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Perturbative description of the fermionic projector: normalization, causality and Furry's theorem

Felix Finster and Jürgen Tolksdorf

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PERTURBATIVE DESCRIPTION OF THE FERMIONIC PROJECTOR: NORMALIZATION, CAUSALITY AND FURRY'S THEOREM

FELIX FINSTER AND JÜRGEN TOLKSDORF

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Abstract. The causal perturbation expansion of the fermionic projector is performed with a contour integral method. Different normalization conditions are analyzed. It is shown that the corresponding light-cone expansions are causal in the sense that they only involve bounded line integrals. For the resulting loop diagrams we prove a generalized Furry theorem.

Contents

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1. Introduction

The causal perturbation theory as developed in [5, 12] gives a perturbative description of the Dirac sea in an external potential (see also [8, Chapter 2]). It is the starting point for a detailed analysis of the fermionic projector in position space [6, 7], which forms the technical core of the fermionic projector approach to quantum field theory (see [10] and the references therein). More recently, the reformulation in terms of causal fermion systems [13] and the non-perturbative construction of the fermionic projector in [15, 16, 17] shed a new light on how the fermionic projector should be normalized. Moreover, the spectral methods used in the non-perturbative construction motivated that the perturbation expansions should be described more efficiently with contour integrals. Finally, the systematic treatment of perturbative quantum field theory in the framework of the fermionic projector in [11] showed that fermion loops are to be described in a specific formalism involving integral kernels L_ℓ to be formed of the contributions to the perturbation expansion in an external potential. This raises the question which of these integral kernels vanish in analogy to Furry's theorem in standard quantum field theory. The goal of the present paper is to treat all these issues in a coherent and conceptually convincing way, also giving a systematic procedure for all computations needed in future applications.

The paper is organized as follows. In Section 2 we recall the definition of the fermionic projector in the Minkowski vacuum and explain the different methods for its normalization, referred to as the *mass normalization* and the *spatial normalization*. In Section 3 we perform the perturbation expansion with contour integral methods, both with mass and spatial normalization. In Section 4 the perturbation expansions are described by the so-called unitary perturbation flow, which is particularly useful if particle and/or anti-particle states are present. In Section 5 we analyze the retarded expansion and the expansion with the Feynman propagator as alternative perturbation expansions. Section 6 is devoted to the light-cone expansion of the resulting Feynman diagrams. It is shown that the light-cone expansions of all diagrams is causal in the sense that it only involves bounded line integrals. In Section 7 we analyze the resulting loop diagrams and prove a generalized Furry theorem which states that certain classes of loop diagrams vanish. Finally, in Appendix A we list the leading orders of the relevant perturbation expansions.

2. The Normalization of the Vacuum Fermionic Projector

We let $(M, \langle ., . \rangle)$ be Minkowski space (with the signature convention $(+ - - -)$). In the vacuum, a completely filled Dirac sea is described by the distribution (for basics see [8, Chapter 2] or [14])

$$P_m(k) = (\slashed{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\,.$$

Taking the Fourier transform, we obtain the distribution

$$P_m(x, y) = \int \frac{d^4k}{(2\pi)^4}\, P_m(k)\, e^{-ik(x-y)}\,,$$

referred to as the *kernel of the fermionic projector* of the vacuum.

2.1. The Mass Normalization and the Spatial Normalization.

The fermionic projector is normalized in two different ways. First, considering the mass m as a variable parameter, one can multiply the fermionic projector with itself,

$$\begin{aligned}
P_m(k)\, P_{m'}(k) &= (\slashed{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\, (\slashed{k}+m')\, \delta(k^2-(m')^2)\, \Theta(-k^0) \\
&= \big(k^2+(m+m')\,\slashed{k}+mm'\big)\, \delta(m^2-(m')^2)\, \delta(k^2-m^2)\, \Theta(-k^0) \\
&= \big(k^2+(m+m')\,\slashed{k}+mm'\big)\, \frac{1}{2m}\, \delta(m-m')\, \delta(k^2-m^2)\, \Theta(-k^0) \\
&= \delta(m-m')\, (\slashed{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\,.
\end{aligned}$$

We thus obtain the distributional identity

$$P_m\, P_{m'} = \delta(m-m')\, P_m\,. \tag{2.1}$$

This resembles idempotence, but it involves a δ-distribution in the mass parameter. We refer to (2.1) as the **mass normalization**.

Alternatively, one can integrate over space. This can be understood from the fact that for a Dirac wave function ψ, the quantity $(\overline{\psi}\gamma^0\psi)(t_0,\vec{x})$ has the interpretation as the probability density for the particle at time t_0 to be at position $\vec{x}$. Integrating over space and polarizing, we obtain the scalar product

$$(\psi|\phi)_{t_0} = \int_{\mathbb{R}^3} \overline{\psi(t_0,\vec{y})}\gamma^0\phi(t_0,\vec{y})\, d^3y\,. \tag{2.2}$$

It follows from current conservation that for any solutions ψ,ϕ of the Dirac equation, this scalar product is independent of the choice of t_0. Since the kernel of the fermionic projector is a solution of the Dirac equation, one is led to evaluating the integral in (2.2) for $\phi(y)=P(y,z)$ and $\overline{\psi(y)}=P(x,y)$.

Lemma 2.1. *For any $t\in\mathbb{R}$, there is the distributional relation*

$$2\pi \int_{\mathbb{R}^3} P\big(x,(t,\vec{y})\big)\, \gamma^0\, P\big((t,\vec{y}),z\big)\, d^3y = -P(x,z)\,. \tag{2.3}$$

Proof. The identity follows by a straightforward computation. First,

$$\begin{aligned}
&\int_{\mathbb{R}^3} P\big(x,(t,\vec{y})\big)\, \gamma^0\, P\big((t,\vec{y}),z\big)\, d^3y \\
&\quad= \int_{\mathbb{R}^3} d^3y \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \int \frac{d^4q}{(2\pi)^4}\, e^{-iq(y-z)}\, P_m(k)\, \gamma^0\, P_m(q) \\
&\quad= \int \frac{d^4k}{(2\pi)^4} \int_{\mathbb{R}} \frac{d\lambda}{2\pi}\, e^{-ikx+iqy}\, P_m(k)\, \gamma^0\, P_m(q)\Big|_{q=(\lambda,\vec{k})}\,.
\end{aligned}$$

Setting $k=(\omega,\vec{k})$, we evaluate the δ-distributions inside the factors P_m,

$$\begin{aligned}
\delta(k^2-m^2)\, \delta(q^2-m^2)\big|_{q=(\lambda,\vec{k})} &= \delta\big(\omega^2-|\vec{k}|^2-m^2\big)\, \delta\big(\lambda^2-|\vec{k}|^2-m^2\big) \\
&= \delta(\lambda^2-\omega^2)\, \delta\big(\omega^2-|\vec{k}|^2-m^2\big)\,.
\end{aligned}$$

This shows that we only get a contribution if $\lambda = \pm\omega$. Using this fact, we can simplify the Dirac matrices according to

$$\begin{aligned}
(\slashed{k}+m)\,\gamma^0\,(\slashed{q}+m) &= (\omega\gamma^0 + \vec{k}\vec{\gamma} + m)\,\gamma^0\,(\pm\omega\gamma^0 + \vec{k}\vec{\gamma} + m) \\
&= (\omega\gamma^0 + \vec{k}\vec{\gamma} + m)\,(\pm\omega\gamma^0 - \vec{k}\vec{\gamma} + m)\,\gamma^0 \\
&= \Big((\pm\omega^2 + |\vec{k}|^2 + m^2)\,\gamma^0 + (1\pm 1)\,\omega\,(\vec{k}\vec{\gamma}) + (1\pm 1)\,m\omega \Big) \\
&= \begin{cases} 2\omega\,(\slashed{k}+m) & \text{in case } + \\ 0 & \text{in case } - \,. \end{cases}
\end{aligned}$$

Hence we only get a contribution if $\lambda = \omega$, giving rise to the identity

$$\delta(\lambda^2 - \omega^2) = \frac{1}{2|\omega|}\,\delta(\lambda - \omega)\,.$$

Putting these formulas together, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} P\big(x, (t,\vec{y})\big)\,\gamma^0\,P\big((t,\vec{y}), z\big)\,d^3y \\
&\quad = \int \frac{d^4k}{(2\pi)^4} \int_{\mathbb{R}} \frac{d\lambda}{2\pi}\, e^{-ik(x-y)}\,\delta(\lambda-\omega)\,\delta(k^2-m^2)\,\frac{2\omega}{2|\omega|}\,(\slashed{k}+m)\,\Theta(-k^0) \\
&\quad = -\frac{1}{2\pi}\int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\,\delta(k^2-m^2)\,(\slashed{k}+m)\,\Theta(-k^0)\,.
\end{aligned}$$

This gives the result. □

We refer to (2.3) as the **spatial normalization** of the fermionic projector.

2.2. **Discussion of the Normalization Method.** Before moving on to interacting systems, we now discuss the normalization methods in general terms. First of all, we point out that in an interacting system, it is in general impossible to keep both the spatial and the mass normalization. Therefore, one must decide whether to use either the mass normalization or the spatial normalization. Historically, the fermionic projector was first constructed using the mass normalization in [5, 12]. The spatial normalization was introduced later when extending the construction to space-times of finite life-time for which the mass normalization cannot be used (see [15]). There are no compelling physical reasons for working with one or the other normalization method. Instead, it is an open question which normalization method should be used. Ultimately, this question can only be answered by physical experiments (for differences between the normalizations see Section 7 below). Nevertheless, there are a few arguments in favor of the spatial normalization:

(a) The spatial integral in (2.3) is closely related to the probability integral for Dirac wave functions. More precisely, the condition 2.3 can be understood by saying that all the states of the fermionic projector should be normalized (up to the irrelevant factor of 2π) with respect to the integral over the probability density

$$\int_{\mathbb{R}^3} (\overline{\psi}\gamma^0\psi)(t,\vec{x})\,d^3x\,. \tag{2.4}$$

Therefore, the spatial normalization condition seems to be adjusted to the probabilistic interpretation of the Dirac wave function.

(b) In the framework of causal fermion systems (as introduced in [13]), the mass normalization is implemented if one chooses the scalar product on the particle space equal to the (suitably rescaled) space-time inner product

$$<\psi|\phi> = \int_M (\overline{\psi}\phi)(x)\, d^4x \,, \tag{2.5}$$

restricted to the occupied fermionic states of the system. However, this procedure only works if the inner product (2.5) is negative definite on the occupied fermionic states. Since it is not clear why this should always be the case, the mass normalization does not seem compatible with the framework of causal fermion systems.

(c) If the image of the fermionic projector is negative definite with respect to (2.5), one can construct a corresponding causal fermion system (at least after introducing a regularization). But this leads to the complicated situation that there are two different scalar products: First, the inner product $-<.|.>$ restricted to the image of the fermionic projector (which coincides with the scalar product $\langle .|. \rangle_{\mathcal{H}}$ on the particle space $\mathcal{H}$ of the corresponding causal fermion system). Second, the scalar product (2.2) obtained by polarizing (2.4) which is needed for the probabilistic interpretation.

   Working with the spatial normalization, on the other hand, it suffices to consider only the scalar product (2.2).

(d) The mass normalization only makes sense in space-times of infinite life-time. The spatial normalization, however, can be used on any globally hyperbolic space-time (for details see [15, 16]).

In view of these arguments, the authors consider the spatial normalization as being more natural and conceptually more convincing.

3. The Causal Perturbation Expansion with Contour Integrals

We now give a convenient method for deducing all the contributions to the causal perturbation expansion including the combinatorial factors. The method is to introduce a resolvent and to recover the fermionic projector as a suitable Cauchy integral.

3.1. **Preliminaries.**

In preparation, we fix our notation and recall a few constructions from [5, 12]. We consider the Dirac equation in an external potential

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \psi(x) = 0 \,. \tag{3.1}$$

Here $\mathcal{B}(x)$ is a matrix-valued potential which we assume to be smooth and symmetric with respect to the spin scalar product, i.e. $\overline{\psi}(\mathcal{B}\phi) = (\overline{\mathcal{B}\psi})\phi$ (where $\overline{\psi} \equiv \psi^* \gamma^0$ is the adjoint spinor). Starting from the plane-wave solutions of the vacuum Dirac equation, the equation in the external potential (3.1) can be solved in a perturbation expansion in $\mathcal{B}$. In the language of Feynman diagrams, this is an expansion in terms of tree diagrams. These diagrams are all well-defined and finite, provided that the potential $\mathcal{B}$ is sufficiently regular and has suitable decay properties at infinity (for details see for example [8, Lemma 2.2.2]). With this in mind, all our perturbation expansions are well-defined on the level of formal power series in $\mathcal{B}$. The questions of convergence of the perturbation expansions can be answered by using non-perturbative constructions (see [15, 16, 17]). Here we shall not consider such convergence questions, but instead we focus on working out the properties of the resulting expansions. For notational simplicity we always restrict attention to one family of Dirac particles of mass m. The generalization to systems of several families or generations of particles is straightforward using the methods in [8, §5.1] and [9, Section 4].

We always denote the objects in the presence of the external field by a tilde. The solutions of the vacuum Dirac equation on the upper respectively lower mass cone are described by the distributions

$$P_\pm = \frac{1}{2}\left(p_m \pm k_m\right), \tag{3.2}$$

where

$$p_m(q) = (\not{q} + m)\,\delta(q^2 - m^2) \tag{3.3}$$

$$k_m(q) = (\not{q} + m)\,\delta(q^2 - m^2)\,\epsilon(q^0), \tag{3.4}$$

where ϵ denotes the step function $\epsilon(x) = 1$ if $x > 0$ and $\epsilon(x) = -1$ otherwise. Moreover, we denote the advanced and retarded Green's functions by

$$s_m^\vee(q) = \frac{\not{q} + m}{q^2 - m^2 - i\varepsilon q_0} \qquad \text{and} \qquad s_m^\wedge(q) = \frac{\not{q} + m}{q^2 - m^2 + i\varepsilon q_0}. \tag{3.5}$$

Using the formula

$$\lim_{\varepsilon \searrow 0}\left(\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon}\right) = 2\pi i\,\delta(x),$$

one immediately verifies that the distribution k_m can be expressed in terms of these Green's functions by

$$k_m = \frac{1}{2\pi i}\left(s_m^\vee - s_m^\wedge\right). \tag{3.6}$$

In particular, the distribution k_m is causal in the sense that it vanishes identically for spacelike separated points. Moreover, the symmetric Green's function s_m is defined by

$$s_m = \frac{1}{2}\left(s_m^\vee + s_m^\wedge\right). \tag{3.7}$$

In the presence of an external potential $\mathcal{B}$, the perturbation expansion for the advanced and retarded Green's functions is unique by causality,

$$\tilde{s}_m^\vee = \sum_{n=0}^\infty (-s_m^\vee \mathcal{B})^n s_m^\vee, \qquad \tilde{s}_m^\wedge = \sum_{n=0}^\infty (-s_m^\wedge \mathcal{B})^n s_m^\wedge. \tag{3.8}$$

Using (3.6), we also have a unique perturbation expansion for the causal fundamental solution,

$$\tilde{k}_m = \frac{1}{2\pi i}(\tilde{s}_m^\vee - \tilde{s}_m^\wedge). \tag{3.9}$$

Using the identities

$$s_m^\vee = s_m + i\pi k_m, \qquad s_m^\wedge = s_m - i\pi k_m, \tag{3.10}$$

one can write the above perturbation series as operator product expansions. More precisely,

$$\tilde{k}_m = \sum_{\beta=0}^\infty (i\pi)^{2\beta}\, b_m^<\, k_m\, (b_m k_m)^{2\beta}\, b_m^>, \tag{3.11}$$

where the factors $b_m^\bullet$ are defined by

$$b_m^< = \sum_{n=0}^\infty (-s_m \mathcal{B})^n, \qquad b_m = \sum_{n=0}^\infty (-\mathcal{B} s_m)^n \mathcal{B}, \qquad b_m^> = \sum_{n=0}^\infty (-\mathcal{B} s_m)^n. \tag{3.12}$$

In the following constructions, we need to multiply the operator products in (3.11). These products have a mathematical meaning as distributions in the involved mass parameters,

$$p_m \, p_{m'} = k_m \, k_{m'} = \delta(m-m') \, p_m \tag{3.13}$$
$$p_m \, k_{m'} = k_m \, p_{m'} = \delta(m-m') \, k_m \tag{3.14}$$
$$k_m \, b^>_m \, b^<_{m'} \, k_{m'} = \delta(m-m') \Big(p_m + \pi^2 \, k_m \, b_m \, p_m \, b_m \, k_m \Big) \,. \tag{3.15}$$

Since these formulas all involve a common prefactors $\delta(m-m')$, we can introduce a convenient notation by leaving out this factor and omitting the mass indices. For clarity, we denote this short notation with a dot, i.e. symbolically

$$A \cdot B = C \qquad \text{stands for} \qquad A_m \, B_{m'} = \delta(m-m') \, C_m \,. \tag{3.16}$$

With this short notation, the multiplication rules can be written in the compact form

$$p \cdot p = k \cdot k = p \,, \qquad p \cdot k = k \cdot p = k \,, \qquad k \, b^> \cdot b^< k = p + \pi^2 pbpbp \,. \tag{3.17}$$

In all the subsequent calculations, the operator products are well-defined provided that the potential $\mathscr{B}$ is sufficiently smooth and has suitable decay properties at infinity (for details see [8, Lemma 2.2.2]). However, all infinite series are to be understood merely as formal power series in the potential $\mathscr{B}$.

3.2. **The Fermionic Projector with Mass Normalization.** Writing (3.11) as

$$\tilde{k} = \sum_{\beta=0}^\infty (i\pi)^{2\beta} \, b^< \, k \, (bk)^{2\beta} \, b^> \,, \tag{3.18}$$

powers of the operator $\tilde{k}$ with the product (3.16) are well-defined using the multiplication rules (3.17). This makes it possible to develop a spectral calculus for $\tilde{k}$. In particular, in [12] the operator P^sea is constructed as the projection operator on the negative spectral subspace of $\tilde{k}$. We now give an equivalent construction using contour integrals, which gives a more systematic procedure for computing all the contributions to the expansion.

We introduce the resolvent by

$$\tilde{R}_\lambda = (\tilde{k} - \lambda)^{-1} \,. \tag{3.19}$$

We choose a contour Γ_- which encloses the point -1 in contour-clockwise direction and does not enclose the points 1 and 0. We set

$$P^\text{sea}_\text{res} = -\frac{1}{2\pi i} \oint_{\Gamma_-} \tilde{R}_\lambda \, d\lambda \,. \tag{3.20}$$

This formula is to be understood as an operator product expansion, as we now explain. We write $\tilde{k}$ as

$$\tilde{k} = k + \Delta k \,,$$

where k is the corresponding distribution in the vacuum. Then $\tilde{R}_\lambda$ can be computed as a Neumann series,

$$\tilde{R}_\lambda = (k - \lambda + \Delta k)^{-1} = (1 + R_\lambda \, \Delta k)^{-1} \cdot R_\lambda = \sum_{n=0}^\infty (-R_\lambda \, \Delta k)^n \cdot R_\lambda \,. \tag{3.21}$$

According to (3.17), the operator k has the eigenvalues ± 1 and 0 with corresponding spectral projectors $(p \pm k)/2$ and $\mathbb{1} - p$. Hence we can write the free resolvent as

$$R_\lambda = \frac{p+k}{2} \left(\frac{1}{1-\lambda} \right) + \frac{p-k}{2} \left(\frac{1}{-1-\lambda} \right) - \frac{\mathbb{1} - p}{\lambda} \,.$$

Substituting this formula in (3.21), to every order in perturbation theory we obtain a meromorphic function in λ having poles only at $\lambda = 0$ and $\lambda = \pm 1$. Thus the contour integral (3.20) can be computed with residues, and the result is independent of the choice of Γ_-. In this way, we obtain a perturbation expansion for $P^{\text{sea}}_{\text{res}}$.

Proposition 3.1. *The perturbation expansion $P^{\text{sea}}_{\text{res}}$ has the properties*

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, P^{\text{sea}}_{\text{res}} = 0 \tag{3.22}$$

$$\big(P^{\text{sea}}_{\text{res}}\big)^* = P^{\text{sea}}_{\text{res}} \tag{3.23}$$

$$P^{\text{sea}}_{\text{res}} \cdot P^{\text{sea}}_{\text{res}} = P^{\text{sea}}_{\text{res}} \,. \tag{3.24}$$

In view of our notation of omitting the factors $\delta(m - m')$ introduced before (3.17), the idempotence relation (3.24) agrees precisely with the normalization (2.1). Therefore, $P^{\text{sea}}_{\text{res}}$ is the **fermionic projector with mass normalization**. The notation for the index "res" has evolved historically and has a twofold meaning. It was first introduced in [7] to denote the operator $\tilde{p}^{\text{res}}$ obtained by applying to $\tilde{k}$ the so-called *residual argument* (see also the proof of Theorem 6.4 below). In [12], the index "res" denoted the operators obtained by *rescaling* the Dirac sea. Using the same notation with a different meaning was motivated by the fact that the residual argument and the rescaling procedure gave rise to very similar operator product expansions. What was then considered a surprising coincidence can in fact be understood systematically by the symmetry consideration in Section 3.4.

In preparation for the proof of Proposition 3.1, we prove a spectral calculus for contour integrals which generalizes (3.20). To this end, we let f should be a holomorphic function defined on an an open neighborhood of the points ± 1. We define $f(\tilde{p}^{\text{res}})$ by inserting the function $f(\lambda)$ into the contour integral (3.20) and integrating around both spectral points ± 1,

$$f\big(\tilde{p}^{\text{res}}\big) := -\frac{1}{2\pi i} \oint_{\Gamma_+ \cup \Gamma_-} f(\lambda)\, \tilde{R}_\lambda \, d\lambda \,, \tag{3.25}$$

where Γ_+ is a contour which encloses $+1$ in counter-clockwise orientation (and does not enclose -1 or 0).

Theorem 3.2. (functional calculus) *For any functions f, g which are holomorphic in discs around ± 1 which contain the contours $\Gamma_\pm$,*

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, f\big(\tilde{p}^{\text{res}}\big) = 0 \tag{3.26}$$

$$f\big(\tilde{p}^{\text{res}}\big)^* = \overline{f}\big(\tilde{p}^{\text{res}}\big)^* \tag{3.27}$$

$$f\big(\tilde{p}^{\text{res}}\big) \cdot g\big(\tilde{p}^{\text{res}}\big) = (fg)\big(\tilde{p}^{\text{res}}\big) \,. \tag{3.28}$$

Proof. Since the image of the operator $\tilde{k}$ lies in the kernel of the Dirac operator, we know that

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{R}_\lambda = (i\partial\!\!\!/ + \mathcal{B} - m) \big(-\lambda^{-1} \big).$$

Taking the contour integral (3.25) gives (3.26).

The operators p_m, k_m and s_m are obviously symmetric (see (3.3), (3.4) and (3.7)). According to (3.11), the operator $\tilde{k}_m$ is also symmetric. Hence the resolvent $\tilde{R}_\lambda$ defined by (3.19) has the property

$$\tilde{R}_\lambda^* = \tilde{R}_{\overline{\lambda}} .$$

The relation (3.27) follows by taking the adjoint of (3.25) and reparametrizing the integral.

The starting point for proving (3.28) is the resolvent identity

$$\tilde{R}_\lambda \cdot \tilde{R}_{\lambda'} = \frac{1}{\lambda - \lambda'} \left(\tilde{R}_\lambda - \tilde{R}_{\lambda'} \right) . \tag{3.29}$$

We set $\Gamma = \Gamma_+ \cup \Gamma_-$ and denote the corresponding contour for λ' by Γ'. Since the integral (3.25) is independent of the precise choice of the contour, we may choose

$$\Gamma = \partial B_\delta(1) \cup \partial B_\delta(-1) \qquad \text{and} \qquad \Gamma' = \partial B_{2\delta}(1) \cup \partial B_{2\delta}(-1)$$

for sufficiently small $\delta < 1/2$. Then Γ does not enclose any point of Γ', implying that

$$\oint_\Gamma \frac{f(\lambda)}{\lambda - \lambda'}\, d\lambda = 0 \qquad \text{for all } \lambda' \in \Gamma' . \tag{3.30}$$

On the other hand, Γ' encloses every point of Γ, so that

$$\oint_{\Gamma'} f(\lambda)\, g(\lambda')\, \frac{\tilde{R}_\lambda}{\lambda - \lambda'}\, d\lambda' = -2\pi i\, f(\lambda)\, g(\lambda)\, \tilde{R}_\lambda \qquad \text{for all } \lambda \in \Gamma . \tag{3.31}$$

Combining (3.29) with (3.30) and (3.31), we obtain

$$\begin{aligned} f\big(\tilde{p}^{\text{res}}\big) \cdot g\big(\tilde{p}^{\text{res}}\big) &= -\frac{1}{4\pi^2} \oint_\Gamma f(\lambda)\, d\lambda \oint_{\Gamma'} g(\lambda')\, d\lambda' \;\frac{1}{\lambda - \lambda'} \left(\tilde{R}_\lambda - \tilde{R}_{\lambda'} \right) \\ &= -\frac{1}{2\pi i} \oint_\Gamma f(\lambda)\, g(\lambda)\, \tilde{R}_\lambda\, d\lambda = (fg)\big(\tilde{p}^{\text{res}}\big) . \end{aligned}$$

This concludes the proof. □

Proof of Proposition 3.1. Follows immediately from Theorem 3.2 if we choose the functions f and g to be identically zero in a neighborhood of $+1$ and be identically equal to one in a neighborhood of -1. □

3.3. **The Fermionic Projector with Spatial Normalization.**

We now turn attention to the spatial normalization integral in (2.3). For convenience, we introduce the short notation

$$(C_1 \,|_t\, C_2)(x,z) \equiv 2\pi \int_{\mathbb{R}^3} C_1\big(x, (t,\vec{y})\big)\, \gamma^0\, C_2\big((t,\vec{y}), z\big)\, d^3y . \tag{3.32}$$

We define

$$P^{\text{sea}} = -\frac{1}{2\pi i} \oint_{\Gamma_-} (-\lambda)\, \tilde{R}_\lambda\, d\lambda . \tag{3.33}$$

with Γ_- and $\tilde{R}_\lambda$ as in (3.20).

Proposition 3.3. *The expansion P^{sea} has the properties*

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, P^{\text{sea}} = 0 \tag{3.34}$$

$$2\pi \int_{\mathbb{R}^3} P^{\text{sea}}\big(x, (t,\vec{y})\big)\, \gamma^0\, P^{\text{sea}}\big((t,\vec{y}), z\big)\, d^3y = -P^{\text{sea}}(x,z) . \tag{3.35}$$

The remainder of this section is devoted to the proof of this proposition. For the spatial integral in (3.35) we introduce the short notation $|_t$, i.e.

$$(A\,|_t\, B)(x,z) := 2\pi \int_{\mathbb{R}^3} A\big(x,(t,\vec{y})\big)\, \gamma^0\, B\big((t,\vec{y}),z\big)\, d^3y\,. \tag{3.36}$$

We begin with a preparatory lemma.

Lemma 3.4. *For any $t_0 \in \mathbb{R}$, the distribution* (3.9) *has the property*

$$\tilde{k}_m\,|_{t_0}\, \tilde{k}_m = \tilde{k}_m\,.$$

Proof. Clearly, it suffices to prove the relation when evaluated by a test function f. Then $\tilde{\phi} := \tilde{k}_m(f)$ is a smooth solution of the Dirac equation with spatially compact support. Therefore, it suffices to show that for any such solution,

$$\tilde{\phi}(t,\vec{x}) = 2\pi \int_{\mathbb{R}^3} \tilde{k}_m(t,\vec{x}; t_0, \vec{y})\, \gamma^0\, \tilde{\phi}_0(\vec{y})\, d^3y\,.$$

Since $\tilde{\phi}$ and $\tilde{k}_m$ satisfy the Dirac equation, it suffices to prove this equation in the case $t > t_0$. In this case, the equation simplifies in view of (3.9) to

$$\tilde{\phi}(x) = i \int_{\mathbb{R}^3} \tilde{s}^\wedge_m(x,y)\, \gamma^0\, \tilde{\phi}_0(y)\big|_{y=(t_0,\vec{y})}\, d^3y\,,$$

where we set $x = (t,\vec{x})$. This identity is derived as follows: We choose a non-negative function $\eta \in C^\infty(\mathbb{R})$ with $\eta|_{[t_0,t]} \equiv 1$ and $\eta_{(-\infty,t_0-1)} \equiv 0$. We also consider η as a function on the time variable in space-time. Then

$$\tilde{\phi}(x) = (\eta\tilde{\phi})(x) = \tilde{s}^\wedge_m\big((i\partial\!\!\!/ + \mathcal{B} - m)(\eta\tilde{\phi})\big) = \tilde{s}^\wedge_m\big(i\gamma^0\, \dot{\eta}\, \tilde{\phi}\big)\,,$$

where we used the defining equation of the Green's function $\tilde{s}^\wedge_m(i\partial\!\!\!/_x + \mathcal{B} - m) = \mathbb{1}$ together with the fact that $\tilde{\phi}$ is a solution of the Dirac equation. To conclude the proof, we choose a sequence η_l such that the sequence of derivatives $\dot{\eta}_l$ converges as $l \to \infty$ in the distributional sense to the δ-distribution δ_{t_0} supported at t_0. Then

$$\begin{aligned}\tilde{s}^\wedge_m\big(i\gamma^0\, \dot{\eta}\, \tilde{\phi}\big)\big)(x) &= \int \Big(\tilde{s}^\wedge_m(x,y)\big(i\gamma^0\, \dot{\eta}(y^0)\, \tilde{\phi}(y)\big)\Big)\, d^4y \\ &\to \int_{\mathbb{R}^3} \Big(\tilde{s}^\wedge_m(x,y)\big(i\gamma^0 \tilde{\phi}\big)\Big)\,\big|_{y=(t_0,\vec{y})}\, d^3y\,,\end{aligned}$$

giving the result. □

Proof of Proposition 3.3. The relation (3.34) follows similar as in Proposition 3.1. In order to prove (3.35), we integrate the relations

$$\tilde{R}_\lambda \cdot (\tilde{k} - \lambda) = \mathbb{1} = (\tilde{k} - \lambda) \cdot \tilde{R}_\lambda\,,$$

to obtain

$$\oint_\Gamma \tilde{R}_\lambda \cdot \tilde{k}\, d\lambda = \oint_\Gamma \tilde{R}_\lambda\, \lambda\, d\lambda = \oint_\Gamma \tilde{k}\, \tilde{R}_\lambda\, d\lambda\,.$$

As a consequence,

$$P^{\text{sea}}\,|_t\, P^{\text{sea}} = -\frac{1}{4\pi^2} \oint_\Gamma d\lambda \oint_{\Gamma'} d\lambda'\, \tilde{R}_\lambda \cdot \tilde{k}\,|_t\, \tilde{k} \cdot \tilde{R}_{\lambda'}\,,$$

and applying Lemma 3.4 for $t_0 = t$ gives

$$P^{\text{sea}}\,|_t\, P^{\text{sea}} = -\frac{1}{4\pi^2} \oint_\Gamma d\lambda \oint_{\Gamma'} d\lambda'\, \tilde{R}_\lambda \cdot \tilde{k} \cdot \tilde{R}_{\lambda'} = -\frac{1}{4\pi^2} \oint_\Gamma \lambda\, d\lambda \oint_{\Gamma'} d\lambda'\, \tilde{R}_\lambda \cdot \tilde{R}_{\lambda'}\,.$$

Now we can again apply (3.29) and (3.30) (which remains valid if the integrand involves an additional factor λ) as well as (3.31). We thus obtain

$$P^{\text{sea}} \,|_t\, P^{\text{sea}} = -\frac{1}{2\pi i} \oint_\Gamma \lambda\, \tilde{R}_\lambda \, d\lambda = -P^{\text{sea}} \,,$$

concluding the proof. □

The resulting perturbation expansion agrees with the expansion given in [5, Section 3] (although at that time the spatial normalization property was not considered).

3.4. **A Symmetry between the Mass and the Spatial Normalizations.** We now want to compute the spatial normalization integral (3.32) for general operator products involving p_m, k_m and s_m (see (3.3), (3.4) and (3.7)). If both operators in the product map to solutions of the Dirac equation, it follows from the conservation of the Dirac current that the integral is independent of the time t. If the operator product involves a Green's function, however, the product will in general depend on t. For example, the integral

$$2\pi \int_{\mathbb{R}^2} p_m(x, (t, \vec{y}))\, \gamma^0\, s_m((t, \vec{y}), z)$$

depends on whether t lies to the future or past of the space-time point z. As a convenient notation, we write $|_{-\infty}$ if the time t at which the integral (3.32) is performed lies in the past of x and z. Similarly, $|_{+\infty}$ denotes the inner product if the time t in (3.32) lies in the future of both x and z. With this notation, we have the following computation rules.

Lemma 3.5. *For all $t \in \mathbb{R}$,*

$$k_m \,|_t\, k_m = k_m = p_m \,|_t\, p_m \tag{3.37}$$

$$k_m \,|_t\, p_m = p_m = p_m \,|_t\, k_m \tag{3.38}$$

$$\mp k_m \,|_{\pm\infty}\, s_m = i\pi\, k_m = \pm s_m \,|_{\pm\infty}\, k_m \tag{3.39}$$

$$\mp p_m \,|_{\pm\infty}\, s_m = i\pi\, p_m = \pm s_m \,|_{\pm\infty}\, p_m \tag{3.40}$$

$$s_m \,|_{\pm\infty}\, s_m = \pi^2\, k_m \,. \tag{3.41}$$

Proof. The first equation in (3.37) coincides with Lemma 3.4. In order to prove the second equation in (3.37), we write

$$p_m = k_m\, \epsilon \,,$$

where $\epsilon(p) = \epsilon(p^0)$ is the operator which multiplies the upper and lower mass shell by $+1$ and -1, respectively. Then

$$p_m \,|_t\, p_m = \epsilon\, k_m \,|_t\, k_m\, \epsilon = \epsilon\, k_m\, \epsilon = k_m \,.$$

The relations (3.38) follows similarly.

In order to prove the remaining rules (3.39)–(3.41), one uses (3.10) to rewrite s_m in terms of k_m and a causal Green's function. We then arrange that the causal Green's function vanishes by using that t lies in the future respectively past of x and z. For example,

$$k_m \,|_{+\infty}\, s_m = k_m \,|_{+\infty}\, (s_m^\vee - i\pi k_m) = -i\pi\, k_m \,|_{+\infty}\, k_m = -i\pi\, k_m$$

$$p_m \,|_{+\infty}\, s_m = \epsilon\, k_m \,|_{+\infty}\, s_m = -i\pi\, \epsilon\, k_m = -i\pi\, p_m$$

$$s_m \,|_{+\infty}\, s_m = (s_m^\wedge + i\pi k_m) \,|_{+\infty}\, (s_m^\vee - i\pi k_m) = \pi^2\, k_m \,|_{+\infty}\, k_m = \pi^2\, k_m \,.$$

The other relations are derived similarly. □

Lemma 3.6. *For all $t \in \mathbb{R}$,*

$$k_m\, b^>_m \,|_t\, b^<_{m'}\, k_m = k_m + \pi^2\, k_m\, b_m\, k_m\, b_m\, k_m\,. \tag{3.42}$$

Proof. Since the operator product $b^<_{m'}\, k_m$ is a solution of the Dirac equation in the external potential $\mathcal{B}$, it follows from current conservation that the product on the left of (3.42) is independent of t. In particular,

$$k_m\, b^>_m \,|_t\, b^<_{m'}\, k_m = \frac{1}{2}\, k_m\, b^>_m \left(|_{+\infty} + |_{-\infty} \right) b^<_{m'}\, k_m\,. \tag{3.43}$$

Computing the operator products in this way, the contributions by (3.39) and (3.40) drop out. Thus we only get a contribution if the factors $b^>_m$ and $b^<_m$ either both contain no factor s_m or both contain at least one factor s_m. Using the computation rules (3.37) and (3.41) gives the result. □

Comparing the computation rules (3.37), (3.38) and (3.42) for the spatial normalization integrals with the corresponding rules for the operator products in (3.17), one obtains agreement when applying the following replacement rules:

$$|_t \longrightarrow \cdot \tag{3.44}$$

$$p_m \longrightarrow k_m \tag{3.45}$$

$$k_m \longrightarrow p_m \tag{3.46}$$

$$s_m \longrightarrow s_m \tag{3.47}$$

(where the dot in (3.44) again refers to the short notation (3.16)). The replacement rules (3.45)–(3.47) were already used in the so-called residual argument to introduce the operator $\tilde{p}^{\text{res}}_m$ (cf. [5, eqs (3.16) and (3.17)]). We write symbolically

$$\tilde{k}_m \longrightarrow \tilde{p}^{\text{res}}_m\,. \tag{3.48}$$

Combining the rule (3.44) with Lemma 3.4, one finds that

$$\tilde{p}^{\text{res}} \cdot \tilde{p}^{\text{res}} = \tilde{p}^{\text{res}}$$

(being a short notation for $\tilde{p}^{\text{res}}_m\, \tilde{p}^{\text{res}}_{m'} = \delta(m - m')\, \tilde{p}^{\text{res}}_m$). Thus $\tilde{p}^{\text{res}}$ has the correct mass normalization. This explains why it coincides with the corresponding operator introduced in [12] by a rescaling procedure for the Dirac sea. It can be written similar to (3.20) as the contour integral

$$\tilde{p}^{\text{res}}_m = -\frac{1}{2\pi i} \oint_{\Gamma_+ \cup \Gamma_-} \tilde{R}_\lambda\, d\lambda\,. \tag{3.49}$$

Next, we introduce the operator $\tilde{p}_m$ similar to (3.33) by

$$\tilde{p}_m = -\frac{1}{2\pi i} \left(\oint_{\Gamma_+} - \oint_{\Gamma_-} \right) \lambda\, \tilde{R}_\lambda\, d\lambda\,. \tag{3.50}$$

Repeating the computation in the proof of Proposition 3.3, one sees that it satisfies the spatial normalization condition

$$\tilde{p}_m \,|_t\, \tilde{p}_m = \tilde{k}_m\,.$$

Again applying our replacements rules, we obtain an operator $\tilde{k}^{\text{res}}_m$,

$$\tilde{p}_m \longrightarrow \tilde{k}^{\text{res}}_m\,, \tag{3.51}$$

which satisfies the mass normalization condition

$$\tilde{k}^{\rm res} \cdot \tilde{k}^{\rm res} = \tilde{p}^{\rm res} \,.$$

It can be written similar to (3.20) as the contour integral

$$\tilde{k}^{\rm res}_m = -\frac{1}{2\pi i} \left(\oint_{\Gamma_+} - \oint_{\Gamma_-} \right) \tilde{R}_\lambda \, d\lambda \,. \tag{3.52}$$

Finally, we can write the fermionic projector with mass and spatial normalization as

$$P^{\rm sea}_{\rm res} = \frac{1}{2} \left(\tilde{p}^{\rm res}_m - \tilde{k}^{\rm res}_m \right) \qquad \text{and} \qquad P^{\rm sea} = \frac{1}{2} \left(\tilde{p}_m - \tilde{k}_m \right) . \tag{3.53}$$

This shows that our replacement rules also transform these fermionic projectors into each other; more precisely,

$$P^{\rm sea} \longrightarrow -P^{\rm sea}_{\rm res} \,. \tag{3.54}$$

We have thus found a symmetry in the perturbation expansions with mass and spatial normalization: If in the operator expansions we exchange all operators according to the replacement rules (3.45)–(3.47), then according to (3.54) the fermionic projector with spatial normalization is transformed up to minus the fermionic projector with mass normalization. This symmetry was already observed in [12], but without understanding the underlying reason (3.44).

4. The Unitary Perturbation Flow

4.1. **The Unitary Perturbation Flow with Mass Normalization.** In [12, Section 5] it is shown that there exists an operator U which transforms the vacuum operators p_m and k_m to the corresponding interacting operators with mass normalization $\tilde{p}^{\rm res}_m$ and $\tilde{k}^{\rm res}_m$. For a consistency, we now denote this operator by $U_{\rm res}$. Then

$$\tilde{P}^{\rm sea}_{\rm res} = U_{\rm res} \cdot \left(\frac{p_m - k_m}{2} \right) \cdot U^*_{\rm res} \,. \tag{4.1}$$

The operator $U_{\rm res}$ maps solutions of the vacuum Dirac equation to solutions of the Dirac equation in the potential. This mapping is invertible, and it is an isometry with respect to the indefinite inner product (2.5). For simplicity, we say that $U_{\rm res}$ is *unitary* with respect to the indefinite inner product (2.5). In applications, one considers a family of potentials $\mathcal{B}(\tau)$ (in the simplest case the family $\mathcal{B}(\tau) = \tau \mathcal{B}_0$ which "turns on" the interaction) and considers the corresponding family of unitary transformations $U_{\rm res}(\tau)$. Then $U_{\rm res}(\tau)$ defines a one-parameter family of transformations, the so-called *unitary perturbation flow*. We now give a systematic procedure for computing the unitary perturbation flow to any order in perturbation theory.

Lemma 4.1. *The operators* $\tilde{k}^{\rm res}_m$ *and* $\tilde{p}^{\rm res}_m$ *defined by* (3.52) *and* (3.49) *satisfy the relations*

$$(i\partial\!\!\!/ + \mathcal{B} - m) \, \tilde{p}^{\rm res} = 0 \tag{4.2}$$

$$(\tilde{p}^{\rm res})^* = \tilde{p}^{\rm res} = \tilde{p}^{\rm res} \cdot \tilde{p}^{\rm res} \tag{4.3}$$

$$(\tilde{k}^{\rm res})^* = \tilde{k}^{\rm res} = \tilde{k}^{\rm res} \cdot \tilde{p}^{\rm res} = \tilde{p}^{\rm res} \cdot \tilde{k}^{\rm res} \,. \tag{4.4}$$

Proof. Follows immediately from the functional calculus of Theorem 3.2. □

Our method for computing U_{res} is to "turn on the perturbation adiabatically." Thus for a parameter $\tau \in [0,1]$ we let $\tilde{p}^{\text{res}}(\tau)$ be the spectral projector corresponding to the perturbation operator $\tau\mathcal{B}$. We define $U^*_{\text{res}}(\tau)$ by

$$U^*_{\text{res}}(\tau) = \lim_{N\to\infty} \tilde{p}^{\text{res}}(0)\cdot \tilde{p}^{\text{res}}\Big(\frac{\tau}{N}\Big) \;\cdots\; \tilde{p}^{\text{res}}\Big(\frac{(N-1)\,\tau}{N}\Big)\cdot \tilde{p}^{\text{res}}(\tau)\,. \tag{4.5}$$

Then $U^*_{\text{res}}(\tau)$ satisfies the differential equation

$$\begin{aligned}\frac{d}{d\tau} U^*_{\text{res}}(\tau) &= \lim_{\varepsilon\searrow 0} \frac{U^*_{\text{res}}(\tau+\varepsilon) - U^*_{\text{res}}(\tau)}{\varepsilon} \\ &= \lim_{\varepsilon\searrow 0} U^*_{\text{res}}(\tau)\cdot \frac{\tilde{p}^{\text{res}}(\tau+\varepsilon)-\tilde{p}^{\text{res}}(\tau)}{\varepsilon} = U^*_{\text{res}}(\tau)\cdot\Big(\frac{d}{d\tau}\tilde{p}^{\text{res}}(\tau)\Big)\,.\end{aligned}$$

Noting that $U^*_{\text{res}}(0) = \tilde{p}_{\text{res}}(0)$ (as is obvious from (4.5) and (4.3)), we can solve this differential equation with an ordered exponential,

$$U^*_{\text{res}}(\tau) = \tilde{p}^{\text{res}}(0)\cdot \mathrm{Pexp}\left(\int_0^\tau (\tilde{p}^{\text{res}})'(s)\,ds\right), \tag{4.6}$$

so that

$$\begin{aligned}U^*_{\text{res}}(\tau) &= \tilde{p}^{\text{res}}(0) + \tilde{p}^{\text{res}}(0)\cdot \int_0^\tau (\tilde{p}^{\text{res}})'(s)\,ds \\ &\qquad + \tilde{p}^{\text{res}}(0)\cdot\int_0^\tau ds_1 \int_0^{s_1} ds_2\, (\tilde{p}^{\text{res}})'(s_2)\cdot(\tilde{p}^{\text{res}})'(s_1) + \cdots \\ &= \tilde{p}^{\text{res}}(0)\cdot\tilde{p}^{\text{res}}(\tau) + \tilde{p}^{\text{res}}(0)\cdot \int_0^\tau ds_1\, \big(\tilde{p}^{\text{res}}(s_1) - \tilde{p}^{\text{res}}(0)\big)\cdot(\tilde{p}^{\text{res}})'(s_1) + \cdots\,.\end{aligned} \tag{4.7}$$

We now verify that the resulting operator U_{res} has the required properties.

Proposition 4.2. *The one-parameter family of operators defined by* (4.5) *satisfy the Dirac equation and are unitary,*

$$(i\partial\!\!\!/ + \tau\mathcal{B} - m)\, U_{\text{res}}(\tau) = 0 \tag{4.8}$$

$$U(\tau)\cdot U^*(\tau) = \mathbb{1} = U^*(\tau)\cdot U(\tau)\,. \tag{4.9}$$

Moreover, they map the free fundamental solutions and spectral projectors to the corresponding interacting objects,

$$U_{\text{res}}(\tau)\cdot k\cdot U^*_{\text{res}}(\tau) = \tilde{k}^{\text{res}}(\tau)\,, \qquad U_{\text{res}}(\tau)\cdot p\cdot U^*_{\text{res}}(\tau) = \tilde{p}^{\text{res}}(\tau)\,. \tag{4.10}$$

Proof. The Dirac equation (4.8) is obviously satisfied in view of (4.2) and (4.3) as well as the fact that the operator $U^*_{\text{res}}(\tau)$ in (4.5) involves a factor $\tilde{p}^{\text{res}}(\tau)$ at the very right. In order to show unitarity, it suffices to prove the second equality in (4.9). Differentiating the first equation in (4.3), we know that

$$(\tilde{p}^{\text{res}})'(\tau) = (\tilde{p}^{\text{res}})'(\tau)^*\,,$$

so that we can omit the stars of $\tilde{p}^{\text{res}}$ and its derivatives in all calculations. Next, differentiating the last relation in (4.3) gives

$$(\tilde{p}^{\text{res}})'(\tau)\cdot\tilde{p}^{\text{res}}(\tau) + \tilde{p}^{\text{res}}(\tau)\cdot(\tilde{p}^{\text{res}})'(\tau) = (\tilde{p}^{\text{res}})'(\tau)\,.$$

Multiplying from the left and right by $\tilde{p}^{\text{res}}$ and using (4.3), we obtain the identity

$$\tilde{p}^{\text{res}}(\tau)\cdot(\tilde{p}^{\text{res}})'(\tau)\cdot\tilde{p}^{\text{res}}(\tau) = 0\,.$$

Since the operator $U^*_{\text{res}}(\tau)$ involves a factor $\tilde{p}^{\text{res}}(\tau)$ at the right, it follows that

$$U^*_{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) \cdot U_{\text{res}}(\tau) = 0\,.$$

Thus

$$\frac{d}{d\tau}\Big(U^*_{\text{res}}(\tau) \cdot U_{\text{res}}(\tau) \Big) = 2\, U^*_{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) \cdot U_{\text{res}}(\tau) = 0\,. \tag{4.11}$$

For $\tau = 0$, it follows from (4.5) that

$$U^*_{\text{res}}(0) \cdot U_{\text{res}}(0) = \tilde{p}^{\text{res}}(0) \cdot \tilde{p}^{\text{res}}(0) = p \cdot p = p\,, \tag{4.12}$$

where in the last step we used the calculation rules (3.17). These rules also show that p acts on the free solutions as the identity. Therefore, we can also write (4.12) as $U^*(0) \cdot U(0) = \mathbb{1}$. Integrating (4.11) gives the unitarity (4.9).

The first equation in (4.10) follows similarly from the fact that

$$U^*_{\text{res}}(0) \cdot \tilde{p}^{\text{res}}(0) \cdot U_{\text{res}}(0) = \mathbb{1} \qquad \text{and}$$

$$\frac{d}{d\tau}\Big(U^*_{\text{res}}(\tau)\, \tilde{p}^{\text{res}}(\tau)\, U_{\text{res}}(\tau) \Big) = 3\, U^*_{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) \cdot U_{\text{res}}(\tau) = 0\,.$$

In order to derive the second equation in (4.10), we differentiate (4.4) to obtain

$$\begin{aligned}(\tilde{k}^{\text{res}})'(\tau) \cdot \tilde{p}^{\text{res}}(\tau) + \tilde{k}^{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) &= (\tilde{k}^{\text{res}})'(\tau) \\ &= (\tilde{p}^{\text{res}})'(\tau) \cdot \tilde{k}^{\text{res}}(\tau) + \tilde{p}^{\text{res}}(\tau) \cdot (\tilde{k}^{\text{res}})'(\tau)\,.\end{aligned}$$

Multiplying from the left and right by $\tilde{p}^{\text{res}}$, we can apply (4.3) and (4.4) to get

$$\tilde{k}^{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) \cdot \tilde{p}^{\text{res}}(\tau) = 0 = \tilde{p}^{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) \cdot \tilde{k}^{\text{res}}(\tau) = 0\,.$$

As a consequence,

$$\begin{aligned}&\frac{d}{d\tau}\Big(U^*_{\text{res}}(\tau) \cdot \tilde{k}^{\text{res}}(\tau) \cdot U_{\text{res}}(\tau) \Big) \\ &\quad = U^*_{\text{res}}(\tau) \cdot \Big((\tilde{p}^{\text{res}})'(\tau) \cdot \tilde{k}^{\text{res}}(\tau) \cdot \tilde{p}^{\text{res}}(\tau) + \tilde{p}^{\text{res}}(\tau) \cdot \tilde{k}^{\text{res}}(\tau) \cdot (\tilde{p}^{\text{res}})'(\tau) \Big) \cdot U_{\text{res}}(\tau) = 0\,.\end{aligned}$$

Using that $U^*_{\text{res}}(0) \cdot \tilde{k}^{\text{res}}(0) \cdot U_{\text{res}}(0) = k$, the result follows. □

4.2. **The Unitary Perturbation Flow with Spatial Normalization.** We now want to construct an operator V which introduces the interaction in the case of a spatial normalization, i.e. in analogy to (4.1)

$$\tilde{P}^{\text{sea}} = U\,|_t \left(\frac{p_m - k_m}{2} \right) |_t\, U^*\,,$$

where the adjoint again refers to the indefinite inner product (2.5). Since the fermionic projector with spatial normalization will in general violate the mass normalization condition (i.e. in general $\tilde{P}^{\text{sea}} \tilde{P}^{\text{sea}} \neq \tilde{P}^{\text{sea}}$), the operator U will in general *not* be unitary with respect to (2.5). But it is unitary with respect to the scalar product (2.2), in the following sense: The scalar product (2.2) is time independent on the solution space of the Dirac equation. The operator U maps solutions of the vacuum Dirac equation to the solutions of the Dirac equation in the external potential. Therefore, we have two different solution spaces, and the scalar product (2.2) on these spaces should be considered as two separate objects. By unitarity of U we mean that U is an isometric bijection of the solutions of the vacuum Dirac equation to the solutions of the Dirac equation in the external potential. The simplest way to construct U is to use the symmetry between the mass and spatial normalization of Section 3.4. Applying it to Proposition 4.2 gives the following result.

Proposition 4.3. *The operators $U(\tau)$ obtained from the operator $U^{\text{res}}(\tau)$ by the replacement rules* (3.45)–(3.47) *satisfy the Dirac equation*

$$(i\partial\!\!\!/ + \tau \mathcal{B} - m)\, U(\tau) = 0 \,.$$

Moreover, they map the free fundamental solutions and spectral projectors to the corresponding interacting objects with spatial normalization,

$$U(\tau)\,|_t\, k\,|_t\, U^*(\tau) = \tilde{k}(\tau)\,, \qquad U(\tau)\,|_t\, p\,|_t\, U^*(\tau) = \tilde{p}(\tau)\,.$$

The operators $U(\tau)$ are unitary with respect to the scalar product (2.2)*, meaning that*

$$U(\tau)\,|_t\, U(\tau)^* = \mathbb{1} = U(\tau)^*\,|_t\, U(\tau)$$

(where the star always denotes the adjoint with respect to the inner product (2.5)*).*

4.3. **Geometric Phases.** We finally note that the operator $U_{\text{res}}[\mathcal{B}] := U_{\text{res}}(1)$ is not uniquely determined by its properties (4.8)–(4.10). In particular, for a given potential $\mathcal{B}$, we could have chosen more generally an arbitrary curve $\mathcal{B}(\tau)$ with $0 \leq \tau \leq 1$ in the space of all smooth potentials with $\mathcal{B}(0) = 0$ and $\mathcal{B}(1) = \mathcal{B}$ and could have replaced the definition (4.6) by

$$U^*_{\text{res}} := \tilde{p}^{\text{res}}(0) \cdot \text{Pexp}\left(\int_0^1 \partial_s \big(\tilde{p}^{\text{res}}[\mathcal{B}(s)] \big)\, ds \right).$$

This alternative definition of U_{res} also has all the desired properties. However, it does depend on the choice of the curve $\mathcal{B}(\tau)$. This non-uniqueness can be understood in analogy to the well-known Berry phase [1] as a geometric phase picked up when changing the system adiabatically around a closed circuit. More precisely, in the description of the Berry phase one changes the potential in a Schrödinger operator adiabatically and continually projects onto a specific bound state (which clearly also varies adiabatically). Similarly, in (4.5) the potential $\mathcal{B}(\tau)$ is varied adiabatically. One difference is that in our setting the potential is given in space-time, and τ parametrizes a family of space-times with different potentials. More importantly, in (4.5) we do not project continually onto a bound state, but onto the whole solution space of the Dirac equation. As a consequence, our holonomy is not only a phase of a bound state, but it is a unitary endomorphism of the solution space of the Dirac equation. In view of (4.10), this endomorphism also respects the splitting into generalized positive and negative energy solutions.

In order to illustrate the holonomy, we consider the simplest possible example. Let $\mathcal{B}(\tau)$ a closed loop with $\mathcal{B}(0) = \mathcal{B}(1) = 0$. Then it is shown in Appendix A that in second order perturbation theory,

$$U^*_{\text{res}}(1) = p + \pi^2 \int_0^1 p\, \mathcal{B}'(s)\, p\, \mathcal{B}(s)\, p\, ds + \mathcal{O}(\mathcal{B}^3)\,. \tag{4.13}$$

The integral does not vanish along general loops, giving a non-trivial holonomy. Note that $U^{\text{res}}(1)$ maps the solution space of the vacuum Dirac equation to itself. The integral in (4.13) is anti-symmetric because

$$\begin{aligned}\Big(\int_0^1 p\, \mathcal{B}'(s)\, p\, \mathcal{B}(s)\, p\, ds \Big)^* &= \int_0^1 p\, \mathcal{B}(s)\, p\, \mathcal{B}'(s)\, p\, ds \\ = p\, \mathcal{B}(s)\, p\, \mathcal{B}(s)\, p \Big|_{s=0}^{s=1} &- \int_0^1 p\, \mathcal{B}'(s)\, p\, \mathcal{B}(s)\, p\, ds = - \int_0^1 p\, \mathcal{B}'(s)\, p\, \mathcal{B}(s)\, p\, ds\,,\end{aligned}$$

which means that $U^*_{\text{res}}(1)$ is indeed unitary to second order in perturbation theory. One can also verify by explicit computation that $U^*_{\text{res}}(1)$ is unitary to higher order.

5. Other Perturbation Expansions of the Fermionic Projector

As an alternative to the causal perturbation expansion, one can also consider a retarded expansion in which the potential $\mathcal{B}$ at a space-time point x influences the fermionic wave functions only in the causal future of x. Such a retarded perturbation expansion is physically questionable because it distinguishes a direction of time. Nevertheless, it is useful in certain applications when the system (including all the sea states) is in a fixed configuration in the past. Another possible method is to perform the perturbation expansion exclusively with the Feynman propagator. This method is again physically questionable, this time because it works with the notion of positive and negative frequency which in curved space-time has no observer-independent meaning.

In this section we work out these alternative perturbation expansions from a mathematical point of view and collect some of their properties. This is instructive in comparison with the causal expansion with mass or spatial normalization.

5.1. **The Retarded Perturbation Expansion.** For a Dirac wave function ψ, the retarded perturbation expansion is obtained similar to (3.8) by iteratively applying the retarded Green's function, i.e.

$$\tilde{\psi} = \sum_{n=0}^\infty (-s^\wedge_m\, \mathcal{B})^n\, \psi\,.$$

In order for our notation to harmonize with that for the perturbation flow, we write

$$\tilde{\psi} = U_\wedge \cdot \psi \qquad \text{with} \qquad U_\wedge = \sum_{n=0}^\infty (-s^\wedge_m\, \mathcal{B})^n\, p_m\,.$$

Thinking of the fermionic projector as being composed of bra and ket states, its perturbation expansion is given similar to (4.1) by

$$\tilde{P}^{\text{sea}}_\wedge = U_\wedge \cdot \Big(\frac{p_m - k_m}{2} \Big) \cdot U^*_\wedge\,,$$

where the adjoint $U^*_\wedge$ (taken with respect to the indefinite inner product (2.5)) involves the advanced Green's function,

$$U^*_\wedge = \sum_{n=0}^\infty p_m\, (-\mathcal{B} s^\vee_m)^n\,.$$

Proposition 5.1. *The retarded perturbation expansion of the fermionic projector $\tilde{P}^{\text{sea}}_\wedge$ has the representation*

$$\tilde{P}^{\text{sea}}_\wedge = \frac{1}{2}\, \big(\tilde{p}^\wedge_m - \tilde{k}_m \big) \tag{5.1}$$

with $\tilde{k}_m$ *according to* (3.9) *and*

$$\tilde{p}_m^\wedge := U_\wedge \cdot p_m \cdot U_\wedge^* \, .$$

The spatial normalization condition is satisfied; i.e., using the notation (3.36),

$$\tilde{P}^{\text{sea}}_\wedge \,|_t\, \tilde{P}^{\text{sea}}_\wedge = \tilde{P}^{\text{sea}}_\wedge \qquad \text{for all } t \in \mathbb{R} \, . \tag{5.2}$$

Proof. From (3.6) we have

$$U_\wedge \cdot k_m \cdot U_\wedge^* = \frac{1}{2\pi i} \sum_{n,n'=0}^\infty (-s_m^\wedge \, \mathcal{B})^n \Big(s_m^\vee - s_m^\wedge \Big) (-\mathcal{B} s_m^\vee)^{n'} \, .$$

Using that the sums are telescopic, we obtain

$$\begin{aligned} U_\wedge \cdot k_m \cdot U_\wedge^* &= \frac{1}{2\pi i} \sum_{n'=0}^\infty s_m^\vee (-\mathcal{B} s_m^\vee)^{n'} - \sum_{n=0}^\infty (-s_m^\wedge \, \mathcal{B})^n s_m^\wedge \\ &= \frac{1}{2\pi i} \big(\tilde{s}_m^\vee - \tilde{s}_m^\wedge \big) = \tilde{k}_m \, , \end{aligned}$$

where in the last line we used (3.8) and (3.9). Hence for the operator k_m, the retarded perturbation expansion coincides with the causal perturbation expansion. This proves (5.1).

The spatial normalization condition can be verified in two different ways. The first method uses the fact that, again due to current conservation, it suffices to prove (5.2) for any t. In the limit $t \to -\infty$, $P^{\text{sea}}_\wedge$ goes over to the vacuum fermionic projector, so that we can use (2.3). The second method is to verify the spatial normalization condition directly using the computation rules of Lemma 3.5. Since $P^{\text{sea}}_\wedge$ satisfies the Dirac equation, exactly as in (3.43) we may take the mean of the computation rules at $t = \pm\infty$. We use the short notation

$$| = \frac{1}{2} \big(|_{+\infty} + |_{-\infty} \big).$$

Decomposing the Green's functions according to (3.10), we obtain the computation rules

$$\begin{aligned} (p_m - k_m) \,|\, s_m^\wedge &= (p_m - k_m) \,|\, (s_m - i\pi k_m) = -i\pi \, (p_m - k_m) \\ s_m^\vee \,|\, (p_m - k_m) &= (s_m + i\pi k_m) \,|\, (p_m - k_m) = i\pi \, (p_m - k_m) \\ s_m^\vee \,|\, s_m^\wedge &= (s_m + i\pi k_m) \,|\, (s_m - i\pi k_m) = 2\pi^2 \, k_m \, . \end{aligned}$$

Hence

$$
\begin{aligned}
&4\,\big(\tilde{P}^{\text{sea}}_\wedge \,|\, \tilde{P}^{\text{sea}}_\wedge - \tilde{P}^{\text{sea}}_\wedge\big) = U_\wedge \cdot \big(p_m - k_m\big) \cdot U_\wedge^* \,|\, U_\wedge \cdot \big(p_m - k_m\big) \cdot U_\wedge^* - 4\tilde{P}^{\text{sea}}_\wedge \\
&= \sum_{n=1}^\infty U_\wedge \cdot \big(p_m - k_m\big) \,|\, (-s^\wedge_m\, \mathcal{B})^n \,\big(p_m - k_m\big) \cdot U_\wedge^* \\
&\quad + \sum_{n'=1}^\infty U_\wedge \cdot \big(p_m - k_m\big)\, (-\mathcal{B}\, s^\vee_m)^{n'} \,|\, \big(p_m - k_m\big) \cdot U_\wedge^* \\
&\quad + \sum_{n,n'=1}^\infty U_\wedge \cdot \big(p_m - k_m\big)\, (-\mathcal{B}\, s^\vee_m)^{n'} \,|\, (-s^\wedge_m\, \mathcal{B})^n \big(p_m - k_m\big) \cdot U_\wedge^* \\
&= -\,i\pi \sum_{n=0}^\infty U_\wedge \cdot \big(p_m - k_m\big) \Big[\mathcal{B}\,(-s^\wedge_m\,\mathcal{B})^n - (-\mathcal{B}\, s^\vee_m)^n\, \mathcal{B}\Big] \big(p_m - k_m\big) \cdot U_\wedge^* \qquad (5.3)\\
&\quad + 2\pi^2 \sum_{n,n'=0}^\infty U_\wedge \cdot \big(p_m - k_m\big)\, (-\mathcal{B}\, s^\vee_m)^{n'}\, \mathcal{B}\, k_m\, \mathcal{B}\, (-s^\wedge_m\, \mathcal{B})^n \big(p_m - k_m\big) \cdot U_\wedge^* \,. \qquad (5.4)
\end{aligned}
$$

The differences of the series involving the advanced and retarded Green's can be rewritten with the help of (3.6) as a telescopic sum,

$$\sum_{n=0}^\infty \Big((-s^\vee_m\,\mathcal{B})^n - (-s^\wedge_m\,\mathcal{B})^n\Big) = 2\pi i \sum_{n,n'=0}^\infty (-s^\vee_m\,\mathcal{B})^n(-k_m\,\mathcal{B})(-s^\wedge_m\,\mathcal{B})^{n'} \,.$$

Using this relation, the summands in (5.3) and (5.4) all cancel, giving the result. □

5.2. **The Expansion with Feynman Propagators.** We finally remark that, similar to the retarded Green's function in the retarded perturbation expansion, one can also perform the perturbation expansion with any other Green's function. As an example, we consider the perturbation expansion with the *Feynman propagator* s^+_m, where

$$s^\pm_m(k) := \lim_{\varepsilon \searrow 0} \frac{\not{k} + m}{k^2 - m^2 \mp i\varepsilon} \,. \qquad (5.5)$$

Then

$$\tilde{\psi} = U_- \cdot \psi \qquad \text{with} \qquad U_- = \sum_{n=0}^\infty (-s^-_m\,\mathcal{B})^n\, p_m \,. \qquad (5.6)$$

Consequently,

$$\tilde{P}^{\text{sea}}_{\text{F}} := U_- \cdot \Big(\frac{p_m - k_m}{2}\Big) \cdot U_-^* \,,$$

where

$$U_-^* = \sum_{n=0}^\infty p_m\,(-\mathcal{B} s^+_m)^n \,.$$

Proposition 5.2. *The perturbation expansion with Feynman propagators of the fermionic projector* $P^{\text{sea}}_{\text{F}}$ *has the representation*

$$\tilde{P}^{\text{sea}}_{\text{F}} = \frac{1}{2}\,\big(\tilde{p}^{\text{res}}_m - \tilde{k}^{\text{F}}_m\big) \qquad (5.7)$$

with $\tilde{p}^{\text{res}}_m$ *according to* (3.49) *and*

$$\tilde{k}^{\text{F}}_m := U_- \cdot k_m \cdot U_-^* \,. \qquad (5.8)$$

Moreover, $\tilde{P}^{\text{sea}}_{\text{F}}$ satisfies the mass normalization condition, i.e.

$$\tilde{P}^{\text{sea}}_{\text{F}} \cdot \tilde{P}^{\text{sea}}_{\text{F}} = \tilde{P}^{\text{sea}}_{\text{F}} \,.$$

Proof. Using the relations

$$s^{\pm}_m = s_m \pm i\pi\, p_m \,, \tag{5.9}$$

we have

$$U_- \cdot p_m \cdot U_-^* = \frac{1}{2\pi i} \sum_{n,n'=0}^{\infty} (-s^-_m\, \mathcal{B})^n \left(s^+_m - s^-_m\right) (-\mathcal{B} s^+_m)^{n'} \,.$$

Using that the sums are telescopic, we obtain

$$U_+ \cdot p_m \cdot U_+^* = \frac{1}{2\pi i} \sum_{n'=0}^{\infty} s^+_m (-\mathcal{B} s^+_m)^{n'} - \sum_{n=0}^{\infty} (-s^-_m\, \mathcal{B})^n s^-_m = \frac{1}{2\pi i} \left(\tilde{s}^+_m - \tilde{s}^-_m\right) = \tilde{p}^{\text{res}}_m \,,$$

where in the last step we used (3.8) and (3.9) together with the replacement rules (3.45)–(3.48). This proves (5.7).

The mass normalization condition can be proved in two ways. One method is to verify it by explicit computation very similar as in the proof of Proposition 5.1 by using (5.9) together with the multiplication rules (3.17). Alternatively, one may relate the mass normalization of $P^{\text{sea}}_{\text{F}}$ directly to the statement of Proposition 5.1 by using the symmetry between the mass and the spatial normalizations shown in Section 3.4. Namely, comparing (3.10) with (5.9), one sees that the rules (3.44)–(3.47) give rise to the replacement

$$P^{\text{sea}}_{\wedge} \longrightarrow -P^{\text{sea}}_{\text{F}} \,. \tag{5.10}$$

Hence the spatial normalization of $P^{\text{sea}}_{\wedge}$ corresponds to the mass normalization of $P^{\text{sea}}_{\text{F}}$. ∎

Combining (5.1) (5.7) with the replacement rules (3.48) (5.10), one obtains

$$p^{\wedge}_m \longrightarrow k^{\text{F}}_m \,. \tag{5.11}$$

6. Causality of the Light-Cone Expansion

We now work out a few properties of the fermionic projector in position space. We generalize concepts introduced in [7] and compare the results for the different perturbation expansions.

In general terms, each perturbation expansion expresses the fermionic projector as a sum of operator products of the form

$$P^{\text{sea}} = \sum_{k=0}^{\infty} \sum_{\alpha=0}^{\alpha_{\max}(k)} c_\alpha\, C_{1,\alpha}\, \mathcal{B}\, C_{2,\alpha}\, \mathcal{B} \,\cdots\, \mathcal{B}\, C_{k+1,\alpha} \,,$$

where the factors $C_{l,\alpha}$ are the Green's functions s_m or fundamental solutions p_m, k_m of the free Dirac equation, and the c_α are combinatorial factors. Provided that the potential $\mathcal{B}$ is smooth and has suitable decay properties at infinity, any such operator product is a well-defined tempered distribution on $M \times M$ (for details see [7, Lemma 1.1] or [8, Lemma 2.2.2]). For the following analysis, it is preferable to express p_m and k_m in terms of the distribution $P_\pm$, so that we have sums of operator products of the form

$$C_1\, \mathcal{B}\, C_2\, \mathcal{B} \cdots \mathcal{B}\, C_{k+1} \qquad \text{with} \qquad C_l \in \{s_m, P_+, P_-\} \,. \tag{6.1}$$

Definition 6.1. *An operator product of the form* (6.1) *is a* **low-energy contribution** *if the factors C_l are all in the set $\{s_m, P_+\}$ or are all in $\{s_m, P_-\}$. Conversely, if the factors C_l involve both P_+ and P_-, then the operator product is a* **high-energy contribution**.

Proposition 6.2. *Every high-energy contribution is a smooth function on $M \times M$.*

Proof. The proposition is obtained by a straightforward adaptation of [7, Proof of Theorem 3.4]. More precisely, following the arguments at the beginning of [7, Proof of Theorem 3.4], it suffices to consider an operator product of the form

$$P_+ \,\mathcal{B}\, C_{n-1} \,\mathcal{B} \cdots \mathcal{B}\, C_1 \,\mathcal{B}\, P_- \,.$$

Now one can proceed inductively as explained after [7, eq. (3.28)]. □

Hence the singularities of the fermionic projector in position space are determined exclusively by the low-energy contributions. The singularity structure is described efficiently by the light-cone expansion (for details see [6, 7] or [8, §2.5]).

Definition 6.3. *A distribution $A(x,y)$ on $M \times M$ is of the order $\mathcal{O}((y-x)^{2p})$, $p \in \mathbb{Z}$, if the product*

$$(y-x)^{-2p}\, A(x,y)$$

is a regular distribution (i.e. a locally integrable function). An expansion of the form

$$A(x,y) = \sum_{j=g}^{\infty} A^{[j]}(x,y) \tag{6.2}$$

with $g \in \mathbb{Z}$ is called **light-cone expansion** *if if the distributions $A^{[j]}(x,y)$ are of the order $\mathcal{O}((y-x)^{2j})$, and if A is approximated by the partial sums in the sense that for all $p \geq g$,*

$$A(x,y) - \sum_{j=g}^{p} A^{[j]}(x,y) \qquad \textit{is of the order } \mathcal{O}((y-x)^{2p+2})\,. \tag{6.3}$$

The parameter g gives the leading order of the singularity of $A(x,y)$ on the light cone. We point out that we do not demand that the infinite series in (6.2) converges. Thus, similar to a formal Taylor series, the series in (6.2) is defined via the approximation by the partial sums (6.3).

The following theorem makes a general statement on the structure of the light-cone expansion of the fermionic projector.

Theorem 6.4. *Every contribution to the perturbation expansions of P^{sea}, $P^{\text{sea}}_{\text{res}}$, $P^{\text{sea}}_{\wedge}$ and $P^{\text{sea}}_{\text{F}}$ has a light-cone expansion of the form* (6.2). *These* **light-cone expansions** *are* **causal** *in the following sense:*

(i) *Every $A^{[j]}$ is smooth away from the light cone,*

$$A^{[j]} \in C^\infty\big(\{(x,y) \in M \times M \,|\, (x-y)^2 \neq 0\}\big)\,. \tag{6.4}$$

(ii) *Every $A^{[j]}$ can be decomposed into a singular and a smooth part,*

$$A^{[j]} = A^{[j]}_{\text{sing}} + A^{[j]}_{\text{reg}} \qquad \textit{with} \qquad A^{[j]}_{\text{reg}} \in C^\infty(M \times M)\,,$$

where the singular part $A^{[j]}_{\text{sing}}(x,y)$ only depends on $\mathcal{B}$ and its derivatives along the line segment

$$\overline{xy} = \{(1+\tau)\,x + \tau y \,|\, 0 \leq \tau \leq 1\}\,.$$

In order to explain this notion of "causality," we first point out that if x and y are causally separated, then the above line segment is inside the "causal diamond"

$$\left(J_x^\vee \cap J_y^\wedge\right) \cup \left(J_y^\vee \cap J_x^\wedge\right), \tag{6.5}$$

where $J_x^\vee$ and $J_x^\wedge$ denote the closed future and past light cones centered at x,

$$J_x^\vee = \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \geq 0\}$$
$$J_x^\wedge = \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \leq 0\}\,.$$

On the other hand, if x and y are space-like separated, then $A^{[j]}(x,y)$ is smooth according to (6.4). Thus the above theorem states that the singularities of the fermionic projector propagate on the light cone and depend causally on $\mathcal{B}$.

Proof of Theorem 6.4. The perturbation expansion for the advanced and retarded Green's functions, (3.8), is strictly causal in the sense that it depends on $\mathcal{B}$ only inside the causal diamond (6.5). By (3.9), the same is true for the causal fundamental solution $\tilde{k}_m$. The light-cone expansion of these distributions can be carried out most conveniently with an iterative construction which involves an expansion in the mass parameter (see [6, Lemma 3.1] and [7, Section 2]). The clue for getting the connection to the light-cone expansions of other operator products is to perform a suitable expansion in momentum space (as worked out in first order perturbation theory in [6, Section 3]). Then the so-called residual argument (cf. [7, Section 3.1]) shows that the distribution $\tilde{p}_m^{\text{res}}$ obtained from $\tilde{k}_m$ by the replacements (3.46) and (3.47) is also causal in the sense that it has the properties (i) and (ii) in the statement of the theorem. Our task is to show that all the other operators have the same light-cone expansions as either $\tilde{k}_m$ or $\tilde{p}_m^{\text{res}}$. In view of Proposition 6.2, it suffices to show that all the expansions

$$\tilde{k}_m^{\text{res}} - \tilde{k}_m\,, \qquad \tilde{k}_m^{\text{F}} - \tilde{k}_m$$
$$\tilde{p}_m - \tilde{p}_m^{\text{res}}\,, \qquad \tilde{p}_m^{\wedge} - \tilde{p}_m^{\text{res}}$$

only involve high-energy contribution in the sense of Definition 6.1.

We next show that the expansions $\tilde{k}_m^{\text{res}} - \tilde{k}_m$ and $\tilde{p}_m - \tilde{p}_m^{\text{res}}$ only involve high-energy contributions. To this end, we consider the contour representation of $\tilde{p}_m^{\text{res}}$ and $\tilde{k}_m^{\text{res}}$ (cf. (3.25) and (3.52)),

$$\tilde{p}_m^{\text{res}} = -\frac{1}{2\pi i}\left(\oint_{\Gamma_+} + \oint_{\Gamma_-}\right)\tilde{R}_\lambda\, d\lambda\,, \qquad \tilde{k}_m^{\text{res}} = -\frac{1}{2\pi i}\left(\oint_{\Gamma_+} - \oint_{\Gamma_-}\right)\tilde{R}_\lambda\, d\lambda\,. \tag{6.6}$$

In order to compute the low-energy contributions, we first substitute the Neumann series (3.21). Then we only take into account the contributions where either for all factors R_λ we consider the poles at $\lambda = 0, 1$ (giving the operator products involving P_+) or for all factors R_λ we consider the poles at $\lambda = 0, -1$ (giving the operator products involving P_-). In view of (6.6), the low-energy contribution of $\tilde{k}_m^{\text{res}}$ is obtained from that of $\tilde{p}_m^{\text{res}}$ by flipping the sign of those operator products which involve P_-. Next, since the operator $\tilde{p}_m^{\text{res}}$ is symmetric and every factor p_m or k_m comes with a factor i, we know that the above operator products all involve an odd number of factors P_-. Hence the low-energy contribution of $\tilde{k}_m^{\text{res}}$ is also obtained from that of $\tilde{p}_m^{\text{res}}$ by flipping the sign of each factor P_-. This shows that under the replacements $k_m \longleftrightarrow p_m$, the operator $\tilde{p}_m^{\text{res}}$ transforms to

$$\tilde{p}_m^{\text{res}} \longrightarrow \tilde{k}_m^{\text{res}} + (\text{high-energy contributions})\,.$$

The claim now follows because under the same replacements, we have the transformations (3.48) and (3.51).

It remains to consider the expansions $\tilde{k}^{\mathrm{F}}_m - \tilde{k}_m$ and $\tilde{p}^{\wedge}_m - \tilde{p}^{\mathrm{res}}_m$. These expansions are obtained from each other by applying the replacement rules (3.45)–(3.47) (cf. (3.48) and (5.11)). In view of this symmetry, it suffices to consider the expansion $\tilde{k}^{\mathrm{F}}_m - \tilde{k}_m$; the proof for $\tilde{p}^{\wedge}_m - \tilde{p}^{\mathrm{res}}_m$ is then immediately obtained by applying the replacements $p_m \longleftrightarrow k_m$. Using the definition (5.8) together with the explicit formulas (5.6), we can substitute (5.9) and multiply out to obtain operator products involving factors s_m and p_m as well as one factor k_m. More precisely,

$$\tilde{k}^{\mathrm{F}}_m = \sum_{\alpha,\beta=0}^{\infty} (i\pi)^\alpha \, (-i\pi)^\beta \; b^<_m \, (p_m b_m)^\alpha \, k_m \, (b_m p_m)^\beta \, b^>_m \;, \tag{6.7}$$

where the factors $b^\bullet_m$ are again given by (3.12). Computing modulo high-energy contributions, we may replace pairs of factors p_m by pairs of factors k_m. If $\alpha+\beta$ is odd, this can be done iteratively until we end up with operator products involving exactly one factor p_m, i.e. which are of the form

$$(i\pi)^{\alpha'} \, (-i\pi)^{\beta'} \; b^<_m \, (k_m b_m)^{\alpha'} p_m \, (b_m k_m)^{\beta'} \, b^>_m \qquad \text{with } \alpha'+\beta' \text{ odd} \,. \tag{6.8}$$

Moreover, again computing modulo high-energy contributions, we may exchange two factors p_m and k_m, which means in (6.8) that the factor p_m can be brought to an arbitrary position. Taking the adjoint of (6.8) and bringing the factor p_m back to the old position, we obtain minus (6.8). This shows that all terms with $\alpha+\beta$ odd cancel.

In the remaining case when $\alpha+\beta$ is even, we may replace all factors p_m in (6.7) by k_m. We thus obtain

$$\begin{aligned}\tilde{k}^{\mathrm{F}}_m = \sum_{\substack{\alpha,\beta=0 \\ \alpha+\beta \text{ even}}}^{\infty} & (i\pi)^{\alpha+\beta} \, (-1)^\beta \; b^<_m \, k_m \, (b_m k_m)^{\alpha+\beta} \, b^>_m \\ & \qquad + (\text{high-energy contributions}) \,.\end{aligned}$$

For fixed $2p := \alpha+\beta$, we need to sum over the combinations

$$(\alpha, \beta) = (0, 2p), (1, 2p-1), \ldots, (2p, 0) \,.$$

Of these $2p+1$ combinations, $p+1$ contribute with a plus sign, whereas p of them give a minus sign. Adding up, we obtain precisely the formula (3.11) for $\tilde{k}_m$. ☐

It is quite remarkable that all our perturbation expansions have causal light-cone expansions. It is not known whether there is a simple criterion to decide which operator products have a causal light-cone expansion. Instead, we finally give a simple example for a perturbation expansion that has a light-cone expansion which is *not causal* in the above sense. To this end, we consider the perturbation series

$$\tilde{P}^{\mathrm{sea}} = \sum_{\alpha,\beta=0}^{\infty} (-s^-_m \mathcal{B})^\alpha P_- (-\mathcal{B} s^-_m)^\beta \,. \tag{6.9}$$

This perturbation series differs from our expansion with the Feynman propagator (5.7) in that we are using the same propagator s^-_m on the left and on the right. As a consequence, the operator $\tilde{P}^{\mathrm{sea}}_{\mathrm{F}}$ is not symmetric. But if we are willing to give up symmetry, then the above series is another possible perturbation expansion for the fermionic projector. To first order in $\mathcal{B}$, we get the contribution

$$-s^-_m \mathcal{B} P_- - P_- \mathcal{B} s^-_m \,.$$

Using (5.9), this can be rewritten as

$$\big(- s_m \mathcal{B} P_- - P_- \mathcal{B} s_m \big) + i\pi \big(p_m \mathcal{B} P_- + P_- \mathcal{B} p_m^- \big) \,.$$

The terms in the first bracket coincides precisely with the first order perturbation of the fermionic projector with mass or spatial normalization. The terms in the second bracket, however, are a consequence of the specific form of the perturbation expansion (6.9). As worked out in detail in [8, Lemmas F.3 and F.4], its light-cone expansion involves unbounded line integrals which violate the property (ii) in Theorem 6.4.

7. Fermion Loops

As explained in [11], fermionic loop diagrams are obtained in the fermionic projector approach by considering the fermionic projector $P(x,y)$ for $x = y$ after subtracting suitable singular contributions (see [11, eq. (2.13)])

$$P(x,x) - (\text{singular contributions}) \,. \tag{7.1}$$

Here we do not enter the analysis of the singular contributions (for details see [9, Section 6 and 7]). Instead, we merely consider the contributions to $P(x,x)$ with a simple ultraviolet regularization. Our goal is to show that certain contributions to the perturbation expansion vanish by symmetry, in generalization of Furry's theorem in standard quantum field theory. We anticipate that this symmetry argument also applies to the singular contributions in (7.1) (independent of the detailed regularization method), implying that the corresponding diagrams do not contribute to the fermion loops (7.1).

7.1. A Generalized Furry Theorem.

Recall the definitions of the following Green's functions (see (5.5) and (3.5), (3.7)):

$$s_m^\pm(k) = \lim_{\varepsilon \searrow 0} \frac{\slashed{k} + m}{k^2 - m^2 \mp i\varepsilon} \tag{7.2}$$

$$s_m^\vee(k) = \lim_{\varepsilon \searrow 0} \frac{\slashed{k} + m}{k^2 - m^2 - i\varepsilon k^0} \tag{7.3}$$

$$s_m^\wedge(k) = \lim_{\varepsilon \searrow 0} \frac{\slashed{k} + m}{k^2 - m^2 + i\varepsilon k^0} \tag{7.4}$$

$$s_m(k) = \frac{1}{2} \big(s_m^\vee + s_m^\wedge \big)(k) = \frac{1}{2} \big(s_m^- + s_m^+ \big)(k) = (\slashed{k} + m)\, \frac{\text{PP}}{k^2 - m^2} \,. \tag{7.5}$$

Taking differences of these Green's functions, we obtain the fundamental solutions (cf. (3.4), (3.6) and (3.3), (5.9)),

$$p_m(k) = \frac{1}{2\pi i} \big(s_m^- - s_m^+ \big)(k) = (\slashed{k} + m)\, \delta(k^2 - m^2) \tag{7.6}$$

$$k_m(k) = \frac{1}{2\pi i} \big(s_m^\vee - s_m^\wedge \big)(k) = (\slashed{k} + m)\, \delta(k^2 - m^2)\, \epsilon(k^0) \,. \tag{7.7}$$

Thus we may decompose any Green's function in terms of s_m and a fundamental solution,

$$\begin{aligned} s_m^\pm &= s_m \pm i\pi\, p_m \\ s_m^\vee &= s_m + i\pi\, k_m \\ s_m^\wedge &= s_m - i\pi\, k_m \,. \end{aligned}$$

We consider products involving of the form

$$C_0(x_0,x_1)\,\mathcal{B}_1\,C_1(x_1,x_2)\,\cdots\,C_{n-1}(x_{n-1},x_n)\,\mathcal{B}_n\,C_n(x_n,x_0)\,,$$

where the factors $C_0,\ldots,C_n$ represent any of the above distributions, and the factors $\mathcal{B}_1,\ldots\mathcal{B}_n$ stand for an odd combination of Dirac matrices, i.e.

$$\mathcal{B}_k = \not{A}_k + \gamma^5 \not{B}_k\,.$$

In order for these products to be well-defined, one should consider the regularized distributions, in the simplest case by taking the above formulas for fixed $\varepsilon > 0$. All our arguments apply just as well to the regularized product. For ease in notation, we prefer to work with the unregularized distributions.

Proposition 7.1. *If the factors $C_0,\ldots,C_n$ are all the advanced Green's functions, then the operator product vanishes:*

$$s^\vee(x_0,x_1)\,\mathcal{B}_1\,s^\vee(x_1,x_2)\,\cdots\,s^\vee(x_{n-1},x_n)\,\mathcal{B}_n\,s^\vee(x_n,x_0) = 0\,.$$

The same holds if all the factors are retarded Green's functions.

Proof. The advanced Green's functions are non-zero only if x_1 lies in the future of x_0, x_2 lies in the future of x_1, …, and x_0 lies in the future of x_n. This is impossible. We note that the causality of the Green's function also holds with regularization, as one sees by taking the Fourier transform of $(k^2 - m^2 + i\varepsilon k^0)^{-1}$ with residues.

The argument for the retarded Green's functions is similar. □

The following result generalizes Furry's theorem in standard quantum field theory (see for example [2, p. 331ff]) to more general distributions than the Feynman propagator.

Theorem 7.2. *Suppose that the factors $C_0,\ldots,C_n$ are a selection of any of the distributions $s^\pm_m$, s_m, p_m or k_m. Then*

$$\begin{aligned}&\mathrm{Tr}\big(\mathcal{B}_0\,C_0(x_0,x_1)\,\mathcal{B}_1\,C_1(x_1,x_2)\,\cdots\,C_{n-1}(x_{n-1},x_n)\,\mathcal{B}_n\,C_n(x_n,x_0)\big)\\&\quad= (-1)^{n+k+1}\,\mathrm{Tr}\,\big(C_n(x_0,x_n)\,\mathcal{B}_n\,C_{n-1}(x_n,x_{n-1})\,\cdots\,C_1(x_2,x_1)\,\mathcal{B}_1\,C_0(x_1,x_0)\,\mathcal{B}_0\big)\,,\end{aligned}$$

where k denotes the number of factors k_m.

Proof. Exactly as explained in [2, p. 331], in traces of products of Dirac matrices we may transpose all matrices according to the transformation

$$\gamma \to -\gamma^{\mathrm{T}}\,, \qquad \gamma_{\alpha\beta} \to -\gamma_{\beta\alpha}\,. \tag{7.8}$$

From a more abstract point of view, this transformation can also be understood as follows. Obviously, the matrices γ^{T} again satisfy the anti-commutation relations. Hence they form a representation of the Clifford algebra. Since in dimension four all irreducible representations are equivalent, there is an invertible matrix S such that

$$\gamma = -S\gamma^{\mathrm{T}}S^{-1}\,.$$

Using this identity for all Dirac matrices, all factors S and S^{-1} cancel each other in the trace of operator products.

Applying the transformation (7.8), each factor $\mathcal{B}_k$ gives a minus sign. In the factors C_k we transform the Fourier integral to obtain

$$\begin{aligned} C(x,y) &= \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)\, f(k)\, e^{-ik(x-y)} \\ &\to \int \frac{d^4k}{(2\pi)^4}\, (-\not{k}+m)^{\mathrm{T}}\, f(k)\, e^{-ik(x-y)} \\ &= \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)^{\mathrm{T}}\, f(-k)\, e^{ik(x-y)} = \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)^{\mathrm{T}}\, f(-k)\, e^{-ik(y-x)}\,. \end{aligned}$$

Using the explicit formulas (7.2), (7.5), (7.6) and (7.7), one sees that the corresponding function f is even for $s^\pm_m$, s_m and p_m, but is odd for k_m. This gives the transformations

$$\begin{aligned} s^\pm_m(x,y) &\to s^\pm_m(y,x)^{\mathrm{T}}\,, & s_m(x,y) &\to s_m(y,x)^{\mathrm{T}}\,, \\ p_m(x,y) &\to p_m(y,x)^{\mathrm{T}}\,, & k_m(x,y) &\to -k_m(y,x)^{\mathrm{T}}\,. \end{aligned} \tag{7.9}$$

It follows that

$$\begin{aligned} &\mathrm{Tr}\big(\mathcal{B}_0\, C_0(x_0,x_1)\, \mathcal{B}_1\, C_1(x_1,x_2) \cdots C_{n-1}(x_{n-1},x_n)\, \mathcal{B}_n\, C_n(x_n,x_0)\big) \\ &\quad= \sum_{\substack{\alpha_0,\ldots,\alpha_n \\ \beta_0,\ldots,\beta_n}} (\mathcal{B}_0)_{\alpha_0\beta_0}\, C_0(x_0,x_1)_{\beta_0,\alpha_1}\, (\mathcal{B}_1)_{\alpha_1\beta_1}\, C_1(x_1,x_2)_{\beta_1\alpha_2} \\ &\qquad\qquad \cdots C_{n-1}(x_{n-1},x_n)_{\beta_{n-1}\alpha_n}\, (\mathcal{B}_n)_{\alpha_n\beta_n}\, C_n(x_n,x_0)_{\beta_n\alpha_0} \\ &\quad\overset{(*)}{=} (-1)^{n+k+1} \sum_{\substack{\alpha_0,\ldots,\alpha_n \\ \beta_0,\ldots,\beta_n}} (\mathcal{B}_0)_{\beta_0\alpha_0}\, C_0(x_1,x_0)_{\alpha_1,\beta_0}\, (\mathcal{B}_1)_{\beta_1\alpha_1}\, C_1(x_2,x_1)_{\alpha_2\beta_1} \\ &\qquad\qquad \cdots C_{n-1}(x_n,x_{n-1})_{\alpha_n\beta_{n-1}}\, (\mathcal{B}_n)_{\beta_n\alpha_n}\, C_n(x_0,x_n)_{\alpha_0\beta_n} \\ &\quad= (-1)^{n+k+1}\, \mathrm{Tr}\big(C_n(x_0,x_n)\, \mathcal{B}_n\, C_{n-1}(x_n,x_{n-1}) \cdots C_1(x_2,x_1)\, \mathcal{B}_1\, C_0(x_1,x_0)\, \mathcal{B}_0\big), \end{aligned}$$

where in $(*)$ we substituted (7.8) and (7.9). This concludes the proof. □

We now apply this theorem to perturbation expansions of the fermionic projector.

Corollary 7.3. *Consider the fermionic projector in the presence of an external potential $\mathcal{B}$ which is odd, i.e.*

$$\mathcal{B} = \not{A} + \gamma^5 \not{B}\,. \tag{7.10}$$

Let $\Delta P^{(n,k)}$ be the contribution to the perturbation expansion of the projector of a fixed order n which involves k factors k_m. Then for any any odd matrix $\mathcal{U} = \not{u} + \gamma^5 \not{v}$,

$$\mathrm{Tr}\big(\mathcal{U}\, \Delta P^{(n,k)}(x,x)\big) = 0 \qquad \text{if } n+k \text{ is even}\,.$$

Proof. Since the perturbation expansion is symmetric under transpositions of the factors, it follows that $\Delta P^{(n,k)}$ can be written as a sum of operator products of the form

$$C_0\, \mathcal{B}\, C_1 \cdots C_{n-1}\, \mathcal{B}\, C_n + C_n\, \mathcal{B}\, C_{n-1} \cdots C_1\, \mathcal{B}\, C_0\,.$$

We now apply Theorem 7.2 to obtain the result. □

7.2. **First Order Loop Diagrams.** We now compute the one-loop contribution in an external potential $\mathcal{B}$ and simplify the formulas with the help of Furry's theorem.

Proposition 7.4. *Consider the fermionic projector in the presence of an external potential $\mathcal{B}$ which is odd* (7.10)*. Then to first order in the external potential, the* *vectorial and axial one-loop contributions are the same for all considered perturbation expansions. More precisely, for any odd matrix $\mathcal{U} = \not{a} + \gamma^5 \not{b}$,*

$$\begin{aligned}\mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}_{\mathrm{res}}(x,x)) &= \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)) = \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}_{\wedge}(x,x)) = \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}_{\mathrm{F}}(x,x)) \\ &= -\frac{1}{2}\,\mathrm{Tr}\Big(\not{a}\,\big(p_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,p_m\big)(x,x)\Big).\end{aligned}$$

Proof. To first order in the external potential, we have

$$\begin{aligned}\Delta P^{\mathrm{sea}}_{\mathrm{res}} = \Delta P^{\mathrm{sea}} &= -s_m\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,s_m \\ \Delta P^{\mathrm{sea}}_{\wedge} &= -s_m^{\wedge}\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,s_m^{\vee} = -(s_m - i\pi\,k_m)\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,(s_m + i\pi\,k_m) \\ &= \Delta P^{\mathrm{sea}} + i\pi\,\big(k_m\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,k_m\big) \\ \Delta P^{\mathrm{sea}}_{\mathrm{F}} &= -s_m^{-}\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,s_m^{+} = -(s_m - i\pi\,p_m)\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,(s_m + i\pi\,p_m) \\ &= \Delta P^{\mathrm{sea}} + i\pi\,\big(p_m\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,p_m\big)\end{aligned}$$

(with P_- according to (3.2)). As a consequence, applying Corollary 7.3,

$$\begin{aligned}\mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}_{\mathrm{res}}(x,x)) &= \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)) = -\frac{1}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(p_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,p_m\big)(x,x)\Big) \\ \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}_{\wedge}(x,x)) &= \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)) + i\pi\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,k_m\big)(x,x)\Big) \\ &= \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)) - \frac{i\pi}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,k_m - k_m\,\mathcal{B}\,k_m\big)(x,x)\Big) \\ \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}_{\mathrm{F}}(x,x)) &= \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)) + i\pi\,\mathrm{Tr}\Big(\mathcal{U}\,\big(p_m\,\mathcal{B}\,P_- - P_-\,\mathcal{B}\,p_m\big)(x,x)\Big) \\ &= \mathrm{Tr}(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)) + \frac{i\pi}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(p_m\,\mathcal{B}\,p_m - p_m\,\mathcal{B}\,p_m\big)(x,x)\Big)\,.\end{aligned}$$

This gives the result. ☐

It is worth noting that this contribution differs from the first-order loop diagram in standard quantum field theory obtained using the Feynman propagator.

Lemma 7.5. *Under the assumptions of Proposition 7.4,*

$$\begin{aligned}\mathrm{Tr}(\mathcal{U}(s_m^-\,\mathcal{B}\,s_m^-)(x,x)) &= -2\pi i\,\mathrm{Tr}\Big(\mathcal{U}\,\Delta P^{\mathrm{sea}}(x,x)\Big) &(7.11)\\ &\quad - \pi^2\,\mathrm{Tr}\Big(\mathcal{U}\,\big(p_m\,\mathcal{B}\,p_m - k_m\,\mathcal{B}\,k_m\big)(x,x)\Big). &(7.12)\end{aligned}$$

Proof.

$$\begin{aligned}s_m^-\,\mathcal{B}\,s_m^- &= (s_m - i\pi\,p_m)\,\mathcal{B}\,(s_m - i\pi\,p_m) \\ &= s_m\,\mathcal{B}\,s_m - i\pi\,\big(p_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,p_m\big) - \pi^2\,p_m\,\mathcal{B}\,p_m \\ s_m^{\wedge}\,\mathcal{B}\,s_m^{\wedge} &= (s_m - i\pi\,k_m)\,\mathcal{B}\,(s_m - i\pi\,k_m) \\ &= s_m\,\mathcal{B}\,s_m - i\pi\big(k_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,k_m\big) - \pi^2\,k_m\,\mathcal{B}\,k_m\end{aligned}$$

Subtracting these identities, we obtain

$$s_m^-\,\mathcal{B}\,s_m^- - s_m^{\wedge}\,\mathcal{B}\,s_m^{\wedge} = -2\pi i\big(P_-\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,P_-\big) - \pi^2\,\big(p_m\,\mathcal{B}\,p_m - k_m\,\mathcal{B}\,k_m\big)\,.$$

Evaluating the operator products on the diagonal (x,x), the operator product involving $s_m^{\vee}$ vanishes according to Proposition 7.1. This gives the result. ☐

The term on the right of (7.11) was discussed by Dirac [3], Heisenberg [18] and computed by Uehling [19] in the static situation. The term on the left of (7.11) was first computed in [4]. We point out that the term (7.12) is a high-energy contribution, which clearly vanishes in the static situation.

7.3. **Second Order Loop Diagrams.** To second order, the contributions to the loop diagram depend on the considered perturbation expansion. Only for P^{sea} we get zero (in agreement with the usual loop computation using the Feynman propagator). In all the other perturbation expansions we get non-trivial high-energy contributions. More precisely, we have the following result.

Proposition 7.6. *Consider the fermionic projector in the presence of an external potential $\mathcal{B}$ which is odd (7.10). Then for the contribution of second order in the external potential and any odd matrix $\mathcal{U} = \not{a} + \gamma^5 \not{b}$, one obtains*

$$
\begin{aligned}
&\mathrm{Tr}(\mathcal{U}\,\Delta P^{\text{sea}}(x,x)) = 0 \\
&\mathrm{Tr}(\mathcal{U}\,\Delta P^{\text{sea}}_{\text{res}}(x,x)) = -\frac{\pi^2}{4}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m - p_m\,\mathcal{B}\,p_m\,\mathcal{B}\,k_m \\
&\qquad\qquad + p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m - k_m\,\mathcal{B}\,p_m\,\mathcal{B}\,p_m\big)(x,x)\Big) \\
&\mathrm{Tr}(\mathcal{U}\,\Delta P^{\text{sea}}_{\wedge}(x,x)) = -\frac{i\pi}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,p_m\,\mathcal{B}\,s_m + k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,p_m - p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m \\
&\qquad\qquad - p_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m + s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m - s_m\,\mathcal{B}\,p_m\,\mathcal{B}\,k_m\big)(x,x)\Big) \\
&\mathrm{Tr}(\mathcal{U}\,\Delta P^{\text{sea}}_{\text{F}}(x,x)) = -\frac{i\pi}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,p_m\,\mathcal{B}\,s_m + k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,p_m - p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m \\
&\qquad\qquad - p_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m + s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m - s_m\,\mathcal{B}\,p_m\,\mathcal{B}\,k_m\big)(x,x)\Big) \\
&\qquad\qquad -\frac{\pi^2}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m - k_m\,\mathcal{B}\,p_m\,\mathcal{B}\,p_m \\
&\qquad\qquad - p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m + p_m\,\mathcal{B}\,p_m\,\mathcal{B}\,k_m\big)(x,x)\Big).
\end{aligned}
$$

Proof. We begin with P^{sea}. The contribution of second order in the external potential is given by (cf. the formula for $\tilde{t}$ in [12, Appendix A]),

$$
\begin{aligned}
\Delta P^{\text{sea}} = &\Big(P_-\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,P_-\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,P_-\Big) + \frac{\pi^2}{2}\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \\
&+ \frac{\pi^2}{4}\Big(k_m\,\mathcal{B}\,p_m\,\mathcal{B}\,k_m - k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m - p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m - p_m\,\mathcal{B}\,p_m\,\mathcal{B}\,p_m\Big)\,.
\end{aligned}
$$

Applying Corollary 7.3, in the loops all contributions involving an even number of factors k_m vanish. Thus

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{U}\,\Delta P^{\text{sea}}(x,x)) = &-\frac{1}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m + s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m\big)(x,x)\Big) \\
&+ \frac{\pi^2}{2}\,\mathrm{Tr}\Big(\mathcal{U}\,\big(k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m\big)(x,x)\Big)\,. \qquad (7.13)
\end{aligned}
$$

On the other hand, Proposition 7.1 yields

$$\begin{aligned}
0 &= \big(s_m^\wedge \,\mathcal{B}\, s_m^\wedge \,\mathcal{B}\, s_m^\wedge - s_m^\vee \,\mathcal{B}\, s_m^\vee \,\mathcal{B}\, s_m^\vee\big)(x,x) \\
&= -2i\pi \big(k_m \,\mathcal{B}\, s_m \,\mathcal{B}\, s_m + s_m \,\mathcal{B}\, k_m \,\mathcal{B}\, s_m + s_m \,\mathcal{B}\, s_m \,\mathcal{B}\, k_m\big)(x,x) \\
&\qquad + 2i\pi^3 \big(k_m \,\mathcal{B}\, k_m \,\mathcal{B}\, k_m\big)(x,x)\,.
\end{aligned}$$

Using these relations, all the terms in (7.13) cancel.

The result for $\Delta P^{\text{sea}}_{\text{res}}$ follows similarly from the formula (see [12, Appendix A]),

$$\begin{aligned}
\Delta P^{\text{sea}}_{\text{res}} &= \Big(P_- \,\mathcal{B}\, s_m \,\mathcal{B}\, s_m + s_m \,\mathcal{B}\, P_- \,\mathcal{B}\, s_m + s_m \,\mathcal{B}\, s_m \,\mathcal{B}\, P_-\Big) - \frac{\pi^2}{2}\, p_m \,\mathcal{B}\, p_m \,\mathcal{B}\, p_m \\
&\quad + \frac{\pi^2}{4}\Big(k_m \,\mathcal{B}\, k_m \,\mathcal{B}\, k_m + p_m \,\mathcal{B}\, p_m \,\mathcal{B}\, k_m - p_m \,\mathcal{B}\, k_m \,\mathcal{B}\, p_m + k_m \,\mathcal{B}\, p_m \,\mathcal{B}\, p_m\Big)\,.
\end{aligned}$$

The computation for $\Delta P^{\text{sea}}_\wedge$ and $\Delta P^{\text{sea}}_{\text{F}}$ is analogous. □

Appendix A. The Leading Orders of the Perturbation Expansions

We now give explicit formulas up to third order of the perturbation series with spatial and mass normalization. All explicit computations were carried out with the help of the Mathematica package BasicCausal.m[1].

We first give the expansions of the fermionic projector with spatial and mass normalization. These formulas were first given in [12, Appendix A] (albeit without analyzing the spatial normalization), and we here restate them for the sake of completeness. The operators $\tilde{k}$ and $\tilde{p}$ in (3.18) and (3.50) have the expansions

$$\begin{aligned}
\tilde{k} &= k - s\mathcal{B}k - k\mathcal{B}s + k\mathcal{B}s\mathcal{B}s + s\mathcal{B}k\mathcal{B}s + s\mathcal{B}s\mathcal{B}k - \pi^2 k\mathcal{B}k\mathcal{B}k \\
&\quad - k\mathcal{B}s\mathcal{B}s\mathcal{B}s - s\mathcal{B}k\mathcal{B}s\mathcal{B}s - s\mathcal{B}s\mathcal{B}k\mathcal{B}s - s\mathcal{B}s\mathcal{B}s\mathcal{B}k \\
&\quad + \pi^2\Big(s\mathcal{B}k\mathcal{B}k\mathcal{B}k + k\mathcal{B}s\mathcal{B}k\mathcal{B}k + k\mathcal{B}k\mathcal{B}s\mathcal{B}k + k\mathcal{B}k\mathcal{B}k\mathcal{B}s\Big) + \mathcal{O}(\mathcal{B}^4) \\
\tilde{p} &= p - s\mathcal{B}p - p\mathcal{B}s + p\mathcal{B}s\mathcal{B}s + s\mathcal{B}p\mathcal{B}s + s\mathcal{B}s\mathcal{B}p \\
&\quad + \frac{\pi^2}{2}\Big(-p\mathcal{B}k\mathcal{B}k + k\mathcal{B}p\mathcal{B}k - k\mathcal{B}k\mathcal{B}p - p\mathcal{B}p\mathcal{B}p\Big) \\
&\quad - p\mathcal{B}s\mathcal{B}s\mathcal{B}s - s\mathcal{B}p\mathcal{B}s\mathcal{B}s - s\mathcal{B}s\mathcal{B}p\mathcal{B}s - s\mathcal{B}s\mathcal{B}s\mathcal{B}p \\
&\quad + \frac{\pi^2}{2}\Big(s\mathcal{B}p\mathcal{B}p\mathcal{B}p + p\mathcal{B}s\mathcal{B}p\mathcal{B}p + p\mathcal{B}p\mathcal{B}s\mathcal{B}p + p\mathcal{B}p\mathcal{B}p\mathcal{B}s \\
&\qquad + p\mathcal{B}s\mathcal{B}k\mathcal{B}k - s\mathcal{B}k\mathcal{B}p\mathcal{B}k + s\mathcal{B}k\mathcal{B}k\mathcal{B}p + k\mathcal{B}s\mathcal{B}k\mathcal{B}p - k\mathcal{B}p\mathcal{B}k\mathcal{B}s \\
&\qquad + p\mathcal{B}k\mathcal{B}s\mathcal{B}k + p\mathcal{B}k\mathcal{B}k\mathcal{B}s - k\mathcal{B}p\mathcal{B}s\mathcal{B}k + k\mathcal{B}k\mathcal{B}p\mathcal{B}s + s\mathcal{B}p\mathcal{B}k\mathcal{B}k \\
&\qquad - k\mathcal{B}s\mathcal{B}p\mathcal{B}k + k\mathcal{B}k\mathcal{B}s\mathcal{B}p\Big) + \mathcal{O}(\mathcal{B}^4)\,.
\end{aligned}$$

The operators $\tilde{p}^{\,\text{res}}$ and $\tilde{k}^{\,\text{res}}$ in (3.49) and (3.52) are obtained by the replacements (3.45)–(3.47), (3.48) and (3.51). The fermionic projectors with spatial normalization and mass normalization are given by (3.53).

We come to the computation of the unitary perturbation flow. Before stating our results, we point out to a complication when computing products in (4.5): The

[1] This package is available as an ancillary file on the arXiv.

factors $\tilde{p}^{\text{res}}$ in (4.5) involve different external potentials. As a consequence, instead of (3.15) we need the more general computation rule

$$k_m\, b^>_m[\mathcal{B}]\, b^<_{m'}[\tilde{B}]\, k_{m'} = \delta(m-m')\Big(p_m + \pi^2\, k_m\, b_m[\mathcal{B}]\, p_m\, b_m[\tilde{B}]\, k_m\Big) \tag{A.1}$$

$$+ \frac{\text{PP}}{m-m'}\, k_m\, \big(1 - b^>_m[\mathcal{B}]\big)\, (\mathcal{B} - \tilde{\mathcal{B}})\, \big(1 - b^<_{m'}[\tilde{B}]\big)\, k_{m'}\,, \tag{A.2}$$

where the square brackets clarify the dependence on the external potential and "PP" denotes the principal value (this rule is obtained by a straightforward computation using the multiplication rules in [12, Lemma 2.1]). The summand (A.2) exhibits the fact that the solution space for the potential $\mathcal{B}$ and mass m is in general not orthogonal to the solution space for another potential $\mathcal{B}'$ and a different mass $m' \neq m$. In the usual description of adiabatic processes in Hilbert spaces, the corresponding contributions to a product of the form (4.5) decay like $1/N$ and thus vanish in the limit $N \to \infty$ (this is precisely the reason why (4.5) is unitary). In our perturbative description, the terms $(\mathcal{B} - \tilde{\mathcal{B}})$ in (A.2) also gives the desired factor $1/N$. However, since we are here working in an indefinite inner product space, proving that the summand (A.2) vanishes in (4.5) in the limit $N \to \infty$ requires methods which we do not want to enter here (more precisely, one would have to use "completeness relations" obtained by integrating the mass parameter on a contour C_ε as introduced in [12, Section 5]). Instead, we simply make the summand (A.2) vanish by working in addition to (3.17) with the multiplication rule

$$s \cdot s = \pi^2\, p \tag{A.3}$$

(which corresponds to and harmonizes with (3.41) if a spatial normalization is used). Then the unitary perturbation flow with mass normalization (cf. (4.5)) can be computed in a straightforward manner. One obtains the expansion

$$\begin{aligned}
U_{\text{res}} &= p - s\mathcal{B}p + s\mathcal{B}s\mathcal{B}p - \frac{\pi^2}{4}\Big(k\mathcal{B}k\mathcal{B}p - p\mathcal{B}k\mathcal{B}k + 2\,p\mathcal{B}p\mathcal{B}p\Big) \\
&\quad + \frac{\pi^2}{12}\Big(k\mathcal{B}p\mathcal{B}s\mathcal{B}k - k\mathcal{B}s\mathcal{B}p\mathcal{B}k + 7\,p\mathcal{B}p\mathcal{B}s\mathcal{B}p + 5\,p\mathcal{B}s\mathcal{B}p\mathcal{B}p + 6\,s\mathcal{B}p\mathcal{B}p\mathcal{B}p\Big) \\
&\quad + \frac{\pi^2}{4}\Big(k\mathcal{B}k\mathcal{B}s\mathcal{B}p + k\mathcal{B}s\mathcal{B}k\mathcal{B}p + s\mathcal{B}k\mathcal{B}k\mathcal{B}p\Big) \\
&\quad - \frac{\pi^2}{4}\Big(p\mathcal{B}k\mathcal{B}s\mathcal{B}k + p\mathcal{B}s\mathcal{B}k\mathcal{B}k + s\mathcal{B}p\mathcal{B}k\mathcal{B}k\Big) + \mathcal{O}(\mathcal{B}^4)\,.
\end{aligned}$$

As is verified by a straightforward computation, the formulas of Proposition 4.2 all hold, giving an a-posteriori justification of the rule (A.3). The perturbation flow with spatial normalization U (see Proposition 4.3) is obtained by applying the replacement rules (3.45)–(3.47).

We finally derive (4.13). For a closed loop $(\mathcal{B}(\tau))_{0 \leq \tau \leq 1}$ with $\mathcal{B}(0) = \mathcal{B}(1) = 0$, the Dyson series (4.7) simplifies with the help of the rules (3.17) and the fact that $\tilde{p}^{\text{res}}(0) = p = \tilde{p}^{\text{res}}(1)$ to

$$U^*_{\text{res}}(1) = p + p \cdot \int_0^1 ds_1 \int_0^{s_1} ds_2\, (\tilde{p}^{\text{res}})'(s_2) \cdot (\tilde{p}^{\text{res}})'(s_1) + \mathcal{O}(\mathcal{B}^3)\,.$$

Now we insert the first order expansion

$$(\tilde{p}^{\text{res}})'(\tau) = -s\,\mathcal{B}'(\tau)\,p - p\,\mathcal{B}'(\tau)\,s + \mathcal{O}(\mathcal{B}^2)\,.$$

Again using the rules (3.17) as well as (A.3), we obtain

$$\begin{aligned} U^*_{\text{res}}(1) &= p + p \cdot \int_0^1 ds_1 \int_0^{s_1} ds_2 \, \big(p \, \mathcal{B}'(s_2) \, s \big) \cdot \big(s \, \mathcal{B}'(s_1) \, p \big) + \mathcal{O}(\mathcal{B}^3) \\ &= p + \pi^2 \int_0^1 ds_1 \int_0^{s_1} ds_2 \, p \, \mathcal{B}'(s_2) \, p \, \mathcal{B}'(s_1) \, p + \mathcal{O}(\mathcal{B}^3) \, . \end{aligned}$$

Carrying out the integral over s_2 and using that $\mathcal{B}(0) = 0$, we obtain (4.13).

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References

[1] M.V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. Roy. Soc. London Ser. A **392** (1984), no. 1802, 45–57.

[2] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, Russian ed., John Wiley & Sons, New York-Chichester-Brisbane, 1980, Translation edited by Seweryn Chomet, A Wiley-Interscience Publication.

[3] P.A.M. Dirac, *Discussion of the infinite distribution of electrons in the theory of the positron*, Proc. Camb. Philos. Soc. **30** (1934), 150–163.

[4] R.P. Feynman, *Space-time approach to quantum electrodynamics*, Physical Rev. (2) **76** (1949), 769–789.

[5] F. Finster, *Definition of the Dirac sea in the presence of external fields*, arXiv:hep-th/9705006, Adv. Theor. Math. Phys. **2** (1998), no. 5, 963–985.

[6] ______, *Light-cone expansion of the Dirac sea to first order in the external potential*, arXiv:hep-th/9707128, Michigan Math. J. **46** (1999), no. 2, 377–408.

[7] ______, *Light-cone expansion of the Dirac sea in the presence of chiral and scalar potentials*, arXiv:hep-th/9809019, J. Math. Phys. **41** (2000), no. 10, 6689–6746.

[8] ______, *The Principle of the Fermionic Projector*, hep-th/0001048, hep-th/0202059, hep-th/0210121, AMS/IP Studies in Advanced Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2006.

[9] ______, *An action principle for an interacting fermion system and its analysis in the continuum limit*, arXiv:0908.1542 [math-ph] (2009).

[10] ______, *A formulation of quantum field theory realizing a sea of interacting Dirac particles*, arXiv:0911.2102 [hep-th], Lett. Math. Phys. **97** (2011), no. 2, 165–183.

[11] ______, *Perturbative quantum field theory in the framework of the fermionic projector*, arXiv:1310.4121 [math-ph] (2013).

[12] F. Finster and A. Grotz, *The causal perturbation expansion revisited: Rescaling the interacting Dirac sea*, arXiv:0901.0334 [math-ph], J. Math. Phys. **51** (2010), 072301.

[13] F. Finster, A. Grotz, and D. Schiefeneder, *Causal fermion systems: A quantum space-time emerging from an action principle*, arXiv:1102.2585 [math-ph], Quantum Field Theory and Gravity (F. Finster, O. Müller, M. Nardmann, J. Tolksdorf, and E. Zeidler, eds.), Birkhäuser Verlag, Basel, 2012, pp. 157–182.

[14] F. Finster, J. Kleiner, and J.-H. Treude, *An introduction to the fermionic projector and causal fermion systems*, in preparation.

[15] F. Finster and M. Reintjes, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime*, arXiv:1301.5420 [math-ph] (2013).

[16] ______, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II – Space-times of infinite lifetime*, arXiv:1312.7209 [math-ph] (2013).

[17] F. Finster et al, *The fermionic projector in an external potential: Non-perturbative construction and the Hadamard property*, in preparation.

[18] W. Heisenberg, *Bemerkungen zur Diracschen Theorie des Positrons*, Z. Phys. **90** (1934), 209–231.

[19] E.A. Uehling, *Polarization effects in the positron theory*, Phys. Rev. **48** (1935), 55–63.

Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: finster@ur.de

Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany
E-mail address: Juergen.Tolksdorf@mis.mpg.de