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Spinors on singular spaces and the topology of causal fermion systems

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SPINORS ON SINGULAR SPACES AND THE TOPOLOGY OF CAUSAL FERMION SYSTEMS

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Abstract. We propose causal fermion systems and Riemannian fermion systems as a framework for describing spinors on singular spaces. The underlying topological structures are introduced and analyzed. The connection to the spin condition in differential topology is worked out. The constructions are illustrated by many simple examples like the Euclidean plane, the two-dimensional Minkowski space, a conical singularity, a lattice system as well as the curvature singularity of the Schwarzschild space-time.

Contents

1. Introduction

Causal fermion systems arise in the context of relativistic quantum theory (see the survey articles [10, 14] and the references therein). From the mathematical point of view, they provide a framework for describing generalized space-times (so-called "quantum space-times") which do not need to have the structure of a Lorentzian manifold. Nevertheless, many structures of Lorentzian spin geometry (like causality, a distinguished time direction, spinors, connection and curvature) have a generalized meaning in these space-times (cf. [12] or again the survey article [14]). The present work is the first paper in which the topology of causal fermion systems is analyzed. Moreover, we extend the framework to the Riemannian setting by introducing so-called *topological fermion systems*. We thus obtain a general setting for describing spinors on singular spaces.

A central idea behind topological fermion systems is to encode the geometry of space (or space-time) in a collection of linear operators on a Hilbert space $\mathcal{H}$. In the smooth setting in which space (or space-time) is a differentiable manifold, one chooses $\mathcal{H}$ to be the span of certain spinorial wave functions on the manifold, typically formed of solutions of the Dirac equation. The spatial (or space-time) dependence of the wave functions is then encoded in the so-called *local correlation operators*, which are bounded linear operators on $\mathcal{H}$. Identifying the points of the manifold with the corresponding local correlation operators, we describe space (or space-time) by a subset of $\mathrm{L}(\mathcal{H})$. Finally, taking the push-forward of the volume measure gives rise to a measure on $\mathrm{L}(\mathcal{H})$, the so-called *universal measure*. This leads to the general setting of topological fermion systems that will be introduced in Definition 2.1 below.

Topological fermion systems also allow for the description of non-smooth geometries in which the underlying space (or space-time) is not a differentiable manifold. This can be understood from the fact that the solutions of a partial differential equation often have better regularity than the coefficients of the PDE they solve. As a consequence, in many situations the wave functions, which are solutions of a geometric PDE, remain continuous even when the metric or curvature develop singularities. Since we derive all relevant objects (like the spinor bundle and Clifford structures) from the local correlation operators, our framework remains well-defined even in such singular situations. Moreover, we can describe discrete spaces (like lattices) or other highly singular spaces, and our methods endow such spaces with non-trivial topological data.

Our framework is very flexible because there is a lot of freedom in choosing the wave functions in $\mathcal{H}$. This has the advantage that one can describe many different geometric situations by tailoring $\mathcal{H}$ in regard to the specific application. It is a main purpose of the present paper to explain how this can be done in different examples. We remark that the framework becomes much more rigid if one assumes that the configurations of wave functions are minimizers of causal variational principles (see [9]) or corresponding Riemannian analogs. The analysis of such variational principles is a separate subject which we cannot enter here. Instead, we refer the interested reader to [19, 2, 13] and the references therein.

The paper is organized as follows. In Section 2 we give the general definition of topological fermion systems and explain in simple examples how such systems can be constructed. In Section 3 we introduce the basic structures inherent to topological fermion systems, starting from the most general singular situation and then specializing in several steps until we end up in the smooth setting. In Section 4 we define so-called topological spinor bundles on a topological manifold and work out the connection to the structures on a usual spin manifold. In Section 5 we address the question of whether a causal fermion system determines a distinguished Clifford structure. In Section 6 we present methods for getting topological information on fermion systems for which the underlying space (or space-time) does not even have the structure of a topological manifold. In Section 7 we illustrate our constructions by the examples of the Euclidean plane and two-dimensional Minkowski space. Section 8 is devoted to examples for spinors on singular spaces: In Section 8.1 we consider singularities of the curvature tensor which can be removed by a conformal transformation. In this case, a rescaling of the spinorial wave functions makes it possible to eliminate the singularity. Section 8.2 treats curvature singularities which cannot be removed by a conformal transformation. In Section 8.3 we describe the curvature singularity of the Schwarzschild black hole. Finally, in Section 8.4 we illustrate the topology of singular spaces in the example of a two-dimensional lattice.

We finally point out that all our constructions are meant to be *topological but not differential geometric* in the following sense: Starting from a Lorentzian manifold, getting into the framework of causal fermion systems makes it necessary to introduce an ultraviolet regularization (for details see [18, Section 4]). This means that the system must be "smeared out" on the microscopic scale. As a consequence, the macroscopic geometry of space-time can be seen only on scales which are larger than the regularization length ε. This subtle point is taken care of in the constructions in [12] by working with the notions of "generically time-like separation" and "spin-connectable space-time points." Moreover, the spin connection in [12] gives a parallel transport along a discrete "chain" of points, and the correspondence to the spinorial Levi-Civita connection is obtained by first taking the limit $\varepsilon \searrow 0$ and then letting the number of points of the chain tend to infinity (see [12, Theorem 5.12]). Similarly, the Euclidean sign operator (which will be introduced after (4.1) below) depends essentially on the regularization, so that in the constructions in [12] it is handled with care. On the other hand, the ultraviolet regularization can be regarded as a continuous deformation of the geometry for small distances, having no influence on the topology. With this in mind, we here take the point of view that for analyzing topological questions, one can make use of the local behavior of the causal fermion system in an arbitrarily small neighborhood of a given space-time point. This leads to different types of constructions which we will explore here. In this way, the topological constructions given in this paper complement the differential geometric constructions in [12] and give a different viewpoint on causal fermion systems.

2. Basic Definitions and Simple Examples

Causal fermion system were first introduced in [14]. Here we give a slightly more general definition and explain it afterwards in a few examples.

Definition 2.1. Given a complex Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ (the *"particle space"*) and parameters $\mathfrak{p}, \mathfrak{q} \in \mathbb{N}_0$ with $\mathfrak{p} \leq \mathfrak{q}$, we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$ of finite rank, which (counting with multiplicities) have at most $\mathfrak{p}$ positive and at most $\mathfrak{q}$ negative eigenvalues. On $\mathcal{F}$ we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$), the so-called *universal measure*. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *topological fermion system* of spin signature $(\mathfrak{p}, \mathfrak{q})$.

In the case $\mathfrak{p} = \mathfrak{q}$, we call $(\mathcal{H}, \mathcal{F}, \rho)$ a *causal fermion system* of spin dimension $n := \mathfrak{p}$. If $\mathfrak{p} = 0$, we call $(\mathcal{H}, \mathcal{F}, \rho)$ a *Riemannian fermion system* of spin dimension $n := \mathfrak{q}$.

It should be noted that the assumption $\mathfrak{p} \leq \mathfrak{q}$ merely is a convention, because otherwise one may replace $\mathcal{F}$ by $-\mathcal{F}$.

A basic feature of topological fermion systems is that the geometry and topology are encoded in terms of linear operators on a Hilbert space. The support of the universal measure ρ, defined by

$$\operatorname{supp} \rho = \{x \in \mathcal{F} \mid \rho(U) > 0 \text{ for every open neighborhood } U \text{ of } x\} \subset \mathcal{F}\,,$$

takes the role of the base space, usually referred to as "space" or "space-time." This concept is illustrated by the following examples.

Example 2.2. *(Dirac spheres)*

(i) We choose $\mathcal{H} = \mathbb{C}^2$ with the canonical scalar product. Moreover, let $\hat{M} = S^2 \subset \mathbb{R}^3$ and $d\mu$ the Lebesgue measure on $\hat{M}$. Consider the mapping

$$F : \hat{M} \to \mathrm{L}(\mathcal{H})\,, \qquad F(p) = 2 \sum_{\alpha=1}^{3} p^\alpha \sigma^\alpha + \mathbb{1}\,, \tag{2.1}$$

where σ^α are the three Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

For any $p \in S^2$, the relations

$$\operatorname{tr}\big(F(p)\big) = 2\,, \qquad \operatorname{tr}\big(F(p)^2\big) = 10$$

show that the eigenvalues of $F(p)$ are equal to 1 ± 2. Hence one eigenvalue is positive and one eigenvalue is negative, so that $F(p) \in \mathcal{F}$ if we chose $\mathfrak{p} = \mathfrak{q} = 1$. We introduce the universal measure as the push-forward measure $\rho = F_* \mu$ (i.e. $\rho(\Omega) := \mu(F^{-1}(\Omega))$). Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension one. The support of ρ is homeomorphic to S^2. We refer to this example as a *Dirac sphere*.

(ii) We again choose $\mathcal{H} = \mathbb{C}^2$ with the canonical scalar product. Taking two different parameters $\tau_\pm > 1$, we introduce the mappings

$$F^\pm : \hat{M} \to \mathrm{L}(\mathcal{H})\,, \qquad F(p) = \tau_\pm \sum_{\alpha=1}^{3} p^\alpha \sigma^\alpha + \mathbb{1}\,,$$

and define the universal measure as the sum of the corresponding push-forward measures,

$$\rho = F^+_* \mu + F^-_* \mu\,. \tag{2.3}$$

Then $(\mathcal{H}, \mathcal{F}, \rho)$ is again a causal fermion system of spin dimension one. The support of ρ is homeomorphic to the disjoint union of two spheres.

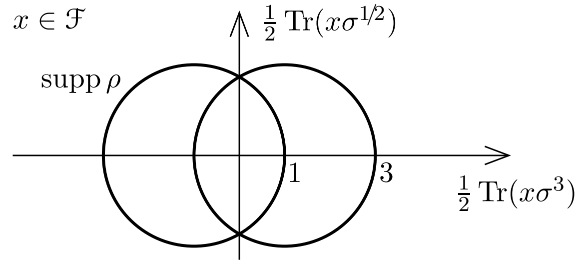

FIGURE 1. Two intersecting Dirac spheres.

(iii) We consider the mappings

$$F^{\pm} : \hat{M} \to \mathrm{L}(\mathcal{H}) \,, \qquad F^{\pm}(p) = 2 \sum_{\alpha=1}^{3} p^\alpha \sigma^\alpha + \mathbb{1} \pm \sigma^3$$

and introduce the universal measure again as the sum of the corresponding push-forward measures (2.3). Then $(\mathcal{H}, \mathcal{F}, \rho)$ is again a causal fermion system of spin dimension one. The support of ρ is homeomorphic to two spheres glued together along circles of latitude (see Figure 1). We refer to this example as *two intersecting Dirac spheres.* ◊

As already becomes clear in these simple examples, there are usually many ways to realize a topological space as the support of a universal measure. The reason is that a topological fermion system encodes more structures than just the topology, so that prescribing only the topology leaves a lot of freedom to modify all the additional structures.

A particular structure on a topological fermion system is the particle space $\mathcal{H}$. The vectors in $\mathcal{H}$ have the interpretation as the wave functions corresponding to the quantum particles of the physical system (this is also the reason for the name "particle space"). More generally, a basic underlying concept is to encode the geometry and topology in a certain family of functions defined on space or in space-time. This is illustrated in the next examples.

Example 2.3. *(Scalar and vector fields on a closed Riemannian manifold)*

(i) Let $(\hat{M}, g)$ be a smooth compact Riemannian manifold without boundary and Δ the Laplace-Beltrami operator acting on complex-valued scalar functions on $\hat{M}$. Then the operator $-\Delta$ with domain $C^\infty(\hat{M})$ is an essentially self-adjoint operator on $L^2(\hat{M})$. It has a purely discrete spectrum lying on the positive real axis. For a given parameter $L > 0$ we let $\mathcal{H}$ be the span of all eigenfunctions corresponding to eigenvalues $\leq L$,

$$\mathcal{H} = \mathrm{rg}\, \chi_{[0,L]}(-\Delta) \subset L^2(\hat{M}) \,.$$

Then $\mathcal{H}$ is a finite-dimensional Hilbert space which, by elliptic regularity theory, consists of smooth functions. Hence, for every $p \in \hat{M}$ the bilinear form $(\psi, \phi) \mapsto -\overline{\psi(p)}\phi(p)$ is well-defined and continuous on $\mathcal{H} \times \mathcal{H}$. By the Fréchet-Riesz theorem, there is a unique linear operator $F(p)$ with the property that

$$-\, \overline{\psi(p)}\phi(p) = \langle \psi | F(p) \phi \rangle_{L^2(\hat{M})} \qquad \text{for all } \psi, \phi \in \mathcal{H} \,. \tag{2.4}$$

This operator has rank at most one and is negative semi-definite. Varying p, we thus obtain a mapping $F : \hat{M} \to \mathcal{F}$ if we choose $\mathfrak{p} = 0$ and $\mathfrak{q} = 1$. Finally, we define $\rho = F_* \mu$ as the push-forward measure of the volume measure (i.e. $\rho(\Omega) := \mu(F^{-1}(\Omega))$. Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a Riemannian fermion system of spin dimension one.

(ii) Let $(\hat{M}, g)$ be a smooth compact Riemannian manifold of dimension k and Δ the covariant Laplacian on smooth vector fields. Complexifying the vector fields and taking the L^2-scalar product

$$\langle u | v \rangle_{L^2} = \int_{\hat{M}} g_{jk} \, \overline{u^j} \, v^k \, d\mu_{\hat{M}} ,$$

the operator $-\Delta$ is essentially self-adjoint and has smooth eigenfunctions. We again set $\mathcal{H} = \mathrm{rg}\, \chi_{[0,L]}(-\Delta)$ and define the operator $F(p) \in \mathrm{L}(\mathcal{H})$ by

$$- g_{jk} \, \overline{u^j(p)} \, v^k(p) = \langle u | F(p) v \rangle_{L^2} \qquad \text{for all } u, v \in \mathcal{H} . \tag{2.5}$$

The operators $F(p)$ are negative semi-definite and have rank at most k. We again introduce the universal measure by $\rho = F_* \mu$. Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a Riemannian fermion system of spin dimension k. ♢

In all applications worked out at present, the functions on space or space-time are *spinors*. In order to get the connection to topological fermion systems, we let $(\hat{M}, g)$ be a spin manifold (Riemannian or Lorentzian) and denote the corresponding spinor bundle by $S\hat{M}$. Then the spinor space $S_p\hat{M}$ at any point $p \in \hat{M}$ is endowed with an inner product, which we denote by

$$\prec .|. \succ_p \,:\, S_p \times S_p \to \mathbb{C} \tag{2.6}$$

and refer to as the *spin scalar product*. Next, we choose $\mathcal{H} \subset \Gamma(\hat{M}, S\hat{M})$ as a subspace of the continuous sections on $\hat{M}$, together with a scalar product $\langle .|. \rangle_{\mathcal{H}}$ (the choice of the scalar product depends on the signature of the metric and the particular application being considered). Then for every $p \in \hat{M}$, we can express the scalar product at a point p in terms of the Hilbert space scalar product,

$$- \prec \psi | \phi \succ_p = \langle \psi | F(p) \phi \rangle_{\mathcal{H}} \qquad \text{for all } \psi, \phi \in \mathcal{H} . \tag{2.7}$$

According to the Riesz representation theorem, this determines a unique linear operator $F(p) \in \mathrm{L}(\mathcal{H})$.

Definition 2.4. The operator $F(p) \in \mathrm{L}(\mathcal{H})$ satisfying (2.7) is referred to as the *local correlation operator* at the point $p \in \hat{M}$.

By construction, the operator $F(p)$ has finite rank (indeed, its rank is at most the dimension of S_p), and its maximal number of positive and negative eigenvalues is determined by the signature of the spin scalar product. Therefore, we can regard $F(p)$ as an element of $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ (for a suitable choice of the spin signature). Varying p, we obtain a mapping $F : \hat{M} \to \mathcal{F}$. Introducing the universal measure as the push-forward of the volume measure $d\mu$ on $\hat{M}$, i.e.

$$\rho(\Omega) := \mu(F^{-1}(\Omega)) ,$$

we obtain a topological fermion system $(\mathcal{H}, \mathcal{F}, \rho)$. The concept behind taking the push-forward measure is that we want to identify the point $p \in \hat{M}$ with its local correlation operator $F(p) \in \mathcal{F}$. Likewise, the manifold $\hat{M}$ should be identified with the subset $F(\hat{M})$ of $\mathcal{F}$. With this in mind, we do not want to work with objects on $\hat{M}$, but instead with corresponding objects on $\mathcal{F}$. Apart from giving a different point of view, this procedure makes it possible to extend the notion of the manifold $\hat{M}$ as well as the objects thereon to a more general setting.

To avoid confusion, we remark that the above-mentioned identification of p with $F(p)$ clearly fails if the mapping F is not injective. For this reason, one usually chooses $\mathcal{H}$ in such a way that F becomes injective. In certain applications, however, it is indeed preferable to work with a mapping F which is not injective. In this case, all points of $\hat{M}$ with the same image are identified when forming the topological fermion system. We will come back to this point in Example 8.3 below.

The simplest setting in which the above construction of topological fermion systems using spinors can be made precise is to choose $\hat{M}$ as a closed Riemannian manifold:

Example 2.5. *(Spinors on a closed Riemannian manifold)* Let $(\hat{M}, g)$ be a compact Riemannian spin manifold of dimension $k \geq 1$. The spinor bundle $S\hat{M}$ is a vector bundle with fibre $S_p\hat{M} \simeq \mathbb{C}^n$ with $n = 2^{[k/2]}$ (see for example [31, 21]). The spin scalar product (2.6) is positive definite. On the smooth sections $\Gamma(S\hat{M})$ of the spinor bundle we can thus introduce the scalar product

$$\langle \psi | \phi \rangle = \int_{\hat{M}} \prec \psi | \phi \succ_p \, d\mu(p) ,$$

where $d\mu = \sqrt{\det g}\, d^k x$ is the volume measure on $\hat{M}$. Forming the completion gives the Hilbert space $L^2(\hat{M}, S\hat{M})$. The Dirac operator $\mathcal{D}$ with domain of definition $\Gamma(S\hat{M})$ is an essentially self-adjoint operator on $L^2(\hat{M}, S\hat{M})$. It has a purely discrete spectrum and finite-dimensional eigenspaces (for details see for example [22]). For a given parameter $L > 0$, we let $\mathcal{H}$ be the space spanned by all eigenvectors whose eigenvalues lie in the interval $[-L, 0]$,

$$\mathcal{H} = \operatorname{rg} \chi_{[-L,0]}(\mathcal{D}) \subset L^2(\hat{M}, S\hat{M}) .$$

Denoting the restriction of the L^2-scalar product to $\mathcal{H}$ by $\langle . | . \rangle_{\mathcal{H}}$, we obtain a finite-dimensional Hilbert space $(\mathcal{H}, \langle . | . \rangle_{\mathcal{H}})$. By elliptic regularity theory, the functions in $\mathcal{H}$ are all smooth.

For every $p \in \hat{M}$ we introduce the local correlation operator by (2.7). This operator is negative semi-definite and has rank at most n. Hence $F(p)$ is an element of $\mathcal{F}$ according to Definition 2.1 if we choose $\mathfrak{p} = 0$ and $\mathfrak{q} = n$. Varying p, we obtain a mapping $F : \hat{M} \to \mathcal{F}$. Finally, we define $\rho = F_* \mu$ as the push-forward measure of the volume measure. Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a Riemannian fermion system of spin dimension n.

This example can readily be extended to an infinite-dimensional particle space by choosing a function $f \in C^0(\mathbb{R})$ and by modifying the above construction to

$$\mathcal{H} = \operatorname{rg} f^2(\mathcal{D}) \subset L^2(\hat{M}, S\hat{M})$$
$$-\prec f(\mathcal{D})\psi | f(\mathcal{D})\phi \succ_p = \langle \psi | F(p) \phi \rangle_{\mathcal{H}} \qquad \text{for all } \psi, \phi \in \mathcal{H} .$$

If f has suitable decay properties at infinity, the operator $f(\mathcal{D})$ maps $\mathcal{H}$ to the continuous functions, so that $F(p) \in \mathrm{L}(\mathcal{H})$ is well-defined. We omit the details for brevity. ◊

Examples of causal fermion systems can be obtained similarly starting from a Lorentzian manifold (see [14, Section 1.1], [12, Section 4 and 5] or [18, Section 4]). In this case, the space $\mathcal{H}$ has the physical interpretation as describing all the occupied

Dirac quantum states of the system, including the so-called Dirac sea (for the connection to physics see [10]). The scalar product on $\mathcal{H}$ is typically deduced from the spatial integral over the Dirac current and is thus closely related to the probabilistic interpretation of the Dirac wave function. The fact that Dirac particles are fermions explains the name *fermion* system. The notion "*causal*" in a causal fermion system can be understood as follows: Taking the product of two operators $x, y \in \mathcal{F}$, we obtain an operator of rank at most $\mathfrak{p}+\mathfrak{q}$. This operator is in general no longer symmetric (because $(xy)^* = yx$, and thus xy is symmetric if and only if x and y commute). Nevertheless, we can consider its characteristic polynomial. We denote its non-trivial zeros (counted with algebraic multiplicities) by $\lambda^{xy}_1, \ldots, \lambda^{xy}_{\mathfrak{p}+\mathfrak{q}}$. For a Riemannian fermion system, these zeros are all real and non-negative. Namely, in this case the operator $-y$ is positive semi-definite, so that its square root $\sqrt{-y}$ is well-defined as a positive semi-definite operator. Using that the spectrum is invariant under cyclic permutations, it follows that

$$xy = -x\,\sqrt{-y}\,\sqrt{-y} \qquad \text{is isospectral to} \qquad \sqrt{-y}\,(-x)\,\sqrt{-y}\,, \tag{2.8}$$

and the last operator product is obviously positive semi-definite. In the case $\mathfrak{p} > 0$, the operator $\sqrt{-y}$ no longer exists as a symmetric operator, so that the argument (2.8) breaks down. It turns out that the λ^{xy}_j will in general be complex, giving rise to the following notion of causality:

Definition 2.6. *(causal structure)* Two points $x, y \in \mathcal{F}$ are called *timelike* separated if the non-trivial zeros $\lambda^{xy}_1, \ldots, \lambda^{xy}_{\mathfrak{p}+\mathfrak{q}}$ of the characteristic polynomial of the operator product xy are are all real. The points x and y are said to be *spacelike* separated if all the λ^{xy}_j are complex and have the same absolute value. In all other cases, the points x and y are said to be *lightlike* separated.

3. Topological Structures

In this section we work out the underlying topological structures. We begin with the most general structures and then specialize the setting in several steps. We first recall a few basic notions from [14]. Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a topological fermion system. On $\mathcal{F}$ we consider the topology induced by the operator norm

$$\|A\| := \sup \big\{ \|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1 \big\}\,.$$

The *base space* M (often referred to as "space" or "space-time") is defined as the support of the universal measure, $M := \operatorname{supp} \rho$. On M we consider the topology induced by $\mathcal{F}$. For every $x \in M$ we define the *spin space* S_x by $S_x = x(\mathcal{H})$; it is a subspace of $\mathcal{H}$ of dimension at most $\mathfrak{p}+\mathfrak{q}$. On S_x we introduce the *spin scalar product* $\prec .|. \succ_x$ by

$$\prec \psi | \phi \succ_x = -\langle \psi | x \phi \rangle_{\mathcal{H}} \qquad \text{for all } \psi, \phi \in S_x\,; \tag{3.1}$$

it is an indefinite inner product of signature (q_x, p_x) with $p_x \leq \mathfrak{p}$ and $q_x \leq \mathfrak{q}$ (the minus sign in (3.1) is needed in order to be consistent with the usual sign conventions for Dirac spinors in Minkowski space; for details see [14]).

3.1. A Sheaf.

The most general setting in which the topology of a topological fermion system of signature $(\mathfrak{p}, \mathfrak{q})$ can be encoded is a that of a sheaf S on M whose stalks $(S_x, \prec .|. \succ_x)$ are indefinite inner product spaces of signature (q_x, p_x). Although this setting is too general for most of our constructions (we will mainly work with topological vector bundles as will be described in Section 3.2 below), we briefly explain how to get the connection to sheaf theory.

For any $x \in M$, we denote the orthogonal projection in $\mathcal{H}$ to the spin space S_x by π_x,

$$\pi_x \,:\, \mathcal{H} \to S_x \,. \tag{3.2}$$

Projecting a given vector $u \in \mathcal{H}$ to the spin spaces gives the mapping

$$\psi_u \,:\, M \to \mathcal{H} \,, \quad x \mapsto \pi_x u \in S_x \,.$$

We refer to ψ_u as the *wave function of the occupied state* u. For any open subset $U \subset M$, we obtain a corresponding wave function by restriction,

$$\psi_u|_U \,:\, U \to \mathcal{H} \qquad \text{with} \qquad \psi(x) \in S_x \quad \text{for all } x \in U \,.$$

We denote the vector space of such wave functions on U by S_U. We let S be the mapping which to an open set U assigns the vector space S_U. Moreover, for an open subset $V \subset U$ we introduce the restriction map as the linear mapping

$$r^U_V \,:\, S_U \to S_V \,, \qquad \psi \mapsto \psi|_V \,.$$

Obviously, these mappings have the following properties:

(I) If U is the empty set, then $S_U = \{0\}$.
(II) The linear map r^U_U is the identity. If $W \subset V \subset U$, then $r^U_W = r^V_W r^U_V$.

This gives the structure of a *presheaf* of complex vector spaces over the topological space M (see [3] or [28, §I.1.2]). Introducing the corresponding *sheaf* by taking the direct limits of the vector spaces S_U (as outlined for example in [28, §I.1.2]) gives the following structure: We define S as the disjoint union of all spin spaces and π as the projection to the base point,

$$S := \dot{\bigcup}_{x \in M} S_x \qquad \text{and} \qquad \pi \,:\, S \to M \,, \;\; S_x \mapsto x \,.$$

Every $\psi \in S_U$ defines the subset $\cup_{x \in U} \psi(x) \subset S$. On S we introduce the topology generated by all these subsets. Then the triple (S, π, M) is a sheaf. The stalks $(S_x, \prec .|. \succ_x)$ are indefinite inner product spaces of signature (q_x, p_x).

Now the cohomology groups $H^r(M, S)$, $r \geq 0$, with coefficients in a sheaf (as defined for example in [28, §I.2.6]) give topological information on the topological fermion system. For example, globally defined continuous sections are naturally isomorphic to elements of the cohomology group $H^0(M, S)$ (see [28, Theorem I.2.6.2]),

$$\Gamma(S) \simeq H^0(M, S) \,.$$

3.2. **A Topological Vector Bundle.** We now show that under a certain regularity assumption (see Definition 3.1), topological fermion systems naturally give rise to topological vector bundles. In preparation, we briefly recall the definition of a topological vector bundle (see [34, 37]) and set up some notation. Let $\mathcal{B}$ and M be topological spaces and $\pi : \mathcal{B} \to M$ a continuous surjective map. Moreover, let Y be a complex vector space and $G \subset \mathrm{GL}(Y)$ a group acting on Y. Then $\mathcal{B}$ is a topological vector bundle with fibre Y and structure group G if every point $x \in M$ has an open neighborhood U

equipped with a homeomorphism $\phi_U : \pi^{-1}(U) \to U \times Y$, called a local trivialization or a bundle chart, such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times Y \\ & \searrow & \downarrow \\ & & U \end{array} \tag{3.3}$$

commutes, where the projection maps are π and the projection onto the first factor, respectively. Furthermore, on overlaps $U \cap V$, we have

$$\phi_U \circ \phi_V^{-1}\big|_{\{x\}\times Y} = g_{UV}(x) \,, \tag{3.4}$$

where $g_{UV} : U \cap V \to G$ is a continuous transition function.

Again setting $M = \operatorname{supp} \rho$, we want to construct a topological vector bundle having the spin space S_x as the fibre at the point $x \in M$. To this end, all the spin spaces must have the same dimension and signature, making it necessary to impose the following condition:

Definition 3.1. The topological fermion system is called *regular* if for all $x \in M$, the operator x has the maximal possible rank $\mathfrak{p} + \mathfrak{q}$.

Clearly, the topological fermion systems of Example 2.2 are all regular. The topological fermion systems in Examples 2.3 and 2.5 are regular if and only if for every $p \in M$, the vectors $\psi(p)$ with $\psi \in \mathcal{H}$ span the fibre at p. We note that most of our constructions can be extended to non-regular topological fermion systems by decomposing M into subsets on which x has fixed rank and a fixed number of positive and negative eigenvalues. For clarity, we postpone this decomposition to Section 6 and for now restrict attention to regular topological fermion systems.

We define $\mathcal{B}$ as the set of pairs

$$\mathcal{B} = \{(x, \psi) \mid x \in M,\ \psi \in S_x\}$$

and let π be the projection onto the first component. Moreover, we let $(Y, \prec.|.\succ)$ be an indefinite inner product space of signature $(\mathfrak{q}, \mathfrak{p})$, and choose $G = \mathrm{U}(\mathfrak{q}, \mathfrak{p})$ as the group of unitary transformations on Y. In order to construct the bundle charts, for any given $x \in M$ we choose a unitary mapping $\sigma : S_x \to Y$. By restricting the projection π_x, (3.2), to S_y, we obtain the mapping

$$\pi_x|_{S_y} \;:\; S_y \to S_x \,.$$

In order to compute its adjoint with respect to the spin scalar product (3.1), for $\psi \in S_x$ and $\phi \in S_y$ we make the computation

$$\begin{aligned} \prec\psi \,|\, \pi_x|_{S_y}\, \phi\succ_x &= -\langle \psi | x\phi\rangle_{\mathcal{H}} = -\langle x\psi|\phi\rangle_{\mathcal{H}} = -\langle \pi_y\, x\, \psi|\phi\rangle_{\mathcal{H}} - \langle y\, (y|_{S_y})^{-1}\, \pi_y\, x\, \psi|\phi\rangle_{\mathcal{H}} \\ &= -\big\langle (y|_{S_y})^{-1}\, \pi_y\, x\, \psi \,\big|\, y\phi\big\rangle_{\mathcal{H}} = \prec (y|_{S_y})^{-1}\, \pi_y\, x\, \psi \,|\, \phi\succ_y \,. \end{aligned}$$

Hence

$$\big(\pi_x|_{S_y}\big)^* = (y|_{S_y})^{-1}\, \pi_y\, x \,.$$

We now introduce the operator

$$T_{xy} = \big(\pi_x|_{S_y}\big)\big(\pi_x|_{S_y}\big)^* = \pi_x\, (y|_{S_y})^{-1}\, \pi_y\, x \;:\; S_x \to S_x \,.$$

By construction, this operator is symmetric and $T_{xx} = \mathbb{1}$. By continuity, there is a neighborhood U of x such that for all $y \in U$, the operator T_{xy} is invertible and has a unique square root ρ_{xy} (defined for example by the power series $\sqrt{T_{xy}} =$

$\sqrt{\mathbb{1} + (T_{xy} - \mathbb{1})} = \mathbb{1} + \frac{1}{2}(T_{xy} - \mathbb{1}) + \cdots)$. Then the mapping $U_{x,y} := \rho_{xy}^{-1} \pi_x|_{S_y} : S_y \to S_x$ is unitary and depends continuously on $y \in U$. We define the bundle chart ϕ_U by

$$\phi_U(y, v) = \big(y, (\sigma \circ U_{x,y})(v)\big) .$$

The commutativity of the diagram (3.3) is clear by construction. The transition functions g_{UV} in (3.4) are in G because we are working with unitary mappings of the fibres throughout. We choose the topology on $\mathcal{B}$ such that all the bundle charts are homeomorphisms.

Definition 3.2. *The topological bundle $\mathcal{B} \to M$ is referred to as the vector bundle associated to the regular topological fermion system $(\mathcal{H}, \mathcal{F}, \rho)$, or simply the* **associated vector bundle**.

3.3. **A Bundle over a Topological Manifold.** In many applications, M has a manifold structure. This motivates us to specialize our setting by assuming that M is a *topological manifold* of dimension $k \geq 1$ (with or without boundary). From the topological point of view, this is a major simplification which excludes many examples (like the intersecting spheres in Example 2.2 (iii)). The main benefit of this simplifying assumption is that one can choose local coordinates and work with partitions of unity. As is made precise in the following theorem, these properties ensure that *every* such bundle can be realized by a topological fermion system. The proof illustrates our concept of encoding the topology of the bundle in a suitable family of sections.

Theorem 3.3. *Let $X \to \hat{M}$ be a bundle over a k-dimensional topological manifold $\hat{M}$, whose fibres are isomorphic to an indefinite inner product space of signature $(\mathfrak{q}, \mathfrak{p})$. Then there is a regular topological fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of signature $(\mathfrak{p}, \mathfrak{q})$ such that the associated vector bundle (see Definition 3.2) is isomorphic to X. If $\hat{M}$ is compact, the particle space $\mathcal{H}$ can be chosen to be finite-dimensional.*

Proof. Let $\{(x_\alpha, U_\alpha)\}$ be an at most countable atlas of $\hat{M}$ such that the bundle has a trivialization on every U_α. Thus we can choose continuous sections $e_\alpha^1, \ldots, e_\alpha^{\mathfrak{p}+\mathfrak{q}}$ on U_α which in every point $p \in U_\alpha$ form a pseudo-orthonormal basis of the fibre, which we denote by $(S_p, \prec . | . \succ_p)$. Next, we let $(Z_i)_{i \in I}$ be an at most countable, locally finite covering of $\hat{M}$ by relatively compact open subsets, which is subordinate to the atlas $\{(x_\alpha, U_\alpha)\}$ (meaning that for every $i \in I$, there is $\alpha = \alpha(i)$ with $Z_i \subset U_{\alpha(i)}$). Starting from the sets $(Z_i)_{i \in I}$, we now want to construct non-empty open sets Ω_i, V_i and W_i with the following properties:

- (i) The V_i are relatively compact and $\overline{\Omega_i} \subset V_i \subset\subset W_i \subset U_{\alpha(i)}$ for all $i \in I$ (where $V \subset\subset W$ stands for $V \subset \overline{V} \subset W$).
- (ii) The family $(V_i)_{i \in I}$ is a locally finite covering of $\hat{M}$.
- (iii) $\overline{\Omega_i} \cap W_j = \varnothing$ for all $i \neq j$.

To this end, we proceed inductively in the index i. If the index set I is finite, we denote it by $I = \{1, \ldots, K\}$ with $K \in \mathbb{N}$. If I is infinite, we set $I = \mathbb{N}$, $K = \infty$ and use the notation $Z_1 \cup \cdots \cup Z_K \equiv \cup_{i \in \mathbb{N}} Z_i$. We let $W_1 = Z_1$ and set $A_1 = \complement(Z_2 \cup \cdots \cup Z_K)$. Then A_1 is a closed (possibly empty) set contained in Z_1. We choose non-empty open sets Ω_1 and V_1 such that $A_1 \subset \Omega_1 \subset\subset V_1 \subset\subset W_1$. For the induction step, assume that Ω_i, V_i and W_i have already been constructed for some i. We set

$$W_{i+1} = Z_{i+1} \setminus (\overline{\Omega_1} \cup \cdots \cup \overline{\Omega_i}) \tag{3.5}$$

$$A_{i+1} = \complement\big(V_1 \cup \cdots \cup V_i \cup Z_{i+2} \cup \cdots \cup Z_K\big) . \tag{3.6}$$

Then A_{i+1} is a closed subset of W_{i+1}. If W_{i+1} is empty, we skip this step and increase i. Otherwise, we choose non-empty open sets Ω_{i+1} and V_{i+1} such that

$$A_{i+1} \subset \Omega_{i+1} \subset\subset V_{i+1} \subset\subset W_{i+1} \,. \tag{3.7}$$

Let us verify that the resulting sets Ω_i, V_i and W_i really have the above properties (i)–(iii). The properties (i) and (iii) are obvious by construction. To prove (ii), let $p \in \hat{M}$. Since $(Z_i)_{i\in I}$ is a locally finite covering, there is an index $i \in I$ such that $p \notin Z_j$ for all $j > i+2$. But then one sees from (3.6) and (3.7) that $p \in V_1 \cup \cdots \cup V_{i+1}$. We conclude that (i)–(iii) hold.

Next, as in the usual construction of the partition of unity, we choose non-negative continuous functions $\eta_i \in C^0(\hat{M}, \mathbb{R}^+_0)$ with $\eta_i|_{V_i} \equiv 1$, $\eta_i|_{W_i \setminus \overline{V_i}} < 1$ and $\operatorname{supp} \eta_i \subset W_i$. We consider the family of compactly supported continuous sections

$$\eta_i \, e^\ell_{\alpha(i)} \qquad \text{and} \qquad \eta_i \, x^j_{\alpha(i)} \, e^\ell_{\alpha(i)} \,, \tag{3.8}$$

where $i \in I$, $\ell = 1, \ldots, \mathfrak{p} + \mathfrak{q}$ and $j = 1, \ldots, k$. Let us verify that these sections are linearly independent. Thus suppose that a linear combination of these functions vanishes. Restricting the functions to Ω_i, by property (iii) all the functions with $j \neq i$ drop out. Thus it remains to show that for any fixed i, the sections in (3.8) restricted to Ω_i are linearly independent. But this follows immediately from the fact that the $e^\ell_{\alpha(i)}$ are linearly independent at every $p \in \Omega_i$, and that the coordinate functions of the chart are linearly independent and not locally constant.

We let $\mathcal{H}_0$ be the vector space spanned by the family of sections (3.8). In order to introduce a scalar product, we represent a function $\phi \in \mathcal{H}_0$ locally in components,

$$\phi(p) = \sum_{\ell=1}^k \phi_\ell\big(x_\alpha(p)\big) \; e^\ell_{\alpha(i)} \qquad \text{for } p \in U_\alpha \,,$$

and introduce the L^2-scalar product

$$\langle \phi | \psi \rangle_{\mathcal{H}} = \sum_{i \in I} \int_{x_{\alpha(i)}(U_{\alpha(i)})} \eta_i(x) \; \sum_{\ell=1}^k \overline{\phi_\ell(x)} \, \psi_\ell(x) \, d^k x \tag{3.9}$$

(thus we define the $e^\ell_{\alpha(i)}$ to be orthonormal and take the measure as a weighted sum of the Lebesgue measures in the charts). This scalar product is well-defined and finite because the functions in $\mathcal{H}_0$ all have compact support and because the sum in (3.9) is locally finite. Moreover, it is clear from our construction that $\mathcal{H}_0$ is *locally finite-dimensional* in the sense that for any compact $K \subset \hat{M}$, the function space

$$\mathcal{H}|_K := \{ \psi|_K \text{ with } \psi \in \mathcal{H}_0 \}$$

is finite-dimensional.

Let us analyze whether $(\mathcal{H}_0, \langle .|. \rangle_{\mathcal{H}})$ is complete. Thus let $\phi_n \in \mathcal{H}_0$ be a Cauchy sequence. Then for every compact $K \subset \hat{M}$, the functions $\phi_n|_K$ are also a L^2-Cauchy sequence. Since the space $\mathcal{H}|_K$ is finite-dimensional, it follows immediately that the sequence $\phi_n|_K$ converges in $\mathcal{H}|_K$. However, the functions ϕ_n need not converge globally in $\mathcal{H}_0$. For this reason, we introduce $\mathcal{H}$ as the completion of $\mathcal{H}_0$,

$$\mathcal{H} := \overline{\mathcal{H}_0}^{\langle .|. \rangle_{\mathcal{H}}} \,.$$

Clearly, $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ is a Hilbert space. Moreover, the functions in $\mathcal{H}$ are again locally finite and $\mathcal{H}|_K = \mathcal{H}_0|_K$. In particular, the functions in $\mathcal{H}$ are all continuous. If $\hat{M}$ is compact, then $\mathcal{H}$ is obviously finite-dimensional.

For any $p \in \hat{M}$, we define the local correlation operator $F(p)$ again by (2.7). Since the functions (3.8) restricted to any point p span the fibre, the operator $F(p)$ has maximal rank $\mathfrak{p}+\mathfrak{q}$. Similar as in (3.9), we choose on $\hat{M}$ the measure $d\mu = \sum_{i \in I} \eta_i(x)\, d^k x$. Introducing the universal measure as the push-forward measure $\rho = F_*(\mu)$, we obtain a regular topological fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin signature $(\mathfrak{p}, \mathfrak{q})$.

It remains to prove that $\hat{M}$ is homeomorphic to $M := \operatorname{supp} \rho \subset \mathcal{F}$, and that the bundle $\mathcal{B} \to M$ is homeomorphic to the bundle $X \to \hat{M}$. First, since $\mathcal{H}$ is locally finite-dimensional and the functions in $\mathcal{H}$ are all continuous, the mapping $F : \hat{M} \to \mathcal{F}$ is continuous. As a consequence, the pre-image of any open neighborhood of a point $q \in F(\hat{M})$ is a non-zero open subset of $\hat{M}$, and thus has non-zero μ-measure. This implies that $F(\hat{M}) \subset \operatorname{supp} \rho$. Thus it suffices to show that the mapping

$$F : \hat{M} \to \operatorname{supp} \rho \subset \mathcal{F} \qquad \text{is a homeomorphism}\,. \tag{3.10}$$

In order to show that F is injective, let $p, q \in \hat{M}$ with $F(p) = F(q)$. Then in view of (2.7), we know that

$$\prec \phi(p) | \psi(p) \succ_p = \prec \phi(q) | \psi(q) \succ_q \qquad \text{for all } \phi, \psi \in \mathcal{H}\,. \tag{3.11}$$

Evaluating these relations for the functions $\eta_i e^\ell_{\alpha(i)}$ in (3.8), we conclude that $\eta_i(p) = \eta_i(q)$ for all i. As a consequence, we can choose the index i such that $p, q \in \overline{V_i}$. Next, evaluating (3.11) for $\phi = \eta_i e^\ell_{\alpha(i)}$ and $\psi = \eta_i x^j_{\alpha(i)} e^\ell_{\alpha(i)}$, we conclude that $x^j_{\alpha(i)}(p) = x^j_{\alpha(i)}$ for all $j = 1, \ldots, k$, implying that $p = q$.

We next prove that the mapping (3.10) is open: For given $p \in \hat{M}$ we choose i such that $p \in V_i$. We let $\mathcal{H}_i \subset \mathcal{H}$ be the subspace spanned by the functions in (3.8), and denote the orthogonal projection to $\mathcal{H}_i$ by $\pi_{\mathcal{H}_i}$. Then for any $q \in \hat{M}$,

$$\begin{aligned}
\|F(p) - F(q)\|_{\mathrm{L}(\mathcal{H})} &= \sup_{u \in \mathcal{H}, \|u\|=1} \langle u | (F(p) - F(q)) u \rangle \\
&\geq \sup_{u \in \mathcal{H}_i, \|u\|=1} \langle u | (F(p) - F(q)) u \rangle = \|\pi_{\mathcal{H}_i} (F(p) - F(q))\, \pi_{\mathcal{H}_i}\|_{\mathrm{L}(\mathcal{H})} \\
&\geq c \left| \big\langle \eta_i e^\ell_{\alpha(i)} \,\big|\, (F(p) - F(q))\, \eta_i e^\ell_{\alpha(i)} \big\rangle \right| + c \sum_{j=1}^k \left| \big\langle \eta_i e^\ell_{\alpha(i)} \,\big|\, (F(p) - F(q))\, \eta_i x^j_{\alpha(i)} e^\ell_{\alpha(i)} \big\rangle \right| \\
&= c \left| \eta_i(p)^2 - \eta_i(q)^2 \right| + c \sum_{j=1}^k \left| \eta_i(p)^2\, x^j_{\alpha(i)}(p) - \eta_i(q)^2\, x^j_{\alpha(i)}(q) \right| ,
\end{aligned}$$

where the constant $c = c(i) > 0$ depends on the scalar products of the vectors in $\mathcal{H}_i$ (here we use that on a finite-dimensional vector space all norms are equivalent). If the left side of this inequality tends to zero, then $\eta(q) \to \eta(p) = 1$ and, so that q lies in W_i. Moreover, $x^j_i(q) \to x^j(p)$, implying that $q \to p$. Hence F^{-1} is continuous. We conclude that $F : \hat{M} \to M$ is a homeomorphism.

Finally, we show that the corresponding bundles $X \to \hat{M}$ and $\mathcal{B} \to M$ are homeomorphic. First, any function ψ in (3.8) is a continuous section of the bundle $X \to M$. Identifying ψ with a vector in $\mathcal{H}$, the mapping $x \mapsto \pi_x \psi$ defines a continuous section in $\mathcal{B} \to M$. This identification of sections can be used to construct a homeomorphism of bundles: For any $u \in X_p$ we choose a function $\psi \in \mathcal{H}_0$ with $\psi(p) = u$. Setting $x = F(p)$, we obtain the vector $\pi_x \psi \in S_x$. As is obvious by construction, the mapping $u \mapsto \pi_x \psi$ does not depend on the choice of ψ and thus defines a mapping $X \to \mathcal{B}$. This mapping is compatible with the projections. Moreover, it is a bijection of the fibres which depends continuously on the base point. Thus it defines a homeomorphism of the bundles. □

In the setting of a topological manifold, it is reasonable to assume that ρ should be a *continuous measure* in the sense that $\rho(\{x\}) = 0$ for all $x \in M$. However, this assumption will not be needed in this paper.

3.4. **A Bundle over a Differentiable Manifold.** In many situations, M is even a *differentiable manifold* (again of dimension $k \geq 1$). We will assume that M is differentiable whenever the tangent space or the tangent bundle are needed in our constructions (more precisely, in Section 4.5, Section 5.3 and some of our examples). In the differentiable setting, it is natural to assume that the universal measure is compatible with the differentiable structure in the following way: First, we always assume that the injection

$$M \hookrightarrow \mathcal{F} \subset \mathrm{L}(\mathcal{H}) \qquad \text{is Fréchet differentiable}\,. \tag{3.12}$$

This assumption makes it possible to identify tangent vectors on M with tangent vectors in $\mathcal{F}$. Moreover, it is a reasonable assumption that the measure ρ should be *absolutely continuous* with respect to the Lebesgue measure in a chart.

4. Topological Spinor Bundles

The vector bundle associated to a regular topological fermion system (see Definition 3.2) is reminiscent of a spinor bundle. In particular, a section ψ of this bundle takes values in the spin spaces, $\psi(x) \in S_x$, and can thus be regarded as a spinorial wave function. However, important structures of a spinor bundle like Clifford multiplication are still missing. We shall now introduce these additional structures and analyze the topological obstructions to their existence. This will make it possible to interpret the vector bundles constructed in Sections 3.2 and 3.3 above as true spinor bundles corresponding in the classical sense of the term to bundles of Clifford algebras and representations of the spin group. We shall see that the topological conditions which the topological fermion system will be required to satisfy (see Theorem 4.5 below) are independent of the standard topological condition for the existence of a spin structure, as expressed by the vanishing of the second Stiefel-Whitney class (see Section 4.5). Thus there are conditions for the existence of spin structures that are specific to topological fermion systems.

4.1. **Clifford Sections.** In this section we only assume that the topological fermion system is *regular* (see Definition 3.1). We denote the space of symmetric linear operators on S_x by $\mathrm{Symm}(S_x) \subset \mathrm{L}(S_x)$. Then for any $x \in M$, the operator $(-x)$ on $\mathcal{H}$ has $\mathfrak{q}$ positive and $\mathfrak{p}$ negative eigenvalues. We denote its positive and negative spectral subspaces by S_x^+ and S_x^-, respectively. In view of (3.1), these subspaces are also orthogonal with respect to the spin scalar product,

$$S_x = S_x^+ \oplus S_x^- \,. \tag{4.1}$$

Moreover, we introduce the *Euclidean sign operator* s_x as a symmetric operator on S_x whose eigenspaces corresponding to the eigenvalues ± 1 are the spaces S_x^+ and S_x^-, respectively. Clearly, for a Riemannian fermion system the Euclidean sign operator is the identity on S_x.

In order to get a connection to the usual Clifford multiplication, we need the notions of a Clifford subspace and a Clifford extension. These notions were first introduced in [12] for causal fermion systems of spin dimension two. We now extend them to general spin dimension.

Definition 4.1. *A subspace $K \subset \mathrm{Symm}(S_x)$ is called a* **Clifford subspace** *of signature (r,s) at the point x (with $r,s \in \mathbb{N}$) if the following conditions hold:*

(i) *For any $u,v \in K$, the anti-commutator $\{u,v\} \equiv uv + vu$ is a multiple of the identity on S_x.*
(ii) *The bilinear form $\langle .,. \rangle$ on K defined by*
$$\frac{1}{2}\{u,v\} = \langle u,v \rangle \, \mathbb{1} \qquad \text{for all } u,v \in K \tag{4.2}$$
is non-degenerate and has signature (r,s).

We denote the set of Clifford subspaces of signature (r,s) at x by $\mathcal{K}_x^{(r,s)}$.

In the setting $\mathfrak{p} = \mathfrak{q}$ of causal fermion systems, a useful method for constructing Clifford subspaces is to begin with the Euclidean sign operator s_x and to add operators which anti-commute with it. This gives rise to the so-called Clifford extensions.

Definition 4.2. *On a causal fermion system, a Clifford subspace K which contains the Euclidean sign operator is referred to as a* **Clifford extension** *(of the Euclidean sign operator s_x).*

Lemma 4.3. *A Clifford extension of dimension m has Lorentzian signature $(1, m-1)$.*

Proof. We choose an orthonormal basis $(s_x, e_1, \ldots, e_{m-1})$ of the Clifford extension K. Choosing an orthonormal eigenvector basis of s_x, we can represent the Euclidean sign operator by the matrix
$$s_x = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} .$$
Due to the anti-commutation relations, in this basis the operators e_j have the matrix representations
$$e_j = \begin{pmatrix} 0 & a_j^* \\ a_j & 0 \end{pmatrix} \qquad \text{and} \qquad a_j^* = -a_j^\dagger$$
(where the dagger denotes transposition and complex conjugation). Hence
$$(e_j^2) = - \begin{pmatrix} a_j^\dagger a_j & 0 \\ 0 & a_j a_j^\dagger \end{pmatrix} .$$
Noting that this matrix is negative definite, we obtain the claim. ☐

We denote the Clifford extensions of signature $(1,r)$ by $\mathcal{K}_x^{s_x, r}$.

In order to introduce global sections of the Clifford algebra, we let $\mathfrak{S}_m$ be the set of m-dimensional subspaces of $\mathrm{L}(\mathcal{H})$ endowed with the topology coming from the metric
$$\mathrm{d}(K,L) = \sup_{u \in K,\, \|u\|=1} \; \inf_{v \in L} \|u - v\| \;+\; \sup_{v \in L,\, \|v\|=1} \; \inf_{u \in K} \|u - v\| .$$

Moreover, we let $\mathcal{C}\ell M$ be a continuous mapping which to every space-time point associates a Clifford subspace,

$$\mathcal{C}\ell M \;:\; M \to \mathfrak{S}_{r+s}\,, \quad x \mapsto \mathcal{C}\ell_x \in \mathcal{K}_x^{(r,s)}\,. \tag{4.3}$$

We refer to $\mathcal{C}\ell M$ as a **Clifford section** of signature (r,s). It is a **section of Clifford extensions** if $\mathcal{C}\ell_x \in \mathcal{K}_x^{s_x,r}$ for all $x \in M$.

Choosing a Clifford section is the main step for getting into the setting in which the elements of S_x can be interpreted as spinors. In the remainder of Section 4, we shall work out this setting in more detail. More precisely, in Section 4.2 we shall analyze topological obstructions for the existence of Clifford sections. In Section 4.3 we will construct the analogs of the usual Pin and Spin groups. In Section 4.4 spinors, spinor bases and bundle charts for spinor bundles will be defined. Finally, in Section 4.5 we will introduce spin structures which associate a tangent vector on a differentiable manifold to a vector in the corresponding Clifford subspace. Altogether, these constructions extend the usual structures of spin geometry to the framework of topological fermion systems.

Before entering the detailed analysis, we now give a brief overview of the different situations of interest. First, recall that regular topological fermion systems $(\mathcal{H}, \mathcal{F}, \rho)$ on a topological manifold $M = \operatorname{supp} \rho$ are distinguished by their spin signature $(\mathfrak{p}, \mathfrak{q})$ and the dimension k of M, which can be chosen independently. When choosing Clifford subspaces or Clifford sections, the signature (r,s) of the Clifford subspace gives additional parameters. If one considers causal fermion systems and Clifford extensions, the signature of the spin scalar product is (n,n) and the signature of the Clifford subspace is $(1, m-1)$, leaving us with two parameters n and m to describe the signatures. In usual spin geometry, the dimension of the Clifford subspace always coincides with the dimension of the manifold, i.e.

$$k = r+s \text{ (in general)} \qquad \text{or} \qquad k = m \text{ (for Clifford extensions)}\,. \tag{4.4}$$

In our setting, these relations are needed if we want to introduce Clifford multiplication with tangent vectors, because then one needs to identify the tangent space T_xM with the corresponding Clifford subspace $\mathcal{C}\ell_x$. This construction will be given in Section 4.5 below. At the present stage, there is no need to impose the relations (4.4). On the contrary, for the sake of having more flexibility it is preferable to carefully distinguish the dimension of the manifold from the dimension of the Clifford subspace and to treat these dimensions as independent parameters.

More specifically, in the case of spin signature $(2,2)$ and Clifford extensions of dimension $m = 4$, one can get the correspondence to Dirac spinors on a Lorentzian manifold. In this case, the corresponding geometric structures are worked out in [12]. Indeed, most of the constructions in [12] apply just as well to the case $m = 5$. In order to model the interactions of the standard model, one should increase the spin signature to $(2\ell, 2\ell)$ with $\ell = 2$ (to get the weak interaction and gravity [11]) or $\ell = 8$ (to also include the strong interaction [6]). These cases of higher spin signature have not yet been studied systematically. If the spin signature is decreased to $(1,1)$, the spin spaces are two-dimensional, making it impossible to represent Dirac spinors. But one can describe Pauli-like spinors in dimensions $m = 1$, 2 or 3. In these lower-dimensional situations, the geometric constructions of [12] simplify considerably. This gives hope that it should be possible to connect the geometric notions to the methods and notions arising in the analysis of causal variational principles (see [9, 19, 2, 13]). We consider

spin signature $(2,2)$	This case is relevant for the description of Dirac spinors (as is explained in [14, 18]). The corresponding Lorentzian quantum geometry is developed in [12]. $m=5$: A Lorentzian Clifford subspace of signature $(1,4)$. Appears naturally in the setting of the Lorentzian quantum geometry in [12]. Is a bit easier to handle than the four-dimensional case. $m=4$: A Lorentzian Clifford subspace with the "physical" signature $(1,3)$.
spin signature $(1,1)$	Is a mathematical simplification. Most analytic work has been done in this case (see [5, 19]). $m=3$: A Lorentzian Clifford subspace of signature $(1,2)$. Appears naturally in the quantum geometry setting, similar to the case $k=5$ in spin signature $(2,2)$. $m=2$: A Lorentzian Clifford subspace of signature $(1,1)$. Seems a good starting point for connecting the Lorentzian quantum geometry with an analysis of the causal action principle.
spin signature $(0,n)$	Riemannian fermion systems of dimension $r+s \geq 2$ have not yet been studied. But it seems an interesting future project to work out the resulting Riemannian quantum geometries.

TABLE 1. Different cases for the spin signature and the dimension of the Clifford subspaces.

this to be a promising starting point for future research. The study of Riemannian fermion systems is also a project for the future. The different cases are summarized in Table 1.

4.2. **Topological Obstructions.** The goal of this section is to determine topological obstructions to the existence of Clifford sections. We shall see that there is an interesting interplay between the conditions that need to be satisfied for the existence of Clifford sections and the usual obstructions to the existence of spin structures on a differentiable manifold as expressed by the usual vanishing of the second Stiefel-Whitney class. The connection between these different conditions will become clear in Section 4.5 below. For the moment, the question whether a Clifford section exists is independent of the usual topological spin condition. This illustrated by the following example which shows that a smooth manifold may fail to admit a section of Clifford extensions even if its tangent bundle is spin.

Example 4.4. *(Non-existence of Clifford sections on the Dirac sphere)* We return to the Dirac sphere of Example 2.2 (i). At a point $x = F(p) \in \operatorname{supp} \rho$, the spin scalar product (3.1) takes the form

$$\prec . | . \succ_x = -\langle ., F(p) . \rangle_{\mathbb{C}^2} \;.$$

By definition, the Euclidean sign operator has the same eigenspaces as (2.1) with eigenvalues ± 1, so that

$$s_x = -p \cdot \sigma$$

(where the dot is a short notation for the sum over the products of components).

We now choose a convenient parametrization of the space $\text{Symm}(S_x)$. Let $|F(p)|$ be the absolute value of the operator $F(p)$, i.e. by (2.1)

$$|F(p)| = \sum_{\alpha=1}^{3} p^\alpha \sigma^\alpha + 2\,\mathbb{1} \equiv p\cdot\sigma + 2\,\mathbb{1}\;.$$

Writing a linear operator in the form $A = |F(p)|^{-1} B |F(p)|$, a direct computation shows that A is symmetric with respect to the inner product $\prec .|. \succ_p$ if and only if B is symmetric with respect to the inner product $\langle ., p\sigma . \rangle_{\mathbb{C}^2}$. Moreover, a short computation yields

$$\begin{aligned} \text{Symm}(S_x) = \big\{ |F(p)|^{-1} (\alpha\, \mathbb{1} + \beta\, p\cdot\sigma + i\, u\cdot\sigma)\, |F(p)| \\ \text{with} \quad \alpha, \beta \in \mathbb{R},\; u \in \mathbb{R}^3 \text{ and } u \perp p \big\}. \end{aligned} \tag{4.5}$$

In order to obtain a two-dimensional Clifford extension at a point p, we need to choose an operator in $\text{Symm}(S_x)$ which anti-commutes with s_x and whose square equals $-\mathbb{1}$. A short computation using (4.5) yields

$$\mathcal{K}_x^{s_x, 1} = \Big\{ \text{span}\big(s_x, i\, |F(p)|^{-1}\, u\cdot\sigma\, |F(p)| \big) \;\Big|\; u \in S^2 \subset \mathbb{R}^3 \text{ with } u \perp p \Big\}\;.$$

We conclude that a *two-dimensional section of Clifford extensions* amounts to finding a tangent unit vector field on the sphere. However, such a vector field does not exist by the well-known "hairy ball theorem".

The existence of general Clifford sections (which may not necessarily be Clifford extensions) depends on the dimension and signature. A *three-dimensional Clifford section* necessarily has signature $(1, 2)$. There is the unique Clifford section

$$\mathcal{C}\ell_x = \big\{ |F(p)|^{-1} (\beta\, p\cdot\sigma + i\, u\cdot\sigma) |F(p)| \text{ with } \alpha, \beta \in \mathbb{R},\; u \in \mathbb{R}^3 \text{ and } u \perp p \big\}\;,$$

which is also a section of Clifford extensions. *Two-dimensional Clifford sections* exist in signature $(0, 2)$, like for example

$$\mathcal{C}\ell_x = \big\{ |F(p)|^{-1} (i\, u\cdot\sigma) |F(p)| \text{ with } u \in \mathbb{R}^3 \text{ and } u \perp p \big\}\;.$$

By applying a unitary transformation $\mathcal{C}\ell_x \to U(p) \mathcal{C}\ell_x U(p)^{-1}$, where U is a continuous family of unitary transformations $U(p) \in \text{U}(S_x)$, one can construct many other Clifford sections of signature $(0, 2)$. In signature $(1, 1)$, however, *no Clifford section exists*, as the following argument shows: Suppose that $\mathcal{C}\ell_x$ were a Clifford section of signature $(1, 1)$. Choosing a positive definite vector unit vector v in $\mathcal{C}\ell_x$, it is of the form

$$v = |F(p)|^{-1} \big(\pm \cosh(\alpha)\, p\cdot\sigma + i \sinh(\alpha)\, u\cdot\sigma \big) |F(p)|$$

with $\alpha \in \mathbb{R}$, and $u \perp p$. Varying p, by continuity we can always choose the plus sign to obtain a global section $v(p)$. A unit vector w in the orthogonal complement of $v(p)$ in $\mathcal{C}\ell_x$ is of the form

$$w = |F(p)|^{-1} \big(\beta\, p\cdot\sigma + i\, t\cdot\sigma \big) |F(p)|$$

with $\beta \in \mathbb{R}$, and where the vector t is tangential to S^2 and has length at least one. In this way, the orthogonal complement of $v(p)$ determines a continuous family of one-dimensional subspaces of the tangent space of S^2, in contradiction to the "hairy ball theorem." ◊

This example illustrates that sections of Clifford extensions do not exist on all topological manifolds. Obstructions to the existence of Clifford sections can be derived for a fairly general class of topological manifolds from classical results on obstructions to the existence of continuous cross-sections of topological fiber bundles over cell complexes [37]. In order to apply these results, we will need to assume that our topological manifold M is homeomorphic to a *finite cell complex*. This will be the case for example if M is a compact topological manifold [30]. Let us recall the following result from [37, Corollary 34.4] which gives the topological obstructions to the existence of continuous cross sections:

Theorem 4.5. *Let $\mathcal{B}$ be a bundle over a finite cell complex K, such that for all $1 \leq q \leq \dim K$, the fibre Y is $(q-1)$-simple. If*

$$H^q\big(K, \mathcal{B}(\pi_{q-1}(Y))\big) = 0 \qquad \forall\, 1 \leq q \leq \dim K\,, \tag{4.6}$$

then $\mathcal{B}$ admits a continuous cross-section.

To explain the notions in this theorem, we first recall that a path-connected space Y is q-simple if the action of $\pi_1(Y, y_0)$ on the q^{th} homotopy group $\pi_q(Y, y_0)$ is trivial for some base point $y_0 \in y$ (and therefore all base points). The assumption that the fibre Y is $(q-1)$-simple implies that the homotopy group $\pi_{q-1}(Y)$ is defined independent of a base point. Finally, $\mathcal{B}(\pi_{q-1}(Y))$ denotes the bundle over K whose fibre at a point $x \in K$ is the homotopy group π_{q-1} of the corresponding fibre $\mathcal{B}(x) \simeq Y$.

For the Clifford extensions that we will consider in this paper, the fibres Y are isomorphic to Lie subgroups of $\mathrm{SU}(1,1)$, $\mathrm{SU}(2,2)$ or $\mathrm{SU}(n)$, and are therefore q-simple, since topological groups are q-simple for all q (see [36, Theorem 7.3.9] or [25, page 281, Section 3C]). Therefore, the only obstructions to the existence of Clifford extensions are given by the cohomological conditions (4.6).

We now formulate these conditions explicitly for causal fermion systems in the cases in which M is a topological manifold M^k of dimension k and the fibre Y is the set of Clifford extensions of dimension m according to Table 1. In the case $m = 5$, the fibre Y in spin dimension $n = 2$ is isomorphic to the circle group S^1 which acts simply transitively on the set of Clifford extensions (see [12, Corollary3.7 and Lemma 3.8]). We thus obtain

$$\pi_0(Y) = 0\,, \qquad \pi_1(Y) = \mathbb{Z}\,, \qquad \pi_n(Y) = 0 \quad \forall\, n \geq 2\,, \tag{4.7}$$

and the obstructions (4.6) reduce to

$$H^2(M^k, \mathbb{Z}) = 0\,. \tag{4.8}$$

Next, in the case $m = 4$ of a four-dimensional Clifford extension, the fibre Y (again in spin dimension $n = 2$) is isomorphic to the product $S^1 \times SO(4, \mathbb{R})$, so that

$$\pi_0(Y) = 0,\ \ \pi_1(Y) = \mathbb{Z} \times \mathbb{Z}_2,\ \ \pi_2(Y) = 0,\ \ \pi_3(Y) = \mathbb{Z} \times \mathbb{Z}. \tag{4.9}$$

The cohomological obstructions (4.6) to the existence of a Clifford extension are therefore given by

$$H^2(M^k, \mathbb{Z} \times \mathbb{Z}_2) = 0 \qquad \text{and} \qquad H^4(M^k, \mathbb{Z} \times \mathbb{Z}) = 0\,. \tag{4.10}$$

The remaining cases of interest are $m = 2$ and 3, with the spin dimension n taken to be equal to one. For $m = 3$, the fibre Y reduces to a point, giving no topological obstructions to the existence of a Clifford extension. In the case $m = 2$, on the other hand, the fibre Y is again a circle, and the obstructions (4.6) are given by

$$H^2(M^k, \mathbb{Z}) = 0. \tag{4.11}$$

We close with two remarks. We first go back to Example 4.4 in which we tried to construct Clifford extensions on the Dirac sphere. In this example, the topological obstruction $H^2(M^2, \mathbb{Z}) = 0$ is violated. This confirms the conclusion that we came to by explicit calculation that Clifford extensions do not exist on the Dirac sphere.

Next, to further illustrate the independence of the usual spin condition from the topological conditions needed for the existence of Clifford extensions, we now give an example of a manifold M^k which admits a Clifford extensions of dimension k, but whose tangent bundle TM is *not* spin. This is indeed the case for the manifold $P_5(\mathbb{R})$, for which $w_2(TP_5(\mathbb{R})) \neq 0$ (see [31], page 87, Example 2.4), but nevertheless the obstruction $H_2(P_5(\mathbb{R}), \mathbb{Z}) = 0$ for the existence of a Clifford extension is satisfied.

4.3. **The Spin Group.** In this section we will show that the usual Pin and Spin groups arise naturally in the context of causal fermion systems by looking at the subgroups of the group of unitary automorphisms of the spin spaces that stabilize Clifford subspaces. So, we again assume that the causal fermion system of signature $(\mathfrak{p}, \mathfrak{q})$ is regular (see Definition 3.1). We denote the group of unitary endomorphisms of S_x by $\mathrm{U}(S_x)$; it is isomorphic to the group $\mathrm{U}(\mathfrak{q}, \mathfrak{p})$. For a given Clifford subspace $K \in \mathcal{K}_x^{(r,s)}$, we introduce the following *stabilizer subgroups*:

$$\begin{aligned} \mathcal{G}_x^K &= \big\{ U \in \mathrm{U}(S_x) \text{ with } UKU^{-1} = K \big\} \\ \mathcal{G}_x^0 &= \big\{ U \in \mathrm{U}(S_x) \text{ with } UvU^{-1} = v \text{ for all } v \in K \big\} \,. \end{aligned} \tag{4.12}$$

It is straightforward to verify that $\mathcal{G}_x^0$ is a normal subgroup of $\mathcal{G}_x^K$. Hence we may form the factor group

$$\mathrm{Pin}_x = \mathcal{G}_x^K / \mathcal{G}_x^0 \,. \tag{4.13}$$

An element $g \in \mathrm{Pin}_x$ also acts on K by conjugation, giving rise to the mapping

$$\mathcal{O}(g) \;:\; K \to K \,, \qquad \mathcal{O}(g) u := g u g^{-1} \,. \tag{4.14}$$

Since unitary transformations do not affect the anti-commutation relations, this mapping is an isometry with respect to the inner product (4.2).

Special mappings in Pin_x can be constructed using Clifford multiplication, as we now explain. Suppose that $v \in K$ is a time-like unit vector (i.e. $v^2 = \mathbb{1}$). Then v is unitary because $vv^* = v^2 = \mathbb{1}$. Using the anti-commutation relations, one verifies that $vKv = K$. Therefore, v can be regarded as an operator in Pin_x,

$$v \in \mathrm{Pin}_x \qquad \text{if } v \in K,\, v^2 = \mathbb{1} \,. \tag{4.15}$$

We next consider a space-like unit vector $v \in K$ (i.e. $v^2 = -\mathbb{1}$). Then the operator iv is unitary because $(iv)(iv)^* = v^2 = \mathbb{1}$, so that

$$iv \in \mathrm{Pin}_x \qquad \text{if } v \in K,\, v^2 = -\mathbb{1} \,. \tag{4.16}$$

Lemma 4.6. *The unitary operators* (4.15) *and* (4.16) *generate* Pin_x.

Proof. For any $U \in \mathcal{G}_x^K$, the resulting operator $\mathcal{O}(U)$ is an isometry of $(K, \langle ., . \rangle)$. As a consequence, $\mathcal{O}(U)$ can be written as a composition of reflections at spacelike or timelike hyperplanes $H_1, \ldots, H_k$ (see [31, Theorem I.2.7]). These reflections can be written as $-\mathcal{O}(w)$, where w is one of the unitary transformations in (4.15) or (4.16) with v the unit normal to the hyperplane. Hence

$$\mathcal{O}(U) = (-1)^k \, \mathcal{O}\big(w_1 \cdots w_k\big).$$

As a consequence,

$$\mathcal{O}\big(U\,(-1)^k\, w_k^* \cdots w_1^*\big) = \mathcal{O}(U)\, \mathcal{O}(U)^{-1} = \mathbb{1}\,.$$

According to the to the definition (4.12), this means that the operator $U\,(-1)^k\, w_k^* \cdots w_1^*$ is in $\mathcal{G}_x^0$. In other words,

$$U = U^0\, w_1 \cdots w_k \qquad \text{with} \qquad U^0 \in \mathcal{G}_x^0\,.$$

Hence U and $w_1 \cdots w_k$ represent the same operator in Spin_x. ☐

Definition 4.7. *The* **spin group** Spin_x *is defined as the subgroup of* Pin_x *generated by even numbers of products of the operators* (4.15) *and* (4.16)*.*

We now explain how the groups Pin_x and Spin_x are related to the usual pin and spin groups defined on a Clifford algebra (see for example [31, 1] or similarly [21] in the Riemannian setting). To this end, we let $\mathcal{C}\ell(K, \langle .,. \rangle)$ be the real Clifford algebra on the inner product space $(K, \langle .,. \rangle)$ (thus it is the algebra generated by vectors in K with the only relation $vw + wv = 2\langle v, w\rangle$; see [31, §I.1], where for convenience we use a different sign convention). In view of the anti-commutation relations (4.2), in our setting $\mathcal{C}\ell(K, \langle .,. \rangle)$ is not only an abstract Clifford algebra, but it comes with a representation by symmetric operators on $(S_x, \prec .|. \succ_x)$. Likewise, the groups Pin_x and Spin_x are not only isomorphic to the usual pin and spin groups (see for example [31, §II.1]), but they come with a unitary representation on the inner product space $(S_x, \prec .|. \succ_x)$.

We conclude by decomposing the representation of $\mathcal{C}\ell(K, \langle .,. \rangle)$ on $(S_x, \prec .|. \succ_x)$ into irreducible components. Denoting this representation by ρ, such a decomposition clearly exists (see [31, Proposition I.5.4]). Moreover, the number ν of inequivalent irreducible representations is given by (see [31, Theorem I.5.7])

$$\nu = \begin{cases} 1 & \text{if } r+s \text{ is even} \\ 2 & \text{if } r+s \text{ is odd} \end{cases}$$

(the fact that $K \subset \mathrm{Symm}(S_x)$ consists of symmetric operators may give constraints for the possible representations, but this is irrelevant for the following argument). For clarity, we treat the two cases after each other.

If $r+s$ is even, we denote the irreducible representation by Δ acting on an inner product space E. We then obtain the decomposition

$$S_x = E \otimes V \qquad \text{and} \qquad \rho(u) = \Delta(u) \otimes \mathbb{1}$$

(where V is an inner product space). Now we can read off the stabilizer groups. If a unitary linear operator on E commutes with all $\Delta(u)$, its eigenspaces are invariant under Δ. Since Δ is irreducible, the eigenspaces are either trivial or they coincide with E. We conclude that

$$\mathcal{G}_x^0 = \{\mathbb{1}\} \otimes \mathrm{U}(V) \tag{4.17}$$

$$\mathcal{G}_x^K = \mathcal{F} \otimes \mathrm{U}(V)\,, \tag{4.18}$$

where $\mathcal{F}$ are the unitary transformations on E which leave K invariant.

If $r+s$ is odd, the above formulas must be modified in a straightforward way by considering the two direct summands. Denoting the two irreducible representations by Δ_ℓ acting on inner product spaces E_ℓ, we obtain

$$S_x = E_1 \otimes V_1 \oplus E_2 \otimes V_2 \qquad \text{and} \qquad \rho(u) = \Delta_1(u) \otimes \mathbb{1} \oplus \Delta_2 \otimes \mathbb{1} \,,$$

where V_1 and V_2 are inner product spaces which might be trivial. Moreover,

$$\mathcal{G}^0_x = \mathbb{1} \otimes \mathrm{U}(V_1) \oplus \mathbb{1} \otimes \mathrm{U}(V_2) \tag{4.19}$$

$$\mathcal{G}^K_x = \big\{ \Delta_1(g) \otimes \mathrm{U}(V_1) \oplus \Delta_2(g) \otimes \mathrm{U}(V_2) \;|\; g \in \mathrm{Pin}(r,s) \big\} \,. \tag{4.20}$$

4.4. **Construction of Bundle Charts.** We now show how the choice of a Clifford section $\mathcal{C}\ell M$ of signature (r,s) gives rise to topological vector bundles with structure groups $\mathrm{SO}(r,s)$ and $\mathcal{G}^0 \times \mathrm{Spin}(r,s)$. In order to choose bundle charts, we fix a matrix representation $(e_1, \ldots, e_{p+q})$ of a Clifford algebra of signature (r,s) on $\mathbb{C}^{\mathfrak{q},\mathfrak{p}}$, which is isomorphic to the Clifford subspaces $\mathcal{C}\ell_x$ coming from our Clifford section. This gives rise to a corresponding Clifford subspace $K \subset \mathrm{L}(\mathbb{C}^{\mathfrak{q},\mathfrak{p}})$ as well as a matrix representation of the group $\mathcal{G}^0$ on $\mathbb{C}^{\mathfrak{q},\mathfrak{p}}$ (see (4.17) respectively (4.19)). For any $x \in M$, we now choose a pseudo-orthonormal spinor basis $(\mathfrak{f}_1, \ldots, \mathfrak{f}_{\mathfrak{p}+\mathfrak{q}})$ of S_x, i.e.

$$\prec \mathfrak{f}_\alpha | \mathfrak{f}_\beta \succ_x = s_\alpha \, \delta_{\alpha\beta} \,,$$

where $s_1, \ldots, s_{\mathfrak{q}} = 1$ and $s_{\mathfrak{q}+1}, \ldots, s_{\mathfrak{p}+\mathfrak{q}} = -1$. This allows us to write the spinors in x in components,

$$S_x \ni \psi = \sum_{\alpha=1}^{\mathfrak{p}+\mathfrak{q}} \psi^\alpha \, \mathfrak{f}_\alpha \,. \tag{4.21}$$

The spinor basis also gives rise to a matrix representation of $\mathcal{C}\ell_x$. A pseudo-orthonormal spinor basis $(\mathfrak{f}_1, \ldots, \mathfrak{f}_{\mathfrak{p}+\mathfrak{q}})$ is called *Clifford compatible* if this matrix representation of $\mathcal{C}\ell_x$ coincides with our fixed Clifford subspace K. A Clifford compatible spinor basis makes it possible to represent the vectors in $\mathcal{C}\ell_x$ as vectors in $\mathbb{R}^{r,s}$ by

$$\mathcal{C}\ell_x \ni u = \sum_{i=1}^{r+s} u^i \, e_i \,. \tag{4.22}$$

By construction, Clifford compatible spinor bases are related to each other by the action of the spin group $\mathcal{G}^0_x \times \mathrm{Spin}_x$, i.e. by transformations of the form

$$\mathfrak{f}_\alpha \to \tilde{\mathfrak{f}}_\alpha = U^{-1} \mathfrak{f}_\alpha \qquad \text{with} \qquad U \in \mathcal{G}^0_x \times \mathrm{Spin}_x \,.$$

This transforms the components in (4.21) according to

$$\psi^\alpha \to \tilde{\psi}^\alpha = U^\alpha_\beta \psi_\beta \qquad \text{with} \qquad U = (U^\alpha_\beta) \in \mathcal{G}^0 \times \mathrm{Spin}(r,s) \subset \mathrm{U}(\mathfrak{q},\mathfrak{p}) \,. \tag{4.23}$$

Likewise, the components of the vector in (4.22) transform to

$$u^i \to \tilde{u}^i = O^i_j \, u^j \qquad \text{with} \qquad O \in \mathrm{SO}(r,s) \,,$$

where O is given by $U e_j U^{-1} = O^i_j e_i$ (and U is the matrix in (4.23)).

At every point $p_0 \in M$, we choose a neighborhood V such that there is a continuous mapping which to every $x \in V$ associates a Clifford compatible spinor basis $(\mathfrak{f}_1(x), \ldots, \mathfrak{f}_{\mathfrak{p}+\mathfrak{q}}(x))$ (such continuous families of spinor bases can be constructed similar as explained in Section 3.2 using projections and polar decompositions in $\mathcal{H}$). Moreover, we choose a chart in V. This gives an atlas of the bundles.

4.5. **Spin Structures.** In this section we will explain that the usual concept of a spin structure on the tangent bundle of a manifold has a counterpart in the context of causal fermion systems, provided that the system satisfies certain assumptions, which we now state. As in Section 4.1, we assume that our topological fermion system is regular of signature $(\mathfrak{p}, \mathfrak{q})$. Moreover, we again assume that there is a Clifford section of signature (r, s) (see (4.3)). In addition, we now assume that the support M^k of the universal measure is a *differentiable manifold* (see Section 3.4) and that the dimension of the manifold coincides with that of the Clifford subspaces (4.4).

Definition 4.8. *A bundle isomorphism*

$$\gamma \,:\, TM \rightarrow \mathcal{C}\ell M$$

is referred to as a **spin structure**.

A spin structure gives rise to the usual Clifford multiplication,

$$\gamma \,:\, T_pM \rightarrow \mathrm{Symm}(S_p) \,,$$

satisfying the anti-commutation relations

$$\gamma(u)\,\gamma(v) + \gamma(v)\,\gamma(u) = 2\,\langle u, v\rangle_x \,,$$

where $\langle .,. \rangle_x$ is the bilinear form in (4.2) of signature (r, s). Denoting this bilinear form by g, we obtain a *pseudo-Riemannian manifold* (M, g) of signature (r, s). We point out that, in contrast to the usual construction on spin manifolds, we do not need to assume an orientation neither of the manifold M nor of the the Clifford subspaces.

The existence of spin structures is subject to the usual topological obstruction:

Theorem 4.9. *A spin structure exists only if the second Stiefel-Whitney class of TM vanishes.*

This theorem follows from a more general result which applies to vector bundles over a topological manifold whose fibres are modules for the action of the spin group $\mathrm{Spin}(r, s)$. We now show for the sake of completeness that in this setting, the usual topological condition guaranteeing the existence of a spin structure through the vanishing of the second Stiefel-Whitney class of the bundle is satisfied on a regular causal fermion systems admitting a Clifford section. Theorem 4.9 will follow as a corollary. We note that the topological arguments appearing in this section are not new. They are essentially an adaptation of the argument given in [31, page 83]. (We refer to [34] and [33] for the construction of characteristic classes and the Serre exact sequence.)

Thus we consider the topological vector bundle $\mathcal{B}$ with structure group $G = \mathrm{Spin(r,s)}$ determined by a given Clifford section $\mathcal{C}\ell M$. To define a spin structure on $\mathcal{B}$, it is convenient to work with the principal bundle $\pi_{P(\mathcal{B})} : P(\mathcal{B}) \rightarrow M$ associated to $\mathcal{B}$. In this setting, a spin structure is a twofold cover

$$\begin{array}{ccc} \tilde{P}(\mathcal{B}) & \stackrel{p}{\rightarrow} & P(\mathcal{B}) \\ & \searrow & \downarrow \\ & & M \end{array}$$

such that $p|_{\tilde{P}_x} \,:\, \tilde{P}_x \rightarrow P_x$, where $\tilde{P}_x = \pi^{-1}_{\tilde{P}(\mathcal{B})}(\{x\})$ and $P_x = \pi^{-1}_{P(\mathcal{B})}(\{x\})$ denote respectively the fibres of $\tilde{P}(\mathcal{B})$ and $P(\mathcal{B})$ over $x \in M$, is a copy of the universal covering map $\tilde{P}_x \simeq \mathrm{Spin}(r, s) \rightarrow P_x \simeq \mathrm{SO}(r, s)$. Note that by constructions in Section 4.1, we know that a Clifford section $\mathcal{C}\ell M$ defines precisely such a twofold cover.

We conclude that a spin structure on $\mathcal{B}$ is given by a cohomology class $[\sigma] \in H^1(P(\mathcal{B}), \mathbb{Z}_2)$. The restriction-induced homomorphism $H^1(P(\mathcal{B}), \mathbb{Z}_2) \to H^1(P_x, \mathbb{Z}_2)$ maps the cohomology class $[\sigma]$ to a generator of $H^1(P_x, \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Therefore, $\mathcal{B}$ admits a spin structure if and only if the sequence

$$0 \to H^1(M, \mathbb{Z}_2) \to H^1(P(\mathcal{B}), \mathbb{Z}_2) \to H^1(P_x, \mathbb{Z}_2) \to 0 \tag{4.24}$$

is exact, and a spin structure on $\mathcal{B}$ is simply a splitting of this sequence. On the other hand, the Serre spectral sequence [33] implies that the sequence

$$0 \to H^1(M, \mathbb{Z}_2) \to H^1(P(\mathcal{B}), \mathbb{Z}_2) \to H^1(P_x, \mathbb{Z}_2) \to H^2(M, \mathbb{Z}_2) \to \cdots\,, \tag{4.25}$$

is exact, with the image of the generator of $H^1(P_x, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ in $H^2(M, \mathbb{Z}_2)$ being the second Stiefel-Whitney class $w_2(\mathcal{B})$. We conclude that $\mathcal{B}$ admits a spin structure if and only if $w_2(\mathcal{B}) = 0$.

5. Tangent Cone Measures and the Tangential Clifford Section

In the previous section, we saw that a Clifford section is essential for giving a topological fermion system the additional structure of a topological spinor bundle. We also worked out the topological obstructions for the existence of a Clifford section. The important remaining question is how to choose a Clifford section. In particular, can the universal measure be used to distinguish a specific Clifford section? We shall now analyze this question, giving an affirmative answer in terms of the so-called *tangential Clifford section*. Before beginning, we point out that all our constructions will be *local* in the sense that they will involve the universal measure only in an arbitrarily small neighborhood of a given point $x \in M$. Consequently, the assumptions needed for the constructions to work will also be local, making it possible to easily verify them in concrete examples by direct computation. Provided that these local conditions are fulfilled at every point $x \in M$, we shall obtain the tangential Clifford section, implying in particular that the topological obstructions of Section 4.2 are fulfilled. In this way, we get a connection between local properties of the topological fermion system and its global topological structure.

5.1. **The Tangent Cone Measures.**

In this section we again assume that the topological fermion system is regular (see Definition 3.1). Moreover, we assume that the universal measure ρ is *locally bounded* in the sense that every $x \in M$ has an open neighborhood U such that $\rho(V) < \infty$ for all measurable $V \subset U$. Finally, we assume that ρ is the *completion of a Borel measure*, meaning that every Borel set in $\mathcal{F}$ is ρ-measurable and that every subset of a set of measure zero is measurable (and clearly also has measure zero). For basic definitions see [24, §7 and §52]. Restricting attention to completions of Borel measures is no major restriction because the universal measures obtained by minimizing the causal action are always of this form (see [9]).

We want to analyze the subset $M \subset \mathcal{F}$ in a neighborhood of a given point $x \in M$. To this end, it is useful to consider a continuous mapping $\mathcal{A}$ from M to the symmetric operators on the spin space at x. We always assume that this mapping vanishes at x, i.e.

$$\mathcal{A} : M \to \text{Symm}(S_x) \qquad \text{with} \qquad \mathcal{A}(x) = 0\,. \tag{5.1}$$

There are different possible choices for $\mathcal{A}$. The simplest choice is

$$\mathcal{A} \,:\, y \mapsto \pi_x\, (y - x)\, x\,. \tag{5.2}$$

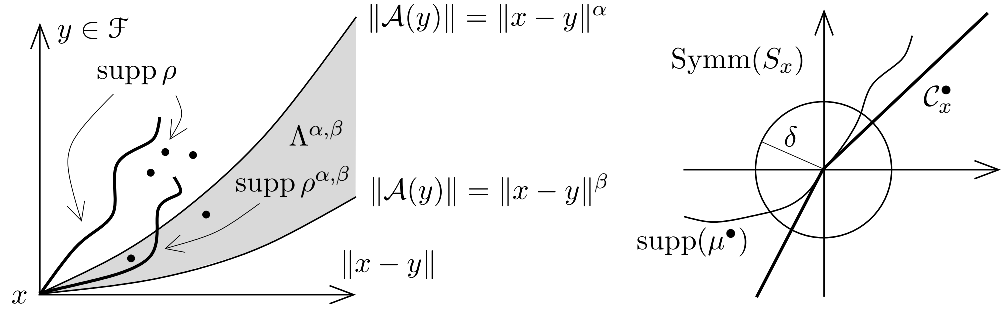

FIGURE 2. The scaling parameters α, β and the tangent cone $\mathcal{C}_x^\bullet$.

Here the factor x on the right is needed for the operator to be symmetric, because

$$\begin{aligned}\prec\!\psi|\mathcal{A}\phi\!\succ_x &\overset{(3.1)}{=} -\langle\psi\,|\,x\,(\pi_x\,(y-x)\,x)\,\phi\rangle_{\mathcal{H}} = -\langle\psi\,|\,x\,\,(y-x)\,x\,\phi\rangle_{\mathcal{H}} \\ &= -\langle\pi_x\,(y-x)\,x\,\psi\,|\,x\,\phi\rangle_{\mathcal{H}} = \prec\!\mathcal{A}\psi|\phi\!\succ_x .\end{aligned}$$

Alternatively, one can consider mappings in which only the operators s_y or π_y enter,

$$\mathcal{A} \;:\; y \mapsto \pi_x\,(s_y - s_x)\,x \tag{5.3}$$

$$\mathcal{A} \;:\; y \mapsto \pi_x\,(\pi_y - \pi_x)\,x\,. \tag{5.4}$$

(where π_x again denotes the orthogonal projection in $\mathcal{H}$ on S_x). Moreover, one might be interested more specifically in the contributions which are block diagonal or block off-diagonal in the decomposition (4.1), which we denote by

$$\mathcal{A}^{\mathrm{d}} := \frac{1}{2}\Big(\mathcal{A} + s_x\,\mathcal{A}\,s_x\Big) \qquad \text{and} \qquad \mathcal{A}^{\mathrm{o}} := \frac{1}{2}\Big(\mathcal{A} - s_x\,\mathcal{A}\,s_x\Big)\,. \tag{5.5}$$

In the applications, the results will depend sensitively on how the functional $\mathcal{A}$ is chosen. However, in the following construction we do not need to specify $\mathcal{A}$.

Thus let $\mathcal{A}$ be any functional (5.1). We introduce the Borel measure μ on Symm(S_x) as the push-forward measure of the universal measure,

$$\mu := \mathcal{A}_* \rho \tag{5.6}$$

(meaning that $\mu(\Omega) := \rho\big(\mathcal{A}^{-1}(\Omega)\big)$). In order to get finer information, one can introduce *scaling parameters* α and β with $0 \leq \alpha < \beta$. Defining the set

$$\Lambda^{\alpha,\beta} = \Big\{ y \in \mathcal{F} \;|\; \|y-x\|^\beta < \|\mathcal{A}(y)\| < \|y-x\|^\alpha \Big\}$$

and multiplying by its characteristic function, we obtain the measure

$$\rho^{\alpha,\beta} := \chi_{\Lambda^{\alpha,\beta}}\,\rho\,.$$

Then we set

$$\mu^{\alpha,\beta}(\Omega) = \mathcal{A}_* \rho^{\alpha,\beta}\,. \tag{5.7}$$

This construction is illustrated on the left of Figure 2.

We now let μ be the measure (5.6), and $\mu^\bullet$ either again the measure (5.6) or one of the measures (5.7) involving scaling parameters. A *conical set* is a set of the form $\mathbb{R}^+ A$ with $A \subset \text{Symm}(S_x)$. We denote the conical sets which are both μ- and $\mu^\bullet$-measurable by $\mathfrak{M}$. For a measurable conical set $A \in \mathfrak{M}$ we consider finite partitions into measurable conical sets of the form

$$A = A_1 \dot{\cup} \cdots \dot{\cup} A_K \qquad \text{with} \qquad K \in \mathbb{N}\,.$$

We denote the set of such partitions by $\mathcal{P}_K$. We define

$$\mu^*_{\text{con}}(A) = \sup_{K\in\mathbb{N}} \sup_{\mathcal{P}_K} \sum_{k=1}^K \limsup_{\delta\searrow 0} \frac{1}{\mu(B_\delta(0))}\, \mu^\bullet\big(B_\delta(0)\cap A_k\big) \tag{5.8}$$

(where $B_\delta(0)\subset \text{Symm}(S_x)$ is the Banach space ball $B_\delta(0)\subset \mathrm{L}(\mathcal{H})$ intersected with $\text{Symm}(S_x)$). For a conical set $A\not\in\mathfrak{M}$ we set $\mu^*_{\text{con}}(A)=\infty$. We remark for clarity that, using that the set $\mathcal{A}^{-1}(B_\delta(0))$ is an open neighborhood of x and that $x\in M:=\operatorname{supp}\rho$, it follows that the measure $\mu^\bullet(B_\delta(0))$ is non-zero.

Lemma 5.1. *μ^*_{con} is an outer measure on the conical sets in* $\text{Symm}(S_x)$ *which is finitely additive.*

Proof. According to the definitions (see [24, §7 and §10]), we need to show that

(i) $\mu^*_{\text{con}}(A\dot\cup B)=\mu^*_{\text{con}}(A)+\mu^*_{\text{con}}(B)$ (finite additivity).

(ii) $\mu^*_{\text{con}}\Big(\bigcup_{n=1}^\infty B_n\Big)\leq \sum_{n=1}^\infty \mu^*_{\text{con}}(B_n)$ (countable subadditivity)

(note that the finite additivity implies that $\mu^*_{\text{con}}(\varnothing)=0$ and also yields the monotonicity $A\subset B\Rightarrow \mu^*_{\text{con}}(A)\leq\mu^*_{\text{con}}(B)$).

To show (ii), we know by the definition of the supremum that for any $B:=\cup_n B_n$ and any $\varepsilon>0$ there is K and a partition $\mathcal{P}_K=\{A_1,\ldots,A_K\}$ of B such that

$$\mu^*_{\text{con}}(B)\leq \varepsilon+\sum_{k=1}^K \limsup_{\delta\searrow 0}\frac{1}{\mu(B_\delta(0))}\,\mu^\bullet\big(B_\delta(0)\cap A_k\big)\,.$$

Using the countable subadditivity of the measure μ, we obtain

$$\begin{aligned}
\mu^*_{\text{con}}(B) &\leq \varepsilon+\sum_{k=1}^K \limsup_{\delta\searrow 0}\frac{1}{\mu(B_\delta(0))}\,\sum_{n=1}^\infty \mu^\bullet\big(B_\delta(0)\cap(A_k\cap B_n)\big)\\
&\leq \varepsilon+\sum_{n=1}^\infty\sum_{k=1}^K \limsup_{\delta\searrow 0}\frac{1}{\mu(B_\delta(0))}\,\mu^\bullet\big(B_\delta(0)\cap(A_k\cap B_n)\big)\leq \varepsilon+\sum_{n=1}^\infty \mu^*_{\text{con}}(B_n)\,,
\end{aligned}$$

where in the last step we used properties of the supremum and 5.8. Since ε is arbitrary, we obtain (ii).

To prove (i), we first note that the inequality $\mu^*_{\text{con}}(A\dot\cup B)\leq\mu^*_{\text{con}}(A)+\mu^*_{\text{con}}(B)$ follows from subadditivity. For the opposite inequality, for given $\varepsilon>0$ we choose partitions $P_K=\{A_1,\ldots,A_K\}$ of A and $P'_{K'}=\{B_1,\ldots,B_{K'}\}$ of B such that

$$\begin{aligned}
\mu^*_{\text{con}}(A) &\leq \varepsilon+\sum_{k=1}^K \limsup_{\delta\searrow 0}\frac{\mu^\bullet\big(B_\delta(0)\cap A_k\big)}{\mu(B_\delta(0))}\\
\mu^*_{\text{con}}(B) &\leq \varepsilon+\sum_{k=1}^{K'} \limsup_{\delta\searrow 0}\frac{\mu^\bullet\big(B_\delta(0)\cap B_k\big)}{\mu(B_\delta(0))}\,.
\end{aligned}$$

Since A and B are disjoint, the family $\{A_1,\ldots,A_K,B_1,\ldots,B_{K'}\}$ is a partition of $A\cup B$. Thus adding the two last inequalities and using (5.8), we obtain

$$\mu^*_{\text{con}}(A)+\mu^*_{\text{con}}(B)\leq 2\varepsilon+\mu^*_{\text{con}}(A\cup B)\,.$$

Since ε is arbitrary, the result follows. ∎

Now we can apply the Carathéodory extension lemma to construct a corresponding measure. To this end, we say that a conical set A is μ_{con}**-measurable** if

$$\mu^*_{\text{con}}(Z) = \mu^*_{\text{con}}(Z \cap A) + \mu^*_{\text{con}}(Z \setminus A) \qquad \text{for every conical set } Z\,. \tag{5.9}$$

By the Carathéodory extension lemma (see for example [24, §12] or [4, Lemma 2.8]), the μ_{con}**-measurable** sets form a σ-algebra, and μ^*_{con} is a measure on this σ-algebra. Moreover, the finite additivity implies that (5.9) is satisfies for all conical Borel sets. We have thus proven the following lemma.

Lemma 5.2. *The* μ_{con}**-measurable** *sets form a σ-algebra $\mathfrak{M}_{\text{con}}$, which contains all conical Borel sets in* $\text{Symm}(S_x)$. *The mapping $\mu^*_{\text{con}} : \mathfrak{M}_{\text{con}} \to [0, \infty]$ is a measure.*

We now apply the last lemma to the measure (5.6). For simplicity, we always restrict the measures to the Borel sets, keeping in mind that we can take the completion whenever needed.

Definition 5.3. *We denote the conical Borel sets of* $\text{Symm}(S_x)$ *by* $\mathfrak{B}_{\text{con}}(x)$. *We denote the measure obtained by applying the above construction with $\mathcal{A}$ according to* (5.1) *by*

$$\mu^\bullet_x \,:\, \mathfrak{B}_{\text{con}}(x) \to [0, \infty]\,.$$

It is referred to as the **tangent cone measures** *corresponding to $\mathcal{A}$. The* **tangent cone** $\mathcal{C}^\bullet_x$ *is defined as the support of the tangent cone measure,*

$$\mathcal{C}^\bullet_x := \text{supp}\, \mu^\bullet_x \subset \text{Symm}(S_x)\,.$$

If $\mu\bullet$ is chosen according to (5.7), *we also write the tangent cone as $\mathcal{C}^{\alpha,\beta}_x$. Likewise, for $\mu^\bullet$ according to* (5.6), *we simply omit the superscript.*

This definition is local in the sense that it depends only on the behavior of the topological fermion system in an arbitrarily small neighborhood of x. Also note that μ_x is a normalized measure, whereas the measure $\mu^\bullet_x$ has total volume at most one. The tangent cone is illustrated on the right of Figure 2.

The next lemma illustrates the usefulness of the scaling parameters.

Lemma 5.4. *We choose $\mathcal{A}$ according to* (5.3) *and consider the scaling parameters in the range*

$$0 \le \alpha < \beta < 2\,.$$

Then the vectors in $\mathcal{C}^{\alpha,\beta}_x$ anti-commute with s_x,

$$\{s_x, v\} = 0 \qquad \text{for all } v \in \mathcal{C}^{\alpha,\beta}_x\,.$$

Proof. Setting $\Delta s = s_x - s_y$, the functional (5.3) becomes

$$\mathcal{A} = s_x\, \Delta s\, x\,.$$

Since the definition of $\mathcal{C}^{\alpha,\beta}$ involves y only in an arbitrarily small neighborhood of x, we want to treat Δs perturbatively. We know that for all y, the operator s_y has the eigenvalues ± 1. A standard perturbation argument yields that to first order, the operator Δs vanishes on the eigenspaces of s_x, i.e.

$$0 = \{s_x, \pi_x\, \Delta s\, \pi_x\} + \mathcal{O}\big((\Delta s)^2\big)\,.$$

Note that here only the projection of Δs to S_x enters. The subtle point is that to second and higher order in perturbation theory, the remaining contributions to Δs, i.e. the operator

$$\Delta s - \pi_x\, \Delta s\, \pi_x$$

also comes into play, and we must make sure that the resulting contribution to the eigenvalues of s_y is dominated by the first order in perturbation theory. Here is where the scaling parameter β enters: For example by expressing the signature operators with contour integrals, one readily sees that for y in a small neighborhood of x, the inequality

$$\|\Delta s\| \leq c\, \|y - x\| \tag{5.10}$$

holds (where the constant c depends on the eigenvalues of x). According to the definition of $\Lambda^{\alpha,\beta}$, the measure $\rho^{\alpha,\beta}$ is non-trivial only for points y for which

$$\|\mathcal{A}(y)\| \geq \|y - x\|^\beta \gg \|\Delta s\|^2 ,$$

where in the last step we used (5.10) and and made $y-x$ sufficiently small. This shows that we may restrict attention to points y for which the first order in perturbation theory is non-zero. Then the second and higher orders vanish in the limit $\delta \searrow 0$. $\square$

5.2. **Construction of a Tangential Clifford Section.** We now address the question whether the geometry of the causal fermion system in a neighborhood of a given point $x \in M$ makes it possible to distinguish a specific Clifford subspace at x. We give a construction which achieves this goal under generic assumptions.

We introduce on $\text{Symm}(S_x)$ the symmetric bilinear form

$$\langle .,. \rangle \;:\; \text{Symm}(S_x) \times \text{Symm}(S_x) \to \mathbb{C}\,, \qquad \langle u, v \rangle = \frac{1}{\mathfrak{p} + \mathfrak{q}}\; \text{Tr}(uv) \tag{5.11}$$

(note that this bilinear form generalizes (4.2) because for two operators u, v in a Clifford subspace K, the formulas agree). In the Riemannian case, this bilinear form (5.11) is positive definite and coincides with the usual Hilbert-Schmidt scalar product. For causal fermion systems, however, the inner product $\langle .,. \rangle$ is indefinite. This makes it necessary to treat these two situations separately. We first give the abstract constructions. A more explicit analysis of the cases shown in Figure 1 will be given afterwards.

We begin with *Riemannian fermion systems*, where the construction is somewhat simpler. At a given point $x \in M$, we consider a tangent cone measure $\mu_x^\bullet$ (either without or for a suitable choice of the scaling parameters). Then $\mu_x^\bullet$ gives rise to a Borel measure on the unit sphere $S_1(0) \subset \text{Symm}(S_x)$ simply by taking the measure of the cone $\mathbb{R}^+\Omega$ over the corresponding set $\Omega \subset S_1(0)$. Denoting the orthogonal projection onto a subspace $U \subset \text{Symm}(S_x)$ by π_U, the function L defined by

$$L(U) = \int_{S_1(0) \subset \text{Symm}(S_x)} \text{Tr}_{\text{Symm}(S_x)}(\pi_U\, \pi_{<e>})\, d\mu_x^\bullet(\mathbb{R}^+ e) \tag{5.12}$$

(where $<e>$ denotes the span of e), is non-negative and tells us about the relative position of the two subspaces U and V. Keeping the dimensions fixed, the functional is maximal if one of the subspaces is contained in the other. This motivates us to maximize over all Clifford subspaces,

$$\text{maximize } L(.) \text{ on } \mathcal{K}_x^{(r,0)}\,. \tag{5.13}$$

Definition 5.5. *For a Riemannian fermion system, the tangent cone measure $\mu_x^\bullet$ is* **non-degenerate** *if for all $x \in M$, the optimization problem* (5.13) *has a unique maximizer $K(x) \in \mathcal{K}_x^{r,s}$, which depends continuously on x. Setting $\mathcal{C}\ell_x = K(x)$ defines the* **tangential Clifford section**.

In the case of *causal fermion systems*, we need to construct a distinguished Clifford extensions $K \in \mathcal{K}_x^{s_x,r}$. Thus our task is to find the vectors which extend s_x to the Clifford subspace K. Since these vectors all anti-commute with s_x, it is useful to introduce the set $\mathfrak{A}(s_x)$ of all symmetric operators on S_x which anti-commute with s_x,

$$\mathfrak{A}(s_x) = \big\{ u \in \text{Symm}(S_x) \,|\, \{u, s_x\} = 0 \big\} \subset \text{Symm}(S_x) \,.$$

Lemma 5.6. *The bilinear form* $\langle .,. \rangle|_{\mathfrak{A}(s_x) \times \mathfrak{A}(s_x)}$ *is negative definite.*

Proof. In view of (3.1), for any $v \in \text{Symm}(S_x)$ the operator xv is a symmetric operator on $\mathcal{H}$. Using that the Hilbert-Schmidt norm is positive, we conclude that

$$\text{Tr}(xv\, xv) > 0 \qquad \text{for all } v \in \text{Symm}(S_x), v \neq 0$$

(here we used that x is invertible on S_x by definition of the spin space). For any $u \in \mathfrak{A}(s_x)$ with $u \neq 0$, we choose $v = |x|^{-\frac{1}{2}} u |x|^{-\frac{1}{2}}$. Then, using that s_x has eigenvalues ± 1 and anti-commutes with u, we obtain

$$\text{Tr}(uu) = \text{Tr}(s_x s_x uu) = -\,\text{Tr}(s_x u s_x u) = -\,\text{Tr}(xvxv) < 0 \,,$$

concluding the proof. ☐

We now consider a tangent cone measure $\mu_x^\bullet$ on the block off-diagonal functional $\mathcal{A}^{\rm o}$ in (5.5). Modifying the functional (5.12) to

$$L(U) = \int_{S_1(0) \cap \mathfrak{A}(s_x)} \text{Tr}_{\text{Symm}(S_x)}(\pi_U \, \pi_{<e>}) \, d\mu_x^\bullet(\mathbb{R}^+ e) \,, \tag{5.14}$$

we can

$$\text{maximize } L(.) \text{ on } \mathcal{K}_x^{s_x,r} \,. \tag{5.15}$$

We note for clarity that the trace in (5.14) is taken effectively only over $\mathfrak{A}(S_x) \subset \text{Symm}(S_x)$, because the operator $\pi_{<e>}$ vanishes on $\mathfrak{A}(S_x)^\perp$. Thus we could also have written the trace in (5.14) as

$$\text{Tr}_{\mathfrak{A}(S_x)}(\pi_{\mathfrak{A}(S_x)} \, \pi_U \, \pi_{\mathfrak{A}(S_x)} \, \pi_{<e>}) \,,$$

but we prefer the more compact notation in (5.14).

Definition 5.7. *For a causal fermion system, the tangent cone measure* $\mu_x^\bullet$ *is* **non-degenerate** *if for all* $x \in M$*, the optimization problem* (5.13) *has a unique maximizer* $K(x) \in \mathcal{K}_x^{s_x,r}$*, which depends continuously on* x*. Setting* $\mathcal{C}\ell_x = K(x)$ *defines the* **tangential Clifford section**.

Clearly, a tangential Clifford section can exist only if the general topological obstructions of Section 4.2 are satisfied. Therefore, if we know that there is a non-degenerate tangent cone measure (see Definitions 5.5 and 5.7), we can infer that there are no topological obstructions to the existence of Clifford sections respectively Clifford extensions. Since the variational principles (5.12) and (5.14) are local (because the tangent cone measure $d\mu_x^\bullet$ only depends on the universal measure in an arbitrarily small neighborhood of x), we thus obtain a method to deduce global topological properties of a causal fermion system from its local behavior at every point $x \in M$.

Examples for the construction of the tangential Clifford section will be given in Sections 7.4 and 7.5 below.

5.3. **Construction of a Spin Structure.** We now assume in addition that M^k is a *differentiable manifold* and that the universal measure ρ is *absolutely continuous* with respect to the Lebesgue measure (see Section 4.5). Moreover, we assume as in (4.4) that the dimension of the manifold coincides with the dimension of the Clifford subspaces,

$$r + s = k\,.$$

Our goal is to construct a spin structure (see Definition 4.8).

We first explain how the tangent cone simplifies under the differentiability and regularity assumptions. For simplicity, we work with the definition of the tangent space $u \in T_xM$ as equivalence classes of curves. For any $u \in T_xM$ we choose a representative $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = u$. Then $\gamma(t)$ is a one-parameter family of linear operators in $\mathcal{F}$. Composing with the operator $\mathcal{A}$, (5.1), we obtain a family of operators in $\mathrm{Symm}(S_x)$. We denote the directional derivative by

$$\mathsf{d}_u \mathcal{A} := \frac{d}{dt} \mathcal{A}\big(\gamma(t)\big)\Big|_{t=0}\,. \tag{5.16}$$

Here the derivative exists in view of our assumption (3.12). Obviously, this definition is independent of the choice of the representative γ. Moreover, $\mathsf{d}_u\mathcal{A}$ is again a symmetric operator on S_x. We thus obtain a mapping

$$\mathsf{d}\mathcal{A} \,:\, T_xM \to \mathrm{Symm}(S_x)\,. \tag{5.17}$$

A short consideration shows that

$$\mathcal{C}_x = \mathsf{d}\mathcal{A}(T_xM)\,,$$

so that the tangent cone simplifies to a plane in $\mathrm{Symm}(S_x)$.

We next assume that $\mathcal{C}\ell M$ is a tangential Clifford section (see Definitions 5.5 and 5.7). Then at every point $x \in M$ we can form the mapping

$$\gamma_x = \pi_{\mathcal{C}\ell_x} \circ \mathsf{d}\mathcal{A} \,:\, T_xM \to \mathcal{C}\ell_x \tag{5.18}$$

(with $\mathsf{d}\mathcal{A}$ according to (5.16) and (5.17)). Then γ gives rise to a spin structure, provided that the mapping γ_x is bijective at every point $x \in M$. We point out that this construction is local, and in applications it is easy to verify whether γ_x is bijective. However, if this local condition is satisfies at every point $x \in M$, this gives rise to global properties of M (see Theorem 4.9).

In applications, it may well happen that the mapping γ_x is *not* bijective. In particular, for causal fermion systems, this is closely related to the problem of distinguishing a direction of time. We will analyze this problem in detail in Section 7.5 in the example of two-dimensional Minkowski space. Here we only explain the difficulty in the case that $\mathcal{A}$ depends only on the signature operator at y. For clarity, we first restate Lemma 5.4 in the differentiable setting and give a different proof.

Lemma 5.8. *Let $\mathcal{A}$ be chosen according to* (5.3). *Then for all $u \in T_xM$,*

$$\big\{ s_x, \mathsf{d}_x\mathcal{A}(u) \big\} = 0\,.$$

Proof. Differentiating the relation $(\pi^+_{\gamma(t)})^2 = \pi^+_{\gamma(t)}$ yields

$$\pi^+_x\, (\mathsf{d}_u \pi^+_x) + (\mathsf{d}_u \pi^+_x)\, \pi^+_x = (\mathsf{d}_u \pi^+_x)\,,$$

where $\mathsf{d}_u \pi^+_x \equiv \partial_t \pi^+_{\gamma(t)}|_{t=0}$. Multiplying from the left and right by π^+_x, we obtain

$$\pi^+_x\, (\mathsf{d}_u \pi^+_x)\, \pi^+_x = 0\,. \tag{5.19}$$

Similarly,

$$\pi_x^- \,(\mathsf{d}_u \pi_x^-)\, \pi_x^- = 0 \,. \tag{5.20}$$

Next, we differentiate the relation $\pi^+_{\gamma(t)} \pi^-_{\gamma(t)} = 0$ to obtain

$$\pi_x^+ \,(\mathsf{d}_u \pi_x^-) + (\mathsf{d}_u \pi_x^+)\, \pi_x^- = 0 \,.$$

Multiplying from the left by π_x^-, we conclude that

$$\pi_x^- \,(\mathsf{d}_u \pi_x^+)\, \pi_x^- = 0 \,. \tag{5.21}$$

Similarly, multiplying from the right by π_x^+, we get

$$\pi_x^+ \,(\mathsf{d}_u \pi_x^-)\, \pi_x^+ = 0 \,. \tag{5.22}$$

Using (5.19)–(5.22) in

$$\pi_x = \pi_x^+ + \pi_x^- \,, \qquad s_x = \pi_x^+ - \pi_x^- \,, \tag{5.23}$$

we obtain

$$\pi_x^+ \,(\mathsf{d}_u s_x)\, \pi_x^+ = 0 = \pi_x^- \,(\mathsf{d}_u s_x)\, \pi_x^- \,.$$

A straightforward calculation using again (5.23) gives the claim. □

Now suppose that $\mathcal{C}\ell_x$ is a Clifford extension (as is always the case if we use the construction leading to Definition 5.7). Then the statement of Lemma 5.8 implies that the image of $\mathsf{d}_x\mathcal{A}$ is orthogonal to the vectors $s_x \in \mathcal{C}\ell_x$. As a consequence, the composition (5.18) is necessarily singular. We will come back to this problem in Section 7.5.

We finally point out that the above construction should be handled with care in the sense that it gives topological, but no geometric information on the fermion system. More specifically, a spin structure $\gamma : TM \to \mathcal{C}\ell M$ induces a Riemannian or Lorentzian metric on M by

$$\langle X, Y \rangle := \frac{1}{2} \,\big\langle \gamma(X), \gamma(y) \big\rangle \,. \tag{5.24}$$

However, it would be too naive to interpret this metric as describing the geometry of space or space-time. Namely, for a causal fermion system constructed on a globally hyperbolic space-time (see [12] or [18, Section 4]), the Lorentzian metric and the spin connection are recovered by considering the closed chain A_{xy} for pairs of points x, y whose distance is much larger than the regularization length (see the constructions in [12]; note that in [12, Theorem 4.7] one first takes the limit $\varepsilon \searrow 0$ and then the limit $N \to \infty$). Moreover, the Clifford extension at x must be chosen as a function of y (as is made precise by the synchronizations and splice maps introduced in [12, Section 3]). In view of these constructions, the linearization of s_x as captured by the mapping $\mathsf{d}\mathcal{A}$ in (5.16) does not encode the macroscopic geometry of the causal fermion system. But it can be used to obtain topological information.

6. The Topology of Discrete and Singular Fermion Systems

In the previous Sections 4 and 5, it was essential that M be a topological manifold (or at least a finite cell complex, because otherwise the cohomology groups cannot be introduced). We now explain how our methods and results can be applied even in cases when M is so singular that it has no manifold structure or is discrete. Our technique is to "extend" M to a larger set $\tilde{M} \subset \mathcal{F}$, and to analyze the topology of the enlarged space. For technical simplicity, we restrict attention to the case that the particle space $\mathcal{H}$ is finite-dimensional.

We begin with a simple method which allows us to associate to M a manifold. For given $r > 0$, we first take an r-neighborhood of M,

$$M_r := B_r(M) \subset \mathcal{F} \tag{6.1}$$

(where we work with the distance function induced by the sup-norm of $\mathrm{L}(\mathcal{H})$). Next, for any p, q with $0 \leq p \leq \mathfrak{p}$ and $0 \leq q \leq \mathfrak{q}$ we define the sets

$$\mathcal{F}^{p,q} = \{x \in \mathcal{F} \,|\, x \text{ has } p \text{ positive and } q \text{ negative eigenvalues}\} .$$

Obviously, these sets form a partition of $\mathcal{F}$,

$$\mathcal{F} = \bigcup_{p,q=0}^{n} \mathcal{F}^{p,q} , \qquad \mathcal{F}^{p,q} \cap \mathcal{F}^{p',q'} = \varnothing \quad \text{if } (p,q) \neq (p',q') .$$

Now we set

$$M_r^{p,q} = M_r \cap \mathcal{F}^{p,q} .$$

If $\mathcal{H}$ is finite-dimensional, the sets $\mathcal{F}^{p,q}$ are manifolds. Hence the sets $M_r^{p,q}$ are open submanifolds of $\mathcal{F}^{p,q}$. Then $SM_r^{p,q}$ is a bundle over a smooth manifold, so that the methods of Sections 4 and 5 apply. Clearly, the construction depends on the choice of the parameter r.

An alternative method is to use the universal measure ρ in the construction of $\tilde{M}$: For a given parameter $\delta > 0$, we define the function $r_\delta : \mathcal{F} \to \mathbb{R}_0^+$ by

$$r_\delta(x) = \sup \big\{ r \in \mathbb{R} \mid \rho\big(B_r(x)\big) < \delta \big\} . \tag{6.2}$$

Moreover, we set

$$M_\delta = \bigcup_{x \in M} B_{r_\delta(x)}(x) \qquad \text{and} \qquad M_\delta^{p,q} = M_\delta \cap \mathcal{F}^{p,q} . \tag{6.3}$$

Again, the sets $M_\delta^{p,q}$ are smooth submanifolds of $\mathcal{F}^{p,q}$.

The above constructions give rise to sets $M_\delta, M_r \subset \mathcal{F}$ (and similarly $M_\delta^{p,q}, M_r^{p,q}$) which carry topological information, but unfortunately these sets are no longer the support of a measure, so that they cannot be regarded as the base spaces of corresponding topological fermion systems. This disadvantage can be removed with the following construction, provided that another measure μ on $\mathcal{F}$ is given. In examples when $\mathcal{F}$ is finite-dimensional, one can choose μ as the Lebesgue measure. The infinite-dimensional situation is definitely more difficult, but one could choose μ for example as a Gaussian measure. Given μ, we can choose a smooth test function $\eta_r : C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)$ and define a measure ρ_r by

$$\rho_r(\Omega) := \int_M \left(\int_\Omega \eta_r(x,y)\, d\mu(y) \right) d\rho(x) . \tag{6.4}$$

A typical example is to choose $\eta_r(x,y) = \eta(\|x-y\|^2/r^2)$ with $\eta \in C_0^\infty([0,1))$. The effect of this construction is that the universal measure is "smeared out" in the sense that the support of ρ_r is an r-neighborhood of $\operatorname{supp} \rho$. Hence the effect on the base space is the same as in the construction (6.1), but with the advantage that $(\mathcal{H}, \mathcal{F}, \rho_r)$ is again a topological fermion system. Modifying (6.4) to

$$\rho_\delta(\Omega) := \int_M \left(\int_\Omega \eta_{r_\delta(x)}(x,y)\, d\mu(y) \right) d\rho(x) , \tag{6.5}$$

one obtains similarly a universal measure ρ_δ whose support coincides with M_δ as defined in (6.3).

The above constructions will be illustrated in Section 8.4 in the example of a lattice system with a non-trivial topology.

7. Examples

In this section, we illustrate our abstract constructions in different examples and indicate potential applications.

7.1. The Euclidean Plane.

The Dirac operator on the Euclidean $\mathbb{R}^2$ can be written as

$$\mathcal{D} = i\sigma^1 \partial_{\zeta_1} + i\sigma^2 \partial_{\zeta_2} \,, \tag{7.1}$$

where we denote the points of $\mathbb{R}^2$ by $\zeta = (\zeta_1, \zeta_2)$. The spinor space $(S_\zeta, \prec.|.\succ_\zeta)$ at a point ζ can be identified with $\mathbb{C}^2$ with the canonical Euclidean scalar product. For clarity, we denote this standard spinor space by $(Y \simeq \mathbb{C}^2, \prec.|.\succ)$. We shall consider eigensolutions of the Dirac operator corresponding to an eigenvalue $m \in \mathbb{R}$,

$$\mathcal{D}\psi = m\psi \,. \tag{7.2}$$

Particular solutions can be written as plane waves

$$\mathfrak{e}_k(\zeta) = (k_1\sigma^1 + k_2\sigma^2 + \mathbb{1})\, \chi\, e^{-ik\, m\zeta} \,, \tag{7.3}$$

where the momentum (which for convenience we rescaled by the mass) lies on the unit sphere, $k := (k_1, k_2) \in S^1 \subset \mathbb{R}^2$, and χ is the fixed spinor $\chi = (1, 0)$ (note that the matrix $k{\cdot}\sigma + 1$ has rank one, and therefore we can choose χ arbitrarily, provided that it does not lie in the kernel of this matrix). A general solution can be written as a linear combination of these plane waves,

$$\psi(\zeta) = \int_{S^1} \hat{\psi}(k)\, \mathfrak{e}_k(\zeta)\, d\nu(k) \,, \tag{7.4}$$

where ν is the normalized Lebesgue measure on the sphere.

We want to introduce $\mathcal{H}$ as the solution space of the Dirac equation (7.2). However, since the solutions (7.4) are in general not square integrable, we cannot use the L^2-scalar product. Instead, we make use of the fact that, in view of (7.4), the solution space can be identified with the space of complex-valued functions on the unit sphere. We can thus take the L^2-scalar product on the sphere,

$$\langle \psi | \phi \rangle_{\mathcal{H}} = \int_{S^1} \overline{\hat{\psi}(k)} \hat{\phi}(k)\, d\nu(k) \,. \tag{7.5}$$

The estimate using the Hölder inequality

$$\begin{aligned} \big|\psi(\zeta) - \psi(\zeta')\big| &\leq \int_{S^1} \big|\hat{\psi}(k)\big|\, \big|\mathfrak{e}_k(\zeta) - \mathfrak{e}_k(\zeta')\big|\, d\nu(k) \\ &\leq \|\hat{\psi}\|_{L^1(S^1, d\nu)} \sup_{k \in S^1} \big|\mathfrak{e}_k(\zeta) - \mathfrak{e}_k(\zeta')\big| \leq |\zeta - \zeta'| \sup_{k \in S^1} \|\hat{\psi}\|_{L^2(S^1, d\nu)} \end{aligned}$$

(where $|\psi|$ denotes the $\mathbb{C}^2$-norm of the spinor) shows that the functions in $\mathcal{H}$ are all continuous. Hence we can introduce the local correlation operator at a point ζ again by (2.7).

The local correlation operators can be described conveniently with the help of the so-called evaluation map, as we now explain (see also [12, Section 4.1] or [18, Section 4]). For any $\zeta \in \mathbb{R}^2$, the *evaluation map* e_ζ is defined by

$$e_\zeta \;:\; \mathcal{H} \to S_\zeta \,, \qquad e_\zeta\, \psi = \psi(\zeta) \,. \tag{7.6}$$

We denote its adjoint by ι_ζ,
$$\iota_\zeta := (e_\zeta)^* \;:\; S_\zeta \to \mathcal{H}\,.$$
Combining the computation
$$\langle \psi \,|\, \iota_\zeta(u) \rangle_{\mathcal{H}} = \prec\! e_\zeta(\psi) \,|\, u \!\succ \,=\, \prec\! \psi(\zeta) \,|\, u \!\succ \overset{(7.4)}{=} \int_{S^1} \overline{\hat{\psi}(k)} \prec\! \mathfrak{e}_k(\zeta) \,|\, u \!\succ d\nu(k)$$
with (7.5), one sees that
$$\widehat{\iota_\zeta(u)}(k) = \prec\! \mathfrak{e}_k(\zeta) | u \!\succ . \tag{7.7}$$
It then follows by definition that the local correlation operators take the form
$$F(\zeta) = -\iota_\zeta\, e_\zeta\,. \tag{7.8}$$
We also introduce the so-called *kernel of the fermionic operator* $P(\zeta', \zeta) \in \mathrm{L}(Y)$ by
$$P(\zeta', \zeta) = -e_{\zeta'}\, \iota_\zeta\,. \tag{7.9}$$

Lemma 7.1. *The kernel of the fermionic operator* (7.9) *is given by*
$$P(\zeta', \zeta) = -(\mathcal{D}_{\zeta'} + \mathbb{1}) \int_{S^1} e^{-ikm(\zeta' - \zeta)}\, d\nu(k) \tag{7.10}$$
$$= i\, m(\zeta' - \zeta)\cdot\sigma\, \frac{J_1(m|\zeta - \zeta'|)}{|\zeta - \zeta'|} - m\, J_0\big(m|\zeta - \zeta'|\big)\,. \tag{7.11}$$

Proof. A short computation using (7.4) and (7.7) gives
$$P(\zeta', \zeta) = -\int_{S^1} \mathfrak{e}_k(\zeta') \otimes \mathfrak{e}_k(\zeta)^* d\nu(k) = \int_{S^1} (k\cdot\sigma + \mathbb{1})\, e^{-ikm(\zeta'-\zeta)}\, d\nu(k)\,,$$
where in the last step we applied (7.3) and simplified the Pauli matrices according to
$$(k\cdot\sigma + \mathbb{1}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (k\cdot\sigma + \mathbb{1}) = (k\cdot\sigma + \mathbb{1})\,.$$
Rewriting the factors k as derivatives, we obtain (7.10). Setting $r = |\zeta - \zeta'|$ and $K = |k|$ and denoting the angle between k and $\zeta - \zeta'$ by φ, we can compute the integral in terms of Bessel functions
$$\int_{S^1} e^{-ikm(\zeta'-\zeta)}\, d\nu(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{iKmr\cos\varphi}\, d\varphi = J_0(mr)\,.$$
Using this equation in (7.10), we can compute the derivative with the help of [35, §10.6.3]. This gives (7.11). ∎

We remark that P can also be written as the distributional Fourier transform
$$P(\zeta', \zeta) = -\int_{\mathbb{R}^2} \frac{d^2k}{\pi}\, (k\cdot\sigma + \mathbb{1})\, \delta(k^2 - 1)\, e^{-ikm(\zeta - \zeta')}\,.$$
In this form, it resembles closely the kernel of the so-called fermionic projector of the vacuum in Minkowski space (see for example [14, Lemma 1.1] or [10, Section 5]). To clarify the notions, we remark that the *"fermionic projector"* differs from the "fermionic operator" in that it involves additional normalization conditions (for details see [20]). Since we here disregard such normalization conditions, we prefer to use the term "fermionic operator" throughout.

Having introduced the local correlation operators (see (2.7) or (7.8)), we can introduce the universal measure by $\rho = F_*\mu$, where $d\mu = d^2\zeta$ is the Lebesgue measure. We thus obtain a Riemannian fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension two. Since for every ζ in $\mathbb{R}^2$, there are two functions $\psi, \phi \in \mathcal{H}$ such that $\psi(\zeta)$ and $\phi(\zeta)$ are linearly independent, we conclude that the Riemannian fermion system is regular. In what follows, we again identify ζ with the corresponding local correlation operator $F(\zeta)$.

Lemma 7.2. *The above Riemannian fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ has the properties*

$$\zeta = -2\,\pi_\zeta \tag{7.12}$$

$$\pi_\zeta\, \zeta'\, \zeta = -m^2 \left(\big|J_1(m|\xi|)\big|^2 + \big|J_0(m|\xi|)\big|^2 \right) \pi_\zeta\,, \tag{7.13}$$

where we used the abbreviation $\xi = \zeta' - \zeta$.

Proof. Let us compute the two non-trivial eigenvalues of $F(\zeta)$. Using (7.3), we obtain

$$\langle \psi | F(\zeta)\, \phi \rangle_{\mathcal{H}} = -\int_{S^1} d\nu(k')\, \overline{\hat{\psi}(k')} \int_{S^1} d\nu(k)\, \hat{\phi}(k)\, e^{i(k'-k)m\zeta}\, \langle (k'\!\cdot\!\sigma + \mathbb{1})\, \chi \,|\, (k\!\cdot\!\sigma + \mathbb{1})\, \chi \rangle_{\mathbb{C}^2}\,.$$

Comparing with (7.5) and simplifying the Pauli matrices by

$$\langle (k'\!\cdot\!\sigma + \mathbb{1})\, \chi \,|\, (k\!\cdot\!\sigma + \mathbb{1})\, \chi \rangle_{\mathbb{C}^2} = 2\left(1 + e^{i\varphi(k',k)}\right),$$

where $\varphi(k', k)$ denotes the angle between k' and k (measured from k' in counter-clockwise direction), we conclude that

$$F(\zeta)\, \psi = -2 \int_{S^1} d\nu(k') \int_{S^1} d\nu(k)\, \hat{\psi}(k)\, e^{i(k'-k)m\zeta} \left(1 + e^{i\varphi(k',k)}\right) \mathfrak{e}_{k'}\,. \tag{7.14}$$

In view of (7.4), this can be written in the shorter form

$$F(\zeta)\, \mathfrak{e}_k = -2 \int_{S^1} d\nu(k')\, \mathfrak{e}_{k'}\, e^{i(k'-k)m\zeta} \left(1 + e^{i\varphi(k',k)}\right). \tag{7.15}$$

Iterating this relation, we obtain

$$\begin{aligned} F(\zeta)\, F(\zeta)\, \mathfrak{e}_k &= 4 \int_{S^1} d\nu(k'')\, \mathfrak{e}_{k''}\, e^{i(k''-k)m\zeta} \int_{S^1} d\nu(k') \left(1 + e^{i\varphi(k'',k')}\right)\left(1 + e^{i\varphi(k',k)}\right) \\ &= 4 \int_{S^1} d\nu(k'')\, \mathfrak{e}_{k''}\, e^{i(k''-k)m\zeta} \left(1 + e^{i\varphi(k'',k)}\right) = -2F(\zeta)\, \mathfrak{e}_k\,, \end{aligned}$$

where in the last line we carried out the integral over $k' \in S^1$. We conclude that $F(\zeta)$ has the eigenvalue -2 with multiplicity two. Identifying ζ with $F(\zeta)$, we thus obtain (7.12).

In preparation for proving (7.13), we form the so-called *closed chain* by taking the product of the kernel of the fermionic operator with its adjoint,

$$A_{\zeta\zeta'} := P(\zeta, \zeta')\, P(\zeta', \zeta)\,.$$

(for the motivation of the name "closed chain" we refer to [8, §3.5]). Using the explicit formula (7.11), we obtain

$$\begin{aligned} A_{\zeta\zeta'} &= \left(i\, m\, \xi\!\cdot\!\sigma\, \frac{J_1(m|\xi|)}{|\xi|} + m\, J_0(m|\xi|) \right) \left(-i\, m\, \xi\!\cdot\!\sigma\, \frac{J_1(m|\xi|)}{|\xi|} + m\, J_0(m|\xi|) \right) \\ &= m^2 \left(\big|J_1(m|\xi|)\big|^2 + \big|J_0(m|\xi|)\big|^2 \right). \end{aligned} \tag{7.16}$$

In particular, the closed chain is a multiple of the identity matrix. As a consequence,

$$\begin{aligned} \pi_\zeta\, \zeta' \zeta &\overset{(7.12)}{=} -\frac{1}{2}\, \zeta\, \zeta'\, \zeta \overset{(7.8)}{=} -\frac{1}{2}\, (\iota_\zeta\, e_\zeta)\, (\iota_{\zeta'}\, e_{\zeta'})\, (\iota_\zeta\, e_\zeta) \overset{(7.9)}{=} -\frac{1}{2}\, \iota_\zeta\, P(\zeta, \zeta')\, P(\zeta', \zeta)\, e_\zeta \\ &= -\frac{m^2}{2} \left(|J_1|^2 + |J_0|^2\right) \iota_\zeta\, e_\zeta \overset{(7.8)}{=} \frac{m^2}{2} \left(|J_1|^2 + |J_0|^2\right) \zeta = -m^2 \left(|J_1|^2 + |J_0|^2\right) \pi_\zeta\,. \end{aligned}$$

This concludes the proof. □

We remark that, combining (7.14) and (7.4), we can write $F(\zeta)$ as the integral operator

$$(\widehat{F(\zeta)\,\psi})(k') = \int_{S^1} F_\zeta(k',k)\,\hat{\psi}(k)\,d\nu(k) \tag{7.17}$$

with the kernel

$$F_\zeta(k',k) = -2e^{i(k'-k)m\zeta}\left(1+e^{i\varphi(k',k)}\right).$$

This makes it possible to compute the trace of $F(\zeta)$ by

$$\mathrm{tr}\left(F(\zeta)\right) = \int_{S^1} F_\zeta(k,k)\,d\nu(k) = -4\,,$$

in agreement with (7.12) and the fact that ζ has rank two.

The main conclusion from Lemma 7.2 is that the operators A chosen by (5.2), (5.3) or (5.4) are all a multiple of the identity. This implies that by analyzing the operator ζ' in a neighborhood of ζ (for example using tangent cone measures corresponding to the mapping (5.4)), it is impossible to distinguish a Clifford subspace at a point ζ. In more technical terms, the non-degeneracy property of Definition 5.5 is necessarily violated. A possible method to avoid this shortcoming is to use the decomposition into left- and right-handed spinors. Before introducing this method in Section 7.3 below, we proceed by adapting the present example to Lorentzian signature.

7.2. **Two-Dimensional Minkowski Space.** We now work out an example in two-dimensional Minkowski space (a similar example in four-dimensional Minkowski space is given in [12, Section 4]). We let (M,g) be two-dimensional Minkowski space. We work in the coordinates $\zeta=(t,x)$ in which $ds^2=dt^2-dx^2$. The Dirac operator can be written as

$$\mathcal{D} = i\gamma^0\partial_t + i\gamma^1\partial_x \tag{7.18}$$

with the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The spinor space $(S_\zeta, \prec.,.\succ_\zeta)$ at a point ζ can be identified with the inner product space $(Y\simeq\mathbb{C}^2, \prec.|.\succ)$, where the spin scalar product is defined by

$$\prec\psi|\phi\succ = \langle\psi|\gamma^0\phi\rangle_{\mathbb{C}^2}\,.$$

The Dirac matrices are obviously symmetric with respect to the spin scalar product.

We consider solutions of the Dirac equation

$$(\mathcal{D}-m)\,\psi = 0\,,$$

where $m>0$ is a given mass parameter. Since the Dirac equation is a linear hyperbolic equation, we know that the initial value problem is well-posed, and that there is a finite speed of propagation. This implies that if a solution has compact support at some time t_0, it will also have compact support at any other time. On such *spatially compact solutions* one can introduce the scalar product

$$(\psi|\phi)_{t_0} = \int_{-\infty}^{\infty} \prec\psi|\gamma^0\phi\succ(t_0,x)\,dx\,. \tag{7.19}$$

The integrand of $(\psi|\psi)_{t_0}$ has the physical interpretation as the probability density of a quantum mechanical particle to be at the position x. Due to current conservation, this scalar product is independent of the time t_0. Therefore, we can simply denote it by $(.|.)$.

Similar as explained in Section 7.1 in the Euclidean setting, the Dirac equation can again be solved by plane wave solutions, which we write as

$$\mathfrak{e}_k(\zeta) = \frac{1}{\sqrt{|k_0|}}\,(k^0\gamma^0 - k^1\gamma^1 + m)\,\chi\, e^{-ik\zeta}\,, \tag{7.20}$$

where the momentum lies on the mass shell

$$k_0^2 - k_1^2 = m^2\,, \tag{7.21}$$

and χ is the fixed spinor $\chi = (1, i)/\sqrt{2}$ (here $k\zeta = k_0\zeta^0 - k_1\zeta^1$ is a Minkowski inner product). A general solution can be written as an integral over the mass shell, which is most conveniently written with a δ-distribution,

$$\psi(\zeta) = \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\,\hat{\psi}(k)\,\mathfrak{e}_k(\zeta)\,\delta\big(k_0^2 - k_1^2 - m^2\big)\,, \tag{7.22}$$

where $\hat{\psi}$ is a complex-valued function on the mass shell. If the function $\hat{\psi}$ satisfies suitable regularity and decay conditions, the wave function ψ will decay at spatial infinity, so that the scalar product (7.19) is finite. More specifically, the scalar product can be expressed by an integral over the mass shell:

Lemma 7.3. *For solutions in the Fourier representation* (7.22)*, the scalar product* (7.19) *becomes*

$$(\psi|\phi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} d^2k\; \delta\big(k_0^2 - k_1^2 - m^2\big)\;\overline{\hat{\psi}(k)}\;\hat{\phi}(q)\,. \tag{7.23}$$

Proof. A direct computation gives

$$\begin{aligned}
(\psi|\phi) &= \int_{-\infty}^{\infty} dx\; e^{-i(q_1-k_1)x} \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\,\delta\big(k_0^2-k_1^2-m^2\big) \int_{\mathbb{R}^2} \frac{d^2q}{(2\pi)^2}\,\delta\big(q_0^2-q_1^2-m^2\big) \\
&\qquad \times \frac{1}{\sqrt{|k_0|\,|q_0|}}\;\overline{\hat{\psi}(k)}\;\hat{\phi}(q)\; \prec\!(k^0\gamma^0 - k^1\gamma^1 + m)\,\chi\,|\,\gamma^0\,(q^0\gamma^0 + q^1\gamma^1 + m)\,\chi\!\succ \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} d^2k\;\delta\big(k_0^2-k_1^2-m^2\big) \sum_{s=\pm 1} \frac{1}{2\sqrt{m^2+k_1^2}} \\
&\qquad \times \frac{1}{|k_0|}\;\overline{\hat{\psi}(k)}\;\hat{\phi}(q)\; \prec\!(k^0\gamma^0 - k^1\gamma^1 + m)\,\chi\,|\,\gamma^0\,(s\,k^0\gamma^0 - k^1\gamma^1 + m)\,\chi\!\succ\,.
\end{aligned}$$

The Dirac matrices can be simplified by

$$\begin{aligned}
&(k^0\gamma^0 - k^1\gamma^1 + m)\,\gamma^0\,(s\,k^0\gamma^0 - k_1\gamma^1 + m) \\
&\quad = \gamma^0(k^0\gamma^0 + k^1\gamma^1 + m)(s\,k^0\gamma^0 - k^1\gamma^1 + m) \\
&\quad = \gamma^0\,\big(s\,k_0^2 + k_1^2 + m^2 + (1+s)\,mk^0\gamma^0 - (1+s)\,\gamma^0\gamma^1\,k^0k^1\big)\,.
\end{aligned}$$

In the case $s = -1$, this expression vanishes in view of (7.21). In the case $s = 1$, on the other hand, we obtain

$$\begin{aligned}
&(k^0\gamma^0 - k^1\gamma^1 + m)\,\gamma^0\,(k^0\gamma^0 - k^1\gamma^1 + m) \\
&\quad = \gamma^0\big(2k_0^2 + 2m\,k^0\gamma^0 - 2\,\gamma^0\gamma^1\,k^0k^1\big) = 2k^0\big(k^0\gamma^0 - k^1\gamma^1 + m\big)\,.
\end{aligned} \tag{7.24}$$

Next, the expectation values of the spinor χ are computed by

$$\prec\!\chi|\gamma^0\chi\!\succ = \langle\chi|\chi\rangle_{\mathbb{C}^2} = 1\,, \qquad \prec\!\chi|\chi\!\succ = 0 = \prec\!\chi|\gamma^1\chi\!\succ\,.$$

Finally, we again use (7.21) to obtain the result. □

With (7.23) we rewrote the scalar product (7.19) coming from the probabilistic interpretation of the Dirac equation in a way which is very similar to the scalar product (7.5). The only difference is that instead of integrating over the circle, we now integrate over the hyperbolas of the mass shell. Since these hyperbolas are non-compact, the estimate used after (7.5) to prove the continuity of the wave functions fails. This makes it necessary to introduce an ultraviolet regularization, as we now explain. First, we consider solutions of negative energy, obtained from (7.22) by integrating only over the lower mass shell,

$$\begin{aligned}\psi(\zeta) &= \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, \hat{\psi}(k)\, \mathfrak{e}_k(\zeta)\, \delta\big(k_0^2 - k_1^2 - m^2\big)\, \Theta(-k^0) \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dp}{2\sqrt{p^2+m^2}}\, \hat{\psi}(k)\, \mathfrak{e}_k(\zeta)\Big|_{k=\left(-\sqrt{p^2+m^2},p\right)}\,.\end{aligned}$$

We denote the scalar product (7.19) on these wave functions by $\langle.|.\rangle_{\mathcal{H}}$. Taking the completion, we obtain the Hilbert space $(\mathcal{H}, \langle.|.\rangle_{\mathcal{H}})$. For a wave function $\psi \in \mathcal{H}$, the evaluation map (7.6) will in general be ill-defined, because ψ need not be continuous. The simplest method to cure this problem is to insert a convergence-generating factor into the Fourier integral. Thus for a given parameter $\varepsilon > 0$ we set

$$e^\varepsilon_\zeta(\psi) = \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\frac{\varepsilon k^0}{2}}\, \hat{\psi}(k)\, \mathfrak{e}_k(\zeta)\, \delta\big(k_0^2 - k_1^2 - m^2\big)\, \Theta(-k^0) \;:\; S_\zeta \to \mathcal{H}\,. \tag{7.25}$$

We define ι^ε_ζ as its adjoint,

$$\iota^\varepsilon_\zeta = \big(e^\varepsilon_\zeta\big)^* \;:\; \mathcal{H} \to S_\zeta\,.$$

Comparing the computation

$$\langle \psi \,|\, \iota^\varepsilon_\zeta(u)\rangle_{\mathcal{H}} = \prec e^\varepsilon_\zeta(\psi)\,|\,u\succ = \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\frac{\varepsilon k^0}{2}}\, \overline{\hat{\psi}(k)} \prec \mathfrak{e}_k(\zeta)|u\succ \delta\big(k_0^2 - k_1^2 - m^2\big)\, \Theta(-k^0)$$

with (7.23), we obtain

$$\widehat{\iota^\varepsilon_\zeta(u)}(k) = e^{\frac{\varepsilon k^0}{2}}\, \Theta(-k^0) \prec \mathfrak{e}_k(\zeta)|u\succ\,. \tag{7.26}$$

Before going on, we remark that there are of course many other ways to introduce an ultraviolet regularization. In generalization of the convergence-generating factor in (7.25), one can work with so-called regularization operators as introduced in [18, Section 4].

We next define the regularized local correlation operators in analogy to (7.8) by

$$F(\zeta) = -\iota^\varepsilon_\zeta\, e^\varepsilon_\zeta\,. \tag{7.27}$$

Moreover, the kernel of the fermionic operator is introduced similar to (7.9) by

$$P(\zeta', \zeta) := -e^\varepsilon_{\zeta'}\, \iota^\varepsilon_\zeta\,. \tag{7.28}$$

The next lemma represents $P(\zeta, \zeta')$ as an integral over the lower mass shell, which is Lorentz invariant except for the convergence-generating factor $e^{\varepsilon k^0}$.

Lemma 7.4. *The kernel of the fermionic operator is given by*

$$
\begin{aligned}
P(\zeta',\zeta) &= \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\varepsilon k^0}\, (k^0\gamma^0 - k^1\gamma^1 + m)\, \delta\big(k_0^2 - k_1^2 - m^2\big)\, \Theta(-k^0)\, e^{-ik(\zeta'-\zeta)} \\
&= \frac{m}{4\pi^2}\, K_0\Big(-im\sqrt{(t'-t+i\varepsilon)^2 - (x'-x)^2}\Big) \\
&\quad + \frac{m}{4\pi^2}\, \frac{K_1\Big(-im\sqrt{(t'-t+i\varepsilon)^2 - (x'-x)^2}\Big)}{\sqrt{(t'-t+i\varepsilon)^2-(x'-x)^2}} \begin{pmatrix} t'-t+i\varepsilon & -x'+x \\ x'-x & -t'+t-i\varepsilon \end{pmatrix} . \qquad (7.29)
\end{aligned}
$$

Proof. Substituting (7.26) and (7.25) into (7.28), we obtain

$$
P(\zeta',\zeta) = -\int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\varepsilon k^0}\, \mathfrak{e}_k(\zeta') \otimes \mathfrak{e}_k(\zeta)^*\, \delta\big(k_0^2-k_1^2-m^2\big)\, \Theta(-k^0)
$$

A short calculation shows that $\chi \otimes \chi^* = (\gamma^0 - \gamma^2)/2$ with

$$
\gamma^2 := -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .
$$

Hence can apply (7.24) together with

$$
\begin{aligned}
&(k^0\gamma^0 - k^1\gamma^1 + m)\, \gamma^2\, (k^0\gamma^0 - k^1\gamma^1 + m) \\
&\qquad = \gamma^2(-k^0\gamma^0 + k^1\gamma^1 + m)(k^0\gamma^0 - k^1\gamma^1 + m) = \gamma^2\left(-k_0^2 + k_1^2 + m^2\right) \overset{(7.21)}{=} 0
\end{aligned}
$$

to obtain

$$
P(\zeta',\zeta) = -\int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\varepsilon k^0}\, \frac{k_0}{|k_0|}\, (k^0\gamma^0 - k^1\gamma^1 + m)\, \delta\big(k_0^2-k_1^2-m^2\big)\, \Theta(-k^0)\, e^{-ik(\zeta'-\zeta)} .
$$

This gives the Fourier integral in the statement of the lemma. It can be computed by

$$
\begin{aligned}
P(\zeta',\zeta) &= (i\partial\!\!\!/_{\zeta'} + m) \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\varepsilon k^0}\, \delta\big(k_0^2-k_1^2-m^2\big)\, \Theta(-k^0)\, e^{-ik(\zeta'-\zeta)} \\
&= (i\partial\!\!\!/_{\zeta'} + m) \int_{-\infty}^{-m} \frac{d\omega}{(2\pi)^2}\, \frac{e^{\varepsilon\omega}}{|k^1|}\, e^{-i\omega(t'-t)}\, \cos(k^1\,(x'-x))\Big|_{k^1=\sqrt{\omega^2-m^2}} \\
&= \frac{1}{4\pi^2}\, (i\partial\!\!\!/_{\zeta'} + m) K_0\Big(-im\sqrt{(t'-t+i\varepsilon)^2-(x'-x)^2}\Big) ,
\end{aligned}
$$

where K_0 is the modified Bessel function (see [35, §10.25]). Using the formulas for derivatives of Bessel functions (see [35, §10.29(ii)]), we obtain (7.29). □

Having introduced the local correlation operators by (7.27), we can introduce the universal measure again by $\rho = F_*(dt\,dx)$ to obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension one. In what follows, we again identify the space-time point ζ with the corresponding local correlation operator $F(\zeta) \in \mathcal{F}$. For the detailed computations, it is convenient to choose pseudo-orthonormal bases $(\mathfrak{f}_\alpha(\zeta))_{\alpha=1,2}$ of the spin spaces S_ζ. To this end, we first need to calculate the nontrivial eigenvalues of $F(\zeta)$. Comparing (7.27) and (7.28), these eigenvalues are the same as those of the matrix $P(\zeta,\zeta)$. Applying Lemma 7.4 and using the symmetry under the transformation $k_1 \to -k_1$, we obtain

$$
\begin{aligned}
P(\zeta,\zeta) &= \int_{\mathbb{R}^2} \frac{d^2k}{(2\pi)^2}\, e^{\varepsilon k^0}\, (k^0\gamma^0 + m)\, \delta\big(k_0^2-k_1^2-m^2\big)\, \Theta(-k^0) \\
&= \int_{-\infty}^{-m} \frac{dk_0}{(2\pi)^2}\, e^{\varepsilon k^0}\, \frac{k^0\gamma^0 + m}{2\sqrt{k_0^2-m^2}} .
\end{aligned}
$$

This matrix has the eigenvalues

$$\nu_{1/2} := \int_{-\infty}^{-m} \frac{dk_0}{(2\pi)^2}\, e^{\varepsilon k^0}\, \frac{\pm k^0 + m}{2\sqrt{k_0^2 - m^2}}\,,$$

and the corresponding eigenvectors $\mathfrak{e}_1 = (1,0)$ and $\mathfrak{e}_2 = (0,1)$ are the canonical pseudo-orthonormal basis of Y (note that $\nu_1 < 0$ and $\nu_2 > 0$). A straightforward computation using again (7.27) and (7.28) shows that corresponding eigenvectors of $F(\zeta)$ are given by

$$\mathfrak{f}_\alpha(\zeta) := \frac{1}{\nu_\alpha}\, \iota_\zeta\, \mathfrak{e}_\alpha\,, \qquad \alpha = 1,2\,. \tag{7.30}$$

Moreover, the computation

$$\begin{aligned}
\prec \mathfrak{f}_\alpha | \mathfrak{f}_\beta \succ_x &= -\langle \mathfrak{f}_\alpha | F^\varepsilon(p)\, \mathfrak{f}_\beta \rangle_{\mathcal{H}} = -\nu_\beta\, \langle \mathfrak{f}_\alpha | \mathfrak{f}_\beta \rangle_{\mathcal{H}} \\
&= -\frac{1}{\nu_\alpha}\, \langle \iota_x \mathfrak{e}_\alpha | \iota_x^\varepsilon \mathfrak{e}_\beta \rangle_{\mathcal{H}} = -\frac{1}{\nu_\alpha} \prec \mathfrak{e}_\alpha | e_x^\varepsilon\, \iota_x^\varepsilon \mathfrak{e}_\beta \succ \\
&\overset{(7.28)}{=} \frac{1}{\nu_\alpha} \prec \mathfrak{e}_\alpha | P(\zeta,\zeta)\, \mathfrak{e}_\beta \succ = \frac{\nu_\beta}{\nu_\alpha} \prec \mathfrak{e}_\alpha | \mathfrak{e}_\beta \succ = s_\alpha\, \delta_{\alpha\beta}
\end{aligned}$$

(with $s_{1/2} = \pm 1$) shows that $(\mathfrak{f}_\alpha(\zeta))$ is indeed a pseudo-orthonormal basis of $(S_\zeta, \prec .|. \succ_\zeta)$.

Lemma 7.5. *In the pseudo-orthonormal spinor bases $\mathfrak{f}_{1/2}(\zeta)$, we have*

$$\zeta = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \qquad \pi_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.31}$$

$$\pi_{\zeta'}\, \zeta = P(\zeta', \zeta) \quad \textit{as defined by (7.28) and given by (7.29)} \tag{7.32}$$

$$\pi_\zeta\, \zeta'\, \zeta = P(\zeta,\zeta')\, P(\zeta',\zeta) \tag{7.33}$$

$$\pi_\zeta\, s_{\zeta'}\, \zeta = -P(\zeta,\zeta') \begin{pmatrix} |\nu_1|^{-1} & 0 \\ 0 & |\nu_2|^{-1} \end{pmatrix} P(\zeta',\zeta) \tag{7.34}$$

$$\pi_\zeta\, \pi_{\zeta'}\, \zeta = P(\zeta,\zeta') \begin{pmatrix} \nu_1^{-1} & 0 \\ 0 & \nu_2^{-1} \end{pmatrix} P(\zeta',\zeta). \tag{7.35}$$

Proof. The equations (7.31) are obvious from the fact that the $(\mathfrak{f}_\alpha(\zeta))_{\alpha=1,2}$ are eigenvectors of ζ. In order to derive the other formulas, we first note that the matrix representation of a general operator $B : S_\zeta \to S'_\zeta$ is computed by

$$B^\alpha_\beta = s_\alpha \prec \mathfrak{f}_\alpha(\zeta') \,|\, B \mathfrak{f}_\beta(\zeta) \succ_{\zeta'} = -s_\alpha\, \langle \mathfrak{f}_\alpha(\zeta') \,|\, F(\zeta') B \mathfrak{f}_\beta(\zeta) \rangle_{\mathcal{H}}\,.$$

In particular,

$$\begin{aligned}
(\pi_{\zeta'}\, \zeta)^\alpha_\beta &= -s_\alpha\, \langle \mathfrak{f}_\alpha(\zeta') \,|\, F(\zeta')\, \pi_{\zeta'}\, \zeta\, \mathfrak{f}_\beta(\zeta) \rangle_{\mathcal{H}} \\
&= -s_\alpha\, \langle \mathfrak{f}_\alpha(\zeta') \,|\, F(\zeta')\, \zeta\, \mathfrak{f}_\beta(\zeta) \rangle_{\mathcal{H}} = -s_\alpha \nu_\alpha \nu_\beta\, \langle \mathfrak{f}_\alpha(\zeta') \,|\, \mathfrak{f}_\beta(\zeta) \rangle_{\mathcal{H}} \\
&= -s_\alpha\, \langle \iota_{\zeta'} \mathfrak{e}_\alpha \,|\, \iota_\zeta \mathfrak{e}_\beta \rangle_{\mathcal{H}} = -s_\alpha \prec \mathfrak{e}_\alpha \,|\, (\iota_{\zeta'})^* \iota_\zeta\, \mathfrak{e}_\beta \succ \overset{(7.28)}{=} s_\alpha \prec \mathfrak{e}_\alpha | P(\zeta',\zeta)\, \mathfrak{e}_\beta \succ\,,
\end{aligned}$$

proving (7.32). The relation (7.33) follows immediately from (7.32).

Next, comparing the matrices in (7.31) and using that they are all diagonal, we obtain

$$(\pi_\zeta\, s_{\zeta'})^\alpha_\beta = (\pi_\zeta\, \zeta')^\alpha_\beta \left(-\frac{1}{|\nu_\beta|} \right) = -P(\zeta,\zeta')^\alpha_\beta\, \frac{1}{|\nu_\beta|}\,.$$

Multiplying by (7.32) gives (7.34). The identity (7.35) is obtained similarly. $\square$

We remark that the relation (7.32) can also be used to define the kernel of the fermionic operator on general topological fermion systems by $P(x, y) = \pi_x y : S_y \to S_x$. This is indeed the procedure in [14, 12] (cf. [14, eq. (1.15)] and [12, eq. (2.7)]). In order to present a somewhat different point of view, we here preferred to define the kernel of the fermionic operator by (7.28), which is also more convenient for doing computations.

7.3. **The Euclidean Plane with Chiral Asymmetry.** We now return to the Euclidean plane as considered in Section 7.1. We modify this example as follows. On Y we introduce the linear operator $\Gamma = \sigma^3$. This operator anti-commutes with the Dirac operator (7.1),

$$\mathcal{D}\Gamma = -\Gamma\mathcal{D}\,.$$

The eigenspaces of Γ give a $\mathbb{Z}_2$-grading of S_x. In analogy to the splitting into right- and left-handed components in four space-time dimensions, we we refer to this grading as the *chiral grading*. Next, we modify the evaluation map by inserting the operator Γ,

$$e_\zeta \,:\, \mathcal{H} \to S_\zeta\,, \qquad e_\zeta\,\psi = (\mathbb{1} + \tau\Gamma)\psi(\zeta)\,, \tag{7.36}$$

where τ is a real parameter. We again denote the adjoint of the evaluation map by ι_ζ and introduce the local correlation operators and the kernel of the fermionic operator by (7.8) and (7.9). In order to obtain an explicit formula for $P(\zeta', \zeta)$, we simply multiply (7.11) from the left and right by $(\mathbb{1} + \tau\Gamma)$,

$$\begin{aligned} &P(\zeta', \zeta) \\ &= -(\mathbb{1} + \tau\Gamma)\left(im(\zeta - \zeta')\cdot\sigma\, \frac{J_1(m|\zeta - \zeta'|)}{|\zeta - \zeta'|} + m\,J_0\big(m|\zeta - \zeta'|\big) \right)(\mathbb{1} + \tau\Gamma)\,. \end{aligned} \tag{7.37}$$

Having introduced the local correlation operators (see (2.7) or (7.8)), we can introduce the universal measure by $\rho = F_*\mu$, where $d\mu = d^2\zeta$ is the Lebesgue measure. We thus obtain a Riemannian fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension two. For the detailed computations, it is convenient to work with an orthonormal basis $(\mathfrak{f}_\alpha(\zeta))_{\alpha=1,2}$ of the spin spaces S_ζ. We again follow the procedure explained before (7.30). Expanding the Bessel functions in (7.37) around zero, one readily finds that the matrix $P(\zeta, \zeta)$ has the eigenvectors $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with corresponding eigenvalues

$$\nu_1 = -m\,(1 + \tau^2)^2 \qquad \text{and} \qquad \nu_2 = -m\,(1 - \tau^2)^2\,.$$

As a consequence, the vectors $\mathfrak{f}_{1/2}(\zeta)$ defined in analogy to (7.30) by

$$\mathfrak{f}_\alpha(\zeta) := \frac{1}{\nu_\alpha}\,\iota_\zeta\,\mathfrak{e}_\alpha\,, \qquad \alpha = 1, 2\,, \tag{7.38}$$

form an orthonormal basis of S_ζ.

Lemma 7.6. *In the orthonormal spinor bases $\mathfrak{f}_{1/2}(\zeta)$, we have*

$$\zeta = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \qquad \pi_\zeta = s_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.39}$$

$$\pi_{\zeta'}\zeta = P(\zeta', \zeta) \quad \textit{as defined by (7.28) and given by (7.37)} \tag{7.40}$$

$$\pi_\zeta\,\zeta'\,\zeta = P(\zeta, \zeta')\,P(\zeta', \zeta) \tag{7.41}$$

$$\pi_\zeta\,s_{\zeta'}\,\zeta = \pi_\zeta\,\pi_{\zeta'}\,\zeta = -P(\zeta, \zeta') \begin{pmatrix} |\nu_1|^{-1} & 0 \\ 0 & |\nu_2|^{-1} \end{pmatrix} P(\zeta', \zeta). \tag{7.42}$$

Proof. Follows exactly as in the proof of Lemma 7.5. □

We finally explain the notion "chiral asymmetry." As one sees best in (7.37), the matrix $(1 + \tau\Gamma)$ inserted in (7.36) multiplies the left- and right-handed components of the fermionic operator by certain prefactors. In the case $\tau \neq 0$, these prefactors are different for the left- and right-handed components. Thus the chiral symmetry is broken. Such a chiral asymmetry is present in a more realistic physical situation in the neutrino sector of the fermionic projector in Minkowski space (for details see [11, Sector 2]).

7.4. **The Spin Structure of the Euclidean Plane with Chiral Asymmetry.** We shall now explore the constructions of Section 5 in the example of the Euclidean plane with chiral asymmetry. This is a preparation for a similar analysis for the two-dimensional Minkowski space, which is a bit more subtle and will be given in Section 7.5 below. Our first step is to construct the tangent cone measure, following the general construction described in Section 5.1. In preparation, we need to specify the functional $\mathcal{A}$ in (5.1). In all the following computations, we work in the basis $(\mathfrak{f}_{1/2}(\zeta))$ of the spin spaces introduced in (7.38). Combining the result of Lemma 7.6 with the explicit form for the kernel of the fermionic operator (7.37), we obtain

$$\pi_\zeta \, s_{\zeta'} \, \zeta = -m \left(J_0(m|\xi|)^2 + J_1(m|\xi|)^2 \right) \begin{pmatrix} (1+\tau^2)^2 & 0 \\ 0 & (1-\tau^2)^2 \end{pmatrix} \tag{7.43}$$

$$\begin{aligned} \pi_\zeta \, \zeta' \, \zeta &= (\mathbb{1} + \tau\Gamma) \Big(im \, \xi \!\cdot\! \sigma \, \frac{J_1(m|\xi|)}{|\xi|} + m \, J_0(m|\xi|) \Big) (\mathbb{1} + \tau\Gamma)^2 \\ &\qquad \times \Big(-im \, \xi \!\cdot\! \sigma \, \frac{J_1(m|\xi|)}{|\xi|} + m \, J_0(m|\xi|) \Big) (\mathbb{1} + \tau\Gamma) \\ &= a(\xi) \, \mathbb{1} + b(\xi) \, \Gamma + c(\xi) \, i \, \xi \!\cdot\! \sigma \, \Gamma \,, \end{aligned} \tag{7.44}$$

where we again set $\xi = \zeta' - \zeta$ and

$$\begin{aligned} a(\xi) &= m^2 \left((1 + 6\tau^2 + \tau^4) \, J_0(m|\xi|)^2 - (1-\tau^2)^2 \, J_1(m|\xi|)^2 \right) \\ b(\xi) &= 4m^2 \, \tau^2 (1+\tau^2) \, J_0(m|\xi|)^2 \\ c(\xi) &= -4m^2 \, \tau (1-\tau^2) \, \frac{J_0(m|\xi|) J_1(m|\xi|)}{|\xi|} \,. \end{aligned}$$

Note that (7.43) is a multiple of the matrices $\mathbb{1}$ and Γ, whereas (7.44) involves in addition the Clifford multiplication by the vector ξ. For the construction of the tangent cone measure, the contribution involving the Clifford multiplication is crucial. Therefore, (7.43) is of no use, leading us to dismiss (5.3) and (5.4). The functional (5.2), on the other hand, looks promising in view of (7.44). The function a, b and c all are all smooth and non-zero at the origin. Namely,

$$\begin{aligned} a &= m^2 \left(1 + 6\tau^2 + \tau^4 \right) + \mathcal{O}\big(|\xi|^2\big) \\ b &= 4m^2 \tau \left(1 + \tau^2 \right) + \mathcal{O}\big(|\xi|^2\big) \\ c &= -2m^3 \tau \big(1 - \tau^2 \big) + \mathcal{O}\big(|\xi|^2\big) \,. \end{aligned}$$

In view of the additional factor ξ in the last summand in (7.44), this means that at $\xi = 0$, the contributions proportional to $\mathbb{1}$ and Γ dominate. It is preferable to remove these contributions, because we want to focus on the Clifford multiplication part. Clearly, on a Riemannian fermion system, where $s_\zeta = \pi_\zeta$, the decomposition (5.5) into block diagonal and off-diagonal parts cannot be used. But we can subtract off the contribution for $\xi = 0$ according to (5.2). We thus obtain

$$\mathcal{A}(\zeta') = \pi_\zeta \, (\zeta' - \zeta) \, \zeta = c_0 \, i \, \xi \cdot \sigma \, \Gamma + \mathcal{O}\big(|\zeta|^2\big) \tag{7.45}$$

with $c_0 = 2m^3 \tau \big(1 - \tau^2\big)$. Restricting attention to the case $\tau \neq 1$, the coefficient c_0 is non-zero.

The resulting tangent cone measure of Lemma 5.2 is most conveniently described as follows. We consider the embedding of S^1 into $\mathrm{Symm}(S_\zeta)$

$$f : S^1 \to \mathrm{Symm}(S_\zeta) \,, \qquad e^{i\varphi} \mapsto \sigma^1 \cos\varphi + \sigma^2 \sin\varphi \,.$$

Moreover, we let μ_{S^1} be the normalized Lebesgue measure on S^1. Then

$$\mu_{\mathrm{con}} = f_* \mu_{S^1} \tag{7.46}$$

(meaning that $\mu_{\mathrm{con}}(\Omega) = \mu_{S^1}(f^{-1}(\Omega))$ for any conical set $\Omega \in \mathfrak{M}_{\mathrm{con}}$).

We next analyze the functional $L(U)$ introduced in (5.12). First, the two-dimensional Clifford subspaces $\mathcal{K}^{(2,0)}_\zeta$ are the two-dimensional subspaces of the space spanned by the three Pauli matrices. Describing them by a unit vector normal to this subspace, we have

$$\mathcal{K}^{(2,0)}_\zeta = \{ K^{(\nu)} \,|\, \nu \in S^2 \subset \mathbb{R}^3 \} \qquad \text{where} \qquad K^{(\nu)} := \{ \vec{x} \cdot \vec{\sigma} \text{ with } \vec{x} \in \mathbb{R}^3, \; \vec{x} \perp \vec{\nu} \} \,.$$

Since $K^{(\nu)} = K^{(-\nu)}$, one sees that $\mathcal{K}^{(2,0)}_\zeta$ is homeomorphic to the real projective plane $\mathbb{P}^2(\mathbb{R})$. Now, using (7.46),

$$\begin{aligned} L\big(K^{(\nu)}\big) &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Tr}_{\mathrm{Symm}(S_x)} \big(\pi_{K^{(\nu)}} \, \pi_{<\sigma^1 \cos\varphi + \sigma^2 \sin\varphi>} \big) \, d\varphi \\ &= \frac{1}{\pi} \int_0^{2\pi} \big(1 - \nu_1^2 \, \cos^2\varphi - \nu_2^2 \sin^2\varphi \big) \, d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \big(2 - \nu_1^2 - \nu_2^2 \big) \, d\varphi \,. \end{aligned}$$

Obviously, this functional is maximal if $\nu_1 = \nu_2 = 0$. As a consequence, there is the unique maximizer

$$K = \{ x \sigma^1 + y \sigma^2 \text{ with } x, y \in \mathbb{R} \} \,.$$

Varying ζ, we obtain a tangential Clifford section,

$$\mathcal{C}\ell_\zeta = K = \{ x \sigma^1 + y \sigma^2 \text{ with } x, y \in \mathbb{R} \} \,. \tag{7.47}$$

We conclude that the tangent cone measure μ_{con} is non-degenerate (see Definition 5.5).

It remains to construct a spin structure. Linearizing (7.45), the mapping $\mathsf{d}\mathcal{A}$ defined by (5.16) is readily computed by

$$\mathsf{d}\mathcal{A}(u) = c_0 \, i \, u \cdot \sigma \, \Gamma \;:\; T_\zeta M \simeq \mathbb{R}^2 \to \mathrm{Symm}(S_\zeta) \,. \tag{7.48}$$

Comparing with (7.47), one sees that the image of $\mathsf{d}\mathcal{A}$ lies in $\mathcal{C}\ell_\zeta$. Hence

$$\gamma_\zeta = \pi_{\mathcal{C}\ell_\zeta} \circ \mathsf{d}\mathcal{A} = \mathsf{d}\mathcal{A} \,, \tag{7.49}$$

where in the last step we used that (as is obvious from (7.48) and (7.47)). In this way, the spin structure gives back the usual Clifford multiplication.

We conclude that for the Euclidean plane with chiral asymmetry, the constructions in Section 5 yield a unique tangential Clifford section and a unique spin structure, giving the usual Clifford multiplication. Apart from providing a consistency check, this result is important because it shows that the constructions in Section 5 also apply to spaces which are sufficiently small perturbations of the Euclidean plane, even if these perturbations destroy the structure of a two-dimensional spin manifold. More precisely, suppose that we perturb the universal measure of the Euclidean plane. Then the resulting topological fermion system will in general no longer have a smooth manifold structure. Nevertheless, provided that the perturbation is so small that the tangent cone measure remains regular in the sense of Definition 5.5, we still have a tangential Clifford section $\mathcal{C}\ell M$, giving rise to a well-defined notion of Clifford multiplication. If the resulting quantum space still has the structure of a two-dimensional differentiable manifold, we can again use (7.49) to obtain a canonical spin structure. In this way, many of our notions and constructions can be carried over to non-smooth or singular spaces.

7.5. **The Spin Structure of Two-Dimensional Minkowski Space.** We now return to the example of two-dimensional Minkowski space introduced in Section 7.2. Our first step is the analysis of the tangent cone measures. In preparation, we need to choose the function $\mathcal{A}$ in (5.1). In all the following computations, we again work in the basis $(\mathfrak{f}_{1/2}(\zeta))$ of the spin spaces introduced in (7.30). For clarity, we write the result of Lemma 7.4 symbolically as

$$\begin{aligned} P(\zeta,\zeta') &= \big((i\xi^0+\varepsilon)\,\gamma^0 - i\xi^1\gamma^1\big)\,\alpha(\xi)+\beta(\xi) \\ P(\zeta',\zeta) &= \big((-i\xi^0+\varepsilon)\,\gamma^0 + i\xi^1\gamma^1\big)\,\alpha(-\xi)+\beta(-\xi)\,, \end{aligned} \tag{7.50}$$

where we again set $\xi=\zeta'-\zeta$ and

$$\begin{aligned} \alpha(\xi) &= \frac{im}{4\pi^2}\,\frac{K_1\Big(-im\sqrt{(-\xi^0+i\varepsilon)^2-(\xi^1)^2}\Big)}{\sqrt{(-\xi^0+i\varepsilon)^2-(\xi^1)^2}} \\ \beta(\xi) &= \frac{m}{4\pi^2}\,K_0\Big(-im\sqrt{(-\xi^0+i\varepsilon)^2-(\xi^1)^2}\Big)\,. \end{aligned} \tag{7.51}$$

The symmetry of the kernel of the fermionic operator yields

$$P(\zeta',\zeta)=P(\zeta,\zeta')^*=\big((-i\xi^0+\varepsilon)\,\gamma^0+i\xi^1\gamma^1\big)\,\overline{\alpha(\xi)}+\overline{\beta(\xi)}\,,$$

implying that

$$\overline{\alpha(\xi)}=\alpha(-\xi) \qquad \text{and} \qquad \overline{\beta(\xi)}=\beta(-\xi)\,. \tag{7.52}$$

These relations can also be verified directly by taking the complex conjugates of the above Bessel functions (and choosing the correct branch of the square root).

The composite expressions in (7.33)–(7.35) all all of the form that a diagonal matrix is multiplied from the left by $P(\zeta,\zeta')$ and from the right by $P(\zeta',\zeta)$. In order to see the general structure, we now compute and expand such expressions in the case that the diagonal matrix is the identity matrix or the matrix γ^0. The corresponding expressions in (7.33)–(7.35) are then readily obtained by taking suitable linear combinations. A straightforward computation yields

$$P(\zeta,\zeta')\,P(\zeta',\zeta)=\begin{pmatrix} (\beta(0)+\varepsilon\alpha(0))^2 & 2i\varepsilon\,\xi^1\,\alpha(0)^2 \\ 2i\varepsilon\,\xi^1\,\alpha(0)^2 & (\beta(0)-\varepsilon\alpha(0))^2 \end{pmatrix}+o(\xi) \tag{7.53}$$

$$P(\zeta,\zeta')\,\gamma^0\,P(\zeta',\zeta)=\begin{pmatrix} (\beta(0)+\varepsilon\alpha(0))^2 & 2i\,\xi^1\,\alpha(0)\,\beta(0) \\ 2i\,\xi^1\,\alpha(0)\,\beta(0) & -(\beta(0)-\varepsilon\alpha(0))^2 \end{pmatrix}+o(\xi) \tag{7.54}$$

(note that $\alpha(0)$ and $\beta(0)$ are real by (7.52)). The off-diagonal matrix elements both have the desired dependence on ξ^1. In (7.53) this dependence drops out in the limit $\varepsilon \searrow 0$. Thus in order for our construction to be independent of the regularization, it is preferable to work with (7.54), which corresponds to choosing $\mathcal{A}$ according to (5.4). If one does not care about the dependence on ε (which is allowed as long as one does not intend taking the limit $\varepsilon \searrow 0$), one can just as well choose $\mathcal{A}$ according to (5.2) or (5.3).

The resulting tangent cone measure of Lemma 5.2 is most conveniently described as follows. We introduce the map

$$f : \{\pm 1\} \to \mathrm{Symm}(S_\zeta) , \qquad f(\pm 1) = \pm \gamma^2 ,$$

where the Dirac matrix γ^2 is defined by

$$\gamma^2 := i \gamma^1 \gamma^0 = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

Moreover, we let μ be the normalized counting measure on $\{\pm 1\}$. Then

$$\mu_{\mathrm{con}} = f_* \mu . \tag{7.55}$$

If one prefers the notation with Dirac measures, one can also write

$$\mu_{\mathrm{con}}(B) = \frac{1}{2} \Big(\delta_{\gamma^2}(B) + \delta_{-\gamma^2}(B) \Big),$$

to be evaluated for conical sets.

We next analyze the variational principle (5.14) and (5.15). Fist of all, the symmetric operators which anti-commute with s_ζ as well as the Clifford extensions can be parametrized by an angle φ,

$$\begin{aligned} \mathfrak{A}(s_\zeta) &= \big\{ \gamma^1 \cos\varphi + \gamma^2 \sin\varphi \text{ with } \varphi \in \mathbb{R} \big\} \\ \mathcal{K}_\zeta^{s_\zeta, 1} &= \Big\{ K^{(\varphi)} \mid \varphi \in \mathbb{R} \Big\} \qquad \text{where} \\ K^{(\varphi)} &:= \{ t\gamma^0 + x(\gamma^1 \cos\varphi + \gamma^2 \sin\varphi) \text{ with } t, x \in \mathbb{R} \} . \end{aligned}$$

Using (7.55), the functional (5.14) becomes

$$L(U) = \frac{1}{2} \sum_{\pm} \mathrm{Tr}_{\mathrm{Symm}(S_x)} \big(\pi_U \, \pi_{\langle \pm \gamma^2 \rangle} \big) = \mathrm{Tr}_{\mathrm{Symm}(S_x)} \big(\pi_U \, \pi_{\langle \gamma^2 \rangle} \big) ,$$

and thus

$$L\big(K^{(\varphi)}\big) = \sin^2 \varphi .$$

Maximizing L, we obtain the unique Clifford extension

$$K = \{ t \gamma^0 + x \gamma^2 \text{ with } t, x \in \mathbb{R} \} .$$

We conclude that the tangent cone measure μ_{con} is non-degenerate (see Definition 5.7).

Linearizing (7.53) and (7.54), the mapping $\mathsf{d}\mathcal{A}$ defined by (5.16) is readily computed by

$$\mathsf{d}\mathcal{A}(u) = c_1 \, u^1 \gamma^2 \;:\; T_\zeta M \simeq \mathbb{R}^2 \to \mathrm{Symm}(S_\zeta) \tag{7.56}$$

with a non-zero constant c_1. Again, the image of $\mathsf{d}\mathcal{A}$ lies in K. However, $\mathsf{d}\mathcal{A}$ is *not* bijective. In the case that $\mathcal{A}$ is chosen according to (5.3), this is precisely the observation made after the proof of Lemma 5.8. More generally, the shortcoming is that the functionals $\mathcal{A}$ considered so far do not distinguish a time direction.

In [12, Section 3.5] a time direction was distinguished in spin dimension two by working in suitable "synchronized" bases of the spin spaces. Here we give a somewhat different construction which does not involve a choice of bases. In preparation, we work out the general structure of the formulas (7.50) and of resulting composite expressions. Since we are interested in the local behavior near $\xi = 0$, it is preferable to expand (7.50) and (7.51) in a Taylor series in ξ^0. Noting that the functions in (7.51) are even in ξ^1, we have

$$\begin{aligned} P(\zeta,\zeta') &= \sum_{p=0}^\infty \Big(i\xi^0 \gamma^0 \, v^0_p + i\xi^1 \gamma^1 \, v^1_p + s_p \Big)\, (i\xi^0)^p \\ P(\zeta',\zeta) &= \sum_{p=0}^\infty \Big(-i\xi^0 \gamma^0 \, v^0_p - i\xi^1 \gamma^1 \, v^1_p + s_p \Big)\, (-i\xi^0)^p \,, \end{aligned} \tag{7.57}$$

where the coefficient functions v^0_p, v^1_p and s_p depend only on $(\xi^1)^2$. The relations (7.52) immediately yield that the functions v^0_p, v^1_p and s_p are real-valued. Hence

$$\begin{aligned} P(\zeta,\zeta')\, P(\zeta',\zeta) = &\sum_{p,p'=0}^\infty (i\xi^0)^p \, (-i\xi^0)^{p'} \, \Big((\xi^0)^2 \, v^0_p \, v^0_{p'} - (\xi^1)^2 \, v^1_p \, v^1_{p'} + s_p \, s_{p'} \\ &+ \xi^0 \xi^1 \, \gamma^0 \gamma^1 \big(v^0_p \, v^1_{p'} - v^1_p \, v^0_{p'} \big) + i\xi^0 \gamma^0 \big(v^0_p s_{p'} - s_p v^0_{p'} \big) + i\xi^1 \gamma^1 \big(v^1_p s_{p'} - s_p v^1_{p'} \big) \Big) \,. \end{aligned} \tag{7.58}$$

We now analyze the symmetry if p and p' are exchanged. Under this replacement, the factor $(i\xi^0)^p \, (-i\xi^0)^{p'}$ changes sign if $p+p'$ is odd, whereas it does not change sign if $p+p'$ is even. As a consequence, the terms in the first line vanish if $p+p'$ is odd, whereas the second line vanishes if $p+p'$ is even. Thus one sees that the closed chain $P(\zeta,\zeta')\, P(\zeta',\zeta)$ remains unchanged under the replacements

$$\xi \to -\xi \qquad \text{and} \qquad \gamma^1 \to -\gamma^1 \,. \tag{7.59}$$

Thus the closed chain does *not* distinguish a time direction. Similarly,

$$\begin{aligned} P(\zeta,\zeta')\, \gamma^0 \, P(\zeta',\zeta) = &\sum_{p,p'=0}^\infty (i\xi^0)^p \, (-i\xi^0)^{p'} \, \gamma^0 \, \Big((\xi^0)^2 \, v^0_p \, v^0_{p'} + (\xi^1)^2 \, v^1_p \, v^1_{p'} + s_p \, s_{p'} \\ &+ \xi^0 \xi^1 \, \gamma^0 \gamma^1 \big(v^0_p \, v^1_{p'} + v^1_p \, v^0_{p'} \big) + i\xi^0 \gamma^0 \big(v^0_p s_{p'} - s_p v^0_{p'} \big) - i\xi^1 \gamma^1 \big(v^1_p s_{p'} + s_p v^1_{p'} \big) \Big) \,. \end{aligned} \tag{7.60}$$

Again considering the symmetry when exchanging p with p', we conclude that the composite expression $P(\zeta,\zeta')\, \gamma^0 \, P(\zeta',\zeta)$ remains unchanged again under the replacements (7.59). Thus again, no time direction is distinguished.

Another method for understanding this shortcoming is to take the trace of composite expressions and use the invariance under cyclic commutations. For example,

$$\mathrm{Tr}\, \big(P(\zeta,\zeta')\, P(\zeta',\zeta) \big) = \mathrm{Tr}\, \big(P(\zeta',\zeta)\, P(\zeta,\zeta') \big) \,,$$

showing that the scalar component of the closed chain is invariant under the replacement $\xi \to -\xi$. By inserting additional matrices into the trace, one can verify the symmetries of all the contributions in (7.58) and (7.60). Moreover, the following consideration shows that it is impossible to distinguish a time direction if $\zeta^1 = 0$: Consider the trace of a composite expression of the form

$$\mathrm{Tr}\, \big(M_0 \, P(\zeta,\zeta') \, M_1 \, P(\zeta',\zeta) \, M_2 \, \cdots \, M_{2k} \, P(\zeta,\zeta') \, M_{2k+1} \, P(\zeta',\zeta) \big) \,, \tag{7.61}$$

where the factors M_ℓ are linear combinations of $\mathbb{1}$ and γ^0 (note that in view of (7.31), these are the only matrices which we can form locally at the point ζ or ζ'). Assuming that $\zeta^1 = 0$, the matrices $P(\zeta,\zeta')$ and $P(\zeta',\zeta)$ are also multiples of $\mathbb{1}$ and γ^0 (see for example (7.57)). As a consequence, all the matrices in (7.61) commute with each other. Thus we can reorder the matrices to obtain (7.61) with the arguments ζ and ζ'

exchanged. Hence (7.61) is symmetric under the replacement $\zeta \to -\zeta$, proving that no time direction is distinguished.

We now give a method for distinguishing a time direction. We first note that taking the commutator of $\mathcal{P}(\zeta,\zeta')$ with γ^0, only the component involving $v^1_\bullet$ remains, i.e. in view of (7.57)

$$\big[\gamma^0, P(\zeta,\zeta')\big] = 2\sum_{p=0}^\infty i\xi^1\gamma^0\gamma^1\, v^1_p\, (i\xi^0)^p\ .$$

We now take the trace of a product which involves two such commutators,

$$\mathcal{B}(\zeta,\zeta') := \mathrm{Tr}\Big(\big[P(\zeta,\zeta'),\gamma^0\big]\, P(\zeta',\zeta)\, \big[P(\zeta,\zeta'),\gamma^0\big]\, P(\zeta',\zeta)\Big)\ .$$

Then of both factors $P(\zeta,\zeta')$ only the component $v^1_\bullet$ enters, whereas the factors $P(\zeta',\zeta)$ also contribute via the components $v^0_\bullet$ and $s_\bullet$. In this way, the symmetry $\zeta \leftrightarrow \zeta'$ is broken. In particular, we may anti-symmetrize to obtain a real-valued functional which is anti-symmetric under the replacement $\xi \to -\xi$,

$$\mathcal{E}(\zeta,\zeta') := \frac{1}{2i}\big(\mathcal{B}(\zeta,\zeta') - \mathcal{B}(\zeta',\zeta)\big)\ . \tag{7.62}$$

A straightforward computation shows that

$$\mathcal{E}(\zeta,\zeta') = 16\,\xi^0\,(\xi^1)^2\,\Big(i\alpha(0)\beta(0)\Big[\beta(0)\,\frac{\partial\alpha(0)}{\partial\xi^0} - \alpha(0)\,\frac{\partial\beta(0)}{\partial\xi^0}\Big] - \varepsilon\alpha(0)^4\Big) + \mathcal{O}\big(\|\xi\|^4\big) \tag{7.63}$$

(where α and β are again the functions (7.51), and $\|\xi\|^2 := |\xi^0|^2 + |\xi^1|^2$). A direct computation shows that the coefficient in the round brackets in (7.63) is indeed non-zero.

Using the notation in Lemma 7.5, we can write the functional $\mathcal{E}(\zeta,\zeta')$ in a simpler form.

Lemma 7.7. *The functional* $\mathcal{E}(\zeta,\zeta')$ *as defined by* (7.62) *is given by*

$$\mathcal{E}(\zeta,\zeta') = ic\,\mathrm{Tr}\left(\zeta\,\pi_{\zeta'}\,\pi_\zeta\,\zeta' - \zeta\,\zeta'\,\pi_\zeta\,\pi_{\zeta'}\right)\,, \tag{7.64}$$

where c is the positive constant

$$c = \frac{4\,\nu_1^2\nu_2^2}{(\nu_2-\nu_1)^2}\ .$$

Proof. Writing out the commutators and cyclically commuting the factors inside the trace, we obtain

$$\mathcal{E}(\zeta,\zeta') = i\,\mathrm{Tr}\Big(P(\zeta,\zeta')\,P(\zeta',\zeta)\,\Big[P(\zeta,\zeta')\,\gamma^0\,P(\zeta',\zeta)\,\gamma^0 - \gamma^0\,P(\zeta,\zeta')\,\gamma^0\,P(\zeta',\zeta)\Big]\Big)\ .$$

Using (7.32), this simplifies to

$$\mathcal{E}(\zeta,\zeta') = i\,\mathrm{Tr}\Big(\zeta'\,\zeta\,\big(\zeta'\,s_{\zeta'}\,\zeta\,s_\zeta - s_\zeta\,\zeta'\,s_{\zeta'}\,\zeta\big)\Big)\ . \tag{7.65}$$

According to (7.31), we can write the matrix product ζs_ζ as a linear combination of ζ and π_ζ,

$$\zeta\,s_\zeta = \frac{\nu_1+\nu_2}{\nu_1-\nu_2}\,\zeta - \frac{2\nu_1\nu_2}{\nu_1-\nu_2}\,\pi_\zeta\ .$$

Substituting this formula into (7.65), multiplying out and cyclically commuting the factors inside the trace, we obtain the result. ∎

The trace in (7.64) is obviously anti-symmetric under permutations of ζ and ζ'. It seems that this trace can be used on general causal fermion systems to distinguish the future from the past. Indeed, when evaluating this trace in spin dimension two, one gets (up to irrelevant prefactors) the same expression which was used in [12, Section 3.5] to distinguish a time direction.

In order to complete the construction of the spin structure, we need to modify (5.4) by adding a term which gives a linear time dependence. According to (7.63), the function $\mathcal{E}$ has this desired linear dependence on ξ^0, but unfortunately it vanishes to second order in the spatial coordinate ξ^1. The last problem can be handled by integrating over a small ball $B_\delta(\zeta')$ (where we again work with the metric in $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ induced by the sup-norm). Thus for for given $\delta > 0$ we choose the functional (5.1) as

$$\mathcal{A}(\zeta') = \pi_\zeta \, \pi_{\zeta'} \, \zeta + s_\zeta \int_{B_\delta(\zeta')} \mathcal{E}(\zeta, \theta) \, d\rho(\theta) + c \,,$$

where the constant c is chosen such that $\mathcal{A}(\zeta) = 0$ (we remark that, just as explained after (7.54), we could also have modified (5.2) or (5.3) instead of (5.4)). Then the derivative of $\mathcal{A}$ becomes in modification of (7.56)

$$\mathsf{d}\mathcal{A}(u) = c_1 \, u^1 \gamma^2 + c_0 \, u^0 \gamma^0 \;:\; T_\zeta M \simeq \mathbb{R}^2 \to \mathrm{Symm}(S_\zeta) \tag{7.66}$$

with a non-zero constant c_0 which depends on the coefficient in (7.63) and on δ. Now $\mathsf{d}\mathcal{A}$ is injective, and its image lies in $\mathcal{C}\ell_\zeta$. Hence the composition

$$\gamma_\zeta = \pi_{\mathcal{C}\ell_\zeta} \circ \mathsf{d}\mathcal{A} = \mathsf{d}\mathcal{A} \tag{7.67}$$

gives the desired spin structure.

We finally point out that the Clifford multiplication induced by the spin structure (7.67) differs from the expected Clifford multiplication by the fact that in (7.66) the component u^1 is multiplied by γ^2 instead of γ^1. This difference is of no relevance as long as topological questions are considered. However, it shows that the Clifford multiplication induced by (7.67) does not have a geometric meaning, exactly as explained at the end of Section 5.3 after (5.24). The significance of the constructions lies in the fact that they are robust to general perturbations. Thus we obtain a canonical spin structure for small perturbations of the two-dimensional Minkowski space, even if the manifold structure ceases to exist.

8. Spinors on Singular Spaces

We now turn attention to examples on curved space or in curved space-time. For the computation of the Dirac operator we shall always use the following convenient method, which makes it unnecessary to compute all the spin coefficients. Instead, the zero order term in the Dirac operator is computed similar to a "covariant divergence." Suppose that the metric of our manifold $(\hat{M}^k, g)$ has signature (r, s), and that the spin scalar product on the corresponding spinor bundle has signature $(\mathfrak{q}, \mathfrak{p})$. Our starting point are symmetric matrices $(\sigma^i)_{i=1,\ldots,k}$ on $\mathbb{C}^{\mathfrak{q},\mathfrak{p}}$ which satisfy the canonical anti-commutation relations in the corresponding Euclidean or Minkowski space,

$$\{\sigma^i, \sigma^j\} = 2 s_i \, \delta^{ij} \,, \qquad \text{with} \qquad s_1, \ldots, s_r = 1, \; s_{r+1}, \ldots, s_k = -1 \,. \tag{8.1}$$

A typical choice are the Pauli matrices or the usual Dirac matrices.

Proposition 8.1. *Let $(\hat{M}^k, g)$ be a spin manifold with a metric of signature (r, s) with $r \leq 1$ and $s \leq 3$. Let $\mathcal{D}$ be the Dirac operator acting on smooth sections $\Gamma(\hat{M}, S\hat{M})$*

of the corresponding spinor bundle (for an arbitrarily chosen spin structure). Assume that there is a local chart (x^i, U) *in which the metric is diagonal, i.e.*

$$ds^2 = f_1\, dx_1^2 + \cdots + f_r\, dx_r^2 \;-\; f_{r+1}\, dx_{r+1}^2 - \cdots - f_k\, dx_k^2$$

(where the coefficient functions $f_i \in C^\infty(U, \mathbb{R}^+)$ *may depend on all* k *variables). Then around every point in* U *there is a local trivialization of the spinor bundle by pseudo-orthonormal bases* $(\mathfrak{e}_\alpha)_{\alpha=1,\ldots,\mathfrak{p}+\mathfrak{q}}$ *such that in this local chart and trivialization, the Dirac operator takes the form*

$$\mathcal{D} = iG^j \frac{\partial}{\partial x^j} + B\,, \tag{8.2}$$

where

$$G^j(x) = f_j(x)^{-\frac{1}{2}}\, \sigma^j \tag{8.3}$$

$$B(x) = \frac{i}{2\sqrt{|\det g|}}\, \partial_j \left(\sqrt{|\det g|}\, G^j \right). \tag{8.4}$$

More generally, if in a chart and local trivialization of the spinor bundle by pseudo-orthonormal bases $(\mathfrak{e}_\alpha)_{\alpha=1,\ldots,\mathfrak{p}+\mathfrak{q}}$ *the matrices* G^j *are of the form*

$$G^j(x) = f_j(x)^{-\frac{1}{2}}\, O^j_k(x)\, \sigma^k \qquad \text{with } O(x) \in \mathrm{SO}(r,s)\,, \tag{8.5}$$

then the Dirac operator is again of the form (8.2) *with* B *according to* (8.4).

Proof. By taking the Cartesian product with a Euclidean or Minkowski space, we obtain a Lorentzian manifold of signature $(1,3)$. Moreover, the spinor bundle of $\hat{M}$ can be recovered as a sub-bundle of the spinor bundle on the Lorentzian manifold (in particular, the spinors of $\hat{M}$ may be recovered as the left- or right-handed component of the four-component Dirac spinors). Then we can use the formalism developed in [7] (see also [8, §1.5]). With (8.3) we have satisfied the anti-commutation relations

$$\{G^j, G^k\} = 2\, g^{jk}\, \mathbb{1}\,.$$

Moreover, the choice (8.3) ensures that the pseudoscalar matrix is constant, and that all derivatives of the G^j are in the span of $\sigma^1, \ldots, \sigma^k$. Then the zero order term of the Dirac operator can be written as (see [7, eqs (41), (42) and (51)])

$$B = G^j E_j \qquad \text{with} \qquad E_j = -\frac{i}{16}\, \mathrm{Tr}\left(G^m \left(\partial_j G^n + \Gamma^n_{jl}\, G^l\right)\right) G_m G_n\,,$$

where Γ^n_{jl} are the Christoffel symbols of the Levi-Civita connection (and the partial derivatives simply act on each component). Hence

$$B = -\frac{i}{16}\, \mathrm{Tr}\left(G_m \left(\nabla_j G_n\right)\right) G^j G^m G^n\,, \tag{8.6}$$

where $\nabla_j G_n \equiv \partial_j G_n - \Gamma^k_{jn} G_k$ is the covariant derivative acting on the components of the spinorial matrix. Using the algebra of the Dirac matrices, one finds that (8.6) has a vectorial component (obtained by using the anti-commutation relations), and an axial component which is totally antisymmetric in the indices j, m, and n. This totally antisymmetric term vanishes for the following reasons: First, since the Levi-Civita connection is torsion-free, we may replace the covariant derivative by a partial derivative. Second, it follows from (8.3) that the matrix $\partial_j G_n$ is a multiple of G_n, implying that the trace $\mathrm{Tr}(G_m(\partial_j G_n))$ is symmetric in the indices m and n.

It remains to compute the vectorial component of (8.6). A short computation yields

$$B = \frac{i}{2}\, \nabla_j G^j \;,$$

and the usual formula for the covariant divergence of a vector field gives the result. □

This proposition also gives a method for constructing the Dirac operator on a manifold. To this end, one takes (8.2) as the definition of the Dirac operator in a local chart and trivialization. Identifying these so-defined Dirac operators in different charts by suitable transformations of the coordinates and spinor frames, one obtains a globally defined Dirac operator. In all the following examples, it will be straightforward to match the Dirac operators in the different charts. However, we point out that in general, this is a non-trivial task, which amounts to verifying that the manifold is spin and to choosing a specific spin structure. In order to keep the following examples as simple as possible, we do not discuss the freedom in choosing different spin structures.

8.1. **Singularities of the Conformal Factor.** We first discuss curvature singularities which can be removed by a conformal transformation of the metric. To this end, we assume that g is a smooth metric, and that

$$\tilde{g} = \lambda^2\, g$$

with a smooth positive function λ. We denote the corresponding spinor bundles and Dirac operators by $S\hat{M}$, $S\tilde{M}$ and $\mathcal{D}$, $\tilde{\mathcal{D}}$. According to [29, 27] there is a fibrewise isometry $\psi \mapsto \iota\psi$ of the spinor bundles such that

$$\tilde{\mathcal{D}}(\iota\psi) = \lambda^{-\frac{k+1}{2}}\, \iota\left(\mathcal{D}(\lambda^{\frac{k-1}{2}}\psi)\right)$$

(where k again denotes the dimension of $\hat{M}$). For ease of notation we usually omit the identification map ι and simply write

$$\tilde{\mathcal{D}}\psi = \lambda^{-\frac{k+1}{2}}\mathcal{D}\big(\lambda^{\frac{k-1}{2}}\psi\big)\,.$$

In particular, solutions of the massless Dirac equation transform conformally,

$$\mathcal{D}\psi = 0 \qquad \Longrightarrow \qquad \tilde{\mathcal{D}}\tilde{\psi} = 0 \quad \text{with} \quad \tilde{\psi} = \lambda^{-\frac{k-1}{2}}\,\psi\,.$$

For the massive Dirac equation $(\mathcal{D} - m)\psi = 0$, the situation is no longer so simple, because the mass parameter brings in a length scale and thus destroys the conformal invariance. Nevertheless, if we consider singularities of the conformal factor λ where $\lambda \searrow 0$, then typically the mass parameter does not affect the behavior of the spinor near the singularity. Hence we expect that the rescaled spinor

$$\lambda^{\frac{k-1}{2}}\,\tilde{\psi} \tag{8.7}$$

has a well-defined limit even if $\lambda \searrow 0$. Since the topological spinor bundle is defined purely in terms of wave functions, this will imply that the topological spinor bundle has a well-defined regular limit even if the metric and curvature become singular.

Clearly, such singularities of the conformal factor can be treated just as well by a conformal rescaling of the metric, as is a common procedure when constructing for example conformal compactifications or Penrose diagrams. Thus at this point, working with topological spinor bundles gives no major benefit. However, the main benefit of working with topological spinor bundles becomes apparent when considering curvature singularities which do *not* come from a conformal transformation of the metric. Such singularities, which we refer to as *genuine curvature singularities*, will be treated in the next section 8.2. Before, we illustrate conformal transformations by two simple examples.

Example 8.2. *(A neck singularity of a 2-d Minkowski cylinder)* On $\hat{M} = \mathbb{R} \times S^1$ we choose coordinates (t, φ) with $t \in \mathbb{R}$ and $0 < \varphi < 2\pi$ and consider the two-dimensional Lorentzian metric

$$ds^2 = dt^2 - R(t)^2 \, d\varphi^2 \,. \tag{8.8}$$

In order to construct the Dirac operator, we use the method of Proposition 8.1. We satisfy the anti-commutation relations (8.1) with the ansatz

$$G^t = \sigma^3 \,, \qquad G^\varphi = \frac{i\sigma^1}{R(t)} \,.$$

In order for these matrices to be symmetric, we need to consider the spin scalar product

$$\prec . | . \succ = \langle . | \sigma^3 . \rangle_{\mathbb{C}^2} \,, \tag{8.9}$$

which is clearly indefinite of signature $(1, 1)$. The matrices G^t and G^φ are of the form (8.3). Thus, using (8.2), the Dirac operator becomes

$$\mathcal{D} = i\sigma^3 \partial_t + \frac{i\dot{R}(t)}{2R(t)} \, \sigma^3 - \frac{1}{R(t)} \, \sigma^1 \partial_\varphi \,.$$

We consider the Dirac equation $(\mathcal{D} - m)\psi = 0$. This equation can be separated by the ansatz

$$\psi = \frac{e^{ik\varphi}}{\sqrt{R(t)}} \begin{pmatrix} \chi_1(t) \\ i\chi_2(t) \end{pmatrix} \quad \text{with} \quad k \in \mathbb{Z} \tag{8.10}$$

to obtain the ODE in time

$$i\dot{\chi}(t) = \begin{pmatrix} m & -k/R(t) \\ -k/R(t) & -m \end{pmatrix} \chi(t) \,. \tag{8.11}$$

Since the matrix on the right is Hermitian, one readily verifies that

$$\frac{d}{dt} \, \|\chi\| = 0 \,, \tag{8.12}$$

implying that the norm of χ is time independent (this conservation law can be identified with current conservation). Since the norm is constant, the equation (8.11) can be understood as describing "oscillations" of the spinor (for more details on this geometric picture we refer to the similar equation [15, eq. (5)] and its reformulation with Bloch vectors in [15, Section 2]). If the function $R(t)$ gets small, the frequency of the oscillations gets larger. In order to control these oscillations, we estimate the matrix in (8.11) to obtain the inequality

$$\left\| \frac{d}{dt} \chi \right\| \leq 2 \left(m + \frac{|k|}{R(t)} \right) \|\chi\| \,.$$

Integration gives the inequality

$$\chi\big|_{t_0}^{t_1} \leq 2 \, \|\chi\| \int_{t_0}^{t_1} \left(m + \frac{|k|}{R(t)} \right) dt \,. \tag{8.13}$$

This shows that if the function $1/R$ is integrable, then the function χ is continuous. As a specific example, one can consider the family of metrics

$$R(t) = (t^2 + \varepsilon^2)^{\frac{1}{4}} \,. \tag{8.14}$$

In the limit $\varepsilon \searrow 0$, the metric becomes singular at $t = 0$, forming a neck singularity. In this limit, the functions χ converge locally uniformly to continuous functions.

In order to construct a causal fermion system, we choose a (for simplicity finite-dimensional) space of Dirac wave functions $\mathcal{H}$ and endow it with a scalar product $\langle .|. \rangle$. We again introduce the local correlation operators F by (2.7). Due to the factor $1/\sqrt{R}$ in (8.10), the Dirac wave functions diverge in the limit $\varepsilon \to 0$. As a consequence, the local correlation operators will also diverge. But we can cure this problem simply by rescaling the local correlation operators according to

$$\tilde{F}(t, \varphi) = R(t)\, F(t, \varphi) \,. \tag{8.15}$$

This rescaling corresponds precisely to the conformal rescaling (8.7) needed to remove the curvature singularity at the cusp of the cylinder. Since the functions χ converge uniformly as $\varepsilon \searrow 0$, we conclude that the rescaled correlation operators converge. Introducing the universal measure by $\rho = \tilde{F}_*(dt\, d\varphi)$, for any $\varepsilon > 0$ we obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension one. This family of causal fermion systems has a regular limit as $\varepsilon \searrow 0$, despite the fact that a curvature singularity forms. $\Diamond$

Example 8.3. *(A conical singularity of a Riemannian surface)* We choose $\hat{M} = \mathbb{R}^2 \setminus \{0\}$ and consider the metric in polar coordinates (r, φ) of the form

$$ds^2 = dr^2 + R(r)^2\, d\varphi^2 \,, \tag{8.16}$$

where $0 < r$ and $0 < \varphi < 2\pi$. We again construct the Dirac operator with the method of Proposition 8.1. In order to satisfy the anti-commutation relations (8.1), we take the ansatz

$$G^r = \sigma^r \,, \qquad G^\varphi = \frac{\sigma^\varphi}{R(r)} \,,$$

where σ^r and σ^φ are the linear combinations of Pauli matrices

$$\sigma^r = \sigma^1 \cos\varphi + \sigma^2 \sin\varphi \,, \qquad \sigma^\varphi = -\sigma^1 \sin\varphi + \sigma^2 \cos\varphi \,.$$

The matrices G^r and G^φ are of the form (8.5). Using (8.2), the Dirac operator is computed by

$$\mathcal{D} = i\sigma^r \partial_r + \frac{i}{R(r)}\, \sigma^\varphi \partial_\varphi + \frac{i}{2} \left(\frac{R'(r)}{R(r)} - 1 \right) \sigma^r \,. \tag{8.17}$$

Note that in the special case $R(r) = r$, the metric (8.16) becomes flat, and the Dirac operator reduces to the Dirac operator (7.1) in Euclidean $\mathbb{R}^2$, written in polar coordinates.

We consider the Dirac equation $\mathcal{D}\psi = \lambda\psi$. This equation can be separated by the ansatz

$$\psi = \frac{1}{\sqrt{R(r)}} \begin{pmatrix} e^{i(k-\frac{1}{2})\varphi}\, \chi_1(r) \\ i e^{i(k+\frac{1}{2})\varphi}\, \chi_2(r) \end{pmatrix} \quad \text{with} \quad k \in \mathbb{Z} + \frac{1}{2} \tag{8.18}$$

to obtain the radial ODE

$$\chi'(r) = \begin{pmatrix} k/R & \lambda \\ -\lambda & -k/R \end{pmatrix} \chi(r) \,.$$

We now choose $R(r)$ as

$$R(r) = \frac{r}{2} \,.$$

The corresponding metric is conical. It cannot be extended to the cusp singularity at $r = 0$. In order to construct a regular topological fermion system, we need to choose at least two wave functions. For simplicity, we choose

$$\lambda = 1 \,, \qquad k = \pm\frac{1}{2}$$

and let $\psi_\pm$ the solution of the Dirac equation which is bounded at the origin. These solutions can be computed explicitly by

$$\begin{aligned}
\psi_+(r,\varphi) &= \frac{1}{r^{\frac{3}{2}}} \begin{pmatrix} r\sin(r) \\ -ie^{i\varphi}\left(\sin(r) - r\cos(r)\right) \end{pmatrix} \\
\psi_-(r,\varphi) &= \frac{1}{r^{\frac{3}{2}}} \begin{pmatrix} e^{-i\varphi}\left(\sin(r) - r\cos(r)\right) \\ ir\sin(r) \end{pmatrix} .
\end{aligned}$$

We let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be the vector space spanned by ψ_+ and ψ_- with the scalar product such that $\psi_\pm$ are orthonormal. Then the local correlation operators (see Definition 2.4) are computed by

$$F(r,\varphi) = -\left(\frac{\sin^2(r)}{r} + \frac{(\sin(r) - r\cos(r))^2}{r^3} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{8.19}$$

Obviously, these local correlation operators tend to zero as $r \searrow 0$. In order to cure this problem, we rescale the local correlation operators similar to (8.15) by setting

$$\tilde{F}(r,\varphi) = \frac{1}{r}\, F(r,\varphi) \,.$$

The function $\tilde{F}$ is continuous and has a non-zero limit at the origin. Thus we can extend it continuously to the origin,

$$\tilde{F} \,:\, \hat{M} \cup \{0\} \simeq \mathbb{R}^2 \to \mathcal{F} \subset \mathrm{L}(\mathcal{H}) \,.$$

Taking the push-forward measure $\rho = \tilde{F}_*(d^2x)$, we obtain a Riemannian fermion system of spin dimension one. We point out that this Riemannian fermion is regular at the origin, although the Riemannian metric of the underlying manifold is ill-defined.

For the just-constructed Riemannian fermion system, the local correlation operators (8.19) do not depend on the angular variable, so that the mapping $\tilde{F}$ is not injective. As a consequence, the space $M := \operatorname{supp}\rho$ is homeomorphic to $\mathbb{R}^+ \cup \{0\}$, meaning that every circle $r = \text{const}$ has been identified with a point. Such identifications are useful in applications in which some degrees of freedom of $\hat{M}$ are irrelevant or should be suppressed. Likewise, such identifications can be arranged to occur at the boundary of $\hat{M}$ or in an asymptotic end, making it possible to describe different types of compactifications.

In order to avoid identifications, one simply extends $\mathcal{H}$ by another wave function, like for example the eigensolution for $\lambda = 1$ and $k = 3/2$ being regular at the origin,

$$\psi = \frac{1}{r^{\frac{5}{2}}} \begin{pmatrix} e^{i\varphi}\left(3r\cos(r) - (3 - r^2)\sin(r)\right) \\ ie^{2i\varphi}\left(3(5 - 2r^2)\sin(r) - (15r - r^3)\cos(r)\right) \end{pmatrix} .$$

A direct computation shows that on the resulting three-dimensional particle space, the rescaled local correlation operators $\tilde{F}$ are continuous at the origin, and the mapping $\tilde{F} : \mathbb{R}^2 \to \mathcal{F}$ is indeed injective. ♢

8.2. **Genuine Singularities of the Curvature Tensor.** We now consider curvature singularities which are genuine in the sense that the singularity cannot be removed by a conformal transformation. The next two examples illustrate that even in such cases, the topological fermion system may be regular and well-behaved.

Example 8.4. *(A genuine singularity on the Lorentzian torus times S^1)* On $\hat{M} = \mathbb{R} \times S^1 \times S^1$ we choose coordinates (t, φ, α) with $t \in \mathbb{R}$ and $0 < \varphi, \alpha < 2\pi$. As the Lorentzian metric on $\hat{M}$ we take the warped product of (8.8) with a metric on S^1,

$$ds^2 = dt^2 - R(t)^2\, d\varphi^2 - S(t)^2\, d\alpha^2 \,.$$

We satisfy the anti-commutation relations (8.1) with the ansatz

$$G^t = \sigma^3 \,, \qquad G^\varphi = \frac{i\sigma^1}{R(t)} \,, \qquad G^\alpha = \frac{i\sigma^2}{S(t)} \,.$$

In order for these matrices to be symmetric, we again consider the spin scalar product (8.9) of signature $(1,1)$. Again using (8.2), the Dirac operator becomes

$$G = i\sigma^3 \partial_t + \frac{i}{2} \left(\frac{\dot{R}(t)}{R(t)} + \frac{\dot{S}(t)}{S(t)} \right) \sigma^3 - \frac{1}{R(t)}\, \sigma^1 \partial_\varphi - \frac{1}{S(t)}\, \sigma^2 \partial_\alpha \,.$$

The Dirac equation $(\mathcal{D} - m)\psi = 0$ can be separated by the ansatz

$$\psi = \frac{e^{ik\varphi + il\alpha}}{\sqrt{R(t) S(t)}} \begin{pmatrix} \chi_1(t) \\ i\chi_2(t) \end{pmatrix} \quad \text{with} \quad k, l \in \mathbb{Z} \,. \tag{8.20}$$

We thus obtain the ODE in time

$$i\dot{\chi}(t) = \begin{pmatrix} m & -\dfrac{k}{R(t)} + \dfrac{il}{S(t)} \\ -\dfrac{k}{R(t)} - \dfrac{il}{S(t)} & -m \end{pmatrix} \chi(t) \,.$$

Since the matrix on the right is Hermitian, the relation (8.12) again holds, whereas (8.13) is modified to

$$\chi\big|_{t_0}^{t_1} \leq 2\, \|\chi\| \int_{t_0}^{t_1} \left(m + \frac{|k|}{R(t)} + \frac{|l|}{S(t)} \right) dt \,.$$

If the functions $1/R$ and $1/T$ are integrable, we infer that χ is continuous.

We let $\mathcal{H}$ be a finite-dimensional space of Dirac wave functions endowed with a scalar product $\langle .|. \rangle$. We again introduce the local correlation operators F by (2.7). For given $\varepsilon > 0$, we again choose the function $R(t)$ according to (8.14). In order for the factor $(R(t)S(t))^{\frac{1}{2}}$ in (8.20) to be regular in the limit $\varepsilon \searrow 0$, we choose

$$S(t) = \frac{1}{R(t)} = (t^2 + \varepsilon^2)^{-\frac{1}{4}} \,.$$

Then the functions χ converge uniformly as $\varepsilon \searrow 0$. Since the prefactors in (8.20) are also regular in this limit, we conclude that the local correlation operators defined by (2.7) also converge. Introducing the universal measure by $\rho = F_*(dt\, d\varphi)$, for any $\varepsilon > 0$ we obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension one. This family of causal fermion systems has a regular limit as $\varepsilon \searrow 0$, despite the fact that a curvature singularity forms.

We point out that in this example, the curvature singularity is genuine (in the sense that it cannot be removed by a conformal transformation). Nevertheless, the corresponding causal fermion systems have a regular limit, even without rescaling the local correlation operators. ◊

Example 8.5. *(A genuine singularity on a cone times S^1)* We consider $\tilde{M} = (\mathbb{R}^2 \setminus \{0\}) \times S^1$, choose polar coordinates (r, φ) in $\mathbb{R}^2$ and the angular coordinate $\alpha \in (0, 2\pi)$ on the factor S^1. We take the warped product metric

$$ds^2 = dr^2 + R(r)^2 \, d\varphi^2 + S(r)^2 \, d\alpha^2 \, .$$

The Dirac operator is computed in analogy to (8.17) by

$$\mathcal{D} = i\sigma^r \partial_r + \frac{i}{R(r)} \, \sigma^\varphi \partial_\varphi + \frac{i}{S(r)} \, \sigma^3 \partial_\alpha + \frac{i}{2} \left(\frac{R'(r)}{R(r)} + \frac{S'(r)}{S(r)} - 1 \right) \sigma^r \, .$$

Employing similar to (8.18) the ansatz

$$\psi = \frac{e^{il\alpha}}{\sqrt{R(r)\, S(r)}} \begin{pmatrix} e^{i(k-\frac{1}{2})\varphi} \, \chi_1(r) \\ ie^{i(k+\frac{1}{2})\varphi} \, \chi_2(r) \end{pmatrix} \quad \text{with} \quad k \in \mathbb{Z} + \frac{1}{2}, \; l \in \mathbb{Z} \, ,$$

we obtain the radial ODE

$$\chi'(r) = \begin{pmatrix} k/R & \lambda - l/S \\ -\lambda - l/S & -k/R \end{pmatrix} \chi(r) \, .$$

We now choose the parameters in such a way that we obtain solutions of the Dirac equation in closed form which are continuous and non-zero at the origin. For the metric functions we take

$$R(r) = \frac{5}{6} \, r \qquad \text{and} \qquad S(r) = \frac{5}{4} \, r \, .$$

Thus the metric on $\mathbb{R}^2 \setminus \{0\}$ is conical (similar to Example 8.3), and the size of the factor S^1 shrinks to zero as $r \searrow 0$. A direct computation shows that the wave functions

$$\begin{aligned} \psi_+(r, \varphi, \alpha) &= \frac{e^{i\alpha}}{r^2} \begin{pmatrix} (1-2r)\sin(r) - r\cos(r) \\ -ie^{i\varphi} \big((2-r)\sin(r) - 2r\cos(r) \big) \end{pmatrix} \\ \psi_-(r, \varphi, \alpha) &= \frac{e^{i\alpha}}{r^2} \begin{pmatrix} e^{-i\varphi} \, \big((2-r)\sin(r) - 2r\cos(r) \big) \\ i\big(1-2r)\sin(r) - r\cos(r) \big) \end{pmatrix} . \end{aligned}$$

are solutions of the Dirac equation corresponding to the angular quantum numbers $k = \pm\frac{1}{2}$ and $l = 1$. Note that the spinors stay bounded as $r \searrow 0$ and do not converge to zero in this limit.

We let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be the vector space spanned by ψ_+ and ψ_- with the scalar product such that $\psi_\pm$ are orthonormal. Then the local correlation operators (see Definition 2.4) have the following expansion near $r = 0$,

$$F(r, \varphi, \alpha) = \begin{pmatrix} 5 & 4e^{-i\varphi} \\ 4e^{-i\varphi} & 5 \end{pmatrix} + \mathcal{O}(r^2) \, .$$

For fixed φ, these local correlation have a well-defined limit as $r \searrow 0$. This allows us to extend F by continuity to a mapping

$$\tilde{F} \;:\; \hat{M} \cup S^1 \times S^1 \simeq [0, \infty) \times S^1 \times S^1 \rightarrow \mathcal{F} \subset \mathrm{L}(\mathcal{H}) \, .$$

Taking the push-forward measure $\rho = \tilde{F}_*(d^2x)$, we obtain a Riemannian fermion system of spin dimension two. With this construction, we have compactified the manifold $\hat{M}$ by glueing an $S^1 \times S^1$ to the singularity point.

We remark that the mapping F is not injective. In order to cure this shortcoming, one simply extends $\mathcal{H}$ by wave functions with other quantum numbers k and l (similar as explained in Example 8.3). Then the compactified manifold can be identified with the support of ρ. $\Diamond$

8.3. **The Curvature Singularity of Schwarzschild Space-Time.** We now explain how causal fermion systems can be used to extend the Schwarzschild geometry by a boundary describing a blow-up of the curvature singularity. In polar coordinates $(t, r, \vartheta, \varphi)$, the line element of the Schwarzschild metric is given by

$$ds^2 = g_{jk}\, dx^j x^k = \frac{\Delta}{r^2}\, dt^2 \;-\; \frac{r^2}{\Delta}\, dr^2 - r^2\, d\vartheta^2 - r^2\, \sin^2\vartheta\, d\varphi^2 \;, \tag{8.21}$$

where

$$\Delta = r^2 - 2Mr \;, \tag{8.22}$$

and M is the mass of the black hole. The variables take values in the range $t \in \mathbb{R}$, $r \in \mathbb{R}^+ \setminus \{r_1\}$, $\vartheta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$, where the zero $r_1 := 2M$ of Δ defines the event horizon of the black hole. The metric has coordinate singularities at $r = r_1$ and $\vartheta = \{0, \pi\}$. Moreover, at $r = 0$ there is the curvature singularity at the center of the black hole. For details on the Schwarzschild geometry we refer for example to [26] or [38].

The Dirac operator in the Schwarzschild geometry can be computed just as explained at the beginning of Section 8. Since we are interested in the curvature singularity at $r = 0$, we restrict attention to the region $r < r_1$ in the *interior of the black hole*. We work in the spinor frame used in [16] in the Kerr-Newman geometry, so that all our formulas are obtained from those in [16] by setting $a = Q = 0$. We let γ^j be the Dirac matrices in Minkowski space in the Weyl representation,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \qquad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix} ,$$

(where $\alpha = 1, 2, 3$, and σ^α are again the Pauli matrices (2.2)). These matrices satisfy the anti-commutation relations (8.1) (with σ^j replaced by γ^j). Moreover, the metric (8.21) is obviously diagonal. Hence Proposition 8.1 applies. We choose the Dirac operator as

$$\mathcal{D} = iG^j \partial_j + B \;,$$

where the Dirac matrices take the form

$$G^0 = \frac{r}{\sqrt{|\Delta|}}\, \gamma^1 \;, \qquad G^1 = -\frac{\sqrt{|\Delta|}}{r}\, \gamma^0 \;, \qquad G^2 = \frac{\gamma^2}{r} \;, \qquad G^3 = \frac{\gamma^3}{r \sin\vartheta} \;,$$

and B is the multiplication operator (8.4). The Dirac equation is

$$\mathcal{D}\Psi = m\Psi \;,$$

where Ψ is a section in the spinor bundle SM. The inner product on the fibre S_xM takes the form

$$\prec\!\Psi|\Phi\!\succ_x = \big\langle \Psi \,\big|\, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Phi \big\rangle_{\mathbb{C}^4} \,.$$

The Dirac equation in the Schwarzschild geometry can be completely separated into ordinary differential equations. We again use a method which immediately generalizes to the Kerr-Newman geometry to give the formulas in [16]. Employing the ansatz

$$\Psi(t,r,\vartheta,\varphi) = \frac{e^{-i\omega t}\, e^{-i(k+\frac{1}{2})\varphi}}{\sqrt{r}\,|\Delta|^{\frac{1}{4}}} \begin{pmatrix} X_-(r)\,Y_-(\vartheta) \\ X_+(r)\,Y_+(\vartheta) \\ X_+(r)\,Y_-(\vartheta) \\ X_-(r)\,Y_+(\vartheta) \end{pmatrix} \tag{8.23}$$

with $\omega \in \mathbb{R}$ and $k \in \mathbb{Z}$ gives two ordinary differential equations for X and Y. The angular equation for Y can be solved explicitly in terms of spin-weighted spherical harmonics. This determines the separation constant λ to take one of the values

$$\lambda = \pm 1, \pm 2, \pm 3, \ldots\,,$$

and the separation constant k must lie in the range

$$-|\lambda| + \frac{1}{2} \;\leq\; k \;\leq\; |\lambda| - \frac{1}{2} \tag{8.24}$$

(see [23] or the detailed computations for the operator K in [17, Appendix A], noting that the separation constants λ and the functions Y coincide with the eigenvalues and eigenfunctions of the operator K in suitable spinor bases). The radial equation becomes

$$\begin{pmatrix} \sqrt{|\Delta|}\,\mathcal{D}_+ & imr - \lambda \\ -imr - \lambda & -\sqrt{|\Delta|}\,\mathcal{D}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0\,,$$

where

$$\mathcal{D}_\pm = \frac{\partial}{\partial r} \;\pm\; i\omega\, \frac{r^2}{\Delta}\,.$$

This equation can be written in the more convenient form

$$\partial_r X = i\omega\, \frac{r^2}{|\Delta|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X + \frac{1}{\sqrt{|\Delta|}} \begin{pmatrix} 0 & \lambda - imr \\ -\lambda - imr & 0 \end{pmatrix} X\,. \tag{8.25}$$

In order to understand the separation ansatz (8.23), one should keep in mind that we restrict attention to the region $r < r_1$ in the interior of the black hole. Then the variable t is spatial, whereas r is the time coordinate. Therefore, the plane-wave $e^{-i\omega t}$ can be used to form the Fourier decomposition of initial data given at some initial time $r_0 < r_1$. The radial equation (8.25) describes the time evolution of each Fourier component. Since the right of this equation is anti-Hermitian, one readily sees that

$$\partial_r |X| = 0\,. \tag{8.26}$$

This corresponds to current conservation for each separated mode.

Suppose that Ψ is a solution of the form (8.23). The angular eigenfunction Y is smooth. Moreover, near the curvature singularity at $r = 0$, the function Δ, (8.22), is smooth and tends linearly to zero. As a consequence, near $r = 0$ the radial equation has the asymptotic form

$$\partial_r X = \frac{\lambda}{\sqrt{2Mr}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X + \mathcal{O}\big(\sqrt{r}\big)\, X\,. \tag{8.27}$$

Since the singularity of the coefficients at $r = 0$ is integrable, a Grönwall estimate similar to (8.13) shows that X can be extended continuously to $r = 0$, and that its norm $|X|$ is bounded away from zero. We conclude that the only singular contribution at the origin is the factor $|\Delta|^{-\frac{1}{4}} r^{-1/2}$ in (8.23). Therefore, we can remove the singularity simply by rescaling the local correlation operators similar to our procedure in Section 8.1. More precisely, we introduce the local correlation operators in modification of (2.7) by

$$-\, r^{\frac{3}{2}} \prec\!\Psi|\Phi\!\succ_{(t,r,\vartheta,\varphi)} = \langle \Psi | F(t,r,\vartheta,\varphi)\, \Phi\rangle_{\mathcal{H}} \qquad \text{for all } \Psi, \Phi \in \mathcal{H}\,. \tag{8.28}$$

It remains to decide of which solutions the space $\mathcal{H}$ should be composed and to choose the Hilbert space scalar product $\langle .|. \rangle_{\mathcal{H}}$. The only subtle point is that we want the mapping F to be injective, making it necessary to choose "sufficiently many" wave functions. We take the span of all wave functions of the form (8.23) for $\omega \in \mathbb{R}$, and $\lambda = \pm 1$, i.e.

$$\Psi = (\Psi^{k\omega\lambda}) \qquad \text{with} \qquad \omega \in \mathbb{R}\,, \lambda \in \{\pm 1, \pm 2\}$$

(and k in the range (8.24)). For the scalar product we simply choose

$$\langle \Psi | \Psi \rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} e^{-\varepsilon^2 \omega^2}\, d\omega \sum_{\lambda=-2}^{2} \sum_{k=-|\lambda|+1/2}^{|\lambda|-1/2} |X^{k\omega\lambda}|^2\,, \tag{8.29}$$

where $\varepsilon > 0$ (we always assume the angular eigenfunctions Y to be normalized; note that by (8.26) the scalar product is independent of r). Polarizing and taking the completion, we obtain a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. The factor $e^{-\varepsilon^2\omega^2}$ in (8.29) can be regarded as a convergence-generating factor describing an ultraviolet regularization on the length scale ε. It ensures that the functions in $\mathcal{H}$ are all continuous, so that the local correlation operators are well-defined by (8.28) for all $r > 0$. Moreover, the continuity of our fundamental solutions at $r = 0$ makes it possible to extend the local correlation operators to $r = 0$,

$$F(t, 0, \vartheta, \varphi) := \lim_{r \searrow 0} F(t, r, \vartheta, \varphi)\,.$$

We thus obtain a mapping

$$F \,:\, \mathbb{R} \times [0, r_1) \times S^2 \to \mathcal{F}\,. \tag{8.30}$$

Again defining the universal measure as the push-forward measure $\rho = F_* \mu$, we obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension two.

Lemma 8.6. *The mapping F,* (8.30)*, is injective.*

Proof. In preparation, we need to construct approximate solutions of the ODE (8.25). Introducing the Regge-Wheeler coordinate u by

$$\frac{du}{dr} = -\frac{r^2}{|\Delta|}\,, \qquad \text{so} \qquad u = r + 2M\, \log|r - 2M|\,,$$

the radial equation can be written as

$$\partial_u X = -i\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X - \frac{\sqrt{|\Delta|}}{r^2} \begin{pmatrix} 0 & \lambda - imr \\ -\lambda - imr & 0 \end{pmatrix} X\,.$$

In order to describe the asymptotics for large ω, we employ the ansatz

$$X = \begin{pmatrix} e^{-i\omega u} & 0 \\ 0 & e^{i\omega u} \end{pmatrix} Z\,.$$

For Z we obtain the equation

$$\partial_u Z = -\frac{\sqrt{|\Delta|}}{r^2} \begin{pmatrix} 0 & (\lambda - imr)\, e^{2i\omega u} \\ (-\lambda - imr)\, e^{-2i\omega u} & 0 \end{pmatrix} Z \, .$$

Due to the oscillatory phase factors $e^{\pm 2i\omega u}$, the right side has no influence on the solution if ω gets large, provide that r stays away from zero. Combining this fact with the observation made after (8.27) that X is continuous at $r=0$ (and this argument is even locally uniform in ω), we conclude that there are solutions with the asymptotics

$$X(u) = \begin{pmatrix} c_1\, e^{-i\omega u} \\ c_2\, e^{i\omega u} \end{pmatrix} + \mathcal{O}\big(\omega^{-1}\big) \, .$$

In view of the factor $e^{-i\omega t}$ in (8.23), we thus obtain solutions which depend on $t+u$ and $t-u$, respectively. Taking superpositions of such solutions for ω in a small neighborhood of some fixed frequency ω_0, we can build up "wave packet solutions," where the first component of X propagates along the curves $t+u = \text{const}$, whereas the second component propagates along the curves $t-u=\text{const}$,

$$X(t,u) = \begin{pmatrix} X_1(t+u) \\ X_2(t-u) \end{pmatrix} + \mathcal{O}\big(\omega_0^{-1}\big) \, . \tag{8.31}$$

We remark for clarity that this estimate is locally uniform in t and u, meaning that (8.31) holds with a fixed error term for all t and u in a compact set. Moreover, the error term clearly depends on the angular momentum mode. But this is of no relevance to us because the particle space only involves the finite number of angular momentum modes $\lambda = -2, \ldots, 2$.

Let $(t,r,\vartheta,\varphi) \neq (\tilde{t},\tilde{r},\tilde{\vartheta},\tilde{\varphi})$ be two distinct space-time points. Then either $(t,r) \neq (\tilde{t},\tilde{r})$ or $(\vartheta,\varphi) \neq (\tilde{\vartheta},\tilde{\varphi})$. In order to treat the first case $(t,r) \neq (\tilde{t},\tilde{r})$, we know that in Regge-Wheeler coordinates either $t+u \neq \tilde{t}+\tilde{u}$ or $t-u \neq \tilde{t}-\tilde{u}$. Thus we can choose a wave packet of the form (8.31) which goes through the point (t,r) but not through the point $(\tilde{t},\tilde{r})$. This shows that the local correlation operators at the two points are necessarily different.

In the remaining case $(t,r) = (\tilde{t},\tilde{r})$ but $(\vartheta,\varphi) \neq (\tilde{\vartheta},\tilde{\varphi})$, we know from the explicit form of the angular eigenfunctions as worked out in [23] or in [17, Lemma A.3] that the span of these functions for eigenvalues in the range $-2 \leq \lambda \leq 2$ contains the constant and linear functions in the Cartesian coordinates (x,y,z) restricted to the sphere $S^2 \subset \mathbb{R}^3$. In particular, the span contains the four functions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 0 \end{pmatrix} .$$

Forming suitable linear combinations, we can construct a spinor which vanishes at (ϑ,φ) but is non-zero at $(\tilde{\vartheta},\tilde{\varphi})$. Taking the spin scalar product with the constant spinor, one concludes that the local correlation operators at the two points are different. □

This lemma allows us to identify the extended space-time $\mathbb{R} \times [0,r_1) \times S^2$ with a subset of $\mathcal{F}$. Thus the causal fermion system describes the whole interior Schwarzschild geometry. Moreover, it includes the singularity at $r=0$ as a boundary of space-time which is diffeomorphic to $\mathbb{R} \times S^2$.

We finally remark that by going over to the Kruskal extension, our construction could readily be extended to the exterior region of the Schwarzschild black hole.

Moreover, using the formulas in [16], the constructions immediately extend to the non-extreme Reissner-Nordström, Kerr and Kerr-Newman geometries.

8.4. **A Lattice System with Non-Trivial Topology.** We now illustrate the constructions in Section 6 by a simple example of a lattice system. Before beginning, we remark that our constructions bear some similarity to idea's by M. Lüscher [32], who considers a lattice on the four-dimensional torus and shows that one can introduce a non-trivial topological charge provided that the field strength of a lattice gauge field is small on the lattice scale (see also the discussion in [39]). However, our construction is different and much more general because we do not need the nearest-neighbor relation. Moreover, we do not assume a connection on the bundle, nor that the corresponding field strength be small. Instead, we need to assume that the distance of the lattice points is small on the "macroscopic length scale" on which the topology of the torus is visible.

We consider the two-dimensional torus $T^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ with the metric induced from the Euclidean metric of $\mathbb{R}^2$. Moreover, for a given parameter $\kappa > 0$ with $2\pi/\kappa \in \mathbb{N}$, we consider the lattice

$$\hat{M} = (\kappa\mathbb{Z})^2/(2\pi\mathbb{Z})^2 \,.$$

Thus $\hat{M}$ is a lattice on T^2 with lattice spacing κ, consisting of $(2\pi/\kappa)^2$ lattice points. We let μ be the normalized counting measure on $\hat{M}$,

$$\mu(\Omega) = \frac{\kappa^2}{(2\pi)^2}\, \#\Omega \,.$$

On the two-dimensional torus there are different spin structures with corresponding Dirac operators. For simplicity, we take the Dirac operator $\mathcal{D}$ obtained from the Dirac operator on $\mathbb{R}^2$ (7.1) by taking the quotient with $(2\pi\mathbb{Z})^2$. Then an eigenvector basis of this Dirac operator is given similar to (7.3) in terms of the plane wave solutions

$$\mathfrak{e}_0^+(\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \mathfrak{e}_0^-(\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.32}$$

$$\mathfrak{e}_k^\pm(\zeta) = \frac{1}{|k|}\left(k_1\sigma^1 + k_2\sigma^2 \pm |k|\mathbb{1}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ik\zeta} \qquad \text{if } k \in \mathbb{Z}^2 \setminus \{0\} \,, \tag{8.33}$$

where now k lies on the dual lattice $\mathbb{Z} \times \mathbb{Z}$. By direct computation one verifies that the eigenvalues of the wave functions $\mathfrak{e}_k^\pm$ are $\pm|k|$.

We choose $\mathcal{H}$ as the vector space spanned by a finite number of plane-wave solutions. The scalar product $\langle .|. \rangle_{\mathcal{H}}$ is defined by imposing that the plane-wave solutions $\mathfrak{e}_k^\pm$ are orthonormal. For computational simplicity, we choose the three-dimensional space

$$\mathcal{H} = \text{span}\big(\mathfrak{e}_0^+, \mathfrak{e}_{(1,0)}^+, \mathfrak{e}_{(0,1)}^+\big) \,, \tag{8.34}$$

but any choice of $\mathcal{H}$ which contains these three vectors would work just as well. For any $p \in \mathcal{T}^2$, we introduce the local correlation operator again by (2.7). We now define the universal measure as the push-forward of the counting measure on the lattice,

$$\rho = \big(F|_{\hat{M}}\big)_* \mu \,.$$

We thus obtain a Riemannian fermion system $(\mathcal{H}, \mathcal{F}, \mu)$ of spin dimension two.

In order to analyze the resulting fermion, it is useful to represent the local correlation operators in the orthonormal basis of the vectors in (8.34). A short computation gives

$$F(\zeta) = -\begin{pmatrix} 1 & e^{-ix} & e^{-iy} \\ e^{ix} & 2 & (1-i)\, e^{ix-iy} \\ e^{iy} & (1+i)\, e^{-ix+iy} & 2 \end{pmatrix}, \tag{8.35}$$

where we denote the components of $\zeta \in T^2$ by (x, y). Moreover, a short computation gives

$$\|F(\zeta) - F(\zeta')\|^2 = 16 - 4\cos(x-x') - 4\cos(y-y') - 8\cos(x-x'+y-y')$$

(where for convenience we work with the Hilbert-Schmidt norm on $\mathrm{L}(\mathcal{H})$). Using the sum rules and the inequality $|\cos\varphi| \leq 1$, we obtain

$$\begin{aligned}\|F(\zeta) - F(\zeta')\| &= 8\sin^2\Big(\frac{x-x'}{2}\Big) + 8\sin^2\Big(\frac{y-y'}{2}\Big) + 16\sin^2\Big(\frac{x-x'}{2} + \frac{y-y'}{2}\Big) \\ &\leq 24\sin^2\Big|\frac{x-x'}{2}\Big| + 32\sin\Big|\frac{x-x'}{2}\Big|\ \sin\Big|\frac{y-y'}{2}\Big| + 24\sin^2\Big|\frac{y-y'}{2}\Big|.\end{aligned}$$

Applying the Schwarz inequality, we conclude that

$$\|F(\zeta) - F(\zeta')\| \leq \sqrt{24}\,\bigg(\sin\Big|\frac{x-x'}{2}\Big| + \sin\Big|\frac{y-y'}{2}\Big|\bigg)\,. \tag{8.36}$$

Moreover, the distance of antipodal points on the torus is computed by

$$\begin{aligned}&\|F(0,0) - F(\pi,0)\| = \|F(0,0) - F(0,\pi)\| = \sqrt{24} \\ &\|F(0,0) - F(\pi,\pi)\| = 4\,.\end{aligned}$$

After these preparations, we can discuss the constructions from Section 6. The matrix representation (8.35) shows in particular that the mapping

$$F : T^2 \to \mathcal{F} \quad \text{is injective}\,.$$

Hence the image $F(T^2)$ is topologically a torus. Taking the image of the lattice $\hat{M}$, we obtain a set of $(2\pi/\kappa)^2$ of points in $\mathcal{F}$. The universal measure of our Riemannian fermion system is the normalized counting measure of these points. In particular, its support are these finite number of points,

$$M := \operatorname{supp}\rho = F(\hat{M}) \subset \mathcal{F}\,.$$

We conclude that the topology of the Riemannian fermion system is trivial.

The situation becomes more interesting when we consider the sets M_r defined by (6.1). For small r, the balls around the points in M do not intersect, so that the topology remains trivial (in view of (8.36), this is the case if $r < \sqrt{24}\,\sin(\kappa/4)$). If, on the other hand, the parameter r is chosen larger than $\sqrt{24}$, then each of these balls contains all M. Then M_r has the trivial topology of a ball. In the intermediate range

$$\sqrt{24}\,\sin\Big(\frac{\kappa}{2}\Big) < r < 2\,, \tag{8.37}$$

the ball around a point in M intersects the neighboring balls, but not the balls around the antipodal points. As a consequence, M_r has the topology of a torus. Even more, M_r can be continuously deformed to the set $F(T^2)$ (more precisely, $F(T^2)$ is a deformation retract of M_r). This implies that M_r has the same bundle topology as $F(T^2)$. In particular, M_r encodes the topological data of the torus. These considerations are illustrated in Figure 3.

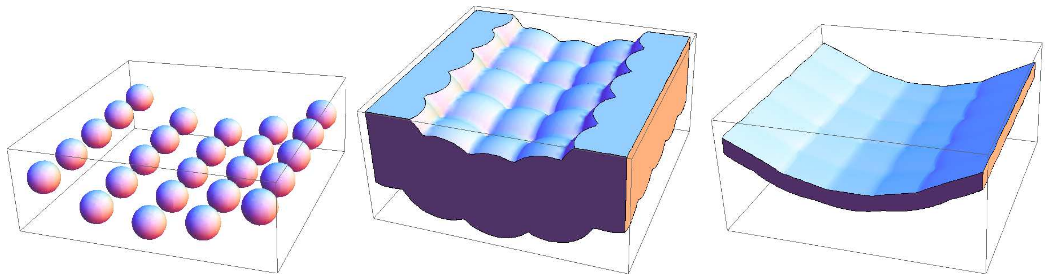

FIGURE 3. The geometry for small r (left), larger r (middle), and when deforming to $F(T^2)$ (right).

For the set M_δ defined in (6.3) the situation is similar, except that we now need to specify the range of δ. In order for the balls to include the nearest neighbors, we need to choose δ larger than the volume of five points. In order to exclude the antipodal points, we need to choose $\delta < 1/2$. Thus in order to recover the topology of the torus, we must choose δ in the range

$$\frac{5\kappa^2}{4\pi^2} < \delta < \frac{1}{2} \,. \tag{8.38}$$

In order to implement the construction (6.4) or (6.5), we must choose a measure ρ on $\mathcal{F}$. A simple method is to choose a basis $F_1, \ldots, F_9$ of the vector space of 3×3-matrices (for example an orthonormal basis with respect to the Hilbert-Schmidt norm) and to represent $F \in \text{Symm}(\mathbb{C}^3)$ by

$$F = \sum_{\alpha=1}^{9} f_\alpha \, F_\alpha \,.$$

Let $d\mu(F)$ be the Lebesgue measure $df_1 \cdots df_9$ multiplied by the Dirac distribution supported on the set $\{\det F = 0\}$. Then μ is a measure supported on the 3×3-matrices of rank at most two. Since $\mathcal{F}$ is a subset of these matrices of positive μ-measure, restricting μ to $\mathcal{F}$ gives a non-trivial measure on $\mathcal{F}$. Choosing $\eta_r(x,y) = \eta(\|x-y\|^2/r^2)$ with $\eta \in C^\infty_0([0,1))$, we can then introduce the measures ρ_r and ρ_δ by (6.4) and (6.5).

To summarize, the constructions of Section 6 make it possible to recover topological information on a lattice, provided that the lattice is sufficiently fine and the parameters r respectively δ are chosen such that the microscopic discrete structure is "smeared out" without affecting the global topological structure. In situations when the lattice spacing κ is very small, the inequalities (8.37) and (8.38) leave a lot of freedom to choose r respectively δ. Thus thinking of a discrete structure on the Planck scale, there is no problem in recovering the global topology of space-time.

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References

[1] H. Baum, *Spinor structures and Dirac operators on pseudo-Riemannian manifolds*, Bull. Polish Acad. Sci. Math. **33** (1985), no. 3-4, 165–171.

[2] Y. Bernard and F. Finster, *On the structure of minimizers of causal variational principles in the non-compact and equivariant settings*, arXiv:1205.0403 [math-ph], Adv. Calc. Var. **7** (2014), no. 1, 27–57.

[3] G.E. Bredon, *Sheaf Theory*, Graduate Texts in Mathematics, vol. 170, Springer-Verlag, New York, 1997.
[4] T. Bröcker, *Analysis II*, Bibliographisches Institut, Mannheim, 1992.
[5] A. Diethert, F. Finster, and D. Schiefeneder, *Fermion systems in discrete space-time exemplifying the spontaneous generation of a causal structure*, arXiv:0710.4420 [math-ph], Int. J. Mod. Phys. A **23** (2008), no. 27/28, 4579–4620.
[6] F. Finster, *The continuum limit of a fermion system involving leptons and quarks: Strong, electroweak and gravitational interactions*, in preparation.
[7] ______, *Local* $\mathrm{U}(2,2)$ *symmetry in relativistic quantum mechanics*, arXiv:hep-th/9703083, J. Math. Phys. **39** (1998), no. 12, 6276–6290.
[8] ______, *The Principle of the Fermionic Projector*, hep-th/0001048, hep-th/0202059, hep-th/0210121, AMS/IP Studies in Advanced Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2006.
[9] ______, *Causal variational principles on measure spaces*, arXiv:0811.2666 [math-ph], J. Reine Angew. Math. **646** (2010), 141–194.
[10] ______, *A formulation of quantum field theory realizing a sea of interacting Dirac particles*, arXiv:0911.2102 [hep-th], Lett. Math. Phys. **97** (2011), no. 2, 165–183.
[11] ______, *The continuum limit of a fermion system involving neutrinos: Weak and gravitational interactions*, arXiv:1211.3351 [math-ph] (2012).
[12] F. Finster and A. Grotz, *A Lorentzian quantum geometry*, arXiv:1107.2026 [math-ph], Adv. Theor. Math. Phys. **16** (2012), no. 4, 1197–1290.
[13] ______, *On the initial value problem for causal variational principles*, arXiv:1303.2964 [math-ph] (2013).
[14] F. Finster, A. Grotz, and D. Schiefeneder, *Causal fermion systems: A quantum space-time emerging from an action principle*, arXiv:1102.2585 [math-ph], Quantum Field Theory and Gravity (F. Finster, O. Müller, M. Nardmann, J. Tolksdorf, and E. Zeidler, eds.), Birkhäuser Verlag, Basel, 2012, pp. 157–182.
[15] F. Finster and C. Hainzl, *Quantum oscillations can prevent the big bang singularity in an Einstein-Dirac cosmology*, arXiv:0809.1693 [gr-qc], Found. Phys. **40** (2010), no. 1, 116–124.
[16] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau, *Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry*, gr-qc/9905047, Comm. Pure Appl. Math. **53** (2000), no. 7, 902–929.
[17] F. Finster and M. Reintjes, *The Dirac equation and the normalization of its solutions in a closed Friedmann-Robertson-Walker universe*, arXiv:0901.0602 [math-ph], Classical Quantum Gravity **26** (2009), no. 10, 105021.
[18] ______, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime*, arXiv:1301.5420 [math-ph] (2013).
[19] F. Finster and D. Schiefeneder, *On the support of minimizers of causal variational principles*, arXiv:1012.1589 [math-ph], Arch. Ration. Mech. Anal. **210** (2013), no. 2, 321–364.
[20] F. Finster and J. Tolksdorf, *Perturbative description of the fermionic projector: Normalization, causality and Furry's theorem*, arXiv:1401.4353 [math-ph] (2014).
[21] T. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Providence, RI, 2000.
[22] N. Ginoux, *The Dirac Spectrum*, Lecture Notes in Mathematics, vol. 1976, Springer-Verlag, Berlin, 2009.
[23] J.N. Goldberg, A.J. Macfarlane, E.T. Newman, F. Rohrlich, and E.C.G. Sudarshan, *Spin-s spherical harmonics and* $\eth$, J. Math. Phys. **8** (1967), 2155–2161.
[24] P.R. Halmos, *Measure Theory*, Springer, New York, 1974.
[25] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.
[26] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, London, 1973.
[27] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986), no. 1, 151–162.
[28] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1978 edition.
[29] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55.

[30] R.C. Kirby and L.C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Princeton University Press, Princeton, N.J., 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.
[31] H.B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.
[32] M. Lüscher, *Topology of lattice gauge fields*, Comm. Math. Phys. **85** (1982), no. 1, 39–48.
[33] J. McCleary, *User's Guide to Spectral Sequences*, Mathematics Lecture Series, vol. 12, Publish or Perish Inc., Wilmington, DE, 1985.
[34] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 76.
[35] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark (eds.), *Digital Library of Mathematical Functions*, National Institute of Standards and Technology from http://dlmf.nist.gov/ (release date 2011-07-01), Washington, DC, 2010.
[36] E.H. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1981, Corrected reprint.
[37] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Landmarks in Mathematics, Princeton University Press, 1951.
[38] R.M. Wald, *General Relativity*, University of Chicago Press, Chicago, IL, 1984.
[39] P. Woit, *Topology and lattice gauge fields*, Nuclear Phys. B **262** (1985), no. 2, 284–298.

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