\Gamma^*} \frac{1}{\sin(\alpha^*)^2} f^{(2)} v^{(2)} \, d\mathcal{H}^1 \\
&+ \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*} u^{(1)} - \frac{b}{\sin(\alpha^*)} u_{\sigma\sigma}^{(2)} - \frac{b}{R^{*2} \sin(\alpha^*)^3} u^{(2)} \right) v^{(2)} \, d\mathcal{H}^1 \\
&= \int_{\Gamma^*} \nabla_{\Gamma^*} u^{(1)} \cdot \nabla_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 - \int_{\Gamma^*} |\sigma^*|^2 u^{(1)} v^{(1)} \, d\mathcal{H}^2 \\
&- \int_{\partial\Gamma^*} \frac{1}{\sin(\alpha^*)^2} f^{(2)} v^{(2)} \, d\mathcal{H}^1 + \int_{\partial\Gamma^*} \frac{b}{\sin(\alpha^*)} u_\sigma^{(2)} v_\sigma^{(2)} \, d\mathcal{H}^1 \\
&+ \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*} - \frac{b}{R^{*2} \sin(\alpha^*)^3} \right) u^{(2)} v^{(2)} \, d\mathcal{H}^1.
\end{aligned}$$

For using the Lemma of Lax-Milgram we define the bilinear form $B : H \times H \longrightarrow \mathbb{R}$ and the functional $F : H \longrightarrow \mathbb{R}$ by

$$\begin{aligned}
B(u,v) &:= \int_{\Gamma^*} \nabla_{\Gamma^*} u^{(1)} \cdot \nabla_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 - \int_{\Gamma^*} |\sigma^*|^2 u^{(1)} v^{(1)} \, d\mathcal{H}^2 \\
&+ \int_{\partial\Gamma^*} \frac{b}{\sin(\alpha^*)} u_\sigma^{(2)} v_\sigma^{(2)} \, d\mathcal{H}^1 + \int_{\partial\Gamma^*} \left(\frac{\cot(\alpha^*)}{R^*} - \frac{b}{R^{*2} \sin(\alpha^*)^3} \right) u^{(2)} v^{(2)} \, d\mathcal{H}^1 \\
F(v) &:= \langle f, v \rangle_{\widetilde{L_2}} .
\end{aligned}$$

B and F are bounded because for all $u, v \in H$ we see

$$\begin{aligned}
|B(u,v)| &\le \int_{\Gamma^*} |\nabla_{\Gamma^*} u^{(1)} \cdot \nabla_{\Gamma^*} v^{(1)}| \, d\mathcal{H}^2 + \frac{2}{R^{*2}} \int_{\Gamma^*} |u^{(1)} v^{(1)}| \, d\mathcal{H}^2 \\
&\quad + \frac{b}{\sin(\alpha^*)} \int_{\partial\Gamma^*} |u_\sigma^{(2)} v_\sigma^{(2)}| \, d\mathcal{H}^1 + \left| \frac{\cot(\alpha^*)}{R^*} - \frac{b}{R^{*2}\sin(\alpha^*)^3} \right| \int_{\partial\Gamma^*} |u^{(2)} v^{(2)}| \, d\mathcal{H}^1 \\
&\le \left\| \nabla_{\Gamma^*} u^{(1)} \right\|_{L_2(\Gamma^*)} \left\| \nabla_{\Gamma^*} v^{(1)} \right\|_{L_2(\Gamma^*)} + c_1 \left\| u^{(1)} \right\|_{L_2(\Gamma^*)} \left\| v^{(1)} \right\|_{L_2(\Gamma^*)} \\
&\quad + c_2 \left\| u_\sigma^{(2)} \right\|_{L_2(\partial\Gamma^*)} \left\| v_\sigma^{(2)} \right\|_{L_2(\partial\Gamma^*)} + c_3 \left\| u^{(2)} \right\|_{L_2(\partial\Gamma^*)} \left\| v^{(2)} \right\|_{L_2(\partial\Gamma^*)} \\
&\le c_4 \left(\left\| \nabla_{\Gamma^*} u^{(1)} \right\|_{L_2(\Gamma^*)} + \left\| u^{(1)} \right\|_{L_2(\Gamma^*)} + \left\| u_\sigma^{(2)} \right\|_{L_2(\partial\Gamma^*)} + \left\| u^{(2)} \right\|_{L_2(\partial\Gamma^*)} \right) \\
&\quad \left(\left\| \nabla_{\Gamma^*} v^{(1)} \right\|_{L_2(\Gamma^*)} + \left\| v^{(1)} \right\|_{L_2(\Gamma^*)} + \left\| v_\sigma^{(2)} \right\|_{L_2(\partial\Gamma^*)} + \left\| v^{(2)} \right\|_{L_2(\partial\Gamma^*)} \right) \\
&= c_4 \left(\left\| u^{(1)} \right\|_{W_2^1(\Gamma^*)} + \left\| u^{(2)} \right\|_{W_2^1(\partial\Gamma^*)} \right) \left(\left\| v^{(1)} \right\|_{W_2^1(\Gamma^*)} + \left\| v^{(2)} \right\|_{W_2^1(\partial\Gamma^*)} \right) \\
&= c_4 \|u\|_H \|v\|_H \\
|F(v)| &\le \int_{\Gamma^*} |f^{(1)} v^{(1)}| \, d\mathcal{H}^2 + \frac{1}{\sin(\alpha^*)^2} \int_{\partial\Gamma^*} |f^{(2)} v^{(2)}| \, d\mathcal{H}^1 \\
&\le \left\| f^{(1)} \right\|_{L_2(\Gamma^*)} \left\| v^{(1)} \right\|_{L_2(\Gamma^*)} + c_5 \left\| f^{(2)} \right\|_{L_2(\partial\Gamma^*)} \left\| v^{(2)} \right\|_{L_2(\partial\Gamma^*)} \\
&\le c_6 \left(\left\| f^{(1)} \right\|_{L_2(\Gamma^*)} + \left\| f^{(2)} \right\|_{L_2(\partial\Gamma^*)} \right) \left(\left\| v^{(1)} \right\|_{L_2(\Gamma^*)} + \left\| v^{(2)} \right\|_{L_2(\partial\Gamma^*)} \right) \\
&\le c_6 \|f\|_{\widetilde{L_2}} \|v\|_H
\end{aligned}$$

by usage of Hölder's inequality. Moreover, we have the energy estimate

$$\begin{aligned}
B(u,u) &= \left\| \nabla_{\Gamma^*} u^{(1)} \right\|_{L_2(\Gamma^*)}^2 - \frac{2}{R^{*2}} \left\| u^{(1)} \right\|_{L_2(\Gamma^*)}^2 \\
&\quad + \frac{b}{\sin(\alpha^*)} \left\| u_\sigma^{(2)} \right\|_{L_2(\partial\Gamma^*)}^2 + \underbrace{\left(\frac{\cot(\alpha^*)}{R^{*2}} - \frac{b}{R^{*2}\sin(\alpha^*)^3} \right)}_{=:\widehat{c}} \left\| u^{(2)} \right\|_{L_2(\partial\Gamma^*)}^2 .
\end{aligned}$$

If $\widehat{c} \ge 0$, we can drop this summand to obtain

$$B(u,u) + c_7 \left\| u^{(1)} \right\|_{L_2(\Gamma^*)}^2 \ge \left\| \nabla_{\Gamma^*} u^{(1)} \right\|_{L_2(\Gamma^*)}^2 + c_8 \left\| u_\sigma^{(2)} \right\|_{L_2(\partial\Gamma^*)}^2$$

and thus we see

$$B(u,u) + C \|u\|_{\widetilde{L_2}}^2 \ge c_9 \left(\left\| u^{(1)} \right\|_{W_2^1(\Gamma^*)}^2 + \left\| u^{(2)} \right\|_{W_2^1(\partial\Gamma^*)}^2 \right) \ge c \|u\|_H^2$$

for some $C, c > 0$. Should $\widehat{c} < 0$ hold, then we can absorb this last summand into $\|u\|_{\widetilde{L_2}}^2$ on the left-hand side and still arrive at the inequality $B(u,u) + C\|u\|_{\widetilde{L_2}}^2 \ge c\|u\|_H^2$. This shows that for $\mu \ge C$ the modified bilinear form

$$B_\mu : H \times H \longrightarrow \mathbb{R} : (u,v) \longmapsto B_\mu(u,v) := B(u,v) + \mu \langle u, v \rangle_{\widetilde{L_2}}$$

satisfies all the assumptions that are necessary to use the Lemma of Lax-Milgram (cf. Section 6.2.1 in [Eva10]). Therefore we know that for each $f \in \widetilde{L_2}$ there exists a unique weak solution $u \in H$ of the modified equation

$$
\begin{aligned}
&-\Delta_{\Gamma^*} v^{(1)} - |\sigma^*|^2 v^{(1)} \\
&+ ⨍_{\Gamma^*} \Delta_{\Gamma^*} v^{(1)} + |\sigma^*|^2 v^{(1)} \, d\mathcal{H}^2 + \mu v^{(1)} =: (L_\mu u)^{(1)} = f^{(1)} \qquad \text{in } \Gamma^* \qquad (6.61)\\
&\sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v^{(1)}) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*} v^{(1)} \\
&-b \sin(\alpha^*) v^{(2)}_{\sigma\sigma} - \frac{b}{{R^*}^2 \sin(\alpha^*)} v^{(2)} + \mu v^{(2)} =: (L_\mu u)^{(2)} = f^{(2)} \qquad \text{on } \partial\Gamma^*. \qquad (6.62)
\end{aligned}
$$

This unique solution u shall be denoted by $u = L_\mu^{-1} f$. A weak solution $u \in H$ of the original problem (6.59)-(6.60) for a right-hand side $f \in \widetilde{L_2}$ is equivalent to a weak solution of (6.61)-(6.62) with a right-hand side $\mu u + f$, i.e. a $u \in H$ satisfying

$$B_\mu(u, v) = \langle \mu u + f, v \rangle_{\widetilde{L_2}} \qquad \forall\, v \in H.$$

Using the weak solvability we obtain $u = L_\mu^{-1}(\mu u + f)$, which can be transformed into $(\mathrm{Id} - K)u = g$ with $g := L_\mu^{-1} f$ and $K := \mu L_\mu^{-1}$. Note that $K : \widetilde{L_2} \to H$ is bounded due to

$$c \|u\|_H^2 \le B_\mu(u, u) = \langle g, u \rangle_{\widetilde{L_2}} \le \|g\|_{\widetilde{L_2}} \|u\|_{\widetilde{L_2}} \le \|g\|_{\widetilde{L_2}} \|u\|_H ,$$

which shows

$$c \|Kg\|_H = c\mu \left\| L_\mu^{-1} g \right\|_H = c\mu \|u\|_H \le \mu \|g\|_{\widetilde{L_2}} .$$

Regarding K as an operator $K : \widetilde{L_2} \longrightarrow H \hookrightarrow \widetilde{L_2}$ it is compact as a composition of a bounded operator and the compact embedding $H \hookrightarrow \widetilde{L_2}$. Fredholm theory as it is used in Theorem 6.2.4 of [Eva10] shows that $u - Ku = g$ has a solution if and only if $\langle g, v \rangle_{\widetilde{L_2}} = 0$ for all $v \in H$ with $v - K^* v = 0$. This condition can be rewritten as $\langle f, v \rangle_{\widetilde{L_2}} = 0$ for all $v \in H$ with $v = K^* v$, because of

$$0 = \langle g, v \rangle_{\widetilde{L_2}} = \left\langle L_\mu^{-1} f, v \right\rangle_{\widetilde{L_2}} = \frac{1}{\mu} \langle Kf, v \rangle_{\widetilde{L_2}} = \frac{1}{\mu} \langle f, K^* v \rangle_{\widetilde{L_2}} = \frac{1}{\mu} \langle f, v \rangle_{\widetilde{L_2}} .$$

The condition $v - K^* v = 0$, however, is equivalent to $B(v, u) = 0$ for all $u \in H$ due to the symmetry of B on H. Note that $B(u, v) = 0$ for all $u \in H$ is the same as finding solutions of

$$
\begin{aligned}
-\Delta_{\Gamma^*} v^{(1)} - |\sigma^*|^2 v^{(1)} &= const. \qquad \text{in } \Gamma^* \\
\sin(\alpha^*)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v^{(1)}) + \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*} v^{(1)} & \\
-b \sin(\alpha^*) v^{(2)}_{\sigma\sigma} - \frac{b}{{R^*}^2 \sin(\alpha^*)} v^{(2)} &= 0 \qquad \text{on } \partial\Gamma^* \\
\int_{\Gamma^*} v^{(1)} \, d\mathcal{H}^2 &= 0,
\end{aligned}
$$

which we already did as we determined $\mathcal{N}(A_0)$ and found these equations to be satisfied exactly for v_1 and v_2 from (6.57). The nullspace element v_0 is omitted, since its first

component is not mean value free as required for H. Summing up we proved (6.59)-(6.60) has a weak solution $u \in H$ if and only if $f \in \widetilde{L_2}$ satisfies $\langle f, v_1 \rangle_{\widetilde{L_2}} = \langle f, v_2 \rangle_{\widetilde{L_2}} = 0$.
The next step is to show that the weak solution is actually a strong solution. Let $f \in X_0$ such that $\int_{\Gamma^*} f^{(1)}\, d\mathcal{H}^2 = 0$ and $\langle f, v_1 \rangle_{\widetilde{L_2}} = \langle f, v_2 \rangle_{\widetilde{L_2}} = 0$. Then we know by Theorem 3.13 that $-A_0$ generates an analytic semigroup and hence there exists some $\mu_0 > 0$ such that $\mu_0 u + A_0 u = f$ has a unique solution $u \in X_1$. The weak solution $u_w \in H$ of $A_0 u_w = f$ also solves $\mu_0 u_w + A_0 u_w = \mu_0 u_w + f =: \widehat{f}$ weakly. We see that $\widehat{f} \in \widetilde{L_2}$ due to $u_w \in H \subseteq \widetilde{L_2}$ and the choice of f. Thus we obtain another $u_s \in X_1$, which also solves $\mu_0 u_s + A_0 u_s = \widehat{f}$. But since this u_s is also a weak solution and hence is unique, it has to coincide with u_w. Thus the solution u_w of $A_0 u_w = f$ is not only in H, but even an element of $X_1 \cap H$. So far we have seen that (6.59)-(6.60) has a solution $u \in X_1$ with $\int_{\Gamma^*} u^{(1)}\, d\mathcal{H}^2 = 0$ for all $f \in X_0$ that satisfy $\int_{\Gamma^*} f^{(1)}\, d\mathcal{H}^2 = 0$ and $\langle f, v_1 \rangle_{\widetilde{L_2}} = \langle f, v_2 \rangle_{\widetilde{L_2}} = 0$.

These considerations regarding the nullspace and the solvability of (6.59)-(6.60) put us into the position of finally start proving the assumptions (a)-(d) from Theorem 6.1.

We turn our attention to assumption (a). We will enclose the set of equilibria $\mathcal{E}$ between a smaller set $\widetilde{\mathcal{E}}$ and a bigger set $\widehat{\mathcal{E}}$ that are C^1-manifolds of dimension 3 and hence $\mathcal{E}$ is a C^1-manifold of dimension 3 as well. The arguments will rely on Theorem 4.B in [Zei85]. To this end define

$$X := \mathbb{R}^3, \qquad Y := X_1 / \mathcal{N}(A_0) \qquad \text{and} \qquad Z := \left\{ v \in X_0 \,\middle|\, \int_{\Gamma^*} v^{(1)}\, d\mathcal{H}^2 = 0 \right\}.$$

Then X and Z are Banach spaces and Y as well, because $\mathcal{N}(A_0)$ is finite dimensional and hence closed. We consider the function

$$F : X \times Y \longrightarrow Z : (t_0, t_1, t_2, w) \longmapsto \begin{pmatrix} H_\Gamma(v^{(1)}) - \overline{H}(v^{(1)}) \\ a + b\varkappa_{\partial D}(v^{(2)}) + \left\langle n_\Gamma(v^{(1)}), n_D \right\rangle \end{pmatrix},$$

where $v = (v^{(1)}, v^{(2)})^T$ shall be given by $v := t_0 v_0 + t_1 v_1 + t_2 v_2 + w$ with $w \in Y$. Then the set of equilibria as given in (6.9) can be written as $\mathcal{E} = \{v \in V \cap X_1 \mid F(v) = 0\}$. We use the orthogonal projection $P : X_0 \longrightarrow \operatorname{span}\{v_1, v_2\}^\perp$, where the orthogonal complement has to be understood with respect to the $\widetilde{L_2}$-inner product, to define

$$\widehat{\mathcal{E}} := \{v \in V \cap X_1 \mid PF(v) = 0\}.$$

Then trivially $\mathcal{E} \subseteq \widehat{\mathcal{E}}$ and PF maps as follows

$$PF : X \times Y \longrightarrow \operatorname{span}\{v_1, v_2\}^\perp \cap Z \subseteq X_0 : (t_0, t_1, t_2, w) \longmapsto PF(v)$$

for $v = t_0 v_0 + t_1 v_1 + t_2 v_2 + w$. The first partial derivative of F with respect to w in $\mathbb{O} := (0,0,0,0) \in X \times Y$, which corresponds to the linearization operator $-A_0$, is given by the calculations done in Section 2.3 and equations (6.11), (6.13)-(6.15) as

$$(D_w F(\mathbb{O})(v))^{(1)} = \Delta_{\Gamma^*} v^{(1)} + |\sigma^*|^2 v^{(1)} - \fint_{\Gamma^*} \Delta_{\Gamma^*} v^{(1)} + |\sigma^*|^2 v^{(1)}\, d\mathcal{H}^2$$

and

$$(D_wF(\mathbb{O})(v))^{(2)} = -\sin(\alpha^*)^2(n_{\partial\Gamma^*}\cdot\nabla_{\Gamma^*}v^{(1)}) - \frac{\sin(\alpha^*)\cos(\alpha^*)}{R^*}v^{(1)} \\ + b\sin(\alpha^*)v^{(2)}_{\sigma\sigma} + \frac{b}{R^{*2}\sin(\alpha^*)}v^{(2)}.$$

Now we will show that

$$D_w(PF)(\mathbb{O}) : {X_1}\big/{\mathcal{N}(A_0)} \longrightarrow \operatorname{span}\{v_1, v_2\}^\perp \cap Z : w \longmapsto D_w(PF)(w)$$

is bijective. First remark that $D_w(PF) = PD_wF$, since P is linear. The injectivity might seem trivial after factorizing out all the functions in $\mathcal{N}(A_0)$. Yet, for the sake of completeness we calculate

$$\begin{aligned} D_w(PF)(\mathbb{O})w = 0 \quad &\Leftrightarrow \quad P(D_wF(\mathbb{O})w) = 0 &&\Leftrightarrow \quad PA_0w = 0 \\ &\Leftrightarrow \quad A_0w \in \mathcal{N}(A_0) &&\Leftrightarrow \quad w \in \mathcal{N}(A_0^2) = \mathcal{N}(A_0) \\ &\Leftrightarrow \quad w = 0 \in {X_1}\big/{\mathcal{N}(A_0)}, \end{aligned}$$

where the fact $\mathcal{N}(A_0) = \mathcal{N}(A_0^2)$ follows from the upcoming Lemma 6.11 and the considerations that follow in the proof of assumption (c). The surjectivity is proved by the facts concerning the solvability of (6.59)-(6.60) from above. Let $f \in \operatorname{span}\{v_1, v_2\}^\perp \cap Z$. Then obviously $\langle f, v_1\rangle_{\widetilde{L_2}} = \langle f, v_2\rangle_{\widetilde{L_2}} = 0$ and we know that there is a solution $u \in X_1$ with $\int_{\Gamma^*} u^{(1)}\, d\mathcal{H}^2 = 0$ of $D_wF(\mathbb{O})u = f$ and $P((D_wF)(\mathbb{O})u) = P(f) = f$. Clearly this u is in ${X_1}\big/{\mathcal{N}(A_0)}$, since contributions of v_0, v_1 and v_2 do not affect $D_wF(\mathbb{O})u = -A_0u = f$. Moreover, $PF(\mathbb{O}) = P(0) = 0$ because $v^* \equiv 0$ corresponds to an SSC. By the same calculations as in Lemma 3.18 we see that F and F_w are continuous in a small neighborhood $U(\mathbb{O}) \subseteq X \times Y$ of $\mathbb{O}$ and so are PF and PF_w. Therefore

$$PF : U(\mathbb{O}) \subseteq X \times Y \longrightarrow \operatorname{span}\{v_1, v_2\}^\perp \cap Z$$

satisfies all assumptions of Theorem 4.B in [Zei85]. So we see that there exist $r_0, r > 0$ such that for every $t \in \mathbb{R}^3$ with $\|t\| \le r_0$ there is exactly one $w(t) \in Y$ for which $\|w(t)\|_Y \le r$ and $PF(t, w(t)) = 0$. Hence

$$\Psi : B_{r_0}(0) \subseteq \mathbb{R}^3 \longrightarrow \mathcal{E} : t = (t_0, t_1, t_2) \longmapsto \Psi(t) := t_0v_0 + t_1v_1 + t_2v_2 + w(t)$$

is the desired parametrization of $\widehat{\mathcal{E}}$ in a neighborhood of $v^* \equiv 0$. Due to the fact that

$$D\Psi(0) = (v_0 + (\partial_{t_0}w)(0), v_1 + (\partial_{t_1}w)(0), v_2 + (\partial_{t_2}w)(0))$$

has full rank, because v_0, v_1 and v_2 are linearly independent and $w(t)$ belongs to Y, which is complementary to $\mathcal{N}(A_0)$, we see that $\widehat{\mathcal{E}}$ is a C^1-manifold with $\dim(\widehat{\mathcal{E}}) = 3$.
Next we try to find a 3-dimensional manifold $\widetilde{\mathcal{E}}$ that is contained in $\mathcal{E}$. We define

$$\widetilde{\mathcal{E}} := \{u \in V \cap X_1 \mid u \text{ parametrizes an SSC}\}.$$

Then $\widetilde{\mathcal{E}} \subseteq \mathcal{E}$ is obvious since for SSCs $F(u) = 0$ holds. For $|x|$, $|y|$, $|H - H^*|$ and $|R - R^*|$ small enough any $u \in \widetilde{\mathcal{E}}$ is given implicitly as the solution of

$$\left\| \Psi(q, u(q)) - \begin{pmatrix} x \\ y \\ H \end{pmatrix} \right\|^2 = R^2 \qquad \forall\, q \in \Gamma^*, \tag{6.63}$$

where Ψ is the curvilinear coordinate system as introduced in (2.22). And since this SC is also stationary, u has to satisfy (6.2). The term $\cos(\alpha)$ can be replaced by $\frac{H}{R}$ and r can be replaced by $r = R\sin(\alpha) = R\sqrt{1 - \left(\frac{H}{R}\right)^2} = \sqrt{R^2 - H^2}$ and so we obtain

$$\frac{H}{R} = \frac{b}{\sqrt{R^2 - H^2}} - a,$$

which is an equation that specifies the relation between R and H. Therefore there is some way of expressing H in terms of R via a C^1-function $H(R)$ and this reduces the degrees of freedom in (6.63) to three. It is again useful to write the curvilinear coordinate system in spherical coordinates. For $q = P(\varphi, \vartheta)$ as in (6.10) we use the tangential correction terms $T(q)$ and $t(q, w)$ defined by

$$T(q) := \begin{pmatrix} \sin(\varphi)\cos(\vartheta) \\ \cos(\varphi)\cos(\vartheta) \\ -\sin(\vartheta) \end{pmatrix} \qquad \text{and} \qquad t(q, w) := -w\eta(\vartheta)\cot(\alpha^*),$$

where $\eta : [0, \pi - \alpha^*] \longrightarrow \mathbb{R} : \vartheta \longmapsto \eta(\vartheta)$ is a mollifier function that satisfies $|\eta(\vartheta)| \in [0, 1]$ and $\eta(\pi - \alpha^*) = 1$. These choices guarantee that

$$\begin{aligned}
\Psi(q, w)|_{\partial\Gamma^*} &= q + wn_{\Gamma^*}(q) + t(q, w)T(q) \\
&= \left.\begin{pmatrix} \cdots \\ \cdots \\ R^*\cos(\vartheta) + H^* \end{pmatrix} + w\begin{pmatrix} \cdots \\ \cdots \\ \cos(\vartheta) \end{pmatrix} + t(q, w)\begin{pmatrix} \cdots \\ \cdots \\ -\sin(\vartheta) \end{pmatrix}\right|_{\vartheta = \pi - \alpha^*} \\
&= \begin{pmatrix} \cdots \\ \cdots \\ -R^*\cos(\alpha^*) + H^* \end{pmatrix} - w\begin{pmatrix} \cdots \\ \cdots \\ \cos(\alpha^*) \end{pmatrix} - w\eta(\pi - \alpha^*)\cot(\alpha^*)\begin{pmatrix} \cdots \\ \cdots \\ -\sin(\alpha^*) \end{pmatrix} \\
&= \begin{pmatrix} \cdots \\ \cdots \\ 0 \end{pmatrix} + w\begin{pmatrix} \cdots \\ \cdots \\ 0 \end{pmatrix} \in \partial\Omega
\end{aligned}$$

as required. Moreover, we see

$$\partial_w\Psi(q, 0) = n_{\Gamma^*}(q) + t_w(q, 0)T(q) = \begin{pmatrix} \sin(\varphi)\sin(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix} + \eta(\vartheta)\cot(\alpha^*)\begin{pmatrix} \sin(\varphi)\cos(\vartheta) \\ \cos(\varphi)\cos(\vartheta) \\ -\sin(\vartheta) \end{pmatrix}.$$

Calculating the derivative of (6.63) in $u \equiv 0$, which corresponds to the parameters $(0,0,R^*) \in \mathbb{R}^3$, we get

$$
\begin{aligned}
&2\left(\Psi(q,u) - \begin{pmatrix} x \\ y \\ H(R) \end{pmatrix}\right) \cdot \partial_w \Psi(q,u)\Bigg|_{u\equiv 0} = 2\left(\Psi(q,0) - \begin{pmatrix} 0 \\ 0 \\ H^* \end{pmatrix}\right) \cdot \partial_w \Psi(q,0) \\
&= 2\left(q - \begin{pmatrix} 0 \\ 0 \\ H^* \end{pmatrix}\right) \cdot \left(\begin{pmatrix} \sin(\varphi)\sin(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix} + \eta(\vartheta)\cot(\alpha^*) \begin{pmatrix} \sin(\varphi)\cos(\vartheta) \\ \cos(\varphi)\cos(\vartheta) \\ -\sin(\vartheta) \end{pmatrix}\right) \\
&= 2R^* \begin{pmatrix} \sin(\varphi)\sin(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix} \cdot \begin{pmatrix} \sin(\varphi)\sin(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix} \\
&+ 2R^*\eta(\vartheta)\cot(\alpha^*) \begin{pmatrix} \sin(\varphi)\sin(\vartheta) \\ \cos(\varphi)\sin(\vartheta) \\ \cos(\vartheta) \end{pmatrix} \cdot \begin{pmatrix} \sin(\varphi)\cos(\vartheta) \\ \cos(\varphi)\cos(\vartheta) \\ -\sin(\vartheta) \end{pmatrix} \\
&= 2R^* + 2R^*\eta(\vartheta)\cot(\alpha^*)0 = 2R^* \neq 0.
\end{aligned}
$$

By the implicit function theorem and the fact that all the terms appearing in (6.63) are smooth, there exists a three parameter family of C^1-functions $u(x,y,R)$ that parametrizes SSCs. For $|x|$, $|y|$ and $|R - R^*|$ sufficiently small all these functions lie inside $\widetilde{\mathcal{E}}$. Hence we found a parametrization

$$
\Phi : (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times (R^* - \varepsilon_3, R^* + \varepsilon_3) \subseteq \mathbb{R}^3 \longrightarrow \widetilde{\mathcal{E}} : (x,y,R) \longmapsto u(x,y,R)
$$

for sufficiently small $\varepsilon_i > 0$ with $i \in \{1,2,3\}$, provided that $D\Psi(0,0,R^*)$ is not degenerated. The fact that $F(u(x,y,R)) = 0$ leads by differentiation to

$$
0 = D_u F(u(0,0,R^*))u_x(0,0,R^*) = D_u F(0)u_x(0,0,R^*) = -A_0 u_x(0,0,R^*),
$$

which proves $u_x(0,0,R^*) \in \mathcal{N}(A_0)$. Similar we show $u_y(0,0,R^*), u_R(0,0,R^*) \in \mathcal{N}(A_0)$. This suggests that $u_x(0,0,R^*)$, $u_y(0,0,R^*)$ and $u_R(0,0,R^*)$ coincide with the functions v_1, v_2 and v_0 from (6.57). In fact, this can be calculated by differentiating

$$
\left\| \Psi(q, u(x,y,R)) - \begin{pmatrix} x \\ y \\ H(R) \end{pmatrix} \right\|^2 - R^2 = 0
$$

with respect to x, y and R and evaluate it in $(0,0,R^*)$. If we use spherical coordinates again, the differentiation with respect to x leads to

$$\begin{aligned}
0 &= 2\left(\Psi(q,0) - \begin{pmatrix}0\\0\\H^*\end{pmatrix}\right) \cdot \left(\partial_w \Psi(q,0)u_x - \begin{pmatrix}1\\0\\0\end{pmatrix}\right)\\
&= 2R^* \begin{pmatrix}\sin(\varphi)\sin(\vartheta)\\ \cos(\varphi)\sin(\vartheta)\\ \cos(\vartheta)\end{pmatrix} \cdot \left(u_x(0,0,R^*) \begin{pmatrix}\sin(\varphi)\sin(\vartheta)\\ \cos(\varphi)\sin(\vartheta)\\ \cos(\vartheta)\end{pmatrix}\right.\\
&\qquad\qquad \left. + u_x(0,0,R^*)\eta(\vartheta)\cot(\alpha^*) \begin{pmatrix}\sin(\varphi)\cos(\vartheta)\\ \cos(\varphi)\cos(\vartheta)\\ -\sin(\vartheta)\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\right)\\
&= 2R^* u_x(0,0,R^*) + 0 - 2R^*\sin(\varphi)\sin(\vartheta)\\
&= 2R^*(u_x(0,0,R^*) - \sin(\varphi)\sin(\vartheta))
\end{aligned}$$

and hence $u_x(0,0,R^*) = \sin(\varphi)\sin(\vartheta)$. Analogously we obtain $u_y(0,0,R^*) = \cos(\varphi)\sin(\vartheta)$ as well as $u_R(0,0,R^*) = H'(R^*)\cos(\vartheta) + R^*$. These functions are known to be linearly independent and therefore the rank of $D\Psi(0,0,R^*)$ is three. Hence $D\Psi(0,0,R^*)$ is non-degenerated and thus the proof of assumption (a) is complete.

Remark 6.10: Actually we even proved a little more than assumption (a). We know by (6.54)-(6.56) that there are three ways to transform the SSC Γ^* into another SSC - namely an x-shift, a y-shift and a radial expansion with a simultaneous shift in z-direction. Knowing $\dim(\mathcal{E}) = 3$ we see that in a small neighborhood of $v^* \equiv 0$ the manifold of equilibria only consists of SSCs. And since we started with an arbitrary SSC Γ^*, we obtain the following result: “Around an SSC the set $\mathcal{E}$ only consists of SSCs”.
Unfortunately, this does not mean that SSCs are the only equilibria of (2.29)-(2.30). Possibly there could be equilibria that are no SSCs, which are isolated or even form a manifold itself. But such equilibria will not lie arbitrary close to an SSC.
This leads to an open question, which we formulate as a conjecture: “SSCs are the only equilibria of (2.29)-(2.30)”. ☐

Assumption (b) is an easy comparison of dimensions. We can see in (2.8) of [PSZ09] that we always have $T_0\mathcal{E} \subseteq \mathcal{N}(A_0)$. This shows that

$$3 = \dim(\mathcal{E}) = \dim(T_0\mathcal{E}) \leq \dim(\mathcal{N}(A_0)) = 3,$$

which leads to $T_0\mathcal{E} = \mathcal{N}(A_0)$ and thus proves assumption (b).

We continue with the proof of assumption (c). To this end the following two lemmas will be helpful.

Lemma 6.11: *Let $P : X_0 \longrightarrow \mathcal{R}(P) = \mathcal{N}(A_0)$ be a projection and $PA_0 = A_0P (= 0)$, then $\mathcal{N}(A_0) = \mathcal{N}(A_0^2)$.*

Proof: The inclusion $\mathcal{N}(A_0) \subseteq \mathcal{N}(A_0^2)$ is trivial. Hence assume $v \in \mathcal{N}(A_0^2)$, then $A_0^2 v = 0$, which means $A_0 v \in \mathcal{N}(A_0)$. P applied to an element of $\mathcal{N}(A_0)$ is the identity and we obtain $A_0 v = P A_0 v = A_0 P v = 0$, which shows $v \in \mathcal{N}(A_0)$. ■

Lemma 6.12: *Assume $\mathcal{N}(A_0) = \mathcal{N}(A_0^2)$. Then $X_0 = \mathcal{N}(A_0) \oplus \mathcal{R}(A_0)$, which means 0 is a semi-simple eigenvalue of A_0.*

Proof: Let $\mu \in (0, \infty)$ satisfy $\mu \notin \sigma(-A_0)$ and define

$$B := (\mu \operatorname{Id} + A_0)^{-1} : X_0 \longrightarrow X_0.$$

The proof is divided into five steps.
1. step: We start by showing that $\frac{1}{\mu}$ is an eigenvalue of B.
Let v be an eigenvector of A_0 corresponding to the eigenvalue 0. Then we see

$$\begin{aligned} Bv = \frac{1}{\mu} v \quad &\Leftrightarrow \quad (\mu \operatorname{Id} + A_0)^{-1} v = \frac{1}{\mu} v \quad \Leftrightarrow \quad \frac{1}{\mu}(\mu \operatorname{Id} + A_0) v = v \\ &\Leftrightarrow \quad v + \frac{1}{\mu} A_0 v = v \quad \Leftrightarrow \quad \frac{1}{\mu} A_0 v = 0, \end{aligned}$$

which shows that $\frac{1}{\mu}$ is an eigenvalue of B with the corresponding eigenvector v.
2. step: The same lines as in step 1 show that $\mathcal{N}\left(\frac{1}{\mu} \operatorname{Id} - B\right) = \mathcal{N}(A_0)$.
3. step: Now we show $\mathcal{N}\left(\frac{1}{\mu} \operatorname{Id} - B\right) = \mathcal{N}\left(\left(\frac{1}{\mu} \operatorname{Id} - B\right)^2\right)$.
The inclusion $\mathcal{N}\left(\frac{1}{\mu} \operatorname{Id} - B\right) \subseteq \mathcal{N}\left(\left(\frac{1}{\mu} \operatorname{Id} - B\right)^2\right)$ is obvious, so let $v \in \mathcal{N}\left(\left(\frac{1}{\mu} \operatorname{Id} - B\right)^2\right)$.
Then we obtain

$$\begin{aligned} 0 = \left(\frac{1}{\mu} \operatorname{Id} - B\right)^2 v &= \left(\frac{1}{\mu^2} \operatorname{Id} - \frac{2}{\mu} B + B^2\right) v \\ &= \left(\frac{1}{\mu^2} \operatorname{Id} - \frac{2}{\mu}(\mu \operatorname{Id} + A_0)^{-1} + (\mu \operatorname{Id} + A_0)^{-2}\right) v. \end{aligned}$$

Applying the operator $(\mu \operatorname{Id} + A_0)^2$ gives

$$\begin{aligned} 0 &= \left(\frac{1}{\mu^2}(\mu \operatorname{Id} + A_0)^2 - \frac{2}{\mu}(\mu \operatorname{Id} + A_0) + \operatorname{Id}\right) v \\ &= \left(\operatorname{Id} + \frac{2}{\mu} A_0 + \frac{1}{\mu^2} A_0^2 - 2 \operatorname{Id} - \frac{2}{\mu} A_0 + \operatorname{Id}\right) v = \frac{1}{\mu^2} A_0^2 v. \end{aligned}$$

This proves $v \in \mathcal{N}(A_0^2) = \mathcal{N}(A_0)$ and with step 2 we get $v \in \mathcal{N}\left(\frac{1}{\mu} \operatorname{Id} - B\right)$.
4. step: Here we show $\mathcal{R}\left(\frac{1}{\mu} \operatorname{Id} - B\right) = \mathcal{R}(A_0)$.
Let $v \in \mathcal{R}\left(\frac{1}{\mu} \operatorname{Id} - B\right)$. Then there exists $u \in X_0$ such that $\left(\frac{1}{\mu} \operatorname{Id} - B\right) u = v$. This can be

transformed into

$$\begin{aligned}
\left(\frac{1}{\mu}\operatorname{Id}-B\right)u = v \qquad &\Leftrightarrow \qquad \frac{1}{\mu}u - v = (\mu\operatorname{Id}+A_0)^{-1}u \\
&\Leftrightarrow \qquad u = (\mu\operatorname{Id}+A_0)\left(\frac{1}{\mu}u - v\right) \\
&\Leftrightarrow \qquad u = u + \frac{1}{\mu}A_0 u - \mu v - A_0 v \\
&\Leftrightarrow \qquad \mu v = A_0\left(\frac{1}{\mu}u - v\right) \\
&\Leftrightarrow \qquad v = A_0\left(\frac{1}{\mu^2}u - \frac{1}{\mu}v\right),
\end{aligned}$$

which proves $v \in \mathcal{R}(A_0)$. The converse inclusion can be shown analogously.
5. step: Due to the compact embedding of $X_1 \hookrightarrow X_0$ the operator

$$B = (\mu\operatorname{Id}+A_0)^{-1} : X_0 \longrightarrow X_1 \hookrightarrow X_0$$

is compact as a composition of a bounded and a compact operator. The spectral theorem for compact operators (cf. Theorem VI.2.5 in [Wer07]) shows

$$X_0 = \mathcal{N}\left(\frac{1}{\mu}\operatorname{Id}-B\right) \oplus \mathcal{R}\left(\frac{1}{\mu}\operatorname{Id}-B\right)$$

and due to step 2 and 4 we get $X_0 = \mathcal{N}(A_0) \oplus \mathcal{R}(A_0)$. ■

Remark 6.13: In Theorem 3.13 of Section 3.1, we saw that $-A_0 \mathrel{\widehat{=}} -A + P$ is the generator of an analytic semigroup, which means that this operator is sectorial. Hence by definition of a sectorial operator (cf. Definition 2.0.1 in [Lun95]) there exists $\omega \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that $\varrho(-A_0) \supseteq S_{\omega,\theta} := \{z \in \mathbb{C} \mid z \neq \omega, |\arg(z-\omega)| < \theta\}$ (see Figure 8). Especially this shows $\varrho(-A_0)$ contains the interval (ω, ∞). Therefore one can always find $\mu \in (0, \infty)$ which satisfies $\mu \notin \sigma(-A_0)$ as required in the proof of Lemma 6.12 by choosing $\mu \in (\omega, \infty) \subseteq \varrho(-A_0)$. And also by the sectoriality of $-A_0$ we know that $\|(\mu\operatorname{Id}+A_0)^{-1}\|_{\mathcal{L}(X_0)} \leq \frac{M}{|\mu-\omega|}$ for all $\mu \in S_{\omega,\theta}$, which justifies the boundedness of B in the 5th step of the previous proof. □

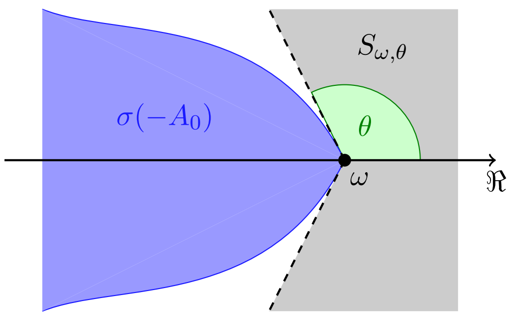

Figure 8: Spectrum of a sectorial operator

By Lemma 6.11 and 6.12 and Remark 6.13 we see that it is enough for 0 to be a semi-simple eigenvalue to find a projection as in the assumptions of Lemma 6.11. Indeed we can find such a projection, which is given by

$$P : X_0 \longrightarrow \mathcal{N}(A_0) : v = (v^{(1)}, v^{(2)}) \longmapsto P(v) := a_0(v)v_0 + a_1(v)v_1 + a_2(v)v_2,$$

where the coefficients a_i are defined as follows

$$\begin{aligned}
a_0(v) &:= \frac{\int\limits_{\Gamma^*} v^{(1)}\, d\mathcal{H}^2}{\int\limits_{\Gamma^*} v_0^{(1)}\, d\mathcal{H}^2} \\
a_1(v) &:= \frac{\left\langle v^{(1)}, v_1^{(1)} \right\rangle_{L_2(\Gamma^*)} + \frac{1}{\sin(\alpha^*)^2} \int\limits_{\partial\Gamma^*} v^{(2)} v_1^{(2)}\, d\mathcal{H}^1}{\left\langle v_1^{(1)}, v_1^{(1)} \right\rangle_{L_2(\Gamma^*)} + \frac{1}{\sin(\alpha^*)^2} \int\limits_{\partial\Gamma^*} v_1^{(2)} v_1^{(2)}\, d\mathcal{H}^1} \\
a_2(v) &:= \frac{\left\langle v^{(1)}, v_2^{(1)} \right\rangle_{L_2(\Gamma^*)} + \frac{1}{\sin(\alpha^*)^2} \int\limits_{\partial\Gamma^*} v^{(2)} v_2^{(2)}\, d\mathcal{H}^1}{\left\langle v_2^{(1)}, v_2^{(1)} \right\rangle_{L_2(\Gamma^*)} + \frac{1}{\sin(\alpha^*)^2} \int\limits_{\partial\Gamma^*} v_2^{(2)} v_2^{(2)}\, d\mathcal{H}^1}
\end{aligned}$$

with v_0, v_1 and v_2 as the elements from (6.57) spanning the nullspace. This projection has the desired properties, which we will prove next.
Obviously $\mathcal{R}(P) = \mathcal{N}(A_0)$ since v_0, v_1 and v_2 span the nullspace of A_0. Moreover, $P|_{\mathcal{N}(A_0)} = \mathrm{Id}_{\mathcal{N}(A_0)}$ or equivalently $P^2 = P$, because $a_i(v_j) = \delta_{ij}$ for $i, j \in \{0, 1, 2\}$ as we calculate now.
First we consider a_0, where $a_0(v_0) = 1$ is obvious and $a_0(v_1) = a_0(v_2) = 0$ follows from

$$\begin{aligned}
\int_{\Gamma^*} v_1^{(1)}\, d\mathcal{H}^2 &= R^* \int_0^{\pi-\alpha^*} \int_0^{2\pi} \sin(\varphi) \sin(\vartheta)^2 d\varphi d\vartheta \\
&= R^* \int_0^{\pi-\alpha^*} \underbrace{\left(\int_0^{2\pi} \sin(\varphi) d\varphi \right)}_{=0} \sin(\vartheta)^2 d\vartheta = 0 \\
\int_{\Gamma^*} v_2^{(1)}\, d\mathcal{H}^2 &= R^* \int_0^{\pi-\alpha^*} \int_0^{2\pi} \cos(\varphi) \sin(\vartheta)^2 d\varphi d\vartheta \\
&= R^* \int_0^{\pi-\alpha^*} \underbrace{\left(\int_0^{2\pi} \cos(\varphi) d\varphi \right)}_{=0} \sin(\vartheta)^2 d\vartheta = 0.
\end{aligned}$$

Next, we trivially have $a_1(v_1) = a_2(v_2) = 1$ and $a_1(v_0) = a_1(v_2) = a_2(v_0) = a_2(v_1) = 0$ is due to

$$\begin{aligned}
\left\langle v_0^{(1)}, v_1^{(1)} \right\rangle_{L_2(\Gamma^*)} &= \int_{\Gamma^*} v_0^{(1)} v_1^{(1)}\, d\mathcal{H}^2 = R^* \int_0^{\pi-\alpha^*} \int_0^{2\pi} (1 + c_\alpha \cos(\vartheta)) \sin(\varphi) \sin(\vartheta)^2 d\varphi d\vartheta \\
&= R^* \int_0^{\pi-\alpha^*} (1 + c_\alpha \cos(\vartheta)) \underbrace{\left(\int_0^{2\pi} \sin(\varphi) d\varphi \right)}_{=0} \sin(\vartheta)^2 d\vartheta = 0 \\
\left\langle v_0^{(1)}, v_2^{(1)} \right\rangle_{L_2(\Gamma^*)} &= \int_{\Gamma^*} v_0^{(1)} v_2^{(1)}\, d\mathcal{H}^2 = R^* \int_0^{\pi-\alpha^*} \int_0^{2\pi} (1 + c_\alpha \cos(\vartheta)) \cos(\varphi) \sin(\vartheta)^2 d\varphi d\vartheta
\end{aligned}$$

$$
\begin{aligned}
&= R^* \int_0^{\pi-\alpha^*} (1 + c_\alpha \cos(\vartheta)) \underbrace{\left(\int_0^{2\pi} \cos(\varphi) d\varphi \right)}_{=0} \sin(\vartheta)^2 d\vartheta = 0 \\
\left\langle v_1^{(1)}, v_2^{(1)} \right\rangle_{L_2(\Gamma^*)} &= \int_{\Gamma^*} v_1^{(1)} v_2^{(1)} \, d\mathcal{H}^2 = R^* \int_0^{\pi-\alpha^*} \int_0^{2\pi} \sin(\varphi)\cos(\varphi)\sin(\vartheta)^3 d\varphi d\vartheta \\
&= R^* \int_0^{\pi-\alpha^*} \underbrace{\left(\int_0^{2\pi} \sin(\varphi)\cos(\varphi) d\varphi \right)}_{=0} \sin(\vartheta)^3 d\vartheta = 0 \\
\left\langle v_2^{(1)}, v_1^{(1)} \right\rangle_{L_2(\Gamma^*)} &= \left\langle v_1^{(1)}, v_2^{(1)} \right\rangle_{L_2(\Gamma^*)} = 0
\end{aligned}
$$

as well as

$$
\begin{aligned}
\int_{\partial\Gamma^*} v_0^{(2)} v_1^{(2)} \, d\mathcal{H}^1 &= \int_0^{2\pi} (1 + c_\alpha \cos(\vartheta)) \cos(\varphi) \sin(\vartheta) R^* \sin(\alpha^*) d\varphi \Big|_{\vartheta=\pi-\alpha^*} \\
&= R^*(1 - c_\alpha \cos(\alpha^*)) \sin(\alpha^*)^2 \int_0^{2\pi} \cos(\varphi) d\varphi = 0 \\
\int_{\partial\Gamma^*} v_0^{(2)} v_2^{(2)} \, d\mathcal{H}^1 &= \int_0^{2\pi} (1 + c_\alpha \cos(\vartheta)) \sin(\varphi) \sin(\vartheta) R^* \sin(\alpha^*) d\varphi \Big|_{\vartheta=\pi-\alpha^*} \\
&= R^*(1 - c_\alpha \cos(\alpha^*)) \sin(\alpha^*)^2 \int_0^{2\pi} \sin(\varphi) d\varphi = 0 \\
\int_{\partial\Gamma^*} v_1^{(2)} v_2^{(2)} \, d\mathcal{H}^1 &= \int_0^{2\pi} \cos(\varphi) \sin(\vartheta) \sin(\varphi) \sin(\vartheta) R^* \sin(\alpha^*) d\varphi \Big|_{\vartheta=\pi-\alpha^*} \\
&= R^* \sin(\alpha^*)^3 \int_0^{2\pi} \cos(\varphi) \sin(\varphi) d\varphi = 0.
\end{aligned}
$$

Furthermore, $PA_0 = 0$ as one can see by

$$
\begin{aligned}
\int_{\Gamma^*} A_0 v^{(1)} \, d\mathcal{H}^2 &= \int_{\Gamma^*} \left(\Delta^B v^{(1)} - \fint_{\Gamma^*} \Delta^B v^{(1)} \, d\mathcal{H}^2 \right) d\mathcal{H}^2 \\
&= \int_{\Gamma^*} \Delta^B v^{(1)} \, d\mathcal{H}^2 - \left(\fint_{\Gamma^*} \Delta^B v^{(1)} \, d\mathcal{H}^2 \right) \int_{\Gamma^*} 1 \, d\mathcal{H}^2 = 0 \\
\left\langle (A_0 v)^{(1)}, v_1^{(1)} \right\rangle_{L_2(\Gamma^*)} &= \int_{\Gamma^*} (A_0 v)^{(1)} v_1^{(1)} \, d\mathcal{H}^2 \\
&= \int_{\Gamma^*} \left(\Delta^B v^{(1)} - \fint_{\Gamma^*} \Delta^B v^{(1)} \, d\mathcal{H}^2 \right) v_1^{(1)} \, d\mathcal{H}^2 \\
&= \int_{\Gamma^*} (\Delta^B v^{(1)}) v_1^{(1)} \, d\mathcal{H}^2 - \left(\fint_{\Gamma^*} \Delta^B v^{(1)} \, d\mathcal{H}^2 \right) \underbrace{\int_{\Gamma^*} v_1^{(1)} \, d\mathcal{H}^2}_{=0} \\
&= \int_{\Gamma^*} \nabla_{\Gamma^*} v^{(1)} \cdot \nabla_{\Gamma^*} v_1^{(1)} - |\sigma^*|^2 v^{(1)} v_1^{(1)} \, d\mathcal{H}^2 \\
&- \int_{\partial\Gamma^*} (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v^{(1)}) v_1^{(2)} \, d\mathcal{H}^1 \\
&= - \int_{\Gamma^*} v^{(1)} \underbrace{\Delta^B v_1^{(1)}}_{=0} \, d\mathcal{H}^2 - \int_{\partial\Gamma^*} (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v^{(1)}) v_1^{(2)} \, d\mathcal{H}^1 \\
&+ \int_{\partial\Gamma^*} (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v_1^{(1)}) v^{(2)} \, d\mathcal{H}^1 \\
&= \int_{\partial\Gamma^*} (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v_1^{(1)}) v^{(2)} - (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} v^{(1)}) v_1^{(2)} \, d\mathcal{H}^1 \\
&= \int_{\partial\Gamma^*} \Bigg(\cot(\alpha^*) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*}) v_1^{(2)} + \frac{b}{\sin(\alpha^*)} (v_1^{(2)})_{\sigma\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \frac{b\varkappa_{\partial D^*}}{\sin(\alpha^*)} \left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \right\rangle v_1^{(2)} \Big) v^{(2)} \, d\mathcal{H}^1 \\
&+ \frac{1}{\sin(\alpha^*)^2} \int_{\partial\Gamma^*} \underbrace{(A_0 v_1)^{(2)}}_{=0} v^{(2)} \, d\mathcal{H}^1 \\
&- \int_{\partial\Gamma^*} \Big(\cot(\alpha^*) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*}) v^{(2)} + \frac{b}{\sin(\alpha^*)} v^{(2)}_{\sigma\sigma} \\
&\qquad - \frac{b\varkappa_{\partial D^*}}{\sin(\alpha^*)} \left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \right\rangle v^{(2)} \Big) v_1^{(2)} \, d\mathcal{H}^1 \\
&- \frac{1}{\sin(\alpha^*)^2} \int_{\partial\Gamma^*} (A_0 v)^{(2)} v_1^{(2)} \, d\mathcal{H}^1 \\
&= - \int_{\partial\Gamma^*} \frac{b}{\sin(\alpha^*)} (v_1^{(2)})_\sigma v_\sigma^{(2)} \, d\mathcal{H}^1 + \int_{\partial\Gamma^*} \frac{b}{\sin(\alpha^*)} v_\sigma^{(2)} (v_1^{(2)})_\sigma \, d\mathcal{H}^1 \\
&- \frac{1}{\sin(\alpha^*)^2} \int_{\partial\Gamma^*} (A_0 v)^{(2)} v_1^{(2)} \, d\mathcal{H}^1 \\
&= \frac{-1}{\sin(\alpha^*)^2} \int_{\partial\Gamma^*} (A_0 v)^{(2)} v_1^{(2)} \, d\mathcal{H}^1
\end{aligned}
$$

and the analogous calculation $\left\langle A_0 v^{(1)}, v_2^{(1)} \right\rangle_{L_2(\Gamma^*)} = \frac{-1}{\sin(\alpha^*)^2} \int_{\partial\Gamma^*} (A_0 v)^{(2)} v_2^{(2)} \, d\mathcal{H}^1$. This shows $PA_0 = 0 (= A_0 P)$ and having found this projection we completed the prove of assumption (c).

The last assumption we have to check for Theorem 6.1 is (d). Here we will see that the eigenvalues of A_0 can be traced back to the eigenvalues of the operator Δ^B. Since we want to show that $\sigma(A_0) \setminus \{0\}$ is contained in the complex right half-plane, we can ignore eigenfunctions corresponding to the eigenvalue 0. Assume that (λ, u) is an eigenpair of A_0 with $\lambda \neq -\frac{2}{R^{*2}} = -|\sigma^*|^2$. Then we first remark that it is not possible for $u^{(1)}$ to be constant, since otherwise

$$
(A_0 u)^{(1)} = \Delta^B u^{(1)} - \fint_{\Gamma^*} \underbrace{\Delta^B u^{(1)}}_{\text{constant}} \, d\mathcal{H}^2 = \Delta^B u^{(1)} - \Delta^B u^{(1)} \underbrace{\fint_{\Gamma^*} 1 \, d\mathcal{H}^2}_{=1} = 0
$$

and u would correspond to the eigenvalue 0, which is not considered.
Due to $\lambda \neq -|\sigma^*|^2$ the constant

$$
c(\lambda, u) := \begin{pmatrix} \frac{1}{\lambda + |\sigma^*|^2} \fint_{\Gamma^*} \Delta^B u^{(1)} \, d\mathcal{H}^2 \\ 0 \end{pmatrix}
$$

is well-defined and the function $\widetilde{u} := u + c(\lambda, u) \not\equiv 0$ is an eigenfunction of Δ^B, as one can see from

$$
\begin{aligned}
\Delta^B \widetilde{u}^{(1)} &= \Delta^B u^{(1)} + \Delta^B c(\lambda, u) = \Delta^B u^{(1)} - \underbrace{\Delta_{\Gamma^*} c(\lambda, u)}_{=0} - |\sigma^*|^2 c(\lambda, u) \\
&= \Delta^B u^{(1)} - \frac{|\sigma^*|^2}{\lambda + |\sigma^*|^2} \fint_{\Gamma^*} \Delta^B u^{(1)} \, d\mathcal{H}^2 \\
&+ \frac{\lambda}{\lambda + |\sigma^*|^2} \fint_{\Gamma^*} \Delta^B u^{(1)} \, d\mathcal{H}^2 - \frac{\lambda}{\lambda + |\sigma^*|^2} \fint_{\Gamma^*} \Delta^B u^{(1)} \, d\mathcal{H}^2
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta^B u^{(1)} - \fint_{\Gamma^*} \Delta^B u^{(1)} \, d\mathcal{H}^2 + \lambda c(\lambda, u) \\
&= A_0 u^{(1)} + \lambda c(\lambda, u) = \lambda u^{(1)} + \lambda c(\lambda, u) = \lambda \widetilde{u}^{(1)}.
\end{aligned}
$$

Obviously, the second component of $\Delta^B \widetilde{u}$ does not change compared to $\Delta^B u$. This argument does not work for $\lambda = -\frac{2}{R^{*2}}$. Therefore we have shown

$$
\sigma(A_0) \subseteq \sigma(\Delta^B) \cup \left\{ -\frac{2}{R^{*2}} \right\}. \tag{6.64}
$$

Remember that we have already proven some statements concerning the eigenvalues of Δ^B. For example we saw in (6.26) that all eigenvalues of Δ^B are real. Since also $-\frac{2}{R^{*2}}$ is in $\mathbb{R}$, all eigenvalues of A_0 are real. With this knowledge the proof of assumption (d) relies on the following argument:
If one real eigenvalue of A_0 would change its sign while varying the parameters (a, b), it would also become 0 at some point. But this would cause $\mathcal{N}(A_0)$ to be higher-dimensional than before. We have already seen that independent of the choice of $a > -1$ and $b > C_{crit}$ the nullspace $\mathcal{N}(A_0)$ is always 3-dimensional. For this reason $\sigma(A_0) \setminus \{0\} \subseteq \mathbb{R}_+ \subseteq \mathbb{C}_+$ has to hold as long as the varied parameters do not violate the condition $a > -1$ and $b > C_{crit}$.

So the strategy to prove (d) will be as follows:

1. Show that the eigenvalues of A_0 depend continuously on the parameters a and b.
2. Find a particular parameter setting (a_0, b_0), where we can easily show that the spectrum of A_0 is contained in $[0, \infty)$.
3. Starting from the particular setting (a_0, b_0), vary the parameters to cover a wider parameter range.

We start by showing the continuous dependence of the eigenvalues on (a, b). Obviously, $\cos(\alpha^*)$, $\sin(\alpha^*)$ and R^* depend continuously on the parameters $a > -1$ and $b > 0$ and so do all coefficients appearing in A_0 and hence also A_0 itself. Therefore we can show

$$
A_0(\widetilde{a}, \widetilde{b}) \xrightarrow[(\widetilde{a},\widetilde{b}) \to (a,b)]{} A_0(a, b) \qquad \text{in } \mathcal{L}(X_1, X_0),
$$

where X_1 is equipped with the graph norm $\|A_0 x\|_{X_0} + \|x\|_{X_1}$. Lemma A.3.1 from [Lun95] shows that

$$
(\lambda \operatorname{Id} - A(\widetilde{a}, \widetilde{b}))^{-1} \xrightarrow[(\widetilde{a},\widetilde{b}) \to (a,b)]{} (\lambda \operatorname{Id} - A(a, b))^{-1} \qquad \text{in } \mathcal{L}(X_0).
$$

Using Theorem 2.25 of [Kat95] we see that $A_0(\widetilde{a}, \widetilde{b}) \xrightarrow[(\widetilde{a},\widetilde{b}) \to (a,b)]{} A_0(a, b)$ in the generalized sense (cf. IV-§ 2 in [Kat95]). In doing so it is important to remark that A_0 is closed, because the resolvent set is not empty. Section IV-§ 3.5 of [Kat95] shows that each finite system of eigenvalues depends continuously on (a, b). We saw in Remark 6.3 that all eigenvalues of Δ^B are isolated and one possible new eigenvalue does not change this fact for A_0. After every eigenvalue of A_0 is isolated, the one-element set $\{\lambda_i\}$ forms such a finite system and therefore depends continuously on the parameters (a, b). This completes the first part of our strategy towards assumption (d).

Now we search for a situation, where we can easily compute the eigenvalues of A_0. We find this in the halfsphere. We choose an arbitrary $a_0 > 0$. By (6.6) we know that for this choice of a_0 an angle $\cos(\alpha^*) = 0$ is always possible. For the moment the parameter $b_0 > 0$ could be chosen arbitrarily since $\cos(\alpha^*) = 0$ simplifies $b_0 > C_{crit}$ to $b_0 > 0$, but for later purpose we choose $b_0 \in (0, 1)$. We set $r^* = \frac{b_0}{a_0}$ and obtain a stationary halfsphere. The reason why we choose Γ^* to be the halfsphere is that by its reflection along the x-y-plane, called $-\Gamma^*$, the resulting surface $\Gamma^* \cup -\Gamma^*$ is smooth.
Due to (6.64) the eigenvalue problem we have to solve is

$$\lambda\varrho = \Delta^B \varrho = \begin{pmatrix} -\frac{1}{R^{*2}\sin(\vartheta)^2}\varrho^{(1)}_{\varphi\varphi} - \frac{1}{R^{*2}}\varrho^{(1)}_{\vartheta\vartheta} - \frac{1}{R^{*2}}\cot(\vartheta)\varrho^{(1)}_{\vartheta} - \frac{2}{R^{*2}}\varrho^{(1)} \\ \frac{1}{R^*}\varrho^{(1)}_{\vartheta}(\pi - \alpha^*) - \frac{b_0}{R^{*2}}(\varrho^{(2)}_{\varphi\varphi} + \varrho^{(2)}) \end{pmatrix}, \tag{6.65}$$

where we have to impose 2π-periodicity in φ and continuity for $\vartheta = 0$. To avoid unnecessary terms we multiply by R^{*2}, add 2ϱ and obtain

$$(R^{*2}\lambda + 2)\varrho = \begin{pmatrix} -\frac{1}{\sin(\vartheta)^2}\varrho^{(1)}_{\varphi\varphi} - \varrho^{(1)}_{\vartheta\vartheta} - \cot(\vartheta)\varrho^{(1)}_{\vartheta} \\ R^*\varrho^{(1)}_{\vartheta}(\pi - \alpha^*) - b_0(\varrho^{(2)}_{\varphi\varphi} + \varrho^{(2)}) + 2\varrho^{(2)} \end{pmatrix}.$$

Then we substitute $\mu := R^{*2}\lambda + 2$ and search for all values μ can attain. Having a reflectional symmetric Γ^* is important but not enough. We also need smoothly reflectable eigenfunctions, i.e. eigenfunctions with $\varrho_\vartheta|_{\vartheta=\pi-\alpha^*} = 0$. To achieve this we have to introduce one more parameter $d \in [0, 1]$ and solve

$$\begin{pmatrix} \mu\varrho^{(1)} \\ d\mu\varrho^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sin(\vartheta)^2}\varrho^{(1)}_{\varphi\varphi} - \varrho^{(1)}_{\vartheta\vartheta} - \cot(\vartheta)\varrho^{(1)}_{\vartheta} \\ R^*\varrho^{(1)}_{\vartheta}(\pi - \alpha^*) - db_0(\varrho^{(2)}_{\varphi\varphi} + \varrho^{(2)}) + 2d\varrho^{(2)} \end{pmatrix} =: \Delta^d \varrho \tag{6.66}$$

on the halfsphere Γ^*, where the domain of Δ^d is given by X_1. For $d = 0$ this reads as

$$\mu\varrho^{(1)} = -\frac{1}{\sin(\vartheta)^2}\varrho^{(1)}_{\varphi\varphi} - \varrho^{(1)}_{\vartheta\vartheta} - \cot(\vartheta)\varrho^{(1)}_{\vartheta}$$

with the boundary condition $\varrho^{(1)}_{\vartheta}(\pi - \alpha^*) = 0$. Together with the 2π-periodicity in φ and the continuity for $\vartheta = 0$ we see that any solution of this problem on the halfsphere Γ^* can be smoothly reflected to a solution of

$$\mu\varrho^{(1)} = -\frac{1}{\sin(\vartheta)^2}\varrho^{(1)}_{\varphi\varphi} - \varrho^{(1)}_{\vartheta\vartheta} - \cot(\vartheta)\varrho^{(1)}_{\vartheta}$$

on the full sphere $\Gamma^* \cup -\Gamma^*$, with periodicity in φ and continuity for $\vartheta = 0$ and $\vartheta = \pi$. Yet, this eigenvalue problem for the Laplace operator on the sphere is already well studied by different authors - for example by [CH68], [Tri72] or chapter XIII in [Jän01]. As each of these sources shows, the eigenvalues of this equation are given as $k(k+1)$ for $k \in \mathbb{N}$. Thus $\mu_k = k(k+1)$ and for λ_k we have the equation $(R^{*2}\lambda_k + 2) = k(k+1)$, which leads to

$$\lambda_k = \frac{k(k+1)-2}{R^{*2}} \qquad \text{for every } k \in \mathbb{N}. \tag{6.67}$$

Obviously, we see $\lambda_k \geq 0$ for $k \geq 1$ and the only eigenvalue that could cause a problem is $\lambda_0 = -\frac{2}{R^{*2}}$. We will see later that although $\lambda_0 = -\frac{2}{R^{*2}}$ is a possible eigenvalue of Δ^B it is not possible as eigenvalue for A_0.

Now we want to increase the parameter d from 0 to 1. We will need the continuous dependence of the eigenvalues on d to argue that while increasing d no eigenvalue can change its sign. This is again due to the three dimensionality of the nullspace. Although we have not included the weight d into the considerations concerning the nullspace previously in this section, the calculations do not change dramatically and we also get that the nullspace is always 3-dimensional for all $d \in [0,1]$. Therefore the continuous dependence of the eigenvalues on d is the next ingredient that we are going to prove.

With

$$(\Delta^d + c\,\mathrm{Id})^{-1} : H(d) := L_2(\Gamma^*) \times L_2(\partial\Gamma^*) \longrightarrow H(d)$$

we denote the inverse operator of $\Delta^d + c\,\mathrm{Id}$, where $H(d)$ shall be equipped with the inner product $\langle u, v\rangle_{H(d)} := \left\langle u^{(1)}, v^{(1)}\right\rangle_{L_2(\Gamma^*)} + d\left\langle u^{(2)}, v^{(2)}\right\rangle_{L_2(\partial\Gamma^*)}$. Moreover, we assume that c is large enough to guarantee that all eigenvalues are positive. Since we only want to show the continuous dependence of the eigenvalues, we do not care for shifts of the operator and the resulting shift of the spectrum. We consider the inverse operator since its spectrum is bounded, which will be important later on. Assuming that we have a solution ϱ of the equation (6.66) we get

$$\begin{aligned}
&\mu\left\langle \varrho^{(1)}, \varrho^{(1)}\right\rangle_{L_2(\Gamma^*)} + d\mu\left\langle \varrho^{(2)}, \varrho^{(2)}\right\rangle_{L_2(\partial\Gamma^*)} \\
&\quad= \left\langle \mu\varrho^{(1)}, \varrho^{(1)}\right\rangle_{L_2(\Gamma^*)} + \left\langle d\mu\varrho^{(2)}, \varrho^{(2)}\right\rangle_{L_2(\partial\Gamma^*)} \\
&\quad= \left\langle (\Delta^d\varrho)^{(1)}, \varrho^{(1)}\right\rangle_{L_2(\Gamma^*)} + \left\langle (\Delta^d\varrho)^{(2)}, \varrho^{(2)}\right\rangle_{L_2(\partial\Gamma^*)} \\
&\quad= \int_{\Gamma^*} -(\Delta_{\Gamma^*}\varrho^{(1)})\varrho^{(1)}\, d\mathcal{H}^2 + \int_{\partial\Gamma^*}\left(-(n_{\partial\Gamma^*}\cdot\nabla_{\Gamma^*}\varrho^{(1)}) + db_0\varrho^{(2)}_{\sigma\sigma} + d\frac{b_0}{R^{*2}}\varrho^{(2)}\right)\varrho^{(2)}\, d\mathcal{H}^1 \\
&\quad= \int_{\Gamma^*}\left\|\nabla_{\Gamma^*}\varrho^{(1)}\right\|^2 d\mathcal{H}^2 - \int_{\partial\Gamma^*} db_0(\varrho^{(2)}_\sigma)^2 + d\frac{b_0}{R^{*2}}(\varrho^{(2)})^2\, d\mathcal{H}^1.
\end{aligned} \tag{6.68}$$

If we denote the eigenvalues of Δ^{-d} by ν, this can be rewritten as

$$\nu = \frac{\left\langle \Delta^{-d}\varrho, \varrho\right\rangle_{H(1)}}{\langle\varrho, \varrho\rangle_{H(d)}}. \tag{6.69}$$

This representation is all we need to apply Courant's maximum-minimum principle (cf. chapter VII §1.4 in [CH68]) to see that for a fixed d the eigenvalues $\nu_k(d)$ can be written as

$$\nu_k(d) = \max_{W\in\Sigma_k}\min_{\varrho\in W\setminus\{0\}} \frac{\left\langle \Delta^{-d}\varrho, \varrho\right\rangle_{H(1)}}{\langle\varrho, \varrho\rangle_{H(d)}},$$

where Σ_k denotes the set of all k-dimensional subspaces of $H(d)$. Now we want to sketch the continuous dependence of $\nu_k(d)$ on d. With $E_k(d) = \operatorname{span}\{\varrho_1(d), \dots, \varrho_k(d)\}$ as the span of the first k eigenfunctions, we estimate

$$\nu_k(d_1) - \nu_k(d_2) \geq \min_{\varrho \in E_k(d_2)\setminus\{0\}} \frac{\left\langle \Delta^{-d_1}\varrho, \varrho \right\rangle_{H(1)}}{\langle \varrho, \varrho \rangle_{H(d_1)}} - \min_{\varrho \in E_k(d_2)\setminus\{0\}} \frac{\left\langle \Delta^{-d_2}\varrho, \varrho \right\rangle_{H(1)}}{\langle \varrho, \varrho \rangle_{H(d_2)}},$$

since the second maximum is attained exactly for $E_k(d_2)$ and the first summand gets smaller if we consider this particular choice. Then we are able to choose $\widehat{\varrho} \in E_k(d_2)$ with

$$\langle \widehat{\varrho}, \widehat{\varrho} \rangle_{H(d_1)} = 1 \tag{6.70}$$

such that the first minimum is attained and get

$$\nu_k(d_1) - \nu_k(d_2) \geq \left\langle \Delta^{-d_1}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)} - \frac{\left\langle \Delta^{-d_2}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)}}{\langle \widehat{\varrho}, \widehat{\varrho} \rangle_{H(d_2)}}.$$

This can be rewritten to

$$\begin{aligned}
\nu_k(d_1) - \nu_k(d_2) &\geq \left\langle \Delta^{-d_1}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)} - \frac{\left\langle \Delta^{-d_2}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)}}{\langle \widehat{\varrho}, \widehat{\varrho} \rangle_{H(d_2)}} \\
&\quad - \left\langle \Delta^{-d_2}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)} + \left\langle \Delta^{-d_2}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)} \\
&= \left\langle (\Delta^{-d_1} - \Delta^{-d_2})\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)} + \left(1 - \frac{1}{\langle \widehat{\varrho}, \widehat{\varrho} \rangle_{H(d_2)}}\right) \left\langle \Delta^{-d_2}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)}.
\end{aligned}$$

The appearing denominator can be written as

$$\begin{aligned}
\langle \widehat{\varrho}, \widehat{\varrho} \rangle_{H(d_2)} &= \left\langle \widehat{\varrho}^{(1)}, \widehat{\varrho}^{(1)} \right\rangle_{L_2(\Gamma^*)} + d_2 \left\langle \widehat{\varrho}^{(2)}, \widehat{\varrho}^{(2)} \right\rangle_{L_2(\partial\Gamma^*)} \\
&\quad + d_1 \left\langle \widehat{\varrho}^{(2)}, \widehat{\varrho}^{(2)} \right\rangle_{L_2(\partial\Gamma^*)} - d_1 \left\langle \widehat{\varrho}^{(2)}, \widehat{\varrho}^{(2)} \right\rangle_{L_2(\partial\Gamma^*)} \\
&= 1 + (d_2 - d_1) \left\langle \widehat{\varrho}^{(2)}, \widehat{\varrho}^{(2)} \right\rangle_{L_2(\partial\Gamma^*)}
\end{aligned}$$

and hence we end up with

$$\begin{aligned}
\nu_k(d_1) - \nu_k(d_2) &\geq \left\langle (\Delta^{-d_1} - \Delta^{-d_2})\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)} \\
&\quad + \left(1 - \frac{1}{1 + (d_2 - d_1) \langle \widehat{\varrho}^{(2)}, \widehat{\varrho}^{(2)} \rangle_{L_2(\partial\Gamma^*)}}\right) \left\langle \Delta^{-d_2}\widehat{\varrho}, \widehat{\varrho} \right\rangle_{H(1)}.
\end{aligned}$$

If we consider the limit $d_2 \longrightarrow d_1$, we first of all observe that the first term on the right-hand side converges to zero which can be see similar as in Chapter 2.3.1 of [Hen06]. It might be noteworthy that the proofs of Theorem 2.3.1 and Theorem 2.3.2 in [Hen06] contain two little mistakes: In the proof of Theorem 2.3.1 the minimum and maximum must be interchanged and in the proof of Theorem 2.3.2 equation (2.22) should estimate the norm $\|A_n\|_{\mathcal{L}(H^{-1}, H_0^1)}$ as this is used in the last line of the proof, but instead it estimates

$\|A_n\|_{\mathcal{L}(L_2,L_2)}$. But the argument previous to (2.22) also justifies this modification and the result remains unchanged. Then we immediately see that

$$\lim_{d_2\to d_1} \nu_k(d_1) - \nu_k(d_2) \geq 0$$

as long as $\left\langle \widehat{\varrho}^{(2)}, \widehat{\varrho}^{(2)} \right\rangle_{L_2(\partial\Gamma^*)}$ remains bounded independent of d. In fact, for an eigenfunction ϱ that satisfies (6.70), the equation (6.69) shows that

$$\left\langle \Delta^{-d_1}\varrho, \varrho \right\rangle_{H(1)} = \nu \left\langle \varrho, \varrho \right\rangle_{H(d_1)} = \nu \leq c < \infty,$$

because the eigenvalues of Δ^{-d_1} are bounded. Yet, controlling $\left\langle \Delta^{-d_1}\varrho, \varrho \right\rangle_{H(1)}$ is due to (6.68) equivalent to controlling the H^1-norm of $\varrho^{(1)}$, given by

$$\int_{\Gamma^*} \left\| \nabla_{\Gamma^*}\varrho^{(1)} \right\|^2 \, d\mathcal{H}^2$$

for all $d \in [0,1]$. Since

$$W_2^1(\Gamma^*) \xrightarrow{\gamma_0} W_2^{\frac{1}{2}}(\partial\Gamma^*) \hookrightarrow L_2(\partial\Gamma^*)$$

this also controls the $L_2(\partial\Gamma^*)$-norm of $\gamma_0\varrho^{(1)} = \varrho^{(2)}$, which is what we need. Interchanging the roles of d_1 and d_2, we also get the converse inequality

$$\lim_{d_2\to d_1} \nu_k(d_2) - \nu_k(d_1) \geq 0.$$

Thus $\lim\limits_{d_2\to d_1} \nu_k(d_2) = \nu_k(d_1)$ and we obtain the continuous dependence of ν_k and therefore also of μ_k on d.

We know that for $d = 0$ all but two eigenvalues are positive and independent of d and the nullspace is always 3-dimensional. If we now increase d from 0 to 1, which leads to Δ^B, no eigenvalue can change its sign. Hence all eigenvalues of Δ^B except 0 and λ_0 are positive in this halfsphere case.

We still have to exclude λ_0 for A_0. If we assume $\lambda_0 \in \sigma(A_0)$ and ϱ_0 to be an eigenfunction corresponding to λ_0, we obtain

$$\begin{aligned}
& & (A_0\varrho_0)^{(1)} &= -\frac{2}{{R^*}^2}\varrho_0^{(1)} \\
&\Rightarrow & \Delta^B\varrho_0^{(1)} - ⨍_{\Gamma^*} \Delta^B\varrho_0^{(1)} \, d\mathcal{H}^2 &= -\frac{2}{{R^*}^2}\varrho_0^{(1)} \\
&\Rightarrow & -\Delta_{\Gamma^*}\varrho_0^{(1)} - \frac{2}{{R^*}^2}\varrho_0^{(1)} - ⨍_{\Gamma^*} \Delta^B\varrho_0^{(1)} \, d\mathcal{H}^2 &= -\frac{2}{{R^*}^2}\varrho_0^{(1)} \\
&\Rightarrow & \Delta_{\Gamma^*}\varrho_0^{(1)} &= -⨍_{\Gamma^*} \Delta^B\varrho_0^{(1)} \, d\mathcal{H}^2 = const. \\
&\Rightarrow & \Delta_{\Gamma^*}\varrho_0^{(1)} &= ⨍_{\Gamma^*} \Delta_{\Gamma^*}\varrho_0^{(1)} + \frac{2}{{R^*}^2}\varrho_0^{(1)} \, d\mathcal{H}^2 \\
&\Rightarrow & \Delta_{\Gamma^*}\varrho_0^{(1)} &= \Delta_{\Gamma^*}\varrho_0^{(1)} \underbrace{⨍_{\Gamma^*} 1 \, d\mathcal{H}^2}_{=1} + \frac{2}{{R^*}^2} ⨍_{\Gamma^*} \varrho_0^{(1)} \, d\mathcal{H}^2 \\
&\Rightarrow & \frac{2}{{R^*}^2} ⨍_{\Gamma^*} \varrho_0^{(1)} \, d\mathcal{H}^2 &= 0 \\
&\Rightarrow & \int_{\Gamma^*} \varrho_0^{(1)} \, d\mathcal{H}^2 &= 0.
\end{aligned}$$

This shows that an eigenfunction ϱ_0 would satisfy $\Delta_{\Gamma^*}\varrho_0^{(1)} = c$ and $\int\limits_{\Gamma^*} \varrho_0^{(1)}\, d\mathcal{H}^2 = 0$. This can be used to calculate

$$\begin{aligned} 0 = c\int_{\Gamma^*} \varrho_0^{(1)}\, d\mathcal{H}^2 &= \int_{\Gamma^*} \varrho_0^{(1)} \Delta_{\Gamma^*}\varrho_0^{(1)}\, d\mathcal{H}^2 \\ &= -\int_{\Gamma^*} \nabla_{\Gamma^*}\varrho_0^{(1)} \cdot \nabla_{\Gamma^*}\varrho_0^{(1)}\, d\mathcal{H}^2 + \int_{\partial\Gamma^*} (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho_0^{(1)})\varrho_0^{(1)}\, d\mathcal{H}^2. \end{aligned}$$

This can be written as

$$\left\|\nabla_{\Gamma^*}\varrho_0^{(1)}\right\|^2_{L_2(\Gamma^*)} = \int_{\partial\Gamma^*} (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho_0^{(1)})\varrho_0^{(1)}\, d\mathcal{H}^2.$$

Utilizing the so far unused second component of $A_0\varrho_0$ we get

$$\begin{aligned} \frac{2}{R^{*2}}\varrho_0^{(2)} &= -(\lambda_0\varrho_0)^{(2)} = -(A_0\varrho_0)^{(2)} \\ &= -(n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho_0^{(1)}) + b_0(\varrho_0^{(2)})_{\sigma\sigma} + \frac{b_0}{R^{*2}}\varrho_0^{(2)} \end{aligned}$$

or equivalently

$$(n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*}\varrho_0^{(1)}) = -\frac{2}{R^{*2}}\varrho_0^{(2)} + b_0(\varrho_0^{(2)})_{\sigma\sigma} + \frac{b_0}{R^{*2}}\varrho_0^{(2)} = \frac{b_0-2}{R^{*2}}\varrho_0^{(2)} + b_0(\varrho_0^{(2)})_{\sigma\sigma}.$$

This can be used to transform the calculation before into

$$\begin{aligned} \left\|\nabla_{\Gamma^*}\varrho_0^{(1)}\right\|^2_{L_2(\Gamma^*)} &= \int_{\partial\Gamma^*} \frac{b_0-2}{R^{*2}}(\varrho_0^{(2)})^2\, d\mathcal{H}^1 + \int_{\partial\Gamma^*} b_0(\varrho_0^{(2)})_{\sigma\sigma}\varrho_0^{(2)}\, d\mathcal{H}^1 \\ &= \frac{b_0-2}{R^{*2}}\left\|\varrho_0^{(2)}\right\|_{L_2(\partial\Gamma^*)} - b_0\int_{\partial\Gamma^*} (\varrho_0^{(2)})^2_\sigma\, d\mathcal{H}^1 \end{aligned}$$

and finally end up with

$$\left\|\nabla_{\Gamma^*}\varrho_0^{(1)}\right\|^2_{L_2(\Gamma^*)} + b_0\left\|(\varrho_0^{(2)})_\sigma\right\|^2_{L_2(\partial\Gamma^*)} = \frac{b_0-2}{R^{*2}}\left\|\varrho_0^{(2)}\right\|_{L_2(\partial\Gamma^*)}. \tag{6.71}$$

Here we reached the point where the choice $b_0 \in (0,1)$ is paying off. Since the numerator is negative, the right-hand side itself is negative. This leads again to a contradiction and shows that $\lambda_0 = -\frac{2}{R^{*2}}$ is not an eigenvalue of A_0. Thus we found the "easy" situation, where every non-zero eigenvalue of A_0 is positive and can come to the last step for proving assumption (d).

Now we can vary the parameters starting from (a_0, b_0) to cover a wide range, where the eigenvalues are positive. We start by noting that all the coefficients appearing in A_0 will not degenerate, because $R^* \neq 0$ and $\sin(\alpha^*) \neq 0$. As we said before the only important restriction comes from the 3-dimensionality of the nullspace $\mathcal{N}(A_0)$. We saw that we can guarantee this dimension as long as

$$b > C_{crit} = -\frac{1}{3}R^*\sin(\alpha^*)^2\cos(\alpha^*) = -\frac{r^*}{3}\sqrt{1-\left(\frac{b}{r^*}-a\right)^2}\left(\frac{b}{r^*}-a\right).$$

This varying process will require several steps and Figure 9 is visualizing the upcoming situation.

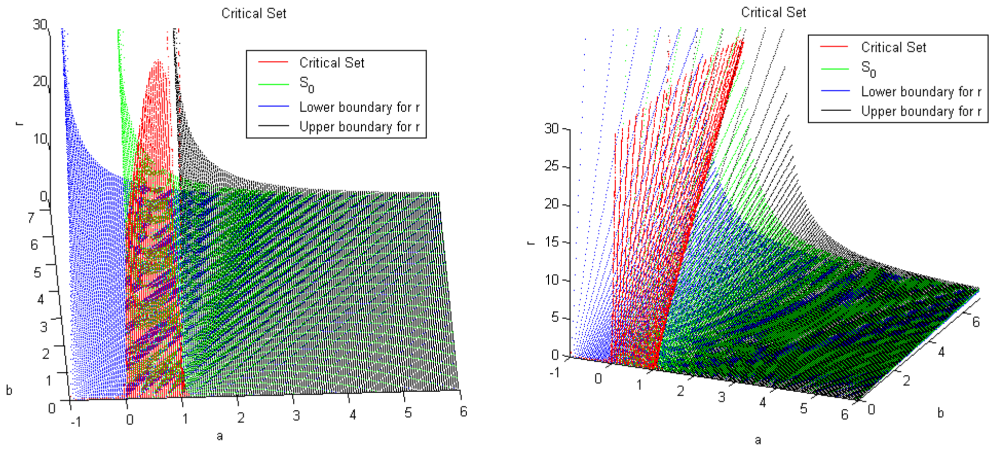

Figure 9: Critical parameter set

First we consider the set corresponding to SSCs with $\cos(\alpha^*) = 0$ given by

$$S_0 := \left\{ (a, b, r^*) \in \mathbb{R}^3 \,\middle|\, a > 0, b > 0, \frac{b}{r^*} = a \right\} = \left\{ \left(a, b, \frac{b}{a}\right) \in \mathbb{R}^3 \,\middle|\, a > 0, b > 0 \right\}.$$

Let $(a_1, b_1) \in (0, \infty) \times (0, \infty)$ be arbitrary and consider the variation

$$(a(t), b(t)) : [0, 1] \longrightarrow (0, \infty) \times (0, \infty) : t \longmapsto (a_0 + t(a_1 - a_0), b_0 + t(b_1 - b_0)).$$

We know that for $\left(a_0, b_0, \frac{b_0}{a_0}\right) \in S_0$ as above the eigenvalues of $A_0(a_0, b_0)$ are all positive and S_0 does not intersect the critical set

$$S_{crit} := \left\{ (a, b, r^*) \in \mathbb{R}^3 \,|\, a > -1, b > 0, r^* \in I_r, b \leq C_{crit} \right\}.$$

Thus the eigenvalues remain positive for all $(a(t), b(t))$ with $t \in [0, 1]$.
Now we consider the SSCs corresponding to $\cos(\alpha^*) > 0$ given by

$$S_+ := \left\{ (a, b, r^*) \in \mathbb{R}^3 \,\middle|\, a > -1, b > 0, \frac{b}{r^*} > a \right\}.$$

Now let $(a_1, b_1, r_1) \in S_+$ be arbitrary and use $b_0 := b_1$ and $a_0 := \frac{b_1}{r_1}$ as a starting point. Then $(a_0, b_0, r_1) = \left(a_0, b_0, \frac{b_0}{a_0}\right) \in S_0$ and therefore the eigenvalues of $A_0(a_0, b_0)$ are positive. While decreasing a_0 to a_1 - which is equivalent to increasing $\cos(\alpha^*)$ from 0 to some positive value - it is still not possible to intersect S_{crit}, since S_{crit} only allows for $\cos(\alpha^*) < 0$. Hence the eigenvalues remain also positive for this variation. This especially covers all cases where $a \leq 0$.
Finally we want to cover all the cases that are left over. For this define the set of all surfaces with $\cos(\alpha^*) < 0$ as

$$S_- := \left\{ (a, b, r^*) \in \mathbb{R}^3 \,\middle|\, a > 0, b > 0, \frac{b}{r^*} < a \right\}$$

and let $(a_2, b_2, r_2) \in S_-$ be given and satisfy

$$b_2 > -\frac{r_2}{3} \sqrt{1 - \left(\frac{b_2}{r_2} - a_2\right)^2} \left(\frac{b_2}{r_2} - a_2\right).$$

Again we try to find a path that connects $(a_2, b_2, r_2) \in S_-$ with a configuration, where we know that all eigenvalues are positive. We remark that due to $(a_2, b_2, r_2) \in S_-$ we know that $r_2 > \frac{b_2}{a_2}$. Decreasing r_2 to $\frac{b_2}{a_2}$ brings us to a configuration in S_0, where we have only positive eigenvalues. During this decreasing process it is not possible that

$$b_2 > -\frac{r_2}{3}\sqrt{1-\left(\frac{b_2}{r_2}-a_2\right)^2}\left(\frac{b_2}{r_2}-a_2\right) = \frac{1}{3}\sqrt{1-\left(\frac{b_2}{r_2}-a_2\right)^2}(a_2r_2-b_2)$$

gets violated, since $\sqrt{1-\left(\frac{b_2}{r_2}-a_2\right)^2} \geq 0$ and $a_2r_2 - b_2$ is decreasing with r_2. This shows that the positivity of the eigenvalues is also valid for (a_2, b_2, r_2). Hence assumption (d) of Theorem 6.1 is satisfied for all SSCs and parameters $(a, b) \in (-1, \infty) \times (0, \infty)$ that satisfy $b > C_{crit}$.

After we checked all assumptions required for the GPLS, we finally apply Theorem 6.1 and obtain the last result of this thesis.

Theorem 6.14 (Stability of spherical caps): *Let $a > -1$, $b > 0$ and $4 < p < \infty$. Moreover, assume Γ^* to be a stationary spherical cap with radius R^* and contact angle α^* that satisfies $b > -\frac{1}{3}R^* \sin(\alpha^*)^2 \cos(\alpha^*)$. Then $\varrho \equiv 0$ is stable in*

$$\widetilde{X} := \left\{ \varrho \in W_p^{2-\frac{2}{p}}(\Gamma^*) \,\middle|\, \varrho|_{\partial\Gamma^*} \in W_p^{3-\frac{3}{p}}(\partial\Gamma^*) \right\}$$

and there exists $\delta > 0$ such that the unique solution $\varrho(t)$ from Theorem 3.22 of the system (2.29)-(2.30) with initial value $\varrho_0 \in \widetilde{X}$ satisfying $\|\varrho_0\|_{W_p^{2-\frac{2}{p}}(\Gamma^*)} + \|\varrho_0|_{\partial\Gamma^*}\|_{W_p^{3-\frac{3}{p}}(\partial\Gamma^*)} < \delta$ exists on $\mathbb{R}^+$ and converges at an exponential rate to some ϱ_∞, which parametrizes a stationary spherical cap as well.

Proof: Reformulating the statement of Theorem 6.1 to the specific case of SCs as presented in this section. ■

List of Figures

List of Notations

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