Sharp Interface Limits for Diffuse Interface Models

Dissertation zur Erlangung des Doktorgrades der
Naturwissenschaften (Dr.rer.nat.)
an der Fakultät für Mathematik
der Universität Regensburg

vorgelegt von

Stefan Schaubeck
aus
Aiglsbach

Juni 2013

Promotionsgesuch eingereicht am: 27.06.2013

Die Arbeit wurde angeleitet von Prof. Dr. H. Abels.

Prüfungsausschuss:

Vorsitzender:	Prof. Dr. Bernd Ammann
1. Gutachter:	Prof. Dr. Helmut Abels
2. Gutachter:	Prof. Dr. Harald Garcke
weiterer Prüfer:	Prof. Dr. Georg Dolzmann

Abstract:
In this thesis we rigorously prove that the Cahn-Larché system with mobility constant $m(\epsilon) \equiv const.$ converges to a modified Hele-Shaw problem as $\epsilon \searrow 0$ where ϵ describes the thickness of the interfacial region. For the proof we construct an approximate solution of the Cahn-Larché system by the method of matched asymptotic expansions. Then we can show that the approximate solutions of Cahn-Larché system converge to the solution of the modified Hele-Shaw problem as $\epsilon \searrow 0$.
For the modified Hele-Shaw problem we prove the existence of a classical solution in a sufficiently small time interval $[0, T]$. By reducing the system to a single evolution equation for the distance function, we show the assertion. Furthermore, we prove an existence result for classical solution to a linearized Hele-Shaw problem used in the higher order expansions.
By the same methods as for the Cahn-Larché system we show the sharp interface limit of a convective Cahn-Hilliard equation with mobility constant $m(\epsilon) = \epsilon$ to an evolution equation for the interface $\Gamma(t)$. Here and for the Cahn-Larché system the main problem is the construction of the approximate solutions.
Finally, we obtain that the surface tension term $-\epsilon \mathrm{div}\,(\nabla c^\epsilon \otimes \nabla c^\epsilon)$ in the "model H" with mobility constant $m(\epsilon) = \epsilon^\theta$, $\theta > 3$, does generally not converge to the mean curvature of the interface.

Contents

1 Introduction

The subject of the present work is the study of sharp interface limits of so-called diffuse interface models. Diffuse interface models describe phase separations and allow a partial mixing of two separated phases in a thin interfacial region on a small length scale $\epsilon > 0$. Sharp interface limit means sending $\epsilon \searrow 0$, that is, the region of mixing becomes arbitrarily thin. More precisely we consider the Cahn-Larché system and a convective Cahn-Hilliard equation with different mobility constants $m(\epsilon)$ where the mobility constant is the inverse of the Peclet number and controls the strength of the diffusion. We rigorously prove that the Cahn-Larché system with mobility constant $m(\epsilon) \equiv$ *const.* and the convective Cahn-Hilliard equation with mobility constant $m(\epsilon) = \epsilon$ converge to certain sharp interface models. In sharp interface models the phases are separated by a surface of lower dimension. If the mobility constants tends to 0 too quickly as $\epsilon \searrow 0$, we can even show a non-convergence result. This is the case for mobility constants of the form $m(\epsilon) = \epsilon^\theta$ with $\theta > 3$.

Diffuse Interface Models: The Cahn-Larché system is a Cahn-Hilliard equation, which takes the elastic effects of the material into account. This model describes phase separation in binary alloys. For example, a different lattice structure of the mixture is a reason to consider elasticity in the Cahn-Hilliard model. We assume that the alloy consists of two components with concentration difference $c : \Omega \times (0, \infty) \to \mathbb{R}$. Here $\Omega \subset \mathbb{R}^d$, $d > 1$, is always a bounded domain with smooth boundary $\partial\Omega$. The elastic effects are described by the deformation vector $\mathbf{u} : \Omega \times (0, \infty) \to \mathbb{R}^d$. For small deformations it is sufficient to consider the linearised strain tensor

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right).$$

In our case the elastic free energy density is described by

$$W(c, \mathcal{E}(\mathbf{u})) = \frac{1}{2} \left(\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star c \right) : \mathcal{C} \left(\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star c \right),$$

where $\mathcal{C} = (\mathcal{C}_{iji'j'})_{i,j,i',j'=1,\dots,d}$ is the elasticity strain tensor and $\mathcal{E}^\star c$ is the stress free strain for concentration c with constant matrix $\mathcal{E}^\star \in \mathbb{R}^{d \times d}$. We require that $\mathcal{C}$ is symmetric and positive definite. This form of the elastic free energy is based on the work of Eshelby [31] and Khachaturyan [45]. Then the total energy of the system is given by $E(c, \mathbf{u}) = E_1(c) + E_2(c, \mathbf{u})$, where

$$E_1(c) \quad = \quad \frac{\epsilon}{2} \int_\Omega |\nabla c(x)|^2 \, dx + \frac{1}{\epsilon} \int_\Omega F(c(x)) \, dx \qquad (1.1)$$

is the Ginzburg-Landau energy and

$$E_2(c, \mathbf{u}) \quad = \quad \int_\Omega W(c(x), \mathcal{E}(\mathbf{u}(x)))\, dx \tag{1.2}$$

is the elastic free energy. Here $F(c)$ is a suitable "double-well" potential taking its global minimum 0 at ± 1, for example $F(c) = (1 - c^2)^2$. The chemical potential $\mu : \Omega \times (0, \infty) \to \mathbb{R}$ is introduced by the first variation of the total energy. Consequently, we consider the following so-called Cahn-Larché system

$$\partial_t c^\epsilon = \Delta \mu^\epsilon \qquad \text{in } \Omega \times (0, \infty), \tag{1.3}$$

$$\mu^\epsilon = \epsilon^{-1} f(c^\epsilon) - \epsilon \Delta c^\epsilon + W_{,c}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon)) \qquad \text{in } \Omega \times (0, \infty), \tag{1.4}$$

$$\operatorname{div} \mathcal{S}^\epsilon = 0 \qquad \text{in } \Omega \times (0, \infty), \tag{1.5}$$

$$\mathcal{S}^\epsilon = W_{,\mathcal{E}}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon)) \qquad \text{in } \Omega \times (0, \infty), \tag{1.6}$$

where $\mathcal{S}^\epsilon$ is the stress tensor. Here we can assume that the equation for the mechanics (1.5) is time independent because the mechanical equilibrium is attained on a much faster time scale than the concentration changing by diffusion. To close the system we require the following boundary and initial values

$$\tfrac{\partial}{\partial n} c^\epsilon = \tfrac{\partial}{\partial n} \mu^\epsilon = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.7}$$

$$\mathbf{u}^\epsilon = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.8}$$

$$c^\epsilon|_{t=0} = c_0^\epsilon \qquad \text{in } \Omega\,, \tag{1.9}$$

where n denotes the unit normal of $\partial\Omega$. For a derivation of the Cahn-Larché system we refer to Garcke [35]. Existence and uniqueness results can be found for example in [35] and [36].

Another interesting task is the rigorous understanding of the sharp interface limit of the so-called "model H". This model consists of the Navier-Stokes system coupled with the Cahn-Hilliard equation and has the following form for fluids with the same density

$$\partial_t v^\epsilon + v^\epsilon \cdot \nabla v^\epsilon - \operatorname{div}(\nu(c^\epsilon) D v^\epsilon) + \nabla p^\epsilon = -\epsilon \operatorname{div}(\nabla c^\epsilon \otimes \nabla c^\epsilon) \quad \text{in } \Omega \times (0, \infty), \tag{1.10}$$

$$\operatorname{div} v^\epsilon = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.11}$$

$$\partial_t c^\epsilon + v^\epsilon \cdot \nabla c^\epsilon = \epsilon^\theta \Delta \mu^\epsilon \quad \text{in } \Omega \times (0, \infty), \tag{1.12}$$

$$\mu^\epsilon = -\epsilon \Delta c^\epsilon + \epsilon^{-1} f'(c^\epsilon) \quad \text{in } \Omega \times (0, \infty), \tag{1.13}$$

where v^ϵ is the velocity field and p^ϵ the pressure. It describes the flow of two viscous fluids like oil and water. Abels et al. [4] showed a convergence result by formally matched asymptotic expansions. But to our knowledge there are no rigorous results known so far for the sharp interface limit. The coupling term in the Cahn-Hilliard equation is of the form $v^\epsilon \cdot \nabla c^\epsilon$. Therefore a first step to handle the convergence problem is to study the sharp interface limit of the convective Cahn-Hilliard equation

$$\partial_t c^\epsilon + v \cdot \nabla c^\epsilon = m(\epsilon)\Delta\mu^\epsilon \qquad \text{in } \Omega \times (0,\infty), \tag{1.14}$$
$$\mu^\epsilon = \epsilon^{-1} f(c^\epsilon) - \epsilon\Delta c^\epsilon \qquad \text{in } \Omega \times (0,\infty), \tag{1.15}$$
$$\tfrac{\partial}{\partial n} c^\epsilon = \tfrac{\partial}{\partial n}\mu^\epsilon = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{1.16}$$
$$c^\epsilon|_{t=0} = c_0^\epsilon \qquad \text{in } \Omega, \tag{1.17}$$

where $v : \Omega \times [0,\infty) \to \mathbb{R}^d$ is a given smooth velocity field. In the following we investigate the sharp interface limit for mobility constants $m(\epsilon) = \epsilon^\theta$ for $\theta = 1$ and $\theta > 3$. Kwek [46] showed the existence of classical solutions to the convective Cahn-Hilliard equation. For the existence of weak solutions and strong solutions locally in time for the "model H" we refer to the results of Abels [1–3].

Sharp Interface Models: In the classical model the components of the alloy or the immiscible fluids fill two disjoint domains $\Omega^+(t), \Omega^-(t) \subset \Omega$ for all times $t \geq 0$. We assume that the two domains are separated by a $(d-1)$-dimensional surface $\Gamma(t)$ such that $\Gamma(t) = \partial\Omega^-(t)$ and $\Gamma(t) \subset \Omega$ at least initially, that is, we do not consider contact angles. Therefore we obtain $\Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$. Then the corresponding sharp interface model to the Cahn-Larché system is a modified Hele-Shaw problem

$$\Delta\mu = 0 \qquad \text{in } \Omega^\pm(t), t > 0, \tag{1.18}$$
$$\operatorname{div} \mathcal{S} = 0 \qquad \text{in } \Omega^\pm(t), t > 0, \tag{1.19}$$
$$V = -\tfrac{1}{2} [\nabla\mu]_{\Gamma(t)} \cdot \nu \qquad \text{on } \Gamma(t), t > 0, \tag{1.20}$$
$$\mu = \sigma\kappa + \tfrac{1}{2}\nu^T \left[W\mathrm{Id} - (\nabla\mathbf{u})^T\mathcal{S}\right]_{\Gamma(t)} \nu \qquad \text{on } \Gamma(t), t > 0, \tag{1.21}$$
$$[\mathcal{S}\nu]_{\Gamma(t)} = [\mathbf{u}]_{\Gamma(t)} = [\mu]_{\Gamma(t)} = 0 \qquad \text{on } \Gamma(t), t > 0, \tag{1.22}$$
$$\tfrac{\partial}{\partial n}\mu = \mathbf{u} = 0 \qquad \text{on } \partial\Omega, t > 0, \tag{1.23}$$
$$\Gamma(0) = \Gamma_{00} \qquad \text{for } t = 0. \tag{1.24}$$

Here the corresponding elastic energy densities have the form $W_-(\mathcal{E}) := W(-1,\mathcal{E})$ and $W_+(\mathcal{E}) := W(1,\mathcal{E})$. Always ν is the unit outer normal of $\partial\Omega^-(t)$, whereas n denotes the unit outer normal of $\partial\Omega$. The normal velocity and the mean curvature of $\Gamma(t)$ are denoted by V and κ, respectively, taken with respect to ν. The constant $\sigma > 0$ describes the surface tension of the interface and $[.]_{\Gamma(t)}$ denotes the jump of a quantity across the interface in direction of ν, i.e., $[f]_{\Gamma(t)}(x) = \lim_{h\to 0}(f(x + h\nu) - f(x - h\nu))$ for $x \in \Gamma(t)$. In Section 4.1 we prove the existence of classical solutions. For the classical Hele-Shaw problem one finds classical solution results in Chen et al.[22] and Escher and Simonett [29]. The global existence of classical solutions and the convergence to spheres are shown in Escher and Simonett [28], provided that the initial value is close to a sphere.
The corresponding sharp interface model to the convective Cahn-Hilliard equation with mobility constant $m(\epsilon) = \epsilon$ is the evolution equation

$$V - v \cdot \nu = 0 \qquad \text{on } \Gamma(t), t > 0, \tag{1.25}$$
$$\Gamma(0) = \Gamma_{00} \qquad \text{for } t = 0. \tag{1.26}$$

That means the motion of the interface $\Gamma(t)$ is independent of μ. Here μ is the solution to the following parabolic boundary problem

$$\partial_t \mu = f'(\pm 1)\Delta\mu - v \cdot \nabla\mu \qquad \text{in } \Omega^{\pm}(t), t > 0, \tag{1.27}$$

$$\mu = \sigma\kappa \qquad \text{on } \Gamma(t), t > 0, \tag{1.28}$$

$$\frac{\partial}{\partial n}\mu = 0 \qquad \text{on } \partial\Omega, t > 0, \tag{1.29}$$

$$\mu|_{t=0} = \mu_{00} \qquad \text{in } \Omega\,. \tag{1.30}$$

For $m(\epsilon) = 1$ we expect a coupled corresponding sharp interface model, see Section 5.1. In the case $m(\epsilon) = \epsilon^\theta$ for $\theta > 3$ we prove a non-convergence result.

Sharp interface limits: In the case of the Cahn-Hilliard equation, there are two kinds of results for the sharp interface limit. Chen [20] showed the convergence of weak solutions to a varifold solution to the corresponding sharp interface model globally in time. He proved that the family of solutions $\{c^\epsilon, \mu^\epsilon\}_{0<\epsilon\leq 1}$ is weakly compact in some functional spaces. Then he obtained the existence of a convergent subsequence. Garcke and Kwak [37] used this method to show the convergence of the Cahn-Larché system (1.3)-(1.6) to the modified Hele-Shaw problem (1.18)-(1.22) with Neumann boundary conditions on $\partial\Omega$ and an angle condition for the interface $\Gamma(t)$. Abels and Röger [6] also applied this method to the "model H". Recently Abels and Lengeler [5] extended this result to fluids with different densities. On the other hand, there is the method Alikakos et al. [10] used in their paper to show the convergence of the Cahn-Hilliard equation to the Hele-Shaw problem. They assumed that the Cahn-Hilliard equation and the Hele-Shaw problem have smooth solutions at least in a sufficiently small time interval $(0, T)$. By formally matched asymptotic expansions they constructed a family of approximate solutions $\{c^\epsilon_A, \mu^\epsilon_A\}_{0<\epsilon\leq 1}$ for the Cahn-Hilliard equation and showed that the difference of the real solution $(c^\epsilon, \mu^\epsilon)$ and approximate solutions converge to 0 as $\epsilon \searrow 0$, provided the initial value c^ϵ_0 of the Cahn-Hilliard equation is chosen suitably. Since the zero order expansion of the approximate solutions is based on the solution to the Hele-Shaw problem, they were able to prove the convergence of the Cahn-Hilliard equation to the Hele-Shaw problem as $\epsilon \searrow 0$. Let us mention that Carlen et al. [19] introduced an alternative method to construct approximate solutions to the Cahn-Hilliard equation. Based on Hilbert expansion they used the ansatz $c^\epsilon(x,t) \approx \sum_{i=1}^N \epsilon^i c_i(x, \Gamma_t^{(N)})$, where $\Gamma_t^{(N)}$ is the Nth order approximate interface. For the Cahn-Larché system a formally matched asymptotic expansion was already done in Leo et al. [48]. One can find some results about the formally matched asymptotic expansion for the quasi-incompressible "model H" in Lowengrub and Truskinovsky [49] and for the incompressible "model H" with different densities and mobility constants $m(\epsilon) = 1, \epsilon$ in [4]. In our work we use the method of Alikakos et al. [10]. By a simpler version of this method we can also show a negative result for the convective Cahn-Hilliard equation with mobility constant $m(\epsilon) = \epsilon^\theta$ for $\theta > 3$. This means we construct approximate solutions $\{c^\epsilon_A\}_{0<\epsilon\leq 1}$ and show that the difference of real solutions and approximate solutions converge to 0 as $\epsilon \searrow 0$ in certain norms. But the approximate solutions c^ϵ_A converge to the “wrong” function. A similar result was obtained by Abels and Lengeler [5] for certain radially symmetric solutions in the case of the “model H”.

Outline of the text: In Chapter 2, we recall the definitions of some function spaces and results from semigroup theory. Moreover, we collect useful inequalities and prove some uniqueness and existence results for ordinary differential equations. Finally, we mention some spectral analysis results proven by Chen [21]. We use these results to prove that the difference of approximate and real solutions for the Cahn-Larché system and convective Cahn-Hilliard equation tends to 0 as $\epsilon \searrow 0$. In Chapter 3, we rigorously prove the sharp interface limit for the Cahn-Larché system. More precisely, we show that the solutions for the Cahn-Larché system (1.3)-(1.9) converge to the solution for the modified Hele-Shaw problem (1.18)-(1.24) as long as smooth solutions exists for the limit system. For that we require suitable initial values c_0^ϵ. We follow the method of Alikakos et al. [10] where the main task is to construct suitable approximate solutions. In Chapter 4, we prove the existence of classical solutions to the modified Hele-Shaw problem (1.18)-(1.24) by using the results of Escher and Simonett [29]. They reduce the system to a single evolution equation for the distance function and prove the existence of a smooth solution for this system. We show that the new appearing differential operator has lower order such that we can apply the same techniques to show the existence of a smooth solution to the new evolution equation for the distance function. Furthermore, we prove an existence result for classical solutions to a linearized Hele-Shaw problem used in the higher order expansions. The proof is based on Alikakos et al. [10], that is, we again reduce the system to a single evolution equation for the distance function. Since the equation is of third order, we add the fourth order differential operator $\epsilon\Delta^2$ to get a solution by known results. Then we verify that the solutions to the new equation converge to a solution to the original equation as $\epsilon \searrow 0$. For that we use an energy method. In Chapter 5, we consider the convective Cahn-Hilliard equation with the mobility constant $m(\epsilon) = \epsilon^\theta$ for $\theta > 3$. In the case $\theta = 0, 1$ we expect that the surface tension term $-\epsilon \operatorname{div}(\nabla c^\epsilon \otimes \nabla c^\epsilon)$ of the “model H” (see (1.10)) converges to the mean curvature functional of the interface. For $\theta > 3$ we show that the term $-\epsilon \operatorname{div}(\nabla c^\epsilon \otimes \nabla c^\epsilon)$ does generally not converge to the mean curvature of the interface, where c^ϵ is here the solution for the convective Cahn-Hilliard equation. The reason is that the convection term $v \cdot \nabla c^\epsilon$ dominates the motion of the interface $\Gamma(t)$. Therefore we can show that the approximate solutions do not have the form $\theta_0(d(x,t)/\epsilon)$ where d is the signed distance function to $\Gamma(t)$ and $\theta_0(x)$ is the “optimal diffuse interface profile”, that is the solution to

$$-w'' + f(w) = 0 \text{ in } \mathbb{R}, \qquad w(0) = 0, \qquad \lim_{z\to\pm\infty} w(z) = \pm 1\,.$$

Finally, in Chapter 6 the sharp interface limit of the convective Cahn-Hilliard equation with mobility constant $m(\epsilon) = \epsilon$ is proven rigorously for suitable initial values c_0^ϵ, that is the solutions to (1.14)-(1.17) converge to (1.25)-(1.30). For the proof we use the same techniques as in Chapter 3.1. In particular, we construct an approximate solution by formally matched asymptotic expansions. The mobility constant $m(\epsilon) = \epsilon$ especially changes the inner expansion for the approximate solutions. By different compatibility conditions for the inner expansion we realize why the motion of the interface $\Gamma(t)$ is independent of μ.

Acknowledgments:
First and foremost, I would like to thank my supervisor Prof. Dr. Helmut Abels for giving me the opportunity to work on the interesting field of sharp interface limits. I am grateful for many motivating discussions and for encouraging in times when work did not make progress. I want to thank the German Research Foundation (DFG) for the financial support within the program “Transport Processes at Fluidic Interfaces”. I also want to mention Dr. Daniel Depner and Dr. Doris Augustin, who carefully read a preliminary version of this dissertation. Finally, I want to thank my colleagues for the nice working environment and especially Carmen for the encouragement, support, and her good mood.

2 Mathematical Background

2.1 Notation

When we write (x.y a), we mean the first line of equation (x.y) and analogously for (x.y b) and so on. In our work the natural numbers including 0 are denoted by $\mathbb{N}$. Moreover, we denote $a \otimes b = (a_i b_j)_{i,j=1}^d$ for $a, b \in \mathbb{R}^d$ and $A : B = \sum_{i,j=1}^d A_{ij} B_{ij}$ for $A, B \in \mathbb{R}^{d\times d}$. The symmetric part of a matrix $A \in \mathbb{R}^{d\times d}$ is denoted by $\mathrm{sym}(A)$, that is, $\mathrm{sym}(A) = \frac{1}{2}(A + A^T)$. We denote

$$\mathrm{diag(x_1, \ldots, x_d)} = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{pmatrix} \in \mathbb{R}^{d\times d}$$

for $x_i \in \mathbb{R}$, $i = 1, \ldots, d$. The cofactor matrix is denoted by $\mathrm{cof}(A)$ for $A \in \mathbb{R}^{d\times d}$. The vector spaces $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ are endowed with the Euclidean norm. By Id we denote both the identity matrix and the identity mapping. The function $\lfloor . \rfloor$ defines the floor function, that is, $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \le x\}$. We write X' for the dual space of a Banach space X and denote the duality product by

$$\langle f, g\rangle \equiv \langle f, g\rangle_{X',X} = f(g) \quad \forall f \in X', g \in X\,.$$

We denote the open ball around $x \in X$ with radius $r > 0$ by $B_r(x)$. The set of all real analytic functions on a given set D is denoted by $C^\omega(D)$. For a sufficient smooth domain $\Omega \subset \mathbb{R}^d$ and an interval $(0, T)$, $T > 0$, we define $\Omega_T = \Omega \times (0, T)$ and $\partial_T\Omega = \partial\Omega \times (0, T)$. Moreover, n denotes the exterior unit normal on $\partial\Omega$. For a hypersurface $\Gamma \subset \Omega$ without boundary such that $\Gamma = \partial\Omega^-$ for a reference domain $\Omega^- \subset \Omega$, the interior domain is denoted by Ω^- and the exterior domain by $\Omega^+ := \Omega\backslash(\Omega^- \cup \Gamma)$, that is Γ separates Ω into an interior and an exterior domain. The exterior unit normal on $\partial\Omega^-$ is denoted by ν. The mean curvature of Γ is denoted by κ with the sign convention that κ is positive, if Γ is curved in direction of ν. For a signed distance function d with respect to Γ, we assume $d < 0$ in Ω^- and $d > 0$ in Ω^+. By this convention we obtain $\nabla d = \nu$ on Γ. Finally, let us mention that we use the Einstein summation convention.

2.2 Basic Assumptions

Unless specified otherwise, $\Omega \subset \mathbb{R}^d$, $d > 1$, is a bounded domain with smooth boundary $\partial\Omega$ and unless noted otherwise, the Landau symbols $\mathcal{O}$ are with respect to the C^0 norm. The "double-well" potential $F : \mathbb{R} \to \mathbb{R}$ is a smooth function taking its global minimum 0 at ± 1. For its derivative $f(c) = F'(c)$ we assume

$$f(\pm 1) = 0, \quad f'(\pm 1) > 0, \quad \int_{-1}^{u} f(s)\,ds = \int_{1}^{u} f(s)\,ds > 0 \quad \forall u \in (-1,1). \tag{2.1}$$

In Chapter 3 and 6 we need an additional assumption

$$cf''(c) \geq 0 \quad \text{if } |c| \geq C_0 \tag{2.2}$$

for some constant $C_0 > 0$. This assumption is not necessary in Chapter 5. The constant elasticity tensor $\mathcal{C} = (\mathcal{C}_{iji'j'})_{i,j,i',j'=1,\ldots,d}$ maps matrices $A \in \mathbb{R}^{d\times d}$ in matrices by the definition

$$(\mathcal{C}A)_{ij} = \sum_{i',j'=1}^{d} \mathcal{C}_{iji'j'} A_{i'j'}\,.$$

In addition, we assume the symmetry properties

$$\mathcal{C}_{iji'j'} = \mathcal{C}_{ijj'i'} = \mathcal{C}_{jii'j'} \quad \text{and} \quad \mathcal{C}_{iji'j'} = \mathcal{C}_{i'j'ij}$$

for all $i, j, i', j' = 1, \ldots, d$. An important assumption is the positive definiteness of $\mathcal{C}$ on symmetric matrices, that is, there exists some constant $c_2 > 0$ such that

$$A : \mathcal{C}A \geq c_2 \left|\mathrm{sym}(A)\right|^2 \quad \forall A \in \mathbb{R}^{d\times d}\,. \tag{2.3}$$

An important consequence of the positive definiteness is the following lemma.

Lemma 2.2.1. *Let the tensor $\mathcal{C}$ be defined as above. Then it holds for all $a, b \in \mathbb{R}^d$*

$$(a \otimes b) : \mathcal{C}\,(a \otimes b) \geq \frac{1}{2} c_2 \left|a \otimes b\right|^2\,. \tag{2.4}$$

Proof: Let $a, b \in \mathbb{R}^d$ be any given vectors. It follows by definition

$$\begin{aligned}
\left|\mathrm{sym}(a \otimes b)\right|^2 &= \frac{1}{4}\,(a \otimes b + b \otimes a) : (a \otimes b + b \otimes a) \\
&= \frac{1}{2} \left|a \otimes b\right|^2 + \frac{1}{2}\,(a \otimes b) : (b \otimes a)\,.
\end{aligned}$$

We show that the second term on the right-hand side is not negative

$$(a \otimes b) : (b \otimes a) = \sum_{i,j=1}^{d} a_i b_j b_i a_j = (a \cdot b)^2 \geq 0\,.$$

Hence the assertion of the lemma follows. □

In the following we also use the constant c_2 instead of $\frac{1}{2}c_2$.

2.3 Function Spaces

In this section we want to recall some definitions of particular function spaces which we need in the following.
We start with the introduction of Sobolev spaces by Fourier transform. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is defined by

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} (1 + |x|)^N \, |\partial_x^\alpha f(x)| < \infty, \forall N \in \mathbb{N}, \alpha \in \mathbb{N}^d \right\}.$$

Then the L^2-Bessel potential space of order $s \in \mathbb{R}$ is defined by

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \langle \xi \rangle^s \hat{f}(\xi) \in L^2(\mathbb{R}^d) \right\}$$

endowed with the norm $\|f\|_{H^s(\mathbb{R}^d)} = \|\langle \xi \rangle^s \hat{f}\|_{L^2(\mathbb{R}^d)}$. Here $\hat{f}$ is the Fourier transform of f and $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. For any non-empty set $\Omega \subset \mathbb{R}^d$ we define

$$H^s(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = F|_\Omega \text{ for some } F \in H^s(\mathbb{R}^d) \right\}.$$

In the present work we always consider bounded domains $\Omega \subset \mathbb{R}^d$ with smooth boundary. Hence we obtain $H^k(\Omega) = W_2^k(\Omega)$ with equivalent norms for all $k \in \mathbb{N}$, cf. [53, Theorem 3.18]. The definition of Sobolev spaces by Fourier transform can be found for example in [53, 56].
Next we introduce Sobolev spaces on the boundary. Let k be any positive integer. First we assume that $\Omega \subset \mathbb{R}^d$ is a $C^{k-1,1}$ hypograph, that is, there exists a $C^{k-1,1}$ function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega = \left\{ x \in \mathbb{R}^d : x_d < \zeta(x') \, \forall x' = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1} \right\}.$$

For $u \in L^2(\partial\Omega)$ we define

$$u_\zeta(x') = u(x', \zeta(x')) \quad \forall x' \in \mathbb{R}^{d-1}.$$

Then, for $0 \le s \le k$, we set

$$H^s(\partial\Omega) = \left\{ u \in L^2(\partial\Omega) : u_\zeta \in H^s(\mathbb{R}^{d-1}) \right\},$$

and equip this space with the inner product

$$(u, v)_{H^s(\partial\Omega)} = (u_\zeta, v_\zeta)_{H^s(\mathbb{R}^{d-1})}.$$

Moreover, we set for $0 \le s \le k$ and $u \in L^2(\partial\Omega)$

$$\|u\|_{H^{-s}(\partial\Omega)} = \left\| u_\zeta \sqrt{1 + |\nabla \zeta|^2} \right\|_{H^{-s}(\mathbb{R}^{d-1})},$$

and then define $H^{-s}(\partial\Omega)$ to be the completion of $L^2(\partial\Omega)$ in this norm.
If Ω is not a hypograph, then we use an open cover $\{U_i\}_{i\in I}$ and a partition of the unity $\{\phi_i\}_{i\in I}$ such that $\phi_i \in C_0^\infty(U_i)$ for all $i \in I$ and define an inner product by

$$(u,v)_{H^s(\partial\Omega)} = \sum_{i\in I} (\phi_i u, \phi_i v)_{H^s(\partial\Omega\cap U_i)} .$$

It can be shown that $H^s(\partial\Omega)$ is independent of the choice of $\{U_i\}_{i\in I}$ and $\{\phi_i\}_{i\in I}$. For more details see Mc Lean [53]. More general, for the definition of Sobolev spaces on Riemannian manifolds, we refer for example to Aubin [16] or Hebey [40].
We continue with the introduction of Hölder spaces. Let $0 < \theta < 1$ be any number, then we define the Hölder space $C^\theta(\overline{\Omega})$ and the little Hölder space $h^\theta(\overline{\Omega})$ by

$$\begin{aligned}
C^\theta(\overline{\Omega}) &= \left\{ f \in C(\overline{\Omega}) : [f]_{C^\theta} = \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\theta} < \infty \right\}, \\
\|f\|_{C^\theta(\overline{\Omega})} &= \|f\|_{C(\overline{\Omega})} + [f]_{C^\theta} , \\
h^\theta(\overline{\Omega}) &= \left\{ f \in C^\theta(\overline{\Omega}) : \lim_{\tau\to 0} \sup_{x,y\in\overline{\Omega}, 0<|x-y|<\tau} \frac{|f(x)-f(y)|}{|x-y|^\theta} = 0 \right\}.
\end{aligned}$$

If $\theta > 0$, $\theta \notin \mathbb{N}$, we define

$$\begin{aligned}
C^\theta(\overline{\Omega}) &= \left\{ f \in C^{[\theta]}(\overline{\Omega}) : \partial^\beta f \in C^{\theta-[\theta]}(\overline{\Omega}), \forall\, |\beta| = [\theta] \right\}, \\
\|f\|_{C^\theta(\overline{\Omega})} &= \|f\|_{C^{[\theta]}(\overline{\Omega})} + \sum_{|\beta|=[\theta]} \left[\partial^\beta f\right]_{C^{\theta-[\theta]}} , \\
h^\theta(\overline{\Omega}) &= \left\{ f \in C^\theta(\overline{\Omega}) : \partial^\beta f \in h^{\theta-[\theta]}(\overline{\Omega}), \forall\, |\beta| = [\theta] \right\},
\end{aligned}$$

where $[\theta]$ is the greatest integer smaller than θ. One can find the definition of the Hölder space $C^\theta(\overline{\Omega})$ in many books, for example see Evans [32] or Alt [11] and the definition of the little Hölder space $h^\theta(\overline{\Omega})$ in Lunardi [50]. Note that $C^\theta(\overline{\Omega})$ is a Banach space, cf. [32], and $h^\theta(\overline{\Omega})$ is a closed subspace of $C^\theta(\overline{\Omega})$.
One can show that $h^\theta(\overline{\Omega})$ is the closure of $C^k(\overline{\Omega})$ in $C^\theta(\overline{\Omega})$ for every $k \in (\theta, \infty]$, cf. [50].
Let M be an m-dimensional sufficiently smooth submanifold of $\mathbb{R}^d$. Then the spaces $C^\theta(M)$ and $h^\theta(M)$, $\theta \in \mathbb{R}_+\backslash\mathbb{N}$, are defined by means of a smooth atlas for M, see Triebel [62].
Little Hölder spaces have been studied by several authors in context with analytic semigroups and maximal regularity, cf. [26, 27, 29, 30, 47].
Furthermore, we define Hölder spaces on the set $[a,b] \times \overline{\Omega}$, $a < b$. For $\alpha > 0$ we set

$$C^{\alpha,0}([a,b]\times\overline{\Omega}) = \left\{ f \in C([a,b]\times\overline{\Omega}) : f(.,x) \in C^\alpha([a,b]), \forall x \in \overline{\Omega}, \|f\|_{C^{\alpha,0}} < \infty \right\},$$

endowed with the norm

$$\|f\|_{C^{\alpha,0}} = \sup_{x\in\overline{\Omega}} \|f(.,x)\|_{C^\alpha([a,b])} .$$

Similarly, we define the space $C^{0,\alpha}([a,b]\times\overline{\Omega})$ with the norm $\|.\|_{C^{0,\alpha}}$. Moreover, we introduce the space $C^{1,2}([a,b]\times\overline{\Omega})$ as follows

$$C^{1,2}([a,b]\times\overline{\Omega}) = \left\{f \in C([a,b]\times\overline{\Omega}) : \exists \partial_t f, \partial_{ij} f \in C([a,b]\times\overline{\Omega}), i,j = 1,\dots d\right\},$$

endowed with the norm

$$\|f\|_{C^{1,2}([a,b]\times\overline{\Omega})} = \|f\|_{C^0} + \sum_{i=1}^{d} \|\partial_i f\|_{C^0} + \|\partial_t f\|_{C^0} + \sum_{i,j=1}^{d} \|\partial_{ij} f\|_{C^0},$$

where $\partial_i = \partial_{x_i}$, $i = 1,\dots,d$. For $0 < \alpha < 2$ the so-called "parabolic" Hölder spaces are defined by

$$\begin{aligned} C^{\alpha/2,\alpha}([a,b]\times\overline{\Omega}) &= C^{\alpha/2,0}([a,b]\times\overline{\Omega}) \cap C^{0,\alpha}([a,b]\times\overline{\Omega}), \\ \|f\|_{C^{\alpha/2,\alpha}([a,b]\times\overline{\Omega})} &= \|f\|_{C^{\alpha/2,0}([a,b]\times\overline{\Omega})} + \|f\|_{C^{0,\alpha}([a,b]\times\overline{\Omega})} \end{aligned}$$

and

$$C^{1+\alpha/2,2+\alpha}([a,b]\times\overline{\Omega}) = \left\{f \in C^{1,2}([a,b]\times\overline{\Omega}) : \exists \partial_t f, \partial_{ij} f \in C^{\alpha/2,\alpha}([a,b]\times\overline{\Omega}), \forall i,j\right\},$$

$$\|f\|_{C^{1+\alpha/2,2+\alpha}([a,b]\times\overline{\Omega})} = \|f\|_{C^0} + \sum_{i=1}^{d} \|\partial_i f\|_{C^0} + \|\partial_t f\|_{C^{\alpha/2,\alpha}} + \sum_{i,j=1}^{d} \|\partial_{ij} f\|_{C^{\alpha/2,\alpha}}.$$

For more information about parabolic Hölder spaces we refer for example to [50]. In the following we often write $\|f\|_{C^n(\Omega)}$ instead of $\|f\|_{C^n(\overline{\Omega})}$ for $f \in C^n(\overline{\Omega})$, $n \in \mathbb{R}$.

2.4 Useful Inequalities

In this section we recall some inequalities which we often use. In the whole section let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open subset with smooth boundary.

Theorem 2.4.1 (Poincaré's inequality)**.** *For fixed $1 \le p \le \infty$ there exists a constant C, depending only on d, p and Ω, such that*

$$\left\| f - \frac{1}{|\Omega|}\int_\Omega f\,dx \right\|_{L^p(\Omega)} \le C\,\|\nabla f\|_{L^p(\Omega)}$$

for each function $f \in W_p^1(\Omega)$.

Proof: For example, see [32, Chapter 5.8, Theorem 1]. □

Moreover, we have a certain interpolation result between the Sobolev space $W_p^k(\Omega)$ and the Lebesgue space $L^q(\Omega)$.

Theorem 2.4.2 (Gagliardo-Nirenberg inequality). *Let $\beta \in \mathbb{N}^d$, $k \in \mathbb{N}$, r, q, and p satisfy*

$$\frac{1}{r} = \frac{|\beta|}{d} + \lambda \left(\frac{1}{p} - \frac{k}{d} \right) + (1-\lambda) \frac{1}{q}, \quad \frac{|\beta|}{d} \leq \lambda \leq 1, \quad 0 \leq |\beta| \leq k-1 \,,$$

then there exists a constant $C > 0$ such that

$$\|f\|_{W^{|\beta_r|}(\Omega)} \leq C \, \|f\|^{\lambda}_{W^k_p(\Omega)} \, \|f\|^{1-\lambda}_{L^q(\Omega)} \,,$$

provided $k - |\beta| - \frac{d}{p}$ is not a negative integer (otherwise it holds for $\lambda = \frac{|\beta|}{k}$).

Proof: We refer to [57, theorem 1.24]. □

In Chapter 3 and 4 we often use the Korn inequality for the strain tensor $\mathcal{E}(\mathbf{u})$.

Theorem 2.4.3 (Korn inequality). *Let $1 < p < \infty$. Then there exists a constant $C = C(p, \Omega)$ such that for any $\boldsymbol{v} \in W^1_{p,0}(\Omega)^d$, it holds*

$$\|\boldsymbol{v}\|_{W^1_{p,0}(\Omega)} \leq C \, \|\mathcal{E}(\boldsymbol{v})\|_{L^p(\Omega)} \,.$$

Proof: We refer to [57, Theorem 1.33]. □

2.5 Interpolation Spaces

One can find a good introduction to the theory of interpolation spaces in Lunardi [52] and Bergh and Löfström [18]. In the following we explain how we construct real interpolation spaces by the K-method and we present some examples, which are used frequently.
Let X, Y be two real or complex Banach spaces. Then (X, Y) is said to be an interpolation couple if X, Y are continuously embedded in a Hausdorff topological vector space Z. In this case the intersection $X \cap Y$ and the sum $X + Y$ are linear subspaces of Z. For every $x \in X + Y$ and $t > 0$ we define

$$K(t, x, X, Y) := \inf_{x=a+b,\, a \in X,\, b \in Y} \left(\|a\|_X + t \, \|b\|_Y \right).$$

From now we write $K(t, x)$ instead of $K(t, x, X, Y)$. For $0 < \theta < 1$, $1 \leq p \leq \infty$ the real interpolation spaces are defined by

$$(X, Y)_{\theta,p} = \left\{ x \in X + Y : t \mapsto t^{-\theta-1/p} K(t, x) \in L^p(0, \infty) \right\},$$

endowed with the norm

$$\|x\|_{(X,Y)_{\theta,p}} = \left\| t^{-\theta-1/p} K(t, x) \right\|_{L^p(0,\infty)} \,,$$

where we use the convention $\frac{1}{p} = 0$ for $p = \infty$. Due to this definition, it can be verified that for $0 < \theta < 1$, $1 \leq p_1 \leq p_2 \leq \infty$ it holds

$$X \cap Y \subset (X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2} \subset X + Y\,.$$

Theorem 2.5.1. *Let $(X_1, Y_1), (X_2, Y_2)$ be interpolation couples. If $T \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2)$, then $T \in \mathcal{L}((X_1, Y_1)_{\theta,p}, (X_2, Y_2)_{\theta,p})$ for every $\theta \in (0, 1)$ and $p \in [1, \infty]$. Moreover, it holds*

$$\|T\|_{\mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_1,X_2)} \|T\|^{\theta}_{\mathcal{L}(Y_1,Y_2)}\,.$$

Proof: See [52, Theorem 1.6.]. □

A consequence of this theorem is the next estimate which is often used.

Corollary 2.5.2. *Let (X, Y) be an interpolation couple. For $0 < \theta < 1$, $1 \leq p \leq \infty$ there exists a constant $C = C(\theta, p) > 0$ such that*

$$\|y\|_{(X,Y)_{\theta,p}} \leq C \|y\|^{1-\theta}_X \|y\|^{\theta}_Y$$

for all $y \in X \cap Y$.

Proof: We refer to [52, Corollary 1.7.]. □

Now we give some examples for real interpolation spaces. For the rest of this section let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. For $s = (1-\theta)s_0 + \theta s_1$, $s_0, s_1 \in \mathbb{R}$, and $0 < \theta < 1$, it holds

$$(H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta,2} = H^s(\Omega) \tag{2.5}$$

with equivalent norms, cf. [53, Theorem B.8]. Moreover, we have

$$(H^{s_0}(\partial\Omega), H^{s_1}(\partial\Omega))_{\theta,2} = H^s(\partial\Omega) \tag{2.6}$$

with equivalent norms, cf. [53, Theorem B.11].
This yields the following elliptic regularity result. Denote by $\Delta_{\partial\Omega}$ the Laplace-Beltrami operator on $\partial\Omega$.

Theorem 2.5.3. *Let $\Omega \subset \mathbb{R}^d$ be as above and $r \in \mathbb{R}$, $r \geq 0$, be any fixed number. Then the operator*

$$-\Delta_{\partial\Omega} + \mathrm{Id} : H^{r+2}(\partial\Omega) \to H^r(\partial\Omega)$$

is an isomorphism.

Proof: For $r \in 2\mathbb{N}$ the assertion directly follows form [17, Bemerkung 1.7.6]. For $r \in \mathbb{R}\backslash 2\mathbb{N}$, $r > 0$, we choose $k \in 2\mathbb{N}$ and $\theta \in (0, 1)$ such that $k < r < k + 2$

and $r = (1-\theta)k + \theta(k+1)$. Since $\left(H^{k+2}(\partial\Omega), H^{k+4}(\partial\Omega)\right)_{\theta,2} = H^{r+2}(\partial\Omega)$ and $\left(H^{k}(\partial\Omega), H^{k+2}(\partial\Omega)\right)_{\theta,2} = H^{r}(\partial\Omega)$, it holds due to Theorem 2.5.1

$$-\Delta_{\partial\Omega} + \mathrm{Id} \in \mathcal{L}(H^{r+2}(\partial\Omega), H^{r}(\partial\Omega)) \text{ and } (-\Delta_{\partial\Omega} + \mathrm{Id})^{-1} \in \mathcal{L}(H^{r}(\partial\Omega), H^{r+2}(\partial\Omega)) .$$

Furthermore, $(-\Delta_{\partial\Omega} + \mathrm{Id})^{-1}|_{H^r}$ is the inverse of $-\Delta_{\partial\Omega} + \mathrm{Id}|_{H^{r+2}}$ because we already know that $(-\Delta_{\partial\Omega} + \mathrm{Id})^{-1}|_{H^k}$ is the inverse of $-\Delta_{\partial\Omega} + \mathrm{Id}|_{H^{k+2}}$. Thus the assertion follows. □

In Section 2.7 we also use the complex interpolation method. Here we omit a detailed definition of the complex interpolation space $(X,Y)_{[\theta]}$ for $\theta \in [0,1]$ and complex Banach spaces X, Y. We refer to [18, 52] for a good introduction of the complex interpolation spaces.

2.6 Some Uniqueness and Existence Results for ODE's

In this section we prove some uniqueness and existence results for ordinary differential equations which we need for the inner expansion of the approximate solutions. For the inner expansion it is important that the solutions are bounded. Therefore we get some conditions on the right-hand side of the ordinary differential equations.

Lemma 2.6.1. *Let $f \in C^\infty(\mathbb{R})$ be given such that the properties (2.1) hold. Then the problem*

$$-w'' + f(w) = 0 \text{ in } \mathbb{R}, \qquad w(0) = 0, \qquad \lim_{z\to\pm\infty} w(z) = \pm 1 \tag{2.7}$$

has a unique solution.
In addition, the following properties hold

$$w'(z) > 0 \qquad \forall z \in \mathbb{R}\,, \tag{2.8}$$
$$\left|w^2(z) - 1\right| + \left|w^{(n)}(z)\right| \le C_n e^{-\alpha|z|} \qquad \forall z \in \mathbb{R},\ n \in \mathbb{N}\setminus\{0\} \tag{2.9}$$

for some constants $C_n > 0$, $n \in \mathbb{N}\setminus\{0\}$, and where α is a fixed constant such that

$$0 < \alpha < \min\left\{\sqrt{f'(-1)}, \sqrt{f'(1)}\right\} .$$

Proof: All solutions to the ordinary differential equation $-w'' + f(w) = 0$ fulfill the equation

$$w' = \pm\sqrt{2\left(E + \int_{-1}^{w} f(s)\,ds\right)},$$

where E is an appropriate constant, for example see [33, Chapter 2 §14]. Since $\int_{-1}^{1} f(s)\,ds = 0$ and $\lim_{z\to\pm\infty} w(z) = \pm 1$, it follows that $E = 0$ and we have a positive sign. Therefore all solutions to (2.7) satisfy the ordinary differential equation

$$w' = \sqrt{2\int_{-1}^{w} f(s)\,ds}, \quad w(0) = 0\,. \tag{2.10}$$

So we can deduce uniqueness and local existence of (2.10). Since ± 1 are stationary solutions to the ODE in (2.10) with initial values ± 1, we conclude that the solution to (2.10) exists globally and satisfies $-1 < w < 1$. Since w grows monotonically, $\lim_{z\to\pm\infty} w(z)$ exists. By contradiction we get $\lim_{z\to\pm\infty} w(z) = \pm 1$, otherwise there exists a constant $c > 0$ and z_0 such that $w'(z) > c$ for all $|z| \geq z_0$ since $\int_{-1}^{u} f(s)\,ds > 0$ for all $u \in (-1, 1)$. Since $\int_{-1}^{w} f(s)\,ds > 0$ for all $z \in \mathbb{R}$, it is not difficult to verify that the solution to (2.10) is a solution to (2.7). Hence (2.7) has a unique solution.
It remains to show the inequalities (2.8) and (2.9). Due to the mean value theorem, we obtain for $s < 1$

$$f(s) = \frac{f(s) - f(1)}{s-1}\,(s-1) = f'(\xi)\,(s-1)$$

for some $\xi \in (s, 1)$. Since $f'(1) > 0$, there exists some constant $c = c(s) \in (0, 1)$ such that $cf'(1) \leq f'(\xi)$ for all $\xi \in (s, 1)$, provided $1 - s > 0$ is small enough and such that $c(s) \to 1$ as $s \to 1$. Therefore there exists some $z_0 > 0$ such that for all $z > z_0$

$$\begin{aligned}(1 - w^2(z))' = -2w(z)w'(z) &= -2w(z)\sqrt{2\int_{1}^{w(z)} f(s)\,ds} \\ &\leq -2w(z)\sqrt{2c(w(z_0))f'(1)\int_{1}^{w(z)} (s-1)\,ds}\end{aligned}$$

since $w(z) \geq 0$ for all $z > 0$ and w grows monotonically. We continue with calculating the integral and use the convergence property of $c(.)$

$$\begin{aligned}(1 - w^2(z))' &\leq -\frac{2\sqrt{c(w(z_0))}w}{1+w}\sqrt{f'(1)}\left(1 - w^2\right) \\ &\leq -(1 - \epsilon(z_0))\sqrt{f'(1)}\left(1 - w^2(z)\right)\end{aligned}$$

for some constant $\epsilon(z_0) > 0$ such that $\epsilon(z_0) \to 0$ as $z_0 \to \infty$. Here we have used that $\lim_{z\to\infty} w(z) = 1$. Therefore Gronwall's inequality yields

$$1 - w^2(z) \leq Ce^{-\alpha z} \quad \forall z \in (0, \infty)$$

for some $C > 0$ and some fixed $0 < \alpha < \sqrt{f'(1)}$. Analogously, we can show

$$1 - w^2(z) \leq Ce^{-\alpha|z|} \quad \forall z \in (-\infty, 0)$$

for some $C > 0$ and some fixed $0 < \alpha < \sqrt{f'(-1)}$.
Using the equation $w' = \sqrt{2\int_1^w f(s)\,ds}$ and the same estimates as above, we can show

$$0 < w' \le C(1-w^2) \le Ce^{-\alpha|z|} \quad \forall z \in \mathbb{R}\backslash[-z_0, z_0]\,,$$

for some $C > 0$. The statement for $w^{(n)}$, $n \ge 2$, follows by induction. We use the following equation

$$w''(z) = \frac{f(w) - f(\pm 1)}{w \mp 1}\,(w \mp 1) = f'(\xi)\frac{w^2 - 1}{w \pm 1}\,,$$

where $\xi \in (w(z), 1)$ and $\xi \in (-1, w(z))$, respectively. □

From now on we denote by θ_0 the unique solution to (2.7) and $\alpha > 0$ is the constant given by Lemma 2.6.1.

Lemma 2.6.2. *Let $U \subset \mathbb{R}^d$ and let $A(z,x)$, $(z,x) \in \mathbb{R} \times U$, be given and smooth. Assume that there exists $A^{\pm}(x)$ such that $A(\pm z, x) - A^{\pm}(x) = \mathcal{O}(e^{-\alpha z})$ as $z \to \infty$. Then, for each $x \in U$, the system*

$$\begin{aligned} &w_{zz}(z,x) - f'(\theta_0(z))\,w(z,x) = A(z,x) \quad \forall z \in \mathbb{R}, \\ &w(0,x) = 0, \quad w(.,x) \in L^\infty(\mathbb{R}) \end{aligned} \tag{2.11}$$

has a solution if and only if

$$\int_{\mathbb{R}} A(z,x)\theta_0'(z)\,dz = 0. \tag{2.12}$$

In addition, if the solution exists, then it is unique and satisfies for every $x \in U$

$$D_z^l\left[w(\pm z, x) + \frac{A^{\pm}(x)}{f'(\pm 1)}\right] = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty, \quad l = 0, 1, 2\,,$$

where α is given as in Lemma 2.6.1. Furthermore, if $A(z,x)$ satisfies for every $x \in U$

$$D_x^m D_z^l\left[A(\pm z, x) - A^{\pm}(x)\right] = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty$$

for all $m = 0, 1, \ldots, M$ and $l = 0, 1, \ldots, L$, then

$$D_x^m D_z^l\left[w(\pm z, x) + \frac{A^{\pm}(x)}{f'(\pm 1)}\right] = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty$$

for all $m = 0, 1, \ldots, M$ and $l = 0, 1, \ldots, L + 2$.

Proof: Let $x \in U$ be given. For simplicity we often write $(w(z), A(z))$ instead of $(w(z,x), A(z,x))$. By using the method of variation of constants, we determine all solutions to the ordinary differential equation $w_{zz}(z) - f'(\theta_0(z))w(z) = A(z)$ with initial value $w(0)=0$, that is, since θ_0' is a solution to the associated homogeneous equation, we set $w(z)=\theta_0'(z)u(z)$ for some function $u:\mathbb{R}\to\mathbb{R}$. Then we obtain the equation

$$u''(z)+2\frac{\theta_0''(z)}{\theta_0'(z)}u'(z)=\frac{A(z)}{\theta_0'(z)}\,.$$

Note that $(\theta_0')^{-2}$ is a solution to the associated homogeneous equation. Hence we get due to [33, Kapitel II. § 11, Satz 3]

$$u'(z)=(\theta_0')^{-2}(z)\left(c+\int_0^z\theta_0'(s)A(s)\,ds\right),$$

where $c=c(x)$ is an arbitrary function independent of z. Therefore all solutions to the ordinary differential equation $w_{zz}(z)-f'(\theta_0(z))w(z)=A(z)$ with initial value $w(0)=0$ have the form

$$w(z)=\theta_0'(z)\int_0^z\left[(\theta_0')^{-2}(r)\left[c+\int_0^r\theta_0'(s)A(s)\,ds\right]\right]dr\,. \tag{2.13}$$

Due to Lemma 2.6.1, we conclude that $w(z)$ is bounded for $z\to\infty$ if and only if

$$c(x)=-\int_0^\infty\theta_0'(s)A(s,x)\,ds\,. \tag{2.14}$$

This can be seen as follows. If c satisfies the equation above, then $w(z)$ converges to $A^+(x)/f'(1)$ as $z\to\infty$ (see below). In particular, $w(z)$ is bounded for $z\in[0,\infty)$. Supposing $c\neq-\int_0^\infty\theta_0'(s)A(s)\,ds$ yields

$$c+\int_0^r\theta_0'(s)A(s)\,ds\nrightarrow 0\quad\text{as } r\to\infty\,,$$

and by l'Hospital's rule we get

$$\theta_0'(z)\int_0^z(\theta_0')^{-2}(r)\,dr=\frac{\int_0^z(\theta_0')^{-2}(r)\,dr}{(\theta_0')^{-1}(z)}\to\infty\quad\text{as } z\to\infty\,,$$

and therefore $|w(z)|\to\infty$ as $z\to\infty$.
Analogously, $w(z)$ is bounded for $z\to-\infty$ if and only if

$$c(x,t)=-\int_0^{-\infty}\theta_0'(s)A(s)\,ds\,.$$

Therefore the system (2.11) has a bounded solution if and only if (2.12) holds. Additionally, the solution is unique.
It remains to show the convergence properties. First we assume that $A(z,x)$ only depends on $x\in U$. Then we obtain

$$w(z)=A(x)\theta_0'(z)\int_0^z\frac{\theta_0(y)-1}{(\theta_0')^2(y)}\,dy\,.$$

Using Taylor expansion gives us

$$\theta_0'' = f(\theta_0) = f'(1)(\theta_0 - 1) + \frac{f''(\xi)}{2}\,(\theta_0 - 1)^2 \tag{2.15}$$

for some $\xi = \xi(\theta_0(z)) \in (\theta_0(z) - 1, 1)$. Replacing the numerator $\theta_0 - 1$ by $\frac{\theta_0''}{f'(1)} - \frac{f''(\xi)(\theta_0-1)^2}{2f'(1)}$ yields

$$\begin{aligned} w(z) &= \frac{A(x)}{f'(1)}\,\theta_0'(z) \int_0^z \frac{\theta_0''(y)}{(\theta_0')^2(y)}\,dy - \frac{A(x)}{2f'(1)}\,\theta_0'(z) \int_0^z \frac{f''(\xi(z))\,(\theta_0(y)-1)^2}{(\theta_0')^2(y)}\,dy \\ &= -\frac{A(x)}{f'(1)}\left(1 - \frac{\theta_0'(z)}{\theta_0'(0)}\right) - \frac{A(x)}{2f'(1)}\,\theta_0'(z) \int_0^z \frac{f''(\xi(z))\,(\theta_0(y)-1)^2}{(\theta_0')^2(y)}\,dy\,. \end{aligned}$$

Since $\theta_0'(z) = \mathcal{O}(e^{-\alpha|z|})$, it is sufficient to show that $\theta_0'(z)\int_0^z f''(\xi)\,(\theta_0-1)^2/(\theta_0')^2 = \mathcal{O}(e^{-\alpha|z|})$. As in the proof of Lemma 2.6.1, we can show $(\theta_0')^2(z) \geq cf'(1)(\theta_0(z)-1)^2$ for all $z \geq z_0$, where c and z_0 are given as in the proof of Lemma 2.6.1. Also we can follow from the proof of Lemma 2.6.1 that $\theta_0'(z) = \mathcal{O}(e^{-\tilde{\alpha}z})$ for some $\widetilde{\alpha} > \alpha$. Therefore we obtain for all $z \geq z_0$

$$\begin{aligned} \left|\theta_0'(z) \int_0^z \frac{f''(\xi(z))\,(\theta_0(y)-1)^2}{(\theta_0')^2(y)}\,dy\right| &\leq C\theta_0'(z) \int_0^z \frac{|f''(\xi(z))|}{cf'(1)}\,dy \\ &\leq C\theta_0'(z)z \leq Ce^{-\tilde{\alpha}z}z \leq Ce^{-\alpha z} \end{aligned}$$

for some constant $C > 0$. We obtain the same statement for z negative with an analogous procedure, too.
For general $A(z,x)$ it is sufficient to consider the case $A(z,x) = \mathcal{O}(e^{-\alpha|z|})$ by linearity. Then we get for $z > 0$ by (2.13) with constant c as in (2.14)

$$|w(z)| = \left|\theta_0'(z) \int_0^z \left[(\theta_0')^{-2}(r)\left[-\int_r^\infty \theta_0'(s)A(s)\,ds\right]\right] dr\right| \leq C\theta_0'(z) \int_0^z \frac{e^{-\alpha y}}{\theta_0'(y)}\,dy\,.$$

We estimate the right-hand side. Note that $\frac{1}{\beta-\alpha}\left(e^{-\alpha z} - e^{-\beta z}\right)$ is the unique solution to

$$v'(z) = -\beta v(z) + e^{-\alpha z} \text{ in } \mathbb{R}, \quad v(0) = 0$$

for any $\beta > \alpha$. Due to [63, II. § 9 IX. Satz], it is sufficient to verify that $\theta_0'(z)\int_0^z \frac{e^{-\alpha y}}{\theta_0'(y)}\,dy$ satisfies for all $z > z_0$ the inequality

$$v'(z) \leq -\beta v(z) + e^{-\alpha z}$$

for some $\beta > \alpha$ and for some $z_0 > 0$. An easy calculation gives us

$$\left(\theta_0'(z) \int_0^z \frac{e^{-\alpha y}}{\theta_0'(y)}\,dy\right)' = \frac{\theta_0''(z)}{\theta_0'(z)}\left(\theta_0'(z) \int_0^z \frac{e^{-\alpha y}}{\theta_0'(y)}\,dy\right) + e^{-\alpha z}\,.$$

So it is sufficient to show

$$\frac{\theta_0''(z)}{\theta_0'(z)} \to -\sqrt{f'(1)} \quad \text{as } z \to \infty \tag{2.16}$$

since $\alpha < \min\left\{\sqrt{f'(-1)}, \sqrt{f'(1)}\right\}$. We apply l'Hospital's rule to obtain

$$\lim_{z\to\infty} \frac{(\theta_0''(z))^2}{(\theta_0'(z))^2} = \lim_{z\to\infty} \frac{\theta_0''(z)\theta_0'''(z)}{\theta_0'(z)\theta_0''(z)} = \lim_{z\to\infty} \frac{f'(\theta_0(z))\theta_0'(z)}{\theta_0'(z)} = f'(1).$$

Since $\theta''(z) < 0$ and $\theta'(z) > 0$ for $z > 0$, (2.16) holds. We can apply the same argumentation for $z \to -\infty$.
Equation (2.11) yields

$$w_{zz} = \mathcal{O}(e^{-\alpha z})\,,$$

and together with (2.16)

$$w_z = \mathcal{O}(e^{-\alpha z})\,.$$

This shows the first convergence property. Differentiating the differential equation for w with respect to z and x, one can verify that the last statement of the lemma is valid for all $m = 0, 1, \ldots, M$ and $l = 0, 1, \ldots, L$. □

Lemma 2.6.3. *Let $U \subset \mathbb{R}^n$ be an open subset and let $B(z, x)$ be a given smooth function defined on $\mathbb{R} \times U$ which satisfies $B(\pm z, x) = \mathcal{O}(e^{-\alpha z})$ as $z \to \infty$. Then for each $x \in U$ the problem*

$$w_{zz} = B \quad \forall z \in \mathbb{R}, \quad w(., x) \in L^\infty(\mathbb{R}) \tag{2.17}$$

has a solution if and only if

$$\int_{\mathbb{R}} B(z, x)\, dz = 0\,. \tag{2.18}$$

In addition, if $w_*(z, x)$ is a solution, then all the solutions can be written as

$$w(z, x) = w_*(z, x) + c(x)\,,$$

where $c(x)$ is an arbitrary function. Furthermore, if $\int_{\mathbb{R}} B(z, x)\, dz = 0$ for all $x \in U$ and

$$D_x^m D_z^l B(\pm z, x) = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty$$

for all $m = 0, 1, \ldots, M$ and $l = 0, 1, \ldots, L$, then there exist functions $w^+(x)$ and $w^-(x)$ such that as $z \to \infty$

$$D_x^m D_z^l \left[w(\pm z, x) - w^\pm(x)\right] = \mathcal{O}(e^{-\alpha z})$$

for all $m = 0, 1, \ldots, M$ and $l = 0, 1, \ldots, L + 2$.

Proof: Assume that the problem (2.17) has a solution. Let $x \in U$ be given. First we show by contradiction that $\lim_{z \to \pm\infty} w_z(z,x) = 0$. Assume there exists a constant $C_1 > 0$ and a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $z_n \to \infty$ as $n \to \infty$ and $|w_z(z_n)| \geq C_1$ for all $n \in \mathbb{N}$. W.l.o.g. assume that $w_z(z_n) \geq C_1$. Then it follows from $w_z(z) = \int_{z_n}^{z} B(s)\, ds + w_z(z_n)$ that

$$w_z(z) \geq -Ce^{-\alpha z_n} + C_1 \geq \frac{1}{2} C_1$$

for $n \in \mathbb{N}$ large enough and for all $z \geq z_n$. By integration this leads to a contradiction to the boundness of w. Analogously, we this holds for $z \to -\infty$. Therefore we conclude

$$\int_{\mathbb{R}} B(z,x)\, dz = \int_{\mathbb{R}} w_{zz}(z,x)\, dz = \lim_{z\to\infty} w_z(s,x)|_{s=-z}^{s=+z} = 0\,.$$

On the other hand assume that (2.18) holds. Then for any constant $c(x)$

$$w(z,x) := \int_0^z \int_0^r B(s,x)\, ds\, dr - z \int_0^\infty B(s,x)\, ds + c(x)$$

is a solution to (2.17) since

$$w_z(z) = \int_0^z B(s,x)\, ds - \int_0^\infty B(s,x)\, ds = \mathcal{O}(e^{-\alpha|z|}) \quad \text{as } z \to \pm\infty\,, \tag{2.19}$$

where the last equality follows from $\int_0^{-\infty} B(z,x)\, dz = \int_0^\infty B(z,x)\, dz$.
Let w_* be a solution to (2.17). Then all solutions to the equation $w_{zz} = B$ have the form $w_* + b(x)z + c(x)$ for any constants $b(x)$ and $c(x)$, that is, all bounded solutions have the form $w_* + c(x)$. The convergence properties for $z \to \pm\infty$ follow as in (2.19) and by differentiating with respect to z and x. □

2.7 Some Results from Semigroup Theory

In Section 3.1 and 6.1 below, we prove some estimates for the concentration c^ϵ in higher norms by semigroup theory. Good references for a systematic treatment of the basic theory are [50, 55, 56]. In this section we only consider the Laplace operator with Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$.
Let X be a complex Banach space with norm $\|.\|$, and $A : D(A) \subset X \to X$ be a closed linear operator with dense domain. We say A is of the type (ϕ, M), $\phi \in (\frac{\pi}{2}, \pi)$, $M > 0$, if and only if

$$\begin{aligned} S_\phi := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \phi\} &\subset \rho(A)\,, \\ \left\|(\lambda \mathrm{Id} - A)^{-1}\right\| &\leq \frac{M}{|\lambda|} \quad \forall \lambda \in S_\phi\,, \end{aligned}$$

where $\rho(A)$ is the resolvent set of A and Id is the identity operator. The operator A is said to be sectorial if there is a constant $\tau \in \mathbb{R}$ such that $A - \tau\mathrm{Id}$ is of the type (ϕ, M) for some $\phi \in (\frac{\pi}{2}, \pi)$ and $M > 0$
Provided A is a sectorial operator with $\sigma(A) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$, we set $D(A^\alpha) = \mathrm{im}(A^{-\alpha})$ for $\alpha > 0$. Here $D(A^\alpha)$ is endowed with the norm $\|.\|_{D(A^\alpha)} = \|A^\alpha.\|$ and $\sigma(A)$ is the spectrum of A, that is, $\sigma(A) = \mathbb{C}\backslash\rho(A)$.
From now let $X = L^p(\Omega)$, $1 < p < \infty$, be endowed with the usual norm $\|.\|_{L^p}$. Define $A_\tau : D(A_\tau) \to X$ by $A_\tau = -\Delta + \tau\mathrm{Id}$ for $\tau \in \mathbb{R}$, with domain $D(A_\tau) = \left\{c \in W_p^2(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = 0\right\}$. Then due to [50, Section 3.1.1], there exists $\lambda \in \mathbb{R}$ such that $-A_\tau$ is a sectorial operator for all $\tau > \lambda$. Later we will apply the results of the following lemma.

Lemma 2.7.1. *Let A_τ be defined as above. Then there exists $\tau \in \mathbb{R}$ such that $-A_\tau^2$ is sectorial with $D(-A_\tau^2) = \left\{c \in W_p^4(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = \frac{\partial}{\partial n} A_\tau c\big|_{\partial\Omega} = 0\right\}$. Furthermore, it holds $W_p^1(\Omega) = D(A_\tau^{\frac{1}{2}})$ with equivalent norms.*

Proof: Since $\sigma(\Delta - \mathrm{Id}) \subset \mathbb{R}$, it follows from Denk et al. [23, 8.2. Theorem] that for every $\phi \in (\frac{\pi}{2}, \pi)$, there exist constants $\tau = \tau(p, \phi) > 0$ and $M = M(p, \phi) > 0$ such that $\Delta - \tau\mathrm{Id}$ is of the type (ϕ, M). In particular, we can choose $\phi > \frac{3}{4}\pi$. Since

$$(\lambda\mathrm{Id} - (-(\Delta - \tau\mathrm{Id})^2)) = -\left(i\sqrt{\lambda}\mathrm{Id} - (\Delta - \tau\mathrm{Id})\right)\left(-i\sqrt{\lambda}\mathrm{Id} - (\Delta - \tau\mathrm{Id})\right)$$

and

$$\left|\arg \pm i\sqrt{\lambda}\right| = \frac{1}{2}\left|\arg \lambda \pm \pi\right|,$$

it follows that $\pm i\sqrt{\lambda} \in S_\phi$ for $|\arg\lambda| \in [0, \frac{\pi}{2}+\delta]$ and $\delta > 0$ small enough, and therefore $-(\Delta - \tau\mathrm{Id})^2$ is also sectorial with $D(-A_\tau^2) = \left\{c \in W_p^4(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = \frac{\partial}{\partial n} A_\tau c\big|_{\partial\Omega} = 0\right\}$. Since A_τ is invertible, it follows $D(A_\tau^{\frac{1}{2}}) = (L^p(\Omega), D(A_\tau))_{[\frac{1}{2}]}$, cf. Seeley [59, Theorem 3], and due to Theorem 4.1 in Seeley [58], it holds $W_p^1(\Omega) = (L^p(\Omega), D(A_\tau))_{[\frac{1}{2}]} = D(A_\tau^{\frac{1}{2}})$ with equivalent norms. □

2.8 Spectral Analysis

In this section we summarize some results proven by Chen [21].
Let f be the derivative of a double-well potential having global minima 0 at ± 1, that is, we assume that $f \in C^\infty(\mathbb{R})$ satisfies (2.1). Let $\gamma \subset \Omega$ be a smooth $(d-1)$-dimensional manifold without boundary and let $r = r(x)$ be the signed distance function satisfying $r < 0$ inside γ and $r > 0$ outside γ. Let $s = s(x)$ be the projection of x on γ along the normal of γ. Then there exits $\delta_0 > 0$ such that $\gamma(2\delta_0) := \left\{x \in \mathbb{R}^d : |r(x)| < 2\delta_0\right\} \subset \Omega$ and such that $\tau : \gamma(2\delta_0) \to (-2\delta_0, 2\delta_0) \times \gamma$ defined by

$\tau(x) = (r(x), s(x))$ is a smooth diffeomorphism where δ_0 only depends on γ and $\partial\Omega$, cf. [42, Kapitel 4.6]. Let $\phi^\epsilon : \Omega \to \mathbb{R}$ be a given function with the expansion

$$\begin{aligned}\phi^\epsilon(x) \quad &= \quad \zeta\left(\tfrac{r(x)}{\delta_0}\right)\left(\theta_0\left(\tfrac{r(x)}{\epsilon}\right) + \epsilon p^\epsilon(s(x))\theta_1\left(\tfrac{r(x)}{\epsilon}\right) + \epsilon^2 q^\epsilon(x)\right)\\ &\quad + \left(1 - \zeta\left(\tfrac{r(x)}{\delta_0}\right)\right)\left(\phi_\epsilon^+(x)\chi_{\{r(x)>0\}} + \phi_\epsilon^-(x)\chi_{\{r(x)<0\}}\right), \end{aligned} \tag{2.20}$$

where $\zeta \in C_0^\infty(\mathbb{R})$ is a cut-off function such that

$$\zeta(z) = 1 \text{ if } |z| < \frac{1}{2}, \qquad \zeta(z) = 0 \text{ if } |z| > 1, \qquad z\zeta'(z) \le 0 \text{ in } \mathbb{R}, \tag{2.21}$$

θ_0 is the unique solution to

$$-\theta_0'' + f(\theta_0) = 0 \text{ in } \mathbb{R}, \qquad \theta_0(0) = 0, \qquad \theta_0(\pm\infty) = \pm 1, \tag{2.22}$$

$\theta_1 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is any function satisfying

$$\int_{\mathbb{R}} \theta_1(\theta_0')^2 f''(\theta_0) = 0, \tag{2.23}$$

and $p^\epsilon(x)$, $q^\epsilon(x)$, ϕ_ϵ^+, and $\phi_\epsilon^-(x)$ are smooth function satisfying

$$\sup_{\epsilon\in(0,1]} |p^\epsilon| + \frac{\epsilon}{\epsilon + |r|}|q^\epsilon| \le C_* \qquad \text{in } \gamma(\delta_0), \tag{2.24}$$

$$\sup_{\epsilon\in(0,1]} |\nabla^\gamma \phi^\epsilon| \le C_* \qquad \text{in } \gamma(\delta_0), \tag{2.25}$$

$$\pm\phi_\epsilon^\pm > 0,\ f'(\phi_\epsilon^\pm) \ge 1/C_* \quad \text{in } \Omega \tag{2.26}$$

for some constant $C_* > 0$ where $\nabla^\gamma = \nabla - \nabla r(\nabla r \cdot \nabla)$ is the tangential gradient along γ. With these conditions we obtain the following proposition.

Proposition 2.8.1. *Assume that (2.1) and (2.20)-(2.26) hold. Then, for any given $\gamma_1 > 0$, there exist constants $\epsilon_0 > 0$ and $C > 0$ which depend only on f, θ_1, C_*, Ω, γ_1, and the C^3 norm of γ such that for every $\epsilon \in (0, \epsilon_0]$, $w \in H^1_{(0)}(\Omega)\setminus\{0\}$, and $\Psi \in H^2(\Omega)$ with $-\Delta\Psi = w$ and $\frac{\partial}{\partial n}\Psi\big|_{\partial\Omega} = 0$, the following inequality holds*

$$\int_\Omega \left(\epsilon|\nabla w|^2 + \epsilon^{-1} f'(\phi^\epsilon) w^2\right) \ge -C\|\nabla\Psi\|^2_{L^2(\Omega)} + \gamma_1\epsilon\|w\|^2_{L^2(\Omega)}. \tag{2.27}$$

Proof: Let γ_1 be any positive constant and let w and Ψ be any given functions as above. First we consider the case $\int_\Omega \epsilon|\nabla w|^2 + \epsilon^{-1} f'(\phi^\epsilon) w^2 \le \gamma_1\epsilon\|w\|^2_{L^2(\Omega)}$. Then by [21, Theorem 3.1.], there exists a constant $C = C(\gamma_1)$ and $\epsilon_1 > 0$ such that for all $\epsilon \in (0, \epsilon_1]$

$$\epsilon\|w\|^2_{L^2(\Omega)} \le C\|\nabla\Psi\|^2_{L^2(\Omega)}.$$

Due to the spectral estimate [21, Theorem 1.1.], there exist a constant C_1 depending on f, θ_1, C_*, Ω, and the C^3 norm of γ such that

$$\begin{aligned}\int_\Omega \left(\epsilon \left|\nabla w\right|^2 + \epsilon^{-1} f'(\phi^\epsilon) w^2\right) &\geq -C_1 \left\|\nabla \Psi\right\|^2_{L^2(\Omega)} \\ &\geq -(C_1 + C\gamma_1) \left\|\nabla \Psi\right\|^2_{L^2(\Omega)} + \gamma_1 \epsilon \left\|w\right\|^2_{L^2(\Omega)} \, .\end{aligned}$$

Hence together with the other case $\int_\Omega \epsilon \left|\nabla w\right|^2 + \epsilon^{-1} f'(\phi^\epsilon) w^2 \geq \gamma_1 \epsilon \left\|w\right\|^2_{L^2(\Omega)}$, the assertion of the lemma follows. □

Remark 2.8.2. *The C^3 norm of γ is defined as follows: Let r be the signed distance function to γ and $M = \sup_{x\in\gamma} |D^2 r(x)|$. Then r is smooth in $\gamma(1/M)$, cf. [42, Kapitel 4.6]. So we define $\|\gamma\|_{C^3} = M + \|r\|_{C^3(\gamma(\delta))}$ where $\delta = \min\{1, 1/2M\}$.*

To estimate the difference between true solutions and approximate solutions for the diffuse interface models, we need the following lemma.

Lemma 2.8.3. *Assume that $f \in C^2(\mathbb{R})$ satisfies (2.2). Then, for any $p \in [2, 3]$, there exists a positive constant C_p depending only on p, $\|f\|_{C^2([-3C_0, 3C_0])}$, and C_0 such that the quantity $\mathcal{N}(c, R) := f(c + R) - f(c) - f'(c)R$ satisfies*

$$R\mathcal{N}(c, R) \geq -C_p \left|R\right|^p \quad \forall c \in [-C_0, C_0], R \in \mathbb{R} \, .$$

Proof: The assertion follows by the mean value theorem and property (2.2). For more details see [10]. □

3 Sharp Interface Limit for Cahn-Larché System

In this chapter we consider the Cahn-Larché system

$$\begin{aligned}
\partial_t c^\epsilon &= \Delta\mu^\epsilon && \text{in } \Omega\times(0,T), && (3.1)\\
\mu^\epsilon &= \epsilon^{-1} f(c^\epsilon) - \epsilon\Delta c^\epsilon + W_{,c}(c^\epsilon,\mathcal{E}(\mathbf{u}^\epsilon)) && \text{in } \Omega\times(0,T), && (3.2)\\
\operatorname{div}\mathcal{S}^\epsilon &= 0 && \text{in } \Omega\times(0,T), && (3.3)\\
\mathcal{S}^\epsilon &= W_{,\mathcal{E}}(c^\epsilon,\mathcal{E}(\mathbf{u}^\epsilon)) && \text{in } \Omega\times(0,T), && (3.4)
\end{aligned}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and $T>0$ is a fixed constant. We close the system with the following boundary and initial conditions

$$\begin{aligned}
\tfrac{\partial}{\partial n} c^\epsilon &= \tfrac{\partial}{\partial n}\mu^\epsilon = 0 && \text{on } \partial\Omega\times(0,T), && (3.5)\\
\mathbf{u}^\epsilon &= 0 && \text{on } \partial\Omega\times(0,T), && (3.6)\\
c^\epsilon|_{t=0} &= c_0^\epsilon && \text{in } \Omega. && (3.7)
\end{aligned}$$

In the whole chapter we assume that the system (3.1)-(3.7) admits a smooth solution for every $\epsilon\in(0,1]$. Provided we choose an appropriate family of initial values $\{c_0^\epsilon\}_{0<\epsilon\le 1}$, we prove that the solutions for the Cahn-Larché system converge as $\epsilon\searrow 0$ to the solution to the Hele-Shaw problem coupled with linearized elasticity

$$\begin{aligned}
\Delta\mu &= 0 && \text{in } \Omega^\pm(t), t>0, && (3.8)\\
\operatorname{div}\mathcal{S} &= 0 && \text{in } \Omega^\pm(t), t>0, && (3.9)\\
V &= -\tfrac{1}{2}\left[\nabla\mu\right]_{\Gamma(t)}\cdot\nu && \text{on } \Gamma(t), t>0, && (3.10)\\
\mu &= \sigma\kappa + \tfrac{1}{2}\nu^T\left[W\mathrm{Id} - (\nabla\mathbf{u})^T\mathcal{S}\right]_{\Gamma(t)}\nu && \text{on } \Gamma(t), t>0, && (3.11)\\
\left[\mathcal{S}\nu\right]_{\Gamma(t)} &= \left[\mathbf{u}\right]_{\Gamma(t)} = \left[\mu\right]_{\Gamma(t)} = 0 && \text{on } \Gamma(t), t>0, && (3.12)\\
\tfrac{\partial}{\partial n}\mu &= \mathbf{u} = 0 && \text{on } \partial\Omega, t>0, && (3.13)\\
\Gamma(0) &= \Gamma_{00} && \text{for } t=0, && (3.14)
\end{aligned}$$

where $\Gamma_{00}\subset\Omega$ is an $(d-1)$-dimensional smooth hypersurface without boundary. For the proof we use the same techniques as Alikakos et al. [10], that is, we construct approximate solutions for the Cahn-Larché system by formally matched asymptotics. Then we show that the difference of the true solutions for the Cahn-Larché system and the approximate solutions converges to zero and finally that the approximate solutions converge to the solution for the elastic Hele-Shaw problem as $\epsilon\searrow 0$.

3.1 Convergence of the Difference of Approximate and True Solutions

In the first part of this chapter we want to show that the difference of approximate solutions having certain properties and true solutions converges to zero as $\epsilon \searrow 0$.
In this section we additionally assume for the double well potential F that $|s|^2 \leq CF(s)$ for all $|s| > C_0$ where $C, C_0 > 0$ are some constants large enough. For example this is valid for $F(s) = (s^2-1)^2$. Later we will see that this assumption is not necessary because we will show $\|c^\epsilon\|_{C^0(\Omega_T)} \leq C_0$ for all $\epsilon > 0$ small enough. Let the total energy of the system be given by $E(c, \mathbf{u}) = E_1(c) + E_2(c, \mathbf{u})$ where $E_1(c)$ is the Ginzburg-Landau, see (1.1), and $E_2(c, \mathbf{u})$ is the elastic free energy, see(1.2).
We start with an energy estimate which we will need later for the convergence proof. We compute for a sufficiently smooth solution $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ to (3.1)-(3.7)

$$\begin{aligned}
\frac{d}{dt}E(c^\epsilon, \mathbf{u}^\epsilon) &= \epsilon \int_\Omega \nabla c^\epsilon \cdot \nabla \partial_t c^\epsilon \, dx + \frac{1}{\epsilon}\int_\Omega f(c^\epsilon)\partial_t c^\epsilon \, dx + \int_\Omega W_{,c}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon))\partial_t c^\epsilon \, dx \\
&\quad + \int_\Omega W_{,\mathcal{E}}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon)) : \mathcal{E}(\partial_t \mathbf{u}^\epsilon) \, dx \\
&= -\epsilon \int_\Omega \Delta c^\epsilon \partial_t c^\epsilon \, dx + \frac{1}{\epsilon}\int_\Omega f(c^\epsilon)\partial_t c^\epsilon \, dx + \int_\Omega W_{,c}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon))\partial_t c^\epsilon \, dx \\
&= \int_\Omega \mu^\epsilon \partial_t c^\epsilon \, dx = \int_\Omega \mu^\epsilon \Delta \mu^\epsilon \, dx = -\int_\Omega |\nabla \mu^\epsilon|^2 \, dx \,,
\end{aligned}$$

where we have used the Neumann boundary conditions for c^ϵ and μ^ϵ on $\partial\Omega$ and $\operatorname{div}(W_{,\mathcal{E}}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon))) = 0$ and therefore $\int_\Omega W_{,\mathcal{E}}(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon)) : \mathcal{E}(\partial_t \mathbf{u}^\epsilon) \, dx = 0$. Integrating over $(0, t)$ yields the following a priori estimate

$$E(c^\epsilon, \mathbf{u}^\epsilon)(t) + \int_0^t \int_\Omega |\nabla \mu^\epsilon|^2 \, dx \, dt = E(c^\epsilon, \mathbf{u}^\epsilon)(0) \tag{3.15}$$

for all $t \geq 0$. The additional assumption for F yields

$$\int_\Omega |c^\epsilon|^2 \, dx \leq C\left(\int_\Omega F(c^\epsilon) \, dx + 1\right) \tag{3.16}$$

for some constant $C > 0$. Since $\mathcal{C}$ is positive definite

$$\begin{aligned}
\frac{1}{2}c_2 \int_\Omega |\mathcal{E}(\mathbf{u}^\epsilon) - \mathcal{E}^\star c^\epsilon|^2 \, dx &\leq \frac{1}{2}\int_\Omega (\mathcal{E}(\mathbf{u}^\epsilon) - \mathcal{E}^\star c^\epsilon) : \mathcal{C}\left(\mathcal{E}(\mathbf{u}^\epsilon) - \mathcal{E}^\star c^\epsilon\right) dx \\
&= \int_\Omega W(c^\epsilon, \mathcal{E}(\mathbf{u}^\epsilon)) \, dx \,.
\end{aligned} \tag{3.17}$$

From (3.15)-(3.17) and the Korn inequality, cf. Section 2.4

$$\|\mathbf{u}\|_{H^1(\Omega)} \leq C \|\mathcal{E}(\mathbf{u})\|_{L^2(\Omega)} \quad \forall \mathbf{u} \in H_0^1(\Omega)^d \,,$$

it follows

$$\sup_{0\le t\le T}\left(\|c^\epsilon(t)\|^2_{L^2(\Omega)}+\epsilon\,\|\nabla c^\epsilon(t)\|^2_{L^2(\Omega)}+\|\mathbf{u}^\epsilon(t)\|^2_{W^1_2(\Omega)}\right)+\|\nabla\mu^\epsilon\|^2_{L^2(0,T;W^1_2(\Omega))}$$
$$\le C\left(E(c^\epsilon,\mathbf{u}^\epsilon)(0)+1\right)$$

for some $C>0$ independent of ϵ. Since we later use the initial condition $(c^\epsilon,\mathbf{u}^\epsilon)(.,0)=(c^\epsilon_A,\mathbf{u}^\epsilon_A)(.,0)$ for every $\epsilon\in(0,1]$ where $(c^\epsilon_A,\mathbf{u}^\epsilon_A)$ is an approximate solution to the Cahn-Larché system, we can verify that $E(c^\epsilon,\mathbf{u}^\epsilon)(0)\le C$ for some $C>0$ independent of ϵ. Using equation (3.3) yields

$$\epsilon\sup_{0\le t\le T}\|\mathbf{u}^\epsilon(t)\|^2_{W^2_2(\Omega)}\le C\epsilon\sup_{0\le t\le T}\|c^\epsilon(t)\|^2_{W^1_2(\Omega)}\le C\,.$$

For an exact verification of the first inequality see Claim 1 in the proof of Theorem 3.1.2 below. Therefore we obtain the following energy estimate

$$\sup_{0\le t\le T}\|c^\epsilon(t)\|^2_{L^2(\Omega)}+\epsilon\sup_{0\le t\le T}\|\nabla c^\epsilon(t)\|^2_{L^2(\Omega)}+\sup_{0\le t\le T}\|\mathbf{u}^\epsilon(t)\|^2_{W^1_2(\Omega)}$$
$$+\,\epsilon\sup_{0\le t\le T}\|\mathbf{u}^\epsilon(t)\|^2_{W^2_2(\Omega)}+\|\nabla\mu^\epsilon\|^2_{L^2(0,T;W^1_2(\Omega))}\le C\,,\quad(3.18)$$

for some constant $C=C(c^\epsilon(0),\mu^\epsilon(0),\mathbf{u}^\epsilon(0))>0$ independent of ϵ. With the help of Proposition 2.8.1, we can prove the following theorem.

Theorem 3.1.1. *Let $\{c^\epsilon_A,\mu^\epsilon_A,\boldsymbol{u}^\epsilon_A\}_{0<\epsilon\le 1}$ be a family of functions in the function space $C^\infty(\overline{\Omega_T})\times C^\infty(\overline{\Omega_T})\times C^\infty(\overline{\Omega_T};\mathbb{R}^d)$ satisfying the system of differential equations*

$$(c^\epsilon_A)_t=\Delta\mu^\epsilon_A\quad in\quad \Omega_T\,,\qquad(3.19)$$
$$\mu^\epsilon_A=-\epsilon\Delta c^\epsilon_A+\epsilon^{-1}f(c^\epsilon_A)+W_{,c}(c^\epsilon_A\,,\mathcal{E}(\boldsymbol{u}^\epsilon_A))+r^\epsilon_A\quad in\quad \Omega_T\,,\qquad(3.20)$$
$$\operatorname{div}W_{,\mathcal{E}}(c^\epsilon_A,\mathcal{E}(\boldsymbol{u}^\epsilon_A))=\boldsymbol{s}^\epsilon_A\quad in\quad \Omega_T\,,\qquad(3.21)$$
$$\tfrac{\partial}{\partial n}c^\epsilon_A=\tfrac{\partial}{\partial n}\mu^\epsilon_A=0\quad on\quad \partial_T\Omega\,,\qquad(3.22)$$
$$\boldsymbol{u}^\epsilon_A=0\quad on\quad \partial_T\Omega\,,\qquad(3.23)$$

where $r^\epsilon_A=r^\epsilon_A(x,t)$ and $\boldsymbol{s}^\epsilon_A=\boldsymbol{s}^\epsilon_A(x,t)$ are functions such that

$$\|r^\epsilon_A\|^2_{L^2(\Omega_T)}+\|\boldsymbol{s}^\epsilon_A\|^2_{L^2(\Omega_T)}\le\frac{1}{2}\epsilon^{pk}\,,\qquad(3.24)$$

$p=\frac{2(d+4)}{d+2}$, and $k\in\mathbb{N}$ such that

$$k>\frac{(4d+10)\,(d+2)}{4\,(d+4)}\,.\qquad(3.25)$$

Also assume that c^ϵ_A satisfies the boundedness condition

$$\sup_{0<\epsilon\le 1}\|c^\epsilon_A\|_{L^\infty(\Omega_T)}\le C_0\qquad(3.26)$$

for some $C_0 > 0$, the energy density f satisfies (2.1) and (2.2), and

$$\phi_t^\epsilon(.) := c_A^\epsilon(., t) \tag{3.27}$$

has the form (2.20). Let $(c^\epsilon, \mu^\epsilon, \boldsymbol{u}^\epsilon)$ be the unique solution to (3.1)-(3.7) with $c_0^\epsilon(x) = c_A^\epsilon(x, 0)$ in Ω. Then there exists a constant $\epsilon_0 = \epsilon_0(C_0, T, \Omega, k, d) \in (0, 1]$ such that, if $\epsilon \in (0, \epsilon_0)$, then

$$\|c^\epsilon - c_A^\epsilon\|_{L^p(\Omega_T)} + \|\boldsymbol{u}^\epsilon - \boldsymbol{u}_A^\epsilon\|_{L^2(0,T;W_2^1(\Omega))} \leq C\epsilon^k$$

for some $C > 0$ independent of ϵ.

Proof: Let $R = c^\epsilon - c_A^\epsilon$ and $\mathbf{u} = \mathbf{u}^\epsilon - \mathbf{u}_A^\epsilon$ be the remainder functions.
It holds for all $t \in [0, T]$

$$\begin{aligned}
\int_\Omega R(., t)\, dx &= \int_0^t \int_\Omega \frac{\partial}{\partial t} \left(c^\epsilon - c_A^\epsilon\right) dx\, ds \\
&= \int_0^t \int_\Omega \Delta \left(\mu^\epsilon - \mu_A^\epsilon\right) dx\, ds \\
&= \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial n} \left(\mu^\epsilon - \mu_A^\epsilon\right) d\mathcal{H}^{d-1}\, ds = 0\,.
\end{aligned}$$

Hence there exists a unique smooth solution $\Psi(x, t)$ to the Neumann boundary Problem

$$-\Delta\Psi(., t) = R(., t) \text{ in } \Omega, \qquad \frac{\partial}{\partial n}\Psi(., t) = 0 \text{ on } \partial\Omega, \qquad \int_\Omega \Psi(., t)\, dx = 0$$

for all $t \in [0, T]$. This can be seen as follows. Applying Poincaré's inequality and the Lax-Milgram theorem, we obtain a unique weak solution $\Psi(., t)$ in the space $\left\{c \in W_2^1(\Omega) : \int_\Omega c\, dx = 0\right\}$. An easy calculation shows that $\Psi(.,)$ is also a weak solution in $W_2^1(\Omega)$. Then by applying the usual regularity theory, we prove that $\Psi(., t)$ is smooth, cf. [53, Chapter 4]. Smoothness with respect to time t follows from the smoothness of R.
Multiplying the equation $\partial_t R - \Delta \left(\mu^\epsilon - \mu_A^\epsilon\right) = 0$ by Ψ and integrating over Ω yields

$$\begin{aligned}
0 &= \int_\Omega \Psi \left(\partial_t R - \Delta \left(\mu^\epsilon - \mu_A^\epsilon\right)\right) dx \\
&= \int_\Omega \Psi \left(-\Delta\partial_t \Psi\right) - \Delta\Psi \left(\mu^\epsilon - \mu_A^\epsilon\right) dx \\
&= \frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla\Psi|^2\, dx - \int_\Omega R \left(\epsilon\Delta R - \epsilon^{-1} \left(f(c^\epsilon) - f(c_A^\epsilon)\right) - W_{,c}(R, \mathbf{u}) + r_A^\epsilon\right) dx \\
&= \frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla\Psi|^2\, dx + \int_\Omega \epsilon \left|\nabla R\right|^2 + \epsilon^{-1} f'(c_A^\epsilon) R^2\, dx \\
&\quad + \int_\Omega \epsilon^{-1} \mathcal{N}(c_A^\epsilon, R) R + W_{,c}(R, \mathbf{u}) R - r_A^\epsilon R\, dx,
\end{aligned} \tag{3.28}$$

where we have used the Neumann boundary conditions for Ψ, R and $\mu^\epsilon - \mu_A^\epsilon$ on $\partial\Omega$, the expressions for μ^ϵ and μ_A^ϵ and $-\Delta\Psi = R$. Here $\mathcal{N}(.,.)$ is defined as in Lemma 2.8.3.
Applying Lemma 2.8.3 yields

$$-\int_\Omega \epsilon^{-1}\mathcal{N}(c_A^\epsilon, R)R\,dx \leq C\epsilon^{-1}\|R\|_{L^p(\Omega)}^p \tag{3.29}$$

for some constant $C = C(p) > 0$. By integration by parts and using (3.3) and (3.21), and the symmetry of $\mathcal{C}$, we obtain

$$\int_\Omega \mathcal{E}(\mathbf{v}) : \mathcal{C}(\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R)\,dx = \int_\Omega \mathbf{v}\cdot\mathbf{s}_A^\epsilon\,dx \quad \forall \mathbf{v} \in H_0^1(\Omega)^d\,.$$

For $\mathbf{v} = \mathbf{u} \in H_0^1(\Omega)^d$ this equation yields since $W_{,c}(R, \mathcal{E}(\mathbf{u}))R = -\mathcal{E}^\star R : \mathcal{C}(\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R)$

$$\begin{aligned}
&\int_\Omega W_{,c}(R, \mathcal{E}(\mathbf{u}))R\,dx \\
&= \int_\Omega (\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R) : \mathcal{C}(\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R)\,dx - \int_\Omega \mathbf{u}\cdot\mathbf{s}_A^\epsilon\,dx \\
&\geq c_2\|\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R\|_{L^2(\Omega)}^2 - \Big(\|\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R\|_{L^2(\Omega)} + \|\mathcal{E}^\star R\|_{L^2(\Omega)}\Big)\|\mathbf{s}_A^\epsilon\|_{L^2(\Omega)} \\
&\geq \frac{c_2}{2}\|\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R\|_{L^2(\Omega)}^2 - C\|\mathbf{s}_A^\epsilon\|_{L^2(\Omega)}^2 - \|\mathcal{E}^\star R\|_{L^2(\Omega)}\|\mathbf{s}_A^\epsilon\|_{L^2(\Omega)}\,,
\end{aligned} \tag{3.30}$$

where we have used the Korn and triangle inequality in the first estimate and Young's inequality in the second estimate. Hölder's inequality gives us the estimate

$$\int_\Omega r_A^\epsilon R\,dx \leq \|r_A^\epsilon\|_{L^2(\Omega)}\|R\|_{L^2(\Omega)}\,. \tag{3.31}$$

By Proposition 2.8.1, there exists some constant $C > 0$ such that

$$\int_\Omega \epsilon|\nabla R|^2 + \epsilon^{-1}f'(c_A^\epsilon)R^2\,dx \geq -C\|\nabla\Psi\|_{L^2(\Omega)}^2 + 2\epsilon\|R\|_{L^2(\Omega)}^2\,. \tag{3.32}$$

Therefore equation (3.28) together with (3.29)-(3.32) and Young's inequality provide us

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\,dx + \epsilon\|R\|_{L^2(\Omega)}^2 + \frac{c_2}{2}\|\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R\|_{L^2(\Omega)}^2 \\
&\quad\leq C\Big(\|\nabla\Psi\|_{L^2(\Omega)}^2 + \epsilon^{-1}\|R\|_{L^p(\Omega)}^p + \epsilon^{-1}\|r_A^\epsilon\|_{L^2(\Omega)}^2 + \epsilon^{-1}\|\mathbf{s}_A^\epsilon\|_{L^2(\Omega)}\Big)
\end{aligned} \tag{3.33}$$

for all $t \in (0, T]$ and some $C = C(f, C_0, p)$. Note that $R(., 0) = 0$ and therefore $\Psi(., 0) = 0$. Then applying Gronwall's inequality yields

$$\begin{aligned}
\sup_{0\leq\tau\leq t}\|\nabla\Psi(., \tau)\|_{L^2(\Omega)}^2 \leq\; & Ce^{Ct}\Big(\epsilon^{-1}\|r_A^\epsilon\|_{L^2(\Omega_t)}^2 \\
&+\epsilon^{-1}\|\mathbf{s}_A^\epsilon\|_{L^2(\Omega_t)}^2 + \epsilon^{-1}\|R\|_{L^p(\Omega_t)}^p\Big)\,.
\end{aligned} \tag{3.34}$$

Integrating inequality (3.33) over $(0,t)$, $t \in [0,T]$, and using (3.34) yields

$$\begin{aligned} &\epsilon \|R\|^2_{L^2(\Omega_t)} + \frac{c_2}{2} \|\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R\|^2_{L^2(\Omega_t)} \\ &\leq C_1 \left(\epsilon^{-1} \|r^\epsilon_A\|^2_{L^2(\Omega_t)} + \epsilon^{-1} \|\mathbf{s}^\epsilon_A\|^2_{L^2(\Omega_t)} + \epsilon^{-1} \|R\|^p_{L^p(\Omega_t)} \right) \end{aligned} \tag{3.35}$$

for some constant $C_1 = C_1(T) > 0$ independent of ϵ and $t \in (0,T]$. Integrating (3.28) over $(0,t)$ yields

$$\epsilon \|\nabla R\|^2_{L^2(\Omega_t)} \leq C \left(\epsilon \|r^\epsilon_A\|^2_{L^2(\Omega_t)} + \|\mathbf{s}^\epsilon_A\|^2_{L^2(\Omega_t)} + \epsilon^{-1} \|R\|^2_{L^2(\Omega_t)} + \epsilon^{-1} \|R\|^p_{L^p(\Omega_t)} \right). \tag{3.36}$$

Since $\int_\Omega R(.,t) = 0$, the Gagliardo-Nirenberg inequality and a Poincaré's inequality (see Section 2.4) implies for $p = 2\frac{d+4}{d+2}$ and for all $t \in [0,T]$

$$\|R\|^p_{L^p(\Omega)} \leq C \|R\|^{\frac{8}{d+2}}_{L^2(\Omega)} \|\nabla R\|^{\frac{2d}{d+2}}_{L^2(\Omega)} .$$

To estimate $\|R\|_{L^2(\Omega)}$ on the right-hand side, we use integration by parts to get

$$\|R\|^2_{L^2(\Omega)} = -\int_\Omega R \Delta \Psi \, dx = \int_\Omega \nabla R \cdot \nabla \Psi \, dx \leq \|\nabla R\|_{L^2(\Omega)} \|\nabla \Psi\|_{L^2(\Omega)} ,$$

where we have used the Neumann boundary conditions for c^ϵ and c^ϵ_A on $\partial\Omega$. Therefore it holds

$$\begin{aligned} \|R\|^p_{L^p(\Omega_t)} &\leq C \int_0^t \|\nabla \Psi(.,\tau)\|^{\frac{4}{d+2}}_{L^2(\Omega)} \|\nabla R(.,\tau)\|^2_{L^2(\Omega)} \, d\tau \\ &\leq C \sup_{0 \leq \tau \leq t} \|\nabla \Psi(.,\tau)\|^{\frac{4}{d+2}}_{L^2(\Omega)} \|\nabla R\|^2_{L^2(\Omega_t)} . \end{aligned} \tag{3.37}$$

In order to treat the term $\|R\|_{L^2(\Omega_t)}$ in (3.36), we define two sets

$$\begin{aligned} A^\epsilon_1 &:= \left\{ t \in [0,T] : \epsilon \|R\|^2_{L^2(\Omega_t)} > 2C_1 \epsilon^{-1} \|R\|^p_{L^p(\Omega_t)} \right\}, \\ A^\epsilon_2 &:= \left\{ t \in [0,T] : \epsilon \|R\|^2_{L^2(\Omega_t)} \leq 2C_1 \epsilon^{-1} \|R\|^p_{L^p(\Omega_t)} \right\}, \end{aligned}$$

where C_1 is the same constant as in (3.35). Furthermore, we set

$$T^\epsilon := \sup \left\{ t \in (0,T] : \|R\|_{L^p(\Omega_t)} \leq \epsilon^k \right\}.$$

1st case: "$T^\epsilon \in A^\epsilon_1$"
Then the definition of A^ϵ_1 and (3.35) yield

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \leq \frac{\epsilon^2}{2C_1} \|R\|^2_{L^2(\Omega_{T^\epsilon})} \leq \frac{1}{2} \left(\|r^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \|\mathbf{s}^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \|R\|^p_{L^p(\Omega_{T^\epsilon})} \right).$$

Therefore we get

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \le \|r^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \|\mathbf{s}^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} \le \frac{1}{2}\epsilon^{pk}\,, \tag{3.38}$$

where we have used (3.24). Hence we conclude $T^\epsilon = T$ by definition of T^ϵ.
2nd case: "$T^\epsilon \in A^\epsilon_2$"
We use inequality (3.37) and apply (3.34), (3.36), and the definition of A^ϵ_2 to obtain

$$\begin{aligned}\|R\|^p_{L^p(\Omega_{T^\epsilon})} \quad &\le \quad C\left(\epsilon^{-1}\|r^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \epsilon^{-1}\|\mathbf{s}^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \epsilon^{-1}\|R\|^p_{L^p(\Omega_{T^\epsilon})}\right)^{\frac{2}{d+2}}\\ &\qquad \times\left(\|r^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \epsilon^{-1}\|\mathbf{s}^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \epsilon^{-4}\|R\|^p_{L^p(\Omega_{T^\epsilon})}\right).\end{aligned}$$

Applying (3.24) and $\|R\|_{L^p(\Omega_{T^\epsilon})} \le \epsilon^k$, yields

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \le C\epsilon^{(-1+pk)\frac{2}{d+2}}\epsilon^{-4+pk} \le C\epsilon^{pk}\epsilon^{\frac{4(d+4)}{(d+2)^2}\left(k-\frac{(4d+10)(d+2)}{4(d+4)}\right)}\,,$$

where we have used the definition of p in the second inequality. By assumption we have $k > \frac{(4d+10)(d+2)}{4(d+4)}$. Therefore there exists $\epsilon_0 \in (0,1]$ such that for all $\epsilon \in (0,\epsilon_0]$, it holds

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \le \frac{1}{2}\epsilon^{pk}\,,$$

provided $T^\epsilon \in A^\epsilon_2$. As in the 1st case, it follows $T^\epsilon = T$.
The estimate for $\mathbf{u}$ follows from

$$\|\mathcal{E}(\mathbf{u})\|_{L^2(\Omega_T)} \le \|\mathcal{E}(\mathbf{u}) - \mathcal{E}^\star R\|_{L^2(\Omega_T)} + \|\mathcal{E}^\star R\|_{L^2(\Omega_T)} \le C\epsilon^{\frac{pk-1}{2}} + C\epsilon^k \le C\epsilon^k$$

by (3.35) and since $pk - 1 \ge 2k$ (use the definitions of p and k). Thus the assertion follows due to the Korn inequality. □

We even get assertions in stronger norms.

Theorem 3.1.2. *Let the assumptions of Theorem 3.1.1 hold. Let $m > 0$ be any fixed integer and assume $\|c^\epsilon_A\|_{W^{m+l+1}_2(\Omega_T)} + \|\mu^\epsilon_A\|_{W^{m+l-1}_2(\Omega_T)} + \|\boldsymbol{u}^\epsilon_A\|_{W^{m+l+1}_2(\Omega_T)} \le \epsilon^{-K(m)}$ for $l > \frac{d+1}{2}$, some integer $K(m)$, and all small $\epsilon > 0$. If k in (3.24) is large enough, then*

$$\|c^\epsilon - c^\epsilon_A\|_{C^m(\Omega_T)} + \|\mu^\epsilon - \mu^\epsilon_A\|_{C^{m-2}(\Omega_T)} + \|\boldsymbol{u}^\epsilon - \boldsymbol{u}^\epsilon_A\|_{C^{m+1}(\Omega_T)} \le \epsilon$$

for all sufficiently small $\epsilon > 0$.

Proof: We show the assertion in the same way as in [10, Theorem 2.3.].
Since for every $m \in \mathbb{N}$

$$W^{m+l}_2(\Omega_T) \hookrightarrow C^m(\overline{\Omega_T}) \quad \text{if } l > \frac{d+1}{2}$$

and since

$$\left(L^2(\Omega_T), W_2^{m+l+1}(\Omega_T)\right)_{\theta,2} = W_2^{m+l}(\Omega_T) \quad \text{for } \theta = \frac{m+l}{m+l+1}\,,$$

cf. Section 2.5, it follows

$$\|c^\epsilon - c_A^\epsilon\|_{C^m(\Omega_T)} \le C\,\|c^\epsilon - c_A^\epsilon\|^{\theta}_{L^2(\Omega_T)}\,\|c^\epsilon - c_A^\epsilon\|^{1-\theta}_{W_2^{m+l+1}(\Omega_T)} \tag{3.39}$$

for some $C>0$. Therefore it is sufficient to show

$$\|c^\epsilon\|^{1-\theta}_{W_2^{m+l+1}(\Omega_T)} \le \epsilon^{-K(m)}$$

for some integer $K(m)$ if k in (3.24) is large enough. The estimate for $\mu^\epsilon - \mu_A^\epsilon$ follows from the equations for the chemical potential (3.2) and (3.20) (see end of the proof). We replace f by $\bar f$ such that $f = \bar f$ in $(-\frac{3}{2}C_0, \frac{3}{2}C_0)$ and $\bar f(c)$ is linear when $|c| > 2C_0$ where C_0 is the same constant as in (3.26). Denote by $(\bar c^\epsilon, \bar{\mathbf{u}}^\epsilon)$ the solution to the modified system with $\bar f$. Define $A : D(A) \to L^p(\Omega)$ by $A = -\Delta + \mathrm{Id}$ with $D(A) = \left\{c \in W_p^2(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = 0\right\}$. W.l.o.g. we assume that in Lemma 2.7.1 the constant $\tau = 1$ since we only consider a finite number of different p's. Otherwise we replace A by $-\Delta + c\mathrm{Id}$ for some $c \in \mathbb{R}$. Therefore $-A^2$ is sectorial with domain $D(-A^2) = \left\{c \in W_p^4(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = \frac{\partial}{\partial n} Ac\big|_{\partial\Omega} = 0\right\}$ and $W_p^1(\Omega) = D(A^{\frac{1}{2}})$ with equivalent norms, see Section 2.7.
Since in general $\bar c^\epsilon \notin D(-A^2)$ (this is the main difference to [10, Theorem 2.3]), we add a function Ψ such that $\bar c^\epsilon + \epsilon^{-1}\Psi \in D(-A^2)$. Define $\Psi(x,t)$ as the unique solution to the Neumann boundary problem

$$-\Delta\Psi(.,t) = W_{,c}(\bar c^\epsilon, \mathcal{E}(\bar{\mathbf{u}}^\epsilon))(.,t) - \frac{1}{|\Omega|}\int_\Omega W_{,c}(\bar c^\epsilon, \mathcal{E}(\bar{\mathbf{u}}^\epsilon))(x,t)\,dx \qquad \text{in } \Omega\,, \tag{3.40}$$

$$\frac{\partial}{\partial n}\Psi(.,t) = 0 \ \text{ on } \partial\Omega\,, \qquad \int_\Omega \Psi(x,t)\,dx = 0\,. \tag{3.41}$$

For the rest of the proof, we use the following estimates for Ψ.
Claim 1: For all $p \ge 2$ and for all $k, n \in \mathbb{N}$, it holds

$$\left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{W_2^1(\Omega)} \le C\left\|\partial_t^k \bar c^\epsilon\right\|_{L^2(\Omega)}\,, \qquad \left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{W_p^{n+2}(\Omega)} \le C\left\|\partial_t^k \bar c^\epsilon\right\|_{W_p^{n+1}(\Omega)}\,, \tag{3.42}$$

$$\left\|\partial_t^k \Psi\right\|_{W_2^1(\Omega)} \le C\left\|\partial_t^k \bar c^\epsilon\right\|_{L^2(\Omega)}\,, \qquad \left\|\partial_t^k \Psi\right\|_{W_p^{n+3}(\Omega)} \le C\left\|\partial_t^k \bar c^\epsilon\right\|_{W_p^{n+1}(\Omega)} \tag{3.43}$$

for some constant $C = C(p,k,n) > 0$.
Since $\mathbf{u} = \partial_t^k \bar{\mathbf{u}}^\epsilon(.,t)$, $t > 0$, is a solution to

$$\begin{aligned} \operatorname{div}\left(\mathcal{C}\mathcal{E}(\mathbf{u})\right) &= \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star \partial_t^k \bar c^\epsilon(.,t)\right) && \text{in } \Omega\,,\\ \mathbf{u} &= 0 && \text{on } \partial\Omega\,, \end{aligned}$$

(see (3.3) and (3.4)) and since $\mathcal{C}$ is positive definite and $\mathcal{C}_{iji'j'} = \mathcal{C}_{jii'j'}$, we obtain

$$\begin{aligned}
\left\|\mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon)\right\|_{L^2(\Omega)}^2 &\leq c_2 \int_\Omega \mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon) : \mathcal{C}\mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon)\, dx = c_2 \int_\Omega \mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon) : \mathcal{C}\mathcal{E}^\star \partial_t^k \bar{c}^\epsilon\, dx \\
&\leq C \left\|\mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon)\right\|_{L^2(\Omega)} \left\|\partial_t^k \bar{c}^\epsilon\right\|_{L^2(\Omega)} .
\end{aligned}$$

Dividing both sides by $\left\|\mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon)\right\|_{L^2(\Omega)}$ and by Korn inequality, cf. Section 2.4, the first assertion follows.
The differential operator div $(\mathcal{C}\mathcal{E}(.))$ is strongly elliptic since $\mathcal{C}$ is positive definite. Then due to to Agmon et al. [9, Theorem 10.5], there exists some constant $C = C(p,n) > 0$ for all $n \in \mathbb{N}$ and $p > 1$ such that

$$\left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{W_p^{n+2}(\Omega)} \leq C\left(\left\|\partial_t^k \bar{c}^\epsilon\right\|_{W_p^{n+1}(\Omega)} + \left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{L^p(\Omega)}\right) . \tag{3.44}$$

To estimate the term $\left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{L^p(\Omega)}$ for $p \geq 2$ on the right-hand side, we use Ehrling's Lemma, cf. [56, Theorem 7.30]. Since $W_p^{n+2}(\Omega)^d$ is compactly imbedded in $L^p(\Omega)^d$ and $L^p(\Omega)^d$ is continuously imbedded in $L^2(\Omega)^d$, we obtain that for all $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$\|\mathbf{u}\|_{L^p(\Omega)} \leq C(\delta)\, \|\mathbf{u}\|_{L^2(\Omega)} + \delta\, \|\mathbf{u}\|_{W_p^{n+2}(\Omega)} ,$$

for every $\mathbf{u} \in W_p^{n+2}(\Omega)^d$. Using this estimate on the right-hand side in (3.44) and choosing $\delta > 0$ small enough, we have

$$\left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{W_p^{n+2}(\Omega)} \leq C\left(\left\|\partial_t^k \bar{c}^\epsilon\right\|_{W_p^{n+1}(\Omega)} + \left\|\partial_t^k \bar{\mathbf{u}}^\epsilon\right\|_{L^2(\Omega)}\right) \leq C \left\|\partial_t^k \bar{c}^\epsilon\right\|_{W_p^{n+1}(\Omega)} ,$$

where the last inequality follows from the first assertion in Claim 1. Thus the second assertion follows.
Multiplying (3.40) by Ψ, integrating the resulting equation over Ω, and using a Poincaré's inequality, cf. Section 2.4 yields

$$\begin{aligned}
\|\Psi\|_{W_2^1(\Omega)} &\leq C\, \|\nabla \Psi\|_{L^2(\Omega)} \leq C\, \|W_{,c}(\bar{c}^\epsilon, \mathcal{E}(\bar{\mathbf{u}}^\epsilon))\|_{L^2(\Omega)} \\
&\leq C\left(\|\bar{\mathbf{u}}^\epsilon\|_{W_2^1(\Omega)} + \|\bar{c}^\epsilon\|_{L^2(\Omega)}\right) \leq C\, \|\bar{c}^\epsilon\|_{L^2(\Omega)} .
\end{aligned}$$

Since $0 = \partial_t^k \int_\Omega \Psi(.,t)\, dx = \int_\Omega \partial_t^k \Psi(.,t)\, dx$ for $k \in \mathbb{N}$, the function $\partial_t^k \Psi$ can be estimated in the same way.
In addition, we obtain due to Agmon et al. [8, Theorem 15.2.] for all $k, n \in \mathbb{N}$

$$\begin{aligned}
\left\|\partial_t^k \Psi\right\|_{W_p^{n+3}(\Omega)} &\leq C\left(\left\|W_{,c}(\partial_t^k \bar{c}^\epsilon, \mathcal{E}(\partial_t^k \bar{\mathbf{u}}^\epsilon))\right\|_{W_p^{n+1}(\Omega)} + \left\|\partial_t^k \Psi\right\|_{L^p(\Omega)}\right) \\
&\leq C\left(\left\|\partial_t^k \bar{c}^\epsilon\right\|_{W_p^{n}(\Omega)} + \left\|\partial_t^k \Psi\right\|_{L^p(\Omega)}\right)
\end{aligned}$$

for some constant $C = C(n,p) > 0$.
We treat the term $\left\|\partial_t^k \Psi\right\|_{L^p(\Omega)}$ in the same way as above. Again by using Ehrling's Lemma for $p \geq 2$, there exists for all $\delta > 0$ a constant $C(\delta) > 0$ such that

$$\begin{aligned}\left\|\partial_t^k \Psi\right\|_{L^p(\Omega)} &\leq C(\delta)\left\|\partial_t^k \Psi\right\|_{L^2(\Omega)} + \delta\left\|\partial_t^k \Psi\right\|_{W_p^{n+3}p(\Omega)} \\ &\leq C(\delta)\left\|\partial_t^k \bar{c}^\epsilon\right\|_{L^2(\Omega)} + \delta\left\|\partial_t^k \Psi\right\|_{W_p^{n+3}p(\Omega)}\,.\end{aligned}$$

Choosing $\delta > 0$ small enough Claim 1 follows.
We set $\Theta := \bar{c}^\epsilon + \epsilon^{-1}\Psi$. Then the Cahn-Hilliard equation can be written as

$$\begin{aligned}\partial_t\Theta + \epsilon A^2\Theta &= A\left(2\epsilon\Theta - \epsilon^{-1}\bar{f}(\bar{c}^\epsilon)\right) + \epsilon^{-1}\bar{f}(\bar{c}^\epsilon) + \epsilon^{-1}\partial_t\Psi - \epsilon\Theta \\ &= \epsilon^{-1}Af_1(\bar{c}^\epsilon,\Theta) + \epsilon^{-1}f_2(\bar{c}^\epsilon,\Theta) + \epsilon^{-1}\partial_t\Psi\,, \qquad (3.45)\end{aligned}$$

where $f_1(c,\Theta) := -\bar{f}(c) + 2\epsilon^2\Theta$ and $f_2(c,\Theta) := \bar{f}(c) - \epsilon^2\Theta$, and where we have used (3.40). Since

$$\frac{\partial}{\partial n}\Theta = \frac{\partial}{\partial n}\bar{c}^\epsilon + \epsilon^{-1}\frac{\partial}{\partial n}\Psi = 0\,,$$

and

$$\begin{aligned}\frac{\partial}{\partial n}A\Theta &= \frac{\partial}{\partial n}\left(-\Delta\bar{c}^\epsilon + \epsilon^{-1}W_{,c}(\bar{c}^\epsilon,\mathcal{E}(\bar{\mathbf{u}}^\epsilon)) - \epsilon^{-1}\frac{1}{|\Omega|}\int_\Omega W_{,c}(\bar{c}^\epsilon,\mathcal{E}(\bar{\mathbf{u}}^\epsilon))(x)\,dx\right) \\ &= \frac{\partial}{\partial n}\left(\epsilon^{-1}\bar{\mu}^\epsilon - \epsilon^{-2}\bar{f}(\bar{c}^\epsilon)\right) = 0\,,\end{aligned}$$

it follows $\Theta \in D(-A^2)$. Therefore we get by semigroup theory

$$\begin{aligned}\Theta(t) &= e^{-\epsilon A^2 t}\Theta(0) \\ &\quad + \epsilon^{-1}\int_0^t e^{-\epsilon A^2(t-\tau)}\left[Af_1(\bar{c}^\epsilon,\Theta)(\tau) + f_2(\bar{c}^\epsilon,\Theta)(\tau) + \partial_t\Psi(\tau)\right]d\tau\,. \qquad (3.46)\end{aligned}$$

Since $\partial_\tau e^{-\epsilon A^2(t-\tau)} = \epsilon A^2 e^{-\epsilon A^2(t-\tau)}$, cf. Pazy [55, Chapter 2, Lemma 4.2], we derive by integration by parts

$$\int_0^t e^{-\epsilon A^2(t-\tau)}\partial_t\Psi(\tau)\,d\tau = e^{-\epsilon A^2(t-\tau)}\Psi(\tau)\Big|_{\tau=0}^{\tau=t} - \int_0^t \epsilon A^2 e^{-\epsilon A^2(t-\tau)}\Psi(\tau)\,d\tau\,.$$

We denote by $\|.\|_p$ the norm of operators from $L^p(\Omega)$ to $L^p(\Omega)$.
If we apply $A^{1/2+\alpha}$, $\alpha \geq 0$, to both sides of (3.46), then we get

$$\begin{aligned}&\left\|A^{1/2+\alpha}\Theta(t)\right\|_{L^p} \\ &\leq \sup_{\tau'>0}\left\|e^{-A^2\tau'}\right\|_p\left(\left\|A^{1/2+\alpha}\Theta(0)\right\|_{L^p} + \epsilon^{-1}\left\|A^{1/2+\alpha}\Psi(0)\right\|_{L^p}\right) + \epsilon^{-1}\left\|A^{1/2+\alpha}\Psi(t)\right\|_{L^p} \\ &\quad + \epsilon^{-1}\left(\sup_{\tau'>0}\left\|(\tau' A^2)^{3/4}e^{-A^2\tau'}\right\|_p\right)\int_0^t\Big[(\epsilon(t-\tau))^{-3/4}\left(\|A^\alpha f_1(\bar{c}^\epsilon,\Theta)(\tau)\|_{L^p}\right. \\ &\quad \left. + \left\|A^{\alpha-1}f_2(\bar{c}^\epsilon,\Theta)(\tau)\right\|_{L^p} + \epsilon\left\|A^{1+\alpha}\Psi(\tau)\right\|_{L^p}\right)\Big]d\tau\,.\end{aligned}$$

Note that for $\beta \in [0,\infty)$, there exists a constant $C = C(\Omega, p, \beta)$ such that

$$\sup_{\tau\geq 0} \left\| (\tau A^2)^\beta e^{-A^2\tau} \right\|_p \leq C\,,$$

cf. [55, Chapter 2, Theorem 6.13]. Therefore we obtain the recurrence inequality

$$\begin{aligned}
&\left\|A^{1/2+\alpha}\Theta\right\|_{L^\infty(0,T;L^p)} \\
&\leq\ C\Big[\left\|A^{1/2+\alpha}\Theta(0)\right\|_{L^p} + \epsilon^{-1}\left\|A^{1/2+\alpha}\Psi(0)\right\|_{L^p} + \epsilon^{-7/4}\Big(\|A^\alpha f_1(\bar{c}^\epsilon,\Theta)\|_{L^\infty(0,T;L^p)} \\
&\quad + \left\|A^{\alpha-1}f_2(\bar{c}^\epsilon,\Theta)\right\|_{L^\infty(0,T;L^p)} + \left\|A^{1+\alpha}\Psi\right\|_{L^\infty(0,T;L^p)}\Big)\Big]\,, \qquad (3.47)
\end{aligned}$$

where we have used $\left\|A^{1/2+\alpha}\Psi(t)\right\|_{L^p} \leq C\left\|A^{1+\alpha}\Psi(t)\right\|_{L^p}$ for some $C > 0$ since $D(A^{\frac{1}{2}}) = W^1_p(\Omega)$. Since $\bar{f}$ has linear growth, there exists a positive constant C such that for any $p \in [1, 2d]$ and any $c \in L^p(\Omega)$

$$\|f_1(c,\Theta)\|_{L^p(\Omega)} + \|f_2(c,\Theta)\|_{L^p(\Omega)} \leq C\left(1 + \|c\|_{L^p(\Omega)} + \|\Theta\|_{L^p(\Omega)}\right). \qquad (3.48)$$

We use (3.47) and (3.48) to estimate $\|\bar{c}^\epsilon\|_{L^\infty(\Omega_T)}$.

Claim 2: There exists some $p > d$, an integer k_0, and a constant $C > 0$ independent of ϵ such that

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;L^\infty)} + \|\bar{c}^\epsilon\|_{L^\infty(0,T;W^1_p)} + \|\Theta\|_{L^\infty(0,T;L^\infty)} + \|\Theta\|_{L^\infty(0,T;W^1_p)} \leq C\epsilon^{-k_0}\,.$$

For the proof we use a bootstrap method. We set $p = p_0 := 2$. Then we already know from the energy estimate (3.18)

$$\epsilon\left(\|\bar{c}^\epsilon\|^2_{L^\infty(0,T;W^1_2)} + \|\Theta\|^2_{L^\infty(0,T;W^1_2)}\right) \leq C\,. \qquad (3.49)$$

We set $\alpha = 0$ in (3.47) and $p_1 = \frac{dp_0}{d-p_0}$ if $d \geq 3$ (in the case $d = 2$ we set $p_1 = 3$). Then it holds

$$\begin{aligned}
&\left\|A^{1/2}\Theta\right\|_{L^\infty(0,T;L^{p_1})} \\
&\leq\ C\Big(\left\|A^{1/2}\Theta(0)\right\|_{L^{p_1}} + \epsilon^{-1}\left\|A^{1/2}\Psi(0)\right\|_{L^{p_1}} + \epsilon^{-7/4}\Big(\|\bar{c}^\epsilon\|_{L^\infty(0,T;L^{p_1})} \\
&\quad + \|\Theta\|_{L^\infty(0,T;L^{p_1})} + \|A\Psi\|_{L^\infty(0,T;L^{p_1})} + 1\Big)\Big)\,,
\end{aligned}$$

where we have used (3.48). For estimating the right-hand side we use Sobolev's imbedding and (3.43) to get

$$\begin{aligned}
\|\bar{c}^\epsilon\|_{L^{p_1}(\Omega)} + \|\Theta\|_{L^{p_1}(\Omega)} + \|A\Psi\|_{L^{p_1}(\Omega)} &\leq C\left(\|\bar{c}^\epsilon\|_{W^1_2(\Omega)} + \|\Theta\|_{W^1_2(\Omega)} + \|\Psi\|_{W^3_2(\Omega)}\right) \\
&\leq C\left(\|\bar{c}^\epsilon\|_{W^1_2(\Omega)} + \|\Theta\|_{W^1_2(\Omega)}\right).
\end{aligned}$$

Therefore there exists some integer k_0 and some constant $C > 0$ such that

$$\left\|A^{1/2}\Theta\right\|_{L^\infty(0,T;L^{p_1})} \le C\epsilon^{-k_0}\,.$$

Since $D(A_{p_1}^{\frac{1}{2}}) = W^1_{p_1}(\Omega)$, it holds

$$\|\Theta\|_{L^\infty(0,T;W^1_{p_1})} \le C\left\|A^{1/2}\Theta\right\|_{L^\infty(0,T;L^{p_1})} \le C\epsilon^{-k_0}\,,$$

where A_{p_1} is the realization of the differential operator A in $L^{p_1}(\Omega)$. Hence by definition of Θ, we get

$$\begin{aligned}\|\overline{c}^\epsilon\|_{L^\infty(0,T;W^1_{p_1})} &\le \|\Theta\|_{L^\infty(0,T;W^1_{p_1})} + \epsilon^{-1}\|\Psi\|_{L^\infty(0,T;W^1_{p_1})}\\ &\le \|\Theta\|_{L^\infty(0,T;W^1_{p_1})} + \epsilon^{-1}\|\Psi\|_{L^\infty(0,T;W^2_2)} \le C\epsilon^{-k_0}\,,\end{aligned}$$

where the last inequality follows from (3.43). (For better clarity we again write k_0, although k_0 is possibly larger than k_0 above.) Again we apply Sobolev's imbedding

$$\begin{aligned}&\|\Theta\|_{L^\infty(0,T;L^{p_2})} \le C\|\Theta\|_{L^\infty(0,T;W^1_{p_1})}\,, \quad \|\overline{c}^\epsilon\|_{L^\infty(0,T;L^{p_2})} \le C\|\overline{c}^\epsilon\|_{L^\infty(0,T;W^1_{p_1})}\,,\\ &\|A\Psi\|_{L^\infty(0,T;L^{p_2})} \le C\|\Psi\|_{L^\infty(0,T;W^3_{p_1})} \le C\|\overline{c}^\epsilon\|_{L^\infty(0,T;W^1_{p_1})}\,,\end{aligned}$$

where $p_2 = \frac{dp_1}{d-p_1}$ (in the case $d > p_1$). Repeating the same procedure step by step, we can show that $\|\overline{c}^\epsilon\|_{L^\infty(0,T;W^1_{p_i})} + \|\Theta\|_{L^\infty(0,T;W^1_{p_i})} \le C\epsilon^{-k_0}$ where $p_i = \frac{dp_{i-1}}{d-p_{i-1}}$, until $p_i > d$ for some finite integer $i = i(d)$. By Sobolev's imbedding Claim 2 follows.
To get estimates in stronger norms we set $\alpha = \frac{1}{2}, 1, \frac{3}{2}, \dots$ in (3.47) where we can control the terms on the right-hand side. To do this, we have to show some regularity estimates. For any $\beta \in \frac{1}{2}\mathbb{N}$ there exists a positive constant $C = C(\beta, k, l, p)$ such that for all $c \in W^k_\infty(0,T; W^{2\beta+l+2}_p(\Omega) \cap W^{2\beta+l+1}_\infty(\Omega))$, $k, l \in \mathbb{N}$

$$\begin{aligned}&\left\|A^\beta\partial_t^k Af_1(c,\Theta)\right\|_{L^\infty(0,T;W^l_p(\Omega))} + \left\|A^\beta\partial_t^k f_2(c,\Theta)\right\|_{L^\infty(0,T;W^l_p(\Omega))}\\ &\le C\left[\left(1 + \|c\|^{2\beta+k+l+1}_{W^k_\infty(0,T;W^{2\beta+l+1}_\infty)}\right)\left(1 + \|c\|_{W^k_\infty(0,T;W^{2\beta+l+2}_p)}\right)\right.\\ &\qquad \left. + \|\Theta\|_{W^k_\infty(0,T;W^{2\beta+l+2}_p)}\right],\end{aligned} \tag{3.50}$$

where we get the term $\|c\|^{2\beta+k+l+1}_{W^k_\infty(0,T;W^{2\beta+l+1}_\infty(\Omega))}$ by chain rule.
By definition of A the function $c = A^n\Theta$, $n \in \mathbb{N}$, is the solution to the elliptic Neumann-boundary problem

$$\begin{aligned}\Delta c - c &= -A^{n+1}\Theta \quad \text{in } \Omega\,,\\ \tfrac{\partial}{\partial n}c &= \tfrac{\partial}{\partial n}\left(A^n\Theta\right) \quad \text{on } \partial\Omega\,.\end{aligned}$$

Therefore [8, Theorem 15.2.] yields

$$\begin{aligned}\|A^n\Theta\|_{W_p^{m+2}(\Omega)} &\le C(\Omega,p,m)\Big(\left\|A^{n+1}\Theta\right\|_{W_p^m(\Omega)} \\ &\qquad + \|A^n\Theta\|_{L^p(\Omega)} + \left\|\tfrac{\partial}{\partial n}\left(A^n\Theta\right)\right\|_{W_p^{m+1-\frac{1}{p}}(\partial\Omega)}\Big)\end{aligned}$$

for all $m \in \mathbb{N}$. Therefore for all $m \in \mathbb{N}$ there exists a constant $C = C(m) > 0$ such that

$$\|\Theta\|_{W_p^{2m}(\Omega)} \le C\|A^m\Theta\|_{L^p(\Omega)} + \sum_{i=0}^{m-1}\left\|\tfrac{\partial}{\partial n}A^i\Theta\right\|_{W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega)}, \tag{3.51}$$

$$\|\Theta\|_{W_p^{2m+1}(\Omega)} \le C\|A^m\Theta\|_{W_p^1(\Omega)} + \sum_{i=0}^{m-1}\left\|\tfrac{\partial}{\partial n}A^i\Theta\right\|_{W_p^{2m-2i-\frac{1}{p}}(\partial\Omega)}, \tag{3.52}$$

where we use the convention that if the upper limit of the summation is less than the lower limit, then the summation is zero. To estimate the boundary terms $\frac{\partial}{\partial n}A^i\Theta$, we use the boundary conditions $\frac{\partial}{\partial n}\Theta = \frac{\partial}{\partial n}A\Theta = 0$. We apply (3.45) $\lfloor i/2\rfloor$-times to obtain

$$\begin{aligned}A^i\Theta &= (-1)^{\lfloor i/2\rfloor}\epsilon^{-\lfloor i/2\rfloor}A^{i-2\lfloor i/2\rfloor}\partial_t^{\lfloor i/2\rfloor}\Theta \\ &\quad + \sum_{j=0}^{\lfloor i/2\rfloor-1}(-1)^j\epsilon^{-j-1}A^{i-2j-2}\partial_t^j\left(\epsilon^{-1}Af_1(\bar{c}^\epsilon,\Theta) + \epsilon^{-1}f_2(\bar{c}^\epsilon,\Theta) + \epsilon^{-1}\partial_t\Psi\right).\end{aligned}$$

Since $\frac{\partial}{\partial n}\partial_t^i\Theta = \frac{\partial}{\partial n}A\partial_t^i\Theta = 0$ for all $i \in \mathbb{N}$, we can neglect the first term on the right-hand side. Hence we can follow

$$\begin{aligned}&\left\|\tfrac{\partial}{\partial n}A^i\Theta\right\|_{W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega)} \\ &\le \epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\left\|\tfrac{\partial}{\partial n}A^{i-2j-2}\partial_t^j\left(Af_1(\bar{c}^\epsilon,\Theta) + f_2(\bar{c}^\epsilon,\Theta) + \partial_t\Psi\right)\right\|_{W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega)} \\ &\le C\epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\left\|A^{i-2j-2}\partial_t^j\left(Af_1(\bar{c}^\epsilon,\Theta) + f_2(\bar{c}^\epsilon,\Theta) + \partial_t\Psi\right)\right\|_{W_p^{2m-2i}(\Omega)}.\end{aligned}$$

We can estimate the right-hand side by inequality (3.50) as follows

$$\begin{aligned}&\left\|\tfrac{\partial}{\partial n}A^i\Theta\right\|_{L^\infty(0,T;W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega))} \\ &\le C\epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\Big[\left(1+\|\bar{c}^\epsilon\|^{2m-3j-3}_{W^j_\infty(0,T;W_\infty^{2m-4j-3}(\Omega))}\right)\left(1+\|\bar{c}^\epsilon\|_{W^j_\infty(0,T;W_p^{2m-4j-2}(\Omega))}\right) \\ &\qquad + \|\Theta\|_{W^j_\infty(0,T;W_p^{2m-4j-2}(\Omega))} + \|\Psi\|_{W^{j+1}_\infty(0,T;W_p^{2m-4j-4}(\Omega))}\Big].\end{aligned} \tag{3.53}$$

We set $m = 2n$ in (3.51) and apply (3.47) with $\alpha = 2n - \frac{1}{2}$, (3.50) with $\beta = 2n - \frac{3}{2}$, $k = l = 0$, and (3.53). Then we get for $n \in \mathbb{N}\backslash\{0\}$

$$\begin{aligned}
&\|\Theta\|_{L^\infty(0,T;W_p^{4n}(\Omega))} \\
&\leq C\|\Theta(0)\|_{W_p^{4n}(\Omega)} + C\epsilon^{-1}\|\Psi(0)\|_{W_p^{4n}(\Omega)} + C\epsilon^{-\frac{7}{4}}\left[\left(1+\|\bar{c}^\epsilon\|^{4n-2}_{L^\infty(0,T;W_\infty^{4n-2})}\right)\right. \\
&\quad\times\left(1+\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n-1})}\right) + \|\Theta\|_{L^\infty(0,T;W_p^{4n-1})} + \left.\|\Psi\|_{L^\infty(0,T;W_p^{4n+1})}\right] \\
&\quad+C\epsilon^{-2n}\sum_{j=0}^{n-2}\left[\left(1+\|\bar{c}^\epsilon\|^{4n-3j-3}_{W_\infty^j(0,T;W_\infty^{4n-4j-3})}\right)\left(1+\|\bar{c}^\epsilon\|_{W_\infty^j(0,T;W_p^{4n-4j-2})}\right)\right. \\
&\quad\left.+\|\Theta\|_{W_\infty^j(0,T;W_p^{4n-4j-2})} + \|\Psi\|_{W_\infty^{j+1}(0,T;W_p^{4n-4j-4})}\right]. \qquad (3.54)
\end{aligned}$$

We set $m = 2n$ in (3.52) and use that $\|A^{2n}\bar{c}^\epsilon\|_{W_p^1(\Omega)} \leq C\left\|A^{2n+1/2}\bar{c}^\epsilon\right\|_{L^p(\Omega)}$. Then we can apply (3.47) with $\alpha = 2n$ and (3.50) with $\beta = 2n-1$, $k = l = 0$. Hence it follows with (3.53)

$$\begin{aligned}
&\|\Theta\|_{L^\infty(0,T;W_p^{4n+1}(\Omega))} \\
&\leq C\|\Theta(0)\|_{W_p^{4n+1}(\Omega)} + C\epsilon^{-1}\|\Psi(0)\|_{W_p^{4n+1}(\Omega)} + C\epsilon^{-\frac{7}{4}}\left[\left(1+\|\bar{c}^\epsilon\|^{4n-1}_{L^\infty(0,T;W_\infty^{4n-1})}\right)\right. \\
&\quad\times\left(1+\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n})}\right) + \|\Theta\|_{L^\infty(0,T;W_p^{4n})} + \left.\|\Psi\|_{L^\infty(0,T;W_p^{4n+2})}\right] \\
&\quad+C\epsilon^{-2n}\sum_{j=0}^{n-2}\left[\left(1+\|\bar{c}^\epsilon\|^{4n-3j-2}_{W_\infty^j(0,T;W_\infty^{4n-4j-2})}\right)\left(1+\|\bar{c}^\epsilon\|_{W_\infty^j(0,T;W_p^{4n-4j-1})}\right)\right. \\
&\quad\left.+\|\Theta\|_{W_\infty^j(0,T;W_p^{4n-4j-1})} + \|\Psi\|_{W_\infty^{j+1}(0,T;W_p^{4n-4j-3})}\right]. \qquad (3.55)
\end{aligned}$$

Repeating the same procedure for $m = 2n+1$ in (3.51) and (3.52) yields

$$\begin{aligned}
&\|\Theta\|_{L^\infty(0,T;W_p^{4n+2}(\Omega))} \\
&\leq C\|\Theta(0)\|_{W_p^{4n+2}(\Omega)} + C\epsilon^{-1}\|\Psi(0)\|_{W_p^{4n+2}(\Omega)} + C\epsilon^{-\frac{7}{4}}\left[\left(1+\|\bar{c}^\epsilon\|^{4n}_{L^\infty(0,T;W_\infty^{4n})}\right)\right. \\
&\quad\times\left(1+\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n+1})}\right) + \|\Theta\|_{L^\infty(0,T;W_p^{4n+1})} + \left.\|\Psi\|_{L^\infty(0,T;W_p^{4n+3})}\right] \\
&\quad+C\epsilon^{-2n}\sum_{j=0}^{n-1}\left[\left(1+\|\bar{c}^\epsilon\|^{4n-3j-1}_{W_\infty^j(0,T;W_\infty^{4n-4j-1})}\right)\left(1+\|\bar{c}^\epsilon\|_{W_\infty^j(0,T;W_p^{4n-4j})}\right)\right. \\
&\quad\left.+\|\Theta\|_{W_\infty^j(0,T;W_p^{4n-4j})} + \|\Psi\|_{W_\infty^{j+1}(0,T;W_p^{4n-4j-2})}\right], \qquad (3.56)
\end{aligned}$$

and

$$
\begin{aligned}
&\|\Theta\|_{L^\infty(0,T;W_p^{4n+3}(\Omega))}\\
&\quad\le\; C\,\|\Theta(0)\|_{W_p^{4n+3}(\Omega)} + C\epsilon^{-1}\,\|\Psi(0)\|_{W_p^{4n+3}(\Omega)} + C\epsilon^{-\frac{7}{4}}\Big[\Big(1+\|\bar c^\epsilon\|^{4n+1}_{L^\infty(0,T;W_\infty^{4n+1})}\Big)\\
&\qquad\times\Big(1+\|\bar c^\epsilon\|_{L^\infty(0,T;W_p^{4n+2})}\Big) + \|\Theta\|_{L^\infty(0,T;W_p^{4n+2})} + \|\Psi\|_{L^\infty(0,T;W_p^{4n+4})}\Big]\\
&\qquad+C\epsilon^{-2n}\sum_{j=0}^{n-1}\Big[\Big(1+\|\bar c^\epsilon\|^{4n-3j}_{W_\infty^j(0,T;W_\infty^{4n-4j})}\Big)\Big(1+\|\bar c^\epsilon\|_{W_\infty^j(0,T;W_p^{4n-4j+1})}\Big)\\
&\qquad+\|\Theta\|_{W_\infty^j(0,T;W_p^{4n-4j+1})} + \|\Psi\|_{W_\infty^{j+1}(0,T;W_p^{4n-4j-1})}\Big]\,. \qquad (3.57)
\end{aligned}
$$

With the last four estimates we can prove the assertion of Claim 3.

Claim 3: Let $p > d$ be as in Claim 2. Then for all $m \in \mathbb{N}$ there exists an integer k_m and a constant $C > 0$ independent of ϵ such that

$$
\begin{aligned}
&\sum_{i=0}^{m}\|\bar c^\epsilon\|_{W_\infty^i(0,T;W_p^{4(m-i)+1}(\Omega))} + \|\bar c^\epsilon\|_{W_\infty^i(0,T;W_\infty^{4(m-i)}(\Omega))}\\
&\qquad\qquad + \|\Theta\|_{W_\infty^i(0,T;W_p^{4(m-i)+1}(\Omega))} + \|\Theta\|_{W_\infty^i(0,T;W_\infty^{4(m-i)}(\Omega))} \le C\epsilon^{-k_m}\,.
\end{aligned}
$$

We proof Claim 3 by induction.
The base case "$m = 0$": See Claim 2.
The inductive step "$m \to m+1$": We set $n = m$ in (3.56) and use the induction hypothesis to obtain

$$
\begin{aligned}
&\|\Theta\|_{L^\infty(0,T;W_p^{4m+2}(\Omega))}\\
&\quad\le\; C\epsilon^{-k_{m+1}} + C\epsilon^{-\frac{7}{4}}\,\|\Psi\|_{L^\infty(0,T;W_p^{4m+3})} + C\epsilon^{-2m}\sum_{j=0}^{m-1}\|\Psi\|_{W_\infty^{j+1}(0,T;W_p^{4m-4j-2})}\\
&\quad\le\; C\epsilon^{-k_{m+1}} + C\epsilon^{-\frac{7}{4}}\,\|\bar c^\epsilon\|_{L^\infty(0,T;W_p^{4m+1})} + C\epsilon^{-2m}\,\|\bar c^\epsilon\|_{W_\infty^m(0,T;W_p^1)}\\
&\qquad+C\epsilon^{-2m}\sum_{j=0}^{m-2}\|\bar c^\epsilon\|_{W_\infty^{j+1}(0,T;W_p^{4m-4j-4})}\\
&\quad\le\; C\epsilon^{-k_{m+1}}
\end{aligned}
$$

for some integer k_{m+1} and where we have used (3.43). Using the definition of Θ, we also get the following estimate for $\bar c^\epsilon$

$$
\begin{aligned}
\|\bar c^\epsilon\|_{L^\infty(0,T;W_p^{4m+2}(\Omega))} &\le \|\Theta\|_{L^\infty(0,T;W_p^{4m+2}(\Omega))} + \epsilon^{-1}\,\|\Psi\|_{L^\infty(0,T;W_p^{4m+2}(\Omega))}\\
&\le C\epsilon^{-k_{m+1}} + \epsilon^{-1}\,\|\bar c^\epsilon\|_{L^\infty(0,T;W_p^{4m+1}(\Omega))} \le \epsilon^{-k_{m+1}}\,.
\end{aligned}
$$

(For better clarity we again write k_{m+1}, although k_{m+1} is possibly larger than above.) By Sobolev's imbedding it holds

$$
\|\bar c^\epsilon\|_{L^\infty(0,T;W_\infty^{4m+1}(\Omega))} + \|\Theta\|_{L^\infty(0,T;W_\infty^{4m+1}(\Omega))} \le C\epsilon^{-k_{m+1}}\,.
$$

Now we set $n = m$ in (3.57) and use the estimates above and the induction hypothesis. Then the same calculation as above yields

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m+3}(\Omega))} + \|\Theta\|_{L^\infty(0,T;W_p^{4m+3}(\Omega))} \le C\epsilon^{-k_{m+1}} \tag{3.58}$$

and

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_\infty^{4m+2}(\Omega))} + \|\Theta\|_{L^\infty(0,T;W_\infty^{4m+2}(\Omega))} \le C\epsilon^{-k_{m+1}}\,. \tag{3.59}$$

In order to estimate $\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_\infty^{4m+3}(\Omega))}$, we need estimates for $\partial_t^i \bar{c}^\epsilon$, $i = 1, \dots, m$, in stronger norms as in the induction hypothesis. To get higher time regularity, we use the Cahn-Larché equation

$$\begin{aligned}
&\|\partial_t \bar{c}^\epsilon\|_{L^\infty(0,T;W^{4m-1}(\Omega)_p)} \\
&\quad\le \epsilon \left\| -\Delta^2 \bar{c}^\epsilon + \epsilon^{-1}\Delta \bar{f}(\bar{c}^\epsilon) + \Delta W_{,c}(\bar{c}^\epsilon, \mathcal{E}(\bar{\mathbf{u}}^\epsilon)) \right\|_{L^\infty(0,T;W_p^{4m-1})} \\
&\quad\le \epsilon \|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m+3})} + \epsilon^{-1}\left(1 + \|\bar{c}^\epsilon\|^{4m}_{L^\infty(0,T;W_\infty^{4m})}\right)\left(1 + \|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m+1})}\right) \\
&\quad\le C\epsilon^{-k_{m+1}}
\end{aligned} \tag{3.60}$$

due to (3.58) and (3.59). By definition of Θ and (3.43) we conclude

$$\|\partial_t \Theta\|_{L^\infty(0,T;W_p^{4m-1})} \le \|\partial_t \bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m-1})} + \epsilon^{-1}\|\partial_t \Psi\|_{L^\infty(0,T;W_p^{4m-1})} \le C\epsilon^{-k_{m+1}}, \tag{3.61}$$

and by Sobolev's imbedding we have

$$\|\Theta\|_{W^1_\infty(0,T;W_\infty^{4m-2})} + \|\bar{c}^\epsilon\|_{W^1_\infty(0,T;W_\infty^{4m-2})} \le C\epsilon^{-k_{m+1}}\,. \tag{3.62}$$

If $m \ge 2$, we differentiate the Cahn-Larché equation with respect to time t and use (3.60)-(3.62) to get

$$\begin{aligned}
\left\|\partial_t^2 \bar{c}^\epsilon\right\|_{L^\infty(0,T;W_p^{4m-5})} &\le \left\| -\epsilon\partial_t \left(\Delta^2 \bar{c}^\epsilon + \epsilon^{-1}\Delta \bar{f}(\bar{c}^\epsilon) + \Delta W_{,c}(\bar{c}^\epsilon, \mathcal{E}(\bar{\mathbf{u}}^\epsilon))\right)\right\|_{L^\infty(0,T;W_p^{4m-5})} \\
&\le C\epsilon^{-k_{m+1}}
\end{aligned}$$

and as above

$$\|\Theta\|_{W^2_\infty(0,T;W_\infty^{4m-5})} + \|\bar{c}^\epsilon\|_{W^2_\infty(0,T;W_\infty^{4m-6})} + \|\Theta\|_{W^2_\infty(0,T;W_\infty^{4m-6})} \le C\epsilon^{-k_{m+1}}\,.$$

Repeating the same procedure step by step, we obtain

$$\begin{aligned}
\sum_{i=0}^{m} &\|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_p^{4(m-i)+3})} + \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_\infty^{4(m-i)+2})} \\
&+ \|\Theta\|_{W^i_\infty(0,T;W_p^{4(m-i)+3})} + \|\Theta\|_{W^i_\infty(0,T;W_\infty^{4(m-i)+2})} \le C\epsilon^{-k_{m+1}}
\end{aligned} \tag{3.63}$$

for some integer k_{m+1} and some constant $C > 0$.
Now we use (3.54) and (3.55) for $n = m+1$ and repeat the same procedure as at the

beginning of inductive step. Instead of the induction hypothesis we consider (3.63). Note that we even get estimates for $\partial_t^{m+1}\bar{c}^\epsilon$ and $\partial_t^{m+1}\Theta$. Then it follows

$$\sum_{i=0}^{m+1} \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_p^{4((m+1)-i)+1}(\Omega))} + \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_\infty^{4((m+1)-i)}(\Omega))} \\ + \|\Theta\|_{W^i_\infty(0,T;W_p^{4((m+1)-i)+1}(\Omega))} + \|\Theta\|_{W^i_\infty(0,T;W_\infty^{4((m+1)-i)}(\Omega))} \leq C\epsilon^{-k_{m+1}}$$

for some integer k_{m+1} and some constant C independent of ϵ. Thus Claim 3 follows. Since $\|\bar{c}^\epsilon\|_{L^\infty(\Omega)} \leq \|\bar{c}^\epsilon_A\|_{L^\infty(\Omega)} + \|\bar{c}^\epsilon - \bar{c}^\epsilon_A\|_{L^\infty(\Omega)} \leq \frac{3}{2}C_0$ for ϵ small enough, we conclude $c^\epsilon = \bar{c}^\epsilon$ and $\mathbf{u}^\epsilon = \bar{\mathbf{u}}^\epsilon$ by uniqueness of the solution to the Cahn-Larché system.
This shows the assertion of the theorem for $c^\epsilon - c^\epsilon_A$. It remains to show the estimates for $\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A$ and $\mu^\epsilon - \mu^\epsilon_A$. As in (3.39) there exists some number $\theta \in (0,1)$ and a constant $C > 0$ such that

$$\begin{aligned} \|\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A\|_{C^{m+1}(\Omega_T)} &\leq C\,\|\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A\|^\theta_{L^2(\Omega_T)}\,\|\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A\|^{1-\theta}_{W_2^{m+l+2}(\Omega_T)} \\ &\leq C\,\|\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A\|^\theta_{L^2(\Omega_T)}\left(\|c^\epsilon\|_{W_2^{m+l+1}(\Omega_T)} + \|\mathbf{u}^\epsilon_A\|_{W_2^{m+l+2}(\Omega_T)}\right)^{1-\theta}, \end{aligned}$$

where we have used (3.42) in the second inequality. We can control the second and the third term as above and the first term is smaller than $C\epsilon^k$ due to Theorem 3.1.1. Choosing k large enough the assertion follows for $\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A$. To estimate $\mu^\epsilon - \mu^\epsilon_A$ we use the equations for the chemical potential (3.2) and (3.20). Then we conclude

$$\begin{aligned} \|\mu^\epsilon - \mu^\epsilon_A\|_{C^{m-2}(\Omega_T)} &\leq \epsilon\,\|\Delta(c^\epsilon - c^\epsilon_A)\|_{C^{m-2}(\Omega_T)} + \epsilon^{-1}\,\|f(c^\epsilon) - f(c^\epsilon_A)\|_{C^{m-2}(\Omega_T)} \\ &\quad + \|W_{,c}(c^\epsilon - c^\epsilon_A, \mathcal{E}(\mathbf{u}^\epsilon - \mathbf{u}^\epsilon_A))\|_{C^{m-2}(\Omega_T)} + \|r^\epsilon_A\|_{C^{m-2}(\Omega_T)}\,. \end{aligned}$$

We already know that the first and the third term are smaller than $\frac{1}{4}\epsilon$ for k in (3.24) large enough and for all $\epsilon > 0$ small enough. We use the chain rule to estimate the second term on the right-hand side. Then for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ such that $n + |\alpha| \leq m - 2$, it follows

$$\begin{aligned} &\|\partial_t^n\partial_x^\alpha(f(c^\epsilon) - f(c^\epsilon_A))\|_{C^0(\Omega_T)} \\ &\leq C\left[\max_{i=1,\dots,n+|\alpha|}\left\|f^{(i)}(c^\epsilon) - f^{(i)}(c^\epsilon_A)\right\|_{C^0(\Omega_T)}\left(\|c^\epsilon_A\|^{n+|\alpha|}_{C^{n+|\alpha|}(\Omega_T)} + 1\right)\right. \\ &\qquad \left. + \max_{i=1,\dots,n+|\alpha|}\left\|f^{(i)}(c^\epsilon)\right\|_{C^0(\Omega_T)}\|c^\epsilon - c^\epsilon_A\|_{C^{n+|\alpha|}(\Omega_T)}\right] \\ &\leq C\epsilon^{-K(m)}\,\|c^\epsilon - c^\epsilon_A\|_{C^0(\Omega_T)} + C\,\|c^\epsilon - c^\epsilon_A\|_{C^{n+|\alpha|}(\Omega_T)} \end{aligned}$$

for some $C = C(\alpha, n) > 0$ and since $\|c - c^\epsilon_A\|_{C^{m-2}(\Omega_T)} \leq \epsilon < 1$ and $\|c^\epsilon\|_{C^{m-2}(\Omega_T)} + \|c^\epsilon_A\|_{C^{m-2}(\Omega_T)} \leq C\epsilon^{-K(m)}$ for some constant $C > 0$ and some integer $K(m)$. Since we even get $\|c^\epsilon - c^\epsilon_A\|_{C^{m-2}(\Omega_T)} \leq \epsilon^3$ and $\|c^\epsilon - c^\epsilon_A\|_{C^0(\Omega_T)} \leq \epsilon^{K(m)+2}$ for k in (3.24) large enough, it holds

$$\epsilon^{-1}\,\|f(c^\epsilon) - f(c^\epsilon_A)\|_{C^{m-2}(\Omega_T)} \leq \frac{1}{4}\epsilon$$

for all $\epsilon > 0$ small enough. For the rest term r_A^ϵ we again apply an interpolation estimate

$$\|r_A^\epsilon\|_{C^{m-2}(\Omega_T)} \leq C \|r_A^\epsilon\|_{L^2(\Omega_T)}^\theta \|r_A^\epsilon\|_{W_2^{m+l-1}(\Omega_T)}^{1-\theta} \leq \frac{1}{4}\epsilon$$

for k in (3.24) large enough and for all $\epsilon > 0$ small enough and where we can estimate $\|r_A^\epsilon\|_{W_2^{m+l-1}(\Omega_T)}$ by (3.20) and by the estimates for c_A^ϵ, μ_A^ϵ, and $\mathbf{u}_A^\epsilon$.
This shows the assertion of the theorem. □

3.2 Asymptotic Expansion

In this section we use a matched asymptotic expansion as in [10] to construct a family of approximate solutions $\{c_A^\epsilon, \mu_A^\epsilon, \mathbf{u}_A^\epsilon\}_{0<\epsilon\leq 1}$ satisfying (3.19)-(3.27).
For a good introduction to the method of matched asymptotic expansions we refer to the books of Nayfeh [54], Kevorkian [43], and Kevorkian and Cole [44]. Away from the interface $\Gamma^\epsilon = \{(x,t) : c^\epsilon(x,t) = 0\}$ we use the original variables to determine the expansion of the solutions $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$. This is called the outer expansion. Near the interface Γ^ϵ we expect that $\nabla d^\epsilon \cdot \nabla c^\epsilon \approx \frac{C}{\epsilon}$ for some constant $C > 0$ and where d^ϵ is the spatial signed distance function to the interface. Therefore we introduce the new variable $z = \frac{d^\epsilon(x,t)}{\epsilon}$ to describe the sharp change near the interface. This is called the inner expansion. We also use a boundary-layer expansion to satisfy the boundary conditions. By the so-called matching conditions we connect the inner and outer expansion and the outer and boundary-layer expansion to obtain suitable approximate solutions $(c_A^\epsilon, \mu_A^\epsilon, \mathbf{u}_A^\epsilon)$ for all $\epsilon \in (0,1]$.
We use the same convention as in [10], more precisely that means the notation $\sum_{i=0}^\infty$ is formal and should be understood only as a finite sum $\sum_{i=0}^K$ plus an error term of order $\mathcal{O}(\epsilon^{K+1})$ where K is a large integer depending on the order of approximation needed. Also the word "smooth" should be understood to mean that all the needed derivatives exist and are continuous. Similarly, the phrase "for all natural integers" should be understood to mean "for all integers needed".
In the following we use the symmetry assumption $\mathcal{C}_{iji'j'} = \mathcal{C}_{ijj'i'}$ for all $i, j, j', i' \in \{1, \ldots, d\}$. Therefore we obtain $\mathcal{C}A = \mathcal{C}A^T$ for all $A \in \mathbb{R}^{d\times d}$ and in particular, $\mathcal{C}\mathcal{E}(\mathbf{u}) = \mathcal{C}\nabla\mathbf{u}$. For the sake of clarity in the asymptotic expansion we use $\mathcal{C}\nabla\mathbf{u}$ instead of $\mathcal{C}\mathcal{E}(\mathbf{u})$.

3.2.1 Representation of the Interface

Assume that $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ is a solution to (3.1)-(3.7) and that

$$\Gamma^\epsilon := \{(x,t) \in \Omega_T : c^\epsilon(x,t) = 0\} = \bigcup_{0<t<T} (\Gamma_t^\epsilon \times \{t\})$$

is a smooth hypersurface. Γ^ϵ is called the interface. Let Q_ϵ^- be the interior of Γ^ϵ and $Q_\epsilon^+ := \Omega_T \backslash (\Gamma^\epsilon \cup Q_\epsilon^+)$. Furthermore, let $d^\epsilon(x,t)$ be the spatial signed distance function to Γ_t^ϵ such that $d^\epsilon < 0$ in Q_ϵ^-. Then d^ϵ is a smooth function and $|\nabla d^\epsilon| = 1$ in a neighborhood of Γ^ϵ, which depends on the curvature of Γ^ϵ. Also we assume that d^ϵ has the expansion

$$d^\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i d^i(x,t)\,,$$

where d^0 is defined in $\overline{\Omega_T}$ and d^i, $i \geq 1$, is defined in a neighborhood of Γ^ϵ. Since d^i is independent of ϵ for all $i \geq 0$, the equation $|\nabla d^\epsilon| = 1$ is equivalent to

$$\nabla d^0 \cdot \nabla d^k = \begin{cases} 1 & \text{if } k = 0\,, \\ 0 & \text{if } k = 1\,, \\ -\frac{1}{2}\sum_{i=1}^{k-1} \nabla d^i \cdot \nabla d^{k-i} & \text{if } k \geq 2\,, \end{cases} \tag{3.64}$$

where all the equations are satisfied in a neighborhood of Γ^ϵ. Then we can assume that d^0 is a spatial signed distance function, and we define

$$\Gamma^0 = \left\{(x,t) \in \Omega_T : d^0(x,t) = 0\right\}\,, \tag{3.65}$$

$$\Gamma^0(\delta) = \left\{(x,t) \in \Omega_T : \left|d^0(x,t)\right| < \delta\right\}\,, \tag{3.66}$$

$$Q_0^+ = \left\{(x,t) \in \Omega_T : d^0(x,t) > 0\right\}\,, \tag{3.67}$$

$$Q_0^- = \left\{(x,t) \in \Omega_T : d^0(x,t) < 0\right\} \tag{3.68}$$

for some constant $\delta > 0$.

Remark 3.2.1. *Only to motivate the construction of the approximate solutions, we need the assumptions that Γ^ϵ is a smooth hypersurface and d^ϵ has a series expansion. But these assumptions are not necessary to show the convergence of the Cahn-Larché system (3.1)-(3.7) to the modified Hele-Shaw problem (3.8)-(3.14). More precisely, in Theorem 3.1.2, 3.2.21, and 3.3.1 these assumptions do not occur. Moreover, the assumption that for all $\epsilon \in (0,1]$ the solution $(c^\epsilon, \mu^\epsilon, \boldsymbol{u}^\epsilon)$ has a series expansion near the interface Γ^ϵ, near the boundary $\partial\Omega$, and away from the interface Γ^ϵ and the boundary $\partial\Omega$ (see Subsection 3.2.2, 3.2.3, and 3.2.5) is also not necessary for the proof of the convergence.*

3.2.2 Outer Expansion

We assume that Γ^0 is known. Then so are d^0, Q_0^+ and Q_0^-. Also we assume that away from the interface Γ^ϵ the solution functions c^ϵ and μ^ϵ have the expansion

$$c^\epsilon(x,t) = c_0^\pm(x,t) + \epsilon c_1^\pm(x,t) + \epsilon^2 c_2^\pm(x,t) + ... \qquad \text{in } Q_0^\pm \backslash \Gamma^0\left(\tfrac{\delta}{2}\right)\,,$$

$$\mu^\epsilon(x,t) = \mu_0^\pm(x,t) + \epsilon \mu_1^\pm(x,t) + \epsilon^2 \mu_2^\pm(x,t) + ... \qquad \text{in } Q_0^\pm \backslash \Gamma^0\left(\tfrac{\delta}{2}\right)\,,$$

$$\mathbf{u}^\epsilon(x,t) = \mathbf{u}_0^\pm(x,t) + \epsilon \mathbf{u}_1^\pm(x,t) + \epsilon^2 \mathbf{u}_2^\pm(x,t) + ... \qquad \text{in } Q_0^\pm \backslash \Gamma^0\left(\tfrac{\delta}{2}\right)\,,$$

where $c_i^{\pm}$, $\mu_i^{\pm}$ and $\mathbf{u}_i^{\pm}$ are appropriate functions defined in $Q_0^{\pm}$ and $\delta > 0$ is a fixed constant independent of ϵ which is to be determined later.
We substitute the expansion for $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ into (3.1)-(3.4) and match the terms with the same power of ϵ. We require for $(x,t) \in Q_0^{\pm}$

$$(c_k^{\pm})_t = \Delta \mu_k^{\pm}\,, \tag{3.69}$$

$$c_k^{\pm} = \begin{cases} \pm 1 & \text{if } k = 0 \\ \frac{\mu_{k-1}^{\pm} - f^{k-1}(c_0^{\pm},\dots,c_{k-1}^{\pm}) + \mathcal{E}^\star : \mathcal{C}\left(\nabla \mathbf{u}_{k-1}^{\pm} - \mathcal{E}^\star c_{k-1}^{\pm}\right) + \Delta c_{k-2}^{\pm}}{f'(c_0^{\pm})} & \text{if } k \geq 1 \end{cases}, \tag{3.70}$$

$$\operatorname{div}\left(\mathcal{C}\nabla \mathbf{u}_k^{\pm}\right) = \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star c_k^{\pm}\right), \tag{3.71}$$

where $c_{-1}^{\pm} = 0$ and f^i is defined such that

$$f(c_0 + \epsilon c_1 + \epsilon^2 c_2 + \dots) = f(c_0) + f'(c_0)\sum_{i=1}^{\infty} \epsilon^i c_i + \sum_{i=2}^{\infty} \epsilon^i f^{i-1}(c_0, \dots, c_{i-1})\,.$$

We obtain this expression by a Taylor series expansion around the point c_0. For example, for $f(c) = c^3 - c$ we get

$$f(c_0 + \epsilon c_1 + \epsilon^2 c_2 + \dots) = f(c_0) + f'(c_0)\sum_{i=1}^{\infty} \epsilon^i c_i + 3\left(c_0 + \frac{1}{3}\sum_{i=1}^{\infty} \epsilon^i c_i\right)\left(\sum_{i=1}^{\infty} \epsilon^i c_i\right)^2.$$

Therefore we set

$$\epsilon^2 f^1(c_0, c_1) = \epsilon^2 3 c_0 c_1^2\,, \quad \epsilon^3 f^2(c_0, c_1, c_2) = \epsilon^3\left(6 c_0 c_1 c_2 + c_1^3\right),$$

and so on.

Remark 3.2.2. *For the outer expansion let us mention the following points.*

1. *When $k = 0$ we only obtain $f(c_0^{\pm}) = 0$. But as in [10] we require $c_0^{\pm} = \pm 1$ for definiteness.*

2. *In order to construct an approximate solution we have required that all outer expansion equations (3.69)-(3.71) are satisfied in $Q_0^{\pm}$ instead of $Q_0^{\pm} \backslash \Gamma^0(\delta/2)$.*

3. *Because of (3.70) we can not require boundary conditions for c_k^{+} on $\partial_T \Omega$. To avoid this problem, we use a boundary-layer expansion. For details see Subsection 3.2.5.*

4. *To determine $\mu_k^{\pm}$ and $\boldsymbol{u}_k^{\pm}$ uniquely, we need boundary conditions on $\partial\Omega$ and Γ^0. We obtain these conditions by the boundary-layer expansion and the inner expansion, see Subsection 3.2.7 and 3.2.8.*

5. *For the inner expansion it is necessary to define* $(c_k^\pm, \mu_k^\pm, \boldsymbol{u}_k^\pm)$ *not only in the domain* $Q_0^\pm$*, but also in* $\Gamma^0(\delta)\backslash Q_0^\pm$*. But it is sufficient to choose any smooth extension of* $(c_k^\pm, \mu^\pm, \boldsymbol{u}_k^\pm)$ *from* $Q_0^\pm$ *to* $Q_0^\pm \cup \Gamma^0(\delta)$*. For the extension of* $c_k^\pm$ *we can use equation (3.70). We can extend* $\mu_k^\pm$ *and* $\boldsymbol{u}_k^\pm$ *in the following way. We use the ansatz* $\mu_k^\pm = \sum_{i=0}^{\bar{K}} (d^0(x,t))^i \, C_{k,i}^\pm(S^0(x,t))$ *in* $\Gamma^0(\delta)\backslash Q_0^\pm$ *for some functions* $C_{k,i}^\pm$ *where* S^0 *is the projection of* x *on* Γ_t^0 *along the normal of* Γ_t^0*. We determine the functions* $C_{k,i}^\pm$ *such that all the normal derivatives of* $\mu^\pm$ *up to order* $\bar{K}$ *match from each side of* Γ_t^0*. Here* $\bar{K}$ *is an integer depending on the order of approximations needed. Because it is sufficient that* $\mu^\pm \in C^{\bar{K}}(\Gamma^0(\delta))$*, this is an appropriate extension. We can extend* $\boldsymbol{u}_k^\pm$ *analogously.*

In order to have bounded solutions in the inner expansion, we need the following definitions

$$O_k^\pm(x,t) := (c_k^\pm)_t - \Delta\mu_k^\pm, \qquad O^\pm := \sum_{i=0}^\infty \epsilon^i O_i^\pm \quad \text{in } Q_0^\pm \cup \Gamma^0(\delta)\,,$$

$$\mathbf{P}_k^\pm(x,t) := \operatorname{div}\left(\mathcal{C}\nabla\mathbf{u}_k^\pm\right) - (\mathcal{C}\mathcal{E}^\star)\,\nabla c_k^\pm\,, \qquad \mathbf{P}^\pm := \sum_{i=0}^\infty \epsilon^i \mathbf{P}_i^\pm \quad \text{in } Q_0^\pm \cup \Gamma^0(\delta)\,.$$

Due to the definition of $(c_k^\pm, \mu_k^\pm, \mathbf{u}_k^\pm)$, it holds that $\mathbf{P}_k^\pm = 0$ and $O_k^\pm = 0$ in $\overline{Q_0^\pm}$ for all $k \geq 0$.

3.2.3 Inner Expansion

As above we assume that $d^0(x,t)$ is known. To understand the behavior of the solution $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ near the interface Γ_t^0, we assume that in $\Gamma^0(\delta)$ the solution $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ has the expansion

$$c^\epsilon(x,t) = \tilde{c}^\epsilon\left(\frac{d^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \tilde{c}^\epsilon(z,x,t) = \sum_{i=0}^\infty \epsilon^i c^i(z,x,t),$$

$$\mu^\epsilon(x,t) = \tilde{\mu}^\epsilon\left(\frac{d^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \tilde{\mu}^\epsilon(z,x,t) = \sum_{i=0}^\infty \epsilon^i \mu^i(z,x,t),$$

$$\mathbf{u}^\epsilon(x,t) = \tilde{\mathbf{u}}^\epsilon\left(\frac{d^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \tilde{\mathbf{u}}^\epsilon(z,x,t) = \sum_{i=0}^\infty \epsilon^i \mathbf{u}^i(z,x,t),$$

where $\tilde{c}^\epsilon, \tilde{\mu}^\epsilon, \tilde{\mathbf{u}}^\epsilon, c^i, \mu^i$ and $\mathbf{u}^i$ are appropriate functions defined in $\mathbb{R} \times \Gamma^0(\delta)$.
To construct a solution in the whole domain Ω_T, we require some matching conditions for the inner and outer expansion. Since $\frac{d^0(x,t)}{\epsilon} \to \infty$ as $\epsilon \to 0$ in Q_0^+, we require the

following inner-outer matching conditions as $z \to \infty$

$$D_x^m D_t^n D_z^l \left[c^k(\pm z, x, t) - c_k^\pm(x,t)\right] = \mathcal{O}(e^{-\alpha z}), \tag{3.72}$$
$$D_x^m D_t^n D_z^l \left[\mu^k(\pm z, x, t) - \mu_k^\pm(x,t)\right] = \mathcal{O}(e^{-\alpha z}), \tag{3.73}$$
$$D_x^m D_t^n D_z^l \left[\mathbf{u}^k(\pm z, x, t) - \mathbf{u}_k^\pm(x,t)\right] = \mathcal{O}(e^{-\alpha z}) \tag{3.74}$$

for all $(x,t) \in \Gamma^0(\delta)$ and all $k, m, n, l \in \left\{0, \ldots, \bar{K}\right\}$ where $\bar{K} > 0$ depends of the order of expansion. Here $\alpha > 0$ is the same constant as in Lemma 2.6.1.

Remark 3.2.3. *In Subsection 3.2.9 we glue together the outer and inner expansions. Then it is necessary that the matching conditions holds for $m, n, l \in \{0, 1, 2\}$ for each order k. Since the equations for $(c_k^\pm, c^k, \mu_k^\pm, \mu^k, \boldsymbol{u}_k^\pm, \boldsymbol{u}^k)$ depend on space and time derivatives and derivatives with respect to z of functions of lower order, it is necessary and sufficient that $m, n, l \in \left\{0, \ldots, \bar{K}\right\}$ where $\bar{K}$ is a constant depending on the order of expansion. To verify the matching conditions we consider the inner expansion equations for $(c^j, \mu^j, \boldsymbol{u}^j)$, which we obtain below, and then we use the results of Section 2.6. One can even verify that the matching conditions are true for all $m, n, l \in \mathbb{N}$.*

Since $c^\epsilon = 0$ on Γ^ϵ, it is natural that we require in our construction

$$c^k(0, x, t) = 0 \quad \forall (x,t) \in \Gamma^0(\delta)$$

for all $k \geq 0$.
As above we substitute the expansion of $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ into (3.1), which yields

$$\epsilon^{-1}\tilde{c}_z^\epsilon d_t^\epsilon + \tilde{c}_t^\epsilon = \epsilon^{-2}\tilde{\mu}_{zz}^\epsilon + 2\epsilon^{-1}\nabla\tilde{\mu}_z^\epsilon \cdot \nabla d^\epsilon + \epsilon^{-1}\tilde{\mu}_z^\epsilon \Delta d^\epsilon + \Delta\tilde{\mu}^\epsilon,$$

into (3.2), which yields

$$\begin{aligned} \tilde{\mu}^\epsilon \;=\; & \epsilon^{-1}\left(f(\tilde{c}^\epsilon) - \tilde{c}_{zz}^\epsilon\right) - \tilde{c}_z^\epsilon \Delta d^\epsilon - 2\nabla\tilde{c}_z^\epsilon \cdot \nabla d^\epsilon - \epsilon\Delta\tilde{c}^\epsilon \\ & -\epsilon^{-1}\mathcal{E}^\star : \mathcal{C}\left(\tilde{\mathbf{u}}_z^\epsilon \otimes \nabla d^\epsilon\right) - \mathcal{E}^\star : \mathcal{C}\left(\nabla\tilde{\mathbf{u}}^\epsilon - \mathcal{E}^\star\tilde{c}^\epsilon\right), \end{aligned}$$

and into (3.3), which yields

$$\begin{aligned} & \epsilon^{-2}\left(\mathcal{C}\left(\tilde{\mathbf{u}}_{zz}^\epsilon \otimes \nabla d^\epsilon\right)\right)\nabla d^\epsilon + \epsilon^{-1}\left(\mathcal{C}_{iji'j'}\,\partial_j(\tilde{\mathbf{u}}_{i'}^\epsilon)_z \partial_{j'} d^\epsilon\right)_{i=1,\ldots,d} \\ & + \epsilon^{-1}\left(\mathcal{C}_{iji'j'}\,(\tilde{\mathbf{u}}_{i'}^\epsilon)_z \partial_{jj'} d^\epsilon\right)_{i=1,\ldots,d} + \epsilon^{-1}\left(\mathcal{C}\nabla\tilde{\mathbf{u}}_z^\epsilon\right)\nabla d^\epsilon + \left(\mathcal{C}_{iji'j'}\,\partial_{jj'}\tilde{\mathbf{u}}_{i'}^\epsilon\right)_{i=1,\ldots,d} \\ & = \epsilon^{-1}\tilde{c}_z^\epsilon\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^\epsilon + \left(\mathcal{C}\mathcal{E}^\star\right)\nabla\tilde{c}^\epsilon \end{aligned}$$

for $(z, x, t) \in S^\epsilon := \{(z, x, t) \in \mathbb{R} \times \Gamma^0(\delta) : z = d^\epsilon(x,t)/\epsilon\}$. We can consider these equations as a system of ordinary differential equations for $(c^i, \mu^i, \mathbf{u}^i)$ with independent variable $z \in \mathbb{R}$, whereas (x,t) are considered as parameters. Of course, it is not clear that the solutions to these ordinary differential equations satisfy the inner-outer matching conditions. Note that these equations have to be satisfied only on S^ϵ. So we can add any terms which vanishes on S^ϵ to enforce the inner-outer matching conditions. We denote these terms by g^ϵ, k^ϵ, h^ϵ, L^ϵ, $\mathbf{l}^\epsilon$, $\mathbf{j}^\epsilon$ and $\mathbf{K}^\epsilon$. We will determine them later. Additionally, let $\eta(z) \in C^\infty(\mathbb{R})$ be an arbitrary fixed function satisfying

$$\eta(z) = \begin{cases} 0 & \text{if } z \leq -1 \\ 1 & \text{if } z \geq 1 \end{cases}, \ \eta'(z) \geq 0 \quad \forall z \in \mathbb{R}, \tag{3.75}$$

$$\text{and} \quad \int_{\mathbb{R}} (\eta(z) - \tfrac{1}{2})\theta_0'(z)\, dz = \int_{\mathbb{R}} z\eta'(z)\theta_0'(z)\, dz = 0, \tag{3.76}$$

where θ_0 is the unique solution to (2.22).

Remark 3.2.4. *If θ_0' is axisymmetric, then we choose η such that $\eta - 1/2$ is point symmetric and η' is axisymmetric. So both equalities of (3.76) are fulfilled. This holds for example for $f(c) = c^3 - c$.*

Furthermore, we set

$$\eta_N^\pm(z) = \eta(-N \pm z), \quad z \in \mathbb{R}, \tag{3.77}$$

where $N > 0$ is a constant to be determined.
From now we consider the following modified equations for $(\tilde{c}^\epsilon, \tilde{\mu}^\epsilon, \tilde{\mathbf{u}}^\epsilon)$

$$\begin{aligned}
\tilde{c}^\epsilon_{zz} - f(\tilde{c}^\epsilon) =& - \mathcal{E}^\star : \mathcal{C}\left(\tilde{\mathbf{u}}^\epsilon_z \otimes \nabla d^\epsilon\right) \\
& - \epsilon\left(\tilde{\mu}^\epsilon + \Delta d^\epsilon \tilde{c}^\epsilon_z + 2\nabla d^\epsilon \cdot \nabla \tilde{c}^\epsilon_z + \mathcal{E}^\star : \mathcal{C}\left(\nabla \tilde{\mathbf{u}}^\epsilon - \mathcal{E}^\star \tilde{c}^\epsilon\right)\right) \\
& - \epsilon^2 \Delta \tilde{c}^\epsilon + g^\epsilon \eta'\left(d^\epsilon - \epsilon z\right) + k^\epsilon \eta'\left(d^\epsilon - \epsilon z\right), \qquad (3.78)\\
\tilde{\mu}^\epsilon_{zz} =& \ \epsilon\left(\tilde{c}^\epsilon_z d^\epsilon_t - 2\nabla \tilde{\mu}^\epsilon_z \cdot \nabla d^\epsilon - \tilde{\mu}^\epsilon_z \Delta d^\epsilon\right) + \epsilon^2\left(\tilde{c}^\epsilon_t - \Delta \tilde{\mu}^\epsilon\right) \\
& + \left(h^\epsilon \eta'' + L^\epsilon \eta'\right)\left(d^\epsilon - \epsilon z\right) - \epsilon^2\left(O^+\eta_N^+ + O^-\eta_N^-\right), \qquad (3.79)\\
\left(\mathcal{C}\left(\tilde{\mathbf{u}}^\epsilon_{zz} \otimes \nabla d^\epsilon\right)\right) \nabla d^\epsilon =& - \epsilon\left(\mathcal{C}_{iji'j'}\, \partial_j (\tilde{\mathbf{u}}^\epsilon_{i'})_z \partial_{j'} d^\epsilon\right)_{i=1,\ldots,d} \\
& - \epsilon\left(\mathcal{C}_{iji'j'}\, (\tilde{\mathbf{u}}^\epsilon_{i'})_z \partial_{jj'} d^\epsilon\right)_{i=1,\ldots,d} - \epsilon\left(\mathcal{C}\nabla \tilde{\mathbf{u}}^\epsilon_z\right) \nabla d^\epsilon \\
& + \epsilon \tilde{c}^\epsilon_z \left(\mathcal{C}\mathcal{E}^\star\right) \nabla d^\epsilon - \epsilon^2\left(\mathcal{C}_{iji'j'}\, \partial_{jj'} \tilde{\mathbf{u}}^\epsilon_{i'}\right)_{i=1,\ldots,d} \\
& + \epsilon^2\left(\mathcal{C}\mathcal{E}^\star\right) \nabla \tilde{c}^\epsilon + M\left(\mathbf{l}^\epsilon \eta'' + \mathbf{K}^\epsilon \eta'\right)\left(d^\epsilon - \epsilon z\right) \\
& + \mathbf{j}^\epsilon \eta''\left(d^\epsilon - \epsilon z\right) + \epsilon^2\left(\mathbf{P}^+\eta_N^+ + \mathbf{P}^-\eta_N^-\right) \qquad (3.80)
\end{aligned}$$

for $z \in \mathbb{R}$ and $(x,t) \in \Gamma^0(\delta)$ and

$$M(x,t) = \left(\mathcal{C}_{iji'j'}\, \partial_{j'} d^0(x,t) \partial_j d^0(x,t)\right)_{i,i'=1}^d. \tag{3.81}$$

We have added the terms $\epsilon^2(O^+\eta_N^+ + O^-\eta_N^-)$ and $\epsilon^2(\mathbf{P}^+\eta_N^+ + \mathbf{P}^-\eta_N^-)$ to satisfy the compatibility conditions for μ^i and $\mathbf{u}^i$. We will see more details in Subsection 3.2.4.

Remark 3.2.5. *It remains to fix the constant N such that the terms $\epsilon^2(O^+\eta_N^+ + O^-\eta_N^-)$ and $\epsilon^2(\boldsymbol{P}^+\eta_N^+ + \boldsymbol{P}^-\eta_N^-)$ do not affect the equations needed for $\tilde{\mu}^\epsilon(\frac{d^\epsilon}{\epsilon}, x, t)$ and $\tilde{\boldsymbol{u}}^\epsilon(\frac{d^\epsilon}{\epsilon}, x, t)$. For that we will see in Subsection 3.2.8 that we can determine d^0 and d^1 independent of $\epsilon^2(O^+\eta_N^+ + O^-\eta_N^-)$ and $\epsilon^2(\boldsymbol{P}^+\eta_N^+ + \boldsymbol{P}^-\eta_N^-)$. So we can set*

$$N := \left\| d^1 \right\|_{C^0(\Gamma^0(\delta))} + 2\,.$$

The so-defined N satisfies the required property. This can be seen as follows. Assume that $\left|z - \frac{d^0+\epsilon d^1}{\epsilon}\right| \leq 1$ and $d^0(x,t) \geq 0$ for $(x,t) \in \Gamma^0(\delta)$. Therefore $(x,t) \in \overline{Q_0^+}$ and so $O^+(x,t) = 0$ and $\boldsymbol{P}^+(x,t) = 0$. Furthermore, it holds

$$z \geq \frac{d^0(x,t)}{\epsilon} + d^1(x,t) - 1 \geq d^1(x,t) - 1 \geq -N + 1\,,$$

and so $\eta_N^-(z) = 0$ by definition of η_N^-, that is, $\epsilon^2(O^+\eta_N^+ + O^-\eta_N^-) = 0$ and $\epsilon^2(\boldsymbol{P}^+\eta_N^+ + \boldsymbol{P}^-\eta_N^-) = 0$. Similarly, this holds when $d^0(x,t) < 0$ and $\left|z - \frac{d^0+\epsilon d^1}{\epsilon}\right| \leq 1$. In Subsection 3.2.9 we set $z = d_\epsilon^K/\epsilon := \sum_{i=0}^K \epsilon^{i-1} d^i$ for some $K \in \mathbb{N}$ and so $\left|\frac{d_\epsilon^K}{\epsilon} - \frac{d^0+\epsilon d^1}{\epsilon}\right| \leq 1$ for all $\epsilon < \epsilon_0$, where $\epsilon_0 > 0$ is small enough.

Moreover, we assume that for $(x,t) \in \Gamma^0(\delta)$ the terms g^ϵ, L^ϵ, h^ϵ, $\mathbf{l}^\epsilon$, and $\mathbf{K}^\epsilon$ have the expansion

$$\begin{aligned}
g^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^{i+1} g^i(x,t), & k^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^i k^i(x,t),\\
L^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^{i+1} L^i(x,t), & h^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^i h^i(x,t),\\
\mathbf{l}^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^i \mathbf{l}^i(x,t), & \mathbf{j}^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^i \mathbf{j}^i(x,t),\\
\mathbf{K}^\epsilon(x,t) &= \sum_{i=0}^\infty \epsilon^{i+1} \mathbf{K}^i(x,t).
\end{aligned}$$

As for the outer expansion we substitute these expansions and the expansions for $\tilde{c}^\epsilon$, $\tilde{\mu}^\epsilon$, $\tilde{\mathbf{u}}^\epsilon$, and d^ϵ into (3.78)-(3.80) and match the terms with the same power of ϵ. Note that $\mathcal{C}\left(\mathbf{u}_{zz}^0 \otimes \nabla d^0\right) \nabla d^0 = M\mathbf{u}_{zz}^0$. We show that M is an invertible matrix. Let $\mathbf{v} \neq 0$ be an arbitrary vector, then it holds for all $(x,t) \in \Gamma^0(\delta)$

$$\mathbf{v} \cdot (M\mathbf{v}) = \left(\mathbf{v} \otimes \nabla d^0\right) : \mathcal{C}\left(\mathbf{v} \otimes \nabla d^0\right) \geq c_2 \left|\mathbf{v} \otimes \nabla d^0\right|^2 > 0\,,$$

due to Lemma 2.2.1. So we get the following ordinary differential equations

$$\left.\begin{aligned}\left(\mathbf{u}^0 - \mathbf{l}^0 d^0 \eta\right)_{zz} &= 0\\ \left(\mathbf{u}^k - \left(\mathbf{l}^k d^0 + \mathbf{l}^0 d^k\right)\eta\right)_{zz} &= D^{k-1}(z,x,t),\, k \geq 1\end{aligned}\right\} z \in \mathbb{R},\ (x,t) \in \Gamma^0(\delta), \tag{3.82}$$

$$\left.\begin{aligned}c_{zz}^0 - f(c^0) &= E^0(z,x,t)\\ c_{zz}^k - f'(c^0)c^k &= (E^k + A^{k-1})(z,x,t),\, k \geq 1\end{aligned}\right\} z \in \mathbb{R},\ (x,t) \in \Gamma^0(\delta), \tag{3.83}$$

$$\left.\begin{aligned}\left(\mu^0 - h^0 d^0 \eta\right)_{zz} &= 0\\ \left(\mu^k - \left(h^k d^0 + h^0 d^k\right)\eta\right)_{zz} &= B^{k-1}(z,x,t),\, k \geq 1\end{aligned}\right\} z \in \mathbb{R},\ (x,t) \in \Gamma^0(\delta), \tag{3.84}$$

where in $\mathbb{R}\times\Gamma^0(\delta)$ D^0, E^0, A^0, and B^0 have the following form

$$\begin{aligned}
D^0 &= M^{-1}\Big[-\left(\mathcal{C}\left(\mathbf{u}^0_{zz}\otimes\nabla d^1\right)\right)\nabla d^0-\left(\mathcal{C}\left(\mathbf{u}^0_{zz}\otimes\nabla d^0\right)\right)\nabla d^1\\
&\quad-\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^0_{i'})_z\partial_{j'}d^0\right)_{i=1,\dots,d}-\left(\mathcal{C}_{iji'j'}\,(\mathbf{u}^0_{i'})_z\partial_{jj'}d^0\right)_{i=1,\dots,d}\\
&\quad-\left(\mathcal{C}\nabla\mathbf{u}^0_z\right)\nabla d^0+c^0_z\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0+\left(\mathbf{j}^1d^0+\mathbf{j}^0d^1\right)\eta''-z\mathbf{j}^0\eta''\Big]\\
&\quad-z\mathbf{l}^0\eta''+\mathbf{K}^0d^0\eta'\,, &(3.85)\\
E^0 &= -\mathcal{E}^\star:\mathcal{C}\left(\mathbf{u}^0_z\otimes\nabla d^0\right)+k^0d^0\eta'\,, &(3.86)\\
A^0 &= -\mu^0-\Delta d^0c^0_z-2\nabla d^0\cdot\nabla c^0_z-\mathcal{E}^\star:\mathcal{C}\left(\nabla\mathbf{u}^0-\mathcal{E}^\star c^0\right)\\
&\quad+g^0d^0\eta'-zk^0\eta'\,, &(3.87)\\
B^0 &= d^0_tc^0_z-\Delta d^0\mu^0_z-2\nabla d^0\cdot\nabla\mu^0_z-zh^0\eta''+L^0d^0\eta'\,, &(3.88)
\end{aligned}$$

and where E^k for $k\geq 1$ and D^{k-1},A^{k-1},B^{k-1} for $k\geq 2$ have the following form

$$\begin{aligned}
D^{k-1} &= M^{-1}\Big[-\left(\mathcal{C}\left(\mathbf{u}^0_{zz}\otimes\nabla d^k\right)\right)\nabla d^0-\left(\mathcal{C}\left(\mathbf{u}^0_{zz}\otimes\nabla d^0\right)\right)\nabla d^k\\
&\quad-\left(\mathcal{C}\left(\mathbf{u}^{k-1}_{zz}\otimes\nabla d^0\right)\right)\nabla d^1-\left(\mathcal{C}\left(\mathbf{u}^{k-1}_{zz}\otimes\nabla d^1\right)\right)\nabla d^0\\
&\quad-\left(\mathcal{C}\left(\mathbf{u}^1_{zz}\otimes\nabla d^{k-1}\right)\right)\nabla d^0-\left(\mathcal{C}\left(\mathbf{u}^1_{zz}\otimes\nabla d^0\right)\right)\nabla d^{k-1}\\
&\quad-\left(\mathcal{C}\left(\mathbf{u}^0_{zz}\otimes\nabla d^1\right)\right)\nabla d^{k-1}-\left(\mathcal{C}\left(\mathbf{u}^0_{zz}\otimes\nabla d^{k-1}\right)\right)\nabla d^1\\
&\quad-\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^{k-1}_{i'})_z\partial_{j'}d^0\right)_{i=1,\dots,d}-\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^0_{i'})_z\partial_{j'}d^{k-1}\right)_{i=1,\dots,d}\\
&\quad-\left(\mathcal{C}_{iji'j'}\,(\mathbf{u}^{k-1}_{i'})_z\partial_{jj'}d^0\right)_{i=1,\dots,d}-\left(\mathcal{C}_{iji'j'}\,(\mathbf{u}^0_{i'})_z\partial_{jj'}d^{k-1}\right)_{i=1,\dots,d}\\
&\quad-\left(\mathcal{C}\nabla\mathbf{u}^{k-1}_z\right)\nabla d^0-\left(\mathcal{C}\nabla\mathbf{u}^0_z\right)\nabla d^{k-1}+c^{k-1}_z\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0\\
&\quad+c^0_z\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^{k-1}+\left(\mathbf{j}^kd^0+\mathbf{j}^{k-1}d^1+\mathbf{j}^1d^{k-1}+\mathbf{j}^0d^k\right)\eta''-z\mathbf{j}^{k-1}\eta''\Big]\\
&\quad+\left(\mathbf{l}^{k-1}d^1+\mathbf{l}^1d^{k-1}\right)\eta''-z\mathbf{l}^{k-1}\eta''+\left(\mathbf{K}^{k-1}d^0+\mathbf{K}^0d^{k-1}\right)\eta'+\mathcal{D}^{k-2}, &(3.89)\\
E^k &= -\mathcal{E}^\star:\mathcal{C}\left(\mathbf{u}^0_z\otimes\nabla d^k+\mathbf{u}^k_z\otimes\nabla d^0\right)+\left(k^kd^0+k^0d^k\right)\eta'\,, &(3.90)\\
A^{k-1} &= -\mu^{k-1}-\left(\Delta d^0c^{k-1}_z+\Delta d^{k-1}c^0_z\right)\\
&\quad-2\left(\nabla d^0\cdot\nabla c^{k-1}_z+\nabla d^{k-1}\cdot\nabla c^0_z\right)+f^{k-1}(c^0,\dots,c^{k-1})\\
&\quad-\mathcal{E}^\star:\mathcal{C}\left(\mathbf{u}^1_z\otimes\nabla d^{k-1}+\mathbf{u}^{k-1}_z\otimes\nabla d^1\right)-\mathcal{E}^\star:\mathcal{C}\left(\nabla\mathbf{u}^{k-1}-\mathcal{E}^\star c^{k-1}\right)\\
&\quad+\left(g^{k-1}d^0+g^0d^{k-1}\right)\eta'+\left(k^{k-1}d^1+k^1d^{k-1}\right)\eta'-zk^{k-1}\eta'+\mathcal{A}^{k-2}, &(3.91)\\
B^{k-1} &= \left(d^{k-1}_tc^0_z+d^0_tc^{k-1}_z\right)-\left(\Delta d^0\mu^{k-1}_z+\Delta d^{k-1}\mu^0_z\right)\\
&\quad-2\left(\nabla d^0\cdot\nabla\mu^{k-1}_z+\nabla d^{k-1}\cdot\nabla\mu^0_z\right)+\left(d^1h^{k-1}+d^{k-1}h^1\right)\eta''\\
&\quad-zh^{k-1}\eta''+\left(L^{k-1}d^0+L^0d^{k-1}\right)\eta'+\mathcal{B}^{k-2}\,. &(3.92)
\end{aligned}$$

Here $\mathcal{D}^{k-2}$, $\mathcal{A}^{k-2}$, and $\mathcal{B}^{k-2}$ have the following form

$$
\begin{aligned}
\mathcal{D}^{k-2} \;=\;& M^{-1}\Bigg[-\sum_{\substack{i,j=0\\ 2\le i+j\le k}}^{k-2}\left(\mathcal{C}\left(\mathbf{u}^i_{zz}\otimes\nabla d^j\right)\right)\nabla d^{k-i-j}\\
&-\sum_{l=1}^{k-2}\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^l_{i'})_z\partial_{j'}d^{k-1-l}\right)_{i=1,\dots,d}\\
&-\sum_{l=1}^{k-2}\left[\left(\mathcal{C}_{iji'j'}\,(\mathbf{u}^l_{i'})_z\partial_{jj'}d^{k-1-l}\right)_{i=1,\dots,d}+\left(\mathcal{C}\nabla\mathbf{u}^l_z\right)\nabla d^{k-1-l}\right]\\
&+\sum_{l=1}^{k-2}c^l_z\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^{k-1-l}+\sum_{l=2}^{k-2}\mathbf{j}^l d^{k-l}\eta''\Bigg]+\sum_{l=1}^{k-2}\mathbf{K}^l d^{k-1-l}\eta'\\
&+\sum_{l=2}^{k-2}\mathbf{l}^l d^{k-l}\eta''-z\mathbf{K}^{k-2}\eta'-M^{-1}\left(\mathcal{C}_{iji'j'}\,\partial_{jj'}\mathbf{u}^{k-2}_{i'}\right)_{i=1,\dots,d}\\
&+M^{-1}\left(\left(\mathcal{C}\mathcal{E}^\star\right)\nabla c^{k-2}\right)+M^{-1}\left(\mathbf{P}^+_{k-2}\eta^+_N+\mathbf{P}^-_{k-2}\eta^-_N\right), \qquad (3.93)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{k-2} \;=\;& \sum_{i=1}^{k-2}\left(-\Delta d^i c^{k-1-i}_z-2\nabla d^i\cdot\nabla c^{k-1-i}_z+d^i g^{k-1-i}\eta'\right)\\
&-\sum_{i=2}^{k-2}\mathcal{E}^\star:\mathcal{C}\left(\mathbf{u}^i_z\otimes\nabla d^{k-i}\right)+\sum_{i=2}^{k-2}k^i d^{k-i}\eta'\\
&-\Delta c^{k-2}-zg^{k-2}\eta', \qquad (3.94)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}^{k-2} \;=\;& \sum_{i=1}^{k-2}\left(-d^i_t c^{k-1-i}_z-\Delta d^i\mu^{k-1-i}_z-2\nabla d^i\cdot\nabla\mu^{k-1-i}_z\right)\\
&+\sum_{i=1}^{k-2}d^i L^{k-1-i}\eta'+\sum_{i=2}^{k-2}d^i h^{k-i}\eta''-zL^{k-2}\eta'\\
&+\left(c^{k-2}_t-\Delta\mu^{k-2}\right)-O^+_{k-2}\eta^+_N-O^-_{k-2}\eta^-_N\,. \qquad (3.95)
\end{aligned}
$$

Here we have used the conventions that if the upper limit of the summation is less than the lower limit, then the summation is zero, that $\mathcal{D}^{-1}=\mathcal{A}^{-1}=\mathcal{B}^{-1}=0$, that $a^{k-1}b^1+a^1b^{k-1}=a^1b^1$ when $k=2$, and that $a^{k-1}b^0c^1+a^1b^0c^{k-1}=a^1b^0c^1$ when $k=2$. We also use these conventions in the following.
Observe that D^{k-1} depends on d^k and $\mathbf{j}^k$. This would result in difficulties in the construction of d^k. To avoid this, we set

$$
\mathbf{j}^k := \begin{cases} 0 & \text{if } k=0\,,\\ \left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^k\right)\right)\nabla d^0+\left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^0\right)\right)\nabla d^k & \text{if } k\ge 1\,.\end{cases} \qquad (3.96)
$$

Actually, we obtain in (3.82a) the equation $\left(\mathbf{u}^0-\mathbf{l}^0d^0\eta\right)_{zz}=M^{-1}(\mathbf{j}^0d^0\eta'')$. But by the definition $\mathbf{j}^0 = 0$, equation ((3.82a) is valid. Since $\mathbf{u}^0_{zz} = d^0\mathbf{l}^0\eta''$, we obtain

$$\begin{aligned} D^0 &= M^{-1}\big[- \big(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^0_{i'})_z\partial_{j'}d^0\big)_{i=1,\dots,d} - \big(\mathcal{C}_{iji'j'}\,(\mathbf{u}^0_{i'})_z\partial_{jj'}d^0\big)_{i=1,\dots,d} \\ &\quad - \big(\mathcal{C}\nabla\mathbf{u}^0_z\big)\,\nabla d^0 + c^0_z\,(\mathcal{C}\mathcal{E}^\star)\,\nabla d^0\big] - z\mathbf{l}^0\eta'' + \mathbf{K}^0 d^0\eta'\,, \qquad (3.97) \end{aligned}$$

and since $\mathbf{u}^{k-1}_{zz} = (d^{k-1}\mathbf{l}^0 + d^0\mathbf{l}^{k-1})\eta'' + D^{k-2}$ for $k \geq 2$, we obtain for $k \geq 2$

$$\begin{aligned} D^{k-1} &= M^{-1}\Big[- \big(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^{k-1}_{i'})_z\partial_{j'}d^0\big)_{i=1,\dots,d} - \big(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}^0_{i'})_z\partial_{j'}d^{k-1}\big)_{i=1,\dots,d} \\ &\quad - \big(\mathcal{C}_{iji'j'}\,(\mathbf{u}^{k-1}_{i'})_z\partial_{jj'}d^0\big)_{i=1,\dots,d} - \big(\mathcal{C}_{iji'j'}\,(\mathbf{u}^0_{i'})_z\partial_{jj'}d^{k-1}\big)_{i=1,\dots,d} \\ &\quad - \big(\mathcal{C}\nabla\mathbf{u}^{k-1}_z\big)\,\nabla d^0 - \big(\mathcal{C}\nabla\mathbf{u}^0_z\big)\,\nabla d^{k-1} + c^{k-1}_z\,(\mathcal{C}\mathcal{E}^\star)\,\nabla d^0 + c^0_z\,(\mathcal{C}\mathcal{E}^\star)\,\nabla d^{k-1} \\ &\quad -d^0\left(\mathcal{C}\left(\mathbf{l}^{k-1}\otimes\nabla d^0\right)\right)\nabla d^1\eta'' - d^0\left(\mathcal{C}\left(\mathbf{l}^{k-1}\otimes\nabla d^1\right)\right)\nabla d^0\eta'' \\ &\quad -d^0\left(\mathcal{C}\left(\mathbf{l}^1\otimes\nabla d^{k-1}\right)\right)\nabla d^0\eta'' - d^0\left(\mathcal{C}\left(\mathbf{l}^1\otimes\nabla d^0\right)\right)\nabla d^{k-1}\eta'' \\ &\quad -d^0\left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^1\right)\right)\nabla d^{k-1}\eta'' - d^0\left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^{k-1}\right)\right)\nabla d^1\eta'' \\ &\quad -z\mathbf{j}^{k-1}\eta''\Big] + \left(\mathbf{l}^{k-1}d^1 + \mathbf{l}^1 d^{k-1}\right)\eta'' - z\mathbf{l}^{k-1}\eta'' + \left(\mathbf{K}^{k-1}d^0 + \mathbf{K}^0 d^{k-1}\right)\eta' \\ &\quad -M^{-1}\left[\left(\mathcal{C}\left(D^{k-2}\otimes\nabla d^0\right)\right)\nabla d^1 + \left(\mathcal{C}\left(D^{k-2}\otimes\nabla d^1\right)\right)\nabla d^0\right] \\ &\quad -M^{-1}\left[\left(\mathcal{C}\left(D^0\otimes\nabla d^{k-1}\right)\right)\nabla d^0 + \left(\mathcal{C}\left(D^0\otimes\nabla d^0\right)\right)\nabla d^{k-1}\right] + \mathcal{D}^{k-2}, \qquad (3.98) \end{aligned}$$

where $\mathcal{D}^{k-2}$ is defined as in (3.93).
To get bounded solutions we will see in the next subsection that it is necessary to require $D^{k-1} = \mathcal{O}(e^{-\alpha|z|})$ and $B^{k-1} = \mathcal{O}(e^{-\alpha|z|})$. This is the reason why we add $\epsilon^2\left(O^+\eta^+_N + O^-\eta_N\right)$ and $\epsilon^2\left(\mathbf{P}^+\eta^+_N + \mathbf{P}^-\eta^-_N\right)$ to handle the terms $c^{k-2}_t - \Delta\mu^{k-2}$ and $-(\mathcal{C}_{iji'j'}\partial_{jj'}\mathbf{u}^{k-2}_{i'}) + (\mathcal{C}\mathcal{E}^\star)\,\nabla c^{k-2}$ in $\mathcal{B}^{k-2}$ and $\mathcal{D}^{k-2}$ (see Lemma 3.2.6 and 3.2.9).

3.2.4 Compatibility Conditions

In this part we study the compatibility conditions of the ordinary differential equations (3.82)-(3.84) in order to have bounded solutions. Additionally, we investigate the behavior of the solutions as $z \to \pm\infty$. It will come out that there exists bounded solutions $(c^k, \mu^k, \mathbf{u}^k)$ for every $k \in \mathbb{N}$. All the assertions of this subsection can be shown as in [10].

Lemma 3.2.6. *Let D^{k-1} and $\mathcal{D}^{k-2}$ be defined as in (3.97), (3.98), and (3.93). Then (3.82b) has a bounded solution for $k = 1$ in $\Gamma^0(\delta)$ if and only if for all $(x,t) \in \Gamma^0(\delta)$, it holds*

$$\begin{aligned} 0 &= -\big(\mathcal{C}_{iji'j'}\left[\partial_j \boldsymbol{u}^0_{i'}\right]\partial_{j'}d^0\big)_{i=1,\dots,d} - \big(\mathcal{C}_{iji'j'}\left[\boldsymbol{u}^0_{i'}\right]\partial_{jj'}d^0\big)_{i=1,\dots,d} \\ &\quad - \left(\mathcal{C}\left[\nabla\boldsymbol{u}^0\right]\right)\nabla d^0 + \left[c^0\right](\mathcal{C}\mathcal{E}^\star)\,\nabla d^0 + M\boldsymbol{l}^0 + M\boldsymbol{K}^0 d^0\,, \qquad (3.99) \end{aligned}$$

and for $k \geq 2$ it has a solution in $\Gamma^0(\delta)$ if and only if for all $(x,t) \in \Gamma^0(\delta)$, it holds

$$\begin{aligned} M\tilde{\mathcal{D}}^{k-2} \;=\; & -\left(\mathcal{C}_{iji'j'}\left[\partial_j \boldsymbol{u}_{i'}^{k-1}\right]\partial_{j'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{iji'j'}\left[\partial_j \boldsymbol{u}_{i'}^{0}\right]\partial_{j'}d^{k-1}\right)_{i=1,\ldots,d} \\ & - \left(\mathcal{C}_{iji'j'}\left[\boldsymbol{u}_{i'}^{k-1}\right]\partial_{jj'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{iji'j'}\left[\boldsymbol{u}_{i'}^{0}\right]\partial_{jj'}d^{k-1}\right)_{i=1,\ldots,d} \\ & - \left(\mathcal{C}\left[\nabla \boldsymbol{u}^{k-1}\right]\right)\nabla d^0 - \left(\mathcal{C}\left[\nabla \boldsymbol{u}^{0}\right]\right)\nabla d^{k-1} + \left[c^{k-1}\right]\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 \\ & + \left[c^0\right]\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^{k-1} + \boldsymbol{j}^{k-1} + M\boldsymbol{l}^{k-1} + M\left(\boldsymbol{K}^{k-1}d^0 + \boldsymbol{K}^0 d^{k-1}\right), \end{aligned} \tag{3.100}$$

where $[.] = .|_{z=-\infty}^{z=+\infty}$ and

$$\tilde{\mathcal{D}}^{k-2}(x,t) = -\int_{\mathbb{R}} \mathcal{D}^{k-2}(z,x,t)\,dz\,.$$

Furthermore, for $k=0$ every bounded solution to (3.82a) has the form

$$\boldsymbol{u}^0(z,x,t) = \tilde{\boldsymbol{u}}^0(x,t) + \boldsymbol{l}^0(x,t)d^0(x,t)\left(\eta(z) - \tfrac{1}{2}\right), \tag{3.101}$$

and if (3.100) is satisfied, then for $k \geq 1$ every solution to (3.82b) has the form

$$\boldsymbol{u}^k(z,x,t) = \tilde{\boldsymbol{u}}^k(x,t) + \left(\boldsymbol{l}^k d^0 + \boldsymbol{l}^0 d^k\right)(x,t)\left(\eta(z) - \tfrac{1}{2}\right) + \boldsymbol{u}_*^{k-1}(z,x,t)\,, \tag{3.102}$$

where $\tilde{\boldsymbol{u}}^k(x,t)$, $k \geq 0$, is an arbitrary function and $\boldsymbol{u}_*^{k-1}(z,x,t)$ is a special solution depending only on functions of order lower than k and is uniquely determined by the normalization

$$\int_{\mathbb{R}} \boldsymbol{u}_*^{k-1}(z,x,t)\theta_0'(z)\,dz = 0 \quad \forall (x,t) \in \Gamma^0(\delta)\,. \tag{3.103}$$

In addition, there exists some $\boldsymbol{u}_{*(k-1)}^{\pm}(x,t)$ depending only on functions of order lower than k such that

$$D_x^m D_t^n D_z^l \left(\boldsymbol{u}_*^{k-1}(\pm z,x,t) - \boldsymbol{u}_{*(k-1)}^{\pm}(x,t)\right) = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty \tag{3.104}$$

for all $m,n,l \geq 0$ and for all $(x,t) \in \Gamma^0(\delta)$ provided $\left(c^i, c_i^{\pm}, \boldsymbol{u}^i, \boldsymbol{u}_i^{\pm}\right)$, $i = 1,\ldots,k-1$, satisfy the matching conditions (3.72) and (3.74).

Proof: The first assertion of the lemma follows from Lemma 2.6.3 and the identities $\int_{\mathbb{R}} z\eta''\,dz = -\int_{\mathbb{R}} \eta'\,dz = -1$ and $\int_{\mathbb{R}} \eta''\,dz = 0$ and the fact that all terms involving the derivatives with respect to z tend to zero exponentially fast.
The second assertion of the lemma follows from the second assertion of Lemma 2.6.3, the inner-outer matching conditions (3.72) and (3.74), and the definition of $\mathbf{P}_{k-2}^{\pm}$ (therefore $D_x^m D_t^n D_z^l \mathcal{D}^{k-2} = \mathcal{O}(e^{-\alpha|z|})$ as $z \to \pm\infty$). □

From now we set

$$k^k(x,t) := \begin{cases} \mathcal{E}^\star : \mathcal{C}\left(\mathbf{l}^0 \otimes \nabla d^0\right) & \text{if } k = 0\,, \\ \mathcal{E}^\star : \mathcal{C}\left(\left(\mathbf{l}^k \otimes \nabla d^0\right) + \left(\mathbf{l}^0 \otimes \nabla d^k\right)\right) & \text{if } k \geq 1 \end{cases} \tag{3.105}$$

for all $(x,t) \in \Gamma^0(\delta)$. By equation (3.101) and (3.102) it is not difficult to obtain for all $k \geq 0$

$$E^k(z,x,t) = -\mathcal{E}^\star : \mathcal{C}\left((\mathbf{u}_*^{k-1})_z \otimes \nabla d^0\right) \tag{3.106}$$

for all $(z,x,t) \in \mathbb{R} \times \Gamma^0(\delta)$ and where $\mathbf{u}_*^{-1} = 0$. In particular, it holds $E^0 = 0$. Thus $c^0(z,x,t) = \theta_0(z)$ is the unique solution to (3.83a) satisfying $\lim_{z\to\infty} c^0(\pm z) = \pm 1$ and the initial condition $c^0(0,x,t) = 0$.

Lemma 3.2.7. *Let $k \geq 1$ be any integer and E^k, A^{k-1}, and $\mathcal{A}^{k-2}$ be defined as in (3.106),(3.91), and (3.94). Then the system*

$$\begin{aligned} &c^k_{zz}(z,x,t) - f'(\theta_0(z))\, c^k(z,x,t) = (E^k + A^{k-1})(z,x,t) \quad \forall z \in \mathbb{R}, \\ &c^k(0,x,t) = 0, \quad c^k(.,x,t) \in L^\infty(\mathbb{R}) \end{aligned} \tag{3.107}$$

has a unique solution for $k = 1$ in $\Gamma^0(\delta)$ if and only if

$$0 \;=\; -\overline{\mu}^0 - \sigma\Delta d^0 - \mathcal{E}^\star : \mathcal{C}\left(\nabla \overline{\boldsymbol{u}}^0\right) + \eta_0 g^0 d^0 \quad \text{in } \Gamma^0(\delta)\,, \tag{3.108}$$

and it has a unique solution for $k \geq 2$ in $\Gamma^0(\delta)$ if and only if for all $(x,t) \in \Gamma^0(\delta)$

$$\begin{aligned} \widetilde{\mathcal{A}}^{k-2} \;=\; & -\overline{\mu}^{k-1} - \sigma\Delta d^{k-1} - \mathcal{E}^\star : \mathcal{C}\nabla\overline{\boldsymbol{u}}^{k-1} + \eta_0\left(g^{k-1}d^0 + g^0 d^{k-1}\right) \\ & +\eta_0 d^0 \mathcal{E}^\star : \mathcal{C}\left(M^{-1}\left[\left(\mathcal{C}\left(\boldsymbol{l}^{k-1} \otimes \nabla d^0\right)\right)\nabla d^1 + \left(\mathcal{C}\left(\boldsymbol{l}^{k-1} \otimes \nabla d^1\right)\right)\nabla d^0\right.\right. \\ & + \left(\mathcal{C}\left(\boldsymbol{l}^1 \otimes \nabla d^{k-1}\right)\right)\nabla d^0 + \left(\mathcal{C}\left(\boldsymbol{l}^1 \otimes \nabla d^0\right)\right)\nabla d^{k-1} + \left(\mathcal{C}\left(\boldsymbol{l}^0 \otimes \nabla d^1\right)\right)\nabla d^{k-1} \\ & \left.\left. + \left(\mathcal{C}\left(\boldsymbol{l}^0 \otimes \nabla d^{k-1}\right)\right)\nabla d^1\right] \otimes \nabla d^0 - \boldsymbol{l}^1 \otimes \nabla d^{k-1} - \boldsymbol{l}^{k-1} \otimes \nabla d^1\right)\,, \end{aligned} \tag{3.109}$$

where

$$\begin{aligned} \overline{\mu}^{k-1}(x,t) &= \frac{1}{2}\int_{\mathbb{R}} \mu^{k-1}(z,x,t)\theta_0'(z)\,dz\,, \\ \sigma &= \frac{1}{2}\int_{\mathbb{R}} \left(\theta_0'(z)\right)^2 dz\,, \\ \overline{\boldsymbol{u}}^{k-1}(x,t) &= \frac{1}{2}\int_{\mathbb{R}} \boldsymbol{u}^{k-1}(z,x,t)\theta_0'(z)\,dz\,, \\ \eta_0 &= \frac{1}{2}\int_{\mathbb{R}} \eta'(z)\theta_0'(z)\,dz\,, \\ \widetilde{\mathcal{A}}^{k-2}(x,t) &= \frac{1}{2}\int_{\mathbb{R}} \left[\Delta d^0 c_z^{k-1} + 2\nabla d^0 \cdot \nabla c_z^{k-1} - f^{k-1}(c^0,\ldots,c^{k-1}) - \mathcal{E}^\star : \mathcal{C}\mathcal{E}^\star c^{k-1}\right. \\ & \left. - \mathcal{A}^{k-2}\right]\theta_0' - \left[c_z^{k-1}\mathcal{E}^\star : \mathcal{C}\left((\mathcal{C}\mathcal{E}^\star)\nabla d^0 \otimes \nabla d^0\right) - \mathcal{E}^\star : \mathcal{C}\left(\mathcal{D}^{k-2} \otimes \nabla d^0\right)\right. \\ & + \mathcal{E}^\star : \mathcal{C}\left(M^{-1}\left[(\mathcal{C}_{iji'j'}\partial_j(\boldsymbol{u}^{k-2}_{*,i'})_z\partial_{j'}d^0)_{i=1,\ldots,d} + (\mathcal{C}_{iji'j'}(\boldsymbol{u}^{k-2}_{*,i'})_z\partial_{jj'}d^0)_{i=1,\ldots,d}\right.\right. \\ & + (\mathcal{C}\nabla(\boldsymbol{u}_*^{k-2})_z)\nabla d^0 + \left(\mathcal{C}\left(D^{k-2} \otimes \nabla d^0\right)\right)\nabla d^1 \\ & \left.\left. + \left(\mathcal{C}\left(D^{k-2} \otimes \nabla d^1\right)\right)\nabla d^0\right] \otimes \nabla d^0\right) + \mathcal{E}^\star : \mathcal{C}\left(D^{k-2} \otimes \nabla d^1\right)\Big]\,\theta_0\,dz\,. \end{aligned}$$

In addition, if (3.108) for $k=1$ or (3.109) for $k \geq 2$ is satisfied and $\left(c^0, c_0^\pm, \mu^0, \mu_0^\pm, \boldsymbol{u}^0, \boldsymbol{u}_0^\pm\right)$,... ,$\left(c^{k-1}, c_{k-1}^\pm, \mu^{k-1}, \mu_{k-1}^\pm, \boldsymbol{u}^{k-1}, \boldsymbol{u}_{k-1}^\pm\right)$ satisfy the matching conditions (3.72)-(3.74), then the unique solution c^k to (3.83b) satisfies the matching condition (3.72) where $c_k^\pm$ is defined by (3.70).

Remark 3.2.8. *In the proof we verify that $\int_{\mathbb{R}} D^0(z)\theta_0(z)\,dz = 0$. Hence $\widetilde{\mathcal{A}}^0$ is independent of d^1 for $k=2$. Note that $\widetilde{\mathcal{A}}^{k-2}$, $k \geq 2$, also depends on c^{k-1}. But later we will see that c^{k-1} can be considered as a quantity only depending on functions of lower order.*

Proof: We prove the first assertion only for $k \geq 2$. For $k=1$ one can apply the same techniques as in the case $k \geq 2$. Due to Lemma 2.6.2 the system (3.107) has a unique solution if and only if

$$0 = \frac{1}{2}\int_{\mathbb{R}} \left(E^k(z,x,t) + A^{k-1}(z,x,t)\right)\theta_0'(z)\,dz$$

for all $(x,t) \in \Gamma^0(\delta)$. Using the equation (3.106) and integration by parts yields

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} E^k(z)\theta_0'(z)\,dz \\
&= \frac{1}{2}\int_{\mathbb{R}} \mathcal{E}^\star : \mathcal{C}\left(D^{k-1}\otimes\nabla d^0\right)\theta_0\,dz - \frac{1}{2}\,\mathcal{E}^\star : \mathcal{C}\left((\mathbf{u}_*^{k-1})_z\otimes\nabla d^0\right)\theta_0\Big|_{z=-\infty}^{z=+\infty} \\
&= \frac{1}{2}\int_{\mathbb{R}} \mathcal{E}^\star : \mathcal{C}\left(D^{k-1}(z)\otimes\nabla d^0\right)\theta_0(z)\,dz
\end{aligned}$$

since $\lim_{z\to\pm\infty}(\mathbf{u}_*^{k-1})_z(z) = 0$. For computing the right-hand side we use (3.98). It holds for all $k \geq 1$

$$\int_{\mathbb{R}} \mathbf{u}_z^{k-1}(z)\theta_0(z)\,dz = \int_{\mathbb{R}} (\mathbf{u}_*^{k-2})_z(z)\theta_0(z)\,dz\,,$$

where we have used (3.102) and

$$\int_{\mathbb{R}} \eta'(z)\theta_0(z)\,dz = \left(\eta - \tfrac{1}{2}\right)\theta_0\Big|_{z=-\infty}^{z=+\infty} - \int_{\mathbb{R}} \left(\eta(z) - \tfrac{1}{2}\right)\theta_0'(z)\,dz = 0$$

due to (3.76). Furthermore, we apply the integrals

$$\begin{aligned}
&\int_{\mathbb{R}} c_z^0(z)\theta_0(z)\,dz = \int_{\mathbb{R}} \theta_0'(z)\theta_0(z)\,dz = 0, && \int_{\mathbb{R}} \eta''(z)\theta_0(z)\,dz = -2\eta_0\,, \\
&\int_{\mathbb{R}} z\eta''(z)\theta_0(z)\,dz = 0, && \int_{\mathbb{R}} D^0(z)\theta_0(z)\,dz = 0\,,
\end{aligned}$$

where the third identity follows from integration by parts and (3.76) and where the last identity follows from (3.97) and the identities above. Therefore we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} E^k(z)\theta_0'(z)\,dz \\
&\quad= \eta_0 d^0\mathcal{E}^\star : \mathcal{C}\left(M^{-1}\left[\left(\mathcal{C}\left(\mathbf{l}^{k-1}\otimes\nabla d^0\right)\right)\nabla d^1 + \left(\mathcal{C}\left(\mathbf{l}^{k-1}\otimes\nabla d^1\right)\right)\nabla d^0\right.\right. \\
&\qquad + \left(\mathcal{C}\left(\mathbf{l}^1\otimes\nabla d^{k-1}\right)\right)\nabla d^0 + \left(\mathcal{C}\left(\mathbf{l}^1\otimes\nabla d^0\right)\right)\nabla d^{k-1} + \left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^1\right)\right)\nabla d^{k-1} \\
&\qquad \left.\left. + \left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^{k-1}\right)\right)\nabla d^1\right]\otimes\nabla d^0\right) - \eta_0\mathcal{E}^\star : \mathcal{C}\left(\left(\mathbf{l}^{k-1}d^1 + \mathbf{l}^1 d^{k-1}\right)\otimes\nabla d^0\right) \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} c_z^{k-1}\theta_0\,dz\,\mathcal{E}^\star : \mathcal{C}\left((\mathcal{C}\mathcal{E}^\star)\nabla d^0\otimes\nabla d^0\right) + \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\int_{\mathbb{R}}\mathcal{D}^{k-2}\theta_0\,dz\otimes\nabla d^0\right) \\
&\qquad - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\int_R \left[M^{-1}\left[(\mathcal{C}_{iji'j'}\partial_j(\mathbf{u}_{*,i'}^{k-2})_z\partial_{j'}d^0)_{i=1,\dots,d} + (\mathcal{C}_{iji'j'}(\mathbf{u}_{*,i'}^{k-2})_z\partial_{jj'}d^0)_{i=1,\dots,d}\right.\right. \\
&\qquad \left.\left. + (\mathcal{C}\nabla(\mathbf{u}_*^{k-2})_z)\nabla d^0\right]\otimes\nabla d^0\right]\theta_0(z)\,dz \\
&\qquad - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(M^{-1}\int_{\mathbb{R}}\left[\left(\mathcal{C}\left(D^{k-2}\otimes\nabla d^0\right)\right)\nabla d^1\right.\right. \\
&\qquad \left.\left. + \left(\mathcal{C}\left(D^{k-2}\otimes\nabla d^1\right)\right)\nabla d^0\right]\theta_0\,dz\otimes\nabla d^0\right).
\end{aligned}
$$

To compute the integral $\frac{1}{2}\int_{\mathbb{R}} A^{k-1}\theta_0'\,dz$, we use the identity (3.91). We apply the definition (3.105) and the identity (3.102) to simplify A^{k-1}

$$
\begin{aligned}
&-\mathcal{E}^\star : \mathcal{C}\left(\mathbf{u}_z^1\otimes\nabla d^{k-1} + \mathbf{u}_z^{k-1}\otimes\nabla d^1\right) + \left(k^{k-1}d^1 + k^1 d^{k-1}\right)\eta' \\
&\quad= -\mathcal{E}^\star : \mathcal{C}\left((\mathbf{u}_*^0)_z\otimes\nabla d^{k-1} + (\mathbf{u}_*^{k-2})_z\otimes\nabla d^1\right) \\
&\qquad - d^0\mathcal{E}^\star : \mathcal{C}\left(\mathbf{l}^1\otimes\nabla d^{k-1} + \mathbf{l}^{k-1}\otimes\nabla d^1\right)\eta' \\
&\qquad + \mathcal{E}^\star : \mathcal{C}\left(d^1\mathbf{l}^{k-1}\otimes\nabla d^0 + d^{k-1}\mathbf{l}^1\otimes\nabla d^0\right)\eta'.
\end{aligned}
$$

As above we can show that

$$
-\int_{\mathbb{R}}\mathcal{E}^\star : \mathcal{C}\left((\mathbf{u}_*^0)_z\otimes\nabla d^{k-1} + (\mathbf{u}_*^{k-2})_z\otimes\nabla d^1\right)\theta_0'\,dz = \int_{\mathbb{R}}\mathcal{E}^\star : \mathcal{C}\left(D^{k-2}\otimes\nabla d^1\right)\theta_0\,dz\,.
$$

Hence we get by definition of σ and η_0

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} A^{k-1}(z)\theta_0'(z)\,dz \\
&\quad= -\overline{\mu}^{k-1} - \sigma\Delta d^{k-1} - \mathcal{E}^\star : \mathcal{C}\nabla\overline{\mathbf{u}}^{k-1} - \eta_0 d^0\mathcal{E}^\star : \mathcal{C}\left(\mathbf{l}^1\otimes\nabla d^{k-1} + \mathbf{l}^{k-1}\otimes\nabla d^1\right) \\
&\qquad + \eta_0\mathcal{E}^\star : \mathcal{C}\left(\left(\mathbf{l}^{k-1}d^1 + \mathbf{l}^1 d^{k-1}\right)\otimes\nabla d^0\right) + \eta_0\left(g^{k-1}d^0 + g^0 d^{k-1}\right) \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}}\mathcal{E}^\star : \mathcal{C}\left(D^{k-2}\otimes\nabla d^1\right)\theta_0\,dz + \frac{1}{2}\int_{\mathbb{R}}\left(-\Delta d^0 c_z^{k-1} - 2\nabla d^0\cdot\nabla c_z^{k-1}\right. \\
&\qquad \left. + f^{k-1}(c^0,\dots,c^{k-1}) + \mathcal{E}^\star : \mathcal{C}\mathcal{E}^\star c^{k-1} + \mathcal{A}^{k-2}\right)\theta_0'\,dz\,.
\end{aligned}
$$

This shows the first assertion for $k \geq 2$.
For all $k \geq 1$ the second assertion of the lemma follows from the second assertion of Lemma 2.6.2 and

$$\begin{aligned} E^k(\pm z) + A^{k-1}(\pm z) &= -\mu^{k-1}(\pm z) + f^{k-1}(c^0(\pm z), \dots, c^{k-1}(\pm z)) \\ &\quad -\mathcal{E}^* : \mathcal{C}\left(\nabla \mathbf{u}^{k-1}(\pm z) - \mathcal{E}^* c^{k-1}(\pm z)\right) \\ &\quad +\mathcal{A}^{k-2}(\pm z) + \mathcal{O}(e^{-\alpha z}) \\ &\to -\mu^{\pm}_{k-1} + f^{k-1}(c^{\pm}_0, \dots, c^{\pm}_{k-1}) \\ &\quad -\mathcal{E}^* : \mathcal{C}\left(\nabla \mathbf{u}^{\pm}_{k-1} - \mathcal{E}^* c^{\pm}_{k-1}\right) - \Delta c^{\pm}_{k-2}, \end{aligned}$$

as $z \to \infty$, since all the terms involving the derivatives with respect to z tend to zero exponentially fast and

$$c^j(\pm z) = c^{\pm}_j + \mathcal{O}(e^{-\alpha z}), \quad \mu^j(\pm z) = \mu^{\pm}_j + \mathcal{O}(e^{-\alpha z}), \quad \mathbf{u}^j(\pm z) = \mathbf{u}^{\pm}_j + \mathcal{O}(e^{-\alpha z}),$$

as $z \to \infty$ and for all $j \in \{0, \dots, k-1\}$. Using the outer expansion (3.70) completes the proof. □

Lemma 3.2.9. *Let B^{k-1} and $\mathcal{B}^{k-2}$ be defined as in (3.92) and (3.95). Then, for $k \geq 1$, the ordinary differential equation (3.84b) has a bounded solution if and only if for all $(x,t) \in \Gamma^0(\delta)$*

$$\begin{aligned} & d^{k-1}_t - \frac{1}{2}\left(\Delta d^0 \left[\mu^{k-1}\right] + \Delta d^{k-1}\left[\mu^0\right]\right) - \nabla d^0 \cdot \left[\nabla \mu^{k-1}\right] \\ & -\nabla d^{k-1} \cdot \left[\nabla \mu^0\right] + \frac{1}{2} h^{k-1} + \frac{1}{2}\left(d^0 L^{k-1} + d^{k-1} L^0\right) = \widetilde{\mathcal{B}}^{k-2}, \end{aligned} \tag{3.110}$$

where $[.] = .\,|^{z=+\infty}_{z=-\infty}$ and

$$\widetilde{\mathcal{B}}^{k-2}(x,t) = -\frac{1}{2} d^0_t(x,t)\left[c^{k-1}(x,t)\right] - \frac{1}{2}\int_{\mathbb{R}} \mathcal{B}^{k-2}(z,x,t)\,dz$$

if $k \geq 2$ and $\widetilde{\mathcal{B}}^{-1} = 0$. In addition, every bounded solution to (3.84a) can be written as

$$\mu^0(z,x,t) = \tilde{\mu}^0(x,t) + d^0(x,t) h^0(x,t)\left(\eta(z) - 1/2\right), \tag{3.111}$$

and if (3.110) is satisfied, then every solution to (3.84b) can be written as

$$\mu^k(z,x,t) = \tilde{\mu}^k(x,t) + \left(d^0 h^k + d^k h^0\right)(x,t)\left(\eta(z) - 1/2\right) + \mu^{k-1}_*(z,x,t), \tag{3.112}$$

where $\tilde{\mu}^k(x,t)$ is an arbitrary function and $\mu^{k-1}_*(z,x,t)$ is a special solution depending only on $(c^{\pm}_i, c^i, \mu^{\pm}_i, \mu^i, d^i, h^i, g^i)$ for $i \leq k-1$ and is uniquely determined by the normalization

$$\int_{\mathbb{R}} \mu^{k-1}_*(z,x,t)\theta_0'(z)\,dz = 0 \quad \forall (x,t) \in \Gamma^0(\delta). \tag{3.113}$$

Furthermore, there exists some $\mu^{\pm}_{*(k-1)}$ *depending only on* $(c_i^{\pm}, c^i, \mu_i^{\pm}, \mu^i, d^i, h^i, g^i)$ *for* $i \leq k-1$ *such that*

$$D_x^m D_t^n D_z^l \left[\mu_*^{k-1}(\pm z, x, t) - \mu^{\pm}_{*(k-1)}(x,t)\right] = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty$$

for all $m, n, l \geq 0$ *and for all* $(x,t) \in \Gamma^0(\delta)$, *provided* $(c^i, c_i^{\pm}, \mu^i, \mu_i^{\pm})$, $i = 1, \dots, k-1$, *satisfy the matching conditions (3.72) and (3.73).*

Proof: The first assertion of the lemma follows from Lemma 2.6.3 and the identities $\int_{\mathbb{R}} \theta_0' \, dz = 2$, $\int_{\mathbb{R}} \eta'' \, dz = 0$, and $\int_{\mathbb{R}} z\eta'' \, dz = -\int_{\mathbb{R}} \eta' \, dz = -1$.
The second assertion follows from the second assertion of Lemma 2.6.3, the inner-outer matching conditions (3.72) and (3.73), and the definition of $O^{\pm}_{k-2}$ (therefore $D_x^m D_t^n D_z^l \mathcal{B}^{k-2} = \mathcal{O}(e^{-\alpha|z|})$ as $z \to \pm\infty$). □

3.2.5 Boundary-Layer Expansion

Let $d_B(x)$ be the signed distance function from x to $\partial\Omega$ where $d_B < 0$ in Ω and $S_B : \left\{x \in \mathbb{R}^d : |d_B| \leq \delta\right\} \to \partial\Omega$ the projection of x to $\partial\Omega$ along the normal of $\partial\Omega$. Furthermore, we define $\partial_T\Omega(\delta) = \{(x,t) \in \Omega_T : -\delta < d_B(x) < 0\}$.
As for the inner expansion we assume that near the boundary $\partial_T\Omega$ the solutions $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ have for every $\epsilon \in (0,1]$ the expansion

$$c^\epsilon(x,t) = c_B^\epsilon\left(\frac{d_B^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad c_B^\epsilon(z,x,t) = 1 + \sum_{i=1}^{\infty} \epsilon^i c_B^i(z,x,t),$$

$$\mu^\epsilon(x,t) = \mu_B^\epsilon\left(\frac{d_B^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \mu_B^\epsilon(z,x,t) = \sum_{i=0}^{\infty} \epsilon^i \mu_B^i(z,x,t),$$

$$\mathbf{u}^\epsilon(x,t) = \mathbf{u}_B^\epsilon\left(\frac{d_B^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \mathbf{u}_B^\epsilon(z,x,t) = \sum_{i=0}^{\infty} \epsilon^i \mathbf{u}_B^i(z,x,t),$$

where $(x,t) \in \overline{\partial_T\Omega(\delta)}$ and $z \in (-\infty, 0]$.
To match the boundary-layer and outer expansion, we require as $z \to -\infty$

$$D_x^m D_t^n D_z^l \left[c_B^k(z,x,t) - c_k^+(x,t)\right] = \mathcal{O}(e^{\alpha z}), \tag{3.114}$$

$$D_x^m D_t^n D_z^l \left[\mu_B^k(z,x,t) - \mu_k^+(x,t)\right] = \mathcal{O}(e^{\alpha z}), \tag{3.115}$$

$$D_x^m D_t^n D_z^l \left[\mathbf{u}_B^k(z,x,t) - \mathbf{u}_k^+(x,t)\right] = \mathcal{O}(e^{\alpha z}) \tag{3.116}$$

for all $(x,t) \in \overline{\partial_T\Omega(\delta)}$ and all $k, m, n, l \in \left\{0, \dots, \bar{K}\right\}$ where $\bar{K}$ depends on the order of the expansion.
Similarly to the inner expansion, we define $M_B(x) = \left(\mathcal{C}_{iji'j'}\, \partial_j d_B(x)\, \partial_{j'} d_B(x)\right)_{i,i'=1}^d$ where an analogous proof shows the invertibility of M_B. Since $\partial\Omega$ is known, we do not require a series expansion for d_B. Also note that d_B is time independent. We

substitute the expansion of $(c^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon)$ into (3.1)-(3.3). Then a calculation similar to the inner expansion gives us

$$
\begin{aligned}
\mathbf{u}^k_{B,zz}(z,x,t) &= \mathbf{A}^{k-1}_B(z,x,t), && k \geq 0\,, && (3.117)\\
c^k_{B,zz}(z,x,t) - f'(1)c^k_B(z,x,t) &= B^{k-1}_B(z,x,t), && k \geq 1\,, && (3.118)\\
\mu^k_{B,zz}(z,x,t) &= C^{k-1}_B(z,x,t), && k \geq 0 && (3.119)
\end{aligned}
$$

for all $(x,t) \in \partial_T\Omega(\delta)$ and $z \in (-\infty, 0)$ where

$$
\begin{aligned}
\mathbf{A}^{k-1}_B &= M^{-1}_B \big[-(\mathcal{C}_{iji'j'}\partial_j(\mathbf{u}^{k-1}_{B,i'})_z \partial_{j'} d_B)_{i=1,\dots,d} - (\mathcal{C}_{iji'j'}(\mathbf{u}^{k-1}_{B,i'})_z \partial_{jj'} d_B)_{i=1,\dots,d} \\
&\quad -(\mathcal{C} : \nabla \mathbf{u}^{k-1}_{B,z})\nabla d_B - \operatorname{div}\left(\mathcal{C} : \nabla \mathbf{u}^{k-2}_B\right) \\
&\quad + c^{k-1}_{B,z}(\mathcal{C} : \mathcal{E}^\star)\nabla d_B + (\mathcal{C} : \mathcal{E}^\star)\nabla c^{k-2}_B \big]\,, && (3.120)\\
B^{k-1}_B &= -\mathcal{E}^\star : \mathcal{C}(\mathbf{u}^k_{B,z} \otimes \nabla d_B) - \mathcal{E}^\star : \mathcal{C}(\nabla \mathbf{u}^{k-1}_B - \mathcal{E}^\star c^{k-1}_B) - \mu^{k-1}_B \\
&\quad + f^{k-1}(c^0_B, \dots, c^{k-1}_B) - 2\nabla d_B \cdot \nabla c^{k-1}_{B,z} - \Delta d_B c^{k-1}_{B,z} - \Delta c^{k-2}_B\,, && (3.121)\\
C^{k-1}_B &= -\Delta d_B \mu^{k-1}_{B,z} - 2\nabla d_B \cdot \nabla \mu^{k-1}_{B,z} - \Delta \mu^{k-2}_B + c^{k-2}_{B,t}\,, && (3.122)
\end{aligned}
$$

where we have assumed that $\mathbf{u}^{-2}_B = \mathbf{u}^{-1}_B = c^{-2}_B = c^{-1}_B = \mu^{-2}_B = \mu^{-1}_B = 0$ and $c^0_B = 1$.

Remark 3.2.10. *Due to equation (3.129) in Lemma 3.2.12 below, we will see that B^{k-1}_B is independent of functions of order k. More precisely, we get the identity $\boldsymbol{u}^k_{B,z} = \int_{-\infty}^z A^{k-1}_B(y)\,dy$. So it is possible to construct c^k_B by functions of order lower than k.*

Since ∇d_B is the unit outer normal to $\partial\Omega$, we obtain $\frac{\partial}{\partial n}\big|_{\partial\Omega} = \epsilon^{-1}\frac{\partial}{\partial z} + \nabla d_B \cdot \nabla$. Also observe that $c^\epsilon_B\left(\frac{d_B(x)}{\epsilon}, x, t\right) = c^\epsilon_B(0,x,t)$ for all $(x,t) \in \partial_T\Omega$ (analogously for μ^ϵ_B and $\mathbf{u}^\epsilon_B$). Therefore to satisfy the boundary conditions on $\partial_T\Omega$, we require

$$
\begin{aligned}
&\mathbf{u}^k_B(0,x,t) = 0 && \forall (x,t) \in \partial_T\Omega, k \geq 0, && (3.123)\\
&c^k_{B,z}(0,x,t) = -\nabla d_B \cdot \nabla c^{k-1}_B(0, S_B(x), t) && \forall (x,t) \in \overline{\partial_T\Omega(\delta)}, k \geq 1, && (3.124)\\
&\mu^k_{B,z}(0,x,t) = -\nabla d_B \cdot \nabla \mu^{k-1}_B(0,x,t) && \forall (x,t) \in \partial_T\Omega, k \geq 0\,. && (3.125)
\end{aligned}
$$

Remark 3.2.11. *The boundary condition (3.124) is necessary only for $x \in \partial_T\Omega$. We use the same boundary condition as in [10]. The boundary condition (3.124) has the advantage that we obtain a unique smooth solution in (x,t) and if for all $k = 0,\dots,j-1$, $\frac{\partial}{\partial n}c^+_k\big|_{\partial_T\Omega} = 0$ and $c^k_B(z,x,t)$, $\mu^k_B(z,x,t)$, and $\boldsymbol{u}^k_B(z,x,t)$ are independent of z (therefore $A^{j-1}(z,x,t)$ is independent of z), then so is $c^j_B(z,x,t)$. For μ^j_B and $\boldsymbol{u}^j_B$ we do not require uniqueness since we specify μ^j_B and $\boldsymbol{u}^j_B$ directly.*

Now let us show that the ordinary differential equations (3.117)-(3.119) with initial values (3.123)-(3.125) have bounded solutions. For convenience we define $\mathbf{F}^{-1} =$

$G^{-2} = G^{-1} = 0$ and

$$\mathbf{F}^k(x,t) = -\int_{-\infty}^{0}\int_{-\infty}^{z} \mathbf{A}_B^k(w,x,t)\,dw\,dz\,, \tag{3.126}$$

$$\begin{aligned} G^k(x,t) &= (\Delta d_B(x) + \nabla d_B(x)\cdot\nabla)\int_{-\infty}^{0}\int_{-\infty}^{z} C_B^k(w,x,t)\,dw\,dz \\ &\quad + \int_{-\infty}^{0} (\Delta\mu_B^k - c_{B,t}^k)(z,x,t)\,dz \end{aligned} \tag{3.127}$$

for all $k \geq 0$ and $(x,t) \in \overline{\partial_T\Omega(\delta)}$.

Lemma 3.2.12. *Let $j \geq 0$ be any integer. Assume that for all $i = 0,\ldots,j-1$, the functions $c_i^+, \boldsymbol{u}_i^+, c_B^i, \boldsymbol{u}_B^i$ are known, smooth, and satisfy the matching conditions (3.114) and (3.116) and the outer-expansion (3.71). Let $\boldsymbol{F}^{j-1}$ be defined as in (3.126) and assume that $\boldsymbol{u}_j^+$ satisfies the boundary condition*

$$\boldsymbol{u}_j^+(x,t) = \boldsymbol{F}^{j-1}(x,t) \quad \forall (x,t) \in \partial_T\Omega\,. \tag{3.128}$$

Also assume that $\boldsymbol{u}_B^{-2} = \boldsymbol{u}_B^{-1} = c_B^{-2} = c_B^{-1} = 0$, and $\boldsymbol{u}_B^i$, $i = 0,\ldots,j-1$, are defined by

$$\boldsymbol{u}_B^i(z,x,t) = \boldsymbol{u}_i^+(x,t) + \int_{-\infty}^{z}\int_{-\infty}^{y} \boldsymbol{A}_B^{i-1}(w,x,t)\,dw\,dy \tag{3.129}$$

for all $z \in (-\infty,0]$ and $(x,t) \in \overline{\partial_T\Omega(\delta)}$, and where $\boldsymbol{A}_B^{i-1}$ is defined as in (3.120). Then for known smooth $\boldsymbol{u}_j^+$ the function $\boldsymbol{u}_B^j$ defined by (3.129) (with $i = j$) satisfies for $k = j$ the boundary-expansion equation (3.117), the boundary condition (3.123), and the matching condition (3.116).

Proof: First let us show that $\mathbf{F}^{j-1}$ and $\mathbf{u}_B^i$, $i = 0,\ldots,j$, are well-defined and smooth. Since $(c_i^+, \mathbf{u}_i^+, c_B^i, \mathbf{u}_B^i)$ satisfy the matching conditions (3.114) and (3.116) and the outer-expansion equation (3.71) for $i = 0,\ldots,j-1$, we obtain

$$\left|\operatorname{div}\left(\mathcal{C} : \nabla\mathbf{u}_B^i\right) - (\mathcal{C} : \mathcal{E}^\star)\nabla c_B^i\right| \leq Ce^{\alpha z} \quad \forall z \in (-\infty,0],\ \forall (x,t) \in \overline{\partial_T\Omega(\delta)}\,,$$

and therefore one concludes by definition of A_B^i and the fact that all terms involving the derivatives with respect to z tend to zero exponentially fast

$$\left|\mathbf{A}_B^i\right| \leq Ce^{\alpha z} \quad \forall z \in (-\infty,0],\ \forall (x,t) \in \overline{\partial_T\Omega(\delta)}$$

for $i = 0,\ldots,j-1$ and some $C > 0$. Therefore the integrals defining $\mathbf{F}^{j-1}$ and $\mathbf{u}_B^i$, $i = 0,\ldots,j$, are well-defined and smooth. The same arguments as above and the definition of $\mathbf{u}_B^j$ yield the matching condition (3.116) for $k = j$. By an easy calculation we obtain (3.117) for $k = j$. Finally, the boundary condition (3.123) immediately follows from the condition (3.128) and the definition of $\mathbf{F}^{j-1}$ in (3.126). □

Remark 3.2.13. *Since $\boldsymbol{A}_B^{-1} = 0$, it follows $\boldsymbol{u}_B^0(z,x,t) = \boldsymbol{u}_0^+(x,t)$ for all $(x,t) \in \partial_T\Omega(\delta)$ and therefore it holds $\boldsymbol{u}_B^1(z,x,t) = \boldsymbol{u}_1^+(x,t)$ for all $(x,t) \in \partial_T\Omega(\delta)$ since $c_B^0(z,x,t) = 1$.*

Lemma 3.2.14. *Let $j \geq 1$ be any integer. Assume that for all $i = 0,\dots,j-1$, the functions $c_i^+, \mu_i^+, c_B^i, \mu_B^i, \boldsymbol{u}_i^+, \boldsymbol{u}_B^i$ are known, smooth, and satisfy the matching conditions (3.114)-(3.116). Then for $k = j$, the boundary-layer expansion equation (3.118) subject to the boundary condition (3.124) has a unique bounded solution c_B^j for $z \in (-\infty, 0]$ and all $(x,t) \in \overline{\partial_T\Omega(\delta)}$. In addition, the solution satisfies the matching condition (3.114) where c_j^+ is defined by* (3.70)*.*

Proof: We can write the boundary-layer expansion equation (3.118) subject to the boundary condition (3.124) in the form

$$\mathbf{x}'(z) = A\mathbf{x}(z) + \mathbf{g}(z)$$

with initial value

$$\mathbf{x}(0) = \mathbf{x}_0\,,$$

where $\mathbf{x}_1(z) = c_B^j(z)$, $\mathbf{x}_2(z) = \mathbf{x}_1'(z)$, $A = \begin{pmatrix} 0 & 1 \\ f'(1) & 0 \end{pmatrix}$, $\mathbf{g}(z) = \begin{pmatrix} 0 \\ B_B^{j-1}(z) \end{pmatrix}$, and $\mathbf{x}_0 = \begin{pmatrix} c_0 \\ -\nabla d_B \cdot c_B^{k-1}(0) \end{pmatrix}$ (c_0 is an arbitrary constant). An easy calculation gives us that $\pm\sqrt{f'(1)}$ are the eigenvalues of A. Let $T \in \mathbb{R}^{2\times 2}$ be the transformation matrix such that

$$D := \begin{pmatrix} -\sqrt{f'(1)} & 0 \\ 0 & \sqrt{f'(1)} \end{pmatrix} = T^{-1}AT\,.$$

By setting $\mathbf{w}(z) = T^{-1}\mathbf{x}(z)$ we obtain

$$\mathbf{w}'(z) = D\mathbf{w}(z) + \mathbf{h}(z) \tag{3.130}$$

with initial value

$$\mathbf{w}(0) = \mathbf{w}_0\,,$$

where $\mathbf{h} = T^{-1}\mathbf{g}$ and $\mathbf{w}_0 = T^{-1}\mathbf{x}_0$. Then the ordinary differential equation (3.130) has the general solution

$$\begin{aligned} \mathbf{w}(z) &= c_1 e^{-\sqrt{f'(1)}z}\mathbf{e}_1 + c_2 e^{\sqrt{f'(1)}z}\mathbf{e}_2 \\ &\quad + \int_0^z \operatorname{diag}(e^{-\sqrt{f'(1)}(z-s)}, e^{\sqrt{f'(1)}(z-s)})\mathbf{h}(s)\,ds\,, \end{aligned}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard unit vectors and $c_1, c_2 \in \mathbb{R}$ any constants. Since $\mathbf{h}(s)$ is bounded for $z \in (-\infty, 0]$, it is not difficult to verify that the solution $\mathbf{w}$ is bounded for $z \in (-\infty, 0]$ if and only if

$$c_1 = -\int_0^{-\infty} e^{\sqrt{f'(1)}s}\mathbf{h}_1(s)\,ds\,.$$

To prove uniqueness of the bounded solution, we consider the initial value

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 = \mathbf{w}(0) = \mathbf{w}_0 = T^{-1}\left(c_0\mathbf{e}_1 + (-\nabla d_B \cdot c_B^{k-1}(0))\mathbf{e}_2\right).$$

Hence c_0 and c_2 are uniquely determined by the equations

$$\begin{aligned} c_0 &= \frac{1}{\mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_1)}\left(c_1 + \left(\nabla d_B \cdot c_B^{k-1}(0)\right)\mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_2)\right), \\ c_2 &= c_0\mathbf{e}_2 \cdot (T^{-1}\mathbf{e}_1) - \left(\nabla d_B \cdot c_B^{k-1}(0)\right)\mathbf{e}_2 \cdot (T^{-1}\mathbf{e}_2). \end{aligned}$$

Here the first equation is well-defined since $\mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_1) \neq 0$. This can be seen as follows

$$\mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_1) = -\frac{\mathbf{e}_1 \cdot (DT^{-1}\mathbf{e}_1)}{\sqrt{f'(1)}} = -\frac{\mathbf{e}_1 \cdot (T^{-1}A\mathbf{e}_1)}{\sqrt{f'(1)}} = -\sqrt{f'(1)}\,\mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_2).$$

Since T is invertible it is not possible that $\mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_1) = \mathbf{e}_1 \cdot (T^{-1}\mathbf{e}_2) = 0$. It remains to verify the matching condition (3.116). First we consider $\mathbf{w}(z)$. Assume that there exists some vector $\mathbf{h}_{-\infty} \in \mathbb{R}^2$ such that $\|\mathbf{h}(z) - \mathbf{h}_{-\infty}\| \leq Ce^{\alpha z}$ for $z \in (-\infty, 0]$ and some $C > 0$ and $\alpha \in (0, \sqrt{f'(1)})$ where c_1 is given as above. Here $\|.\|$ denotes the Euclidean norm of the vector space $\mathbb{R}^2$. We show that

$$\left\|\mathbf{w}(z) - \operatorname{diag}(1/\sqrt{f'(1)}, -1/\sqrt{f'(1)})\,\mathbf{h}_{-\infty}\right\| \leq Ce^{\alpha z} \tag{3.131}$$

for all $z \in (-\infty, 0]$. Since $\lim_{z\to-\infty} e^{\sqrt{f'(1)}z} = 0$, we get

$$\begin{aligned} \frac{1}{\sqrt{f'(1)}} &= e^{-\sqrt{f'(1)}z}\int_{-\infty}^{z} e^{\sqrt{f'(1)}s}\,ds, \\ \frac{1}{-\sqrt{f'(1)}} &= e^{\sqrt{f'(1)}z}\int_{0}^{z} e^{-\sqrt{f'(1)}s}\,ds - \frac{e^{\sqrt{f'(1)}z}}{\sqrt{f'(1)}}. \end{aligned}$$

Using these two equations and the expressions for $\mathbf{w}$ and c_1, we obtain

$$\begin{aligned} &\left\|\mathbf{w}(z) - \operatorname{diag}(1/\sqrt{f'(1)}, -1/\sqrt{f'(1)})\,\mathbf{h}_{-\infty}\right\| \\ &\leq \|c_2\mathbf{e}_2\|\,e^{\sqrt{f'(1)}z} + \left\|e^{-\sqrt{f'(1)}z}\int_{-\infty}^{z} e^{\sqrt{f'(1)}s}(\mathbf{h}_1(s) - \mathbf{h}_{-\infty,1})\,ds\,\mathbf{e}_1\right\| \\ &\quad + \left\|e^{\sqrt{f'(1)}z}\int_{0}^{z} e^{-\sqrt{f'(1)}s}(\mathbf{h}_2(s) - \mathbf{h}_{-\infty,2})\,ds\,\mathbf{e}_2\right\| + \left\|\frac{h_{-\infty,2}}{\sqrt{f'(1)}}\mathbf{e}_2\right\| e^{\sqrt{f'(1)}z}. \end{aligned} \tag{3.132}$$

A similar calculation as above gives us

$$\begin{aligned} &\left\|e^{-\sqrt{f'(1)}z}\int_{-\infty}^{z} e^{\sqrt{f'(1)}s}(\mathbf{h}_1(s) - \mathbf{h}_{-\infty,1})\,ds\,\mathbf{e}_1\right\| \\ &\leq Ce^{-\sqrt{f'(1)}z}\int_{-\infty}^{z} e^{\sqrt{f'(1)}s}e^{\alpha s}\,ds = \frac{C}{\sqrt{f'(1)} + \alpha}\,e^{\alpha z} \end{aligned}$$

and

$$
\begin{aligned}
&\left\| e^{\sqrt{f'(1)}z} \int_0^z e^{-\sqrt{f'(1)}s} (\mathbf{h}_2(s) - \mathbf{h}_{-\infty,2})\, ds\, \mathbf{e}_2 \right\| \\
&\quad \leq \; C e^{\sqrt{f'(1)}z} \int_z^0 e^{-\sqrt{f'(1)}s} e^{\alpha s}\, ds = \frac{C}{\sqrt{f'(1)} - \alpha} \left(e^{\alpha z} - e^{\sqrt{f'(1)}z} \right).
\end{aligned}
$$

By using these estimates, inequality (3.131) follows from (3.132).
Now we come back to $c_B^j(z)$. Since $(c_i^+, \mu_i^+, c_B^i, \mu_B^i, \mathbf{u}_i^+, \mathbf{u}_B^i)$, $i = 0, \ldots, j-1$, satisfy the matching conditions (3.114)-(3.116) and due to the definition of c_j^+, it follows

$$
\left| \frac{B_B^{j-1}(z)}{-f'(1)} - c_j^+ \right| \leq C e^{\alpha z}
$$

for some constant $C > 0$. Since $A^{-1} = 1/f'(1) \left(\begin{smallmatrix} 0 & 1 \\ f'(1) & 0 \end{smallmatrix} \right)$, the equation

$$
\frac{B_B^{j-1}(z)}{-f'(1)}\, \mathbf{e}_1 = -A^{-1} T \mathbf{h}(z)
$$

follows due to the definition of $\mathbf{g}$ and $\mathbf{h}$. Altogether, we obtain

$$
\|\mathbf{h}(z) - \mathbf{h}_{-\infty}\| \leq C e^{\alpha z}
$$

with $\mathbf{h}_{-\infty} = -c_j^+ T^{-1} A \mathbf{e}_1$ and some $C > 0$. Due to $1/\sqrt{f'(1)} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) = -D^{-1}$ an easy calculation gives us $1/\sqrt{f'(1)}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) = -T^{-1} A^{-1} T$. Because of this, we can prove the matching condition (3.116) as follows

$$
\begin{aligned}
\left| c_B^j(z) - c_j^+ \right| \;&\leq\; \left\| \mathbf{x}(z) - c_j^+\, \mathbf{e}_1 \right\| = \left\| T \left(\mathbf{w}(z) + T^{-1} A^{-1} T \mathbf{h}_{-\infty} \right) \right\| \\
&=\; \left\| T \left(\mathbf{w}(z) - \operatorname{diag}(1/\sqrt{f'(1)}, -1/\sqrt{f'(1)})\, \mathbf{h}_{-\infty} \right) \right\| \\
&\leq\; C e^{\alpha z}
\end{aligned}
$$

for some $C > 0$ and where we have used (3.131). The assertion for higher derivatives with respect to x, t, and z can be proved in the same way. □

Lemma 3.2.15. *Let $j \geq 0$ be any integer. Assume that for all $i = 0, \ldots, j-1$, the functions $c_i^+, \mu_i^+, c_B^i, \mu_B^i$ are known, smooth, and satisfy the matching conditions (3.114) and (3.115) and the outer-expansion equation (3.69). Let G^{j-1} be defined as in (3.127) and assume that μ_{j-1}^+ satisfies the boundary condition*

$$
\frac{\partial}{\partial n} \mu_{j-1}^+(x,t) = G^{j-2}(x,t) \quad \forall (x,t) \in \partial_T \Omega\,. \tag{3.133}
$$

Also assume that $\mu_B^{-2} = \mu_B^{-1} = c_B^{-2} = c_B^{-1} = 0$ and μ_B^i, $i = 0, \ldots, j-1$, are defined by

$$
\mu_B^i(z,x,t) = \mu_i^+(x,t) + \int_{-\infty}^z \int_{-\infty}^y C_B^{i-1}(w,x,t)\, dw\, dy \tag{3.134}
$$

for all $z \in (-\infty, 0]$ and $(x,t) \in \overline{\partial_T\Omega(\delta)}$ and where C_B^{i-1} is defined as in (3.122). Then for known smooth μ_j^+ the function μ_B^j defined by (3.134) (with $i = j$) satisfies for $k = j$ the boundary-expansion equation (3.119), the boundary condition (3.125), and the matching condition (3.115).

Proof: We can show that G^{j-2} and μ^i, $i = 0, \dots, j$, are well-defined and that μ_B^j and μ_j^+ satisfy the matching condition (3.115) in the same way as in the proof of Lemma 3.2.12. By direct calculation (3.119) immediately follows. Hence it remains to prove the boundary condition (3.125). We use the definition of μ_B^j to get

$$\begin{aligned}
\mu_{B,z}^j(0,x,t) &= \int_{-\infty}^0 C_B^{j-1}(z,x,t)\,dz \\
&= -\Delta d_B\, \mu_B^{k-1}\Big|_{z=-\infty}^{z=0} - 2\nabla d_B \cdot \nabla\, \mu_B^{k-1}\Big|_{z=-\infty}^{z=0} - \int_{-\infty}^0 \Delta\mu_B^{k-2} - c_{B,t}^{k-2}\,dz \\
&= -\Delta d_B \mu_B^{k-1}(0) - 2\nabla d_B \cdot \nabla \mu_B^{k-1}(0) + \Delta d_B \mu_{k-1}^+ + 2\nabla d_B \cdot \nabla \mu_{k-1}^+ \\
&\quad - \int_{-\infty}^0 \Delta\mu_B^{k-2} - c_{B,t}^{k-2}\,dz
\end{aligned} \tag{3.135}$$

since $\lim_{z\to-\infty} \mu_B^{j-1}(z,x,t) = \mu_{j-1}^+(x,t)$. Equation (3.134) for $i = j-1$ gives us

$$\mu_B^{j-1}(0,x,t) - \mu_{j-1}^+(x,t) = \int_{-\infty}^0 \int_{-\infty}^y C_B^{j-2}(w,x,t)\,dw\,dy\,.$$

Substituting this relation into (3.135) and using the boundary condition (3.133), we get

$$\mu_{B,z}^j(0) + \nabla d_B \cdot \nabla \mu_B^{j-1}(0) = -G^{j-2} + \nabla d_B \cdot \nabla \mu_{j-1}^+ = 0\,,$$

where the last equation follows from (3.133). □

Note that for the required boundary conditions (3.128) and (3.133), we only need functions of lower order. Provided the functions of lower order are known, we have given boundary conditions for μ_k^+ and $\mathbf{u}_k^+$.

3.2.6 Basic Steps for Solving Expansions of each Order

As in [10] we define the unknown functions

$$\mathcal{V}^j \equiv \left(c_j^\pm, c^j, c_B^j, \mu_j^\pm, \mu^j, \mu_B^j, \mathbf{u}_j^\pm, \mathbf{u}^j, \mathbf{u}_B^j, d^j, g^j, L^j, h^j, \mathbf{l}^j, \mathbf{K}^j\right)$$

for each $j \geq 0$ recursively. We call $\mathcal{V}^j$ the jth order expansion.
We assume that $\mathcal{V}^i$ are known for $i = 0, \dots, j-1$, and the corresponding outer, inner, and boundary-layer expansion equations, the inner-outer matching conditions, and the outer-boundary matching conditions are satisfied for $i = 0, \dots, j-1$. Moreover,

we assume that the compatibility conditions (3.99) if $j = 1$ or (3.100) if $j > 1$, (3.108) if $j = 1$ or (3.109) if $j > 1$, and (3.110) are satisfied for $k = j$. In the following we derive the equations for $\mathcal{V}^j$. As in [10] we first construct $(c^j, c_j^\pm, c_B^j)$. Then we continue with $(\mathbf{u}^j, \mathbf{u}_j^\pm, \mathbf{u}_B^j)$ and $(\mu^j, \mu_j^\pm, \mu_B^j)$ and finally, we show how to find d^j. More precisely, we carry out the following steps:

Step 1: After determining $\mathbf{u}_*^{j-1}$ by the known quantities $\mathcal{V}^i$, $i \leq j-1$, we can determine $(c^j, c_j^\pm, c_B^j)$. Therefore we can consider $(c^j, c_j^\pm, c_B^j)$ as known quantities depending only on $\mathcal{V}^i$, $i \leq j-1$.

For Steps 2-9 we assume that d^j is known.

Step 2: We obtain $\mathbf{u}^j$ by equation (3.102).

Step 3: By the matching condition (3.74), we determine $\tilde{\mathbf{u}}^j$ and $\mathbf{l}^j$ in $\Gamma^0(\delta)$.

Step 4: The compatibility condition (3.99) if $j = 0$ or (3.100) if $j > 0$ for $k = j + 1$ yields the boundary condition for $\mathbf{u}_j^\pm$ on Γ^0.

Step 5: The outer expansion (3.71), the boundary condition (3.128), and Step 3 and 4 give us an elliptic boundary problem for $\mathbf{u}_j^\pm$. So we can determine $\mathbf{u}_j^\pm$ uniquely.

Step 6: By solving the compatibility condition (3.108) if $j = 0$ or (3.109) if $j > 0$ for $k = j + 1$ on Γ^0, we can determine $\overline{\mu}^j$ on Γ^0.

Step 7: We obtain $\tilde{\mu}^j = \overline{\mu}^j$ in $\Gamma^0(\delta)$. Then by equation (3.112), μ^j is uniquely determined on Γ^0.

Step 8: The matching condition yields the boundary condition for $\mu_j^\pm$ on Γ^0. Together with the boundary condition (3.133) on $\partial_T\Omega$ and the outer expansion (3.69), we can determine $\mu_j^\pm$ uniquely.

Step 9: Again the matching condition and equation (3.112) prescribe how to define $\tilde{\mu}^j$ and h^j in $\Gamma^0(\delta)$. Therefore μ^j is uniquely determined. It is not difficult to see that the identity for $\overline{\mu}^j$ on Γ^0 in Step 6 is satisfied.

The following step determines d^j.

Step 10: By the compatibility condition (3.110), we obtain an evolution equation for d^j on Γ^0. For $j = 0$ we require that d^0 is a signed spatial distance function and for $j \geq 1$ we require (3.64). Now we have a system of equations which determine $(\mathbf{u}_j^\pm, \mu_j^\pm, d^j)$ uniquely.

Step 11: Note that the compatibility conditions (3.99) if $j = 0$ or (3.100) if $j > 0$, (3.108) if $j = 0$ or (3.109) if $j > 0$, and (3.110) are satisfied on Γ^0 only. We are able to determine g^j, L^j, and $\mathbf{K}^j$ such that these compatibility conditions are satisfied in $\Gamma^0(\delta)$.

Step 12: Finally, by (3.129) and (3.134) we get $\mathbf{u}_B^j$ and μ_B^j. This completes the construction of $\mathcal{V}^j$.

After motivating the construction of $\mathcal{V}^j$ in the Steps 1-12, we verify that $\mathcal{V}^j$ satisfies all the corresponding outer, inner, and boundary-layer expansion equations, the inner-outer matching conditions, and the outer-boundary matching conditions for $k = j$. In addition, we show that the compatibility conditions (3.99) if $j = 0$ or (3.100) if $j > 0$, (3.108) if $j = 0$ or (3.109) if $j > 0$, and (3.110) are also satisfied for $k = j + 1$.

In the next two subsections we carry out Steps 1-12 in detail.

3.2.7 The Zero-th Order Expansion

In this subsection we solve for $\mathcal{V}^0$.
Step 1: We already know the leading order term of the outer and boundary-layer expansion for c^ϵ. Since $E^0 = 0$ we also know the inner expansion. More precisely, it means that $c_0^\pm(x,t) = \pm 1$ for $(x,t) \in Q_0^\pm \cup \Gamma^0(\delta)$, $c_B^0(z,x,t) = 1$ for $(z,x,t) \in (-\infty, 0] \times \overline{\partial_T \Omega(\delta)}$, and $c^0(z,x,t) = \theta_0(z)$ for $(z,x,t) \in \mathbb{R} \times \Gamma^0(\delta)$.
Now we assume that Γ^0 and therefore d^0 are known. We obtain the construction of d^0 below.
Step 2: Since the compatibility condition is satisfied for (3.82a), equation (3.101) yields

$$\mathbf{u}^0(z,x,t) = \tilde{\mathbf{u}}^0(x,t) + d^0(x,t)\mathbf{l}^0(x,t)\left(\eta(z) - 1/2\right) \quad \forall (x,t) \in \Gamma^0(\delta)\,, \tag{3.136}$$

where we define $\tilde{\mathbf{u}}^0$ and $\mathbf{l}^0$ later.
Step 3: From equation (3.136) we get the condition

$$\lim_{z\to\infty} \mathbf{u}^0(\pm z,x,t) = \tilde{\mathbf{u}}^0(x,t) \pm \frac{1}{2} d^0(x,t)\mathbf{l}^0(x,t) \quad \forall (x,t) \in \Gamma^0(\delta)\,.$$

In order to satisfy the matching condition on Γ^0 (note that $d^0 = 0$ on Γ^0), we get the condition

$$\mathbf{u}_0^+(x,t) = \lim_{z\to\infty} \mathbf{u}^0(z,x,t) = \tilde{\mathbf{u}}^0(x,t) = \lim_{z\to\infty} \mathbf{u}^0(-z,x,t) = \mathbf{u}_0^-(x,t) \quad \forall (x,t) \in \Gamma^0\,.$$

To satisfy the matching condition in $\Gamma^0(\delta) \backslash \Gamma^0$, it is necessary and sufficient to define

$$\begin{aligned}
\tilde{\mathbf{u}}^0(x,t) &:= \frac{1}{2}\left(\mathbf{u}_0^+(x,t) + \mathbf{u}_0^-(x,t)\right) && \forall (x,t) \in \Gamma^0(\delta)\,, \\
\mathbf{l}^0(x,t) &:= \frac{1}{d^0(x,t)}\left(\mathbf{u}_0^+(x,t) - \mathbf{u}_0^-(x,t)\right) && \forall (x,t) \in \Gamma^0(\delta)\backslash\Gamma^0\,.
\end{aligned} \tag{3.137}$$

The natural way to define $\mathbf{l}^0$ on Γ^0 is

$$\mathbf{l}^0\big|_{\Gamma^0} := \nabla\left(\mathbf{u}_0^+(x,t) - \mathbf{u}_0^-(x,t)\right) \cdot \nabla d^0 = \frac{\partial}{\partial \nu}\left(\mathbf{u}_0^+(x,t) - \mathbf{u}_0^-(x,t)\right), \tag{3.138}$$

where $\nabla d^0 = \nu$ is the unit outward normal of Γ_t^0.
Step 4: We consider the compatibility condition (3.99) on Γ^0 for $k = 1$. One gets

$$\begin{aligned}
0 \;=\; & -\left(\mathcal{C}_{iji'j'}\left(\partial_j(\mathbf{u}_0^+)_{i'} - \partial_j(\mathbf{u}_0^-)_{i'}\right)\partial_{j'}d^0\right)_{i=1,\dots,d} - \left(\mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right)\right)\nabla d^0 \\
& +2\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 + M\left(\partial_\nu \mathbf{u}_0^+ - \partial_\nu \mathbf{u}_0^-\right),
\end{aligned} \tag{3.139}$$

where we have used the definition of $\mathbf{l}^0$ on Γ^0 and $[\mathbf{u}^0] = 0$ on Γ^0. We can simplify this equation for $(x,t) \in \Gamma^0$. Let $\{\tau_1, \ldots, \tau_{d-1}\}$ be an orthonormal basis of the tangent space of Γ_t^0. Then it holds for all $\mathbf{u} \in C^1(\Omega; \mathbb{R}^d)$

$$\nabla \mathbf{u} = (\partial_\nu \mathbf{u} \otimes \nu) + \sum_{i=1}^{d-1} (\partial_{\tau_i} \mathbf{u} \otimes \tau_i). \tag{3.140}$$

Since $\mathbf{u}_0^+ = \mathbf{u}_0^-$ on Γ^0, we obtain $\left(\partial_{\tau_i} \mathbf{u}_0^+ \otimes \tau_i\right) - \left(\partial_{\tau_i} \mathbf{u}_0^- \otimes \tau_i\right) = 0$ on Γ^0 for all $i = 1, \ldots, d-1$, and therefore we have

$$\nabla \mathbf{u}_0^+ - \nabla \mathbf{u}_0^- = \left(\partial_\nu \mathbf{u}_0^+ \otimes \nu\right) - \left(\partial_\nu \mathbf{u}_0^- \otimes \nu\right) \quad \text{on } \Gamma^0. \tag{3.141}$$

Using this property, we obtain on Γ^0

$$\begin{aligned}
&\left(\mathcal{C}_{iji'j'} \left(\partial_j (\mathbf{u}_0^+)_{i'} - \partial_j (\mathbf{u}_0^-)_{i'}\right) \partial_{j'} d^0\right)_{i=1,\ldots,d} \\
&= \left(\mathcal{C}_{iji'j'} \left(\left(\partial_\nu \mathbf{u}_0^+ \otimes \nu\right)_{i'j} - \left(\partial_\nu \mathbf{u}_0^- \otimes \nu\right)_{i'j}\right) \partial_{j'} d^0\right)_{i=1,\ldots,d} \\
&= \left(\mathcal{C}_{iji'j'} \left(\partial_\nu (\mathbf{u}_0^+)_{i'} - \partial_\nu (\mathbf{u}_0^-)_{i'}\right) \partial_j d^0 \partial_{j'} d^0\right)_{i=1,\ldots,d} \\
&= \left(\mathcal{C} \left(\left(\partial_\nu \mathbf{u}_0^+ \otimes \nu\right) - \left(\partial_\nu \mathbf{u}_0^- \otimes \nu\right)\right)\right) \nu \\
&= \left(\mathcal{C} \left(\nabla \mathbf{u}_0^+ - \nabla \mathbf{u}_0^-\right)\right) \nu \, .
\end{aligned}$$

By the definition of M (see (3.81)), we have

$$\begin{aligned}
M \left(\partial_\nu \mathbf{u}_0^+ - \partial_\nu \mathbf{u}_0^-\right) &= \left(\mathcal{C}_{iji'j'} \left(\partial_\nu (\mathbf{u}_0^+)_{i'} - \partial_\nu (\mathbf{u}_0^-)_{i'}\right) \partial_j d^0 \partial_{j'} d^0\right)_{i=1,\ldots,d} \\
&= \left(\mathcal{C} \left(\nabla \mathbf{u}_0^+ - \nabla \mathbf{u}_0^-\right)\right) \nu \, ,
\end{aligned}$$

where the second equation follows as above. So equation (3.139) turns into

$$0 = \left(2\mathcal{C}\mathcal{E}^\star - \mathcal{C} \left(\nabla \mathbf{u}_0^+ - \nabla \mathbf{u}_0^-\right)\right) \nu \quad \text{on } \Gamma^0 \, .$$

Step 5: Note that $\mathbf{F}^{-1} = 0$. Hence we get the boundary condition $\mathbf{u}_0^+\big|_{\partial_T \Omega} = 0$ due to (3.128). Moreover, we require $\operatorname{div}\left(\mathcal{C}\mathbf{u}_0^\pm\right) = 0$ in $Q_0^\pm$ due to the outer expansion. Therefore we can determine uniquely $\mathbf{u}_0^\pm$ by solving the elliptic boundary problem

$$\begin{aligned}
\operatorname{div}\left(\mathcal{C}\nabla \mathbf{u}_0^\pm\right) &= 0 && \text{in } Q_0^\pm \, , \\
\left[(\mathcal{C}\nabla \mathbf{u}_0^\pm - \mathcal{C}\mathcal{E}^\star c_0^\pm)\nu\right]_{\Gamma_t^0} = \left[\mathbf{u}_0^\pm\right]_{\Gamma_t^0} &= 0 && \text{on } \Gamma_t^0, t \in [0,T] \, , \\
\mathbf{u}_0^+ &= 0 && \text{on } \partial\Omega \times [0,T] \, .
\end{aligned}$$

Step 6: Due to the definitions of $\mathbf{u}^0$ in Step 2 and $\overline{\mathbf{u}}^0$ in Lemma 3.2.7 and the property $\int_{\mathbb{R}} (\eta - 1/2)\, \theta_0'(z)\, dz = 0$, we can conclude that $\tilde{\mathbf{u}}^0 = \overline{\mathbf{u}}^0$ in $\Gamma^0(\delta)$. Then the compatibility condition (3.108) on Γ^0 for $k = 1$ is equivalent to

$$\overline{\mu}^0(x,t) = -\sigma \Delta d^0 - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla \mathbf{u}_0^+ + \nabla \mathbf{u}_0^-\right) \quad \forall (x,t) \in \Gamma^0 \, .$$

It is a known fact that

$$\Delta d^0 = \mathrm{div}_{\Gamma_t^0} \nu = -\kappa_{\Gamma_t^0} \,,$$

where $\kappa_{\Gamma_t^0}$ is the mean curvature of Γ_t^0, cf. [24]. Then we get

$$\overline{\mu}^0(x,t) = \sigma \kappa_{\Gamma_t^0} - \frac{1}{2} \mathcal{E}^\star : \mathcal{C} \left(\nabla \mathbf{u}_0^+ + \nabla \mathbf{u}_0^- \right) \quad \forall (x,t) \in \Gamma^0 \,.$$

Step 7: Since the compatibility condition is satisfied for (3.84a), equation (3.111) yields

$$\mu^0(z,x,t) = \tilde{\mu}^0(x,t) + d^0(x,t) h^0(x,t) \left(\eta(z) - 1/2 \right) \quad \forall (x,t) \in \Gamma^0(\delta) \,, \qquad (3.142)$$

where we define $\tilde{\mu}^0$ and h^0 later. As above we obtain $\tilde{\mu}^0 = \overline{\mu}^0$. Note that $d^0 = 0$ on Γ^0, and therefore we obtain

$$\mu^0(z,x,t) = \sigma \kappa_{\Gamma_t^0} - \frac{1}{2} \mathcal{E}^\star : \mathcal{C} \left(\nabla \mathbf{u}_0^+ + \nabla \mathbf{u}_0^- \right) \quad \forall (x,t) \in \Gamma^0 \,.$$

Step 8: The equation above and the matching condition lead to

$$\mu_0^\pm(x,t) = \lim_{z \to \infty} \mu^0(\pm z,x,t) = \sigma \kappa_{\Gamma_t^0} - \frac{1}{2} \mathcal{E}^\star : \mathcal{C} \left(\nabla \mathbf{u}_0^+ + \nabla \mathbf{u}_0^- \right) \quad \forall (x,t) \in \Gamma^0 \,.$$

Since $G^{-1} = 0$, we obtain $\frac{\partial}{\partial n} \mu_0^+ \big|_{\partial_T \Omega} = 0$ (see (3.133)), and since $c_0^\pm = \pm 1$, the outer expansion for $k = 0$ reads $\Delta \mu_0^\pm = 0$ in $Q_0^\pm$. Together with the boundary condition on Γ_t^0, we can determine uniquely $\mu_0^\pm$ by solving the elliptic boundary problem

$$\begin{aligned}
\Delta \mu_0^\pm &= 0 && \text{in } Q_0^\pm \,, \\
\mu_0^\pm &= \sigma \kappa_{\Gamma_t^0} - \tfrac{1}{2} \mathcal{E}^\star : \mathcal{C} \left(\nabla \mathbf{u}_0^+ + \nabla \mathbf{u}_0^- \right) && \text{on } \Gamma_t^0, t \in [0,T] \,, \\
\tfrac{\partial}{\partial n} \mu_0^+ &= 0 && \text{on } \partial\Omega \times [0,T] \,.
\end{aligned}$$

Step 9: From equation (3.142) we get the condition

$$\lim_{z \to \infty} \mu^0(\pm z,x,t) = \overline{\mu}^0(x,t) \pm \frac{1}{2} d^0(x,t) h^0(x,t) \quad \forall (x,t) \in \Gamma^0(\delta) \,.$$

So in order to satisfy the matching condition $\lim_{z \to \infty} \mu^0(\pm z,x,t) = \mu_0^\pm(x,t)$, it is necessary and sufficient to take

$$\begin{aligned}
\overline{\mu}^0(x,t) = \tilde{\mu}^0(x,t) &:= \frac{1}{2} \left(\mu_0^+(x,t) + \mu_0^-(x,t) \right) && \forall (x,t) \in \Gamma^0(\delta), \\
h^0(x,t) &:= \frac{1}{d^0(x,t)} \left(\mu_0^+(x,t) - \mu_0^-(x,t) \right) && \forall (x,t) \in \Gamma^0(\delta) \backslash \Gamma^0. \qquad (3.143)
\end{aligned}$$

The natural way to define h^0 on Γ^0 is

$$h^0 \,|_{\Gamma^0} := \nabla d^0 \cdot \nabla \left(\mu_0^+(x,t) - \mu_0^-(x,t) \right) = \frac{\partial}{\partial \nu} \left(\mu_0^+(x,t) - \mu_0^-(x,t) \right) . \qquad (3.144)$$

Note that the definition of $\overline{\mu}^0$ satisfies the identity for $\overline{\mu}^0$ in Step 6.
Step 10: On Γ^0 the compatibility condition (3.110) for $k=1$ reads

$$\begin{aligned} d_t^0(x,t) &= \frac{1}{2}\Delta d^0\left[\mu^0\right] + \nabla d^0\cdot\left[\nabla\mu^0\right] - \frac{1}{2}h^0 \\ &= \frac{1}{2}\left(\frac{\partial}{\partial\nu}\mu_0^+ - \frac{\partial}{\partial\nu}\mu_0^-\right) \quad \forall (x,t)\in\Gamma^0, \end{aligned}$$

where we have used the equations $[\mu^0] = \left(\mu_0^+ - \mu_0^-\right) = 0$ and $\nabla d^0\cdot[\nabla\mu^0] = \nabla d^0\cdot\left(\nabla\mu_0^+ - \nabla\mu_0^-\right) = h^0$ on Γ^0. Note that the normal velocity of Γ_t^0 is given by $-d_t^0$ and the unit outer normal ν by ∇d^0. Therefore Γ^0, $\mu_0 := \mu_0^+\chi_{\{d^0\geq 0\}} + \mu_0^-\chi_{\{d^0<0\}}$, and $\mathbf{u}_0 := \mathbf{u}_0^+\chi_{\{d^0\geq 0\}} + \mathbf{u}_0^-\chi_{\{d^0<0\}}$ satisfy the problem (3.8)-(3.14). Equation (3.11) is satisfied since $\nu^T\left[W\mathrm{Id} - \nabla(\mathbf{u}_0)^T\mathcal{S}\right]_{\Gamma_t^0}\nu = -\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) : \mathcal{C}\mathcal{E}^\star$ on Γ^0. We will show this in the proof of Lemma 3.2.16 below.
Step 11: Until now we fulfill the compatibility conditions (3.99),(3.108), and (3.110) for $k=1$ only on Γ^0. To satisfy the compatibility conditions in $\Gamma^0(\delta)\backslash\Gamma^0$ we set

$$g^0(x,t) := \frac{1}{2\eta_0 d^0}\left(\mu_0^+ + \mu_0^- + 2\sigma\Delta d^0 + \mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right)\right), \tag{3.145}$$

$$L^0(x,t) := -\frac{1}{d^0}\left(2d_t^0 - \left(\Delta d^0 + 2\nabla d^0\cdot\nabla\right)\left(\mu_0^+ - \mu_0^-\right) + h^0\right), \tag{3.146}$$

$$\begin{aligned} \mathbf{K}^0(x,t) := \frac{1}{d^0}M^{-1}(&\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}_0^+ - \mathbf{u}_0^-)_{i'}\,\partial_{j'}d^0\right)_{i=1,\ldots,d} \\ &+ \left(\mathcal{C}_{iji'j'}\left(\mathbf{u}_0^+ - \mathbf{u}_0^-\right)_{i'}\partial_{jj'}d^0\right)_{i=1,\ldots,d} \\ &+ \left(\mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right)\right)\nabla d^0 - 2\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 - M\mathbf{l}^0) \end{aligned} \tag{3.147}$$

for $(x,t)\in\Gamma^0(\delta)\backslash\Gamma^0$. Since the numerators of g^0, L^0, and $\mathbf{K}^0$ vanish on Γ^0 we can extend g^0, L^0, and $\mathbf{K}^0$ smoothly to Γ^0 by

$$g^0(x,t) := \frac{1}{2\eta_0}\nabla d^0\cdot\nabla\left(\mu_0^+ + \mu_0^- + 2\sigma\Delta d^0 + \mathcal{E}^\star : \mathcal{C}(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-)\right), \tag{3.148}$$

$$L^0(x,t) := -\nabla d^0\cdot\nabla\left(2d_t^0 - \left(\Delta d^0 + 2\nabla d^0\cdot\nabla\right)\left(\mu_0^+ - \mu_0^-\right) + h^0\right), \tag{3.149}$$

$$\begin{aligned} \mathbf{K}^0(x,t) := M^{-1}\nabla d^0\cdot\nabla(&\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}_0^+ - \mathbf{u}_0^-)_{i'}\,\partial_{j'}d^0\right)_{i=1,\ldots,d} \\ &- \left(\mathcal{C}_{iji'j'}\left(\mathbf{u}_0^+ - \mathbf{u}_0^-\right)_{i'}\partial_{jj'}d^0\right)_{i=1,\ldots,d} \\ &\left(\mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right)\right)\nabla d^0 - 2\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 - M\mathbf{l}^0) \end{aligned} \tag{3.150}$$

for $(x,t)\in\Gamma^0$.
Step 12: Observe that $\mathbf{A}_B^{-1} = 0$ and $C_B^{-1} = 0$. Then Lemma 3.2.12 and 3.2.15 yield $\mathbf{u}_B^0(z,x,t) = \mathbf{u}_0^+(x,t)$ and $\mu_B^0(z,x,t) = \mu_0^+(x,t)$ for all $(x,t)\in\overline{\partial_T\Omega(\delta)}$.
After motivating the construction of $\mathcal{V}^0$, we obtain the following result.

Lemma 3.2.16. *Let $\Gamma_{00}\subset\subset\Omega$ be a given smooth hypersurface without boundary and assume that the Hele-Shaw problem (3.8)-(3.14) admits a smooth solution $(\mu,\boldsymbol{u},\Gamma)$ in the time interval $[0,T]$. Let d^0 be the signed distance from x to*

Γ_t such that $d^0 < 0$ inside of Γ_t, and let δ be a small positive constant such that dist$(\Gamma_t, \partial\Omega) > 2\delta$ for all $t \in [0,T]$, d^0 is smooth in $\Gamma(2\delta) := \{(x,t) \in \Omega_T : |d^0| < 2\delta\}$ and $\mu^\pm := \mu|_{Q_0^\pm}$ and $\boldsymbol{u}^\pm := \boldsymbol{u}|_{Q_0^\pm}$ have a smooth extension to $Q_0^\pm \cup \Gamma(2\delta)$ where $Q_0^\pm := \{(x,t) \in \Omega_T : \pm d^0 > 0\}$. Define the hypersurface Γ^0 by

$$\Gamma^0 = \Gamma\,,$$

the outer expansion functions $(c_0^\pm, \mu_0^\pm, \boldsymbol{u}_0^\pm)$ in $Q_0^\pm \cup \Gamma^0(\delta)$ by

$$c_0^\pm(x,t) = \pm 1, \qquad \mu_0^\pm(x,t) = \mu^\pm(x,t), \qquad \boldsymbol{u}_0^\pm(x,t) = \boldsymbol{u}^\pm(x,t),$$

the inner expansion functions $(c^0, \mu^0, \boldsymbol{u}^0)$ in $\mathbb{R} \times \Gamma^0(\delta)$ by

$$\begin{aligned} &c^0(z,x,t) = \theta_0(z), \quad \mu^0(z,x,t) = \mu^+(x,t)\eta(z) + \mu^-(x,t)(1-\eta(z)),\\ &\boldsymbol{u}^0(z,x,t) = \boldsymbol{u}^+(x,t)\eta(z) + \boldsymbol{u}^-(x,t)(1-\eta(z)), \end{aligned}$$

and the boundary expansion functions $(c_B^0, \mu_B^0, \boldsymbol{u}_B^0)$ in $(-\infty, 0] \times \overline{\partial_T\Omega(\delta)}$ by

$$c_B^0(z,x,t) = 1, \qquad \mu_B^0(z,x,t) = \mu_0^+(x,t), \qquad \boldsymbol{u}_B^0(z,x,t) = \boldsymbol{u}^+(x,t).$$

Furthermore, define $h^0(x,t)$ by (3.143) and (3.144), $g^0(x,t)$ by (3.145) and (3.148), $L^0(x,t)$ by (3.146) and (3.149), $\boldsymbol{l}^0(x,t)$ by (3.137) and (3.138), and $\boldsymbol{K}^0(x,t)$ by (3.147) and (3.150), $\boldsymbol{j}^0(x,t)$ by (3.96), and $k^0(x,t)$ by (3.105). Then, for $k = 0$, the outer expansion equations (3.69)-(3.71), the inner-expansion equations (3.82)-(3.84), the boundary-layer-expansion equations (3.117),(3.119), the inner-outer matching conditions (3.72)-(3.74), the outer boundary matching conditions (3.114)-(3.116), and the boundary conditions (3.123)-(3.125) are all satisfied. In addition, the compatibility conditions (3.99), (3.108), and (3.110) for $k = 1$ are also satisfied.

Proof: We verify all the properties by direct calculation.
To (3.69)-(3.71): The outer expansion equations (3.69)-(3.71) are satisfied by definition of $(c_0^\pm, \mu_0^\pm, \mathbf{u}_0^\pm)$ and the equations for $(\mu^\pm, \mathbf{u}^\pm)$.
To (3.82): We consider $\Gamma^0(\delta)\backslash\Gamma^0$ and Γ^0 separately. Then we obtain

$$\begin{aligned} \left(\mathbf{u}^0 - \mathbf{l}^0 d^0 \eta\right)_{zz} &= \left(\mathbf{u}_0^+\eta + \mathbf{u}_0^-(1-\eta) - \left(\mathbf{u}_0^+ - \mathbf{u}_0^-\right)\eta\right)_{zz} \\ &= (\mathbf{u}_0^+)_{zz} = 0 \qquad\qquad \text{in } \Gamma^0(\delta)\backslash\Gamma^0\,, \\ \left(\mathbf{u}^0 - \mathbf{l}^0 d^0 \eta\right)_{zz} &= \left(\mathbf{u}_0^+\eta + \mathbf{u}_0^-(1-\eta)\right)_{zz} = (\mathbf{u}_0^- - \mathbf{u}_0^-)\eta'' = 0 \qquad \text{on } \Gamma^0 \end{aligned}$$

since $\mathbf{u}_0^+ = \mathbf{u}_0^-$ on Γ^0.
To (3.83): The definitions of c^0 and θ_0 yield

$$c_{zz}^0 - f(c^0) = \theta_0'' - f(\theta_0) = 0 = E^0 \quad \text{in } \Gamma^0(\delta)\,.$$

To (3.84): The proof for μ^0 is analogous to $\mathbf{u}^0$.
To (3.117),(3.119): The assertions immediately follow from the definitions of μ_B^0

and $\mathbf{u}_B^0$.
To (3.72): It follows from Lemma 2.6.1.
To (3.73): Since for $z > 0$

$$\begin{aligned} \mu^0(+z,x,t) - \mu_0^+(x,t) &= (1-\eta(z))\left(\mu_0^-(x,t) - \mu_0^+(x,t)\right), \\ \mu^0(-z,x,t) - \mu_0^-(x,t) &= \eta(-z)\left(\mu_0^+(x,t) - \mu_0^-(x,t)\right), \end{aligned}$$

the assertion follows from (3.75) for all $(x,t) \in \Gamma^0(\delta)$.
To (3.74): The proof for $\mathbf{u}^0$ and $\mathbf{u}_0^\pm$ is analogous to μ^0 and $\mu_0^\pm$.
To (3.114)-(3.116): The assertions immediately follow from the definitions of c_B^0, μ_B^0, and $\mathbf{u}_B^0$.
To (3.123)-(3.125): The assertions immediately follow from the definitions of c_B^0, μ_B^0, and $\mathbf{u}_B^0$.
To (3.99): Due to the definition of $\mathbf{K}^0$ we have for all $(x,t) \in \Gamma^0(\delta)\backslash\Gamma^0$

$$\begin{aligned} &-\left(\mathcal{C}_{iji'j'}\left[\partial_j \mathbf{u}_{i'}^0\right]\partial_{j'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{iji'j'}\left[\mathbf{u}_{i'}^0\right]\partial_{jj'}d^0\right)_{i=1,\ldots,d} \\ &-\left(\mathcal{C}\left[\nabla\mathbf{u}^0\right]\right)\nabla d^0 + \left[c^0\right](\mathcal{C}\mathcal{E}^\star)\nabla d^0 + M\mathbf{l}^0 + M\mathbf{K}^0 d^0 \\ &\quad= -\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}_0^+ - \mathbf{u}_0^-)_{i'}\,\partial_{j'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{iji'j'}\left(\mathbf{u}_0^+ - \mathbf{u}_0^-\right)_{i'}\partial_{jj'}d^0\right)_{i=1,\ldots,d} \\ &\qquad -\left(\mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right)\right)\nabla d^0 + 2\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 + M\mathbf{l}^0 + M\mathbf{K}^0 d^0 = 0\,. \end{aligned}$$

On Γ^0 we obtain

$$\begin{aligned} &-\left(\mathcal{C}_{iji'j'}\left[\partial_j \mathbf{u}_{i'}^0\right]\partial_{j'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{iji'j'}\left[\mathbf{u}_{i'}^0\right]\partial_{jj'}d^0\right)_{i=1,\ldots,d} \\ &-\left(\mathcal{C}\left[\nabla\mathbf{u}^0\right]\right)\nabla d^0 + \left[c^0\right](\mathcal{C}\mathcal{E}^\star)\nabla d^0 + M\mathbf{l}^0 + M\mathbf{K}^0 d^0 \\ &\quad= -\left(\mathcal{C}_{iji'j'}\,\partial_j(\mathbf{u}_0^+ - \mathbf{u}_0^-)_{i'}\,\partial_{j'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{iji'j'}\left(\mathbf{u}_0^+ - \mathbf{u}_0^-\right)_{i'}\partial_{jj'}d^0\right)_{i=1,\ldots,d} \\ &\qquad -\left(\mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right)\right)\nabla d^0 + 2\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 + M\left(\partial_\nu\mathbf{u}_0^+ - \partial_\nu\mathbf{u}_0^-\right) \\ &\quad= \left(2\mathcal{C}\mathcal{E}^\star - \mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right)\right)\nu = 0\,, \end{aligned}$$

where the second equation follows in the same way as the calculation in Step 4 since $\mathbf{u}_0^+ = \mathbf{u}_0^-$ on Γ^0 and where the last equation follows from $[\mathcal{S}\nu]_{\Gamma_t^0} = 0$.
To (3.108): In $\Gamma^0(\delta)\backslash\Gamma^0$ we obtain

$$\begin{aligned} &-\overline{\mu}^0 - \sigma\Delta d^0 - \mathcal{E}^\star : \mathcal{C}\left(\nabla\overline{\mathbf{u}}^0\right) + \eta_0 d^0 g^0 \\ &\quad= -\frac{1}{2}\mu_0^+\int_{\mathbb{R}}\eta\theta_0'\,dz - \frac{1}{2}\mu_0^-\int_{\mathbb{R}}(1-\eta)\,\theta_0'\,dz - \sigma\Delta d^0 \\ &\qquad -\frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\left(\mathbf{u}_0^+\int_{\mathbb{R}}\eta\theta_0'\,dz + \mathbf{u}_0^-\int_{\mathbb{R}}(1-\eta)\,\theta_0'\,dz\right)\right) \\ &\qquad +\frac{\eta_0 d^0}{2\eta_0 d^0}\left(\mu_0^+ + \mu_0^- + 2\sigma\Delta d^0 + \mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right)\right) \\ &\quad= -\frac{1}{2}\mu_0^+ - \frac{1}{2}\mu_0^- - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) + \frac{1}{2}\mu_0^+ + \frac{1}{2}\mu_0^- \\ &\qquad +\frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) = 0\,, \end{aligned}$$

where we have used the the definitions of $\overline{\mu}^0$ and $\overline{\mathbf{u}}^0$ in Lemma 3.2.7, the properties of θ_0, and (3.76). On Γ^0 we get

$$\begin{aligned}
&-\overline{\mu}^0 - \sigma\Delta d^0 - \mathcal{E}^\star : \mathcal{C}\left(\nabla\overline{\mathbf{u}}^0 - \mathcal{E}^\star\overline{c}^0\right) + \eta_0 d^0 g^0 \\
&= -\frac{1}{2}\mu_0^+ - \frac{1}{2}\mu_0^- + \sigma\kappa_{\Gamma_t^0} - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) \\
&= -\frac{1}{2}\nu^\top\left[W\mathrm{Id} - (\nabla\mathbf{u}_0)^T\mathcal{S}\right]_{\Gamma_t^0}\nu - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right)
\end{aligned}$$

since $\Delta d^0 = -\kappa_{\Gamma_t^0}$ and $d^0 = 0$ on Γ^0 and where we have applied the boundary condition (3.11). Let us show that the term on the right-hand side vanishes. First note that

$$\begin{aligned}
\nu^T\left[W\mathrm{Id}\right]_{\Gamma_t^0}\nu &= \frac{1}{2}\left(\mathcal{C}\left(\nabla\mathbf{u}_0^+ - \mathcal{E}^\star\right) : \left(\nabla\mathbf{u}_0^+ - \mathcal{E}^\star\right) - \mathcal{C}\left(\nabla\mathbf{u}_0^- + \mathcal{E}^\star\right) : \left(\nabla\mathbf{u}_0^- + \mathcal{E}^\star\right)\right) \\
&= \frac{1}{2}\mathcal{C}\nabla\mathbf{u}_0^+ : \nabla\mathbf{u}_0^+ - \frac{1}{2}\mathcal{C}\nabla\mathbf{u}_0^- : \nabla\mathbf{u}_0^- - \mathcal{C}\nabla\mathbf{u}_0^+ : \mathcal{E}^\star - \mathcal{C}\nabla\mathbf{u}_0^- : \mathcal{E}^\star .
\end{aligned}$$

Due to the definition of $\mathcal{S}$ we get

$$\begin{aligned}
\nu^T\left[(\nabla\mathbf{u}_0)^T\mathcal{S}\right]_{\Gamma_t^0}\nu &= \nu^T(\nabla\mathbf{u}_0^+)^T(\mathcal{C}\nabla\mathbf{u}_0^+)\nu - \nu^T(\nabla\mathbf{u}_0^-)^T(\mathcal{C}\nabla\mathbf{u}_0^-)\nu \\
&\quad -\nu^T(\nabla\mathbf{u}_0^+)^T(\mathcal{C}\mathcal{E}^\star)\nu - \nu^T(\nabla\mathbf{u}_0^-)^T(\mathcal{C}\mathcal{E}^\star)\nu \\
&= \left(\partial_\nu\mathbf{u}_0^+\otimes\nu\right) : \mathcal{C}\nabla\mathbf{u}_0^+ - \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\nabla\mathbf{u}_0^- \\
&\quad - \left(\partial_\nu\mathbf{u}_0^+\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star - \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star \\
&= \nabla\mathbf{u}_0^+ : \mathcal{C}\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^- : \mathcal{C}\nabla\mathbf{u}_0^+ \\
&\quad + \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\nabla\mathbf{u}_0^+ - \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\nabla\mathbf{u}_0^- \\
&\quad - \left(\partial_\nu\mathbf{u}_0^+\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star - \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star ,
\end{aligned}$$

where the last equality follows from (3.141) since $[\mathbf{u}_0]_{\Gamma_t^0} = 0$ on Γ_t^0. Together we have

$$\begin{aligned}
&\nu^T\left[W\mathrm{Id} - \nabla(\mathbf{u}_0)^T\mathcal{S}\right]_{\Gamma_t^0}\nu \\
&= -\frac{1}{2}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right) : \mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right) - \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right) \\
&\quad - \left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) : \mathcal{C}\mathcal{E}^\star + \left(\partial_\nu\mathbf{u}_0^+\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star + \left(\partial_\nu\mathbf{u}_0^-\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star \\
&= -\frac{1}{2}\left(\left(\partial_\nu\mathbf{u}_0^+ + \partial_\nu\mathbf{u}_0^-\right)\otimes\nu\right) : \mathcal{C}\left(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-\right) - \left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) : \mathcal{C}\mathcal{E}^\star \\
&\quad + \left(\left(\partial_\nu\mathbf{u}_0^+ + \partial_\nu\mathbf{u}_0^-\right)\otimes\nu\right) : \mathcal{C}\mathcal{E}^\star \\
&= -\left(\nabla\mathbf{u}_0^+ + \nabla\mathbf{u}_0^-\right) : \mathcal{C}\mathcal{E}^\star ,
\end{aligned}$$

where we have used (3.141) in the second equation and $[\mathcal{S}\nu]_{\Gamma_t^0} = 0$ in the last equation. Therefore we fulfill the compatibility condition (3.108) on Γ^0.

To (3.110): In $\Gamma^0(\delta)\backslash\Gamma^0$ we directly obtain by the definition of L^0

$$\begin{aligned} & d_t^0 - \frac{1}{2}\Delta d^0\left[\mu^0\right] - \nabla d^0 \cdot \left[\nabla \mu^0\right] + \frac{1}{2}h^0 + \frac{1}{2}d^0 L^0 \\ = \ & d_t^0 - \frac{1}{2}\Delta d^0\left(\mu_0^+ - \mu_0^-\right) - \nabla d^0 \cdot \nabla\left(\mu_0^+ - \mu_0^-\right) + \frac{1}{2}h^0 \\ & -\frac{1}{2}\left(2d_t^0 - \left(\Delta d^0 + 2\nabla d^0 \cdot \nabla\right)\left(\mu_0^+ - \mu_0^-\right) + h^0\right) = 0\,. \end{aligned}$$

On Γ^0 it holds due to the interface condition (3.10)

$$\begin{aligned} & d_t^0 - \frac{1}{2}\Delta d^0\left[\mu^0\right] - \nabla d^0 \cdot \left[\nabla \mu^0\right] + \frac{1}{2}h^0 + \frac{1}{2}d^0 L^0 \\ = \ & -V - \left(\frac{\partial}{\partial\nu}\mu_0^+ - \frac{\partial}{\partial\nu}\mu_0^-\right) + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\mu_0^+ - \frac{\partial}{\partial\nu}\mu_0^-\right) = 0 \end{aligned}$$

since $d_t^0 = -V$ on Γ_t^0. This completes the proof. □

3.2.8 The Higher-Order Expansions

Let $j \geq 1$ be an integer. Assume that $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$ are known and that the matching conditions for $k = 0, \ldots, j-1$ and the compatibility conditions of Lemma 3.2.6, 3.2.7, and 3.2.9 are satisfied for $k = j$.
Step 1: After determining $\mathbf{u}_*^{j-1}$, we can calculate c^j in $\mathbb{R}\times\Gamma^0(\delta)$ by equation (3.107). Equation (3.70) gives us $c_j^\pm$ in $Q_0^\pm$ directly. The proof of Lemma 3.2.14 shows how we can obtain c_B^j in $(-\infty, 0] \times \overline{\partial_T\Omega(\delta)}$. So we know that c^j, $c_j^\pm$, and c_B^j are known functions which only depend on $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$.
For Step 2-9 we assume that d^j is known. The construction of d^j is shown below.
Step 2: Equation (3.102) yields for all $(x,t) \in \Gamma^0(\delta)$

$$\mathbf{u}^j(z,x,t) = \tilde{\mathbf{u}}^j(x,t) + \left(\mathbf{l}^j d^0 + \mathbf{l}^0 d^j\right)(x,t)\left(\eta(z) - 1/2\right) + \mathbf{u}_*^{j-1}(z,x,t)\,, \quad (3.151)$$

where we define $\tilde{\mathbf{u}}^j$ and $\mathbf{l}^j$ later.
Step 3: Due to the definition of η, we obtain from equation (3.151) and Lemma 3.2.6

$$\lim_{z\to\infty} \mathbf{u}^j(\pm z, x, t) = \tilde{\mathbf{u}}^j(x,t) \pm \frac{1}{2}\left(\mathbf{l}^j d^0 + \mathbf{l}^0 d^j\right) + \mathbf{u}_{*(j-1)}^\pm(x,t) \quad \forall (x,t) \in \Gamma^0(\delta).$$

By the inner-outer matching condition we get on Γ^0

$$\mathbf{u}_j^\pm(x,t) = \tilde{\mathbf{u}}^j(x,t) \pm \frac{1}{2}\mathbf{l}^0 d^j + \mathbf{u}_{*(j-1)}^\pm(x,t)\,. \quad (3.152)$$

For satisfying the inner-outer matching condition on $\Gamma^0(\delta)\backslash\Gamma^0$, it is necessary and sufficient to define

$$\begin{aligned}
\tilde{\mathbf{u}}^j(x,t) &:= \frac{1}{2}\left(\mathbf{u}_j^+ + \mathbf{u}_j^- - \mathbf{u}_{*(j-1)}^+ - \mathbf{u}_{*(j-1)}^-\right) \quad \text{in } \Gamma^0(\delta)\,, \\
\mathbf{l}^j(x,t) &:= \begin{cases} \frac{1}{d^0}\left(-d^j\mathbf{l}^0 + \mathbf{u}_j^+ - \mathbf{u}_j^- - \mathbf{u}_{*(j-1)}^+ + \mathbf{u}_{*(j-1)}^-\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0 \\ \nabla d^0\cdot\nabla\left(-d^j\mathbf{l}^0 + \mathbf{u}_j^+ - \mathbf{u}_j^- - \mathbf{u}_{*(j-1)}^+ + \mathbf{u}_{*(j-1)}^-\right) & \text{on } \Gamma^0. \end{cases}
\end{aligned}$$

Note that the numerator in the definition of $\mathbf{l}^j$ vanishes on Γ^0. So the definition of $\mathbf{l}^j$ is natural on Γ^0.

Step 4: On Γ^0 the compatibility condition (3.100) reads for $k = j+1$

$$\begin{aligned}
M\tilde{\mathcal{D}}^{j-1} &= -\left(\mathcal{C}_{ili'l'}\left[\partial_l\mathbf{u}_{i'}^j\right]\partial_{l'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{ili'l'}\left[\partial_l\mathbf{u}_{i'}^0\right]\partial_{l'}d^j\right)_{i=1,\ldots,d} \\
&\quad - \left(\mathcal{C}_{ili'l'}\left[\mathbf{u}_{i'}^j\right]\partial_{ll'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{ili'l'}\left[\mathbf{u}_{i'}^0\right]\partial_{ll'}d^j\right)_{i=1,\ldots,d} \\
&\quad - \left(\mathcal{C}\left[\nabla\mathbf{u}^j\right]\right)\nabla d^0 - \left(\mathcal{C}\left[\nabla\mathbf{u}^0\right]\right)\nabla d^j + \left[c^j\right]\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 \\
&\quad + \left[c^0\right]\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^j + \mathbf{j}^j + M\mathbf{l}^j + M\mathbf{K}^0 d^j \\
&= -\left(\mathcal{C}_{ili'l'}\left(\partial_l(\mathbf{u}_j^+)_{i'} - \partial_l(\mathbf{u}_j^-)_{i'}\right)\partial_{l'}d^0\right)_{i=1,\ldots,d} \\
&\quad - \left(\mathcal{C}_{ili'l'}\left(\partial_l(\mathbf{u}_0^+)_{i'} - \partial_l(\mathbf{u}_0^-)_{i'}\right)\partial_{l'}d^j\right)_{i=1,\ldots,d} \\
&\quad -d^j\left(\mathcal{C}_{ili'l'}\mathbf{l}_{i'}^0\partial_{ll'}d^0\right)_{i=1,\ldots,d} - \left(\mathcal{C}_{ili'l'}\left((\mathbf{u}_{*(j-1)}^+)_{i'} - (\mathbf{u}_{*(j-1)}^-)_{i'}\right)\partial_{ll'}d^0\right)_{i=1,\ldots,d} \\
&\quad - \left(\mathcal{C}\left(\nabla\mathbf{u}_j^+ - \nabla\mathbf{u}_j^-\right)\right)\nabla d^0 + \left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^j\right)\right)\nabla d^0 + \left(c_j^+ - c_j^-\right)\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^0 \\
&\quad +2\left(\mathcal{C}\mathcal{E}^\star\right)\nabla d^j - M\left(\partial_\nu(\mathbf{l}^0 d^j)\right) + M\left(\partial_\nu\mathbf{u}_j^+ - \partial_\nu\mathbf{u}_j^-\right) \\
&\quad -M\left(\partial_\nu\mathbf{u}_{*(j-1)}^+ - \partial_\nu\mathbf{u}_{*(j-1)}^-\right) + M\mathbf{K}^0 d^j\,, \qquad (3.153)
\end{aligned}$$

where $\nu = \nabla d^0$ is the unit outward normal of Γ_t^0 and since $[\mathbf{u}^j] = \mathbf{l}^0 d^j + \mathbf{u}_{*(j-1)}^+ - \mathbf{u}_{*(j-1)}^-$ on Γ^0, $\mathbf{u}_0^+ = \mathbf{u}_0^-$ on Γ^0, and

$$-\left(\mathcal{C}\left[\nabla\mathbf{u}^0\right]\right)\nabla d^j + \mathbf{j}^j = \left(\mathcal{C}\left(\mathbf{l}^0\otimes\nabla d^j\right)\right)\nabla d^0$$

due to (3.96), (3.138), and (3.141). To simplify this equation for $(x,t)\in\Gamma^0$, we use an analogous calculation as in Step 4 in Subsection 3.2.7. Let $\tau_1,\ldots\tau_{d-1}$ be an orthonormal basis of the tangent space of Γ_t^0. Then equation (3.152) yields for all $i = 1,\ldots,d-1$

$$\left(\partial_{\tau_i}\mathbf{u}_j^+\otimes\tau_i\right) - \left(\partial_{\tau_i}\mathbf{u}_j^-\otimes\tau_i\right) = \partial_{\tau_i}\left(\mathbf{l}^0 d^j + \mathbf{u}_{*(j-1)}^+ - \mathbf{u}_{*(j-1)}^-\right)\otimes\tau_i$$

and therefore by equation (3.140) we get

$$\begin{aligned}
\nabla\mathbf{u}_j^+ - \nabla\mathbf{u}_j^- &= \left(\partial_\nu\mathbf{u}_j^+\otimes\nu\right) - \left(\partial_\nu\mathbf{u}_j^-\otimes\nu\right) \\
&\quad + \sum_{i=1}^{d-1}\partial_{\tau_i}\left(\mathbf{l}^0 d^j + \mathbf{u}_{*(j-1)}^+ - \mathbf{u}_{*(j-1)}^-\right)\otimes\tau_i\,. \qquad (3.154)
\end{aligned}$$

We define the matrix $B(d^j) \in \mathbb{R}^{d\times d}$ by

$$B(d^j) := \sum_{i=1}^{d-1} \partial_{\tau_i} \left(\mathbf{l}^0 d^j + \mathbf{u}^+_{*(j-1)} - \mathbf{u}^-_{*(j-1)} \right) \otimes \tau_i \,.$$

Using (3.154), we obtain on Γ^0_t

$$\begin{aligned}
&\left(\mathcal{C}_{ili'l'} \left(\partial_l (\mathbf{u}_j^+)_{i'} - \partial_l (\mathbf{u}_j^-)_{i'}\right) \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&= \left(\mathcal{C}_{ili'l'} \left((\partial_\nu \mathbf{u}_j^+ \otimes \nu)_{i'l} - (\partial_\nu \mathbf{u}_j^- \otimes \nu)_{i'l}\right) \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&\quad + \left(\mathcal{C}_{ili'l'} B(d^j)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&= \left(\mathcal{C}_{ili'l'} \left(\partial_\nu (\mathbf{u}_j^+)_{i'} - \partial_\nu (\mathbf{u}_j^-)_{i'}\right) \partial_l d^0 \partial_{l'} d^0\right)_{i=1,\dots,d} + \left(\mathcal{C}_{ili'l'} B(d^j)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&= \left(\mathcal{C}\left(\left(\partial_\nu \mathbf{u}_j^+ \otimes \nu\right) - \left(\partial_\nu \mathbf{u}_j^- \otimes \nu\right)\right)\right)\nu + \left(\mathcal{C}_{ili'l'} B(d^j)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&= \left(\mathcal{C}\left(\nabla \mathbf{u}_j^+ - \nabla \mathbf{u}_j^-\right)\right)\nu + \left(\mathcal{C}_{ili'l'} B(d^j)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} - \left(\mathcal{C} B(d^j)\right)\nu \,.
\end{aligned}$$

Due to the definition of M (see (3.81)) and (3.154), we get on Γ^0_t

$$\begin{aligned}
M\left(\partial_\nu \mathbf{u}_j^+ - \partial_\nu \mathbf{u}_j^-\right) &= \left(\mathcal{C}_{ili'l'} \left(\partial_\nu (\mathbf{u}_j^+)_{i'} - \partial_\nu (\mathbf{u}_j^-)_{i'}\right) \partial_l d^0 \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&= \left(\mathcal{C}\left(\left(\partial_\nu \mathbf{u}_j^+ \otimes \nu\right) - \left(\partial_\nu \mathbf{u}_j^- \otimes \nu\right)\right)\right)\nu \\
&= \left(\mathcal{C}\left(\nabla \mathbf{u}_j^+ - \nabla \mathbf{u}_j^-\right)\right)\nu - \left(\mathcal{C} B(d^j)\right)\nu \,.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&\left(\mathcal{C}_{ili'l'} \left(\partial_l (\mathbf{u}_j^+)_{i'} - \partial_l (\mathbf{u}_j^-)_{i'}\right) \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&+ \left(\mathcal{C}\left(\nabla \mathbf{u}_j^+ - \nabla \mathbf{u}_j^-\right)\right)\nabla d^0 - M\left(\partial_\nu \mathbf{u}_j^+ - \partial_\nu \mathbf{u}_j^-\right) \\
&\quad = \left(\mathcal{C}\left(\nabla \mathbf{u}_j^+ - \nabla \mathbf{u}_j^-\right)\right)\nu + \left(\mathcal{C}_{ili'l'} B(d^j)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \,. \qquad (3.155)
\end{aligned}$$

Furthermore, we use the definition of $B(d^j)$ and M (see (3.81)) to obtain

$$\begin{aligned}
&\left(\mathcal{C}_{ili'l'} B(d^j)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} + M\left(\partial_\nu (\mathbf{l}^0 d^j)\right) + M\left(\partial_\nu \mathbf{u}^+_{*(j-1)} - \partial_\nu \mathbf{u}^-_{*(j-1)}\right) \\
&= \sum_{k=1}^{d-1} \partial_{\tau_k} d^j \left(\mathcal{C}_{ili'l'} (\mathbf{l}^0 \otimes \tau_k)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} + \partial_\nu d^j \left(\mathcal{C}_{ili'l'} (\mathbf{l}^0 \otimes \nabla d^0)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&\quad + \sum_{k=1}^{d-1} d^j \left(\mathcal{C}_{ili'l'} (\partial_{\tau_k} \mathbf{l}^0 \otimes \tau_k)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} + d^j \left(\mathcal{C}_{ili'l'} (\partial_\nu \mathbf{l}^0 \otimes \nabla d^0)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&\quad + \sum_{k=1}^{d-1} \left(\mathcal{C}_{ili'l'} (\partial_{\tau_k} (\mathbf{u}^+_{*(j-1)} - \mathbf{u}^-_{*(j-1)}) \otimes \tau_k)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&\quad + \left(\mathcal{C}_{ili'l'} (\partial_\nu (\mathbf{u}^+_{*(j-1)} - \mathbf{u}^-_{*(j-1)}) \otimes \nabla d^0)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&= \left(\mathcal{C}\left(\mathbf{l}^0 \otimes \nabla d^0\right)\right)\nabla d^j + d^j \left(\mathcal{C}_{ili'l'} (\nabla \mathbf{l}^0)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \\
&\quad + \left(\mathcal{C}_{ili'l'} \left(\nabla (\mathbf{u}^+_{*(j-1)} - \mathbf{u}^-_{*(j-1)})\right)_{i'l} \partial_{l'} d^0\right)_{i=1,\dots,d} \qquad (3.156)
\end{aligned}$$

since $\nu = \nabla d^0$ on Γ^0. Altogether, the compatibility condition (3.153) turns on Γ^0 with (3.155) and (3.156) into

$$\left(\mathcal{C}\left(\nabla \mathbf{u}_j^+ - \nabla \mathbf{u}_j^-\right)\right)\nu = B_{j-1}\nabla d^j + \mathbf{b}_{j-1}d^j + \mathbf{c}_{j-1} \quad \text{on } \Gamma^0,$$

where $B_{j-1} = B_{j-1}(x,t) \in \mathbb{R}^{d\times d}$, $\mathbf{b}_{j-1} = \mathbf{b}_{j-1}(x,t) \in \mathbb{R}^d$, and $\mathbf{c}_{j-1} = \mathbf{c}_{j-1}(x,t) \in \mathbb{R}^d$ only depend on the known functions $\mathcal{V}^0,\ldots,\mathcal{V}^{j-1}$.
Step 5: Let $\mathbf{F}^{j-1}$ be defined as in (3.126). Then equation (3.128) gives us a boundary condition for the outer expansion $\mathbf{u}_j^+$ on $\partial_T\Omega$. Together with the outer expansion (3.71) and the conditions on Γ^0, the functions $\mathbf{u}_j^\pm$ satisfies the following boundary value problem for each $t > 0$

$$\begin{aligned}
\operatorname{div}\left(\mathcal{C}\nabla \mathbf{u}_j^\pm\right) &= \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star c_j^\pm\right) && \text{in } Q_0^\pm\,,\\
\left[(\mathcal{C}\nabla \mathbf{u}_j^\pm)\nu\right]_{\Gamma_t^0} &= B_{j-1}\nabla d^j + \mathbf{b}_{j-1}d^j + \mathbf{c}_{j-1} && \text{on } \Gamma_t^0, t>0\,,\\
\left[\mathbf{u}_j^\pm\right]_{\Gamma_t^0} &= \mathbf{l}^0 d^j + \left[\mathbf{u}_{*(j-1)}^\pm\right]_{\Gamma_t^0} && \text{on } \Gamma_t^0, t>0\,,\\
\mathbf{u}_j^+ &= \mathbf{F}^{j-1} && \text{on } \partial\Omega, t>0\,.
\end{aligned}$$

In Section 4.2 we show the solvability of this system.
Step 6: We consider the compatibility condition (3.109) for $k = j+1$. Since $\int_{\mathbb{R}}(\eta - 1/2)\theta_0'\,dz = \int_{\mathbb{R}}\mathbf{u}_*^{j-1}\theta_0'\,dz = 0$, we get due to the definition of $\overline{\mathbf{u}}_j$ in Lemma 3.2.7 and (3.151) in $\Gamma^0(\delta)$

$$\overline{\mathbf{u}}^j(x,t) \;=\; \frac{1}{2}\int_{\mathbb{R}}\mathbf{u}^j(z)\theta_0'\,dz = \tilde{\mathbf{u}}^j = \frac{1}{2}\left(\mathbf{u}_j^+ + \mathbf{u}_j^-\right) - \frac{1}{2}\left(\mathbf{u}_{*(j-1)}^+ + \mathbf{u}_{*(j-1)}^-\right)\,,$$

where the last equation follows from the definition of $\tilde{\mathbf{u}}^j$ in Step 3. Then the compatibility condition (3.109) for $k = j+1$ reads on Γ^0

$$\overline{\mu}^j(x,t) = -\sigma\Delta d^j - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla \mathbf{u}_j^+ + \nabla \mathbf{u}_j^-\right) + \eta_0 d^j g^0 - \mathfrak{A}^{j-1}\,, \tag{3.157}$$

where we set for $(x,t) \in \Gamma^0(\delta)$

$$\mathfrak{A}^{j-1}(x,t) \;=\; \widetilde{\mathcal{A}}^{j-1}(x,t) - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla \mathbf{u}_{*(j-1)}^+ + \nabla \mathbf{u}_{*(j-1)}^-\right)(x,t)\,.$$

Note that $\mathfrak{A}^{j-1}$ only depends on the known functions $\mathcal{V}^0,\ldots,\mathcal{V}^{j-1}$.
Step 7: Lemma 3.2.9 gives us an equation for μ^j in $\mathbb{R}\times\Gamma^0(\delta)$

$$\mu^j(z,x,t) = \tilde{\mu}^j(x,t) + \left(d^0h^j + d^jh^0\right)(x,t)\left(\eta(z) - \tfrac{1}{2}\right) + \mu_*^{j-1}(z,x,t)\,, \tag{3.158}$$

where μ_*^{j-1} only depends on $\mathcal{V}^0,\ldots,\mathcal{V}^{j-1}$ and satisfies (3.113) with $k=j$. We define $\tilde{\mu}^j$ and h^j later. As in the construction of the zero order functions, it follows

$$\tilde{\mu}^j(x,t) = \frac{1}{2}\int_{\mathbb{R}}\mu^j(z)\theta_0'(z)\,dz = \overline{\mu}^j(x,t) \quad \forall (x,t) \in \Gamma^0(\delta)\,.$$

The restriction of (3.158) on Γ^0 and equation (3.157) give us

$$\begin{aligned}\mu^j(z,x,t) &= -\sigma\Delta d^j - \frac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_j^+ + \nabla\mathbf{u}_j^-\right) + d^j\left(\eta_0 g^0 + h^0\left(\eta - \tfrac{1}{2}\right)\right) \\ &\quad -\mathfrak{A}^{j-1} + \mu_*^{j-1} \quad \forall (x,t)\in\Gamma^0. \end{aligned} \tag{3.159}$$

So μ^j is uniquely determined on Γ^0.

Step 8: We consider $z\to\pm\infty$ in (3.159) and use the inner-outer matching condition to get on Γ^0

$$\begin{aligned}\mu_j^\pm\big|_{\Gamma^0} = \lim_{z\to\pm\infty}\mu^j(z,.) &= -\sigma\Delta d^j - \tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_j^+ + \nabla\mathbf{u}_j^-\right) \\ &\quad + d^j\left(\eta_0 g^0 \pm \tfrac{1}{2}h^0\right) - \mathfrak{A}^{j-1} + \mu_{*(j-1)}^\pm, \end{aligned} \tag{3.160}$$

where $\eta(z)=1$ and $\eta(-z)=0$ for $z>1$. Then the function $\mu_j^\pm$ is uniquely determined by the outer expansion (3.69) and the boundary condition (3.133), that is, $\mu_j^\pm$ is the solution to the elliptic boundary problem

$$\begin{aligned} \Delta\mu_j^\pm &= \partial_t c_j^\pm && \text{in } Q_0^\pm\,, \\ \mu_j^\pm &= -\sigma\Delta d^j - \tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla\mathbf{u}_j^+ + \nabla\mathbf{u}_j^-\right) && \\ &\quad + d^j\left(-\eta_0 g^0 \pm \tfrac{1}{2}h^0\right) - \mathfrak{A}^{j-1} + \mu_{*(j-1)}^\pm && \text{on } \Gamma_t^0, t\geq 0\,, \\ \tfrac{\partial}{\partial n}\mu_j^+ &= G^{j-1} && \text{on } \partial\Omega, t\geq 0\,. \end{aligned}$$

Step 9: Sending z in (3.158) to $\pm\infty$ and using the matching condition yields

$$\mu_j^\pm(x,t) = \overline{\mu}^j(x,t) \pm \frac{1}{2}\left(d^0h^j + d^jh^0\right)(x,t) + \mu_{*(j-1)}^\pm(x,t) \quad \forall (x,t)\in\Gamma^0(\delta)\,.$$

Hence it is necessary and sufficient to take h^j and $\tilde{\mu}^j$ as

$$\begin{aligned}\overline{\mu}^j = \tilde{\mu}^j &:= \frac{1}{2}\left(\mu_j^+ + \mu_j^- - \mu_{*(j-1)}^+ - \mu_{*(j-1)}^-\right) \quad \text{in } \Gamma^0(\delta), \\ h^j &:= \begin{cases} \frac{1}{d^0}\left(-d^jh^0 + \mu_j^+ - \mu_j^- - \mu_{*(j-1)}^+ + \mu_{*(j-1)}^-\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0 \\ \nabla d^0\cdot\nabla\left(-d^jh^0 + \mu_j^+ - \mu_j^- - \mu_{*(j-1)}^+ + \mu_{*(j-1)}^-\right) & \text{on } \Gamma^0. \end{cases}\end{aligned}$$

Consider (3.160) to recognize that the numerator in the definition of h^j vanishes on Γ^0. So the definition of h^j is natural. Note that this definition of $\overline{\mu}^j$ coincides with (3.157). To see this use (3.160) again.

By Steps 2-9 we can determine $\mu^j, \mu_j^\pm, \mathbf{u}^j, \mathbf{u}_j^\pm, h^j$, and $\mathbf{l}^j$ depending on d^j. The next step shows us how we can determine d^j.

Step 10: We consider the compatibility condition (3.110) on Γ^0 for $k=j+1$. Note that $[\mu^0]=0$ and $d^0=0$ and use the definition of h^j on Γ^0. Then we have

$$\begin{aligned} d_t^j &= \frac{1}{2}\Delta d^0\left[\mu^j\right] + \nabla d^0\cdot\left[\nabla\mu^j\right] + \nabla d^j\cdot\left[\nabla\mu^0\right] \\ &\quad -\frac{1}{2}\nabla d^0\cdot\nabla\left(-d^jh^0 + \mu_j^+ - \mu_j^- - \mu_{*(j-1)}^+ + \mu_{*(j-1)}^-\right) - \frac{1}{2}d^jL^0 + \widetilde{\mathcal{B}}^{j-1} \\ &= \frac{1}{2}\left(\Delta d^0h^0 + \nabla d^0\cdot\nabla h^0 - L^0\right)d^j + \frac{1}{2}\left(\tfrac{\partial}{\partial\nu}\mu_j^+ - \tfrac{\partial}{\partial\nu}\mu_j^-\right) + d_*^{j-1}, \end{aligned}$$

where $\nu = \nabla d^0$ is the unit outward normal of Γ_t^0, $[\mu^j] = h^0 d^j + \mu^+_{*(j-1)} - \mu^-_{*(j-1)}$, $\nabla d^0 \cdot [\nabla \mu^j] = \frac{\partial}{\partial n}\mu_j^+ - \frac{\partial}{\partial n}\mu_j^-$, $\nabla d^j \cdot [\nabla \mu^0] = \nabla d^j \cdot \nabla d^0 h^0$ on Γ^0, $\nabla d^j \cdot \nabla d^0 = -\frac{1}{2}\sum_{i=1}^{j-1} \nabla d^i \cdot \nabla d^{j-i}$, and

$$\begin{aligned} d_*^{j-1}(x,t) &= \frac{1}{2}\Delta d^0 \left(\mu^+_{*(j-1)} + \mu^-_{*(j-1)}\right) - \frac{1}{2}\nabla d^0 \cdot \nabla \left(\mu^+_{*(j-1)} - \mu^-_{*(j-1)}\right) \\ &\quad -\frac{3}{4}\sum_{i=1}^{j-1} \nabla d^i \cdot \nabla d^{j-i} h^0 + \widetilde{\mathcal{B}}^{j-1}. \end{aligned}$$

Here we can see that d_*^{j-1} only depends on the known functions $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$.
Altogether, $(d^j, \mu_j^\pm, \mathbf{u}_j^\pm)$ satisfies the problem

$$\begin{aligned} \Delta \mu_j^\pm &= a^{1\pm}_{j-1} && \text{in } Q_0^\pm, && (3.161)\\ \mu_j^\pm &= -\sigma \Delta d^j - \tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla \mathbf{u}_j^+ + \nabla \mathbf{u}_j^-\right) && && \\ &\quad - a^{2\pm}_{j-1} d^j + a^{3\pm}_{j-1} && \text{on } \Gamma^0, && (3.162)\\ \tfrac{\partial}{\partial n}\mu_j^+ &= a^4_{j-1} && \text{on } \partial_T\Omega, && (3.163)\\ \operatorname{div}\left(\mathcal{C}\mathcal{E}(\mathbf{u}_j^\pm)\right) &= a^{5\pm}_{j-1} && \text{in } Q_0^\pm, && (3.164)\\ \left(\mathcal{C}\left[\nabla \mathbf{u}_j^\pm\right]_{\Gamma_t^0}\right)\nu_{\Gamma_t^0} &= a^6_{j-1}\nabla d^j + a^7_{j-1} d^j + a^8_{j-1} && \text{on } \Gamma^0, && (3.165)\\ \left[\mathbf{u}_j^\pm\right]_{\Gamma_t^0} &= a^9_{j-1} d^j + a^{10}_{j-1} && \text{on } \Gamma^0, && (3.166)\\ \mathbf{u}_j^+ &= a^{11}_{j-1} && \text{on } \partial_T\Omega, && (3.167)\\ \partial_t d^j &= a^{12}_{j-1} d^j + \tfrac{1}{2}\left[\tfrac{\partial}{\partial \nu}\mu_j^\pm\right] + a^{13}_{j-1} && \text{on } \Gamma^0, && (3.168)\\ \nabla d^0 \cdot \nabla d^j &= a^{14}_{j-1} && \text{in } \Gamma^0(\delta), && (3.169)\\ d^j(x,0) &= 0 && \text{in } \Gamma_0, && (3.170) \end{aligned}$$

where a^i_{j-1}, $i = 1, \ldots, 12$ only depends on the known functions $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$. In Chapter 4.2 we show that this problem has a smooth solution.
Assume that $(d^j, \mu_j^\pm, \mathbf{u}_j^\pm)$ solve problem (3.161)-(3.170). Going through Step 2-9 again, we obtain $\mu^j, \mathbf{u}^j, h^j$, and $\mathbf{l}^j$. In Lemma 3.2.17 below we verify that $\mu_j^\pm, \mu^j, \mathbf{u}_j^\pm$, and $\mathbf{u}^j$ satisfy the matching conditions (3.73) and (3.74) and the compatibility conditions (3.100) and (3.110) for $k = j+1$.
Notice that in the derivation of (3.161)-(3.170) we need the inner expansions only for $(x,t) \in \Gamma^0$ where $O^+_{j-1} = O^-_{j-1} = 0$ and $\mathbf{P}^+_{j-1} = \mathbf{P}^-_{j-1} = 0$ by the definitions of $c^\pm_{j-1}, \mu^\pm_{j-1}, \mathbf{u}^\pm_{j-1}, O^\pm_{j-1}$, and $\mathbf{P}^\pm_{j-1}$. Therefore the solution d^1 of (3.161)-(3.170) is independent of the terms $\epsilon^2(O_0^+\eta_N^+ + O_0^-\eta_N^-)$ and $\epsilon^2(\mathbf{P}_0^+\eta_N^+ + \mathbf{P}_0^-\eta_N^-)$ which we added in (3.79) and (3.80). In particular, d^1 is independent of the constant N. So we can define $N := \|d^1\|_{C^0(\Gamma^0(\delta))} + 2$, see Remark 3.2.5.
Step 11: We can define g^j in $\Gamma^0(\delta)\backslash\Gamma^0$ in a unique way such that the compatibility condition (3.109) is satisfied for $k = j+1$. Since (3.109) already holds on Γ^0, we can extend g^j smoothly to Γ^0. Similarly, we can define uniquely $\mathbf{K}^j$ and L^j to satisfy the

compatibility conditions (3.100) and (3.110).
Step 12: By Lemma 3.2.12 and Lemma 3.2.15 we immediately get $\mathbf{u}_B^j$ and μ_B^j.
After going through Step 1-12, it holds.

Lemma 3.2.17. *Let $j \geq 1$ be an arbitrary integer and assume $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$ are given and satisfy the matching conditions (3.72)-(3.74) and (3.114)-(3.116) for all $k = 0, \ldots, j-1$. Furthermore, let the compatibility conditions (3.99), (3.108), and (3.110) if $j = 1$ or the compatibility conditions (3.100), (3.109), and (3.110) for $k = j$ if $j > 1$ be satisfied. Then there exists*

$$\mathcal{V}^j = \left(c_j^\pm, c^j, c_B^j, \mu_j^\pm, \mu^j, \mu_B^j, \boldsymbol{u}_j^\pm, \boldsymbol{u}^j, \boldsymbol{u}_B^j, d^j, g^j, L^j, h^j, \boldsymbol{l}^j, \boldsymbol{K}^j\right)$$

satisfying, for $k = j$, the outer expansion equations (3.69)-(3.71), the inner expansion equations (3.82)-(3.84), the boundary-layer expansion equations (3.117)-(3.119) and (3.123)-(3.125), the inner-outer matching conditions (3.72)-(3.74), the outer-boundary matching conditions (3.114)-(3.116), and (3.64). In addition, the compatibility conditions (3.100), (3.109), and (3.110) for $k = j + 1$ are also satisfied.

Proof: We choose $(\mu_j^\pm, \mathbf{u}_j^\pm, d^j)$ as the solution to the linearized Hele-Shaw problem (3.161)-(3.170). Then we define $(c_j^\pm, c^j, c_B^j)$ as in Step 1, $\mathbf{u}^j$ as in Step 2, $(\tilde{\mathbf{u}}^j, \mathbf{l}^j)$ as in Step 3, μ^j as in Step 7, $(\tilde{\mu}^j, h^j)$ as in Step 9, $(g^j, L^j, \mathbf{K}^j)$ as in Step 11, and $(\mu_B^j, \mathbf{u}_B^j)$ as in Step 12. Then we can verify the required conditions. Furthermore, we can verify $\tilde{\mu}^j = \overline{\mu}^j$ and $\tilde{\mathbf{u}}^j = \overline{\mathbf{u}}^j$ where $\overline{\mu}^j$ and $\overline{\mathbf{u}}^j$ are defined as in Lemma 3.2.7. By the interface condition (3.160) we conclude that the identity for $\overline{\mu}^j$ in Step 6 coincides with the definition in Step 9.
To (3.69)-(3.71): The outer expansion equations are satisfied by definition of $(c_j^\pm, \mu_j^\pm, \mathbf{u}_j^\pm)$.
To (3.82)-(3.84): The inner expansion equations are satisfied by definition of $(c^j, \mu^j, \mathbf{u}^j)$ and Lemma 3.2.6 - 3.2.9.
To (3.117)-(3.119) and (3.123)-(3.125): The assertions immediately follow from the definitions of $(c_B^j, \mu_B^j, \mathbf{u}_B^j)$ and the results of Subsection 3.2.5.
To (3.72): Due to Lemma 3.2.7 the inner-outer matching condition is satisfied.
To (3.73): Since

$$d^0 h^j = -d^j h^0 + \mu_j^+ - \mu_j^- - \mu_{*(j-1)}^+ + \mu_{*(j-1)}^- \quad \text{in } \Gamma^0(\delta)\backslash\Gamma^0$$

by definition of h^j, the assertion follows from the definition of μ^j in Step 7, the definition of $\tilde{\mu}^j$ in Step 9, and Lemma 3.2.9. On Γ^0 the matching condition follows from (3.159) and the interface condition (3.160). In both cases we use $\eta(\pm z) - \frac{1}{2} = \pm\frac{1}{2}$ for $z > 1$ and $D_x^m D_t^n D_z^l \left[\mu_*^{j-1}(\pm z) - \mu_{*(j-1)}^\pm\right] = \mathcal{O}(e^{-\alpha z})$ as $z \to \infty$ for all $m, n, l \geq 0$ and $(x, t) \in \Gamma^0(\delta)$ due to Lemma 3.2.9.
To (3.74): Due to the definition of $\mathbf{l}^j$ and $\tilde{\mathbf{u}}^j$, the assertion immediately follows from

Lemma 3.2.6 in $\Gamma^0(\delta)\backslash\Gamma^0$. On Γ^0 it holds for $z > 1$

$$\begin{aligned}
&\mathbf{u}^j(z,x,t) - \mathbf{u}_j^+(x,t) \\
&= \frac{1}{2}\left(\mathbf{u}_j^+ + \mathbf{u}_j^- - \mathbf{u}_{*(j-1)}^+ - \mathbf{u}_{*(j-1)}^-\right) + \frac{1}{2}\mathbf{l}^0 d^j + \mathbf{u}_*^{j-1} - \mathbf{u}_j^+ \\
&= -\frac{1}{2}\left(\mathbf{u}_j^+ - \mathbf{u}_j^-\right) - \frac{1}{2}\left(\mathbf{u}_{*(j-1)}^+ + \mathbf{u}_{*(j-1)}^-\right) + \frac{1}{2}\mathbf{l}^0 d^j + \mathbf{u}_*^{j-1} \\
&= -\mathbf{u}_{*(j-1)}^+ + \mathbf{u}_*^{j-1}
\end{aligned}$$

since $\left[\mathbf{u}_j^\pm\right]_{\Gamma_t^0} = \mathbf{l}^0 d^j + \left[\mathbf{u}_{*(j-1)}^\pm\right]_{\Gamma_t^0}$. For $z < -1$ we obtain an analogous result. Then the assertion follows from Lemma 3.2.6.

To (3.114)-(3.116): Due to Lemma 3.2.12 - 3.2.15, the assertions hold.

To (3.64): The equation is satisfied by definition of d^j.

To (3.100): By definition of $\mathbf{K}^j$ the compatibility condition holds in $\Gamma^0(\delta)\backslash\Gamma^0$. On Γ^0 the assertion follows from the interface condition $\left[\left(\mathcal{C}\nabla\mathbf{u}_j^\pm\right)\nu\right]_{\Gamma_t^0} = B_{j-1}\nabla d^j + \mathbf{b}_{j-1}$ and the same calculation as in Step 4.

To (3.109): In $\Gamma^0(\delta)\backslash\Gamma^0$ we satisfy the compatibility condition by definition of g^j. On Γ^0 the compatibility condition is satisfied by the definition of $\tilde{\mu}^j = \overline{\mu}^j$ in Step 9, the interface condition (3.160) for $\mu_j^\pm$, and the definition of $\mathfrak{A}^{j-1}$ in Step 6. Here note that $\overline{\mathbf{u}}^j = \tilde{\mathbf{u}}^j$ (see Step 6).

To (3.110): In $\Gamma^0(\delta)\backslash\Gamma^0$ we satisfy the compatibility condition by definition of L^j. On Γ^0 the compatibility condition is satisfied by the interface condition (3.168) where a_{j-1}^{12} and a_{j-1}^{13} are defined by Step 10. Also we apply the inner-outer matching condition (3.73). □

As consequence we obtain recursively.

Theorem 3.2.18. *Let $(\mu, \boldsymbol{u}, \Gamma)$ be a smooth solution for the Hele-Shaw problem (3.8)-(3.14). Then, for any fixed integer $K > 0$, there exist $\mathcal{V}^0, \dots, \mathcal{V}^K$ such that the outer expansion equations (3.69)-(3.71), the inner expansion equations (3.82)-(3.84), the boundary-layer expansion equations (3.117)-(3.119) and (3.123)-(3.125), the inner-outer matching conditions (3.72)-(3.74), and the outer-boundary matching conditions (3.114)-(3.116) are satisfied for $k = 0, \dots, K$. In addition, $(\mu_0^\pm, \boldsymbol{u}_0^\pm, \Gamma^0)$ coincides with $(\mu, \boldsymbol{u}, \Gamma)$.*

Remark 3.2.19. *In Theorem 3.2.18 we only consider a fixed $K > 0$ because the extensions of $\mu_k^\pm$ and $\boldsymbol{u}_k^\pm$ from $Q_0^\pm$ to $Q_0^\pm \cup \Gamma^0(\delta)$ only lie in $C^{\bar{K}}$ when we use the ansatz of Remark 3.2.2. For smooth extensions this restriction is not necessary.*

3.2.9 Construction of an Approximate Solution

By using the inner, outer, and boundary-layer expansions we construct approximate solutions. Our aim is to keep the error term r_A^ϵ in Theorem 3.1.1 as small as possible. Let $(\mu, \mathbf{u}, \Gamma)$ be a smooth solution for the Hele-Shaw problem (3.8)-(3.14) for given

smooth hypersurface Γ_{00} and some $T > 0$. Let $K \geq 3$ be an arbitrary fixed integer and $\mathcal{V}^0, \ldots, \mathcal{V}^K$ be defined as in Theorem 3.2.18.
Define

$$
\begin{aligned}
d_\epsilon^K(x,t) &= \sum_{i=0}^{K} \epsilon^i d^i(x,t)\,, \quad \forall (x,t) \in \Gamma^0(\delta), && (3.171)\\
\Gamma_\epsilon^K &= \left\{(x,t) \in \Gamma^0(\delta) : d_\epsilon^K(x,t) = 0\right\}. && (3.172)
\end{aligned}
$$

We obtain that d_ϵ^K is a K-th order approximate distance function. This means that d_ϵ^K vanishes on Γ_ϵ^K and

$$
\left|\nabla d_\epsilon^K\right|^2 = 1 + \sum_{\substack{1\leq i,j\leq K\\ i+j\geq K+1}} \epsilon^{i+j} \nabla d^j \cdot \nabla d^i \quad \forall (x,t) \in \Gamma^0(\delta)\,, \tag{3.173}
$$

where we have used (3.64).
Now we are able to define a suitable inner, outer and, boundary-layer approximate solution $\left(c_I^K, \mu_I^K, \mathbf{u}_I^K\right)$, $\left(c_O^K, \mu_O^K, \mathbf{u}_O^K\right)$, and $\left(c_\partial^K, \mu_\partial^K, \mathbf{u}_\partial^K\right)$ by

$$
\begin{aligned}
c_I^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i\; c^i(z,x,t)\big|_{z=d_\epsilon^K/\epsilon} && \forall (x,t) \in \Gamma^0(\delta)\,,\\
\mu_I^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i\; \mu^i(z,x,t)\big|_{z=d_\epsilon^K/\epsilon} && \forall (x,t) \in \Gamma^0(\delta)\,,\\
\mathbf{u}_I^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i\; \mathbf{u}^i(z,x,t)\big|_{z=d_\epsilon^K/\epsilon} && \forall (x,t) \in \Gamma^0(\delta)\,.
\end{aligned}
$$

We define the outer approximate solution by

$$
\begin{aligned}
c_O^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i \left(c_i^+(x,t)\chi_{\overline{Q_0^+}} + c_i^-(x,t)\chi_{Q_0^-}\right) && \forall (x,t) \in \Omega_T\,,\\
\mu_O^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i \left(\mu_i^+(x,t)\chi_{\overline{Q_0^+}} + \mu_i^-(x,t)\chi_{Q_0^-}\right) && \forall (x,t) \in \Omega_T\,,\\
\mathbf{u}_O^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i \left(\mathbf{u}_i^+(x,t)\chi_{\overline{Q_0^+}} + \mathbf{u}_i^-(x,t)\chi_{Q_0^-}\right) && \forall (x,t) \in \Omega_T
\end{aligned}
$$

and the boundary-layer approximate solution by

$$
\begin{aligned}
c_\partial^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i \, c_B^i(z,x,t)\big|_{z=d_B/\epsilon} - \epsilon^K c_B^K(0,x,t) && \forall (x,t)\in \overline{\partial_T\Omega(\delta)}\,,\\
\mu_\partial^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i \, \mu_B^i(z,x,t)\big|_{z=d_B/\epsilon} - \epsilon^K \mu_B^K(0,x,t) && \forall (x,t)\in \overline{\partial_T\Omega(\delta)}\,,\\
\mathbf{u}_\partial^K(x,t) &:= \sum_{i=0}^{K} \epsilon^i \, \mathbf{u}_B^i(z,x,t)\big|_{z=d_B/\epsilon} && \forall (x,t)\in \overline{\partial_T\Omega(\delta)}\,.
\end{aligned}
$$

The next step is to combine the inner and outer approximate solutions. For that we choose a smooth cut-off function $\zeta \in C_0^\infty(\mathbb{R})$ as in (2.21) and define

$$
c_A^K := \begin{cases} c_\partial^K & \text{in } \overline{\partial_T\Omega(\delta/2)}\,,\\ c_\partial^K \zeta(d_B/\delta) + c_O^K(1-\zeta(d_B/\delta)) & \text{in } \partial_T\Omega(\delta)\backslash\overline{\partial_T\Omega(\delta/2)}\,,\\ c_O^K & \text{in } \Omega_T\backslash(\partial_T\Omega(\delta)\cup\Gamma^0(\delta))\,,\\ c_I^K\zeta(d^0/\delta) + c_O^K(1-\zeta(d^0/\delta)) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0(\delta/2)\,,\\ c_I^K & \text{in } \Gamma^0(\delta/2)\,, \end{cases} \tag{3.174}
$$

where d_B is the signed distance function to $\partial\Omega$. We similarly define μ_A^K and $\mathbf{u}_A^K$. Since the cut-off function ζ is smooth this also holds for $(c_A^K, \mu_A^K, \mathbf{u}_A^K)$.
An important question is how good our approximate solution $(c_A^K, \mu_A^K, \mathbf{u}_A^K)$ is. For that we consider $(c_I^K, \mu_I^K, \mathbf{u}_I^K)$, $(c_O^K, \mu_O^K, \mathbf{u}_O^K)$, and $(c_\partial^K, \mu_\partial^K, \mathbf{u}_\partial^K)$ separately. Notice that $\left|\frac{d_\epsilon^K}{\epsilon} - \frac{d^0+\epsilon d^1}{\epsilon}\right| = \left|\sum_{i=2}^K \epsilon^{i-1} d^i\right| \le 1$ for all ϵ small enough, and hence by Remark 3.2.5, $O_j^+\eta_N^+ + O_j^-\eta_N^-\big|_{z=d_\epsilon^K/\epsilon} = 0$ and $\mathbf{P}_j^+\eta_N^+ + \mathbf{P}_j^-\eta_N^-\big|_{z=d_\epsilon^K/\epsilon} = 0$, for $j = 0, \ldots, K-2$. By the inner expansion equations (3.84) we obtain for all $(x,t)\in\Gamma^0(\delta)$ and $z = d_\epsilon^K/\epsilon$

$$
\begin{aligned}
&((c_I^K)_t - \Delta\mu_I^K)(x,t)\\
&= -\frac{\left|\nabla d_\epsilon^K\right|^2 - 1}{\epsilon^2}\sum_{i=0}^K \epsilon^i \mu_{zz}^i + \frac{1}{\epsilon}\sum_{\substack{0\le i,j\le K\\ i+j\ge K}} \epsilon^{i+j}\left(c_z^i d_t^j - 2\nabla\mu_z^i\cdot\nabla d^j - \mu_z^i\Delta d^j\right)\\
&\quad + \sum_{i=K-1}^K \epsilon^i\left(c_t^i - \Delta\mu^i\right) - \epsilon^{K-2}h^K d^0\eta'' + \frac{1}{\epsilon^2}\sum_{\substack{0\le i\le K\\ 0\le j\le K-1\\ i+j\ge K+1}} \epsilon^{i+j} d^i h^j \eta''\\
&\quad -\epsilon^{K-2}L^{K-1}d^0\eta' + \frac{1}{\epsilon}\sum_{\substack{0\le i\le K\\ 0\le j\le K-2\\ i+j\ge K}} \epsilon^{i+j} d^i L^j\eta' \quad \forall (x,t)\in\Gamma^0(\delta)\,,
\end{aligned} \tag{3.175}
$$

where we have used (3.173). Furthermore, we have by the inner expansion equations (3.83) for all $(x,t) \in \Gamma^0(\delta)$ and $z = d_\epsilon^K/\epsilon$

$$
\begin{aligned}
&(\mu_I^K + \epsilon\Delta c_I^K - \epsilon^{-1} f(c_I^K) - W_{,c}(c_I^K, \mathcal{E}(\mathbf{u}_I^K)))(x,t) \\
&= \epsilon^K \mu^K + \epsilon \sum_{i=K-1}^{K} \epsilon^i \Delta c^i - \frac{1 - \left|\nabla d_\epsilon^K\right|^2}{\epsilon} \sum_{i=0}^{K} \epsilon^i c_{zz}^i \\
&\quad + \sum_{\substack{0\le i,j\le K \\ i+j\ge K}} \epsilon^{i+j} \left(2\nabla c_z^i \cdot \nabla d^j + c_z^i \Delta d^j\right) - \epsilon^K f^K(c^0, \dots, c^K) \\
&\quad + \sum_{\substack{0\le i,j\le K \\ i+j\ge K+1}} \epsilon^{i+j-1} \mathcal{E}^\star : \mathcal{C}(\mathbf{u}_z^i \otimes \nabla d^j) + \epsilon^K \mathcal{E}^\star : \mathcal{C}\nabla \mathbf{u}^K - \epsilon^K \mathcal{E}^\star : \mathcal{C}\mathcal{E}^\star c^K \\
&\quad + \epsilon^{K-1} g^{K-1} d^0 \eta' - \sum_{\substack{0\le i\le K-2 \\ 0\le j\le K \\ i+j\ge K}} \epsilon^{i+j} g^i d^j \eta' + \epsilon^{K-1} k^K d^0 \eta' \\
&\quad - \sum_{\substack{0\le i\le K-1 \\ 0\le j\le K \\ i+j\ge K+1}} \epsilon^{i+j-1} k^i d^j \eta' \qquad \forall (x,t) \in \Gamma^0(\delta)
\end{aligned} \tag{3.176}
$$

and by the inner expansion equations (3.82) for all $(x,t) \in \Gamma^0(\delta)$ and $z = d_\epsilon^K/\epsilon$

$$
\begin{aligned}
&(\operatorname{div}\left(\mathcal{C}\nabla \mathbf{u}_I^K\right) - \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star c_I^K\right))(x,t) \\
&= \sum_{\substack{0\le i,j,k\le K \\ i+j+k\ge K+1}} \epsilon^{i+j+k-2} (\mathcal{C}(\mathbf{u}^i \otimes \nabla d^j)) \nabla d^k + \sum_{\substack{0\le i,j\le K \\ i+j\ge K}} \epsilon^{i+j-1} (\mathcal{C}_{klk'l'} \partial_l (\mathbf{u}_{k'}^i)_z \partial_{l'} d^j)_{k=1,\dots,d} \\
&\quad + \sum_{\substack{0\le i,j\le K \\ i+j\ge K}} \epsilon^{i+j-1} (\mathcal{C}_{klk'l'} (\mathbf{u}_{k'}^i)_z \partial_{ll'} d^j)_{k=1,\dots,d} + \sum_{\substack{0\le i,j\le K \\ i+j\ge K}} \epsilon^{i+j-1} (\mathcal{C}\nabla \mathbf{u}_z^i) \nabla d^j \\
&\quad + \sum_{i=K-1}^{K} \epsilon^i (\mathcal{C}_{klk'l'} \partial_{ll'} \mathbf{u}_{k'}^i)_{k=1,\dots,d} - \sum_{\substack{0\le i,j\le K \\ i+j\ge K}} \epsilon^{i+j-1} c_z^i (\mathcal{C}\mathcal{E}^\star) \nabla d^j \\
&\quad - \sum_{i=K-1}^{K} \epsilon^i (\mathcal{C}\mathcal{E}^\star) \nabla c^i - \sum_{\substack{0\le i\le K \\ 0\le j\le K-1 \\ i+j\ge K+1}} \epsilon^{i+j-2} d^i M \mathbf{l}^j \eta'' + \epsilon^{K-2} d^0 M \mathbf{l}^K \eta'' \\
&\quad - \sum_{\substack{0\le i\le K \\ 0\le j\le K-2 \\ i+j\ge K}} \epsilon^{i+j-1} d^i M \mathbf{K}^j \eta' + \epsilon^{K-2} d^0 M \mathbf{K}^{K-1} \eta' \\
&\quad - \sum_{\substack{0\le i\le K \\ 0\le j\le K-1 \\ i+j\ge K+1}} \epsilon^{i+j-2} d^i \mathbf{j}^j \eta'' + \epsilon^{K-2} d^0 \mathbf{j}^K \eta'' \quad \forall (x,t) \in \Gamma^0(\delta)\,.
\end{aligned} \tag{3.177}
$$

By definition we obtain for the outer expansions in $Q_0^+ \cup Q_0^-$

$$(c_O^K)_t - \Delta \mu_O^K = 0\,, \tag{3.178}$$

$$\mu_O^K + \epsilon \Delta c_O^K - \epsilon^{-1} f(c_O^K) - W_{,c}(c_O^K, \mathcal{E}(\mathbf{u}_O^K)) = \epsilon^K \mu_K^{\pm} - \epsilon^K f^K(c_0^{\pm}, \ldots, c_K^{\pm})$$
$$+\epsilon^K \mathcal{E}^\star : \mathcal{C}(\mathcal{E}(\mathbf{u}_K^{\pm}) - \mathcal{E}^\star c_K^{\pm}) + \sum_{i=K-1}^{K} \epsilon^{i+1} \Delta c_i^{\pm}\,, \tag{3.179}$$

$$\operatorname{div}\left(\mathcal{C}\mathcal{E}(\mathbf{u}_O^K)\right) - \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star c_O^K\right) = 0\,. \tag{3.180}$$

Finally, we get for the boundary-layer expansion in $\partial\Omega(\delta) \times (0,T)$ and $z = d_B/\epsilon$

$$\begin{aligned}
&((c_\partial^K)_t - \Delta \mu_\partial^K)(x,t) \\
&= \epsilon^{K-1}\left(c_{B,z}^K d_{B,t} - 2\nabla \mu_{B,z}^K \cdot \nabla d_B - \mu_{B,z}^K \Delta d_B\right) + \sum_{i=K-1}^{K} \epsilon^i \left(c_{B,t}^i - \Delta \mu_B^i\right) \\
&\quad -\epsilon^K \left(c_{B,t}^K(0) - \Delta \mu_B^K(0)\right) \quad \forall (x,t) \in \partial\Omega(\delta) \times (0,T)\,,
\end{aligned} \tag{3.181}$$

where $(c_B^K(0), \mu_B^K(0)) = (c_B^K, \mu_B^K)(0,x,t)$ for all $(x,t) \in \overline{\partial\Omega(\delta)} \times [0,T]$. For the chemical potential equation we get in $\partial\Omega(\delta) \times [0,T]$ with $z = d_B/\epsilon$

$$\begin{aligned}
&(\mu_\partial^K + \epsilon \Delta c_\partial^K - \epsilon^{-1} f(c_\partial^K) - W_{,c}(c_\partial^K, \mathcal{E}(\mathbf{u}_\partial^K)))(x,t) \\
&= \epsilon^K \mu_B^K + \epsilon^K 2 \nabla c_{B,z}^K \cdot \nabla d_B + \epsilon^K c_{B,z}^K \Delta d_B + \sum_{i=K}^{K+1} \epsilon^i \Delta c_B^{i-1} \\
&\quad -\epsilon^K f^K(c_B^0, \ldots, c_B^K - c_B^K(0)) + \epsilon^K \mathcal{E}^\star : \mathcal{C} \nabla \mathbf{u}_B^K - \epsilon^K \mathcal{E}^\star : \mathcal{C}\mathcal{E}^\star c_B^K \\
&\quad -\epsilon^K \left(\mu_B^K(0) + \epsilon \Delta \mu_B^K(0) - \epsilon^{-1} f'(\theta_0) c_B^K(0) - \mathcal{E}^\star : \mathcal{C}\mathcal{E}^\star c_B^K(0)\right)
\end{aligned} \tag{3.182}$$

for all $(x,t) \in \partial\Omega(\delta) \times [0,T]$. For the equation of the stress tensor we obtain in $\partial\Omega(\delta) \times [0,T]$ with $z = d_B/\epsilon$

$$\begin{aligned}
&(\operatorname{div}\left(\mathcal{C}\nabla \mathbf{u}_\partial^K\right) - \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star c_\partial^K\right))(x,t) \\
&= \epsilon^{K-1}\left(\mathcal{C}_{iji'j'} \partial_j (\mathbf{u}_{B,i'}^K)_z \partial_{j'} d_B\right)_{i=1,\ldots,d} + \epsilon^{K-1}\left(\mathcal{C}_{iji'j'} (\mathbf{u}_{B,i'}^K)_z \partial_{jj'} d_B\right)_{i=1,\ldots,d} \\
&\quad +\epsilon^{K-1}\left(\mathcal{C}\nabla \mathbf{u}_{B,z}^K\right) \nabla d_B + \sum_{i=K-1}^{K} \epsilon^i \operatorname{div}\left(\mathcal{C}\nabla \mathbf{u}_B^i\right) - \epsilon^{K-1}\left(\mathcal{C}\mathcal{E}^\star\right) \nabla d_B c_{B,z}^K \\
&\quad -\sum_{i=K-1}^{K} \epsilon^i \left(\mathcal{C}\mathcal{E}^\star\right) \nabla c_B^i + \epsilon^K (\mathcal{C}\mathcal{E}^\star) \nabla c_B^K(0) \quad \forall (x,t) \in \partial\Omega(\delta) \times [0,T]\,.
\end{aligned} \tag{3.183}$$

Additionally, we check the boundary conditions on $\partial\Omega \times (0,T)$. Consider the boundary conditions (3.123)-(3.125) and note the extra terms $\epsilon^K c_B^K(0,x,t)$ and $\epsilon^K \mu_B^K(0,x,t)$ added in the definitions of $c_\partial^K(x,t)$ and $\mu_\partial^K(x,t)$. Then we obtain

$$\mathbf{u}_\partial^K = 0 \qquad \text{and} \qquad \frac{\partial}{\partial n} c_\partial^K = \frac{\partial}{\partial n} \mu_\partial^K = 0 \qquad \text{on } \partial\Omega \times (0,T)\,.$$

It remains to show how good the approximate solutions $(c_A^K, \mu_A^K, \mathbf{u}_A^K)$ are in the domains $\Gamma^0(\delta)\backslash\Gamma^0(\delta/2)$ and $\partial_T\Omega(\delta)\backslash\partial_T\Omega(\delta/2)$ where we have glued together the inner and outer approximate solutions and the boundary-layer and outer approximate solutions. By definition $|d^0(x,t)| \in [\delta/2, \delta)$ for $(x,t) \in \Gamma^0(\delta)\backslash\Gamma^0(\delta/2)$. So for sufficiently small ϵ the property $\left|d_\epsilon^K\right| = \left|\sum_{i=0}^K \epsilon^i d^i\right| \geq \delta/4$ is valid for all $(x,t) \in \Gamma^0(\delta)\backslash\Gamma^0(\delta/2)$. Applying the matching conditions (3.72) yields

$$\begin{aligned} &\left\|c_A^K - c_O^K\right\|_{C^2(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} \\ &= \left\|\zeta(d^0/\delta)\sum_{i=0}^K \epsilon^i \left(c^i(d_\epsilon^K/\epsilon, x, t) - c_i^\pm(x,t)\right)\right\|_{C^2(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} \\ &= \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{4\epsilon}})\,, \end{aligned} \tag{3.184}$$

and analogously we get

$$\left\|\mu_A^K - \mu_O^K\right\|_{C^2(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} = \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{4\epsilon}}), \tag{3.185}$$

$$\left\|\mathbf{u}_A^K - \mathbf{u}_O^K\right\|_{C^2(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} = \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{4\epsilon}}). \tag{3.186}$$

By using the outer-boundary matching condition (3.114), a similar statement holds

$$\begin{aligned} &\left\|c_A^K - c_O^K\right\|_{C^2(\partial_T\Omega(\delta)\backslash\partial_T\Omega(\delta/2))} \\ &= \left\|\zeta(d_B/\delta)\left(\sum_{i=1}^K \epsilon^i \left(c_B^i(d_B/\epsilon) - c_i^+\right) - \epsilon^K c_B^K(0)\right)\right\|_{C^2(\partial_T\Omega(\delta)\backslash\partial_T\Omega(\delta/2))} \\ &= \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{2\epsilon}}) + \mathcal{O}(\epsilon^K)\,. \end{aligned} \tag{3.187}$$

A similar relation holds for $(\mu_A^K, \mathbf{u}_A^K)$ and $(\mu_O^K, \mathbf{u}_O^K)$. Since the equations (3.1)-(3.3) contain second space derivatives and a first time derivative, we have used the C^2-norm. Therefore by (3.173)-(3.187) the approximate solution $(c_A^K, \mu_A^K, \mathbf{u}_A^K)$ fulfills the following equations

$$\begin{aligned} (c_A^K)_t - \Delta\mu_A^K =: e_K(x,t) &= \mathcal{O}(\epsilon^{K-2}) && \text{in } \Omega\times(0,T)\,, \\ \mu_A^K + \epsilon\Delta c_A^K - \epsilon^{-1}f(c_A^K) - W_{,c}(c_A^K, \mathcal{E}(\mathbf{u}_A^K)) &= \mathcal{O}(\epsilon^{K-1}) && \text{in } \Omega\times(0,T)\,, \\ \operatorname{div}\left(\mathcal{C}\mathcal{E}(\mathbf{u}_A^K)\right) - \operatorname{div}\left(\mathcal{C}\mathcal{E}^\star c_A^K\right) &= \mathcal{O}(\epsilon^{K-2}) && \text{in } \Omega\times(0,T)\,, \\ \tfrac{\partial}{\partial n}c_A^K = \tfrac{\partial}{\partial n}\mu_A^K &= 0 && \text{on } \partial\Omega\times(0,T)\,, \\ \mathbf{u}_A^K &= 0 && \text{on } \partial\Omega\times(0,T)\,. \end{aligned}$$

Here and in the following the Landau symbols are with respect to the C^0-norm unless noted otherwise. In the same way as in [10], we modify c_A^K and μ_A^K so that the error term e_K vanishes. We set $c_A^\epsilon = c_A^K - \frac{1}{|\Omega|}\int_0^t\int_\Omega e_K(\xi,\tau)\,d\xi\,d\tau$ and $\mu_A^\epsilon = \mu_\epsilon^K - \hat{e}_K(x,t)$

where $\hat{e}_K(x,t)$ is the solution to the elliptic problem

$$\Delta \hat{e}_K(x,t) = e_K(x,t) - \frac{1}{|\Omega|}\int_\Omega e_K(\xi,t)\,d\xi \quad \text{in } \Omega_T\,,$$
$$\frac{\partial}{\partial n}\hat{e}_K = 0 \quad \text{on } \partial_T\Omega\,, \qquad \int_\Omega \hat{e}_K(\xi,t)\,d\xi = 0 \quad \forall t\in[0,T]\,.$$

Note that $\hat{e}_K = \mathcal{O}(\epsilon^{K-2})$ since $e_K = \mathcal{O}(\epsilon^{K-2})$. In addition, we define $\mathbf{u}_A^\epsilon = \mathbf{u}_A^K$. Therefore $(c_A^\epsilon, \mu_A^\epsilon, \mathbf{u}_A^\epsilon)$ satisfies

$$\begin{aligned}
(c_A^\epsilon)_t - \Delta\mu_A^\epsilon &= 0 && \text{in } \Omega_T\,, && (3.188)\\
\mu_A^\epsilon + \epsilon\Delta c_A^\epsilon - \epsilon^{-1}f(c_A^\epsilon) - W_{,c}(c_A^\epsilon, \mathcal{E}(\mathbf{u}_A^\epsilon)) &= \mathcal{O}(\epsilon^{K-3}) && \text{in } \Omega_T\,, && (3.189)\\
\operatorname{div}(\mathcal{C}\mathcal{E}(\mathbf{u}_A^\epsilon)) - \operatorname{div}(\mathcal{C}\mathcal{E}^\star c_A^\epsilon) &= \mathcal{O}(\epsilon^{K-2}) && \text{in } \Omega_T\,, && (3.190)\\
\tfrac{\partial}{\partial n}c_A^\epsilon = \tfrac{\partial}{\partial n}\mu_A^\epsilon &= 0 && \text{on } \partial_T\Omega\,, && (3.191)\\
\mathbf{u}_A^\epsilon &= 0 && \text{on } \partial_T\Omega\,. && (3.192)
\end{aligned}$$

Remark 3.2.20. *With (3.24) and (3.25) we can specify the size of K.*

$$K-3 \geq \frac{pk}{2} > d+2,5\,.$$

In particular, it is sufficient to calculate the 8th order term of the expansion in two dimensions and the 9th order term in three dimensions.

We summarize the results of this subsection in the following theorem.

Theorem 3.2.21. *Let $\Gamma_{00}\subset\Omega$ be a given smooth hypersurface without boundary and let $(\mu_0, \boldsymbol{u}_0, \Gamma^0)$ be a smooth solution to the Hele-Shaw problem (3.8)-(3.14) for $t\in[0,T]$ with initial value Γ_{00} such that $\Gamma^0\subset\Omega\times[0,T]$. Then for every $K>3$, there exists a positive constant ϵ_0 such that for every $\epsilon\in(0,\epsilon_0]$ there exists an approximate solution $(c_A^\epsilon, \mu_A^\epsilon, \boldsymbol{u}_A^\epsilon)$ satisfying (3.188)-(3.192). Additionally, it holds as $\epsilon\searrow 0$*

$$\begin{aligned}
\|\mu_A^\epsilon - \mu_0\|_{C^0(\Omega_T)} &= \mathcal{O}(\epsilon)\,,\\
\left\|c_A^\epsilon(x,t) - \theta_0(d^0(x,t)/\epsilon + d^1(x,t))\right\|_{C^0(\Gamma^0(\delta))} &= \mathcal{O}(\epsilon)\,,\\
\|c_A^\epsilon \mp 1\|_{C^0(Q_0^\pm\setminus\Gamma^0(\delta/2))} &= \mathcal{O}(\epsilon)\,,\\
\|\boldsymbol{u}_A^\epsilon - \boldsymbol{u}_0\|_{C^0(\Omega_T)} &= \mathcal{O}(\epsilon)\,.
\end{aligned}$$

Proof: The construction of an approximate solution $(c_A^\epsilon, \mu_A^\epsilon, \mathbf{u}_A^\epsilon)$ satisfying (3.188)-(3.192) is described above.
Due to the construction of μ_A^ϵ and (3.185), it follows as $\epsilon\searrow 0$

$$\|\mu_A^\epsilon - \mu_0\|_{C^0(\Omega_T\setminus(\Gamma^0(\delta/2)\cup\partial_T\Omega(\delta/2)))} = \mathcal{O}(\epsilon)\,.$$

So it remains to consider the domains $\Gamma^0(\delta/2)$ and $\partial_T\Omega(\delta/2)$. By triangle inequality it holds

$$\|\mu_A^\epsilon - \mu_0\|_{C^0(\Gamma^0(\delta/2))} \leq \left\|\mu_A^\epsilon - \mu^0\right\|_{C^0(\Gamma^0(\delta/2))} + \left\|\mu^0 - \mu_0\right\|_{C^0(\Gamma^0(\delta/2))}\,.$$

By definition it follows $\|\mu_A^\epsilon - \mu^0\|_{C^0(\Gamma^0(\delta/2))} \leq C\epsilon$. To estimate the second term, Lemma 3.2.16 gives us the exact definition of μ^0. Therefore we have

$$\begin{aligned}\left\|\mu^0 - \mu_0\right\|_{C^0(\Gamma^0(\delta/2)\cap Q_0^+)} &= \left\|(1-\eta(d_\epsilon^K/\epsilon))(\mu_0^+ - \mu_0^-)\right\|_{C^0(\Gamma^0(\delta/2)\cap Q_0^+)} \\ &= \left\|\chi_{\{d_\epsilon^K \leq \epsilon\}}(\mu_0^+ - \mu_0^-)\right\|_{C^0(\Gamma^0(\delta/2)\cap Q_0^+)} \\ &\leq C\epsilon\,,\end{aligned}$$

where the second equality follows from (3.75) and the last inequality from $\mu_0^+ = \mu_0^-$ on Γ^0 and $\left\{d_\epsilon^K \leq \epsilon\right\} \subset \{d^0 \leq C\epsilon\}$ for some $C > 0$. The proof for $\Gamma^0(\delta/2) \cap Q_0^-$ is done in the same way. Since $\mu_B^0 = \mu_0^+$ in $\partial_T\Omega(\delta/2)$, the construction of μ_A^ϵ yields

$$\|\mu_A^\epsilon - \mu_0\|_{C^0(\partial_T\Omega(\delta/2))} \leq C\epsilon$$

for some $C > 0$ independent of ϵ. We analogously show

$$\|\mathbf{u}_A^\epsilon - \mathbf{u}_0\|_{C^0(\Omega_T)} + \|c_A^\epsilon \mp 1\|_{C^0(Q_0^\pm\setminus\Gamma^0(\delta/2))} = \mathcal{O}(\epsilon)\,.$$

To estimate the last term, we again consider the domains $\Gamma^0(\delta/2)$ and $\Gamma^0(\delta)\setminus\Gamma^0(\delta/2)$ separately. We use that $c_A^\epsilon = c_A^\epsilon - c_A^K + c_A^K$ and apply $c_A^K = c_K^I$ in $\Gamma^0(\delta/2)$ and the triangle inequality to obtain

$$\begin{aligned}&\left\|c_A^\epsilon - \theta_0(d^0/\epsilon + d^1)\right\|_{C^0(\Gamma^0(\delta/2))} \\ &\leq \left\|c_A^\epsilon - c_A^K\right\|_{C^0(\Gamma^0(\delta/2))} + \epsilon\left\|\sum_{i=1}^K \epsilon^{i-1}c^i\right\|_{C^0(\Gamma^0(\delta/2))} \\ &\quad + \left\|\theta_0\left(\frac{d^0}{\epsilon} + d^1 + \epsilon\sum_{i=2}^K \epsilon^{i-2}d^i\right) - \theta_0\left(\frac{d^0}{\epsilon} + d^1\right)\right\|_{C^0(\Gamma^0(\delta/2))} \\ &\leq C\epsilon^{K-2} + C\epsilon + C\epsilon \end{aligned} \tag{3.193}$$

since θ_0 is a Lipschitz function. In $\Gamma^0(\delta)\setminus\Gamma^0(\delta/2)$ we write the difference $c_A^\epsilon - \theta_0(d^0/\epsilon + d^1)$ as

$$\begin{aligned}&c_A^\epsilon - \theta_0(d^0/\epsilon + d^1) \\ &\quad = \zeta(d^0/\delta)\left(\theta_0(d_\epsilon^K/\epsilon) - \theta_0(d^0/\epsilon + d^1)\right) \\ &\qquad + \left(c_A^\epsilon - \left[\zeta(d^0/\delta)\theta_0(d_\epsilon^K/\epsilon) + (1-\zeta(d^0/\delta))(2\chi_{Q_0^+} - 1)\right]\right) \\ &\qquad + (1-\zeta(d^0/\delta))\left((2\chi_{Q_0^+} - 1) - \theta_0(d^0/\epsilon + d^1)\right).\end{aligned}$$

On the right-hand side the first term can be estimated as in (3.193). For the second term we use the definition for c_A^K in $\Gamma^0(\delta)\backslash\Gamma^0(\delta/2)$

$$\begin{aligned}
&\left\| c_A^\epsilon - \left[\zeta(d^0/\delta)\theta_0(d_\epsilon^K/\epsilon) + (1-\zeta(d^0/\delta))(2\chi_{Q_0^+} - 1)\right]\right\|_{C^0(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} \\
\leq\ & \left\| c_A^\epsilon - c_A^K \right\|_{C^0(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} \\
&+ \left\| c_A^K - \left[\zeta(d^0/\delta)\theta_0(d_\epsilon^K/\epsilon) + (1-\zeta(d^0/\delta))(2\chi_{Q_0^+} - 1)\right]\right\|_{C^0(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} \\
\leq\ & C\epsilon^{K-2} + C\epsilon\,.
\end{aligned}$$

To estimate the third term on the right-hand side, we use that $|d^0 + \epsilon d^1| \geq \delta/4$ in $\Gamma^0(\delta)\backslash\Gamma^0(\delta/2)$ for all ϵ small enough. Then applying the property (2.9) yield

$$\left\|(2\chi_{Q_0^+} - 1) - \theta_0(d^0/\epsilon + d^1)\right\|_{C^0(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} \leq Ce^{-\frac{\alpha\delta}{8\epsilon}}$$

for some constant $C > 0$. This completes the proof. □

3.3 Convergence Result

Theorem 3.3.1. *Let Ω be a smooth domain and Γ_{00} be a smooth hypersurface in Ω without boundary. Assume that the Hele-Shaw problem (3.8)-(3.14) has a smooth solution $(\mu, \boldsymbol{u}, \Gamma)$ on a time interval $[0,T]$ such that $\Gamma_t \subset \Omega$ for all $t \in [0,T]$ where $\Gamma := \bigcup_{0\leq t\leq T}(\Gamma_t \times \{t\})$. Then there exists a family of smooth functions $\{c_0^\epsilon(x)\}_{0<\epsilon\leq 1}$ which are uniformly bounded in $\epsilon \in (0,1]$ and $x \in \overline{\Omega}$, such that if $(c^\epsilon, \boldsymbol{u}^\epsilon)$ satisfies the Cahn-Larché equation*

$$c_t^\epsilon - \Delta\left(-\epsilon\Delta c^\epsilon + \epsilon^{-1}f(c^\epsilon) + W_{,c}(c^\epsilon, \mathcal{E}(\boldsymbol{u}^\epsilon))\right) = 0 \qquad \text{in } \Omega_T, \tag{3.194}$$

$$\operatorname{div} W_{,\mathcal{E}}(c^\epsilon, \mathcal{E}(\boldsymbol{u}^\epsilon)) = 0 \qquad \text{in } \Omega_T, \tag{3.195}$$

$$\tfrac{\partial}{\partial n}c^\epsilon = \tfrac{\partial}{\partial n}\left(-\epsilon\Delta c^\epsilon + \epsilon^{-1}f(c^\epsilon) + W_{,c}(c^\epsilon, \mathcal{E}(\boldsymbol{u}^\epsilon))\right) = 0 \qquad \text{on } \partial_T\Omega, \tag{3.196}$$

$$\boldsymbol{u}^\epsilon = 0 \qquad \text{on } \partial_T\Omega, \tag{3.197}$$

$$c^\epsilon\big|_{t=0} = c_0^\epsilon \qquad \text{on } \Omega\,, \tag{3.198}$$

then

$$\lim_{\epsilon\to 0} c^\epsilon(x,t) = \begin{cases} -1 & \text{if } (x,t) \in Q^- \\ 1 & \text{if } (x,t) \in Q^+ \end{cases} \text{ uniformly on compact subsets,}$$

$$\lim_{\epsilon\to 0}\left(-\epsilon\Delta c^\epsilon + \epsilon^{-1}f(c^\epsilon) + W_{,c}(c^\epsilon, \mathcal{E}(\boldsymbol{u}^\epsilon))\right)(x,t) = \mu(x,t) \text{ uniformly on } \overline{\Omega_T}\,,$$

$$\lim_{\epsilon\to 0}\boldsymbol{u}^\epsilon(x,t) = \boldsymbol{u}(x,t) \text{ uniformly on } \overline{\Omega_T}\,,$$

where Q^+ and Q^- are respectively the exterior (in Ω_T) and interior of Γ.

Proof: Let $(c_A^\epsilon, \mu_A^\epsilon, \mathbf{u}_A^\epsilon)$ be the approximate solution constructed in Theorem 3.2.21. Then by Theorem 3.1.2 and 3.2.21, we obtain

$$\begin{aligned} \lim_{\epsilon\to 0} \|c^\epsilon \mp 1\|_{C^0(Q^\pm\setminus\Gamma(\delta/2))} &= 0\,, \\ \lim_{\epsilon\to 0} \|\mu^\epsilon - \mu\|_{C^0(\Omega_T)} &= 0\,, \\ \lim_{\epsilon\to 0} \|\mathbf{u}^\epsilon - \mathbf{u}\|_{C^0(\Omega_T)} &= 0 \end{aligned}$$

for every $\delta > 0$ small enough, as long as $\Phi_t^\epsilon(.) = c_A^\epsilon(., t)$ has the form (2.20).
Hence we have to check that $\Phi_t^\epsilon(.) = c_A^\epsilon(., t)$ has the form (2.20) where $r = r_t(x)$ is the signed distance function to $\Gamma_t^{\epsilon K} := \left\{x \in \Omega : d_\epsilon^K(x,t) = 0\right\}$. Observe that $\nabla d_\epsilon^K = \nabla d^0 + \epsilon \sum_{i=1}^K \epsilon^{i-1} \nabla d^i$. Therefore by implicit function theorem $\Gamma_t^{\epsilon K}$ is a smooth hypersurface for all $\epsilon > 0$ small enough since $\nabla d^0 = \nu_{\Gamma_t^0}$ on Γ^0. Moreover, the C^3 norm of $\Gamma_t^{\epsilon K}$ can be bounded independent of $\epsilon \in (0,1]$ and $t \in [0,T]$ (for the definition of the C^3 norm, see Remark 2.8.2). This can be seen as follows. Let $g : U' \subset \mathbb{R}^{d-1} \to \mathbb{R}$ and $U \subset \mathbb{R}^d$ such that $\Gamma^0 \cap U = (U', g(U'))$, that is the manifold Γ^0 is locally (possibly after rotation) the graph of g. As in Section 2.8 there exists $a > 0$ such that $\tau : \Gamma^0 \times (-a, a) \to \mathrm{im}(\tau)$ defined by $\tau(x_0, r) = x_0 + r\nu_{\Gamma_0}(x_0)$ is a smooth diffeomorphism. Furthermore, define the function $F : U' \times (-a,a) \to \mathbb{R}$ by $F(x', r) = r + \sum_{i=1}^K \epsilon^i d^i((x', g(x')) + r\nu_{\Gamma_0}(x', g(x')))$. Since

$$\frac{\partial}{\partial r} F(x,r) = 1 + \sum_{i=1}^K \epsilon^i \nabla d^i((x', g(x')) + r\nu_{\Gamma_0}(x', g(x'))) \cdot \nu_{\Gamma_0}(x', g(x')) > 0$$

for all $(x', r) \in U' \times (-a, a)$ and for all $\epsilon > 0$ small enough, we can apply the implicit function theorem and obtain $f : U' \to (-a, a)$ such that $F(x', f(x')) = 0$. Here we can define f in the whole space U' since we find for every $x' \in U'$ a number r such that $F(x', r) = 0$ provided $\epsilon > 0$ is small enough since $F(x', -a) < 0$ and $F(x', a) > 0$ for every $x' \in U'$ and for all $\epsilon > 0$ small enough. So it holds $(x', g(x')) + f(x')\nu_{\Gamma_0}(x', g(x')) \in \Gamma_t^{\epsilon K}$ for all $x' \in U'$ due to $d^0((x', g(x')) + f(x')\nu_{\Gamma_0}(x', g(x'))) = f(x')$. Since

$$f(x') = -\sum_{i=1}^K \epsilon^i d^i((x', g(x')) + f(x')\nu_{\Gamma_0}(x', g(x')))$$

and

$$\nabla_{x'} f(x') = -\frac{1}{\partial_r F(x', f(x'))} (\nabla_{x'} F)(x', f(x'))\,,$$

we can verify by direct calculation that $\|f\|_{C^3(U')} \le C\epsilon$ for some $C > 0$ independent of ϵ. As above there exists some $\tilde{a} > 0$ such that $\tilde{\tau} : U' \times (-\tilde{a}, \tilde{a}) \to \mathrm{im}(\tilde{\tau})$ defined by

$$\tilde{\tau}(x', r) = x + r\nu_{\Gamma_t^{\epsilon K}}(x) \text{ for } x = (x', g(x')) + f(x')\nu_{\Gamma_0}(x', g(x'))\,.$$

is a smooth diffeomorphism. Since

$$\nu_{\Gamma_t^{\epsilon K}}(x) = \frac{1}{|\nabla d_\epsilon^K(x)|}\nabla d_\epsilon^K(x)$$

for $x = (x', g(x')) + f(x')\nu_{\Gamma_0}(x', g(x'))$ and for all $x \in U'$, we can conclude by (3.173)

$$\left\| \nu_{\Gamma_t^{\epsilon K}}((x', g(x')) + f(x')\nu_{\Gamma_0}(x', g(x'))) \right\|_{C^3(U')} \leq C$$

for some $C > 0$ independent of ϵ. Hence it follows that there exists some $C > 0$ independent of ϵ such that $\|\tilde{\tau}(x', r)\|_{C^3(U' \times (-\tilde{a}, \tilde{a}))} \leq C$. So we can conclude $\|r_t\|_{C^3(\mathrm{im}(\tilde{\tau}))} \leq C$ since $r_t(x) = (\tilde{\tau}^{-1}(x))_d$ and $\tilde{\tau} \circ \tilde{\tau}^{-1}(x) = x$ for all $x \in \mathrm{im}(\tilde{\tau})$. Since there is an integer $L > 0$ such that $\Gamma^0 \subset \bigcup_{i=1}^L U_i$ and $\Gamma^0 \cap \Gamma_i$, $i = 1, \ldots, L$, can be described by a graph, the C^3 norm of $\Gamma_t^{\epsilon K}$ can be bounded independent of ϵ. We set $\delta_0 = \delta/2$ where δ is as in Section 3.2. Hence $\Gamma^{\epsilon K}(\delta_0) \subset \Gamma^0(\delta)$ for every ϵ small enough. The construction of c_A^ϵ in Section 3.2 yields for all $(x,t) \in \Omega_T \backslash \partial_T \Omega(\delta)$

$$\begin{aligned} c_A^\epsilon &= c_A^K + R_A^\epsilon \\ &= \zeta(r_t(x)/\delta_0) c_I^K + (1 - \zeta(r_t(x)/\delta_0))\, c_O^K \\ &\quad + \left(\zeta(d^0(x,t)/\delta) - \zeta(r_t(x)/\delta_0)\right) \left(c_I^K - c_O^K\right) + R_A^\epsilon \end{aligned}$$

for some function $R_A^\epsilon(t)$ such that $|R_A^\epsilon(t)| \leq C\epsilon^{K-2}$ for some $C > 0$ and for all $t \in [0,T]$. Since $\zeta(r_t(x)/\delta_0) - \zeta(d^0(x,t)/\delta) = 0$ in $\Gamma^{\epsilon K}(\delta_0/2) \cap \Gamma^0(\delta_0) = \Gamma^{\epsilon K}(\delta_0/2)$ (for ϵ small enough) and due to the inner-outer matching condition, we obtain

$$\left\| \left(\zeta(d^0(x,t)/\delta) - \zeta(r_t(x)/\delta_0)\right) \left(c_I^K - c_O^K\right) \right\|_{C^1(\Gamma^{\epsilon K}(\delta_0))} \leq \epsilon^2$$

for all $\epsilon > 0$ small enough. Moreover, note that $c_B^K(x,t) = 1 + \sum_{i=1}^K \epsilon^i c^i(\frac{d_B}{\epsilon}, x, t)$ with $\left\| c^i(\frac{d_B(.)}{\epsilon}, ., .) \right\|_{C^0(\overline{\partial_T \Omega(\delta)})} \leq C$ for all $i = 1, \ldots, K$. Hence in order to satisfy (2.23)-(2.26), we can replace c_A^ϵ by

$$\zeta(r_t(x)/\delta_0) c_I^K(x,t) + (1 - \zeta(r_t(x)/\delta_0))\, c_O^K(x,t)$$

for all $(x,t) \in \Omega_T$ and for $K \geq 4$. Obviously, we set $\phi_t^\pm = c_O^K$ in Ω_T. Due to the construction of c_O^K in Section 3.2, condition (2.26) is satisfied for C_* large enough. So it is sufficient to verify that c_I^K satisfies (2.25) and that for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0) \subset \Gamma^0(\delta)$

$$c^0(\tfrac{d_\epsilon^K}{\epsilon}, x, t) + \epsilon c^1(\tfrac{d_\epsilon^K}{\epsilon}, x, t) = \theta_0(\tfrac{r_t(x)}{\epsilon}) + \epsilon p^\epsilon(S_t(x), t)\theta_1(\tfrac{r_t(x)}{\epsilon}) + \epsilon^2 \overline{q}^\epsilon(x,t)$$

such that θ_1 satisfies (2.23), and $p^\epsilon(x,t)$ and $\overline{q}^\epsilon(x,t)$ satisfy (2.24). Here $S_t(x)$ is the projection from x to $\Gamma_t^{\epsilon K}$ along the normal of $\Gamma_t^{\epsilon K}$.
Before we verify these three conditions, we estimate $r_t - d_\epsilon^K(.,t)$. For $x = S_t(x) +$

$r_t(x)\nu_{\Gamma_t^{\epsilon K}} \in \Gamma_t^{\epsilon K}(\delta_0)$ we obtain

$$\begin{aligned}
2(r_t(x) - d_\epsilon^K(x,t)) &= \int_0^{r_t(x)} 2 - 2\tfrac{d}{d\tau} d_\epsilon^K(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}}, t)\, d\tau \\
&= \int_0^{r_t(x)} 2 - 2\nabla r_t(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}}) \cdot \nabla d_\epsilon^K(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}}, t)\, d\tau \\
&= \int_0^{r_t(x)} \left|\nabla r_t(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}}) - \nabla d_\epsilon^K(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}}, t)\right|^2 d\tau \\
&\quad - \int_0^{r_t(x)} \left(\left|\nabla d_\epsilon^K(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}})\right|^2 - 1 \right) d\tau
\end{aligned} \tag{3.199}$$

since $\nabla r_t(S_t(x) + \tau\nu_{\Gamma_t^{\epsilon K}}) = \nabla r_t(S_t(x)) = \nu_{\Gamma_t^{\epsilon K}}$ on $\Gamma_t^{\epsilon K}$. Using $\nabla^{\Gamma_t^{\epsilon K}} d_\epsilon^K = 0$ on $\Gamma_t^{\epsilon K}$ and $\left\| \left|\nabla d_\epsilon^K\right|^2 - 1 \right\|_{C^0(\Gamma^{\epsilon K}(\delta_0))} \leq C\epsilon^{K+1}$, we get

$$\left|\nabla^{\Gamma_t^{\epsilon K}} d_\epsilon^K(x,t)\right| \leq C r_t(x) \quad \text{in } \Gamma_t^{\epsilon K}(\delta_0) \tag{3.200}$$

and

$$\left|\nabla d_\epsilon^K - \nabla r_t\right| \leq C\epsilon^{K+1} \quad \text{on } \Gamma_t^{\epsilon K}$$

for some $C > 0$ and where $\nabla^{\Gamma_t^{\epsilon K}} = \nabla - \nabla r_t(\nabla r_t \cdot \nabla)$ is the tangential gradient on $\Gamma_t^{\epsilon K}$. Hence it follows from the estimate above

$$\left|\nabla d_\epsilon^K(x,t) - \nabla r_t(x)\right| \leq C r_t(x) + C\epsilon^{K+1} \quad \text{in } \Gamma_t^{\epsilon K}(\delta_0)$$

for some $C > 0$ independent of ϵ and $t \in [0,T]$. Applying this estimate to (3.199) yields for all $x \in \Gamma_t^{\epsilon K}(\delta_0)$

$$\left|r_t(x) - d_\epsilon^K(x,t)\right| \leq C \int_0^{|r_t(x)|} \tau^2\, d\tau + C\epsilon^{K+1} = C\left|r_t(x)\right|^3 + C\epsilon^{K+1}. \tag{3.201}$$

By using this estimate, one can conclude another useful property for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$

$$\begin{aligned}
|r_t(x)| &\leq \left|d_\epsilon^K(x,t)\right| + \left|r_t(x) - d_\epsilon^K(x,t)\right| \leq \left|d_\epsilon^K(x,t)\right| + C\left|r_t(x)\right|^3 + C\epsilon^{K+1} \\
&\leq \left|d_\epsilon^K(x,t)\right| + C\delta_0^2\left|r_t(x)\right| + C\epsilon^{K+1}.
\end{aligned}$$

Choosing δ_0 small enough yields for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$

$$|r_t(x)| \leq C\left|d_\epsilon^K(x,t)\right| + C\epsilon^{K+1} \tag{3.202}$$

for some $C > 0$ independent of ϵ and $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$.
Using (3.200), we can verify (2.25) as follows. Since $c^0(z,x,t) = \theta_0(z)$, we get for all

$(x,t) \in \Gamma^{\epsilon K}(\delta_0)$

$$\begin{aligned}
\left|\nabla^{\Gamma_t^{\epsilon K}} c_I^K(x,t)\right| &= \left| \left(\left[\nabla^{\Gamma_t^{\epsilon K}} + \frac{1}{\epsilon} \left(\nabla^{\Gamma_t^{\epsilon K}} d_\epsilon^K \right) \frac{\partial}{\partial z} \right] \sum_{i=0}^{K} \epsilon^i c^i(z,x,t) \right) \Big|_{z=d_\epsilon^K/\epsilon} \right| \\
&\leq C + \frac{|r_t(x)|}{\epsilon} \theta_0'(\tfrac{d_\epsilon^K(x,t)}{\epsilon}) \\
&\leq C + C \frac{|d_\epsilon^K|}{\epsilon} e^{-\frac{\alpha |d_\epsilon^K|}{\epsilon}} \leq C\,, \qquad (3.203)
\end{aligned}$$

where we have used (3.202) and the fact that the function $h : [0,\infty) \to \mathbb{R}$ defined by $h(z) = ze^{-\alpha z}$ is a bounded function (h takes its absolute maximum at $z = \frac{1}{\alpha}$).
We continue with condition (2.23). To this end we solve for c^1. For $(x,t) \in \Gamma^0$ the equation for c^1 in (3.83) reads

$$\begin{aligned}
c_{zz}^1 - f'(\theta_0)c^1 &= -\mathcal{E}^\star : \mathcal{C}((\mathbf{u}_*^0)_z \otimes \nabla d^0) - \mu^0 - \Delta d^0 c_z^0 \\
&\quad -\mathcal{E}^\star : \mathcal{C}(\nabla \mathbf{u}^0 - \mathcal{E}^\star \theta_0) - zk^0\eta' \\
&= -\mathcal{E}^\star : \mathcal{C}((\mathbf{u}_*^0)_z \otimes \nabla d^0) + \sigma \Delta d^0 + \tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}(\nabla \mathbf{u}_0^+ + \nabla \mathbf{u}_0^-) \\
&\quad -\Delta d^0 \theta_0' - \mathcal{E}^\star : \mathcal{C}(\nabla \mathbf{u}_0^+ \eta + \nabla \mathbf{u}_0^-(1-\eta) - \mathcal{E}^\star \theta_0) \\
&\quad -z\mathcal{E}^\star : \mathcal{C}(\mathbf{l}^0 \otimes \nabla d^0)\eta' \\
&= -\mathcal{E}^\star : \mathcal{C}((\mathbf{u}_*^0)_z \otimes \nabla d^0) + \sigma \Delta d^0 - \Delta d^0 \theta_0' \\
&\quad +\tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}(\nabla \mathbf{u}_0^+ - \nabla \mathbf{u}_0^-) - \mathcal{E}^\star : \mathcal{C}(\nabla \mathbf{u}_0^+ - \nabla \mathbf{u}_0^-)\eta \\
&\quad +\mathcal{E}^\star : (\mathcal{C}\mathcal{E}^\star)\theta_0 - z\mathcal{E}^\star : \mathcal{C}(\mathbf{l}^0 \otimes \nabla d^0)\eta'\,, \qquad (3.204)
\end{aligned}$$

where we have used the definitions of $(c^0, \mu^0, \mathbf{u}^0, k^0)$. To handle the term on the right-hand side, we calculate $(\mathbf{u}_*^0)_z$. Here we use the ordinary differential equation (3.82) on Γ^0. By (3.104) we obtain $\lim_{z\to-\infty} \partial_z \mathbf{u}_*^0(z) = 0$, and therefore we get

$$\begin{aligned}
(\mathbf{u}_*^0)_z &= \int_{-\infty}^{z} (\mathbf{u}_*^0)_{zz}(s)\, ds \\
&= \int_{-\infty}^{z} M^{-1} \big[-(\mathcal{C}_{iji'j'} \partial_j (\mathbf{u}_{i'}^0)_z \partial_{j'} d^0)_{i=1,\dots,d} - (\mathcal{C}_{iji'j'} (\mathbf{u}_{i'}^0)_z \partial_{jj'} d^0)_{i=1,\dots,d} \\
&\quad -(\mathcal{C}\nabla \mathbf{u}_z^0)\nabla d^0 + \theta_0'(\mathcal{C}\mathcal{E}^\star)\nabla d^0 \big] - s\mathbf{l}^0 \eta''\, ds\,.
\end{aligned}$$

The matching condition $\lim_{z\to-\infty}(\theta_0(z), \mathbf{u}^0(z)) = (-1, \mathbf{u}_0^-)$ (see (3.72) and (3.74)) and the equation $\int_{-\infty}^{z} s\eta''(s)\, ds = z\eta'(z) - \eta(z)$ yield

$$\begin{aligned}
(\mathbf{u}_*^0)_z &= M^{-1} \big[-(\mathcal{C}_{iji'j'} \partial_j (\mathbf{u}_0^+ \eta + \mathbf{u}_0^-(1-\eta))_{i'} \partial_{j'} d^0)_{i=1,\dots,d} \\
&\quad +(\mathcal{C}_{iji'j'} \partial_j \mathbf{u}_{0,i'}^- \partial_{j'} d^0)_{i=1,\dots,d} \\
&\quad -(\mathcal{C}_{iji'j'} (\mathbf{u}_0^+ \eta + \mathbf{u}_0^-(1-\eta))_{i'} \partial_{jj'} d^0)_{i=1,\dots,d} \\
&\quad +(\mathcal{C}_{iji'j'} \mathbf{u}_{0,i'}^- \partial_{jj'} d^0)_{i=1,\dots,d} - (\mathcal{C}\nabla(\mathbf{u}_0^+ \eta + \mathbf{u}_0^-(1-\eta)))\nabla d^0 \\
&\quad +(\mathcal{C}\nabla \mathbf{u}_0^-)\nabla d^0 + \theta_0(\mathcal{C}\mathcal{E}^\star)\nabla d^0 + (\mathcal{C}\mathcal{E}^\star)\nabla d^0 \big] - z\eta'\mathbf{l}^0 + \eta\mathbf{l}^0\,,
\end{aligned}$$

where we have used the definition of $\mathbf{u}^0$. Since $[\mathcal{S}\nu]_{\Gamma_t^0} = \left[\mathbf{u}_0^\pm\right]_{\Gamma_t^0} = 0$, we can conclude

$$\begin{aligned}(\mathbf{u}_*^0)_z &= M^{-1}\left[-(\mathcal{C}_{iji'j'}(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-)_{i'j}\partial_{j'}d^0)_{i=1,\dots,d}\,\eta \right.\\ &\quad -(\mathcal{C}(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-))\nabla d^0\eta + \tfrac{1}{2}\theta_0(\mathcal{C}(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-))\nabla d^0\\ &\quad \left.+\tfrac{1}{2}(\mathcal{C}(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-))\nabla d^0\right] - z\eta'\mathbf{l}^0 + \eta\mathbf{l}^0 \quad \text{on } \Gamma^0 .\end{aligned}$$

The interface condition $\left[\mathbf{u}_0^\pm\right]_{\Gamma_t^0} = 0$ also yields $\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^- = \frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-)\otimes\nabla d^0$ on Γ^0. Furthermore, note that $M = (\mathcal{C}_{iji'j'}\partial_j d^0\partial_{j'}d^0)_{i,i'=1}^d$. So we obtain

$$\begin{aligned}(\mathbf{u}_*^0)_z(z) &= -2\eta(z)\frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-) + \frac{1}{2}\theta_0(z)\frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-)\\ &\quad +\frac{1}{2}\frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-) - z\eta'(z)\mathbf{l}^0 + \eta\mathbf{l}^0\\ &= -\eta(z)\frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-) + \frac{1}{2}\theta_0(z)\frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-)\\ &\quad +\frac{1}{2}\frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-) - z\eta'(z)\mathbf{l}^0 \quad \text{on } \Gamma^0 ,\end{aligned}$$

where we have used the definition of $\mathbf{l}^0$ in the last equation. We insert this equation into (3.204) and use $\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^- = \frac{\partial}{\partial\nu}(\mathbf{u}_0^+ - \mathbf{u}_0^-)\otimes\nabla d^0$ on Γ^0 to obtain

$$c_{zz}^1 - f'(\theta_0)c^1 = \sigma\Delta d^0 - \theta_0'\Delta d^0 + \theta_0\left(\mathcal{E}^\star : (\mathcal{C}\mathcal{E}^\star) - \tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-)\right).$$

Since $[\mathcal{S}\nu]_{\Gamma_t^0} = 0$ and $\mathrm{div}\mathcal{S} = 0$ in $Q_0^\pm$, we obtain $\mathcal{C}\mathcal{E}^\star - \frac{1}{2}\mathcal{C}(\nabla\mathbf{u}_0^+ - \nabla\mathbf{u}_0^-) = [\mathcal{S}]_{\Gamma_t^0} = 0$. Therefore c^1 satisfies the following ordinary differential equation on Γ^0

$$c_{zz}^1 - f'(\theta_0)c^1 = \sigma\Delta d^0 - \theta_0'\Delta d^0 ,$$

and so $c^1(z,x,t) = \Delta d^0(x,t)\theta_1(z)$ on Γ^0 where θ_1 satisfies

$$\theta_1'' - f'(\theta_0)\theta_1 = \sigma - \theta_0' \text{ in } \mathbb{R}, \quad \theta_1(0) = 0, \quad \theta_1 \in L^\infty(\mathbb{R}).$$

Since $\int_{\mathbb{R}}(\sigma - \theta_0')\theta_0' = 0$ by definition of σ, Lemma 2.6.2 yields that θ_1 exists and is unique. By integration by parts we show that θ_1 satisfies the property (2.23)

$$\begin{aligned}0 &= \int_{\mathbb{R}}\theta_0''(\sigma - \theta_0')\,dz = \int_{\mathbb{R}}\theta_0''\left(\theta_1'' - f'(\theta_0)\theta_1\right)\,dz\\ &= -\int_{\mathbb{R}}\theta_1'\left(\theta_0'' - f(\theta_0)\right)'\,dz + \int_{\mathbb{R}} f''(\theta_0)(\theta_0')^2\theta_1\,dz = \int_{\mathbb{R}} f''(\theta_0)(\theta_0')^2\theta_1\,dz . \qquad (3.205)\end{aligned}$$

It is natural to set for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$

$$\begin{aligned}p^\epsilon &= \Delta d^0(x,t),\\ \overline{q}^\epsilon &= \epsilon^{-2}\left(\theta_0\left(\tfrac{d_\epsilon^K}{\epsilon}\right) - \theta_0\left(\tfrac{r_t(x)}{\epsilon}\right)\right) + \epsilon^{-1}\left(c^1\left(\tfrac{d_\epsilon^K}{\epsilon}, x, t\right) - p^\epsilon(S_t(x),t)\,\theta_1\left(\tfrac{r_t(x)}{\epsilon}\right)\right).\end{aligned}$$

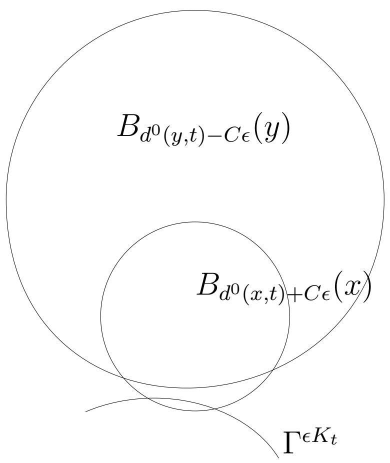
Then due to construction, it holds in $\Gamma^{\epsilon K}(\delta_0)$

$$c^0\left(\tfrac{d_\epsilon^K}{\epsilon}, x, t\right) + \epsilon c^1\left(\tfrac{d_\epsilon^K}{\epsilon}, x, t\right) = \theta_0\left(\tfrac{r_t(x)}{\epsilon}\right) + \epsilon p^\epsilon(S_t(x), t)\, \theta_1\left(\tfrac{r_t(x)}{\epsilon}\right) + \epsilon^2 \overline{q}^\epsilon(x,t)\,.$$

It remains to verify condition (2.24). We denote by $S_t^0(x)$ the projection from x to Γ_t^0 along the normal of Γ_t^0. By geometric arguments we estimate $S_t(x) - S_t^0(x)$ for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$. Let $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$ be any point. W.l.o.g. we assume $d^0(x,t) \geq 0$. By definition of $\Gamma_t^{\epsilon K}$ there exists some $C > 0$ independent of (x,t) such that $B_{d^0(x,t)+C\epsilon}(x) \cap \Gamma_t^{\epsilon K} \neq \emptyset$. Moreover, there exists some $y \in \Omega_0^+$ such that $\overline{B_{d^0(y,t)}(y)} \cap \Gamma_t^0 = \{S_t^0(x)\}$ and $d^0(y,t) - d^0(x,t) \geq \delta_0$ for δ_0 small enough. In particular, $B_{d^0(x,t)}(x) \subset B_{d^0(y,t)}(y)$ and $\partial B_{d^0(y,t)}(y) \cap \partial B_{d^0(x,t)}(x) = \{S_t^0(x)\}$. Again by definition of $\Gamma_t^{\epsilon K}$ there exists some $C > 0$ independent of (x,t) such that $B_{d^0(y,t)-C\epsilon}(y) \cap \Gamma_t^{\epsilon K} = \emptyset$. Then it holds $S_t^0(x), S_t(x) \in B_{d^0(x,t)+C\epsilon}(x) \backslash B_{d^0(y,t)-C\epsilon}(y)$. As shown below, we obtain

$$\left|S_t(x) - S_t^0(x)\right| \leq C\epsilon \quad \forall (x,t) \in \Gamma^{\epsilon K}(\delta_0) \tag{3.206}$$

for some C independent of ϵ and (x,t).



Let us prove the estimate (3.206). Let $z \in B_{d^0(x,t)+C\epsilon}(x) \backslash B_{d^0(y,t)-C\epsilon}(y)$ and let $p \in \Omega$ be the orthogonal projection of z onto the line $\{x + t(x-y) : t \in \mathbb{R}\}$. For convenience we right $\overline{ab}$ instead of $|a-b|$ for $a, b \in \mathbb{R}^d$. By the construction of $B_{d^0(x,t)}(x)$ and $B_{d^0(y,t)}(y)$, it holds $d^0(y,t) = \overline{xy} + d^0(x,t)$. By the Pythagoras' theorem we can follow

$$\overline{pz}^2 + \overline{px}^2 = \overline{xz}^2 \leq (d^0(x,t) + C\epsilon)^2 \tag{3.207}$$

and

$$\overline{pz}^2 + \overline{py}^2 = \overline{yz}^2 \geq (d^0(x,t) + \overline{xy} - C\epsilon)^2$$

since $z \in B_{d^0(x,t)+C\epsilon}(x) \backslash B_{d^0(y,t)-C\epsilon}(y)$. Using these estimates, we get

$$\begin{aligned} & (d^0(x,t) + C\epsilon)^2 + \overline{xy}^2 + 2\overline{xy}\,\overline{px} \\ \geq\; & \overline{pz}^2 + \overline{px}^2 + \overline{xy}^2 + 2\overline{xy}\,\overline{px} = \overline{pz}^2 + (\overline{px} + \overline{xy})^2 = \overline{pz}^2 + \overline{py}^2 \\ \geq\; & (d^0(x,t) + C\epsilon)^2 + 2(d^0(x,t) + C\epsilon)(\overline{xy} - 2C\epsilon) + (\overline{xy} - 2C\epsilon)^2\,. \end{aligned}$$

Hence we obtain

$$2\overline{xy}\,\overline{px} \geq 2d^0(x,t)\overline{xy} - 4C\epsilon\left(d^0(x,t) - \overline{xy}\right),$$

and therefore

$$\overline{px} \geq d^0(x,t) - \frac{2C}{\overline{xy}}\,\epsilon\left(d^0(x,t) - \overline{xy}\right).$$

Since $p \in B_{d^0(x,t)+C\epsilon}(x)$ and $\overline{xy} \geq \delta_0$ by definition of y, we conclude

$$\left|\overline{px} - d^0(x,t)\right| \leq C\epsilon\,, \tag{3.208}$$

for some $C > 0$ independent of ϵ. Applying (3.207), one can conclude

$$\overline{pz} \leq C\epsilon$$

for some $C > 0$ independent of ϵ. Hence by triangle inequality it holds for all $z, w \in B_{d^0(x,t)+C\epsilon}(x)\backslash B_{d^0(y,t)-C\epsilon}(y)$

$$|z - w| = \overline{wz} \leq \overline{p_w w} + \overline{p_z z} + \overline{p_w p_z} \leq C\epsilon$$

for some $C > 0$ independent of ϵ and where p_w and p_z are the orthogonal projection of w and z onto the line $\{x + t(x-y) : t \in \mathbb{R}\}$. Here we have used $\overline{p_w p_z} \leq C\epsilon$ due to the inequality (3.208).
For all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$ we can conclude by triangle inequality

$$\begin{aligned} |\overline{q}^\epsilon(x,t)| \;\leq\; & \epsilon^{-2}\left|\theta_0\left(\tfrac{d_\epsilon^K}{\epsilon}\right) - \theta_0\left(\tfrac{r_t(x)}{\epsilon}\right)\right| + \epsilon^{-1}\left|c^1\left(\tfrac{d_\epsilon^K}{\epsilon}, x, t\right) - c^1\left(\tfrac{r_t(x)}{\epsilon}, x, t\right)\right| \\ & + \epsilon^{-1}\left|c^1\left(\tfrac{r_t(x)}{\epsilon}, x, t\right) - c^1\left(\tfrac{r_t(x)}{\epsilon}, S_t^0(x), t\right)\right| \\ & + \epsilon^{-1}\left|c^1\left(\tfrac{r_t(x)}{\epsilon}, S_t^0(x), t\right) - p^\epsilon(S_t(x), t)\,\theta_1\left(\tfrac{r_t(x)}{\epsilon}\right)\right|. \end{aligned} \tag{3.209}$$

We apply the mean value theorem to the first term on the right-hand side to obtain

$$\begin{aligned} \left|\theta_0\left(\tfrac{d_\epsilon^K}{\epsilon}\right) - \theta_0\left(\tfrac{r_t(x)}{\epsilon}\right)\right| \;\leq\; & \epsilon^{-1}\left|d_\epsilon^K(x,t) - r_t(x)\right|\theta_0'\left(\tfrac{\Theta d_\epsilon^K + (1-\Theta)r_t}{\epsilon}\right) \\ \leq\; & C\epsilon^{-1}|r_t|^3 e^{-\frac{\alpha|r_t|}{\epsilon}} = C\frac{|r_t|^2}{\epsilon^2}e^{-\frac{\alpha|r_t|}{\epsilon}}\epsilon\,|r_t| \end{aligned}$$

for some $\Theta = \Theta(x,t) \in (0,1)$ and where we have used (3.201) and $\left|d_\epsilon^K\right| \leq C|r_t| + C\epsilon^{K+1}$, see (3.202). Since the function $h : [0,\infty) \to \mathbb{R}$ defined by $h(z) = z^2 e^{-\alpha z}$ is bounded (it takes its absolute maximum at $z = \frac{2}{\alpha}$), it follows

$$\left|\theta_0\left(\tfrac{d_\epsilon^K}{\epsilon}\right) - \theta_0\left(\tfrac{r_t(x)}{\epsilon}\right)\right| \leq C\epsilon\,|r_t| \quad \text{in } \Gamma^{\epsilon K}(\delta_0)$$

for some $C >$ independent of ϵ and $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$.
We obtain the same result for the second term in (3.209) by the same arguments

$$\left|c^1\left(\tfrac{d_\epsilon^K}{\epsilon}\right) - c^1\left(\tfrac{r_t(x)}{\epsilon}\right)\right| \leq C\epsilon\,|r_t| \quad \text{in } \Gamma^{\epsilon K}(\delta_0)$$

for some $C >$ independent of ϵ and $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$.
Again applying the mean value theorem to the third term in (3.209), yields for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$

$$\begin{aligned}\left|c^1\left(\tfrac{r_t(x)}{\epsilon}, x, t\right) - c^1\left(\tfrac{r_t(x)}{\epsilon}, S_t^0(x), t\right)\right| &\leq C\left|x - S_t^0(x)\right| \\ &\leq |r_t(x)| + \left|d^0(x,t) - r_t(x)\right| \\ &\leq C\,|r_t(x)| + C\epsilon\,,\end{aligned}$$

where we have used $\left|d^0 - d_\epsilon^K\right| \leq C\epsilon$ and (3.201).
For the last term in (3.209) we use that $c^1(z,x,t) = \Delta d^0(x,t)\theta_1(z)$ on Γ^0. Then we conclude for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$

$$\begin{aligned}&\left|c^1\left(\tfrac{r_t(x)}{\epsilon}, S_t^0(x), t\right) - p^\epsilon(S_t(x),t)\,\theta_1\left(\tfrac{r_t(x)}{\epsilon}\right)\right| \\ &\leq \left|\Delta d^0(S_t^0(x),t) - \Delta d^0(S_t(x),t)\right| \left|\theta_1\left(\tfrac{r_t(x)}{\epsilon}\right)\right| \\ &\leq C\left|S_t(x) - S_t^0(x)\right| \leq C\epsilon\,,\end{aligned}$$

where the last inequality follows from (3.206). Altogether, we get the estimate

$$|\overline{q}^\epsilon(x,t)| \leq C(1 + \epsilon^{-1}\,|r_t(x)|) \quad \forall (x,t) \in \Gamma^{\epsilon K}(\delta_0)\,.$$

Hence p^ϵ and $\overline{q}^\epsilon$ satisfy (2.24). This completes the proof. □

Remark 3.3.2. *The initial value $c_0^\epsilon(x) = c_A^\epsilon(x,0)$ is independent of the solutions for the modified Hele-Shaw problem and the linearized Hele-Shaw problem. This can be seen as follows. By solving the first order partial differential equation (3.64) with Cauchy data $d^k(x,0) = 0$ on Γ_{00}, we can directly determine d^k for all $k \in \mathbb{N}$. Hence one can find $\mathcal{V}^k$ for $t = 0$ and for all $k \in \mathbb{N}$.*

4 Classical Solutions of Sharp Interface Models

We show that the Hele-Shaw problem (3.8)-(3.14) and the linearized Hele-Shaw problem (3.161)-(3.170) have smooth solutions. For the proof we transform these problems to fixed domains. To this end we apply the Hansawa transformation. In the following we describe the construction of the Hansawa transformation and summarize some facts about it.

We assume that the domain $\Omega \subset \mathbb{R}^d$ has a smooth boundary $\partial\Omega$. Let $\Sigma \subset \Omega$ be a smooth $(d-1)$-dimensional reference manifold without boundary such that $\Sigma = \partial\Omega^-$ for a reference domain $\Omega^- \subset \Omega$. We set $\Omega^+ := \Omega \backslash \overline{\Omega^-}$, that is, Σ separates Ω into an interior domain Ω^- and an exterior domain Ω^+. Denote by ν_Σ the unit normal of Σ that points outside Ω^-. We observe that the function

$$\mathcal{N} : \Sigma \times (-a_0, a_0) \to \mathbb{R}^d : (x, \lambda) \mapsto x + \lambda \nu_\Sigma(x)$$

is a smooth diffeomorphism onto its image $B_{a_0}(\Sigma) := \operatorname{im}(\mathcal{N}) \subset \Omega$ provided $0 < a_0 < \operatorname{dist}(\Sigma, \partial\Omega)$ is small enough where a_0 depends on the maximal curvature of Σ and $\operatorname{dist}(\Sigma, \partial\Omega)$, cf. [41, Kapitel 4.6]. For the inverse of $\mathcal{N}$ we have the decomposition $\mathcal{N}^{-1} = (S, d_\Sigma)$ where $d_\Sigma(y)$ is the signed distance from y to Σ and $S(y)$ is the orthogonal projection of y onto Σ. For some "height function" $h : \Sigma \times [0, T] \to \mathbb{R}$, we define the map

$$\theta_h : \Sigma \times [0, T] \to \Omega : (x, t) \mapsto x + h(x, t)\nu_\Sigma(x) \,.$$

Then for every $t \in [0, T]$ the function $\theta_h(., t) : \Sigma \to \Omega$ is injective provided $|h(x, t)| < a_0$ for all $(x, t) \in \Sigma \times [0, T]$, and we define for $t \in [0, T]$

$$\Gamma_{h(t)} := \{\theta_h(x, t) : x \in \Sigma\} \,.$$

Then $\Gamma_{h(t)}$ separates Ω into an interior domain $\Omega^-_{h(t)}$ and an exterior domain $\Omega^+_{h(t)}$ such that $\Gamma_{h(t)} = \partial\Omega^-_{h(t)}$ and $\Omega^+_{h(t)} = \Omega \backslash \overline{\Omega^-_{h(t)}}$. Furthermore, we use the definitions

$$\Gamma_{h,T} := \bigcup_{t \in (0,T]} (\Gamma_{h(t)} \times \{t\}) \quad \text{and} \quad \Omega^\pm_{h,T} := \bigcup_{t \in (0,T]} (\Omega^\pm_{h(t)} \times \{t\}) \,.$$

Note that for fixed $t \in [0, T]$, $\Gamma_{h(t)}$ is the zero-level set of the function

$$\Phi_h : B_{a_0}(\Sigma) \times [0, T] \to \mathbb{R} : (x, t) \mapsto d_\Sigma(x) - h(S(x), t) \,.$$

To transform the Hele-Shaw problem (3.8)-(3.14) and the linearized Hele-Shaw problem (3.161)-(3.170) to fixed domains, we extend for fixed $t \in [0,T]$ the diffeomorphism $\theta_h(.,t) : \Sigma \to \mathrm{im}(\theta_h(.,t))$ to Ω by the so-called "Hansawa transformation" that was first introduced by E. I. Hanzawa [39] for multi-dimensional Stefan problems. One can apply the Hansawa transformation not only for the Stefan problem, for example see [28] or [7]. We choose $a \in (0, a_0/4)$ and fix some $\chi \in C^\infty(\mathbb{R}, [0,1])$ such that $\chi(\lambda) = 1$ if $|\lambda| \leq a$, and $\chi(\lambda) = 0$ if $|\lambda| \geq 3a$ and such that $\sup|\chi'(\lambda)| < 1/a$. Then we define

$$\Theta_h(x,t) := x + \chi(d_\Sigma(x))h(S(x),t)\nu_\Sigma(S(x))\,. \tag{4.1}$$

Since $\mathcal{N} : \Sigma \times (-a_0, a_0) \to B_{a_0}(\Sigma)$ is a smooth diffeomorphism and the function $[\lambda \mapsto \lambda + \chi(\lambda)h(s,t)]$ is strictly increasing for all $s \in \Sigma \times [0,T]$ provided $\|h(.,t)\|_{C^0} \leq a$ for all $t \in [0,T]$ (note that $\sup|\chi'(\lambda)| < 1/a$), we can conclude that

$$\Theta_h(.,t) \in \mathrm{Diff}^\delta(\Omega,\Omega) \cap \mathrm{Diff}^\delta(\Omega^\pm, \Omega^\pm_{h(t)}) \quad \forall t \in [0,T]$$

if $h(.,t) \in C^\delta(\Sigma)$, $\delta \in [1,\infty]$. Furthermore, it holds for all $t \in [0,T]$

$$\Theta_h|_{\Sigma\times[0,T]} = \theta_h \quad \text{and} \quad \Theta_h(.,t)|_{B_b(\partial\Omega)} = \mathrm{Id}_{B_b(\partial\Omega)}$$

for some sufficiently small $b > 0$ where $B_b(\partial\Omega) = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < b\}$.

4.1 Classical Solution of the Modified Hele-Shaw Problem

Our proof is based on a paper of Escher and Simonett [29].
We consider a smooth domain $\Omega \subset \mathbb{R}^d$ which is divided into two parts $\Omega^+(t)$ and $\Omega^-(t)$ with a common interface $\Gamma(t)$, $t > 0$, that is, $\partial\Omega^-(t) = \Gamma(t)$, $\partial\Omega^-(t) \cap \partial\Omega = \emptyset$, and $\partial\Omega^+(t) = \partial\Omega \cup \Gamma(t)$ for all $t \in [0,T]$. Then we seek for a solution to the problem

$$\begin{aligned}
\Delta\mu &= 0 && \text{in } \Omega^\pm(t), t \in (0,T), && (4.2)\\
\mathrm{div}\,\mathcal{S} &= 0 && \text{in } \Omega^\pm(t), t \in (0,T), && (4.3)\\
V &= -\tfrac{1}{2}\,[\nabla\mu]_{\Gamma(t)} \cdot \nu && \text{on } \Gamma(t), t \in (0,T), && (4.4)\\
\mu &= \sigma\kappa + \tfrac{1}{2}\nu^T \left[W\mathrm{Id} - (\nabla\mathbf{u})^T\mathcal{S}\right]_{\Gamma(t)} \nu && \text{on } \Gamma(t), t \in (0,T), && (4.5)\\
[\mathcal{S}\nu]_{\Gamma(t)} &= [\mathbf{u}]_{\Gamma(t)} = [\mu]_{\Gamma(t)} = 0 && \text{on } \Gamma(t), t \in (0,T), && (4.6)\\
\tfrac{\partial}{\partial n}\mu &= \mathbf{u} = 0 && \text{on } \partial\Omega, t \in (0,T), && (4.7)\\
\Gamma(0) &= \Gamma_0 && \text{for } t = 0. && (4.8)
\end{aligned}$$

As above let $\Sigma \subset \Omega$ be a smooth $(d-1)$-dimensional reference manifold without boundary such that $\Sigma = \partial\Omega^-$ for a reference domain Ω^- with a smooth boundary sufficiently close to $\Omega^-(0)$. We define $\Omega^+ := \Omega\backslash\overline{\Omega^-}$. For some sufficiently small $0 < a < a_0/4$ and given $\alpha \in (0,1)$ let

$$\mathfrak{A} := \left\{h \in C^{2+\alpha}(\Sigma) : \|h\|_{C^1} < a\right\}, \tag{4.9}$$

where a_0 and a are given as above. In the sequel we need the spaces

$$\mathcal{V} := h^{2+\alpha_0}(\Sigma) \cap \mathfrak{A} \quad \text{and} \quad \mathcal{U} := h^{2+\beta}(\Sigma) \cap \mathfrak{A}$$

for some fixed $\alpha_0, \beta \in (\alpha, 1)$, $\beta < \alpha_0$, and where the little Hölder spaces are defined as in Section 2.3.
We assume that

$$\Gamma(t) = \Gamma_{h(t)} = \left\{ \theta_{h(t)}(x) : x \in \Sigma \right\}$$

for some $h : [0, T] \to \mathfrak{A}$ and fixed $T > 0$.
Then for given $h_0 \in \mathfrak{A}$ we can reformulate the Hele-Shaw system (4.2)-(4.8) into

$$\begin{aligned}
\Delta\mu &= 0 && \text{in } \Omega^{\pm}_{h,T}\,, && (4.10)\\
\operatorname{div}\mathcal{S} &= 0 && \text{in } \Omega^{\pm}_{h,T}\,, && (4.11)\\
V &= -\tfrac{1}{2}\,[\nabla\mu]_{\Gamma(t)} \cdot \nu && \text{on } \Gamma_{h,T}, && (4.12)\\
\mu &= \sigma\kappa + \tfrac{1}{2}\nu^T \left[W\mathrm{Id} - (\nabla\mathbf{u})^T\mathcal{S}\right]_{\Gamma(t)} \nu && \text{on } \Gamma_{h,T}, && (4.13)\\
[\mathcal{S}\nu]_{\Gamma(t)} &= [\mathbf{u}]_{\Gamma(t)} = [\mu]_{\Gamma(t)} = 0 && \text{on } \Gamma_{h,T} && (4.14)\\
\tfrac{\partial}{\partial n}\mu &= \mathbf{u} = 0 && \text{on } \partial_T\Omega, && (4.15)\\
h(.,0) &= h_0 && \text{on } \Sigma, && (4.16)
\end{aligned}$$

where $\Omega^{\pm}_{h,T}$ and $\Gamma_{h,T}$ are defined as above. As main result of this section we obtain:

Theorem 4.1.1. *Let $h_0 \in \mathcal{V}$ be given. Then, for sufficiently small $T > 0$, the problem (4.10)-(4.16) possesses a unique classical solution $(\mu, \boldsymbol{u}, h)$, i.e.*

$$\begin{aligned}
\mu(.,t) &= (\mu^+(.,t), \mu^-(.,t)) \in C^\infty(\overline{\Omega^+_{h(t)}}) \times C^\infty(\overline{\Omega^-_{h(t)}})\,, && t \in (0,T]\,,\\
\boldsymbol{u}(.,t) &= (\boldsymbol{u}^+(.,t), \boldsymbol{u}^-(.,t)) \in C^\infty(\overline{\Omega^+_{h(t)}}, \mathbb{R}^d) \times C^\infty(\overline{\Omega^-_{h(t)}}, \mathbb{R}^d)\,, && t \in (0,T]\,,\\
h &\in C([0,T], \mathcal{V}) \cap C^\infty(\Sigma \times (0,T))\,.
\end{aligned}$$

Moreover, the interface depends analytically on the time variable.

We want to express the normal velocity V of $\Gamma(t)$ and the outer unit normal $\nu_{\Gamma(t)}$ by h. Since $\Gamma(t)$ is the zero-level set of the function $\Phi_h(.,t)$, the normal velocity V of $\Gamma(t)$ at the point $y = \theta_h(x,t) = \Theta_h(x,t)$ is given by

$$V(x,t) = -\left.\frac{\partial_t \Phi_h(y,t)}{|\nabla_y \Phi_h(y,t)|}\right|_{y=\Theta_h(x,t)} = \left.\frac{\partial_t h(x,t)}{|\nabla_y \Phi_h(y,t)|}\right|_{y=\Theta_h(x,t)},$$

and the outer unit normal field on $\Gamma(t)$ is given by $\nu_{\Gamma(t)} = \nabla_y \Phi_h(.,t) / |\nabla_y \Phi_h(.,t)|$. As a consequence equation (4.12) takes the form

$$\partial_t h(x,t) = -\frac{1}{2}\left(\nabla\mu^+(y,t) - \nabla\mu^-(y,t)\right) \cdot \nabla_y \Phi_h(y,t)\Big|_{y=\Theta_h(x,t)}\,. \tag{4.17}$$

To describe $\mu^\pm$ as a function of h, we introduce the following operators. For a function $u^\pm \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ and $h \in \mathfrak{A}$ we define the transformed differential operators

$$\begin{aligned} A^\pm(h)u^\pm &:= (\Delta(u^\pm \circ \Theta_h^{-1})) \circ \Theta_h\,, \\ B^\pm(h)u^\pm &:= \gamma^\pm((\nabla(u^\pm \circ \Theta_h^{-1}) \cdot \nabla \Phi_h) \circ \Theta_h)\,, \end{aligned}$$

where $\gamma^\pm$ denotes the restriction operator from $\Omega^\pm$ to Σ, and we define for a function $\mathbf{w}^\pm \in C^2(\Omega^\pm)^d \cap C^1(\overline{\Omega^\pm})^d$

$$\begin{aligned} C^\pm(h)\mathbf{w}^\pm &:= (\mathrm{div}(\mathcal{C}\nabla(\mathbf{w}^\pm \circ \Theta_h^{-1}))) \circ \Theta_h\,, \\ D^\pm(h)\mathbf{w}^\pm &:= \gamma^\pm(((\mathcal{C}\nabla(\mathbf{w}^\pm \circ \Theta_h^{-1}))\nabla\Phi_h/\,|\nabla\Phi_h|) \circ \Theta_h)\,, \\ E^\pm(h)\mathbf{w}^\pm &:= \tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}\gamma^\pm(\nabla(\mathbf{w}^\pm \circ \Theta_h^{-1}) \circ \Theta_h)\,, \\ F(h) &:= 2((\mathcal{C}\mathcal{E}^\star)\nabla\Phi_h/\,|\nabla\Phi_h|) \circ \Theta_h\,. \end{aligned}$$

Note that we again use the symmetry of $\mathcal{C}$, that is, $\mathcal{C}\mathcal{E}(\mathbf{w}) = \mathcal{C}\nabla\mathbf{w}$. Furthermore, we set for $u = (u^+, u^-) \in C^2(\overline{\Omega^+}) \times C^2(\overline{\Omega^-})$ and $\mathbf{w} = (\mathbf{w}^+, \mathbf{w}^-) \in C^2(\overline{\Omega^+})^d \times C^2(\overline{\Omega^-})^d$

$$\begin{aligned} &A(h)u := (A^+(h)u^+, A^-(h)u^-), \qquad B(h)u := B^+(h)u^+ - B^-(h)u^-, \\ &C(h)\mathbf{w} := (C^+(h)\mathbf{w}^+, C^-(h)\mathbf{w}^-), \quad D(h)\mathbf{w} := D^+(h)\mathbf{w}^+ - D^-(h)\mathbf{w}^-, \\ &E(h)\mathbf{w} := E^+(h)\mathbf{w}^+ + E^-(h)\mathbf{w}^-\,. \end{aligned} \tag{4.18}$$

The transformed mean curvature operator is defined by

$$H(h) := \kappa_h \circ \Theta_h \quad \text{on } \Sigma$$

for $h \in \mathfrak{A}$ and where κ_h is the mean curvature of Γ_h. Assume that $\Gamma_0 = \Gamma_{h_0}$ for some $h_0 \in \mathfrak{A}$. Then we are able to express the motion equation (4.17) by an evolution equation on Σ

$$\partial_t h + \frac{1}{2}B(h)u(h) = 0, \quad h(0) = h_0\,, \tag{4.19}$$

where $u(h)$ is the solution to the transformed Laplace equation

$$A(h)u = 0 \qquad \text{in } \Omega^\pm\,, \tag{4.20}$$
$$u = \sigma H(h) - E(h)\mathbf{w} \quad \text{on } \Sigma\,, \tag{4.21}$$
$$\tfrac{\partial}{\partial n}u = 0 \qquad \text{on } \partial\Omega\,, \tag{4.22}$$

where $\mathbf{w}(h)$ is the solution to the transformed equation to linearized elasticity

$$C(h)\mathbf{w} = 0 \qquad \text{in } \Omega^\pm\,, \tag{4.23}$$
$$D(h)\mathbf{w} = F(h) \quad \text{on } \Sigma\,, \tag{4.24}$$
$$[\mathbf{w}]_\Sigma = 0 \qquad \text{on } \Sigma\,, \tag{4.25}$$
$$\mathbf{w} = 0 \qquad \text{on } \partial\Omega\,. \tag{4.26}$$

Remark 4.1.2. *As in Subsection 3.2.7 (see Step 10), one can show that*

$$\tfrac{1}{2}\nu^T \left[W\mathrm{Id} - (\nabla \boldsymbol{u})^T \mathcal{S}\right]_{\Gamma(t)} \nu = -\tfrac{1}{2}\mathcal{E}^\star : \mathcal{C}\left(\nabla \boldsymbol{u}^+ + \nabla \boldsymbol{u}^-\right)$$

on $\Gamma(t)$*. This explains the construction of the operator* $E(h)$*.*

Now we want to reduce the coupled equations (4.19)-(4.26) to a single evolution equation for the height function function h. First we summarize some results of Escher and Simonett [29].

Lemma 4.1.3. *There exist functions*

$$P \in C^\omega(\mathcal{U}, \mathcal{L}(h^{3+\alpha}(\Sigma), h^{1+\alpha}(\Sigma))) \quad \text{and} \quad K \in C^\omega(\mathcal{U}, h^{1+\beta}(\Sigma))$$

such that

$$H(h) = P(h)h + K(h) \quad \text{for} \quad h \in h^{3+\alpha}(\Sigma)\,.$$

Proof: We use a localization system $\{(U_l, \varphi_l) : 1 \le l \le L\}$ for Σ, that is, $\Sigma = \bigcup_{l=1}^L U_l$ and $\varphi_l : (-a,a)^{d-1} \to U_l$ is a smooth parametrization of U_l for $a > 0$ small enough. Then one can show that $H(h)$ has a local representation $P(h)h + K(h)$ such that $P(h)$ and $Q(h)$ have the required properties. For more details see [29, Chapter 3] or [28, Lemma 3.1]. □

Lemma 4.1.4. *Let* $h \in \mathcal{U}$ *be given and let* $\sigma \in [\alpha, \beta]$ *be fixed. Then the elliptic boundary value problem*

$$\begin{aligned} A^\pm(h)u^\pm &= 0 \quad \text{in } \Omega^\pm\,, \\ u^\pm &= g \quad \text{on } \Sigma\,, \\ \tfrac{\partial}{\partial n} u^+ &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $u^\pm = T^\pm(h)g \in h^{1+\sigma}(\overline{\Omega^\pm})$ *for each* $g \in h^{1+\sigma}(\Sigma)$ *and*

$$\left[h \mapsto T^\pm(h)\right] \in C^\omega(\mathcal{U}, \mathcal{L}(h^{1+\sigma}(\Sigma), h^{1+\sigma}(\overline{\Omega^\pm})))\,.$$

Moreover, it holds

$$\left[h \mapsto B^\pm(h)\right] \in C^\omega(\mathcal{U}, \mathcal{L}(h^{1+\sigma}(\overline{\Omega^\pm}), h^\sigma(\Sigma)))\,.$$

Proof: First one shows that

$$(A^\pm, B^\pm) \in C^\omega(\mathcal{U}, \mathcal{L}(h^{1+\sigma}(\overline{\Omega^\pm}), h^{\sigma-1}(\overline{\Omega^\pm}) \times h^\sigma(\Sigma)))\,.$$

Then for given $h \in \mathcal{U}$, we prove that

$$\begin{aligned} (A^-, \gamma^-) &\in \mathrm{Isom}(h^{1+\sigma}(\overline{\Omega^-}), h^{\sigma-1}(\overline{\Omega^-}) \times h^{1+\sigma}(\Sigma))\,, \\ (A^+, \gamma^+, \tfrac{\partial}{\partial n}) &\in \mathrm{Isom}(h^{1+\sigma}(\overline{\Omega^+}), h^{\sigma-1}(\overline{\Omega^+}) \times h^{1+\sigma}(\Sigma) \times h^\sigma(\Sigma))\,. \end{aligned}$$

For $s < 0$ one can find the definition of $h^s(\overline{\Omega})$ in [62]. Finally, it holds

$$\big[h \mapsto T^{\pm}(h)\big] \in C^{\omega}(\mathcal{U}, \mathcal{L}(h^{1+\sigma}(\Sigma), h^{1+\sigma}(\overline{\Omega^{\pm}})))\,.$$

We can show these assertions as in Lemma 4.1.5 - 4.1.7 below or see Lemma 2.2 and 2.3 in [29]. □

The following lemmas describe the properties of the new operators $C^{\pm}, D^{\pm}, E^{\pm}$, and F.

Lemma 4.1.5. *Let $(C^{\pm}, D^{\pm}, E^{\pm}, F)$ be defined as above. Then we have the following properties*

1. $(C^{\pm}, D^{\pm}, E^{\pm}) \in C^{\omega}(\mathcal{U}, \mathcal{L}(h^{2+\beta}(\overline{\Omega^{\pm}})^d, h^{\beta}(\overline{\Omega^{\pm}})^d \times h^{1+\beta}(\Sigma)^d \times h^{1+\beta}(\Sigma)^d))\,.$
2. $F \in C^{\omega}(\mathcal{U}, h^{1+\beta}(\Sigma)^d)\,.$

Proof: To 1.: The proof works similarly as Lemma 2.2 in [29]. It is generally known that for differentiable functions $u : \Omega \to \mathbb{R}$ and $\mathbf{v} : \Omega \to \mathbb{R}^d$

$$\begin{aligned}(\nabla u) \circ \Theta_h &= g^{ij}\partial_i(u \circ \Theta_h)\partial_j\Theta_h\,,\\ (\operatorname{div} \mathbf{v}) \circ \Theta_h &= g^{ij}\partial_i(\mathbf{v} \circ \Theta_h) \cdot \partial_j\Theta_h\,,\end{aligned}$$

where

$$(g_{ij})_{i,j=1}^d = G := D\Theta_h^T D\Theta_h = (\partial_i\Theta_h \cdot \partial_j\Theta_h)_{i,j=1}^d$$

and

$$\big(g^{ij}\big)_{i,j=1}^d = G^{-1}\,.$$

The mappings

$$[(u,v) \mapsto uv] \;:\; h^{\delta}(\overline{\Omega}) \times h^{\delta}(\overline{\Omega}) \to h^{\delta}(\overline{\Omega})\,, \tag{4.27}$$

$$[(u,v) \mapsto uv] \;:\; h^{\delta}(\Sigma) \times h^{\delta}(\Sigma) \to h^{\delta}(\Sigma) \tag{4.28}$$

are bilinear and continuous for all $\delta \in \mathbb{R}_+ \backslash \mathbb{N}$, cf. [47] (using the definition in (2.3) of $h^{\delta}(\overline{\Omega})$, one can easily verify this property by direct calculation). Hence it follows $g_{ij} \in h^{1+\beta}(\overline{\Omega})$ and therefore $g^{ij} \in h^{1+\beta}(\overline{\Omega})$ as well. The fact that

$$[h \mapsto \Theta_h] \in C^{\omega}(\mathcal{U}, h^{2+\beta}(\overline{\Omega})^d)\,,$$

see [29, proof of Lemma 2.2.], yields that $[h \mapsto g_{ij}] \in C^{\omega}(\mathcal{U}, h^{1+\beta}(\overline{\Omega}))$, and since matrix inversion is analytic as well, it holds $[h \mapsto g^{ij}] \in C^{\omega}(\mathcal{U}, h^{1+\beta}(\overline{\Omega}))$. Furthermore, it is not difficult to see that $[h \mapsto \nabla\Phi_h] \in C^{\omega}(\mathcal{U}, h^{1+\beta}(\Sigma)^d)$. So with (4.27) and (4.28), we can conclude that

$$\begin{aligned}&C^{\pm}(h) = C_{ij}(h)\partial_i\partial_j + C_i(h)\partial_i\,, \quad D^{\pm}(h) = D_i(h)\gamma^{\pm}\partial_i\,,\\ &E^{\pm}(h) = E_i(h)\gamma^{\pm}\partial_i\end{aligned} \tag{4.29}$$

for some $C_{ij}(h) \in h^{1+\beta}(\overline{\Omega})^{d\times d}$, $C_i(h) \in h^{\beta}(\overline{\Omega})^{d\times d}$, $D_i(h) \in h^{1+\beta}(\Sigma)^{d\times d}$, and $E_i(h) \in h^{1+\beta}(\Sigma)^{d\times d}$ and where the coefficient matrices satisfy

$$
\begin{aligned}
&[h \mapsto (C_{ij}(h), C_i(h), D_i(h)), E_i(h)] \\
&\qquad\qquad \in C^{\omega}(\mathcal{U}, h^{1+\beta}(\overline{\Omega})^{d\times d} \times h^{\beta}(\overline{\Omega})^{d\times d} \times h^{1+\beta}(\Sigma)^{d\times d} \times h^{1+\beta}(\Sigma)^{d\times d}) .
\end{aligned}
$$

Applying (4.27) and (4.28) again, the assertion follows from (4.29).
To 2.: Since $[h \mapsto \nabla\Phi_h] \in C^{\omega}(\mathcal{U}, h^{1+\beta}(\Sigma)^d)$, the assertion for F follows immediately.
□

For better clarity we define the following function spaces

$$
\begin{aligned}
X^{2+\beta} &:= h^{2+\beta}(\overline{\Omega^+})^d \times h^{2+\beta}(\overline{\Omega^-})^d , \\
Y^{\beta} &:= h^{\beta}(\overline{\Omega^+})^d \times h^{\beta}(\overline{\Omega^-})^d \times h^{1+\beta}(\Sigma)^d \times h^{2+\beta}(\Sigma)^d \times h^{2+\beta}(\partial\Omega)^d .
\end{aligned}
$$

Then the following lemma holds.

Lemma 4.1.6. *Let $h \in \mathcal{U}$ be given. Then it holds*

$$
(C(h), D(h), [\,.\,]_{\Sigma}, \gamma(\,.\,)) \in \mathrm{Isom}(X^{2+\beta}, Y^{\beta}) . \tag{4.30}
$$

Proof: We can transform the elliptic boundary value problem

$$
\begin{aligned}
C(h)\mathbf{w} &= \mathbf{f} && \text{in } \Omega^{\pm} , && (4.31)\\
D(h)\mathbf{w} &= \mathbf{m} && \text{on } \Sigma , && (4.32)\\
[\mathbf{w}]_{\Sigma} &= \mathbf{g} && \text{on } \Sigma , && (4.33)\\
\mathbf{w} &= \mathbf{k} && \text{on } \partial\Omega && (4.34)
\end{aligned}
$$

by the Hansawa transformation back into the system

$$
\begin{aligned}
\operatorname{div} \mathcal{C}\nabla\mathbf{w}^{\pm} &= \mathbf{f}_h^{\pm} && \text{in } \Omega_h^{\pm} , && (4.35)\\
\nu \cdot \mathcal{C}\left[\nabla\mathbf{w}^{\pm}\right]_{\Gamma_h} &= \mathbf{m}_h && \text{on } \Gamma_h , && (4.36)\\
\left[\mathbf{w}^{\pm}\right]_{\Gamma_h} &= \mathbf{g}_h && \text{on } \Gamma_h , && (4.37)\\
\mathbf{w}^{+} &= \mathbf{k}_h && \text{on } \partial\Omega , && (4.38)
\end{aligned}
$$

where $(\mathbf{f}_h^{\pm}, \mathbf{m}_h, \mathbf{g}_h, \mathbf{k}_h) := (\mathbf{f}^{\pm} \circ \Theta_h^{-1}, \mathbf{m} \circ \Theta_h^{-1}, \mathbf{g} \circ \Theta_h^{-1}, \mathbf{k} \circ \Theta_h^{-1})$. Since $\Theta_h = \mathrm{Id}$ near the boundary $\partial\Omega$, we know $\mathbf{k}_h = \mathbf{k}$ on $\partial\Omega$.
First we show that the boundary value problem has a weak solution $\mathbf{w}_h^{\pm} \in H^1(\Omega_h^{\pm})^d$, that is,

$$
\begin{aligned}
&\int_{\Omega_h^+} \mathcal{C}\nabla\mathbf{w}_h^+ : \nabla\mathbf{u}\,dx + \int_{\Omega_h^-} \mathcal{C}\nabla\mathbf{w}_h^- : \nabla\mathbf{u}\,dx \\
&= \; -\int_{\Omega_h^+} \mathbf{f}_h^+ \cdot \mathbf{u}\,dx - \int_{\Omega_h^-} \mathbf{f}_h^- \cdot \mathbf{u}\,dx - \int_{\Gamma_h} \mathbf{m}_h \cdot \mathbf{u}\,d\mathcal{H}^{d-1}
\end{aligned}
$$

for all $\mathbf{u} \in H_0^1(\Omega)^d$, and $\left[\mathbf{w}_h^\pm\right]_{\Gamma_h} = \mathbf{g}_h$ and $\mathbf{w}_h^+\big|_{\partial\Omega} = \mathbf{k}$.
We choose functions $\widetilde{\mathbf{w}}^\pm \in H^1(\Omega_h^\pm)^d$ such that

$$\left[\widetilde{\mathbf{w}}^\pm\right]_{\Gamma_h} = \mathbf{g}_h \quad \text{on } \Gamma_h \qquad\qquad \text{and} \qquad\qquad \widetilde{\mathbf{w}}^+ = \mathbf{k} \quad \text{on } \partial\Omega\,.$$

Then we define a bilinear functional $a(.,.) : H_0^1(\Omega)^d \times H_0^1(\Omega)^d \to \mathbb{R}$ by

$$a(\mathbf{v},\mathbf{w}) := \int_\Omega \mathcal{C}\nabla\mathbf{v} : \nabla\mathbf{w}\,dx \quad \forall \mathbf{v},\mathbf{w} \in H_0^1(\Omega)^d\,.$$

Due to the positive definiteness of $\mathcal{C}$ (see (2.3)) and Korn's inequality, $a(.,.)$ is a coercive continuous bilinear form. Next we define $F \in H^{-1}(\Omega)^d$ by

$$\begin{aligned}\langle F,\mathbf{u}\rangle_{H^{-1},H_0^1} \;:=\; & -\int_{\Omega_h^+} \mathbf{f}_h^+ \cdot \mathbf{u}\,dx - \int_{\Omega_h^-} \mathbf{f}_h^- \cdot \mathbf{u}\,dx - \int_{\Gamma_h} \mathbf{m}_h \cdot \mathbf{u}\,d\mathcal{H}^{d-1} \\ & -\int_{\Omega_h^+} \mathcal{C}\nabla\widetilde{\mathbf{w}}^+ : \nabla\mathbf{u}\,dx - \int_{\Omega_h^-} \mathcal{C}\nabla\widetilde{\mathbf{w}}^- : \nabla\mathbf{u}\,dx\,.\end{aligned}$$

The Lax-Milgram theorem gives us a unique solution $\mathbf{v} \in H_0^1(\Omega)^d$ to the problem

$$a(\mathbf{v},\mathbf{w}) = \langle F,\mathbf{w}\rangle \quad \forall \mathbf{w} \in H_0^1(\Omega)^d\,.$$

It is not difficult to verify that $\mathbf{w}_h^\pm := \mathbf{v} + \widetilde{\mathbf{w}}^\pm \in H^1(\Omega_h^\pm)^d$ is a solution to

$$\begin{aligned}&\int_{\Omega_h^+} \mathcal{C}\nabla\mathbf{w}_h^+ : \nabla\mathbf{u}\,dx + \int_{\Omega_h^-} \mathcal{C}\nabla\mathbf{w}_h^- : \nabla\mathbf{u}\,dx \\ &\quad = -\int_{\Omega_h^+} \mathbf{f}_h^+ \cdot \mathbf{u}\,dx - \int_{\Omega_h^-} \mathbf{f}_h^- \cdot \mathbf{u}\,dx - \int_{\Gamma_h} \mathbf{m}_h \cdot \mathbf{u}\,d\mathcal{H}^{d-1}\end{aligned}$$

for all $\mathbf{u} \in H_0^1(\Omega)^d$. In addition, it holds

$$\left[\mathbf{w}_h^\pm\right]_{\Gamma_h} = \mathbf{g}_h \quad \text{on } \Gamma_h \qquad\qquad \text{and} \qquad\qquad \mathbf{w}_h^+ = \mathbf{k} \quad \text{on } \partial\Omega\,.$$

Since $h \in C^2(\Sigma)$ and therefore Γ_h is C^2, we obtain by [53, Theorem 4.20]

$$\mathbf{w}_h^\pm \in H^2(\Omega_h^\pm)^d\,.$$

Since $\Theta_h \in C^{2+\beta}(\overline{\Omega})^d$, it also follows

$$\mathbf{w}^\pm := \mathbf{w}_h^\pm \circ \Theta_h \in H^2(\Omega^\pm)^d\,.$$

Then an easy calculation gives us that $\mathbf{w} = (\mathbf{w}^+, \mathbf{w}^-)$ is the unique solution to (4.31)-(4.34).
It remains to show that $\mathbf{w}^\pm \in h^{2+\beta}(\overline{\Omega^\pm})^d$ for $(\mathbf{f}^\pm, \mathbf{m}, \mathbf{g}, \mathbf{k}) \in h^\beta(\overline{\Omega^\pm})^d \times h^{1+\beta}(\Sigma)^d \times h^{2+\beta}(\Sigma)^d \times h^{2+\beta}(\partial\Omega)^d$.
For the solution $\mathbf{w}_h^\pm$ to the transformed system (4.35)-(4.38), it directly follows

from Theorem 5.21 in [38] that $\mathbf{w}_h^\pm \in C^{2+\beta}(\Omega_h^\pm)^d$. Since $\Theta_h \in C^{2+\beta}$, it also holds $\mathbf{w}^\pm = \mathbf{w}_h^\pm \circ \Theta_h \in C^{2+\beta}(\Omega^\pm)^d$.
To get boundary regularity we apply some results for parabolic boundary value problems in [25]. For that we extend the elliptic problem to a parabolic problem. We assume that there exists open sets $G_1 \subsetneq G_2 \subset \Omega$ such that $\Sigma \cap G_2 \subset \{x \in \mathbb{R}^d : x_d = 0\}$ and $\Omega^+ \cap G_2 \subset \{x \in \mathbb{R}^d : x_d \geq 0\}$. Otherwise we replace the reference manifold Σ, G_1, and G_2 by $\tilde{\Sigma}$, $\tilde{G}_1$, and $\tilde{G}_2$ such that $\tilde{\Sigma} \cap \tilde{G}_2$ is a hyperplane, that is, $\tilde{\Sigma} \cap \tilde{G}_2 \subset \left\{x \in \mathbb{R}^d : \sum_{i=1}^d a_1 x_i = c\right\}$ for some $a_i, c \in \mathbb{R}$, $i = 1, \dots, d$, and such that the new height function $\tilde{h} : \tilde{\Sigma} \to \mathbb{R}$ also satisfies $\|\tilde{h}\|_{C^1} \leq a$ for some sufficiently small $a > 0$ (see (4.9)). We fix two cut-off functions $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi = 0$ in $\mathbb{R}\backslash(0,T)$ for some $T > 0$ and $\varphi \not\equiv 0$ and $\lambda \in C_0^\infty(\mathbb{R}^d)$ such that $\lambda = 1$ in G_1 and $\lambda = 0$ in $\mathbb{R}^d\backslash G_2$. For $\mathbf{v}^\pm(x,t) := \varphi(t)\lambda(x)\mathbf{w}^\pm(x) \in C_0^\infty(0,\infty; H^2(\Omega^\pm)^d)$, it holds

$$\begin{aligned}
C^\pm(h)\mathbf{v}^\pm &= \varphi\lambda C^\pm(h)\mathbf{w}^\pm + \left(C^\pm(h)\mathbf{v}^\pm - \varphi\lambda C^\pm(h)\mathbf{w}^\pm\right) \\
&= \varphi\lambda\mathbf{f}^\pm + \left(C^\pm(h)\mathbf{v}^\pm - \varphi\lambda C^\pm(h)\mathbf{w}^\pm\right) && \text{in } \Omega^\pm \times (0,\infty)\,, \\
D(h)\mathbf{v} &= \varphi\lambda D(h)\mathbf{w} + (D(h)\mathbf{v} - \varphi\lambda D(h)\mathbf{w}) \\
&= \varphi\lambda\mathbf{m} + (D(h)\mathbf{v} - \varphi\lambda D(h)\mathbf{w}) && \text{on } \Sigma \times (0,\infty)\,.
\end{aligned}$$

Therefore $\mathbf{v}^\pm$ is a solution to the parabolic boundary value problem

$$\begin{aligned}
\partial_t \mathbf{v}^\pm + C^\pm(h)\mathbf{v}^\pm &= \mathbf{f}_1^\pm && \text{in } \Omega^\pm \times (0,\infty)\,, \\
D(h)\mathbf{v} &= \mathbf{m}_1 && \text{on } \Sigma \times (0,\infty)\,, \\
\left[\mathbf{v}^\pm\right]_\Sigma &= \mathbf{g}_1 && \text{on } \Sigma \times (0,\infty)\,, \\
\mathbf{v}^+ &= 0 && \text{on } \partial\Omega \times (0,\infty)\,, \\
\mathbf{v}^\pm\big|_{t=0} &= 0 && \text{in } \Omega\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{f}_1^\pm &= \varphi'\lambda\mathbf{w}^\pm + \varphi\lambda\mathbf{f}^\pm + \left(C^\pm(h)(\varphi\lambda\mathbf{w}^\pm) - \varphi\lambda C^\pm(h)\mathbf{w}^\pm\right), \\
\mathbf{m}_1 &= \varphi\lambda\mathbf{m} + (D(h)(\varphi\lambda\mathbf{w}) - \varphi\lambda D(h)\mathbf{w})\,, \\
\mathbf{g}_1 &= \varphi\lambda\mathbf{g}\,.
\end{aligned}$$

Note that the operators $C^\pm(h)$ and $D(h)$ are time independent since we consider a time independent height function h. Define $\mathbf{u} \in C_0^\infty(0,\infty; H^2(\mathbb{R}_+^d)^{2d})$ by

$$\mathbf{u}(x', x_d, t) = \begin{pmatrix} \mathbf{v}^+(x', x_d, t) \\ \mathbf{v}^-(x', -x_d, t) \end{pmatrix} \quad \forall x' \in \mathbb{R}^{d-1}, \forall x_d \geq 0\,,$$

where we extend $\mathbf{v}^\pm$ to the half space $\mathbb{R}_\pm^d$ by zero. Then by definition of λ, $\mathbf{u}$ is a solution to the parabolic boundary problem

$$\partial_t \mathbf{u} + L\mathbf{u} = \mathbf{f}_2 \quad \text{in } \left\{x \in \mathbb{R}^d : x_d > 0\right\} \times (0,\infty)\,, \tag{4.39}$$

$$B\mathbf{u} = \mathbf{m}_2 \quad \text{on } \left\{x \in \mathbb{R}^d : x_d = 0\right\} \times (0,\infty)\,, \tag{4.40}$$

$$\mathbf{u}|_{t=0} = 0 \quad \text{in } \mathbb{R}_+^d\,, \tag{4.41}$$

where

$$L = \begin{pmatrix} C^+(h) & 0 \\ 0 & C_d^-(h) \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ D^+(h) & -D_d^-(h) \end{pmatrix},$$

$$\mathbf{f}_2(x', x_d, t) = \begin{pmatrix} \mathbf{f}_1^+(x', x_d, t) \\ \mathbf{f}_1^-(x', -x_d, t) \end{pmatrix}, \quad \mathbf{m}_2(x', t) = \begin{pmatrix} \mathbf{g}_1(x', t) \\ \mathbf{m}_1(x', t) \end{pmatrix},$$

where $C_d^-(h)$ is defined as $C^-(h)$ except ∂_{x_d} is replaced by $-\partial_{x_d}$. $D_d^-(h)$ is defined analogously. Note that L is strongly elliptic for $h = 0$ since $\mathcal{C}$ is positive definite. For $h \neq 0$ the principal part of L stays strongly elliptic since the Hansawa transformation is a diffeomorphism. One can find a detailed explanation at the end of the proof. Due to (4.29), we can apply Theorem VI.21. in [25]. We verify the complementing conditions at the end of the proof. Then we get the estimate

$$\begin{aligned} \|\mathbf{u}\|_{C^{1+\beta/2,2+\beta}(\Omega_T^+)} \leq \; & C \Big(\left\|(\mathbf{f}_1^+, \mathbf{f}_1^-)\right\|_{C^{\beta/2,\beta}(\Omega_T^+ \times \Omega_T^-)} \\ & + \|\mathbf{g}_1\|_{C^{1+\beta/2,2+\beta}(\Sigma_T)} + \|\mathbf{m}_1\|_{C^{(1+\beta)/2,1+\beta}(\Sigma_T)} \Big) \end{aligned}$$

for some $C > 0$ independent of $\mathbf{f}_1^\pm$, $\mathbf{g}_1$, and $\mathbf{m}_1$. Here we write $\Omega^\pm$ instead of $\mathbb{R}^d_\pm$ since all occurring functions have support in $\Omega^\pm$. An easy calculation gives us

$$\begin{aligned} &\left\|C^\pm(h)(\varphi\lambda\mathbf{w}^\pm) - \varphi\lambda C^\pm(h)\mathbf{w}^\pm\right\|_{C^\beta(\Omega^\pm)} \leq C \left\|\mathbf{w}^\pm\right\|_{C^{1+\beta}(\Omega^\pm)}, \\ &\|D(h)(\varphi\lambda\mathbf{w}) - \varphi\lambda D(h)\mathbf{w}\|_{C^{1+\beta}(\Sigma)} \leq C \left(\left\|\mathbf{w}^+\right\|_{C^{1+\beta}(\Omega^+)} + \left\|\mathbf{w}^-\right\|_{C^{1+\beta}(\Omega^-)} \right) \end{aligned}$$

for some constant $C = C(\varphi, \lambda, \Theta_h)$. Hence we get by definition of $\mathbf{f}_1^\pm$, $\mathbf{g}_1$, and $\mathbf{m}_1$

$$\begin{aligned} \|\mathbf{u}\|_{C^{1+\beta/2,2+\beta}(\Omega_T^+)} \leq \; & C \Big(\left\|\mathbf{f}^+\right\|_{C^\beta(\Omega^+)} + \left\|\mathbf{f}^-\right\|_{C^\beta(\Omega^-)} + \left\|\mathbf{w}^+\right\|_{C^{1+\beta}(\Omega^+)} \\ & + \left\|\mathbf{w}^-\right\|_{C^{1+\beta}(\Omega^-)} + \|\mathbf{g}\|_{C^{2+\beta}(\Sigma)} + \|\mathbf{m}\|_{C^{1+\beta}(\Sigma)} \Big) \end{aligned}$$

for some constant $C = C(\varphi, \lambda, \Theta_h)$. Due to $(C(\overline{\Omega}), C^2(\overline{\Omega}))_{\frac{1+\beta}{2},\infty} = C^{1+\beta}(\overline{\Omega})$ (see [52, 1.4.3 Exercises]), the continuous imbedding $W^1_p(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p > d$, and the Gagliardo-Nirenberg inequality (see Section 2.4), it holds

$$\left\|\mathbf{w}^\pm\right\|_{C^{1+\beta}(\Omega^\pm)} \leq C \left\|\mathbf{w}^\pm\right\|^{\frac{1-\beta}{2}}_{W^1_p(\Omega^\pm)} \left\|\mathbf{w}^\pm\right\|^{\frac{1+\beta}{2}}_{C^2(\Omega^\pm)} \leq C \left\|\mathbf{w}^\pm\right\|^{\gamma}_{H^1(\Omega^\pm)} \left\|\mathbf{w}^\pm\right\|^{1-\gamma}_{C^2(\Omega^\pm)}$$

for some $\gamma \in (0, 1)$. Therefore by definition of $\mathbf{u}$, we obtain

$$\begin{aligned} \left\|\mathbf{w}^\pm\right\|_{C^{2+\beta}(G_1^\pm)} \leq \; & C \Big(\left\|\mathbf{f}^+\right\|_{C^\beta(\Omega^+)} + \left\|\mathbf{f}^-\right\|_{C^\beta(\Omega^-)} + \left\|\mathbf{w}^+\right\|^{\gamma}_{H^1(\Omega^+)} \left\|\mathbf{w}^+\right\|^{1-\gamma}_{C^2(\Omega^+)} \\ & + \left\|\mathbf{w}^-\right\|^{\gamma}_{H^1(\Omega^-)} \left\|\mathbf{w}^-\right\|^{1-\gamma}_{C^2(\Omega^-)} + \|\mathbf{g}\|_{C^{2+\beta}(\Sigma)} + \|\mathbf{m}\|_{C^{1+\beta}(\Sigma)} \Big), \end{aligned}$$

where $G_1^\pm = G_1 \cap \Omega^\pm$. Note that, if we have to replace Σ, G_1, and G_2 by $\tilde{\Sigma}$, $\tilde{G}_1$, and $\tilde{G}_2$ as described above, then this estimate is also valid for $G_1^\pm$. The reason for this is as follows. We can choose a smooth height function $\tilde{h} : \tilde{\Sigma} \to \mathbb{R}$ such that $\Theta_{\tilde{h}}(\tilde{\Sigma}) = \Sigma$. W.l.o.g. we also assume $\Theta_{\tilde{h}}(\tilde{G}_1) = G_1$. Then it holds for every smooth function $f : \Omega \to \mathbb{R}$ and every $\theta \geq 0$

$$\left\| f \circ \Theta_{\tilde{h}}^{-1} \right\|_{C^\theta(G_1)} \leq \tilde{C} \left\| f \right\|_{C^\theta(\tilde{G}_1)} \leq C \left\| f \circ \Theta_{\tilde{h}}^{-1} \right\|_{C^\theta(G_1)}$$

for some constants $\tilde{C} = \tilde{C}(\theta, \tilde{h}, G_1) > 0$ and $C = C(\theta, \tilde{h}, G_1) > 0$.
On the boundary $\partial\Omega$ we get analogous estimates by [25, Theorem VI.21.]. Then by partition of the unity and transformation, we obtain

$$\begin{aligned} \left\| \mathbf{w}^\pm \right\|_{C^{2+\beta}(\Omega^\pm)} \leq\; & C \Big(\left\| \mathbf{f}^+ \right\|_{C^\beta(\Omega^+)} + \left\| \mathbf{f}^- \right\|_{C^\beta(\Omega^-)} + \left\| \mathbf{w}^+ \right\|_{H^1(\Omega^+)} + \left\| \mathbf{w}^- \right\|_{H^1(\Omega^-)} \\ & + \left\| \mathbf{g} \right\|_{C^{2+\beta}(\Sigma)} + \left\| \mathbf{m} \right\|_{C^{1+\beta}(\Sigma)} + \left\| \mathbf{k} \right\|_{C^{2+\beta}(\partial\Omega)} \Big), \end{aligned}$$

where we have used Young's inequality

$$\left\| \mathbf{w}^\pm \right\|_{H^1(\Omega^\pm)}^\gamma \left\| \mathbf{w}^\pm \right\|_{C^2(\Omega^\pm)}^{1-\gamma} \leq C(\epsilon) \left\| \mathbf{w}^\pm \right\|_{H^1(\Omega^\pm)} + \epsilon \left\| \mathbf{w}^\pm \right\|_{C^2(\Omega^\pm)}$$

for any $\epsilon > 0$. We refer to Theorem 4.20 and Theorem 4.16 in [53] or see the calculation above to estimate $\|\mathbf{w}^\pm\|_{H^1(\Omega^\pm)} \leq C(\left\| \mathbf{f}^\pm \right\|_{L^2(\Omega^\pm)} + \|\mathbf{g}\|_{H^{1/2}(\Sigma)} + \|\mathbf{m}\|_{H^{1/2}(\Sigma)} + \|\mathbf{k}\|_{H^{1/2}(\partial\Omega)})$. Hence we obtain

$$\begin{aligned} \left\| \mathbf{w}^\pm \right\|_{C^{2+\beta}(\Omega^\pm)} \leq\; & C \Big(\left\| \mathbf{f}^+ \right\|_{C^\beta(\Omega^+)} + \left\| \mathbf{f}^- \right\|_{C^\beta(\Omega^-)} + \left\| \mathbf{g} \right\|_{C^{2+\beta}(\Sigma)} \\ & + \left\| \mathbf{m} \right\|_{C^{1+\beta}(\Sigma)} + \left\| \mathbf{k} \right\|_{C^{2+\beta}(\partial\Omega)} \Big), \end{aligned} \tag{4.42}$$

where C is independent of $\mathbf{f}^\pm$, $\mathbf{g}$, $\mathbf{m}$, and $\mathbf{k}$. In particular, we conclude

$$\mathbf{w}^\pm \in C^{2+\beta}(\overline{\Omega^\pm})^d .$$

It remains to show that $\mathbf{w}^\pm \in h^{2+\beta}(\overline{\Omega^\pm})^d$. For the proof we use an approximation argument. Since $h^\theta(\overline{\Omega})$ is the closure of $C^k(\overline{\Omega})$ in $C^\theta(\overline{\Omega})$ for every $k \in (\theta, \infty)$, see Section 2.3, there exists a sequence $\left(\mathbf{f}_n^\pm, \mathbf{m}_n, \mathbf{g}_n, \mathbf{k}_n\right)_{n \in \mathbb{N}} \subset C^\gamma(\overline{\Omega^\pm})^d \times C^{1+\gamma}(\Sigma)^d \times C^{2+\gamma}(\Sigma)^d \times C^{2+\gamma}(\partial\Omega)^d$ for $\gamma \in (\beta, 1)$, such that

$$\begin{aligned} &\mathbf{f}_n^\pm \to \mathbf{f}^\pm \text{ in } C^\beta(\overline{\Omega^\pm})^d , && \mathbf{m}_n \to \mathbf{m} \text{ in } C^{1+\beta}(\Sigma)^d , \\ &\mathbf{g}_n \to \mathbf{g} \text{ in } C^{2+\beta}(\Sigma)^d , && \mathbf{k}_n \to \mathbf{k} \text{ in } C^{2+\beta}(\partial\Omega)^d , \end{aligned}$$

as $n \to \infty$. First assume that we have more regularity for h such that $h \in C^{1+\gamma}(\Sigma)$, that is, $C_{ij} \in C^{1+\gamma}(\overline{\Omega})^{d \times d}$, $C_i \in C^\gamma(\overline{\Omega})^{d \times d}$, and $D_i \in C^{1+\gamma}(\Sigma)^{d \times d}$ in (4.29). Let $\mathbf{w}_n^\pm$ be the solution to (4.31)-(4.34) with right-hand side $(\mathbf{f}_n^\pm, \mathbf{m}_n, \mathbf{g}_n, \mathbf{k}_n)$ instead of $(\mathbf{f}^\pm, \mathbf{m}, \mathbf{g}, \mathbf{k})$. Then the same arguments as above yield $\mathbf{w}_n^\pm \in C^{2+\gamma}(\overline{\Omega^\pm})^d$ for all $n \in \mathbb{N}$, and by (4.42) we obtain

$$\begin{aligned} \left\| \mathbf{w}^\pm - \mathbf{w}_n^\pm \right\|_{C^{2+\beta}(\Omega^\pm)} \leq\; & C \Big(\left\| \mathbf{f}^+ - \mathbf{f}_n^+ \right\|_{C^\beta(\Omega^+)} + \left\| \mathbf{f}^- - \mathbf{f}_n^- \right\|_{C^\beta(\Omega^-)} \\ & + \left\| \mathbf{g} - \mathbf{g}_n \right\|_{C^{2+\beta}(\Sigma)} + \left\| \mathbf{m} - \mathbf{m}_n \right\|_{C^{1+\beta}(\Sigma)} \\ & \left\| \mathbf{k} - \mathbf{k}_n \right\|_{C^{2+\beta}(\partial\Omega)} \Big) \to 0 , \end{aligned}$$

as $n \to \infty$ since $\mathbf{w} - \mathbf{w}_n$ is the solution to (4.31)-(4.34) with right-hand side $(\mathbf{f} - \mathbf{f}_n, \mathbf{m} - \mathbf{m}_n, \mathbf{g} - \mathbf{g}_n, \mathbf{k} - \mathbf{k}_n)$. Therefore we have

$$\mathbf{w}^{\pm} \in h^{2+\beta}(\overline{\Omega^{\pm}})^d$$

provided $C_{ij} \in C^{1+\gamma}(\overline{\Omega})^{d\times d}$, $C_i \in C^{\gamma}(\overline{\Omega})^{d\times d}$, and $D_i \in C^{1+\gamma}(\Sigma)^{d\times d}$.
Now consider the original case, that is, $C_{ij} \in h^{1+\beta}(\overline{\Omega})^{d\times d}$, $C_i \in h^{\beta}(\overline{\Omega})^{d\times d}$, and $D_i \in h^{1+\beta}(\Sigma)^{d\times d}$ in (4.29), then there exists a sequence $\left(C_{ij}^n, C_i^n, D_i^n\right)_{n\in\mathbb{N}} \subset C^{1+\gamma}(\overline{\Omega})^{d\times d} \times C^{\gamma}(\overline{\Omega})^{d\times d} \times C^{1+\gamma}(\overline{\Sigma})^{d\times d}$ such that

$$\begin{aligned} &C_{ij}^n \to C_{ij} \text{ in } C^{1+\beta}(\overline{\Omega})^{d\times d}, \quad C_i^n \to C_i \text{ in } C^{\beta}(\overline{\Omega})^{d\times d}, \\ &D_i^n \to D_i \text{ in } C^{1+\beta}(\Sigma)^{d\times d}, \end{aligned} \tag{4.43}$$

as $n \to \infty$. Let $\mathbf{w}_n^{\pm}$ be the solutions to (4.31)-(4.34) where (C_{ij}, C_i, D_i) are replaced by $\left(C_{ij}^n, C_i^n, D_i^n\right)$. As above we can show that $\mathbf{w}_n^{\pm} \in h^{2+\beta}(\overline{\Omega^{\pm}})^d$ and

$$\begin{aligned} \left\|\mathbf{w}_n^{\pm}\right\|_{C^{2+\beta}(\Omega^{\pm})} \leq\ & C\Big(\left\|\mathbf{f}^{+}\right\|_{C^{\beta}(\Omega^{+})} + \left\|\mathbf{f}^{-}\right\|_{C^{\beta}(\Omega^{-})} + \|\mathbf{g}\|_{C^{2+\beta}(\Sigma)} \\ & + \|\mathbf{m}\|_{C^{1+\beta}(\Sigma)} + \|\mathbf{k}\|_{C^{2+\beta}(\partial\Omega)}\Big) \end{aligned} \tag{4.44}$$

for some $C = C(\left\|C_{ij}^n\right\|_{C^{1+\beta}(\Omega)}, \|C_i^n\|_{C^{\beta}(\Omega)}, \|D_i^n\|_{C^{1+\beta}(\Sigma)}) > 0$. Due to the convergence properties (4.43), we can choose the constant C in (4.44) independent of $n \in \mathbb{N}$. Observe that $\mathbf{o}^{\pm} = \mathbf{w}^{\pm} - \mathbf{w}_n^{\pm}$ is the solution to the boundary value problem

$$\begin{aligned} C^{\pm}(h)\mathbf{o}^{\pm} &= (C_{ij}^n - C_{ij})\partial_{ij}\mathbf{w}_n^{\pm} + (C_i^n - C_i)\partial_i\mathbf{w}_n^{\pm} && \text{in } \Omega^{\pm}, \\ D(h)\mathbf{o} &= (D_i^n - D_i)\left[\gamma^{\pm}(\partial_i\mathbf{w}_n^{\pm})\right]_{\Sigma} && \text{on } \Sigma, \\ \left[\mathbf{o}^{\pm}\right]_{\Sigma} &= 0 && \text{on } \Sigma, \\ \mathbf{o}^{+} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Therefore it holds

$$\begin{aligned} \left\|\mathbf{w}^{\pm} - \mathbf{w}_n^{\pm}\right\|_{C^{2+\beta}(\Omega^{\pm})} \leq\ & C\Big(\left\|(C_{ij}^n - C_{ij})\partial_{ij}\mathbf{w}_n^{+} + (C_i^n - C_i)\partial_i\mathbf{w}_n^{+}\right\|_{C^{\beta}(\Omega^{+})} \\ & + \left\|(C_{ij}^n - C_{ij})\partial_{ij}\mathbf{w}_n^{-} + (C_i^n - C_i)\partial_i\mathbf{w}_n^{-}\right\|_{C^{\beta}(\Omega^{-})} \\ & + \left\|(D_i^n - D_i)\left[\gamma^{\pm}(\partial_i\mathbf{w}_n^{\pm})\right]\right\|_{C^{1+\beta}(\Sigma)}\Big), \end{aligned}$$

where C is the same constant as in (4.42), in particular independent of $n \in \mathbb{N}$. We use (4.44) and (4.43) to obtain

$$\left\|\mathbf{w}^{\pm} - \mathbf{w}_n^{\pm}\right\|_{C^{2+\beta}(\Omega^{\pm})} \longrightarrow 0 \quad \text{as } n \to \infty,$$

and in particular, it holds

$$\mathbf{w}^{\pm} \in h^{2+\beta}(\overline{\Omega^{\pm}})^d$$

since $h^{2+\beta}(\overline{\Omega^\pm})$ is a closed subspace in $C^{2+\beta}(\overline{\Omega^\pm})$.
In the following we show that the operator L also is strongly elliptic for $h \neq 0$. For a function $\mathbf{u} \in C^1(\overline{\Omega^+})^d \cap C^2(\Omega^+)^d$, it holds for the principal part $(C^+(h))^p$ of $C^+(h)$

$$(C^+(h))^p \mathbf{u} = \partial_k \left(\mathcal{C}_{ii'} \partial_i \Theta_{h,k}^{-1} \circ \Theta_h \partial_{i'} \Theta_{h,l}^{-1} \circ \Theta_h \right) \partial_l \mathbf{u} \,,$$

where $\mathcal{C}_{ii'} = (\mathcal{C}_{iji'j'})_{jj'=1}^d$. Thus for all $\xi, \eta \in \mathbb{R}^d$, it follows

$$\mathcal{C}_{iji'j'} \partial_i \Theta_{h,k}^{-1} \circ \Theta_h \partial_{i'} \Theta_{h,l}^{-1} \circ \Theta_h \xi_k \xi_l \eta_j \eta_{j'} \geq c_2 \left| D\Theta_h^{-T} \circ \Theta_h \, \xi \right|^2 |\eta|^2 \geq C \, |\xi|^2 \, |\eta|^2$$

for some $C = C(h) > 0$ and where we have used that Θ_h is a diffeomorphism. For $(C_d^-(h))^p$ an analogous result is valid.
Finally, we show that the parabolic system (4.39)-(4.41) satisfies the complementing condition. Denote by L^p, B^p, $(C^+(h))^p$, and $(C_d^-(h))^p$ the principal parts of the operators L, B, $C^+(h)$, and $C_d^-(h)$. We have to verify that the system

$$L^p \mathbf{w} = 0 \quad \text{in } \left\{ x \in \mathbb{R}^d : x_d > 0 \right\} , \tag{4.45}$$

$$B^p \mathbf{w} = 0 \quad \text{on } \left\{ x \in \mathbb{R}^d : x_d = 0 \right\} \tag{4.46}$$

has no solution of the form

$$\mathbf{w}(x) = e^{i\xi' \cdot (x' - x_0')} (\mathbf{u}(x_d), \mathbf{v}(x_d)) \,, \tag{4.47}$$

where $x = (x', x_d)$, $\xi' \in \mathbb{R}^{d-1}$ is nonzero, $x_0' \in \mathbb{R}^{d-1}$ is a fixed arbitrary vector, and $\mathbf{u}, \mathbf{v} : \mathbb{R} \to \mathbb{C}^d$ are arbitrary functions such that $(\mathbf{u}(x_d), \mathbf{v}(x_d))$ tends to 0 exponentially as $x_d \to \infty$. In the following we set $x_0' = 0$. For $x_0' \neq 0$ we only have to replace $e^{i\xi' \cdot x'}$ by $e^{i\xi' \cdot (x' - x_0')}$ in the calculation below. We can assume that in (4.1) the cut-off function $\chi \equiv 1$ in G_2. Since for the solution $\mathbf{u}$ to (4.39)-(4.41), it holds $\operatorname{supp} \mathbf{u} \subset G_2 \times (0, \infty)$, we can use the Hansawa transformation

$$\Theta_h(x) = x - h(x') e_d \,, \quad \Theta_h^{-1}(x) = x + h(x') e_d \quad \forall x \in \mathbb{R}^d \,,$$

where $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$. Therefore it holds

$$D\Theta_h^{-1}(x) = (\delta_{ij} + \delta_{id} \partial_j h(x'))_{ij=1}^d \,.$$

Hence we obtain for the principal part of $C^+(h)$

$$\begin{aligned} (C^+(h))^p \mathbf{w} &= \left(\mathcal{C}_{jklm} \partial_k ((\partial_i \mathbf{w}_l) \circ \Theta_h^{-1}) \partial_m \Theta_{h,i}^{-1} \right)_{j=1,\ldots,d} \circ \Theta_h \\ &= (\mathcal{C}_{jklm} (\partial_{km} \mathbf{w}_l + \partial_{kd} \mathbf{w}_l \partial_m h + (\partial_{dm} \mathbf{w}_l + \partial_{dd} \mathbf{w}_l \partial_m h) \partial_k h))_{j=1,\ldots,d} \end{aligned} \tag{4.48}$$

for $\mathbf{w} \in C^2(\mathbb{R}^d, \mathbb{R}^d)$. We have an analogous result for $(C_d^-(h))^p \mathbf{w}$ where h is replaced by $-h$ and ∂_k and ∂_m are replaced by $-\partial_k$ and $-\partial_m$ when $k, m = d$.
Since $\Gamma_h \subset G_2$ is given by $\{(x', -h(x')) : (x', 0) \in G_2\}$, the unit outer normal has the form

$$\nu_{\Gamma_h}(x') = \frac{1}{\left| \begin{pmatrix} \nabla_{x'} h(x') \\ 1 \end{pmatrix} \right|} \begin{pmatrix} \nabla_{x'} h(x') \\ 1 \end{pmatrix}.$$

Thus the operator $D^+(h)\mathbf{w}$ can be written as

$$D^+(h)\mathbf{w}(x)$$
$$= \frac{1}{|(\nabla_{x'}h(x'),1)|}\left(\mathcal{C}_{jklm}(\partial_m \mathbf{w}_l + \partial_d \mathbf{w}_l \partial_m h)\partial_k h + \mathcal{C}_{jdlm}(\partial_m \mathbf{w}_l + \partial_d \mathbf{w}_l \partial_m h)\right)_{j=1,\ldots,d}(x').$$

For the operator $D^-(h)\mathbf{w}$, it follows

$$D^-(h)\mathbf{w}(x)$$
$$= \frac{1}{|(\nabla_{x'}h(x'),1)|}\left(\mathcal{C}_{jklm}(\partial_m \mathbf{w}_l - \partial_d \mathbf{w}_l \partial_m h)\partial_k h + \mathcal{C}_{jdlm}(\partial_m \mathbf{w}_l - \partial_d \mathbf{w}_l \partial_m h)\right)_{j=1,\ldots,d}(x'),$$

where we replace ∂_m by $-\partial_m$ when $m = d$. Note that $(D^+(h))^p = D^+(h)$ and $(D^-_d(h))^p = D^-_d(h)$. Assume that we have a solution to (4.45) and (4.46) of the form (4.47). We multiply (4.45) by $\overline{\mathbf{w}}$ and integrate over $(0,\infty)$ with respect to x_d. Since $(\mathbf{u}(x_d), \mathbf{v}(x_d))$ tends to 0 exponentially, the following integrals are well-defined. Then we obtain

$$\begin{aligned} 0 &= \int_0^\infty \left(L^p(e^{i\xi'\cdot x'}(\mathbf{u},\mathbf{v}))\right) e^{-i\xi'\cdot x'}(\overline{\mathbf{u}},\overline{\mathbf{v}})\,dx_d \\ &= \int_0^\infty \left((C^+(h))^p(e^{i\xi'\cdot x'}\mathbf{u})\right) e^{-i\xi'\cdot x'}\overline{\mathbf{u}} + \left((C^-_d(h))^p(e^{i\xi'\cdot x'}\mathbf{v})\right) e^{-i\xi'\cdot x'}\overline{\mathbf{v}}\,dx_d, \end{aligned} \tag{4.49}$$

where $(\overline{\mathbf{u}},\overline{\mathbf{v}})$ is the complex conjugate of $(\mathbf{u},\mathbf{v})$. We use the relation (4.48) and apply integration by parts

$$\begin{aligned} &\int_0^\infty \left((C^+(h))^p(e^{i\xi'\cdot x'}\mathbf{u})\right) e^{-i\xi'\cdot x'}\overline{\mathbf{u}}\,dx_d \\ &= \int_0^\infty \sum_{k=1}^{d-1}\sum_{j,l,m=1}^{d} \mathcal{C}_{jklm}\left(\partial_{km}(e^{i\xi'\cdot x'}\mathbf{u}_l) + \partial_{kd}(e^{i\xi'\cdot x'}\mathbf{u}_l)\partial_m h\right) e^{-i\xi'\cdot x'}\overline{\mathbf{u}}_j\,dx_d \\ &\quad -\int_0^\infty \sum_{j,l,m=1}^{d} \mathcal{C}_{jdlm}\left(\partial_m(e^{i\xi'\cdot x'}\mathbf{u}_l) + \partial_d(e^{i\xi'\cdot x'}\mathbf{u}_l)\partial_m h\right)\partial_d(e^{-i\xi'\cdot x'}\overline{\mathbf{u}}_j)\,dx_d \\ &\quad -\int_0^\infty \sum_{j,k,l,m=1}^{d} \mathcal{C}_{jklm}\left(\partial_m(e^{i\xi'\cdot x'}\mathbf{u}_l) + \partial_d(e^{i\xi'\cdot x'}\mathbf{u}_l)\partial_m h\right)\partial_k h\,\partial_d(e^{-i\xi'\cdot x'}\overline{\mathbf{u}}_j)\,dx_d \\ &\quad -|(\nabla_{x'}h(x'),1)|\left(D^+(h)(e^{i\xi'\cdot x'}\mathbf{u}(0))\right) e^{-i\xi'\cdot x'}\overline{\mathbf{u}}(0)\,, \end{aligned} \tag{4.50}$$

where we have used $\lim_{x_d\to\infty}\mathbf{u}(x_d) = 0$. By direct calculation we obtain for the first integrand on the right-hand side of (4.50)

$$\begin{aligned} &\sum_{k=1}^{d-1}\sum_{j,l,m=1}^{d} \mathcal{C}_{jklm}\left(\partial_{km}(e^{i\xi'\cdot x'}\mathbf{u}_l) + \partial_{kd}(e^{i\xi'\cdot x'}\mathbf{u}_l)\partial_m h\right) e^{-i\xi'\cdot x'}\overline{\mathbf{u}}_j \\ &\quad = -(\overline{\mathbf{u}}\otimes(\xi',0)) : \mathcal{C}(\mathbf{u}\otimes(\xi',0)) + i(\overline{\mathbf{u}}\otimes(\xi',0)) : \mathcal{C}(\mathbf{u}'\otimes e_d) \\ &\qquad +i(\overline{\mathbf{u}}\otimes(\xi',0)) : \mathcal{C}(\mathbf{u}'\otimes\nabla h)\,, \end{aligned} \tag{4.51}$$

for the second integrand on the right-hand side

$$
\begin{aligned}
-\sum_{j,l,m=1}^{d} & \mathcal{C}_{jdlm}\left(\partial_m(e^{i\xi'\cdot x'}\mathbf{u}_l)+\partial_d(e^{i\xi'\cdot x'}\mathbf{u}_l)\partial_m h\right)\partial_d(e^{-i\xi'\cdot x'}\overline{\mathbf{u}}_j) \\
= \; & -i(\overline{\mathbf{u}}'\otimes e_d):\mathcal{C}(\mathbf{u}\otimes(\xi',0))-(\overline{\mathbf{u}}'\otimes e_d):\mathcal{C}(\mathbf{u}'\otimes e_d) \\
& -(\overline{\mathbf{u}}'\otimes e_d):\mathcal{C}(\mathbf{u}'\otimes\nabla h)\,, \qquad (4.52)
\end{aligned}
$$

and for the third integrand on the right-hand side

$$
\begin{aligned}
-\sum_{j,k,l,m=1}^{d} & \mathcal{C}_{jklm}\left(\partial_m(e^{i\xi'\cdot x'}\mathbf{u}_l)+\partial_d(e^{i\xi'\cdot x'}\mathbf{u}_l)\partial_m h\right)\partial_k h\partial_d(e^{i\xi'\cdot x'}\mathbf{u}_j) \\
= \; & -i(\overline{\mathbf{u}}'\otimes\nabla h):\mathcal{C}(\mathbf{u}\otimes(\xi',0))-(\overline{\mathbf{u}}'\otimes\nabla h):\mathcal{C}(\mathbf{u}'\otimes e_d) \\
& -(\overline{\mathbf{u}}'\otimes\nabla h):\mathcal{C}(\mathbf{u}'\otimes\nabla h)\,. \qquad (4.53)
\end{aligned}
$$

We insert (4.51)-(4.53) in (4.50) and use the symmetry of $\mathcal{C}$ to get

$$
\begin{aligned}
\int_0^\infty & \left((C^+(h))^p(e^{i\xi'\cdot x'}\mathbf{u})\right)e^{-i\xi'\cdot x'}\overline{\mathbf{u}}\,dx_d \\
= \; & -\int_0^\infty(\mathbf{u}\otimes(\xi',0)-i\mathbf{u}'\otimes e_d-i\mathbf{u}'\otimes\nabla h): \\
& \quad :\mathcal{C}\left(\overline{\mathbf{u}}\otimes(\xi',0)-\overline{i\mathbf{u}'}\otimes e_d-\overline{i\mathbf{u}'}\otimes\nabla h\right)\,dx_d \\
& -|(\nabla_{x'}h(x'),1)|\left(D^+(h)(e^{i\xi'\cdot x'}\mathbf{u}(0))\right)e^{-i\xi'\cdot x'}\overline{\mathbf{u}}(0)\,. \qquad (4.54)
\end{aligned}
$$

An analogous calculation yields

$$
\begin{aligned}
\int_0^\infty & \left((C_d^-(h))^p(e^{i\xi'\cdot x'}\mathbf{v})\right)e^{-i\xi'\cdot x'}\overline{\mathbf{v}}\,dx_d \\
= \; & -\int_0^\infty(\mathbf{v}\otimes(\xi',0)+i\mathbf{v}'\otimes e_d+i\mathbf{v}'\otimes\nabla h): \\
& \quad :\mathcal{C}\left(\overline{\mathbf{v}}\otimes(\xi',0)+\overline{i\mathbf{v}'}\otimes e_d+\overline{i\mathbf{v}'}\otimes\nabla h\right)\,dx_d \\
& +|(\nabla_{x'}h(x'),1)|\left(D_d^-(h)(e^{i\xi'\cdot x'}\mathbf{v}(0))\right)e^{-i\xi'\cdot x'}\overline{\mathbf{v}}(0)\,. \qquad (4.55)
\end{aligned}
$$

Since $B(e^{i\xi'\cdot x'}(\mathbf{u}(0),\mathbf{v}(0)))=0$, it follows $\mathbf{u}(0)=\mathbf{v}(0)$ and $D^+(h)(e^{i\xi'\cdot x'}\mathbf{u}(0))=D_d^-(h)(e^{i\xi'\cdot x'}\mathbf{v}(0))$, and therefore we conclude

$$
\left(D^+(h)(e^{i\xi'\cdot x'}\mathbf{u}(0))\right)e^{i\xi'\cdot x'}\mathbf{u}(0)-\left(D_d^-(h)(e^{i\xi'\cdot x'}\mathbf{v}(0))\right)e^{i\xi'\cdot x'}\mathbf{v}(0)=0\,. \qquad (4.56)
$$

By the positive definiteness of the tensor $\mathcal{C}$ and by (4.49) together with (4.54)-(4.56), it follows

$$
\begin{aligned}
0 \;\leq\; & -c_2\int_0^\infty\Big[|\mathrm{sym}(\mathbf{u}\otimes(\xi',0)-i\mathbf{u}'\otimes e_d-i\mathbf{u}'\otimes\nabla h)|^2 \\
& \quad +|\mathrm{sym}(\mathbf{v}\otimes(\xi',0)+i\mathbf{v}'\otimes e_d+i\mathbf{v}'\otimes\nabla h)|^2\Big]\,dx_d\,.
\end{aligned}
$$

Since h is independent of x_d, it follows $\partial_d h(x) = 0$. Hence we conclude $\mathbf{u}_d' \equiv 0$ due to $(\mathrm{sym}(\mathbf{u} \otimes (\xi', 0) - i\mathbf{u}' \otimes e_d - i\mathbf{u}' \otimes \nabla h))_{dd} = -i\mathbf{u}_d'$. By $\mathbf{u}(0) = 0$, it holds $\mathbf{u}_d \equiv 0$. Therefore $(\mathrm{sym}(\mathbf{u} \otimes (\xi', 0) - i\mathbf{u}' \otimes e_d - i\mathbf{u}' \otimes \nabla h))_{dj} = -i\mathbf{u}_j'$ for all $j = 1, \dots, d$. Thus it follows $\mathbf{u} \equiv 0$ and analogously $\mathbf{v} \equiv 0$. Hence the system (4.45) and (4.46) has no non-trivial solutions with $\lim_{x_d \to \infty} \mathbf{u}(x_d) = \lim_{x_d \to \infty} \mathbf{v}(x_d) = 0$.
This completes the proof of the theorem. □

Lemma 4.1.7. *Let $h \in \mathcal{U}$ be given. Then the elliptic boundary value problem*

$$\begin{aligned} C^\pm(h)\boldsymbol{w}^\pm &= 0 && \text{in } \Omega^\pm\,, \\ D(h)\boldsymbol{w} &= \boldsymbol{m} && \text{on } \Sigma\,, \\ \left[\boldsymbol{w}^\pm\right]_\Sigma &= 0 && \text{on } \Sigma\,, \\ \boldsymbol{w}^+ &= 0 && \text{on } \partial\Omega \end{aligned}$$

has a unique solution $\boldsymbol{w}^\pm = U^\pm(h)\boldsymbol{m} \in h^{2+\beta}(\overline{\Omega^\pm})^d$ for each $\boldsymbol{m} \in h^{1+\beta}(\Sigma)^d$, and

$$\left[h \mapsto U^\pm(h)\right] \in C^\omega(\mathcal{U}, \mathcal{L}(h^{1+\beta}(\Sigma)^d, h^{2+\beta}(\overline{\Omega^\pm})^d))\,.$$

Proof: It follows from Lemma 4.1.6 that there is a unique solution

$$\mathbf{w}^\pm = U^\pm(h)\mathbf{m} = (C(h), D(h), [\,.\,]_\Sigma, \gamma(\,.\,))^{-1}(0, \mathbf{m}, 0, 0)$$

in $h^{2+\beta}(\overline{\Omega^\pm})^d$. Then the rest of the proof is done the same way as Lemma 2.3 in [29] since

$$(U^+(h), U^-(h)) = e \circ (C(h), D(h), [\,.\,]_\Sigma, \gamma(\,.\,))^{-1}\,,$$

where e is the evaluation map, i.e. $e(V)(\mathbf{g}) := V(0, \mathbf{g}, 0, 0)$, for $V \in \mathcal{L}(Y^\beta, X^{2+\beta})$. □

Now we are able to reduce the coupled system (4.19)-(4.26) for $(u, \mathbf{w}, h)$ to a single evolution equation for the height function h. We define the operators

$$\mathcal{A} : \mathcal{U} \to \mathcal{L}(h^{3+\alpha}(\Sigma), h^\alpha(\Sigma)) : h \mapsto \frac{1}{2}\sigma B(h)T(h)P(h)$$

and

$$\mathcal{F} : \mathcal{U} \to h^\beta(\Sigma) : h \mapsto -\frac{1}{2}\sigma B(h)T(h)K(h) + \frac{1}{2}B(h)T(h)E(h)U(h)F(h)\,,$$

where

$$\begin{aligned} &B(h) = B^+(h) - B^-(h), \quad T(h) = (T^+(h), T^-(h)), \\ &U(h) = (U^+(h), U^-(h)). \end{aligned} \tag{4.57}$$

It follows from Lemma 4.1.3-4.1.7 that the operators $\mathcal{A}$ and $\mathcal{F}$ are well defined. Moreover, the principal part $\mathcal{A}$ is a quasilinear non-local operator of third order. Therefore we consider the nonlinear evolution equation on $h^\alpha(\Sigma)$ for given $h_0 \in \mathcal{V}$

$$\partial_t h + \mathcal{A}(h)h = \mathcal{F}(h), \quad h(0) = h_0 \,. \tag{4.58}$$

We use the theory of abstract quasilinear evolution equations of parabolic type developed by Amann [14] to solve equation (4.58). Before we do this, we have to investigate the principal part $\mathcal{A}$. For this reason, let E_0 and E_1 be Banach spaces such that E_1 is densely injected in E_0 and let $\mathcal{H}(E_1, E_0)$ denote the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$ is the generator of a strongly continuous analytic semigroup on E_0. Then we obtain the following result:

Theorem 4.1.8. *Let $h \in \mathcal{U}$ be given. Then it holds*

$$\mathcal{A}(h) \in \mathcal{H}(h^{3+\alpha}(\Sigma), h^\alpha(\Sigma)) \,.$$

Proof: One can find the details of the proof in [29, Theorem 4.1]. □

With the help of Theorem 4.1.8, we can prove Theorem 4.1.1.

Proof of Theorem 4.1.1: Let $h_0 \in \mathcal{V}$ be given. Then the existence of a unique solution

$$h \in C([0,T], \mathcal{V}) \cap C((0,T], h^{3+\alpha}(\Sigma)) \cap C^1((0,T], h^\alpha(\Sigma))$$

is a consequence of Theorem 12.1 in [14]. Also we get from [14, Chapter 12] that the semiflow $(t,x) \mapsto h(t,x)$ is analytic for $t > 0$ since

$$(\mathcal{A}, \mathcal{F}) \in C^\omega(\mathcal{U}, \mathcal{H}(h^{3+\alpha}(\Sigma), h^\alpha(\Sigma)) \times h^\beta(\Sigma)) \,,$$

in particular, we conclude $h \in C^\omega((0,T), h^{3+\alpha}(\Sigma))$. It remains to show that $h \in C^\infty(\Sigma \times (0,T))$. Since $h(\tau) \in h^{3+\alpha}(\Sigma)$ for all $\tau > 0$, we can start with the more regular initial value $h_1 := h(\tau)$. Then we get for the initial value h_1 a new solution $h(.,h_1)$ such that

$$h(.,h_1) \in C([\tau, t_1^+], \mathcal{V}_1) \cap C((\tau, t_1^+], h^{4+\alpha_1}(\Sigma)) \cap C^1((\tau, t_1^+], h^{1+\alpha_1}(\Sigma)) \,,$$

where $\alpha_1 \in (0,\alpha)$, $\mathcal{V}_1 = \mathcal{V} \cap h^{3+\alpha}(\Sigma)$, and $[\tau, t_1^+]$ is the maximal interval of existence. Here we have used that

$$(\mathcal{A}, \mathcal{F}) \in C^\omega(\mathcal{U} \cap h^{3+\beta_1}(\Sigma), \mathcal{H}(h^{4+\alpha_1}(\Sigma), h^{1+\alpha_1}(\Sigma)) \times h^{1+\beta_1}(\Sigma))$$

for $\alpha_1 < \beta_1 < \alpha$. We show by contradiction that $t_1^+ > T$. Assume that $t_1^+ \leq T$. Since $h(.,h_1) = h(.)$ on $[\tau, t_1^+)$, Theorem 12.5 in [14] leads the assumption to a contradiction. Now we can apply a bootstrapping argument to show $h \in C^\infty(\Sigma \times (0,T))$.
Therefore $(\mu^\pm, \mathbf{u}^\pm, h) = (\tilde{\mu}^\pm \circ \Theta_h^{-1}, \tilde{\mathbf{u}}^\pm \circ \Theta_h^{-1}, h)$ is the desired solution where

$$\begin{aligned} \tilde{\mathbf{u}}^\pm &= U^\pm(h)F(h) \,, \\ \tilde{\mu}^\pm &= \sigma T^\pm(h)H(h) - E(h)(\tilde{\mathbf{u}}^+, \tilde{\mathbf{u}}^-) \,. \end{aligned}$$

For more details see [29, Section 4]. For that observe that the operator F in [29] has the same properties as our operator $\mathcal{F}$ and the operator Φ in [29] coincides with $\mathcal{A}$ in this section. $\square$

4.2 Classical Solution of the Linearized Hele-Shaw Problem

In this section we show that the linearized Hele-Shaw problem (3.161)-(3.170) has a smooth solution provided the interface Γ^0 and the boundary $\partial\Omega$ are smooth. Note that Γ^0 is given in contrast to Section 4.1 where the interface is unknown.
For the proof we use an energy method as in [10]. First we transform the system to the fixed smooth hypersurface Γ_{00} and then reduce it to a single evolution equation for d^j, $j \geq 1$.
For better legibility we only write in the following $(d, \Gamma, \Gamma_0, \Omega^{\pm}(t), \Omega_0^{\pm})$ instead of $(d^j, \Gamma^0, \Gamma_{00}, \Omega_0^{\pm}(t), \Omega_{00}^{\pm})$.
We choose the reference manifold $\Sigma = \Gamma_0$ since Γ_0 is smooth, and we assume that

$$\Gamma(t) = \Gamma_{h(t)} = \{x + h(x,t)\nu_{\Gamma_0}(x) : x \in \Gamma_0\}$$

for all $t \in [0,T]$ and for some height function h with $h(.,0) = 0$. Since $\Gamma(t)$ is smooth for all $t \geq 0$, we can assume that $h \in C^\infty([0,T]; C^\infty(\Gamma_0))$. Furthermore, we assume that $\|h(.,t)\|_{C^1(\Gamma_0)} < a$ for all $t \in [0,T]$ where a is given as in Section 4.1. Obviously, it holds

$$\Omega^{\pm}(t) = \Omega^{\pm}_{h(t)} \quad \forall t \in [0,T]\,.$$

For the Hansawa transformation Θ_h we obtain the following properties

$$\Theta_h(.,t) \in \mathrm{Diff}^\infty(\Omega,\Omega) \cap \mathrm{Diff}^\infty(\Omega_0^{\pm}, \Omega^{\pm}(t)) \quad \text{and} \quad \Theta_h(\Gamma_0, t) = \Gamma(t)$$

for all $t \in [0,T]$. As in Section 4.1 we define the transformed differential operators for a function $u^{\pm} \in C^2(\Omega_0^{\pm}) \cap C^1(\overline{\Omega_0^{\pm}})$

$$\begin{aligned} A^{\pm}(h)u^{\pm} &:= (\Delta(u^{\pm} \circ \Theta_h^{-1})) \circ \Theta_h\,, \\ B^{\pm}(h)u^{\pm} &:= \gamma^{\pm}((\nabla(u^{\pm} \circ \Theta_h^{-1}) \cdot \nabla\Phi_h / |\nabla\Phi_h|) \circ \Theta_h)\,, \end{aligned}$$

where $\gamma^{\pm}$ denotes the restriction operator from $\Omega_0^{\pm}$ to Γ_0. (Here we add the factor $1/|\nabla\Phi_h|$ in the operator $B^{\pm}$.) Note that the outer unit normal field on $\Gamma(t)$ is given by $\nu_{\Gamma(t)}(.,t) = \nabla_x\Phi_h(.,t)/|\nabla_x\Phi_h(.,t)|$. For a function $\mathbf{w}^{\pm} \in C^2(\Omega_0^{\pm})^d \cap C^1(\overline{\Omega_0^{\pm}})^d$ we define

$$\begin{aligned} C^{\pm}(h)\mathbf{w}^{\pm} &:= (\mathrm{div}(\mathcal{C}\nabla(\mathbf{w}^{\pm} \circ \Theta_h^{-1}))) \circ \Theta_h\,, \\ D^{\pm}(h)\mathbf{w}^{\pm} &:= \gamma^{\pm}(((\mathcal{C}\nabla(\mathbf{w}^{\pm} \circ \Theta_h^{-1}))\nabla\Phi_h / |\nabla\Phi_h|) \circ \Theta_h) \end{aligned}$$

and for a function $d \in C^2(\Gamma_0)$ the operator P by

$$P(h)d := \Delta_{\Gamma(t)}(d \circ \Theta_h^{-1}) \circ \Theta_h \,,$$

where $\Delta_{\Gamma(t)}$ denotes the Laplace-Beltrami operator on $\Gamma(t)$.
Since $h(.,t) \in C^\infty(\Gamma_0)$ is given for all $t \in [0,T]$, we get the following properties

$$A^\pm(h(.,t)) \in \mathcal{L}(H^m(\Omega_0^\pm), H^{m-2}(\Omega_0^\pm)) \qquad \forall m \in \mathbb{N}, m \geq 2 \,, \tag{4.59}$$
$$B^\pm(h(.,t)) \in \mathcal{L}(H^m(\Omega_0^\pm), H^{m-3/2}(\Gamma_0) \qquad \forall m \in \mathbb{N}, m \geq 2 \,, \tag{4.60}$$
$$C^\pm(h(.,t)) \in \mathcal{L}(H^m(\Omega_0^\pm)^d, H^{m-2}(\Omega_0^\pm)^d) \qquad \forall m \in \mathbb{N}, m \geq 2 \,, \tag{4.61}$$
$$D^\pm(h(.,t)) \in \mathcal{L}(H^m(\Omega_0^\pm)^d, H^{m-3/2}(\Gamma_0)^d) \qquad \forall m \in \mathbb{N}, m \geq 2 \,, \tag{4.62}$$
$$P(h(.,t)) \in \mathcal{L}(H^{m-1/2}(\Gamma_0), H^{m-5/2}(\Gamma_0)) \qquad \forall m \in \mathbb{N}, m \geq 3 \,. \tag{4.63}$$

Here we have used that the trace operator $\gamma^\pm$ is a linear operator from $H^m(\Omega_0^\pm) \to H^{m-1/2}(\Gamma_0)$ for all $m \in \mathbb{N} \setminus \{0\}$, cf. [56, Theorem 7.40]. We define $A(h), B(h), C(h)$, and $D(h)$ as in (4.18). Now we reduce the system (4.2)-(4.8) to a single evolution equation by the same technique as in Section 4.1.

Lemma 4.2.1. *Let $m \geq 2$ be any integer and $t \in [0,T]$ be given and fixed. Then the elliptic boundary value problem*

$$\begin{aligned} A^\pm(h)u^\pm &= f^\pm \ \textit{in } \Omega_0^\pm \,, \\ u^\pm &= g^\pm \ \textit{on } \Gamma_0 \,, \\ \tfrac{\partial}{\partial n} u^+ &= 0 \quad \textit{on } \partial\Omega \end{aligned}$$

has a unique solution $u^\pm = S^\pm(h)f^\pm \in H^m(\Omega_0^\pm)$ for each $f^\pm \in H^{m-2}(\Omega_0^\pm)$ and $g^\pm \equiv 0$ and

$$S^\pm(h) \in \mathcal{L}(H^{m-2}(\Omega_0^\pm), H^m(\Omega_0^\pm)) \,.$$

Moreover, it has a unique solution $u^\pm = T^\pm(h)g^\pm \in H^m(\Omega_0^\pm)$ for each $g^\pm \in H^{m-1/2}(\Gamma_0)$ and $f^\pm \equiv 0$ and

$$T^\pm(h) \in \mathcal{L}(H^{m-1/2}(\Gamma_0), H^m(\Omega_0^\pm)) \,.$$

Proof: Let $m \geq 2$ be any integer and $f^\pm \in H^{m-2}(\Omega_0^\pm)$ and $g^\pm \in H^{m-1/2}(\Gamma_0)$ be any functions. First we consider the domain Ω_0^+. For the proof we use similar techniques as in Lemma 4.1.6. We transform boundary value problem back into

$$\Delta u^+ = f_h^+ \text{ in } \Omega^+(t) \,, \tag{4.64}$$
$$u^+ = g_h^+ \text{ on } \Gamma(t) \,, \tag{4.65}$$
$$\tfrac{\partial}{\partial n} u^+ = 0 \quad \text{on } \partial\Omega \,, \tag{4.66}$$

where $(f_h^+, g_h^+) = (f^+ \circ \Theta_h^{-1}, g^+ \circ \Theta_h^{-1})$. To get a unique weak solution, that is

$$\int_{\Omega^+(t)} \nabla u^+ \cdot \nabla v \, dx = -\int_{\Omega^+(t)} f_h^+ v \, dx \quad \forall v \in H_D^1(\Omega^+(t)) \quad \text{and} \quad u^+\big|_{\Gamma(t)} = g_h^+ \,,$$

we define the continuous bilinear functional $a(.,.) : H^1_D(\Omega^+(t)) \times H^1_D(\Omega^+(t)) \to \mathbb{R}$ by

$$a(u,v) := \int_{\Omega^+(t)} \nabla u \cdot \nabla v \, dx \,,$$

where

$$H^1_D(\Omega^+(t)) = \left\{ v \in H^1(\Omega^+(t)) : v|_{\Gamma(t)} = 0 \right\} .$$

Since $\mathcal{H}^{d-1}(\Gamma(t)) > 0$, we can apply Poincaré's inequality

$$\|v\|_{H^1(\Omega^+(t))} \le C \, \|\nabla v\|_{L^2(\Omega^+(t))}$$

for all $v \in H^1_D(\Omega^+(t))$, cf. [57, Theorem 1.32]. Hence $a(.,.)$ is coercive. Furthermore, we choose a function $\tilde{u}^+ \in H^1(\Omega^+(t))$ such that

$$\tilde{u}^+\big|_{\Gamma(t)} = g_h^+ \,,$$

and we define $F \in (H^1_D(\Omega^+(t)))'$ by

$$\langle F, v\rangle_{(H^1_D)', H^1_D} := -\int_{\Omega^+(t)} f_h^+ v \, dx - \int_{\Omega^+(t)} \nabla \tilde{u}^+ \cdot \nabla v \, dx \,.$$

Then the Lax-Milgram theorem gives us a unique solution $w^+ \in H^1_D(\Omega^+(t))$ to the problem

$$a(w^+, v) = \langle F, v\rangle \quad \forall v \in H^1_D(\Omega^+(t)) \,.$$

It is not difficult to verify that $u_h^+ := w^+ + \tilde{u}^+ \in H^1(\Omega^+(t))$ is a weak solution to (4.64)-(4.66).
To get higher regularity up to the boundary, we refer to Theorem 4.16 and Theorem 4.18 in [53].
We can tread the domain $\Omega^-(t)$ analogously or we apply the standard regularity results of Evans [32] or Renardy and Rogers [56] since we have a Dirichlet problem in $\Omega^-(t)$. Finally, we obtain that $(u^+, u^-) = (u_h^+ \circ \Theta_h, u_h^- \circ \Theta_h) \in H^m(\Omega^+) \times H^m(\Omega^-)$ is the unique desired function. □

Lemma 4.2.2. *Let $m \ge 2$ be any integer and $t \in [0,T]$ be given and fixed. Then the elliptic boundary value problem*

$$\begin{aligned} C^\pm(h)\boldsymbol{w}^\pm &= \boldsymbol{f}^\pm \quad \text{in } \Omega_0^\pm \,, \\ D(h)\boldsymbol{w} &= \boldsymbol{m} \quad \text{on } \Gamma_0 \,, \\ \left[\boldsymbol{w}^\pm\right]_{\Gamma_0} &= \boldsymbol{g} \quad \text{on } \Gamma_0 \,, \\ \boldsymbol{w}^+ &= \boldsymbol{k} \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $\boldsymbol{w}^\pm = U^\pm(h)(\boldsymbol{f}^+, \boldsymbol{f}^-, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{k}) \in H^m(\Omega_0^\pm)^d$ for each $(\boldsymbol{f}^+, \boldsymbol{f}^-, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{k}) \in H^{m-2}(\Omega_0^+)^d \times H^{m-2}(\Omega_0^-)^d \times H^{m-3/2}(\Gamma_0)^d \times H^{m-1/2}(\Gamma_0)^d \times H^{m-1/2}(\partial\Omega)^d$.
Moreover, $U^\pm(h)$ is a linear bounded operator.

Proof: We obtain a unique solution $\mathbf{w}^{\pm} \in H^2(\Omega_0^{\pm})^d$ in the same way as in in the proof of Lemma 4.1.6. To get higher regularity, we refer to [53, Theorem 4.20].
□

We define the operators $B(h), S(h), T(h)$, and $U(h)$ as in (4.57).
Note that

$$\begin{aligned}\Delta d &= \operatorname{div}(\nabla d) = \operatorname{div}(\nabla_{\Gamma(t)} d + \partial_{\nu_{\Gamma(t)}} d\, \nu_{\Gamma(t)}) \\ &= \operatorname{div}_{\Gamma(t)}(\nabla_{\Gamma(t)} d) + \operatorname{div}((\nabla d^0 \cdot \nabla d)\nabla d^0)\,,\end{aligned}$$

where we have used that $\nu_{\Gamma(t)} \cdot \nabla_{\Gamma(t)}(\partial_{\nu_{\Gamma(t)}} d) = 0$ since $\nabla_{\Gamma(t)} f \in \mathrm{T}_x\Gamma(t)$ for $f \in C^1(\Gamma(t))$. Using (3.169), we obtain

$$\Delta d = \Delta_{\Gamma(t)} d + a^{15}_{j-1}\,,$$

where a^{15}_{j-1} only depends on the known functions $\mathcal{V}_0, \ldots, \mathcal{V}_{j-1}$.
As in [10] we can reduce the coupled problem (3.161)-(3.170) to a single evolution equation for $p(x,t) = d(\Theta_h(x,t),t)$ on Γ_0. Since

$$\partial_t p = \partial_t(d \circ \Theta_h) = (\partial_t d) \circ \Theta_h + \partial_t \Theta_h \cdot (\nabla d) \circ \Theta_h\,,$$

we obtain

$$\partial_t p + \frac{1}{2}\sigma \mathcal{A}(h) p = \mathcal{F}(h)(p) \quad \text{on } \Gamma_0\,, \quad p(0) = 0 \tag{4.67}$$

for all $t \in [0,T]$ and

$$\nabla d^0 \cdot \nabla d = a^{14}_{j-1} \quad \text{in } \Gamma(\delta)\,.$$

Here

$$\mathcal{A}(h) = B(h)T(h)P(h) \in \mathcal{L}(H^{m-1/2}(\Gamma_0), H^{m-7/2}(\Gamma_0)) \quad \forall m \in \mathbb{N}, m \geq 4$$

and

$$\begin{aligned}\mathcal{F}(h)(p) &= \partial_t\Theta_h \cdot \nabla_h p + a^{12}_{j-1,h} p + a^{13}_{j-1,h} - \frac{1}{2}\sigma B(h)T(h) a^{15}_{j-1,h} \\ &\quad - \frac{1}{2} B(h)T(h)(a^{2+}_{j-1,h}p, a^{2-}_{j-1,h}p) + \frac{1}{2} B(h)T(h)(a^{3+}_{j-1,h}, a^{3-}_{j-1,h}) \\ &\quad - \frac{1}{4} B(h)T(h)(\mathcal{E}^\star : \mathcal{C}\nabla_h U(h)(0,0,a^6_{j-1,h}\nabla_h p + a^7_{j-1,h}p, a^9_{j-1,h}p, 0)) \\ &\quad - \frac{1}{4} B(h)T(h)(\mathcal{E}^\star : \mathcal{C}\nabla_h (U(h)(a^{5+}_{j-1,h}, a^{5-}_{j-1,h}, a^8_{j-1,h}, a^{10}_{j-1,h}, a^{11}_{j-1,h}))) \\ &\quad + \frac{1}{2} B(h)S(h)(a^{1+}_{j-1,h}, a^{1-}_{j-1,h})\,,\end{aligned} \tag{4.68}$$

where $\nabla_h u = \nabla(u \circ \Theta_h^{-1}) \circ \Theta_h$ for $u \in C^1(\Omega_0^{\pm})$ and $a^i_{j-1,h} = a^i_{j-1} \circ \Theta_h$ for $i \geq 1$. For a function $q \in C^1(\Gamma_0)$, we define $\nabla_h q$ by

$$\nabla_h q = (\nabla_{\Gamma(t)}(q \circ \Theta_h^{-1})) \circ \Theta_h + a^{14}_{j-1,h}\,.$$

This definition is natural since for $p(x,t) = d(\Theta_h(x,t),t)$, it holds

$$\nabla_h p = (\nabla_{\Gamma(t)} d) \circ \Theta_h + a^{14}_{j-1,h} = (\nabla d) \circ \Theta_h,$$

when d is a solution to (3.161)-(3.170).
We seek for a smooth solution $p \in C^\infty(\Gamma_0 \times [0,T])$ to (4.67).
As main result of this section, we obtain the following theorem.

Theorem 4.2.3. *For given $h \in C^\infty([0,\infty), C^\infty(\Gamma_0))$ with $h(.,0) = 0$ there exists a unique classical solution $p \in C^\infty(\Gamma_0 \times [0,T])$ to (4.67) on a sufficiently small interval of existence $[0,T]$.*

The evolution equation (4.67) is of third order. Thus we add an fourth order operator to get a smooth solution by standard arguments. For all $\epsilon > 0$ we consider the parabolic evolution equation

$$\partial_t p^\epsilon + \epsilon \Delta^2_{\Gamma_0} p^\epsilon + \tfrac{1}{2}\sigma \mathcal{A}(0) p^\epsilon = \tfrac{1}{2}\sigma(\mathcal{A}(0) - \mathcal{A}(h)) p^\epsilon + \mathcal{F}(h)(p^\epsilon) \quad \text{in } \Gamma_0 \times (0,T), \tag{4.69}$$

$$p^\epsilon = 0 \quad \text{on } \Gamma_0\,. \tag{4.70}$$

By semigroup theory one can show

$$p^\epsilon \in C^\infty(\Gamma_0 \times (0,T])$$

for all $\epsilon \in (0,1]$, cf. [61, Chapter 15.1, Exercises 5 and 6]. Since we have no boundary conditions for p^ϵ, we even get

$$p^\epsilon \in C^\infty(\Gamma_0 \times [0,T])$$

for all $\epsilon \in (0,1]$, cf. Lunardi [51, Corollary 2.3]. In the following we will see that the solutions p^ϵ, $\epsilon > 0$, converge to a solution to (4.67) as $\epsilon \to 0$. Hence we want to control the norm of p^ϵ independently of ϵ. This is the reason why we expand the operator $\mathcal{A}(h)$ to $\mathcal{A}(0) + (\mathcal{A}(h) - \mathcal{A}(0))$. The details are explained below.

Lemma 4.2.4. *Let $q \in C^\infty(\Gamma_0 \times \mathbb{R})$ be an arbitrary function and let $\mathcal{A}$ be defined as above. Then it holds for all $t \geq 0$*

$$\begin{aligned}&\left| \int_0^t \int_{\Gamma_0} (\mathcal{A}(0) - \mathcal{A}(h)) q \Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} \, ds \right| \\ &\quad \leq \; C\big(\sup_{\tau \in [0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)} \big) \, \|q\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))}\end{aligned} \tag{4.71}$$

for some continuous $C(.) > 0$ with $C(0) = 0$ that is independent of q.

Proof: For the proof we use the definition of the operator $\mathcal{A}(h) = B(h)T(h)P(h)$ and apply integration by parts after coordinate transformation.

By change of variables and definition of $B(h)$, we obtain for an arbitrary function $q \in C^\infty(\Gamma_0 \times \mathbb{R})$

$$
\begin{aligned}
&\int_{\Gamma_0} B(h)T(h)P(h)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} \\
&= \int_{\Gamma(t)} (B(h)T(h)P(h)q) \circ \Theta_h^{-1} (\Delta_{\Gamma_0} q) \circ \Theta_h^{-1} \left|\det(\partial_{\tau_i}\Theta_h^{-1} \cdot \partial_{\tau_j}\Theta_h^{-1})_{i,j=1}^{d-1}\right|^{\frac{1}{2}} \, d\mathcal{H}^{d-1} \\
&= \int_{\Gamma(t)} \nu_{\Gamma(t)} \cdot \nabla \left[(T^+(h)P(h)q) \circ \Theta_h^{-1} - (T^-(h)P(h)q) \circ \Theta_h^{-1}\right] \\
&\qquad \times (\Delta_{\Gamma_0} q) \circ \Theta_h^{-1} \left|\det(\partial_{\tau_i}\Theta_h^{-1} \cdot \partial_{\tau_j}\Theta_h^{-1})_{i,j=1}^{d-1}\right|^{\frac{1}{2}} \, d\mathcal{H}^{d-1} ,
\end{aligned}
$$

where $\{\tau_1(x), \ldots, \tau_{d-1}(x)\}$ is an orthonormal basis of $T_x\Gamma(t)$. We continue with integration by parts and use the Neumann boundary condition for $T^+(h)$ on $\partial\Omega$ (note that $\Theta_h = \mathrm{Id}$ in an open neighborhood of $\partial\Omega$). Furthermore, we use that $\Delta\left((T^\pm(h)g) \circ \Theta_h^{-1}\right) = 0$ for all $g \in H^{1/2}(\Gamma_0)$ to get

$$
\begin{aligned}
&\int_{\Gamma_0} B(h)T(h)P(h)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} \\
&= -\int_{\Omega^+(t)} \nabla((T^+(h)P(h)q) \circ \Theta_h^{-1}) \\
&\qquad \cdot \nabla \left[(T^+(0)\Delta_{\Gamma_0} q) \circ \Theta_h^{-1} \left|\det(\partial_{\tau_i}\Theta_h^{-1} \cdot \partial_{\tau_j}\Theta_h^{-1})_{i,j=1}^{d-1}\right|^{\frac{1}{2}}\right] dx \\
&\quad -\int_{\Omega^-(t)} \nabla((T^-(h)P(h)q) \circ \Theta_h^{-1}) \\
&\qquad \cdot \nabla \left[(T^-(0)\Delta_{\Gamma_0} q) \circ \Theta_h^{-1} \left|\det(\partial_{\tau_i}\Theta_h^{-1} \cdot \partial_{\tau_j}\Theta_h^{-1})_{i,j=1}^{d-1}\right|^{\frac{1}{2}}\right] dx ,
\end{aligned}
$$

where we use a smooth extension for the orthonormal basis $\{\tau_1(x), \ldots, \tau_{d-1}(x)\}$. Again we change the variables to obtain

$$
\begin{aligned}
&\int_{\Gamma_0} B(h)T(h)P(h)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} \\
&= -\int_{\Omega_0^+ \cup \Omega_0^-} \nabla_h T^\pm(h)P(h)q \cdot \Big(\nabla\Big[(T^\pm(0)\Delta_{\Gamma_0} q) \circ \Theta_h^{-1} \\
&\qquad \left|\det(\partial_{\tau_i}\Theta_h^{-1} \cdot \partial_{\tau_j}\Theta_h^{-1})_{i,j=1}^{d-1}\right|^{\frac{1}{2}}\Big]\Big) \circ \Theta_h \left|\det D\Theta_h\right| dx ,
\end{aligned}
$$

where $\int_{\Omega_0^+ \cup \Omega_0^-} T^\pm = \int_{\Omega_0^+} T^+ + \int_{\Omega_0^-} T^-$. Since $h(.,0) = 0$ and therefore $\Theta_h(.,0) = \mathrm{Id}$, we obtain for a smooth function $f = f(x,t)$

$$
\begin{aligned}
|(\nabla_h - \nabla)f(x,t)| &\leq C(\|h(.,t)\|_{C^1(\Gamma_0)}) \, |\nabla f(x,t)| , \\
\left|\nabla \left|\det(\partial_{\tau_i}\Theta_h^{-1} \cdot \partial_{\tau_j}\Theta_h^{-1})_{i,j=1}^{d-1}\right|^{\frac{1}{2}}(x,t)\right| &\leq C(\|h(.,t)\|_{C^2(\Gamma_0)}) , \\
||\det D\Theta_h(x,t)| - 1| &\leq C(\|h(.,t)\|_{C^1(\Gamma_0)})
\end{aligned}
$$

for some continuous $C(.) > 0$ independent of f and t such that $C(0) = 0$. Hence we conclude

$$\left| \int_0^t \int_{\Gamma_0} B(h)T(h)P(h)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} + \int_{\Omega_0^+ \cup \Omega_0^-} \nabla(T^\pm(h)P(h)q) \cdot \nabla(T^\pm(0)P(0)q) \, dx \, ds \right|$$
$$\leq C\big(\sup_{\tau \in [0,t]} \|h(.,\tau)\|_{C^2} \big) \Big(\left\|T^+(h)P(h)q\right\|_{L^2(0,t;H^1)} \left\|T^+(0)P(0)q\right\|_{L^2(0,t;H^1)}$$
$$+ \left\|T^-(h)P(h)q\right\|_{L^2(0,t;H^1)} \left\|T^-(0)P(0)q\right\|_{L^2(0,t;H^1)} \Big)$$

for some continuous $C(.) > 0$ such that $C(0) = 0$. Due to Lemma 4.2.1 and (4.63), there exists a constant $C > 0$ independent of $t \in [0,T]$ such that

$$\left\|T^\pm(h(.,t))P(h(.,t))q(.,t)\right\|_{H^1(\Omega_0^\pm)} \leq C \left\|q(,.t)\right\|_{H^{5/2}(\Gamma_0)} .$$

Hence we get

$$\left| \int_0^t \int_{\Gamma_0} B(h)T(h)P(h)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} + \int_{\Omega_0^+ \cup \Omega_0^-} \nabla(T^\pm(h)P(h)q) \cdot \nabla(T^\pm(0)P(0)q) \, dx \, ds \right|$$
$$\leq C\big(\sup_{\tau \in [0,t]} \|h(.,\tau)\|_{C^2(\Gamma_0)} \big) \left\|q\right\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))} \tag{4.72}$$

for some continuous $C(.) > 0$ such that $C(0) = 0$.
Now we show an analogous estimate for $B(0)T(0)P(0)$. As above we obtain by integration by parts

$$\int_{\Gamma_0} B(0)T(0)P(0)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} = - \int_{\Omega_0^+ \cup \Omega_0^-} \left|\nabla T^\pm(0)P(0)q\right|^2 dx \, .$$

Therefore using integration by parts again, we have

$$\int_{\Gamma_0} B(0)T(0)P(0)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} + \int_{\Omega_0^+ \cup \Omega_0^-} \nabla T^\pm(h)P(h)q \cdot \nabla T^\pm(0)P(0)q \, dx$$
$$= \int_{\Omega_0^+ \cup \Omega_0^-} \nabla((T^\pm(h)P(h) - T^\pm(0)P(0))q) \cdot \nabla(T^\pm(0)P(0)q) \, dx$$
$$= \int_{\Gamma_0} (P(0) - P(h))q \, \nu_{\Gamma_0} \cdot \gamma^+ \big(\nabla T^+(0)P(0)q\big) \, d\mathcal{H}^{d-1}$$
$$- \int_{\Gamma_0} (P(0) - P(h))q \, \nu_{\Gamma_0} \cdot \gamma^- \big(\nabla T^-(0)P(0)q\big) \, d\mathcal{H}^{d-1}$$

since $\Delta T^\pm(0) = 0$ in $\Omega_0^\pm$, $\frac{\partial}{\partial n} T^+(0) = 0$ on $\partial\Omega$, and $\gamma^\pm(T^\pm(0)f) = \gamma^\pm(T^\pm(h)f) = f$ on Γ_0 for any $f \in H^{1/2}(\Gamma_0)$.
Since it holds for all $u, v \in H^{1/2}(\Gamma_0)$

$$\left| \int_{\Gamma_0} u \, v \, d\mathcal{H}^{d-1} \right| \leq C \left\|u\right\|_{H^{1/2}(\Gamma_0)} \left\|v\right\|_{H^{-1/2}(\Gamma_0)}$$

for some $C > 0$, cf. [53, page 98], and since the Steklov-Poincaré operator

$$\nu_{\Gamma_0} \cdot \gamma^{\pm}(\nabla T^{\pm}(0)) : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$$

is bounded, cf. [53, page 145f], we obtain

$$\left| \int_{\Gamma_0} B(0)T(0)P(0)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} + \int_{\Omega_0^+ \cup \Omega_0^-} \nabla T^{\pm}(h)P(h)q \cdot \nabla T^{\pm}(0)P(0)q \, dx \right| \\ \leq \|(P(0) - P(h))q\|_{H^{1/2}(\Gamma_0)} \|P(0)q\|_{H^{1/2}(\Gamma_0)} . \tag{4.73}$$

By definition of $P(0)$ we have

$$\|P(0)q(.,t)\|_{H^{1/2}(\Gamma_0)} \leq C \|q(.,t)\|_{H^{5/2}(\Gamma_0)} \tag{4.74}$$

for some $C > 0$ independent of t, and as above there exists some continuous $C(.) > 0$ with $C(0) = 0$ such that

$$\|(P(0) - P(h))q(.,t)\|_{H^{1/2}(\Gamma_0)} \leq C(\|h(.,t)\|_{C^3(\Gamma_0)}) \|q(.,t)\|_{H^{5/2}(\Gamma_0)} . \tag{4.75}$$

Thus the assertion of the lemma follows from (4.72)-(4.75). □

In the next lemma we investigate the commutator of $\mathcal{A}(h)$ and ∂_τ.

Lemma 4.2.5. *Let $\{\tau_1(x), \ldots, \tau_{d-1}(x)\}$ be an orthonormal basis of $T_x\Gamma_0$, $\alpha \in \mathbb{N}^{d-1}$ be given, and $q \in C^\infty(\Gamma_0 \times \mathbb{R})$ be an arbitrary function. Then it holds for all $t \in [0, T]$*

$$\left| \int_0^t \int_{\Gamma_0} (\mathcal{A}(h)\partial_\tau^\alpha - \partial_\tau^\alpha \mathcal{A}(h))q \, \Delta_{\Gamma_0} \partial_\tau^\alpha q \, d\mathcal{H}^{d-1} \, ds \right| \\ \leq \; C \|q\|_{L^2(0,t;H^{2+|\alpha|}(\Gamma_0))} \|q\|_{L^2(0,t;H^{5/2+|\alpha|}(\Gamma_0))} \tag{4.76}$$

for some $C = C(\alpha) > 0$ independent of t and q and where $\partial_\tau^\alpha = \partial_{\tau_1}^{\alpha_1} \ldots \partial_{\tau_{d-1}}^{\alpha_{d-1}}$.

Proof: We fix any $t \in [0, T]$. First we consider the case $|\alpha| = 1$. Let $\mathcal{T} \in C_0^\infty(\Omega)$ be a smooth extension of a tangent vector $\tau(x) \in T_x\Gamma_0$. Then for $u \in H^3(\Omega_0^{\pm})$ we obtain by product rule

$$\begin{aligned} \partial_\tau B^{\pm}(h)u &= \partial_\tau \left[\nu_{\Gamma(t)} \circ \Theta_h \cdot \gamma^{\pm}(\nabla(u \circ \Theta_h^{-1}) \circ \Theta_h) \right] \\ &= B^{\pm}(h)(\mathcal{T} \cdot \nabla u) - \nu_{\Gamma(t)} \circ \Theta_h \cdot \gamma^{\pm}((D\Theta_h^{-T} \circ \Theta_h)(D\mathcal{T})^T \nabla u) \\ &\quad + \nu_{\Gamma(t)} \circ \Theta_h \cdot \gamma^{\pm}((\mathcal{T} \cdot \nabla(D\Theta_h^{-T} \circ \Theta_h)_{ij})_{i,j=1}^d \nabla u) \\ &\quad + \partial_\tau(\nu_{\Gamma(t)} \circ \Theta_h) \cdot \gamma^{\pm}((D\Theta_h^{-T} \circ \Theta_h)\nabla u) . \end{aligned}$$

Hence it follows

$$\left\| \partial_\tau B^{\pm}(h)u - B^{\pm}(h)(\mathcal{T} \cdot \nabla u) \right\|_{H^{1/2}(\Gamma_0)} \leq C \|u\|_{H^2(\Omega_0^{\pm})} , \tag{4.77}$$

where we can choose $C > 0$ independently of u and $t \in [0, T]$, since $[0, T]$ is a compact interval.
Let $p \in H^{5/2}(\Gamma_0)$ be an arbitrary function. Then $w = \mathcal{T} \cdot \nabla T^+(h)p - T^+(h)\partial_\tau p$ is the solution to

$$\begin{aligned} A^+(h)w = A^+(h)(\mathcal{T} \cdot \nabla T^+(h)p) & \quad \text{in } \Omega_0^+ \,, \\ w = 0 & \quad \text{on } \Gamma_0 \,, \\ \tfrac{\partial}{\partial n} w = 0 & \quad \text{on } \partial\Omega \end{aligned}$$

due to the properties of $T^+(h)$. Hence there exists a constant $C > 0$ independent of p such that

$$\left\| \mathcal{T} \cdot \nabla T^+(h)p - T^+(h)\partial_\tau p \right\|_{H^2(\Omega_0^+)} \;\leq\; C \left\| A^+(h)(\mathcal{T} \cdot \nabla T^+(h)p) \right\|_{L^2(\Omega_0^+)} \,.$$

To estimate the right-hand side, we use $A^+(h)T^+(h)p = 0$ to get

$$\begin{aligned} & A^+(h)(\mathcal{T} \cdot \nabla T^+(h)p) \\ &= A^+(h)(\mathcal{T} \cdot \nabla T^+(h)p) - \mathcal{T} \cdot \nabla (A^+(h)T^+(h)p) \\ &= \left(\Delta \left(\mathcal{T} \cdot \nabla T^+(h)p\right) \circ \Theta_h^{-1}\right) \circ \Theta_h - \left(\mathcal{T} \circ \Theta_h^{-1} \cdot \nabla \Delta \left((T^+(h)p) \circ \Theta_h^{-1}\right)\right) \circ \Theta_h \,. \end{aligned}$$

By product rule the right-hand side contains terms which only depend on first and second order partial derivatives of $T^+(h)p$. Thus by definition of $T^+(h)$, we get the estimate

$$\left\| \mathcal{T} \cdot \nabla T^+(h)p - T^+(h)\partial_\tau p \right\|_{H^2(\Omega_0^+)} \leq C \left\| p \right\|_{H^{3/2}(\Gamma_0)} \,, \tag{4.78}$$

where we can choose $C > 0$ independent of p and $t \in [0, T]$ since $[0, T]$ is a compact interval. We get an analogous result for $\mathcal{T} \cdot \nabla T^-(h) - T^-(h)\partial_\tau$

$$\left\| \mathcal{T} \cdot \nabla T^-(h)p - T^-(h)\partial_\tau p \right\|_{H^2(\Omega_0^-)} \leq C \left\| p \right\|_{H^{3/2}(\Gamma_0)} \,. \tag{4.79}$$

In addition, we get for an arbitrary function $p \in H^{9/2}(\Gamma_0)$ by using charts for Γ_0 and the definition of $P(h)$

$$\left\| \partial_\tau P(h)p - P(h)(\partial_\tau p) \right\|_{H^{3/2}(\Gamma_0)} \leq C \left\| p \right\|_{H^{7/2}(\Gamma_0)} \tag{4.80}$$

for some $C > 0$ independent of $t \in [0, T]$ and p.
Since we have the relation

$$\begin{aligned} \partial_\tau \mathcal{A}(h) - \mathcal{A}(h)\partial_\tau \;=\; & \partial_\tau B(h)T(h)P(h) - B(h)(\mathcal{T} \cdot \nabla T(h))P(h) \\ & + B(h)(\mathcal{T} \cdot \nabla T(h))P(h) - B(h)T(h)\partial_\tau P(h) \\ & + B(h)T(h)\partial_\tau P(h) - B(h)T(h)P(h)\partial_\tau \,, \end{aligned}$$

it holds due to (4.60) and Lemma 4.2.1

$$\begin{aligned}
&\left|\int_{\Gamma_0} (\mathcal{A}(h)\partial_\tau - \partial_\tau \mathcal{A}(h))q\, \Delta_{\Gamma_0}\partial_\tau q\, d\mathcal{H}^{d-1}\right| \\
&\leq \|(\mathcal{A}(h)\partial_\tau - \partial_\tau\mathcal{A}(h))q\|_{L^2(\Gamma_0)} \|\Delta_{\Gamma_0}\partial_\tau q\|_{L^2(\Gamma_0)} \\
&\leq C\Big[\|(\partial_\tau B(h) - B(h)\mathcal{T}\cdot\nabla)T(h)P(h)q\|_{L^2(\Gamma_0)} \\
&\quad + \|(\mathcal{T}\cdot\nabla T(h) - T(h)\partial_\tau)P(h)q\|_{H^2(\Omega_0^+)\times H^2(\Omega_0^-)} \\
&\quad + \|\partial_\tau P(h) - P(h)\partial_\tau q\|_{H^{3/2}(\Gamma_0)}\Big] \|q\|_{H^3(\Gamma_0)} \\
&\leq C\|q\|_{H^{7/2}(\Gamma_0)} \|q\|_{H^3(\Gamma_0)} \qquad (4.81)
\end{aligned}$$

for $C > 0$ independent of $t \in [0, T]$ and q and where we have used (4.77)-(4.80) in the last inequality.
It is not difficult to prove the case $|\alpha| > 1$ by induction where we use the same estimates as above. □

Lemma 4.2.6. *Let $\{\tau_1(x), \dots, \tau_{d-1}(x)\}$ be an orthonormal basis of $T_x\Gamma_0$, $\alpha \in \mathbb{N}^{d-1}$ be given, and $p, q \in C^\infty(\Gamma_0 \times \mathbb{R})$ be arbitrary functions. Then it holds for all $t \in [0, T]$*

$$\begin{aligned}
&\left|\int_0^t \int_{\Gamma_0} \partial_\tau^\alpha \mathcal{F}(h)p\, \partial_\tau^\alpha q\, d\mathcal{H}^{d-1}\, ds\right| \\
&\leq C(\|p\|_{L^2(0,t;H^{5/2+|\alpha|}(\Gamma_0))} \|q\|_{L^2(0,t;H^{|\alpha|}(\Gamma_0))} + 1) \qquad (4.82)
\end{aligned}$$

for some $C = C(\alpha) > 0$ independent of t and p, q and where $\partial_\tau^\alpha = \partial_{\tau_1}^{\alpha_1} \dots \partial_{\tau_{d-1}}^{\alpha_{d-1}}$.

Proof: It is sufficient to estimate the terms of $\mathcal{F}(h)p$ in (4.68) which depend on p. The other terms can be estimated by some constant $C > 0$ independent of $t \in [0, T]$ since $[0, T]$ is a compact interval.
For arbitrary $\alpha \in \mathbb{N}^{d-1}$, $p, q \in C^\infty(\Gamma_0 \times \mathbb{R})$, and $t \in [0, T]$, it holds

$$\begin{aligned}
&\left|\int_0^t \int_{\Gamma_0} \partial_\tau^\alpha(\partial_t\Theta_h \cdot \nabla_h p + a_{j-1,h}^{12} p)\partial_\tau^\alpha q\, d\mathcal{H}^{d-1}\, ds\right| \\
&\leq C\|p\|_{L^2(0,t;H^{1+|\alpha|}(\Gamma_0))} \|q\|_{L^2(0,t;H^{|\alpha|}(\Gamma_0))}, \qquad (4.83)
\end{aligned}$$

where we can choose $C = C(\alpha) > 0$ independent of $t \in [0, T]$ and p, q.
Due to the property (4.60) and Lemma 4.2.1, we get

$$\begin{aligned}
&\left|\int_0^t \int_{\Gamma_0} \partial_\tau^\alpha B(h)T(h)(a_{j-1,h}^{2+}p, a_{j-1,h}^{2-}p)\, \partial_\tau^\alpha q\, d\mathcal{H}^{d-1}\, ds\right| \\
&\leq \left\|\partial_\tau^\alpha B(h)T(h)(a_{j-1,h}^{2+}p, a_{j-1,h}^{2-}p)\right\|_{L^2(0,t;L^2(\Gamma_0))} \|q\|_{L^2(0,t;H^{|\alpha|}(\Gamma_0))} \\
&\leq C\|p\|_{L^2(0,t;H^{3/2+|\alpha|}(\Gamma_0))} \|q\|_{L^2(0,t;H^{|\alpha|}(\Gamma_0))} \qquad (4.84)
\end{aligned}$$

for some $C > 0$ independent of $t \in [0, T]$ and p, q.

$$
\begin{aligned}
&\Big| \int_0^t \int_{\Gamma_0} \partial_\tau^\alpha B(h) T(h) (\mathcal{E}^\star : \mathcal{C} \nabla_h \\
&\qquad\qquad U(h)(0, 0, a^6_{j-1,h} \nabla_h p + a^7_{j-1,h} p, a^9_{j-1,h} p, 0))\, \partial_\tau^\alpha q \, d\mathcal{H}^{d-1}\, ds \Big| \\
\leq &\; C \left\| U(h)(0, 0, a^6_{j-1,h} \nabla_h p + a^7_{j-1,h} p, a^9_{j-1,h} p, 0) \right\|_{L^2(0,t;H^{3+|\alpha|}(\Omega_0^+) \times H^{3+|\alpha|}(\Omega_0^-))} \\
&\; \times \|q\|_{L^2(0,t;H^{|\alpha|})} \\
\leq &\; C \|p\|_{L^2(0,t;H^{5/2+|\alpha|}(\Gamma_0))} \|q\|_{L^2(0,t;H^{|\alpha|}(\Gamma_0))} \qquad (4.85)
\end{aligned}
$$

for some $C > 0$ independent of $t \in [0, T]$ and p. Therefore the assertion follows from (4.83)-(4.85). ☐

Lemma 4.2.7. *For all $t \in [0, T]$, $\mathcal{A}(h)(t)$ is a linear bounded operator from $H^{5/2}(\Gamma_0)$ to $H^{-1/2}(\Gamma_0)$.*

Proof: By definition $P(h) : H^{5/2}(\Gamma_0) \to H^{1/2}(\Gamma_0)$ is a bounded linear operator. It remains to show that $B^\pm(h) T^\pm(h) : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ is a well-defined bounded linear operator. Let $g \in H^{1/2}(\Gamma_0)$ be arbitrary. Then by chain rule $w = T^+(h) g$ is the unique solution to the elliptic boundary value problem

$$
\begin{aligned}
\mathcal{P} w = 0 &\quad \text{in } \Omega_0^+ , \\
w = g &\quad \text{on } \Gamma_0 , \\
\tfrac{\partial}{\partial n} w = 0 &\quad \text{on } \partial\Omega ,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{P} w &= \operatorname{div}(A_h \nabla w) - \operatorname{div} A_h \cdot \nabla w + \Delta \Theta_h^{-1} \circ \Theta_h \cdot \nabla w , \\
A_h &= D\Theta_h^{-1} \circ \Theta_h D\Theta_h^{-T} \circ \Theta_h .
\end{aligned}
$$

Since $\Gamma(t)$ is the zero-level set of $d^0 \circ \Theta_h^{-1}$, it follows that

$$
\frac{D\Theta_h^{-T} \nu_{\Gamma_0} \circ \Theta_h^{-1}}{\left| D\Theta_h^{-T} \nu_{\Gamma_0} \circ \Theta_h^{-1} \right|} = \nu_{\Gamma(t)}
$$

and

$$
B^+(h) w = \gamma^+ \left(\frac{1}{\left| D\Theta_h^{-T} \nu_{\Gamma_0} \circ \Theta_h^{-1} \right|} \nu_{\Gamma_0} \cdot A_h \nabla w \right)
$$

for any $w \in H^2(\Omega_0^+)$. Therefore the operator $B^+(h) T^+(h) : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$

defined by

$$
\begin{aligned}
&\left\langle B^+(h)T^+(h)w, u\right\rangle_{H^{-1/2},H^{1/2}} \\
&= -\int_{\Omega_0^+} \left(A_h \nabla T^+(h)w\right)\cdot \nabla\left(T^+(0)u/\left|D\Theta_h^{-T}\nu_{\Gamma_0}\circ\Theta_h^{-1}\right|\right)dx \\
&\quad -\int_{\Omega_0^+}\left(\operatorname{div} A_h\cdot\nabla T^+(h)w - \Delta\Theta_h^{-1}\circ\Theta_h\cdot\nabla T^+(h)w\right) \\
&\qquad \times T^+(0)u/\left|D\Theta_h^{-T}\nu_{\Gamma_0}\circ\Theta_h^{-1}\right|dx \qquad (4.86)
\end{aligned}
$$

is for all $u, w \in H^{1/2}(\Gamma_0)$ well-defined, bounded, and linear since $A_h = \mathrm{Id}$ on $\partial\Omega$ and $\frac{\partial}{\partial n}T^+(h)w = 0$ on $\partial\Omega$.
Since $T^-(h)$ is the solution operator for a Dirichlet problem, we obtain that $B^-(h)T^-(h)$ is a Steklov-Poincaré operator and therefore $B^+(h)T^+(h) : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ is well-defined bounded linear operator, cf. [53, Theorem 4.21]. Of course, we can also prove the assertion for $B^-(h)T^-(h)$ in the same way as above. □

Using the lemmas above, we can show that p^ϵ is uniformly bounded for all $\epsilon \in (0,1]$.

Theorem 4.2.8. *Let p^ϵ be the solution to (4.69) and (4.70). Then there exists some $T > 0$ such that for every $m \in \mathbb{N}$, it holds*

$$\sup_{0\le t\le T} \|p^\epsilon(.,t)\|_{H^m(\Gamma_0)} + \|p^\epsilon\|_{L^2(0,T;H^{m+3/2}(\Gamma_0))} \le C \qquad (4.87)$$

for some constant $C = C(m) > 0$ independent of ϵ.

Proof: The proof is based on energy estimates. Multiplying both sides of (4.69) by p^ϵ and integrating over Γ_0 and $(0,t)$ yields for all $t \in [0,T]$

$$
\begin{aligned}
&\frac{1}{2}\|p^\epsilon(.,t)\|^2_{L^2(\Gamma_0)} + \epsilon\int_0^t \|\Delta_{\Gamma_0}p^\epsilon\|^2_{L^2(\Gamma_0)}\,dt \\
&= -\frac{1}{2}\sigma\int_0^t \langle \mathcal{A}(h)p^\epsilon, p^\epsilon\rangle_{H^{-1/2},H^{1/2}}\,dt + \int_0^t\int_{\Gamma_0}\mathcal{F}(h)p^\epsilon p^\epsilon\,d\mathcal{H}^{d-1}\,dt\,,
\end{aligned}
$$

where we have used integration by parts and $p^\epsilon(.,0) = 0$. By Lemma 4.2.7, we obtain

$$\langle \mathcal{A}(h)p^\epsilon, p^\epsilon\rangle_{H^{-1/2},H^{1/2}} \le C\|p^\epsilon\|_{H^{5/2}(\Gamma_0)}\|p^\epsilon\|_{H^{1/2}(\Gamma_0)}$$

for some $C > 0$ independent of ϵ. Due Lemma 4.2.6 with $p = p^\epsilon$, $q = p^\epsilon$, and $\alpha = 0$, it follows

$$\int_0^t\int_{\Gamma_0}\mathcal{F}(h)p^\epsilon p^\epsilon\,d\mathcal{H}^{d-1}\,dt \le C\left(\|p^\epsilon\|_{L^2(0,t;H^{5/2}(\Gamma_0))}\|p^\epsilon\|_{L^2(0,t;L^2(\Gamma_0))} + 1\right)$$

for some $C > 0$ independent of ϵ. Hence we conclude

$$\sup_{\tau\in[0,t]}\|p^\epsilon(.,\tau)\|^2_{L^2(\Gamma_0)} \le C\left(\|p^\epsilon\|_{L^2(0,t;H^{5/2}(\Gamma_0))}\|p^\epsilon\|_{L^2(0,t;H^{1/2}(\Gamma_0))} + 1\right) \qquad (4.88)$$

for some $C > 0$ independent of ϵ.
Since $\operatorname{div}(\nabla T^{\pm}(0)\Delta_{\Gamma_0} q) = 0$ in $\Omega_0^{\pm}$ and $\frac{\partial}{\partial n} T^{+}(0)P(0)q = 0$ on $\partial\Omega$, it holds for every $q \in C^{\infty}(\Gamma_0)$

$$
\begin{aligned}
-\int_{\Gamma_0} \mathcal{A}(0)q\Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} &= -\int_{\Gamma_0} \nu_{\Gamma_0} \cdot \left[\nabla T^{+}(0)(\Delta_{\Gamma_0} q) - \nabla T^{-}(0)(\Delta_{\Gamma_0} q)\right] \Delta_{\Gamma_0} q \, d\mathcal{H}^{d-1} \\
&= \int_{\Omega_0^+} \left|\nabla T^{+}(0)(\Delta_{\Gamma_0} q)\right|^2 dx + \int_{\Omega_0^-} \left|\nabla T^{-}(0)(\Delta_{\Gamma_0} q)\right|^2 dx \\
&\geq C \left\|\Delta_{\Gamma_0} q\right\|^2_{H^{1/2}(\Gamma_0)} \qquad (4.89)
\end{aligned}
$$

for some $C > 0$. Here the last inequality can be shown as follows. For all $u \in H^1(\Omega_0^{\pm})$ there exists a constant $C > 0$ independent of u such that

$$
\|u\|_{H^1(\Omega_0^{\pm})} \leq C \left(\|\nabla u\|_{L^2(\Omega_0^{\pm})} + h(u) \right),
$$

where

$$
h(u) = \int_{\Gamma_0} u \, d\mathcal{H}^{d-1},
$$

cf. Fröhlich [34, Korollar 2.3]. Now choose $u = T^{\pm}(\Delta_{\Gamma_0} q)$ and note that $h(T^{\pm}(\Delta_{\Gamma_0} q)) = 0$. Then due to the continuity of the trace operator $\gamma^{\pm} : H^1(\Omega_0^{\pm}) \to H^{1/2}(\Gamma_0)$, the last inequality follows in (4.89).
Multiplying both sides of (4.69) by $-\Delta_{\Gamma_0} p^{\epsilon}$ and integrating over Γ_0 and $(0,t)$, we obtain

$$
\begin{aligned}
0 &= \int_0^t \int_{\Gamma_0} -p_t^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} - \epsilon \Delta_{\Gamma_0}^2 p^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} - \frac{1}{2}\sigma \mathcal{A}(0) p^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} \, d\mathcal{H}^{d-1} \, ds \\
&\quad - \int_0^t \int_{\Gamma_0} \frac{1}{2}\sigma(\mathcal{A}(0) - \mathcal{A}(h)) p^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} + \mathcal{F}(h)(p^{\epsilon}) \Delta_{\Gamma_0} p^{\epsilon} \, d\mathcal{H}^{d-1} \, ds \\
&= \int_0^t \frac{d}{dt} \|\nabla_{\Gamma_0} p^{\epsilon}\|^2_{L^2(\Gamma_0)} + \epsilon \int_0^t \|\nabla_{\Gamma_0} \Delta_{\Gamma_0} p^{\epsilon}\|^2_{L^2(\Gamma_0)} - \frac{1}{2}\sigma \int_0^t \int_{\Gamma_0} \mathcal{A}(0) p^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} \, d\mathcal{H}^{d-1} \\
&\quad - \int_0^t \int_{\Gamma_0} \frac{1}{2}\sigma(\mathcal{A}(0) - \mathcal{A}(h)) p^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} + \mathcal{F}(h)(p^{\epsilon}) \Delta_{\Gamma_0} p^{\epsilon} \, d\mathcal{H}^{d-1} \, ds,
\end{aligned}
$$

where we have applied integration by parts and the fact $\nabla_{\Gamma_0} f$ is perpendicular to the mean curvature vector $\kappa_{\Gamma_0} \nu_{\Gamma_0}$ for every $f : \Gamma_0 \to \mathbb{R}$ smooth enough. By inequality (4.89) with $q = p^{\epsilon}$ and $p^{\epsilon}(.,0) = 0$, we get

$$
\begin{aligned}
&\sup_{0 \leq \tau \leq t} \|\nabla_{\Gamma_0} p^{\epsilon}\|^2_{L^2(\Gamma_0)} + \epsilon \int_0^t \|\nabla_{\Gamma_0} \Delta_{\Gamma_0} p^{\epsilon}\|^2_{L^2(\Gamma_0)} \, ds + \int_0^t \|\Delta_{\Gamma_0} p^{\epsilon}\|^2_{H^{1/2}(\Gamma_0)} \, ds \\
&\quad \leq \; C \left| \int_0^t \int_{\Gamma_0} \frac{1}{2}\sigma(\mathcal{A}(0) - \mathcal{A}(h)) p^{\epsilon} \Delta_{\Gamma_0} p^{\epsilon} + \mathcal{F}(h)(p^{\epsilon}) \Delta_{\Gamma_0} p^{\epsilon} \, d\mathcal{H}^{d-1} \, ds \right|,
\end{aligned}
$$

where $C > 0$ is independent of ϵ.
Applying Lemma 4.2.4 with $q = p^\epsilon$ and 4.2.6 with $p = p^\epsilon$, $q = \Delta_{\Gamma_0} p^\epsilon$, and $\alpha = 0$ yields

$$\begin{aligned}
&\sup_{0\le\tau\le t} \|\nabla_{\Gamma_0} p^\epsilon\|^2_{L^2(\Gamma_0)} + \epsilon \int_0^t \|\nabla_{\Gamma_0}\Delta_{\Gamma_0} p^\epsilon\|^2_{L^2(\Gamma_0)}\, ds + \int_0^t \|\Delta_{\Gamma_0} p^\epsilon\|^2_{H^{1/2}(\Gamma_0)}\, ds \\
&\quad\le\ \tilde{C}\big(\sup_{\tau\in[0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)}\big)\, \|p^\epsilon\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))} \\
&\qquad + C\, \|p^\epsilon\|_{L^2(0,t;H^{5/2}(\Gamma_0))}\, \|p^\epsilon\|_{L^2(0,t;H^2(\Gamma_0))} + C
\end{aligned} \tag{4.90}$$

for some continuous $\tilde{C}(.) \ge 0$ with $\tilde{C}(0) = 0$ and independent of ϵ . For the left-hand side we use the elliptic estimate

$$\|p^\epsilon\|^2_{H^{5/2}(\Gamma_0)} \le C \left(\|\Delta_{\Gamma_0} p^\epsilon\|^2_{H^{1/2}(\Gamma_0)} + \|p^\epsilon\|^2_{H^{1/2}(\Gamma_0)} \right), \tag{4.91}$$

see Theorem 2.5.3, and for the right-hand side we apply the interpolation

$$\big(H^1(\Gamma_0), H^{5/2}(\Gamma_0)\big)_{\frac{2}{3},2} = H^2(\Gamma_0)\,,$$

see (2.6). Altogether we obtain by (4.88) and (4.90)

$$\begin{aligned}
&\sup_{0\le\tau\le t} \|p^\epsilon\|^2_{H^1(\Gamma_0)} + \epsilon \int_0^t \|\nabla_{\Gamma_0}\Delta_{\Gamma_0} p^\epsilon\|^2_{L^2(\Gamma_0)}\, ds + \int_0^t \|p^\epsilon\|^2_{H^{5/2}(\Gamma_0)}\, ds \\
&\quad\le\ \tilde{C}\big(\sup_{\tau\in[0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)}\big)\, \|p^\epsilon\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))} \\
&\qquad + C\, \|p^\epsilon\|^{\frac{5}{3}}_{L^2(0,t;H^{5/2}(\Gamma_0))}\, \|p^\epsilon\|^{\frac{1}{3}}_{L^2(0,t;H^1(\Gamma_0))} + C
\end{aligned} \tag{4.92}$$

for some continuous $\tilde{C}(.) \ge 0$ with $\tilde{C}(0) = 0$ and independent of ϵ.
We choose $t > 0$ such that $\tilde{C}\big(\sup_{\tau\in[0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)}\big) \le \frac{1}{2}$. Such $t > 0$ exists because $h \in C^\infty([0,T]; C^\infty(\Gamma_0))$. Furthermore, we apply Young's inequality. Then we can find some constant $C > 0$ independent of ϵ such that

$$\sup_{0\le\tau\le t} \|p^\epsilon\|^2_{H^1(\Gamma_0)} + \int_0^t \|p^\epsilon\|^2_{H^{5/2}(\Gamma_0)}\, ds \le C \left(\|p^\epsilon\|^2_{L^2(0,t;H^1(\Gamma_0))} + 1 \right).$$

By Gronwall's inequality, we conclude

$$\sup_{0\le\tau\le t} \|p^\epsilon\|^2_{H^1(\Gamma_0)} + \int_0^t \|p^\epsilon\|^2_{H^{5/2}(\Gamma_0)}\, ds \le C\,. \tag{4.93}$$

To obtain higher space regularity we apply ∇_{Γ_0} to the differential equation (4.69), multiply both sides by $-\nabla_{\Gamma_0}\Delta_{\Gamma_0} p^\epsilon$, and integrate over Γ_0 and $(0,t)$. Then we get by integration by parts

$$\frac{1}{2}\frac{d}{dt}\int_0^t \|\Delta_{\Gamma_0}p^\epsilon\|^2_{L^2(\Gamma_0)} + \epsilon\int_0^t \left\|\Delta^2_{\Gamma_0}p^\epsilon\right\|^2_{L^2(\Gamma_0)} - \frac{1}{2}\sigma\int_0^t\int_{\Gamma_0}(\mathcal{A}(0)\nabla_{\Gamma_0}p^\epsilon)\cdot\Delta_{\Gamma_0}\nabla_{\Gamma_0}p^\epsilon$$
$$= \frac{1}{2}\sigma\int_0^t\int_{\Gamma_0}\nabla_{\Gamma_0}(\mathcal{A}(h)p^\epsilon)\cdot(\nabla_{\Gamma_0}\Delta_{\Gamma_0} - \Delta_{\Gamma_0}\nabla_{\Gamma_0})p^\epsilon\, d\mathcal{H}^{d-1}\, ds$$
$$+ \frac{1}{2}\sigma\int_0^t\int_{\Gamma_0}(\nabla_{\Gamma_0}\mathcal{A}(h)p^\epsilon - \mathcal{A}(h)\nabla_{\Gamma_0}p^\epsilon)\cdot\Delta_{\Gamma_0}\nabla_{\Gamma_0}p^\epsilon\, d\mathcal{H}^{d-1}\, ds$$
$$+ \frac{1}{2}\sigma\int_0^t\int_{\Gamma_0}(\mathcal{A}(h)\nabla_{\Gamma_0} - \mathcal{A}(0)\nabla_{\Gamma_0})p^\epsilon\cdot\Delta_{\Gamma_0}\nabla_{\Gamma_0}p^\epsilon\, d\mathcal{H}^{d-1}\, ds$$
$$+ \int_0^t\int_{\Gamma_0}\nabla_{\Gamma_0}\mathcal{F}(h)p^\epsilon\cdot\nabla_{\Gamma_0}\Delta_{\Gamma_0}p^\epsilon\, d\mathcal{H}^{d-1}\, ds\,.$$

By (4.89) with $q = \partial_{\tau_i}p^\epsilon$, $i = 1, \ldots, d-1$, Lemma 4.2.4, Lemma 4.2.5, Lemma 4.2.6, and the elliptic estimate (4.91), we obtain

$$\begin{aligned}
&\sup_{0\le\tau\le t}\|p^\epsilon\|^2_{H^2(\Gamma_0)} + \epsilon\int_0^t \left\|\Delta^2_{\Gamma_0}p^\epsilon\right\|^2_{L^2(\Gamma_0)} ds + \int_0^t \|\nabla_{\Gamma_0}p^\epsilon\|^2_{H^{5/2}(\Gamma_0)} ds \\
&\le\ \tilde{C}\big(\sup_{\tau\in[0,t]}\|h(.,\tau)\|_{C^3(\Gamma_0)}\big)\,\|\nabla_{\Gamma_0}p^\epsilon\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))} \\
&\qquad + C\,\|p^\epsilon\|_{L^2(0,t;H^{7/2})}\,\|p^\epsilon\|_{L^2(0,t;H^3)} + C\sup_{0\le\tau\le t}\|p^\epsilon\|^2_{L^2(\Gamma_0)} + C
\end{aligned}$$

for the same $\tilde{C}(.) > 0$ as in (4.92). Therefore we can choose the same t as above such that $\tilde{C}\big(\sup_{\tau\in[0,t]}\|h(.,\tau)\|_{C^3(\Gamma_0)}\big) \le \frac{1}{2}$. Since $\big(H^2(\Gamma_0), H^{7/2}(\Gamma_0)\big)_{\frac{2}{3},2} = H^3(\Gamma_0)$, it holds

$$\|u\|_{H^3(\Gamma_0)} \le C\,\|u\|^{\frac{1}{3}}_{H^2(\Gamma_0)}\,\|u\|^{\frac{2}{3}}_{H^{7/2}(\Gamma_0)}$$

for any u smooth enough and some C independent of u. Hence by Gronwall's and Young's inequalities and (4.93), it follows

$$\sup_{0\le\tau\le t}\|p^\epsilon\|^2_{H^2(\Gamma_0)} + \int_0^t \|p^\epsilon\|^2_{H^{7/2}(\Gamma_0)} \le C$$

for the same t as in (4.93) and for some C independent of ϵ.
Next, we apply Δ_{Γ_0} to the differential equation (4.69) and test with $\Delta^2_{\Gamma_0}p^\epsilon$. Applying this procedure again and again, the assertion of the theorem follows. □

Similarly, we can obtain higher order time estimates for p^ϵ independent of ϵ.

Corollary 4.2.9. *Let p^ϵ be the solutions to (4.69) and (4.70). Then there exists some $T > 0$ such that for every $m, n \in \mathbb{N}$, it holds*

$$\|p^\epsilon\|_{H^n(0,T;H^{m-1/2}(\Gamma_0))} \le C \tag{4.94}$$

for some $C = C(m,n) > 0$ independent of $\epsilon \in (0,1]$.

Proof: By definition $\mathcal{F}(h(.,t)) : H^{5/2}(\Gamma_0) \to H^{1/2}(\Gamma_0)$ is a bounded operator for all $t \in [0,T]$, $\mathcal{A}(h)(t) : H^{5/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ is bounded by Lemma 4.2.7 for all $t \in [0,T]$, and $\Delta^2_{\Gamma_0} : H^4(\Gamma_0) \to L^2(\Gamma_0)$ is bounded. Therefore equation (4.69) yields

$$\|\partial_t p^\epsilon\|_{L^2(0,T;H^{-1/2}(\Gamma_0))} \leq C$$

for some $C > 0$ independent of ϵ.
Getting higher order space estimates for $\partial_t p^\epsilon$, we use the properties $\mathcal{A}(h)(t) \in \mathcal{L}(H^{m-1/2}(\Gamma_0), H^{m-7/2}(\Gamma_0))$ and $\mathcal{F}(h(.,t)) : H^{m-1/2}(\Gamma_0) \to H^{m-5/2}(\Gamma_0)$ is bounded for all $t \in [0,T]$ and all $m \in \mathbb{N}$, $m \geq 4$, and Theorem 4.2.8. Then it follows

$$\|\partial_t p^\epsilon\|_{L^2(0,T;H^{m-1/2}(\Gamma_0))} \leq C \tag{4.95}$$

for all $m \in \mathbb{N}$ and some $C = C(m) > 0$ independent of ϵ.
Differentiating equation (4.69) with respect to t and using estimate (4.95), we obtain

$$\left\|\partial_t^2 p^\epsilon\right\|_{L^2(0,T;H^{m-1/2}(\Gamma_0))} \leq C$$

for all $m \in \mathbb{N}$ and some $C = C(m) > 0$ independent of ϵ.
Repeating this procedure any number of times, the assertion follows. □

Now we are able to show that the solution functions p^ϵ converge to a weak solution to (4.67) as $\epsilon \searrow 0$.

Lemma 4.2.10. *There exists*

$$p \in H^1(0,T;H^{-1/2}(\Gamma_0) \cap L^2(0,T;H^{5/2}(\Gamma_0))$$

such that p is the unique weak solution to (4.67) in the sense of $D'((0,T);H^{-1/2}(\Gamma_0))$, that is

$$\begin{aligned}&\langle -\int_0^T p(t)\varphi'(t)\,dt, f\rangle_{H^{-1/2},H^{1/2}} + \frac{1}{2}\sigma\langle\int_0^T \mathcal{A}(h)p(t)\varphi(t)\,dt, f\rangle_{H^{-1/2},H^{1/2}}\\&= \langle\int_0^T \mathcal{F}(h)p(t)\varphi(t)\,dt, f\rangle_{H^{-1/2},H^{1/2}}\end{aligned}$$

for all $\varphi \in C_0^\infty(0,T)$ and $f \in H^{1/2}(\Gamma_0)$, and we require $p(0) = 0$.

Proof: Since $L^2(0,T;H^{5/2}(\Gamma_0))$ is a Hilbert space, there exists a function $p \in L^2(0,T;H^{5/2}(\Gamma_0))$ such that $p^\epsilon \rightharpoonup p$ in $L^2(0,T;H^{5/2}(\Gamma_0))$ as $\epsilon \searrow 0$ (actually for a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\epsilon_n \to 0$ as $n \to \infty$). In the following we show that p is a weak solution. Let $f \in H^{1/2}(\Gamma_0)$ and $\varphi \in C_0^\infty(0,T)$ be arbitrary. Then it holds

$$\begin{aligned}&-\int_0^T \langle p^\epsilon(t), \varphi'(t)f\rangle_{H^{-1/2},H^{1/2}}\,dt + \epsilon\int_0^T \left\langle\Delta^2_{\Gamma_0}p^\epsilon(t), \varphi(t)f\right\rangle_{H^{-1/2},H^{1/2}}\,dt\\&+\frac{1}{2}\sigma\int_0^T \langle\mathcal{A}(h)p^\epsilon(t), \varphi(t)f\rangle_{H^{-1/2},H^{1/2}}\,dt = \int_0^T \langle\mathcal{F}(h)p^\epsilon(t), \varphi(t)f\rangle_{H^{-1/2},H^{1/2}}\,dt\end{aligned}$$

since

$$-\int_0^T \langle p^\epsilon(t), \varphi'(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt = \int_0^T \langle \partial_t p^\epsilon(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt\,.$$

Sending $\epsilon \searrow 0$ and using $p^\epsilon \rightharpoonup p$ in $L^2(0,T;H^{5/2}(\Gamma_0))$, we get

$$\begin{aligned}\int_0^T \langle p^\epsilon(t), \varphi'(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt &= \int_0^T \int_{\Gamma_0} p^\epsilon(t)\varphi'(t) f\, d\mathcal{H}^{d-1} dt \\ &\to \int_0^T \int_{\Gamma_0} p(t)\varphi'(t) f\, d\mathcal{H}^{d-1} dt \\ &= \int_0^T \langle p(t), \varphi'(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt\,,\end{aligned}$$

as $\epsilon \searrow 0$. By using Corollary 4.2.9, we obtain

$$\begin{aligned}\epsilon \int_0^T \left\langle \Delta^2_{\Gamma_0} p^\epsilon(t), \varphi(t) f\right\rangle_{H^{-1/2},H^{1/2}} dt &\le \epsilon \int_0^T \left\|\Delta^2_{\Gamma_0} p^\epsilon(t)\right\|_{H^{-1/2}} \|\varphi(t) f\|_{H^{1/2}}\, dt \\ &\le \epsilon \|p^\epsilon\|_{L^2(0,T;H^{7/2})} \|\varphi f\|_{L^2(0,T;H^{1/2})} \longrightarrow 0\,,\end{aligned}$$

as $\epsilon \searrow 0$. Let $\mathcal{A}(h)'(t) \in \mathcal{L}(H^{1/2}(\Gamma_0), H^{-5/2}(\Gamma_0))$ be the adjoint operator of $\mathcal{A}(h)(t)$ for all $t \in [0,T]$. Then it follows

$$\begin{aligned}\int_0^T \langle \mathcal{A}(h)(t) p^\epsilon(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt &= \int_0^T \langle p^\epsilon(t), \varphi(t)\mathcal{A}(h)'(t) f\rangle_{H^{5/2},H^{-5/2}}\, dt \\ &\to \int_0^T \langle p(t), \varphi(t)\mathcal{A}(h)'(t) f\rangle_{H^{5/2},H^{-5/2}}\, dt \\ &= \int_0^T \langle \mathcal{A}(h)(t) p(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt\,,\end{aligned}$$

as $\epsilon \searrow 0$. We split the term $\mathcal{F}(h)$ into two parts $\mathcal{F}_1(h)$ and $\mathcal{F}_2(h)$ where $\mathcal{F}_1(h)$ consists of all terms which contains p^ϵ, and $\mathcal{F}_2(h) = \mathcal{F}(h) - \mathcal{F}_1(h)$. It is not difficult to show that $\mathcal{F}_1(h(.,t))$ is a linear operator for all $t \in [0,T]$. We treat the term $\mathcal{F}_1(h)p^\epsilon$ as well as the term $\mathcal{A}(h)p^\epsilon$. Note that the adjoint operator $\mathcal{F}_1(h)' \in \mathcal{L}(H^{-1/2}(\Gamma_0), H^{-5/2}(\Gamma_0))$. Since $\mathcal{F}_2(h)$ does not depend on p^ϵ, we obtain

$$\int_0^T \langle \mathcal{F}(h) p^\epsilon(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt \to \int_0^T \langle \mathcal{F}(h) p(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\, dt\,,$$

as $\epsilon \searrow 0$. Therefore it follows

$$\begin{aligned}&\langle -\int_0^T p(t)\varphi'(t)\, dt, f\rangle_{H^{-1/2},H^{1/2}} + \frac{1}{2}\sigma\langle \int_0^T \mathcal{A}(h) p(t)\varphi(t)\, dt, f\rangle_{H^{-1/2},H^{1/2}} \\ &= \langle \int_0^T \mathcal{F}(h) p(t)\varphi(t)\, dt, f\rangle_{H^{-1/2},H^{1/2}}\,.\end{aligned}$$

That is (4.67) holds in the sense of $D'((0,T);H^{-1/2}(\Gamma_0))$ and $\partial_t p = -\frac{1}{2}\sigma\mathcal{A}(h)p + \mathcal{F}(h)(p) \in L^2(0,T;H^{-1/2}(\Gamma_0))$. Next we show $p(0)=0$. Since

$$H^1(0,T;H^{-1/2}(\Gamma_0)) \cap L^2(0,T;H^{1/2}(\Gamma_0)) \subset C([0,T];L^2(\Gamma_0)),$$

cf. [56, Lemma 11.4], the initial condition $p(0)=0$ is well-defined. For all $\varphi \in C^1([0,T])$ with $\varphi(T)=0$ and $f \in H^{1/2}(\Gamma_0)$, it holds

$$\begin{aligned}(p(0),\varphi(0)f)_{L^2(\Gamma_0)} &= -\int_0^T \langle \partial_t p(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt \\ &\quad -\int_0^T \langle p(t), \varphi'(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt\,.\end{aligned}$$

Therefore the following relation is valid

$$\begin{aligned}(p(0),\varphi(0)f)_{L^2(\Gamma_0)} &= -\int_0^T \langle \mathcal{F}(h)p(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt \\ &\quad +\frac{1}{2}\sigma\int_0^T \langle \mathcal{A}(h)p(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt \\ &\quad -\int_0^T \langle p(t), \varphi'(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt\,.\end{aligned}$$

Using $p^\epsilon \rightharpoonup p$ in $L^2(0,T;H^{5/2}(\Gamma_0))$, we show as above

$$\begin{aligned}(p(0),\varphi(0)f)_{L^2(\Gamma_0)} &= \lim_{\epsilon\searrow 0}\Big[-\int_0^T \langle \mathcal{F}(h)p^\epsilon(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt \\ &\quad +\frac{1}{2}\sigma\int_0^T \langle \mathcal{A}(h)p^\epsilon(t), \varphi(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt \\ &\quad -\int_0^T \langle p^\epsilon(t), \varphi'(t) f\rangle_{H^{-1/2},H^{1/2}}\,dt\Big] \\ &= \lim_{\epsilon\searrow 0}\Big[-\epsilon\int_0^T \big\langle \Delta^2_{\Gamma_0} p^\epsilon(t), \varphi(t) f\big\rangle_{H^{-1/2},H^{1/2}}\,dt \\ &\quad +(p^\epsilon(0),\varphi(0)f)_{L^2(\Gamma_0)}\Big] = 0\,.\end{aligned}$$

Since the embedding $H^{1/2}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is dense and $f \in H^{1/2}(\Gamma_0)$ is arbitrary, the statement $p(0)=0$ follows.
It remains to show uniqueness. Assume that p_1 and p_2 are two weak solutions to (4.67). We set $p := p_1 - p_2$. First we show

$$\begin{aligned}\|\nabla_{\Gamma_0} p(.,t)\|^2_{L^2(\Gamma_0)} &= \|\nabla_{\Gamma_0} p(.,s)\|^2_{L^2(\Gamma_0)} \\ &\quad -2\int_s^t \langle \partial_t p(.,\tau), \Delta_{\Gamma_0} p(.,\tau)\rangle_{H^{-1/2},H^{1/2}}\,d\tau \qquad (4.96)\end{aligned}$$

for all $s,t \in [0,T]$. Due to [13, Chapter III, Theorem 4.10.2], we have the continuous embedding

$$\begin{aligned} & H^1(0,T;H^{-1/2}(\Gamma_0)) \cap L^2(0,T;H^{5/2}(\Gamma_0)) \\ & \quad \hookrightarrow \; C([0,T];(H^{-1/2}(\Gamma_0),H^{5/2}(\Gamma_0))_{\frac{1}{2},\frac{1}{2}}) = C([0,T];H^1(\Gamma_0))\,, \end{aligned}$$

where the last equations holds due to (2.6). Hence the relation (4.96) is well-defined (possibly we have to redefine p on a measure zero set of $[0,T]$). We approximate p by a sequence of smooth functions $(p_n)_{n\in\mathbb{N}}$. Then we show that (4.96) holds for all p_n, $n \in \mathbb{N}$, by integration by parts. Finally, we send $n \to \infty$. For more details see proof of Theorem 3 of Chapter 5.9 in [32], which works analogously.
We multiply both sides of (4.67) by $\Delta_{\Gamma_0} p$ and integrate over Γ_0 and $(0,t)$. Using (4.96), $p(.,0) = 0$, and the definition of $\mathcal{A}(h)p \in H^{-1/2}(\Gamma_0)$ in (4.86), we obtain in the same way as in the proof of Theorem 4.2.8

$$\begin{aligned} & \sup_{\tau\in[0,t]} \|p(.,\tau)\|^2_{H^1(\Gamma_0)} + \int_0^t \|\Delta_{\Gamma_0} p(.,\tau)\|_{H^{1/2}(\Gamma_0)}\, d\tau \\ & \leq \left| \int_0^t \frac{1}{2}\sigma \langle (\mathcal{A}(0) - \mathcal{A}(h))\, p, \Delta_{\Gamma_0} p\rangle_{H^{-1/2},H^{1/2}}\, ds + \int_0^t \int_{\Gamma_0} \mathcal{F}_1(h)(p)\Delta_{\Gamma_0} p \, dx\, ds \right| , \end{aligned}$$

where $\mathcal{F}_1(h)$ contains all terms of $\mathcal{F}(h)$ which depend on p. We can estimate the first term on the right-hand side by Lemma 4.2.4. (Possibly we have to approximate p by smooth functions $(p_n)_{n\in\mathbb{N}}$ such that $p_n \to p$ in $L^2(0,t;H^{5/2}(\Gamma_0))$ as $n \to \infty$. Then Lemma 4.2.4 is valid for all p_n, $n \in \mathbb{N}$. By sending $n \to \infty$, we can also apply Lemma 4.2.4 to p.) The second term on the right-hand side can be estimated by Lemma 4.2.6. Note that we only consider $\mathcal{F}_1$. Moreover, we use the elliptic estimate (4.91) to get

$$\begin{aligned} & \sup_{\tau\in[0,t]} \|p(,.\tau)\|^2_{H^1(\Gamma_0)} + \int_0^t \|p(.,\tau)\|_{H^{5/2}(\Gamma_0)}\, d\tau \\ & \leq \tilde{C}\big(\sup_{\tau\in[0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)} \big) \|p\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))} + \|p\|_{L^2(0,t;H^{5/2}(\Gamma_0))} \|p\|_{L^2(0,t;H^2(\Gamma_0))} \\ & \leq \tilde{C}\big(\sup_{\tau\in[0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)} \big) \|p\|^2_{L^2(0,t;H^{5/2}(\Gamma_0))} + \|p\|^{\frac{5}{3}}_{L^2(0,t;H^{5/2}(\Gamma_0))} \|p\|^{\frac{1}{3}}_{L^2(0,t;H^1(\Gamma_0))} \end{aligned}$$

for some continuous $\tilde{C}(.) \geq 0$ with $\tilde{C}(0) = 0$. Choosing $t > 0$ such that $\tilde{C}\big(\sup_{\tau\in[0,t]} \|h(.,\tau)\|_{C^3(\Gamma_0)}\big) \leq \frac{1}{2}$ and applying Young's and Gronwall's inequality yield

$$\sup_{\tau\in[0,t]} \|p(,.\tau)\|^2_{H^1(\Gamma_0)} + \int_0^t \|p(.,\tau)\|_{H^{5/2}(\Gamma_0)}\, d\tau \leq 0\,.$$

Therefore it holds $p_1 = p_2$. □

Now we are able to show the main theorem of this section.

Proof of Theorem 4.2.3: It remains to show higher regularity of the solution p, which we obtain in Lemma 4.2.10. By Corollary 4.2.9 we get a uniform boundness of the solutions p^ϵ in $H^n(0,T;H^{m-1/2}(\Gamma_0))$ for any given integer n, m, and therefore it holds

$$p^\epsilon \rightharpoonup p \quad \text{in } H^n(0,T;H^{m-1/2}(\Gamma_0))\,,$$

as $\epsilon \to 0$, since $H^n(0,T;H^{m-1/2}(\Gamma_0))$ is a Hilbert space. Possibly the convergence is only valid for a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\epsilon_n \to 0$ as $n \to \infty$. $\square$

5 Nonconvergence in the Case of Small Mobility Constants

As before we assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and $\Gamma_0 \subset \Omega$ is a smooth hypersurface without boundary. Let Ω_0^- be the interior of Γ_0 and $\Omega_0^+ = \Omega \backslash \overline{\Omega_0^+}$. Let d^0 denote the signed distance to Γ_0 such that $d^0 < 0$ in Ω_0^-. Moreover, we assume that there exists some constant $C_0 \geq 1$ such that $F(c)$ is monotonically increasing for $c \geq C_0$ (that is $f(c) \geq 0$) and monotonically decreasing for $c \leq -C_0$ (that is $f(c) \leq 0$). For example, this holds for $F(c) = \frac{1}{8}\left(1-c^2\right)^2$. We consider the convective Cahn-Hilliard equation

$$\partial_t c^\epsilon + v \cdot \nabla c^\epsilon = m(\epsilon)\Delta\mu^\epsilon \qquad \text{in } \Omega \times (0,\infty)\,, \tag{5.1}$$

$$\mu^\epsilon = -\epsilon\Delta c^\epsilon + \epsilon^{-1} f(c^\epsilon) \qquad \text{in } \Omega \times (0,\infty)\,, \tag{5.2}$$

$$\tfrac{\partial}{\partial n} c^\epsilon = \tfrac{\partial}{\partial n}\mu^\epsilon = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{5.3}$$

for a given velocity field $v \in C_b^0(\mathbb{R}; C_b^4(\overline{\Omega}))$ with $\operatorname{div} v = 0$ and $v \cdot n_{\partial\Omega} = 0$ on $\partial\Omega$. Here the mobility constant $m(\epsilon)$ has the form $m(\epsilon) = \epsilon^\theta$ for some $\theta \geq 0$. To close the system, we choose the special initial value

$$c^\epsilon|_{t=0} = \zeta\left(\tfrac{d^0}{\delta}\right)\theta_0\left(\tfrac{d^0}{\epsilon}\right) + \left(1 - \zeta\left(\tfrac{d^0}{\delta}\right)\right)\left(2\chi_{\{d^0 \geq 0\}} - 1\right) \qquad \text{in } \Omega\,, \tag{5.4}$$

where θ_0 is the unique solution to the problem (2.7) and ζ is the same cut-off function as in (2.21). The constant $\delta > 0$ is determined later.

5.1 Motivation

First we give some motivation for this chapter. We consider the so-called "model H"

$$\begin{aligned}
\partial_t v^\epsilon + v^\epsilon \cdot \nabla v^\epsilon - \operatorname{div}(\nu(c^\epsilon) D v^\epsilon) + \nabla p^\epsilon &= -\epsilon \operatorname{div}(\nabla c^\epsilon \otimes \nabla c^\epsilon) && \text{in } \Omega \times (0,\infty),\\
\operatorname{div} v^\epsilon &= 0 && \text{in } \Omega \times (0,\infty),\\
\partial_t c^\epsilon + v^\epsilon \cdot \nabla c^\epsilon &= \epsilon^\theta \Delta\mu^\epsilon && \text{in } \Omega \times (0,\infty),\\
\mu^\epsilon &= -\epsilon\Delta c^\epsilon + \epsilon^{-1} f'(c^\epsilon) && \text{in } \Omega \times (0,\infty),
\end{aligned}$$

together with Dirichlet boundary conditions for v^ϵ and Neumann boundary conditions for c^ϵ and μ^ϵ. Here v^ϵ is the velocity field, $Dv^\epsilon = \frac{1}{2}(\nabla v^\epsilon + (\nabla v^\epsilon)^T)$, p^ϵ is the pressure, and $\nu(c^\epsilon) > 0$ is the viscosity of the mixture.

Using the method of formally matched asymptotic expansions, Abels et al. [4] showed that the solutions to the "model H" converge to the solutions to the following sharp interface models

$$\begin{aligned}
\partial_t v + v\cdot\nabla v - \operatorname{div}(\nu^\pm Dv) + \nabla p &= 0 && \text{in } \Omega^\pm(t), t>0\,,\\
\operatorname{div} v &= 0 && \text{in } \Omega^\pm(t), t>0\,,\\
\Delta\mu &= 0 && \text{in } \Omega^\pm(t), t>0\,,\\
[v]_{\Gamma(t)} &= 0 && \text{on } \Gamma(t), t>0\,,\\
-\left[\nu_{\Gamma(t)}\cdot(\nu^\pm Dv - p\mathrm{Id})\right]_{\Gamma(t)} &= \sigma\kappa\nu_{\Gamma(t)} && \text{on } \Gamma(t), t>0\,,\\
V - \nu_{\Gamma(t)}\cdot v &= -\left[\nu_{\Gamma(t)}\cdot\nabla\mu\right]_{\Gamma(t)} && \text{on } \Gamma(t), t>0\,,\\
\mu &= \sigma\kappa_{\Gamma(t)} && \text{on } \Gamma(t), t>0\,,
\end{aligned}$$

when $\theta = 0$ and

$$\begin{aligned}
\partial_t v + v\cdot\nabla v - \operatorname{div}(\nu^\pm Dv) + \nabla p &= 0 && \text{in } \Omega^\pm(t), t>0\,,\\
\operatorname{div} v &= 0 && \text{in } \Omega^\pm(t), t>0\,,\\
[v]_{\Gamma(t)} &= 0 && \text{on } \Gamma(t), t>0\,,\\
-\left[\nu_{\Gamma(t)}\cdot(\nu^\pm Dv - p\mathrm{Id})\right]_{\Gamma(t)} &= \sigma\kappa\nu_{\Gamma(t)} && \text{on } \Gamma(t), t>0\,,\\
V - \nu_{\Gamma(t)}\cdot v &= 0 && \text{on } \Gamma(t), t>0\,,
\end{aligned}$$

when $\theta = 1$. Here $\nu^\pm > 0$ are viscosity constants and V denotes the normal velocity of $\Gamma(t)$.

In this chapter we want to investigate what happens for "large" θ. For that we consider the simplified diffuse interface model (5.1)-(5.3). In particular, we consider the limit $\epsilon \to 0$ of the surface tension tensor in the Navier-Stokes equation of the "model H". More precisely, we calculate for $\varphi \in C^\infty([0,T];\mathcal{D}(\Omega)^d)$

$$\lim_{\epsilon\to 0}\int_0^T \langle H^\epsilon, \varphi\rangle\, dt\,,$$

where

$$\langle H^\epsilon, \varphi\rangle := \epsilon\int_\Omega \nabla c^\epsilon\otimes\nabla c^\epsilon : \nabla\varphi\, dx\,.$$

For $\theta = 0, 1$ we expect for $\varphi \in C^\infty_{0,\sigma}$ that

$$\langle H^\epsilon, \varphi\rangle \longrightarrow 2\sigma\int_{\Gamma(t)} \nu_{\Gamma(t)}\otimes\nu_{\Gamma(t)} : \nabla\varphi\, d\mathcal{H}^{d-1} = -2\sigma\int_{\Gamma(t)} \kappa\nu_{\Gamma(t)}\cdot\varphi\, d\mathcal{H}^{d-1}\,,$$

as $\epsilon \to 0$ since $\nu_{\Gamma(t)}\otimes\nu_{\Gamma(t)} : \nabla\varphi = -\operatorname{div}_{\Gamma(t)}\varphi$ due to $\operatorname{div}\varphi = 0$. Here and in the following $\sigma \in \mathbb{R}$ is defined as

$$\sigma = \frac{1}{2}\int_{\mathbb{R}} \left(\theta_0'(z)\right)^2 dz\,.$$

Note that this definition coincides with the definition of σ in Chapter 3.

But for sufficiently large θ we obtain a different result.

5.2 Nonconvergence Result

First we investigate the flow of the velocity field v and prove some properties which we will need later.

Lemma 5.2.1. *Let $v : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^d$ be a given smooth velocity field such that $v \cdot n|_{\partial\Omega} = 0$. Then there exists a unique global solution y to the problem*

$$\frac{d}{dt} y(t; y_0) = v\left(y(t; y_0), t\right), \quad y(0; y_0) = y_0$$

for all $y_0 \in \overline{\Omega}$. In particular, the flow $X(., t) =: X_t : \Omega \to \Omega$ defined by $X(y_0, t) = y(t; y_0)$ is a C^4-diffeomorphism for all $t \in \mathbb{R}$.
In addition, if $\operatorname{div} v = 0$ in Ω, then it holds

$$\det\left(DX_t(x)\right) = 1 \quad \textit{in } \Omega_T \tag{5.5}$$

and

$$\left|DX_t^{-T} \circ X_t \nabla d^0\right|^2 = \det(\partial_{\tau_i} X_t \cdot \partial_{\tau_j} X_t)_{i,j=1}^{d-1} \quad \textit{on } \Gamma_0\,, \tag{5.6}$$

where $\{\tau_1(x), \ldots, \tau_{d-1}(x)\}$ is an orthonormal basis of $T_x\Gamma_0$.

Proof: By the Picard-Lindelöf theorem, there exists a unique solution $y(.; y_0) : I_{max} \to \mathbb{R}^d$ for every $y_0 \in \Omega$ where I_{max} is the maximal interval of existence. Since $v \cdot n|_{\partial\Omega} = 0$, there exists a unique global solution $y(.; y_0)$ such that $y(t; y_0) \in \partial\Omega$ for all $t \in \mathbb{R}$ when $y_0 \in \partial\Omega$, cf. [15, § 35. 4. Bemerkung]. By the uniqueness of the solutions, it follows $y(t; y_0) \in \Omega$ for all $t \in I_{max}$ when $y_0 \in \Omega$. In particular, every solution $y(.; y_0)$ is bounded for $y_0 \in \overline{\Omega}$ and therefore it holds $I_{max} = \mathbb{R}$. Since $v(., t) \in C^4(\overline{\Omega})$ for all $t \in \mathbb{R}$, it follows from [63, III. §13 XI. Corollar] that $X_t \in C^4(\Omega)^d$. Let us show that X_t is invertible for all $t \in \mathbb{R}$. Let $t_0 \in \mathbb{R}$ be any time. Define $X_{t_0}^{-1} : \overline{\Omega} \to \overline{\Omega}$ by $X_{t_0}^{-1}(x) = \tilde{y}(-t_0; x)$ where $\tilde{y}(.; x)$ is the solution to

$$\tilde{y}'(t) = v(\tilde{y}(t), t + t_0) \text{ in } \mathbb{R}\,, \quad \tilde{y}(0) = x\,.$$

Claim: $X_{t_0}(X_{t_0}^{-1}(x)) = x$ for all $x \in \overline{\Omega}$.
By definition of $X_{t_0}^{-1}$, it holds

$$X_{t_0}(X_{t_0}^{-1}(x)) = y(t_0; \tilde{y}(-t_0; x))\,.$$

Since $y(. + t_0; \tilde{y}(-t_0; x)) : \mathbb{R} \to \mathbb{R}$ and $\tilde{y}$ are both the solution to

$$y'(t) = v(y(t), t + t_0) \text{ in } \mathbb{R}\,, \quad y(-t_0) = \tilde{y}(-t_0; x)\,,$$

it follows $\tilde{y}(t; x) = y(t + t_0; \tilde{y}(-t_0; x))$ for all $t \in \mathbb{R}$ by uniqueness of the solutions. In particular, it follows $y(t_0; \tilde{y}(-t_0; x)) = \tilde{y}(0; x) = x$. This shows the claim.
Analogously, one can show $X_{t_0}^{-1}(X_{t_0}(x)) = x$ for all $x \in \overline{\Omega}$. Hence $X_{t_0}^{-1}$ is the inverse

of X_{t_0} and $X_{t_0}^{-1} \in C^4(\Omega)^d$ by the same arguments as for X_{t_0}.
Due to [24, Satz 5.2], it holds

$$\frac{d}{dt}\det(DX_t(x)) = \operatorname{div} v(X,t)|_{X=X_t(x)} \det(DX_t(x))\,.$$

Since $\operatorname{div} v = 0$ and $X_0 = \mathrm{Id}$, we obtain

$$\det(DX_t(x)) = 1 \quad \forall (x,t) \in \Omega_T\,.$$

Using this property, we can verify the last assertion of the lemma. Since $X_t^{-1}(X_t(x)) = x$ for all $x \in \Omega$, it follows by differentiating with respect to x

$$\mathrm{Id} = DX_t^{-1} \circ X_t\, DX_t \quad \text{in } \Omega\,.$$

Due to Cramer's rule and the last equation, it follows

$$DX_t^{-T} \circ X_t = \frac{1}{\det(DX_t^T)} \operatorname{cof}(DX_t^T)^T = \operatorname{cof}(DX_t)\,.$$

Therefore we get in a neighborhood of $\Gamma(0)$

$$\begin{aligned} \left|DX_t^{-T} \circ X_t \nabla d^0\right|^2 &= \nabla d^0 \cdot \left(\operatorname{cof}(DX_t^T)\operatorname{cof}(DX_t)\right) \nabla d^0 \\ &= \nabla d^0 \cdot \left(\operatorname{cof}(DX_t^T DX_t)\right) \nabla d^0\,. \end{aligned}$$

Let Q be the change-of-basis matrix taking the orthonormal basis $\{\tau_1, \ldots, \tau_{d-1}, \nu_{\Gamma_0}\}$ to the standard basis $\{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$. Then Cramer's rule yields

$$Q^T = Q^{-1} = \frac{1}{\det Q} \operatorname{cof} Q^T = \operatorname{cof} Q^T\,,$$

and therefore it holds on Γ_0

$$\begin{aligned} \nabla d^0 \cdot \left(\operatorname{cof}(DX_t^T DX_t)\right) \nabla d^0 &= (Qe_d) \cdot \left(\operatorname{cof}(DX_t^T DX_t)\right) (Qe_d) \\ &= e_d \cdot \left(\operatorname{cof} Q^T \operatorname{cof}(DX_t^T DX_t) \operatorname{cof} Q\right) e_d \\ &= \left(\operatorname{cof}(Q^T DX_t^T DX_t\, Q)\right)_{dd} \\ &= \det(\partial_{\tau_i} X_t \cdot \partial_{\tau_j} X_t)_{i,j=1}^{d-1}\,, \end{aligned}$$

where the last equality follows due to the definition of the cofactor matrix. This completes the proof of the lemma. □

As main result of this chapter, we obtain:

Theorem 5.2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, $\Gamma_0 \subset \Omega$ a smooth hypersurface without boundary and let $(c^\epsilon, \mu^\epsilon)$ be the solution to the* *convective Cahn-Hilliard equation (5.1)-(5.3) with initial condition (5.4). Then, for every $T > 0$ and for all $\varphi \in C^\infty([0,T];\mathcal{D}(\Omega)^d)$, it holds*

$$\int_0^T \langle H^\epsilon, \varphi\rangle \, dt \longrightarrow 2\sigma \int_0^T \int_{\Gamma(t)} \left|\nabla(d^0(X_t^{-1}))\right| \nu_{\Gamma(t)} \otimes \nu_{\Gamma(t)} : \nabla\varphi \, d\mathcal{H}^{d-1} \, dt \,,$$

as $\epsilon \searrow 0$ and where the evolving hypersurface $\Gamma(t)$ is the solution to the evolution equation

$$V(x,t) = \nu_{\Gamma(t)}(x,t) \cdot v(x,t) \text{ on } \Gamma(t), t \in (0,T), \quad \Gamma(0) = \Gamma_0,$$

where V is the normal velocity of $\Gamma(t)$. Moreover, it holds

$$\|c^\epsilon - (2\chi_{Q^+} - 1)\|^2_{L^2(\Omega_T)} = \mathcal{O}(\epsilon) \,,$$

as $\epsilon \searrow 0$.

Remark 5.2.3. *In general $\left|\nabla(d^0(X_t^{-1}))\right| = \left|DX_t^{-T} \nabla d^0 \circ X_t^{-1}\right| \neq 1$. This can be shown as follows. By choosing a suitable interface Γ_0, it is sufficient to show that in general DX_t^{-T} is not length preserving. We show this by a counterexample. Let $\Omega \subset \mathbb{R}^2$ be the interior of the ellipse defined by the equation $\frac{x_1^2}{2} + \frac{x_2^2}{4} = 1$. For the velocity field $v : \overline{\Omega} \to \mathbb{R}$, we set $v(x_1,x_2) := (x_2, -2x_1)$. Note that $\operatorname{div} v = 0$ in Ω and $v \cdot n_{\partial\Omega} = 0$ on $\partial\Omega$ since $(2x_1, x_2)$ for $(x_1,x_2) \in \partial\Omega$ is a normal on $\partial\Omega$. Then the function $y : \mathbb{R} \to \Omega$ defined by $y(t) = \big(\sin(\sqrt{2}t), \sqrt{2}\cos(\sqrt{2}t)\big)$ is a solution to*

$$y'(t) = v(y(t)) \text{ in } \mathbb{R} \,, \quad y(0) = (0, \sqrt{2}) \,.$$

Since the velocity field v is independent of the time t, it follows $X_t^{-1} = X_{-t}$ where X_t is the flow of the ordinary differential equation $y' = v(y)$. Differentiating the identity $X_{-t} \circ X_t = \operatorname{Id}$ with respect to t, yields

$$0 = DX_{-t}(X_t(x)) \, v(X_t(x)) - v(x) \quad \forall x \in \Omega$$

since $\partial_t X_t = v(X_t)$. Using our special solution above, we obtain $X_{\frac{\pi}{2\sqrt{2}}}(0,\sqrt{2}) = (1,0)$. Hence we conclude

$$0 = DX_{-\frac{\pi}{2\sqrt{2}}}(1,0) \, v(1,0) - v(0,\sqrt{2}) = DX^{-1}_{\frac{\pi}{2\sqrt{2}}}(1,0) \binom{0}{-2} - \binom{\sqrt{2}}{0} \,.$$

Thus there exists a vector $w \in \mathbb{R}^2$ and $(x,t) \in \Omega \times \mathbb{R}$ such that $\left|DX_t^{-1}(x)w\right| \neq |w|$ and therefore $DX_t^{-T}(x)$ is also not length preserving.

The strategy of the proof of the theorem is the following: First we construct a family of approximate solutions $\{c_A^\epsilon\}_{0<\epsilon\leq 1}$ and estimate the difference $\nabla(c^\epsilon - c_A^\epsilon)$. Then we show that H^ϵ converges to an approximate functional H_A^ϵ as $\epsilon \searrow 0$ when $\theta > 3$. Finally, we prove the assertion of the theorem for H_A^ϵ.
We start with the observation that $\Gamma(t) := X_t(\Gamma_0)$ is the solution to the evolution equation.

Lemma 5.2.4. *Let $\Gamma_0 \subset \Omega$ be a given smooth hypersurface without boundary. Then the evolving hypersurface $\Gamma_t := \Gamma(t) := X_t(\Gamma_0) \subset \Omega$ is the solution to the problem*

$$V(x,t) = \nu(x,t) \cdot v(x,t) \quad \text{on } \Gamma_t, t > 0, \quad \Gamma(0) = \Gamma_0,$$

where V is the normal velocity and ν the unit outward normal to Γ_t.

Proof: The initial condition $\Gamma(0) = \Gamma_0$ is satisfied since $X_t(x)|_{t=0} = x$ for all $x \in \Omega$. Let $x_0 \in \Gamma_{t_0}$, $t_0 \in (0,T)$, be arbitrary. Then there exists $\tilde{x}_0 \in \Gamma_0$ such that $x_0 = X_{t_0}(\tilde{x}_0)$. By definition of the normal velocity, we obtain

$$\begin{aligned} V(x_0,t_0) &= \frac{d}{dt}X_t(x)\bigg|_{(x,t)=(\tilde{x}_0,t_0)} \cdot \nu(x_0,t_0) = v(X_{t_0}(\tilde{x}_0),t_0) \cdot \nu(x_0,t_0) \\ &= v(x_0,t_0) \cdot \nu(x_0,t_0)\,. \end{aligned}$$

This completes the proof. □

Let $d : \Omega \times \mathbb{R} \to \mathbb{R}$ be the signed distance to Γ_t satisfying $d(.,t) < 0$ inside Γ_t and $d(.,t) > 0$ outside Γ_t. Note that $d^0(x) = d(x,0)$ for all $x \in \Omega$. Let $S_0(x)$ be the projection of x on Γ_0 along the normal of Γ_0. As in Section 2.8 there exists a constant $\delta > 0$ such that $\Gamma_0(\delta) := \{x \in \Omega : |d^0(x))| < \delta\} \subset \Omega$ and $\tau_0 : \Gamma_0(\delta) \to (-\delta,\delta) \times \Gamma_0$ defined by $\tau_0(x) = (d^0(x), S_0(x))$ is a smooth diffeomorphism. Furthermore, we define Γ, $\Gamma(\delta)$, and $Q^\pm$ as in (3.65)-(3.68).

Lemma 5.2.5. *For $e : \bigcup_{t\in[0,T]} X_t(\Gamma_0(\delta)) \times \{t\} \to \mathbb{R}$ defined by $e(x,t) := d^0\left(X_t^{-1}(x)\right)$ the following properties hold:*

1. *$\frac{d}{dt}e(x,t) = -v(x,t) \cdot \nabla e(x,t)$ for all $(x,t) \in \bigcup_{t\in[0,T]} X_t(\Gamma_0(\delta)) \times \{t\}$.*
2. *$e(x,t)$ is a level set function for Γ_t, that is, $e(x,t) = 0$ if and only if $x \in \Gamma_t$.*

Proof: By definition of δ, the function e is differentiable with respect to x in $X_t(\Gamma_0(\delta))$ for all $t \in [0,T]$.
To 1: It holds for all $x \in \Omega$

$$X_t(X_t^{-1}(x)) = x\,.$$

Differentiating with respect to t and x, we get the identities

$$0 = DX_t(X_t^{-1}(x))\partial_t X_t^{-1}(x) + \partial_t X_t(X_t^{-1}(x)) \tag{5.7}$$
$$\text{and } \mathrm{Id} = DX_t(X_t^{-1}(x))DX_t^{-1}(x). \tag{5.8}$$

Hence we get by the definition of e

$$\begin{aligned} \frac{d}{dt}e(x,t) &= \frac{d}{dt}\left(d^0(X_t^{-1}(x))\right) = \nabla d^0(X_t^{-1}(x)) \cdot \partial_t X_t^{-1}(x) \\ &= -\nabla d^0(X_t^{-1}(x)) \cdot DX_t^{-1}\partial_t X_t(X_t^{-1}(x)) \\ &= -\nabla d^0(X_t^{-1}(x)) \cdot DX_t^{-1} v(X_t(X_t^{-1}(x)),t) \\ &= -\nabla\left(d^0(X_t^{-1}(x))\right) \cdot v(x,t) \\ &= -\nabla e(x,t) \cdot v(x,t)\,, \end{aligned}$$

where we have used (5.7) and (5.8) in the third equation.
To 2: The following equivalence transformations hold since $X_t : \Omega \to \Omega$ is a diffeomorphism

$$d^0(X_t^{-1}(x)) = 0 \Leftrightarrow X_t^{-1}(x) \in \Gamma_0 \Leftrightarrow \exists y \in \Gamma_0 \text{ s.t. } X_t^{-1}(x) = y \\ \Leftrightarrow \exists y \in \Gamma_0 \text{ s.t. } x = X_t(y) \Leftrightarrow x \in X_t(\Gamma_0)\,.$$

This shows that e is a level set function for Γ_t. □

As in Chapter 3.1 let θ_0 be the solution to (2.22) and let ζ be a cut-off function as in (2.21). Then we define

$$c_A^\epsilon(x,t) := \begin{cases} \pm 1 & \text{in } \overline{Q^\pm} \cap \bigcup_{t\in[0,T]} \overline{X_t(\Omega\backslash\Gamma_0(\delta))} \times \{t\}\,, \\ \zeta\left(\frac{e}{\delta}\right)\theta_0\left(\frac{e}{\epsilon}\right) \pm \left(1-\zeta\left(\frac{e}{\delta}\right)\right) & \text{in } Q^\pm \cap \bigcup_{t\in[0,T]} X_t(\Gamma_0(\delta)\backslash\Gamma_0\left(\frac{\delta}{2}\right)) \times \{t\}, \\ \theta_0\left(\frac{e}{\epsilon}\right) & \text{in } \bigcup_{t\in[0,T]} X_t(\Gamma_0\left(\frac{\delta}{2}\right)) \times \{t\}\,. \end{cases}$$

Note that $c_A^\epsilon(.,0) = c^\epsilon(.,0)$ since $e(.,0) = d^0$ and $\partial_t c_A^\epsilon + v \cdot \nabla c_A^\epsilon = 0$ in Ω_T since $\partial_t e + v \cdot \nabla e = 0$ (this is the reason why we use $\zeta\left(\frac{e}{\delta}\right)$ instead of $\zeta\left(\frac{d}{\delta}\right)$ as we have used in Subsection 3.2.9). Moreover, c_A^ϵ and Δc_A^ϵ satisfy Neumann boundary conditions on $\partial\Omega$, since $c_A^\epsilon = 1$ in a neighborhood of the boundary $\partial\Omega$.
Furthermore, we define for all $\varphi \in \mathcal{D}(\Omega)^d$ the functional $H_A^\epsilon : \mathcal{D}(\Omega)^d \to \mathbb{R}$ by

$$\langle H_A^\epsilon, \varphi\rangle = \epsilon \int_\Omega \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon : \nabla\varphi \, dx\,.$$

Lemma 5.2.6. *Let c_A^ϵ be defined as above. Then there exists some constant $C > 0$ independent of ϵ and $\epsilon_0 \in (0,1]$ such that the estimates*

$$\|\Delta c_A^\epsilon(.,t)\|_{L^2(\Omega)} \leq C\epsilon^{-\frac{3}{2}}\,, \tag{5.9}$$

$$\|\nabla c_A^\epsilon(.,t)\|_{L^2(\Omega)} \leq C\epsilon^{-\frac{1}{2}}\,, \tag{5.10}$$

$$\|f(c_A^\epsilon(.,t))\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}}\,, \tag{5.11}$$

$$\|c_A^\epsilon(.,t) - (2\chi_{Q^+}(.,t) - 1)\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}} \tag{5.12}$$

hold for all $t \in [0,T]$ and $\epsilon \in (0,\epsilon_0)$.

Proof: We obtain for all $(x,t) \in \bigcup_{t\in[0,T]} X_t(\Gamma_0\left(\frac{\delta}{2}\right)) \times \{t\}$

$$\Delta c_A^\epsilon(x,t) = \epsilon^{-2}\,|\nabla e|^2\,\theta_0''\left(\tfrac{e}{\epsilon}\right) + \epsilon^{-1}\Delta e\,\theta_0'\left(\tfrac{e}{\epsilon}\right).$$

Hence there exists some constant $C > 0$ independent of ϵ and $t \in [0,T]$ such that

$$\|\Delta c_A^\epsilon(.,t)\|_{L^2(X_t(\Gamma_0(\delta/2)))} \leq C\left(\epsilon^{-2}\left\|\theta_0''\left(\tfrac{e}{\epsilon}\right)\right\|_{L^2(X_t(\Gamma_0(\delta/2)))} + \epsilon^{-1}\right).$$

Using $\theta_0''(z) \le Ce^{-\alpha|z|}$ (see Lemma 2.6.1) for all $z \in \mathbb{R}$ and for some $C > 0$, we conclude

$$\begin{aligned}
\|\theta_0''(e/\epsilon)\|^2_{L^2(X_t(\Gamma_0(\delta/2)))} &\le C \int_{X_t(\Gamma_0(\delta/2))} e^{-2\alpha\left|d^0(X_t^{-1}(x))/\epsilon\right|}\, dx \\
&= C \int_{\Gamma_0(\delta/2)} e^{-2\alpha\left|d^0(x)/\epsilon\right|} \det|DX_t(x)|\, dx \\
&= C \int_{\Gamma_0(\delta/2)} e^{-2\alpha\left|d^0(x)/\epsilon\right|}\, dx\,,
\end{aligned}$$

where we have used (5.5). Using the identity $\int_{\Gamma_0(\delta/2)} f(x)\, dx = \int_{-\delta/2}^{\delta/2} \int_{\Gamma_0^r} f(x)\, d\mathcal{H}^{n-1}\, dr$ for all integrable functions f where $\Gamma_0^r = \{x \in \Omega : x = s + r\,\nu_{\Gamma_0}(s), s \in \Gamma_0\}$ for $r \in \mathbb{R}$ (we will show this identity at the end of the proof), one gets

$$\begin{aligned}
\|\theta_0''(e/\epsilon)\|^2_{L^2(X_t(\Gamma_0(\delta/2)))} &\le C \int_{-\delta/2}^{\delta/2} \int_{\Gamma_0^r} e^{-2\alpha\left|d^0/\epsilon\right|}\, d\mathcal{H}^{n-1}\, dr \\
&= C \int_{-\delta/2}^{\delta/2} e^{-2\alpha|r/\epsilon|} \int_{\Gamma_0^r} 1\, d\mathcal{H}^{n-1}\, dr \\
&\le C\epsilon
\end{aligned}$$

for some $C = C(\Gamma_0) > 0$ independent of ϵ and $t \in [0, T]$. Again using Lemma 2.6.1, we obtain in $\bigcup_{t\in[0,T]} X_t(\Gamma_0(\delta)\backslash\Gamma_0\left(\frac{\delta}{2}\right)) \times \{t\}$

$$\begin{aligned}
\Delta c_A^\epsilon(x,t) &= \epsilon^{-2}\,|\nabla e|^2\,\theta_0''\left(\tfrac{e}{\epsilon}\right)\zeta\left(\tfrac{e}{\delta}\right) + \epsilon^{-1}\Delta e\,\theta_0'\left(\tfrac{e}{\epsilon}\right)\zeta\left(\tfrac{e}{\delta}\right) + 2\epsilon^{-1}\theta_0'\left(\tfrac{e}{\epsilon}\right)\nabla e\cdot\nabla\left(\zeta\left(\tfrac{e}{\delta}\right)\right) \\
&\quad + \left(\theta_0\left(\tfrac{e}{\epsilon}\right) - (2\chi_{\overline{Q^+}} - 1)\right)\Delta\left(\zeta\left(\tfrac{e}{\delta}\right)\right) \\
&= \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{4\epsilon}})\,.
\end{aligned}$$

Altogether, this shows (5.9) for all $\epsilon > 0$ small enough.
The second estimate can be shown by the same arguments.
Using $f(\pm 1) = 0$ and the Taylor expansion, it follows

$$f(c_A^\epsilon) = f'((2\chi_{\overline{Q^+}} - 1) + \Theta(c_A^\epsilon - 2\chi_{\overline{Q^+}} + 1))\left(c_A^\epsilon - (2\chi_{\overline{Q^+}} - 1)\right)$$

for some $\Theta = \Theta(x,t) \in (0,1)$. For all $(x,t) \in \bigcup_{t\in[0,T]} X_t(\Gamma_0(\frac{\delta}{2})) \times \{t\}$, there exists some constant $C > 0$ independent of x and t such that

$$\left|c_A^\epsilon - (2\chi_{\overline{Q^+}} - 1)\right| = \left|\theta_0\left(\tfrac{e}{\epsilon}\right) - (2\chi_{\overline{Q^+}} - 1)\right| \le Ce^{-\frac{\alpha|e|}{2\epsilon}}$$

due to Lemma 2.6.1. In $\bigcup_{t\in[0,T]} X_t(\Gamma_0(\delta)\backslash\Gamma_0\left(\frac{\delta}{2}\right)) \times \{t\}$ we get due to the definition of c_A^ϵ

$$\left|c_A^\epsilon - (2\chi_{\overline{Q^+}} - 1)\right| \le Ce^{-\frac{\alpha\delta}{4}}\,.$$

In $\bigcup_{t\in[0,T]} \overline{X_t(\Omega\backslash\Gamma_0(\delta))} \times \{t\}$ we have $c_A^\epsilon = 2\chi_{\overline{Q^+}} - 1$. Then we can apply the same estimate as above to prove the third and also the forth assertion.
It remains to show the integral identity $\int_{\Gamma_0(\delta/2)} f(x)\,dx = \int_{-\delta/2}^{\delta/2}\int_{\Gamma_0^r} f(x)\,d\mathcal{H}^{n-1}\,dr$. We choose an relatively open set $U \subset \Gamma_0$ such that U can be described as a graph, i.e. (possibly after rotation) there exists an open set $D \subset \mathbb{R}^{d-1}$ and a function $g : D \to \mathbb{R}$ such that $U = \{(y, g(y)) : y \in D\}$. Define the sets $U(\delta)$ and U_r, $r \in (-\delta, \delta)$, by

$$U(\delta) = \{x + r\nu_{\Gamma_0}(x) : x \in U, r \in (-\delta,\delta)\}\,, \qquad U_r = \{x + r\nu_{\Gamma_0}(x) : x \in U\}\,.$$

Then the function $\Phi : (-\delta,\delta)\times D \to U(\delta)$ defined by $\Phi(r, y) = (y, g(y))+r\nu_{\Gamma_0}(y, g(y))$ is a smooth diffeomorphism. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary integrable function. By coordinate transformation, we obtain

$$\begin{aligned}\int_{U(\delta)} f(x)\,dx &= \int_{-\delta}^{\delta}\int_D (f\circ\Phi)(r,y)\,|\det(D\Phi(r,y))|\,dy\,dr\\ &= \int_{-\delta}^{\delta}\int_D (f\circ\Phi)(r,y)\,\left|\det(D\Phi(r,y)^T D\Phi(r,y))\right|^{\frac{1}{2}}\,dy\,dr\,.\end{aligned}$$

We continue with calculating $\det(D\Phi(r,y)^T D\Phi(r,y))$. For all $i \in \{1,\dots d-1\}$, it follows

$$\partial_r\Phi(r,y)\cdot\partial_{y_i}\Phi(r,y) = \nu_{\Gamma_0}(y,g(y))\cdot[(e_i,\partial_i g(y)) + \partial_i(\nu_{\Gamma_0}(y,g(y)))] = 0\,,$$

where $e_i \in \mathbb{R}^{d-1}$ is the i-th standard unit vector. Here we have used that

$$\nu_{\Gamma_0}(y,g(y))\cdot(e_i,\partial_i g(y)) = \tfrac{1}{\sqrt{|\nabla g|^2+1}}\begin{pmatrix}-\nabla g\\ 1\end{pmatrix}\cdot\begin{pmatrix}e_i\\ \partial_i g\end{pmatrix} = 0$$

and $\nu_{\Gamma_0}(y,g(y))\cdot\partial_i(\nu_{\Gamma_0}(y,g(y))) = 0$. Hence we get

$$\det(D\Phi(r,y)^T D\Phi(r,y)) = \det\left(\begin{array}{c|c}1 & 0\\ \hline 0 & D_y\Phi^T D_y\Phi\end{array}\right) = \det(D_y\Phi^T(r,y)D_y\Phi(r,y))\,.$$

Therefore we obtain the identity

$$\begin{aligned}\int_{U(\delta)} f(x)\,dx &= \int_{-\delta}^{\delta}\int_D (f\circ\Phi)(r,y)\,\left|\det(D_y\Phi(r,y)^T D_y\Phi(r,y))\right|^{\frac{1}{2}}\,dy\,dr\\ &= \int_{-\delta}^{\delta}\int_{U_r} f(x)\,d\mathcal{H}^{d-1}\,dr\,,\end{aligned}$$

where the last equality follows from $\Phi(r,D) = U_r$ for all $r \in (-\delta,\delta)$, that is, $\Phi(r,.) : D \to U_r$ is a chart for U_r. Using partition of the unity, the assertion follows. This completes the proof of the lemma. □

Lemma 5.2.7. *Let c_ϵ^A be defined as above and let c^ϵ be the unique solution to (5.1)-(5.3) with initial condition (5.4). Then, for $\theta > 3$, there exists some constant $C > 0$ independent of ϵ and $\epsilon_0 > 0$ such that*

$$\epsilon \left\|\nabla(c^\epsilon - c_A^\epsilon)\right\|^2_{L^2(\Omega_T)} \le C\epsilon^{\frac{\theta-3}{2}}\,, \tag{5.13}$$

$$\textit{and}\ \ \|c^\epsilon - c_A^\epsilon\|^2_{L^2(\Omega_T)} \le C\epsilon^{\theta-2} \tag{5.14}$$

for all $\epsilon \in (0, \epsilon_0]$.

Proof: Let $R = c^\epsilon - c_A^\epsilon$ be the remainder. Since $\partial_t c_A^\epsilon + v \cdot \nabla c_A^\epsilon = 0$ in Ω_T, it holds

$$\begin{aligned}
\int_\Omega R(.,t)\,dx &= \int_0^t \int_\Omega \partial_t R\,dx\,dt = -\int_0^t \int_\Omega v \cdot \nabla R\,dx\,dt + \epsilon^\theta \int_0^t \int_\Omega \Delta\mu^\epsilon\,dx\,dt \\
&= \int_0^t \int_\Omega \mathrm{div} v\,R\,dx\,dt - \int_0^t \int_{\partial\Omega} v \cdot nR\,d\mathcal{H}^{d-1}\,dt \\
&\quad + \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial n}\mu^\epsilon\,d\mathcal{H}^{d-1}\,dt = 0
\end{aligned}$$

for all $t \in [0,T]$. Hence we can find a unique solution $\Psi : \Omega_T \to \mathbb{R}$ to the problem

$$-\Delta\Psi(.,t) = R(.,t) \text{ in } \Omega\,, \qquad \frac{\partial}{\partial n}\Psi(.,t) = 0 \text{ on } \partial\Omega\,, \qquad \int_\Omega \Psi(.,t) = 0$$

for every $t \in [0,T]$. We multiply the difference of the differential equations for c^ϵ and c_A^ϵ by Ψ and integrate the resulting equation over Ω. Then we get for all $t \in (0,t)$

$$\begin{aligned}
0 &= \int_\Omega \Psi\left[\partial_t R + v \cdot \nabla R + \epsilon^{\theta+1}\Delta^2 R + \epsilon^{\theta+1}\Delta^2 c_A^\epsilon - \epsilon^{\theta-1}\Delta f(c^\epsilon)\right]dx \\
&= \int_\Omega \Psi(-\Delta\partial_t\Psi) - \nabla\Psi \cdot vR + \epsilon^{\theta+1}\Delta\Psi\Delta R\,dx \\
&\quad + \int_\Omega \epsilon^{\theta+1}\Delta\Psi\Delta c_A^\epsilon - \epsilon^{\theta-1}\Delta\Psi f(c^\epsilon)\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\,dx + \int_\Omega -\nabla\Psi \cdot vR + \epsilon^{\theta+1}|\nabla R|^2\,dx \\
&\quad + \int_\Omega -\epsilon^{\theta+1}R\Delta c_A^\epsilon + \epsilon^{\theta-1}Rf(c^\epsilon)\,dx\,,
\end{aligned}$$

where we have used the Neumann boundary conditions $\frac{\partial}{\partial n}\Delta c_A^\epsilon = \frac{\partial}{\partial n}\Psi = \frac{\partial}{\partial n}f(c^\epsilon) = 0$, $v \cdot n = 0$ and $\frac{\partial}{\partial n}\Delta c^\epsilon = -\epsilon^{-1}\frac{\partial}{\partial n}\mu^\epsilon + \epsilon^{-2}\frac{\partial}{\partial n}f(c^\epsilon) = 0$ on $\partial\Omega$ and $\mathrm{div}\,v = 0$ in Ω.
By the assumptions $f(c^\epsilon) \ge 0$ for $c^\epsilon \ge C_0 \ge 1 \ge c_A^\epsilon$ and $f(c^\epsilon) \le 0$ for $c^\epsilon \le -C_0 \le -1 \le c_A^\epsilon$, we obtain

$$\int_{\{x\in\Omega\,:\,|c^\epsilon(x,t)|\ge C_0\}} f(c^\epsilon)R\,dx \ge 0\,.$$

Hence Hölder's and Young's inequalities yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\,dx + \epsilon^{\theta+1}\int_\Omega |\nabla R|^2\,dx + \epsilon^{\theta-1}\int_{\{x\in\Omega\,:\,|c^\epsilon(x,t)|\geq C_0\}} f(c^\epsilon)R\,dx \\
&\leq \left|\int_\Omega \nabla\Psi\cdot vR\,dx\right| + \epsilon^{\theta+1}\|R\|_{L^2(\Omega)}\|\Delta c_A^\epsilon\|_{L^2(\Omega)} \\
&\quad + \epsilon^{\theta-1}\left|\int_{\{x\in\Omega\,:\,|c^\epsilon(x,t)|<C_0\}} f(c^\epsilon)R\,dx\right|. \qquad (5.15)
\end{aligned}
$$

We estimate the right-hand side. By integration by parts and due to $v\cdot n=0$ on $\partial\Omega$, the identity

$$
\int_\Omega \partial_{ij}\Psi v_j\partial_i\Psi\,dx = -\int_\Omega \partial_i\Psi\partial_j v_j\partial_i\Psi\,dx - \int_\Omega \partial_i\Psi v_j\partial_{ij}\Psi\,dx = -\int_\Omega \partial_i\Psi v_j\partial_{ij}\Psi\,dx
$$

yields

$$
\int_\Omega \nabla\Psi\cdot\big(D^2\Psi\,v\big)\,dx = 0\,.
$$

Therefore we obtain the following estimate for the first term in (5.15) on the right-hand side

$$
\begin{aligned}
\int_\Omega \nabla\Psi\cdot vR\,dx &= -\int_\Omega \nabla\Psi\cdot v\Delta\Psi\,dx = \int_\Omega \nabla\Psi\cdot\big(D^2\Psi\,v\big) + \nabla v:(\nabla\Psi\otimes\nabla\Psi)\,dx \\
&= \int_\Omega \nabla v:(\nabla\Psi\otimes\nabla\Psi)\,dx \leq \|\nabla v\|_{L^\infty(\Omega_T)}\|\nabla\Psi\|^2_{L^2(\Omega)}\,, \qquad (5.16)
\end{aligned}
$$

for all $t\in[0,T]$ and where we have used the boundary condition $\frac{\partial}{\partial n}\Psi=0$ on $\partial\Omega$. Using Taylor expansion yields for the last term in (5.15) on the right-hand side, we get for all $\epsilon\in(0,\epsilon_0)$

$$
\begin{aligned}
\int_{\{x\in\Omega\,:\,|c^\epsilon(x,t)|<C_0\}} f(c^\epsilon)R\,dx &= \int_{\{x\in\Omega\,:\,|c^\epsilon(x,t)|<C_0\}} f(c_A^\epsilon)R + f'(c_A^\epsilon+\Theta R)R^2\,dx \\
&\leq \|f(c_A^\epsilon)\|_{L^2(\Omega)}\|R\|_{L^2(\Omega)} + C\|R\|^2_{L^2(\Omega)} \\
&\leq C\epsilon^{\frac{1}{2}}\|R\|_{L^2(\Omega)} + C\|R\|^2_{L^2(\Omega)} \qquad (5.17)
\end{aligned}
$$

for some $\Theta=\Theta(x,t)\in(0,1)$ and some constant $C>0$ independent of ϵ and $t\in[0,T]$. Here we have used inequality (5.11) with the same constant $\epsilon_0>0$. Therefore estimate (5.15) turns with (5.16), (5.17), and (5.9) into

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\,dx + \frac{\epsilon^{\theta+1}}{2}\int_\Omega |\nabla R|^2\,dx \\
&\leq C_1\Big(\|\nabla\Psi\|^2_{L^2(\Omega)} + \epsilon^{\theta-\frac{1}{2}}\|R\|_{L^2(\Omega)} + \epsilon^{\theta-\frac{1}{2}}\|R\|_{L^2(\Omega)} + \epsilon^{\theta-1}\|R\|^2_{L^2(\Omega)}\Big)
\end{aligned}
$$

for some $C_1 > 0$ independent of ϵ and $t \in [0,T]$. Since

$$\|R\|_{L^2(\Omega)}^2 = -\int_\Omega R\Delta\Psi\, dx = \int_\Omega \nabla R \cdot \nabla \Psi\, dx \leq \|\nabla R\|_{L^2(\Omega)} \|\nabla \Psi\|_{L^2(\Omega)} \,,$$

we obtain by Young's inequality

$$\epsilon^{\theta-\frac{1}{2}} \|R\|_{L^2(\Omega)} \leq \epsilon^{\frac{3\theta-3}{2}} + C\|\nabla\Psi\|_{L^2(\Omega)}^2 + \frac{\epsilon^{\theta+1}}{16C_1}\|\nabla R\|_{L^2(\Omega)}^2$$

and

$$\epsilon^{\theta-1}\|R\|_{L^2(\Omega)}^2 \leq \frac{\epsilon^{\theta+1}}{8C_1}\|\nabla R\|_{L^2(\Omega)}^2 + C\epsilon^{\theta-3}\|\nabla\Psi\|_{L^2(\Omega)}^2 \,.$$

Using the last two inequalities and $\theta > 3$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\, dx + \frac{\epsilon^{\theta+1}}{4}\int_\Omega |\nabla R|^2\, dx \leq C\left(\|\nabla\Psi\|_{L^2(\Omega)}^2 + \epsilon^{\frac{3\theta-3}{2}}\right) \tag{5.18}$$

for some $C > 0$ independent of ϵ. Since $R(.,0) = 0$, it follows $\Psi(.,0) = 0$. Then the Gronwall inequality yields

$$\sup_{0\leq t\leq T}\|\nabla\Psi(.,t)\|_{L^2(\Omega)}^2 \leq C\epsilon^{\frac{3\theta-3}{2}}$$

for some $C = C(T) > 0$ independent of ϵ. Integrating (5.18) over $(0,T)$, yields

$$\epsilon^{\theta+1}\|\nabla R\|_{L^2(\Omega_T)}^2 \leq C\left(\|\nabla\Psi\|_{L^2(\Omega_T)}^2 + \epsilon^{\frac{3\theta-3}{2}}\right) \leq C\epsilon^{\frac{3\theta-3}{2}}$$

for some $C > 0$ independent of ϵ. Furthermore, it follows

$$\|R\|_{L^2(\Omega_T)}^2 \leq \|\nabla\Psi\|_{L^2(\Omega_T)}\|\nabla R\|_{L^2(\Omega_T)} \leq C\epsilon^{\frac{3\theta-3}{4}}\epsilon^{\frac{\theta-5}{4}} = C\epsilon^{\theta-2} \,.$$

Hence the assertions of the lemma follow. □

Now we can show that H^ϵ converges to H_A^ϵ as ϵ goes to zero.

Lemma 5.2.8. *Let H^ϵ and H_A^ϵ be defined as above and let $\theta > 3$. Then it holds for all $\varphi \in C^\infty([0,T];\mathcal{D}(\Omega)^d)$*

$$\left|\int_0^T \langle H^\epsilon - H_A^\epsilon, \varphi\rangle\, dt\right| \longrightarrow 0 \,,$$

as $\epsilon \searrow 0$.

Proof: We choose any $\varphi \in C^\infty([0,T];\mathcal{D}(\Omega)^d)$ and set $R = c^\epsilon - c_A^\epsilon$. Then it holds by triangle inequality

$$\begin{aligned}
&\epsilon\left|\int_{\Omega_T}(\nabla c^\epsilon \otimes \nabla c^\epsilon - \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon) : \nabla\varphi\, dx\right| \\
&\leq \epsilon\left|\int_{\Omega_T}(\nabla c^\epsilon \otimes \nabla R) : \nabla\varphi\, dx\right| + \epsilon\left|\int_{\Omega_T}(\nabla R \otimes \nabla c_A^\epsilon) : \nabla\varphi\, dx\right| \\
&\leq \epsilon\,\|\nabla\varphi\|_{L^\infty(\Omega_T)}\,\|\nabla R\|_{L^2(\Omega_T)}\left(\|\nabla c^\epsilon\|_{L^2(\Omega_T)} + \|\nabla c_A^\epsilon\|_{L^2(\Omega_T)}\right).
\end{aligned}$$

Due to Lemma 5.2.6, we have

$$\|\nabla c_A^\epsilon\|^2_{L^2(\Omega_T)} \leq C\epsilon^{-1}$$

for some $C > 0$ independent of ϵ. Since

$$\|\nabla c^\epsilon\|_{L^2(\Omega_T)} \leq \|\nabla c_A^\epsilon\|_{L^2(\Omega_T)} + \|\nabla R\|_{L^2(\Omega_T)}$$

and due to Lemma 5.2.7, it follows

$$\left|\int_0^T \langle H^\epsilon - H_A^\epsilon, \varphi\rangle\, dt\right| \leq C\epsilon^{\frac{\theta-3}{4}}\left(1 + \epsilon^{\frac{\theta-3}{4}}\right)$$

for some constant $C = C(\varphi) > 0$ and for all ϵ small enough. Since $\theta > 3$ the assertion follows. □

Lemma 5.2.9. *Let H_A^ϵ and c_A^ϵ be defined as above. Then it holds for all $\varphi \in \mathcal{D}(\Omega)^d$ and $t \in [0,T]$*

$$\langle H_A^\epsilon, \varphi\rangle \longrightarrow 2\sigma\int_{\Gamma(t)}\left|\nabla(d^0(X_t^{-1}))\right|\nu_{\Gamma(t)} \otimes \nu_{\Gamma(t)} : \nabla\varphi\, d\mathcal{H}^{d-1},$$

as $\epsilon \searrow 0$.

Proof: Let $\varphi \in \mathcal{D}(\Omega)^d$ and $t \in [0,T]$ be arbitrary. Observe that

$$|\nabla c_A^\epsilon| = \left|\epsilon^{-1}\zeta\left(\tfrac{e}{\delta}\right)\theta_0'\left(\tfrac{e}{\epsilon}\right)\nabla e + \left(\theta_0\left(\tfrac{e}{\epsilon}\right) - \left(2\chi_{\overline{Q^+}} - 1\right)\right)\nabla\left(\zeta\left(\tfrac{e}{\delta}\right)\right)\right| \leq C\epsilon^{-1}e^{-\frac{\alpha\delta}{4\epsilon}}$$

in $\bigcup_{t\in[0,T]} X_t(\Gamma_0(\delta)\backslash\Gamma_0(\frac{\delta}{2})) \times \{t\}$ and $\nabla c_A^\epsilon = 0$ in $\bigcup_{t\in[0,T]}\overline{X_t(\Omega\backslash\Gamma_0(\delta))} \times \{t\}$. Hence we can replace c_A^ϵ by $\theta_0(\frac{e}{\epsilon})$ in the whole domain $\overline{\Omega_T}$ since the remainder decays exponentially as $\epsilon \to 0$.
Since $X_t : \Omega \to \Omega$ is a diffeomorphism, we obtain by coordinate transformation

$$\begin{aligned}
\langle H_A^\epsilon, \varphi\rangle &= \epsilon\int_\Omega \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon : \nabla\varphi\, dx \\
&= \epsilon\int_\Omega \nabla c_A^\epsilon \circ X_t \otimes \nabla c_A^\epsilon \circ X_t : \nabla\varphi \circ X_t\,|\det(DX_t)|\, dx\,.
\end{aligned}$$

Due to Lemma 5.2.1, it holds

$$\det(DX_t(x)) = 1 \quad \forall (x,t) \in \Omega_T\,, \tag{5.19}$$

and the identity (5.8) yields

$$\nabla c_A^\epsilon \circ X_t = DX_t^{-T} \circ X_t \, \nabla (c_A^\epsilon \circ X_t)\,. \tag{5.20}$$

By equations (5.19) and (5.20), we conclude

$$\langle H_A^\epsilon, \varphi\rangle = \epsilon \int_\Omega M\, \nabla(c_A^\epsilon \circ X_t) \otimes M\, \nabla(c_A^\epsilon \circ X_t) : \nabla\varphi \circ X_t \, dx\,,$$

where $M = M(x,t) := (DX_t^{-T} \circ X_t)(x)$. Using $c_A^\epsilon = \theta_0(d^0 \circ X_t^{-1}/\epsilon)$ yields

$$\langle H_A^\epsilon, \varphi\rangle = \epsilon^{-1} \int_\Omega \Big(\theta_0'\Big(\tfrac{d^0}{\epsilon}\Big)\Big)^2 M\, \nabla d^0 \otimes M\, \nabla d^0 : \nabla\varphi \circ X_t \, dx\,. \tag{5.21}$$

Now we consider the limit $\epsilon \searrow 0$.
Claim: Let $f \in C^1(\overline{\Omega})$ be an arbitrary function. Then it holds

$$\epsilon^{-1} \int_\Omega \Big(\theta_0'\Big(\tfrac{d^0}{\epsilon}\Big)\Big)^2 f\, dx \longrightarrow 2\sigma \int_{\Gamma_0} f \, d\mathcal{H}^{d-1}\,, \tag{5.22}$$

as $\epsilon \searrow 0$.
Proof of the claim: Since there exists some constant $C > 0$ such that

$$\Big|\theta_0'\Big(\tfrac{d^0(x)}{\epsilon}\Big)\Big| \le C e^{-\frac{\alpha |d^0(x)|}{\epsilon}} \quad \forall x \in \Omega\,,$$

it is sufficient to consider the domain $\Gamma_0(\delta)$ instead of Ω. Hence we have the identity

$$\epsilon^{-1} \int_{\Gamma_0(\delta)} \Big(\theta_0'\Big(\tfrac{d^0}{\epsilon}\Big)\Big)^2 f\, dx = \epsilon^{-1} \int_{-\delta}^{\delta} \big(\theta_0'\big(\tfrac{r}{\epsilon}\big)\big)^2 \int_{\Gamma_0^r} f\, d\mathcal{H}^{d-1}\, dr\,,$$

where $\Gamma_0^r = \{x \in \Omega : x = s + r\nu_{\Gamma_0}(s), s \in \Gamma_0\}$, see at the end of the proof of Lemma 5.2.6. Define $\Psi_f : \mathbb{R} \to \mathbb{R}$ by

$$\Psi_f(r) = \begin{cases} \int_{\Gamma_0^r} f\, d\mathcal{H}^{d-1} & \text{if } r \in (-\delta,\delta) \\ 0 & \text{if } r \in \mathbb{R}\backslash(-\delta,\delta)\,, \end{cases}$$

$\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(r) = \frac{1}{2\sigma}\left(\theta_0'(r)\right)^2\,,$$

and $\varphi_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\epsilon(r) = \epsilon^{-1}\varphi\Big(\frac{r}{\epsilon}\Big)$$

for $\epsilon \in (0,1]$. Then φ is a one-dimensional mollifier. Therefore we obtain

$$\begin{aligned}\epsilon^{-1}\int_{\Gamma_0(\delta)}\left(\theta_0'\left(\tfrac{d^0}{\epsilon}\right)\right)^2 f\,dx &= 2\sigma\int_{\mathbb{R}}\varphi_\epsilon(r)\Psi_f(r)\,dr\\ &= 2\sigma\int_{\mathbb{R}}\varphi_\epsilon(r)\Psi_f(-(0-r))\,dr\\ &= 2\sigma\varphi_\epsilon * \tilde{\Psi}_f(0)\,,\end{aligned}$$

where $\tilde{\Psi}_f(r) = \Psi(-r)$ and $\varphi_\epsilon * \tilde{\Psi}_f(0)$ denotes convolution of φ_ϵ and $\tilde{\Psi}_f(0)$. To show the convergence, it is necessary to estimate $|\Psi_f(r) - \Psi_f(0)|$ for $r \le \delta$. By definition of Ψ_f, we obtain

$$\begin{aligned}|\Psi_f(r)-\Psi_f(0)| &= \left|\int_{\Gamma_0^r} f\,d\mathcal{H}^{d-1} - \int_{\Gamma_0} f\,d\mathcal{H}^{d-1}\right|\\ &= \left|\int_{\Gamma_0} f\circ\tau_r\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12} - f\,d\mathcal{H}^{d-1}\right|\\ &\le \left|\int_{\Gamma_0}(f\circ\tau_r - f)\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12}d\mathcal{H}^{d-1}\right|\\ &\quad+\left|\int_{\Gamma_0} f\left(1-\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12}\right)d\mathcal{H}^{d-1}\right|,\end{aligned}$$

where we have used the transformation $\tau_r : \Gamma_0 \to \Gamma_0^r$ defined by $\tau_r(x) = x + r\nu_{\Gamma_0}(x)$ and where $\{\tau_1(x),\dots,\tau_{d-1}(x)\}$ is an orthonormal basis of $T_x\Gamma_0$, cf. [12, Aufgabe 53]. We separately estimate the two terms on the right-hand side. The fundamental theorem of calculus yields

$$\begin{aligned}&\left|\int_{\Gamma_0}(f\circ\tau_r - f)\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12}d\mathcal{H}^{d-1}\right|\\ &= \left|\int_{\Gamma_0}\left(\int_0^r\nabla f(x+s\nu_{\Gamma_0})\cdot\nu_{\Gamma_0}\,ds\right)\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12}d\mathcal{H}^{d-1}\right|\\ &\le C\,|r|\,\|f\|_{C^1(\overline{\Omega})}\end{aligned}$$

for some $C = C(\Gamma_0) > 0$ independent of r. We continue with the second term. Note that

$$\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r = \delta_{ij} + r\left(\tau_i\cdot\partial_{\tau_j}\nu_{\Gamma_0} + \tau_j\cdot\partial_{\tau_i}\nu_{\Gamma_0} + r\partial_{\tau_i}\nu_{\Gamma_0}\cdot\partial_{\tau_j}\nu_{\Gamma_0}\right).$$

Hence we can conclude

$$\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right| = 1 + \mathcal{O}(r)\,,$$

and therefore it follows

$$1-\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12} = \frac{1-\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|}{1+\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac12}} = \mathcal{O}(r)\,.$$

Thus we get the following estimate

$$\left|\int_{\Gamma_0} f\left(1-\left|\det(\partial_{\tau_i}\tau_r\cdot\partial_{\tau_j}\tau_r)_{i,j=1}^{d-1}\right|^{\frac{1}{2}}\right)d\mathcal{H}^{d-1}\right|\le C\,|r|\,\|f\|_{C^0(\overline{\Omega})}$$

for some $C=C(\Gamma_0)>0$ independent of r. Hence we obtain

$$|\Psi_f(r)-\Psi_f(0)|\le C\,|r|\,\|f\|_{C^1(\overline{\Omega})}\ .$$

Applying this estimate, we can prove the assertion

$$\begin{aligned}
&\left|\epsilon^{-1}\int_{\Gamma_0(\delta)}\left(\theta_0'\left(\tfrac{d^0}{\epsilon}\right)\right)^2 f\,dx-2\sigma\int_{\Gamma_0} f\,d\mathcal{H}^{d-1}\right|\\
&= 2\sigma\left|\varphi_\epsilon*\tilde{\Psi}_f(0)-\tilde{\Psi}_f(0)\right|\\
&= 2\sigma\left|\int_{\mathbb{R}}\varphi_\epsilon(y)\tilde{\Psi}_f(-y)\,dy-\tilde{\Psi}_f(0)\right|\\
&= 2\sigma\left|\int_{\mathbb{R}}\varphi_\epsilon(y)\left(\tilde{\Psi}_f(-y)-\tilde{\Psi}_f(0)\right)dy\right|\\
&\le 2\sigma\int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\varphi_\epsilon(y)\left|\tilde{\Psi}_f(-y)-\tilde{\Psi}_f(0)\right|dy+2\sigma\int_{\mathbb{R}\setminus(-\sqrt{\epsilon},\sqrt{\epsilon})}\varphi_\epsilon(y)\left|\tilde{\Psi}_f(-y)-\tilde{\Psi}_f(0)\right|dy\\
&\le C\epsilon^{\frac{1}{2}}\|f\|_{C^1(\overline{\Omega})}\int_{\mathbb{R}}\varphi_\epsilon(y)\,dy+C\,\|f\|_{C^0(\overline{\Omega})}\int_{\mathbb{R}\setminus(-\sqrt{\epsilon},\sqrt{\epsilon})}\varphi_\epsilon(y)\,dy\\
&\le C\epsilon^{\frac{1}{2}}\|f\|_{C^1(\overline{\Omega})}+C\,\|f\|_{C^0(\overline{\Omega})}\int_{\mathbb{R}\setminus(-\sqrt{\epsilon},\sqrt{\epsilon})}\epsilon^{-1}e^{-\frac{2\alpha|y|}{\epsilon}}\,dy\\
&\le C\epsilon^{\frac{1}{2}}\|f\|_{C^1(\overline{\Omega})}+C\,\|f\|_{C^0(\overline{\Omega})}\,\epsilon^{-1}e^{-\frac{2\alpha}{\sqrt{\epsilon}}}
\end{aligned}$$

for some $C=C(\Gamma_0)>0$ independent of ϵ and where we have used $\int_{\mathbb{R}}\varphi_\epsilon(z)\,dz=1$ for all $\epsilon\in(0,1]$. Hence the claim follows.
The relation (5.21) and the property (5.22) yield

$$\langle H_A^\epsilon,\varphi\rangle\longrightarrow 2\sigma\int_{\Gamma_0} M\nabla d^0\otimes M\nabla d^0:\varphi\circ X_t\,d\mathcal{H}^{d-1}\,,$$

as $\epsilon\searrow 0$.
We apply coordinate transformation to the right-hand side. Note that due to Lemma 5.2.1, it holds

$$|M|^2=\left|DX_t^{-T}\circ X_t\nabla d^0\right|^2=\det(\partial_{\tau_i}X_t\cdot\partial_{\tau_j}X_t)_{i,j=1}^{d-1}\quad\text{on }\Gamma_0\,.$$

Then we obtain

$$\begin{aligned}
&2\sigma\int_{\Gamma_0} DX_t^{-T}\circ X_t\nabla d^0\otimes DX_t^{-T}\circ X_t\nabla d^0:\varphi\circ X_t\,d\mathcal{H}^{d-1}\\
&= 2\sigma\int_{\Gamma(t)}\left|\det(\partial_{\tau_i}X_t\cdot\partial_{\tau_j}X_t)_{i,j=1}^{d-1}\circ X_t^{-1}\right|^{\frac{1}{2}}\frac{\nabla e}{|\nabla e|}\otimes\frac{\nabla e}{|\nabla e|}:\nabla\varphi\,d\mathcal{H}^{d-1}
\end{aligned}$$

since $X_t(\Gamma_0) = \Gamma(t)$ and $DX_t^{-T} \nabla d^0 \circ X_t^{-1} = \nabla e$. Since $\nabla e / |\nabla e| = \nu_{\Gamma(t)}$ and

$$\left| \det(\partial_{\tau_i} X_t \cdot \partial_{\tau_j} X_t)_{i,j=1}^{d-1} \circ X_t^{-1} \right|^{\frac{1}{2}} = \left| DX_t^{-T} \nabla d^0 \circ X_t^{-1} \right| = \left| \nabla \big(d^0(X_t^{-1}) \big) \right| ,$$

the assertion of the lemma follows. □

Proof of Theorem 5.2.2: The first assertion of the theorem immediately follows by Lemma (5.2.8) and (5.2.9).
The second assertion is a consequence of Lemma 5.2.6 and Lemma 5.2.7 since $\theta > 3$. □

6 Sharp Interface Limit for Convective Cahn-Hilliard Equation

We assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$. In the whole chapter let $v : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^d$ be a smooth velocity field such that $\operatorname{div} v = 0$ in $\Omega \times \mathbb{R}$ and $v = 0$ on $\partial\Omega \times \mathbb{R}$. As in Chapter 5 we consider the convective Cahn-Hilliard equation. But here we choose the mobility constant $m(\epsilon) = \epsilon$, that is, $\theta = 1$

$$\partial_t c^\epsilon + v \cdot \nabla c^\epsilon = \epsilon \Delta \mu^\epsilon \qquad \text{in } \Omega \times (0,T)\,, \tag{6.1}$$

$$\mu^\epsilon = -\epsilon \Delta c^\epsilon + \epsilon^{-1} f(c^\epsilon) \qquad \text{in } \Omega \times (0,T)\,, \tag{6.2}$$

$$\tfrac{\partial}{\partial n} c^\epsilon = \tfrac{\partial}{\partial n} \mu^\epsilon = 0 \qquad \text{on } \partial\Omega \times (0,T)\,, \tag{6.3}$$

$$c^\epsilon|_{t=0} = c_0^\epsilon \qquad \text{in } \Omega\,. \tag{6.4}$$

We assume that this convective Cahn-Hilliard equation has a smooth solution $(c^\epsilon, \mu^\epsilon)$ for all $\epsilon \in (0,1]$ and for some $T \in (0,\infty)$. By the same techniques as in Chapter 3 we prove that for an appropriate family of initial values $\{c_0^\epsilon\}_{0<\epsilon\le 1}$ the solutions to the convective Cahn-Hilliard equations converge as $\epsilon \searrow 0$ to the geometric evolution equation

$$V - v \cdot \nu = 0 \qquad \text{on } \Gamma(t), t > 0, \tag{6.5}$$

$$\Gamma(0) = \Gamma_{00} \qquad \text{for } t = 0, \tag{6.6}$$

where $\Gamma_{00} \subset \Omega$ is a given smooth hypersurface without boundary. Hence the motion of $\Gamma(t)$ is independent of μ where μ is the solution to the parabolic boundary problem

$$\partial_t \mu = f'(\pm 1) \Delta \mu - v \cdot \nabla \mu \qquad \text{in } \Omega^\pm(t), t > 0, \tag{6.7}$$

$$\mu = \sigma \kappa \qquad \text{on } \Gamma(t), t > 0, \tag{6.8}$$

$$\tfrac{\partial}{\partial n} \mu = 0 \qquad \text{on } \partial\Omega, t > 0, \tag{6.9}$$

$$\mu|_{t=0} = \mu_{00} \qquad \text{in } \Omega \tag{6.10}$$

for some given initial value μ_{00}. Here $V = V(x,t)$ denotes the normal velocity of the interface $\Gamma(t)$.

Remark 6.0.1. *We can assume that the system (6.5)-(6.10) admits a smooth solution (μ, Γ). This can be seen as follows.*

1. *Since the velocity field v is smooth, we already see in Lemma 5.2.1 and 5.2.4 that $\Gamma(t) = X_t(\Gamma_{00})$ is a global smooth solution to (6.5) and (6.6) where $X_t = X(.,t) : \Omega \to \Omega$ satisfies*

$$\frac{d}{dt} X(.,t) = v(X(.,t))\,, \quad X(.,0) = \mathrm{Id}\,.$$

   Let us also mention that $\Gamma(t) \subset \Omega$ for all $t \in \mathbb{R}$.

2. *Moreover, we can assume that the parabolic boundary problem (6.7)-(6.10) admits smooth solutions $\mu^+ = \mu|_{\Omega^+(t)}$ and $\mu^- = \mu|_{\Omega^-(t)}$ in the time interval $[0,T]$. This can be seen as follows. By Lagrangian coordinates $u(x,t) = \mu(X_t(x),t)$ we transform the system to the fixed domains $\Omega^+(0) = \Omega^+_{00}$ and $\Omega^-(0) = \Omega^-_{00}$. Then we obtain a parabolic problem of the form*

$$\begin{aligned} \partial_t u + A^\pm(t)u &= f^\pm(t) && \text{in } \Omega^\pm_{00}, t > 0, \\ u &= g^\pm(t) && \text{on } \Gamma_{00}, t > 0, \\ B(t)u &= 0 && \text{on } \partial\Omega, t > 0, \end{aligned}$$

   where $A^\pm(t)$ is an elliptic second order differential operator and $B(t)$ is a first order differential operator acting on the boundary $\partial\Omega$. We choose the initial value $u|_{t=-1} = 0$ and a smooth extension of $(A^\pm(t), f^\pm(t), g^\pm(t), B(t))$ to $[-1,\infty)$. Then by [14, 14.7 Corollary], we get $u^\pm \in C^\infty(\overline{\Omega^\pm} \times (-1,T))$, if $g^\pm = 0$. Setting $\mu_{00} = u(.,0)$, we get $u^\pm \in C^\infty(\overline{\Omega^\pm} \times [0,T))$. In the case $g^\pm \neq 0$ we choose smooth extensions $G^\pm$ of $g^\pm$ to $\Omega^\pm_{00}$ such that $G^+ = 0$ in a neighborhood of $\partial\Omega$ and replace $f^\pm(t)$ by $f^\pm(t) - (\partial_t + A(t))G^\pm(t)$. Now $w^\pm = u^\pm + G^\pm$ is the required solution where $u^\pm$ is the solution with new right-hand side $f^\pm(t) - (\partial_t + A(t))G^\pm(t)$ and $g^\pm = 0$.
   In the whole section we assume that μ_{00} is an given initial value such that the solution μ to (6.7)-(6.10) satisfies $\mu^\pm \in C^\infty(\overline{\Omega^\pm(t)} \times [0,T])$.

3. *Since we later set $c_0^\epsilon = c_A^\epsilon(.,0)$, we will see in the construction of approximate solutions $\{c_A^\epsilon\}_{0<\epsilon\leq 1}$ that the initial values c_0^ϵ depend on Γ_{00} and μ_{00} or for given c_0^ϵ the initial value μ_{00} depends on c_0^ϵ. But the leading order term of c_0^ϵ is independent of μ_{00}. For more details see Subsection 6.2.6 and 6.2.7 below.*

6.1 Convergence of the Difference of Approximate and True Solutions

We start with an energy estimate. Now we assume that there exists a constant $C_0 > 2$ such that

$$|c|^2 \leq CF(c) \quad \forall\, |c| \geq C_0 \tag{6.11}$$

for some $C > 0$. Later we will see that $\|c^\epsilon\|_{C^0(\Omega_T)} < 2$ for all $\epsilon > 0$ small enough, and therefore this assumption is not necessary since we can modify $F(c)$ for $|c| > 2$ such that (6.11) holds and since c^ϵ is also the unique solution for the modified F. Let $E(c)$ be the Ginzburg-Landau energy, that is,

$$E(c) = \frac{\epsilon}{2} \int_\Omega |\nabla c(x)|^2 \, dx + \frac{1}{\epsilon} \int_\Omega F(c(x)) \, dx \, .$$

By differentiating the Ginzburg-Landau energy with respect to t and integration by parts, we conclude

$$\begin{aligned} \frac{d}{dt} E(c^\epsilon) &= \epsilon \int_\Omega \nabla c^\epsilon \cdot \nabla c^\epsilon_t \, dx + \frac{1}{\epsilon} \int_\Omega f(c^\epsilon) c^\epsilon_t \, dx \\ &= -\epsilon \int_\Omega \Delta c^\epsilon c^\epsilon_t \, dx + \frac{1}{\epsilon} \int_\Omega f(c^\epsilon) c^\epsilon_t \, dx \\ &= \int_\Omega \mu^\epsilon c^\epsilon_t \, dx = \epsilon \int_\Omega \mu^\epsilon \Delta \mu^\epsilon \, dx - \int_\Omega v \cdot \nabla c^\epsilon \mu^\epsilon \, dx \, , \end{aligned} \tag{6.12}$$

where we have used the boundary condition $\frac{\partial}{\partial n} c^\epsilon = 0$. To handle the second term on the right-hand side, we use equation (6.2)

$$\begin{aligned} \int_\Omega v \cdot \nabla c^\epsilon \mu^\epsilon \, dx &= -\epsilon \int_\Omega v \cdot \nabla c^\epsilon \Delta c^\epsilon \, dx + \epsilon^{-1} \int_\Omega v \cdot \nabla c^\epsilon f(c^\epsilon) \, dx \\ &= \epsilon \int_\Omega \nabla (v \cdot \nabla c^\epsilon) \cdot \nabla c^\epsilon \, dx + \epsilon^{-1} \int_\Omega v \cdot \nabla f(c^\epsilon) \, dx \\ &= \epsilon \int_\Omega \nabla v : (\nabla c^\epsilon \otimes \nabla c^\epsilon) \, dx + \frac{\epsilon}{2} \int_\Omega v \cdot \nabla \left(|\nabla c^\epsilon|^2 \right) dx \\ &= \epsilon \int_\Omega \nabla v : (\nabla c^\epsilon \otimes \nabla c^\epsilon) \, dx \, , \end{aligned}$$

where we have used the Neumann boundary condition $\frac{\partial}{\partial n} c^\epsilon = 0$ on $\partial\Omega$, $\operatorname{div} v = 0$, and $v = 0$ on $\partial\Omega$. Hence there exists some constant $C > 0$ such that

$$\left| \int_\Omega v \cdot \nabla c^\epsilon \mu^\epsilon \, dx \right| \leq C \, \|\nabla v\|_{L^\infty(\Omega_T)} \, E(c^\epsilon) \, .$$

Using this estimate, we conclude by (6.12)

$$\frac{d}{dt} E(c^\epsilon) + \epsilon \, \|\nabla \mu^\epsilon\|^2_{L^2(\Omega)} \leq C \, \|\nabla v\|_{L^\infty(\Omega_T)} \, E(c^\epsilon) \, . \tag{6.13}$$

We can apply Gronwall's inequality to get

$$\sup_{\tau \in [0,T]} E(c^\epsilon(\tau)) \leq C E(c^\epsilon(0))$$

for some $C = C(T,v) > 0$. Integrating (6.13) over $(0,T)$ and using the previous estimate, we obtain

$$\epsilon \|\nabla \mu^\epsilon\|^2_{L^2(\Omega_T)} \leq C E(c^\epsilon(0))$$

for some $C = C(T,v) > 0$. Finally, by using (6.11), we conclude by definition of the Ginzburg-Landau energy $E(c^\epsilon)$

$$\sup_{\tau\in[0,t]} \left(\|c^\epsilon(.,\tau)\|^2_{L^2(\Omega)} + \epsilon \|\nabla c^\epsilon(.,\tau)\|^2_{L^2(\Omega)} \right) + \epsilon \|\nabla \mu^\epsilon\|^2_{L^2(\Omega_t)} \leq C\left(E(c^\epsilon(0)) + 1\right) \quad (6.14)$$

for some constant $C = C(T,v) > 0$ independent of ϵ. In the following we consider the initial condition $c^\epsilon(.,0) = c^\epsilon_A(.,0)$ for every $\epsilon \in (0,1]$ where c^ϵ_A is an approximate solution to the convective Cahn-Hilliard equation. Then it is possible to verify $E(c^\epsilon_A(0)) \leq C$ for some $C > 0$ independent of ϵ.
Now we prove that the difference of approximate solutions and true solutions converges to zero as $\epsilon \searrow 0$.

Theorem 6.1.1. *Let $\{c^\epsilon_A, \mu^\epsilon_A\}_{0<\epsilon\leq 1}$ be a family of functions in $C^\infty(\overline{\Omega_T}) \times C^\infty(\overline{\Omega_T})$ satisfying the system of differential equations*

$$(c^\epsilon_A)_t + v\cdot\nabla c^\epsilon_A = \epsilon\Delta\mu^\epsilon_A \qquad \text{in } \Omega_T\,, \quad (6.15)$$

$$\mu^\epsilon_A = -\epsilon\Delta c^\epsilon_A + \epsilon^{-1} f(c^\epsilon_A) + r^\epsilon_A \qquad \text{in } \Omega_T\,, \quad (6.16)$$

$$\frac{\partial}{\partial n} c^\epsilon_A = \frac{\partial}{\partial n}\mu^\epsilon_A = 0 \qquad \text{on } \partial_T\Omega\,, \quad (6.17)$$

where $r^\epsilon_A = r^\epsilon_A(x,t)$ is a function such that

$$\|r^\epsilon_A\|^2_{L^2(\Omega_T)} \leq \epsilon^{pk}\,, \quad (6.18)$$

$p = \frac{2(d+4)}{d+2}$, and $k \in \mathbb{N}$ such that

$$k > \frac{(d+2)^2}{d+4}\,. \quad (6.19)$$

Also assume that c^ϵ_A satisfies the boundedness condition

$$\sup_{0<\epsilon\leq 1} \|c^\epsilon_A\|_{L^\infty(\Omega_T)} \leq C_0 \quad (6.20)$$

for some $C_0 > 0$, f satisfies (2.1) and (2.2), and

$$\phi^\epsilon_t(.) := c^\epsilon_A(.,t) \quad (6.21)$$

has the form (2.20). Let $(c^\epsilon, \mu^\epsilon)$ be the unique solution to (6.1)-(6.3) with $c^\epsilon_0(x) = c^\epsilon_A(x,0)$ in Ω. Then there exists a constant $\epsilon_0 = \epsilon_0(C_0, T, \Omega, k, d) \in (0,1]$ such that if $\epsilon \in (0,\epsilon_0)$, then

$$\|c^\epsilon - c^\epsilon_A\|_{L^2(\Omega_T)} \leq \epsilon^k\,.$$

Proof: Let $R = c^\epsilon - c_A^\epsilon$ be the remainder function and let $\Psi(x,t)$ be the unique smooth solution to the Neumann boundary problem

$$-\Delta\Psi(.,t) = R(.,t) \text{ in } \Omega, \qquad \frac{\partial}{\partial n}\Psi(.,t) = 0 \text{ on } \partial\Omega, \qquad \int_\Omega \Psi(.,t)\,dx = 0$$

for all $t \in [0,T]$. The existence and uniqueness of Ψ follows as in the proof of Lemma 5.2.7.

Again as in the proof of Lemma 5.2.7, it follows by testing the differential equations for c^ϵ and c_A^ϵ with R

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\,dx + \epsilon\int_\Omega \epsilon\,|\nabla R|^2 + \epsilon^{-1} f'(c_A^\epsilon)R^2\,dx \\ = \int_\Omega \nabla\Psi\cdot vR\,dx - \int_\Omega \mathcal{N}(c_A^\epsilon, R)R - \epsilon r_A^\epsilon R\,dx\,, \end{aligned} \tag{6.22}$$

where $\mathcal{N}(c_A^\epsilon, R) \equiv f(c_A^\epsilon + R) - f(c_A^\epsilon) - f'(c_A^\epsilon)R$. Due to (5.16), we obtain

$$\int_\Omega \nabla\Psi\cdot vR\,dx \leq \|\nabla v\|_{L^\infty(\Omega)}\|\nabla\Psi\|^2_{L^2(\Omega)}\,. \tag{6.23}$$

Applying Lemma 2.8.3 yields

$$-\int_\Omega \mathcal{N}(c_A^\epsilon, R)R\,dx \leq C_p\|R\|^p_{L^p(\Omega)}\,. \tag{6.24}$$

By Hölder's inequality, it holds

$$\epsilon\int_\Omega r_A^\epsilon R\,dx \leq \epsilon\,\|r_A^\epsilon\|_{L^2(\Omega)}\|R\|_{L^2(\Omega)}\,, \tag{6.25}$$

and due to Proposition 2.8.1, we obtain

$$\epsilon\int_\Omega \epsilon\,|\nabla R|^2 + \epsilon^{-1}f'(c_A^\epsilon)R^2\,dx \geq -C\epsilon\,\|\nabla\Psi\|^2_{L^2(\Omega)} + 2\epsilon^2\|R\|^2_{L^2(\Omega)}\,. \tag{6.26}$$

Hence using (6.23)-(6.26) and Young's estimate, equation (6.22) implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\,dx + \epsilon^2\|R\|^2_{L^2(\Omega)} \leq C\left(\|\nabla\Psi\|^2_{L^2(\Omega)} + \|r_A^\epsilon\|^2_{L^2(\Omega)} + \|R\|^p_{L^p(\Omega)}\right) \tag{6.27}$$

for some $C = C(v) > 0$. By Gronwall's inequality, it follows

$$\sup_{0\leq\tau\leq t}\|\nabla\Psi(.,\tau)\|^2_{L^2(\Omega)} \leq Ce^{Ct}\left(\|r_A^\epsilon\|^2_{L^2(\Omega_t)} + \|R\|^p_{L^p(\Omega_t)}\right) \tag{6.28}$$

since $R(.,0) = \Psi(.,0) = 0$. Integrating (6.27) over $(0,t)$ and using (6.28) yields

$$\epsilon^2\|R\|^2_{L^2(\Omega_t)} \leq C_1\left(\|r_A^\epsilon\|^2_{L^2(\Omega_t)} + \|R\|^p_{L^p(\Omega_t)}\right) \tag{6.29}$$

for some $C_1 = C_1(T, v, p) > 0$ independent of ϵ and $t \in [0,T]$. Integrating (6.22) over $(0,t)$ and using (6.23)-(6.25) yields

$$\begin{aligned}\epsilon^2 \|\nabla R\|^2_{L^2(\Omega)} &\leq C\left(\|R\|^2_{L^2(\Omega)} + \|R\|^p_{L^p(\Omega)} + \|\nabla\Psi\|^2_{L^2(\Omega)} + \epsilon^2 \|r^\epsilon_A\|^2_{L^2(\Omega)}\right)\\ &\leq C\left(\|R\|^2_{L^2(\Omega)} + \|R\|^p_{L^p(\Omega)} + \epsilon^2 \|r^\epsilon_A\|^2_{L^2(\Omega)}\right) \qquad (6.30)\end{aligned}$$

for some $C = C(v) > 0$ and where the last inequality follows by (6.28). As in the proof of Theorem 3.1.1, it holds for all $t \in (0,T]$

$$\|R\|^p_{L^p(\Omega_t)} \leq C \sup_{0\leq\tau\leq t} \|\nabla\Psi(.,\tau)\|^{\frac{4}{d+2}}_{L^2(\Omega)} \|\nabla R\|^2_{L^2(\Omega_t)}\,. \qquad (6.31)$$

Define the sets A^ϵ_1 and A^ϵ_2 by

$$\begin{aligned}A^\epsilon_1 &:= \left\{t\in[0,T] : \epsilon^2 \|R\|^2_{L^2(\Omega_t)} > 2C_1 \|R\|^p_{L^p(\Omega_t)}\right\},\\ A^\epsilon_2 &:= \left\{t\in[0,T] : \epsilon^2 \|R\|^2_{L^2(\Omega_t)} \leq 2C_1 \|R\|^p_{L^p(\Omega_t)}\right\},\end{aligned}$$

where C_1 is the same constant as in (6.29). Moreover, we define

$$T^\epsilon := \sup\left\{t\in(0,T] : \|R\|_{L^p(\Omega_t)} \leq \epsilon^k\right\}.$$

1st case: "$T^\epsilon \in A^\epsilon_1$"
Then the definition of A^ϵ_1 and (6.29) yields

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \leq \frac{\epsilon^2}{2C_1}\|R\|^2_{L^2(\Omega_{T^\epsilon})} \leq \frac{1}{2}\left(\|r^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} + \|R\|^p_{L^p(\Omega_{T^\epsilon})}\right),$$

and thus we conclude

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \leq \|r^\epsilon_A\|^2_{L^2(\Omega_{T^\epsilon})} \leq \frac{1}{2}\epsilon^{pk}\,. \qquad (6.32)$$

It follows $T^\epsilon = T$ by definition of T^ϵ.
2nd case: "$T^\epsilon \in A^\epsilon_2$"
Here we use the estimate (6.31) and then (6.28), (6.30), and the definition of A^ϵ_2

$$\begin{aligned}\|R\|^p_{L^p(\Omega_{T^\epsilon})} &\leq C\left(\|r^\epsilon_A\|^2_{L^2(\Omega_t)} + \|R\|^p_{L^p(\Omega_t)}\right)^{\frac{2}{d+2}}\left(\epsilon^{-4}\|R\|^p_{L^p(\Omega)} + \|r^\epsilon_A\|^2_{L^2(\Omega)}\right)\\ &\leq C\epsilon^{\frac{2pk}{d+2}}\epsilon^{pk-4}\\ &= C\epsilon^{pk}\epsilon^{\frac{4(d+4)}{(d+2)^2}\left(k-\frac{(d+2)^2}{d+4}\right)}.\end{aligned}$$

In the last equality we have used $p = \frac{2(d+4)}{d+2}$. Since $k > \frac{(d+2)^2}{d+4}$, there exists $\epsilon_0 \in (0,1]$ such that for all $\epsilon \in (0,\epsilon_0]$, it holds

$$\|R\|^p_{L^p(\Omega_{T^\epsilon})} \leq \frac{1}{2}\epsilon^{pk}\,,$$

provided $T^\epsilon \in A_2^\epsilon$. Also in the 2nd case, it follows $T^\epsilon = T$. This shows the assertion of the theorem. $\square$

We even get estimates in stronger norms.

Lemma 6.1.2. *Let the assumptions of Theorem 6.1.1 hold. Let $m > 0$ be any fixed integer and assume $\|c_A^\epsilon\|_{W_2^{m+l+1}(\Omega_T)} + \|\mu_A^\epsilon\|_{W_2^{m+l-1}(\Omega_T)} \le \epsilon^{-K(m)}$ for $l > \frac{d+1}{2}$, some integer $K(m)$, and for all small ϵ. If k in (6.18) is large enough, then*

$$\|c^\epsilon - c_A^\epsilon\|_{C^m(\Omega_T)} + \|\mu^\epsilon - \mu_A^\epsilon\|_{C^{m-2}(\Omega_T)} \le \epsilon$$

for all sufficiently small $\epsilon > 0$.

Proof: We show the assertion in the same way as Theorem 3.1.2.
As in (3.39), there exists some $\theta \in (0,1)$ such that

$$\|c^\epsilon - c_A^\epsilon\|_{C^m(\Omega_T)} \le C \, \|c^\epsilon - c_A^\epsilon\|_{L^2(\Omega_T)}^\theta \, \|c^\epsilon - c_A^\epsilon\|_{W_2^{m+l+1}(\Omega_T)}^{1-\theta} ,$$

for some $C > 0$. Therefore it is sufficient to show

$$\|c^\epsilon\|_{W_2^{m+l+1}(\Omega_T)} \le \epsilon^{-K(m)} ,$$

for some integer $K(m)$ if k in (6.18) is large enough. As in Theorem 3.1.2, the estimate for $\mu^\epsilon - \mu_A^\epsilon$ follows from the equations for the chemical potential (6.2) and (6.16).
We replace f by $\bar{f}$ where $f = \bar{f}$ in $(-\frac{3}{2}C_0, \frac{3}{2}C_0)$ and $\bar{f}(c)$ is linear when $|c| > 2C_0$ where C_0 is the same constant as in (6.20). Denote by $\bar{c}^\epsilon$ the solution to the modified system with $\bar{f}$. Define $A : D(A) \to L^p(\Omega)$ by $A = -\Delta + \mathrm{Id}$ with $D(A) = \left\{c \in W_p^2(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = 0\right\}$. W.l.o.g. we assume that in Lemma 2.7.1 the constant is $\tau = 1$ since we only consider a finite number of different p's. Otherwise we replace A by $-\Delta + c\mathrm{Id}$ for some $c \in \mathbb{R}$. Therefore $-A^2$ is sectorial with $D(-A^2) = \left\{c \in W_p^4(\Omega) : \frac{\partial}{\partial n} c\big|_{\partial\Omega} = \frac{\partial}{\partial n} Ac\big|_{\partial\Omega} = 0\right\}$ and $W_p^1(\Omega) = D(A^{\frac{1}{2}})$ with equivalent norms, cf. Lemma 2.7.1.
Then the convective Cahn-Hilliard equation can be written as

$$\begin{aligned} \bar{c}_t^\epsilon + \epsilon^2 A^2 \bar{c}^\epsilon &= A(-\bar{f}(\bar{c}^\epsilon) + 2\epsilon^2 \bar{c}^\epsilon) + \bar{f}(\bar{c}^\epsilon) - v \cdot \nabla \bar{c}^\epsilon - \epsilon^2 \bar{c}^\epsilon \\ &= A f_1(\bar{c}^\epsilon) + f_2(\bar{c}^\epsilon) , \end{aligned} \tag{6.33}$$

where $f_1(c) := -\bar{f}(c) + 2\epsilon^2 c$ and $f_2(c) := -v \cdot \nabla c + \bar{f}(c) - \epsilon^2 c$.
Since

$$\frac{\partial}{\partial n} A \bar{c}^\epsilon = -\frac{\partial}{\partial n} \Delta \bar{c}^\epsilon + \frac{\partial}{\partial n} \bar{c}^\epsilon = \epsilon^{-1} \frac{\partial}{\partial n} \bar{\mu}^\epsilon - \epsilon^{-2} \frac{\partial}{\partial n} \bar{f}(\bar{c}^\epsilon) = 0 ,$$

it follows $\bar{c}^\epsilon \in D(-A^2)$. Hence the following proof is a bit simpler than in Theorem 3.1.2. Then by semigroup theory, we obtain

$$\bar{c}^\epsilon(t) = e^{-\epsilon^2 A^2 t} \bar{c}^\epsilon(0) + \int_0^t e^{-\epsilon^2 A^2 (t-\tau)} \left[A f_1(\bar{c}^\epsilon(\tau)) + f_2(\bar{c}^\epsilon(\tau))\right] d\tau . \tag{6.34}$$

In the following we denote by $\|.\|_p$ the norm of operators from $L^p(\Omega)$ to $L^p(\Omega)$. We apply $A^{1/2+\alpha}$, $\alpha \geq 0$, to both sides of (6.34) and obtain

$$\begin{aligned}
&\left\| A^{1/2+\alpha}\bar{c}^\epsilon(t)\right\|_{L^p} \\
&\quad \leq \left(\sup_{\tau'>0}\left\|e^{-A^2\tau'}\right\|_p\right)\left\|A^{1/2+\alpha}\bar{c}^\epsilon(0)\right\|_{L^p} + \left(\sup_{\tau'>0}\left\|(\tau' A^2)^{3/4}e^{-A^2\tau'}\right\|_p\right) \\
&\qquad \times \int_0^t \left[\left(\epsilon^2(t-\tau)\right)^{-3/4}\left(\|A^\alpha f_1(\bar{c}^\epsilon(\tau))\|_{L^p} + \left\|A^{\alpha-1}f_2(\bar{c}^\epsilon(\tau))\right\|_{L^p}\right)\right] d\tau\,.
\end{aligned}$$

Note that for $\beta \in [0,\infty)$, there exists a constant $C = C(\Omega, p, \beta)$ such that

$$\sup_{\tau\geq 0}\left\|(\tau A^2)^\beta e^{-A^2\tau}\right\|_p \leq C\,,$$

cf. [55, Chapter 2, Theorem 6.13]. Therefore we obtain the recurrence inequality

$$\begin{aligned}
\left\|A^{1/2+\alpha}\bar{c}^\epsilon\right\|_{L^\infty(0,T;L^p)} \;\leq\; & C\Big[\left\|A^{1/2+\alpha}\bar{c}^\epsilon(0)\right\|_{L^p} + \epsilon^{-3/2}\Big(\|A^\alpha f_1(\bar{c}^\epsilon)\|_{L^\infty(0,T;L^p)} \\
& + \left\|A^{\alpha-1}f_2(\bar{c}^\epsilon)\right\|_{L^\infty(0,T;L^p)}\Big)\Big]\,. \qquad (6.35)
\end{aligned}$$

Since $\bar{f}$ has linear growth, there exists a positive constant C such that for any $p \in [1, 2d]$ and any $c \in L^p(\Omega)$

$$\|f_1(c)\|_{L^p} \leq C\left(1 + \|c\|_{L^p}\right). \qquad (6.36)$$

For estimating the term $A^{-1}f_2(\bar{c}^\epsilon)$, we first consider $A^{-1}(v\cdot\nabla\bar{c}^\epsilon)$ (this is the main difference to Theorem 3.1.2 and [10, Theorem 2.3]).
Claim 1: For all $p \in [2,\infty)$ there exists a constant $C = C(p, v) > 0$ independent of ϵ such that

$$\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)(.,t)\right\|_{W^1_p(\Omega)} \leq C\,\|\bar{c}^\epsilon(.,t)\|_{L^p(\Omega)}\ ,$$

for all $t \in [0,T]$.
Since $\operatorname{div} v = 0$, we obtain $v\cdot\nabla\bar{c}^\epsilon = \operatorname{div}(\bar{c}^\epsilon v)$ and therefore by definition of A

$$\begin{aligned}
\int_\Omega A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\,dx &= \int_\Omega \Delta A^{-1}(v\cdot\nabla\bar{c}^\epsilon) + \operatorname{div}(\bar{c}^\epsilon v)\,dx \\
&= \int_{\partial\Omega}\frac{\partial}{\partial n}A^{-1}(v\cdot\nabla\bar{c}^\epsilon) + \bar{c}^\epsilon v\cdot n\,d\mathcal{H}^{d-1} = 0 \qquad (6.37)
\end{aligned}$$

since $\frac{\partial}{\partial n}A^{-1}(\,.\,) = v\cdot n = 0$ on $\partial\Omega$. Hence there exists a unique smooth solution to the Neumann boundary problem

$$\begin{aligned}
&-\Delta\Psi(.,t) = A^{-1}(v\cdot\nabla\bar{c}^\epsilon)(.,t) \text{ in } \Omega\,, \\
&\frac{\partial}{\partial n}\Psi(.,t) = 0 \text{ on } \partial\Omega\,, \quad \int_\Omega \Psi(x,t)\,dx = 0
\end{aligned} \qquad (6.38)$$

for all $t \in [0,T]$. Multiplying (6.38a) by Ψ, integrating the resulting equation over Ω, and using a Poincaré's inequality yield

$$\|\Psi\|^2_{W^1_2(\Omega)} \leq C\,\|\nabla\Psi\|^2_{L^2(\Omega)} \leq C\,\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^2(\Omega)}\|\Psi\|_{L^2(\Omega)}\ ,$$

and therefore we get

$$\|\Psi\|_{W^1_2(\Omega)} \leq C\,\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^2(\Omega)}\ . \tag{6.39}$$

To get an estimate for Ψ in L^p-norm, we apply [8, Theorem 15.2.] to (6.38)

$$\|\Psi\|_{W^1_p(\Omega)} \leq C\left(\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^p(\Omega)} + \|\Psi\|_{L^p(\Omega)}\right)$$

for some $C = C(p) > 0$. To estimate the second term on the right-hand side, we use Ehrling's lemma, cf. [56, Theorem 7.30]. Since $W^1_p(\Omega)$ is compactly imbedded in $L^p(\Omega)$ and $L^p(\Omega)$ is continuously imbedded in $L^2(\Omega)$, we obtain

$$\|c\|_{L^p(\Omega)} \leq C(\delta)\,\|c\|_{L^2(\Omega)} + \delta\,\|c\|_{W^1_p(\Omega)}$$

for every $c \in W^1_p(\Omega)$ and some constant $C(\delta) > 0$. Choosing $\delta > 0$ small enough, we get

$$\begin{aligned}\|\Psi\|_{W^1_p(\Omega)} &\leq C\left(\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^p(\Omega)} + \|\Psi\|_{L^2(\Omega)}\right)\\ &\leq C\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^p(\Omega)}\end{aligned} \tag{6.40}$$

for some $C = C(p) > 0$ and where the second estimate follows from (6.39). Applying this estimate, we can show Claim 1 as follows. Since $A^{-1}(v\cdot\nabla\bar{c}^\epsilon) = -\mathrm{div}\,(\nabla\Psi)$ and $A = -\Delta + \mathrm{Id}$, the function $c = A^{-1}(v\cdot\nabla\bar{c}^\epsilon)$ solves the following Neumann boundary problem for all $t \in [0,T]$

$$-\Delta c = \mathrm{div}\,(\bar{c}^\epsilon v) + \mathrm{div}\,(\nabla\Psi)\ \text{ in }\Omega\,, \qquad \frac{\partial}{\partial n}c = 0 \text{ on } \partial\Omega \tag{6.41}$$

since $v\cdot\nabla\bar{c}^\epsilon = \mathrm{div}\,(\bar{c}^\epsilon v)$. Due to Simader and Sohr [60, Theorem 1.4], we conclude

$$\begin{aligned}\left\|\nabla A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^p(\Omega)} &\leq C\left(\|\bar{c}^\epsilon\|_{L^p(\Omega)} + \|\nabla\Psi\|_{L^p(\Omega)}\right)\\ &\leq C\left(\|\bar{c}^\epsilon\|_{L^p(\Omega)} + \left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^p(\Omega)}\right)\end{aligned}$$

for some $C = C(p,v) > 0$ and where we have used (6.40) in the second inequality. Due to (6.37), we can apply a Poincaré's inequality to obtain

$$\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{W^1_p(\Omega)} \leq C\left(\|\bar{c}^\epsilon\|_{L^p(\Omega)} + \left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^p(\Omega)}\right)$$

for some $C = C(p,v) > 0$. To handle the second term on the right-hand side we apply the Ehrling's lemma in the same way as above. This yields

$$\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{W^1_p(\Omega)} \leq C\left(\|\bar{c}^\epsilon\|_{L^p(\Omega)} + \left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^2(\Omega)}\right). \tag{6.42}$$

To estimate the second term on the right-hand side, we use the energy method. We multiply the equation $(-\Delta + \mathrm{Id})A^{-1}(v\cdot\nabla\bar{c}^\epsilon) = \mathrm{div}\,(\bar{c}^\epsilon v)$ by $A^{-1}(v\cdot\nabla\bar{c}^\epsilon)$, integrate the resulting equation over Ω, and apply integration by parts to conclude

$$\begin{aligned}\left\|\nabla A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|^2_{L^2(\Omega)} + \left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|^2_{L^2(\Omega)} &= -\int_\Omega \bar{c}^\epsilon v\cdot\nabla(A^{-1}(v\cdot\nabla\bar{c}^\epsilon))\,dx \\ &\leq \|\bar{c}^\epsilon v\|_{L^2(\Omega)}\left\|\nabla A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|_{L^2(\Omega)},\end{aligned}$$

where we have used the boundary conditions $\frac{\partial}{\partial n}A^{-1}(\,.\,) = v\cdot n = 0$ on $\partial\Omega$. Thus it holds

$$\left\|A^{-1}(v\cdot\nabla\bar{c}^\epsilon)\right\|^2_{W^1_2(\Omega)} \leq C\,\|\bar{c}^\epsilon\|^2_{L^2(\Omega)}$$

for some $C > 0$. Therefore we can estimate the second term on the right-hand side in (6.42). Thus Claim 1 follows.
Due to Claim 1, estimate (6.36), and the definitions of f_2, there exists some constant $C > 0$ such that

$$\|f_1(\bar{c}^\epsilon)\|_{L^p(\Omega)} + \left\|A^{-1}f_2(\bar{c}^\epsilon)\right\|_{L^p(\Omega)} \leq C\left(1 + \|c\|_{L^p(\Omega)}\right) \tag{6.43}$$

for all $p \in [2, 2d]$. We use (6.35) and (6.43) to estimate $\|\bar{c}^\epsilon\|_{L^\infty}$.
Claim 2: There exists some $p > d$, an integer k_0, and a constant $C > 0$ independent of ϵ such that

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;L^\infty)} + \|\bar{c}^\epsilon\|_{L^\infty(0,T;W^1_p)} \leq C\epsilon^{-k_0}.$$

For the proof we use a bootstrap method. First we set $p = p_0 := 2$. Then we already known from the energy estimate (6.14)

$$\epsilon\,\|\bar{c}^\epsilon\|^2_{L^\infty(0,T;W^1_2)} \leq C. \tag{6.44}$$

Then we set $\alpha = 0$ in (6.35) and $p_1 = \frac{dp_0}{d-p_0}$ (in the case $d = 2$ we choose $p_1 = 3$). Then it holds

$$\begin{aligned}\left\|A^{1/2}\bar{c}^\epsilon\right\|_{L^\infty(0,T;L^{p_1})} &\leq C\left(\left\|A^{1/2}c^\epsilon_A(0)\right\|_{L^{p_1}} + \epsilon^{-3/2}\left(\|\bar{c}^\epsilon\|_{L^\infty(0,T;L^{p_1})} + 1\right)\right) \\ &\leq C\left(\left\|A^{1/2}c^\epsilon_A(0)\right\|_{L^{p_1}} + \epsilon^{-3/2}\left(\|\bar{c}^\epsilon\|_{L^\infty(0,T;W^1_2)} + 1\right)\right),\end{aligned}$$

where we have used (6.43) in the first inequality and Sobolev's imbedding in the second inequality. Since $D(A^{\frac{1}{2}}_{p_1}) = W^1_{p_1}(\Omega)$, it holds

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;W^1_{p_1})} \leq C\left\|A^{1/2}\bar{c}^\epsilon\right\|_{L^\infty(0,T;L^{p_1})},$$

where A_{p_1} denotes the realization of the differential operator A in $L^{p_1}(\Omega)$. Hence by Sobolev's imbedding, we have

$$\|\overline{c}^\epsilon\|_{L^\infty(0,T;L^{p_2})} \leq C\,\|\overline{c}^\epsilon\|_{L^\infty(0,T;W^1_{p_1})} \leq C\left\|A^{1/2}\overline{c}^\epsilon\right\|_{L^\infty(0,T;L^{p_1})},$$

where $p_2 = \frac{dp_1}{d-p_1}$ (in the case $d > p_1$). Repeating the same procedure step by step, we can show that $\|\overline{c}^\epsilon\|_{L^\infty(0,T;L^{p_i})} \leq C\epsilon^{-k_0}$ where $p_i = \frac{dp_{i-1}}{d-p_{i-1}}$, until $p_i > d$ for some finite integer $i = i(N)$. Then by Sobolev's imbedding we obtain $\|\overline{c}^\epsilon\|_{L^\infty(0,T;L^\infty)} \leq C\epsilon^{-k_0}$. This shows Claim 2.

To get estimates in stronger norms we set $\alpha = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ in (6.35) where we can control the terms on the right-hand side. To do this, we need some inequalities. For any $\beta \in \frac{1}{2}\mathbb{N}$ there exists a positive constant $C = C(\beta, k, l, p)$ such that for all $c \in W^k_\infty(0,T; W^{2\beta+l+2}_p(\Omega) \cap W^{2\beta+l+1}_\infty(\Omega))$, $k, l \in \mathbb{N}$

$$\begin{aligned} &\left\|A^\beta\partial_t^k A f_1(c)\right\|_{L^\infty(0,T;W^l_p(\Omega))} + \left\|A^\beta\partial_t^k f_2(c)\right\|_{L^\infty(0,T;W^l_p(\Omega))} \\ &\leq\; C\left(\left(1 + \|c\|^{2\beta+k+l+1}_{W^k_\infty(0,T;W^{2\beta+l+1}_\infty(\Omega))}\right)\left(1 + \|c\|_{W^k_\infty(0,T;W^{2\beta+l+2}_p(\Omega))}\right)\right), \end{aligned} \tag{6.45}$$

where we get the term $\|c\|^{2\beta+k+l+1}_{W^k_\infty(0,T;W^{2\beta+l+1}_\infty(\Omega))}$ by chain rule.

By definition of A the function $c = A^n\overline{c}^\epsilon$, $n \in \mathbb{N}$, is the solution to the elliptic Neumann boundary problem

$$\begin{aligned} \Delta c - c &= -A^{n+1}\overline{c}^\epsilon && \text{in } \Omega\,, \\ \frac{\partial}{\partial n}c &= \frac{\partial}{\partial n}\left(A^n\overline{c}^\epsilon\right) && \text{on } \partial\Omega\,. \end{aligned}$$

Then [8, Theorem 15.2.] gives us the estimate

$$\begin{aligned} \|A^n\overline{c}^\epsilon\|_{W^{m+1}_p(\Omega)} \;\leq\; & C(\Omega,p,m)\Bigg(\left\|A^{n+1}\overline{c}^\epsilon\right\|_{W^{m-1}_p(\Omega)} \\ & + \|A^n\overline{c}^\epsilon\|_{L^p(\Omega)} + \left\|\frac{\partial}{\partial n}\left(A^n\overline{c}^\epsilon\right)\right\|_{W_p^{m-\frac{1}{p}}(\partial\Omega)}\Bigg) \end{aligned}$$

for all $m \in \mathbb{N}$. Therefore for all $m \in \mathbb{N}$ there exists a constant $C = C(m) > 0$ such that

$$\|\overline{c}^\epsilon\|_{W^{2m}_p(\Omega)} \;\leq\; C\,\|A^m\overline{c}^\epsilon\|_{L^p(\Omega)} + \sum_{i=0}^{m-1}\left\|\frac{\partial}{\partial n}A^i\overline{c}^\epsilon\right\|_{W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega)}, \tag{6.46}$$

$$\|\overline{c}^\epsilon\|_{W^{2m+1}_p(\Omega)} \;\leq\; C\,\|A^m\overline{c}^\epsilon\|_{W^1_p(\Omega)} + \sum_{i=0}^{m-1}\left\|\frac{\partial}{\partial n}A^i\overline{c}^\epsilon\right\|_{W_p^{2m-2i-\frac{1}{p}}(\partial\Omega)}, \tag{6.47}$$

where we use the convention that if the upper limit of the summation is less than the lower limit, then the summation is zero. To estimate the boundary terms $\frac{\partial}{\partial n}A^i\overline{c}^\epsilon$, we use the boundary conditions $\frac{\partial}{\partial n}\overline{c}^\epsilon = \frac{\partial}{\partial n}A\overline{c}^\epsilon = 0$. We apply (6.33) $\lfloor i/2 \rfloor$-times to get

$$\begin{aligned}
A^i\overline{c}^\epsilon &= (-1)^{\lfloor i/2\rfloor}\epsilon^{-2\lfloor i/2\rfloor}A^{i-2\lfloor i/2\rfloor}\partial_t^{\lfloor i/2\rfloor}\overline{c}^\epsilon \\
&\quad + \sum_{j=0}^{\lfloor i/2\rfloor-1}(-1)^j\epsilon^{-2j-2}A^{i-2j-2}\partial_t^j\left(Af_1(\overline{c}^\epsilon)+f_2(\overline{c}^\epsilon)\right),
\end{aligned}$$

where $\lfloor . \rfloor$ defines the floor function. Since $\frac{\partial}{\partial n}\partial_t^i\overline{c}^\epsilon = \frac{\partial}{\partial n}A\partial_t^i\overline{c}^\epsilon = 0$ for all $i \in \mathbb{N}$, we can neglect the first term on the right-hand side. Hence we get

$$\begin{aligned}
&\left\|\tfrac{\partial}{\partial n}A^i\overline{c}^\epsilon\right\|_{W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega)} \\
&\leq \epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\left\|\tfrac{\partial}{\partial n}A^{i-2j-2}\partial_t^j\left(Af_1(\overline{c}^\epsilon)+f_2(\overline{c}^\epsilon)\right)\right\|_{W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega)} \\
&\leq C\epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\left\|A^{i-2j-2}\partial_t^j\left(Af_1(\overline{c}^\epsilon)+f_2(\overline{c}^\epsilon)\right)\right\|_{W_p^{2m-2i}(\Omega)}.
\end{aligned}$$

We can estimate the right-hand side by inequality (6.45) as follows

$$\begin{aligned}
&\left\|\tfrac{\partial}{\partial n}A^i\overline{c}^\epsilon\right\|_{L^\infty(0,T;W_p^{2m-1-2i-\frac{1}{p}}(\partial\Omega))} \\
&\leq C\epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\left\|A^{i-2j-2}\partial_t^j\left(Af_1(\overline{c}^\epsilon)+f_2(\overline{c}^\epsilon)\right)\right\|_{L^\infty(0,T;W_p^{2m-2i}(\Omega))} \\
&\leq C\epsilon^{-i}\sum_{j=0}^{\lfloor i/2\rfloor-1}\left[\left(1+\|\overline{c}^\epsilon\|^{2m-3j-3}_{W^j_\infty(0,T;W_\infty^{2m-4j-3}(\Omega))}\right)\right. \\
&\qquad \left.\left(1+\|\overline{c}^\epsilon\|_{W^j_\infty(0,T;W_p^{2m-4j-2}(\Omega))}\right)\right]. \qquad (6.48)
\end{aligned}$$

We set $m = 2n$ in (6.46) and apply (6.35) with $\alpha = 2n - \frac{1}{2}$, (6.45) with $\beta = 2n - \frac{3}{2}$, $k = l = 0$, and (6.48). Then we get for $n \in \mathbb{N}\setminus\{0\}$

$$\begin{aligned}
&\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n}(\Omega))} \\
&\leq C\left\|A^{2n}\overline{c}^\epsilon(.,0)\right\|_{L^p(\Omega)} + C\epsilon^{-\frac{3}{2}}\left(1+\|\overline{c}^\epsilon\|^{4n-2}_{L^\infty(0,T;W_\infty^{4n-2}(\Omega))}\right) \\
&\quad\times\left(1+\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n-1}(\Omega))}\right) + C\epsilon^{-2n}\sum_{j=0}^{n-2}\left[\left(1+\|\overline{c}^\epsilon\|^{4n-3j-3}_{W^j_\infty(0,T;W_\infty^{4n-4j-3}(\Omega))}\right)\right. \\
&\quad\left.\times\left(1+\|\overline{c}^\epsilon\|_{W^j_\infty(0,T;W_p^{4n-4j-2}(\Omega))}\right)\right]. \qquad (6.49)
\end{aligned}$$

Moreover, we set $m = 2n$ in (6.47) and use that $\|A^{2n}\overline{c}^\epsilon\|_{W_p^1(\Omega)} \leq C\left\|A^{2n+1/2}\overline{c}^\epsilon\right\|_{L^p(\Omega)}$. Then we can apply (6.35) with $\alpha = 2n$ and (6.45) with $\beta = 2n-1$, $k = l = 0$. Hence it follows with (6.48)

$$
\begin{aligned}
&\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n+1}(\Omega))} \\
&\quad\le\ C\left\|A^{2n+1/2}\overline{c}^\epsilon(.,0)\right\|_{L^p(\Omega)} + C\epsilon^{-\frac{3}{2}}\left(1+\|\overline{c}^\epsilon\|^{4n-1}_{L^\infty(0,T;W_\infty^{4n-1}(\Omega))}\right) \\
&\qquad\times\left(1+\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n}(\Omega))}\right) + C\epsilon^{-2n}\sum_{j=0}^{n-2}\left[\left(1+\|\overline{c}^\epsilon\|^{4n-3j-2}_{W^j_\infty(0,T;W_\infty^{4n-4j-2}(\Omega))}\right)\right. \\
&\qquad\left.\times\left(1+\|\overline{c}^\epsilon\|_{W^j_\infty(0,T;W_p^{4n-4j-1}(\Omega))}\right)\right]. \qquad (6.50)
\end{aligned}
$$

Repeating the same procedure for $m = 2n+1$ in (6.46) and (6.47) yields

$$
\begin{aligned}
&\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n+2}(\Omega))} \\
&\quad\le\ C\left\|A^{2n+1}\overline{c}^\epsilon(.,0)\right\|_{L^p(\Omega)} + C\epsilon^{-\frac{3}{2}}\left(1+\|\overline{c}^\epsilon\|^{4n}_{L^\infty(0,T;W_\infty^{4n}(\Omega))}\right) \\
&\qquad\times\left(1+\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n+1}(\Omega))}\right) + C\epsilon^{-2n}\sum_{j=0}^{n-1}\left[\left(1+\|\overline{c}^\epsilon\|^{4n-3j-1}_{W^j_\infty(0,T;W_\infty^{4n-4j-1}(\Omega))}\right)\right. \\
&\qquad\left.\times\left(1+\|\overline{c}^\epsilon\|_{W^j_\infty(0,T;W_p^{4n-4j}(\Omega))}\right)\right], \qquad (6.51)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n+3}(\Omega))} \\
&\quad\le\ C\left\|A^{2n+3/2}\overline{c}^\epsilon(.,0)\right\|_{L^p(\Omega)} + C\epsilon^{-\frac{3}{2}}\left(1+\|\overline{c}^\epsilon\|^{4n+1}_{L^\infty(0,T;W_\infty^{4n+1}(\Omega))}\right) \\
&\qquad\times\left(1+\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4n+2}(\Omega))}\right) + C\epsilon^{-2n}\sum_{j=0}^{n-1}\left[\left(1+\|\overline{c}^\epsilon\|^{4n-3j}_{W^j_\infty(0,T;W_\infty^{4n-4j}(\Omega))}\right)\right. \\
&\qquad\left.\times\left(1+\|\overline{c}^\epsilon\|_{W^j_\infty(0,T;W_p^{4n-4j+1}(\Omega))}\right)\right]. \qquad (6.52)
\end{aligned}
$$

With the last four estimates we can prove the assertion of the lemma.
Claim 3: Let $p > d$ be as in Claim 2. Then for all $m \in \mathbb{N}$ there exists an integer k_m and a constant $C > 0$ independent of ϵ such that

$$\sum_{i=0}^{m}\|\overline{c}^\epsilon\|_{W^i_\infty(0,T;W_p^{4(m-i)+1}(\Omega))} + \|\overline{c}^\epsilon\|_{W^i_\infty(0,T;W_\infty^{4(m-i)}(\Omega))} \le C\epsilon^{-k_m}\,.$$

We proof Claim 3 by mathematical induction.
The base case "$m = 0$": See Claim 2.
The inductive step "$m \to m+1$":
The induction hypothesis applied to (6.51) with $m = n$ yields

$$\|\overline{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m+2}(\Omega))} \le C\epsilon^{-k_{m+1}}$$

for some integer k_{m+1} and some constant $C > 0$. Using this estimate and again the induction hypothesis for (6.52) with $m = n$ yields

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m+3}(\Omega))} \leq C\epsilon^{-k_{m+1}} \tag{6.53}$$

for some integer k_{m+1} (for better clarity we again write k_{m+1}) and some constant $C > 0$. Since $p > d$ we can apply Sobolev's imbedding to get

$$\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_\infty^{4m+2}(\Omega))} \leq C\epsilon^{-k_{m+1}} . \tag{6.54}$$

In order to estimate $\|\bar{c}^\epsilon\|_{L^\infty(0,T;W_\infty^{4m+3}(\Omega))}$, we need estimates for $\partial_t^i \bar{c}^\epsilon$, $i = 1, \dots, m$, in higher norms as in the induction hypothesis. To get higher time regularity, we use (6.33), which yields

$$\begin{aligned} \|\partial_t \bar{c}^\epsilon\|_{L^\infty(0,T;W_p^{4m-1})} &\leq \epsilon^2 \left\|A^2 \bar{c}^\epsilon\right\|_{L^\infty(0,T;W_p^{4m-1})} \\ &\quad + \|Af_1(\bar{c}^\epsilon) + f_2(\bar{c}^\epsilon)\|_{L^\infty(0,T;W_p^{4m-1})} \leq C\epsilon^{-k_{m+1}} , \end{aligned} \tag{6.55}$$

due to (6.53), (6.45), and the induction hypothesis. By Sobolev's imbedding it holds

$$\|\bar{c}^\epsilon\|_{W^1_\infty(0,T;W_\infty^{4m-2})} \leq C\epsilon^{-k_{m+1}} .$$

In the case $m \geq 2$ we differentiate (6.33) with respect to time t and use (6.55), (6.45), and the induction hypothesis to conclude

$$\begin{aligned} \left\|\partial_t^2 \bar{c}^\epsilon\right\|_{L^\infty(0,T;W_p^{4m-5})} &\leq \epsilon^2 \left\|\partial_t A^2 \bar{c}^\epsilon\right\|_{L^\infty(0,T;W_p^{4m-1})} \\ &\quad + \|\partial_t Af_1(\bar{c}^\epsilon) + \partial_t f_2(\bar{c}^\epsilon)\|_{L^\infty(0,T;W_p^{4m-1})} \leq C\epsilon^{-k_{m+1}} , \end{aligned}$$

and by Sobolev's imbedding we have

$$\|\bar{c}^\epsilon\|_{W^2_\infty(0,T;W_\infty^{4m-6})} \leq C\epsilon^{-k_{m+1}} .$$

Repeating the same procedure step by step, we obtain

$$\sum_{i=0}^{n} \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_p^{4(m-i)+3}(\Omega))} + \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_\infty^{4(m-i)+2}(\Omega))} \leq C\epsilon^{-k_{m+1}} \tag{6.56}$$

for some integer k_{m+1} and some constant $C > 0$.
Now we use (6.49) and (6.50) for $n = m + 1$ and repeat the same procedure as at the beginning of the inductive step. Instead of the induction hypothesis we consider (6.56). Then it follows

$$\sum_{i=0}^{m+1} \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_p^{4((m+1)-i)+1}(\Omega))} + \|\bar{c}^\epsilon\|_{W^i_\infty(0,T;W_\infty^{4((m+1)-i)}(\Omega))} \leq C\epsilon^{-k_{m+1}}$$

for some integer k_{m+1} and some constant C independent of ϵ. Thus Claim 3 follows. Since $\|\bar{c}^\epsilon\|_{L^\infty(\Omega)} \leq \|\bar{c}^\epsilon_A\|_{L^\infty(\Omega)} + \|\bar{c}^\epsilon - \bar{c}^\epsilon_A\|_{L^\infty(\Omega)} \leq \frac{3}{2}C_0$ for ϵ small enough, we conclude $c^\epsilon = \bar{c}^\epsilon$ by uniqueness of the solution to the convective Cahn-Hilliard equation. This shows the assertion of the lemma. □

6.2 Asymptotic Expansion

As in Section 3.2 we apply matched asymptotic expansion to construct approximate solutions $\{c_A^\epsilon, \mu_A^\epsilon\}_{0<\epsilon\leq 1}$ satisfying (6.15)-(6.17).
We use the same assumptions and definitions as in Subsection 3.2.1. In particular, we assume that

$$\Gamma^\epsilon := \{(x,t) \in \Omega_T : c^\epsilon(x,t) = 0\} = \bigcup_{0<t<T} (\Gamma_t^\epsilon \times \{t\})$$

is a smooth hypersurface and the spatial signed distance function d^ϵ has the expansion

$$d^\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i d^i(x,t)\,,$$

where d^0 is defined on $\overline{\Omega_T}$ and d^i, $i \geq 1$, is defined in a neighborhood of Γ^ϵ. Therefore we get the following conditions

$$\nabla d^0 \cdot \nabla d^k = \begin{cases} 1 & \text{if } k = 0\,, \\ 0 & \text{if } k = 1\,, \\ -\frac{1}{2}\sum_{i=1}^{k-1} \nabla d^i \cdot \nabla d^{k-i} & \text{if } k \geq 2\,, \end{cases} \tag{6.57}$$

where all the equations are satisfied in a neighborhood of Γ^ϵ. Observe that only for motivating the construction of the approximate solutions these assumptions are necessary.

6.2.1 Outer Expansion

We assume that c^ϵ and μ^ϵ have the same expansions as in Section 3.2.2, that is,

$$\begin{aligned} c^\epsilon(x,t) &= c_0^\pm(x,t) + \epsilon c_1^\pm(x,t) + \epsilon^2 c_2^\pm(x,t) + ... && \text{in } Q_0^\pm \backslash \Gamma^0\left(\tfrac{\delta}{2}\right), \\ \mu^\epsilon(x,t) &= \mu_0^\pm(x,t) + \epsilon \mu_1^\pm(x,t) + \epsilon^2 \mu_2^\pm(x,t) + ... && \text{in } Q_0^\pm \backslash \Gamma^0\left(\tfrac{\delta}{2}\right), \end{aligned}$$

where $c_i^\pm$ and $\mu_i^\pm$ are appropriate functions defined in $Q_0^\pm$ and $\delta > 0$ is a fixed constant independent of ϵ which is to be determined later. Then by substituting it into (6.1), we require

$$(c_k^\pm)_t + v \cdot \nabla c_k^\pm = \Delta \mu_{k-1}^\pm \quad \text{in } Q_0^\pm\,, \tag{6.58}$$

and into (6.2), we require

$$c_k^\pm = \begin{cases} \pm 1 & \text{, if } k = 0 \\ \frac{\mu_{k-1}^\pm - f^{k-1}(c_0^\pm, \ldots, c_{k-1}^\pm) + \Delta c_{k-2}^\pm}{f'(c_0^\pm)} & \text{, if } k \geq 1 \end{cases} \quad \text{in } Q_0^\pm\,, \tag{6.59}$$

where $\mu_{-1}^\pm = c_{-1}^\pm = 0$ and f^i is defined as in Subsection 3.2.2.

Remark 6.2.1. *To construct $\mu_k^\pm$, we later insert the expression for $c_{k+1}^\pm$ into (6.58) and obtain a parabolic equation for $\mu_k^\pm$. To get a unique solution it is natural to require for the initial value*

$$\mu_0^\pm(x,0) = \mu_{00}(x) \quad \text{in } \Omega_{00}^\pm .$$

For $k \geq 1$ we choose any initial values such that the solution $\mu_k^\pm$ is smooth, that is, $\mu_k^\pm \in C^\infty(\overline{\Omega^\pm(t)} \times [0,T])$. Due to Remark 6.0.1 this always is possible. Note that different initial values $\mu_k^\pm(.,0)$ induce different initial values $c_0^\epsilon = c_A^\epsilon(.,0)$. But the leading order term of $c_A^\epsilon(.,0)$ is independent of the choice of the initial values, see Subsection 6.2.6 and 6.2.7.

As in Subsection 3.2.2 we need an extension of $(c_k^\pm, \mu_k^\pm)$ from $Q_0^\pm$ to $Q_0^\pm \cup \Gamma^0(\delta)$. We can handle this as in Remark 3.2.2.
For the compatibility conditions we need the following definitions

$$O_k^\pm(x,t) := \left(c_k^\pm\right)_t + v \cdot \nabla c_k^\pm, \quad O^\pm := \sum_{i=1}^\infty \epsilon^i O_i^\pm \quad \text{in } Q_0^\pm \cup \Gamma^0(\delta),$$

$$P_k^\pm(x,t) := \Delta \mu_k^\pm, \quad P^\pm := \sum_{i=1}^\infty \epsilon^i P_i^\pm \quad \text{in } Q_0^\pm \cup \Gamma^0(\delta).$$

Obviously, for all $k \geq 0$ it holds $O_k^\pm - P_{k-1}^\pm = 0$ in $\overline{Q_0^\pm}$ where $P_{-1}^\pm = 0$.

6.2.2 Inner Expansion

Again we assume that c^ϵ and μ^ϵ have the same expansion in $\Gamma^0(\delta)$ as in Section 3.2.3, that is,

$$c^\epsilon(x,t) = \tilde{c}^\epsilon\left(\frac{d^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \tilde{c}^\epsilon(z,x,t) = \sum_{i=0}^\infty \epsilon^i c^i(z,x,t),$$

$$\mu^\epsilon(x,t) = \tilde{\mu}^\epsilon\left(\frac{d^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \tilde{\mu}^\epsilon(z,x,t) = \sum_{i=0}^\infty \epsilon^i \mu^i(z,x,t),$$

where $\tilde{c}^\epsilon, \tilde{\mu}^\epsilon, c^i$ and μ^i are appropriate functions defined in $\mathbb{R} \times \Gamma^0(\delta)$.
We require the same inner-outer matching conditions as $z \to \infty$

$$D_x^m D_t^n D_z^l \left[c^k(\pm z, x, t) - c_k^\pm(x,t)\right] = \mathcal{O}(e^{-\alpha z}), \tag{6.60}$$

$$D_x^m D_t^n D_z^l \left[\mu^k(\pm z, x, t) - \mu_k^\pm(x,t)\right] = \mathcal{O}(e^{-\alpha z}) \tag{6.61}$$

for all $(x,t) \in \Gamma^0(\delta)$ and all $k, m, n, l \in \left\{0, \ldots, \bar{K}\right\}$ where $\bar{K}$ depends on the order of expansion. Also it is natural that we require

$$c^k(0,x,t) = 0 \quad \forall (x,t) \in \Gamma^0(\delta), \, \forall k \geq 0 .$$

Substituting the expansion of $(c^\epsilon, \mu^\epsilon)$ into (6.1), we obtain

$$\begin{aligned} &\epsilon^{-1}\tilde{c}_z d^\epsilon_t + \tilde{c}_t + \epsilon^{-1}\tilde{c}_z v\cdot\nabla d^\epsilon + v\cdot\nabla\tilde{c} \\ &\quad = \epsilon\left(\epsilon^{-2}\tilde{\mu}_{zz} + 2\epsilon^{-1}\nabla\tilde{\mu}_z\cdot\nabla d^\epsilon + \epsilon^{-1}\tilde{\mu}_z\Delta d^\epsilon + \Delta\tilde{\mu}\right), \end{aligned}$$

and into (6.2), we get

$$\tilde{\mu} = -\epsilon^{-1}\left(\tilde{c}_{zz} - f(\tilde{c})\right) - \tilde{c}_z\Delta d^\epsilon - 2\nabla\tilde{c}_z\cdot\nabla d^\epsilon - \epsilon\Delta\tilde{c}$$

for all $(z,x,t)\in S^\epsilon := \{(z,x,t)\in\mathbb{R}\times\Gamma^0(\delta) : z = d^\epsilon(x,t)/\epsilon\}$. Let $\eta, \eta_N^\pm \in C^\infty(\mathbb{R})$ be defined as in (3.75)-(3.77) with the same constant N. For satisfying the compatibility and matching conditions, we consider the modified equations

$$\begin{aligned} \tilde{c}_{zz} - f(\tilde{c}) &= \epsilon\left(-\tilde{\mu} - \Delta d^\epsilon\tilde{c}_z - 2\nabla d^\epsilon\cdot\nabla\tilde{c}_z\right) - \epsilon^2\Delta\tilde{c} \\ &\quad + g^\epsilon\eta'\left(d^\epsilon - \epsilon z\right), \qquad (6.62)\\ \tilde{\mu}_{zz} &= \left(\tilde{c}_z d^\epsilon_t + \tilde{c}_z v\cdot\nabla d^\epsilon\right) - \epsilon\left(2\nabla\tilde{\mu}_z\cdot\nabla d^\epsilon + \tilde{\mu}_z\Delta d^\epsilon\right) \\ &\quad + \epsilon\left(\tilde{c}_t + v\cdot\nabla\tilde{c}\right) - \epsilon^2\left(\Delta\tilde{\mu}\right) + \left(h^\epsilon\eta'' + L^\epsilon\eta'\right)\left(d^\epsilon - \epsilon z\right) \\ &\quad - \epsilon\left(O^+\eta_N^+ + O^-\eta_N^-\right) + \epsilon^2\left(P^+\eta_N^+ + P^-\eta_N^-\right) \qquad (6.63) \end{aligned}$$

for $z\in\mathbb{R}$ and $(x,t)\in\Gamma^0(\delta)$ and where g^ϵ, L^ϵ, and h^ϵ have the following expansions in $\Gamma^0(\delta)$

$$g^\epsilon(x,t) = \sum_{i=0}^\infty \epsilon^{i+1}g^i(x,t), \qquad L^\epsilon(x,t) = \sum_{i=0}^\infty \epsilon^i L^i(x,t),$$
$$h^\epsilon(x,t) = \sum_{i=0}^\infty \epsilon^i h^i(x,t).$$

We get the following ordinary differential equations by equating the ϵ^k terms

$$\left.\begin{aligned} c^0_{zz} - f(c^0) &= 0 \\ c^k_{zz} - f'(c^0)c^k &= A^{k-1}(z,x,t),\ k\ge 1 \end{aligned}\right\}\quad z\in\mathbb{R}, \qquad (6.64)$$

$$\left.\begin{aligned} \left(\mu^0 - h^0 d^0\eta\right)_{zz} &= B^0(z,x,t) \\ \left(\mu^k - \left(h^k d^0 + h^0 d^k\right)\eta\right)_{zz} &= B^k(z,x,t),\ k\ge 1 \end{aligned}\right\}\quad z\in\mathbb{R} \qquad (6.65)$$

for all $(x,t)\in\Gamma^0(\delta)$ and where for $k\ge 0$

$$\begin{aligned} A^{k-1}(z,x,t) &= -\mu^{k-1} - \left(\Delta d^0 c_z^{k-1} + \Delta d^{k-1}c_z^0\right) \\ &\quad -2\left(\nabla d^0\cdot\nabla c_z^{k-1} + \nabla d^{k-1}\cdot\nabla c_z^0\right) + f^{k-1}(c^0,\ldots,c^{k-1}) \\ &\quad + \left(g^{k-1}d^0 + g^0 d^{k-1}\right)\eta' + \mathcal{A}^{k-2}, \qquad (6.66)\\ B^k(z,x,t) &= c_z^k(d_t^0 + v\cdot\nabla d^0) + c_z^0(d_t^k + v\cdot\nabla d^k) + (L^k d^0 + L^0 d^k)\eta' + \mathcal{B}^{k-1}, \qquad (6.67) \end{aligned}$$

where

$$
\begin{aligned}
\mathcal{A}^{k-2}(z,x,t) &= \sum_{i=1}^{k-2}\left(-\Delta d^i c_z^{k-1-i} - 2\nabla d^i \cdot \nabla c_z^{k-1-i} + d^i g^{k-1-i}\eta'\right) \\
&\quad -\Delta c^{k-2} - z g^{k-2}\eta', \qquad (6.68) \\
\mathcal{B}^{k-1}(z,x,t) &= \sum_{i=1}^{k-1}\left(d_t^i c_z^{k-i} + v\cdot\nabla d^i c_z^{k-i}\right) \\
&\quad +\sum_{i=0}^{k-1}\left(-\Delta d^i \mu_z^{k-i-1} - 2\nabla d^i\cdot\nabla\mu_z^{k-1-i}\right) \\
&\quad +\sum_{i=1}^{k-1}\left(d^i L^{k-i}\eta' + d^i h^{k-i}\eta''\right) \\
&\quad -zL^{k-1}\eta' - zh^{k-1}\eta' + c_t^{k-1} + v\cdot\nabla c^{k-1} - \Delta\mu^{k-2} \\
&\quad -O_{k-1}^{+}\eta_N^{+} - O_{k-1}^{-}\eta_N^{-} + P_{k-2}^{+}\eta_N^{+} + P_{k-2}^{-}\eta_N^{-} \qquad (6.69)
\end{aligned}
$$

for all $(x,t)\in\Gamma^0(\delta)$. We have used the convention that if the upper limit of the summation is less than the lower limit, then the summation is zero, that $A^{-1} = c^{-1} = \mu^{-1} = 0$, that $a^k b^0 + a^0 b^k = a^0 b^0$ when $k=0$, that $a^{k-1}b^0 + a^0 b^{k-1} = a^0 b^0$ when $k=1$, and that $a^{k-1}b^1 + a^1 b^{k-1} = a^1 b^1$ when $k=2$.
Since we require $c^0(0,x,t)=0$ and $\lim_{z\to\infty} c^0(\pm z,x,t) = \pm 1$, we immediately see that $c^0(z,x,t)=\theta_0(z)$ for all $z\in\mathbb{R}$ and $(x,t)\in\Gamma^0(\delta)$ where θ_0 is the unique solution to (2.7).
By the terms $O_{k-1}^{\pm}$ and $P_{k-2}^{\pm}$ the right-hand side B^k is $\mathcal{O}(e^{-\alpha|z|})$ as $z\to\pm\infty$ provided the inner-outer matching conditions are satisfied for all functions. This is necessary to get bounded solutions, see Lemma 2.6.3.

Remark 6.2.2. 1. *Note that the right-hand side of (6.65) depends on terms of order k. More precisely, B^k depends on c^k and d^k. But it is shown later that c^k and d^k can be expressed in terms of order lower than k. Therefore we can solve for μ^k. We do not write B^{k-1} instead of B^k because by the compatibility condition for μ^k and therefore by B^k, we obtain an equation to solve for d^k. For more details see the next subsection.*

2. *In contrast to the outer expansion, we can not choose the initial values for μ^k because we also solve the ordinary differential equations for $t=0$ with given right-hand side.*

6.2.3 Compatibility Conditions

In this subsection we verify that the ordinary differential equations (6.64) and (6.65) have bounded solutions provided some compatibility conditions are satisfied.

Lemma 6.2.3. *Let $k \geq 1$ be any integer and A^{k-1} and $\mathcal{A}^{k-2}$ be defined as in (6.66) and (6.68). Then the system*

$$\begin{aligned} &c^k_{zz}(z,x,t) - f'(\theta_0(z))\, c^k(z,x,t) = A^{k-1}(z,x,t) \quad \forall z \in \mathbb{R}, \\ &c^k(0,x,t) = 0, \quad c^k(.,x,t) \in L^\infty(\mathbb{R}) \end{aligned} \tag{6.70}$$

has a unique solution for all $(x,t) \in \Gamma^0(\delta)$ if and only if

$$\widetilde{\mathcal{A}}^{k-2} \;=\; -\overline{\mu}^{k-1} - \sigma \Delta d^{k-1} + \eta_0 \left(g^{k-1} d^0 + g^0 d^{k-1} \right) \quad in\ \Gamma^0(\delta), \tag{6.71}$$

where

$$\begin{aligned} \overline{\mu}^{k-1}(x,t) &= \frac{1}{2} \int_{\mathbb{R}} \mu^{k-1}(z,x,t) \theta_0'(z)\, dz\,, \\ \sigma &= \frac{1}{2} \int_{\mathbb{R}} \left(\theta_0'(z) \right)^2 dz\,, \\ \eta_0 &= \frac{1}{2} \int_{\mathbb{R}} \eta'(z) \theta_0'(z)\, dz\,, \\ \widetilde{\mathcal{A}}^{k-2}(x,t) &= \frac{1}{2} \int_{\mathbb{R}} \big(\Delta d^0 c_z^{k-1} + 2 \nabla d^0 \cdot \nabla c_z^{k-1} - f^{k-1}(c^0, \ldots, c^{k-1}) \\ &\qquad - \mathcal{A}^{k-2} \big) \theta_0'(z)\, dz\,, \end{aligned}$$

where in the last definition there is no term involving d^0 when $k = 1$.
In addition, if (6.71) is satisfied and $\left(c^0, c_0^\pm, \mu^0, \mu_0^\pm\right), \ldots, \left(c^{k-1}, c_{k-1}^\pm, \mu^{k-1}, \mu_{k-1}^\pm\right)$ satisfy the matching conditions (6.60) and (6.61), then the unique solution c^k of (6.64b) satisfies the matching condition (6.60) where $c_k^\pm$ is given by (6.59).

Proof: The proof is a special case of Lemma 3.2.7 or see [10]. □

Lemma 6.2.4. *Let $k \geq 0$ be any integer and let B^k and $\mathcal{B}^{k-1}$ be defined as in (6.67) and (6.69). Then (6.65) has a bounded solution for $k = 0$ in $\Gamma^0(\delta)$ if and only if*

$$d_t^0 + v \cdot \nabla d^0 + \frac{1}{2} L^0 d^0 = 0 \quad in\ \Gamma^0(\delta)\,, \tag{6.72}$$

and for $k \geq 1$ it has a bounded solution if and only if

$$d_t^k + v \cdot \nabla d^k + \frac{1}{2}(L^k d^0 + L^0 d^k) = \widetilde{\mathcal{B}}^{k-1} \quad in\ \Gamma^0(\delta)\,, \tag{6.73}$$

where

$$\widetilde{\mathcal{B}}^{k-1}(x,t) = -\frac{1}{2} \left[c^k \right] (d_t^0 + v \cdot \nabla d^0) - \frac{1}{2} \int_{\mathbb{R}} \mathcal{B}^{k-1}\, dz \quad in\ \Gamma^0(\delta)\,,$$

where $[.] = .\,|_{-\infty}^{+\infty}$. *In addition, if (6.72) for* $k=0$ *or (6.73) for* $k \geq 1$ *is satisfied, then every solution to (6.65) can be written as*

$$\begin{aligned} \mu^0(z,x,t) &= \tilde{\mu}^0(x,t) + (d^0h^0)(x,t)(\eta(z) - \tfrac{1}{2}) + \mu_*^0(z,x,t) && \text{if } k = 0, \\ \mu^k(z,x,t) &= \tilde{\mu}^k(x,t) + (d^0h^k + d^kh^0)(x,t)\,(\eta(z) - \tfrac{1}{2}) + \mu_*^k(z,x,t) && \text{if } k \geq 1 \end{aligned} \tag{6.74}$$

for all $(x,t) \in \Gamma^0(\delta)$. *Here* $\tilde{\mu}^0(x,t)$ *and* $\tilde{\mu}^k(x,t)$ *are arbitrary functions,* $\mu_*^0(z,x,t)$ *is a special solution depending only on* (c^0, d^0, L^0), *and* $\mu_*^k(z,x,t)$ *is a special solution depending only on* $(c_i^\pm, c^i, \mu_i^\pm, \mu^i, d^i, h^i, g^i, L^i)$ *for* $i \leq k-1$ *and* (c^k, d^k, L^k). *Furthermore,* μ_*^k, $k \geq 0$, *is uniquely determined by the condition*

$$\int_{\mathbb{R}} \mu_*^k(z,x,t)\theta_0'(z)\,dz = 0 \quad \forall (x,t) \in \Gamma^0(\delta)\,. \tag{6.75}$$

Moreover, μ_*^k, $k \geq 0$, *satisfies*

$$D_x^m D_t^n D_z^l \left[\mu_*^k(\pm z,x,t) - \mu_{*(k)}^\pm(x,t)\right] = \mathcal{O}(e^{-\alpha z}) \quad \text{as } z \to \infty$$

for some $\mu_{*(k)}^\pm$ *depending only on* (c^0, d^0, L^0) *if* $k=0$, *or* $(c_i^\pm, c^i, \mu_i^\pm, \mu^i, d^i, h^i, g^i)$ *for* $i \leq k-1$ *and* $(c_k^\pm, c^k, d^k, L^k)$ *if* $k \geq 1$.

Proof: We can use the same argumentation as in Lemma 3.2.9, or see [10]. □

6.2.4 Boundary-Layer Expansion

Let d_B, S_B and $\partial_T\Omega(\delta)$ be defined as in Subsection 3.2.5. Near the boundary $\partial_T\Omega$ we assume that the solutions $(c^\epsilon, \mu^\epsilon)$ have for every $\epsilon \in (0,1]$ the form

$$c^\epsilon(x,t) = c_B^\epsilon\left(\frac{d_B^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad c_B^\epsilon(z,x,t) = 1 + \sum_{i=1}^{\infty} \epsilon^i c_B^i(z,x,t),$$

$$\mu^\epsilon(x,t) = \mu_B^\epsilon\left(\frac{d_B^\epsilon(x,t)}{\epsilon}, x, t\right), \qquad \mu_B^\epsilon(z,x,t) = \sum_{i=0}^{\infty} \epsilon^i \mu_B^i(z,x,t),$$

where $(x,t) \in \overline{\partial_T\Omega(\delta)}$ and $z \in (-\infty, 0]$.
Also we require the outer-boundary matching conditions

$$D_x^m D_t^n D_z^l \left[c_B^k(z,x,t) - c_k^+(x,t)\right] = \mathcal{O}(e^{\alpha z})\,, \tag{6.76}$$

$$D_x^m D_t^n D_z^l \left[\mu_B^k(z,x,t) - \mu_k^+(x,t)\right] = \mathcal{O}(e^{\alpha z})\,, \tag{6.77}$$

as $z \to -\infty$ and for all $(x,t) \in \overline{\partial_T\Omega(\delta)}$ and all $k,m,n,l \in \{0,\ldots,\bar{K}\}$ where $\bar{K}$ depends on the order of expansion.

To satisfy the convective Cahn-Hilliard equations (6.1) and (6.2) near $\partial_T\Omega$, it is sufficient to require that

$$c^k_{B,zz}(z,x,t) - f'(1)c^k_B(z,x,t) = A^{k-1}_B(z,x,t), \qquad k \geq 1\,, \tag{6.78}$$
$$\mu^k_{B,zz}(z,x,t) = B^{k-1}_B(z,x,t), \qquad k \geq 0 \tag{6.79}$$

for all $(x,t) \in \partial_T\Omega$ and $z \in (-\infty, 0)$ where

$$\begin{aligned} A^{k-1}_B(z,x,t) &:= -\mu^{k-1}_B + f^{k-1}(c^0_B,\dots,c^{k-1}_B) - 2\nabla d_B \cdot \nabla c^{k-1}_{B,z} \\ &\quad -\Delta d_B c^{k-1}_{B,z} - \Delta c^{k-2}_B\,, \end{aligned} \tag{6.80}$$
$$\begin{aligned} B^{k-1}_B(z,x,t) &:= v \cdot \nabla d_B c^k_{B,z} + c^{k-1}_{B,t} + v \cdot \nabla c^{k-1}_B - \Delta d_B \mu^{k-1}_{B,z} \\ &\quad -2\nabla d_B \cdot \nabla \mu^{k-1}_{B,z} - \Delta \mu^{k-2}_B\,, \end{aligned} \tag{6.81}$$

where we have assumed that $c^{-2}_B = c^{-1}_B = \mu^{-2}_B = \mu^{-1}_B = 0$ and $c^0_B = 1$.
To enforce the boundary conditions $\frac{\partial}{\partial n}c^\epsilon = \frac{\partial}{\partial n}\mu^\epsilon = 0$, we require as in Subsection 3.2.5 that

$$c^k_{B,z}(0,x,t) = -\nabla d_B \cdot c^{k-1}_B(0,S_B(x),t) \qquad \forall (x,t) \in \overline{\partial_T\Omega(\delta)}, k \geq 1, \tag{6.82}$$
$$\mu^k_{B,z}(0,x,t) = -\nabla d_B \cdot \mu^{k-1}_B(0,x,t) \qquad \forall (x,t) \in \partial_T\Omega, k \geq 0. \tag{6.83}$$

Lemma 6.2.5. *Let $j \geq 1$ be any integer. Assume that for all $i = 0,\dots,j-1$, the functions $c^+_i, \mu^+_i, c^i_B, \mu^i_B$ are known, smooth, and satisfy the matching conditions (6.76) and (6.77). Then for $k = j$, the boundary-layer equation (6.78) subject to the boundary condition (6.82) has a unique bounded solution c^j_B for $z \in (-\infty, 0]$ and all $(x,t) \in \overline{\partial_T\Omega(\delta)}$. In addition, the solution satisfies the matching condition (6.76) where c^+_j is defined by (6.59).*

Proof: It is the same proof as for Lemma 3.2.14 or [10, Lemma 4.5]. □

We set for all $k \geq 0$

$$\begin{aligned} G^{k-1}(x,t) &= (\Delta d_B(x) + \nabla d_B(x) \cdot \nabla) \int_{-\infty}^0 \int_{-\infty}^z B^{k-1}_B(w,x,t)\,dw\,dz \\ &\quad - \int_{-\infty}^0 (c^k_{B,t} + v \cdot \nabla c^k_B - \Delta \mu^{k-1}_B)(z,x,t)\,dz \end{aligned} \tag{6.84}$$

for all $(x,t) \in \overline{\partial_T\Omega(\delta)}$.

Remark 6.2.6. *Observe that B^{k-1}_B and G^{k-1} depend on c^k_B. It is shown later that c^k_B can be expressed in terms of order lower than k. Therefore we can consider B^{k-1}_B and G^{k-1} as functions depending only on terms of order lower than k. In addition, it follows $B^{-1}_B = G^{-1} = 0$ since $c^0_B = 1$.*

Lemma 6.2.7. *Let $j \geq 0$ be any integer. Assume that for all $i = 0, \ldots, j-1$, the functions $c_i^+, \mu_i^+, c_B^i, \mu_B^i$ and c_B^j are known, smooth, and satisfy the matching conditions (6.76) and (6.77) and the outer expansion $c_{i,t}^+ + v \cdot \nabla c_i^+ - \Delta\mu_{i-1}^+ = 0$. Let G^{j-1} be defined as in (6.84) and assume that μ_{j-1}^+ satisfies the boundary condition*

$$\frac{\partial}{\partial n}\mu_{j-1}^+(x,t) = G^{j-2}(x,t) \quad \forall (x,t) \in \partial_T\Omega\,. \tag{6.85}$$

Also assume that $\mu_B^{-1} = c_B^{-1} = 0$ and μ_B^i, $i = 0, \ldots, j-1$ are defined by

$$\mu_B^i(z,x,t) = \mu_i^+(x,t) + \int_{-\infty}^{z}\int_{-\infty}^{y} B_B^{i-1}(w,x,t)\,dw\,dy \tag{6.86}$$

for all $z \in (-\infty, 0]$ and $(x,t) \in \overline{\partial_T\Omega(\delta)}$ and where B_B^i is defined as in (6.81). Then for known smooth μ_j^+ the function μ_B^j defined by (6.86) (with $i = j$) satisfies for $k = j$ the boundary-expansion equation (6.79), the boundary condition (6.83), and the matching condition (6.77).

Proof: We only prove that μ_B^j satisfies (6.83). The other assertions of the lemma can be shown as in the proof of Lemma 3.2.15. Observe that $v \cdot \nabla d_B = 0$ on $\partial_T\Omega$, and hence we obtain for all $(x,t) \in \partial_T\Omega$

$$\begin{aligned}
\mu_{B,z}^j(0,x,t) &= \int_{-\infty}^{0} B_B^{j-1}(z,x,t)\,dz \\
&= \int_{-\infty}^{0} -\Delta d_B \mu_{B,z}^{j-1} - 2\nabla d_B \cdot \nabla \mu_{B,z}^{j-1} + c_{B,t}^{j-1} + v \cdot \nabla c_B^{j-1} - \Delta\mu_B^{j-2}\,dz \\
&= (\Delta d_B + \nabla d_B \cdot \nabla)(\mu_{j-1}^+ - \mu_B^{j-1}(0,x,t)) \\
&\quad + \nabla d_B \cdot \nabla(\mu_{j-1}^+ - \mu_B^{j-1}(0,x,t)) + \int_{-\infty}^{0} c_{B,t}^{j-1} + v \cdot \nabla c_B^{j-1} - \Delta\mu_B^{j-2}\,dz
\end{aligned}$$

since $\lim_{z\to\infty} \mu_B^{j-1}(-z,x,t) = \mu_{j-1}^+(x,t)$. From (6.86) with $i = j-1$, we get

$$\mu_{j-1}^+(x,t) - \mu_B^{j-1}(0,x,t) = -\int_{-\infty}^{0}\int_{-\infty}^{y} B_B^{j-2}(w,x,t)\,dw\,dy\,,$$

and we have

$$\nabla d_B \cdot \mu_{j-1}^+ = \tfrac{\partial}{\partial n}\mu_{j-1}^+ = G^{j-2} \quad \text{on } \partial_T\Omega\,.$$

Hence it follows

$$\begin{aligned}
&\mu_{B,z}^j(0,x,t) + \nabla d_B \cdot \nabla\mu_B^{j-1}(0,x,t) \\
&\quad = G^{j-2} - (\Delta d_B + \nabla d_B \cdot \nabla)\int_{-\infty}^{0}\int_{-\infty}^{y} B_B^{j-2}(w,x,t)\,dw\,dy \\
&\qquad + \int_{-\infty}^{0} c_{B,t}^{j-1} + v \cdot \nabla c_B^{j-1} - \Delta\mu_B^{j-2}\,dz = 0
\end{aligned}$$

by the defintion of G^{j-2}. □

Remark 6.2.8. *As in the inner expansion the functions (c_B^k, μ_B^k) are also determined for $t = 0$, that is, we cannot choose an initial value for μ_B^k as for the outer expansion.*

6.2.5 Basic Steps of Solving Expansions of each Order

For each $j \geq 0$ we recursively define the jth order expansion

$$\mathcal{V}^j \equiv (c_j^{\pm}, c^j, c_B^j, \mu_j^{\pm}, \mu^j, \mu_B^j, d^j, g^j, h^j, L^j)\,.$$

We carry out the same steps as in [10] and Subsection 3.2.6 but in another sequence due to the mobility constant $m(\epsilon) = \epsilon$. More precisely, we carry out the following steps:
Step 1: We determine $(c^j, c_j^{\pm}, c_B^j)$. Therefore we can consider $(c^j, c_j^{\pm}, c_B^j)$ as known quantities depending only on $\mathcal{V}^i$, $i \leq j-1$.
Step 2: (Step 2 corresponds to Step 6 in [10].) By the compatibility conditions (6.72) and (6.73) (with $k = j$), we obtain an evolution equation for d^j on Γ^0. Then we can uniquely determine d^0 and Γ^0 with the initial condition $\Gamma^0(0) = \Gamma_{00}$ since we require that d^0 is the signed distance function to Γ^0. For $j \geq 1$ we get a unique d^j together with (6.57) and the initial condition $d^j(x, 0) = 0$ on $\Gamma^0(0)$. Hence we can consider d^j as known quantities depending only on $\mathcal{V}^i$, $i \leq j-1$.
Step 3: We determine L^j such that the compatibility conditions (6.72) and (6.73) (with $k = j$) are satisfied on $\Gamma^0(\delta)$.
Step 4-7 corresponds to Step 2-5 in [10] and to Step 6-9 in Subsection 3.2.6, that is, we determine μ^j and $\mu_j^{\pm}$. Note that for $\mu_j^{\pm}$ we have to solve a parabolic differential equation with initial value μ_{00} instead of an elliptic differential equation.
Step 8 and 9 corresponds to Step 7 and 8 in [10] and to Step 11 and 12 in Subsection 3.2.6. This means that we determine g^j to satisfy the compatibility condition (6.71) in $\Gamma^0(\delta)$. Moreover, we solve for μ_B^j. Observe that we have already determined L^j.

After motivating the construction of $\mathcal{V}^j$ in the Steps 1-9, we verify that $\mathcal{V}^j$ satisfies all the corresponding outer, inner, and boundary-layer expansion equations, the inner-outer matching conditions, and the outer-boundary matching conditions for $k = j$. In addition we show that the compatibility condition (6.72) or (6.73) is satisfied for $k = j$ and the compatibility condition (6.71) for $k = j + 1$.

6.2.6 The Zero-th Order Expansion

We carry out Step 1-9 for $j = 0$.
Step 1: We already know that $c_0^{\pm}(x, t) = \pm 1$ for $(x, t) \in Q_0^{\pm} \cup \Gamma^0(\delta)$, $c^0(z, x, t) = \theta_0(z)$ for $(z, x, t) \in \mathbb{R} \times \Gamma^0(\delta)$, and $c_B^0(z, x, t) = 1$ for $(z, x, t) \in (-\infty, 0] \times \overline{\partial_T \Omega(\delta)}$.
Since we require $c_A^{\epsilon}(., 0) = c_0^{\epsilon}$, one sees that the leading order term of c_0^{ϵ} is independent of μ_{00} and vice versa.
Step 2: The compatibility condition (6.72) reads for all $(x, t) \in \Gamma^0$

$$d_t^0(x, t) + v(x, t) \cdot \nabla d^0(x, t) = 0\,.$$

It is known that the normal velocity of Γ_t^0 is given by $-d_t$ and the outer unit normal of Γ_t^0 by ∇d^0. Hence Γ^0 is uniquely determined by the evolution equation

$$V(x,t) = \nu_{\Gamma_t^0}(x,t) \cdot v(x,t) \quad \forall (x,t) \in \Gamma^0, \quad \Gamma^0(0) = \Gamma_{00}.$$

Note that due to Lemma 5.2.4 the hypersurface Γ_t^0 exists globally and $\Gamma_t^0 \subset \Omega$ for all $t \in \mathbb{R}$. More precisely, Γ_t^0 is given by $\Gamma_t^0 = X_t(\Gamma_{00})$ for all $t \in \mathbb{R}$ where X_t is defined by Lemma 5.2.1.

So from now d^0 and Γ^0 are known quantities.

Step 3: To satisfy the compatibility condition (6.72) in $\Gamma^0(\delta)$, we set

$$L^0(x,t) := \begin{cases} -\frac{2}{d^0}\left(d_t^0 + v \cdot \nabla d^0\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0 \\ -2\nabla d^0 \cdot \nabla\left(d_t^0 + v \cdot \nabla d^0\right) & \text{on } \Gamma^0\,. \end{cases} \tag{6.87}$$

Since the numerator vanishes on Γ^0, L^0 is smooth.

Step 4: The compatibility condition (6.71) for $k = 1$ and for all $(x,t) \in \Gamma^0$ is

$$\overline{\mu}^0(x,t) = -\sigma \Delta d^0(x,t) = \sigma \kappa_{\Gamma_t^0}(x,t)\,, \tag{6.88}$$

where $\kappa_{\Gamma_t^0}$ is the mean curvature of Γ_t^0.

Step 5: By equation (6.74) we obtain for all $(x,t) \in \Gamma^0(\delta)$

$$\mu^0(z,x,t) = \tilde{\mu}^0(x,t) + d^0(x,t)h^0(x,t)\left(\eta(z) - 1/2\right) + \mu_*^0(z,x,t)\,, \tag{6.89}$$

where we define $\tilde{\mu}^0$ and h^0 later. Due to the definitions of $\overline{\mu}^0$ and μ_*^0 in Lemma 6.2.3 and 6.2.4 and due to the fact $\int_{\mathbb{R}} (\eta - 1/2)\,\theta_0'(z), dz = 0$, it follows that $\tilde{\mu}^0 = \overline{\mu}^0$ in $\Gamma^0(\delta)$. By Step 2 equation (6.65) reads for all $(x,t) \in \Gamma^0$

$$\mu_{zz}^0(z,x,t) = 0\,.$$

Since all solutions for this linear differential equation have the form $c_1(x,t)z + c_2(x,t)$ for some constants $c_1(x,t), c_2(x,t) \in \mathbb{R}$, all bounded solutions are constant functions with respect to z, in particular $\mu_*^0(z,x,t) = c_2(x,t)$ for some $c_2(x,t)$. By (6.75) and since $\theta_0'(z) > 0$ for all $z \in \mathbb{R}$, it follows $\mu_*^0(z,x,t) = 0$ for all $(x,t) \in \Gamma^0$. Therefore we require $\mu^0(z,x,t) = \overline{\mu}^0(x,t) = \sigma\kappa_{\Gamma_t^0}$ for all $(x,t) \in \Gamma^0$.

Step 6: The equation

$$\mu_0^{\pm}(x,t) = \lim_{z\to\infty} \mu^0(\pm z,x,t) = \sigma\kappa_{\Gamma_t^0} \quad \forall (x,t) \in \Gamma^0$$

follows from the matching condition (6.61). The outer expansions (6.58) and (6.59) yield for $k = 1$

$$\partial_t c_1^{\pm} + v \cdot \nabla c_1^{\pm} = \Delta \mu_0^{\pm} \quad \text{and} \quad c_1^{\pm} = \frac{\mu_0^{\pm} - f^0(c_0^{\pm})}{f'(c_0^{\pm})} \quad \text{in } Q_0^{\pm}. \tag{6.90}$$

So we obtain the parabolic equation

$$\partial_t \mu_0^{\pm} = f'(\pm 1)\Delta\mu_0^{\pm} - v \cdot \nabla \mu_0^{\pm} \quad \text{in } Q_0^{\pm}. \tag{6.91}$$

In the construction of $\mathcal{V}^1$ we define $c_1^\pm$ by the second equation in (6.90). Hence the first equation in (6.90) and (6.91) are equivalent. Therefore the initial value $c_0^\epsilon = c_A^\epsilon(.,0)$ depends on μ_{00} and vice versa.
Note that $G^{-1} = 0$ due to Remark 6.2.6. With the boundary conditions $\mu_0^\pm\big|_{\Gamma_t^0} = \sigma\kappa_{\Gamma_t^0}$ and $\frac{\partial}{\partial n}\mu_0^+\big|_{\partial_T\Omega} = G^{-1} = 0$, we determine $\mu_0^\pm$ uniquely by solving the parabolic equation

$$\begin{aligned}
\partial_t\mu_0^\pm &= f'(\pm 1)\Delta\mu_0^\pm - v\cdot\nabla\mu_0^\pm && \text{in } Q_0^\pm,\\
\mu_0^\pm &= \sigma\kappa_{\Gamma_t^0} && \text{on } \Gamma_t^0, t\in(0,T],\\
\tfrac{\partial}{\partial n}\mu_0^+ &= 0 && \text{on } \partial\Omega\times(0,T],\\
\mu_0^\pm\big|_{t=0} &= \mu_{00} && \text{in } \Omega\,.
\end{aligned}$$

Therefore Γ^0 together with $\mu_0 := \mu_0^+\chi_{\{d^0\geq 0\}} + \mu_0^-\chi_{\{d^0<0\}}$ satisfies the sharp interface problem (6.5)-(6.10).
Step 7: Since $\overline{\mu}^0 = \tilde{u}^0$ in $\Gamma^0(\delta)$, equation (6.89) gives us the relation

$$\lim_{z\to\infty}\mu^0(\pm z,x,t) = \overline{\mu}^0(x,t) \pm \frac{1}{2}d^0(x,t)h^0(x,t) + \mu_{*(0)}^\pm \quad \forall (x,t)\in\Gamma^0(\delta)\,,$$

where $\mu_{*(0)}^\pm(x,t) = \lim_{z\to\infty}\mu_*^0(\pm z,x,t)$ for all $(x,t)\in\Gamma^0(\delta)$. To enforce the matching condition $\lim_{z\to\infty}\mu^0(\pm z,x,t) = \mu_0^\pm(x,t)$, it is necessary and sufficient to define

$$\overline{\mu}^0(x,t) = \tilde{\mu}^0(x,t) \;:=\; \frac{1}{2}\left(\mu_0^+ + \mu_0^- - \mu_{*(0)}^+ - \mu_{*(0)}^-\right) \quad \text{in } \Gamma^0(\delta), \tag{6.92}$$

$$h^0(x,t) \;:=\; \begin{cases} \frac{1}{d^0}\left(\mu_0^+ - \mu_0^- - \mu_{*(0)}^+ + \mu_{*(0)}^-\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0\\ \nabla d^0\cdot\nabla\left(\mu_0^+ - \mu_0^- - \mu_{*(0)}^+ + \mu_{*(0)}^-\right) & \text{on } \Gamma^0. \end{cases} \tag{6.93}$$

Note that the so-defined $\tilde{\mu}^0 = \overline{\mu}^0$ satisfies (6.88) by definition of $\mu_0^\pm$ and since $\mu_{*(0)}^\pm = 0$ on Γ^0.
Step 8: To satisfy the compatibility condition (6.71) in $\Gamma^0(\delta)$, it is necessary and sufficient to take

$$g^0(x,t) := \begin{cases} \frac{1}{2\eta_0 d^0}\left(\mu_0^+ + \mu_0^- - \mu_{*(0)}^+ - \mu_{*(0)}^- + 2\sigma\Delta d^0\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0\\ \frac{1}{2\eta_0}\nabla d^0\cdot\nabla\left(\mu_0^+ + \mu_0^- - \mu_{*(0)}^+ - \mu_{*(0)}^- + 2\sigma\Delta d^0\right) & \text{on } \Gamma^0. \end{cases} \tag{6.94}$$

Note that the numerator vanishes on Γ^0 since $\mu_{*(0)}^+ = \mu_{*(0)}^- = 0$ and $\mu_0^\pm = \sigma\kappa_{\Gamma_t^0} = -\sigma\Delta d^0$ on Γ^0.
Step 9: Since $B_B^{-1} = 0$ in $\overline{\partial_T\Omega(\delta)}$, it follows from (6.86) that $c_B^0(z,x,t) = \mu_0^+(x,t)$ for all $z\in(-\infty,0]$ and $(x,t)\in\overline{\partial_T\Omega(\delta)}$.
This determines the construction of $\mathcal{V}^0$, and we obtain the following result.

Lemma 6.2.9. *Let $\Gamma_{00}\subset\subset\Omega$ be a given smooth hypersurface without boundary and let μ_{00} be a given smooth initial value. Assume that the parabolic boundary problem*

(6.7)-(6.10) coupled with the evolution equation (6.5)-(6.6) admits a smooth solution (μ, Γ) *in the time interval* $[0, T]$*. Let* d^0 *be the signed distance from* x *to* Γ_t *such that* $d^0 < 0$ *inside of* Γ_t*, and let* δ *be a small constant such that* $dist(\Gamma_t, \partial\Omega) > 2\delta$ *for all* $t \in [0, T]$*,* d^0 *is smooth in* $\Gamma(2\delta) := \{(x,t) \in \Omega_T \mid |d^0| < 2\delta\}$*, and* $\mu^\pm := \mu|_{Q_0^\pm}$ *has a smooth extension to* $Q_0^\pm \cup \Gamma(2\delta)$ *where* $Q_0^\pm := \{(x,t) \in \Omega_T \mid \pm d^0 > 0\}$*. Define the hypersurface* Γ^0 *by*

$$\Gamma^0 = \Gamma\,,$$

the outer expansion functions $(c_0^\pm, \mu_0^\pm)$ *in* $Q_0^\pm \cup \Gamma^0(\delta)$ *by*

$$c_0^\pm(x,t) = \pm 1 \quad \textit{and} \quad \mu_0^\pm(x,t) = \mu^\pm(x,t),$$

the inner expansion functions (c^0, μ^0) *in* $\mathbb{R} \times \Gamma^0(\delta)$ *by*

$$\begin{aligned} c^0(z,x,t) &= \theta_0(z), \\ \mu^0(z,x,t) &= (\mu_0^+ - \mu_{*(0)}^+)(x,t)\,\eta(z) + (\mu_0^- - \mu_{*(0)}^-)(x,t)(1-\eta(z)) + \mu_*^0(z,x,t)\,, \end{aligned}$$

and the boundary expansion functions (c_B^0, μ_B^0) *in* $(-\infty, 0] \times \overline{\partial_T\Omega(\delta)}$ *by*

$$c_B^0(z,x,t) = 1 \quad \textit{and} \quad \mu_B^0(z,x,t) = \mu_0^+(x,t),$$

where $\mu_{*(0)}^\pm$ *and* μ_*^0 *are defined by Lemma 6.2.4. Furthermore, define* L^0 *by (6.87),* h^0 *by (6.93), and* g^0 *by (6.94).*
Then, for $k = 0$*, the outer expansion equation (6.59), the inner expansion equations (6.64) and (6.65), the boundary-layer expansion equation (6.79), the inner-outer matching conditions (6.60) and (6.61), the outer-boundary matching conditions (6.76) and (6.77), the boundary conditions (6.82) and (6.83) and the compatibility condition (6.72) are all satisfied. In addition, for* $k = 1$*, the outer expansion equation (6.58) where* $c_1^\pm$ *is defined by (6.59) (with* $k = 1$*) and the compatibility condition (6.71) are also satisfied.*

Proof: Since L^0 only depends on d^0 the functions μ_*^0 and $\mu_{*(0)}^\pm$ are well-defined by Lemma 6.2.4, in particular, μ^0 is also well-defined.
By direct calculations we verify the claimed properties.
To (6.59): It is satisfied by definition of $c_0^\pm$.
To (6.64): Since $c^0(z,x,t) = \theta_0(z)$ for all $z \in \mathbb{R}$ and $(x,t) \in \Gamma^0(\delta)$, the assertion immediately follows.
To (6.65): For all $(x,t) \in \Gamma^0(\delta)\backslash\Gamma^0$ it follows due to the definition of μ^0 and h^0

$$\left(\mu^0 - h^0 d^0 \eta\right)_{zz}(z,x,t) = (\mu_0^-(x,t) - \mu_{*(0)}^-(x,t) + \mu_*^0(z,x,t))_{zz} = B^0(z,x,t)\,,$$

where the last equation is satisfied by definition of μ_*^0. On Γ^0 we obtain since $d^0 = 0$ on Γ^0

$$\begin{aligned} \left(\mu^0 - h^0 d^0 \eta\right)_{zz}(z,x,t) &= \left(\mu_0^-(x,t) + \mu_*^0(z,x,t)\right)_{zz} \\ &= B^0(z,x,t)\,, \end{aligned}$$

where we have used the identities $\mu_0^- = \mu_0^+$ and $\mu_{*(0)}^- = \mu_{*(0)}^+ = 0$ (see Step 4) on Γ^0 in the first equality and the definition of μ_*^0 in the second equality.
To (6.79): Since $B_B^{-1} = 0$, the assertion follows from the definition of μ_B^0 since μ_B^0 is independent of z.
To (6.60): Since $c^0 = \theta_0$, the assertion holds due to Lemma 2.6.1.
To (6.61): It holds for all $z \geq 1$

$$\begin{aligned}
\mu^0(z,x,t) - \mu_0^+(x,t) &= \left(\mu_0^- - \mu_0^+\right)(1-\eta(z)) - \mu_{*(0)}^+\eta(z) \\
&\quad - \mu_{*(0)}^-(1-\eta(z)) + \mu_*^0(z,x,t) \\
&= -\mu_{*(0)}^+(x,t) + \mu_*^0(z,x,t)\,, \\
\mu^0(-z,x,t) - \mu_0^-(x,t) &= \left(\mu_0^+ - \mu_0^-\right)\eta(-z) - \mu_{*(0)}^+\eta(-z) \\
&\quad - \mu_{*(0)}^-(1-\eta(-z)) + \mu_*^0(-z,x,t) \\
&= -\mu_{*(0)}^-(x,t) + \mu_*^0(-z,x,t)
\end{aligned}$$

by (3.75). Then the matching conditions follow from Lemma 6.2.4.
To (6.76), (6.77): It is a direct consequence of the definitions of c_B^0 and μ_B^0.
To (6.82), (6.83): Since c_B^0 and μ_B^0 are independent of z, the initial conditions are satisfied.
To (6.72): In $\Gamma^0(\delta)\backslash\Gamma^0$ the compatibility condition is satisfied by definition of L^0. Note that $d^0 = 0$ on Γ^0. Hence we conclude

$$\partial_t d^0 + v\cdot\nabla d^0 - \frac{1}{2}L^0 d^0 = \partial_t d^0 + v\cdot\nabla d^0 = 0 \quad \text{on } \Gamma^0$$

due to the definition of the hypersurface Γ_t^0.
To (6.58): The definition of $c_1^\pm$ yields

$$\Delta\mu_0^\pm - \partial_t c_1^\pm - v\cdot\nabla c_1^\pm = \Delta\mu_0^\pm - \frac{1}{f'(\pm 1)}\left(\partial_t\mu_0^\pm - v\cdot\nabla\mu_0^\pm\right) = 0 \quad \text{in } Q_0^\pm\,,\,,$$

where the last equation follows from $\mu_0^\pm = \mu^\pm$ in $Q_0^\pm$.
To (6.71): Due to the definition of $\overline{\mu}^0$ in Lemma 6.2.3, we get in $\Gamma^0(\delta)\backslash\Gamma^0$

$$\begin{aligned}
&-\overline{\mu}^0(x,t) - \sigma\Delta d^0(x,t) + \eta_0 d^0(x,t)g^0(x,t) \\
&= -\frac{1}{2}\left(\mu_0^+ - \mu_{*(0)}^+\right)\int_{\mathbb{R}}\eta(z)\theta_0'(z)\,dz - \frac{1}{2}\left(\mu_0^- - \mu_{*(0)}^-\right)\int_{\mathbb{R}}(1-\eta(z))\,\theta_0'(z)\,dz \\
&\quad + \frac{1}{2}\int_{\mathbb{R}}\mu_*^0(z,x,t)\theta_0'(z)\,dz - \sigma\Delta d^0 + \frac{\eta_0 d^0}{2\eta_0 d^0}\left(\mu_0^+ + \mu_0^- - \mu_{*(0)}^+ - \mu_{*(0)}^- + 2\sigma\Delta d^0\right) \\
&= -\frac{1}{2}\left(\mu_0^+ - \mu_{*(0)}^+\right) - \frac{1}{2}\left(\mu_0^- - \mu_{*(0)}^-\right) + \frac{1}{2}\left(\mu_0^+ + \mu_0^- - \mu_{*(0)}^+ - \mu_{*(0)}^-\right) = 0
\end{aligned}$$

where we have used

$$\int_{\mathbb{R}}\eta(z)\theta_0'(z)\,dz = \int_{\mathbb{R}}(\eta(z) - \tfrac{1}{2})\theta_0'(z)\,dz + \frac{1}{2}\int_{\mathbb{R}}\theta_0'(z)\,dz = 1$$

and

$$\int_{\mathbb{R}} (1-\eta(z))\theta_0'(z)\,dz = \int_{\mathbb{R}} (\tfrac{1}{2}-\eta(z))\theta_0'(z)\,dz + \frac{1}{2}\int_{\mathbb{R}} \theta_0'(z)\,dz = 1$$

due to (3.76) and $\lim_{z\to\infty}\theta_0(\pm z) = \pm 1$.
On Γ^0 we use that $d^0 = 0$ and $\mu^-_{*(0)}(x,t) = \mu^+_{*(0)}(x,t) = 0$ (see Step 4). Hence we get

$$-\overline{\mu}^0(x,t) - \sigma\Delta d^0(x,t) + \eta_0 d^0(x,t) g^0(x,t) = -\frac{1}{2}\left(\mu_0^+ + \mu_0^-\right) + \sigma\kappa_{\Gamma_t^0} = 0 \quad \text{on } \Gamma^0$$

since $\Delta d^0 = -\kappa_{\Gamma_t^0}$ and $\mu_0^+ + \mu_0^- = \sigma\kappa_{\Gamma_t^0}$ on Γ^0.
This completes the proof. □

6.2.7 The Higher-Order Expansions

Let $j \geq 1$ be an integer. Assume that $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$ are known and the inner-outer and outer-boundary matching conditions for $k = 0, \ldots, j-1$ and the compatibility condition (6.71) for $k = j$ are satisfied.
Step 1: Since the compatibility condition (6.71) is satisfied, we can determine c^j as solution to the system (6.70) in $\Gamma^0(\delta)$. By equations (6.59), (6.78), and (6.82) we obtain $c_j^\pm$ in $Q_0^\pm$ and c_B^j in $(-\infty, 0] \times \overline{\partial_T\Omega(\delta)}$. So we can assume that c^j, $c_j^\pm$, and c_B^j are known functions depending on the known quantities $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$.
Note that $(c^j, c_j^\pm, c_B^j)$ depends on $(\mu^{j-1}, \mu_{j-1}^\pm, \mu_B^{j-1})$. Hence the initial value $c_0^\epsilon = c_A^\epsilon(.,0)$ depends on μ_{00}.
Step 2: The compatibility condition (6.73) for $k = j$ reads for all $(x,t) \in \Gamma^0$

$$\partial_t d^j + v \cdot \nabla d^j + \frac{1}{2} L^0 d^j = \widetilde{\mathcal{B}}^{j-1}\,. \tag{6.95}$$

By the initial condition

$$d^j(.,0) = 0 \quad \text{on } \Gamma_{00}\,, \tag{6.96}$$

we can determine d^j on Γ^0 uniquely. By the equation

$$\nabla d^0 \cdot \nabla d^j = -\frac{1}{2}\sum_{i=1}^{j-1} \nabla d^i \cdot \nabla d^{j-i} \quad \text{in } \Gamma^0(\delta)\,, \tag{6.97}$$

one can solve for d^j in $\Gamma^0(\delta)$. A detailed description to construct d^j is given in the following. Therefore d^j can be considered as a known function depending on the known quantities $\mathcal{V}^0, \ldots, \mathcal{V}^{j-1}$.
Since $c_0^\pm = \pm 1$ in $\Gamma^0(\delta)$, it holds $O_0^+ = O_0^- = 0$ in $\Gamma^0(\delta)$, in particular, $O_0^+\eta_N^+ + O_0^-\eta_N^- = 0$ in $\Gamma^0(\delta)$. That is $\mathcal{V}^0$ and $\mathcal{V}^1$ are independent of N, in particular, d^1 is independent of N. Therefore once we find d^1, we can fix $N = \|d^1\|_{C^0(\Gamma^0(\delta))} + 2$ (see Remark 3.2.5).
One can construct d^j satisfying (6.95)-(6.97) as follows. Let X_t be defined as in

Lemma 5.2.1 (observe that $\Gamma_t^0 = X_t(\Gamma_{00})$). For every fixed $x \in \Gamma_{00}$ we solve the ordinary differential equation

$$\frac{d}{dt}p(x,t) = -\frac{1}{2}L^0(X_t(x),t)p(x,t) + \widetilde{\mathcal{B}}^{j-1}(X_t(x),t) \text{ in } [0,T], \quad p(x,0) = 0\,.$$

Once we obtain p on $\Gamma_{00} \times [0,T]$, we fix any $t \in [0,T]$ and use $p(x,t)$, $x \in \Gamma_{00}$, as Cauchy data for the linear first-order partial differential equation

$$\big(DX_t^{-1}(X_t(x))\nabla d^0(X_t(x),t)\big) \cdot \nabla p(x,t) = -\frac{1}{2}\sum_{i=1}^{j-1}(\nabla d^i \cdot \nabla d^{j-i})(X_t(x),t)$$

to get p in $\bigcup_{t\in[0,T]} X_t^{-1}(\Gamma_t^0(\delta)) \times \{t\}$. Note that Γ_{00} is non characteristic for all $t \in [0,T]$. This can be seen as follows. Since $\Gamma_t^0 = X_t(\Gamma_{00})$ for all $t \in [0,T]$, it holds $d^0(X_t(x),t) = 0$ if and only if $x \in \Gamma_{00}$. Hence $(DX_t)^T\nabla d^0(X_t,t) = \nabla(d^0(X_t,t))$ is parallel to $\nu_{\Gamma_{00}}$ for all $t \in [0,T]$. Then it follows

$$\begin{aligned}&\big(DX_t^{-1}(X_t)\nabla d^0(X_t,t)\big) \cdot \big((DX_t)^T\nabla d^0(X_t,t)\big)\\ =\;& \nabla d^0(X_t,t) \cdot \big(DX_tDX_t^{-1}(X_t)\nabla d^0(X_t,t)\big) = \nabla d^0(X_t,t)\cdot\nabla d^0(X_t,t) = 1\end{aligned}$$

since $\mathrm{Id} = D(X_t^{-1}(X_t)) = DX_t^{-1}(X_t)DX_t$ for all $t \in [0,T]$. Therefore Γ_{00} is non-characteristic for all $t \in [0,T]$. After constructing p in $\bigcup_{t\in[0,T]} X_t^{-1}(\Gamma_t^0(\delta)) \times \{t\}$, we set $d^j(x,t) = p(X_t^{-1}(x),t)$ for all $(x,t) \in \Gamma^0(\delta)$. Indeed the so-defined d^j is the desired function. By chain rule we obtain for all $(x,t) \in \Gamma^0$

$$\begin{aligned}\partial_t d^j(x,t) &= \frac{d}{dt}(p(X_t^{-1}(x)),t) = \partial_t p(X_t^{-1},t) + \partial_t X_t^{-1}\cdot\nabla p(X_t^{-1},t)\\ &= -\frac{1}{2}L^0(x,t)d^j(x,t) + \widetilde{\mathcal{B}}^{j-1}(x,t) - v(x,t)\cdot(DX_t^{-1})^T\nabla p(X_t^{-1}(x),t)\\ &= -\frac{1}{2}L^0(x,t)d^j(x,t) + \widetilde{\mathcal{B}}^{j-1}(x,t) - v(x,t)\cdot\nabla d^j(x,t)\,,\end{aligned}$$

where the third equality follows since $-v(x,t) = DX_t(X_t^{-1}(x))\partial_t X_t^{-1}(x)$ by (5.7), and therefore we get

$$-DX_t^{-1}(x)v(x,t) = \partial_t X_t^{-1}(x)$$

since $DX_t^{-1}(x)$ is the inverse of $DX_t(X_t^{-1}(x))$ by (5.8). Hence d^j is a solution to (6.95). Since $X_0 = \mathrm{Id}$, the initial condition (6.96) is also satisfied. It remains to verify (6.97). We get for all $(x,t) \in \Gamma^0(\delta)$

$$\begin{aligned}\nabla d^0(x,t)\cdot\nabla d^j(x,t) &= \nabla d^0\cdot\nabla(p(X_t^{-1}(x),t)) = \nabla d^0\cdot\big((DX_t^{-1})^T\nabla p(X_t^{-1},t)\big)\\ &= -\frac{1}{2}\sum_{i=1}^{j-1}(\nabla d^i\cdot\nabla d^{j-i})(x,t)\,.\end{aligned}$$

Therefore $d^j(x,t) = p(X_t^{-1}(x),t)$ is a solution to (6.95)-(6.97).

Step 3: To satisfy the compatibility condition (6.73) in $\Gamma^0(\delta)$, it is necessary and sufficient to define L^j by

$$L^j(x,t) := \begin{cases} \frac{2}{d^0}\left(\widetilde{\mathcal{B}}^{j-1} - d_t^j - v\cdot\nabla d^k - \frac{1}{2}L^0 d^k\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0 \\ 2\nabla d^0\cdot\nabla\left(\widetilde{\mathcal{B}}^{j-1} - d_t^j - v\cdot\nabla d^k - \frac{1}{2}L^0 d^k\right) & \text{on } \Gamma^0. \end{cases}$$

Since the numerator vanishes on Γ^0, L^j is smooth.

Step 4: Since $d^0 = 0$ on Γ^0, the compatibility condition (6.71) reads for $k = j+1$ on Γ^0

$$\overline{\mu}^j(x,t) = -\sigma\Delta d^j(x,t) + \eta_0(d^j g^0)(x,t) - \widetilde{\mathcal{A}}^{j-1}(x,t) \quad \text{on } \Gamma^0. \tag{6.98}$$

Step 5: By Lemma 6.2.4 μ^j has the following form in $\mathbb{R}\times\Gamma^0(\delta)$

$$\mu^j(z,x,t) = \tilde{\mu}^j(x,t) + \left(d^0h^j + d^jh^0\right)(x,t)\left(\eta(z) - \tfrac{1}{2}\right) + \mu_*^j(z,x,t), \tag{6.99}$$

where we define $\tilde{\mu}^j$ and h^j later. Here μ_*^j only depends on the known quantities $\mathcal{V}^0,\ldots,\mathcal{V}^{j-1}$ (observe that (c^j,d^j,L^j) only depends on $\mathcal{V}^0,\ldots,\mathcal{V}^{j-1}$ and therefore B^j, too) and satisfies the condition (6.75). As for the zero-th order expansions, it follows $\tilde{\mu}^j = \frac{1}{2}\int_{\mathbb{R}}\mu^j\theta_0' = \overline{\mu}^j$ in $\Gamma^0(\delta)$. Restricting (6.99) on Γ^0 and using (6.98) yields

$$\mu^j(z,x,t) = -\sigma\Delta d^j + d^j\left(\eta_0 g^0 + h^0\left(\eta(z) - \tfrac{1}{2}\right)\right) - \widetilde{\mathcal{A}}^{j-1} + \mu_*^j(z,x,t) \tag{6.100}$$

for all $(z,x,t)\in\mathbb{R}\times\Gamma^0$.

Step 6: Sending z in (6.100) to $\pm\infty$ and using the inner-outer matching condition, we obtain on Γ^0

$$\mu_j^\pm(x,t) = \lim_{z\to\pm\infty}\mu^j(z,x,t) = -\sigma\Delta d^j + d^j\left(\eta_0 g^0 \pm \tfrac{1}{2}h^0\right) - \widetilde{\mathcal{A}}^{j-1} + \mu_{*(j)}^\pm, \tag{6.101}$$

where $\lim_{z\to\infty}\eta(z) = 1$ and $\lim_{z\to\infty}\eta(-z) = 0$. The outer expansion equations reads for $k = j+1$ in $Q_0^\pm$

$$\partial_t c_{j+1}^\pm + v\cdot\nabla c_{j+1}^\pm = \Delta\mu_j^\pm \quad \text{and} \quad c_{j+1}^\pm = \frac{\mu_j^\pm - f^j(c_0^\pm,\ldots,c_j^\pm) + \Delta c_{j-1}^\pm}{f'(c_0^\pm)}. \tag{6.102}$$

Hence for $\mu_j^\pm$ we obtain the equation

$$\partial_t\mu_j^\pm = f'(\pm1)\Delta\mu_j^\pm - v\cdot\nabla\mu_j^\pm + a_{j-1}^\pm \quad \text{in } Q_0^\pm, \tag{6.103}$$

where $a_{j-1}^\pm = (\partial_t + v\cdot\nabla)(f^j(c_0^\pm,\ldots,c_j^\pm) - \Delta c_{j-1}^\pm)$. In the construction of $\mathcal{V}^{j+1}$ we define $c_{j+1}^\pm$ by the second equation in (6.102). Hence the first equation in (6.102) and (6.103) are equivalent. Therefore the initial value $c_0^\epsilon = c_A^\epsilon(.,0)$ depends on $\mu_j^\pm(.,0)$ and vice versa.

Note that G^{j-1} only depends on $\mathcal{V}^0,\ldots,\mathcal{V}^{j-1}$ and c_B^j, which we already known. With the boundary conditions on Γ^0 given by (6.101) and $\frac{\partial}{\partial n}\mu_j^+\big|_{\partial_T\Omega} = G^{j-1}$, we determine $\mu_j^\pm$ as solution to

$$\begin{aligned}
\partial_t \mu_j^\pm &= f'(\pm 1)\Delta\mu_j^\pm - v\cdot\nabla\mu_j^\pm + a_{j-1}^\pm && \text{in } Q_0^\pm, && (6.104)\\
\mu_j^\pm &= -\sigma\Delta d^j + d^j\left(\eta_0 g^0 \pm \tfrac{1}{2}h^0\right) - \widetilde{\mathcal{A}}^{j-1} + \mu_{*(j)}^\pm && \text{on } \Gamma_t^0, t\in(0,T], && (6.105)\\
\tfrac{\partial}{\partial n}\mu_j^+ &= G^{j-1} && \text{on } \partial\Omega\times(0,T], && (6.106)\\
\mu_j^\pm\big|_{t=0} &= \mu_0^j && \text{in } \Omega\,, && (6.107)
\end{aligned}$$

where we choose the initial value μ_0^j such that the solution $\mu_j^\pm$ satisfy $\mu_k^\pm \in C^\infty(\overline{\Omega^\pm(t)}\times[0,T])$, see Remark 6.2.1.

Step 7: Sending z in (6.99) to $\pm\infty$ and using the inner-outer matching condition $\lim_{z\to\infty}\mu^j(\pm z,x,t) = \mu_j^\pm(x,t)$ yields

$$\mu_j^\pm(x,t) = \tilde{\mu}^j(x,t) \pm \tfrac{1}{2}\left(d^0h^j + d^jh^0\right)(x,t) + \mu_{*(j)}^\pm(x,t) \quad \text{in } \Gamma^0(\delta)\,.$$

Hence it is necessary and sufficient to define $\tilde{\mu}^j$ and h^j by

$$\overline{\mu}^j(x,t) = \tilde{\mu}^j(x,t) := \frac{1}{2}\left(\mu_j^+ + \mu_j^- - \mu_{*(j)}^+ - \mu_{*(j)}^-\right) \quad \text{in } \Gamma^0(\delta),$$

$$h^j(x,t) := \begin{cases} \frac{1}{d^0}\left(-d^jh^0 + \mu_j^+ - \mu_j^- - \mu_{*(j)}^+ + \mu_{*(j)}^-\right) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0 \\ \nabla d^0\cdot\nabla\left(-d^jh^0 + \mu_j^+ - \mu_j^- - \mu_{*(j)}^+ + \mu_{*(j)}^-\right) & \text{on } \Gamma^0. \end{cases}$$

Note that the numerator in the definition of h^j vanishes on Γ^0. We can verify this by equation (6.101). Therefore h^j is smooth. Moreover, the so-defined $\tilde{\mu}^j = \overline{\mu}^j$ satisfies (6.98) where we have used (6.101) again.

Step 8: We can define g^j for all $(x,t)\in\Gamma^0(\delta)\backslash\Gamma^0$ by

$$g^j(x,t) := \frac{1}{\eta_0 d_0}\left(-\eta_0 d^j g^0 + \sigma\Delta d^j + \tilde{\mu}^j + \mathcal{A}^{j-1}\right) \quad \text{in } \Gamma^0(\delta)\backslash\Gamma^0$$

such that the compatibility condition (6.71) for $k = j+1$ is satisfied . Note that (6.71) is satisfied on Γ^0 (see (6.98)), and therefore we can extend g^j to Γ^0 such that g^j is smooth in $\Gamma^0(\delta)$ by

$$g^j(x,t) := \frac{1}{\eta_0}\nabla d^0\cdot\left(-\eta_0 d^j g^0 + \sigma\Delta d^j + \tilde{\mu}^j + \mathcal{A}^{j-1}\right) \quad \text{on } \Gamma^0.$$

Step 9: By equation (6.86) we obtain μ_B^j.

In summary the following lemma holds.

Lemma 6.2.10. *Let $j\geq 1$ be an arbitrary integer and assume that $\mathcal{V}^0,\dots,\mathcal{V}^{j-1}$ are known and satisfy the inner-outer matching conditions (6.60) and (6.61), the* *outer-boundary matching conditions (6.76) and (6.77) for all $k = 0, \ldots, j-1$, and the compatibility condition (6.71) for $k = j$. Then there exists*

$$\mathcal{V}^j = (c_j^{\pm}, c^j, c_B^j, \mu_j^{\pm}, \mu^j, \mu_B^j, d^j, h^j, g^j, L^j)$$

satisfying, for $k = j$, the compatibility condition (6.73), the outer expansion equation (6.59), the inner expansion equations (6.64) and (6.65), the boundary-layer expansion equations (6.78), (6.79), (6.82), and (6.83), the inner-outer matching conditions (6.60) and (6.61), and the outer-boundary matching conditions (6.76) and (6.77). In addition, for $k = j+1$, the outer expansion equation (6.58) where $c_{j+1}^{\pm}$ is defined by (6.59) (with $k = j+1$) and the compatibility condition (6.71) are also satisfied.

Proof: Define $c_j^{\pm}$, c^j, and c_B^j as in Step 1, d^j as the unique solution to problem (6.95)-(6.97), and $\mu_j^{\pm}$ as the unique solution to (6.104)-(6.107). Furthermore, define L^j as in Step 3, μ^j as in Step 5, $\tilde{\mu}^j$ and h^j as in Step 7, g^j as in Step 8, and μ_B^j by (6.86). By definition of μ^j and $\overline{\mu}^j$ in Lemma 6.2.3, one concludes $\tilde{\mu}^j = \int_{\mathbb{R}} \mu^j(z)\, dz = \overline{\mu}^j$ in $\Gamma^0(\delta)$. Using Step 1-9, it is not difficult to show that $\mathcal{V}^j$ satisfies the required conditions. Details are omitted. □

As consequence we obtain recursively.

Theorem 6.2.11. *Let (μ, Γ) be a smooth solution to (6.5)-(6.10). Then, for any fixed integer $K > 0$, there exist $\mathcal{V}^0, \ldots, \mathcal{V}^K$ such that the outer expansion equations (3.69)-(3.71), the inner expansion equations (3.82)-(3.84), the boundary-layer expansion equations (3.117)-(3.119) and (3.123)-(3.125), the inner-outer matching conditions (3.72)-(3.74), and the outer-boundary matching conditions (3.114)-(3.116) are satisfied for $k = 0, \ldots, K$. In addition, $(\mu_0^{\pm}, \Gamma^0)$ coincides with (μ, Γ).*

6.2.8 Construction of an Approximate Solution

The construction of an approximate solution is done in the same way as in Subsection 3.2.9, that is, we connect the inner, outer, and boundary-layer expansions.
Let (μ, Γ) be a smooth solution to the parabolic boundary problem (6.7)-(6.10) coupled with the evolution equation (6.5)-(6.6) in the time interval $[0, T]$ for given smooth hypersurface Γ_{00} without boundary and suitable initial value μ_{00}. Let $K \geq 2$ be an arbitrary fixed integer. We define d_ϵ^K and Γ_ϵ^K by

$$\begin{aligned} d_\epsilon^K(x,t) &= \sum_{i=0}^{K} \epsilon^i d^i(x,t)\,, \quad \forall (x,t) \in \Gamma^0(\delta), \\ \Gamma_\epsilon^K &= \left\{ (x,t) \in \Gamma^0(\delta) : d_\epsilon^K(x,t) = 0 \right\}. \end{aligned}$$

Furthermore, we construct an approximate solution c_A^K by

$$c_A^K(x,t) := \begin{cases} c_\partial^K & \text{in } \overline{\partial_T\Omega(\delta/2)}\,, \\ c_\partial^K\zeta(d_B/\delta) + c_O^K(1-\zeta(d_B/\delta)) & \text{in } \partial_T\Omega(\delta)\backslash\overline{\partial_T\Omega(\delta/2)}\,, \\ c_O^K & \text{in } \Omega_T\backslash(\partial_T\Omega(\delta)\cup\Gamma^0(\delta))\,, \\ c_I^K\zeta(d^0/\delta) + c_O^K(1-\zeta(d^0/\delta)) & \text{in } \Gamma^0(\delta)\backslash\Gamma^0(\delta/2)\,, \\ c_I^K & \text{in } \Gamma^0(\delta/2)\,, \end{cases}$$

where d_B is the signed distance function to $\partial\Omega$ and ζ is a smooth cut-off function as in (2.21) and where

$$\begin{aligned} c_O^K(x,t) &:= \sum_{i=0}^K \epsilon^i \left(c_i^+(x,t)\chi_{\overline{Q_0^+}} + c_i^-(x,t)\chi_{Q_0^-}\right) && \forall (x,t)\in\Omega_T\,, \\ c_I^K(x,t) &:= \sum_{i=0}^K \epsilon^i\, c^i(z,x,t)\big|_{z=d_\epsilon^K/\epsilon} && \forall (x,t)\in\Gamma^0(\delta)\,, \\ c_\partial^K(x,t) &:= \sum_{i=0}^K \epsilon^i\, c_B^i(z,x,t)\big|_{z=d_B/\epsilon} - \epsilon^K c_B^K(0,x,t) && \forall (x,t)\in\overline{\partial_T\Omega(\delta)}\,. \end{aligned}$$

We define μ_A^K similarly. By the same calculation as in Subsection 3.2.9, we obtain the following the error terms. In $\Gamma^0(\delta/2)$ we use equation (6.65) and set $z = \frac{d_\epsilon^K}{\epsilon}$ to get

$$\begin{aligned} &((c_I^K)_t + v\cdot\nabla c_I^K - \epsilon\Delta\mu_I^K)(x,t) \\ &= \frac{1-\left|\nabla d_\epsilon^K\right|^2}{\epsilon}\sum_{i=0}^K \epsilon^i\mu_{zz}^i + \sum_{\substack{0\le i,j\le K\\ i+j\ge K+1}} \epsilon^{i+j-1}\big(c_z^i d_t^j + v\cdot\nabla d^j c_z^i - 2\nabla\mu_z^i\cdot\nabla d^j - \mu_z^i\Delta d^j\big) \\ &\quad +\epsilon^K(c_t^K + v\cdot\nabla c^K) - \sum_{i=K-1}^K \epsilon^{i+1}\Delta\mu^i - \epsilon^{K-1}h^K d^0\eta'' + \sum_{\substack{0\le i\le K\\ 0\le j\le K-1\\ i+j\ge K+1}} \epsilon^{i+j-1}d^i h^j\eta'' \\ &\quad -\epsilon^{K-1}L^K d^0\eta' + \sum_{\substack{0\le i\le K\\ 0\le j\le K-1\\ i+j\ge K+1}} \epsilon^{i+j-1}d^i L^j\eta' = \mathcal{O}(\epsilon^{K-1}) \quad \forall (x,t)\in\Gamma^0(\delta)\,. \end{aligned}$$

Here we have used that

$$\left|\nabla d_\epsilon^K\right|^2 = 1 + \sum_{\substack{1\le i,j\le K\\ i+j\ge K+1}} \epsilon^{i+j}\nabla d^j\cdot\nabla d^i$$

and $O_j^+\eta_N^+ + O_j^-\eta_N^-\big|_{z=d_\epsilon^K/\epsilon} = 0$ and $\mathbf{P}_j^+\eta_N^+ + \mathbf{P}_j^-\eta_N^-\big|_{z=d_\epsilon^K/\epsilon} = 0$ for $j = 0,\dots,K-1$ and for all $\epsilon > 0$ small enough (see Remark 3.2.5). Equation (6.64) with $z = \frac{d_\epsilon^K}{\epsilon}$

yields

$$
\begin{aligned}
&(\mu_I^K + \epsilon\Delta c_I^K - \epsilon^{-1} f(c_I^K))(x,t) \\
&= \epsilon^K \mu^K + \epsilon \sum_{i=K-1}^{K} \epsilon^i \Delta c^i - \frac{1-\left|\nabla d_\epsilon^K\right|^2}{\epsilon} \sum_{i=0}^{K} \epsilon^i c_{zz}^i \\
&\quad + \sum_{\substack{0\le i,j\le K\\ i+j\ge K}} \epsilon^{i+j}\left(2\nabla c_z^i \cdot \nabla d^j + c_z^i \Delta d^j\right) - \epsilon^K f^K(c^0,\dots,c^K) \\
&\quad + \epsilon^{K-1} g^{K-1} d^0 \eta' - \sum_{\substack{0\le i\le K-2\\ 0\le j\le K\\ i+j\ge K}} \epsilon^{i+j} g^i d^j \eta' = \mathcal{O}(\epsilon^{K-1}) \qquad \forall (x,t)\in \Gamma^0(\delta)\,.
\end{aligned}
$$

For the outer expansion we use equations (6.58) and (6.59) to obtain in $Q_0^+ \cup Q_0^-$

$$
\begin{aligned}
&(c_O^K)_t + v\cdot \nabla c_O^K - \epsilon\Delta\mu_O^K = \epsilon^{K+1}\Delta\mu_K^\pm = \mathcal{O}(\epsilon^{K+1})\,, \\
&\mu_O^K + \epsilon\Delta c_O^K - \epsilon^{-1} f(c_O^K) = \epsilon^K \mu_K^\pm - \epsilon^K f^K(c_0^\pm,\dots,c_K^\pm) + \sum_{i=K-1}^{K} \epsilon^{i+1}\Delta c_i^\pm = \mathcal{O}(\epsilon^K)\,,
\end{aligned}
$$

For the boundary-layer expansion we consider equation (3.118) and set $z = \frac{d_B}{\epsilon}$

$$
\begin{aligned}
&((c_\partial^K)_t + v\cdot\nabla c_\partial^K - \epsilon\Delta\mu_\partial^K)(x,t) \\
&= \epsilon^K\left(c_{B,t}^K + v\cdot\nabla c_B^K - 2\nabla\mu_B^K\cdot\nabla d_B - \mu_{B,z}^K \Delta d_B\right) - \sum_{i=K-1}^{K} \epsilon^{i+1}\Delta\mu_B^i \\
&\quad -\epsilon^K\left(c_{B,t}^K(0) + v\cdot\nabla c_B^K(0) - \epsilon\Delta c_B^K(0)\right) = \mathcal{O}(\epsilon^K) \quad \forall (x,t)\in\partial\Omega(\delta)\times(0,T)\,,
\end{aligned}
$$

where $(c_B^K(0), \mu_B^K(0)) = (c_B^K(0,x,t), \mu_B^K(0,x,t))$ for all $(x,t)\in\overline{\partial\Omega(\delta)}$. Equation (6.78) with $z = \frac{d_B}{\epsilon}$ yields

$$
\begin{aligned}
&(\mu_\partial^K + \epsilon\Delta c_\partial^K - \epsilon^{-1} f(c_\partial^K))(x,t) \\
&= \epsilon^K\mu_B^K + \epsilon^K 2\nabla c_{B,z}^K\cdot\nabla d_B + \epsilon^K c_{B,z}^K \Delta d_B + \sum_{i=K}^{K+1} \epsilon^i \Delta c_B^{i-1} \\
&\quad -\epsilon^K f^K(c_B^0,\dots,c_B^K - c^K(0)) + \epsilon^{K-1} f'(\theta_0) c_B^K(0) - \epsilon^K\left(\mu_B^K(0) + \epsilon\Delta c_B^K(0)\right) \\
&= \mathcal{O}(\epsilon^{K-1}) \quad \forall (x,t)\in\partial\Omega(\delta)\times[0,T)\,.
\end{aligned}
$$

Note that for $t=0$ the error estimate is also valid for the chemical potential equations, that is, in each case the second equation.
By construction of c_∂^K and μ_∂^K, we get on the boundary $\partial_T\Omega$

$$
\tfrac{\partial}{\partial n} c_\partial^K(x,t) = \tfrac{\partial}{\partial n}\mu_\partial^K(x,t) = 0 \quad \forall (x,t)\in\partial_T\Omega\,.
$$

For the regions where we glue together the inner and outer expansions and the boundary-layer and outer expansions, we obtain the same estimates as in Subsection 3.2.9. More precisely, this means

$$\begin{aligned} \left\|c_A^K - c_O^K\right\|_{C^2(\Gamma^0(\delta)\backslash\Gamma^0(\delta/2))} &= \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{4\epsilon}})\,, \\ \left\|c_A^K - c_O^K\right\|_{C^2(\partial_T\Omega(\delta)\backslash\partial_T\Omega(\delta/2))} &= \mathcal{O}(\epsilon^{-2}e^{-\frac{\alpha\delta}{4\epsilon}}) + \mathcal{O}(\epsilon^K)\,. \end{aligned}$$

For μ_A^K the analogous estimates are valid. Hence $(c_A^\epsilon, \mu_A^\epsilon)$ are approximate solutions to the convective Cahn-Hilliard equation with error terms $\mathcal{O}(\epsilon^{K-1})$. Additionally, we can modify (c_A^K, μ_A^K) in the same way as in Subsection 3.2.9 such that $(c_A^\epsilon, \mu_A^\epsilon)$ is a solution to

$$(c_A^\epsilon)_t + v\cdot\nabla c_A^\epsilon - \epsilon\Delta\mu_A^\epsilon = 0 \qquad \text{in } \Omega_T\,, \tag{6.108}$$

$$\mu_A^\epsilon + \epsilon\Delta c_A^\epsilon - \epsilon^{-1}f(c_A^\epsilon) = \mathcal{O}(\epsilon^{K-2}) \qquad \text{in } \Omega_T\,, \tag{6.109}$$

$$\frac{\partial}{\partial n}c_A^\epsilon = \frac{\partial}{\partial n}\mu_A^\epsilon = 0 \qquad \text{on } \partial_T\Omega\,. \tag{6.110}$$

Remark 6.2.12. *Now we can specify the order of expansion which we need. From (6.18) and (6.19) we obtain the condition*

$$K - 2 \geq \frac{pk}{2} > d + 2\,.$$

In particular, it is sufficient to calculate the 7th order term of expansion in two dimensions and the 8th order term in three dimensions.

In summary, the following theorem is valid.

Theorem 6.2.13. *Let $\Gamma_{00} \subset \Omega$ be a given smooth hypersurface without boundary and $\mu_{00} : \Omega \to \mathbb{R}$ be a given smooth function. Assume (μ_0, Γ^0) is a smooth solution to (6.5)-(6.10) with initial values μ_{00} and Γ_{00} in the time interval $[0,T]$. Then for every $K > 1$, there exists a positive constant ϵ_0 such that for every $\epsilon \in (0, \epsilon_0]$ there exists an approximate solution $(c_A^\epsilon, \mu_A^\epsilon)$ satisfying (6.108)-(6.110). Additionally, it holds*

$$\begin{aligned} \|\mu_A^\epsilon - \mu_0\|_{C^0(\Omega_T)} &= \mathcal{O}(\epsilon)\,, \\ \left\|c_A^\epsilon(x,t) - \theta_0(d^0(x,t)/\epsilon + d^1(x,t))\right\|_{C^0(\Gamma^0(\delta))} &= \mathcal{O}(\epsilon)\,, \\ \|c_A^\epsilon \mp 1\|_{C^0(Q_0^\pm\backslash\Gamma^0(\delta/2))} &= \mathcal{O}(\epsilon)\,, \end{aligned}$$

as $\epsilon \searrow 0$.

Proof: We can proof the assertions in the same way as in Theorem 3.2.21 or see [10, Theorem 4.12]. $\square$

6.3 Convergence Result

As main result of this chapter, we obtain the following theorem.

Theorem 6.3.1. *Let Ω be a smooth domain and $\Gamma_{00} \subset \Omega$ be a smooth hypersurface without boundary and let $\mu_{00}|_{\Omega_{00}^{\pm}} : \Omega_0^{\pm} \to \mathbb{R}$ and $v : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^d$ be smooth functions with* $\operatorname{div} v = 0$ *in* $\Omega \times \mathbb{R}$ *and* $v = 0$ *on* $\partial\Omega \times \mathbb{R}$. *Assume that the parabolic boundary problem (6.7)-(6.10) has a smooth solution μ in the time interval $[0,T]$ where $\Gamma(t)$, $t \in [0,T]$ is given by (6.5)-(6.6). Then there exists a family of smooth functions $\{c_0^\epsilon(x)\}_{0<\epsilon<1}$ which are uniformly bounded in $\epsilon \in (0,1]$ and $x \in \overline{\Omega}$, such that if c^ϵ satisfies the Cahn-Hiliard equation*

$$\begin{aligned} c_t^\epsilon + v \cdot \nabla c^\epsilon - \epsilon\Delta\left(-\epsilon\Delta c^\epsilon + \epsilon^{-1} f(c^\epsilon)\right) = 0 &\quad \text{in } \Omega_T, \\ \frac{\partial}{\partial n} c^\epsilon = \frac{\partial}{\partial n}\left(-\epsilon\Delta c^\epsilon + \epsilon^{-1} f(c^\epsilon)\right) = 0 &\quad \text{on } \partial_T\Omega, \\ c^\epsilon|_{t=0} = c_0^\epsilon &\quad \text{in } \Omega\,, \end{aligned} \tag{6.111}$$

then

$$\lim_{\epsilon\to 0} c^\epsilon(x,t) = \begin{cases} -1 & \text{if } (x,t) \in Q^- \\ 1 & \text{if } (x,t) \in Q^+ \end{cases} \text{ uniformly on compact subsets,}$$

$$\lim_{\epsilon\to 0}\left(-\epsilon\Delta c^\epsilon + \epsilon^{-1} f(c^\epsilon)\right)(x,t) = \mu(x,t) \text{ uniformly on } \overline{\Omega_T}\,,$$

where Q^+ and Q^- are respectively the exterior (in Ω_T) and interior of Γ.

Proof: Let $(c_A^\epsilon, \mu_A^\epsilon)$ be the approximate solution constructed in Theorem 6.2.13. Then Lemma 6.1.2 and Theorem 6.2.13 yield

$$\begin{aligned} \lim_{\epsilon\to 0} \|c^\epsilon \mp 1\|_{C^0(Q^\pm \setminus \Gamma(\delta/2))} &= 0\,, \\ \lim_{\epsilon\to 0} \|\mu^\epsilon - \mu\|_{C^0(\Omega_T)} &= 0 \end{aligned}$$

for any $\delta > 0$ small enough, as long as $\Phi_t^\epsilon(.) = c_A^\epsilon(.,t)$ has the form (2.20) where $r = r_t(x)$ is the signed distance function to $\Gamma_t^{\epsilon K} := \left\{x \in \Omega : d_\epsilon^K(x,t) = 0\right\}$. As in Theorem 3.3.1 it can be shown that $\Gamma_t^{\epsilon K}$ is a smooth hypersurface for all $\epsilon > 0$ small enough and the C^3 norm of $\Gamma_t^{\epsilon K}$ is independent of ϵ. We set $\delta_0 = \delta/2$ where δ is defined as in Section 6.2. By the same arguments as in the proof of Theorem 3.3.1, we can replace c_A^ϵ by

$$\zeta(r_t(x)/\delta_0) c_I^K(x,t) + (1 - \zeta(r_t(x)/\delta_0))\, c_O^K(x,t)$$

for all $(x,t) \in \Omega_T$ and where ζ is defined as in (2.21). Then we can show as in the proof of Theorem 3.3.1 that the conditions (2.25) and (2.26) are satisfied (see (3.203)). So it is sufficient to verify that for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0) \subset \Gamma^0(\delta)$ (for all $\epsilon > 0$ small enough) the following identity holds

$$c^0(\tfrac{d_\epsilon^K}{\epsilon}, x, t) + \epsilon c^1(\tfrac{d_\epsilon^K}{\epsilon}, x, t) = \theta_0(\tfrac{r_t(x)}{\epsilon}) + \epsilon p^\epsilon(S_t(x), t)\theta_1(\tfrac{r_t(x)}{\epsilon}) + \epsilon^2 \overline{q}^\epsilon(x,t)\,, \tag{6.112}$$

where θ_1 satisfies (2.23) and $p^\epsilon(x,t)$ and $\overline{q}^\epsilon(x,t)$ satisfy (2.24). Here $S_t(x)$ is the projection from x to $\Gamma_t^{\epsilon K}$ along the normal of $\Gamma_t^{\epsilon K}$ and c^0 and c^1 are the functions obtained by the inner expansion in Section 6.2.
For $(x,t) \in \Gamma^0$ the equation for c^1 in (6.64) reads

$$c^1_{zz} - f'(\theta_0)c^1 = -\mu^0 - \Delta d^0 \theta_0' = \sigma \Delta d^0 - \Delta d^0 \theta_0' \,,$$

and therefore we obtain $c^1(z,x,t) = \Delta d^0(x,t)\theta_1(z)$ where θ_1 satisfies

$$\theta_1'' - f'(\theta_0)\theta_1 = \sigma - \theta_0' \quad \text{in } \mathbb{R}\,, \quad \theta_1(0) = 0\,, \quad \theta_1 \in L^\infty(\mathbb{R})\,.$$

Now we can prove that θ_1 satisfies (2.23) as in the proof of Theorem 3.3.1 (see (3.205)) or [10, Theorem 5.1].
We define p^ϵ and $\overline{q}^\epsilon$ by

$$\begin{aligned} p^\epsilon(x,t) &:= \Delta d^0(x,t)\,, \\ \overline{q}^\epsilon(x,t) &:= \epsilon^{-2}\left(\theta_0\left(\tfrac{d^K_\epsilon}{\epsilon}\right) - \theta_0\left(\tfrac{r_t(x)}{\epsilon}\right)\right) \\ &\quad + \epsilon^{-1}\left(c^1\left(\tfrac{d^K_\epsilon}{\epsilon}, x, t\right) - p^\epsilon(S_t(x),t)\,\theta_1\left(\tfrac{r_t(x)}{\epsilon}\right)\right) \end{aligned}$$

for all $(x,t) \in \Gamma^{\epsilon K}(\delta_0)$. Then the identity (6.112) holds due to the definitions of p^ϵ and $\overline{q}^\epsilon$. Finally, one can verify condition (2.24) again as in the proof of Theorem 3.3.1 or [10, Theorem 5.1]. $\square$

Bibliography

[1] H. Abels. "Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities". In: *Comm. Math. Phys.* 289.1 (2009), pp. 45–73. ISSN: 0010-3616. DOI: 10.1007/s00220-009-0806-4. URL: http://dx.doi.org/10.1007/s00220-009-0806-4.

[2] H. Abels. "On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities". In: *Arch. Ration. Mech. Anal.* 194.2 (2009), pp. 463–506. ISSN: 0003-9527. DOI: 10.1007/s00205-008-0160-2. URL: http://dx.doi.org/10.1007/s00205-008-0160-2.

[3] H. Abels. "Strong well-posedness of a diffuse interface model for a viscous, quasi-incompressible two-phase flow". In: *SIAM J. Math. Anal.* 44.1 (2012), pp. 316–340. ISSN: 0036-1410. DOI: 10.1137/110829246. URL: http://dx.doi.org/10.1137/110829246.

[4] H. Abels, H. Garcke, and G. Grün. "Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities". In: *Math. Models Methods Appl. Sci.* 22.3 (2012), pp. 1150013, 40. ISSN: 0218-2025. DOI: 10.1142/S0218202511500138. URL: http://dx.doi.org/10.1142/S0218202511500138.

[5] H. Abels and D. Lengeler. "On Sharp Interface Limits for Diffuse Interface Models for Two-Phase Flows". In: *ArXiv e-prints* (Dec. 2012). arXiv:1212.5582 [math.AP].

[6] H. Abels and M. Röger. "Existence of weak solutions for a non-classical sharp interface model for a two-phase flow of viscous, incompressible fluids". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26.6 (2009), pp. 2403–2424. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2009.06.002. URL: http://dx.doi.org/10.1016/j.anihpc.2009.06.002.

[7] H. Abels and M. Wilke. "Well-Posedness and Qualitative Behaviour of Solutions for a Two-Phase Navier-Stokes-Mullins-Sekerka System". In: *ArXiv e-prints* (Dec. 2012). arXiv:1201.0179 [math.AP].

[8] S. Agmon, A. Douglis, and L. Nirenberg. "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I". In: *Comm. Pure Appl. Math.* 12 (1959), pp. 623–727. ISSN: 0010-3640.

[9] S. Agmon, A. Douglis, and L. Nirenberg. "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II". In: *Comm. Pure Appl. Math.* 17 (1964), pp. 35–92. ISSN: 0010-3640.

[10] N. D. Alikakos, P. W. Bates, and X. Chen. "Convergence of the Cahn-Hilliard equation to the Hele-Shaw model". In: *Arch. Rational Mech. Anal.* 128.2 (1994), pp. 165–205. ISSN: 0003-9527. DOI: 10.1007/BF00375025. URL: http://dx.doi.org/10.1007/BF00375025.

[11] H. W. Alt. *Lineare Funktionalanalysis - Eine anwendungsorientierte Einführung.* 5. überarb. Aufl. Berlin: Springer DE, 2006. ISBN: 978-3-540-34186-4.

[12] H. W. Alt. *Vorlesung "Analysis III" - Lecture "Analysis 3". Wintersemester 2001.* URL: http://www.iam.uni-bonn.de/~alt/index/lehre-diplom.html#ws2001-analysis3.

[13] H. Amann. *Linear and quasilinear parabolic problems. Vol. I.* Vol. 89. Monographs in Mathematics. Abstract linear theory. Boston, MA: Birkhäuser Boston Inc., 1995, pp. xxxvi+335. ISBN: 3-7643-5114-4. DOI: 10.1007/978-3-0348-9221-6. URL: http://dx.doi.org/10.1007/978-3-0348-9221-6.

[14] H. Amann. "Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems". In: *Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992).* Vol. 133. Teubner-Texte Math. Stuttgart: Teubner, 1993, pp. 9–126.

[15] V. I. Arnold. *Gewöhnliche Differentialgleichungen* -. 2. Aufl. Berlin: Springer DE, 2001. ISBN: 978-3-540-66890-9.

[16] T. Aubin. *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Berlin: Springer-Verlag, 1998, pp. xviii+395. ISBN: 3-540-60752-8.

[17] C. Bär. *Geometrische Analysis. Wintersemester 2007/08.* 2011. URL: http://geometrie.math.uni-potsdam.de/documents/baer/skripte/skript-GeomAna.pdf.

[18] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Berlin: Springer-Verlag, 1976, pp. x+207.

[19] E. A. Carlen, M. C. Carvalho, and E. Orlandi. "Approximate solutions of the Cahn-Hilliard equation via corrections to the Mullins-Sekerka motion". In: *Arch. Ration. Mech. Anal.* 178.1 (2005), pp. 1–55. ISSN: 0003-9527. DOI: 10.1007/s00205-005-0366-5. URL: http://dx.doi.org/10.1007/s00205-005-0366-5.

[20] X. Chen. "Global asymptotic limit of solutions of the Cahn-Hilliard equation". In: *J. Differential Geom.* 44.2 (1996), pp. 262–311. ISSN: 0022-040X. URL: http://projecteuclid.org/getRecord?id=euclid.jdg/1214458973.

[21] X. Chen. "Spectrum for the Allen-Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces". In: *Comm. Partial Differential Equations* 19.7-8 (1994), pp. 1371–1395. ISSN: 0360-5302. DOI: 10.1080/03605309408821057. URL: http://dx.doi.org/10.1080/03605309408821057.

[22] X. Chen, J. Hong, and F. Yi. "Existence, uniqueness, and regularity of classical solutions of the Mullins-Sekerka problem". In: *Comm. Partial Differential Equations* 21.11-12 (1996), pp. 1705–1727. ISSN: 0360-5302. DOI: 10.1080/03605309608821243. URL: http://dx.doi.org/10.1080/03605309608821243.

[23] R. Denk, M. Hieber, and J. Prüss. "$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type". In: *Mem. Amer. Math. Soc.* 166.788 (2003), pp. viii+114. ISSN: 0065-9266.

[24] C. Eck, H. Garcke, and P. Knabner. *Mathematische Modellierung* -. 2. überarb. Aufl. Berlin: Springer DE, 2011. ISBN: 978-3-642-18423-9.

[25] S. D. Eidelman and N. V. Zhitarashu. *Parabolic boundary value problems.* Vol. 101. Operator Theory: Advances and Applications. Translated from the Russian original by Gennady Pasechnik and Andrei Iacob. Basel: Birkhäuser Verlag, 1998, pp. xii+298. ISBN: 3-7643-2972-6. DOI: 10.1007/978-3-0348-8767-0. URL: http://dx.doi.org/10.1007/978-3-0348-8767-0.

[26] J. Escher and B.-V. Matioc. "On periodic Stokesian Hele-Shaw flows with surface tension". In: *European J. Appl. Math.* 19.6 (2008), pp. 717–734. ISSN: 0956-7925. DOI: 10.1017/S0956792508007699. URL: http://dx.doi.org/10.1017/S0956792508007699.

[27] J. Escher, U. F. Mayer, and G. Simonett. "The surface diffusion flow for immersed hypersurfaces". In: *SIAM J. Math. Anal.* 29.6 (1998), 1419–1433 (electronic). ISSN: 0036-1410. DOI: 10.1137/S0036141097320675. URL: http://dx.doi.org/10.1137/S0036141097320675.

[28] J. Escher and G. Simonett. "A center manifold analysis for the Mullins-Sekerka model". In: *J. Differential Equations* 143.2 (1998), pp. 267–292. ISSN: 0022-0396. DOI: 10.1006/jdeq.1997.3373. URL: http://dx.doi.org/10.1006/jdeq.1997.3373.

[29] J. Escher and G. Simonett. "Classical solutions for Hele-Shaw models with surface tension". In: *Adv. Differential Equations* 2.4 (1997), pp. 619–642. ISSN: 1079-9389.

[30] J. Escher and G. Simonett. "Classical solutions of multidimensional Hele-Shaw models". In: *SIAM J. Math. Anal.* 28.5 (1997), pp. 1028–1047. ISSN: 0036-1410. DOI: 10.1137/S0036141095291919. URL: http://dx.doi.org/10.1137/S0036141095291919.

[31] J. D. Eshelby. "Elastic inclusions and inhomogeneities". In: *Progress in Solid Mechanics, Vol. II.* Amsterdam: North-Holland, 1961, pp. 87–140.

[32] L. C. Evans. *Partial differential equations.* Vol. 19. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 1998, pp. xviii+662. ISBN: 0-8218-0772-2.

[33] O. Forster. *Analysis* 2. *Differentialrechnung im* $\mathbf{R}^n$. Gewöhnliche Differentialgleichungen, Rororo Vieweg: Mathematik Grundkurs, No. 31. Verlag Vieweg, Braunschweig, 1977, pp. viii+163. ISBN: 3-499-27031-5.

[34] A. Fröhlich. *Stokes- und Navier-Stokes-Gleichungen in gewichteten Funktionenräumen.* Aachen: Shaker Verlag, 2001. ISBN: 3-8265-8680-8.

[35] H. Garcke. "On Cahn-Hilliard systems with elasticity". In: *Proc. Roy. Soc. Edinburgh Sect. A* 133.2 (2003), pp. 307–331. ISSN: 0308-2105. DOI: 10.1017/S0308210500002419. URL: http://dx.doi.org/10.1017/S0308210500002419.

[36] H. Garcke. "On Mathematical Models For Phase Separation In Elastically Stressed Solids". Habilitation thesis. University of Bonn, 2000.

[37] H. Garcke and D. J. C. Kwak. "On asymptotic limits of Cahn-Hilliard systems with elastic misfit". In: *Analysis, modeling and simulation of multiscale problems.* Berlin: Springer, 2006, pp. 87–111. DOI: 10.1007/3-540-35657-6_4. URL: http://dx.doi.org/10.1007/3-540-35657-6_4.

[38] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Vol. 2. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2005, pp. xii+302. ISBN: 88-7642-168-8.

[39] E. Hanzawa. "Classical solutions of the Stefan problem". In: *Tôhoku Math. J. (2)* 33.3 (1981), pp. 297–335. ISSN: 0040-8735. DOI: 10.2748/tmj/1178229399. URL: http://dx.doi.org/10.2748/tmj/1178229399.

[40] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities.* Vol. 5. Courant Lecture Notes in Mathematics. New York: New York University Courant Institute of Mathematical Sciences, 1999, pp. x+309. ISBN: 0-9658703-4-0; 0-8218-2700-6.

[41] S. Hildebrandt. *Analysis 1* -. 2. korr. Aufl. Berlin: Springer DE, 2005. ISBN: 978-3-540-25368-6.

[42] S. Hildebrandt. *Analysis 2.* Berlin: Springer DE, 2003. ISBN: 978-3-642-18972-2.

[43] J. Kevorkian. *Partial differential equations.* Second. Vol. 35. Texts in Applied Mathematics. Analytical solution techniques. New York: Springer-Verlag, 2000, pp. xii+636. ISBN: 0-387-98605-7.

[44] J. Kevorkian and J. D. Cole. *Multiple scale and singular perturbation methods.* Vol. 114. Applied Mathematical Sciences. New York: Springer-Verlag, 1996, pp. viii+632. ISBN: 0-387-94202-5. DOI: 10.1007/978-1-4612-3968-0. URL: http://dx.doi.org/10.1007/978-1-4612-3968-0.

[45] A. G. Khachaturyan. "Some questions concerning the theory of phase transitions in solids". In: *Soviet Physics Solid State* 8 (1967), pp. 2263 –2268.

[46] K. H. Kwek. "On the Cahn-Hilliard type equation". Ph. D. thesis. Georgia Institute of Technology, 1991.

[47] J. LeCrone. "Elliptic operators and maximal regularity on periodic little-Hölder spaces". In: *J. Evol. Equ.* 12.2 (2012), pp. 295–325. ISSN: 1424-3199. DOI: 10.1007/s00028-011-0133-z. URL: http://dx.doi.org/10.1007/s00028-011-0133-z.

[48] P.H. Leo, J.S. Lowengrub, and H.J. Jou. "A diffuse interface model for microstructural evolution in elastically stressed solids". In: *Acta Materialia* 46.6 (1998), pp. 2113 –2130. ISSN: 1359-6454. DOI: 10.1016/S1359-6454(97)00377-7. URL: http://www.sciencedirect.com/science/article/pii/S1359645497003777.

[49] J. Lowengrub and L. Truskinovsky. "Quasi-incompressible Cahn-Hilliard fluids and topological transitions". In: *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 454.1978 (1998), pp. 2617–2654. ISSN: 1364-5021. DOI: 10.1098/rspa.1998.0273. URL: http://dx.doi.org/10.1098/rspa.1998.0273.

[50] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Progress in Nonlinear Differential Equations and their Applications, 16. Basel: Birkhäuser Verlag, 1995, pp. xviii+424. ISBN: 3-7643-5172-1. DOI: 10.1007/978-3-0348-9234-6. URL: http://dx.doi.org/10.1007/978-3-0348-9234-6.

[51] A. Lunardi. "C^∞ regularity for fully nonlinear abstract evolution equations". In: *Differential equations in Banach spaces (Bologna, 1985).* Vol. 1223. Lecture Notes in Math. Berlin: Springer, 1986, pp. 176–185. DOI: 10.1007/BFb0099192. URL: http://dx.doi.org/10.1007/BFb0099192.

[52] A. Lunardi. *Interpolation theory.* Second. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2009, pp. xiv+191. ISBN: 978-88-7642-342-0; 88-7642-342-0.

[53] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge: Cambridge University Press, 2000, pp. xiv+357. ISBN: 0-521-66332-6; 0-521-66375-X.

[54] A. H. Nayfeh. *Perturbation methods.* Wiley Classics Library. Reprint of the 1973 original. New York: Wiley-Interscience [John Wiley & Sons], 2000, pp. xiv+425. ISBN: 0-471-39917-5. DOI: 10.1002/9783527617609. URL: http://dx.doi.org/10.1002/9783527617609.

[55] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Vol. 44. Applied Mathematical Sciences. New York: Springer-Verlag, 1983, pp. viii+279. ISBN: 0-387-90845-5. DOI: 10.1007/978-1-4612-5561-1. URL: http://dx.doi.org/10.1007/978-1-4612-5561-1.

[56] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*. Second. Vol. 13. Texts in Applied Mathematics. New York: Springer-Verlag, 2004, pp. xiv+434. ISBN: 0-387-00444-0.

[57] T. Roubíček. *Nonlinear partial differential equations with applications*. Vol. 153. International Series of Numerical Mathematics. Basel: Birkhäuser Verlag, 2005, pp. xviii+405. ISBN: 978-3-7643-7293-4; 3-7643-7293-1.

[58] R. Seeley. "Interpolation in L^p with boundary conditions". In: *Studia Math.* 44 (1972). Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I, pp. 47–60. ISSN: 0039-3223.

[59] R. Seeley. "Norms and domains of the complex powers $A_B z$". In: *Amer. J. Math.* 93 (1971), pp. 299–309. ISSN: 0002-9327.

[60] C. G. Simader and H. Sohr. "A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains". In: *Mathematical problems relating to the Navier-Stokes equation*. Vol. 11. Ser. Adv. Math. Appl. Sci. World Sci. Publ., River Edge, NJ, 1992, pp. 1–35.

[61] M. E. Taylor. *Partial differential equations III. Nonlinear equations*. Vol. 117. Applied Mathematical Sciences. New York: Springer, 1996. ISBN: 0-387-94652-7.

[62] H. Triebel. *Theory of function spaces*. Vol. 78. Monographs in Mathematics. Basel: Birkhäuser Verlag, 1983, p. 284. ISBN: 3-7643-1381-1. DOI: 10.1007/978-3-0346-0416-1. URL: http://dx.doi.org/10.1007/978-3-0346-0416-1.

[63] W. Walter. *Gewöhnliche Differentialgleichungen*. Seventh. Springer-Lehrbuch. [Springer Textbook]. Eine Einführung. [An introduction]. Berlin: Springer-Verlag, 2000. ISBN: 3-540-67642-2.