Shape and topology optimization in fluids using a phase field approach and an application in structural optimization

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Claudia Hecht

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Prüfungsausschuss:	Vorsitzender:	Prof. Dr. U. Bunke
	1. Gutachter:	Prof. Dr. H. Garcke
	2. Gutachter:	Prof. Dr. H. Abels
	weiterer Prüfer:	Prof. Dr. G. Dolzmann

Abstract

We consider the problem of shape and topology optimization in fluid mechanics with a general objective functional. A phase field approach is introduced and discussed in terms of well-posedness and first order necessary optimality conditions. The state constraints are either given by the Stokes or the stationary Navier-Stokes equations. We find that minimizers of the diffuse interface setting have a converging subsequence as the interface thickness tends to zero. If this sequence fulfills a certain convergence rate or the total potential power is minimized in a Stokes flow, we obtain that the limit element is a minimizer of the sharp interface formulation. Additionally, we can derive in both, the Stokes and stationary Navier-Stokes setting, optimality conditions of the sharp interface model which can be verified to be the limit of corresponding optimality systems of the phase field model. Finally, we also apply this approach to structural optimization, where we want to find the optimal material distribution of two given elastic materials for a general objective functional. Using the techniques developed before, we can derive convergence results of a phase field approach similar to the fluid mechanical setting and discuss both the diffuse and the sharp interface formulation with regard to well-posedness and necessary optimality conditions.

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Contents

1 Introduction

The mathematical problem of shape optimization in fluids is to minimize some objective functional depending on the solution of a system of partial differential equations describing the fluid mechanics in an unknown bounded set. The control is represented by the shape of the set. If the topology of the set is not prescribed in advance, we refer to those problems as shape and topology optimization. Another application of shape and topology optimization can be found in structural optimization. There we try to find an optimal configuration of two different elastic materials in some fixed container, where optimal once again means that a certain objective functional depending on the behaviour of the elastic materials is minimized.

Applications of shape and topology optimization reach from optimizing guitars, crashworthiness of transport vehicles and tunnel design to biomechanical applications such as bone remodelling. Structural optimization has turned out to be helpful in solving automative design problems in order to maximize the stiffness of vehicles for instance or reduce the stresses to improve durability. In the context of fluid mechanics, we can find utilizations in the paper production industry ([HMT99]) or in biomedical engineering such as considering blood flow or the study of lungs and kidneys ([ABH05, BBB$^+$12, Bej00]). Moreover, we find a lot of work done in car design using the ideas of shape optimization in fluid dynamics, see for instance [GO05, DHM04, HS93, HH09]. However, it seems that airplane optimization plays the biggest role in applications, in particular in terms of optimization of wings and airfoils. A small percentage of a wing's drag minimization already yields a large profit for the industry. Just to mention a few works recently done on this topic we refer to [MP01, HMTT00, JMP98, Ang83, JMP98, GISS12, SSS11] and also to existing software like [BNS09, FD12]. In fact, there are many more application fields and we mention for instance the overviews in [MP04, Ben03, HM03, JT08, Bej00].
Newton already discussed the problem of finding the shape of an object's surface in a fluid having the least resistance, which can be considered as a first study of the drag minimizing problem, see [New63]. Nowadays, applications of this problem may be seeking the optimal shape of a harbor, while trying to minimize the incoming waves or optimizing wings or airfoils of an airplane as mentioned above. Another classical example in the field of finding optimal geometric forms, although not in fluid dynamics, is Plateau's problem, which consists of finding a set of least area among all sets with a given boundary and is motivated by Plateau's experiments with soap films. One important contribution concerning this problem was made by De Giorgi, see [DG61], by considering this problem in the context of sets of finite perimeter. He was one main contributor to the theory of sets of finite perimeter, also known as Caccioppoli sets, see [DG54]. Caccioppoli sets define the framework for our sharp interface model, too. Thus the fluid region, which will be the control in our problem, is chosen to be a Caccioppoli set. Using Caccioppoli sets as admissible space in shape optimization is a commonly used approach, see for instance [DZ01, SZ92, AB93, BHJ96].

One of the first treatments of shape optimization in a general setting appeared in [CZ73], where a finite element model is introduced. For a good review of the first approaches towards shape optimization we refer to [CH81]. In the context of optimal control theory, thus having partial differential equations as part of the model, the first discussions

of topology optimization were mainly aimed at structural optimization in elastic bodies, see for instance [BK88, KS86, EO01, Ben03, HP05] and included references. One of the first approaches of finding the optimal material distribution in presence of two materials can be found in [Tho92]. But the problem of finding optimal structures in mechanical engineering dates at least back to the beginning of the 20th century where Michell [Mic04] considered optimal truss layouts. In the field of fluid mechanics there are mainly works considering optimal shapes, while fixing the topology. Pioneering works in this branch were in particular [Pir73, Pir74, Sim91, Sim80] and we refer to [MP04, GI04, DZ01, HM03, SS10, PS10, JLM03] for some recent works. Most of those works concentrate on numerical methods or on local deformation of some given fluid domain and calculating shape derivatives. One of the typical examples considered there is the drag minimizing problem. Regarding the general problem of finding the optimal form of a fluid region to minimize a given cost functional, the approach of pure shape optimization with fixed topology may not be the best choice, since the optimal topology is a priori unknown. As indicated in our numerical examples in Section 10, it is for instance not clear how many pipes are optimal to transport a fluid. Thus topological changes should be allowed to get optimal profiles, which leads to the field of topology optimization in fluid dynamics.
This is still a young research field receiving growing attention in recent years. One of the first models for treating topology optimization in fluids has been introduced by [BP03] and is described below. For some recent results on specific topics of topology optimization in fluid dynamics we refer for instance to [DLL+11, GHHS05, KMP11, BKW07, Wik08], which mainly concentrate on numerical aspects and on the discussion of the model introduced in [BP03]. In the work of [BP03] the Stokes equations describe the fluid and the minimized objective functional is the drag functional. Moreover, an additional penalization term α, which is called inverse permeability, is introduced both in the state equations and in the objective functional. The Stokes equations then read in the strong formulation

$$\alpha(\rho)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0$$

and the corresponding objective functional is given by

$$\frac{1}{2}\int_\Omega \alpha(\rho)\,|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x.$$

This penalization term α should be very small in the fluid region, and very large outside, so that the velocity $\boldsymbol{u}$ vanishes in the limit outside of the fluid region. The design variable ρ is then a function with values between 0 and 1, where 0 corresponds to non-presence of fluid and 1 to the presence of fluid. We will use a similar approach in this work. In [BP03] the inverse permeability α is a fixed function, interpolating between two fixed finite values. We will consider the case, that the interpolation function α_ε depends on some parameter $\varepsilon > 0$, and as $\varepsilon \searrow 0$ the penalization term will vanish in the fluid domain, and will be infinite outside of the fluid. If α_ε is interpreted as inverse permeability of some porous medium outside the fluid this means that the permeability of the porous material tends to zero as $\varepsilon \searrow 0$. In [Evg05] the finiteness and fixed choice of α is interpreted as “not allowing real topology changes of the fluid region”, since the permeability of the porous medium is never zero, thus no “real walls” can appear. In our sharp interface model, the permeability of the porous material will be zero and so, following the definition of

[Evg05], we allow "real" topology changes. Another discussion of topology optimization in fluids can be found in [Evg05, Evg06]. There, the same model as in [BP03] is considered, but the interpolation function α now interpolates between zero and infinity, and thus giving rise to "real" topological changes. Moreover, the existence of black-and-white solutions (thus there exist only fluid or non-fluid regions, and no values in between) is shown. Yet, there are drawbacks in their work. Using the approach of [Evg05], it is not clear if this problem is still well-posed in case of objective functionals other than the drag functional. Moreover, the convergence of dark-grey-and-light-grey solutions (thus for the design variable only values close to 0 and close to 1 exist) to black-and-white solutions in the strong topology can only be shown under some additional assumption. Additionally, it is indicated in [Evg05] that numerics may pose problems in this setting.

As it turns out, none of the problems mentioned above arise in our approach. Since we approximate the black-and-white solutions by a diffuse interface problem, namely a phase field approach, numerics can be carried out with known techniques and existence of the approximating problem can even be ensured with a general objective functional. The convergence of the diffuse interface solutions to the sharp interface black-and-white solutions in the strong L^1-topology can be shown for the problem of minimizing the total potential power without additional assumptions. Besides, we can consider a general objective functional and under certain assumption we can still prove convergence of minimizers. Moreover, the approach of [Evg05] could not directly be generalized to the stationary Navier-Stokes equations, see [Evg06], but instead a relaxation of the incompressibility constraint and a filter have to be introduced in order to get a well-posed problem. Again this is no problem in our approach and we can handle the nonlinear state equations without major changes in the model.

We suggest a phase field model to describe the topology optimization problem. The idea is to have a small interfacial layer between the fluid region and the region outside the fluid, rather than the boundary of the fluid region being a free hypersurface. Phase field models are currently widely used in different research fields. The diffuse interface approach was already proposed by van der Waals [vdW79]. Later on, this theory was generalized by Ginzburg and Landau [GL50], and researches like Cahn, Hilliard and Allen, see [CH58, AC79, Cah61, CH59], applied it to microstructural evolution processes like spinodal decomposition and domain coarsening kinetics in binary alloys. Recently, applications of phase field models can be found in fields like phase transformations, solidification processes, grain growth and lately also in image segmentation, see for instance [BCM04]. For a good review we refer to [Che02]. Finally, phase field models were also coupled to hydrodynamic models and have for example applications in binary mixtures [LT98], spinodal decomposition, mixing and interfacial stretching or nucleation of droplets. A good overview can be found in [AMW98, LT98].
One of the first researchers using a phase field formulation for topology optimization were Bourdin and Chambolle in [BC03], where the compliance is minimized while regularizing with a perimeter term. After introducing a so-called fictious material relaxation, the perimeter is approximated by the Ginzburg-Landau energy, and Γ-convergence of the resulting energy is shown. In the main part of this thesis, we will consider fluid dynamics, which already implies a different setting as in [BC03]. Besides, our approach for shape optimization in fluid mechanics regularizes the state equations and the perimeter functional

at the same time. In [BC03], Γ-convergence of their reduced objective functional was proven by using the Γ-convergence result of Modica and Mortola [Mod87, MM77] and by showing that their functional is a continuous perturbation of the Ginzburg-Landau energy. However, this strategy cannot be used in our setting, because we have a problem in which both the objective functional and the state constraints depend on the phase field parameter $\varepsilon > 0$. In the last part we will apply the techniques developed before to structural optimization. There, the state equations are independent of the phase field variable $\varepsilon > 0$, but in contrast to [BC03] we consider a general objective functional and in addition to Γ-convergence of the reduced objective functionals we can show that certain first optimality systems of the phase field problem converge to necessary optimality conditions of the sharp interface formulation. So far, similar considerations have only been carried out by formal asymptotics, compare [BFSGS13]. For further results on applications of phase field models in topology optimization we refer to [WZ04, BGS+12, KNT10, BC06]. To the author's knowledge, a phase field approach has not been applied to topology optimization in fluids before.

One major advantage of our phase field approach for topology optimization is the regularity of the phase field variable describing the fluid region, which allows more efficient numerical calculations and gives rise to better analysis, while providing all relevant geometric properties. Moreover, topological changes can be handled without much effort, as well as numerics can be carried out quite easily, since we can apply the well-developed methods for phase field models. Besides, parametrization of a fixed domain by a family of diffeomorphisms, as it is done for instance in [AJ05, MP04, NPT09, MP01], naturally limits the admissible solutions. Using a phase field approach we certainly enlarge the set of possible solutions. Since we can show that the phase field formulation yields an appropriate approximation for the sharp interface problem, it is a serious alternative in this setting. In particular, most of the classical shape calculus results cannot be carried out rigorously, since the lack of regularity of the minimizing sets result in heuristic calculations and well-posedness, thus the existence of minimizers, is often not guaranteed. Our results are all verified and consistent without imposing unverified assumptions and give rise to a more systematic approach to topology optimization in fluid dynamics. But if the assumptions necessary for deriving the classical shape optimization results are fulfilled, we can show that our first order optimality condition results are equivalent to those of the known literature, see for instance [BFCLS97, BFCLS96, GMZ08, MP04, Pir73, Pir74, Sim91, SS10, ADDM13, AJVG11, HHS13].

In the main part of this study, we will consider a general objective functional depending on the velocity $\boldsymbol{u}$ of some fluid (Part I and II), namely

$$\int_\Omega f\left(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}\right)\,\mathrm{d}x.$$

In the third part it can additionally depend on the pressure p of the fluid, thus we minimize

$$\int_\Omega f\left(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}\right)\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x.$$

We handle this case in a separate part because certain difficulties occur, see also discussion in Section 18.1. The state equations are either the Stokes equations (Part I and III), given in the strong formulation by

$$-\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0,$$

or the stationary Navier-Stokes equations (Part II and discussion at the end of Part III) which are given by

$$-\mu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0.$$

We will show that our phase field approach is a good approximation of the sharp interface model in the sense that minimizers of the phase field formulation converge to a minimizer of the sharp interface setting under certain assumptions. In the particular situation of minimizing the total potential power in a Stokes flow we can even prove Γ-convergence of the reduced objective functionals. The sharp interface formulation is, as already mentioned above, a topology optimization problem where the admissible sets are Caccioppoli sets and the objective functional is penalized by adding a perimeter term $P_\Omega(E)$, which is the perimeter of the fluid region E inside the fixed container Ω. This perimeter penalization is a typical ansatz to overcome the general ill-posedness of the problem, see for instance [Pet99, AB93, BC03, Mur77].
The notion of Γ-convergence was introduced by De Giorgi [DG75] and has since been extended to a broad range of applications. One of the most important results concerning Γ-convergence of functionals is certainly the result of Modica and Mortola [MM77, Mod87], who proved that the Ginzburg-Landau energy

$$E_\varepsilon(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x$$

Γ-converges to some multiple of the perimeter functional

$$P_\Omega(\{\varphi = 1\})$$

in $L^1(\Omega)$ as $\varepsilon \searrow 0$, compare also the results in [Ste88, Alb00]. This will be an important ingredient for our approach, too, since the phase field models presented in this thesis will have the Ginzburg-Landau energy as a penalization term in the objective functional and the sharp interface problems contain the perimeter functional. Γ-convergence results have already been discussed for phase field approaches in topology optimization of materials, where we mainly point out the work of [BC03, WZ04, Gar08, PRW11]. In the last part, where we consider structural optimization, we state comparable results as in [BC03, WZ04] concerning Γ-convergence of the reduced objective functionals but with a very generally formulated problem.
We emphasize that in the fluid dynamical problems considered here, the state equations depend on the phase field parameter ε, which describes the interface thickness. This is not the case in the cited works above, but changes the mathematical considerations drastically.
In the specific setting of minimizing the total potential power in a Stokes flow we will prove Γ-convergence of the reduced objective functionals. For a general objective functional and the stationary Navier-Stokes equations, we can show under certain assumptions on the sequence of minimizers of the phase field problems that those minimizers approximate a minimizer of the sharp interface problem. We can even show that certain first order optimality conditions of the phase field model and the sharp interface model converge to

each other, both in the fluid dynamical problems and in structural optimization. This is an important result and has to the author's knowledge not been shown in a comparable setting before.

One aim of this work is to show well-posedness of the phase field approach and to show that it approximates the classical topology optimization problem, which is a sharp interface model. Additionally, we derive first order necessary optimality conditions for both models and show that the optimality system derived in the diffuse interface model approximates the optimality system of the sharp interface model.

This dissertation is organized in four parts as follows:

Part I: In the first part, we start by considering a general shape and topology optimization problem, where the objective functional depends on the velocity of the fluid described by the Stokes equations. We first introduce the phase field approach describing the shape and topology optimization problem on a diffuse interface level in Section 5. After discussing well-posedness of this problem, we formulate the sharp interface shape and topology optimization problem in Section 6, discuss its well-posedness, and show that minimizers of the phase field model have a converging subsequence if the interface thickness tends to zero. If this converging subsequence fulfills a certain convergence rate, we find that the limit element is a minimizer of the sharp interface model and that the minimal functional values converge, too. In the specific situation of minimizing the total potential power we can even obtain the stronger result of Γ-convergence of the reduced objective functionals associated to the phase field problems, see Section 6.3. Finally, we discuss in Section 6.4 possible extensions of the obtained statements and whether a more general result could be expected.

To discuss first order optimality conditions we first derive the classical variational inequality for the phase field model in Section 7.1 by parametric variations. But we also vary the design variable in Section 7.2, which is the phase field variable in this approach, by deforming the domain with a suitable transformation. Thus we apply the same idea as used in shape calculus. This will lead to suitable optimality conditions that approximate the optimality conditions of the sharp interface model. Moreover, we can show that under suitable assumptions those conditions can also be derived from the variational inequality directly, see Section 7.3.
Similarly, we start discussing optimality conditions for the sharp interface model in Section 8. First of all we calculate shape derivatives in Section 8.1, while assuming more regularity on the minimizer than we can actually prove. We remark that those regularity assumptions are typically imposed in the field of shape calculus. As a result we see that we arrive in the same results as already known in literature. Then we generalize this result by calculating first order optimality conditions, that can be shown without additional assumptions, see Section 8.2, but are under suitable assumptions equivalent to the shape derivatives calculated before, compare Section 8.3.

Finally, in Section 9 we even show that the optimality conditions of the phase field model derived in Section 7.2 converge, as the thickness of the interface tends to zero, to the optimality conditions of the sharp interface model derived in Section 8.2 if the convergence rate on the sequence of minimizers mentioned above is satisfied. Thus we have obtained a consistent approximation of the sharp interface model by means of our phase field approach.

Part II: In the second part we consider the stationary Navier-Stokes equations as a system describing the fluid motion, while still regarding the general objective functional depending on the velocity of the fluid that has been considered in the first part. Due to the fact that uniqueness of a solution to the stationary Navier-Stokes equations can only be shown under additional assumptions, we have to impose some restrictions on the objective functional in Section 11.2, which are for instance in case of the drag minimization problem equivalent to “smallness of data or large viscosity”, as in classical results. After that we introduce the phase field model and discuss its well-posedness, see Section 12. In Section 13 we consider the sharp interface model and examine the state equations with regard to solvability and uniqueness. Analogously to the first part we show that there exists a subsequence of the sequence of minimizers of the phase field problems, that converges as the thickness of the interface tends to zero. Assuming this subsequence satisfies a certain convergence rate we prove that the limit element is a minimizer of the sharp interface model in Section 14.

Then we derive first order optimality conditions for the phase field model in Section 15, first in form of a variational inequality, and then again by varying the domain by means of a transformation, and show thereafter that we can directly derive the latter from the variational inequality, too. Then we calculate classical shape derivatives of the sharp interface problem in Section 16.1, while imposing additional regularity assumptions on a minimizing set, that have not been verified in general. But we can derive optimality conditions without stating additional assumptions in Section 16.2, and show that those are equivalent to the results obtained by shape calculus if imposing the regularity assumptions used in Section 16.1.
We complete the discussion in Section 17 by showing that the optimality system of the diffuse interface model approximates optimality conditions of the sharp interface model if the convergence rate stated above is satisfied for the sequence of minimizers.

Part III: In Part III we generalize the objective functional of the previous parts by adding a pressure depending term. The main discussion is carried out while assuming the Stokes equations as a fluid model, but in Section 22 we point out that we could also apply the same generalized objective functional to a Navier-Stokes setting as in Part II.
At the beginning, we discuss the correct formulation of the model in this setting, see Section 18.1. Then we examine the phase field model with regard to well-posedness and optimality conditions in Section 19. The same discussion is carried out for the sharp interface model in Section 20, where in addition we briefly consider the general well-posedness of the pressure in measurable sets in Section 20.1.
After discussing both models independently, in Section 21 we prove convergence of minimizers of the phase field model to a minimizer of the sharp interface model

if the interface thickness tends to zero under the constraint that the sequence of minimizers fulfills a certain convergence rate. Besides, we show convergence of the optimality system as in the previous parts.

Part IV: In the last part, we will apply the methods and techniques developed before to a rather general problem in structural optimization. To be more precise, the problem is to find an optimal configuration in a mixture of two elastic materials. The first discussions are concerned with the formulation and mathematical considerations of an appropriate phase field approach for this problem, see Section 24. In Section 25 we introduce the corresponding sharp interface problem. In contrast to the fluid dynamical problems examined in the first three parts of this work, we can even directly show existence of a minimizer for the sharp interface problem. Besides, we use the ideas of the previous parts to obtain first order optimality conditions by geometric variations for both the sharp and the diffuse model. The subject of Section 26 is to connect the phase field formulation to the sharp interface problem. This is first done in terms of Γ-convergence of the reduced objective functionals. We remark that this is possible here without additional assumptions on the convergence rate of the minimizers or on the specific form of the objective functionals, as it was necessary in the first parts. Finally, we also prove that the first order optimality conditions for the phase field model derived before by geometric variations are an approximation of necessary optimality conditions in the sharp interface setting. In particular, we hereby generalise findings from literature where this result has already been indicated by formal asymptotics for certain objective functionals.

2 Notation and assumptions

We start our discussion by introducing some notation that we will use in the following and by giving the general assumptions that have to be fulfilled throughout the first three parts in this work, if not stated differently. In the last part we are considering structural optimization, thus the general setting changes in comparison to the first three parts where the problems are based upon fluid mechanics. Thus we will change the setting in the last part and details concerning this notation and assumptions can be found in Section 23.

First of all we define the fixed domain Ω, which will be some container in the following discussion, and the external forces that are given in this setting.

(A1) $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain with outer unit normal $\boldsymbol{n}$ and $d \in \{2,3\}$, such that $\mathbb{R}^d \setminus \overline{\Omega}$ is connected.

(A2) External forces:
$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is the applied body force and $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ the given boundary function such that $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}x = 0$.

We will propose a phase field model that approximates the sharp interface topology optimization problem. Therefore, we approximate the perimeter functional by the Ginzburg-Landau energy, where we still have to choose the potential that is used in this energy. Here we will focus on a double obstacle potential, leading to a so-called "sharp-diffuse" interface. For further discussions of different potentials we refer to [BE91].

(A3) Potential:
ψ is assumed to be an obstacle potential of the following form:

$$\psi : \mathbb{R} \to \overline{\mathbb{R}}, \quad \psi(\varphi) := \begin{cases} \psi_0(\varphi), & \text{if } |\varphi| \le 1 \\ +\infty, & \text{if } |\varphi| > 1 \end{cases}$$

where
$$\psi_0(\varphi) := \frac{1}{2}\left(1 - \varphi^2\right).$$

Let us denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the extended real numbers equipped with the standard order topology together with the usual convention $0^{-1} = \infty$ and $\infty \cdot 0 = 0$.

Moreover, we introduce an interpolation function α_ε as in [BP03], which appears in our phase field model in the objective functional and in the state equations. This interpolation function can be considered as inverse permeability, as it will have large values outside the fluid region (namely $\alpha_\varepsilon(-1) = \overline{\alpha}_\varepsilon$) and will be zero in the fluid region (modelled by $\alpha_\varepsilon(1) = 0$), where we remark that our design variable φ will have values 1 in the fluid region and -1 outside of the fluid. This inverse permeability can be considered, as the name already indicates, as inverse permeability of some porous medium being located outside the fluid. Hence, in the pure non-fluid domain, the state equation can be considered as a Darcy flow through porous medium with permeability $\overline{\alpha}_\varepsilon^{-1}$. In the pure fluid phase, we will obtain the classical Stokes equations or Navier-Stokes equations, depending on the

model. Moreover, we ensure by adding a term including α_ε to the objective functional that the velocity of the fluid is zero outside the fluid domain in the limit case $\varepsilon \searrow 0$. A more detailed description of the phase field approach can be found in Section 5.1.
One important property is that α_ε depends on the phase field parameter ε that models the thickness of the interface, and as ε tends to zero, α_ε will interpolate between infinity and zero, thus allowing “real” walls and topological changes, compare also discussion in [BP03, Evg05, Evg06].

(A4) Inverse permeability/interpolation function:
Let $\alpha_\varepsilon : [-1,1] \to [0, \overline{\alpha}_\varepsilon]$ be a decreasing, surjective and twice continuously differentiable function for $\varepsilon > 0$.
Let $\alpha_0 : [-1,1] \to [0,+\infty]$ be given as a decreasing, surjective, continuous function with $\alpha_0(0) < +\infty$.

It is required that $\overline{\alpha}_\varepsilon > 0$ is chosen such that $\lim_{\varepsilon\searrow 0} \overline{\alpha}_\varepsilon = +\infty$ and α_ε converges pointwise to α_0. Additionally, we impose $\alpha_\delta(x) \geq \alpha_\varepsilon(x)$ if $\delta \leq \varepsilon$ for all $x \in [-1,1]$ and a growth condition of the form $\overline{\alpha}_\varepsilon = \mathrm{o}\left(\varepsilon^{-2/3}\right)$.

Remark 2.1. *In space dimension $d = 2$ one could weaken the growth condition for $\overline{\alpha}_\varepsilon$ stated in Assumption **(A4)**. Applying the imbedding theorem $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for all $p < \infty$ we could establish the same results by only imposing $\overline{\alpha}_\varepsilon = o\left(\varepsilon^{-1+\delta}\right)$ for some $\delta \in (0,1)$, see also Remark 6.3.*

We denote $\mathbb{R}^d$-valued functions and spaces consisting of $\mathbb{R}^d$-valued functions in boldface.

Let us introduce the following function space:

$$\boldsymbol{U} := \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega, \operatorname{div} \boldsymbol{u} = 0\right\}$$

and the corresponding homogeneous space:

$$\boldsymbol{V} := \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0\right\}.$$

Throughout this work, we will sometimes work with the following auxiliary spaces:

$$\boldsymbol{H}_{\boldsymbol{g}}^1(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega\right\}$$

and

$$L_0^2(\Omega) = \left\{q \in L^2(\Omega) \mid \int_\Omega q \,\mathrm{d}x = 0\right\}.$$

We will be considering a general objective functional depending on the velocity of the fluid and its derivative in the first two parts. Our basic assumptions on this objective functional are the following:

(A5) Objective functional:
We choose $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ as a Carathéodory function, thus fulfilling

i) $f(\cdot, v, A) : \Omega \to \mathbb{R}$ is measurable for each $v \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, and

ii) $f(x,\cdot,\cdot):\mathbb{R}^d\times\mathbb{R}^{d\times d}\to\mathbb{R}$ is continuous for almost every $x\in\Omega$.

Let $p\geq 2$ for $d=2$ and $2\leq p\leq {}^{2d}\!/_{d-2}$ for $d=3$ and assume that there are $a\in L^1(\Omega)$, $b_1,b_2\in L^\infty(\Omega)$ such that for almost every $x\in\Omega$ it holds

$$|f(x,v,A)|\leq a(x)+b_1(x)|v|^p+b_2(x)|A|^2,\quad \forall v\in\mathbb{R}^d, A\in\mathbb{R}^{d\times d}. \tag{2.1}$$

Additionally, assume that the functional

$$F:\boldsymbol{H}^1(\Omega)\to\mathbb{R}$$

$$F(\boldsymbol{u}):=\int_\Omega f(x,\boldsymbol{u}(x),\mathrm{D}\boldsymbol{u}(x))\,\mathrm{d}x \tag{2.2}$$

is weakly lower semicontinuous, $F|_{\boldsymbol{U}}$ is bounded from below, and F is radially unbounded in $\boldsymbol{U}$, which means

$$\lim_{k\to\infty}\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}=+\infty \implies \lim_{k\to\infty}F(\boldsymbol{u}_k)=+\infty \tag{2.3}$$

for any sequence $(\boldsymbol{u}_k)_{k\in\mathbb{N}}\subseteq\boldsymbol{U}$.

Remark 2.2. *Remark that condition* (2.1) *implies that*

$$\boldsymbol{H}^1(\Omega)\ni\boldsymbol{u}\mapsto\int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}(x))\,\mathrm{d}x$$

is continuous. To be precise, (2.1) *is a necessary and sufficient condition such that*

$$L^p(\Omega)^d\times L^2(\Omega)^{d\times d}\ni(v,A)\mapsto f(\cdot,v,A)\in L^1(\Omega)$$

is a well-defined Nemytskii operator. If this operator exists as a Nemytskii operator, it is a continuous operator. For details we refer to [AZ90, Trö09, Sho97].

Remark 2.3. *Assume that $F|_{\boldsymbol{U}}$ is bounded from below, where $F:\boldsymbol{H}^1(\Omega)\to\mathbb{R}$ is defined in* (2.2)*, and that $f:\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\to\mathbb{R}$ is smooth. Then the convexity of*

$$\mathbb{R}^{d\times d}\ni A\mapsto f(x,v,A)\in\mathbb{R}$$

is according to [Eva98, Theorem 1, Chapter 8.2] sufficient for the weakly lower semicontinuity of F.
Besides, the coercivity of $F|_{\boldsymbol{U}}$, i.e.

$$\exists c_1,c_2>0:\quad F(\boldsymbol{u})\geq c_1\|\mathrm{D}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}-c_2\quad\forall\boldsymbol{u}\in\boldsymbol{U},$$

is sufficient for (2.3).

The assumptions stated above are the basic assumptions used throughout this work. When deriving optimality conditions we need to impose more regularity on the objective functional and the external force term, such that we can differentiate:

Additional assumptions necessary for Sections 7-9, 15-17, 19.3, 20.3 and 21.2:

(A6) External body force:
Assume $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$.

(A7) Objective functional:
Assume that $x \mapsto f(x, v, A) \in \mathbb{R}$ is in $W^{1,1}(\Omega)$ for all $(v, A) \in \mathbb{R}^d, \mathbb{R}^{d\times d}$ and the partial derivatives $\mathrm{D}_2 f(x, \cdot, A)$, $\mathrm{D}_3 f(x, v, \cdot)$ exist for all $v \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ and a.e. $x \in \Omega$. Let $p \geq 2$ for $d = 2$ and $2 \leq p \leq 2d/d-2$ for $d = 3$ and assume that there are $\hat{a} \in L^1(\Omega)$, $\hat{b}_1, \hat{b}_2 \in L^\infty(\Omega)$ such that for almost every $x \in \Omega$ it holds

$$\mathrm{D}_{(2,3)} f(x, v, A) \leq \hat{a}(x) + \hat{b}_1(x)\,|v|^{p-1} + \hat{b}_2(x)\,|A| \quad \forall v \in \mathbb{R}^d, A \in \mathbb{R}^{d\times d}. \tag{2.4}$$

Remark 2.4. *Notice, that the exponent $p \geq 2$ in Assumption **(A5)** and Assumption **(A7)** is chosen in dependency of the space dimension $d \in \{2,3\}$ such that such that $\boldsymbol{H}^1(\Omega)$ imbeds continuously into $\boldsymbol{L}^p(\Omega)$.*

Notation Here and subsequently, we will use the notation

$$\mathrm{D}_{(2,3)} f(x, v, A)$$

and

$$\mathrm{D}_i f(x, v, A) \quad i \in \{1,2,3\}$$

as the differential of

$$\mathbb{R}^d \times \mathbb{R}^{d\times d} \ni (v, A) \mapsto f(x, v, A)$$

for fixed $x \in \Omega$ and of

$$\Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \ni (x, \boldsymbol{v}, A) \mapsto f(x, \boldsymbol{v}, A)$$

with respect to the i-th variable, respectively, whereas we write

$$\mathrm{D}_u f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\boldsymbol{v} := \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,(\boldsymbol{v}, \mathrm{D}\boldsymbol{v}).$$

Remark 2.5. *We notice that*

$$L^p(\Omega)^d \ni v \mapsto \mathrm{D}_2 f(\cdot, v, A) \in L^{p/p-1}(\Omega) \quad \forall A \in L^2(\Omega)^{d\times d}$$
$$L^2(\Omega)^{d\times d} \ni A \mapsto \mathrm{D}_3 f(\cdot, v, A) \in L^2(\Omega) \quad \forall v \in L^p(\Omega)^d$$

are well-defined Nemytskii operators if and only if (2.4) *is fulfilled. In this case we also obtain that the operator*

$$L^p(\Omega)^d \times L^2(\Omega)^{d\times d} \ni (v, A) \mapsto f(\cdot, v, A) \in L^1(\Omega)$$

*is continuously Fréchet differentiable. And so, if the objective functional fulfills additionally Assumption **(A7)**, we find that*

$$F : \boldsymbol{H}^1(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x$$

is continuously Fréchet differentiable and that its directional derivative is given in the following form:

$$\mathrm{D}F(\boldsymbol{u})(\boldsymbol{v}) = \int_\Omega \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,(\boldsymbol{v}, \mathrm{D}\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

For details concerning Nemytskii operators we refer to [AZ90, Trö09, Sho97].

The design variable in the phase field model will be a function $\varphi : \Omega \to \mathbb{R}$ having $H^1(\Omega)$-regularity in order to have a derivative which is in $L^2(\Omega)$. This ensures that the Ginzburg-Landau energy, given by $\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x$, which will be part of the objective functional, is finite. We impose additionally an integral constraint, thus $\fint_\Omega \varphi\,\mathrm{d}x \leq \beta$ for some fixed constant $\beta \in (-1,1)$, where we denote by $\fint_\Omega \varphi\,\mathrm{d}x := |\Omega|^{-1}\int_\Omega \varphi\,\mathrm{d}x$ the mean value of φ in Ω. This means, that we prescribe a maximal amount of fluid that can be used in the shape and topology optimization process. This condition could without much effort be replaced by a condition of the form $\fint_\Omega \varphi\,\mathrm{d}x = \beta$, or by imposing both a maximal and a minimal value $-\beta_1 \leq \fint_\Omega \varphi\,\mathrm{d}x \leq \beta_2$ with $\beta_1, \beta_2 \in (-1,1)$.
Due to the choice of the double obstacle potential in the Ginzburg-Landau energy, we have additionally the constraint for the phase field variable to be between -1 and 1, and thus we arrive in the following admissible set for the optimal control problem in the phase field formulation:

$$\Phi_{ad} := \left\{\varphi \in H^1(\Omega) \mid \fint_\Omega \varphi\,\mathrm{d}x \leq \beta, |\varphi| \leq 1 \text{ a.e. in } \Omega\right\}$$

for some fixed constant $\beta \in (-1,1)$.

Since we will introduce a Lagrange multiplier for the integral constraint, we use the enlarged admissible set

$$\overline{\Phi}_{ad} = \left\{\varphi \in H^1(\Omega) \mid |\varphi| \leq 1 \text{ a.e. in } \Omega\right\}$$

for necessary optimality conditions of first order.

To formulate the sharp interface model, we have to introduce some notation. Therefore, let $E \subset \Omega$ be some measurable set. Then we define

$$\boldsymbol{U}^E := \{\boldsymbol{u} \in \boldsymbol{U} \mid \boldsymbol{u} = \boldsymbol{0} \text{ a.e. in } \Omega \setminus E\}$$

where this may be an empty set, for example in the case that $\mathcal{H}^{d-1}\left(\{\boldsymbol{g} \neq \boldsymbol{0}\} \cap (\Omega \setminus E)\right) > 0$.
To continue, we denote by

$$\boldsymbol{V}^E := \{\boldsymbol{v} \in \boldsymbol{V} \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. in } \Omega \setminus E\}$$

the corresponding homogeneous vector space.
If φ is an element in $L^1(\Omega)$, we see that

$$E^\varphi := \{x \in \Omega \mid \varphi(x) = 1\}$$

is a measurable set and so we can introduce the notation

$$\boldsymbol{U}^\varphi := \boldsymbol{U}^{E^\varphi} \text{ and } \boldsymbol{V}^\varphi := \boldsymbol{V}^{E^\varphi}.$$

The corresponding admissible sets of the optimization problem in the sharp interface model are then defined as follows:

$$\Phi_{ad}^0 := \left\{\varphi \in BV(\Omega, \{\pm 1\}) \mid \fint_\Omega \varphi\,\mathrm{d}x \leq \beta,\ \boldsymbol{U}^\varphi \neq \emptyset\right\}$$

and

$$\overline{\Phi}_{ad}^{0} := \{\varphi \in BV(\Omega, \{\pm 1\}) \mid \boldsymbol{U}^{\varphi} \neq \varnothing\}.$$

The additional constraint $\boldsymbol{U}^{\varphi} \neq \varnothing$ is necessary, since for a general $\varphi \in BV(\Omega, \{\pm 1\})$ this space may be empty, for instance if the prescribed boundary data and the condition $\boldsymbol{u}|_{\Omega \setminus E^{\varphi}} = \boldsymbol{0}$ are conflicting, as already mentioned above.

Now we can introduce the functionals that we will be considering in the following and whose minimization defines the topology optimization problem. In the phase field model the objective functional, depending on the phase field parameter $\varepsilon > 0$, is defined by

$$J_{\varepsilon} : L^{1}(\Omega) \times \boldsymbol{H}^{1}(\Omega) \to \overline{\mathbb{R}}$$

$$J_{\varepsilon}(\varphi, \boldsymbol{u}) := \frac{1}{2}\int_{\Omega} \alpha_{\varepsilon}(\varphi)|\boldsymbol{u}|^{2}\,\mathrm{d}x + \int_{\Omega} f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \frac{\gamma\varepsilon}{2}\int_{\Omega}|\nabla\varphi|^{2}\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_{\Omega}\psi(\varphi)\,\mathrm{d}x \tag{2.5}$$

if $\varphi \in \Phi_{ad}$ and $\boldsymbol{u} = \boldsymbol{S}_{\varepsilon}(\varphi)$ and $J_{\varepsilon}(\varphi, \boldsymbol{u}) := +\infty$ otherwise. The solution operator $\boldsymbol{S}_{\varepsilon}$ for the penalized Stokes equations is defined in Lemma 5.1.

Moreover, we will see in Section 6.2 that minimizers of the reduced functionals

$$j_{\varepsilon} : L^{1}(\Omega) \to \overline{\mathbb{R}}, \quad j_{\varepsilon}(\varphi) := J_{\varepsilon}(\varphi, \boldsymbol{S}_{\varepsilon}(\varphi))$$

converge as $\varepsilon \searrow 0$ under certain assumptions to a minimizer of

$$j_{0} : L^{1}(\Omega) \to \overline{\mathbb{R}}, \quad j_{0}(\varphi) := J_{0}(\varphi, \boldsymbol{S}_{0}(\varphi))$$

where $J_{0} : L^{1}(\Omega) \times \boldsymbol{H}^{1}(\Omega) \to \overline{\mathbb{R}}$ is defined as

$$J_{0}(\varphi, \boldsymbol{u}) := \begin{cases} \int_{\Omega} f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_{0} P_{\Omega}(E^{\varphi}) & \text{if } \varphi \in \Phi_{ad}^{0} \text{ and } \boldsymbol{u} = \boldsymbol{S}_{0}(\varphi), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

We remark that the solution operator for the Stokes equations $\boldsymbol{S}_{0}$ is defined in Lemma 6.1.

In the second part, we will consider the stationary Navier-Stokes equations instead of the Stokes equations as state constraints. Analogously, we then define

$$J_{\varepsilon}^{N} : L^{1}(\Omega) \times \boldsymbol{H}^{1}(\Omega) \to \overline{\mathbb{R}}$$

$$J_{\varepsilon}^{N}(\varphi, \boldsymbol{u}) := \frac{1}{2}\int_{\Omega} \alpha_{\varepsilon}(\varphi)|\boldsymbol{u}|^{2}\,\mathrm{d}x + \int_{\Omega} f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \frac{\gamma\varepsilon}{2}\int_{\Omega}|\nabla\varphi|^{2}\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_{\Omega}\psi(\varphi)\,\mathrm{d}x \tag{2.7}$$

if $\boldsymbol{u} \in \boldsymbol{S}_{\varepsilon}^{N}(\varphi)$ and $\varphi \in \Phi_{ad}$, and $J_{\varepsilon}^{N}(\varphi, \boldsymbol{u}) = +\infty$ otherwise, where $\boldsymbol{S}_{\varepsilon}^{N}$ is the solution operator for the penalized stationary Navier-Stokes equations defined in Lemma 12.1. We will see that those equations may not be uniquely solvable, and so the solution operator

$\boldsymbol{S}_\varepsilon^N$ is in general set-valued. Moreover, we will prove in Section 14 that minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ converge under certain assumptions to a minimizer of J_0^N as $\varepsilon \searrow 0$, where J_0^N is defined by

$$J_0^N : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$$

$$J_0^N(\varphi, \boldsymbol{u}) := \begin{cases} \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\, \mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) & \text{if } \varphi \in \Phi_{ad}^0 \text{ and } \boldsymbol{u} \in \boldsymbol{S}_0^N(\varphi), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.8}$$

Here $\boldsymbol{S}_0^N$ is the solution operator for the stationary Navier-Stokes equations defined in Lemma 13.1.

As we will derive in Sections 7.2, 8, 15.2 and 16 first order optimality conditions for minimizers of J_ε and J_0, or J_ε^N and J_0^N, respectively, by varying the domain Ω with transformations, we introduce here the admissible transformations and its corresponding velocity fields:

Definition 2.1 ($\mathcal{V}_{ad}$, $\mathcal{T}_{ad}$, $\overline{\mathcal{V}}_{ad}$, $\overline{\mathcal{T}}_{ad}$)**.** The space $\mathcal{V}_{ad}$ of admissible velocity fields is defined as the set of all $V \in C\left([-\tau,\tau]; C\left(\overline{\Omega}, \mathbb{R}^d\right)\right)$, where $\tau > 0$ is some fixed, small constant, such that it holds:

- **(V1)** **(V1a)** $V(t,\cdot) \in C^2\left(\overline{\Omega}, \mathbb{R}^d\right)$,
  (V1b) $\exists C > 0$: $\|V(\cdot,y) - V(\cdot,x)\|_{C([-\tau,\tau],\mathbb{R}^d)} \le C|x-y| \ \forall x,y \in \overline{\Omega}$,
- **(V2)** $V(t,x) \cdot \boldsymbol{n}(x) = 0 \quad$ on $\partial\Omega$,
- **(V3)** $\operatorname{div} V(t,\cdot) = 0$,
- **(V4)** $V(t,x) = \boldsymbol{0}$ for a.e. $x \in \partial\Omega$ with $\boldsymbol{g}(x) \neq \boldsymbol{0}$.

We will often use the notation $V(t) = V(t,\cdot)$.
We say that V is an element in $\overline{\mathcal{V}}_{ad}$ if $V \in C\left([-\tau,\tau]; C\left(\overline{\Omega}, \mathbb{R}^d\right)\right)$ fulfills **(V1)**, **(V2)** and **(V4)**.
Then the space $\mathcal{T}_{ad}$ of admissible transformations for the domain is defined as solutions of the ordinary differential equation

$$\partial_t T_t(x) = V(t, T_t(x)) \tag{2.9a}$$
$$T_0(x) = x \tag{2.9b}$$

for $V \in \mathcal{V}_{ad}$, which gives some $T : (-\tilde{\tau}, \tilde{\tau}) \times \overline{\Omega} \to \overline{\Omega}$, with $0 < \tilde{\tau}$ small enough.
Similarly, we say that a transformation T is in $\overline{\mathcal{T}}_{ad}$ if it solves (2.9) together with some velocity field V which is an element in $\overline{\mathcal{V}}_{ad}$.

We see then directly that $\mathcal{V}_{ad} \subseteq \overline{\mathcal{V}}_{ad}$ and $\mathcal{T}_{ad} \subseteq \overline{\mathcal{T}}_{ad}$.

Remark 2.6. *Let $V \in \overline{\mathcal{V}}_{ad}$ and $T \in \overline{\mathcal{V}}_{ad}$ be the transformation associated to V by (2.9). Then T inherits the following properties:*

- $T(\cdot,x) \in C^1\left([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d\right)$ *for all* $x \in \overline{\Omega}$,
- $\exists c > 0, \forall x,y \in \overline{\Omega},\ \|T(\cdot,x) - T(\cdot,y)\|_{C^1([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d)} \leq c\,|x-y|$,
- $\forall t \in [-\tilde{\tau},\tilde{\tau}],\ x \mapsto T_t(x) = T(t,x) : \overline{\Omega} \to \overline{\Omega}$ *is bijective,*
- $\forall x \in \overline{\Omega},\ T^{-1}(\cdot,x) \in C\left([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d\right)$,
- $\exists c > 0, \forall x,y \in \overline{\Omega},\ \left\|T^{-1}(\cdot,x) - T^{-1}(\cdot,y)\right\|_{C([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d)} \leq c\,|x-y|$.

This follows from [DZ91, Theorem 2.3, Remark 2.4].

We will discuss the admissible velocities and transformations in more detail in Section 3.3.

Before starting the introduction and discussion of the precise models, we want to give an example for an interpolation function α_ε fulfilling Assumption **(A4)**:

Example 2.1. Let $\overline{\alpha}_\varepsilon = \varepsilon^{-1/2}$ and

$$\alpha_\varepsilon(x) := \frac{1}{1+x+\overline{\alpha}_\varepsilon^{-1}} - \frac{1}{2\left(2+\overline{\alpha}_\varepsilon^{-1}\right)} - \frac{x}{2\left(2+\overline{\alpha}_\varepsilon^{-1}\right)}.$$

Then α_ε converges pointwise in $(-1,1)$ to

$$\alpha_0(x) = \frac{1}{1+x} - \frac{1}{4} - \frac{x}{4}$$

as $\varepsilon \searrow 0$ and **(A4)** is fulfilled.

We give now some examples for objective functionals, for which the assumptions above are fulfilled:

Example 2.2. Typical examples for f would be a tracking-type functional for the velocity

$$f_1(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) = \frac{1}{2}\left|\boldsymbol{u}-\boldsymbol{u}_d(x)\right|^2 + \frac{1}{2}\left|\mathrm{D}\boldsymbol{u}-\mathrm{D}\boldsymbol{u}_d(x)\right|^2$$

for some given $\boldsymbol{u}_d \in \boldsymbol{H}^1(\Omega)$, or

$$f_2(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) = \left|\operatorname{curl}\boldsymbol{u}\right|^2$$

(see for example [GMZ08]) and

$$f_3(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) = \left|\operatorname{curl}\boldsymbol{u}\right|^2 + \left|\boldsymbol{u}-\boldsymbol{u}_d(x)\right|^2.$$

To see that f_2 fulfills (2.3), we use [Tem01, Remark 1.6, Appendix I] or [BF13, Chapter IV], which implies that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq c_0(\Omega)\,\|\operatorname{curl}\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + c(\boldsymbol{g},\Omega) \quad \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Example 2.3. In the following we will often use the following functional:

$$f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) = \frac{\mu}{2} |\mathrm{D}\boldsymbol{u}|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u}$$

where $\boldsymbol{f}$ is the external force given by Assumption **(A2)** and $\mu > 0$ is the viscosity. Then minimizing $\int_\Omega \left(\frac{\mu}{2} |\nabla \boldsymbol{u}|^2 - \boldsymbol{f} \cdot \boldsymbol{u}\right) \mathrm{d}x$ can be interpreted as minimizing the total potential power, minimizing the viscous drag or minimizing the dissipated power while maximizing the flow velocities at the applied forces. In the special case $\boldsymbol{f} \equiv \boldsymbol{0}$, the problem can in a Stokes flow also be considered as minimizing the average pressure drop (compare [BP03, Appendix A]).
We will use this example in this thesis to give some explicit formulae for the derived optimality systems. As this is a typical objective functional used for shape optimization in fluids, we thus can compare our calculations to known results, such as [Sim80, SS10, MP04, PS10] et al.

Remark 2.7. *Beyond, we give some remarks on why some of the assumptions are necessary and when we can drop them:*

1. *The condition of $\mathbb{R}^d \setminus \overline{\Omega}$ being connected arises due to technical reasons, in particular when defining solenoidal extensions of the boundary data, see for instance Lemma 11.3. Anyhow, we could establish the same result for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ by using for instance a generalized version of Lemma 11.3, which can be found in [Gal11, Lemma IX.4.2]. But then, we would need additional smallness assumptions on $\int_{\Gamma_i} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}x$, if Γ_i are the connected components of $\partial\Omega$, and on the integral along Γ_i of the gradient of the fundamental solution of the Laplace's equation, to guarantee existence and uniqueness of the equations appearing in this work. Anyhow, one could transfer our results to this case. On details concerning this topic we refer the reader to [Gal11].*

2. *The condition $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}x = 0$ stated in Assumption **(A2)** is the compatibility condition, which is a direct result when stating for $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ that $\operatorname{div} \boldsymbol{u} = 0$ as well as $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$, since we get due to Gauß' theorem*

$$0 = \int_\Omega \operatorname{div} \boldsymbol{u} \, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}x.$$

3. *The radially unboundedness of the objective functional stated in (2.3) is not necessary for the well-posedness of the phase field model. But the sharp interface model won't be well-defined without this assumptions, since we cannot guarantee existence of minimizers if the objective functional does not control the velocity, compare also Remark 6.4.*
   This property will also play an essential role in the proof of Theorem 6.1, where we show convergence of the minimizers.

3 Mathematical background

3.1 Functions of bounded variation

We give here a brief introduction to the theory of functions of bounded variation. For more details and the proofs of the statements that we use here we refer the reader to the books [AFP00, EG92, Giu77, Pfe12]. We start by stating the basic definitions, which are taken from [AFP00].

Definition 3.1 (Functions of bounded variation, $BV(\Omega)$)**.** We say that $\varphi \in L^1(\Omega)$ is a function of bounded variation in Ω if the distributional derivative of φ is representable by a finite Radon measure in Ω, i.e. if

$$\int_\Omega \varphi \partial_i \phi \,\mathrm{d}x = -\int_\Omega \phi \,\mathrm{d}\mathrm{D}_i\varphi \qquad \forall \phi \in C_0^\infty(\Omega), \quad i = 1, \dots, d$$

for some $\mathbb{R}^d$-valued measure $\mathrm{D}\varphi = (\mathrm{D}_1\varphi, \dots, \mathrm{D}_d\varphi)$ in Ω.
The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.
Functions in $BV(\Omega)$ only having the values 1 or -1 a.e. in Ω will be referred to by $BV(\Omega, \{\pm 1\})$.
We denote the total variation of the measure $\mathrm{D}\varphi$ by $|\mathrm{D}\varphi|$, thus it holds

$$|\mathrm{D}\varphi|(E) := \sup\left\{ \sum_{n=0}^{\infty} |\mathrm{D}\varphi(E_n)| \mid E_n \text{ measurable and pairwise disjoint}, E = \bigcup_{n=0}^{\infty} E_n \right\}$$

for all measurable $E \subset \Omega$.

Moreover we define the norm of a function of bounded variation by

$$\|\varphi\|_{BV(\Omega)} := \|\varphi\|_{L^1(\Omega)} + |\mathrm{D}\varphi|(\Omega).$$

In the following, we remark some properties of functions of bounded variation:

Properties. 1. *The total variation of some $\varphi \in BV(\Omega)$ is given by*

$$|\mathrm{D}\varphi|(\Omega) = \sup\left\{ \int_\Omega \varphi \operatorname{div} \phi \,\mathrm{d}x \mid \phi \in C_0^\infty(\Omega)^d, \|\phi\|_\infty \le 1 \right\}$$

and moreover

$$BV(\Omega) \ni \varphi \mapsto |\mathrm{D}\varphi|(\Omega)$$

is lower semicontinuous with respect to the $L^1(\Omega)$-topology.
For the proof of this statement we refer the reader to [AFP00, Proposition 3.6].

2. *The space $BV(\Omega)$ equipped with the norm $\|\cdot\|_{BV(\Omega)}$ defines a Banach space and the imbedding*

$$BV(\Omega) \hookrightarrow L^1(\Omega)$$

is compact. This follows from [AFP00, Theorem 3.23].

3. *We say that a sequence* $(\varphi_n)_{n\in\mathbb{N}} \subseteq BV(\Omega)$ *converges weakly-$*$ in* $BV(\Omega)$ *to* $\varphi \in BV(\Omega)$, *if* $(\varphi_n)_{n\in\mathbb{N}}$ *converges to* φ *in* $L^1(\Omega)$ *and*

$$\lim_{n\to\infty} \int_\Omega \phi \,\mathrm{d}\mathrm{D}\varphi_n = \int_\Omega \phi \,\mathrm{d}\mathrm{D}\varphi \qquad \forall \phi \in C_0(\Omega).$$

With this definition, we see that $(\varphi_n)_{n\in\mathbb{N}} \subseteq BV(\Omega)$ *converges weakly-$*$ to* φ *in* $BV(\Omega)$ *if and only if* $(\varphi_n)_{n\in\mathbb{N}}$ *is bounded in* $BV(\Omega)$ *and converges to* φ *in* $L^1(\Omega)$, *see [AFP00, Proposition 3.13].*

Now we proceed with defining sets of finite perimeter, which are also called Caccioppoli sets.

Definition 3.2 (Sets of finite perimeter, Caccioppoli sets)**.** Let E be a Lebesgue-measurable set in $\mathbb{R}^d$. For any $\Omega \subseteq \mathbb{R}^d$ the perimeter of E in Ω, denoted by $P_\Omega(E)$, is the total variation of χ_E in Ω, i.e.

$$P_\Omega(E) := \sup\left\{ \int_E \operatorname{div}\phi \,\mathrm{d}x \mid \phi \in C_0^1(\Omega)^d, \ \|\phi\|_\infty \le 1 \right\}.$$

We say that E is a set of finite perimeter in Ω (or Caccioppoli set) if $P_\Omega(E) < \infty$.

We state some properties of Caccioppoli sets:

Properties. 1. *For any set* E *of finite perimeter in* Ω *the distributional derivative* $\mathrm{D}\chi_E$ *of the characteristic function* χ_E *is an* $\mathbb{R}^d$*-valued finite Radon measure in* Ω *with*

$$P_\Omega(E) = |\mathrm{D}\chi_E|(\Omega) \tag{3.1}$$

and it holds

$$\int_E \operatorname{div}\phi \,\mathrm{d}x = -\int_\Omega \nu_E \cdot \phi \,\mathrm{d}\,|\mathrm{D}\chi_E| \qquad \forall \phi \in C_0^1(\Omega)^d$$

where $\mathrm{D}\chi_E = \nu_E\,|\mathrm{D}\chi_E|$ *is the polar decomposition of* $\mathrm{D}\chi_E$ *given by the Radon-Nikodým theorem (see for instance [AFP00, Corollary 1.29]). Hence,* $\nu_E : \Omega \to \mathbb{R}$ *is a* $|\mathrm{D}\chi_E|$*-measurable function such that* $|\nu_E| = 1$ $|\mathrm{D}\chi_E|$*-a.e..*
We call ν_E *the* generalised unit normal *for the Caccioppoli set* E*.*

2. *We say that the Caccioppoli sets* $(E_n)_{n\in\mathbb{N}}$ converge in measure *to* E *in* Ω *if*

$$\lim_{n\to\infty} |\Omega \cap (E_n \Delta E)| = 0$$

where the symmetric difference for two sets A, B *is defined by*

$$A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

This corresponds to $L^1(\Omega)$*-convergence of the characteristic functions. For more details see [AFP00, Remark 3.37].*

3. *It holds:* $P_\Omega(E) = P_\Omega(F)$ *if* $|\Omega \cap (E\Delta F)| = 0$ *and* $E \mapsto P_\Omega(E)$ *is lower semicontinuous with respect to convergence in measure in* Ω*, see [AFP00, Proposition 3.38].*

4. *For Lebesgue-measurable sets $(E_n)_{n\in\mathbb{N}}$ with*

$$\sup_{n\in\mathbb{N}} P_\Omega(E_n) < \infty$$

there exists a subsequence that converges in measure in Ω, since $|\Omega| < \infty$. The proof of this statement can be found in [AFP00, Theorem 3.39].

5. *For a better understanding of the perimeter of some set E we notice that*

$$P_\Omega(E) = \mathcal{H}^{d-1}\left(\Omega \cap \partial^* E\right) \tag{3.2}$$

where $\partial^* E$ is the essential boundary of E and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. For details we refer to [AFP00, Definition 3.60]. We just note, that for an open set E with $\partial E \cap \Omega \in C^2$ it holds $\partial^* E \cap \Omega = \partial E \cap \Omega$, see [Pfe12], and thus

$$P_\Omega(E) = \mathcal{H}^{d-1}\left(\Omega \cap \partial E\right) \tag{3.3}$$

whereas for a general set E we only have

$$P_\Omega(E) \leq \mathcal{H}^{d-1}\left(\partial E \cap \Omega\right).$$

For a fuller treatment we refer the reader to [EG92, AFP00].

Further, we will at some point later on need the following trace theorem, which is taken from [EG92, Theorem 1, Section 5.3]:

Theorem 3.1. *Assume U is bounded, open and has a Lipschitz boundary. Then there exists a bounded linear mapping*

$$T : BV(U) \to L^1(\partial U)$$

such that

$$\int_U \varphi \operatorname{div} \phi \,\mathrm{d}x = -\int_U \phi \,\mathrm{d}\mathrm{D}\varphi + \int_{\partial U} (\phi \cdot \nu)\, T\varphi \,\mathrm{d}x$$

for all $\varphi \in BV(U)$ and $\phi \in C^1(\mathbb{R}^d, \mathbb{R}^d)$.
The function $T\varphi$, which is uniquely defined up to sets of measure zero, is called trace of φ on ∂U.

Another important theorem will be the following, see [AFP00, Theorem 2.39]:

Theorem 3.2 (Reshetnyak continuity)**.** *Let $\mu, (\mu_n)_{n\in\mathbb{N}}$ be $\mathbb{R}^m$-valued finite Radon-measures in Ω; if $\lim_{n\to\infty} |\mu_n|(\Omega) = |\mu|(\Omega)$ then*

$$\lim_{n\to\infty} \int_\Omega f\left(x, \frac{\mu_n}{|\mu_n|}(x)\right) \mathrm{d}|\mu_n|(x) = \int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) \mathrm{d}|\mu|(x)$$

for every continuous and bounded function $f : \Omega \times S^{m-1} \to \mathbb{R}$.

For proving convergence of minimizers of the phase field model in Section 6.2 and Section 14, we have to approximate Caccioppoli sets by smooth sets. This is done by applying the following lemma, which is taken from [Mod87, Lemma 1]:

Lemma 3.1. *Let E be a measurable subset of Ω. If E and $\Omega \setminus E$ both contain a non-empty open ball, then there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of open subset of Ω such that*

1. *$\partial E_n \cap \Omega \in C^2$ for n large enough,*
2. *$\lim_{n\to\infty} |E_n \Delta E| = 0, \qquad \lim_{n\to\infty} P_\Omega(E_n) = P_\Omega(E)$,*
3. *$|E_n| = |E|$ for n large enough.*

Moreover, we get the following convergence rate:

$$|E_n \Delta E| = \mathcal{O}\left(n^{-1}\right). \tag{3.4}$$

The convergence rate is not stated in the formulation of [Mod87, Lemma 1], but is a direct consequence of the construction and can be seen for instance in [Mod87, (12)].

We remark that we cannot expect to find a sequence of smooth sets that approximates some arbitrary Caccioppoli set E from the interior or from the exterior, as stated in [Pfe12, Section 7.1].

3.2 Crack free Caccioppoli sets

The sharp interface formulations introduced later will deal with functionals that are defined on $\boldsymbol{U}^E$ or $\boldsymbol{V}^E$ for some Caccioppoli set $E \subset \Omega$, thus functions in $\boldsymbol{H}^1(\Omega)$, which are zero almost everywhere in $\Omega \setminus E$. This set $\Omega \setminus E$ will correspond to the part of Ω, which models the non-presence of fluid.

One question that arises during these considerations is, if the set $\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v}|_{\Omega\setminus E} = \boldsymbol{0}\right\}$ can be identified with $\boldsymbol{H}_0^1(\mathrm{int}(E))$. This can be answered positive in some sense, as the following discussion shows.

Definition 3.3 (Crack free sets due to [DZ07])**.** A subset of E of $\mathbb{R}^d$ such that $\partial E \neq \emptyset$ is called crack free if $\mathrm{int}\, E = \mathrm{int}\, \overline{E}$.

Remark 3.1. *[Remarks about crack free sets]*

- *The following equivalence relations hold due to [DZ07]:*

$$\mathrm{int}\, E = \mathrm{int}\, \overline{E} \iff \partial \overline{E} = \partial E \iff \overline{\mathbb{R}^d \setminus \overline{E}} = \overline{\mathbb{R}^d \setminus E} \iff b_E = b_{\overline{E}}$$

  where $b_E(x) := d_E(x) - d_{\mathbb{R}^n \setminus E}(x)$ is the oriented distance function. Here $d_A(x) = \inf\{|y-x| \mid y \in A\}$ denotes the ordinary distance function to some subset $A \neq \emptyset$ of $\mathbb{R}^d$.

- *Due to [DZ07, Theorem 6.3] it holds for crack free sets $E \subset \Omega$, that*

$$\left\{\psi \in H_0^1(\Omega) \mid (1-\chi_E)\,\psi = 0 \text{ a.e.}\right\} = H_0^1\left(\mathrm{int}\, \overline{E}\right).$$

  This implies for crack free sets $E \subset \Omega$:

$$\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. in } \Omega \setminus E\right\} = \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid (1-\chi_E)\,\boldsymbol{v} = \boldsymbol{0} \text{ a.e.}\right\} = \boldsymbol{H}_0^1\left(\mathrm{int}\, \overline{E}\right).$$

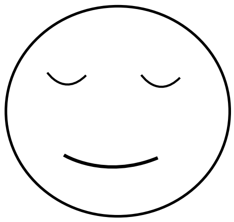


Figure 1: A set with cracks

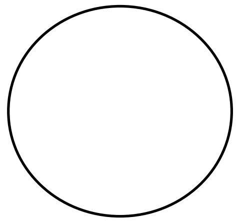


Figure 2: A crack free set

Here we are considering Caccioppoli sets, and so we state:

Lemma 3.2. *For every set $E \subset \Omega$ of bounded variation there exists a crack free representative E_c (i.e. $\operatorname{int} E_c = \operatorname{int} \overline{E}_c$) such that $|E \Delta E_c| = 0$ and thus $P_\Omega(E) = P_\Omega(E_c)$.*

Proof. We define

$$E_c = \operatorname{int}_* E = \left\{ x \in \mathbb{R}^d \mid \lim_{r \searrow 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1 \right\}$$

as the measure theoretic interior of E, compare [DZ11, Pfe12]. Then we have by [DZ11, Theorem 3.3, Chapter 5] that $|E \Delta E_c| = 0$ and hence $P_\Omega(E_c) = P_\Omega(E)$, see Section 3.1. Besides, we otain by using the calculation rules of [Pfe12, DZ11] that $\operatorname{int}\left(\overline{\operatorname{int}_* E}\right) \subset \operatorname{int}(\operatorname{int}_* E) \subset \operatorname{int}\left(\overline{\operatorname{int}_* E}\right)$ and hence $\operatorname{int}_* E$ is a crack free representative of E. □

Remark 3.2. *Lemma 3.2 together with the remarks stated before imply for our case that every Caccioppoli set E has a crack free representative E_c, for which it holds in particular*

$$\left\{ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{0} \ \text{a.e. in } \Omega \setminus E_c \right\} = \boldsymbol{H}_0^1(\operatorname{int} E_c) = \boldsymbol{H}_0^1\left(\operatorname{int} \overline{E_c}\right).$$

Example 3.1. One example, which is taken from [DZ01, Section 2.3, Chapter 3], for a cracked set is shown in Figure 1. Figure 2 shows the crack free set, which would be the corresponding representative of Figure 1 constructed by Lemma 3.2.

Example 3.2. Assume we have a Caccioppoli set E that looks like the interior of the rectangle indicated in Figure 3 and Ω is some domain that contains the whole rectangle. Then any function $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v} = \boldsymbol{0}$ in $\Omega \setminus E$ would be zero on the outer boundary of E. But even though the line inside E is not contained in E, we don't know if $\boldsymbol{v} = \boldsymbol{0}$ there, since this is a set of measure zero.
The same holds true for the situation illustrated in Figure 4. We notice there, that cracks can even change the number of connected components of a set, thus a priori the terminology "connected components" of some Caccioppoli set E is not well-defined.

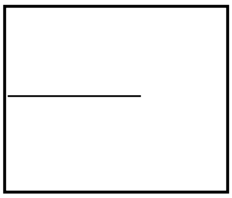


Figure 3: A set with cracks

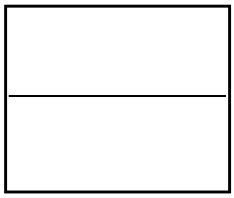


Figure 4: Cracks can alter the connectedness of a set

3.3 Introduction to shape calculus

The idea of shape calculus is to consider functionals J that depend on some bounded set $E \subset \mathbb{R}^d$ and map into $\mathbb{R}$. Often, and in particular in our situation, one wants to minimize the functional $J(E)$. One idea to obtain some first order and second order optimality conditions is to vary a set E_0 along a transformation T, for example by some $T \in \mathcal{T}_{ad}$, and obtain deformed sets $E_t := T_t(E_0)$. Then one sees that for a local optimal shape E_0 it holds $J(E_0) \leq J(E_t)$ for all t small enough, if T is chosen suitably. This leads to first order necessary optimality conditions, since we see

$$\partial_t|_{t=0} J(E_t) = 0.$$

This motivates the formal introduction of the so called *shape derivative*

$$\mathrm{D}J(E_0)\,[V] := \partial_t|_{t=0} J(E_t) = \lim_{t \to 0} \frac{J(T_t(E_0)) - J(E_0)}{t}$$

if V is the velocity field associated to the transformation T, see Definition 2.1.
This method of calculating the shape derivative is often referred to as *velocity (speed) method*. Another typical approach is the *method of perturbation of the identity operator*, where the transformation is simply defined by $\tilde{T}_t(x) := x + tV(x)$ and $V \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is chosen appropriately. It can be shown that for first order calculus both methods are equivalent if $V \mapsto \mathrm{D}J(E_0)\,[V]$ inherits certain regularity properties, see [DZ91, DZ01, SZ92]. In this case, one can show that $\mathrm{D}J(E_0)\,[V] = \mathrm{D}J(E_0)\,[V(0)]$, whereas otherwise the shape derivative may depend on the acceleration $\partial_t|_{t=0} V(t)$. Besides, the results based on perturbations of the identity can directly be recovered by applying the velocity method to appropriate vector fields. For details concerning this topic we refer for instance to [DZ01, Chapter 7]. Thus we only concentrate in the following on the velocity method.
Moreover, it follows from (2.9) that the shape derivative only depends on V, and this justifies the notation $\mathrm{D}J(E_0)\,[V]$.

For more details and a good introduction to the theory of shape calculus we refer the reader to [DZ01, SZ92].

Definition 3.4 (Shape derivative, material derivative)**.** Let U be a bounded open set and choose some admissible transformation $T \in \overline{\mathcal{T}}_{ad}$. Assume $u_t = u(T_t(U)) \in L^2(T_t(U))$ is a function depending on the set $T_t(U)$, for example as a solution to a partial differential equation solved in $T_t(U)$. If the limit $\partial_t|_{t=0}\,(u_t \circ T_t)$ exists in some space, for instance in $L^2(U)$, then we call the limit function the *material derivative* and denote it by

$$\dot{u}\,[V] := \partial_t|_{t=0}\,(u_t \circ T_t)$$

if V is the velocity associated to T.
If $u \in H^1(U)$, then we can define the *shape derivative* by

$$u'\,[V] := \dot{u}\,[V] - \nabla u \cdot V(0). \tag{3.5}$$

Now we give some general calculation rules:

Lemma 3.3. *Let $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$. Then it holds*

$$\partial_t|_{t=0} \det \mathrm{D}T_t = \operatorname{div} V(0), \tag{3.6}$$

$$\partial_t \det \mathrm{D}T_t = (\operatorname{div} V\,(t,\cdot)) \circ T_t \cdot \det \mathrm{D}T_t(t,\cdot) \quad \forall t \in (-\tilde{\tau}, \tilde{\tau}) \tag{3.7}$$

and

$$\partial_t|_{t=0}\,(\mathrm{D}T_t)^{-1} = -\mathrm{D}V(0). \tag{3.8}$$

Proof. Formula (3.6) can be calculated as in [SS10, Lemma 3.1]. For (3.8) we use [SS10, Lemma 3.10] and get

$$\partial_t|_{t=0}\,(\mathrm{D}T_t)^{-1} = -\mathrm{D}T_0^{-1}\,(\partial_t|_{t=0}\mathrm{D}T_t)\,\mathrm{D}T_0^{-1} = -\mathrm{D}\,(\partial_t|_{t=0}T_t) = -\mathrm{D}V(0).$$

Finally, we obtain (3.7) by direct calculations, using in particular (2.9a) and the chain rule. □

Remark 3.3. *We remark that for $V \in \mathcal{V}_{ad}$ it holds per definition* $\operatorname{div} V(t,\cdot) = 0$ *for all t and, due to* (3.7)*, we thus obtain*

$$\partial_t \det \mathrm{D}T_t = 0 \quad \forall t \in (-\tilde{\tau}, \tilde{\tau})\,.$$

In particular, we obtain for any $\varphi \in L^1(\Omega)$ and t small enough

$$\partial_t \int_\Omega \varphi \circ T_t^{-1}\,\mathrm{d}x = \partial_t \int_\Omega \varphi \det \mathrm{D}T_t\,\mathrm{d}x = 0$$

and so a transformation by T_t will not change the volume of φ. This property will be important later on.

One important formula in the following will be the shape derivative of the perimeter functional:

Lemma 3.4. *Let $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$ be an admissible transformation and $E \subset \Omega$ be a Caccioppoli set. Then:*

$$\mathrm{D}P_\Omega(E)\,[V] := \partial_t|_{t=0}P_\Omega\,(T_t(E)) = \int_\Omega (\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu)\,\mathrm{d}\,|\mathrm{D}\chi_E| \tag{3.9}$$

where ν is the generalised unit normal on E.
Moreover, if E is an open set such that $\partial E \cap \Omega \in C^2$, this can be further rewritten as

$$\mathrm{D}P_\Omega(E)\,[V] = \int_{\partial E \cap \Omega} \kappa V(0) \cdot \nu\,\mathrm{d}x \tag{3.10}$$

where $\kappa = \operatorname{div}_{\partial E \cap \Omega}\nu$ denotes the mean curvature of $\partial E \cap \Omega$.

Proof. First we note, that [Giu77, 10.2] implies

$$\mathrm{D}P_\Omega(E)\,[V] = \partial_t|_{t=0} \int_\Omega \mathrm{d}\,|\mathrm{D}\chi_{E_t}| = \int_{\Omega\cap\partial^* E} \nu \cdot \dot{H}_0 \nu \,\mathrm{d}x$$

with

$$\dot{H}_0 := \partial_t|_{t=0} H(t), \quad H(t) := |\det \mathrm{D}T_t|\,(\mathrm{D}T_t)^{-1}$$

if $E_t = T_t(E)$. We start reformulating this expression by using (3.2) and (3.1):

$$\int_{\Omega\cap\partial^* E} \nu \cdot \dot{H}_0 \nu \,\mathrm{d}x = \int_\Omega \nu \cdot \dot{H}_0 \nu \,\mathrm{d}\,|\mathrm{D}\chi_E|\,.$$

Using the calculation rules of Lemma 3.3 we see

$$\partial_t|_{t=0}\,|\det \mathrm{D}T_t|\,(\mathrm{D}T_t)^{-1} = \operatorname{div} V(0) - \mathrm{D}V(0)$$

and so inserted in the formula above we have

$$\mathrm{D}P_\Omega(E)\,[V] = \int_\Omega \left(\operatorname{div} V(0)\, \underbrace{\nu\cdot\nu}_{=1} - \nu \cdot \nabla V(0)\nu \right) \mathrm{d}\,|\mathrm{D}\chi_E|$$

which proves (3.9).

Now assume that E is an open set with $\partial E \cap \Omega \in C^2$. Then we get with (3.3)

$$\int_\Omega (\operatorname{div} V(0) - \nu\cdot\nabla V(0)\nu)\,\mathrm{d}\,|\mathrm{D}\chi_E| = \int_{\partial E\cap\Omega} (\operatorname{div} V(0) - \nu\cdot\nabla V(0)\nu)\,\mathrm{d}x$$

and using the definition of surface divergence and the tangential Stokes formula (see for instance [DZ01, Chapter 8, Section 5.5]) this can be rearranged to

$$\int_{\partial E\cap\Omega} (\operatorname{div} V(0) - \nu\cdot\nabla V(0)\nu)\,\mathrm{d}x = \int_{\partial E\cap\Omega} \operatorname{div}_{\partial E\cap\Omega} V(0)\,\mathrm{d}x = \int_{\partial E\cap\Omega} \kappa V(0)\cdot\nu\,\mathrm{d}x$$

which proves (3.10).

□

We notice some properties of the admissible transformations $T \in \overline{\mathcal{T}}_{ad}$:

Lemma 3.5. *Let $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$. Then for small t it holds*

- $T_t(\Omega) = \Omega$,
- $x \in \partial\Omega \implies T_t(x) \in \partial\Omega$,
- $T_t(x) = x$ *if* $\boldsymbol{g}(x) \neq \mathbf{0}$.

If moreover $V \in \mathcal{V}_{ad}$ and thus $T \in \mathcal{T}_{ad}$ we get

$$\partial_t \int_\Omega \varphi \circ T_t^{-1}\,\mathrm{d}x = 0 \text{ for all } \varphi \in L^1(\Omega). \tag{3.11}$$

Proof. The fact, that T maps boundary points to boundary points follows from [DZ01, Remark 5.1, Remark 5.2], and this implies then the first statement, see also Remark 2.6. Moreover, for a.e. $x \in \partial\Omega$ with $\boldsymbol{g}(x) \neq \mathbf{0}$ we have per definition of $\overline{\mathcal{V}}_{ad}$ already $V(t,x) = 0$ and so $T_t(x) = x$.
Finally we see from Remark 3.3 that (3.11) holds for all $\varphi \in L^1(\Omega)$ if $V \in \mathcal{V}_{ad}$.
For details we refer to [DZ01, Chapter 7] and [SZ92, Chapter 2]. $\square$

Remark 3.4. *Using Lemma 3.5, we obtain for $T \in \mathcal{T}_{ad}$ directly that $\varphi \circ T_t^{-1} \in \Phi_{ad}$ and $\varphi \circ T_t^{-1} \in \Phi_{ad}^0$ if $\varphi \in \Phi_{ad}$ or $\varphi \in \Phi_{ad}^0$, respectively. Similarly, we find for $T \in \overline{\mathcal{T}}_{ad}$ that $\varphi \circ T_t^{-1} \in \overline{\Phi}_{ad}$ and $\varphi \circ T_t^{-1} \in \overline{\Phi}_{ad}^0$ if $\varphi \in \overline{\Phi}_{ad}$ or $\varphi \in \overline{\Phi}_{ad}^0$, respectively.*

In the following, our shape functional will also depend on functions in $\boldsymbol{H}^1(\Omega)$. Thus we will quite often use the following special transformation, which is also used in [BHW06]:

Lemma 3.6. *Assume $T \in \overline{\mathcal{T}}_{ad}$.*
Let $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ and define

$$\boldsymbol{v}_t := \left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{v} \circ T_t^{-1}.$$

Then it holds $\operatorname{div} \boldsymbol{v}_t = 0$ and

$$\dot{\boldsymbol{v}}[V] := \partial_t|_{t=0}\,(\boldsymbol{v}_t \circ T_t)$$

is well defined in $\boldsymbol{H}^1(\Omega)$.
Moreover, we have for any Caccioppoli set $E \subset \Omega$ with $E_t := T_t(E)$ the following inclusions:

$$\boldsymbol{v} \in \boldsymbol{V}^E \implies \boldsymbol{v}_t \in \boldsymbol{V}^{E_t},$$

$$\boldsymbol{v} \in \boldsymbol{U}^E \implies \boldsymbol{v}_t \in \boldsymbol{U}^{E_t}$$

and

$$\boldsymbol{v} \in \boldsymbol{U} \implies \boldsymbol{v}_t \in \boldsymbol{U}.$$

Proof. Let $\zeta \in C_0^\infty(\Omega)$ be arbitrary. Then we can calculate, using change of variables and the divergence theorem

$$\begin{aligned}
&\int_\Omega \operatorname{div}\left(\left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{v} \circ T_t^{-1}\right)\zeta\,\mathrm{d}y = \int_{T_t(\Omega)} \operatorname{div}\left(\left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{v} \circ T_t^{-1}\right)\zeta\,\mathrm{d}y = \\
&= -\int_{T_t(\Omega)} \left(\left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{v} \circ T_t^{-1}\right)\cdot \nabla\zeta\,\mathrm{d}y = \\
&= -\int_\Omega \left(\det \mathrm{D}T_t^{-1}\right)(x)\,(\mathrm{D}T_t)(x)\,\boldsymbol{v}(x)\cdot(\nabla\zeta)(T_t(x))\det \mathrm{D}T_t(x)\,\mathrm{d}x = \\
&= -\int_\Omega \boldsymbol{v}(x)\cdot\nabla(\zeta\circ T_t)(x)\,\mathrm{d}x = \int_{\Omega = T_t^{-1}(\Omega)} (\operatorname{div}\boldsymbol{v})(x)\,\zeta(T_t(x))\,\mathrm{d}x = \\
&= \int_\Omega \left(\det \mathrm{D}T_t^{-1}\right)(\operatorname{div}\boldsymbol{v})\left(T_t^{-1}(y)\right)\zeta(y)\,\mathrm{d}y.
\end{aligned}$$

Since this identity holds for any $\zeta \in C_0^\infty(\Omega)$, it follows from the fundamental lemma of calculus of variations that

$$\operatorname{div} \boldsymbol{v}_t = \left(\det \mathrm{D}T_t^{-1}\right)(\operatorname{div}\boldsymbol{v}) \circ T_t^{-1}.$$

Due to $T_t(x) \in \Omega$ for every $x \in \Omega$ we deduce

$$\operatorname{div} \boldsymbol{v}_t\,(x) = \left(\det \mathrm{D}T_t^{-1}(x)\right) \underbrace{(\operatorname{div} \boldsymbol{v})\left(T_t^{-1}\,(x)\right)}_{=0} = 0$$

which holds for almost every $x \in \Omega$. Besides, T_t maps boundary points of Ω to boundary points of Ω, which implies directly $\boldsymbol{v}_t|_{\partial\Omega} = \mathbf{0}$ if $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$. On the other hand, using the special form of $T \in \overline{\mathcal{T}}_{ad}$, in particular property **(V4)**, we also get $\boldsymbol{v}_t|_{\partial\Omega} = \boldsymbol{g}$ if $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}$.

Now assume that $\boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$ and let $E_t := T_t(E)$. We get for every $x \in \Omega \setminus E_t = \Omega_t \setminus E_t = T_t\,(\Omega \setminus E)$ the following identity

$$\boldsymbol{v}_t\,(x) = \left(\det \mathrm{D}T_t^{-1}\,(x)\right)(\mathrm{D}T_t\,(x))\,\underbrace{\boldsymbol{v}\,\underbrace{\left(T_t^{-1}\,(x)\right)}_{\in\Omega\setminus E}}_{=\mathbf{0}} = \mathbf{0}.$$

This guarantees $\boldsymbol{v}_t \in \boldsymbol{V}^{E_t}$ if $\boldsymbol{v} \in \boldsymbol{V}^E$ and $\boldsymbol{v}_t \in \boldsymbol{U}^{E_t}$ if $\boldsymbol{v} \in \boldsymbol{U}^E$. Next we calculate

$$\boldsymbol{v}_t\,(T_t\,(x)) - \boldsymbol{v}\,(x) = \left(\det \mathrm{D}T_t^{-1}\right)\mathrm{D}T_t\boldsymbol{v} - \boldsymbol{v} = \left(\det \mathrm{D}T_t^{-1}\right)\mathrm{D}T_t\boldsymbol{v} - \left[\left(\det \mathrm{D}T_t^{-1}\right)\mathrm{D}T_t\right]|_{t=0}\boldsymbol{v}$$

and so the limit $\partial_t|_{t=0}\,(\boldsymbol{v}_t \circ T_t)$ is defined in $\boldsymbol{H}^1(\Omega)$ which implies the statement. □

Moreover direct calculations give:

Lemma 3.7. *Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\operatorname{div} \boldsymbol{u} = 0$. Then*

$$\operatorname{div}\,(\operatorname{div} V(0)\boldsymbol{u} + \mathrm{D}\boldsymbol{u}V(0) - \mathrm{D}V(0)\boldsymbol{u}) = 0 \qquad \forall V \in \overline{\mathcal{V}}_{ad},$$

where this identitiy has to be understood in the distributional sense.

Proof. We introduce the notation $V(0) = (V_i)_{i=1}^d$ and $\boldsymbol{u} = (u_i)_{i=1}^d$. Then we see:

$$\begin{aligned}
&\operatorname{div}\,(\operatorname{div} V\,(0)\,\boldsymbol{u} - \mathrm{D}V\,(0)\,\boldsymbol{u}) = \sum_{i,j,k=1}^d \partial_i\,(\partial_k V_k u_i - \partial_j V_i u_j) = \\
&= \sum_{i,j,k=1}^d (\partial_i\partial_k V_k)\,u_i - (\partial_j V_i)\,(\partial_i u_j) - (\partial_i\partial_j V_i)\,u_j = -\sum_{i,j=1}^d (\partial_j V_i)\,(\partial_i u_j)
\end{aligned} \tag{3.12}$$

where we used in particular $\operatorname{div} \boldsymbol{u} = 0$. Besides, we obtain for some arbitrary test function $\zeta \in C_0^\infty(\Omega)$ that

$$\begin{aligned}
\int_\Omega \mathrm{D}\boldsymbol{u}V(0)\cdot\nabla\zeta\,\mathrm{d}x &= \sum_{i,j=1}^d \int_\Omega \partial_i u_j V_i \partial_j \zeta\,\mathrm{d}x = -\sum_{i,j=1}^d \int_\Omega u_j\partial_i V_i\partial_j\zeta + u_j V_i \partial_i\partial_j\zeta\,\mathrm{d}x = \\
&= \sum_{i,j=1}^d \int_\Omega u_j\partial_j\partial_i V_i\zeta + u_j\partial_j V_i\partial_i\zeta\,\mathrm{d}x = \\
&= \sum_{i,j=1}^d \int_\Omega u_j\partial_j\partial_i V_i\zeta - \partial_i u_j\partial_j V_i\zeta - u_j\partial_i\partial_j V_i\zeta\,\mathrm{d}x = \\
&= \sum_{i,j=1}^d \int_\Omega -\partial_i u_j\partial_j V_i\zeta\,\mathrm{d}x.
\end{aligned}$$

As this identity holds for all $\zeta \in C_0^\infty(\Omega)$ we find that

$$\operatorname{div}\,(\mathrm{D}\boldsymbol{u}V(0)) = \sum_{i,j=1}^{d} \partial_i u_j \partial_j V_i$$

where this identity is to be understood in the distributional sense. Combining this result with (3.12) yields then the statement. □

3.4 Introduction to Γ-convergence

A very useful tool when considering minimization problems of a sequence of functionals is the notion of Γ-convergence. This will be the framework that we will use to connect the phase field model with the sharp interface model in certain situations later on.

The following definitions and statements are taken from [DM93].

Definition 3.5. Assume that X is a topological space which satisfies the first axiom of countability or X is a Banach space whose dual X' is separable, endowed with its weak topology. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of functions from X to $\overline{\mathbb{R}}$. Then we say that $(F_n)_{n\in\mathbb{N}}$ Γ-converges to $F : X \to \overline{\mathbb{R}}$ if the following two conditions are satisfied:

$\liminf$**-condition** :

For every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ converging to x in X it is

$$F(x) \le \liminf_{n\to\infty} F_n(x_n); \tag{3.13}$$

$\limsup$**-condition** :

For every $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x in X such that

$$F(x) \ge \limsup_{n\to\infty} F_n(x_n). \tag{3.14}$$

We then call F the Γ-limit of $(F_n)_{n\in\mathbb{N}}$.

Remark 3.5. *Notice, that an infinite dimensional Banach space endowed with its weak topology does not satisfy the first axiom of countability. But by [DM93, Proposition 8.10] the characterisation of the* Γ*-limit by* (3.13)-(3.14) *still holds true if the dual is separable.*

We notice:

Remark 3.6.
1. *Because of* (3.13) *condition* (3.14) *is equivalent to stating the existence of some* $(x_n)_{n\in\mathbb{N}}$ *converging to* $x \in X$ *such that*

$$\lim_{n\to\infty} F_n(x_n) = F(x).$$

2. *In the following we will call the sequence* $(x_n)_{n\in\mathbb{N}}$ *of condition* (3.14) *sometimes* "recovery sequence"*, and refer to* (3.13) *by* "lower semicontinuity"*.*

3. *In some sense,* (3.13) *states that* F *is a lower bound on* $(F_n)_{n\in\mathbb{N}}$*, whereas* (3.14) *ensures that this bound is optimal.*

We state some properties that will be important in the following and can be found in [DM93]:

Properties. *1. If $(F_n)_{n\in\mathbb{N}}$ Γ-converges to F and G is a continuous function on X, then $(F_n + G)_{n\in\mathbb{N}}$ Γ-converges to $(F + G)$.*

2. Every Γ-limit is lower semicontinuous.

The most important point, why Γ-convergence is a suitable framework for minimizing functionals, is given by the following theorem:

Theorem 3.3. *Let X be a topological space which satisfies the first axiom of countability or a Banach space whose dual X' is separable. Assume $(F_n)_{n\in\mathbb{N}}$ Γ-converges to F in X. Moreover, let $(x_n)_{n\in\mathbb{N}}$ be minimizers of $(F_n)_{n\in\mathbb{N}}$. Then:*

- *every limit point of $(x_n)_{n\in\mathbb{N}}$ is a minimizer of $(F_n)_{n\in\mathbb{N}}$;*
- *if $(x_n)_{n\in\mathbb{N}}$ converges to x, then x is a minimizer of F and $\lim_{n\to\infty} F_n(x_n) = F(x)$.*

Proof. See proof of [DM93, Corollary 7.20]. □

Part I
Stokes flow

In this first part, we consider the shape and topology optimization problem of minimizing a general objective functional depending on the velocity of some fluid, which is described by the Stokes equations, by varying the region which is filled with fluid. For the sharp interface model, we will use a perimeter penalization to overcome the general ill-posedness of the problem. Moreover, we introduce a phase field approach describing the optimization problem. In this diffuse interface setting we use the Ginzburg-Landau energy as penalization, since this is a diffuse interface approximation of the perimeter functional, see [Mod87] and discussion in Section 6. Simultaneously, we relax the non-permeability condition of the non-fluid region.

We start by reviewing some of the standard facts from fluid mechanics. We then introduce the phase field model in Section 5 and establish some results on this problem, such as well-posedness of the state equations, which are given by a penalized Stokes system, and the existence of minimizers for the overall optimization problem.
After that we propose in Section 6.1 the sharp interface model describing the shape and topology optimization problem and show unique solvability of the state equations. We find that the sequence of minimizers of the phase field problem has a subsequence that converges as the thickness of the interface tends to zero. If this convergence fulfills a certain rate with respect to the phase field parameter or if the objective functional is the total potential power, we find that the limit element is a minimizer of the sharp interface model.
In Sections 7 and 8 we develop optimality conditions in both the diffuse and the sharp interface setting, and prove then in Section 9 that certain optimality conditions of the phase field model are an approximation of an optimality system derived for the sharp interface model.

4 Important facts related to fluid mechanics

We start by recalling some basic facts from fluid mechanics that will be used frequently in the following.

The subsequent technical lemma will be needed quite often during our considerations:

Lemma 4.1. *Let U be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $\boldsymbol{v}_* \in \boldsymbol{H}^{\frac{1}{2}}(\partial U)$ satisfy*

$$\int_{\partial U} \boldsymbol{v}_* \cdot \boldsymbol{n} \,\mathrm{d}x = 0$$

where $\boldsymbol{n}$ denotes the outer unit normal on U. Then there exists a vector field $\boldsymbol{V} \in \boldsymbol{H}^1(U)$ such that

$$\operatorname{div} \boldsymbol{V} = 0, \qquad \boldsymbol{V}|_{\partial U} = \boldsymbol{v}_*, \qquad \|\boldsymbol{V}\|_{\boldsymbol{H}^1(U)} \le c(U) \|\boldsymbol{v}_*\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial U)}.$$

Proof. This follows from combining [Soh01, Lemma II.2.1.1] and the results on the extension operator of [Soh01, Section II.1.2] and can also be found for instance in [Gal11, Exercise III.3.5]. □

Besides, we will often use the following result, which is taken from [Soh01, Lemma II.2.1.1]:

Lemma 4.2. *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then we have: For each $g \in L^2(U)$ with $\int_U g \,\mathrm{d}x = 0$ there exists at least one $\boldsymbol{v} \in \boldsymbol{H}_0^1(U)$ satisfying*

$$\operatorname{div} \boldsymbol{v} = g, \quad \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(U)} \leq C(U) \|g\|_{L^2(U)}.$$

One important inequality in the following will be:

Lemma 4.3. *Assume $U \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Then it holds*

$$\left\| p - \fint_U p \,\mathrm{d}x \right\|_{L^2(U)} \leq c(U) \|\nabla p\|_{\boldsymbol{H}^{-1}(U)} \quad \forall p \in L^2(U). \tag{4.1}$$

Proof. Follows by applying [Soh01, Lemma II.1.5.4] to $q := p - \fint_U p \,\mathrm{d}x$. □

Considering the pressure in fluid mechanics we will use in general the following result:

Lemma 4.4. *Assume $U \subset \mathbb{R}^d$ is any bounded open set and let $\boldsymbol{F} \in \boldsymbol{H}^{-1}(U)$ such that*

$$\langle \boldsymbol{F}, \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1}(U)} = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(U), \operatorname{div} \boldsymbol{v} = 0.$$

Then there exists some $p \in L^2_{loc}(U)$ such that

$$\boldsymbol{F} = \nabla p.$$

This function p is defined uniquely up to constants, which can be chosen in every connected component of U.
Moreover, if U is a Lipschitz domain, we even get $p \in L^2(U)$ and

$$\left\| p - \fint_U p \,\mathrm{d}x \right\|_{L^2(U)} \leq c(U) \|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(U)}. \tag{4.2}$$

Proof. The existence of p in $L^2(U)$ and $L^2_{loc}(U)$, respectively, is given by [Tem77, Proposition 1.1, Proposition 1.2]. For a Lipschitz domain U, (4.1) gives additionally estimate (4.2).

□

5 Phase field model

In this section, we formulate the phase field model for a general shape and topology optimization problem in a Stokes flow. Thus we introduce the problem formulation and then discuss its well-posedness by showing unique solvability of the state equations as well as the existence of minimizers for the overall optimization problem.

5.1 Problem formulation

The overall problem is given by the following optimal control problem:

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u})} J_\varepsilon(\varphi,\boldsymbol{u}) := \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)\,|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \\ + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x\end{aligned} \tag{5.1}$$

with

$$(\varphi,\boldsymbol{u}) \in \Phi_{ad}\times\boldsymbol{U}$$

s.t.

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{V}. \tag{5.2}$$

Thus in this approach $\varphi \in \Phi_{ad} = \left\{\varphi \in H^1(\Omega) \mid ⨍_\Omega \varphi\,\mathrm{d}x \le \beta, |\varphi| \le 1 \text{ a.e. in } \Omega\right\}$ will be the design variable, where $\{x\in\Omega \mid \varphi(x) = 1\}$ models the fluid region, and $\{x\in\Omega \mid \varphi(x) = -1\}$ the non-presence of fluid. Since we are considering a phase field model, we have in particular an interface between fluid region and the region without fluid which thickness is proportional to a small parameter $\varepsilon > 0$. As ε tends to zero this interface will vanish and we arrive in a sharp interface setting.

As we will see later, we use a perimeter penalization in the sharp interface model to overcome the general ill-posedness of the problem, see Section 6. This is a common approach in shape and topology optimization, see [Pet99, AB93, BC03, DZ01]. It is a well-known result, that the perimeter can be approximated in the diffuse interface setting by a multiple of the Ginzburg-Landau energy, which is given by

$$E_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$$

$$E_\varepsilon(\varphi) := \begin{cases} \frac{\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x, & \text{if } \varphi\in H^1(\Omega),\\ +\infty, & \text{otherwise}\end{cases}$$

for some potential ψ. To be precise, $(E_\varepsilon)_{\varepsilon>0}$ Γ-converges in $L^1(\Omega)$ to the perimeter, multiplied by some constant depending on the potential, as $\varepsilon \searrow 0$. We will discuss this in more detail in Section 6, compare also [Mod87, MM77]. Here we choose as potential the so-called obstacle potential, see Assumption **(A3)**, which ensures in particular that $|\varphi| \le 1$ almost everywhere in Ω. This leads to a so-called "sharp diffuse interface", in contrast to the "diffuse diffuse interface", where the phase field variable φ could take values outside of the interval $[-1,1]$, too. For details concerning different choices of potentials we refer to [BE91]. We remark that a smooth double-well potential would be possible, too, in this setting, if the proofs are adapted.

Thus we add the Ginzburg-Landau energy to the objective functional in (5.1) to end up in a perimeter penalized sharp interface model and ensure hereby the existence of minimizers for the phase field model. The constant $\gamma > 0$ is an arbitrary parameter that can be considered as a weighting parameter of the Ginzburg-Landau energy, and thus of the perimeter penalization in the sharp interface model.
The second term in (5.1) is the general objective functional that we want to minimize and is assumed to fulfill Assumption **(A5)**.

The fluid is described by a weak formulation of a penalized Stokes system (5.2), similar to [BP03]. Here $\mu > 0$ denotes the viscosity of some fluid, whose velocity is denoted by $\boldsymbol{u}$, and $\boldsymbol{f}$ is an external force acting in Ω. We remark in particular that in the fluid part, thus inside $\{\varphi = 1\}$, α_ε will be zero, and thus (5.2) reduces to the weak formulation of the classical Stokes equations. In the pure non-fluid region, thus inside $\{\varphi = -1\}$, the state system (5.2) can be considered as a Darcy flow through porous medium with permeability $\alpha_\varepsilon(-1)^{-1}$, see [BP03, SP80]. Since $\alpha_\varepsilon(-1)^{-1}$ tends due to Assumption **(A4)** to zero as $\varepsilon \searrow 0$, we will obtain in the limit $\varepsilon \searrow 0$ a model where the region outside the fluid is impermeable.
Besides, we want the velocity of the fluid in the sharp interface setting to be defined on the whole of Ω and define its value to be zero outside the fluid region. From this viewpoint, the first term $\frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x$ appearing in the objective functional J_ε can be considered as a penalization term for the permeability through the non-fluid region and will vanish as $\varepsilon \searrow 0$. Hereby, this term ensures that in the limit the velocity is zero if no fluid is present and that the medium outside the fluid is impermeable.

This means, we approximate the topology optimization problem simultaneously in two ways: First of all, we use a phase field approximation and arrive in a diffuse interface problem. At the same time, we relax the condition of impermeability of the non-fluid domain. But since we penalize the velocity being non-zero outside the fluid domain, we arrive in the limit $\varepsilon \searrow 0$ not only in a sharp interface problem, but also in impermeable walls, where we only have fluid domains and non-fluid domains, and nothing in between.

In this section we assume $\varepsilon > 0$ to be fixed and discuss the optimal control problem (5.1) – (5.2) for this arbitrary chosen $\varepsilon > 0$.

We want to discuss well-posedness of the constraints (5.2), which will allow us to define a solution operator called $\boldsymbol{S}_\varepsilon$, see Lemma 5.1, and therefore we can then define a reduced objective functional. For this objective functional we will then consider the sharp interface limit as $\varepsilon \searrow 0$, see Section 6, and obtain under certain assumptions that minimizers of the phase field problem approximate a minimizer of the sharp interface formulation describing the free boundary topology optimization problem.

5.2 Existence results

First of all, we want to consider the state equations (5.2). This is obviously a weak formulation of the following system:

$$\alpha_\varepsilon(\varphi)\,\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{5.3a}$$
$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{5.3b}$$
$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega. \tag{5.3c}$$

The well-posedness of this system is established by the next lemma:

Lemma 5.1. *For every $\varphi \in L^1(\Omega)$ with $|\varphi(x)| \leq 1$ a.e. in Ω there exists a unique $\boldsymbol{u} \in \boldsymbol{U}$ such that* (5.3) *is fulfilled in the sense of* (5.2)*. Moreover, the solution $\boldsymbol{u}$ fulfills*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega, \overline{\alpha}_\varepsilon, \mu)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}\right). \tag{5.4}$$

This defines a solution operator for the constraints, which will be denoted by

$$\boldsymbol{S}_\varepsilon : \overline{\Phi}_{ad} \to \boldsymbol{U}, \qquad \boldsymbol{S}_\varepsilon(\varphi) := \boldsymbol{u} \quad \text{if } \boldsymbol{u} \text{ solves } (5.2).$$

Remark 5.1. *Remark that for any $\boldsymbol{u} \in \boldsymbol{U}$ and $\varphi \in \overline{\Phi}_{ad}$ fulfilling* (5.2) *there exists some pressure $p \in L^2(\Omega)$, which is unique up to a constant, such that*

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - \int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \tag{5.5}$$

holds. This follows from Lemma 4.4. But since the objective functional is in this part independent of the pressure of the fluid, we will drop this variable in the following.

Proof. The proof is based on Lax-Milgram's theorem applied to the following bilinear form

$$a : \boldsymbol{V}\times\boldsymbol{V} \to \mathbb{R}$$
$$a(\boldsymbol{v}_1, \boldsymbol{v}_2) = \int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{v}_1\cdot\boldsymbol{v}_2 + \mu\nabla\boldsymbol{v}_1\cdot\nabla\boldsymbol{v}_2\,\mathrm{d}x$$

for some fixed $\varphi \in L^1(\Omega)$ with $|\varphi(x)| \leq 1$ a.e. in Ω. To this end, we first reduce the problem into a homogeneous one by choosing some $\boldsymbol{w} \in \boldsymbol{U}$ such that

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega)\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} \tag{5.6}$$

which exists for example due to Lemma 4.1, and see that $\boldsymbol{u} \in \boldsymbol{U}$ solves (5.2) if and only if $\boldsymbol{u}_0 := \boldsymbol{u} - \boldsymbol{w} \in \boldsymbol{V}$ solves

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}_0\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}_0\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \langle\boldsymbol{F}, \boldsymbol{v}\rangle_{\boldsymbol{H}^{-1}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

where $\boldsymbol{F} \in \boldsymbol{V}'$ is defined by

$$\langle\boldsymbol{F}, \boldsymbol{v}\rangle_{\boldsymbol{H}^{-1}(\Omega)} := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - \alpha_\varepsilon(\varphi)\boldsymbol{w}\cdot\boldsymbol{v} - \mu\nabla\boldsymbol{w}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Now using Poincaré's inequality and $\alpha_\varepsilon \geq 0$, we see that a is a coercive bilinear form on $\boldsymbol{V} \times \boldsymbol{V}$, and thus Lax-Milgram's theorem A.2 gives the existence of a unique $\boldsymbol{u}_0 \in \boldsymbol{V}$ such that it holds

$$a(\boldsymbol{u}_0, \boldsymbol{v}) = \langle \boldsymbol{F}, \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

together with

$$\|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega) \|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

Therefore, $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$ is the desired solution of (5.2) which fulfills moreover

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} &\leq \|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \leq \\
&\leq c(\Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \overline{\alpha}_\varepsilon \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} + \mu \|\nabla \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \right) + \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \overset{(5.6)}{\leq} \\
&\leq c(\Omega, \overline{\alpha}_\varepsilon, \mu) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} \right)
\end{aligned}$$

and this proves the statement. □

Remark 5.2. *We notice that* (5.4) *only gives an estimate on $\boldsymbol{S}_\varepsilon(\varphi)$ depending on ε. So we don't get a uniform bound on $\|\boldsymbol{S}_\varepsilon(\varphi)\|_{\boldsymbol{H}^1(\Omega)}$ independent of ε, which will be necessary in some sense for the sharp interface convergence, see proof of Theorem 6.1. And so we will need the coercivity condition on the functional stated in Assumption* ***(A5)****.*

Moreover, we can show the existence of minimizers for the stated minimization problem. We will show this existence result without assuming the coercivity of the objective functional with respect to the velocity as it is assumed in Assumption **(A5)**. Thus to get well-posedness of the diffuse interface problem for fixed $\varepsilon > 0$, this assumption is not necessary. But we will see, that we could neither show convergence of minimizers of the phase field problem as ε tends to zero, nor expect to find existence of a minimizer of the sharp interface problem, compare considerations in Section 6, if the objective functional is not radially unbounded in the sense of (2.3).

Theorem 5.1. *There exists at least one minimizer of* (5.1) – (5.2)*.*

Proof. From the boundedness assumption in Assumption **(A5)** we deduce that $J_\varepsilon : \Phi_{ad} \times \boldsymbol{U} \to \mathbb{R}$ is bounded from below by a constant. Thus we can choose an admissible minimizing sequence $(\varphi_k, \boldsymbol{u}_k)_{k\in\mathbb{N}} \subset \Phi_{ad} \times \boldsymbol{U}$, which gives in particular that $\boldsymbol{u}_k = \boldsymbol{S}_\varepsilon(\varphi_k)$ for all $k \in \mathbb{N}$. We use (5.4) to deduce a uniform bound on $\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}$ from the state equations.
Moreover, the uniform bound in $(J_\varepsilon(\varphi_k, \boldsymbol{u}_k))_{k\in\mathbb{N}}$ implies that $\sup_{k\in\mathbb{N}} \|\nabla\varphi_k\|_{L^2(\Omega)} < \infty$. Besides, $\varphi_k \in \Phi_{ad}$ for all $k \in \mathbb{N}$, and so $\|\varphi_k\|_{L^\infty(\Omega)} \leq 1\ \forall k \in \mathbb{N}$.

Now we split the proof into several parts:

- *1st step:* Convergence results:

  Because of the bounds derived above we can choose a subsequence of $(\boldsymbol{u}_k, \varphi_k)_{k\in\mathbb{N}}$, denoted by the same, such that

$$\begin{aligned}
\boldsymbol{u}_k &\rightharpoonup \boldsymbol{u}_0 \quad \text{in } \boldsymbol{H}^1(\Omega)\,,\\
\boldsymbol{u}_k &\to \boldsymbol{u}_0 \quad \text{in } \boldsymbol{L}^2(\Omega)\,,\\
\varphi_k &\rightharpoonup \varphi_0 \quad \text{in } H^1(\Omega)\,,\\
\varphi_k &\to \varphi_0 \quad \text{in } L^2(\Omega)\,,
\end{aligned} \tag{5.7}$$

  for some element $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{U} \times \Phi_{ad}$. Here we used that Φ_{ad} and $\boldsymbol{U}$ are closed and convex and thus weakly closed subspaces of $H^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)$, respectively.

- *2nd step:* Next we show, that $\boldsymbol{u}_0 = \boldsymbol{S}_\varepsilon(\varphi_0)$.

  Therefore we use similar techniques as in [BP03]. We want to show first of all

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_k)\,\boldsymbol{u}_k \cdot \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_0)\,\boldsymbol{u}_0 \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

  For this purpose, let $\boldsymbol{v} \in \boldsymbol{V}$ be fixed and write

$$\begin{aligned}
\int_\Omega (\alpha_\varepsilon(\varphi_k)\boldsymbol{u}_k - \alpha_\varepsilon(\varphi_0)\boldsymbol{u}_0)\cdot \boldsymbol{v}\,\mathrm{d}x = &\int_\Omega (\alpha_\varepsilon(\varphi_k)\,\boldsymbol{u}_k - \alpha_\varepsilon(\varphi_k)\,\boldsymbol{u}_0)\cdot \boldsymbol{v}\,\mathrm{d}x +\\
&+ \int_\Omega (\alpha_\varepsilon(\varphi_k)\,\boldsymbol{u}_0 - \alpha_\varepsilon(\varphi_0)\boldsymbol{u}_0)\cdot \boldsymbol{v}\,\mathrm{d}x.
\end{aligned} \tag{5.8}$$

  To consider the first term we use that $\alpha_\varepsilon \leq \overline{\alpha}_\varepsilon$ together with the strong convergence of $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ to $\boldsymbol{u}_0$ in $\boldsymbol{L}^2(\Omega)$ to see

$$\int_\Omega (\alpha_\varepsilon(\varphi_k)\,\boldsymbol{u}_k - \alpha_\varepsilon(\varphi_k)\,\boldsymbol{u}_0)\cdot \boldsymbol{v}\,\mathrm{d}x \leq C\,\|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \xrightarrow{k\to\infty} 0.$$

  For the second term we notice that for a subsequence $\lim_{k\to\infty}\varphi_k(x) = \varphi_0(x)$ almost everywhere in Ω, which implies

$$\alpha_\varepsilon(\varphi_k)\boldsymbol{u}_0 \cdot \boldsymbol{v} \xrightarrow{k\to\infty} \alpha_\varepsilon(\varphi_0)\boldsymbol{u}_0\cdot \boldsymbol{v} \quad \text{a.e. in } \Omega.$$

  We use again the boundedness of α_ε to get

$$|(\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi_0))\,\boldsymbol{u}_0\cdot\boldsymbol{v}| \leq C\,|\boldsymbol{u}_0\cdot\boldsymbol{v}|$$

  pointwise almost everywhere in Ω.
  From Lebesgue's convergence theorem, we get therefrom that the second term in (5.8) also tends to zero as $k \to \infty$.

  Then we can take the limit $k \to \infty$ in the weak formulation of the state equation (5.2) and see that it holds

$$\int_\Omega \alpha_\varepsilon(\varphi_0)\,\boldsymbol{u}_0\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}_0\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{V}.$$

In particular, this gives $\boldsymbol{u}_0 = \boldsymbol{S}_\varepsilon(\varphi_0)$ and thus $(\varphi_0,\boldsymbol{u}_0)$ is admissible for (5.1) – (5.2).

- *3rd step:* We show weakly lower semicontinuity of J_ε:

  To consider the objective functional, we argument analogously to [BP03] and find that

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_k)\,|\boldsymbol{u}_k|^2\,\mathrm{d}x - \int_\Omega \alpha_\varepsilon(\varphi_0)\,|\boldsymbol{u}_0|^2\,\mathrm{d}x &= \int_\Omega \alpha_\varepsilon(\varphi_k)\left(|\boldsymbol{u}_k|^2 - |\boldsymbol{u}_0|^2\right)\mathrm{d}x +\\ &+ \int_\Omega (\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi_0))\,|\boldsymbol{u}_0|^2\,\mathrm{d}x.\end{aligned}$$

  Considering the first part, $\alpha_\varepsilon \le \overline{\alpha}_\varepsilon$ implies

$$\begin{aligned}\left|\int_\Omega \alpha_\varepsilon(\varphi_k)\left(|\boldsymbol{u}_k|^2 - |\boldsymbol{u}_0|^2\right)\mathrm{d}x\right| &\le \overline{\alpha}_\varepsilon \int_\Omega |\boldsymbol{u}_k\cdot\boldsymbol{u}_k - \boldsymbol{u}_0\cdot\boldsymbol{u}_0|\,\mathrm{d}x =\\ &= \overline{\alpha}_\varepsilon \int_\Omega |\boldsymbol{u}_k - \boldsymbol{u}_0|\,|\boldsymbol{u}_k + \boldsymbol{u}_0|\,\mathrm{d}x \le\\ &\le \overline{\alpha}_\varepsilon \underbrace{\|\boldsymbol{u}_k + \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}}_{\le C} \|\boldsymbol{u}_k - \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \xrightarrow{k\to\infty} 0.\end{aligned}$$

  For the second term we use similar to the second step of this proof pointwise convergence of $(\varphi_k)_{k\in\mathbb{N}}$ to φ_0, continuity and boundedness of α_ε together with Lebesgue's convergence theorem to conclude

$$\int_\Omega (\alpha_\varepsilon(\varphi_k) - \alpha_\varepsilon(\varphi_0))\,|\boldsymbol{u}_0|^2\,\mathrm{d}x \xrightarrow{k\to\infty} 0$$

  which leads to

$$\lim_{k\to\infty}\int_\Omega \alpha_\varepsilon(\varphi_k)\,|\boldsymbol{u}_k|^2\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_0)\,|\boldsymbol{u}_0|^2\,\mathrm{d}x.$$

  This gives us in view of the lower semicontinuity of the objective functional stated in Assumption **(A5)** and by using

$$\lim_{k\to\infty}\int_\Omega \psi(\varphi_k)\,\mathrm{d}x = \lim_{k\to\infty}\frac{1}{2}\int_\Omega \left(1-\varphi_k^2\right)\mathrm{d}x \overset{(5.7)}{=} \frac{1}{2}\int_\Omega\left(1-\varphi_0^2\right)\mathrm{d}x = \int_\Omega \psi(\varphi_0)\,\mathrm{d}x$$

  the estimate

$$J_\varepsilon(\varphi_0,\boldsymbol{u}_0) \le \liminf_{k\to\infty} J_\varepsilon(\varphi_k,\boldsymbol{u}_k)$$

  which implies that $(\varphi_0,\boldsymbol{u}_0)$ minimizes J_ε.

□

Thus we have shown that the phase field model, which is given by (5.1) – (5.2), is well-defined in the sense that we have a well-defined solution operator for the constraints and have guaranteed existence of a minimizer for the overall optimization problem.

6 Sharp interface limit

The considerations of the previous section justify the definition of a reduced objective functional $j_\varepsilon(\varphi) := J_\varepsilon(\varphi, \boldsymbol{S}_\varepsilon(\varphi))$, which is extended to the whole space $L^1(\Omega)$ as follows:

$$j_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$$

$$j_\varepsilon(\varphi) := \begin{cases} J_\varepsilon(\varphi, \boldsymbol{S}_\varepsilon(\varphi)) & \text{if } \varphi \in \Phi_{ad}, \\ +\infty & \text{otherwise,} \end{cases} \tag{6.1}$$

which means

$$j_\varepsilon(\varphi) = \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 \,\mathrm{d}x + \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) \,\mathrm{d}x + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi) \,\mathrm{d}x \tag{6.2}$$

with $\boldsymbol{u} = \boldsymbol{S}_\varepsilon(\varphi)$ if $\varphi \in \Phi_{ad}$ and $j_\varepsilon(\varphi) := +\infty$ otherwise.

One natural question arising when considering phase field models is whether this model converges to some sharp interface free boundary problem as the thickness of the interface tends to zero. In our setting this means, that we are interested in finding a functional related to $(j_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \searrow 0$. We will find that under certain assumptions, minimizers of $(j_\varepsilon)_{\varepsilon>0}$ converge to a minimizer of a sharp interface model describing shape and topology optimization in a Stokes flow formulated in a Caccioppoli setting with a perimeter penalization. This sharp interface problem will be introduced and discussed in the next subsection.

6.1 Sharp interface model

We will restrict our attention in this subsection to the sharp interface problems arising as limit of the phase field model described in the previous section as the thickness of the interface and the permeability of the porous medium tends to zero, compare Section 6.2.

To identify the limit of $(j_\varepsilon)_{\varepsilon>0}$ we begin with defining a solution operator $\boldsymbol{S}_0$ that corresponds to the limiting problem of (5.3). The resulting system in the strong formulation for some $\varphi \in \overline{\Phi}_{ad}^0$ will be the following:

$$-\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } E^\varphi, \tag{6.3a}$$
$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{6.3b}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Omega \setminus E^\varphi, \tag{6.3c}$$
$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{6.3d}$$

where we recall that E^φ is given by

$$E^\varphi := \{x \in \Omega \mid \varphi(x) = 1\}.$$

This means, that in the fluid domain E^φ we will have a formulation of the Stokes equations, and outside of the fluid domain we define the velocity to be zero. In this way, we can still define the velocity in Ω, which is one key ingredient in this model.

Let us start by considering well-posedness of this system. Therefore, we notice particularly that (6.3) may not be well-defined, for instance if $\mathcal{H}^{d-1}\left(\{\boldsymbol{g} \neq \boldsymbol{0}\} \cap (\Omega \setminus E^\varphi)\right) > 0$, since we then have inconsistent equations which cannot be fulfilled. We exclude this case by stating that the solution space $\boldsymbol{U}^\varphi$ is not empty, which implies especially that the conditions $\boldsymbol{u} = \boldsymbol{0}$ in $\Omega \setminus E^\varphi$ and $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$ are not conflicting. This condition is included the definition of the admissible set $\overline{\Phi}_{ad}^0 \subseteq \Phi_{ad}^0$.

In fact, choosing $\varphi \in \overline{\Phi}_{ad}^0$, we can find a unique velocity solving the state equations (6.3) in the following sense:

Lemma 6.1. *For every $\varphi \in L^1(\Omega)$ such that $\boldsymbol{U}^\varphi \neq \emptyset$ there exists a unique $\boldsymbol{u} \in \boldsymbol{U}^\varphi$ such that* (6.3) *is fulfilled in the following sense:*

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}^\varphi. \tag{6.4}$$

This defines a solution operator denoted by

$$\boldsymbol{S}_0 : \overline{\Phi}_{ad}^0 \to \boldsymbol{U}, \qquad \boldsymbol{S}_0(\varphi) := \boldsymbol{u} \in \boldsymbol{U}^\varphi \quad \textit{if } \boldsymbol{u} \textit{ fulfills } (6.4).$$

Proof. We fix some arbitrary $\varphi \in L^1(\Omega)$ with $\boldsymbol{U}^\varphi \neq \emptyset$ and choose $\widehat{\boldsymbol{u}} \in \boldsymbol{U}^\varphi$. We reformulate the non-homogeneous problem (6.3) in analogy with Lemma 5.1 into a homogeneous one by noticing that $\boldsymbol{u} \in \boldsymbol{U}^\varphi$ fulfills (6.4) if and only if $\boldsymbol{w} := \boldsymbol{u} - \widehat{\boldsymbol{u}} \in \boldsymbol{V}^\varphi$ solves

$$\mu \int_\Omega \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \left\langle \widehat{\boldsymbol{f}}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}^\varphi \tag{6.5}$$

where we defined $\widehat{\boldsymbol{f}} \in (\boldsymbol{V}^\varphi)'$ by

$$\left\langle \widehat{\boldsymbol{f}}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)} := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} - \mu \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x.$$

We remark that $\boldsymbol{V}^\varphi$ is a closed subspace of $\boldsymbol{V}$ and thus in particular a Hilbert space. Proceeding similarly to Lemma 5.1 we define the bilinear form

$$a^\varphi : \boldsymbol{V}^\varphi \times \boldsymbol{V}^\varphi \to \mathbb{R}$$

by

$$a^\varphi(\boldsymbol{v}_1, \boldsymbol{v}_2) = \mu \int_\Omega \nabla \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_2 \,\mathrm{d}x$$

which is due to Poincaré's inequality a coercive bilinear form on $\boldsymbol{V}^\varphi \times \boldsymbol{V}^\varphi$.
Thus, we get with Lax-Milgram's theorem A.2 the existence and uniqueness of $\boldsymbol{w} \in \boldsymbol{V}^\varphi$ such that

$$a^\varphi(\boldsymbol{w}, \boldsymbol{v}) = \left\langle \widehat{\boldsymbol{f}}, \boldsymbol{v} \right\rangle_{\boldsymbol{H}^{-1}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}^\varphi.$$

Therefore $\boldsymbol{u} := \boldsymbol{w} + \widehat{\boldsymbol{u}} \in \boldsymbol{U}^\varphi$ fulfills (6.4). □

Remark 6.1. *Remark that we do not get a continuous dependence of $\boldsymbol{S}_0(\varphi)$ on φ, since even Lax-Milgram's theorem only implies an estimate of the form*

$$\|\boldsymbol{S}_0(\varphi)\|_{\boldsymbol{H}^1(\Omega)} \leq c(\boldsymbol{g}, \Omega, \boldsymbol{f}, \widehat{\boldsymbol{u}})$$

where $\widehat{\boldsymbol{u}} \in \boldsymbol{U}^\varphi$ is a solenoidal extension of the boundary data $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$, which is zero in $\Omega \smallsetminus E^\varphi$. But for this extension, we do not get a uniform estimate only depending on Ω and $\boldsymbol{g}$, but merely an estimate of the form

$$\|\widehat{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega, E^\varphi)\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}$$

and it is not clear how the constant $c(\Omega, E^\varphi)$ depends on E^φ, compare also Remark 5.2. On some first results about the dependence of this constant on E^φ, and thus on φ, we refer the reader to [BRW06].

To clarify that Φ_{ad}^0 is not an empty set we give here some examples for which $\boldsymbol{U}^\varphi \neq \emptyset$:

Lemma 6.2. *Let $\varphi \in BV(\Omega, \{\pm 1\})$ such that $\mathcal{H}^{d-1}(\{\boldsymbol{g} \neq \boldsymbol{0}\} \cap \{\varphi = -1\}) = 0$ and assume* $\operatorname{int}(E^\varphi)$ *is some fixed open set with Lipschitz boundary. Moreover, let $(E_i)_{i=1}^N$ be the connected components of* $\operatorname{int}(E^\varphi)$, *$\int_{\partial \overline{E_i}} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}x = 0$ for all $i \in \{1, \ldots, N\}$.*
Then there exists at least one element $\boldsymbol{u} \in \boldsymbol{U}^\varphi$, which fulfills moreover

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega, \varphi)\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}.$$

Remark 6.2. *For every $\varphi \in BV(\Omega)$, where Ω has due to Assumption **(A1)** Lipschitz boundary, we can define a trace $T\varphi \in L^1(\partial\Omega)$, see Theorem 3.1. Thus the set*

$$\{x \in \partial\Omega \mid \boldsymbol{g}(x) \neq \boldsymbol{0}, \varphi(x) = -1\}$$

is well-defined, up to sets of measure zero, for any $\varphi \in BV(\Omega)$.

Proof of Lemma 6.2. We take some general $\varphi \in BV(\Omega, \{\pm 1\})$ fulfilling the assumptions of the statement. Then we get an element in $\boldsymbol{U}^\varphi$ by applying Lemma 4.1 to every connected component E_i of E^φ to get functions $\boldsymbol{v}_i \in \boldsymbol{H}^1(E_i)$ such that

$$\operatorname{div} \boldsymbol{v}_i = 0, \quad \boldsymbol{v}_i|_{\partial E_i \cap \partial\Omega} = \boldsymbol{g}, \quad \boldsymbol{v}_i|_{\partial E_i \cap \Omega} = \boldsymbol{0}.$$

We can extend those functions to $\boldsymbol{v}_i \in \boldsymbol{H}^1(\Omega)$ by zero, since we have zero boundary condition in Ω. Defining

$$\boldsymbol{u} := \sum_{i=1}^N \boldsymbol{v}_i$$

we get the desired vector field $\boldsymbol{u} \in \boldsymbol{U}^\varphi$. $\square$

One of the most important tools in the subsequent discussion will be the following technical lemma. This will be used essentially for the sharp interface convergence result (see Section 6.2).

Lemma 6.3. *Let* $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)$ *with* $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ *be such that for* $\varepsilon \searrow 0$

$$\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \quad \|\varphi_\varepsilon - \varphi_0\|_{L^1(E^{\varphi_0}\cap\{\varphi_\varepsilon<0\})} = \mathcal{O}(\varepsilon) \tag{6.6}$$

with $\varphi_0 \in BV(\Omega, \{\pm 1\})$, $\boldsymbol{U}^{\varphi_0} \neq \emptyset$ *and* $|\varphi_\varepsilon| \leq 1$ *pointwise almost everywhere in* Ω. *Then there exists a subsequence of* $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ *(denoted by the same) such that*

$$\lim_{\varepsilon\searrow 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega \alpha_0(\varphi_0)|\boldsymbol{u}_0|^2 \,\mathrm{d}x = 0$$

where $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$.

Proof. We split the proof into several steps:

- *1st step:* First of all we choose a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges pointwise almost everywhere in Ω to φ_0. Then we take some $\delta > 0$, such that $\varepsilon < \delta$ for ε small enough and notice that due to Assumption **(A4)** it holds $\alpha_\delta \leq \alpha_\varepsilon$ pointwise, and therefore we arrive in the pointwise estimate

$$\alpha_\delta(\varphi_0(x)) = \lim_{\varepsilon\searrow 0} \alpha_\delta(\varphi_\varepsilon(x)) \leq \liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)). \tag{6.7}$$

  This gives, as $\delta \searrow 0$,

$$\begin{aligned}\alpha_0(\varphi_0(x)) = \lim_{\delta\searrow 0} \alpha_\delta(\varphi_0(x)) = \lim_{\delta\searrow 0}\left(\lim_{\varepsilon\searrow 0} \alpha_\delta(\varphi_\varepsilon(x))\right) \leq \lim_{\delta\searrow 0}\left(\liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x))\right) = \\ = \liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x))\end{aligned} \tag{6.8}$$

  for almost every $x \in \Omega$. On the other hand we deduce from $\alpha_\varepsilon \leq \alpha_0$ pointwise almost everywhere

$$\limsup_{\varepsilon\searrow 0}(\alpha_\varepsilon(\varphi_\varepsilon(x))) \leq \limsup_{\varepsilon\searrow 0}(\alpha_0(\varphi_\varepsilon(x))) = \alpha_0(\varphi_0(x)).$$

  We sum up the estimates to obtain

$$\alpha_0(\varphi_0(x)) \leq \liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) \leq \limsup_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) \leq \alpha_0(\varphi_0(x))$$

  which holds for almost every $x \in \Omega$ and implies

$$\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) = \alpha_0(\varphi_0(x)) \quad \text{for a.e. } x \in \Omega. \tag{6.9}$$

  This will be used later.

- *2nd step:* Now we show, that for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v}|_{\Omega\setminus E^{\varphi_0}} = \boldsymbol{0}$ it holds

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 \,\mathrm{d}x = 0.$$

  To this end, we notice first for almost every $x \in \Omega$ that due to (6.9),

$$\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x))|\boldsymbol{v}(x)|^2 = 0. \tag{6.10}$$

To apply Lebesgue's convergence theorem and deduce the convergence in $L^1(\Omega)$ we estimate in several steps. Since α_ε is decreasing we find

$$\alpha_\varepsilon\left(\varphi_\varepsilon\left(x\right)\right)\left|\boldsymbol{v}(x)\right|^2 \le \alpha_\varepsilon\left(0\right)\left|\boldsymbol{v}(x)\right|^2 \le \alpha_0(0)\left|\boldsymbol{v}(x)\right|^2$$

for almost every $x \in \{\varphi_\varepsilon \ge 0\}$ where we used $\lim_{\varepsilon\searrow 0}\alpha_\varepsilon\left(0\right) = \alpha_0(0) < \infty$, see Assumption **(A4)**. From this bound and the pointwise convergence (6.10) we obtain thanks to Lebesgue's convergence theorem

$$\lim_{\varepsilon\searrow 0}\int_{\{\varphi_\varepsilon\ge 0\}}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{v}\right|^2\,\mathrm{d}x = 0. \tag{6.11}$$

To consider the part of Ω where φ_ε is non-positive, we deduce from $\boldsymbol{v}|_{\Omega\setminus E^{\varphi_0}} = \boldsymbol{0}$ that $\{x\in\Omega \mid \boldsymbol{v}(x)\ne\boldsymbol{0}\} \subseteq \{x\in\Omega \mid \varphi_0(x)=1\}$ and thus we get for almost every $x\in\{\varphi_\varepsilon<0\}$ the estimate

$$\alpha_\varepsilon\left(\varphi_\varepsilon(x)\right)\left|\boldsymbol{v}(x)\right|^2 \le \overline{\alpha}_\varepsilon\underbrace{\left|\varphi_\varepsilon(x)-\varphi_0(x)\right|}_{\ge 1}\left|\boldsymbol{v}(x)\right|^2\chi_{\{\varphi_0=1\}}(x). \tag{6.12}$$

Due to the pointwise estimate $|\varphi_\varepsilon|\le 1, |\varphi_0|\le 1$ we have

$$\overline{\alpha}_\varepsilon\int_\Omega\chi_{E^{\varphi_0}\cap\{\varphi_\varepsilon<0\}}\left|\varphi_\varepsilon-\varphi_0\right|\left|\boldsymbol{v}\right|^2\,\mathrm{d}x \le C\overline{\alpha}_\varepsilon\left\|\varphi_0-\varphi_\varepsilon\right\|^{\frac{2}{3}}_{L^1(E^{\varphi_0}\cap\{\varphi_\varepsilon<0\})}\left\|\boldsymbol{v}\right\|^2_{\boldsymbol{L}^6(\Omega)}. \tag{6.13}$$

We combine

$$\left\|\varphi_\varepsilon-\varphi_0\right\|_{L^1(E^{\varphi_0}\cap\{\varphi_\varepsilon<0\})} = \mathcal{O}\left(\varepsilon\right) \tag{6.14}$$

and $\overline{\alpha}_\varepsilon = \mathrm{o}\left(\varepsilon^{-2/3}\right)$, see Assumption **(A4)**, to get therefrom

$$\lim_{\varepsilon\searrow 0}\int_\Omega\chi_{\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\}}\overline{\alpha}_\varepsilon\left|\varphi_\varepsilon-\varphi_0\right|\left|\boldsymbol{v}\right|^2\,\mathrm{d}x = 0. \tag{6.15}$$

And so, in view of (6.12)

$$\lim_{\varepsilon\searrow 0}\int_{\{\varphi_\varepsilon<0\}}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{v}\right|^2\,\mathrm{d}x = 0$$

which gives combined with (6.11) finally

$$\lim_{\varepsilon\searrow 0}\int_\Omega\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{v}\right|^2\,\mathrm{d}x = 0.$$

We notice that for every $\varepsilon>0$ the velocity field $\boldsymbol{u}_\varepsilon\in\boldsymbol{U}$ is the unique solution of

$$\min_{\boldsymbol{v}\in\boldsymbol{U}}F_\varepsilon\left(\boldsymbol{v}\right) := \int_\Omega\left(\frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{v}\right|^2+\frac{\mu}{2}\left|\nabla\boldsymbol{v}\right|^2-\boldsymbol{f}\cdot\boldsymbol{v}\right)\mathrm{d}x$$

since the state equation (5.2) is the first order optimality condition for this optimization problem, which is necessary and sufficient for the convex optimization problem of minimizing the functional F_ε over $\boldsymbol{U}$.

We proceed by defining

$$F_0(\boldsymbol{v}) := \int_\Omega \left(\frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla \boldsymbol{v}|^2 - \boldsymbol{f}\cdot\boldsymbol{v}\right) \mathrm{d}x$$

and notice, that the unique minimizer of F_0 in $\boldsymbol{U}$ is $\boldsymbol{S}_0(\varphi_0)$, since again the state equations are the necessary and sufficient first order optimality conditions for the convex optimization problem $\min_{v\in\boldsymbol{U}} F_0(\boldsymbol{v})$. We use the functionals $(F_\varepsilon)_{\varepsilon>0}$ to show that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded:

- *3rd step:* From $\boldsymbol{U}^{\varphi_0} \neq \emptyset$ we know that can choose some $\boldsymbol{u}_0 \in \boldsymbol{U}^{\varphi_0} \subset \boldsymbol{U}$ and obtain, because $\boldsymbol{u}_\varepsilon$ are minimizers of F_ε, the estimate

$$\begin{aligned}
&\int_\Omega \left(\frac{\mu}{2}|\nabla \boldsymbol{u}_\varepsilon|^2 + \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon\right) \mathrm{d}x = F_\varepsilon(\boldsymbol{u}_\varepsilon) \le F_\varepsilon(\boldsymbol{u}_0) = \\
&= \int_\Omega \left(\frac{\mu}{2}|\nabla \boldsymbol{u}_0|^2 + \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_0|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_0\right) \mathrm{d}x \le \\
&\le \int_\Omega \left(\frac{\mu}{2}|\nabla \boldsymbol{u}_0|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_0\right) \mathrm{d}x + \left(\limsup_{\varepsilon\searrow 0} \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_0|^2 \,\mathrm{d}x + c\right)
\end{aligned} \tag{6.16}$$

  for some constant $c \ge 0$ and $\varepsilon > 0$ small enough.
  To see that $\limsup_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_0|^2 \,\mathrm{d}x < \infty$ we can use the second step of this proof. And so from (6.16), the inequalities of Poincaré and Young and the boundary condition on $\boldsymbol{u}_\varepsilon$ we find a constant $C > 0$ independent of ε such that

$$\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} < C.$$

The result of the previous step implies in particular the existence of a subsequence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$, which will be denoted by the same, that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\boldsymbol{u}_0 \in \boldsymbol{U}$. To see that $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$, we next claim that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges in $\boldsymbol{U}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology to F_0 as $\varepsilon \searrow 0$.

- *4th step:* We will see, that the constant sequence defines a recovery sequence for $(F_\varepsilon)_{\varepsilon>0}$. Choosing $\boldsymbol{v} \in \boldsymbol{U}$ we can assume that $F_0(\boldsymbol{v}) < \infty$, otherwise it would hold trivially

$$\limsup_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}) \le F_0(\boldsymbol{v}).$$

  Therefore, we can assume $\int_\Omega \alpha_0(\varphi_0)|\boldsymbol{v}|^2 \,\mathrm{d}x < \infty$ and so $\boldsymbol{v} \in \boldsymbol{U}^{\varphi_0}$. Due to the second step of this proof this yields

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 \,\mathrm{d}x = 0.$$

  As the remaining terms of $(F_\varepsilon)_{\varepsilon>0}$ are independent of ε this already implies

$$\limsup_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}) \le F_0(\boldsymbol{v}).$$

- *5th step:* Let $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0} \subseteq \boldsymbol{U}$ be an arbitrary sequence that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{v} \in \boldsymbol{U}$. Due to the compact imbedding of $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^2(\Omega)$ we certainly have a subsequence of $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$, which will be denoted by the same, that converges pointwise almost everywhere in Ω to $\boldsymbol{v}$. From this convergence, the pointwise convergence of $\alpha_\varepsilon(\varphi_\varepsilon)$ that was proven in (6.9) and Fatou's lemma we see

$$\begin{aligned}\int_\Omega \alpha_0(\varphi_0)|\boldsymbol{v}|^2\,\mathrm{d}x &= \int_\Omega \left(\liminf_{\varepsilon\searrow 0}\alpha_\varepsilon(\varphi_\varepsilon)\right)\left(\liminf_{\varepsilon\searrow 0}|\boldsymbol{v}_\varepsilon|^2\right)\mathrm{d}x \le \\ &\le \int_\Omega \liminf_{\varepsilon\searrow 0}\left(\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}_\varepsilon|^2\right)\mathrm{d}x \le \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}_\varepsilon|^2\,\mathrm{d}x\end{aligned} \tag{6.17}$$

  which yields

$$F_0(\boldsymbol{v}) \le \liminf_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon)$$

  since the remaining terms are weakly lower semicontinuous in $\boldsymbol{H}^1(\Omega)$.

This proves that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F_0 as $\varepsilon \searrow 0$ in $\boldsymbol{U}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology. In view of standard results for Γ-convergence, see Theorem 3.3, we see therefrom that the limit point of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is the unique minimizer of F_0, and thus $\boldsymbol{u}_0$ minimizes F_0 in $\boldsymbol{U}$. We find that the first order optimality conditions for the convex optimization problem $\min_{\boldsymbol{v}\in\boldsymbol{U}} F_0(\boldsymbol{u})$ are exactly given by the state equations (6.4). Thus, the minimizer $\boldsymbol{u}_0 \in \boldsymbol{U}$ of F_0 fulfills (6.4) and hence $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$.
Due to the Γ-convergence result we have additionally $\lim_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) = F_0(\boldsymbol{u}_0)$ and so

$$\lim_{\varepsilon\searrow 0}\left[\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\mu}{2}|\nabla\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\right] = \int_\Omega \frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{u}_0|^2 + \frac{\mu}{2}|\nabla\boldsymbol{u}_0|^2\,\mathrm{d}x.$$

This gives us in view of (6.17) and by using Lemma A.1 the convergences

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{u}_0|^2\,\mathrm{d}x, \qquad \lim_{\varepsilon\searrow 0}\int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}_0|^2\,\mathrm{d}x$$

and finally proves the statement of the lemma.

□

Remark 6.3. *If we are in space dimension* $d = 2$ *we can use that* $\boldsymbol{H}^1(\Omega)$ *is imbedded in* $\boldsymbol{L}^{p'}(\Omega)$ *for any* $1 \le p' < \infty$. *Hence we can replace* (6.13) *for some* $1 < p < \infty$ *by*

$$\overline{\alpha}_\varepsilon \int_\Omega \chi_{E^{\varphi_0}\cap\{\varphi_\varepsilon<0\}}|\varphi_\varepsilon - \varphi_0||\boldsymbol{v}|^2\,\mathrm{d}x \le C\overline{\alpha}_\varepsilon \|\varphi_0 - \varphi_\varepsilon\|_{L^1(E^{\varphi_0}\cap\{\varphi_\varepsilon<0\})}^{1/p} \left\||\boldsymbol{v}|^2\right\|_{\boldsymbol{L}^{p'}(\Omega)}^{1/p'}$$

where $p' = \frac{p}{p-1}$. *Thus to conclude* (6.15) *from* (6.14) *it is sufficient to assume* $\overline{\alpha}_\varepsilon = o\left(\varepsilon^{-1/p}\right)$ *for any* $p \in (1, +\infty)$. *And so the condition* $\overline{\alpha}_\varepsilon = o\left(\varepsilon^{-2/3}\right)$ *claimed in Assumption* ***(A4)*** *can be weakened if* $d = 2$, *see also Remark 2.1.*

Sharp interface problem We will see in the next subsection, that minimizers of $(j_\varepsilon)_{\varepsilon>0}$ converge under certain assumptions to a minimizer of the following sharp interface shape and topology optimization problem:

$$\min_{(\varphi,\boldsymbol{u})} J_0(\varphi,\boldsymbol{u}) := \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \tag{6.18}$$

with

$$(\varphi,\boldsymbol{u}) \in \Phi^0_{ad}\times \boldsymbol{U}^\varphi$$

s.t.

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{V}^\varphi. \tag{6.19}$$

Thus we minimize the objective functional given by Assumption **(A5)** and a multiple of the perimeter of the fluid region in Ω to the objective functional. This ensures by the compact imbedding $BV(\Omega)\hookrightarrow L^1(\Omega)$ a certain compactness property for the optimization problem, which is in general useful for proving existence of a minimizer, compare also discussion in [DZ01]. The constant $\gamma>0$ is a fixed, arbitrary variable which can be used for weighting the perimeter term in the objective functional. The new appearing constant $c_0>0$ is due to technical reasons and depends only on the choice of the potential ψ. It is given by

$$c_0 := \int_{-1}^{1}\sqrt{2\psi(x)}\,\mathrm{d}x = \int_{-1}^{1}\sqrt{(1-x^2)}\,\mathrm{d}x = \frac{\pi}{2}.$$

This constant appears in the Γ-convergence of the Ginzburg-Landau energy to the perimeter functional, see also proof of Theorem 6.1, and details can be found for instance in [BE91, Mod87].

This sharp interface model can be seen as a generalization of perimeter penalized shape optimization problems to fluid dynamics, which is so far mainly used in structural topological optimization, see also Part IV. It is more general than the existing models in topology and shape optimization in fluid dynamics, see for instance [Evg05, Evg06, Sim91, PS10], since the admissible domains are Caccioppoli sets, and thus in particular topological changes are included in the optimization problem. In contrast to [BP03], the region outside the fluid is here not modeled as a porous medium, but rather we really have zero permeability and arrive therefrom in black-and-white solutions.

Recalling the definition of the admissible set for the sharp interface

$$\Phi^0_{ad} = \left\{\varphi\in BV(\Omega,\{\pm1\}) \mid \fint_\Omega \varphi\,\mathrm{d}x \le \beta,\ \boldsymbol{U}^\varphi \neq \emptyset\right\}$$

we notice that this set is closed in a certain sense:

Remark 6.4. *Assume we have a sequence $(\varphi_k)_{k\in\mathbb{N}}\subseteq\Phi^0_{ad}$ which converges to φ in $L^1(\Omega)$ and let $(\boldsymbol{u}_k)_{\in\mathbb{N}}$, $\boldsymbol{u}_k\in\boldsymbol{U}^{\varphi_k}$, be bounded in $\boldsymbol{H}^1(\Omega)$ uniformly in $k\in\mathbb{N}$. Then there exists a subsequence (which is denoted by the same) such that $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some element $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$, and it follows directly $\boldsymbol{u}\in\boldsymbol{U}^\varphi$, which shows that $\boldsymbol{U}^\varphi\neq\emptyset$. This shows $\varphi\in\Phi^0_{ad}$ and yields that Φ^0_{ad} is closed in a certain sense. But as already*

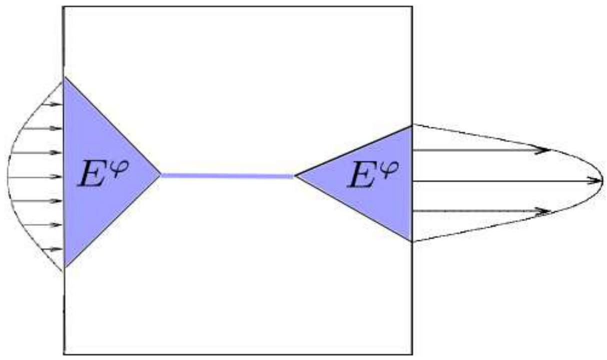


Figure 5: Critical case with $\boldsymbol{U}^\varphi = \emptyset$

mentioned, we have to use the radially unboundedness of the objective functional with respect to the velocity in $\boldsymbol{H}^1(\Omega)$*, see* (2.3)*. Otherwise, the critical situation illustrated in Figure 5 could occur as a limit case, and in this case it would hold* $\boldsymbol{U}^\varphi = \emptyset$*.*
We only can exclude this case by imposing a bound on the $\boldsymbol{H}^1$*-norm of the solutions to the state equations, and we do this here by assuming* (2.3)*. As already discussed in Remark 6.1, the state equations do not imply a bound on the solution independent of the set* E^φ*, mainly due to the non-homogeneous boundary data.*
In the analysis we use the radially unboundedness crucially in the fourth step of the proof of Theorem 6.1.

Moreover we notice, that variations in Φ^0_{ad} are possible, even though it is not a convex set. Reasonable variations of some element $\varphi \in \Phi^0_{ad}$ are geometric variations, thus deformations of the fluid region $\{\varphi = 1\}$ by suitable transformations. This will be done in Section 7.2.

6.2 Convergence of minimizers

The first step in showing that the phase field model gives a good approximation of the sharp interface setting described above is to consider minimizers of $(j_\varepsilon)_{\varepsilon>0}$ and show that they converge, under certain assumptions, to a minimizer of the sharp interface model, see Theorem 6.1.

To be precise, we will see in this section, that minimizers of $(j_\varepsilon)_{\varepsilon>0}$ converge in $L^1(\Omega)$ to a minimizer of

$$j_0 : L^1(\Omega) \to \overline{\mathbb{R}}, \quad j_0(\varphi) := J_0(\varphi, \boldsymbol{S}_0(\varphi))$$

where

$$J_0 : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$$

is given by

$$J_0(\varphi, \boldsymbol{u}) := \begin{cases} \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) & \text{if } \varphi \in \Phi^0_{ad} \text{ and } \boldsymbol{u} = \boldsymbol{S}_0(\varphi), \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\boldsymbol{S}_0$ is the solution operator of the Stokes equations defined in Lemma 6.1.

Due to Lemma 6.1 and Assumption **(A5)** we see that $J_0(\varphi, \boldsymbol{u})$ is well-defined.

Thus we can prove the main result of this section:

Theorem 6.1. *Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be minimizers of $(j_\varepsilon)_{\varepsilon>}$. Then there exists a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$, which is denoted by the same, and an element $\varphi_0 \in L^1(\Omega)$ such that*

$$\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0. \tag{6.20}$$

If it holds

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{6.21}$$

then we obtain moreover

$$\lim_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi_0) \tag{6.22}$$

and φ_0 is a minimizer of j_0.

Remark 6.5. *In particular, Theorem 6.1 implies that if* (6.21) *is fulfilled, then the sharp interface problem is well-posed in the sense, that there exists a least one minimizer of* (6.18)-(6.19)*. This has not been shown so far and is still an open problem for the general shape optimization problem in fluid dynamics, compare also discussion in Section 6.4.*

Proof. We split the proof into several parts:

- *1st step:* Assume we have an arbitrary $\varphi \in L^1(\Omega)$ chosen such that $j_0(\varphi) < \infty$. Then we will start by approximating E^φ by smooth sets as follows:

  We use the same idea as for example [Ste88, Theorem 1] uses as regularization argument. Let $(E_k)_{k\in\mathbb{N}}$ be the subsets of Ω approximating E^φ given by Lemma 3.1 which fulfill in particular $\partial E_k \cap \Omega \in C^2$. Then we define $\varphi_k := 2\chi_{E_k} - 1$. Due to the approximation properties in Lemma 3.1 it follows that $\lim_{k\to\infty} P_\Omega(E_k) = P_\Omega(E^\varphi)$ and $\lim_{k\to\infty} \|\varphi_k - \varphi\|_{L^1(\Omega)} = 0$ with the convergence rate

  $$\|\varphi_k - \varphi\|_{L^1(\Omega)} = \mathcal{O}\left(k^{-1}\right). \tag{6.23}$$

  Besides, Lemma 3.1 tells us that

  $$|E_k| = |E^\varphi| \quad \forall k \gg 1. \tag{6.24}$$

- *2nd step:* Let $(\varphi_k)_{k\in\mathbb{N}}$ be the sequence approximating $\varphi \in L^1(\Omega)$ given by the first step. This means in particular $\partial E_k \cap \Omega \in C^2$ for any $k \in \mathbb{N}$ where $E_k = \{\varphi_k = 1\}$. We will construct for every k large enough a recovery sequence $\left(\varphi_\varepsilon^k\right)_{\varepsilon>0} \subset L^1(\Omega)$ converging to φ_k in $L^1(\Omega)$ as $\varepsilon \searrow 0$ such that

  $$\limsup_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon^k\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon^k\right)\right) \mathrm{d}x \le \gamma c_0 P_\Omega(E_k) \tag{6.25}$$

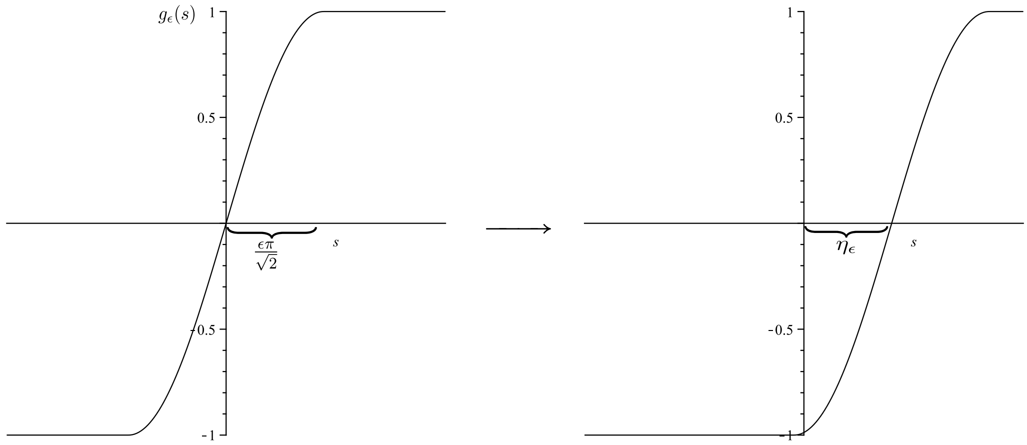


Figure 6: Profile of recovery sequence $\left(\varphi_\varepsilon^k\right)_{\varepsilon>0}$

analog as it is done for example in [Ste88, p. 222 ff], [Mod87, Proposition 2] or [BE91, Proposition 3.11]. Thus we fix first of all $k \gg 1$.

We define for $\varepsilon > 0$ small enough the function $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$g_\varepsilon(s) := \begin{cases} -1, & s < -\varepsilon\frac{\pi}{\sqrt{2}} \\ \sin\left(\frac{s}{\sqrt{2}\varepsilon}\right) & |s| \le \varepsilon\frac{\pi}{\sqrt{2}} \\ 1 & s > \varepsilon\frac{\pi}{\sqrt{2}} \end{cases}$$

and let

$$\tilde{\varphi}_\varepsilon^k(x) := g_\varepsilon(d(x))$$

with d being the signed distance function to $\Gamma := \partial E_k \cap \partial(\Omega \setminus E_k)$, which means $d(x) = d(x,\Gamma)$ for $x \in E_k$ and $d(x) = -d(x,\Gamma)$ otherwise.
Due to our assumptions, Γ defines a C^2-submanifold and thus the signed distance function d to Γ is due to [KP81] a C^2-function. This corresponds to the construction used in [Ste88] adapted to the obstacle potential ψ with $\psi_0(x) = \frac{1}{2}\left(1 - x^2\right)$.

To fulfill the integral constraint, it may be necessary to shift the profile by a constant $\eta_\varepsilon > 0$. Here we choose $\eta_\varepsilon := \varepsilon\frac{\pi}{\sqrt{2}} = \mathcal{O}(\varepsilon)$ to ensure $\varphi_\varepsilon^k(x) = -1$ if $\varphi(x) < 0$. Thus we define

$$\varphi_\varepsilon^k(x) := g_\varepsilon(d(x) - \eta_\varepsilon)$$

see Figure 6. Then we get pointwise $g_\varepsilon(d(x) - \eta_\varepsilon) \le \varphi_k(x)$ and so in particular

$$\int_\Omega \varphi_\varepsilon^k(x)\,\mathrm{d}x \le \int_\Omega \varphi_k(x)\,\mathrm{d}x \overset{(6.24)}{=} \int_\Omega \varphi\,\mathrm{d}x \le \beta\,|\Omega|$$

which means, that the integral constraint is fulfilled for φ_ε^k.

Now we use calculations that can be found in more detail in [Mod87, Ste88, BE91] to obtain

$$\lim_{\varepsilon \searrow 0} \left\| \varphi_\varepsilon^k - \varphi_k \right\|_{L^1(\Omega)} = 0$$

and that (6.25) holds. Moreover, we get therefrom the following convergence rate:

$$\left\| \varphi_\varepsilon^k - \varphi_k \right\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon). \tag{6.26}$$

- *3rd step:* Let $\varphi \in L^1(\Omega)$ with $j_0(\varphi) < \infty$ be chosen as in the first step. We will show now that there exists a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ converging to φ in $L^1(\Omega)$ as $\varepsilon \searrow 0$ such that

$$\limsup_{\varepsilon \searrow 0} j_\varepsilon(\varphi_\varepsilon) \leq j_0(\varphi).$$

To this end, let $(\varphi_k)_{k\in\mathbb{N}}$ be the sequence approximating φ given by the first step, and for every $k \gg 1$ let $\left(\varphi_\varepsilon^k\right)_{\varepsilon>0}$ the sequences approximating φ_k as $\varepsilon \searrow 0$ given by the second step. Then we choose a diagonal sequence $\left(\varphi_{\varepsilon_k}^k\right)_{k\in\mathbb{N}}$ that converges to φ in $L^1(\Omega)$ and fulfills per construction

$$\limsup_{k\to\infty} \int_\Omega \left(\frac{\gamma\varepsilon_k}{2} \left|\nabla\varphi_{\varepsilon_k}^k\right|^2 + \frac{\gamma}{\varepsilon_k}\psi\left(\varphi_{\varepsilon_k}^k\right)\right) \mathrm{d}x \leq \gamma c_0 P_\Omega(E^\varphi)$$

which follows from (6.25) and $\lim_{k\to\infty} P_\Omega(E_k) = P_\Omega(E^\varphi)$. Besides, we conclude from (6.23) and (6.26) the following convergence rate

$$\left\| \varphi_{\varepsilon_k}^k - \varphi \right\|_{L^1(\Omega)} = \mathcal{O}\left(k^{-1}\right).$$

We continue with defining $\boldsymbol{u}_k = \boldsymbol{S}_{\varepsilon_k}\left(\varphi_{\varepsilon_k}^k\right)$ and see that $\boldsymbol{U}^\varphi \neq \emptyset$ since $j_0(\varphi) < \infty$. From Lemma 6.3 we thus get (after possibly choosing a subsequence) that $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{u} = \boldsymbol{S}_0(\varphi)$ and it holds

$$\lim_{k\to\infty} \int_\Omega \alpha_{\varepsilon_k}\left(\varphi_{\varepsilon_k}^k\right) |\boldsymbol{u}_k|^2 \,\mathrm{d}x = \int_\Omega \alpha_0(\varphi) |\boldsymbol{u}|^2 \,\mathrm{d}x = 0.$$

Using the continuity of the objective functional we end up with

$$\limsup_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}^k) \leq j_0(\varphi).$$

- *4th step:* Next we will show that for any sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)$ converging to an arbitrary element $\varphi \in L^1(\Omega)$ such that

$$\|\varphi_\varepsilon - \varphi\|_{L^1(\{x\in\Omega | \varphi(x)=1, \varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{6.27}$$

it holds

$$j_0(\varphi) \leq \liminf_{\varepsilon \searrow 0} j_\varepsilon(\varphi_\varepsilon).$$

Without loss of generality we assume $\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty$ and can therefore assume $\varphi \in BV(\Omega, \{\pm 1\})$ and $\int_\Omega \varphi \leq \beta|\Omega|$. Moreover we denote $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$.
From Assumption **(A5)** and $\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty$ we know that there exists a subsequence, denoted by the same, such that $\left(\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\right)_{\varepsilon>0}$ is bounded uniformly

in $\varepsilon > 0$. So we obtain for a subsequence, which is still indexed by $\varepsilon > 0$, that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some element $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Furthermore, we see that

$$\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty \implies \liminf_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x < \infty.$$

At the same time we can assume that (after choosing a subsequence) $(\varphi_\varepsilon)_{\varepsilon>0}$ and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converge pointwise almost everywhere in Ω, and as a consequence we get similar to (6.17) with Fatou's Lemma

$$\int_\Omega \alpha_0(\varphi)|\boldsymbol{u}|^2 \,\mathrm{d}x \le \liminf_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x < \infty$$

and thus in particular $\boldsymbol{u} = \boldsymbol{0}$ a.e. in $\Omega \setminus E^\varphi$ where we used $\lim_{\varepsilon\searrow 0}\alpha_\varepsilon(\varphi_\varepsilon(x)) = \alpha_0(\varphi(x))$ a.e. in Ω, which follows as in (6.8)-(6.9).
We have $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$, which gives us $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$, and as a consequence $\boldsymbol{u} \in \boldsymbol{U}$. Altogether this implies $\boldsymbol{u} \in \boldsymbol{U}^\varphi$, and thus $\boldsymbol{U}^\varphi \neq \emptyset$ together with $j_0(\varphi) < \infty$.

According to [Mod87, Proposition 1] we have, after rescaling in ε,

$$\gamma c_0 P_\Omega(E^\varphi) \le \liminf_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right) \mathrm{d}x.$$

After those preparation, we choose a subsequence $(j_{\varepsilon_k}(\varphi_{\varepsilon_k}))_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon).$$

We will now apply Lemma 6.3 to deduce the convergence of a subsequence of $(\boldsymbol{u}_{\varepsilon_k})_{k\in\mathbb{N}}$ in $\boldsymbol{H}^1(\Omega)$. For this purpose, we use in particular the convergence rate of $(\varphi_{\varepsilon_k})_{k\in\mathbb{N}}$ stated in (6.27). Thus, we obtain the existence of a subsequence $\left(\boldsymbol{u}_{\varepsilon_{k(l)}}\right)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty}\left\|\boldsymbol{u}_{\varepsilon_{k(l)}} - \boldsymbol{u}\right\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{l\to\infty}\int_\Omega \alpha_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right)\left|\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 \mathrm{d}x = 0$$

where $\boldsymbol{u} = \boldsymbol{S}_0(\varphi)$.
Plugging these results together we end up with

$$\begin{aligned} j_0(\varphi) &= \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \le \liminf_{l\to\infty} j_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right) = \\ &= \lim_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) \end{aligned}$$

and finish the fourth step.

- *5th step:* We use the results of the previous steps to finally prove the statement. First of all we see, that the existence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0} \subset \Phi_{ad}$ of $(j_\varepsilon)_{\varepsilon>0}$ with $j_\varepsilon(\varphi_\varepsilon) < \infty$ follows from Theorem 5.1.

Let now $\widetilde{\varphi}_\varepsilon \subseteq L^1(\Omega)$ be the sequence constructed in the third step corresponding to some arbitrary $\widetilde{\varphi} \in \Phi^0_{ad}$ (for instance we can take some element constructed in Lemma 6.2). Then, as we have shown, there exists a constant $C > 0$ independent of ε such that

$$j_\varepsilon\left(\widetilde{\varphi}_\varepsilon\right) < C.$$

Since φ_ε is a minimizer of j_ε for every $\varepsilon > 0$ we deduce

$$j_\varepsilon\left(\varphi_\varepsilon\right) \leq j_\varepsilon\left(\widetilde{\varphi}_\varepsilon\right) < C$$

and so we can conclude

$$\int_\Omega \left(\frac{\gamma\varepsilon}{2} \left|\nabla\varphi_\varepsilon\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right) \mathrm{d}x < C. \tag{6.28}$$

Using the arguments of [Mod87, Proposition 3, case a)], compare also [Ste88, Proposition 3, Remark (1.35)], we get from this uniform estimate that $\left(\varphi_\varepsilon\right)_{\varepsilon>0}$ has a subsequence that converges in $L^1(\Omega)$ to an element $\varphi_0 \in L^1(\Omega)$.

For the next step we assume that the sequence of minimizers $\left(\varphi_\varepsilon\right)_{\varepsilon>0}$ fulfills additionally (6.21). Then we see by the fourth step of this proof, that

$$j_0\left(\varphi_0\right) \leq \liminf_{\varepsilon\searrow 0} j_\varepsilon\left(\varphi_\varepsilon\right). \tag{6.29}$$

Taking another arbitrary admissible $\varphi \in L^1(\Omega)$, $j_0(\varphi) < \infty$, we find again by the third step of this proof, that there exists a sequence $\left(\widehat{\varphi}_\varepsilon\right)_{\varepsilon>0} \subset L^1(\Omega)$ converging in $L^1(\Omega)$ to φ as $\varepsilon \searrow 0$ such that

$$\limsup_{\varepsilon\searrow 0} j_\varepsilon\left(\widehat{\varphi}_\varepsilon\right) \leq j_0\left(\varphi\right).$$

And thus, by the minimizing property of φ_ε and (6.29), we end up with

$$j_0\left(\varphi_0\right) \leq \liminf_{\varepsilon\searrow 0} j_\varepsilon\left(\varphi_\varepsilon\right) \leq \limsup_{\varepsilon\searrow 0} j_\varepsilon\left(\widehat{\varphi}_\varepsilon\right) \leq j_0\left(\varphi\right) \tag{6.30}$$

which implies

$$j_0\left(\varphi_0\right) \leq j_0\left(\varphi\right) \quad \forall\varphi \in L^1(\Omega).$$

And thus φ_0 minimizes j_0. It remains to prove (6.22). But for this purpose we choose $\varphi \equiv \varphi_0$ in the previous considerations and obtain then from (6.30) that

$$j_0\left(\varphi_0\right) \leq \liminf_{\varepsilon\searrow 0} j_\varepsilon\left(\varphi_\varepsilon\right) \leq \limsup_{\varepsilon\searrow 0} j_\varepsilon\left(\widehat{\varphi}_\varepsilon\right) \leq j_0\left(\varphi_0\right) \tag{6.31}$$

and thus

$$\lim_{\varepsilon\searrow 0} j_\varepsilon\left(\varphi_\varepsilon\right) = j_0\left(\varphi_0\right).$$

This finally proves the statement of the theorem.

□

Remark 6.6. *In [Mod87, Ste88, BE91] the integral constraint reads* $\int_\Omega \varphi = \beta\,|\Omega|$*, instead of requiring* $\fint_\Omega \varphi\,\mathrm{d}x \leq \beta\,|\Omega|$*. The consequence is, that in this case* η_ε *is to be chosen between* $\left[0, \frac{\varepsilon\pi}{\sqrt{2}}\right]$ *in the second step of the proof of Theorem 6.1, and then the same result could be established.*

This shows that our phase field problem, which was introduced and discussed in Section 5, is an approximation of the sharp interface model (6.18)-(6.19) describing topology optimization in a Stokes flow in the sense, that minimizers of the phase field problem converge under certain assumptions to a minimizer of the sharp interface problem. In this situation, we then obtain as a consequence existence of a minimizer of (6.18)-(6.19). In Section 6.4 we will discuss whether we can expect to obtain a stronger result. In particular we will discuss condition (6.21) and the general well-posedness of the sharp interface problem.
But before starting this discussion, we will restrict in the next subsection to a special choice for the objective functional, namely the total potential power. We will find, that in this case we can even prove that the reduced objective functionals Γ-converge and we can drop the growth condition on the minimizers.

6.3 Minimizing the total potential power

Now we want to restrict to the special case of

$$f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) = \frac{\mu}{2} |\mathrm{D}\boldsymbol{u}|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u} \tag{6.32}$$

thus we minimize the total potential power, see also Example 2.3. Remark that in this setting (5.1) – (5.2) is equivalent to

$$\begin{aligned} \min_{(\varphi,\boldsymbol{u})\in\Phi_{ad}\times\boldsymbol{U}} J_\varepsilon(\varphi, \boldsymbol{u}) := \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi) |\boldsymbol{u}|^2 \,\mathrm{d}x + \int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}|^2 - \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}x + \\ + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x \end{aligned} \tag{6.33}$$

since the state equations (5.2) correspond then exactly to the necessary and sufficient first order optimality conditions for the convex minimization problem $\min_{\boldsymbol{u}\in\boldsymbol{U}} J_\varepsilon(\varphi, \cdot)$. Nevertheless, we want to stick to our notation and consider the reduced objective functional. Using this specific objective functional, we can generalise the result of Theorem 6.1 inasmuch as we can even prove that $(j_\varepsilon)_{\varepsilon>0}$ Γ-converges in $L^1(\Omega)$ to j_0.

Theorem 6.2. *Assume the objective functional is given by* (6.32). *Then the reduced objective functionals* $(j_\varepsilon)_{\varepsilon>0}$ *Γ-converge in* $L^1(\Omega)$ *to* j_0 *as* $\varepsilon \searrow 0$.

And as consequence we obtain directly the following results:

Corollary 6.1. *Assume the objective functional is given by* (6.32) *and that* $(\varphi_\varepsilon)_{\varepsilon>0}$ *are minimizers of* $(j_\varepsilon)_{\varepsilon>0}$. *Then there exists a subsequence of* $(\varphi_\varepsilon)_{\varepsilon>0}$, *which is denoted by the same, and an element* $\varphi_0 \in L^1(\Omega)$, *such that*

$$\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \qquad \lim_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi_0)$$

and moreover φ_0 *is a minimizer of* j_0.

Remark 6.7. *We remark that for this objective functional, we obtain in particular the existence of a minimizer of the sharp interface problem* (6.18)-(6.19) *without any additional assumption.*

We start by proving that Corollary 6.1 is a direct consequence of Theorem 6.2.

Proof of Corollary 6.1. We can use the same compactness result as we have established in the 5th step of the proof of Theorem 6.1 to deduce the existence of a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges in $L^1(\Omega)$. After that, we obtain by the Γ-convergence result of Theorem 6.2 and standard results on Γ-convergence, compare Theorem 3.3, directly the statement of the Corollary. □

Now we proof the stated theorem:

Proof of Theorem 6.2. Again we want to split the proof into two steps, namely we first prove the existence of a recovery sequence and show afterwards the lower semicontinuity property.

- *1st step:* Assume we have an arbitrary $\varphi \in L^1(\Omega)$ such that $j_0(\varphi) < \infty$. Then we can follow the arguments of the third step in the proof of Theorem 6.1 to obtain a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ such that

$$\limsup_{\varepsilon \searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi).$$

- *2nd step:* Next take an arbitrary sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ converging to an element $\varphi \in L^1(\Omega)$. We want to show, that

$$j_0(\varphi) \le \liminf_{\varepsilon \searrow 0} j_\varepsilon(\varphi_\varepsilon).$$

  This is the crucial difference to Theorem 6.1. Since we do not assume any additional conditions on the sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ we cannot apply Lemma 6.3.

  But by the same method as in the fourth step of the proof of Theorem 6.1 we find that we may assume without loss of generality $\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty$, and thus we find that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $\boldsymbol{H}^1(\Omega)$, where we denote $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$. By [Mod87, Proposition 1] we see that

$$\gamma c_0 P_\Omega(E^\varphi) \le \liminf_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right) \mathrm{d}x.$$

  Now we choose a subsequence $(j_{\varepsilon_k}(\varphi_{\varepsilon_k}))_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon).$$

  Since $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ are uniformly bounded in $\boldsymbol{H}^1(\Omega)$ we find that there exists a subsequence $\left(\boldsymbol{u}_{\varepsilon_{k(l)}}\right)_{l\in\mathbb{N}}$ of $(\boldsymbol{u}_{\varepsilon_k})_{k\in\mathbb{N}}$ that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\boldsymbol{u} \in \boldsymbol{U}$. As we find by Fatou's lemma that

$$\int_\Omega \alpha_0(\varphi)|\boldsymbol{u}|^2 \,\mathrm{d}x \le \liminf_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x < \infty \tag{6.34}$$

  we obtain moreover $\boldsymbol{u} \in \boldsymbol{U}^\varphi$.

  Denoting $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi)$ we see

$$\boldsymbol{u}_0 = \mathrm{argmin}_{\boldsymbol{v}\in\boldsymbol{U}^\varphi} \int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 - \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x$$

  and moreover, by the weak convergence of $\left(\boldsymbol{u}_{\varepsilon_{k(l)}}\right)_{l\in\mathbb{N}}$ to $\boldsymbol{u} \in \boldsymbol{U}^\varphi$ and making use of (6.34),

$$\int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x \le \liminf_{l\to\infty}\left[\int_\Omega \alpha_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right)\left|\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 + \frac{\mu}{2}\left|\nabla\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_{\varepsilon_{k(l)}}\,\mathrm{d}x\right]$$

  and thus

$$\int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}_0|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_0\,\mathrm{d}x \le \int_\Omega \frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x \le$$
$$\le \liminf_{l\to\infty}\left[\int_\Omega \alpha_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right)\left|\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 + \frac{\mu}{2}\left|\nabla\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_{\varepsilon_{k(l)}}\,\mathrm{d}x\right].$$

Plugging those results together we find

$$j_0(\varphi) = \int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}_0|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_0 \,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \leq \liminf_{l\to\infty} j_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right) =$$
$$= \lim_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon)$$

and can thus finish the second step.

In particular, this shows the statement of the theorem. □

6.4 Further discussions on possible generalizations

In Theorem 6.1 we had to make an assumption on the convergence rate with respect to $\varepsilon > 0$ of the converging subsequence of the sequence of minimizers of $(j_\varepsilon)_{\varepsilon>0}$. We want to discuss now what happens for a general converging subsequence of minimizers, denoted by $(\varphi_\varepsilon)_{\varepsilon>0}$, converging to some limit element φ_0. Then, following the arguments of Theorem 6.1, we still obtain that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$, where $\boldsymbol{u}_\varepsilon := \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$, converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\boldsymbol{u} \in \boldsymbol{U}^{\varphi_0}$ and

$$J_0(\varphi_0, \boldsymbol{u}) \le \liminf_{\varepsilon \searrow 0} J_\varepsilon(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon).$$

To deduce therefrom, that $j_0(\varphi_0) \le \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon)$, we still have to show, that $\boldsymbol{u} = \boldsymbol{S}_0(\varphi_0)$. But with regard to the stated convergence results and the state equations corresponding to $\boldsymbol{u}_\varepsilon$ we only obtain

$$\Lambda(\boldsymbol{v}) + \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

where $\Lambda \in \boldsymbol{V}'$ is given by $\Lambda(\boldsymbol{v}) := \lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v}\,\mathrm{d}x$ for all $\boldsymbol{v} \in \boldsymbol{V}$. To show that $\boldsymbol{u} = \boldsymbol{S}_0(\varphi_0)$ it thus remains to show, that $\Lambda(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}^{\varphi_0}$.

To get at least an impression of what Λ might be, we assume for simplicity for the following discussion that the state equations are given by the Laplace equation instead of the Stokes system and assume homogeneous boundary data. Thus $\boldsymbol{u}_\varepsilon \in \boldsymbol{H}^1_0(\Omega)$ is assumed to solve

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v}\,\mathrm{d}x + \int_\Omega \nabla \boldsymbol{u}_\varepsilon \cdot \nabla \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega). \tag{6.35}$$

Besides, we still assume that $\boldsymbol{u}_\varepsilon$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some element $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$ and deduce moreover from $\sup_{\varepsilon>0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x < \infty$ as in Theorem 6.1 that $\boldsymbol{u} = \boldsymbol{0}$ a.e. in $\{\varphi_0 = -1\}$. Besides, we can prove as in (6.7) – (6.9) that we have the pointwise convergence $\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon) = \alpha_0(\varphi_0)$.

Applying the idea of [Gri11, Section 2.3.1], we define $\boldsymbol{u}^\kappa_\varepsilon := \left(|\boldsymbol{u}_\varepsilon|^2 + \kappa\right)^{-\frac{1}{2}} \boldsymbol{u}_\varepsilon$, where $\kappa > 0$ is an arbitrary constant and notice, that $\boldsymbol{u}^\kappa_\varepsilon \in \boldsymbol{H}^1(\Omega)$. Then we insert $\boldsymbol{v} \equiv \boldsymbol{u}^\kappa_\varepsilon$ as a test function into (6.35) and obtain

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}^\kappa_\varepsilon\,\mathrm{d}x + \int_\Omega \underbrace{|\nabla \boldsymbol{u}_\varepsilon|^2 \frac{\kappa}{\left(|\boldsymbol{u}_\varepsilon|^2 + \kappa\right)^{\frac{3}{2}}}}_{\ge 0}\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}^\kappa_\varepsilon\,\mathrm{d}x \le \|\boldsymbol{f}\|_{\boldsymbol{L}^1(\Omega)} \underbrace{\|\boldsymbol{u}^\kappa_\varepsilon\|_{\boldsymbol{L}^\infty(\Omega)}}_{\le 1} \tag{6.36}$$

wherefrom we obtain that

$$\sup_{\varepsilon,\kappa>0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}^\kappa_\varepsilon\,\mathrm{d}x < \infty.$$

Since moreover $\|\boldsymbol{u}^\kappa_\varepsilon\|_{\boldsymbol{L}^\infty(\Omega)} \le 1$, we obtain that there exists a subsequence of $(\boldsymbol{u}^\kappa_\varepsilon)_{\kappa>0}$, denoted by the same, that converges weakly-$*$ in $\boldsymbol{L}^\infty(\Omega)$ as $\kappa \searrow 0$. As the limit element coincides with the pointwise limit, we deduce that $(\boldsymbol{u}_\varepsilon^\kappa)_{\kappa>0}$ converges weakly-$*$ in $\boldsymbol{L}^\infty(\Omega)$ to

$$\Omega \ni x \mapsto \begin{cases} \frac{\boldsymbol{u}_\varepsilon}{|\boldsymbol{u}_\varepsilon|}, & \text{if } \boldsymbol{u}_\varepsilon \neq \boldsymbol{0}, \\ \boldsymbol{0}, & \text{else} \end{cases}$$

as $\kappa \searrow 0$. And so we find:

$$\sup_{\varepsilon>0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon| \, \mathrm{d}x = \sup_{\varepsilon>0} \lim_{\kappa\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}_\varepsilon^\kappa \, \mathrm{d}x < \infty. \tag{6.37}$$

Now we define the measures $m_\varepsilon \in \mathcal{M}(\Omega)$ by

$$m_\varepsilon(F) := \int_F \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon| \, \mathrm{d}x$$

where $\mathcal{M}(\Omega)$ is the space of Radon measures, cf. [AFP00, Definition 1.40]. Then we can deduce from (6.37) and the weak-$*$ compactness cirterion for finite Radon measures given by [AFP00, Theorem 1.59] that there exists a subsequence, which will be denoted by the same, which converges weakly-$*$ in the space $\mathcal{M}(\Omega)$ to a Radon measure $m_0 \in \mathcal{M}(\Omega)$, which means

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon| \zeta \, \mathrm{d}x = \lim_{\varepsilon\searrow 0} \int_\Omega \zeta \, \mathrm{d}m_\varepsilon = \lim_{\varepsilon\searrow 0} \int_\Omega \zeta \, \mathrm{d}m_0 \quad \forall \zeta \in C_0(\Omega).$$

Now we apply Egorov's theorem, see for instance [AFP00, Theorem 1.34], to find from the pointwise convergence of $\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon$ to zero that there exists for all $\tau > 0$ some open subset $E_\tau \subset \Omega$ such that $(\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges uniformly on E_τ to zero. Using the lower semicontinuity of $\mathcal{M}(\Omega) \ni \nu \mapsto |\nu|(A)$ for any open set A with respect to the weak-$*$ convergence of Radon measures, cf. [AFP00, Corollary 1.60], we can deduce from

$$\lim_{\varepsilon\searrow 0} m_\varepsilon(E_\tau) = 0 \quad \forall \tau > 0$$

that

$$m_0(E_\tau) \le \liminf_{\varepsilon\searrow 0} m_\varepsilon(E_\tau) = 0$$

and thus $m_0(E_\tau) = 0$. Using finally the σ-additivity of the measure m_0 we arrive in

$$m_0\left(\bigcup_\tau E_\tau\right) \le \sum_\tau m_0(E_\tau) = 0.$$

Now we define $\widetilde{E}^{\varphi_0} := \bigcup_\tau E_\tau \cap \operatorname{int}(E^{\varphi_0})$, which differs from another representative of E^{φ_0} only by a set of zero $\mathcal{L}^d$-measure. Then we find

$$m_0\left(\widetilde{E}^{\varphi_0}\right) = 0.$$

Using the weak-$*$-convergence of $(m_\varepsilon)_{\varepsilon>0}$ in $\mathcal{M}(\Omega)$ we can deduce from this

$$\begin{aligned} \left|\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \, \mathrm{d}x\right| &\le \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon| |\boldsymbol{v}| \, \mathrm{d}x = \int_\Omega |\boldsymbol{v}| \, \mathrm{d}m_\varepsilon \xrightarrow{\varepsilon\searrow 0} \int_\Omega |\boldsymbol{v}| \, \mathrm{d}m_0 = \\ &= \int_{\widetilde{E}^{\varphi_0}} |\boldsymbol{v}| \, \mathrm{d}m_0 = 0 \end{aligned}$$

which holds for any $\boldsymbol{v} \in \boldsymbol{C}_0(\Omega)$ with $\boldsymbol{v}|_{\Omega \setminus \widetilde{E}^{\varphi_0}} = \mathbf{0}$. This yields

$$\lim_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{C}_0(\Omega), \boldsymbol{v}|_{\Omega \setminus \widetilde{E}^{\varphi_0}} = \mathbf{0}.$$

Thus, we can deduce from the state equation (6.35) that the weak $\boldsymbol{H}_0^1(\Omega)$ limit $\boldsymbol{u}$ of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ fulfills the following equation

$$\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{C}_0(\Omega),\, \boldsymbol{v}|_{\Omega \setminus \widetilde{E}^{\varphi_0}} = \mathbf{0}. \tag{6.38}$$

As $\widetilde{E}^{\varphi_0}$ is an open set, we can define as usual $\boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}\right)$ as the closure of $\boldsymbol{C}_0^\infty\left(\widetilde{E}^{\varphi_0}\right)$ with respect to the $\boldsymbol{H}^1(\Omega)$-norm. Denoting by $e_0 : \boldsymbol{C}_0^\infty\left(\widetilde{E}^{\varphi_0}\right) \to \boldsymbol{C}_0^\infty(\Omega)$ the extension by zero, we then see that we can extend e_0 uniquely to $\boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}\right)$. Then we define as in [DZ01] the following space

$$\boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}, \Omega\right) \coloneqq e_0\left(\boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}\right)\right).$$

Using this notation, we obtain from (6.38) that it holds

$$\int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}, \Omega\right). \tag{6.39}$$

We recall, that the state equation in our sharp interface formulation would require that the preceeding identity holds for all $\boldsymbol{v} \in \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \mathbf{0} \text{ a.e. in } \Omega \setminus E^{\varphi_0}\right\}$, but in general we have the strict inclusion

$$\boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}, \Omega\right) \subsetneq \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \mathbf{0} \text{ a.e. in } \Omega \setminus E^{\varphi_0}\right\}$$

even though $\widetilde{E}^{\varphi_0}$ is a representative of E^{φ_0}. More precisely, we obtain from [DZ01, Theorem 6.2, Chapter 8] that

$$\boldsymbol{H}_0^1\left(\widetilde{E}^{\varphi_0}, \Omega\right) = \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \mathbf{0} \text{ q.e. in } \Omega \setminus \widetilde{E}^{\varphi_0}\right\}.$$

Here, $\boldsymbol{q} = \mathbf{0}$ q.e. (quasi-everywhere) in $\Omega \setminus \widetilde{E}^{\varphi_0}$ means that the set $\{x \in \Omega \setminus \widetilde{E}^{\varphi_0} \mid \boldsymbol{q}(x) \neq \mathbf{0}\}$ has capacity zero. The notion of capacity and some of its properties are briefly discussed below.
Thus we obtain for this setting, that $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ fulfills $\boldsymbol{u} = \mathbf{0}$ a.e. in $\Omega \setminus E^{\varphi_0}$ and besides

$$\int_\Omega \boldsymbol{v} \,\mathrm{d}m_0 + \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \tag{6.40}$$

where $m_0|_{E^{\varphi_0}}$ is a Radon measure which is singular to the d-dimensional Lebesgue measure.

Phenomena like this have already been observed in literature when considering the pure shape optimization problem in the sharp interface formulation. We want to illustrate this on the classical example of periodically perforated domains with suitable critical size described in [CM82, Example 2.9] in the slightly modified version of [DMM87]. For this purpose, let $d = 2$ and Ω is the ball around zero with radius 2. We define for a given constant $c > 0$ and any $n \in \mathbb{N}$ the sets

$$F_n \coloneqq \overline{\bigcup_{i \in \mathbb{N}} F_n^i}$$

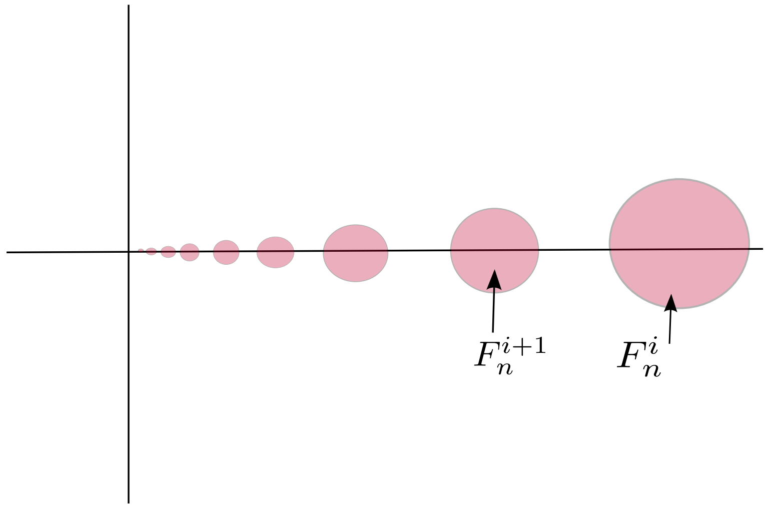


Figure 7: Sketch of F_n^i for some fixed n

where

$$F_n^i := \left\{ x \in \mathbb{R}^d \mid \left\| x - \left(\exp\left(-\frac{i}{h} \right), 0 \right) \right\| \leq \exp\left(-\frac{i}{h} - \frac{\pi h}{c} \right) \right\},$$

see Figure 7 for a sketch of those sets.

In particular we define $E_n := \Omega \setminus F_n$, which is then the set in which we want to solve our state equation. We see that

$$P_\Omega(E_n) = \sum_{i=1}^{\infty} 2\pi \exp\left(-\frac{i}{n} - \frac{\pi n}{c} \right) = \frac{2\pi \exp^{-\pi n/c}}{-1 + \exp^{n^{-1}}} \xrightarrow{n \to \infty} 0,$$

and

$$\|\chi_\Omega - \chi_{E_n}\|_{L^1(\Omega)} = \bigcup_{i \in \mathbb{N}} \int_{E_n^i} 1 \,\mathrm{d}x = \frac{\pi \exp\left(-2\pi n/c\right)}{-1 + \exp^{2/n}} \xrightarrow{n \to \infty} 0. \tag{6.41}$$

It was shown in [DMM87, Example 2.2] that for given $\boldsymbol{g} \in \boldsymbol{H}^1(\Omega)$ solutions of the Dirichlet problem

$$\begin{aligned} -\Delta \boldsymbol{w}_n &= \mathbf{0} && \text{in } E_n, \\ \boldsymbol{w}_n &= \boldsymbol{g} && \text{on } \partial\Omega, \\ \boldsymbol{w}_n &= \mathbf{0} && \text{on } \partial E_n \cap \Omega, \end{aligned}$$

converge weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$. Moreover, the limit element $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ is given as the weak solution of

$$\begin{aligned} \nu \boldsymbol{w} - \Delta \boldsymbol{w} &= \mathbf{0} && \text{in } \Omega, \\ \boldsymbol{w} &= \boldsymbol{g} && \text{on } \partial\Omega, \end{aligned}$$

in the following sense: $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{L}^2(\Omega, \nu)$, $\boldsymbol{w} - \boldsymbol{g} \in \boldsymbol{H}_0^1(\Omega)$ and

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{v} \,\mathrm{d}\nu + \int_\Omega \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^2(\Omega, \nu).$$

Here we denote by $\boldsymbol{L}^2(\Omega,\nu)$ the Lebesgue space of order two with respect to the measure ν, which is defined by

$$\nu(F) = \int_{F_S} \frac{c}{x_1}\,\mathrm{d}x_1, \qquad F_S := \{x_1 \in (0,1) \mid (x_1,0) \in F\}\,.$$

Here we find again, that ν is a measure, which is singular to the two-dimensional Lebesgue measure. Defining now

$$\widetilde{\Omega} := \Omega \setminus \{(x_1,0) \mid x_1 \in (0,1)\}$$

we find directly, using the notation introduced above, that

$$\int_\Omega \nabla \boldsymbol{w} \cdot \nabla \boldsymbol{v}\,\mathrm{d}x = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{H}_0^1\left(\widetilde{\Omega},\Omega\right)$$

thus the test functions here have in particular zero boundary conditions on the positive x-axis. We observe, that this is a comparable situation to the state equations that we obtained as a limit system using the phase field approximation, see (6.39) and (6.40).

In the example of perforated holes described above, we obtain that the limit function solves not the Laplace equation, but the Laplace equation is disturbed by a measure which has support on a set with zero Lebesgue measure, but with positive capacity. For compact sets $K \subset \Omega$ the (harmonic) capacity of K with respect to Ω is given by

$$cap_\Omega(K) := \inf\left\{ \int_\Omega |\nabla\zeta|^2\,\mathrm{d}x \mid \zeta \in C_0^\infty(\Omega), \zeta \geq 1 \text{ on } K \right\}.$$

This definition can be extended to arbitrary sets, see for instance [EG92]. We say that a set $E \subset \mathbb{R}^d$ has zero capacity if $cap_\Omega(E\cap\Omega) = 0$, and we say that a property holds quasi everywhere (q.e.) in E if it holds for all $x \in E$ except for a subset of E with capacity zero. We do not want to go into detail to capacity theory and refer the reader for instance to [EG92, DMM87, BDM93] for further discussions on this topic. We just remark that zero capacity implies that the set also has Lebesgue measure zero, but this does not hold true conversely. Thus describing sets up to zero capacity gives in general more information, and actually the measures that appear as perturbance of the Dirichlet problems in shape optimization can be shown to vanish on every set of zero capacity, see for instance [BDM91] and references therein.

We have to remark that a behaviour as described above leads to problems concerning the analysis of the sharp interface formulation of the shape optimization problem (6.18)-(6.19). Thus even the existence of a minimizer in this general setting is still an open problem, compare also discussion in [BG04]. Even for the Laplace equation with Dirichlet boundary conditions there are still open questions concerning well-posedness of the shape optimization problem in a general setting as described in (6.18)-(6.19). We refer for instance to [BZ95], where the existence of optimal shapes for certain objective functionals with an elliptic system as a constraint has been shown in the class of subsets fulfilling a capacitary constraint called (r,c) capacity density condition. In [BDM93] the shape optimization problem is solved in the class of quasi open sets, but in particular a monotonicity with respect to set inclusions for the objective functional has to be assumed. Even for shape optimization problems with a perimeter constraint there are to the author's knowledge so far only results under restrictions on the objective functional, such as being decreasing with respect to set inclusions, see for instance [BBH09]. In our setting, the natural objective functional fulfilling a certain monotonicity property is the total potential power which is considered in Section 6.3. And in this setting, as shown in Section 6.3, we actually can generalize the result of Theorem 6.1.

One possibility to overcome the problem of ill-posed formulations is the introduction of a relaxed formulation. This has been done for instance in [BDM91] where the problem of solving

$$\min_{A\subset\Omega, A \text{ open}} \int_\Omega f\left(x, u_A(x)\right) \mathrm{d}x$$

$$\text{s.t. } u_A \in H_0^1(A), \quad -\Delta u_A = f \text{ in } A$$

is, roughly speaking, replaced by

$$\min_{\nu\in\mathcal{M}_0(\Omega)} \int_\Omega f\left(x, u_\nu(x)\right) \mathrm{d}x$$

$$\text{s.t. } u_\nu \in H_0^1(\Omega)\cap L^2(\Omega,\nu), \quad u_\nu\nu - \Delta u_\nu = f \text{ in } \left(H_0^1(\Omega)\cap L^2(\Omega,\nu)\right)'$$

where $\mathcal{M}_0(\Omega)$ here denotes the set of all nonnegative Borel measures on Ω that vanish on sets with capacity zero. Then the problem is well-posed, and if the original problem has a solution, there is a correspondence to a measure which is then also a solution of the relaxed problem.

Now all of those results mentioned above have one main ingredient in common: they all make essential use of elements from capacity theory. As discussed above, even with our approach we could end up with state equations that need capacitary quantities to be described correctly. But by the formulation (6.18)-(6.19) we only control our admissible sets up to sets of Lebesgue measure zero, whereas we cannot control sets with zero Lebesgue measure but positive capacity. Thus it seems like a possible extension of our results have to involve elements from capacity theory. But it is not clear how to couple the Caccioppoli setting or even the phase field formulation to relaxations as in [BDM91] or capacitary constraints as in [BZ95].

Additionally, even though we have to make a certain assumption to get the desired results, this may not always be too restrictive. We will always obtain that the sequence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0}$ of the phase field problems has a subsequence that converges in $L^1(\Omega)$ to some limit element $\varphi_0 \in L^1(\Omega)$. Then we have to assume in general that this convergence takes place at a certain rate depending on $\varepsilon > 0$, in particular we need for an arbitrary objective functional additionally

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}\left(\varepsilon\right).$$

But this convergence rate actually is a natural one, as the so called “optimal profiles”, which are used as recovery sequences for proving Γ-convergence of the Ginzburg-Landau energy and can also be found in the proof of Theorem 6.1, fulfill this convergence rate, q.v. Theorem 6.1. Moreover, comparable convergence rates can often be observed in numerical simulations, more precisely the thickness of the interface behaves as $\mathcal{O}(\varepsilon)$.

7 Optimality conditions for the phase field model

We come back to discuss the phase field model (5.1) – (5.2). Therefore, we choose again some arbitrary $\varepsilon > 0$. For this section assume that $\varphi_\varepsilon \in \Phi_{ad}$ is a fixed minimizer of (5.1) – (5.2), which exists due to Theorem 5.1. Furthermore, let us denote by $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ the solution to the penalized Stokes equation (5.2) corresponding to φ_ε. The aim will be to derive first order necessary optimality conditions for (5.1) – (5.2) which are fulfilled in φ_ε.
To this end, we start by deriving the "classical" variational inequality in the adjoint formulation in Section 7.1, since the phase field model stated in the form (5.1) – (5.2) can be considered as an optimal control problem. For a fuller treatment of the general theory of optimal control problems we refer for instance to [Trö09]. But then we also consider geometric variations, thus variations of the region filled with fluid by suitable transformations, since this will also be the approach for deriving optimality conditions in the sharp interface model and we want to compare later on both optimality systems, see Sections 8 and 9.

Throughout this section we state additionally Assumptions **(A6)** and **(A7)**, which ensure differentiability of the objective functional and enough regularity on the external force term.

7.1 Variational inequality

As already mentioned above, we will start by considering (5.1) – (5.2) as a classical optimal control problem and derive optimality conditions by varying φ in Φ_{ad} to arrive in a variational inequality.

Therefore, we first of all show a kind of Lipschitz continuity of the solution operator $\boldsymbol{S}_\varepsilon$:

Lemma 7.1. *Assume that $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ solves (5.2) for the fixed minimizer $\varphi_\varepsilon \in \Phi_{ad}$. Then there exists some $L = L(\Omega, \boldsymbol{g}, \boldsymbol{f}, \alpha_\varepsilon, \mu) > 0$ such that*

$$\|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \le L \|\varphi - \varphi_\varepsilon\|_{L^\infty(\Omega)} \qquad \forall \varphi \in \overline{\Phi}_{ad}, \boldsymbol{u} = \boldsymbol{S}_\varepsilon(\varphi).$$

Proof. We choose some $\varphi \in \overline{\Phi}_{ad}$ and denote $\boldsymbol{u} = \boldsymbol{S}_\varepsilon(\varphi)$. Subtracting (5.2) corresponding to $\boldsymbol{u}_\varepsilon$ from (5.2) corresponding to $\boldsymbol{u}$ and testing with $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{u}_\varepsilon \in \boldsymbol{V}$ yields

$$(\alpha_\varepsilon(\varphi)\boldsymbol{u} - \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \mu(\nabla \boldsymbol{v}, \nabla \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = 0$$

and so we have

$$\mu \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} = -(\alpha_\varepsilon(\varphi)\boldsymbol{u} - \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}. \tag{7.1}$$

Since α_ε is twice continuously differentiable, we can use the Taylor expansion pointwise almost everywhere to conclude that there exists for almost every $x \in \Omega$ some ξ_x between $\varphi_\varepsilon(x)$ and $\varphi(x)$ such that

$$\alpha_\varepsilon(\varphi_\varepsilon(x)) = \alpha_\varepsilon(\varphi(x)) + \alpha'_\varepsilon(\varphi(x))(\varphi_\varepsilon(x) - \varphi(x)) + \frac{1}{2}\alpha''_\varepsilon(\xi_x)(\varphi_\varepsilon(x) - \varphi(x))^2.$$

Inserting this into (7.1) we get

$$\mu \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \le \underbrace{-\left(\alpha_\varepsilon(\varphi)\boldsymbol{v},\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}}_{\le 0} + (\underbrace{\alpha'_\varepsilon(\varphi)}_{|\cdot|\le C}(\varphi_\varepsilon-\varphi)\boldsymbol{u}_\varepsilon,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \\ + (\underbrace{\frac{1}{2}\alpha''_\varepsilon(\xi_x)}_{|\cdot|\le C}(\varphi_\varepsilon-\varphi)^2\boldsymbol{u}_\varepsilon,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}. \tag{7.2}$$

In the next step we use Hölder's inequality and the boundedness of $\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}$ by some constant only depending on ε, Ω and the given data, which is given for example by using (5.4), and arrive in

$$\mu \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \le C\left(\|\varphi-\varphi_\varepsilon\|_{L^\infty(\Omega)} + \|\varphi-\varphi_\varepsilon\|^2_{L^\infty(\Omega)}\right)\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

Due to $\boldsymbol{v}\in\boldsymbol{H}^1_0(\Omega)$ we can use Poincaré's inequality to conclude

$$\mu \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \le C\|\varphi-\varphi_\varepsilon\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$$

where we made use of the pointwise constraints $|\varphi_\varepsilon|\le 1$, $|\varphi|\le 1$. This gives

$$\|\nabla\boldsymbol{u}-\nabla\boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} = \|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le C\|\varphi-\varphi_\varepsilon\|_{L^\infty(\Omega)}$$

and proves the statement.

□

As we know that $\varphi_\varepsilon\in\Phi_{ad}$ is a minimizer of j_ε and Φ_{ad} is a convex set we see that it holds

$$j_\varepsilon(\varphi_\varepsilon)\le j_\varepsilon(\varphi_\varepsilon+t(\varphi-\varphi_\varepsilon)) \quad \forall\varphi\in\Phi_{ad},\, 0<t\ll 1.$$

This leads to first order optimality conditions for (5.1) – (5.2) in form of a variational inequality, which are standard for optimal control problems, see [Trö09], if $\boldsymbol{S}_\varepsilon$ is differentiable at φ_ε in direction $(\varphi-\varphi_\varepsilon)$. The remainder of this subsection will be devoted to the derivation of the exact formulation of this first order optimality conditions with the help of an adjoint variable.

We start these considerations by differentiating the solution operator $\boldsymbol{S}_\varepsilon$:

Lemma 7.2. *Let $\varphi\in\overline{\Phi}_{ad}$. Then the directional derivative*

$$\partial_t|_{t=0}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon+t(\varphi-\varphi_\varepsilon)) = \mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) =: \boldsymbol{u}\in\boldsymbol{V}$$

exists in $\boldsymbol{H}^1(\Omega)$, is well-defined, and is given as the unique weak solution to

$$\begin{aligned} \alpha'_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\boldsymbol{u}_\varepsilon+\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}-\mu\Delta\boldsymbol{u}+\nabla p &= \boldsymbol{0} && \textit{in } \Omega, && (7.3a)\\ \operatorname{div}\boldsymbol{u} &= 0 && \textit{in } \Omega, && (7.3b)\\ \boldsymbol{u} &= \boldsymbol{0} && \textit{on } \partial\Omega, && (7.3c)\end{aligned}$$

where $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$. Here we denote by $\partial_t|_{t=0}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon+t(\varphi-\varphi_\varepsilon)) = \lim_{t\to 0,t>0}\frac{1}{t}\left(\boldsymbol{S}_\varepsilon(\varphi_\varepsilon+t(\varphi-\varphi_\varepsilon))-\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)\right)$ the one-sided directional derivative.

Proof. We split the proof into several steps:

- *1st step:* We show that there exists a unique weak solution of (7.3).
For this purpose, we define the bilinear form $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ as follows

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, \mathrm{d}x.$$

This bilinear form is due to Poincaré's inequality coercive and continuous on $\boldsymbol{V} \times \boldsymbol{V} \subseteq \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$, since $\alpha_\varepsilon \geq 0$.
Applying Lax-Milgram's theorem A.2, we can deduce existence and uniqueness of a solution $\boldsymbol{u} \in \boldsymbol{V}$ for the following equation:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \left(-\alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

which fulfills moreover the following a priori estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{1}{c} \left\|\alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{L}^2(\Omega)}$$

with $c = c(\mu, \Omega)$. From (5.4) this can be estimated by

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega, \mu, \alpha_\varepsilon, \boldsymbol{g}, \boldsymbol{f}). \tag{7.4}$$

- *2nd step:* We now show that $\mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) = \partial_t|_{t=0}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon))$ is well-defined and given by (7.3).

Let us first consider the mapping $H^1(\Omega) \ni \varphi \mapsto \alpha_\varepsilon(\varphi) \in L^3(\Omega)$ as a Nemytskii operator. Therefore, we extend $\alpha_\varepsilon : [-1,1] \to [0, \overline{\alpha}_\varepsilon]$ to a continuous differentiable function defined on $\mathbb{R}$ such that $\alpha_\varepsilon, \alpha_\varepsilon' \in L^\infty(\mathbb{R})$. In order to show that this operator is Fréchet differentiable we want to apply the statements of [AZ90] and note that, since α_ε and α_ε' are uniformly bounded,

$$L^6(\Omega) \ni h \mapsto \alpha_\varepsilon'(\varphi)\, h \in L^6(\Omega)$$

is a well-defined operator from $H^1(\Omega) \subseteq L^6(\Omega)$ to $L^6(\Omega)$ for every $\varphi \in H^1(\Omega)$. And so the results from [AZ90] give that α_ε defines a Fréchet differentiable Nemytskii operator as a mapping from $H^1(\Omega)$ into $L^3(\Omega)$. This yields for arbitrary $\varphi \in H^1(\Omega)$ the existence of some r_α such that

$$\begin{aligned} &\alpha_\varepsilon(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon)) = \alpha_\varepsilon(\varphi_\varepsilon) + t\alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) + r_\alpha(t) \\ &\text{with } \lim_{t \searrow 0} \frac{\|r_\alpha(t)\|_{L^3(\Omega)}}{t} = 0. \end{aligned} \tag{7.5}$$

We introduce for fixed $\varphi \in \overline{\Phi}_{ad}$ and $t \in (0,1)$ the notation

$$\widetilde{\boldsymbol{u}}_t := \boldsymbol{S}_\varepsilon(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon)), \quad \boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$$

and

$$\widehat{\boldsymbol{u}}_t := \frac{1}{t}(\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon) - \boldsymbol{u} \in \boldsymbol{V}$$

where $\boldsymbol{u}$ denotes the solution of (7.3).
From (5.4) we arrive in

$$\|\widetilde{\boldsymbol{u}}_t\|_{\boldsymbol{H}^1(\Omega)} \le C\left(\Omega, \mu, \boldsymbol{g}, \boldsymbol{f}, \overline{\alpha}_\varepsilon\right). \tag{7.6}$$

We are reduced to proving $\|\widehat{\boldsymbol{u}}_t\|_{\boldsymbol{H}^1(\Omega)} \to 0$ for $t \searrow 0$ for establishing the statement of the lemma.
Therefore, we use (7.5) to conclude

$$\frac{1}{t}\left(\underbrace{\alpha_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right)}_{=\alpha_\varepsilon(\varphi_\varepsilon)+\alpha_\varepsilon'(\varphi_\varepsilon)t(\varphi-\varphi_\varepsilon)+r_\alpha(t)}\widetilde{\boldsymbol{u}}_t - \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\right) - \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon - \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u} =$$
$$= \alpha_\varepsilon\left(\varphi_\varepsilon\right)\widehat{\boldsymbol{u}}_t + \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\widehat{\boldsymbol{u}}_t + \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\boldsymbol{u} + \frac{r_\alpha(t)}{t}\widetilde{\boldsymbol{u}}_t$$

and deduce that $\widehat{\boldsymbol{u}}_t$ fulfills

$$\begin{aligned}\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\widehat{\boldsymbol{u}}_t\cdot\boldsymbol{v} + \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\widehat{\boldsymbol{u}}_t\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\widehat{\boldsymbol{u}}_t\cdot\nabla\boldsymbol{v}\,\mathrm{d}x =\\ = \int_\Omega -\alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\boldsymbol{u}\cdot\boldsymbol{v} - \frac{r_\alpha(t)}{t}\widetilde{\boldsymbol{u}}_t\cdot\boldsymbol{v}\,\mathrm{d}x\end{aligned} \tag{7.7}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$. Considering

$$\boldsymbol{h}(t) := -\alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\boldsymbol{u} - \frac{1}{t}r_\alpha(t)\,\widetilde{\boldsymbol{u}}_t \in \boldsymbol{H}^{-1}(\Omega)$$

we will show in the next step that $\|\boldsymbol{h}(t)\|_{\boldsymbol{H}^{-1}(\Omega)} \to 0$ as $t \searrow 0$. For this reason we use that $\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ is bounded by a constant $c = c\left(\Omega, \mu, \alpha_\varepsilon, \boldsymbol{g}\right)$, see (7.4), which gives

$$\left\|\alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\boldsymbol{u}\right\|_{\boldsymbol{L}^2(\Omega)} \le C|t|\,\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le C|t| \xrightarrow{t\searrow 0} 0. \tag{7.8}$$

In view of (7.5) and the uniform boundedness of $\|\widetilde{\boldsymbol{u}}_t\|_{\boldsymbol{H}^1(\Omega)}$, which is given by (7.6), we can estimate

$$\begin{aligned}\left\|\frac{r_\alpha(t)}{t}\widetilde{\boldsymbol{u}}_t\right\|_{\boldsymbol{L}^2(\Omega)} &\le \frac{\|r_\alpha(t)\|_{L^3(\Omega)}}{|t|}\|\widetilde{\boldsymbol{u}}_t\|_{\boldsymbol{L}^6(\Omega)} \le C\frac{\|r_\alpha(t)\|_{L^3(\Omega)}}{|t|_{H^1(\Omega)}}\|\widetilde{\boldsymbol{u}}_t\|_{\boldsymbol{H}^1(\Omega)} \le\\ &\le C\frac{\|r_\alpha(t)\|_{L^3(\Omega)}}{|t|_{H^1(\Omega)}} \xrightarrow{t\searrow 0} 0.\end{aligned} \tag{7.9}$$

Altogether this implies

$$\|\boldsymbol{h}(t)\|_{\boldsymbol{H}^{-1}(\Omega)} \xrightarrow{t\searrow 0} 0. \tag{7.10}$$

Now we use Poincaré's inequality for $\widehat{\boldsymbol{u}}_t \in \boldsymbol{H}_0^1(\Omega)$ to deduce from Lemma 7.1 the following estimate:

$$
\begin{aligned}
\int_\Omega \alpha_\varepsilon' (\varphi - \varphi_\varepsilon)\, t\, |\widehat{\boldsymbol{u}}_t|^2 \, \mathrm{d}x &\le c(\alpha_\varepsilon)\, |t|\, \|\widehat{\boldsymbol{u}}_t\|^2_{\boldsymbol{L}^2(\Omega)} \le c(\alpha_\varepsilon, \Omega)\, |t|\, \|\nabla \widehat{\boldsymbol{u}}_t\|^2_{\boldsymbol{L}^2(\Omega)} \le \\
&\le c(\alpha_\varepsilon, \Omega)\, |t| \left(\left\| \frac{1}{t} \nabla (\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon) \right\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \right)^2 \le \\
&\le c(\alpha_\varepsilon, \Omega)\, |t| \left(L \frac{1}{|t|} \|\varphi - \varphi_\varepsilon\|_{L^\infty(\Omega)} |t| + \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \right)^2
\end{aligned}
$$

and hence we obtain

$$
\lim_{t \searrow 0} \left| \int_\Omega \alpha_\varepsilon' (\varphi - \varphi_\varepsilon)\, t\, |\widehat{\boldsymbol{u}}_t|^2 \, \mathrm{d}x \right| = 0. \tag{7.11}
$$

Testing (7.7) with $\widehat{\boldsymbol{u}}_t \in \boldsymbol{V}$ yields

$$
\int_\Omega \alpha_\varepsilon (\varphi_\varepsilon) |\widehat{\boldsymbol{u}}_t|^2 + \alpha_\varepsilon' (\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\, t\, |\widehat{\boldsymbol{u}}_t|^2 + \mu \|\nabla \widehat{\boldsymbol{u}}_t\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{h}(t), \widehat{\boldsymbol{u}}_t \rangle_{\boldsymbol{H}^{-1}(\Omega)}
$$

and from that

$$
\mu \|\nabla \widehat{\boldsymbol{u}}_t\|^2_{\boldsymbol{L}^2(\Omega)} \le \langle \boldsymbol{h}(t), \widehat{\boldsymbol{u}}_t \rangle_{\boldsymbol{H}^{-1}(\Omega)} - \int_\Omega \alpha_\varepsilon' (\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\, t\, |\widehat{\boldsymbol{u}}_t|^2 \, \mathrm{d}x.
$$

Thus, by (7.10), (7.11) and Young's inequality

$$
\lim_{t \searrow 0} \|\nabla \widehat{\boldsymbol{u}}_t\|_{\boldsymbol{L}^2(\Omega)} = 0
$$

which gives with Poincaré's inequality

$$
\lim_{t \searrow 0} \|\widehat{\boldsymbol{u}}_t\|_{\boldsymbol{H}^1(\Omega)} = 0.
$$

This finishes the second step and completes the proof of the lemma.

□

Remark 7.1. *Let $\widetilde{\alpha}_\varepsilon : \mathbb{R} \to [-\delta, \overline{\alpha}_\varepsilon + \delta]$, with $0 < \delta < \frac{\mu}{C_P}$, be an extension of $\alpha_\varepsilon : [-1,1] \to [0, \overline{\alpha}_\varepsilon]$ such that $\widetilde{\alpha}_\varepsilon \in C^2(\mathbb{R})$. Here $C_P = C_P(\Omega)$ is the constant from the Poincaré inequality, thus it is chosen such that*

$$
\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \le C_P \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega).
$$

Then we can extend the solution operator $\boldsymbol{S}_\varepsilon : \overline{\Phi}_{ad} \to \boldsymbol{U}$, defined in Lemma 5.1, to $\widetilde{\boldsymbol{S}}_\varepsilon : L^6(\Omega) \to \boldsymbol{U}$ such that $\widetilde{\boldsymbol{S}}_\varepsilon|_{\overline{\Phi}_{ad}} = \boldsymbol{S}_\varepsilon$. Moreover, one can show by an application of the implicit function theorem, using similar arguments as in [Hec11, Section 5], see also [Trö09], that $\widetilde{\boldsymbol{S}}_\varepsilon$ is Fréchet differentiable, and $\mathrm{D}\widetilde{\boldsymbol{S}}_\varepsilon (\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) = \mathrm{D}\boldsymbol{S}_\varepsilon (\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)$ for all $\varphi, \varphi_\varepsilon \in \overline{\Phi}_{ad}$.
As a consequence, we obtain that the reduced objective functional $j_\varepsilon : \overline{\Phi}_{ad} \to \mathbb{R}$ can be extended to a Fréchet differentiable functional $\widetilde{j}_\varepsilon : H^1(\Omega) \to \mathbb{R}$.

After proving the existence of directional derivatives of the solution operator, we can derive first order optimality conditions for the optimal control problem, which is given in its reduced form by

$$\min_{\varphi\in\Phi_{ad}} j_\varepsilon(\varphi). \tag{7.12}$$

As $\varphi_\varepsilon \in \Phi_{ad}$ is chosen as global minimizer of the reduced objective functional j_ε we find due to the convexity of the admissible set Φ_{ad} for any $\varphi \in \Phi_{ad}$ and any $t \in [0,1]$:

$$j_\varepsilon(\varphi_\varepsilon) \le j(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon)).$$

Using the differentiability result for $\boldsymbol{S}_\varepsilon$ of Lemma 7.2 we can deduce therefrom

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \ge 0 \quad \forall \varphi \in \Phi_{ad}. \tag{7.13}$$

Denoting $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ and $\boldsymbol{u} = \mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)$ we can apply chain rule to calculate

$$\begin{aligned}\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) = {} & \frac{1}{2}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{u}_\varepsilon \,\mathrm{d}x + \\ & + \int_\Omega \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u}, \mathrm{D}\boldsymbol{u}) \,\mathrm{d}x + \\ & + \gamma\varepsilon \int_\Omega \nabla\varphi_\varepsilon \cdot \nabla(\varphi - \varphi_\varepsilon) \,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \,\mathrm{d}x.\end{aligned} \tag{7.14}$$

To rewrite this expression into a more convenient variational inequality we define an adjoint equation. For this purpose we introduce the Lagrangian $\mathcal{L}_\varepsilon$ and calculate formally to deduce the adjoint system. On more details about the Lagrangian ansatz we refer to [Trö09], where this approach is further discussed. Thus, we define the Lagrangian

$$\mathcal{L}_\varepsilon : \Phi_{ad} \times \boldsymbol{U} \times \boldsymbol{V} \to \mathbb{R}$$

by

$$\mathcal{L}_\varepsilon(\varphi, \boldsymbol{u}, \boldsymbol{q}) := J_\varepsilon(\varphi, \boldsymbol{u}) - \int_\Omega (\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p - \boldsymbol{f})\cdot\boldsymbol{q} \,\mathrm{d}x.$$

The variational inequality is then formally derived by

$$\begin{aligned}\mathrm{D}_\varphi\mathcal{L}_\varepsilon(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{q}_\varepsilon)(\varphi - \varphi_\varepsilon) = {} & \frac{1}{2}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x + \gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla(\varphi - \varphi_\varepsilon) \,\mathrm{d}x + \\ & + \frac{\gamma}{\varepsilon}\int_\Omega \psi_0'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \,\mathrm{d}x - \\ & - \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon \,\mathrm{d}x \ge 0 \quad \forall\varphi \in \Phi_{ad}\end{aligned}$$

and the adjoint equation can be deduced by

$$\begin{aligned}\mathrm{D}_{\boldsymbol{u}}\mathcal{L}_\varepsilon(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{q}_\varepsilon)(\boldsymbol{u}) = {} & \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u} \,\mathrm{d}x + \int_\Omega \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u}, \mathrm{D}\boldsymbol{u}) \,\mathrm{d}x - \\ & - \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon \,\mathrm{d}x - \underbrace{\int_\Omega -\mu\Delta\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon \,\mathrm{d}x}_{=\int_\Omega -\mu\boldsymbol{u}\cdot\Delta\boldsymbol{q}_\varepsilon \,\mathrm{d}x} = 0 \quad \forall\boldsymbol{u} \in \boldsymbol{V}.\end{aligned}$$

This implies the following adjoint system:

$$\begin{aligned} \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon &= \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon + \mathrm{D}_u f(\cdot,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon) && \text{in } \Omega, && (7.15\text{a})\\ \operatorname{div}\boldsymbol{q}_\varepsilon &= 0 && \text{in } \Omega, && (7.15\text{b})\\ \boldsymbol{q}_\varepsilon &= \boldsymbol{0} && \text{on } \partial\Omega. && (7.15\text{c}) \end{aligned}$$

As the calculations above are only formally, we come now in the next lemma and theorem to proving its validity. We start by showing that the adjoint system, stated in the form of (7.15), is well-posed.

Lemma 7.3. *There exists a unique* $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ *such that* (7.15) *is fulfilled in the following weak formulation:*

$$\begin{aligned} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{q}_\varepsilon\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x + \\ + \int_\Omega \mathrm{D}_{(2,3)}f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{v},\mathrm{D}\boldsymbol{v})\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{V} \end{aligned} \tag{7.16}$$

where $\pi_\varepsilon \in L^2(\Omega)$ *fulfills* (7.15a) *in the distributional sense and is unique up to a constant.*

Proof. Existence and uniqueness of a solution for (7.16) follows as in Lemma 7.2 by applying Lax-Milgram's theorem A.2. Besides, the pressure $\pi_\varepsilon \in L^2(\Omega)$, which won't play an important role in the following, is given by Lemma 4.4. □

We can transform the variational inequality now into an adjoint formulation. After additionally introducing a Lagrange multiplier for the integral constraint we end up with the following optimality system for (5.1) – (5.2):

Theorem 7.1. *The following optimality system is fulfilled for any minimizer* $(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon)$ *of* (5.1) – (5.2)*:*

$$\begin{aligned} \left(\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon) - \alpha_\varepsilon'(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \lambda_\varepsilon, \varphi-\varphi_\varepsilon\right)_{L^2(\Omega)} + \\ + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla(\varphi-\varphi_\varepsilon)\right)_{\boldsymbol{L}^2(\Omega)} \geq 0 \quad \forall\varphi\in\overline{\Phi}_{ad}, \end{aligned} \tag{7.17}$$

$$\left.\begin{aligned} \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon - \mu\Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= \boldsymbol{f} && \textit{in } \Omega,\\ \operatorname{div}\boldsymbol{u}_\varepsilon &= 0 && \textit{in } \Omega,\\ \boldsymbol{u}_\varepsilon &= \boldsymbol{g} && \textit{on } \partial\Omega,\\ \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon &= \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon + \mathrm{D}_u f(\cdot,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon) && \textit{in } \Omega,\\ \operatorname{div}\boldsymbol{q}_\varepsilon &= 0 && \textit{in } \Omega,\\ \boldsymbol{q}_\varepsilon &= \boldsymbol{0} && \textit{on } \partial\Omega,\\ \lambda_\varepsilon\left(\int_\Omega\varphi_\varepsilon\,\mathrm{d}x - \beta|\Omega|\right) = 0,\ \lambda_\varepsilon\geq 0,& \\ \int_\Omega\varphi_\varepsilon\,\mathrm{d}x \leq \beta|\Omega|, \quad |\varphi_\varepsilon|\leq 1 \ \textit{a.e. in } \Omega,& \end{aligned}\right\} \tag{7.18}$$

where $\lambda_\varepsilon \in \mathbb{R}^+$ denotes a Lagrange multiplier for the integral constraint.
Here, $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$ and $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ are weak solutions of the state equations and adjoint system, respectively.

Proof. We start by showing the existence of a Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}^+$ for the integral constraint $\int_\Omega \varphi_\varepsilon \,\mathrm{d}x \leq \beta\,|\Omega|$. Therefore we want to use the results of [KZ79] and rewrite therefore our problem as follows: using the notation of [KZ79] we introduce

$$C = \left\{\varphi \in H^1(\Omega) \mid |\varphi| \leq 1 \text{ a.e. in } \Omega\right\}, \quad g(\varphi) := -\int_\Omega \varphi \,\mathrm{d}x + \beta\,|\Omega|,$$

$$K = \mathbb{R}^+,\, Y = \mathbb{R},\, X = H^1(\Omega) \cap L^\infty(\Omega)$$

and see, in the notation of [KZ79],

$$C(\varphi_\varepsilon) = \{\lambda\,(\varphi - \varphi_\varepsilon) \mid \lambda \geq 0, \varphi \in C\},$$

$$K(g(\varphi_\varepsilon)) = \left\{k - \lambda\left(-\int_\Omega \varphi_\varepsilon \,\mathrm{d}x + \beta\,|\Omega|\right) \mid k \in \mathbb{R}^+, \lambda \geq 0\right\},$$

$$g'(\varphi_\varepsilon)\varphi = -\int_\Omega \varphi \,\mathrm{d}x.$$

We show that φ_ε is a regular point in the sense of [KZ79], which means it has to hold

$$g'(\varphi_\varepsilon)C(\varphi_\varepsilon) - K(g(\varphi_\varepsilon)) = Y. \tag{7.19}$$

This is in our setting equivalent to

$$\left\{-\lambda_1 \int_\Omega (\varphi - \varphi_\varepsilon) - \left[k - \lambda_2\left(-\int_\Omega \varphi_\varepsilon \,\mathrm{d}x + \beta\,|\Omega|\right)\right] \mid \lambda_1 \geq 0, \lambda_2 \geq 0, k \in \mathbb{R}^+, \varphi \in C\right\} = \mathbb{R}.$$

If it holds $\int_\Omega \varphi_\varepsilon \,\mathrm{d}x < \beta\,|\Omega|$, we see that

$$\left\{-k + \lambda_2\left(-\int_\Omega \varphi_\varepsilon \,\mathrm{d}x + \beta\,|\Omega|\right) \mid k, \lambda_2 \in \mathbb{R}^+\right\} = \mathbb{R}^- + \mathbb{R}^+ = \mathbb{R}$$

and (7.19) follows directly. For the case $\int_\Omega \varphi_\varepsilon \,\mathrm{d}x = \beta\,|\Omega|$ we choose for instance $\varphi \equiv -1 \in C$ and see

$$\left\{-\lambda_1 \int_\Omega \varphi \,\mathrm{d}x - k \mid \lambda_1, k \in \mathbb{R}^+\right\} = \mathbb{R}$$

to deduce (7.19). Our goal is to apply [KZ79, Theorem 3.1]. We notice, that in this work the objective functional, in our case j_ε, is assumed to be Fréchet differentiable. But essentially, when having a look into the details of the proof of [KZ79, Theorem 3.1], it will turn out that this is not necessary for our special situation. The first utilization of the Fréchet differentiability of the objective functional in the proof of [KZ79, Theorem 3.1] is when deducing that it holds

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)\varphi \geq 0 \quad \forall \varphi \in T(\Phi_{ad}, \varphi_\varepsilon) \tag{7.20}$$

where

$$T(\Phi_{ad}, \varphi_\varepsilon) = \left\{\varphi \in \Phi_{ad} \mid \varphi = \lim_{n\to\infty} \frac{\varphi^n - \varphi_\varepsilon}{t_n}, t_n \searrow 0, \varphi^n \in \Phi_{ad}\right\}$$

is the sequential tangent cone on the admissible set Φ_{ad} in φ_ε. Choosing an arbitrary $\varphi = \lim_{n\to\infty} \frac{\varphi^n - \varphi_\varepsilon}{t_n} \in T(\Phi_{ad}, \varphi_\varepsilon)$ we know due to (7.13) that it holds

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi^n - \varphi_\varepsilon) \geq 0 \quad \forall \varphi^n \in \Phi_{ad}$$

which yields

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)\varphi = \lim_{n\to\infty} \frac{1}{t_n} \mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi^n - \varphi_\varepsilon) \geq 0$$

and we thus can conclude (7.20). In the remaining considerations of the proof of [KZ79, Theorem 3.1], the only derivatives of j_ε that appear are of the form

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)\varphi, \qquad \varphi \in C(\varphi_\varepsilon).$$

Due to the definition of the convex cone $C(\varphi_\varepsilon)$, those derivatives are according to Lemma 7.2 well-defined in our setting, and moreover $\mathrm{D}j_\varepsilon(\varphi_\varepsilon)\lambda\varphi = \lambda \mathrm{D}j_\varepsilon(\varphi_\varepsilon)\varphi$ for all $\lambda \in \mathbb{R}^+$ and $\varphi \in C(\varphi_\varepsilon)$, compare also discussion in Remark 7.1. Thus we can deduce as in [KZ79, Theorem 3.1] the existence of some $\lambda_\varepsilon \in \mathbb{R}^+$ such that

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) + \lambda_\varepsilon \int_\Omega (\varphi - \varphi_\varepsilon)\,\mathrm{d}x \geq 0 \quad \forall \varphi \in \overline{\Phi}_{ad} \tag{7.21}$$

together with the complementarity condition

$$\lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon \,\mathrm{d}x - \beta |\Omega| \right) = 0. \tag{7.22}$$

Next, we want to reformulate the directional derivative (7.14) of j_ε with help of the adjoint variable $\boldsymbol{q}_\varepsilon$, which is given by Lemma 7.3. To this end, we denote $\boldsymbol{u} := \mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)$, given by (7.3), and insert the adjoint state $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ as a test function into the linearized equation (7.3). This gives

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} \cdot \boldsymbol{q}_\varepsilon \,\mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{q}_\varepsilon \,\mathrm{d}x = 0.$$

Similarly, we use the linearized state $\boldsymbol{u} \in \boldsymbol{V}$ as a test function for the adjoint equation (7.16) and obtain

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon \cdot \boldsymbol{u}\,\mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{q}_\varepsilon \cdot \nabla \boldsymbol{u}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}\,\mathrm{d}x +& \\ + \int_\Omega \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x.&\end{aligned}$$

Comparing these two equalities we see directly

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon = -\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}\,\mathrm{d}x - \int_\Omega \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x. \tag{7.23}$$

Inserting this into formula (7.14) we end up in

$$
\begin{aligned}
\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) = {} & \frac{1}{2}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{u}_\varepsilon\,\mathrm{d}x + \\
& + \int_\Omega \mathrm{D}_{(2,3)}f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \\
& + \gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_\varepsilon)\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x = \\
& = \left(\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon) - \alpha_\varepsilon'(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon, \varphi-\varphi_\varepsilon\right)_{L^2(\Omega)} + \\
& + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla(\varphi-\varphi_\varepsilon)\right)_{\boldsymbol{L}^2(\Omega)}
\end{aligned}
$$

and the statement follows with (7.21) and (7.22). □

Example 7.1. Using the total potential power as an objective functional, which is introduced in Example 2.3, thus

$$
f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) = \frac{\mu}{2}|\mathrm{D}\boldsymbol{u}|^2 - \boldsymbol{f}(x)\cdot\boldsymbol{u}
$$

the directional derivative of the reduced objective functional (7.14) reads

$$
\begin{aligned}
\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) = {} & \frac{1}{2}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{u}_\varepsilon\,\mathrm{d}x + \\
& + \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u} - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x + \\
& + \gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_\varepsilon)\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi_0'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x
\end{aligned}
$$

with $\boldsymbol{u} = \mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)$. Due to the state equations (5.2) of $\boldsymbol{u}_\varepsilon$ this can be further simplified to

$$
\begin{aligned}
\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) = {} & \frac{1}{2}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x + \gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_\varepsilon)\,\mathrm{d}x + \\
& + \frac{\gamma}{\varepsilon}\int_\Omega \psi_0'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x
\end{aligned}
$$

and so for this special form of the objective functional and state equations, we don't even need an adjoint state for the optimality conditions and end up with a very simple form for the optimality conditions.

7.2 Geometric variations

In this subsection we come to deriving optimality conditions for the phase field problem (5.1) – (5.2) with a different ansatz. The goal is to obtain an optimality system that corresponds to optimality conditions of the sharp interface problem, see Section 8.2. In the sharp interface setting optimality conditions are obtained by geometric variations instead of parametric variations, thus by variations of the domain along suitable transformations. Therefore, we apply this idea to the phase field model in the current subsection. We will then later on see that under certain assumptions the obtained optimality system really converges, as $\varepsilon \searrow 0$, to the optimality system derived in the sharp interface model, see Section 9.

We begin with choosing minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in \Phi_{ad} \times \boldsymbol{U}$ of the phase field model (5.1) – (5.2) and fix $\varepsilon > 0$. Since (5.1) – (5.2) is equivalent to the reduced problem

$$\min_{\varphi \in \Phi_{ad}} j_\varepsilon(\varphi) \tag{7.24}$$

where the reduced objective functional j_ε is defined in (6.1), this is equivalent to stating φ_ε is a solution of (7.24) and $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$.

As indicated above, we will vary the domain Ω by suitable transformations. Therefore we define for further considerations

$$\varphi_\varepsilon(t) := \varphi_\varepsilon \circ T_t^{-1}, \quad \Omega_t := T_t(\Omega) \tag{7.25}$$

$$\boldsymbol{u}_\varepsilon(t) = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon(t))$$

where the transformation T is chosen in $\overline{\mathcal{T}}_{ad}$ and V denotes the corresponding velocity field in $\overline{\mathcal{V}}_{ad}$.

Then we get the following result concerning differentiability with respect to t:

Lemma 7.4. *The mapping $\mathbb{R} \supset I \ni t \mapsto \boldsymbol{u}_\varepsilon(t) \circ T_t \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$ (where I is a small interval around 0) and $\dot{\boldsymbol{u}}_\varepsilon[V] := \partial_t|_{t=0}(\boldsymbol{u}_\varepsilon(t) \circ T_t) \in \boldsymbol{H}_0^1(\Omega)$ is given as the unique solution to*

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\dot{\boldsymbol{u}}_\varepsilon[V] \cdot \boldsymbol{z} + \mu \nabla \dot{\boldsymbol{u}}_\varepsilon[V] \cdot \nabla \boldsymbol{z} \, \mathrm{d}x = \\
&= \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \, \mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T \nabla \boldsymbol{z} \, \mathrm{d}x + \\
&+ \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}) \, \mathrm{d}x - \\
&- \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \operatorname{div} V(0) \, \mathrm{d}x + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z} \, \mathrm{d}x + \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z} \, \mathrm{d}x - \\
&- \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)\boldsymbol{z} \, \mathrm{d}x
\end{aligned} \tag{7.26}$$

which has to hold for every $\boldsymbol{z} \in \boldsymbol{V}$, together with

$$\operatorname{div} \dot{\boldsymbol{u}}_\varepsilon[V] = \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0). \tag{7.27}$$

Remark 7.2. *We remark that here and subsequently we write for two matrices $A, B \in \mathbb{R}^{d\times d}$ the matrix product by*

$$AB = (AB)_{i,j=1}^d = \left(\sum_{k=1}^d A_{ik}B_{kj}\right)_{i,j=1}^d$$

and if $v \in \mathbb{R}^d$ is a vector we denote by

$$Av = A \cdot v = (A \cdot v)_{i=1}^d = \left(\sum_{j=1}^d A_{ij}v_j\right)_{i=1}^d$$

the vector-matrix product. Moreover, for two vectors $v, w \in \mathbb{R}^d$ we have the inner product $v \cdot w = \sum_{i=1}^d v_i w_i$. As usual we will sometimes write for the standard matrix inner product $A \cdot B$ instead of $A : B$ when no confusion can arise. The notation

$$\mathrm{D}V\boldsymbol{w} = \nabla V \cdot \boldsymbol{w}$$

for functions $V \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ stands for the directional derivative of V along $\boldsymbol{w}$. And finally, we recall for convenience that we denote by $\nabla V = (\partial_i V_j)_{i,j=1}^d$ the transpose of the total derivative of $V \in \boldsymbol{H}^1(\Omega)$, and so we have for $\varphi \in H^1(\Omega)$:

$$\nabla V \nabla \varphi = \left(\sum_{j=1}^d \partial_i V_j \partial_j \varphi\right)_{i=1}^d .$$

Proof. The main idea of the proof is to apply the implicit function theorem. To this end we define the function

$$F : I \times \boldsymbol{H}^1_{\boldsymbol{g}}(\Omega) \to \boldsymbol{V}' \times L^2_0(\Omega)$$

with

$$F(t, \boldsymbol{u}) := (F_1(t, \boldsymbol{u}), F_2(t, \boldsymbol{u})) \in \boldsymbol{V}' \times L^2_0(\Omega)$$

by

$$\begin{aligned} F_1(t, \boldsymbol{u})(\boldsymbol{z}) := &\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} \cdot \left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t \,\mathrm{d}x + \\ &+ \int_\Omega \mu \mathrm{D}T_t^{-T}\nabla\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t \,\mathrm{d}x - \\ &- \int_\Omega \boldsymbol{f} \circ T_t \cdot \left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t \,\mathrm{d}x \end{aligned} \tag{7.28}$$

and

$$F_2(t, \boldsymbol{u}) = \left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{u}\right)\det \mathrm{D}T_t.$$

The function F_2 is motivated by the identity $\left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{v}\right) \circ T_t^{-1} = \operatorname{div}\left(\boldsymbol{v} \circ T_t^{-1}\right)$. This function is well-defined, since for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ fulfilling $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}$ we have due to Gauß' theorem

$$\begin{aligned} \int_\Omega \left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{v}\right)\det \mathrm{D}T_t \,\mathrm{d}x &= \int_\Omega \operatorname{div}\left(\boldsymbol{v} \circ T_t^{-1}\right) \circ T_t \det \mathrm{D}T_t \,\mathrm{d}x = \int_\Omega \operatorname{div}\left(\boldsymbol{v} \circ T_t^{-1}\right)\mathrm{d}x = \\ &= \int_{\partial\Omega} \boldsymbol{v} \circ T_t^{-1} \cdot \boldsymbol{n}\,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{g} \circ T_t^{-1} \cdot \boldsymbol{n}\,\mathrm{d}x = 0 \end{aligned}$$

where we used, that $T_t(x) = x$ if $\boldsymbol{g}(x) \neq \mathbf{0}$, see Lemma 3.5, and $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \,\mathrm{d}x = 0$, see Assumption **(A2)**. Therefore, it follows $F_2(t, \boldsymbol{u}) \in L_0^2(\Omega)$ for all $\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{g}}^1(\Omega)$.

We choose $\boldsymbol{z} \in \boldsymbol{V}$ arbitrary and define $\boldsymbol{z}_t := \left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{z} \circ T_t^{-1}$. One obtains from Lemma 3.6 that $\boldsymbol{z}_t \in \boldsymbol{V}$ and recalling $\varphi_\varepsilon(t) = \varphi_\varepsilon \circ T_t^{-1}$ we can therefore calculate

$$\begin{aligned}
F_1\left(t, \boldsymbol{u}_\varepsilon(t) \circ T_t\right)(\boldsymbol{z}) &= \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon(t) \circ T_t\right) \boldsymbol{u}_\varepsilon(t) \circ T_t \cdot \boldsymbol{z}_t \circ T_t \det \mathrm{D}T_t \,\mathrm{d}x + \\
&+ \int_\Omega \mu \left(\nabla \boldsymbol{u}_\varepsilon(t)\right) \circ T_t : (\nabla \boldsymbol{z}_t) \circ T_t \cdot \det \mathrm{D}T_t \,\mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f} \circ T_t \cdot \boldsymbol{z}_t \circ T_t \det \mathrm{D}T_t \,\mathrm{d}x = \\
&= \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon(t)\right) \boldsymbol{u}_\varepsilon(t) \cdot \boldsymbol{z}_t \,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon(t) : \nabla \boldsymbol{z}_t \,\mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f} \cdot \boldsymbol{z}_t \,\mathrm{d}x = 0
\end{aligned} \tag{7.29}$$

and

$$F_2\left(t, \boldsymbol{u}_\varepsilon(t) \circ T_t\right) = \underbrace{\operatorname{div} \boldsymbol{u}_\varepsilon(t)}_{=0} \circ T_t (\det \mathrm{D}T_t) = 0.$$

To apply the implicit function theorem we have to shift F, since $\boldsymbol{H}_{\boldsymbol{g}}^1(\Omega)$ is not a linear space. Therefore, we fix $\boldsymbol{G} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{G}|_{\partial\Omega} = \boldsymbol{g}$ and define

$$(G_1, G_2) = G : I \times \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}' \times L_0^2(\Omega)$$

by

$$G(t, \boldsymbol{v}) := F\left(t, \boldsymbol{v} + \boldsymbol{G}\right).$$

Then we obtain from the considerations above

$$G\left(t, \boldsymbol{u}_\varepsilon(t) \circ T_t - \boldsymbol{G}\right) = F\left(t, \boldsymbol{u}_\varepsilon(t)\right) = 0 \quad \forall t \in I.$$

Moreover we have

$$\mathrm{D}_u G_1\left(0, \boldsymbol{u}_\varepsilon - \boldsymbol{G}\right)(\boldsymbol{v})(\boldsymbol{z}) = \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right) \boldsymbol{v} \cdot \boldsymbol{z} \,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{z} \,\mathrm{d}x$$

and

$$\mathrm{D}_u G_2\left(0, \boldsymbol{u}_\varepsilon - \boldsymbol{G}\right) \boldsymbol{v} = \operatorname{div} \boldsymbol{v}.$$

Applying Lax-Milgram's theorem A.2 and using Lemma 4.2 we see that $\mathrm{D}_u G\left(0, \boldsymbol{u}_\varepsilon + \boldsymbol{G}\right)$ is an isomorphism as a mapping from $\boldsymbol{H}_0^1(\Omega)$ into $\boldsymbol{V}' \times L_0^2(\Omega)$.
Thus the implicit function theorem implies differentiability of $t \mapsto (\boldsymbol{u}_\varepsilon(t) \circ T_t - \boldsymbol{G}) \in \boldsymbol{H}^1(\Omega)$ at $t = 0$, and thus $t \mapsto \boldsymbol{u}_\varepsilon(t) \circ T_t$ is differentiable. At the same time, the implicit function theorem tells us that

$$\begin{aligned}
\partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t) \circ T_t\right) = \partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t) \circ T_t - \boldsymbol{G}\right) &= -\mathrm{D}_u G\left(0, \boldsymbol{u}_\varepsilon - \boldsymbol{G}\right)^{-1} \partial_t G\left(0, \boldsymbol{u}_\varepsilon - \boldsymbol{G}\right) = \\
&= -\mathrm{D}_u G\left(0, \boldsymbol{u}_\varepsilon - \boldsymbol{G}\right)^{-1} \partial_t F\left(0, \boldsymbol{u}_\varepsilon\right)
\end{aligned} \tag{7.30}$$

and this gives the stated result.

□

The preceding lemma ensures thus the differentiability such that we can now formulate first order necessary optimality conditions for (7.24). To this end we notice that $\varphi_\varepsilon(t) \in \Phi_{ad}$ if $T \in \mathcal{T}_{ad}$, compare Remark 3.4. Thus, $\varphi_\varepsilon(t)$, which is given by (7.25), defines for any $t \in I \subseteq \mathbb{R}$ small enough an admissible comparison function and therefrom we arrive in

$$j_\varepsilon\left(\varphi_\varepsilon\right) \leq j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) \quad \forall T \in \mathcal{T}_{ad},\, |t| \ll 1$$

and from this

$$\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = 0 \quad \forall T \in \mathcal{T}_{ad}. \tag{7.31}$$

This is the idea for deriving the desired first order optimality conditions for problem (5.1) – (5.2), as the following lemma states:

Lemma 7.5. *For any minimizer* $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in \Phi_{ad} \times \boldsymbol{U}$ *of* (5.1) – (5.2) *we have the following necessary optimality conditions:*

$$\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\, \mathrm{d}x, \tag{7.32}$$

$$\lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon \, \mathrm{d}x - \beta\, |\Omega| \right) = 0 \tag{7.33}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ *with velocity* $V \in \overline{\mathcal{V}}_{ad}$, *where* $\lambda_\varepsilon \in \mathbb{R}^+$ *is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:*

$$\begin{aligned}
\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = &\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right) \left(\boldsymbol{u}_\varepsilon \cdot \dot{\boldsymbol{u}}_\varepsilon[V] + \frac{1}{2}\left|\boldsymbol{u}_\varepsilon\right|^2 \operatorname{div} V(0)\right) \mathrm{d}x + \\
+ &\int_\Omega \left[\mathrm{D} f\left(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon\right)\left(V(0), \dot{\boldsymbol{u}}_\varepsilon[V], \mathrm{D}\dot{\boldsymbol{u}}_\varepsilon[V] - \mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0)\right) + \right.\\
+ &\left. f\left(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon\right) \operatorname{div} V(0)\right] \mathrm{d}x + \\
+ &\int_\Omega \left(\frac{\gamma\varepsilon}{2} \left|\nabla\varphi_\varepsilon\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right) \operatorname{div} V(0) - \gamma\varepsilon \nabla\varphi_\varepsilon \cdot \nabla V(0) \nabla\varphi_\varepsilon \, \mathrm{d}x
\end{aligned} \tag{7.34}$$

where $\dot{\boldsymbol{u}}_\varepsilon[V] \in \boldsymbol{H}_0^1(\Omega)$ *is given as the solution of* (7.26)-(7.27).

Proof. We consider the individual terms of the functional j_ε and calculate, while using the calculation rules of Section 3.3. Starting with the first term of (5.1) we see

$$\begin{aligned}
\partial_t|_{t=0} \int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon(t)\right) \left|\boldsymbol{u}_\varepsilon(t)\right|^2 \mathrm{d}x &= \partial_t|_{t=0} \int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon(t)\circ T_t\right) \left|\boldsymbol{u}_\varepsilon(t)\circ T_t\right|^2 \det \mathrm{D}T_t \, \mathrm{d}x = \\
&= \int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right) \left(2\boldsymbol{u}_\varepsilon \cdot \dot{\boldsymbol{u}}_\varepsilon[V] + \left|\boldsymbol{u}_\varepsilon\right|^2 \operatorname{div} V(0)\right) \mathrm{d}x.
\end{aligned}$$

Next, we have for the part of j_ε involving the objective functional

$$\begin{aligned}
&\partial_t|_{t=0} \int_{\Omega_t} f\left(x, \boldsymbol{u}_\varepsilon(t), \mathrm{D}\boldsymbol{u}_\varepsilon(t)\right) \mathrm{d}x = \partial_t|_{t=0} \int_\Omega f\left(x, \boldsymbol{u}_\varepsilon(t), \mathrm{D}\boldsymbol{u}_\varepsilon(t)\right) \circ T_t \det \mathrm{D}T_t \, \mathrm{d}x = \\
&= \int_\Omega \left[\partial_t|_{t=0} \left(f\left(T_t(x), \boldsymbol{u}_\varepsilon(t)\circ T_t, \left(\mathrm{D}\boldsymbol{u}_\varepsilon(t)\right)\circ T_t\right)\right) + f\left(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon\right) \operatorname{div} V(0)\right] \mathrm{d}x = \\
&= \int_\Omega \left[\mathrm{D} f\left(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon\right)\left(V(0), \dot{\boldsymbol{u}}_\varepsilon[V], \mathrm{D}\dot{\boldsymbol{u}}_\varepsilon[V] - \mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0)\right) + \right.\\
&\left. + f\left(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon\right) \operatorname{div} V(0)\right] \mathrm{d}x.
\end{aligned}$$

In the same manner we can see that

$$\begin{aligned}\partial_t|_{t=0}\int_{\Omega_t}\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon(t)\right|^2\,\mathrm{d}x &= \frac{\gamma\varepsilon}{2}\int_\Omega \partial_t|_{t=0}\left|\nabla\varphi_\varepsilon(t)\right|^2\circ T_t\det \mathrm{D}T_t\,\mathrm{d}x = \\ &= \frac{\gamma\varepsilon}{2}\int_\Omega \partial_t|_{t=0}\left|\nabla T_t^{-1}\nabla\varphi_\varepsilon\right|^2\det\mathrm{D}T_t\,\mathrm{d}x = \\ &= -\gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x+\frac{\gamma\varepsilon}{2}\int_\Omega\left|\nabla\varphi_\varepsilon\right|^2\operatorname{div}V(0)\,\mathrm{d}x.\end{aligned}$$

Finally, the differential of the potential term reads as

$$\partial_t|_{t=0}\int_{\Omega_t}\frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon(t)\right)\,\mathrm{d}x=\partial_t|_{t=0}\int_\Omega\frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\det\mathrm{D}T_t\,\mathrm{d}x=\int_\Omega\frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\operatorname{div}V(0)\,\mathrm{d}x.$$

Plugging all these terms together we end up with formula (7.34).

We continue with choosing $T\in\overline{\mathcal{T}}_{ad}$ and introducing the notation

$$g:[-t_0,t_0]\to\mathbb{R},\quad g(t):=-\int_\Omega\varphi_\varepsilon\circ T_t^{-1}\,\mathrm{d}x+\beta\left|\Omega\right|$$

for some $t_0>0$ small enough. To obtain a Lagrange multiplier λ_ε for the integral constraint we distinguish between two cases.

1st case: If $g(0)>0$, thus $\int_\Omega\varphi_\varepsilon\,\mathrm{d}x<\beta\left|\Omega\right|$, we find for t small enough and any transformation $T\in\overline{\mathcal{T}}_{ad}$ that $\int_\Omega\varphi_\varepsilon\circ T_t^{-1}\,\mathrm{d}x<\beta\left|\Omega\right|$, and so $\varphi_\varepsilon\circ T_t^{-1}\in\Phi_{ad}$, compare also Remark 3.4. Thus,

$$j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)\geq j_\varepsilon\left(\varphi_\varepsilon\right)\quad\forall|t|\ll 1$$

and so

$$\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=0.$$

Hence, (7.32) – (7.33) is fulfilled for $\lambda_\varepsilon=0$. Therefore, so we can assume for the following considerations that $\int_\Omega\varphi_\varepsilon\,\mathrm{d}x=\beta\left|\Omega\right|$.

2nd case: By the considerations of the first case, we can assume without loss of generality that $g(0)=0$. We follow now a similar idea as in [BGH98, Proof of Proposition 1.17]. Since $\int_\Omega\varphi_\varepsilon\,\mathrm{d}x=\beta\left|\Omega\right|$, we may find some $W\in\overline{\mathcal{V}}_{ad}$ with associated transformation $S\in\overline{\mathcal{T}}_{ad}$ such that

$$-\int_\Omega\varphi_\varepsilon\operatorname{div}W(0)\,\mathrm{d}x=1.$$

We define $g:=[-t_0,t_0]\times[-s_0,s_0]\to\mathbb{R}$ by

$$g(t,s):=-\int_\Omega\varphi_\varepsilon\circ T_t^{-1}\circ S_s^{-1}\,\mathrm{d}x+\beta\left|\Omega\right|$$

for $s_0>0$ small enough. We want to use the implicit function theorem to find a function $t\mapsto s(t)$ such that $g(t,s(t))=0$. To this end, we notice that by assumption it holds $g(0,0)=g(0)=0$ and besides

$$\partial_s|_{s=0}g(0,s)=-\partial_s|_{s=0}\int_\Omega\varphi_\varepsilon\det\mathrm{D}S_s\,\mathrm{d}x=-\int_\Omega\varphi_\varepsilon\operatorname{div}W(0)\,\mathrm{d}x=1\neq0.\tag{7.35}$$

Moreover, since $V, W \in \overline{\mathcal{V}}_{ad}$ and thus $V(t), W(s) \in C^2\left(\overline{\Omega}, \mathbb{R}^d\right)$ for all $|t| \ll 1$ and $|s| \ll 1$, we see directly that g is continuously differentiable. And so the implicit function theorem yields the existence of some $\tau_0 > 0$ and a continuously differentiable function $s : [-\tau_0, \tau_0] \to \mathbb{R}$ such that

$$g(t, s(t)) = 0, \quad \forall t \in (-\tau_0, \tau_0), \qquad s'(0) = -\partial_s g(0,0)^{-1} \partial_t g(0,0).$$

The last identity can in view of (7.35) be rewritten as

$$s'(0) = -g'(0). \tag{7.36}$$

In particular, we obtain that $\varphi_\varepsilon \circ T_t^{-1} \circ S_{s(t)}^{-1} \in \Phi_{ad}$ for all $t \in (-\tau_0, \tau_0)$, compare Remark 3.4, and so

$$j_\varepsilon \left(\varphi_\varepsilon \circ T_t^{-1} \circ S_{s(t)}^{-1}\right) \geq j_\varepsilon \left(\varphi_\varepsilon\right)$$

holds for all t small enough. From this, we see

$$0 = \partial_t|_{t=0} j_\varepsilon \left(\varphi_\varepsilon \circ T_t^{-1} \circ S_{s(t)}^{-1}\right) = \partial_t|_{t=0} j_\varepsilon \left(\varphi_\varepsilon \circ \left(S_{s(t)} \circ T_t\right)^{-1}\right). \tag{7.37}$$

Introducing the notation $\widetilde{T}_t := S_{s(t)} \circ T_t$, we find from $S, T \in \overline{\mathcal{T}}_{ad}$ that $\widetilde{T} \in \overline{\mathcal{T}}_{ad}$ with $\partial_t|_{t=0} \widetilde{T}_t = W(0)s'(0) + V(0)$. Now we notice, that by (7.26)-(7.27) and (7.34) the expression $\partial_t|_{t=0} j_\varepsilon \left(\varphi_\varepsilon \circ T_t^{-1}\right)$ only depends on $\partial_t|_{t=0} T_t$ and that $C^1(\Omega) \ni \partial_t|_{t=0} T_t \mapsto j_\varepsilon(\varphi_\varepsilon \circ T_t^{-1})$ is linear. Thus, (7.37) reads as

$$\partial_s|_{s=0} j_\varepsilon \left(\varphi_\varepsilon \circ S_s^{-1}\right) s'(0) + \partial_t|_{t=0} j_\varepsilon \left(\varphi_\varepsilon \circ T_t^{-1}\right) = 0.$$

Defining

$$\lambda_\varepsilon := \partial_s|_{s=0} j_\varepsilon \left(\varphi_\varepsilon \circ S_s^{-1}\right) \in \mathbb{R} \tag{7.38}$$

we thus have

$$\partial_t|_{t=0} j_\varepsilon \left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon s'(0) = \lambda_\varepsilon g'(0) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\, \mathrm{d}x$$

where we made use of (7.36). This shows, that (7.32) is fulfilled for λ_ε, if λ_ε is defined by (7.38). As $g(0) = 0$, condition (7.33) holds trivially. And so it remains to show that $\lambda_\varepsilon \geq 0$. To this end, we recall that $\int_\Omega \varphi_\varepsilon = \beta\,|\Omega|$ and by the particular choice of $W \in \overline{\mathcal{V}}_{ad}$ we have

$$\partial_s|_{s=0} \left(\int_\Omega \varphi_\varepsilon \circ S_s^{-1}\, \mathrm{d}x\right) = \int_\Omega \varphi_\varepsilon \operatorname{div} W(0) = -1 < 0.$$

Thus, any $s > 0$ small enough fulfills $\int_\Omega \varphi_\varepsilon \circ S_s^{-1}\, \mathrm{d}x \leq \beta\,|\Omega|$, which yields that $\varphi_\varepsilon \circ S_s^{-1} \in \Phi_{ad}$. Hence,

$$j_\varepsilon \left(\varphi_\varepsilon \circ S_s^{-1}\right) \geq j_\varepsilon \left(\varphi_\varepsilon\right) \quad \forall 0 < s \ll 1$$

and thus we obtain

$$\lambda_\varepsilon = \partial_s|_{s=0} j_\varepsilon \left(\varphi_\varepsilon \circ S_s^{-1}\right) \geq 0.$$

So we have shown, that $\lambda_\varepsilon \in \mathbb{R}^+$ is a Lagrange multiplier for the integral constraint.

We finally remark that $\lambda_\varepsilon \in \mathbb{R}^+$ does not depend on the choice of the transformation $T \in \overline{\mathcal{T}}_{ad}$ or on its velocity field $V \in \overline{\mathcal{V}}_{ad}$. This can be seen in the definition of λ_ε, see (7.38), since the transformation $S \in \overline{\mathcal{T}}_{ad}$ is chosen independently of T and V. $\square$

So far, we have developed necessary optimality conditions for minimizers of (5.1) – (5.2) in Lemma 7.5. We will later see, that those optimality conditions converge under certain assumptions to optimality conditions derived in the sharp interface model as $\varepsilon \searrow 0$, see Sections 8.2 and 9.

But before considering the sharp interface model and the convergence of the optimality systems, we want to reformulate these conditions under some more regularity assumptions on the data and $\partial\Omega$ to get a formula that is more convenient.

Lemma 7.6. *Assume that $\partial\Omega \in C^2$, $\mathrm{D}_u f(\cdot, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) \in \boldsymbol{L}^2(\Omega)$ for $\boldsymbol{u}_\varepsilon \in \boldsymbol{H}^2(\Omega)$ and let the boundary data $\boldsymbol{g} \in \boldsymbol{H}^{\frac{3}{2}}(\partial\Omega)$.*

For any minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in \Phi_{ad} \times \boldsymbol{U}$ of (5.1) – (5.2) we have the following necessary optimality conditions:

$$-\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x = \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right), \tag{7.39}$$

$$\lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x - \beta\,|\Omega|\right) = 0 \tag{7.40}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$ with some Lagrange multiplier $\lambda_\varepsilon \geq 0$ for the integral constraint. The derivative $\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right)$ is given by (7.34) and can be reformulated as follows:

$$\begin{aligned}
&\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right)\mathrm{d}x + \\
&+ \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right)\mathrm{d}x + \int_{\partial\Omega} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) V(0)\cdot\boldsymbol{n}\,\mathrm{d}x + \\
&+ \mu \int_{\partial\Omega} \left(\partial_{\boldsymbol{n}}\boldsymbol{q}_\varepsilon \cdot \partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\right)\left(V(0)\cdot\boldsymbol{n}\right)\mathrm{d}x - \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\,\boldsymbol{n}\cdot\left(\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\left(V(0)\cdot\boldsymbol{n}\right)\right)\mathrm{d}x + \\
&+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x
\end{aligned} \tag{7.41}$$

where the adjoint variable $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ is given as the unique weak solution of

$$\begin{aligned}
\alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon &= \mathrm{D}_u f(\cdot, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) + \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{q}_\varepsilon &= 0 && \text{in } \Omega,\\
\boldsymbol{q}_\varepsilon &= \boldsymbol{0} && \text{on } \partial\Omega.
\end{aligned} \tag{7.42}$$

Remark 7.3. *Existence and uniqueness of a weak solution $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ of (7.42) follows from Lemma 7.3.*

Remark 7.4. *The definition of $\overline{\mathcal{V}}_{ad}$ implies for $V \in \overline{\mathcal{V}}_{ad}$ in particular $V(0)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, and so (7.41) can be simplified to*

$$\begin{aligned}\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = &\int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right) \operatorname{div}\left(\left|\boldsymbol{u}_\varepsilon\right|^2 V(0)\right) \mathrm{d}x + \\ &+ \int_\Omega \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right) \mathrm{d}x + \\ &+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right) \operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon \,\mathrm{d}x.\end{aligned} \tag{7.43}$$

Since we want to transfer the calculations of Lemma 7.6 to the sharp interface setting in Section 8.3, where Ω will be replaced by some set $E \subseteq \Omega$, we will carry out the calculations without using $V(0) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Proof. Since $\partial\Omega \in C^2$ we can apply regularity theory for the Stokes equations, see for instance [Soh01, Gal11, Tem77], to deduce from the state equations (7.26)-(7.27) and (5.2) that $\boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon[V]$ are in $\boldsymbol{H}^2(\Omega)$. Moreover, we get $\boldsymbol{H}^2$-regularity for $\boldsymbol{q}_\varepsilon$ by regularity theory applied to (7.42).

We define

$$\boldsymbol{u}_\varepsilon'[V] := \dot{\boldsymbol{u}}_\varepsilon[V] - \nabla\boldsymbol{u}_\varepsilon \cdot V(0)$$

which is a well-defined function in $\boldsymbol{H}^1(\Omega)$, because of our regularity results.

- *1st step:* We first prove that $\boldsymbol{u}_\varepsilon'[V]$ solves the following system in the weak formulation:

$$\begin{aligned}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon'[V] - \mu\Delta\boldsymbol{u}_\varepsilon'[V] + \nabla p_\varepsilon'[V] &= \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\boldsymbol{u}_\varepsilon && \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}_\varepsilon'[V] &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon'[V] &= -\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\left(V(0)\cdot\boldsymbol{n}\right) && \text{on } \partial\Omega.\end{aligned} \tag{7.44}$$

  To this end, we observe that for arbitrary $\boldsymbol{z} \in \boldsymbol{C}_0^\infty(\Omega)$ with $\operatorname{div}\boldsymbol{z} = 0$ it holds

$$\begin{aligned}\int_\Omega \nabla\boldsymbol{u}_\varepsilon'[V] : \nabla\boldsymbol{z}\,\mathrm{d}x = \int_\Omega \nabla\dot{\boldsymbol{u}}_\varepsilon[V] : \nabla\boldsymbol{z}\,\mathrm{d}x - \int_\Omega \mathrm{D}^2\boldsymbol{u}_\varepsilon V(0) : \nabla\boldsymbol{z}\,\mathrm{d}x - \\ - \int_\Omega \nabla V(0)\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z}\,\mathrm{d}x.\end{aligned}$$

  Besides, we have

$$\begin{aligned}&\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon'[V]\cdot\boldsymbol{z}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\dot{\boldsymbol{u}}_\varepsilon[V]\cdot\boldsymbol{z}\,\mathrm{d}x - \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left(\nabla\boldsymbol{u}_\varepsilon\cdot V(0)\right)\cdot\boldsymbol{z}\,\mathrm{d}x = \\ &= \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\dot{\boldsymbol{u}}_\varepsilon[V]\cdot\boldsymbol{z}\,\mathrm{d}x + \int_\Omega \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\left(\boldsymbol{u}_\varepsilon\cdot\boldsymbol{z}\right)\mathrm{d}x + \\ &+ \int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left(\operatorname{div}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z}V(0)\right)\cdot\boldsymbol{u}_\varepsilon\,\mathrm{d}x.\end{aligned}$$

  Thus, using (7.26) we get

$$
\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon'[V] \cdot \boldsymbol{z} \,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon'[V] : \nabla \boldsymbol{z} \,\mathrm{d}x = \\
&= \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T \nabla \boldsymbol{z} \,\mathrm{d}x - \int_\Omega \mu \mathrm{D}^2 \boldsymbol{u}_\varepsilon V(0) : \nabla \boldsymbol{z} \,\mathrm{d}x + \\
&+ \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \left(\operatorname{div} V(0) \boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} \right) \mathrm{d}x - \\
&- \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \operatorname{div} V(0) \,\mathrm{d}x + \int_\Omega \left(\nabla \boldsymbol{f} \cdot V(0) \right) \cdot \boldsymbol{z} + \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z} \,\mathrm{d}x + \\
&+ \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \left(\operatorname{div} V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) - \mathrm{D}V(0)\boldsymbol{z} \right) \cdot \boldsymbol{u}_\varepsilon \,\mathrm{d}x + \\
&+ \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon) \left(\mathrm{D}\varphi_\varepsilon V(0) \right) \left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{z} \right) \mathrm{d}x.
\end{aligned}
$$

Next we use that

$$
\operatorname{div} \left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) \right) = 0
$$

which follows from $\operatorname{div} \boldsymbol{z} = 0$, see Lemma 3.7, and see, as $\boldsymbol{z} \in \boldsymbol{C}_0^\infty(\Omega)$,

$$
\left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) \right)|_{\partial\Omega} = V(0)\mathrm{D}\boldsymbol{z}.
$$

Now we take $\left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) \right)$ for some $\boldsymbol{z} \in \boldsymbol{C}_0^\infty(\Omega)$ with $\operatorname{div} \boldsymbol{z} = 0$ as a test function for the state equations (5.2) (in the strong formulation), integrate over Ω and use the divergence theorem (note that we do not have zero boundary conditions in the test function) to get

$$
\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon \cdot \left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) \right) \mathrm{d}x + \\
&+ \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} - \mathrm{D}\boldsymbol{z} V(0) \right) \mathrm{d}x = \\
&= \int_\Omega \boldsymbol{f} \cdot \left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) \right) \mathrm{d}x + \int_{\partial\Omega} \mu \left(\mathrm{D}\boldsymbol{z} V(0) \right) \cdot \partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon \,\mathrm{d}x.
\end{aligned}
$$

Furthermore, we calculate

$$
\begin{aligned}
&\int_\Omega \boldsymbol{f} \cdot \left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z} V(0) \right) \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{z} \operatorname{div} V(0) - \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z} \,\mathrm{d}x + \\
&+ \int_{\partial\Omega} (\boldsymbol{z} \cdot \boldsymbol{f}) \left(V(0) \cdot \boldsymbol{n} \right) \mathrm{d}x - \int_\Omega (\boldsymbol{z} \cdot \boldsymbol{f}) \operatorname{div} V(0) \,\mathrm{d}x - \int_\Omega \boldsymbol{z} \cdot \left(\nabla \boldsymbol{f} \cdot V(0) \right) \mathrm{d}x = \\
&= - \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z} \,\mathrm{d}x - \int_\Omega \boldsymbol{z} \cdot \left(\nabla \boldsymbol{f} \cdot V(0) \right) \mathrm{d}x.
\end{aligned}
$$

Besides, we get with chain rule the following identities

$$
\int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \left(\mathrm{D}\boldsymbol{z} V(0) \right) \mathrm{d}x = \mu \int_\Omega \nabla \boldsymbol{u}_\varepsilon : \nabla V(0) \nabla \boldsymbol{z} \,\mathrm{d}x + \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}^2 \boldsymbol{z} V(0) \,\mathrm{d}x
$$

and

$$
\begin{aligned}
&\int_\Omega \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}^2 \boldsymbol{z} V(0) \,\mathrm{d}x = \int_\Omega \sum_{i,j,k=1}^d \partial_i u_j \left(\partial_k \partial_i z_j \right) \left(V(0) \right)_k \mathrm{d}x = \\
&= - \int_\Omega \mathrm{D}^2 \boldsymbol{u}_\varepsilon V(0) : \nabla \boldsymbol{z} \,\mathrm{d}x - \int_\Omega \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \operatorname{div} V(0) \,\mathrm{d}x + \int_{\partial\Omega} \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \left(V(0) \cdot \boldsymbol{n} \right) \mathrm{d}x.
\end{aligned}
$$

Hence, we have established

$$\begin{aligned}&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot(\operatorname{div}V(0)\boldsymbol{z}-\mathrm{D}V(0)\boldsymbol{z}+\mathrm{D}\boldsymbol{z}V(0))\,\mathrm{d}x+\int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon:\nabla V(0)\nabla\boldsymbol{z}\,\mathrm{d}x+\\&+\int_\Omega\mu\nabla\boldsymbol{u}_\varepsilon:\nabla(\operatorname{div}V(0)\boldsymbol{z}-\mathrm{D}V(0)\boldsymbol{z})\,\mathrm{d}x-\int_\Omega\mu\nabla\boldsymbol{u}_\varepsilon:\nabla\boldsymbol{z}\operatorname{div}V(0)\,\mathrm{d}x-\\&-\mu\int_\Omega \mathrm{D}^2\boldsymbol{u}_\varepsilon V(0):\nabla\boldsymbol{z}\,\mathrm{d}x=-\int_\Omega\boldsymbol{f}\cdot\mathrm{D}\boldsymbol{V}(0)\boldsymbol{z}\,\mathrm{d}x-\int_\Omega\boldsymbol{z}\cdot(\nabla\boldsymbol{f}\cdot V(0))\,\mathrm{d}x+\\&+\int_{\partial\Omega}(\nabla\boldsymbol{z}\cdot V(0))\cdot\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon-(\nabla\boldsymbol{u}_\varepsilon:\nabla\boldsymbol{z})\,(V(0)\cdot\boldsymbol{n})\,\mathrm{d}x.\end{aligned}$$

We continue by noticing that on $\partial\Omega$ it holds $\boldsymbol{z}=\boldsymbol{0}$ and so $\nabla\boldsymbol{z}$ has no tangential components. Similar we see, that for every $x\in\partial\Omega$ where $\boldsymbol{u}_\varepsilon(x)\neq\boldsymbol{0}$ we find $V(0,x)=0$ and so $\boldsymbol{u}_\varepsilon(x)$ has also no tangential components on $\{x\in\partial\Omega\mid V(0,x)\neq 0\}$. For almost every $x\in\partial\Omega$ such that $V(0,x)\neq 0$ we thus find

$$\nabla\boldsymbol{z}=\partial_{\boldsymbol{n}}\boldsymbol{z}\boldsymbol{n},\quad \nabla\boldsymbol{u}_\varepsilon=\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\boldsymbol{n}$$

hence

$$\partial_i z_j=n_i\partial_{\boldsymbol{n}}z_j,\quad \partial_i u_{\varepsilon,j}=n_i\partial_{\boldsymbol{n}}u_{\varepsilon,j}$$

and therefrom

$$(\nabla\boldsymbol{z}\cdot V(0))\cdot\nabla\boldsymbol{u}_\varepsilon\cdot\boldsymbol{n}=(V(0)\cdot\boldsymbol{n})\,\partial_{\boldsymbol{n}}\boldsymbol{z}\cdot\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon=\nabla\boldsymbol{u}_\varepsilon:\nabla\boldsymbol{z}\,(V(0)\cdot\boldsymbol{n})\,.$$

This leads to

$$\int_{\partial\Omega}(\nabla\boldsymbol{z}\cdot V(0))\cdot\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon-(\nabla\boldsymbol{u}_\varepsilon:\nabla\boldsymbol{z})(V(0)\cdot\boldsymbol{n})\,\mathrm{d}x=0.$$

And so we arrive in

$$\begin{aligned}&\int_\Omega\alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}'_\varepsilon[V]\cdot\boldsymbol{z}\,\mathrm{d}x+\int_\Omega\mu\nabla\boldsymbol{u}'_\varepsilon[V]:\nabla\boldsymbol{z}\,\mathrm{d}x=\\&=\int_\Omega\alpha'_\varepsilon(\varphi_\varepsilon)\,(\mathrm{D}\varphi_\varepsilon V(0))\,(\boldsymbol{u}_\varepsilon\cdot\boldsymbol{z})\,\mathrm{d}x\quad\forall\boldsymbol{z}\in\boldsymbol{C}_0^\infty(\Omega),\,\operatorname{div}\boldsymbol{z}=0.\end{aligned}\tag{7.45}$$

According to [Soh01, Tem77] the space $\{\boldsymbol{z}\in\boldsymbol{C}_0^\infty(\Omega)\mid\operatorname{div}\boldsymbol{u}=0\}$ is dense in $\boldsymbol{V}$ and from this we can deduce that (7.45) holds true for all $\boldsymbol{z}\in\boldsymbol{V}$.

Moreover, we have $\operatorname{div}\boldsymbol{u}_\varepsilon=0$ and as a consequence we find that

$$\operatorname{div}\boldsymbol{u}'_\varepsilon[V]=\operatorname{div}\left(\dot{\boldsymbol{u}}_\varepsilon[V]-\nabla\boldsymbol{u}_\varepsilon\cdot V(0)\right)=\operatorname{div}\dot{\boldsymbol{u}}_\varepsilon[V]-\nabla\boldsymbol{u}_\varepsilon:\mathrm{D}V(0)\overset{(7.27)}{=}0\tag{7.46}$$

and since $\dot{\boldsymbol{u}}_\varepsilon[V]=\boldsymbol{0}$ on $\partial\Omega$ we see

$$\boldsymbol{u}'_\varepsilon[V]=-\nabla\boldsymbol{u}_\varepsilon\cdot V(0)\qquad\text{a.e. on }\partial\Omega.$$

As already discussed above, $\nabla\boldsymbol{u}_\varepsilon$ has no tangential components on $\{x\in\partial\Omega\mid V(0,x)\neq0\}$, which gives

$$\boldsymbol{u}'_\varepsilon[V]=-\nabla\boldsymbol{u}_\varepsilon\cdot V(0)=-\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\,(V(0)\cdot\boldsymbol{n})\qquad\text{a.e. on }\partial\Omega.$$

Here we made in particular use of $V(0,x)=0$ for every $x\in\{\boldsymbol{g}\neq\boldsymbol{0}\}$, which follows from $V\in\overline{\mathcal{V}}_{ad}$. This finishes the first step.

- *2nd step:* We next reformulate (7.34) by using the results of the first step.

  Our calculations start with observing

$$
\begin{aligned}
&\int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot \dot{\boldsymbol{u}}_{\varepsilon}[V] \,\mathrm{d}x = \\
&= \int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x + \int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot (\nabla \boldsymbol{u}_{\varepsilon} \cdot V(0)) \,\mathrm{d}x = \\
&= \int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x + \int_{\Omega} \frac{1}{2} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \nabla |\boldsymbol{u}_{\varepsilon}|^2 \cdot V(0) \,\mathrm{d}x
\end{aligned} \tag{7.47}
$$

  which implies

$$
\begin{aligned}
&\int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon}) \left(\boldsymbol{u}_{\varepsilon} \cdot \dot{\boldsymbol{u}}_{\varepsilon}[V] + \frac{1}{2} |\boldsymbol{u}_{\varepsilon}|^2 \operatorname{div} V(0) \right) \mathrm{d}x = \\
&= \int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x + \int_{\Omega} \frac{1}{2} \alpha_{\varepsilon}(\varphi_{\varepsilon}) \operatorname{div} \left(|\boldsymbol{u}_{\varepsilon}|^2 V(0) \right) \mathrm{d}x.
\end{aligned} \tag{7.48}
$$

  We now proceed by testing the adjoint equation (7.42) with $\boldsymbol{u}_{\varepsilon}'[V]$ while taking the boundary data of $\boldsymbol{u}_{\varepsilon}'[V]$ into account and obtain

$$
\begin{aligned}
&\int_{\Omega} \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{q}_{\varepsilon} \cdot \boldsymbol{u}_{\varepsilon}'[V] + \mu \nabla \boldsymbol{q}_{\varepsilon} \cdot \nabla \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x + \int_{\partial\Omega} \pi_{\varepsilon} \boldsymbol{u}_{\varepsilon}'[V] \cdot \boldsymbol{n} \,\mathrm{d}x = \\
&= \int_{\Omega} \mathrm{D}_u f(x, \boldsymbol{u}_{\varepsilon}, \mathrm{D}\boldsymbol{u}_{\varepsilon})\, \boldsymbol{u}_{\varepsilon}'[V] + \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x + \\
&+ \mu \int_{\partial\Omega} \boldsymbol{u}_{\varepsilon}'[V] \cdot (\partial_{\boldsymbol{n}} \boldsymbol{q}_{\varepsilon}) \,\mathrm{d}x - \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_{\varepsilon}, \mathrm{D}\boldsymbol{u}_{\varepsilon})\, \boldsymbol{n} \cdot \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x = \\
&= \int_{\Omega} \mathrm{D}_u f(x, \boldsymbol{u}_{\varepsilon}, \mathrm{D}\boldsymbol{u}_{\varepsilon})\, \boldsymbol{u}_{\varepsilon}'[V] + \alpha_{\varepsilon}(\varphi_{\varepsilon})\, \boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{u}_{\varepsilon}'[V] \,\mathrm{d}x - \\
&- \mu \int_{\partial\Omega} (\partial_{\boldsymbol{n}} \boldsymbol{u}_{\varepsilon} \cdot \partial_{\boldsymbol{n}} \boldsymbol{q}_{\varepsilon})\, V(0) \cdot \boldsymbol{n} \,\mathrm{d}x + \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_{\varepsilon}, \mathrm{D}\boldsymbol{u}_{\varepsilon})\, \boldsymbol{n} \cdot (\partial_{\boldsymbol{n}} \boldsymbol{u}_{\varepsilon} (V(0) \cdot \boldsymbol{n})) \,\mathrm{d}x.
\end{aligned} \tag{7.49}
$$

  Let $x \in \partial\Omega$ be such that $\boldsymbol{g}(x) = \mathbf{0}$, then we find that $\nabla \boldsymbol{u}_{\varepsilon}$ has no tangential components, which means $\nabla \boldsymbol{u}_{\varepsilon} = \boldsymbol{n} \partial_{\boldsymbol{n}} \boldsymbol{u}_{\varepsilon}$, that is component-by-component $\partial_i u_{\varepsilon,j} = n_i \partial_{\boldsymbol{n}} u_{\varepsilon,j}$, and we obtain

$$
\boldsymbol{n} \cdot \partial_n \boldsymbol{u}_{\varepsilon} = \sum_{i=1}^{d} n_i \partial_{\boldsymbol{n}} u_{\varepsilon,i} = \operatorname{div} \boldsymbol{u}_{\varepsilon}.
$$

  From this we find

$$
\begin{aligned}
\int_{\partial\Omega} \pi_{\varepsilon} \boldsymbol{u}_{\varepsilon}'[V] \cdot \boldsymbol{n} \,\mathrm{d}x &= \int_{\partial\Omega} \pi_{\varepsilon} \left(-\partial_{\boldsymbol{n}} \boldsymbol{u}_{\varepsilon} \cdot \boldsymbol{n} \cdot (V(0) \cdot \boldsymbol{n}) \right) \mathrm{d}x = \\
&= \int_{\partial\Omega \cap \{\boldsymbol{g}=\mathbf{0}\}} \pi_{\varepsilon} \left(-\operatorname{div} \boldsymbol{u}_{\varepsilon} (V(0) \cdot \boldsymbol{n}) \right) \mathrm{d}x = 0
\end{aligned} \tag{7.50}
$$

  and so (7.49) reads as

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{q}_\varepsilon \cdot \boldsymbol{u}_\varepsilon'[V] + \mu \nabla \boldsymbol{q}_\varepsilon \cdot \nabla \boldsymbol{u}_\varepsilon'[V]\, \mathrm{d}x = \\
&= \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \boldsymbol{u}_\varepsilon'[V] + \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}_\varepsilon'[V]\, \mathrm{d}x - \\
&- \mu \int_{\partial\Omega} (\partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon \cdot \partial_{\boldsymbol{n}} \boldsymbol{q}_\varepsilon)\, V(0) \cdot \boldsymbol{n}\, \mathrm{d}x + \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \boldsymbol{n} \cdot (\partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon\, (V(0) \cdot \boldsymbol{n}))\, \mathrm{d}x.
\end{aligned} \tag{7.51}$$

In a similar way we now insert the adjoint state $\boldsymbol{q}_\varepsilon$ as a test function for the state equations (7.44) of $\boldsymbol{u}_\varepsilon'[V]$ and arrive in

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon'[V] \cdot \boldsymbol{q}_\varepsilon + \mu \nabla \boldsymbol{u}_\varepsilon'[V] \cdot \nabla \boldsymbol{q}_\varepsilon\, \mathrm{d}x = \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\, (\mathrm{D}\varphi_\varepsilon V(0))\, (\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon)\, \mathrm{d}x. \tag{7.52}$$

Comparing (7.51) and (7.52) we find

$$\begin{aligned}
\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon \cdot \boldsymbol{u}_\varepsilon'[V]\, \mathrm{d}x = &- \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \boldsymbol{u}_\varepsilon'[V]\, \mathrm{d}x + \\
&+ \mu \int_{\partial\Omega} (\partial_{\boldsymbol{n}} \boldsymbol{q}_\varepsilon \cdot \partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon)\, V(0) \cdot \boldsymbol{n}\, \mathrm{d}x - \\
&- \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \boldsymbol{n} \cdot (\partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon\, (V(0) \cdot \boldsymbol{n}))\, \mathrm{d}x + \\
&+ \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\, (\mathrm{D}\varphi_\varepsilon V(0))\, (\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon)\, \mathrm{d}x.
\end{aligned} \tag{7.53}$$

Substituting (7.53) into (7.34) and using (7.48) we have

$$\begin{aligned}
&\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2} \alpha_\varepsilon(\varphi_\varepsilon)\, \mathrm{div}\left(|\boldsymbol{u}_\varepsilon|^2\, V(0)\right) \mathrm{d}x + \\
&+ \int_\Omega [\mathrm{D}f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, (V(0), \dot{\boldsymbol{u}}_\varepsilon[V], \mathrm{D}\dot{\boldsymbol{u}}_\varepsilon[V] - \mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0)) + \\
&+ f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \mathrm{div}\, V(0)]\, \mathrm{d}x + \\
&+ \int_\Omega \left(\frac{\gamma\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon} \psi(\varphi_\varepsilon)\right) \mathrm{div}\, V(0) - \gamma\varepsilon \nabla\varphi_\varepsilon \cdot \nabla V(0) \nabla\varphi_\varepsilon\, \mathrm{d}x - \\
&- \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \boldsymbol{u}_\varepsilon'[V]\, \mathrm{d}x + \mu \int_{\partial\Omega} (\partial_{\boldsymbol{n}} \boldsymbol{q}_\varepsilon \cdot \partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon)\, V(0) \cdot \boldsymbol{n}\, \mathrm{d}x - \\
&- \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\, \boldsymbol{n} \cdot (\partial_{\boldsymbol{n}} \boldsymbol{u}_\varepsilon\, (V(0) \cdot \boldsymbol{n}))\, \mathrm{d}x + \\
&+ \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\, (\mathrm{D}\varphi_\varepsilon V(0))\, (\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon)\, \mathrm{d}x.
\end{aligned}$$

To simplify this expression, we insert $\dot{\boldsymbol{u}}_\varepsilon[V] = \boldsymbol{u}_\varepsilon'[V] + \nabla \boldsymbol{u}_\varepsilon \cdot V(0)$ and calculate

$$
\begin{aligned}
&\int_\Omega \mathrm{D}f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\left(V(0),\dot{\boldsymbol{u}}_\varepsilon\left[V\right],\mathrm{D}\dot{\boldsymbol{u}}_\varepsilon\left[V\right]-\mathrm{D}\boldsymbol{u}_\varepsilon\mathrm{D}V(0)\right)\,\mathrm{d}x=\\
&=\int_\Omega \mathrm{D}f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\big(V(0),\boldsymbol{u}'_\varepsilon\left[V\right]+\nabla\boldsymbol{u}_\varepsilon\cdot V(0),\mathrm{D}\boldsymbol{u}'_\varepsilon\left[V\right]+\mathrm{D}^2\boldsymbol{u}_\varepsilon V(0)\big)\,\mathrm{d}x=\\
&=\int_\Omega \mathrm{D}_{(2,3)}f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\big(\boldsymbol{u}'_\varepsilon\left[V\right],\mathrm{D}\boldsymbol{u}'_\varepsilon\left[V\right]\big)\,\mathrm{d}x+\\
&+\int_\Omega\left(\mathrm{D}f\right)\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\big(\cdot,\nabla\boldsymbol{u}_\varepsilon,\mathrm{D}^2\boldsymbol{u}_\varepsilon\big)V(0)\,\mathrm{d}x=\\
&=\int_\Omega \mathrm{D}_{(2,3)}f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\big(\boldsymbol{u}'_\varepsilon\left[V\right],\mathrm{D}\boldsymbol{u}'_\varepsilon\left[V\right]\big)\,\mathrm{d}x+\int_\Omega \mathrm{D}\left(f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\right)V(0)\,\mathrm{d}x
\end{aligned}
$$

wherefrom we arrive in the following expression

$$
\begin{aligned}
&\int_\Omega \mathrm{D}f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\left(V(0),\dot{\boldsymbol{u}}_\varepsilon\left[V\right],\mathrm{D}\dot{\boldsymbol{u}}_\varepsilon\left[V\right]-\mathrm{D}\boldsymbol{u}_\varepsilon\mathrm{D}V(0)\right)\,\mathrm{d}x+\\
&+\int_\Omega f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\,\mathrm{div}\,V(0)\,\mathrm{d}x=\\
&=\int_\Omega\big[\mathrm{D}_{(2,3)}f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\big(\boldsymbol{u}'_\varepsilon\left[V\right],\mathrm{D}\boldsymbol{u}'_\varepsilon\left[V\right]\big)+\,\mathrm{div}\,\left(f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)V(0)\right)\big]\,\mathrm{d}x.
\end{aligned}
$$

And so we conclude

$$
\begin{aligned}
&\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=\int_\Omega\frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\,\mathrm{div}\,\big(\left|\boldsymbol{u}_\varepsilon\right|^2V(0)\big)\,\mathrm{d}x+\\
&+\int_\Omega\alpha'_\varepsilon\left(\varphi_\varepsilon\right)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\left(\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\right)\,\mathrm{d}x+\int_\Omega\,\mathrm{div}\,\left(f\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)V(0)\right)\,\mathrm{d}x+\\
&+\mu\int_{\partial\Omega}\left(\partial_{\boldsymbol{n}}\boldsymbol{q}_\varepsilon\cdot\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\right)V(0)\cdot\boldsymbol{n}\,\mathrm{d}x-\int_{\partial\Omega}\left(\mathrm{D}_3f\right)\left(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon\right)\boldsymbol{n}\cdot\left(\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon\left(V(0)\cdot\boldsymbol{n}\right)\right)\,\mathrm{d}x+\\
&+\int_\Omega\left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2+\frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right)\mathrm{div}\,V(0)-\gamma\varepsilon\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x.
\end{aligned}
$$

Hence, we have shown the statement.

□

Now we consider again our example of minimizing the total potential power:

Example 7.2. Using the energy introduced in Example 2.3, thus

$$
f\left(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}\right)=\frac{\mu}{2}\left|\mathrm{D}\boldsymbol{u}\right|^2-\boldsymbol{f}(x)\cdot\boldsymbol{u}
$$

we find that the variation of the minimizer φ_ε along a transformation $T\in\overline{\mathcal{T}}_{ad}$ with velocity $V\in\overline{\mathcal{V}}_{ad}$ given by formula (7.34) reads as

$$
\begin{aligned}
&\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=\int_\Omega\alpha_\varepsilon\left(\varphi_\varepsilon\right)\bigg(\boldsymbol{u}_\varepsilon\cdot\dot{\boldsymbol{u}}_\varepsilon\left[V\right]+\frac{1}{2}\left|\boldsymbol{u}_\varepsilon\right|^2\,\mathrm{div}\,V(0)\bigg)\,\mathrm{d}x+\\
&+\int_\Omega\big[-\mathrm{D}\boldsymbol{f}V(0)-\boldsymbol{f}\cdot\dot{\boldsymbol{u}}_\varepsilon\left[V\right]+\mu\mathrm{D}\boldsymbol{u}_\varepsilon\cdot\left(\mathrm{D}\dot{\boldsymbol{u}}_\varepsilon\left[V\right]-\mathrm{D}\boldsymbol{u}_\varepsilon\mathrm{D}V(0)\right)+\\
&+\left(\frac{\mu}{2}\left|\mathrm{D}\boldsymbol{u}_\varepsilon\right|^2-\boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon\right)\mathrm{div}\,V(0)\bigg]\,\mathrm{d}x+\\
&+\int_\Omega\left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2+\frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right)\mathrm{div}\,V(0)-\gamma\varepsilon\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x.
\end{aligned}
\tag{7.54}
$$

Under the additional regularity assumptions stated in Lemma 7.6 this can in view of (7.41) be reformulated to

$$\begin{aligned}\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right) \mathrm{d}x+\\ + \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\mathrm{D}\varphi_\varepsilon V(0))(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon)\, \mathrm{d}x + \int_{\partial\Omega}\left(\frac{\mu}{2}|\nabla \boldsymbol{u}_\varepsilon|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon\right) V(0)\cdot\boldsymbol{n}\, \mathrm{d}x+\\ + \int_{\partial\Omega}\left(\mu\partial_{\boldsymbol{n}}\boldsymbol{q}_\varepsilon \cdot \partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon - \mu|\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon|^2\right)(V(0)\cdot\boldsymbol{n})\, \mathrm{d}x+\\ + \int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\, \mathrm{d}x.\end{aligned} \tag{7.55}$$

Inserting $\boldsymbol{u}_\varepsilon'[V]$, given by (7.44), as a test function in the adjoint system (7.42) and using the state equations for $\boldsymbol{u}_\varepsilon$ we find due to the special choice of the objective functional

$$\begin{aligned}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\mathrm{D}\varphi_\varepsilon V(0))(\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon) + \int_{\partial\Omega}\left(\mu\partial_{\boldsymbol{n}}\boldsymbol{q}_\varepsilon\cdot\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon - \mu|\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon|^2\right)(V(0)\cdot\boldsymbol{n}) =\\ = -\int_{\partial\Omega}\mu|\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon|^2 (V(0)\cdot\boldsymbol{n})\, \mathrm{d}x\end{aligned}$$

and so (7.55) can be rewritten as

$$\begin{aligned}\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right) \mathrm{d}x+\\ + \int_{\partial\Omega}\left(-\frac{\mu}{2}|\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon\right) V(0)\cdot\boldsymbol{n}\, \mathrm{d}x+\\ + \int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\, \mathrm{d}x.\end{aligned} \tag{7.56}$$

We see that again the formula for the derivative $\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right)$ reduces to a very simple structure where no adjoint state is needed. Inserting the assumption $V(0)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, which follows from $V \in \overline{\mathcal{V}}_{ad}$, we can even simplify the expression (7.56) to

$$\begin{aligned}\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right) \mathrm{d}x+\\ + \int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\, \mathrm{d}x\end{aligned} \tag{7.57}$$

and observe that we can compare this optimality criteria easily to those obtained by considering the variational inequality, see also Example 7.1, where also no adjoint variable was necessary and a very similar formula was found for first order calculus.
The similarity of those optimality criteria is actually no coincidence, as the next subsection will show for the general setting.

7.3 Linking the optimality criteria

The aim of this subsection is to discuss the connection between the optimality systems derived in the previous two subsections (see Lemma 7.6 and Theorem 7.1). More precisely, we will show in the following calculations that (7.39) – (7.40) together with (7.43) can be deduced directly from the variational inequality of Theorem 7.1.

Therefore, we assume again that φ_ε is a fixed minimizer of j_ε with $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ for some chosen $\varepsilon > 0$.

Due to Theorem 7.1 we find

$$\begin{aligned}\left(\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon) - \alpha_\varepsilon'(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \lambda_\varepsilon, \varphi - \varphi_\varepsilon\right)_{L^2(\Omega)} + \\ + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla(\varphi-\varphi_\varepsilon)\right)_{\boldsymbol{L}^2(\Omega)} \geq 0 \qquad \forall\varphi\in\overline{\Phi}_{ad}\end{aligned} \tag{7.58}$$

where $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ is given by (7.17) and $\lambda_\varepsilon \in \mathbb{R}^+$ is a Lagrange multiplier for the integral constraint fulfilling the complementarity condition

$$\lambda_\varepsilon\left(\int_\Omega \varphi_\varepsilon \,\mathrm{d}x - \beta|\Omega|\right) = 0.$$

We notice that $\boldsymbol{q}_\varepsilon$ coincides with the adjoint state defined in Section 7.2, which is given by (7.42), since the system has a unique solution and (7.42) equals the adjoint system in (7.17).

Let $T\in\overline{\mathcal{T}}_{ad}$ and define

$$\varphi_t := \varphi_\varepsilon\circ T_{-t}\in\overline{\Phi}_{ad}.$$

Inserting φ_t for $t>0$ small enough as a comparison function in (7.58) we get

$$\begin{aligned}\left(\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon) - \alpha_\varepsilon'(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \lambda_\varepsilon, \varphi_t - \varphi_\varepsilon\right)_{L^2(\Omega)} + \\ + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla(\varphi_t-\varphi_\varepsilon)\right)_{\boldsymbol{L}^2(\Omega)} \geq 0.\end{aligned}$$

Dividing by $t>0$ and letting $t\searrow 0$ gives

$$\begin{aligned}\left(\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon) - \alpha_\varepsilon'(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \lambda_\varepsilon, \partial_t|_{t=0}\varphi_t\right)_{L^2(\Omega)} + \\ + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla\partial_t|_{t=0}\varphi_t\right)_{\boldsymbol{L}^2(\Omega)} \geq 0.\end{aligned}$$

Using $\partial_t|_{t=0}\varphi_t = -\mathrm{D}\varphi_\varepsilon V(0)$ we find

$$\begin{aligned}&\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla\partial_t|_{t=0}\varphi_t\,\mathrm{d}x = \partial_t|_{t=0}\frac{1}{2}\int_\Omega|\nabla\varphi_t|^2\,\mathrm{d}x = \partial_t|_{t=0}\frac{1}{2}\int_\Omega|\nabla(\varphi_\varepsilon\circ T_{-t})|^2\,\mathrm{d}x = \\ &= \frac{1}{2}\int_\Omega|\nabla\varphi_\varepsilon|^2\operatorname{div}V(0) - 2\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x.\end{aligned}$$

Besides, using chain rule and the divergence theorem, we obtain

$$\int_\Omega \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon)\,\partial_t|_{t=0}\varphi_t\,\mathrm{d}x = -\frac{\gamma}{\varepsilon}\int_\Omega \mathrm{D}(\psi_0(\varphi_\varepsilon))\cdot V(0)\,\mathrm{d}x = \frac{\gamma}{\varepsilon}\int_\Omega \psi_0(\varphi_\varepsilon)\,\operatorname{div} V(0)\,\mathrm{d}x - \\ - \int_{\partial\Omega}\frac{\gamma}{\varepsilon}\psi_0(\varphi_\varepsilon)\underbrace{V(0)\cdot \boldsymbol{n}}_{=0}\,\mathrm{d}x = \frac{\gamma}{\varepsilon}\int_\Omega \psi_0(\varphi_\varepsilon)\,\operatorname{div} V(0)\,\mathrm{d}x.$$

By using change of variables we can deduce

$$\int_\Omega \frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)\,|\boldsymbol{u}_\varepsilon|^2\,\partial_t|_{t=0}\varphi_t\,\mathrm{d}x = \frac{1}{2}\partial_t|_{t=0}\int_\Omega (\alpha_\varepsilon(\varphi_t))\,|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = \\ = \frac{1}{2}\partial_t|_{t=0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\boldsymbol{u}_\varepsilon\circ T_t|^2\det \mathrm{D}T_t\,\mathrm{d}x = \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2\,V(0)\right)\mathrm{d}x$$

and similar we get

$$\lambda_\varepsilon\partial_t|_{t=0}\int_\Omega \varphi_t\,\mathrm{d}x = \lambda_\varepsilon\partial_t|_{t=0}\int_\Omega \varphi_\varepsilon\det\mathrm{D}T_t\,\mathrm{d}x = \lambda_\varepsilon\int_\Omega \operatorname{div} V(0)\varphi_\varepsilon\,\mathrm{d}x$$

which follows with Lemma 3.3. And finally we see

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\partial_t|_{t=0}\varphi_t\,\mathrm{d}x = -\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\mathrm{D}\varphi_\varepsilon V(0)\,\mathrm{d}x.$$

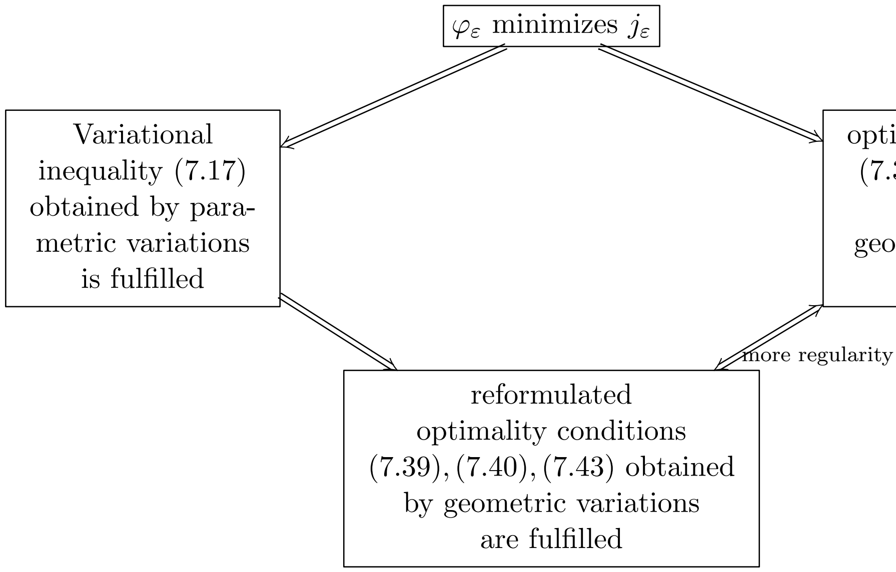
Thus we have deduced from the variational inequality (7.58) the following equation

$$\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)\operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2\,V(0)\right)\mathrm{d}x + \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\mathrm{D}\varphi_\varepsilon V(0)\,\mathrm{d}x + \\ + \int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x = -\lambda_\varepsilon\int_\Omega \operatorname{div} V(0)\varphi_\varepsilon\,\mathrm{d}x \tag{7.59}$$

by doing the same calculations as above again with t replaced by $-t$.

By noticing that the optimality condition (7.32) together with (7.43), see Lemma 7.6, equals (7.59) we have shown that (7.32) – (7.33) together with (7.43) can be derived from the variational inequality of Theorem 7.1 directly.

Summary To summarize the results of Section 7, we see that any minimizer $\varphi_\varepsilon \in \Phi_{ad}$ of j_ε fulfills the variational inequality stated in Theorem 7.1. Moreover, we have derived under the general assumptions of this section the optimality conditions of Lemma 7.5, which were obtained by varying the minimizer φ_ε along a transformation of Ω. We have shown in Lemma 7.6 that these optimality conditions are equivalent to (7.39) – (7.40) together with (7.43) if we state more regularity on the data and Ω. But as we have seen in this subsection, (7.43) can be derived directly from the variational inequality of Theorem 7.1, and thus (7.39) – (7.40) together with (7.43) is fulfilled for any minimizer of j_ε. We summarize these results in the following diagram:


φ_ε minimizes j_ε
Variational inequality (7.17) obtained by parametric variations is fulfilled
optimality conditions (7.32), (7.33), (7.34) obtained by geometric variations are fulfilled
more regularity
reformulated optimality conditions (7.39), (7.40), (7.43) obtained by geometric variations are fulfilled

8 Optimality conditions for the sharp interface model

For this section we assume that $\varphi_0 \in \Phi^0_{ad}$ is a fixed minimizer of j_0 with $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$. The existence of such a minimizer is so far only guaranteed under certain assumptions, see also Theorem 6.1 and discussion in Section 6.4.

We will derive in this section first order necessary optimality conditions for this minimizer of j_0. Notice that, as φ_0 minimizes j_0, the pair $(\varphi_0, \boldsymbol{u}_0)$ solves the optimization problem (6.18) – (6.19) and so we calculate in this section necessary optimality conditions for the sharp interface formulation of the shape and topology optimization problem. Again, as done for the phase field model in Section 7, we will do this in two ways. The first one is the classical approach, which means in this case calculating shape derivatives. But to do this, we have to impose more regularity, not only on Ω and the objective functional, but in particular on the minimizing set E^{φ_0}. This regularity may not be true in general. Thus we use a second ansatz in Section 8.2, which also relies on geometric variations. Hereby, we obtain first order necessary optimality conditions without stating additional assumptions, which means that they hold true for our given minimizer. Moreover, we can show that the optimality conditions derived in Section 8.2 are equivalent to the classical shape derivatives if the regularity assumptions that are necessary for deriving the shape derivatives are fulfilled, see Section 8.3.

Throughout the following section we state as in the previous section additionally Assumptions **(A6)** and **(A7)**, which ensure differentiability of the objective functional and enough regularity on the external force term.

8.1 Shape derivative approach

We start by deriving optimality conditions in the context of shape calculus. This is a classical approach for shape optimization problems and is based on local smooth variations of the minimizing set. The best general reference here is [DZ01, SZ92].

The minimization problem

$$\min_{\varphi \in L^1(\Omega)} j_0(\varphi)$$

can equivalently be formulated as

$$\min_{\varphi \in \Phi^0_{ad}} j_0(\varphi). \tag{8.1}$$

Thus we minimize the functional $\Phi^0_{ad} \ni \varphi \mapsto j_0(\varphi)$, though it only depends on E^{φ}. Heuristically one could therefore rewrite this as

$$\min_{E \subset \Omega, |E| \leq 0.5(1+\beta)|\Omega|} J_S(E) := j_0(2\chi_E - 1) = J_0\left(2\chi_E - 1, \boldsymbol{S}_0(2\chi_E - 1)\right). \tag{8.2}$$

Therefore, minimizing J_S can be considered as a classical shape optimization problem.

In this subsection we want to derive first order optimality conditions for (8.1) by considering this as an shape optimization problem and calculating shape derivatives, as it is done for instance in [DZ01, SZ92, Sim91, Sim80, BFCLS96, SS10, Pir73] and a lot more work that can be found in literature concerning shape sensitivity analysis in fluid dynamics. We follow the works mentioned above in stating more regularity assumptions than we have actually shown so far to ensure the correctness of the following analysis. Hence, we define

$$E_0 := \operatorname{int}\left(\{x \in \Omega \mid \varphi_0(x) = 1\}\right)$$

and assume

$$E_0 \text{ is a fixed open subset of } \Omega \text{ with } \partial E_0 \in C^2, \tag{8.3a}$$

$$E_0 \text{ has } N < \infty \text{ connected components}, \tag{8.3b}$$

$$\boldsymbol{g} \in \boldsymbol{H}^{\frac{3}{2}}(\partial\Omega), \tag{8.3c}$$

$$\mathrm{D}_u f\,(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \in \boldsymbol{L}^2(E_0), \text{ if } \boldsymbol{u}_0 \in \boldsymbol{H}^2(E_0). \tag{8.3d}$$

In the following we denote by $C_1, \ldots, C_N$ the connected components of E_0.

To apply the classical approach of shape calculus, see for instance [DZ01, SZ92], we choose $T \in \overline{\mathcal{T}}_{ad}$ and define

$$E_t := T_t(E_0).$$

Since we have non-homogeneous boundary data we first have to ensure that we can still find a solution to the state equations in the transformed region $T_t(E_0)$. This is a consequence of the fact that $\boldsymbol{U}^{E_0} \neq \emptyset$ as the following lemma shows:

Lemma 8.1. *The solution space $\boldsymbol{U}^{E_t}$ is not empty for t small enough.*

Proof. We have that $(\varphi_0, \boldsymbol{u}_0)$ are a solution to the optimization problem (6.18) – (6.19) since φ_0 minimizes j_0. This gives in particular $\boldsymbol{u}_0 \in \boldsymbol{U}^{\varphi_0}$ and thus $\boldsymbol{U}^{E_0} \neq \emptyset$. Defining

$$\boldsymbol{v}_t := (\det \mathrm{D}T_t)\,\mathrm{D}T_t^{-1}\boldsymbol{u}_0 \circ T_t^{-1}$$

we see with Lemma 3.6 that $\operatorname{div} \boldsymbol{v}_t = 0$, since $\operatorname{div} \boldsymbol{u}_0 = 0$. Due to $\boldsymbol{u}_0 \in \boldsymbol{U}^{E_0}$ and the choice of $T \in \overline{\mathcal{T}}_{ad}$ we get additionally that $\boldsymbol{v}_t \in \boldsymbol{U}^{E_t}$ and thus $\boldsymbol{U}^{E_t} \neq \emptyset$, compare Lemma 3.6. □

In the following we will use the notation

$$E_t = T_t(E_0), \quad \Omega_t = T_t(\Omega), \quad \boldsymbol{u}_t = \boldsymbol{S}_0(2\chi_{E_t} - 1).$$

We define for every t with $|t| \ll 1$, an associated pressure $p_t \in L^2(E_t)$ as follows: Lemma 4.4 gives the existence of some $p_t \in L^2(E_t)$ such that it holds in the distributional sense

$$\nabla p_t = \boldsymbol{f} + \mu\Delta\boldsymbol{u}_t \quad \text{ in } E_t.$$

By imposing $\int_{T_t(C_i)} p_t \,\mathrm{d}x = 0$ for every $i = 1, \ldots, N$ this function is uniquely defined. In particular, we make use of the fact that for t small enough the connected components of $T_t(E_0)$ will be given by $T_t(C_i)$, $i = 1, \ldots, N$.
Since E_t has C^2-regularity we can then prove the following regularity result for the solution $(\boldsymbol{u}_t, p_t)$ of the Stokes equations in E_t:

Lemma 8.2. *From the regularity assumptions* (8.3) *it follows $\boldsymbol{u}_t \in \boldsymbol{H}^2(E_t)$ for t small enough. Moreover, we obtain $p_t \in H^1(E_t)$. In particular, this holds true for $t = 0$.*

Proof. We see that $\boldsymbol{u}_t = \boldsymbol{S}_0\left(2\chi_{E_t} - 1\right)$ solves together with p_t the following classical Stokes equations in E_t:

$$\begin{aligned} -\mu\Delta\boldsymbol{u}_t + \nabla p_t &= \boldsymbol{f} && \text{in } E_t,\\ \operatorname{div}\boldsymbol{u}_t &= 0 && \text{in } E_t,\\ \boldsymbol{u}_t &= \boldsymbol{g} && \text{on } \partial\Omega \cap \partial E_t,\\ \boldsymbol{u}_t &= \boldsymbol{0} && \text{on } \Omega \cap \partial E_t, \end{aligned}$$

which are well-defined due to Lemma 8.1.
Then the statement follows from classical regularity results for the Stokes equations, see for instance [Gal11, Tem77]. □

As we want to derive first order optimality conditions we have to differentiate the solution operator of the state equations with respect to the transformation. Therefore, we show the following differentiability result:

Lemma 8.3. *The mappings* $\mathbb{R} \supseteq I \ni t \mapsto \boldsymbol{u}_t \circ T_t \in \boldsymbol{H}^2(E_0)$ *and* $\mathbb{R} \supseteq I \ni t \mapsto p_t \circ T_t \in H^1(E_0)$ *are differentiable at* $t = 0$*, where* $0 \in I \subseteq \mathbb{R}$ *is a small interval.*

Proof. To show the stated result we want to apply an implicit function argument. The problem hereby is the special form of the fluid dynamic equations since the range of $\boldsymbol{H}^1_0(E_0) \ni \boldsymbol{v} \mapsto \operatorname{div}\boldsymbol{v}$ is $L^2_0(E_0)$. But neither $(\operatorname{div}\boldsymbol{u}_t)\circ T_t$ nor $\nabla\boldsymbol{v} : \mathrm{D}T_t^{-1}$ for some $\boldsymbol{v} \in \boldsymbol{H}^1_0(E_0)$ is in general an element of $L^2_0(E_0)$ and so it is not so obvious how to define the function on which to apply the implicit function theorem.
For this reason we will apply a different implicit function statement, namely Theorem A.3, in a similar way as it is applied in [Sim91]. To this end we choose some $\boldsymbol{b} \in \boldsymbol{H}^2(E_0)$ such that $\boldsymbol{b}|_{\partial\Omega\cap\partial E_0} = \boldsymbol{g}$, $\boldsymbol{b}|_{\Omega\cap\partial E_0} = \boldsymbol{0}$ and $\operatorname{div}\boldsymbol{b} = 0$, which is possible due to the results of [Gal11, Section III.3]. The assumptions necessary for applying those results are fulfilled, since we have $\partial E_0 \in C^2$ and the thanks to E_0 being a minimizer of J_0^S we find $\boldsymbol{U}^{E_0} \neq \varnothing$ and thus $\int_{\partial\Omega\cap\partial E_0} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}x = 0$.
Then we define

$$F : I \times \left(\boldsymbol{H}^2(E_0) \cap \boldsymbol{H}^1_0(E_0)\right) \times H^1(E_0) \to \boldsymbol{L}^2(E_0) \times H^1(E_0) \times \mathbb{R}^N$$

by

$$\begin{aligned} F(t,\boldsymbol{v},p) = \Bigg(&-\mu\sum_{i,j,k=1}^{d}\left(\mathrm{D}T_t^{-T}\right)_{ij}\partial_j\left(\left(\mathrm{D}T_t^{-T}\right)_{ik}\partial_k\boldsymbol{v}\right) + \mathrm{D}T_t^{-T}\nabla p,\\ &\mathrm{D}T_t^{-1} : \nabla\boldsymbol{v}, \left(\int_{C_i} p\det \mathrm{D}T_t\,\mathrm{d}x\right)_{i=1}^{N}\Bigg). \end{aligned}$$

We see that $F(t,\cdot) \in \mathcal{L}\left(\left(\boldsymbol{H}^2(E_0)\cap\boldsymbol{H}^1_0(E_0)\right)\times\boldsymbol{H}^1(E_0), \boldsymbol{L}^2(E_0)\times H^1(E_0)\times\mathbb{R}^N\right)$ for all $t \in I$. An easy computation shows for all $t \in I$ that

$$\begin{aligned} F(t,\boldsymbol{u}_t\circ T_t - \boldsymbol{b}, p_t\circ T_t) = \Bigg(&(-\mu\Delta\boldsymbol{u}_t + \nabla p_t)\circ T_t, (\operatorname{div}\boldsymbol{u}_t)\circ T_t, \left(\int_{T_t(C_i)} p_t\,\mathrm{d}x\right)_{i=1}^{N}\Bigg) -\\ &- F(t,\boldsymbol{b},0) = (\boldsymbol{f}\circ T_t, 0, 0) - F(t,\boldsymbol{b},0). \end{aligned}$$

Thanks to the regularity of $T \in \overline{\mathcal{T}}_{ad}$ it follows directly that

$$I \ni t \mapsto F(t,\cdot) \in \mathcal{L}\left(\left(\boldsymbol{H}^2(E_0) \cap \boldsymbol{H}_0^1(E_0)\right) \times H^1(E_0), \boldsymbol{L}^2(E_0) \times H^1(E_0) \times \mathbb{R}^N\right)$$

is differentiable at $t = 0$. We have moreover differentiability of

$$I \ni t \mapsto (\boldsymbol{f} \circ T_t, 0, 0) - F(t, \boldsymbol{b}, 0) \in \boldsymbol{L}^2(E_0) \times H^1(E_0) \times \mathbb{R}^N$$

at $t = 0$. Next we observe

$$F(0, \boldsymbol{v}, q) = \left(-\mu\Delta\boldsymbol{v} + \nabla q, \operatorname{div}\boldsymbol{v}, \left(\int_{C_i} q \, \mathrm{d}x\right)_{i=1}^N\right).$$

To establish (A.1) we denote $\left(\boldsymbol{r}_1, r_2, (s_i)_{i=1}^N\right) := F(0, \boldsymbol{v}, q)$ for arbitrary $(\boldsymbol{v}, q) \in \left(\boldsymbol{H}^2(E_0) \cap \boldsymbol{H}_0^1(E_0)\right) \times H^1(E_0)$. We find that $r_2 = \operatorname{div}\boldsymbol{v}$ for $\boldsymbol{v} \in \boldsymbol{H}_0^1(E_0)$ and thus $\int_{C_i} r_2 \, \mathrm{d}x = 0$ for all $i = 1, \ldots, N$. At the same time, $(\boldsymbol{v}, q)$ solve the following general non-homogeneous Stokes system:

$$\begin{aligned} -\Delta\boldsymbol{v} + \nabla q &= \boldsymbol{r}_1 && \text{in } E_0, \\ \operatorname{div}\boldsymbol{v} &= r_2 && \text{in } E_0, \\ \boldsymbol{v} &= \boldsymbol{0} && \text{on } \partial E_0, \end{aligned}$$

together with $\int_{C_i} q \, \mathrm{d}x = s_i$, $i = 1, \ldots, N$. Using standard results for the Stokes equations, see for instance [Tem77, Proposition 2.2], this yields the following estimate:

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{H}^2(E_0)} + \|q\|_{H^1(E_0)} &\leq c_0(\mu, E_0)\left(\|\boldsymbol{r}_1\|_{\boldsymbol{L}^2(E_0)} + \|r_2\|_{H^1(E_0)} + \sum_{i=1}^N |s_i|\right) = \\ &= c_0(\mu, E_0) \, \|F(0, \boldsymbol{v}, q)\|_{\boldsymbol{L}^2(E_0) \times H^1(E_0) \times \mathbb{R}^N} \end{aligned}$$

which implies (A.1). And so we can finally apply Theorem A.3 to get differentiability of $t \mapsto \boldsymbol{u}_t \circ T_t \in \boldsymbol{H}^2(E_0)$, $t \mapsto p_t \circ T_t \in H^1(E_0)$ and deduce the statement.

□

Thanks to this lemma we now have all assumptions fulfilled to use [Sim80, Theorem 3.1, Theorem 3.2] and conclude the formula for the shape derivative $\boldsymbol{u}'[V] := \partial_t|_{t=0}(\boldsymbol{u}_t \circ T_t) - \nabla\boldsymbol{u}_0 \cdot V(0)$:

Lemma 8.4. *The shape derivative $\boldsymbol{u}'[V] \in \boldsymbol{H}^1(E_0)$ is given as the unique weak solution of*

$$\begin{aligned} -\mu\Delta\boldsymbol{u}'[V] + \nabla p'[V] &= \boldsymbol{0} && \textit{in } E_0, && \text{(8.4a)} \\ \operatorname{div}\boldsymbol{u}'[V] &= 0 && \textit{in } E_0, && \text{(8.4b)} \\ \boldsymbol{u}'[V] &= -\left(V(0) \cdot \nu\right) \partial_\nu \boldsymbol{u}_0 && \textit{on } \partial E_0 \cap \Omega, && \text{(8.4c)} \\ \boldsymbol{u}'[V] &= \boldsymbol{0} && \textit{on } \partial E_0 \cap \partial\Omega, && \text{(8.4d)} \end{aligned}$$

where ν denotes the outer unit normal on E_0.

Proof. In view of the differentiability result of Lemma 8.3 we can directly apply [Sim80, Theorem 3.1, Theorem 3.2], since

$$\left(\boldsymbol{H}^1(E_t) \times L_0^2(E_t)\right) \ni (\boldsymbol{u}, p) \mapsto -\mu\Delta\boldsymbol{u} + \nabla p \in \mathcal{D}'(E_t)$$

is obviously differentiable. Moreover, existence and uniqueness of a solution to (8.4) follows with arguments similar to those in the proof of Lemma 8.3 by applying Lax-Milgram's theorem A.2 once we show $\int_{\partial E_0 \cap \Omega} (V(0) \cdot \nu)\, \partial_\nu \boldsymbol{u}_0 \cdot \nu \,\mathrm{d}x = 0$. But this can be deduced as in [Sim91] by observing that $\boldsymbol{u}_0 = \boldsymbol{0}$ on $\partial E_0 \cap \Omega$ and so $\nabla \boldsymbol{u}_0 = \nu \partial_\nu \boldsymbol{u}_0$, thus $\partial_i u_{0,j} = \nu_i \partial_\nu u_{0,j}$, on $\partial E_0 \cap \Omega$ wherefrom we obtain

$$\nu \cdot \partial_\nu \boldsymbol{u}_0 = \sum_{i=1}^d \nu_i \cdot \partial_\nu u_{0,i} = \sum_{i=1}^d \partial_i u_{0,i} = \operatorname{div} \boldsymbol{u}_0 = 0.$$

□

Now we can finally calculate the shape derivative $\mathrm{D}J_S(E_0)[V] = \partial_t|_{t=0} J_S(T_t(E_0))$ of J_S:

Theorem 8.1. *Since E_0 is assumed to minimize J_S, the following necessary optimality conditions are fulfilled*

$$\mathrm{D}J_S(E_0)[V] = -\lambda_0 \int_\Omega (2\chi_{E_0} - 1) \operatorname{div} V(0)\,\mathrm{d}x \quad \forall V \in \overline{\mathcal{V}}_{ad} \tag{8.5}$$

for some Lagrange multiplier $\lambda_0 \geq 0$, which fulfills moreover

$$\lambda_0 \left(\int_\Omega (2\chi_{E_0} - 1) - \beta\,|\Omega| \right) = 0 \tag{8.6}$$

and the shape derivative is given by

$$\begin{aligned} \mathrm{D}J_S(E_0)[V] = &\int_{\partial E_0} \left(\mu \partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0 - (\mathrm{D}_3 f)(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\, \nu \cdot \partial_\nu \boldsymbol{u}_0\right) V(0) \cdot \nu \,\mathrm{d}x + \\ &+ \int_{E_0} \mathrm{D}\left(f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\right) V(0)\,\mathrm{d}x + \int_\Omega f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \operatorname{div} V(0)\,\mathrm{d}x + \\ &+ \gamma c_0 \int_{\partial E_0 \cap \Omega} \kappa V(0) \cdot \nu \,\mathrm{d}x. \end{aligned} \tag{8.7}$$

Here κ denotes the mean curvature of $\partial E_0 \cap \Omega$, ν the outer unit normal on E_0 and $\boldsymbol{q}_0 \in \boldsymbol{H}_0^1(E_0)$ the adjoint state given as strong solution of

$$-\mu\Delta\boldsymbol{q}_0 + \nabla\pi_0 = \mathrm{D}_u f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \qquad \text{in } E_0, \tag{8.8a}$$

$$\operatorname{div} \boldsymbol{q}_0 = 0 \qquad \text{in } E_0, \tag{8.8b}$$

$$\boldsymbol{q}_0 = \boldsymbol{0} \qquad \text{on } \partial E_0. \tag{8.8c}$$

Remark 8.1. *Note that $\lambda_0 \geq 0$ is a Lagrange multiplier for the volume constraint $|E_0| \leq 0.5(1+\beta)\,|\Omega|$ and thus (8.6), which can be rewritten as*

$$\lambda_0 \left(|E_0| - \frac{(1+\beta)}{2} |\Omega| \right) = 0,$$

is the associated complementarity condition.

Remark 8.2. *Again existence and uniqueness of a solution to* (8.8) *can be shown directly by Lax-Milgram's theorem A.2.*

Proof. Let us first observe that we obtain due to the stated regularity of E_0 from (3.10)

$$\partial_t|_{t=0} P_\Omega(E_t) = \int_{\partial E_0 \cap \Omega} \kappa V(0) \cdot \nu \,\mathrm{d}x.$$

We recall for the sake of convenience, cf. (8.2), that J_S is given by

$$J_S(E_t) = \int_{\Omega_t} f(x, \boldsymbol{u}_t, \mathrm{D}\boldsymbol{u}_t)\,\mathrm{d}x + \gamma c_0 P_\Omega(E_t).$$

In order to prove (8.7) we have to calculate $\partial_t|_{t=0} J_S(E_t)$. To this end, we use chain rule and change of variables to compute

$$\begin{aligned}
&\partial_t|_{t=0} \int_{\Omega_t} f(x, \boldsymbol{u}_t, \mathrm{D}\boldsymbol{u}_t)\,\mathrm{d}x = \partial_t|_{t=0} \int_{\Omega} f(x, \boldsymbol{u}_t, \mathrm{D}\boldsymbol{u}_t) \circ T_t \det \mathrm{D}T_t \,\mathrm{d}x = \\
&= \int_\Omega \left[\partial_t|_{t=0} \left(f\left(T_t(x), \boldsymbol{u}_t \circ T_t, \mathrm{D}\left(\boldsymbol{u}_t \circ T_t\right) \mathrm{D}T_t^{-1}\right)\right) + f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \operatorname{div} V(0)\right] \mathrm{d}x = \\
&= \int_\Omega [\mathrm{D}f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)(V(x), \dot{\boldsymbol{u}}_0[V], \mathrm{D}\dot{\boldsymbol{u}}_0[V] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)) + \\
&+ f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \operatorname{div} V(0)]\,\mathrm{d}x
\end{aligned}$$

where we use the notation $\dot{\boldsymbol{u}}_0[V](x) = \partial_t|_{t=0}(\boldsymbol{u}_t \circ T_t)(x)$.
Inserting $\dot{\boldsymbol{u}}[V] = \boldsymbol{u}'[V] + \nabla \boldsymbol{u}_0 \cdot V(0)$, see (3.5), we obtain

$$\begin{aligned}
&\int_\Omega \mathrm{D}f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)(V(x), \dot{\boldsymbol{u}}_0[V], \mathrm{D}\dot{\boldsymbol{u}}_0[V] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0))\,\mathrm{d}x = \\
&= \int_\Omega \left[\mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\left(\boldsymbol{u}'[V], \mathrm{D}\boldsymbol{u}'[V]\right) + \mathrm{D}f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\left(\cdot, \nabla \boldsymbol{u}_0, \mathrm{D}^2 \boldsymbol{u}_0\right) V(0)\right] \mathrm{d}x.
\end{aligned}$$

Using additionally $\boldsymbol{u}'[V] = \boldsymbol{u}_0 = \mathbf{0}$ in $\Omega \setminus E_0$ yields the rewritten equation

$$\begin{aligned}
&\partial_t|_{t=0} \int_{\Omega_t} f(x, \boldsymbol{u}_t, \mathrm{D}\boldsymbol{u}_t)\,\mathrm{d}x = \int_{E_0} \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\left(\boldsymbol{u}'[V], \mathrm{D}\boldsymbol{u}'[V]\right)\mathrm{d}x + \\
&+ \int_{E_0} \mathrm{D}f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\left(\cdot, \nabla \boldsymbol{u}_0, \mathrm{D}^2 \boldsymbol{u}_0\right) V(0)\,\mathrm{d}x + \int_\Omega f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \operatorname{div} V(0)\,\mathrm{d}x = \\
&= \int_{E_0} \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\left(\boldsymbol{u}'[V], \mathrm{D}\boldsymbol{u}'[V]\right)\mathrm{d}x + \\
&+ \int_{E_0} \mathrm{D}\left(f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)\right) V(0)\,\mathrm{d}x + \int_\Omega f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \operatorname{div} V(0)\,\mathrm{d}x.
\end{aligned}$$

We can multiply the adjoint equation (8.8a) with $\boldsymbol{u}'[V]$, integrate over E_0 and use integration by parts (notice that $\boldsymbol{u}'[V]$ has non-homogeneous boundary conditions on ∂E_0) to arrive in

$$\begin{aligned}
&\int_{E_0} \mu \nabla \boldsymbol{q}_0 \cdot \nabla \boldsymbol{u}'[V]\,\mathrm{d}x \underbrace{- \int_{\partial E_0} \mu \partial_\nu \boldsymbol{q}_0 \cdot \boldsymbol{u}'[V]\,\mathrm{d}x}_{= \int_{\partial E_0} \mu \partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0 (V(0)\cdot\nu)\,\mathrm{d}x} + \int_{\partial E_0} \pi_0 \boldsymbol{u}'[V] \cdot \nu \,\mathrm{d}x = \\
&= \int_{E_0} \mathrm{D}_u f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \boldsymbol{u}'[V]\,\mathrm{d}x \underbrace{- \int_{\partial E_0} (\mathrm{D}_3 f)(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \nu \cdot \boldsymbol{u}'[V]\,\mathrm{d}x}_{= \int_{\partial E_0} (\mathrm{D}_3 f)(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \nu \cdot \partial_\nu \boldsymbol{u}_0 (V(0)\cdot\nu)\,\mathrm{d}x}.
\end{aligned} \tag{8.9}$$

As shown in (7.50) we find

$$\int_{\partial E_0} \pi_0 \boldsymbol{u}'[V] \cdot \nu \, \mathrm{d}x = -\int_{\partial E_0} \pi_0 \partial_\nu \boldsymbol{u}_0 \cdot \nu \left(V(0) \cdot \nu\right) \mathrm{d}x = 0$$

while making use of $\operatorname{div} \boldsymbol{u}_0 = 0$. Inserting the adjoint state $\boldsymbol{q}_0$ as a test function in the linearized equation (8.4a) we end up with

$$\int_{E_0} \mu \nabla \boldsymbol{u}'[V] \cdot \nabla \boldsymbol{q}_0 \, \mathrm{d}x = 0. \tag{8.10}$$

Comparison of (8.9) and (8.10) implies

$$\begin{aligned}\int_{\partial E_0} \mu \left(\partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0\right) \left(V(0) \cdot \nu\right) \mathrm{d}x - \int_{\partial E_0} \left(\mathrm{D}_3 f\right)\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \nu \cdot \partial_\nu \boldsymbol{u}_0 \left(V(0) \cdot \nu\right) \mathrm{d}x = \\ = \int_{E_0} \mathrm{D}_u f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \boldsymbol{u}'[V] \, \mathrm{d}x = \int_{E_0} \mathrm{D}_{(2,3)} f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \left(\boldsymbol{u}'[V], \mathrm{D}\boldsymbol{u}'[V]\right).\end{aligned} \tag{8.11}$$

Inserting this in the calculations above we obtain

$$\begin{aligned}\partial_t|_{t=0} \int_{\Omega_t} f\left(x, \boldsymbol{u}_t, \mathrm{D}\boldsymbol{u}_t\right) \mathrm{d}x &= \int_{\partial E_0} \mu \left(\partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0\right) \left(V(0) \cdot \nu\right) \mathrm{d}x - \\ &- \int_{\partial E_0} \left(\mathrm{D}_3 f\right)\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \nu \cdot \partial_\nu \boldsymbol{u}_0 \left(V(0) \cdot \nu\right) \mathrm{d}x + \\ &+ \int_{E_0} \mathrm{D}\left(f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right)\right) V(0) \, \mathrm{d}x + \int_\Omega f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \operatorname{div} V(0) \, \mathrm{d}x\end{aligned}$$

and (8.7) is proved.

We now turn to the existence of a Lagrange multiplier $\lambda_0 \in \mathbb{R}^+$ for the integral constraint. Therefore we can proceed in the same way as in Lemma 7.5 and construct the multiplier explicitly. □

Remark 8.3. *Assuming that $f\left(x, \boldsymbol{u}(x), \mathrm{D}\boldsymbol{u}(x)\right) = 0$ for a.e. $x \in \Omega \setminus E_0$ if $\boldsymbol{u} = \boldsymbol{0}$ a.e. in $\Omega \setminus E_0$, then formula (8.7) would read as*

$$\begin{aligned}\mathrm{D}J_S\left(E_0\right)[V] = &\int_{\partial E_0} \left(\mu \partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0 - \left(\mathrm{D}_3 f\right)\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \nu \cdot \partial_\nu \boldsymbol{u}_0\right) V(0) \cdot \nu \, \mathrm{d}x + \\ &+ \int_{\partial E_0} f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) V(0) \cdot \nu \, \mathrm{d}x + \gamma c_0 \int_{\partial E_0 \cap \Omega} \kappa V(0) \cdot \nu \, \mathrm{d}x.\end{aligned}$$

The resulting formula is then in Hadamard-form, and thus in a representation formula that was proposed by Zolésio, and is not explicitly depending on the shape derivative or material derivative any more, and hence is more convenient for computation, see for instance [Epp09]. For the general structure theorem in shape calculus we refer to [DZ01, SZ92] and included references.

This proves a general formula for the shape derivative in a Stokes flow. We want to apply this now to an explicit example and compare it to existing results.

Example 8.1. Using the total potential power as an objective functional, which is introduced in Example 2.3, thus

$$f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) = \frac{\mu}{2} |\mathrm{D}\boldsymbol{u}|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u}$$

we obtain from (8.7)

$$\begin{aligned}\mathrm{D}J_S(E_0)[V] = &\int_{\partial E_0} \left(\mu \partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0 - \mu |\partial_\nu \boldsymbol{u}_0|^2\right) V(0) \cdot \nu \,\mathrm{d}x + \\ &+ \int_{\partial E_0} \left(\frac{\mu}{2} |\partial_\nu \boldsymbol{u}_0|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u}_0\right) V(0) \cdot \nu \,\mathrm{d}x + \gamma c_0 \int_{\partial E_0 \cap \Omega} \kappa V(0) \cdot \nu \,\mathrm{d}x.\end{aligned}$$

Due to the special structure of the objective functional this equation can be further simplified by inserting the linearized state $\boldsymbol{u}'[V]$ as a test function in the adjoint equation (8.8), using the state equations (6.3) written for $\boldsymbol{u}_0$ and making use of the particular form of the objective functional f. Then similar calculations as in Example 7.2 yield the rewritten equation

$$\mathrm{D}J_S(E_0)[V] = \int_{\partial E_0} \left(-\frac{\mu}{2} |\partial_\nu \boldsymbol{u}_0|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_0\right) V(0) \cdot \nu \,\mathrm{d}x + \gamma c_0 \int_{\partial E_0 \cap \Omega} \kappa V(0) \cdot \nu \,\mathrm{d}x.$$

This result coincides with the common known results found in the literature, see for instance [Sim91, Pir73].
Moreover, we find that this formula looks similar to (7.56), where with the same approach optimality criteria for the phase field model were derived. As we will show in the next subsections that the shape derivatives are equivalent to optimality conditions that are the limit of corresponding optimality systems of the phase field model for $\varepsilon \searrow 0$, we find that the similarity does not only hold true for this example, but is a result that will be verified in a more general setting.

8.2 Geometric variations

As already indicated at the beginning of this section, we want to derive first order optimality conditions for the sharp interface problem (6.18) – (6.19) without stating the unverified regularity assumptions on the minimizing set that were necessary in Section 8.1. Again we want to use the idea of deforming the minimizing set by a suitable transformation as in the previous subsection and in Section 7.2. But as we have no additional regularity on the set on which we solve the Stokes equations, we get no regularity of the state variable, namely the solution of the Stokes equations corresponding to the minimizing set E_0, and so the shape derivative is not well defined in $\boldsymbol{H}^1(\Omega)$ or $\boldsymbol{H}^1(E_0)$. Therefore, the considerations and calculations of Section 8.1 cannot by used and we have to apply different techniques.
In particular, we will result in optimality conditions, that can under certain assumptions be verified to be the limit system of the optimality conditions obtained in the phase field model, see Sections 7.2 and 9.

Thus we start developing first order optimality conditions for the limit problem

$$\min_{\varphi\in L^1(\Omega)} j_0(\varphi) \tag{8.12}$$

without additional regularity assumptions.
To this end we fix for the rest of this subsection

$$E_0 := \{x \in \Omega \mid \varphi_0(x) = 1\}$$

where we recall that $\varphi_0 \in \Phi_{ad}^0$ was chosen as minimizer of j_0. For the following considerations, let us introduce the notation

$$\varphi_0(t) = \varphi_0 \circ T_t^{-1}, \quad \Omega_t = T_t(\Omega)$$
$$\boldsymbol{u}_0(t) = \boldsymbol{S}_0(\varphi_0(t)), \quad E_0(t) = T_t(E_0) = E^{\varphi_0(t)}$$

for some given transformation $T \in \overline{\mathcal{T}}_{ad}$.

Remark 8.4. *With the choice of $\varphi_0 \in \operatorname{argmin}_{\varphi\in\Phi_{ad}^0} j_0(\varphi)$ and $T \in \overline{\mathcal{T}}_{ad}$ we find $\varphi_0(t) \in \overline{\Phi}_{ad}^0$, since we have $\left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{u} \circ T_t^{-1} \in \boldsymbol{U}^{\varphi_0(t)}$ for any $\boldsymbol{u} \in \boldsymbol{U}^{\varphi_0}$ and thus $\boldsymbol{U}^{\varphi_0(t)} \neq \varnothing$ follows from $\boldsymbol{U}^{\varphi_0} \neq \varnothing$, cp. Lemma 3.6. This ensures that $\boldsymbol{u}_0(t)$ is well-defined.*

Let us first examine the differentiability of the transformed state variable in direction of the transformation.

Lemma 8.5. *The mapping $\mathbb{R} \supset I \ni t \mapsto \boldsymbol{u}_0(t) \circ T_t \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$ (where I is a small interval around 0) and $\dot{\boldsymbol{u}}_0[V] = \partial_t|_{t=0}(\boldsymbol{u}_0(t) \circ T_t) \in \boldsymbol{H}^1(\Omega)$ with $\dot{\boldsymbol{u}}_0[V]|_{\Omega\setminus E_0} = \boldsymbol{0}$ is given as the unique solution to*

$$\begin{aligned}
\int_{E_0} \mu\nabla\dot{\boldsymbol{u}}_0[V] : \nabla\boldsymbol{z}\,\mathrm{d}x &= \int_{E_0} \mu \mathrm{D}V(0)^T\nabla\boldsymbol{u}_0 : \nabla\boldsymbol{z}\,\mathrm{d}x + \int_{E_0}\mu\nabla\boldsymbol{u}_0 : \mathrm{D}V(0)^T\nabla\boldsymbol{z}\,\mathrm{d}x + \\
&+ \int_{E_0}\mu\nabla\boldsymbol{u}_0 : \nabla\left(\operatorname{div}V(0)\,\boldsymbol{z} - \mathrm{D}V(0)\,\boldsymbol{z}\right)\mathrm{d}x - \int_{E_0}\mu\nabla\boldsymbol{u}_0 : \nabla\boldsymbol{z}\operatorname{div}V(0)\,\mathrm{d}x + \\
&+ \int_{E_0}(\nabla\boldsymbol{f}\cdot V(0))\cdot\boldsymbol{z}\,\mathrm{d}x + \int_{E_0}\boldsymbol{f}\cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x
\end{aligned} \tag{8.13}$$

which has to hold for all $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$*, together with*

$$\operatorname{div} \dot{\boldsymbol{u}}_0 [V] = \nabla \boldsymbol{u}_0 : \mathrm{D}V(0). \tag{8.14}$$

Remark 8.5. *Here and in the following calculations we will again use the notation and conventions outlined in Remark 7.2.*

Proof. The basic idea of the proof is to apply some implicit function argument. But as in Lemma 8.3 we cannot apply the standard version of the implicit function theorem, since

$$\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v}|_{\Omega \smallsetminus E_0} = \boldsymbol{0}\right\} \ni \boldsymbol{v} \mapsto \operatorname{div} \boldsymbol{v}$$

won't be surjective as a mapping into $L_0^2(\Omega)$ or $L_0^2(E_0)$, as we don't have enough regularity of E_0 (see counterexample in [Gal11]). So we will use again Theorem A.3, but now applied to a different setting. We define

$$F : I \times \boldsymbol{V}^{E_0} \to \left(\boldsymbol{V}^{E_0}\right)' \times L^2(\Omega)$$

by

$$\begin{aligned} F(t,\boldsymbol{v})\boldsymbol{z} := &\int_{E_0} \mu \nabla \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \mathrm{d}x - \\ &- \int_{E_0} \mu \nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right) \cdot \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x = \\ &= \int_{E_0} \mu \det \mathrm{D}T_t^{-1} \mathrm{D}T_t \cdot \mathrm{D}T_t^{-1} \cdot \nabla \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x - \\ &- \int_{E_0} \mu \nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right) \cdot \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x \,\mathrm{d}x \quad \forall \boldsymbol{z} \in \boldsymbol{V}^{E_0}. \end{aligned}$$

Then we observe with Lemma 3.6 that due to $\boldsymbol{u}_0(t) \in \boldsymbol{U}^{E_0(t)}$ and $T \in \overline{\mathcal{T}}_{ad}$ it follows $(\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right) \boldsymbol{u}_0(t) \circ T_t \in \boldsymbol{U}^{E_0}$. Moreover, for $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$ arbitrary we get $\boldsymbol{z}_t := \left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\, \boldsymbol{z} \circ T_t^{-1} \in \boldsymbol{V}^{E_0(t)}$ and thus we find

$$\begin{aligned} &\int_{E_0} \mu \left(\nabla \boldsymbol{u}_0(t)\right)(T_t) \cdot \left(\nabla \boldsymbol{z}_t\right)(T_t) \cdot \det \mathrm{D}T_t \,\mathrm{d}x - \int_{E_0} \boldsymbol{f} \circ T_t \cdot \boldsymbol{z}_t \circ T_t \cdot \det \mathrm{D}T_t \,\mathrm{d}x = \\ &= \int_{E_0(t)} \mu \nabla \boldsymbol{u}_0(t) \cdot \nabla \boldsymbol{z}_t \,\mathrm{d}x - \int_{E_0(t)} \boldsymbol{f} \cdot \boldsymbol{z}_t \,\mathrm{d}x = 0. \end{aligned}$$

Next we choose some $\boldsymbol{G} \in \boldsymbol{U}^{E_0}$ which is possible since φ_0 is a minimizer of j_0 with $j_0(\varphi_0) < \infty$ and therefore in particular $\boldsymbol{U}^{E_0} \neq \emptyset$.
Then we see by direct calculation that it holds

$$\begin{aligned} &F\left(t, (\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right) \boldsymbol{u}_0(t) \circ T_t - \boldsymbol{G}\right) \boldsymbol{z} = \int_{E_0} \boldsymbol{f} \circ T_t \cdot \boldsymbol{z}_t \circ T_t \cdot \det \mathrm{D}T_t \,\mathrm{d}x - F(t, \boldsymbol{G})\, \boldsymbol{z} = \\ &= \int_{E_0} \boldsymbol{f} \circ T_t \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x - F(t, \boldsymbol{G})\, \boldsymbol{z} =: \widetilde{F}(t)\boldsymbol{z} \end{aligned}$$

which defines

$$\widetilde{F}(t) \in \left(\boldsymbol{V}^{E_0}\right)'.$$

Summarizing, we have

$$F(t,\cdot) \in \mathcal{L}\left(\boldsymbol{V}^{E_0}, \left(\boldsymbol{V}^{E_0}\right)'\right) \quad \forall t \in I$$

and

$$F\left(t, (\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t) \circ T_t - \boldsymbol{G}\right) = \widetilde{F}(t) \quad \forall t \in I.$$

Due to the differentiability assumptions on the transformation $T \in \overline{\mathcal{T}}_{ad}$ we observe that

$$I \ni t \mapsto F(t,\cdot) \in \mathcal{L}\left(\boldsymbol{V}^{E_0}, \left(\boldsymbol{V}^{E_0}\right)'\right)$$

as well as $I \ni t \mapsto \widetilde{F}(t) \in \left(\boldsymbol{V}^{E_0}\right)'$ are differentiable at $t = 0$. To apply Theorem A.3 we still have to show condition (A.1). Therefore, we see that for all $\boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{V}^{E_0}$ it holds

$$F(0,\boldsymbol{v})\boldsymbol{z} = \int_{E_0} \mu \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{z} \, \mathrm{d}x. \tag{8.15}$$

Thus for fixed $\boldsymbol{v} \in \boldsymbol{V}^{E_0}$ we can estimate, using Poincaré's inequality,

$$\|F(0,\boldsymbol{v})\|_{\left(\boldsymbol{V}^{E_0}\right)'} = \sup_{\boldsymbol{z}\in\boldsymbol{V}^{E_0}\setminus\{\boldsymbol{0}\}} \frac{|F(0,\boldsymbol{v})\boldsymbol{z}|}{\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)}} \geq \frac{\mu \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \geq c(\Omega) \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}.$$

which implies (A.1).
And so we can apply Theorem A.3 to get differentiability of

$$I \ni t \mapsto \left((\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t) \circ T_t - \boldsymbol{G}\right) \in \boldsymbol{H}^1(\Omega)$$

and thus of $t \mapsto \boldsymbol{u}_0(t) \circ T_t \in \boldsymbol{H}^1(\Omega)$ at $t = 0$. Besides, we obtain that $\dot{\boldsymbol{u}}_0[V]$ is the unique solution of

$$F\left(0, \operatorname{div} V(0)\boldsymbol{u}_0 - \mathrm{D}V(0)\boldsymbol{u}_0 + \dot{\boldsymbol{u}}_0[V]\right) = \widetilde{F}'(0) - \partial_t|_{t=0} F\left(t, \boldsymbol{u}_0 - \boldsymbol{G}\right).$$

This reads

$$\begin{aligned} F\left(0, \dot{\boldsymbol{u}}_0[V]\right)\boldsymbol{z} &= \widetilde{F}'(0)\boldsymbol{z} - \partial_t F\left(0, \boldsymbol{u}_0 - \boldsymbol{G}\right)\boldsymbol{z} - F\left(0, \operatorname{div} V(0)\boldsymbol{u}_0 - \mathrm{D}V(0)\boldsymbol{u}_0\right)\boldsymbol{z} = \\ &= \partial_t|_{t=0} \int_{E_0} \left(\boldsymbol{f} \circ T_t \cdot \boldsymbol{z}_t \circ T_t \cdot \det \mathrm{D}T_t\right) \mathrm{d}x - \partial_t F\left(0, \boldsymbol{u}_0\right)\boldsymbol{z} \quad \forall \boldsymbol{z} \in \boldsymbol{V}^{E_0} \end{aligned}$$

which yields with (8.15) the stated result.

□

After those preparatory steps, we can now deduce our main result. To this end we notice as in Section 7.2 that due to construction $\varphi_0(t)$ are admissible comparison functions for j_0 if $T \in \mathcal{T}_{ad}$, see Remark 3.4, and so

$$j_0(\varphi_0) \leq j_0\left(\varphi_0 \circ T_t^{-1}\right) \quad \forall T \in \mathcal{T}_{ad}\ |t| \ll 1,$$

which implies

$$\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = 0 \quad \forall T \in \mathcal{T}_{ad}.$$

This is the main idea used for the necessary optimality conditions of problem (8.12), as shown in the following theorem:

Theorem 8.2. *For every minimizer $\varphi_0 \in \Phi_{ad}^0$ with $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0) \in \boldsymbol{U}^{\varphi_0}$ of* (8.12) *we have the following necessary optimality conditions:*

$$\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = -\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x, \tag{8.16}$$

$$\lambda_0 \left(\int_\Omega \varphi_0 \,\mathrm{d}x - \beta\,|\Omega| \right) = 0 \tag{8.17}$$

which holds for all $T \in \overline{\mathcal{T}}_{ad}$ with velocity field $V \in \overline{\mathcal{V}}_{ad}$. Here $\lambda_0 \geq 0$ is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:

$$\begin{aligned}\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = &\int_\Omega \left[\mathrm{D}f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right)\left(V(0), \dot{\boldsymbol{u}}_0[V], \mathrm{D}\dot{\boldsymbol{u}}_0[V] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)\right) + \right.\\ &+ \left. f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \operatorname{div} V(0)\right] \mathrm{d}x + \\ &+ \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right) \mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|\end{aligned} \tag{8.18}$$

with ν being the generalised unit normal on $E_0 = \{\varphi_0 = 1\}$. Moreover $\dot{\boldsymbol{u}}_0[V] \in \boldsymbol{H}_0^1(\Omega)$ with $\dot{\boldsymbol{u}}_0[V] = \boldsymbol{0}$ a.e. in $\Omega \setminus E_0$ is given as the unique solution to (8.13)-(8.14).

Proof. Analysis similar to that in the proof of Lemma 7.5 and applying Lemma 3.4 shows (8.18). The existence of a Lagrange multiplier $\lambda_0 \in \mathbb{R}^+$ for the integral constraint can be deduced by the same method as in Lemma 7.5, which reduces basically to an explicit construction. □

To show the similarity to the already obtained optimality conditions, we consider again the example of minimizing the total potential power of Example 2.3.

Example 8.2. Let

$$f\left(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}\right) = \frac{\mu}{2}\left|\mathrm{D}\boldsymbol{u}\right|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u},$$

then we get from (8.18) the following expression:

$$\begin{aligned}\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = &\int_\Omega -\mathrm{D}\boldsymbol{f} V(0) - \boldsymbol{f} \cdot \dot{\boldsymbol{u}}_0[V] + \mu \mathrm{D}\boldsymbol{u}_0 : \left(\mathrm{D}\dot{\boldsymbol{u}}_0[V] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)\right) \mathrm{d}x + \\ &+ \int_\Omega \left(\frac{\mu}{2}\left|\mathrm{D}\boldsymbol{u}_0\right|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_0 \right) \operatorname{div} V(0)\,\mathrm{d}x + \\ &+ \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right) \mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|.\end{aligned}$$

It is worth pointing out the similarity to the formula obtained in the phase field setting for this example, which is calculated in (7.54). As one may suppose by the resemblance of those equations, we will show in Section 9 even for the general objective functional that under suitable addition assumptions $\lim_{\varepsilon \searrow 0} \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right)$ is fulfilled for all $T \in \overline{\mathcal{T}}_{ad}$.

8.3 Linking the optimality criteria

In the previous two subsections we have derived two different versions of optimality criteria for minimizers of j_0. In Section 8.1 we have calculated the classical shape derivatives, under some additional regularity assumptions on the minimizer, whereas the optimality conditions derived in Theorem 8.2 of Section 8.2 are verified without any additional assumptions. Anyhow, both systems result from geometric variations.

In this subsection, we will show that, stating the regularity assumptions made in Section 8.1, both optimality conditions are equivalent.

Lemma 8.6. *Assume that* $E_0 := \operatorname{int}\left(\{x \in \Omega \mid \varphi_0(x) = 1\}\right)$ *is a well-defined open subset of* Ω *and that the regularity assumptions* (8.3) *are fulfilled.*
Then we have

$$\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = \mathrm{D}J_S(E_0)\,[V] \tag{8.19}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ *with velocity* $V \in \overline{\mathcal{V}}_{ad}$ *where* $\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right)$ *is given by* (8.18) *and* $\mathrm{D}J_S(E_0)\,[V]$ *by* (8.7).
This means, that the optimality conditions of Theorem 8.1 and of Theorem 8.2 are equivalent.

Proof. The proof follows closely the arguments of Lemma 7.6 and we only give the main ideas here. We start with observing that standard regularity theory for the Stokes equations, cf. [Tem77, Gal11, Soh01], imply $\boldsymbol{q}_0$, $\boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_0\,[V]$ are in $\boldsymbol{H}^2(E_0)$, where the adjoint state $\boldsymbol{q}_0$ is given in Theorem 8.1.

Thus, we find that the shape derivative

$$\boldsymbol{u}_0'\,[V] := \dot{\boldsymbol{u}}_0\,[V] - \nabla \boldsymbol{u}_0 \cdot V(0)$$

is a function in $\boldsymbol{H}^1(E_0)$.

- *1st step:* We see that $\boldsymbol{u}_0'\,[V]$ solves

$$\begin{aligned} -\mu\Delta \boldsymbol{u}_0'\,[V] + \nabla p_0'\,[V] &= \boldsymbol{0} && \text{in } E_0,\\ \operatorname{div} \boldsymbol{u}_0'\,[V] &= 0 && \text{in } E_0,\\ \boldsymbol{u}_0'\,[V] &= -\partial_\nu \boldsymbol{u}_0\,(V(0)\cdot\nu) && \text{on } \partial E_0, \end{aligned}$$

  in the following sense:

$$\mu \int_{E_0} \nabla \boldsymbol{u}_0'\,[V] : \nabla \boldsymbol{z}\,\mathrm{d}x = 0 \quad \forall \boldsymbol{z} \in \boldsymbol{H}_0^1(E_0),\ \operatorname{div}\boldsymbol{z} = 0,$$

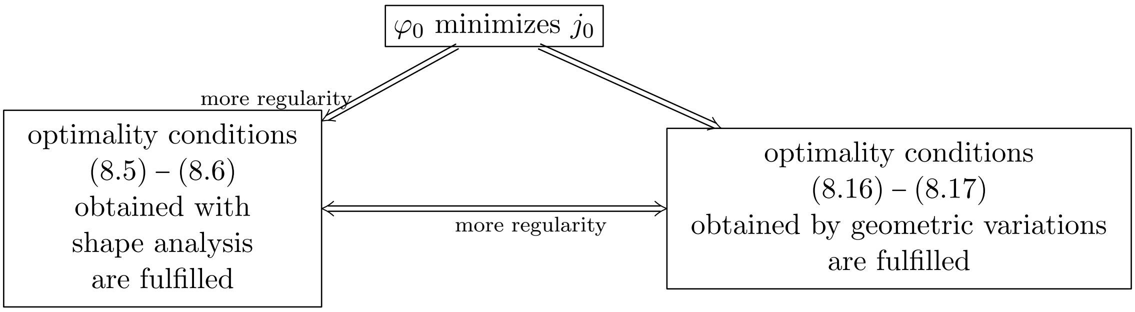
  where ν denotes the outer unit normal on E_0. This can be shown by following the arguments of Lemma 7.6.

- *2nd step:* We now follow the arguments of the second step of the proof of Lemma 7.6, with integrals over Ω replaced by integrals over E_0 when considering the linearized and adjoint state equations, to reformulate (8.18) to (8.7). Moreover, we can deduce

$$\int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right)\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right| = \int_{\partial E_0 \cap \Omega} \kappa V(0)\cdot\nu\,\mathrm{d}x$$

  from Lemma 3.4. □

Summarizing the results of this section, we have calculated on the one hand classical shape derivatives, which only hold true under some unverified regularity assumptions on the minimizing set, but on the other hand we also have derived analytically verified optimality conditions by geometric variations. Additionally, we have also shown that those optimality criteria are equivalent if imposing the regularity assumptions necessary for deriving shape derivatives. This shows, that we have derived two different optimality conditions and that the obtained results are conform with each other and with the analysis found in literature. These statements are summarized by the following diagram:



9 Convergence of the optimality system

We have already shown that a subsequence of minimizers of $(j_\varepsilon)_{\varepsilon>0}$, denoted by $(\varphi_\varepsilon)_{\varepsilon>0}$, converges in $L^1(\Omega)$ to some limit element $\varphi_0 \in L^1(\Omega)$ as $\varepsilon \searrow 0$. If additionally the growth condition (6.21) is fulfilled, we even find that φ_0 is a minimizer of j_0 and $\lim_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi_0)$, see Theorem 6.1.

The aim of this section is to show that in this situation the optimality conditions for $(j_\varepsilon)_{\varepsilon>0}$ derived in Section 7.2 converge to the optimality conditions of j_0 derived in Section 8.2. We remark in particular, that we do not assume any additional regularity assumption on the minimizing set, the data or Ω to prove the convergence of the optimality systems, whereas this was necessary in Lemma 7.6 and Section 8. So the following result is consistent with what we have shown before. But as we are considering first order optimality conditions we assume again differentiability of the body force and the objective functional which given by Assumptions **(A6)** and **(A7)**.

Theorem 9.1. *Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be the sequence of minimizers of $(j_\varepsilon)_{\varepsilon>0}$ converging to $\varphi_0 \in L^1(\Omega)$ given by Theorem 6.1. Assume moreover that*

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega \mid \varphi_0(x)=1, \varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon). \tag{9.1}$$

Then the limit element φ_0 is a minimizer of j_0. Moreover it holds

$$\lim_{\varepsilon\searrow 0} \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) \quad \forall T \in \overline{\mathcal{T}}_{ad}. \tag{9.2}$$

If

$$|\{\varphi_0 = 1\}| > 0 \tag{9.3}$$

then we have additionally the following convergence results:

$$\begin{aligned}
\varphi_\varepsilon &\xrightarrow{\varepsilon\searrow 0} \varphi_0 && \text{in } L^1(\Omega), && (9.4a)\\
\boldsymbol{u}_\varepsilon &\xrightarrow{\varepsilon\searrow 0} \boldsymbol{u}_0 && \text{in } \boldsymbol{H}^1(\Omega), && (9.4b)\\
\dot{\boldsymbol{u}}_\varepsilon[V] &\xrightarrow{\varepsilon\searrow 0} \dot{\boldsymbol{u}}_0[V] && \text{in } \boldsymbol{H}^1(\Omega), && (9.4c)\\
\lambda_\varepsilon &\xrightarrow{\varepsilon\searrow 0} \lambda_0 && \text{in } \mathbb{R}, && (9.4d)\\
j_\varepsilon(\varphi_\varepsilon) &\xrightarrow{\varepsilon\searrow 0} j_0(\varphi_0) && \text{in } \mathbb{R}, && (9.4e)
\end{aligned}$$

where $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$, $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$, $(\lambda_\varepsilon)_{\varepsilon>0} \subseteq \mathbb{R}^+$ are Lagrange multipliers for the integral constraint defined due to Lemma 7.5, $\lambda_0 \in \mathbb{R}^+$ is a Lagrange multiplier such that it holds (8.16) – (8.17)*, and thus is a Lagrange multiplier for the integral constraint in the sharp interface according to Theorem 8.2.*

Remark 9.1. *We remark that condition* (9.3) *is only necessary to prove convergence of the Lagrange multipliers $(\lambda_\varepsilon)_{\varepsilon>0}$, whereas the other statements would hold true even if* (9.3) *is not fulfilled. But as $|\{\varphi_0 = 1\}| = 0$ means that there is no fluid present at all (up* *to sets of measure zero),* (9.3) *is not a restrictive assumption. For instance in the case of non-homogeneous boundary data, thus if* $\mathcal{H}^{d-1}(\{x \in \partial\Omega \mid \boldsymbol{g}(x) \neq \mathbf{0}\}) > 0$*, we find that* $|\{x \in \Omega \mid \varphi_0(x) = 1\}| > 0$*, and thus* (9.3) *is fulfilled, since* $\boldsymbol{u}_0 \in \boldsymbol{U}^{E_0}$ *fulfills* $\boldsymbol{u}_0|_{\partial\Omega} = \boldsymbol{g}$ *and* $\boldsymbol{u}_0|_{\{\varphi_0=-1\}} = \mathbf{0}$.

Proof. We assume for the following considerations that (9.1) is fulfilled. The existence of a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges to a minimizer φ_0 of j_0 in $L^1(\Omega)$ follows from Theorem 6.1. In fact, we even obtain therefrom directly (9.4e). Moreover, by using (9.1) we can apply Lemma 6.3 to obtain, after possibly choosing a subsequence

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \alpha_0(\varphi_0)|\boldsymbol{u}_0|^2\,\mathrm{d}x = 0 \tag{9.5}$$

and

$$\lim_{\varepsilon\searrow 0}\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0$$

which shows (9.4b).
From the second step in the proof of Lemma 6.3 we even find

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v}|_{\Omega\setminus E^{\varphi_0}} = \mathbf{0}. \tag{9.6}$$

This result will be used later on in this proof. We proceed by defining the auxiliary functions $\boldsymbol{w}_\varepsilon := (-\operatorname{div} V(0) + \mathrm{D}V(0))\,\boldsymbol{u}_\varepsilon$ for all $\varepsilon > 0$ and obtain from the regularity of V and (9.4b) directly that $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{w}_0 := (-\operatorname{div} V(0) + \mathrm{D}V(0))\,\boldsymbol{u}_0$.
We recall, that $\dot{\boldsymbol{u}}_\varepsilon[V] \in \boldsymbol{H}^1_0(\Omega)$ is due Lemma 7.5 given as the unique solution of

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\dot{\boldsymbol{u}}_\varepsilon[V]\cdot\boldsymbol{z} + \mu\nabla\dot{\boldsymbol{u}}_\varepsilon[V] : \nabla\boldsymbol{z}\,\mathrm{d}x = \int_\Omega \mu \mathrm{D}V(0)^T\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z}\,\mathrm{d}x +\\
&+ \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T\nabla\boldsymbol{z}\,\mathrm{d}x + \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \nabla\left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right)\mathrm{d}x -\\
&- \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z}\operatorname{div} V(0)\,\mathrm{d}x + \int_\Omega (\nabla\boldsymbol{f}\cdot V(0))\cdot\boldsymbol{z}\,\mathrm{d}x +\\
&+ \int_\Omega \boldsymbol{f}\cdot\mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x - \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x \quad \forall \boldsymbol{z}\in\boldsymbol{V}
\end{aligned} \tag{9.7}$$

together with

$$\operatorname{div}\dot{\boldsymbol{u}}_\varepsilon[V] = \nabla\boldsymbol{u}_\varepsilon : \mathrm{D}V(0) \tag{9.8}$$

where we use again the notation outlined in Remark 7.2. The main idea of the proof is to use the approach of Lemma 6.3, i.e. we show that $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ are the unique minimizers of functionals which Γ-converge as $\varepsilon \searrow 0$ in the weak $\boldsymbol{H}^1(\Omega)$-topology. To this end, we define for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\begin{aligned}
F_\varepsilon(\boldsymbol{v}) := &\int_\Omega \left(\frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2\right)\mathrm{d}x - R_\varepsilon(\boldsymbol{v}) +\\
&+ \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\mathrm{D}V(0)\boldsymbol{v}\,\mathrm{d}x - D_\varepsilon(\boldsymbol{w}_\varepsilon)(\boldsymbol{v})
\end{aligned}$$

where $R_\varepsilon \in \boldsymbol{H}^{-1}(\Omega)$ is given by

$$\begin{aligned} R_\varepsilon(\boldsymbol{z}) :=& \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T \nabla \boldsymbol{z} \,\mathrm{d}x + \\ &+ \mu \int_\Omega \nabla \boldsymbol{u}_\varepsilon : \nabla \left(\operatorname{div} V(0) \boldsymbol{z} - \mathrm{D}V(0) \boldsymbol{z} \right) \mathrm{d}x - \\ &- \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \operatorname{div} V(0) \,\mathrm{d}x + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z} + \boldsymbol{f} \cdot \mathrm{D}V(0) \boldsymbol{z} \,\mathrm{d}x \end{aligned}$$

and $D_\varepsilon(\boldsymbol{w}_\varepsilon) \in \boldsymbol{H}^{-1}(\Omega)$ is defined by

$$D_\varepsilon(\boldsymbol{w}_\varepsilon)(\boldsymbol{z}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{w}_\varepsilon \cdot \boldsymbol{z} + \mu \nabla \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{z} \,\mathrm{d}x.$$

Additionally, we define

$$F_0(\boldsymbol{v}) := \int_\Omega \left(\frac{1}{2} \alpha_0 \left(\varphi_0\right) |\boldsymbol{v}|^2 + \frac{\mu}{2} |\nabla \boldsymbol{v}|^2 \right) \mathrm{d}x - R_0(\boldsymbol{v}) - D_0(\boldsymbol{w}_0)(\boldsymbol{v})$$

where

$$\begin{aligned} R_0(\boldsymbol{z}) :=& \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z} \,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_0 : \mathrm{D}V(0)^T \nabla \boldsymbol{z} \,\mathrm{d}x + \\ &+ \mu \int_\Omega \nabla \boldsymbol{u}_0 : \nabla \left(\operatorname{div} V(0) \boldsymbol{z} - \mathrm{D}V(0) \boldsymbol{z} \right) \mathrm{d}x - \\ &- \int_\Omega \mu \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z} \operatorname{div} V(0) \,\mathrm{d}x + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z} \,\mathrm{d}x + \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0) \boldsymbol{z} \,\mathrm{d}x \end{aligned}$$

and

$$D_0(\boldsymbol{w}_0)(\boldsymbol{z}) = \int_\Omega \alpha_0(\varphi_0) \boldsymbol{w}_0 \cdot \boldsymbol{z} + \mu \nabla \boldsymbol{w}_0 \cdot \nabla \boldsymbol{z} \,\mathrm{d}x.$$

We remark that $(R_\varepsilon)_{\varepsilon>0} \subseteq \boldsymbol{H}^{-1}(\Omega)$ and $R_0 \in \boldsymbol{H}^{-1}(\Omega)$. From the already proven convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{u}_0$ we find that $(R_\varepsilon)_{\varepsilon>0}$ converges to R_0 (strongly) in $\boldsymbol{H}^{-1}(\Omega)$.

Next we see, that due to Lemma 3.7 it holds

$$\operatorname{div} \left(\operatorname{div} V(0) \boldsymbol{u}_\varepsilon + \mathrm{D}\boldsymbol{u}_\varepsilon V(0) - \mathrm{D}V(0) \boldsymbol{u}_\varepsilon \right) = 0$$

and so

$$\operatorname{div} \boldsymbol{w}_\varepsilon = \operatorname{div} \left(-\operatorname{div} V(0) \boldsymbol{u}_\varepsilon + \mathrm{D}V(0) \boldsymbol{u}_\varepsilon \right) = \operatorname{div} \left(\mathrm{D}\boldsymbol{u}_\varepsilon V(0) \right) = \mathrm{D}\boldsymbol{u}_\varepsilon : \nabla V(0)$$

where we used for the last step $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$. This implies

$$\operatorname{div} \left(\dot{\boldsymbol{u}}_\varepsilon [V] - \boldsymbol{w}_\varepsilon \right) = 0.$$

And so we can conclude from $\dot{\boldsymbol{u}}_\varepsilon [V] |_{\partial\Omega} = \boldsymbol{w}_\varepsilon|_{\partial\Omega} = \boldsymbol{0}$ that $(\dot{\boldsymbol{u}}_\varepsilon [V] - \boldsymbol{w}_\varepsilon) \in \boldsymbol{V}$. In particular, we can insert $(\dot{\boldsymbol{u}}_\varepsilon [V] - \boldsymbol{w}_\varepsilon) \in \boldsymbol{V}$ as a test function into (9.7) and end up with

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2 \mathrm{d}x + \int_\Omega \mu\left|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2 \mathrm{d}x = R_\varepsilon\left(\dot{\boldsymbol{u}}_\varepsilon[V]-\boldsymbol{w}_\varepsilon\right) - \\
&- \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot \mathrm{D}V(0)\left(\dot{\boldsymbol{u}}_\varepsilon[V]-\boldsymbol{w}_\varepsilon\right)\mathrm{d}x - D_\varepsilon\left(\dot{\boldsymbol{u}}_\varepsilon[V]\right)(\boldsymbol{w}_\varepsilon) \le \\
&\le \|R_\varepsilon\|_{\boldsymbol{H}^{-1}(\Omega)}\left(\|\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{w}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\right) + \\
&+ C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\right)^{\frac12}\left(\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2\right)^{\frac12} + \left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2\right)^{\frac12}\right) + \\
&+ C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2\right)^{\frac12}\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2\right)^{\frac12} + \mu\|\nabla\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned} \tag{9.9}$$

By observing

$$\begin{aligned}
\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2 \mathrm{d}x &= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|\left(-\operatorname{div}V(0)+\mathrm{D}V(0)\right)\boldsymbol{u}_\varepsilon\right|^2 \mathrm{d}x \le \\
&\le C\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \mathrm{d}x.
\end{aligned} \tag{9.10}$$

we find thanks to Young's inequality from (9.9)

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2 \mathrm{d}x + \int_\Omega \frac{\mu}{2}\left|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2 \mathrm{d}x \le \\
&\le \underbrace{\|R_\varepsilon\|_{\boldsymbol{H}^{-1}(\Omega)}}_{\le C}\left(\|\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{H}^1(\Omega)} + \underbrace{\|\boldsymbol{w}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}}_{\le C}\right) + \underbrace{C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\right)}_{\le C} + \\
&+ \mu\underbrace{\|\nabla\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}}_{\le C}\|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned} \tag{9.11}$$

And so, by using again Young's inequality together with Poincaré's inequality (notice $\dot{\boldsymbol{u}}_\varepsilon[V]$ has zero boundary data on $\partial\Omega$) we end up having a uniform bound on $\|\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{H}^1(\Omega)}$ and

$$\sup_{\varepsilon>0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2 \mathrm{d}x < \infty. \tag{9.12}$$

This directly implies the existence of a subsequence of $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$, denoted by the same, that converges weakly in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon\searrow 0$.

After these preparatory steps we notice that $(\dot{\boldsymbol{u}}_\varepsilon[V]-\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ are the unique minimizers in $\boldsymbol{V}$ of the convex functionals $(F_\varepsilon)_{\varepsilon>0}$, and similarly $(\dot{\boldsymbol{u}}_0[V]-\boldsymbol{w}_0)$ is the unique minimizer of F_0 in $\boldsymbol{V}$. This follows by observing that the linearized state equations (9.7) – (9.8) and (8.13) – (8.14) are the necessary and sufficient optimality conditions for these convex optimization problems, see also discussion in Lemma 6.3.

We continue by proving that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F_0 in $\boldsymbol{V}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology as $\varepsilon\searrow 0$. For this purpose, we will follow closely the arguments of Lemma 6.3 and only point out the steps which differ from the corresponding parts in the proof of Lemma 6.3. We conclude in several steps:

Claim: For any $\boldsymbol{v} \in \boldsymbol{V}$ it holds $\limsup_{\varepsilon \searrow 0} F_\varepsilon(\boldsymbol{v}) \leq F_0(\boldsymbol{v})$.

Proof: Without loss of generality we can assume $F_0(\boldsymbol{v}) < \infty$, which gives $\int_\Omega \alpha_0(\varphi_0) |\boldsymbol{v}|^2 < \infty$. As we know $\alpha_0(\varphi_0) \in \{0, \infty\}$ a.e. in Ω this already implies $\boldsymbol{v} = \boldsymbol{0}$ in $\Omega \setminus E^{\varphi_0}$. Using (9.6) we deduce therefrom

$$\lim_{\varepsilon \searrow 0} \int_\Omega \frac{1}{2} \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}|^2 \,\mathrm{d}x = \int_\Omega \alpha_0(\varphi_0) |\boldsymbol{v}|^2 \,\mathrm{d}x = 0$$

and applying Hölder's inequality we get moreover

$$\left| \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)\boldsymbol{v} \,\mathrm{d}x \right| \leq C \left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}|^2 \right)^{\frac{1}{2}} \xrightarrow{\varepsilon \searrow 0} 0.$$

Similarly, we get due to (9.10) that

$$\left| \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{w}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x \right| \leq C \left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \xrightarrow{\varepsilon \searrow 0} 0.$$

Combining these results with the convergence of $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{w}_0$ in $\boldsymbol{H}^1(\Omega)$ we deduce the claim.

Claim: Let $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0} \subset \boldsymbol{V}$ be such that $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{v}$ as $\varepsilon \searrow 0$. Then:

$$F_0(\boldsymbol{v}) \leq \liminf_{\varepsilon \searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon).$$

Proof: We assume $\liminf_{\varepsilon \searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon) < \infty$, otherwise the claim would be trivial. Following the arguments of Lemma 6.3, in particular the calculation in (6.17), we can deduce

$$\int_\Omega \alpha_0(\varphi_0) |\boldsymbol{v}|^2 \,\mathrm{d}x \leq \liminf_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}_\varepsilon|^2 \,\mathrm{d}x.$$

Next we choose a subsequence such that

$$\lim_{k \to \infty} \int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k}) |\boldsymbol{v}_{\varepsilon_k}|^2 \,\mathrm{d}x = \liminf_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}_\varepsilon|^2 \,\mathrm{d}x.$$

By Hölder's inequality we find for this subsequence

$$\begin{aligned} &\left| \int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k}) \boldsymbol{u}_{\varepsilon_k} \cdot \mathrm{D}V(0)\boldsymbol{v}_{\varepsilon_k} \,\mathrm{d}x \right| \leq \\ &\leq C \left(\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k}) |\boldsymbol{u}_{\varepsilon_k}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \underbrace{\left(\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k}) |\boldsymbol{v}_{\varepsilon_k}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}}_{<C} \end{aligned}$$

which gives in view of (9.5),

$$\lim_{k \to \infty} \left| \int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k}) \boldsymbol{u}_{\varepsilon_k} \cdot \mathrm{D}V(0)\boldsymbol{v}_{\varepsilon_k} \,\mathrm{d}x \right| = 0.$$

Thus, we obtain

$$\liminf_{\varepsilon \searrow 0} \left| \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)\boldsymbol{v}_\varepsilon \,\mathrm{d}x \right| \leq \lim_{k \to \infty} \left| \int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k}) \boldsymbol{u}_{\varepsilon_k} \cdot \mathrm{D}V(0)\boldsymbol{v}_{\varepsilon_k} \,\mathrm{d}x \right| = 0$$

and therefrom

$$\liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot \mathrm{D}V(0)\boldsymbol{v}_\varepsilon\,\mathrm{d}x = 0.$$

Similarly, we find by means of (9.10)

$$0 = \int_\Omega \alpha_0(\varphi_0)\boldsymbol{w}_0\cdot\boldsymbol{v}\,\mathrm{d}x = \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{w}_\varepsilon\cdot\boldsymbol{v}_\varepsilon\,\mathrm{d}x.$$

Now we can use the strong convergence of $(R_\varepsilon)_{\varepsilon>0}$ to R_0 in $\boldsymbol{H}^{-1}(\Omega)$ and the weakly lower semicontinuity of the remaining terms to deduce the statement.

Combining the previous two claims, we can conclude that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F_0 in $\boldsymbol{V}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology. And so standard results for Γ-convergence, which are stated in Theorem 3.3, imply:

Claim: If $\boldsymbol{v}_\varepsilon\in\boldsymbol{V}$ minimizes F_ε for every $\varepsilon>0$ and the sequence $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{v}_0$, then $\boldsymbol{v}_0$ minimizes F_0 and $\lim_{\varepsilon\searrow 0}F_\varepsilon(\boldsymbol{v}_\varepsilon) = F_0(\boldsymbol{v}_0)$.

We will use this result to show the remaining statements of the theorem. To this end, we recall that $(\dot{\boldsymbol{u}}_\varepsilon[V]-\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some element in $\boldsymbol{H}^1(\Omega)$, which has to be a minimizer of F_0 due to the claim above. But since F_0 is a strictly convex function, the minimizer $\dot{\boldsymbol{u}}_0[V]-\boldsymbol{w}_0$ is the only one, and thus $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\dot{\boldsymbol{u}}_0[V]$ and

$$\lim_{\varepsilon\searrow 0}F_\varepsilon\left(\dot{\boldsymbol{u}}_\varepsilon[V]-\boldsymbol{w}_\varepsilon\right) = F_0\left(\dot{\boldsymbol{u}}_0[V]-\boldsymbol{w}_0\right). \tag{9.13}$$

By

$$\left|\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{w}_\varepsilon\cdot\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x\right| \overset{(9.10)}{\le} C\underbrace{\left(\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}}_{\xrightarrow{(9.5)}0}\underbrace{\left(\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}}_{\overset{(9.12)}{<}C} \tag{9.14}$$

we also have

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{w}_\varepsilon\cdot\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x = 0.$$

Thanks to the convergence of $(R_\varepsilon)_{\varepsilon>0}$ to R_0 in $\boldsymbol{H}^{-1}(\Omega)$, the strong convergence of $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ this yields in view of (9.13)

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\left[\int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2+\frac{\mu}{2}\left|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2\,\mathrm{d}x+\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\cdot\mathrm{D}V(0)\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x\right]=\\&=\int_\Omega \frac{1}{2}\alpha_0\left(\varphi_0\right)\left|\dot{\boldsymbol{u}}_0[V]\right|^2+\frac{\mu}{2}\left|\nabla\dot{\boldsymbol{u}}_0[V]\right|^2\,\mathrm{d}x.\end{aligned}$$

Applying again (9.12) and (9.5) we find similar to (9.14)

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\cdot\mathrm{D}V(0)\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x = 0$$

wherefrom we arrive in

$$\lim_{\varepsilon\searrow 0}\left[\int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right|^2+\frac{\mu}{2}\left|\nabla\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right|^2\right]=\int_\Omega \frac{1}{2}\alpha_0\left(\varphi_0\right)\left|\dot{\boldsymbol{u}}_0\left[V\right]\right|^2+\frac{\mu}{2}\left|\nabla\dot{\boldsymbol{u}}_0\left[V\right]\right|^2\,\mathrm{d}x.$$

Thus, using Lemma A.1, we can deduce the strong convergence of $\left(\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ and

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right|^2\,\mathrm{d}x=\int_\Omega \frac{1}{2}\alpha_0(\varphi_0)\left|\dot{\boldsymbol{u}}_0\left[V\right]\right|^2\,\mathrm{d}x=0.$$

We continue this proof by considering the terms in the optimality system arising from the Ginzburg-Landau energy. To this end we observe that

$$\lim_{\varepsilon\searrow 0}j_\varepsilon\left(\varphi_\varepsilon\right)=j_0(\varphi_0),\quad \lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2\,\mathrm{d}x=0$$

together with (9.4b) imply

$$\lim_{\varepsilon\searrow 0}\int_\Omega\left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2+\frac{\gamma}{\varepsilon}\psi\left(\varphi\right)\right)\mathrm{d}x=\gamma c_0 P_\Omega\left(E_0\right).$$

Using the same calculations as in [Gar08, Proof of Theorem 4.2] we can deduce therefrom

$$\lim_{\varepsilon\searrow 0}\gamma\int_\Omega\left(\frac{\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2+\frac{1}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right)\operatorname{div}V(0)\,\mathrm{d}x=\gamma c_0\int_\Omega \operatorname{div}V(0)\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|$$

and

$$\lim_{\varepsilon\searrow 0}\gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x=\gamma c_0\int_\Omega \nu\cdot\nabla V(0)\nu\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|$$

where ν is as usual the generalised unit normal on E_0. The proof in [Gar08] uses ideas of [LM89] and is based on using the auxiliary function $\phi(\varphi)=2\int_{-1}^{\varphi}\sqrt{\psi\left(z\right)}\,\mathrm{d}z$. The unit normal ν is then approximated by smooth functions ν_δ, where ν_δ should also be good approximations of $\nu_\varepsilon:=-\frac{\nabla\phi(\varphi_\varepsilon)}{|\nabla\phi(\varphi_\varepsilon)|}$. Then, applying the Reshetnyak continuity theorem, see Theorem 3.2, gives the result. For more details we refer the reader to [Gar08, Proof of Theorem 4.2].

To finish the proof of (9.2) we deduce from (9.5) and (9.12)

$$\left|\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\cdot\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\,\mathrm{d}x\right|\le\left(\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2\right)^{\frac{1}{2}}\left(\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right|^2\right)^{\frac{1}{2}}\xrightarrow{\varepsilon\searrow 0}0.$$

At the same time, (9.2) and the regularity of $V\in\overline{\mathcal{V}}_{ad}$ imply

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2\operatorname{div}V(0)\,\mathrm{d}x=0.$$

Due to the proven convergence results of $\left(\boldsymbol{u}_\varepsilon\right)_{\varepsilon>0}$ and $\left(\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right)_{\varepsilon>0}$ we thus obtain

$$\lim_{\varepsilon\searrow 0}\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=\partial_t|_{t=0}j_0\left(\varphi_0\circ T_t^{-1}\right).\tag{9.15}$$

It remains to consider the Lagrange multipliers $\left(\lambda_\varepsilon\right)_{\varepsilon>0}$. In view of (7.32), we see that the left-hand side of

$$\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=-\lambda_\varepsilon\int_\Omega \varphi_\varepsilon\operatorname{div}V(0)\,\mathrm{d}x$$

converges for every $T \in \overline{\mathcal{T}}_{ad}$ with velocity field $V \in \overline{\mathcal{V}}_{ad}$ as $\varepsilon \searrow 0$. We choose similar to [Gar08] a specific velocity field $V \in \overline{\mathcal{V}}_{ad}$ such that it holds

$$\int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x > 0.$$

This is possible, since $\varphi_0 \in \Phi_{ad}$ and thus $\{\varphi_0 = 1\} \subsetneq \Omega$, and due to (9.3) it holds $\{\varphi_0 = -1\} \subsetneq \Omega$. Then we deduce from (9.15) that

$$\lim_{\varepsilon\searrow 0} -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x = \lim_{\varepsilon\searrow 0} \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right).$$

But since

$$\lim_{\varepsilon\searrow 0} \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x = \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x > 0$$

it follows therefrom that $(\lambda_\varepsilon)_{\varepsilon>0}$ converges in $\mathbb{R}$, and we call the limit element $\lambda_0 \in \mathbb{R}^+$. Additionally, we know then that $\lambda_0 \in \mathbb{R}^+$ fulfills (8.16) – (8.17). This finally finishes the proof. □

We give a brief conclusion of the results we have shown in this part. Starting in Section 5 we have introduced a phase field approach for general shape and topology optimization problems in a Stokes flow and showed its well-posedness. Moreover, we have shown that the phase field model approximates a sharp interface model as the phase field parameter ε tends to zero in the following sense: In Section 6 we have shown that under suitable assumptions on the sequence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0}$ a subsequence of the latter converges in $L^1(\Omega)$ to a minimizer of the sharp interface model and the minimal functional values then converge, too. We have generalised this result in the case of minimizing the total potential power in Section 6.3. More precisely, in this setting we could show Γ-convergence of the corresponding reduced objective functionals. Additionally, we have discussed optimality criteria for both the diffuse and the sharp interface formulation independently. In this section, we have finally shown, that the optimality conditions of the phase field model approximate the optimality system of the sharp interface problem if we assume some convergence rate for the minimizers $(\varphi_\varepsilon)_{\varepsilon>0}$. This implies, that we have found a consistent approximation for the shape and topology optimization problem, where even numerical results using first order optimality conditions can be expected to bring good results. First numerical results using the proposed phase field ansatz will briefly be shown in the next section. An outlook and a discussion on whether it may be possible to extend or generalize the stated results has been given in Section 6.4.

Figure 8: Shape minimizer for the diffuser problem

10 Examples

In this section, we want to discuss briefly some numerical results which were calculated by Christian Kahle from University of Hamburg. We will see, that we arrive with our ansatz in the same results as obtained in known literature. We consider the typical problem of minimizing the total potential power, thus the objective functional is given by

$$f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) := \frac{\mu}{2} |\mathrm{D}\boldsymbol{u}|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u}(x).$$

We assume to have no body force, which means $\boldsymbol{f} \equiv \mathbf{0}$. The settings are adapted from [BP03], where the detailed description of the prescribed data and geometry can be found. To solve the problems, the phase field formulation, given by (5.1) – (5.2), is used, and in particular the first order optimality conditions given by Theorem 7.1 are one key ingredient for the method.

The first example is the diffuser problem, compare Figure 8, with a parabolic inflow profile at the left-hand side and a parabolic outflow profile at the right-hand side. Using this data, the optimization leads to the result shown in Figure 8, where the red domain corresponds to the fluid domain.
As a second example we consider a double pipe setting, which is described in more detail in [BP03]. At the left-hand side of the geometry we have two inflow regions and and the right-hand side there are two outflow regions. In particular, a different behaviour for different lengths of the domain Ω can be observed. For a wide domain, we obtain a connected fluid domain, see Figure 9, whereas for narrower domains Ω we find as a solution two fluid domains connecting the in- and outflows, see Figure 10. The same results have been observed in known literature, see for instance [BP03, DLL$^+$11]. This example indicates the importance of allowing topological changes in shape optimization problems in fluid dynamics, since even in this standard setting it is a priori unknown which topology brings a better result in terms of minimal values of the objective functional.

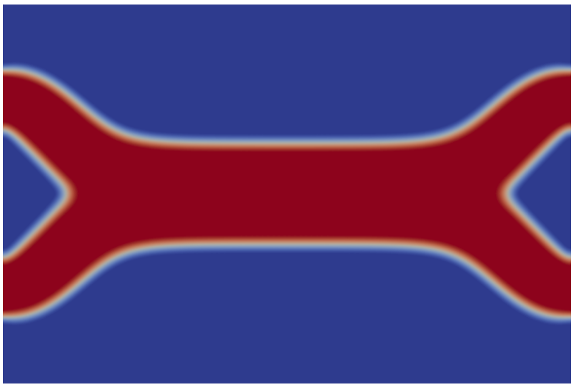


Figure 9: Solution for the double pipe problem, if we have a wide domain $\Omega = (0, 1.5) \times (0, 1)$.

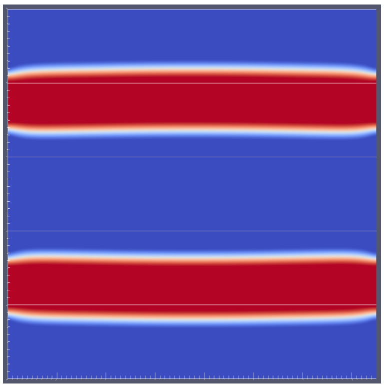


Figure 10: Solution for the double pipe problem, if we have a smaller domain $\Omega = (0, 1) \times (0, 1)$.

Part II
Stationary Navier-Stokes flow

In this part, we want to alter the problem of Part I by using the stationary Navier-Stokes equations as state constraints instead of the Stokes equations, in both the phase field and the sharp interface model. So the convective acceleration $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$, where $\boldsymbol{u}$ denotes the velocity of the fluid, will not be neglected in the state equations any more, which leads to a nonlinear system. Analytically, this gives rise to several difficulties. First, we certainly have to deal with some technical aspects if we have nonlinear state equations. But most important, the steady-state Navier-Stokes equations do not inherit a unique solution in general. Uniqueness has so far only been shown for small data or large viscosity, cf. [Soh01, Gal11, Tem77].

Nevertheless, we can study the same aspects as in the previous part, namely well-posedness and first order optimality conditions of the phase field model and the sharp interface model. We will then establish a relation between both models. Since, as indicated above, the state equations are not uniquely solvable, we will not be able to consider some reduced objective functionals as we have done in the first part. However, we are still able to deduce convergence of the minimizers of the diffuse interface model to a minimizer of the sharp interface model, if the minimizers fulfill a certain convergence rate. We find that then simultaneously the corresponding optimality systems converge.

11 The stationary Navier-Stokes equations

In this section we review some of the standard facts on the stationary Navier-Stokes equations. We will touch only a few aspects of the theory that will be of importance for us. This will motivate the additional assumptions introduced in Section 11.2 that are required for this part. We will see that those conditions are necessary because of the possibility of non-uniqueness of solutions to the stationary Navier-Stokes equations.

11.1 Introduction to known results

Let us start by introducing a notation for the nonlinear convective term arising in the stationary Navier-Stokes equations. We denote by

$$b : \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \mathbb{R}$$

the following trilinear form

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \sum_{i,j=1}^{d} \int_\Omega u_i \partial_i v_j w_j \, \mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}x.$$

Using the restriction on the space dimension $d \in \{2, 3\}$, the imbedding theorems and classical results, we see that this trilinear form fulfills the following properties:

Lemma 11.1. *The form b is well-defined and continuous in the space*

$$\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1_0(\Omega).$$

Moreover we have:

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le K_\Omega \|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{u},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \tag{11.1}$$

with

$$K_\Omega = \begin{cases} \frac{2\sqrt{2}|\Omega|^{1/6}}{3} & \text{if } d = 3, \\ \frac{|\Omega|^{1/2}}{2} & \text{if } d = 2. \end{cases}$$

Additionally, the following properties are satisfied:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0 \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega),\ \operatorname{div}\boldsymbol{u} = 0,\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{11.2}$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega),\ \operatorname{div}\boldsymbol{u} = 0,\ \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega). \tag{11.3}$$

Proof. The stated continuity and estimate (11.1) can be found in [Gal11, Lemma IX.1.1] and (11.2) – (11.3) are considered in [Gal11, Lemma IX.2.1]. □

Besides, we have the following important continuity property:

Lemma 11.2. *Let* $(\boldsymbol{u}_n)_{n\in\mathbb{N}}, (\boldsymbol{v}_n)_{n\in\mathbb{N}}, \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ *be such that*

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}, \quad \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \qquad \text{in } \boldsymbol{H}^1(\Omega)$$

where $\boldsymbol{v}_n|_{\partial\Omega} = \boldsymbol{v}|_{\partial\Omega}$ *for all* $n \in \mathbb{N}$.
Then

$$\lim_{n\to\infty} b(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}) = b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega).$$

Moreover, one can show that

$$\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \ni (\boldsymbol{u},\boldsymbol{v}) \mapsto b(\boldsymbol{u},\cdot,\boldsymbol{v}) \in \boldsymbol{H}^{-1}(\Omega) \tag{11.4}$$

is strongly continuous.

Proof. We apply the idea of [Zei97, Lemma 72.5]. Therefore, we first notice that $\boldsymbol{H}^1(\Omega)$ is compactly imbedded in $\boldsymbol{L}^3(\Omega)$ and continuously in $\boldsymbol{L}^6(\Omega)$, since we are restricted to space dimensions $d \in \{2,3\}$. Thus having weak converging sequences $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{v}_n)_{n\in\mathbb{N}} \subseteq \boldsymbol{H}^1(\Omega)$ as given in the statement, we find that they converge strongly in $\boldsymbol{L}^3(\Omega)$ and are bounded in $\boldsymbol{L}^6(\Omega)$. So we can estimate for any $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ with Hölder's inequality

$$\begin{aligned} |b(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}) - b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\le |b(\boldsymbol{u}_n-\boldsymbol{u},\boldsymbol{v}_n,\boldsymbol{w})| + |b(\boldsymbol{u},\boldsymbol{v}_n-\boldsymbol{v},\boldsymbol{w})| \le \\ &\le \underbrace{\|\boldsymbol{u}_n-\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}}_{\to 0} \underbrace{\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{w}\|_{\boldsymbol{L}^6(\Omega)}}_{\le C} + |b(\boldsymbol{u},\boldsymbol{v}_n-\boldsymbol{v},\boldsymbol{w})|. \end{aligned}$$

With similar estimates and making use of the imbedding theorems and Poincaré's inequality we see that $b(\boldsymbol{u},\cdot,\boldsymbol{w}) \in \boldsymbol{H}^{-1}(\Omega)$ and so the weak convergence of $(\boldsymbol{v}_n-\boldsymbol{v})_{n\in\mathbb{N}} \subseteq \boldsymbol{H}_0^1(\Omega)$ in $\boldsymbol{H}^1(\Omega)$ to zero implies

$$\lim_{n\to\infty} b(\boldsymbol{u},\boldsymbol{v}_n-\boldsymbol{v},\boldsymbol{w}) = 0$$

and so we have shown

$$\lim_{n\to\infty} b(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}) = b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega).$$

The strong continuity stated in (11.4) follows in a similar way, see for instance [Zei97, Lemma 72.5].

□

We continue with a technical lemma that will be needed quite often and is taken from [Gal11, Lemma IX.4.2].

Lemma 11.3. *Let U be a bounded Lipschitz domain in $\mathbb{R}^d$ such that $\mathbb{R}^d \setminus \overline{U}$ is connected and let $\boldsymbol{v}_* \in \boldsymbol{H}^{\frac{1}{2}}(\partial U)$ satisfy*

$$\int_{\partial U} \boldsymbol{v}_* \cdot \boldsymbol{n} \, \mathrm{d}x = 0$$

where $\boldsymbol{n}$ denotes here the outer unit normal on U.
Then for any $\eta > 0$ there exists some $\delta = \delta(\eta, \boldsymbol{v}_*, \boldsymbol{n}, U) > 0$ and a vector field $\boldsymbol{V} = \boldsymbol{V}(\delta)$ such that

$$\boldsymbol{V} \in \boldsymbol{H}^1(U), \qquad \operatorname{div} \boldsymbol{V} = 0, \qquad \boldsymbol{V} = \boldsymbol{v}_* \text{ on } \partial U$$

and verifying

$$\left| \int_U \boldsymbol{u} \cdot \nabla \boldsymbol{V} \cdot \boldsymbol{u} \, \mathrm{d}x \right| \le \eta \, \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(U)} \tag{11.5}$$

for all $\boldsymbol{u} \in \boldsymbol{H}^1_0(U)$.

The next lemma corresponds to [Gal11, Theorem IX.4.1].

Lemma 11.4. *Let U be a be a bounded Lipschitz domain of $\mathbb{R}^d$ such that $\mathbb{R}^d \setminus \overline{U}$ is connected and let*

$$\boldsymbol{v}_* \in \boldsymbol{H}^{\frac{1}{2}}(\partial U), \quad \boldsymbol{F} \in \boldsymbol{H}^{-1}(U)$$

with $\boldsymbol{v}_*$ satisfying

$$\int_{\partial U} \boldsymbol{v}_* \cdot \boldsymbol{n} \, \mathrm{d}x = 0$$

where $\boldsymbol{n}$ denotes the outer unit normal on U.
Then there exists a constant $\tilde{c}_3 = \tilde{c}_3(d, U)$ such that if

$$\|\boldsymbol{v}_*\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial U)} \le \tilde{c}_3 \frac{\mu}{2}$$

any solution $\boldsymbol{v} \in \boldsymbol{H}^1(U)$ of

$$\begin{aligned} -\mu\Delta\boldsymbol{v} + \boldsymbol{v} \cdot \nabla\boldsymbol{v} + \nabla p &= \boldsymbol{F} && \text{in } U \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } U \\ \boldsymbol{v} &= \boldsymbol{v}_* && \text{on } \partial U \end{aligned}$$

in the following sense

$$\int_\Omega \mu \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{w} \, \mathrm{d}x + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) = \langle \boldsymbol{F}, \boldsymbol{w} \rangle_{\boldsymbol{H}^{-1}(U)} \quad \forall \boldsymbol{w} \in \boldsymbol{V} \tag{11.6}$$

verifies the following estimate:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(U)} \le \frac{\tilde{c}_4}{\mu} \left(\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}(U)} + \|\boldsymbol{v}_*\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\partial U)} + \mu \, \|\boldsymbol{v}_*\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial U)} \right)$$

with $\tilde{c}_4 = \tilde{c}_4(d, U)$.

Furthermore, we have the following standard uniqueness result for the stationary Navier-Stokes equation, compare [Gal11, Theorem IX.2.1]:

Lemma 11.5. *Let U be a bounded Lipschitz domain in $\mathbb{R}^d$, and let $\boldsymbol{v} \in \boldsymbol{H}^1(U)$ be a solution to*

$$\begin{aligned} -\mu\Delta\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p &= \boldsymbol{F} && \text{in } U \\ \operatorname{div}\boldsymbol{v} &= 0 && \text{in } U \\ \boldsymbol{v} &= \boldsymbol{v}_* && \text{on } \partial U \end{aligned}$$

for $\boldsymbol{F}$ and $\boldsymbol{v}_*$ as in Lemma 11.4. If we denote by $\boldsymbol{w}$ another solution corresponding to the same data, $\boldsymbol{v} \equiv \boldsymbol{w}$, provided that

$$\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(U)} < \frac{\mu}{K_U}$$

where

$$K_U = \begin{cases} \frac{2\sqrt{2}|U|^{1/6}}{3} & \text{if } d = 3, \\ \frac{|U|^{1/2}}{2} & \text{if } d = 2. \end{cases}$$

For further details and an extensive discussion of the stationary Navier-Stokes equations we refer the reader for instance to the books [Gal11, Tem77, Soh01].

We will now discuss the equations in a formulation, in which they will occur in our sharp interface model later on. The important point here is that we will have to solve the stationary Navier-Stokes equations in a set of finite perimeter $E \subset \Omega$ and not any more in open Lipschitz sets as in the discussion above. However, we will work again in the given Lipschitz container Ω, as we have done it for the Stokes equations in the first part, and assume the velocity being zero outside of E. One observes, that the constant appearing in the estimate of Lemma 11.4 depends on the domain U on which the state equations are soved. In order to get a better insight on how the constant depends on the domain, we restate the lemma in a different formulation here and give some details on the proof.

Lemma 11.6. *Assume $E \subseteq \Omega$ is an arbitrary Lipschitz domain and let $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ be the boundary function given by Assumption **(A2)**. Then there exists some constant $c_3 = c_3(E)$, such that if*

$$\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} \leq c_3\frac{\mu}{2}$$

we obtain for any $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ fulfilling

$$\boldsymbol{u}|_{\Omega\setminus E} = \boldsymbol{0}, \quad \operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g} \tag{11.7}$$

and

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \qquad \forall\boldsymbol{v}\in\boldsymbol{V}^E \tag{11.8}$$

the following estimate

$$\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{c_4}{\mu}\left(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{g}\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} + \mu\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}\right) \tag{11.9}$$

for some $c_4 = c_4(E)$.

Remark 11.1. *Remark that for an arbitrary set $E \subset \Omega$ there may not even be a solenoidal velocity field $\boldsymbol{u}$ with $\boldsymbol{u} = \boldsymbol{0}$ in $\Omega \setminus E$ and $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$, consider for instance the case $\mathcal{H}^{d-1}\left(\left\{x \in \partial\Omega \mid x \notin \overline{E}, \boldsymbol{g}(x) \neq \boldsymbol{0}\right\}\right) > 0$. In this case, there is no solution to* (11.8) *and so the statement of the lemma is trivial.*

Proof. Assume that $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ solves (11.7)-(11.8). We define the trace operator

$$T : \boldsymbol{H}^E(\mathrm{div}) \to \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$$

with

$$\boldsymbol{H}^E(\mathrm{div}) := \left\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \mathrm{div}\,\boldsymbol{v} = 0,\ \boldsymbol{v}|_{\Omega\setminus E} = \boldsymbol{0} \text{ a.e.}\right\}$$

as the restriction of the classical trace operator defined on $\boldsymbol{H}^1(\Omega)$ to the closed subspace $\boldsymbol{H}^E(\mathrm{div})$. We see that T is continuous and we denote the continuity constant by $c_T(E)$. Applying the results from Lemma 4.1 one sees quiet easily that $\mathcal{R}(T)$ is closed, as E is assumed to have a Lipschitz boundary. Using the closed graph theorem, in much the same way as the proof of [Soh01, Lemma II.2.1.1] we see that there exists a continuous inverse

$$T^{-1} : \mathcal{R}(T) \to \boldsymbol{H}^E(\mathrm{div})/\mathcal{N}(T)$$

from the image of T to $\boldsymbol{H}^E(\mathrm{div})$ modulus the null space of T, where

$$\left\|T^{-1}\right\|_{\mathcal{L}(\mathcal{R}(T),\boldsymbol{H}^E(\mathrm{div})/\mathcal{N}(T))} \le c_T(E)^{-1}.$$

In particular, considering our given boundary function $\boldsymbol{g}$, we deduce from (11.7) that $\boldsymbol{g} \in \mathcal{R}(T)$ and so

$$\|[\boldsymbol{u}]\|_{\boldsymbol{H}^1(\Omega)} \le c_T(E)^{-1}\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}$$

where $[\boldsymbol{u}]$ is the equivalence class of $\boldsymbol{u}$ in the quotient space $\boldsymbol{H}^E(\mathrm{div})/\mathcal{N}(T)$.
This implies, that we can select a representative $\boldsymbol{u}_d \in [\boldsymbol{u}]$ such that

$$\|\boldsymbol{u}_d\|_{\boldsymbol{H}^1(\Omega)} \le c_T(E)^{-1}\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}. \tag{11.10}$$

Now we proceed similar to [Gal11, Theorem IX.4.1] and define $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{u}_d \in \boldsymbol{V}^E$ to see

$$\mu\int_\Omega \nabla\boldsymbol{w}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}) + b(\boldsymbol{w},\boldsymbol{u}_d,\boldsymbol{v}) + b(\boldsymbol{u}_d,\boldsymbol{w},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - \mu\nabla\boldsymbol{u}_d\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - \\ - b(\boldsymbol{u}_d,\boldsymbol{u}_d,\boldsymbol{v})$$

which holds for all $\boldsymbol{v} \in \boldsymbol{V}^E$. Inserting $\boldsymbol{v} \equiv \boldsymbol{w}$ and using Lemma 11.1 we obtain therefrom

$$\mu\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{w},\boldsymbol{u}_d,\boldsymbol{w}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w} - \mu\nabla\boldsymbol{u}_d\cdot\nabla\boldsymbol{w}\,\mathrm{d}x - b(\boldsymbol{u}_d,\boldsymbol{u}_d,\boldsymbol{w})$$

and so we can estimate, while using (11.1),

$$\mu\|\nabla\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\Omega)} \le c(\Omega)\Big(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \mu\|\nabla\boldsymbol{u}_d\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{u}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \\ + \|\nabla\boldsymbol{u}_d\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\Big)\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}$$

and with (11.10) therefore

$$\mu\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \le c(\Omega,E)\left(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \mu\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} + \|\boldsymbol{g}\|^2_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\right).$$

If we choose $\|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)} \le c(\Omega,E)^{-1}\frac{\mu}{2}$ we can deduce therefrom (11.9). This proves the statement. □

This lemma already indicates some problem: To show uniqueness of a solution to (11.8) by using similar results as Lemma 11.6 and Lemma 11.5, one would have to impose a smallness condition on the right-hand side of (11.9), and thus bound the data $\boldsymbol{f}$ and $\boldsymbol{g}$ by the viscosity μ and the constant $c_4(E)$. But since we don't have a fixed set E, but rather vary the sets to find a minimizer of some functional, and it is not well-understood how c_4 depends on E, we cannot control this constant. Thus the uniqueness imposes a problem here which we will have to deal with in the following.

11.2 Additional assumptions on data and objective functional

The state equations in the sharp interface model will be given by (11.8). When we pass to the limit $\varepsilon \searrow 0$ in the phase field model, we have to consider the state equations, too, and obtain a sequence of velocities depending on the phase field parameter ε. Under suitable assumptions, one can show that the sequence converges to a velocity field solving the sharp interface state equation (11.8). To ensure that the limit element coincides with a given velocity solving (11.8) we need uniqueness of a solution to (11.8). This is important, since the objective functional may have a different value for two different solutions of (11.8).
For a fixed set E, one could simply assume that the right-hand side of (11.9) is smaller than $\frac{\mu}{K_\Omega}$ and then obtain with a result similar to Lemma 11.5 uniqueness of the solution. But here we want to vary the fluid region E in order to minimize a certain objective functional. This means in particular, that the constant $c_4(E)$ of (11.9) will change as E changes, and it is not clear how c_4 depends on E. For some recent results concerning this constant we refer for instance to [BRW06].

To overcome this problem, we control the velocity by the objective functional and ensure in this way that $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ is small enough for the minimizing set E, if $\boldsymbol{u}$ solves (11.8). Thus, we make the following additional assumption throughout this part:

(A8) We assume, that the body force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, the boundary term $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$, the viscosity μ and the objective functional f are chosen such that:

1. there exists some constant $C_u \in \mathbb{R}$ fulfilling

$$J_0^N(\varphi, \boldsymbol{u}) \le C_u \implies \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega} \tag{11.11}$$

   for all $\varphi \in \Phi_{ad}^0$ and $\boldsymbol{u} \in \boldsymbol{S}_0^N(\varphi)$; and

2. there exists at least one $\varphi_0 \in \Phi_{ad}^0$ and $\boldsymbol{u}_0 \in \boldsymbol{S}_0^N(\varphi_0)$ with

$$J_0^N(\varphi_0, \boldsymbol{u}_0) \le C_u, \tag{11.12}$$

where the solution operator $\boldsymbol{S}_0^N$ for the stationary Navier-Stokes equations is defined in Lemma 13.1 and the functional J_0^N is given by (13.18).

This requirement will imply unique solvability of the state equations in a neighborhood of the minimizer of J_0^N, see Corollary 14.1, which will be crucial for the convergence proof, see Theorem 14.1.

Before starting the considerations concerning the phase field approach we want to discuss Assumption **(A8)** on the example of minimizing the total potential power, see Example 2.3. We will see that in this case Assumption **(A8)** is equivalent to the usual "smallness of data or high viscosity" stated in literature concerning uniqueness of the stationary Navier-Stokes equations, cf. [Gal11, Tem77, Zei97].

Example 11.1. Let's consider the problem of minimizing the total potential power, which leads to the following objective functional in the sharp interface formulation:

$$J_0^N(\varphi, \boldsymbol{u}) := \int_\Omega \frac{\mu}{2} |\nabla \boldsymbol{u}|^2 - \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi).$$

We take some arbitrary set E_0, maybe with Lipschitz regularity, such that $\varphi_0 := (2\chi_{E_0} - 1) \in \Phi_{ad}^0$ and use for instance Lemma 6.2 to construct an element $\boldsymbol{u} \in \boldsymbol{U}^{E_0}$. Without loss of generality we may assume homogeneous boundary data, thus $\boldsymbol{g} = \boldsymbol{0}$. Otherwise, we reduce the problem to a homogeneous one as in the proof of Lemma 11.6. Then we can, due to Lemma 11.6, choose $\boldsymbol{f}$ small enough or μ large enough, respectively, depending on E_0, such that for a solution $\boldsymbol{u} \in \boldsymbol{U}^{E_0}$ of (11.9) corresponding to E_0 it holds

$$\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \le \left(\frac{1}{3}\left(\frac{\mu}{2K_\Omega}\right)^2 - \frac{8}{3\mu^2}\left(\beta_d |\Omega|^{\frac{1}{d}} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\right)^2 - \frac{4}{3\mu}\gamma c_0 P_\Omega(E_0)\right)^{\frac{1}{2}} \tag{11.13}$$

where the right-hand side will be positive for μ large enough.
Now we define

$$C_u := \frac{\mu}{4}\left(\frac{\mu}{2K_\Omega}\right)^2 - \frac{1}{\mu}\left(\beta_d |\Omega|^{\frac{1}{d}} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\right)^2$$

with

$$\beta_d := \begin{cases} \frac{2}{\sqrt{3}}, & \text{if } d = 3, \\ \frac{1}{\sqrt{2}}, & \text{if } d = 2, \end{cases}$$

being the constant arising in Poincaré's inequality, see [Gal11, (II.5.5)], in the following sense:

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le \beta_d |\Omega|^{1/d} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{11.14}$$

With the help of Hölder's inequality we get

$$\begin{aligned} J_0^N(\varphi_0, \boldsymbol{u}_0) &= \frac{\mu}{2} \|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}^2 - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_0 \, \mathrm{d}x + \gamma c_0 P_\Omega(E_0) \le \\ &\le \frac{\mu}{2} \|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} + \gamma c_0 P_\Omega(E_0). \end{aligned}$$

This gives us in view of Poincaré's inequality (11.14) and by using Young's inequality the estimate

$$\begin{aligned} J_0^N(\varphi_0, \boldsymbol{u}_0) &\le \frac{\mu}{2} \|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}^2 + \beta_d |\Omega|^{\frac{1}{d}} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} + \gamma c_0 P_\Omega(E_0) \le \\ &\le \frac{3\mu}{4} \|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}^2 + \frac{1}{\mu}\left(\beta_d |\Omega|^{\frac{1}{d}} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\right)^2 + \gamma c_0 P_\Omega(E_0) \le \\ &\le \frac{\mu}{4}\left(\frac{\mu}{2K_\Omega}\right)^2 - \frac{1}{\mu}\left(\beta_d |\Omega|^{\frac{1}{d}} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\right)^2 = C_u. \end{aligned}$$

Thus we have verified (11.12).

Taking on the other hand an arbitrary $(\varphi, \boldsymbol{u}) \in \Phi_{ad}^0 \times \boldsymbol{U}$ such that

$$J_0^N(\varphi, \boldsymbol{u}) \le C_u$$

we see, using again Poincaré's inequality (11.14) and Young's inequality, that

$$\begin{aligned}\frac{\mu}{2}\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2 &\le C_u - \gamma c_0 P_\Omega(E_0) + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x \le \\ &\le C_u + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le C_u + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\beta_d|\Omega|^{\frac{1}{d}}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \\ &\le C_u + \frac{1}{\mu}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\beta_d|\Omega|^{\frac{1}{d}}\right)^2 + \frac{\mu}{4}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2.\end{aligned}$$

This implies

$$\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2 \le \frac{4}{\mu}C_u + \frac{4}{\mu^2}\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\beta_d|\Omega|^{\frac{1}{d}}\right)^2 \le \left(\frac{\mu}{2K_\Omega}\right)^2$$

and therefrom

$$\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}$$

which yields (11.11).

Hence, using this choice of the data, Assumption **(A8)** would be fulfilled for the problem of minimizing the total potential power.
In particular, we notice that in this setting, Assumption **(A8)** reduces to choosing the external force $\boldsymbol{f}$ and the viscosity μ small and large enough, respectively, such that (11.13) is fulfilled. And so we find, that Assumption **(A8)** is a comparable condition as imposed in classical literature to ensure uniqueness of a solution to the stationary Navier-Stokes equations.

12 Phase field model

In this section, we will restrict our attention to the phase field model describing shape and topology optimization in a stationary Navier-Stokes flow. After formulating the problem, we develop existence and uniqueness theory for the state equations and proceed the study by showing the existence of a minimizer for the overall optimization problem.

12.1 Problem formulation

This section is devoted to the study of the following general optimization problem:

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u})} J_\varepsilon^N(\varphi,\boldsymbol{u}) := \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \\ + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x\end{aligned} \tag{12.1}$$

with

$$(\varphi,\boldsymbol{u}) \in \Phi_{ad}\times\boldsymbol{U}$$

s.t.

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{V}. \tag{12.2}$$

This problem corresponds to (5.1) – (5.2) of Part I, where we replaced the penalized Stokes equations in the constraints by penalized stationary Navier-Stokes equations. Again the design variable is given by $\varphi \in \Phi_{ad} = \left\{\varphi\in H^1(\Omega) \mid \fint_\Omega \varphi\,\mathrm{d}x \le \beta,\, |\varphi|\le 1 \text{ a.e. in } \Omega\right\}$, which models the presence and non-presence of fluid in the following sense: The set $\{x\in\Omega \mid \varphi(x)=1\}$ is the set which is filled with fluid, and so the constraints (12.2) reduce in this part to the stationary Navier-Stokes equations. In $\{x\in\Omega \mid \varphi(x)=-1\}$ the equations can be considered as a Darcy flow through porous medium whose permeability is given by $\alpha_\varepsilon(-1)^{-1}$. As ε tends to zero, the permeability will tend to zero and the penalization term in the penalized stationary Navier-Stokes equations including the inverse permeability will vanish. In particular, the velocity of the fluid will then be zero in $\{x\in\Omega \mid \varphi(x)=-1\}$. In this sense, the set $\{x\in\Omega \mid \varphi(x)=-1\}$ represents the region where no fluid is present.
Since we consider a phase field model, we have additionally an interface between fluid and non-fluid whose thickness is proportional to ε, and thus vanishes as ε tends to zero.

For details concerning the phase field approximation and the penalization term ensuring vanishing permeability of the walls in the limit $\varepsilon \searrow 0$ we refer to the discussion in Section 5.1.

We will be concerned in the following with well-posedness of the constraints (12.2) and define a solution operator called $\boldsymbol{S}_\varepsilon^N$, see Lemma 12.1. Since in general we might not have a *unique* solution for an arbitrary $\varphi\in\overline{\Phi}_{ad}$, the solution operator may be set valued, and so we cannot reformulate the problem into minimizing a reduced objective functional as it was possible in Section 5.
Afterwards, we show existence of minimizers for the optimal control problem (12.1) – (12.2).

12.2 Existence results

We start by examining the state equation (12.2). This is obviously a weak formulation of the following system:

$$\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{12.3a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{12.3b}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \partial\Omega. \tag{12.3c}$$

We show:

Lemma 12.1. *For every $\varphi \in L^1(\Omega)$ with $|\varphi(x)| \le 1$ a.e. in Ω there exists at least one $\boldsymbol{u} \in \boldsymbol{U}$ such that* (12.3) *is fulfilled in the sense of* (12.2).
This defines a set-valued solution operator for the constraints, which will be denoted by

$$\boldsymbol{S}_\varepsilon^N(\varphi) := \{\boldsymbol{u} \in \boldsymbol{U} \mid \boldsymbol{u} \text{ solves } (12.2)\} \qquad \forall \varphi \in \overline{\Phi}_{ad}.$$

Remark 12.1. *Remark that for any $\boldsymbol{u} \in \boldsymbol{U}$ and $\varphi \in \overline{\Phi}_{ad}$ fulfilling* (12.2) *there exists some $p \in L^2(\Omega)$, which is unique up to a constant, such that*

$$\int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \tag{12.4}$$

holds for all $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$. This follows from Lemma 4.4. But as the pressure is not appearing in the objective functional in this setting, we will drop this variable in the following considerations.

Proof. For showing the existence of a velocity field $\boldsymbol{u} \in \boldsymbol{U}$ satisfying (12.2) we apply the arguments of [Zei97, Theorem 72.A], which is an application of the theory on pseudo-monotone operators. To this end, we fix $\varphi \in L^1(\Omega)$ with $|\varphi| \le 1$ a.e. in Ω.
At first, we rewrite the non-homogeneous problem into a homogeneous one analogously to [Tem77, Theorem 1.5, Chapter II] by defining $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$ as a solution of

$$\operatorname{div}\boldsymbol{\psi} = 0 \qquad \text{in } \Omega,$$

$$\boldsymbol{\psi} = \boldsymbol{g} \qquad \text{on } \partial\Omega,$$

such that

$$b(\boldsymbol{v},\boldsymbol{\psi},\boldsymbol{v}) \le \frac{\mu}{2}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{12.5}$$

The existence of such a function $\boldsymbol{\psi}$ follows from Lemma 11.3. Then $\boldsymbol{u} \in \boldsymbol{U}$ solves (12.2) if and only if $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{\psi} \in \boldsymbol{V}$ fulfills

$$\int_\Omega \alpha_\varepsilon(\varphi)\hat{\boldsymbol{u}}\cdot\boldsymbol{v} + \mu\nabla\hat{\boldsymbol{u}}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\hat{\boldsymbol{u}},\hat{\boldsymbol{u}},\boldsymbol{v}) + b(\hat{\boldsymbol{u}},\boldsymbol{\psi},\boldsymbol{v}) + b(\boldsymbol{\psi},\hat{\boldsymbol{u}},\boldsymbol{v}) = \langle \hat{\boldsymbol{f}},\boldsymbol{v}\rangle_{\boldsymbol{H}^{-1}(\Omega)} \tag{12.6}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ where we defined

$$\hat{\boldsymbol{f}} := \boldsymbol{f} + \mu\Delta\boldsymbol{\psi} - \boldsymbol{\psi}\cdot\nabla\boldsymbol{\psi} - \alpha_\varepsilon(\varphi)\boldsymbol{\psi} \in \boldsymbol{H}^{-1}(\Omega).$$

Then we can deduce that the linear operator $A : \boldsymbol{V} \to \boldsymbol{V}'$, which is given by

$$A(\boldsymbol{v})(\boldsymbol{w}) := \int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{v}\cdot\boldsymbol{w} + \mu\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{w}\,\mathrm{d}x + b(\boldsymbol{v},\boldsymbol{\psi},\boldsymbol{w}) + b(\boldsymbol{\psi},\boldsymbol{v},\boldsymbol{w}) \qquad \forall \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V},$$

is monotone because

$$\langle A\boldsymbol{v} - A\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w} \rangle_{\boldsymbol{V}'} = \underbrace{\int_\Omega \alpha_\varepsilon(\varphi) |\boldsymbol{v} - \boldsymbol{w}|^2 \, \mathrm{d}x}_{\geq 0} + \mu \|\nabla(\boldsymbol{v} - \boldsymbol{w})\|^2_{\boldsymbol{L}^2(\Omega)} \underbrace{+ b(\boldsymbol{v} - \boldsymbol{w}, \boldsymbol{\psi}, \boldsymbol{v} - \boldsymbol{w})}_{\overset{(12.5)}{\geq} -\frac{\mu}{2}\|\nabla(\boldsymbol{v}-\boldsymbol{w})\|^2_{\boldsymbol{L}^2(\Omega)}} +$$
$$+ \underbrace{b(\boldsymbol{\psi}, \boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w})}_{\overset{(11.2)}{=} 0} \geq \frac{\mu}{2} \|\nabla(\boldsymbol{v} - \boldsymbol{w})\|^2_{\boldsymbol{L}^2(\Omega)} \geq 0 \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}. \tag{12.7}$$

Thus, A is a monotone and linear operator and therefore pseudo-monotone (compare Remark A.5).
Defining $B : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$B(\boldsymbol{v})(\boldsymbol{w}) = b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$$

we see that B is strongly continuous (see Lemma 11.2) and thus $A+B$ is due to Remark A.5 pseudo-monotone. Moreover, since both B and A are bounded, we get that $A + B$ is a bounded operator, and from

$$B(\boldsymbol{v})(\boldsymbol{v}) = b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) = 0$$

and estimate (12.7) we see that $A + B : \boldsymbol{V} \to \boldsymbol{V}'$ is coercive.
For this reason, we can apply the main theorem on pseudo-monotone operators (see Theorem A.4) to get the existence of some $\hat{\boldsymbol{u}} \in \boldsymbol{V}$ such that (12.6) is fulfilled, which implies that $\boldsymbol{u} := \hat{\boldsymbol{u}} + \boldsymbol{\psi} \in \boldsymbol{U}$ fulfills (12.2).

□

In general we won't have a unique solution $\boldsymbol{u}$ of (12.2). But under an additional assumption, which will be fulfilled for example for minimizers of J_ε^N if ε is small enough, see Corollary 14.1, we can show uniqueness:

Lemma 12.2. *Assume that there exists a solution $\boldsymbol{u} \in \boldsymbol{U}$ of* (12.2) *such that it holds*

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}. \tag{12.8}$$

Then this is the only solution of (12.2)*.*

Proof. Assume $\boldsymbol{u} \in \boldsymbol{U}$ fulfills (12.2) and it holds (12.8). Moreover, assume $\widehat{\boldsymbol{u}} \in \boldsymbol{U}$ is another solution of (12.2). Similar to [Gal11, Theorem IX.2.1] we define $\boldsymbol{z} := \widehat{\boldsymbol{u}} - \boldsymbol{u}$ and see that $\boldsymbol{z}$ satisfies

$$\int_\Omega \alpha_\varepsilon(\varphi) \boldsymbol{z} \cdot \boldsymbol{v} \, \mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{v} \, \mathrm{d}x + b(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{u}}, \boldsymbol{v}) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Using the trilinearity of b this can be rewritten as

$$\int_\Omega \alpha_\varepsilon(\varphi) \boldsymbol{z} \cdot \boldsymbol{v} \, \mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{v} \, \mathrm{d}x + b(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}) + b(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Inserting $\boldsymbol{z} \in \boldsymbol{V}$ as a test function and using Lemma 11.1 we obtain therefrom

$$\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{z}|^2 \,\mathrm{d}x + \mu \int_\Omega |\nabla \boldsymbol{z}|^2 \,\mathrm{d}x + b(\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{z}) = 0.$$

This gives us in view of $\alpha_\varepsilon \geq 0$ and (11.1)

$$\mu \|\nabla \boldsymbol{z}\|^2_{\boldsymbol{L}^2(\Omega)} \leq K_\Omega \|\nabla \boldsymbol{z}\|^2_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.$$

Finally, we see from (12.8) that

$$\underbrace{\left(\mu - K_\Omega \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\right)}_{>0} \|\nabla \boldsymbol{z}\|^2_{\boldsymbol{L}^2(\Omega)} \leq 0$$

which implies together with Poincaré's inequality $\boldsymbol{z} \equiv \boldsymbol{0}$ and thus the stated uniqueness. □

Remark 12.2. *This uniqueness result corresponds to the uniqueness result of the classical Navier-Stokes equations given in Lemma 11.5.*

Let us now analyze the overall optimization problem given by (12.1)-(12.2). After having considered the state constraints, we can deduce well-posedness of the problem as the next theorem will show.

We remark that for the proof of the following theorem we do not use the radially unboundedness of the objective functional stated in Assumption **(A5)**. But we cannot show convergence of minimizers of the phase field model to a minimizer of the sharp interface model if Assumption **(A5)** is not fulfilled. Thus even though the phase field model would inherit solutions, it is not clear what happens in the sharp interface limit if the objective functional is not radially unbounded with respect to the velocity, compare also discussion in Section 5.2.

Theorem 12.1. *There exists at least one minimizer of* (12.1) – (12.2)*.*

Proof. This can be shown analogously to Theorem 5.1, where we just have to deal with the nonlinearity here. Thus we only point out the differences in the following.

We start by choosing an admissible minimizing sequence $(\varphi_k, \boldsymbol{u}_k)_{k\in\mathbb{N}} \subseteq \Phi_{ad} \times \boldsymbol{U}$, which means in particular that $\boldsymbol{u}_k \in \boldsymbol{S}^N_\varepsilon(\varphi_k)$. We use the state equation (12.2) to deduce a uniform bound on $\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}$ as follows:

Let $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega)$ be such that $\operatorname{div} \boldsymbol{\psi} = 0$, $\boldsymbol{\psi}|_{\partial\Omega} = \boldsymbol{g}$ and $b(\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{v}) \leq \frac{\mu}{2} \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}$ for all $\boldsymbol{v} \in \boldsymbol{V}$, which can be chosen due to Lemma 11.3. Then we see that $\hat{\boldsymbol{u}}_k := \boldsymbol{u}_k - \boldsymbol{\psi} \in \boldsymbol{V}$ is a solution to (12.6) with φ replaced by φ_k. Testing this equation with $\boldsymbol{v} = \hat{\boldsymbol{u}}_k$ it follows

$$\begin{aligned}
&\underbrace{(\alpha_\varepsilon(\varphi_k)\hat{\boldsymbol{u}}_k, \hat{\boldsymbol{u}}_k)_{\boldsymbol{L}^2(\Omega)}}_{\geq 0} + \mu \|\nabla \hat{\boldsymbol{u}}_k\|^2_{\boldsymbol{L}^2(\Omega)} + \underbrace{b(\hat{\boldsymbol{u}}_k, \boldsymbol{\psi}, \hat{\boldsymbol{u}}_k)}_{\geq -\frac{\mu}{2}\|\nabla \hat{\boldsymbol{u}}_k\|^2_{\boldsymbol{L}^2(\Omega)}} = \left\langle \hat{\boldsymbol{f}}_k, \hat{\boldsymbol{u}}_k \right\rangle_{\boldsymbol{H}^{-1}(\Omega)} = \\
&= (\boldsymbol{f}, \hat{\boldsymbol{u}}_k)_{\boldsymbol{L}^2(\Omega)} - \mu \int_\Omega \nabla \boldsymbol{\psi} \cdot \nabla \hat{\boldsymbol{u}}_k \,\mathrm{d}x - b(\boldsymbol{\psi}, \boldsymbol{\psi}, \hat{\boldsymbol{u}}_k) - \int_\Omega \alpha_\varepsilon(\varphi_k)\boldsymbol{\psi} \cdot \hat{\boldsymbol{u}}_k \,\mathrm{d}x.
\end{aligned} \tag{12.9}$$

Now using the inequalities of Poincaré and Young we can deduce therefrom the existence of some constant $c > 0$ such that

$$\|\nabla \hat{\boldsymbol{u}}_k\|^2_{\boldsymbol{L}^2(\Omega)} \leq c\left(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \mu \|\nabla \boldsymbol{\psi}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\psi}\|^4_{\boldsymbol{H}^1(\Omega)} + \overline{\alpha}^2_\varepsilon \|\boldsymbol{\psi}\|^2_{\boldsymbol{L}^2(\Omega)}\right). \tag{12.10}$$

Applying again Poincaré's inequality and inserting $\boldsymbol{u}_k = \hat{\boldsymbol{u}}_k + \boldsymbol{\psi}$ we obtain therefrom a bound on $\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}$ uniform in $k \in \mathbb{N}$. Moreover, the uniform bound on $\left(J^N_\varepsilon(\varphi_k, \boldsymbol{u}_k)\right)_{k\in\mathbb{N}}$ implies that $\sup_{k\in\mathbb{N}} \|\nabla\varphi_k\|_{L^2(\Omega)} < \infty$ and so we get, after possibly choosing subsequences, the following convergence results:

$$\begin{aligned}
\boldsymbol{u}_k &\rightharpoonup \boldsymbol{u}_0 && \text{in } \boldsymbol{H}^1(\Omega),\\
\boldsymbol{u}_k &\to \boldsymbol{u}_0 && \text{in } \boldsymbol{L}^2(\Omega),\\
\varphi_k &\rightharpoonup \varphi_0 && \text{in } H^1(\Omega),\\
\varphi_k &\to \varphi_0 && \text{in } L^2(\Omega),
\end{aligned}$$

for some limit elements $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{U} \times \Phi_{ad}$.

Next we show that $\boldsymbol{u}_0 \in \boldsymbol{S}^N_\varepsilon(\varphi_0)$. To see this, we proceed as in the proof of Theorem 5.1 while making use of continuity properties of b (see Lemma 11.2).

We follow the arguments of the proof of Theorem 5.1 to get the lower semicontinuity of J^N_ε and therefrom

$$J^N_\varepsilon(\varphi_0, \boldsymbol{u}_0) \leq \liminf_{k\to\infty} J^N_\varepsilon(\varphi_k, \boldsymbol{u}_k)$$

which proves that $(\varphi_0, \boldsymbol{u}_0)$ is a minimizer of (12.1) – (12.2).

□

13 Sharp interface model

Based on the considerations of the previous section we extend the functional J_ε^N to

$$J_\varepsilon^N : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$$

by (2.5), which means

$$J_\varepsilon^N(\varphi, \boldsymbol{u}) = \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x \tag{13.1}$$

if $\varphi \in \Phi_{ad}$ and $\boldsymbol{u} \in \boldsymbol{S}_\varepsilon^N(\varphi)$, and $J_\varepsilon^N(\varphi, \boldsymbol{u}) := +\infty$ otherwise. This is the objective functional J_ε^N defined in (12.1) extended to the whole space $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$.

Remark 13.1. *We denote this functional by J_ε^N, where the N indicates the nonlinearity included here, to distinguish this functional from J_ε of Section 6, where the Stokes equations where considered as state equations.*

The aim of this section is to derive the sharp interface limit for $\varepsilon \searrow 0$ of the diffuse problem of minimizing J_ε^N in $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$. To this end, we start by introducing and investigating the sharp interface model that will correspond to the phase field model of the previous section as ε tends to zero. We will see that this problem describes a general sharp interface shape and topology optimization problem in a stationary Navier-Stokes flow and is a nonlinear version of the problem description in a Stokes flow, compare Section 6.
We first consider the state equations that correspond to system (12.3) in the limit $\varepsilon \searrow 0$. The resulting system in the strong formulation for some $\varphi \in \overline{\Phi}_{ad}^0$ will be:

$$\begin{aligned}
-\mu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } E^\varphi, && (13.2a)\\
\operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega, && (13.2b)\\
\boldsymbol{u} &= \boldsymbol{0} && \text{in } \Omega\setminus E^\varphi, && (13.2c)\\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \partial\Omega, && (13.2d)
\end{aligned}$$

where we recall that E^φ represents the region filled with fluid and is given by

$$E^\varphi := \{x \in \Omega \mid \varphi(x) = 1\}.$$

Apparently, system (13.2) may not be well-posed for arbitrary sets E^φ, since for instance (13.2c) and (13.2d) may be inconsistent with one another. As a consequence, we can only expect to find a solution of this system if at least the solution space $\{\boldsymbol{v} \in \boldsymbol{U} \mid \boldsymbol{v}|_{\Omega\setminus E^\varphi} = \boldsymbol{0}\}$ is not empty.
Due to the nonlinearity in the equation we have to deal additionally with some technical difficulties. So we can only show the existence of a solution to (13.2) for $\varphi \in \overline{\Phi}_{ad}^0$ fulfilling an additional assumption.

Lemma 13.1. *Let $\varphi \in L^1(\Omega)$ be such that there exists some $\boldsymbol{w} \in \boldsymbol{U}^\varphi$ and some $c > 0$, $c < \mu$, with*

$$\left|\int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{v}\,\mathrm{d}x\right| \le c\,\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}^\varphi. \tag{13.3}$$

Then there exists some $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ *such that* (13.2) *is fulfilled in the following weak sense:*

$$\mu \int_{\Omega} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{\varphi}. \tag{13.4}$$

This defines a set-valued solution operator denoted by

$$\boldsymbol{S}_0^N(\varphi) := \{\boldsymbol{u} \in \boldsymbol{U}^{\varphi} \mid (13.4) \text{ is fulfilled for } \boldsymbol{u}\} \quad \forall \varphi \in \overline{\Phi}_{ad}^0$$

which may be empty if there is no $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ *such that* (13.3) *is fulfilled.*

Remark 13.2. *We point out that* (13.3) *is sufficient but not necessary for the existence of a solution to* (13.4)*, so* $\boldsymbol{S}_0^N(\varphi)$ *may be non-empty for* $\varphi \in \overline{\Phi}_{ad}^0$ *even if* (13.3) *is not fulfilled.*

Proof. We fix some arbitrary $\varphi \in L^1(\Omega)$ with $\boldsymbol{U}^{\varphi} \neq \varnothing$ and choose $\boldsymbol{w} \in \boldsymbol{U}^{\varphi}$ due to (13.3) which gives in particular a constant $0 < c < \mu$ with

$$b(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v}) \leq c \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{\varphi}. \tag{13.5}$$

This estimate will play an essential role in this existence theorem. We now proceed analogously to the proof of Lemma 12.1 and only point out the main steps here. We reformulate the non-homogeneous problem (6.3) into a homogeneous one by noticing that $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ fulfills (13.4) if and only if $\widehat{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{w} \in \boldsymbol{V}^{\varphi}$ solves

$$\int_{\Omega} \mu \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x + b(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{u}}, \boldsymbol{v}) + b(\widehat{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{w}, \widehat{\boldsymbol{u}}, \boldsymbol{v}) = \langle \widehat{\boldsymbol{f}}, \boldsymbol{v} \rangle_{\boldsymbol{H}^{-1}(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{\varphi} \tag{13.6}$$

where we defined

$$\widehat{\boldsymbol{f}} := \boldsymbol{f} + \mu \Delta \boldsymbol{w} - \boldsymbol{w} \cdot \nabla \boldsymbol{w} \in (\boldsymbol{V}^{\varphi})'.$$

We remark that $\boldsymbol{V}^{\varphi}$ is a closed subspace of $\boldsymbol{V}$, and thus in particular a Hilbert space. Defining the operators

$$A^{\varphi} : \boldsymbol{V}^{\varphi} \to (\boldsymbol{V}^{\varphi})'$$

$$A^{\varphi}(\boldsymbol{v}_1)(\boldsymbol{v}_2) := \int_{\Omega} \mu \nabla \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_2 \,\mathrm{d}x + b(\boldsymbol{v}_1, \widehat{\boldsymbol{u}}, \boldsymbol{v}_2) + b(\widehat{\boldsymbol{u}}, \boldsymbol{v}_1, \boldsymbol{v}_2)$$

and

$$B^{\varphi} : \boldsymbol{V}^{\varphi} \to (\boldsymbol{V}^{\varphi})'$$

$$B^{\varphi}(\boldsymbol{v}_1)(\boldsymbol{v}_2) := b(\boldsymbol{v}_1, \boldsymbol{v}_1, \boldsymbol{v}_2)$$

we obtain from (13.5) that A is a monotone and linear operator and therefore pseudo-monotone. Besides, we find that B is strongly continuous, and thus $A + B$ defines a pseudo-monotone operator. Finally, coerciveness of $A + B$ follows as in the proof of Lemma 12.1. Thus, we can apply the main theorem on pseudo-monotone operators, see Theorem A.4, to get the existence of $\widehat{\boldsymbol{u}} \in \boldsymbol{V}^{\varphi}$ such that it holds (13.6), which then implies that $\boldsymbol{u} := \widehat{\boldsymbol{u}} + \boldsymbol{w} \in \boldsymbol{U}^{\varphi}$ is a solution of (13.4). □

Similar to the phase field setting we don't have a unique solution of the state equation (13.4). But under an additional constraint, which will be fulfilled for minimizers of our overall optimization problem introduced later, see Lemma 13.5, we can deduce uniqueness, as the following lemma shows:

Lemma 13.2. *Assume that there exists a solution $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ of* (13.4) *such that it holds*

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}. \tag{13.7}$$

Then this is the only solution of (13.4)*.*

Proof. Follows as in Lemma 12.2. □

Corresponding to Lemma 6.3 we get the following lemma, which will be one key ingredient for our convergence result (see Section 14):

Lemma 13.3. *Let $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)$, $|\varphi_\varepsilon| \leq 1$ a.e., with $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon)$ for all $\varepsilon > 0$ be given such that*

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \quad \|\varphi_\varepsilon - \varphi_0\|_{L^1(E^{\varphi_0} \cap \{\varphi_\varepsilon < 0\})} = \mathcal{O}(\varepsilon), \tag{13.8}$$

$$\sup_{\varepsilon > 0} \|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} < \infty, \tag{13.9}$$

where $\varphi_0 \in BV(\Omega, \{\pm 1\})$ and $\boldsymbol{U}^{\varphi_0} \neq \emptyset$.

Then there exists a subsequence of $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ (denoted by the same) and some $\boldsymbol{u}_0 \in \boldsymbol{S}_0^N(\varphi_0)$ such that

$$\lim_{\varepsilon \searrow 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}x = \int_\Omega \alpha_0(\varphi_0) |\boldsymbol{u}_0|^2 \, \mathrm{d}x = 0. \tag{13.10}$$

Proof. We skip some details which can be found in the proof of Lemma 6.3 and mainly point out the differences that occur when dealing with the nonlinearity in the state equation.

We start by choosing a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges pointwise almost everywhere to φ_0 in Ω. Then we get analog to (6.8) – (6.9) that it holds

$$\lim_{\varepsilon \searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) = \alpha_0(\varphi_0(x)) \qquad \text{for a.e. } x \in \Omega. \tag{13.11}$$

Moreover, we see as in the second step of the proof of Lemma 6.3 that we can deduce

$$\lim_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}|^2 \, \mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v}|_{\Omega \setminus E^{\varphi_0}} = \boldsymbol{0}$$

from the convergence rate given by (13.8) and the convergence rate on α_ε given by Assumption **(A4)**.

Next we notice that $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$ are for all $\varepsilon > 0$ the unique solution of

$$\min_{\boldsymbol{v} \in \boldsymbol{U}} FP_\varepsilon(\boldsymbol{v}) := \int_\Omega \left(\frac{1}{2} \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}|^2 + \frac{\mu}{2} |\nabla \boldsymbol{v}|^2 + \boldsymbol{u}_\varepsilon \cdot \nabla \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} - \boldsymbol{f} \cdot \boldsymbol{v} \right) \mathrm{d}x$$

since the state equations (12.2) are the necessary and sufficient first order optimality conditions for these optimization problems.

From the boundedness of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ we can find a subsequence that converges weakly in $\boldsymbol{H}^1(\Omega)$ and pointwise almost everywhere to some limit element $\boldsymbol{u}_0 \in \boldsymbol{U}$ as $\varepsilon \searrow 0$.

We then define

$$FP_0(\boldsymbol{v}) := \int_\Omega \left(\frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{v}|^2 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{v} + \frac{\mu}{2}|\nabla \boldsymbol{v}|^2 - \boldsymbol{f} \cdot \boldsymbol{v} \right) \mathrm{d}x$$

and claim, that $(FP_\varepsilon)_{\varepsilon>0}$ Γ-converges to FP_0 as $\varepsilon \searrow 0$ in $\boldsymbol{U}$ equipped with the weak $\boldsymbol{H}^1(\Omega)$ topology. We notice particularly that $FP_0 \not\equiv \infty$ as $\boldsymbol{U}^{\varphi_0} \neq \emptyset$.

Using the continuity properties of the trilinear form b (compare Lemma 11.2) we get with similar arguments as in the fourth step of the proof of Lemma 6.3 that for any $\boldsymbol{v} \in \boldsymbol{U}$ it holds

$$\limsup_{\varepsilon \searrow 0} FP_\varepsilon(\boldsymbol{v}) \le FP_0(\boldsymbol{v})$$

and thus the constant sequence defines a recovery sequence.

For showing the lower semicontinuity condition, let $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0} \subseteq \boldsymbol{U}$ be an arbitrary sequence that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{v} \in \boldsymbol{U}$.
The compact imbedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ tells us that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges strongly in $\boldsymbol{L}^3(\Omega)$ to $\boldsymbol{u}_0$ and $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$ converges strongly in $\boldsymbol{L}^3(\Omega)$ to $\boldsymbol{v}$. Additionally, both sequences converge weakly in $\boldsymbol{L}^6(\Omega)$, as $\boldsymbol{H}^1(\Omega)$ is also continuously imbedded in $\boldsymbol{L}^6(\Omega)$ due to the restriction on the space dimension $d \in \{2,3\}$. And so we find

$$\begin{aligned}
&|b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v}_\varepsilon) - b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{v})| = \\
&= |b(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0, \boldsymbol{u}_\varepsilon, \boldsymbol{v}_\varepsilon) + b(\boldsymbol{u}_0, \boldsymbol{u}_\varepsilon, \boldsymbol{v}_\varepsilon - \boldsymbol{v}) + b(\boldsymbol{u}_0, \boldsymbol{u}_\varepsilon - \boldsymbol{u}_0, \boldsymbol{v})| \le \\
&\le \underbrace{\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{L}^3(\Omega)}}_{\xrightarrow{\varepsilon \searrow 0} 0} \underbrace{\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}_\varepsilon\|_{\boldsymbol{L}^6(\Omega)}}_{\le C} + \underbrace{\|\boldsymbol{u}_0\|_{\boldsymbol{L}^6(\Omega)} \|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}}_{\le C} \underbrace{\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}}_{\xrightarrow{\varepsilon \searrow 0} 0} + \\
&+ |b(\boldsymbol{u}_0, \boldsymbol{u}_\varepsilon - \boldsymbol{u}_0, \boldsymbol{v})|.
\end{aligned} \tag{13.12}$$

Observing that $b(\boldsymbol{u}_0, \cdot, \boldsymbol{v}) \in \boldsymbol{H}^{-1}(\Omega)$ we obtain moreover from the weak convergence of $(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0)_{\varepsilon>0}$ in $\boldsymbol{H}^1_0(\Omega)$ to zero that it holds $\lim_{\varepsilon \searrow 0} b(\boldsymbol{u}_0, \boldsymbol{u}_\varepsilon - \boldsymbol{u}_0, \boldsymbol{v}) = 0$ and can thus deduce

$$\lim_{\varepsilon \searrow 0} |b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v}_\varepsilon) - b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{v})| = 0.$$

The remaining terms can be considered as in the proof of Lemma 6.3 and we obtain

$$FP_0(\boldsymbol{v}) \le \liminf_{\varepsilon \searrow 0} FP_\varepsilon(\boldsymbol{v}_\varepsilon)$$

which proves that $(FP_\varepsilon)_{\varepsilon>0}$ Γ-converges to FP_0 as $\varepsilon \searrow 0$ in $\boldsymbol{U}$ equipped with the weak $\boldsymbol{H}^1(\Omega)$-topology.

Applying standard results on Γ-convergence, see Theorem 3.3, we can conclude that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is converging weakly in $\boldsymbol{H}^1(\Omega)$ to the unique minimizer of FP_0, which implies

that $\boldsymbol{u}_0$ minimizes FP_0. But, considering the necessary and sufficient first order optimality conditions for this convex optimization problem, this implies that $\boldsymbol{u}_0$ fulfills the state equation (13.4) and this implies $\boldsymbol{u}_0 \in \boldsymbol{S}_0^N(\varphi_0)$.
Besides, the Γ-convergence result gives then $\lim_{\varepsilon\searrow 0} FP_\varepsilon(\boldsymbol{u}_\varepsilon) = FP_0(\boldsymbol{u}_0)$. Applying estimates as in (13.12) we find

$$\lim_{\varepsilon\searrow 0} b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) = b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{u}_0)$$

and so, analog to Lemma 6.3, we get with Lemma A.1 the convergences (13.10). This proves the lemma.

□

We state another variant of this lemma, where the uniform bound on the velocities is not part of the assumption, but instead we have more information about the limit element of the phase field variables:

Lemma 13.4. *Let $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)$, $|\varphi_\varepsilon| \le 1$ a.e., with $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon)$ for all $\varepsilon > 0$ be given such that*

$$\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \quad \|\varphi_\varepsilon - \varphi_0\|_{L^1(E^{\varphi_0}\cap\{\varphi_\varepsilon<0\})} = \mathcal{O}(\varepsilon). \tag{13.13}$$

Moreover, let the limit element $\varphi_0 \in BV(\Omega, \{\pm 1\})$ be such that there exists some $\boldsymbol{u} \in \boldsymbol{U}^{\varphi_0}$ and a constant $0 < \overline{c} < \mu$ fulfilling

$$\left|\int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x\right| \le \overline{c} \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{13.14}$$

Then there exists a subsequence of $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ (denoted by the same) and some $\boldsymbol{u}_0 \in \boldsymbol{S}_0^N(\varphi_0)$, such that

$$\lim_{\varepsilon\searrow 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}x = \int_\Omega \alpha_0(\varphi_0) |\boldsymbol{u}_0|^2 \, \mathrm{d}x = 0.$$

Proof. We want to apply Lemma 13.3 and thus have to show that there exists a uniform bound on $\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}$. To do this, let $\boldsymbol{u} \in \boldsymbol{U}^{\varphi_0}$ be chosen such that (13.14) is fulfilled. We obtain from the state equations (12.2), written for $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon)$, that for $\boldsymbol{w}_\varepsilon := \boldsymbol{u}_\varepsilon - \boldsymbol{u} \in \boldsymbol{V}$ it holds

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{w}_\varepsilon \cdot \boldsymbol{v} + \mu \nabla \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{v} \, \mathrm{d}x + b(\boldsymbol{w}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{w}_\varepsilon, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{w}_\varepsilon, \boldsymbol{v}) = \\ = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} - \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u} \cdot \boldsymbol{v} - \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, \mathrm{d}x - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}.\end{aligned}$$

We can insert $\boldsymbol{w}_\varepsilon \in \boldsymbol{V}$ as a test function into this equation and obtain with similar calculations as in [Gal11, Theorem IX.4.1]

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{w}_\varepsilon|^2 \, \mathrm{d}x + \mu \|\nabla \boldsymbol{w}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{w}_\varepsilon, \boldsymbol{u}, \boldsymbol{w}_\varepsilon) = \\ = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}_\varepsilon - \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u} \cdot \boldsymbol{w}_\varepsilon - \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{w}_\varepsilon \, \mathrm{d}x + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_\varepsilon).\end{aligned}$$

Applying the inequalities of Young, Hölder and Poincaré this gives with (13.14)

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2 \,\mathrm{d}x + \mu\|\nabla \boldsymbol{w}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2 \,\mathrm{d}x +$$
$$+\left(\limsup_{\varepsilon\searrow 0}\frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}|^2\,\mathrm{d}x + c\right) + \mu\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} +$$
$$+ C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}\|\nabla\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + \overline{c}\|\nabla\boldsymbol{w}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)}$$

which holds for $C, c \geq 0$ independent of ε and $\varepsilon > 0$ small enough.
Thus we get, after applying Young's inequality, a constant $C > 0$ independent of $\varepsilon > 0$, such that

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \|\nabla\boldsymbol{u}_\varepsilon\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\left(\limsup_{\varepsilon\searrow 0}\frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}|^2\,\mathrm{d}x + 1\right) \tag{13.15}$$

for all $\varepsilon > 0$ small enough.

Using the considerations of the second step in the proof of Lemma 6.3 we find that

$$\limsup_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}|^2\,\mathrm{d}x = 0$$

and so we can deduce from (13.15) and Poincaré's inequality that there exists a constant $C > 0$ independent of ε such that

$$\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} < C.$$

We can now complete the proof by applying Lemma 13.3.

□

After this discussion of the state equations for the sharp interface model and some technical lemmas, let us now formulate the sharp interface problem in its complete form:

Sharp interface problem We will see, that if a subsequence of a sequence of minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ converges with a certain rate, then it converges to a minimizer of the following sharp interface shape and topology optimization problem as ε tends to zero:

$$\min_{(\varphi,\boldsymbol{u})} J_0^N(\varphi,\boldsymbol{u}) := \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \tag{13.16}$$

with

$$(\varphi,\boldsymbol{u}) \in \Phi^0_{ad}\times\boldsymbol{U}^\varphi$$

s.t.

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{V}^\varphi. \tag{13.17}$$

Again $c_0 = \frac{\pi}{2} > 0$ is a constant arising from the Γ-convergence of the Ginzburg-Landau energy to the perimeter functional and $\gamma > 0$ is some arbitrary weighting parameter of the perimeter term. For details concerning this model we refer to the discussion at the end of Section 6.1. We just remark that this formulation allows in particular topological changes and is thus more general than the models for shape and topology optimization in fluid dynamics that can be found in literature so far.

In view of the results of this subsection we can define the functional corresponding to $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ as $\varepsilon \searrow 0$ by extending J_0^N to

$$J_0^N : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$$

by

$$J_0^N(\varphi, \boldsymbol{u}) := \begin{cases} \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) & \text{if } \varphi \in \Phi_{ad}^0 \text{ and } \boldsymbol{u} \in \boldsymbol{S}_0^N(\varphi), \\ +\infty & \text{otherwise.} \end{cases} \tag{13.18}$$

Before we restrict our considerations to the convergence of minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ to a minimizer of J_0^N we want to state an important property of minimizers of J_0^N:

Lemma 13.5. *Every minimizer $(\varphi, \boldsymbol{u})$ of J_0^N, so in particular $\boldsymbol{u} \in \boldsymbol{S}_0^N(\varphi)$, fulfills*

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}. \tag{13.19}$$

In particular, this implies by Lemma 13.2 that $\boldsymbol{S}_0^N(\varphi) = \{\boldsymbol{u}\}$.

Remark 13.3. *The existence of minimizers for J_0^N under certain assumptions will follow from Theorem 14.1. But in general the existence of minimizers is still an open problem, compare discussion in Section 6.4.*

Proof. Assume to have an arbitrary minimizer $(\varphi, \boldsymbol{u})$ of J_0^N. Let $(\varphi_c, \boldsymbol{u}_c)$ be such that

$$J_0^N(\varphi_c, \boldsymbol{u}_c) \le C_u$$

which are given by Assumption **(A8)**. Then it holds, since $(\varphi, \boldsymbol{u})$ minimize J_0^N in particular

$$J_0^N(\varphi, \boldsymbol{u}) \le J_0^N(\varphi_c, \boldsymbol{u}_c) \le C_u$$

and so by (11.11) we deduce

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}$$

which proves (13.19). □

Remark 13.4. *Using the results of Lemma 13.5, we see in particular that for a minimizer $(\varphi, \boldsymbol{u}) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ of J_0^N, the state equations* (13.2) *corresponding to φ have due to Lemma 13.2 always a* unique *solution, thus $\boldsymbol{S}_0^N(\varphi) = \{\boldsymbol{u}\}$.*
This will play an essential role when showing that minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ converge to a minimizer of J_0^N, see Theorem 14.1.

Thus we have finished this section by introducing a sharp interface model for shape and topology optimization in a stationary Navier-Stokes flow, namely (13.16)-(13.17), and have discussed an important property for minimizers of the sharp interface problem. The goal of the next section will be to connect the phase field model, which was introduced in Section 12, to this sharp interface model. Hereby, we obtain the existence of a minimizer for (13.16)-(13.17) if the minimizers of the phase field model fulfill a certain convergence rate. The general well-posedness of this problem is not guaranteed, compare for instance [BG04].

14 Convergence of minimizers

In this section, we will finally show that any subsequence of a sequence of minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ converges as $\varepsilon \searrow 0$. If this sequence fulfills additionally a certain convergence rate, we find that the limit element is a minimizer of J_0^N. We point out, that the convergence of both the design variable $(\varphi_\varepsilon)_{\varepsilon>0}$ and the velocities as solutions of the state equations are then in the strong $L^1(\Omega)$- and $\boldsymbol{H}^1(\Omega)$ topology, respectively.
In particular, we can deduce from this result directly the existence of minimizers of J_0^N if the stated assumptions are fulfilled.

Theorem 14.1. *Let $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ be minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$. Then there exists a subsequence, denoted by the same, and an element $(\varphi_0, \boldsymbol{u}_0) \in L^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ such that*

$$\varphi_\varepsilon \overset{\varepsilon\searrow 0}{\to} \varphi_0 \qquad \text{in } L^1(\Omega),$$
$$\boldsymbol{u}_\varepsilon \overset{\varepsilon\searrow 0}{\rightharpoonup} \boldsymbol{u}_0 \qquad \text{in } \boldsymbol{H}^1(\Omega).$$

If it holds

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{14.1}$$

then we obtain additionally that

$$\boldsymbol{u}_\varepsilon \xrightarrow{\varepsilon\searrow 0} \boldsymbol{u}_0 \qquad \text{in } \boldsymbol{H}^1(\Omega).$$

Moreover, $(\varphi_0, \boldsymbol{u}_0)$ is then a minimizer of J_0^N and

$$\lim_{\varepsilon\searrow 0} J_\varepsilon^N(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) = J_0^N(\varphi_0, \boldsymbol{u}_0). \tag{14.2}$$

Remark 14.1. *The existence of minimizers $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ of J_ε^N for every $\varepsilon > 0$ follows by Theorem 12.1. Thus, using the statement of Theorem 14.1, it follows in particular the existence of a minimizer for J_0^N if* (14.1) *is fulfilled for a sequence of minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$.*

Proof. We split the proof into several parts and use ideas of Theorem 6.1:

- *1st step:* Assume that $(\varphi, \boldsymbol{u}) \in L^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ is an arbitrary pair such that

  $$J_0^N(\varphi, \boldsymbol{u}) \le C_u$$

  and thus, due to (11.11), in particular

  $$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}. \tag{14.3}$$

  By Lemma 13.2 we hence get $\boldsymbol{S}_0^N(\varphi) = \{\boldsymbol{u}\}$.

  Let $(E_k)_{k\in\mathbb{N}}$ be the subsets of Ω approximating E^φ with $\partial E_k \cap \Omega \in C^2$, which are given by Lemma 3.1. Then we define $\varphi_k := 2\chi_{E_k} - 1$. Due to the approximation properties described in Lemma 3.1 it follows that $\lim_{k\to\infty}\|\varphi_k - \varphi\|_{L^1(\Omega)} = 0$, $\lim_{k\to\infty} P_\Omega(E_k) = P_\Omega(E^\varphi)$ and

  $$\|\varphi_k - \varphi\|_{L^1(\Omega)} = \mathcal{O}\left(k^{-1}\right). \tag{14.4}$$

- *2nd step:* Let $(\varphi_k)_{k\in\mathbb{N}}$ be the sequence approximating $\varphi \in L^1(\Omega)$ given by the first step and denote $E_k := \{\varphi_k = 1\}$. Then we choose for every $k \gg 1$ a sequence $\left(\varphi_\varepsilon^k\right)_{\varepsilon>0} \subseteq L^1(\Omega)$ converging to φ_k in $L^1(\Omega)$ as $\varepsilon \searrow 0$ such that

$$\limsup_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2} \left|\nabla\varphi_\varepsilon^k\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon^k\right)\right) \mathrm{d}x \le \gamma c_0 P_\Omega\left(E_k\right) \tag{14.5}$$

and

$$\left\|\varphi_\varepsilon^k - \varphi_k\right\|_{L^1(\Omega)} = \mathcal{O}\left(\varepsilon\right) \tag{14.6}$$

by following the arguments of the second step in the proof of Theorem 6.1.

- *3rd step:* Let $(\varphi, \boldsymbol{u}) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ be the pair chosen in the first step. We will show now, that there exists a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \in L^1(\Omega)$ converging to φ in $L^1(\Omega)$ and $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}_\varepsilon^N\left(\varphi_\varepsilon\right)$ converging to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ such that

$$\limsup_{\varepsilon\searrow 0} J_\varepsilon^N\left(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon\right) \le J_0^N\left(\varphi, \boldsymbol{u}\right).$$

To this end, let $(\varphi_k)_{k\in\mathbb{N}}$ be the sequence approximating φ given by the first step and for every $k \gg 1$ let $\left(\varphi_\varepsilon^k\right)_{\varepsilon>0}$ be the sequences approximating φ_k as $\varepsilon \searrow 0$ given by the second step.

Choosing a diagonal sequence $\left(\varphi_{\varepsilon_k}^k\right)_{k\in\mathbb{N}}$ that converges to φ in $L^1(\Omega)$, we see that this sequence fulfills per construction

$$\limsup_{k\to\infty} \int_\Omega \left(\frac{\gamma\varepsilon_k}{2} \left|\nabla\varphi_{\varepsilon_k}^k\right|^2 + \frac{\gamma}{\varepsilon_k}\psi\left(\varphi_{\varepsilon_k}^k\right)\right) \mathrm{d}x \le \gamma c_0 P_\Omega\left(E^\varphi\right).$$

Furthermore, (14.4) and (14.6) yield the following convergence rate for $k \to \infty$:

$$\left\|\varphi_{\varepsilon_k}^k - \varphi\right\|_{L^1(\Omega)} = \mathcal{O}\left(k^{-1}\right).$$

Then we choose some $\boldsymbol{u}_k \in \boldsymbol{S}_{\varepsilon_k}^N\left(\varphi_{\varepsilon_k}^k\right)$. By using (11.1), we observe that (14.3) implies (13.14) and so we can apply Lemma 13.4 to find that, after possible choosing a subsequence, $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{u}_0 \in \boldsymbol{S}_0^N(\varphi) = \{\boldsymbol{u}\}$, thus $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, and it holds

$$\lim_{k\to\infty} \int_\Omega \alpha_{\varepsilon_k}\left(\varphi_{\varepsilon_k}^k\right) \left|\boldsymbol{u}_k\right|^2 \mathrm{d}x = \int_\Omega \alpha_0\left(\varphi\right) \left|\boldsymbol{u}\right|^2 \mathrm{d}x = 0.$$

Using the continuity of the objective functional we end up with

$$\limsup_{k\to\infty} J_{\varepsilon_k}^N(\varphi_{\varepsilon_k}^k, \boldsymbol{u}_k) \le J_0^N(\varphi, \boldsymbol{u}).$$

- *4th step:* Next we will show that for any sequence $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ such that $(\varphi_\varepsilon)_{\varepsilon>0}$ converges strongly in $L^1(\Omega)$ to some $\varphi \in L^1(\Omega)$ fulfilling

$$\|\varphi_\varepsilon - \varphi\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{14.7}$$

and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$ it holds

$$J_0^N(\varphi,\boldsymbol{u}) \le \liminf_{\varepsilon\searrow 0} J_\varepsilon^N(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon).$$

Without loss of generality we assume $\liminf_{\varepsilon\searrow 0} J_\varepsilon^N(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon) < \infty$ and $\varphi\in BV(\Omega,\{\pm1\})$ with $\fint_\Omega \varphi\,\mathrm{d}x \le \beta$.

We can assume that (after choosing a subsequence) $(\varphi_\varepsilon)_{\varepsilon>0}$ and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converge pointwise almost everywhere in Ω, and thus using Fatou's Lemma, we see

$$\int_\Omega \alpha_0(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x \le \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x < \infty$$

and so in particular $\boldsymbol{u}=\boldsymbol{0}$ a.e. in $\Omega\setminus E^\varphi$. For more details on this estimate we refer to the fourth step in the proof of Theorem 6.1.
Thanks to $\boldsymbol{u}_\varepsilon\in\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon)$ we see $\boldsymbol{u}_\varepsilon\in\boldsymbol{U}$ for all $\varepsilon>0$ and deduce $\boldsymbol{u}\in\boldsymbol{U}$. Altogether this implies $\boldsymbol{u}\in\boldsymbol{U}^\varphi$ and thus $\boldsymbol{U}^\varphi\neq\emptyset$.

Using [Mod87, Proposition 1] we get after rescaling in ε that

$$\gamma c_0 P_\Omega(E^\varphi) \le \liminf_{\varepsilon\searrow 0}\int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\mathrm{d}x.$$

We choose then a subsequence $\left(J_{\varepsilon_k}^N(\varphi_{\varepsilon_k},\boldsymbol{u}_{\varepsilon_k})\right)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} J_{\varepsilon_k}^N(\varphi_{\varepsilon_k},\boldsymbol{u}_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} J_\varepsilon^N(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon).$$

With the help of the convergence rate (14.7) and using

$$\sup_{k\in\mathbb{N}}\|\boldsymbol{u}_{\varepsilon_k}\|_{\boldsymbol{H}^1(\Omega)} < \infty$$

which follows from the weak convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$, we thus can apply Lemma 13.3 and get a subsequence $\left(J_{\varepsilon_{k(l)}}^N\left(\varphi_{\varepsilon_{k(l)}},\boldsymbol{u}_{\varepsilon_{k(l)}}\right)\right)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty}\left\|\boldsymbol{u}_{\varepsilon_{k(l)}}-\boldsymbol{u}\right\|_{\boldsymbol{H}^1(\Omega)} = 0,\quad \lim_{l\to\infty}\int_\Omega \alpha_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right)\left|\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2\mathrm{d}x = \int_\Omega\alpha_0(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x = 0.$$

Plugging these results together we end up with

$$\begin{aligned} J_0^N(\varphi,\boldsymbol{u}) &= \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \le \liminf_{l\to\infty} J_{\varepsilon_{k(l)}}^N\left(\varphi_{\varepsilon_{k(l)}},\boldsymbol{u}_{\varepsilon_{k(l)}}\right) = \\ &= \lim_{k\to\infty} J_{\varepsilon_k}^N(\varphi_{\varepsilon_k},\boldsymbol{u}_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} J_\varepsilon^N(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon) \end{aligned}$$

and finish the fourth step.

- *5th step:* Now let $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ be minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$.

  By Assumption **(A8)** we know that there exists some $(\widetilde{\varphi}, \widetilde{\boldsymbol{u}}) \in \Phi_{ad}^0 \times \boldsymbol{U}$ with $\widetilde{\boldsymbol{u}} \in \boldsymbol{S}_0^N(\widetilde{\varphi})$ and

  $$J_0^N(\widetilde{\varphi}, \widetilde{\boldsymbol{u}}) \le C_u. \tag{14.8}$$

  This gives in view of (11.11) in particular

  $$\|\nabla \widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}$$

  and thus by Lemma 11.1

  $$b(\boldsymbol{v}, \widetilde{\boldsymbol{u}}, \boldsymbol{v}) \le \frac{\mu}{2} \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

  From (14.8) we find that we can apply the third part of this proof and obtain a sequence $(\widetilde{\varphi}_\varepsilon, \widetilde{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging in $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ to $(\widetilde{\varphi}, \widetilde{\boldsymbol{u}})$ such that

  $$\limsup_{\varepsilon \searrow 0} J_\varepsilon^N(\widetilde{\varphi}_\varepsilon, \widetilde{\boldsymbol{u}}_\varepsilon) \le J_0^N(\widetilde{\varphi}, \widetilde{\boldsymbol{u}}) \le C_u \tag{14.9}$$

  and in particular

  $$\sup_{\varepsilon>0} J_\varepsilon^N(\widetilde{\varphi}_\varepsilon, \widetilde{\boldsymbol{u}}_\varepsilon) < \infty.$$

  From the fact that $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ minimize J_ε^N for every $\varepsilon > 0$ we know that

  $$J_\varepsilon^N(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \le J_\varepsilon^N(\widetilde{\varphi}_\varepsilon, \widetilde{\boldsymbol{u}}_\varepsilon) < C \tag{14.10}$$

  where $C > 0$ is a constant independent of $\varepsilon > 0$. Therefrom

  $$\sup_{\varepsilon>0} \int_\Omega \left(\frac{\gamma\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon) \right) \mathrm{d}x < \infty \tag{14.11}$$

  and by Assumption **(A5)** also

  $$\sup_{\varepsilon>0} \|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} < \infty. \tag{14.12}$$

  Now using the arguments of [Mod87, Proposition 3, case a)] we get from (14.11) that $(\varphi_\varepsilon)_{\varepsilon>0}$ has a subsequence, denoted by the same, that converges in $L^1(\Omega)$ to an element $\varphi_0 \in L^1(\Omega)$.
  Besides, we find that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ has a subsequence that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{u}_0 \in \boldsymbol{U}$.
  If we assume, that the sequence of minimizers fulfills the convergence rate (14.1) we see using the fourth step of this proof, that it holds

  $$J_0^N(\varphi_0, \boldsymbol{u}_0) \le \liminf_{\varepsilon \searrow 0} J_\varepsilon^N(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon). \tag{14.13}$$

We want to show, that $(\varphi_0, \boldsymbol{u}_0)$ are a minimizer for J_0^N. For this purpose, let $(\varphi, \boldsymbol{u})$ be another arbitrary pair. To show that

$$J_0^N(\varphi_0, \boldsymbol{u}_0) \leq J_0^N(\varphi, \boldsymbol{u})$$

we can assume without loss of generality that $J_0^N(\varphi, \boldsymbol{u}) \leq C_u$, since by (14.9), (14.10) and (14.13) we have

$$J_0^N(\varphi_0, \boldsymbol{u}_0) \leq C_u. \tag{14.14}$$

Consequently, the third step of this proof guarantees the existence of a sequence $(\overline{\varphi}_\varepsilon, \overline{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging to $(\varphi, \boldsymbol{u})$ in $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ such that

$$\limsup_{\varepsilon \searrow 0} J_\varepsilon^N(\overline{\varphi}_\varepsilon, \overline{\boldsymbol{u}}_\varepsilon) \leq J_0^N(\varphi, \boldsymbol{u}).$$

Combining those result, we obtain

$$J_0^N(\varphi_0, \boldsymbol{u}_0) \leq \liminf_{\varepsilon \searrow 0} J_\varepsilon^N(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \leq \limsup_{\varepsilon \searrow 0} J_\varepsilon^N(\overline{\varphi}_\varepsilon, \overline{\boldsymbol{u}}_\varepsilon) \leq J_0^N(\varphi, \boldsymbol{u}) \tag{14.15}$$

the second inequality being a consequence of $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ minimizing J_ε^N for every $\varepsilon > 0$. As $(\varphi, \boldsymbol{u})$ has been arbitrary this implies that $(\varphi_0, \boldsymbol{u}_0)$ is a minimizer of J_0^N.

To deduce the statement of the theorem, it remains to show the strong convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ and (14.2). For this purpose, we use again (14.1) and consequently can apply Lemma 13.3 to deduce that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ and

$$\lim_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}x = 0. \tag{14.16}$$

By the third step of this proof and (14.14) we find a sequence $(\widehat{\varphi}_\varepsilon, \widehat{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ converging to $(\varphi_0, \boldsymbol{u}_0)$ strongly in $L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ such that

$$\limsup_{\varepsilon \searrow 0} J_\varepsilon^N(\widehat{\varphi}_\varepsilon, \widehat{\boldsymbol{u}}_\varepsilon) \leq J_0^N(\varphi_0, \boldsymbol{u}_0).$$

Then we see similar to (14.15) by applying (14.13) that

$$\begin{aligned} J_0^N(\varphi_0, \boldsymbol{u}_0) &\leq \liminf_{\varepsilon \searrow 0} J_\varepsilon^N(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \leq \limsup_{\varepsilon \searrow 0} J_\varepsilon^N(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \leq \limsup_{\varepsilon \searrow 0} J_\varepsilon^N(\widehat{\varphi}_\varepsilon, \widehat{\boldsymbol{u}}_\varepsilon) \leq \\ &\leq J_0^N(\varphi_0, \boldsymbol{u}_0) \end{aligned}$$

and can finally deduce (14.2).

□

Using this result, we can now show that for a minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ of J_ε^N the state equations corresponding to φ_ε have a *unique* solution if $\varepsilon > 0$ is small enough and (14.1) is fulfilled, as the following corollary shows:

Corollary 14.1. *Assume* $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ *are minimizer of* J_ε^N *such that* (14.1) *is fulfilled. Then, for* $\varepsilon > 0$ *small enough, it holds*

$$\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon) = \{\boldsymbol{u}_\varepsilon\}\,.$$

This means, that the solution of (12.3) *corresponding to* φ_ε *is* unique. *Moreover, we have*

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}. \tag{14.17}$$

Proof. It follows from Theorem 14.1, that

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} < \delta$$

for some $0 < \delta < \frac{\mu}{2K_\Omega}$, if $\varepsilon > 0$ is small enough, where $\boldsymbol{u} \in \boldsymbol{S}_0^N(\varphi)$ and $(\varphi, \boldsymbol{u})$ is some minimizer of J_0^N. Due to Lemma 13.5 we know that it holds

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}$$

and hence we have

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \delta + \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{2K_\Omega} + \frac{\mu}{2K_\Omega} = \frac{\mu}{K_\Omega}$$

and the statement follows from Lemma 12.2. □

So we have shown that also in a stationary Navier-Stokes flow the proposed phase field model (12.1)-(12.2) approximates the sharp interface model (13.16) – (13.17) in the following sense: We can show, that a subsequence of any sequence of minimizers of the phase field problem converges. If this convergence fulfills a certain rate, we find that those minimizers converge in the strong topology to a minimizer of the sharp interface problem as the phase field parameter $\varepsilon > 0$, modelling the thickness of the interface, tends to zero.

15 Optimality conditions for the phase field model

In the following, let again $\varepsilon > 0$ be fixed. We choose for this section $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ as minimizer of J_ε^N such that it holds

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}. \tag{15.1}$$

In particular, this implies by Lemma 12.2 directly $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon) = \{\boldsymbol{u}_\varepsilon\}$.

Remark 15.1. *We point out, that due to Corollary 14.1 we obtain under certain assumptions and for $\varepsilon > 0$ small enough that* (15.1) *is fulfilled for the minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$.*

Similar to Section 7, we will in this section derive optimality conditions for (12.1) – (12.2). We begin with considering the problem as a classical optimal control problem and deduce a variational inequality, see Section 15.1. Afterwards, we will use the ideas of shape calculus and deduce optimality conditions by transformation of the domain. This will yield an optimality system that can be connected to an optimality criteria obtained for the sharp interface problem later on (cf. Section 16.2). The connection between this optimality system and the variational inequality derived before is pointed out in Section 17.
Throughout the following section we state additionally Assumptions **(A6)** and **(A7)**, which ensure differentiability of the objective functional and enough regularity on the external force term.

15.1 Variational inequality

Considering (12.1) – (12.2) as an optimal control problem, we want to derive a variational inequality by parametric variations in the admissible set Φ_{ad}. A good introduction and deeper discussion of general optimal control problems and standard methods in this topic can be found in [Trö09].

To deduce the variational inequality, the standard procedure involves the definition of a reduced cost functional, and for this purpose a single-valued solution operator is essential. By the assumptions made above, we have unique solvability of the state equations corresponding to the minimizer φ_ε. To ensure uniqueness even in a neighborhood of this minimizer we start by showing a kind of Lipschitz continuity of the solution operator $\boldsymbol{S}_\varepsilon^N$:

Lemma 15.1. *There exists some $L = L(\Omega, \boldsymbol{g}, \boldsymbol{f}, \varepsilon, \mu) > 0$ such that*

$$\|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \le L\, \|\varphi - \varphi_\varepsilon\|_{L^\infty(\Omega)} \qquad \forall \varphi \in \overline{\Phi}_{ad}, \boldsymbol{u} \in \boldsymbol{S}_\varepsilon^N(\varphi).$$

Proof. We choose some $\varphi \in \overline{\Phi}_{ad}$ and $\boldsymbol{u} \in \boldsymbol{S}_\varepsilon^N(\varphi)$. Subtracting (12.2), written for $\boldsymbol{u}_\varepsilon$, from (12.2) written for $\boldsymbol{u}$ and testing with $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{u}_\varepsilon \in \boldsymbol{V}$ yields

$$\int_\Omega (\alpha_\varepsilon(\varphi)\, \boldsymbol{u} - \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon) \cdot \boldsymbol{v} \,\mathrm{d}x + \mu\, \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{v}) = 0$$

which can be rewritten as follows

$$\int_\Omega (\alpha_\varepsilon(\varphi)\, \boldsymbol{u} - \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon) \cdot \boldsymbol{v} \,\mathrm{d}x + \mu\, \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{u}_\varepsilon, \boldsymbol{v}, \boldsymbol{v}) = 0$$

and so by (11.2)

$$\int_\Omega \left(\alpha_\varepsilon\left(\varphi\right)\boldsymbol{u} - \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\right)\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\left\|\nabla\boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)} + b\left(\boldsymbol{v},\boldsymbol{u}_\varepsilon,\boldsymbol{v}\right) = 0.$$

Due to (11.1) and (15.1) we observe that

$$b\left(\boldsymbol{v},\boldsymbol{u}_\varepsilon,\boldsymbol{v}\right) \leq K_\Omega\left\|\nabla\boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)}\left\|\nabla\boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{L}^2(\Omega)} < \mu\left\|\nabla\boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)}.$$

And so we have some $c > 0$ such that

$$c\left\|\nabla\boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)} \leq -\int_\Omega\left(\alpha_\varepsilon\left(\varphi\right)\boldsymbol{u} - \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\right)\cdot\boldsymbol{v}\,\mathrm{d}x. \tag{15.2}$$

Now the remaining considerations can be carried out as in the proof of Lemma 7.1, while using (15.1), which gives the necessary bound on $\left\|\boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{H}^1(\Omega)}$. □

This result gives in particular that (15.1) implies uniqueness of solutions to (12.2) not only for φ_ε, but even in a neighborhood of φ_ε:

Corollary 15.1. *There exists a neighborhood $N(\varphi_\varepsilon) \subseteq \overline{\Phi}_{ad}$ with respect to the $L^\infty(\Omega)$–topology of φ_ε such that for any $\varphi \in N(\varphi_\varepsilon)$ and $\boldsymbol{u} \in \boldsymbol{S}^N_\varepsilon(\varphi)$ it holds*

$$\left\|\nabla\boldsymbol{u}\right\| < \frac{\mu}{K_\Omega}$$

and thus, due to Lemma 12.2, $\boldsymbol{S}^N_\varepsilon\left(\varphi\right) = \{\boldsymbol{u}\}$.

Proof. Follows directly from Lemma 15.1 and (15.1). □

According to this corollary, we obtain for any $\varphi \in \overline{\Phi}_{ad}$ and t small enough that the solution space $\boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right)$ will always consist of one element. So it makes sense to define a reduced objective functional in a neighborhood of φ_ε by

$$j^N_\varepsilon\left(\varphi\right) := J^N_\varepsilon\left(\varphi, \boldsymbol{S}^N_\varepsilon\left(\varphi\right)\right) \qquad \forall\varphi \in N(\varphi_\varepsilon) \tag{15.3}$$

and we see that it holds

$$j^N_\varepsilon\left(\varphi_\varepsilon\right) \leq j^N_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) \qquad \forall\varphi \in \Phi_{ad},\ 0 < t \ll 1 \tag{15.4}$$

which will leed to a variational inequality similar to that in Section 7.

Using that $\boldsymbol{S}^N_\varepsilon$ is in this small neighborhood of φ_ε a single-valued operator mapping to $\boldsymbol{H}^1(\Omega)$, we can differentiate this operator:

Lemma 15.2. *Let $\varphi \in \overline{\Phi}_{ad}$ be given. Then the directional derivative*

$$\partial_t|_{t=0}\boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) = \mathrm{D}\boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right) =: \boldsymbol{u} \in \boldsymbol{V}$$

exists in $\boldsymbol{H}^1(\Omega)$, is well-defined, and is given as the unique weak solution to

$$\alpha'_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon + \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}_\varepsilon + \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{in } \Omega, \tag{15.5a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{15.5b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{15.5c}$$

where $\boldsymbol{u}_\varepsilon = \boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon\right)$. Here we denote by $\partial_t|_{t=0}\boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) = \lim_{t\to 0, t>0}\frac{1}{t}\left(\boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) - \boldsymbol{S}^N_\varepsilon\left(\varphi_\varepsilon\right)\right)$ the one-sided directional derivative.

Proof. First we notice, that the formulation is well-defined, since due to Corollary 15.1 $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon))$ consists for any $\varphi \in \overline{\Phi}_{ad}$ only of one element for t small enough.

Then we apply analysis similar to that in the proof of Lemma 7.2. We only point out the main steps and differences occurring due to the nonlinear state equations.
The unique solvability of (15.5) follows by applying Lax-Milgram's theorem A.2 to the following bilinear form

$$a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$$
$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x + b(\boldsymbol{u}, \boldsymbol{u}_\varepsilon, \boldsymbol{v}) + b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}, \boldsymbol{v}).$$

This bilinear form is obviously continuous. Moreover, coercivity follows from (15.1) and (11.1), since

$$|b(\boldsymbol{u}, \boldsymbol{u}_\varepsilon, \boldsymbol{u})| \le K_\Omega \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2(\Omega)} \overset{(15.1)}{<} \mu \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}$$

and so we see with Poincaré's inequality

$$a(\boldsymbol{u}, \boldsymbol{u}) = \underbrace{\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, |\boldsymbol{u}|^2 \,\mathrm{d}x}_{\ge 0} + \mu \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{u}, \boldsymbol{u}_\varepsilon, \boldsymbol{u}) \ge c \|\boldsymbol{u}\|^2_{\boldsymbol{H}_0^1(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{V}$$

for some $c = c(\mu, \Omega) > 0$.
Applying Lax-Milgram's theorem A.2 yields thus existence and uniqueness of a solution $\boldsymbol{u} \in \boldsymbol{V}$ for the following equation:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \left(-\alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

fulfilling additionally the following a priori estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le \frac{1}{c} \left\|\alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\, \boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{L}^2(\Omega)}$$

which can, due to (15.1) and Poincaré's inequality be estimated by

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le c(\Omega, \mu, \alpha_\varepsilon, \boldsymbol{g}). \tag{15.6}$$

For proving that $\partial_t|_{t=0} \boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon))$ is well defined and given by (15.5), we use the same arguments as in the proof of Lemma 7.2, while including the nonlinearity b. For the sake of completeness we point out again the main steps.
The trilinear form $b : \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \to \mathbb{R}$ is due to Lemma 11.1 continuous, thus $\boldsymbol{H}^1(\Omega) \ni \boldsymbol{u} \mapsto b(\boldsymbol{u}, \boldsymbol{u}, \cdot) \in \boldsymbol{H}^{-1}(\Omega)$ is Fréchet differentiable and we can write for $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ arbitrary

$$b(\boldsymbol{u}, \boldsymbol{u}, \cdot) = b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \cdot) + (b(\boldsymbol{u} - \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \cdot) + b(\boldsymbol{u}_\varepsilon, \boldsymbol{u} - \boldsymbol{u}_\varepsilon, \cdot)) + r_b(\boldsymbol{u})$$
$$\frac{\|r_b(\boldsymbol{u})\|_{\boldsymbol{H}^{-1}(\Omega)}}{\|\boldsymbol{u} - \boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}} \xrightarrow{\|\boldsymbol{u}-\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} \to 0} 0. \tag{15.7}$$

In addition, one obtains as in (7.5) for $\varphi \in H^1(\Omega)$ arbitrary

$$\begin{aligned} &\alpha_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) = \alpha_\varepsilon\left(\varphi_\varepsilon\right) + t\alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right) + r_\alpha\left(t\right) \\ &\text{with } \lim_{t \searrow 0} \frac{\left\|r_\alpha\left(t\right)\right\|_{L^3(\Omega)}}{t} = 0. \end{aligned} \tag{15.8}$$

We define for the fixed $\varphi \in \overline{\Phi}_{ad}$ and $t \in (0,1)$ small enough the following functions:

$$\widetilde{\boldsymbol{u}}_t := \boldsymbol{S}_\varepsilon^N\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right), \quad \boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon^N\left(\varphi_\varepsilon\right), \quad \widehat{\boldsymbol{u}}_t := \frac{1}{t}\left(\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon\right) - \boldsymbol{u} \in \boldsymbol{V}$$

where $\boldsymbol{u}$ denotes the solution of (15.5).

After some calculations, we see that $\widehat{\boldsymbol{u}}_t$ solves

$$\begin{aligned} &\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\widehat{\boldsymbol{u}}_t \cdot \boldsymbol{v} + \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\widehat{\boldsymbol{u}}_t \cdot \boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\widehat{\boldsymbol{u}}_t \cdot \nabla\boldsymbol{v}\,\mathrm{d}x + b\left(\widehat{\boldsymbol{u}}_t, \boldsymbol{u}_\varepsilon, \boldsymbol{v}\right) + \\ &+ b\left(\boldsymbol{u}_\varepsilon, \widehat{\boldsymbol{u}}_t, \boldsymbol{v}\right) = \int_\Omega -\alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\boldsymbol{u} \cdot \boldsymbol{v} - \frac{r_\alpha\left(t\right)}{t}\widetilde{\boldsymbol{u}}_t \cdot \boldsymbol{v} - \frac{r_b\left(\widetilde{\boldsymbol{u}}_t\right)\left(\boldsymbol{v}\right)}{t} \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{15.9}$$

Defining

$$\boldsymbol{h}(t) := -\alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\boldsymbol{u} - \frac{1}{t}r_\alpha\left(t\right)\widetilde{\boldsymbol{u}}_t - \frac{1}{t}r_b\left(\widetilde{\boldsymbol{u}}_t\right) \in \boldsymbol{H}^{-1}\left(\Omega\right)$$

we see from (7.8), (7.9) and

$$\begin{aligned} \left\|\frac{1}{t}r_b\left(\widetilde{\boldsymbol{u}}_t\right)\right\|_{\boldsymbol{H}^{-1}(\Omega)} &= \frac{\left\|r_b\left(\widetilde{\boldsymbol{u}}_t\right)\right\|_{\boldsymbol{H}^{-1}(\Omega)}}{\left\|\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{H}^1(\Omega)}} \frac{\left\|\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{H}^1(\Omega)}}{|t|} \overset{\text{Lemma 15.1}}{\leq} \\ &\leq C\frac{\left\|r_b\left(\widetilde{\boldsymbol{u}}_t\right)\right\|_{\boldsymbol{H}^{-1}(\Omega)}}{\left\|\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{H}^1(\Omega)}} \frac{|t|\left\|\varphi - \varphi_\varepsilon\right\|_{L^\infty(\Omega)}}{|t|} = \\ &= C\frac{\left\|r_b\left(\widetilde{\boldsymbol{u}}_t\right)\right\|_{\boldsymbol{H}^{-1}(\Omega)}}{\left\|\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{H}^1(\Omega)}} \xrightarrow{\left\|\widetilde{\boldsymbol{u}}_t - \boldsymbol{u}_\varepsilon\right\|_{\boldsymbol{H}^1(\Omega)} \to 0} 0 \end{aligned}$$

which follows from (15.7), that it holds

$$\left\|\boldsymbol{h}(t)\right\|_{\boldsymbol{H}^{-1}(\Omega)} \xrightarrow{t \searrow 0} 0. \tag{15.10}$$

Applying again the Lipschitz continuity of $\boldsymbol{S}_\varepsilon^N$ stated in in Lemma 15.1 we obtain, as in (7.11),

$$\lim_{t \searrow 0}\left|\int_\Omega \alpha_\varepsilon'\left(\varphi - \varphi_\varepsilon\right)t\left|\widehat{\boldsymbol{u}}_t\right|^2\,\mathrm{d}x\right| = 0. \tag{15.11}$$

Testing (15.9) with $\widehat{\boldsymbol{u}}_t$ yields

$$\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\widehat{\boldsymbol{u}}_t\right|^2 + \alpha_\varepsilon'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)t\left|\widehat{\boldsymbol{u}}_t\right|^2 + \mu\left\|\nabla\widehat{\boldsymbol{u}}_t\right\|_{\boldsymbol{L}^2(\Omega)}^2 + b\left(\widehat{\boldsymbol{u}}_t, \boldsymbol{u}_\varepsilon, \widehat{\boldsymbol{u}}_t\right) = \left\langle\boldsymbol{h}(t), \widehat{\boldsymbol{u}}_t\right\rangle_{\boldsymbol{H}^{-1}(\Omega)}.$$

This gives in view of (15.1) and (11.1) the estimate

$$c \|\nabla \widehat{\boldsymbol{u}}_t\|^2_{\boldsymbol{L}^2(\Omega)} \le \langle \boldsymbol{h}(t), \widehat{\boldsymbol{u}}_t \rangle_{\boldsymbol{H}^{-1}(\Omega)} - \int_\Omega \alpha'_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\, t\, |\widehat{\boldsymbol{u}}_t|^2 \,\mathrm{d}x$$

for some $c > 0$. This implies by using Young's inequality, (15.10) and (15.11)

$$\lim_{t \searrow 0} \|\nabla \widehat{\boldsymbol{u}}_t\|_{\boldsymbol{L}^2(\Omega)} = 0$$

and so by Poincaré's inequality

$$\lim_{t \searrow 0} \|\widehat{\boldsymbol{u}}_t\|_{\boldsymbol{H}^1(\Omega)} = 0.$$

Hence, we can conclude the statement of the lemma.

□

Remark 15.2. *We extend $\alpha_\varepsilon : [-1,1] \to [0, \overline{\varepsilon}]$ to a C^2-function $\widetilde{\alpha}_\varepsilon : \mathbb{R} \to [-\delta, \overline{\alpha}_\varepsilon + \delta]$, with $0 < \delta$ suitable small, compare Remark 7.1. Then we can apply the implicit function theorem similar to [Hec11, Trö09], to obtain that there is a $L^\infty(\Omega)$-neighborhood U of φ_ε such that $\boldsymbol{S}^N_\varepsilon$ is a well-defined single-valued operator on U. Additionally we then obtain that $\boldsymbol{S}^N_\varepsilon : U \subset L^\infty(\Omega) \to \boldsymbol{U}$ is Fréchet-differentiable at φ_ε, compare also Remark 7.1. Hence, the reduced objective functional $j^N_\varepsilon : U \cap \overline{\Phi}_{ad} \subset L^\infty(\Omega) \cap H^1(\Omega) \to \mathbb{R}$ can be shown to be Fréchet differentiable at φ_ε.*

With the help of the differentiability result for $\boldsymbol{S}^N_\varepsilon$ of Lemma 15.2 we get from (15.4) that

$$\mathrm{D}j^N_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \ge 0 \quad \forall \varphi \in \Phi_{ad}. \tag{15.12}$$

We thus have derived a variational inequality as first order optimality condition for the optimal control problem (12.1) – (12.2). The remainder of this subsection will be devoted to stating this variational inequality in a more convenient form and introducing a Lagrange multiplier for the integral constraint as we have done in Section 7.1.

Using the results of [KZ79] one can establish as in Theorem 7.1 the existence of a Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}^+$ for the integral constraint such that it holds

$$\mathrm{D}j^N_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) + \lambda_\varepsilon \int_\Omega (\varphi - \varphi_\varepsilon) \,\mathrm{d}x \ge 0 \quad \forall \varphi \in \overline{\Phi}_{ad}$$

together with the complementarity condition

$$\lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon \,\mathrm{d}x - \beta |\Omega| \right) = 0.$$

By the chain rule and $\boldsymbol{u}_\varepsilon = \boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon)$, $\boldsymbol{u} = \mathrm{D}\boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)$ we have the following formula:

$$\begin{aligned}
\mathrm{D}j^N_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) = {} & \frac{1}{2} \int_\Omega \alpha'_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \boldsymbol{u} \cdot \boldsymbol{u}_\varepsilon \,\mathrm{d}x + \\
& + \int_\Omega \mathrm{D}_{(2,3)} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u}, \mathrm{D}\boldsymbol{u}) \,\mathrm{d}x + \\
& + \gamma\varepsilon \int_\Omega \nabla \varphi_\varepsilon \cdot \nabla(\varphi - \varphi_\varepsilon) \,\mathrm{d}x + \frac{\gamma}{\varepsilon} \int_\Omega \psi'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \,\mathrm{d}x.
\end{aligned} \tag{15.13}$$

In order to rewrite this expression with the help of an adjoint variable, we introduce an adjoint system in the following lemma. This could be deduced similar to Section 7 with the help of a Lagrangian.

Lemma 15.3. *There exists a unique $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}_0^1(\Omega)$ such that*

$$\begin{aligned}\alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\boldsymbol{u}_\varepsilon^T\cdot\boldsymbol{q}_\varepsilon - \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon &= \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon + && \text{(15.14a)}\\ &+ \mathrm{D}_u f(\cdot,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon) \qquad \textit{in } \Omega, && \text{(15.14b)}\\ \operatorname{div}\boldsymbol{q}_\varepsilon &= 0 \qquad \textit{in } \Omega, && \text{(15.14c)}\\ \boldsymbol{q}_\varepsilon &= \boldsymbol{0} \qquad \textit{on } \partial\Omega, && \text{(15.14d)}\end{aligned}$$

is fulfilled in the following weak formulation:

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{q}_\varepsilon\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{v},\boldsymbol{u}_\varepsilon,\boldsymbol{q}_\varepsilon) - b(\boldsymbol{u}_\varepsilon,\boldsymbol{q}_\varepsilon,\boldsymbol{v}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x + \\ + \int_\Omega \mathrm{D}_{(2,3)} f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{v},\mathrm{D}\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{V}.\end{aligned}$$

Proof. Follows as in Lemma 15.2 by applying Lax-Milgram's theorem A.2 and making use of (15.1). □

Following the calculations similar to those in the proof of Theorem 7.1 we arrive finally in the following variational inequality as first order optimality conditions:

Theorem 15.1. *The following optimality system is fulfilled for any minimizer $(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon)$ of J_ε^N:*

$$\begin{aligned}\left(\frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)\,|\boldsymbol{u}_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi_0'(\varphi_\varepsilon) - \alpha_\varepsilon'(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \lambda_\varepsilon, \varphi-\varphi_\varepsilon\right)_{L^2(\Omega)} + \\ + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla(\varphi-\varphi_\varepsilon)\right)_{\boldsymbol{L}^2(\Omega)} \geq 0 \quad \forall\varphi\in\overline{\Phi}_{ad}\end{aligned} \tag{15.15}$$

$$\left.\begin{aligned}\alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon - \mu\Delta\boldsymbol{u}_\varepsilon + \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon &= \boldsymbol{f} \qquad \textit{in } \Omega,\\ \operatorname{div}\boldsymbol{u}_\varepsilon &= 0 \qquad \textit{in } \Omega,\\ \boldsymbol{u}_\varepsilon &= \boldsymbol{g} \qquad \textit{on } \partial\Omega,\\ \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\boldsymbol{u}_\varepsilon^T\cdot\boldsymbol{q}_\varepsilon - \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon &= \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon + \\ &+ \mathrm{D}_u f(\cdot,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon) \qquad \textit{in } \Omega,\\ \operatorname{div}\boldsymbol{q}_\varepsilon &= 0 \qquad \textit{in } \Omega,\\ \boldsymbol{q}_\varepsilon &= \boldsymbol{0} \qquad \textit{on } \partial\Omega,\\ \lambda_\varepsilon\left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x - \beta|\Omega|\right) &= 0,\ \lambda_\varepsilon \geq 0,\\ \int_\Omega \varphi_\varepsilon\,\mathrm{d}x \leq \beta|\Omega|, &\quad |\varphi_\varepsilon| \leq 1 \ \textit{a.e. in } \Omega,\end{aligned}\right\} \tag{15.16}$$

where $\lambda_\varepsilon \in \mathbb{R}^+$ is a Lagrange multiplier for the integral constraint.
Here, $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$ and $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ are weak solutions of the state equations and adjoint system, respectively.

This gives us a first version of optimality conditions for the phase field model, if considered as an optimal control problem. But since it is an approximation for the shape and topology optimization problem, we next want to derive optimality conditions by geometric variations.

15.2 Geometric variations

In this section we want to derive different optimality conditions in the same way as we did in Section 7.2 by geometric variations. In the end we want to obtain an optimality system for which we can consider the limit $\varepsilon \searrow 0$ and hope to arrive in an optimality system for the sharp interface, see Section 16.2. This limit is considered in Section 17.

We recall, that we have a fixed minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ of J_ε^N, and $\varepsilon > 0$ is assumed to be fixed and small enough such that $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon) = \{\boldsymbol{u}_\varepsilon\}$ and (15.1) is fulfilled.

We will obtain optimality criteria by deforming the domain Ω along suitable transformations. For this purpose, we choose some $T \in \overline{\mathcal{T}}_{ad}$ and denote in the following by $V \in \overline{\mathcal{V}}_{ad}$ its velocity field. Let us introduce the notation

$$\varphi_\varepsilon(t) := \varphi_\varepsilon \circ T_t^{-1}, \quad \Omega_t := T_t(\Omega).$$

We choose elements solving the state equations corresponding to $\varphi_\varepsilon(t)$:

$$\boldsymbol{u}_\varepsilon(t) \in \boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon(t)).$$

The latter is possible since the choice of $T \in \overline{\mathcal{T}}_{ad}$ implies by Remark 3.4 for $\varphi_\varepsilon \in \Phi_{ad}$ that $\varphi_\varepsilon(t) \in \overline{\Phi}_{ad}$, see also Lemma 12.1.

So far, it is not clear if $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon(t)) = \{\boldsymbol{u}_\varepsilon(t)\}$, even though this holds true for $t = 0$. But the implicit function theorem will guarantee uniqueness for small t, thus $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon(t)) = \{\boldsymbol{u}_\varepsilon(t)\}$ for t small enough, and will give us at the same time differentiability of $t \mapsto (\boldsymbol{u}_\varepsilon(t) \circ T_t)$ at $t = 0$, as the following lemma shows:

Lemma 15.4. *For t small enough, we have $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon(t)) = \{\boldsymbol{u}_\varepsilon(t)\}$, thus the state equations* (12.2) *corresponding to $\varphi_\varepsilon(t)$ have a* unique *solution if t is small enough.*

Moreover, we get that the mapping $\mathbb{R} \supset I \ni t \mapsto \boldsymbol{u}_\varepsilon(t) \circ T_t \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$ (where I is a small interval around 0) and $\dot{\boldsymbol{u}}_\varepsilon[V] := \partial_t|_{t=0}(\boldsymbol{u}_\varepsilon(t) \circ T_t)$ is given as the unique weak solution to

$$\begin{aligned}
&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\dot{\boldsymbol{u}}_\varepsilon[V] \cdot \boldsymbol{z} + \mu \nabla \dot{\boldsymbol{u}}_\varepsilon[V] \cdot \nabla \boldsymbol{z}\, \mathrm{d}x + b(\boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon[V], \boldsymbol{z}) + b(\dot{\boldsymbol{u}}_\varepsilon[V], \boldsymbol{u}_\varepsilon, \boldsymbol{z}) = \\
&= \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z}\, \mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T \nabla \boldsymbol{z}\, \mathrm{d}x + \\
&+ \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla\left(\operatorname{div} V(0) \boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right) \mathrm{d}x - \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \operatorname{div} V(0)\, \mathrm{d}x + \\
&+ b(\mathrm{D}V(0)\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{z}) - b(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \mathrm{D}V(0)\boldsymbol{z}) + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}\, \mathrm{d}x + \\
&+ \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\, \mathrm{d}x - \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)\boldsymbol{z}\, \mathrm{d}x
\end{aligned} \tag{15.17}$$

which has to hold for all $\boldsymbol{z} \in \boldsymbol{V}$, together with

$$\operatorname{div} \dot{\boldsymbol{u}}_\varepsilon[V] = \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0). \tag{15.18}$$

Remark 15.3. *In the following discussion and calculations we will follow the notation specified in Remark 7.27.*

Proof. We apply the arguments of Lemma 7.4 after changing the definition of the function F to

$$F : I \times \boldsymbol{H}^1_{\boldsymbol{g}}(\Omega) \to \boldsymbol{V}' \times L^2_0(\Omega)$$
$$F(t,\boldsymbol{u}) := (F_1(t,\boldsymbol{u}), F_2(t,\boldsymbol{u})) \in \boldsymbol{V}' \times L^2_0(\Omega)$$

where we define

$$\begin{aligned} F_1(t,\boldsymbol{u})(\boldsymbol{z}) := & \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}\cdot\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t + \\ & + \int_\Omega \mu \mathrm{D}T_t^{-T}\nabla\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t\,\mathrm{d}x + \\ & + \int_\Omega \boldsymbol{u}\cdot \mathrm{D}T_t^{-T}\nabla\boldsymbol{u}\cdot\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t\,\mathrm{d}x - \\ & - \int_\Omega \boldsymbol{f}\circ T_t\cdot\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det \mathrm{D}T_t\,\mathrm{d}x \end{aligned}$$

and

$$F_2(t,\boldsymbol{u}) = \left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{u}\right)\det \mathrm{D}T_t.$$

We observe that

$$F\left(t, \boldsymbol{u}_\varepsilon(t)\circ T_t\right) = 0.$$

Besides we find that $\mathrm{D}_u F(0,\boldsymbol{u}_\varepsilon)$ is for all $\boldsymbol{u}\in\boldsymbol{H}^1_0(\Omega)$ given by

$$\mathrm{D}_u F_1(0,\boldsymbol{u}_\varepsilon)(\boldsymbol{u})(\boldsymbol{z}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}\cdot\boldsymbol{z} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{z} + \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{z} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}_\varepsilon\cdot\boldsymbol{z}\,\mathrm{d}x \quad \forall \boldsymbol{z}\in\boldsymbol{V}$$

and

$$\mathrm{D}_u F_2(0,\boldsymbol{u}_\varepsilon)\boldsymbol{u} = \operatorname{div}\boldsymbol{u}.$$

Thus we can use Lemma 4.2 and (15.1) to obtain in much the same way as in Lemma 15.2 from Lax-Milgram's theorem A.2 that $\mathrm{D}_u F(0,\boldsymbol{u}) : \boldsymbol{H}^1_0(\Omega)\to\boldsymbol{V}'\times L^2_0(\Omega)$ is an isomorphism. As a consequence, we can apply the implicit function theorem to

$$G : I\times\boldsymbol{H}^1_0(\Omega)\to\boldsymbol{V}'\times L^2_0(\Omega)$$
$$G(t,\boldsymbol{v}) := F(t,\boldsymbol{v}+\boldsymbol{G}),$$

which fulfills

$$G\left(t,\boldsymbol{u}_\varepsilon(t)\circ T_t - \boldsymbol{G}\right) = 0 \quad \forall t\in I$$

for some fixed chosen $\boldsymbol{G}\in\boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{G}|_{\partial\Omega} = \boldsymbol{g}$. From this we obtain existence and uniqueness of a function $t\mapsto\boldsymbol{u}(t)$ such that $G(t,\boldsymbol{u}(t)) = 0$ for all $t\in I$ in a small interval I around zero. But since $G(t,\boldsymbol{w}_\varepsilon(t)\circ T_t - \boldsymbol{G}) = 0$ for all $t\in I$ and for all $\boldsymbol{w}_\varepsilon(t)\in\boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon(t))$, this yields already $\boldsymbol{u}(t) = \boldsymbol{u}_\varepsilon(t)\circ T_t - \boldsymbol{G} = \boldsymbol{w}_\varepsilon(t)\circ T_t - \boldsymbol{G}$ for all $\boldsymbol{w}_\varepsilon(t)\in\boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon(t))$ and thus $\boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon(t)) = \{\boldsymbol{u}_\varepsilon(t)\}$ and the first statement of the lemma follows.
The implicit function theorem gives more in this setting, namely the differentiability of $t\mapsto(\boldsymbol{u}_\varepsilon(t)\circ T_t - \boldsymbol{G})\in\boldsymbol{H}^1(\Omega)$ at $t=0$ and thus of $t\mapsto(\boldsymbol{u}_\varepsilon(t)\circ T_t)$ as a mapping from I to $\boldsymbol{H}^1(\Omega)$ at $t=0$ together with

$$\begin{aligned} \partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t)\circ T_t\right) &= \partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t)\circ T_t - \boldsymbol{G}\right) = -\mathrm{D}_u G\left(0,\boldsymbol{u}_\varepsilon - \boldsymbol{G}\right)^{-1}\partial_t G\left(0,\boldsymbol{u}_\varepsilon-\boldsymbol{G}\right) = \\ &= -\mathrm{D}_u F\left(0,\boldsymbol{u}_\varepsilon\right)^{-1}\partial_t F\left(0,\boldsymbol{u}_\varepsilon\right) \end{aligned}$$

wherefrom we can deduce the statement.
For details we refer to Lemma 7.4. □

Using this result, we can now proceed to deriving first order optimality conditions by using the reduced functional

$$j_\varepsilon^N(\varphi_\varepsilon(t)) := J_\varepsilon^N(\varphi_\varepsilon(t), \boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon(t)))$$

which is due to Lemma 15.4 for t small enough well-defined.
We notice, that $\varphi_\varepsilon(t) \in \Phi_{ad}$ if $T \in \mathcal{T}_{ad}$, see Remark 3.4, and so those functions define admissible comparison functions, which implies

$$j_\varepsilon^N(\varphi_\varepsilon) \leq j_\varepsilon^N\left(\varphi_\varepsilon \circ T_t^{-1}\right) \quad \forall T \in \mathcal{T}_{ad}$$

and hence

$$\partial_t|_{t=0} j_\varepsilon^N\left(\varphi_\varepsilon \circ T_t^{-1}\right) = 0 \quad \forall T \in \mathcal{T}_{ad}.$$

As a result, we can derive analogously as in Lemma 7.5 first order optimality conditions for the problem (12.1) – (12.2):

Lemma 15.5. *For any minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in \Phi_{ad} \times \boldsymbol{U}$ of (12.1) – (12.2) there exists some Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}^+$ for the integral constraint such that the following necessary optimality conditions hold true:*

$$\partial_t|_{t=0} j_\varepsilon^N\left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\, \mathrm{d}x, \tag{15.19}$$

$$\lambda_\varepsilon\left(\int_\Omega \varphi_\varepsilon\, \mathrm{d}x - \beta|\Omega|\right) = 0 \tag{15.20}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$, where this derivative is given by the following formula:

$$\begin{aligned}\partial_t|_{t=0} j_\varepsilon^N\left(\varphi_\varepsilon \circ T_t^{-1}\right) = &\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left(\boldsymbol{u}_\varepsilon \cdot \dot{\boldsymbol{u}}_\varepsilon[V] + \frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 \operatorname{div} V(0)\right) \mathrm{d}x + \\ &+ \int_\Omega \left[\mathrm{D}f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)(V(0), \dot{\boldsymbol{u}}_\varepsilon[V], \mathrm{D}\dot{\boldsymbol{u}}_\varepsilon[V] - \mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0)) + \right. \\ &\left. + f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) \operatorname{div} V(0)\right] \mathrm{d}x + \\ &+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right) \operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\, \mathrm{d}x\end{aligned} \tag{15.21}$$

and $\dot{\boldsymbol{u}}_\varepsilon[V] \in \boldsymbol{H}_0^1(\Omega)$ is given as the solution of (15.17)-(15.18).

Proof. Those calculations can be carried out exactly as in Lemma 7.5 and the existence of a Lagrange multiplier follows as in Lemma 7.5. □

In Section 17 we will see that the optimality conditions derived in Lemma 15.5 converge in the setting of Theorem 6.1 to the corresponding optimality system in the sharp interface, which is derived in Section 16.2, as $\varepsilon \searrow 0$.

But before that, we want to reformulate the optimality condition under more restrictive regularity assumptions:

Lemma 15.6. *Assume that $\partial\Omega \in C^2$, $\mathrm{D}_u f(\cdot, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) \in \boldsymbol{L}^2(\Omega)$ for $\boldsymbol{u}_\varepsilon \in \boldsymbol{H}^2(\Omega)$ and the boundary data $\boldsymbol{g} \in \boldsymbol{H}^{\frac{3}{2}}(\partial\Omega)$.*

For any minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in \Phi_{ad} \times \boldsymbol{U}$ of (12.1) – (12.2) we have the following necessary optimality conditions:

$$\partial_t|_{t=0} j_\varepsilon^N \left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x, \tag{15.22}$$

$$\lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x - \beta|\Omega|\right) = 0 \tag{15.23}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$ and some Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}^+$ for the integral constraint. The expression $\partial_t|_{t=0} j_\varepsilon^N\left(\varphi_\varepsilon \circ T_t^{-1}\right)$ is given by (15.21) and can be reformulated as follows:

$$\begin{aligned}
&\partial_t|_{t=0} j_\varepsilon^N \left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right)\mathrm{d}x+\\
&+ \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right)\mathrm{d}x + \int_{\partial\Omega} f(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) V(0)\cdot\boldsymbol{n}\,\mathrm{d}x+\\
&+ \mu \int_{\partial\Omega} (\partial_n \boldsymbol{q}_\varepsilon \cdot \partial_n \boldsymbol{u}_\varepsilon)(V(0)\cdot\boldsymbol{n})\,\mathrm{d}x - \int_{\partial\Omega} (\mathrm{D}_3 f)(x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon)\boldsymbol{n}\cdot\left(\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon (V(0)\cdot\boldsymbol{n})\right)\mathrm{d}x+\\
&+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x
\end{aligned} \tag{15.24}$$

and the adjoint variable $\boldsymbol{q}_\varepsilon \in \boldsymbol{V}$ is given as the unique weak solution of

$$\begin{aligned}
\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\boldsymbol{u}_\varepsilon^T \cdot \boldsymbol{q}_\varepsilon - \boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon &= \mathrm{D}_u f(\cdot, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{q}_\varepsilon &= 0 && \text{in } \Omega,\\
\boldsymbol{q}_\varepsilon &= \boldsymbol{0} && \text{on } \partial\Omega.
\end{aligned} \tag{15.25}$$

Remark 15.4. *In particular $V \in \overline{\mathcal{V}}_{ad}$ implies $V(0)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, and so (15.24) can be simplified to*

$$\begin{aligned}
\partial_t|_{t=0} j_\varepsilon^N \left(\varphi_\varepsilon \circ T_t^{-1}\right) &= \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right)\mathrm{d}x+\\
&+ \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\left(\mathrm{D}\varphi_\varepsilon V(0)\right)\left(\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right)\mathrm{d}x+\\
&+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x.
\end{aligned} \tag{15.26}$$

As we want to transfer the analysis of the proof of Lemma 15.6 to the sharp interface setting in Section 16.3, we will carry out the calculations without using the assumption $V(0)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. The same has been done in the proof of Lemma 7.6, on which the arguments here are based.

Proof. We follow closely the analysis carried out in the proof of Lemma 7.6 and only point out the main steps and differences that occur due to the nonlinear state equations.
We start by applying regularity theory, which can be found in [Soh01, Gal11, Tem77], to deduce from the regularity of the data stated in the assumptions of the lemma that $\boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon[V]$ and $\boldsymbol{q}_\varepsilon$ are in $\boldsymbol{H}^2(\Omega)$ and define

$$\boldsymbol{u}'_\varepsilon[V] = \dot{\boldsymbol{u}}_\varepsilon[V] - \nabla\boldsymbol{u}_\varepsilon \cdot V(0)$$

which is then a function in $\boldsymbol{H}^1(\Omega)$.

Then we choose an arbitrary $\boldsymbol{z} \in \boldsymbol{C}_0^\infty(\Omega)$ with $\operatorname{div}\boldsymbol{z} = 0$ and see that

$$\begin{aligned} b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}'_\varepsilon[V], \boldsymbol{z}\right) &= b\left(\boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon[V], \boldsymbol{z}\right) - \int_\Omega \boldsymbol{u}_\varepsilon \cdot \nabla\left(\nabla\boldsymbol{u}_\varepsilon \cdot V(0)\right) \cdot \boldsymbol{z}\,\mathrm{d}x = \\ &= b\left(\boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon[V], \boldsymbol{z}\right) - \int_\Omega \boldsymbol{u}_\varepsilon \cdot \left(\mathrm{D}^2\boldsymbol{u}_\varepsilon V(0)\right) \cdot \boldsymbol{z}\,\mathrm{d}x - \int_\Omega \boldsymbol{u}_\varepsilon \cdot \left(\nabla V(0)\nabla\boldsymbol{u}_\varepsilon\right) \cdot \boldsymbol{z}\,\mathrm{d}x. \end{aligned}$$

We may now integrate by parts and use $\boldsymbol{z}|_{\partial\Omega} = \boldsymbol{0}$ to obtain

$$\begin{aligned} &\int_\Omega \boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{u}_\varepsilon \operatorname{div} V(0)\boldsymbol{z}\,\mathrm{d}x = \int_\Omega \sum_{i,j,k=1}^d u_i\partial_i u_j \partial_k V(0)_k z_j\,\mathrm{d}x = \\ &= -\int_\Omega \sum_{i,j,k=1}^d \partial_k u_i \partial_i u_j V(0)_k z_j\,\mathrm{d}x - \int_\Omega \sum_{i,j,k=1}^d u_i \partial_k\partial_i u_j V(0)_k z_j\,\mathrm{d}x - \\ &- \int_\Omega \sum_{i,j,k=1}^d u_i\partial_i u_j V(0)_k \partial_k z_j\,\mathrm{d}x = \\ &= -\int_\Omega \left(\nabla\boldsymbol{u}_\varepsilon \cdot V(0)\right) \cdot \nabla\boldsymbol{u}_\varepsilon \cdot \boldsymbol{z}\,\mathrm{d}x - \int_\Omega \boldsymbol{u}_\varepsilon \cdot \left(\mathrm{D}^2\boldsymbol{u}_\varepsilon V(0)\right) \cdot \boldsymbol{z}\,\mathrm{d}x - \int_\Omega \boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{u}_\varepsilon \cdot \mathrm{D}\boldsymbol{z} V(0)\,\mathrm{d}x \end{aligned}$$

which implies

$$\begin{aligned} -\int_\Omega \boldsymbol{u}_\varepsilon \cdot \left(\mathrm{D}^2\boldsymbol{u}_\varepsilon V(0)\right) \cdot \boldsymbol{z}\,\mathrm{d}x - \int_\Omega \left(\nabla\boldsymbol{u}_\varepsilon \cdot V(0)\right) \cdot \nabla\boldsymbol{u}_\varepsilon \cdot \boldsymbol{z}\,\mathrm{d}x - b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \mathrm{D}V(0)\boldsymbol{z}\right) = \\ = b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z}V(0)\right). \end{aligned}$$

Using this relation we obtain for all $\boldsymbol{z} \in \boldsymbol{C}_0^\infty(\Omega)$ with $\operatorname{div}\boldsymbol{z} = 0$ that

$$\begin{aligned} &b\left(\boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon[V], \boldsymbol{z}\right) + b\left(\dot{\boldsymbol{u}}_\varepsilon[V], \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right) - b\left(\mathrm{D}V(0)\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right) + b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \mathrm{D}V(0)\boldsymbol{z}\right) = \\ &= b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}'_\varepsilon[V], \boldsymbol{z}\right) + b\left(\boldsymbol{u}'_\varepsilon[V], \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right) + \int_\Omega \boldsymbol{u}_\varepsilon \cdot \left(\mathrm{D}^2\boldsymbol{u}_\varepsilon V(0)\right) \cdot \boldsymbol{z}\,\mathrm{d}x + \\ &+ \int_\Omega \boldsymbol{u}_\varepsilon \cdot \left(\nabla V(0)\nabla\boldsymbol{u}_\varepsilon\right) \cdot \boldsymbol{z}\,\mathrm{d}x + \int_\Omega \left(\nabla\boldsymbol{u}_\varepsilon \cdot V(0)\right) \cdot \nabla\boldsymbol{u}_\varepsilon \cdot \boldsymbol{z}\,\mathrm{d}x - \\ &- b\left(\mathrm{D}V(0)\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right) + b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \mathrm{D}V(0)\boldsymbol{z}\right) = \\ &= b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}'_\varepsilon[V], \boldsymbol{z}\right) + b\left(\boldsymbol{u}'_\varepsilon[V], \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right) - b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z} + \mathrm{D}\boldsymbol{z}V(0)\right). \end{aligned} \tag{15.27}$$

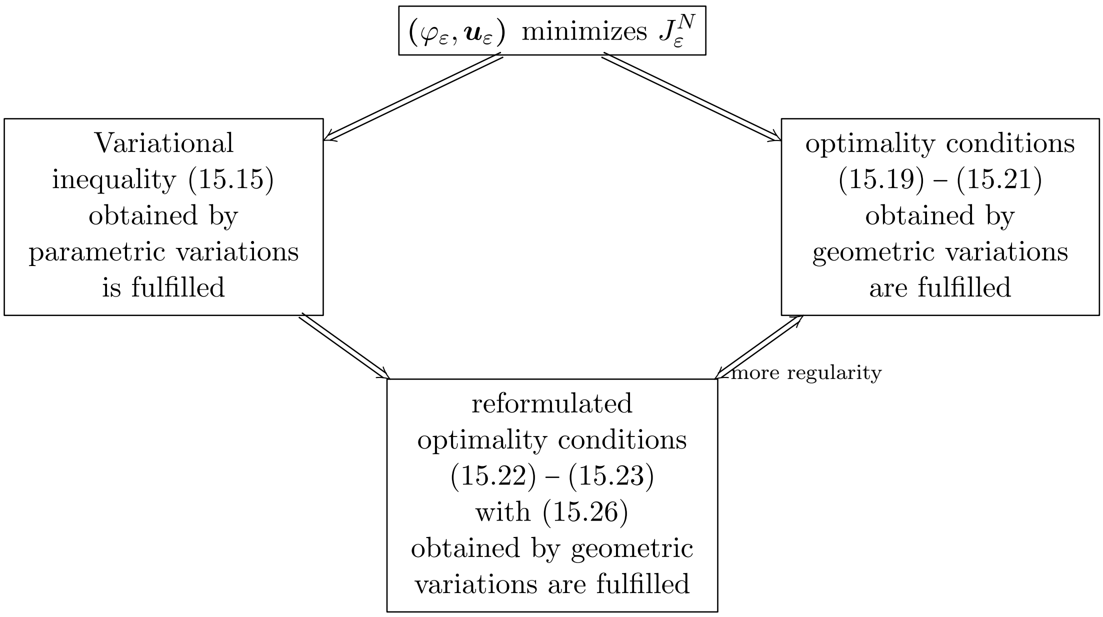
Now we can apply analysis similar to that in the proof of Lemma 7.6, while making in particular use of (15.17) and (15.27). From these arguments we then arrive in the following system for $\boldsymbol{u}'_\varepsilon[V]$:

$$\begin{aligned}
\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}'_\varepsilon[V] - \mu\Delta\boldsymbol{u}'_\varepsilon[V] + \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u}'_\varepsilon[V] + \boldsymbol{u}'_\varepsilon[V]\cdot\nabla\boldsymbol{u}_\varepsilon + & \\
+\nabla p'_\varepsilon[V] &= \alpha'_\varepsilon(\varphi_\varepsilon)(\mathrm{D}\varphi_\varepsilon V(0))\boldsymbol{u}_\varepsilon && \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u}'_\varepsilon[V] &= 0 && \text{in } \Omega,\\
\boldsymbol{u}'_\varepsilon[V] &= -\partial_{\boldsymbol{n}}\boldsymbol{u}_\varepsilon(V(0)\cdot\boldsymbol{n}) && \text{on } \partial\Omega.
\end{aligned}$$

Hence, we obtain with the same calculations as done in the second step of the proof of Lemma 7.6 the stated result. □

15.3 Linking the optimality criteria

Analogously as in Section 7.3 one can show that the variational inequality (15.15) given by Theorem 15.1 implies (15.22) – (15.23) with (15.26).
Moreover, we have shown in Lemma 15.6, that (15.21) is equivalent to (15.26), if we assume more regularity. But on the other hand, we can derive both the variational inequality and conditions (15.19)-(15.21) without any additional regularity, compare Theorem 15.1 and Lemma 15.5.
So we can summarize the results of Section 15 by the following diagram, which is fulfilled if (15.1) holds for the minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ of J_ε^N and thus is in particular fulfilled if $\varepsilon > 0$ is small enough and $(\varphi_\varepsilon)_{\varepsilon>0}$ is a sequence fulfilling (14.1).



16 Optimality conditions for the sharp interface model

For this section we assume that $(\varphi_0, \boldsymbol{u}_0) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ is a minimizer of J_0^N, and thus by Lemma 13.5 in particular $\{\boldsymbol{u}_0\} = \boldsymbol{S}_0^N(\varphi_0)$. The existence of such a minimizer is for example guaranteed in the setting of Theorem 6.1.

The aim of this section is to derive first order optimality conditions for J_0^N, thus necessary conditions that have to be fulfilled for the minimizer $(\varphi_0, \boldsymbol{u}_0)$ of J_0^N. Therefore, we start by calculating shape derivatives in Section 16.1, which is formally only possible when assuming more regularity on the data. Particularly, we need regularity of the minimizing set E^{φ_0} of the sharp interface problem which has not been shown to be valid in this general setting. And so we consider additionally another ansatz, which does not need additional unverified regularity. This will be done in Section 16.2 and is also based on geometric variations. That those optimality conditions are equivalent if assuming the regularity assumption necessary for calculating the shape derivatives is then shown in Section 16.3.

For this purpose we have to assume for the remainder of this section Assumptions **(A6)** and **(A7)** to ensure differentiability of the objective functional and the external force term.

16.1 Shape derivative approach

The first ansatz for deriving optimality conditions is to reformulate the problem

$$\min_{(\varphi, \boldsymbol{u}) \in L^1(\Omega) \times \boldsymbol{H}^1(\Omega)} J_0^N(\varphi, \boldsymbol{u})$$

analogously as in Section 8 formally into the shape optimization problem

$$\min_{E \subset \Omega, |E| \leq 0.5(1+\beta)|\Omega|} J_S^N(E) := J_0^N\left(2\chi_E - 1, \boldsymbol{S}_0^N(2\chi_E - 1)\right) \tag{16.1}$$

as soon as we know that $\boldsymbol{S}_0^N(2\chi_E - 1)$ only consists of one element. This is fulfilled for E^{φ_0} if $(\varphi_0, \boldsymbol{u}_0)$ is a minimizer of J_0^N, but is so far not known for an arbitrary set E. As already discussed in Lemma 13.1 and Remark 13.2 we cannot even guarantee the existence of a solution for arbitrary φ. But as a consequence of the implicit function theorem, we will obtain existence and uniqueness of a solution to the state equations for small geometric perturbations of the minimizer φ_0. So we start our investigations by fixing the minimizing set

$$E_0 := \operatorname{int}\left(\{x \in \Omega \mid \varphi_0(x) = 1\}\right),$$

choosing a transformation $T \in \overline{\mathcal{T}}_{ad}$ and introducing the notation

$$\Omega_t := T_t(\Omega), \qquad E_t := T_t(E_0).$$

As already indicated above, we have to make some regularity assumptions in order to use the correct shape calculus. So we assume for the rest of this subsection:

$$E_0 \text{ is a fixed open subset of } \Omega \text{ with } \partial E_0 \in C^2, \tag{16.2a}$$
$$E_0 \text{ has } N < \infty \text{ connected components}, \tag{16.2b}$$
$$\boldsymbol{g} \in \boldsymbol{H}^{\frac{3}{2}}(\partial\Omega), \tag{16.2c}$$
$$\mathrm{D}_u f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \in \boldsymbol{L}^2(E_0), \text{ if } \boldsymbol{u}_0 \in \boldsymbol{H}^2(E_0). \tag{16.2d}$$

In the following, we will denote by $C_1, \dots, C_N$ the connected components of E_0.

Then we can choose a pressure $p_0 \in L^2(E_0)$ associated to $\boldsymbol{u}_0$ as follows: Due to Lemma 4.4 there exists a unique $p_0 \in L^2(E_0)$ with $\int_{C_i} p_0 \,\mathrm{d}x = 0$ for every $i = 1, \dots, N$, such that it holds in the distributional sense

$$\nabla p_0 = \boldsymbol{f} + \mu\Delta\boldsymbol{u}_0 - \boldsymbol{u}_0 \cdot \nabla\boldsymbol{u}_0 \quad \text{in } E_0.$$

We can deduce from the regularity assumptions (16.2) additional regularity of the minimizer $\boldsymbol{u}_0 \in \boldsymbol{S}_0^N(\varphi_0)$ and its associated pressure p_0:

Lemma 16.1. *Under the regularity assumptions* (8.3) *it follows* $\boldsymbol{u}_0 \in \boldsymbol{H}^2(E_0)$ *and* $p_0 \in H^1(E_0)$.

Proof. This follows as in Lemma 8.2 by applying regularity results for the stationary Navier-Stokes equations. These can be found for instance in [Gal11, Tem77]. □

Now we show that the state equations (13.2) inherit a unique solution for small deformations of E_0. This is shown by an application of the implicit function theorem, and implies additionally directly a differentiability result that will be essential for calculating the shape derivative later on.

Lemma 16.2. *There exists a small interval $I \subset \mathbb{R}$, $0 \in I$, such that for every $t \in I$ there exists a unique solution $\boldsymbol{u}_t \in \boldsymbol{U}^{E_t}$ to the state equations* (13.2) *written for E_t, thus $\boldsymbol{S}_0^N(2\chi_{E_t} - 1) = \{\boldsymbol{u}_t\}$. Additionally, we obtain a unique associated pressure $p_t \in L^2(E_t)$ fulfilling*

$$\nabla p_t = \boldsymbol{f} + \mu\Delta\boldsymbol{u}_t - \boldsymbol{u}_t \cdot \nabla\boldsymbol{u}_t \quad \text{in } E_t \tag{16.3}$$

in the distributional sense together with $\int_{T_t(C_i)} p_t \,\mathrm{d}x = 0$ for all $i = 1, \dots, N$. The functions inherit the regularity $(\boldsymbol{u}_t, p_t) \in \boldsymbol{H}^2(E_0) \times H^1(E_0)$ for all $t \in I$.

Moreover, the mappings $I \ni t \mapsto \boldsymbol{u}_t \circ T_t \in \boldsymbol{H}^2(E_0)$ and $I \ni t \mapsto p_t \circ T_t \in H^1(E_0)$ are differentiable at $t = 0$.

Proof. In contrast to Lemma 8.3, we cannot apply Theorem A.3, since our constraints are not linear any more. To overcome the difficulty arising from the incompressibility condition, see discussion in the proof of Lemma 8.3, we alter the function on which to apply the implicit function theorem analogously to [BFCLS96]. Denoting by $(C_i)_{i=1}^N$ the connected components of E_0 we define the function

$$F : I \times \left(\boldsymbol{H}^2(E_0) \cap \boldsymbol{H}^1_{\boldsymbol{g}}(E_0)\right) \times H^1(E_0) \to \boldsymbol{L}^2(E_0) \times \left(\bigtimes_{i=1}^N \left(H^1(C_i) \cap L^2_0(C_i)\right)\right) \times \mathbb{R}^N$$

by

$$F(t,\boldsymbol{v},p) = \Bigg(-\mu \sum_{i,j,k=1}^{d} \left(\mathrm{D}T_t^{-T}\right)_{ij} \partial_j \left(\left(\mathrm{D}T_t^{-T}\right)_{ik} \partial_k \boldsymbol{v}\right) + \boldsymbol{v}\cdot \mathrm{D}T_t^{-T}\nabla \boldsymbol{v} + \mathrm{D}T_t^{-T}\nabla p - \boldsymbol{f}\circ T_t,$$
$$\left(\left(\mathrm{D}T_t^{-1} : \nabla \boldsymbol{v}\big|_{C_i} - \fint_{C_i} \mathrm{D}T_t^{-1} : \nabla \boldsymbol{v}\,\mathrm{d}x\right)_{i=1}^{N}, \left(\int_{C_i} p \det \mathrm{D}T_t \,\mathrm{d}x\right)_{i=1}^{N}\right).$$

Using this definition, we can ensure that

$$\left(\mathrm{D}T_t^{-1} : \nabla \boldsymbol{v} - \fint_{C_i} \mathrm{D}T_t^{-1} : \nabla \boldsymbol{v}\,\mathrm{d}x\right) \in L_0^2(C_i) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{\boldsymbol{g}}^1(E_0).$$

Moreover, direct calculations give for all $\boldsymbol{u} \in \boldsymbol{H}^2(E_t) \cap \boldsymbol{H}_{\boldsymbol{g}}^1(E_t)$ and $p \in H^1(E_t)$

$$\begin{aligned} F\left(t,\boldsymbol{u}\circ T_t, p\circ T_t\right) = &\left(\left(-\mu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u} + \nabla p - \boldsymbol{f}\right)\circ T_t,\right.\\ &\left(\left(\operatorname{div}\boldsymbol{u}\right)\circ T_t|_{C_i} - \fint_{C_i} \left(\operatorname{div}\boldsymbol{u}\right)\circ T_t\,\mathrm{d}x\right)_{i=1}^{N}, \left(\int_{T_t(C_i)} p\,\mathrm{d}x\right)_{i=1}^{N}) \end{aligned} \tag{16.4}$$

where we used in particular that $\boldsymbol{u} \in \boldsymbol{H}_{\boldsymbol{g}}^1(E_t)$ implies $\boldsymbol{u}\circ T_t \in \boldsymbol{H}_{\boldsymbol{g}}^1(E_0)$, compare Lemma 3.5.

Next we fix some $\boldsymbol{G} \in \boldsymbol{H}^2(E_0)$ with $\boldsymbol{G}|_{\partial\Omega\cap\partial E_0} = \boldsymbol{g}$, $\boldsymbol{G}|_{\Omega\cap\partial E_0} = \boldsymbol{0}$ and $\operatorname{div}\boldsymbol{G} = 0$, which can be found by applying the results of [Gal11, Section III.3] and using $\partial E_0 \in C^2$.
Then we define

$$G : I \times \left(\boldsymbol{H}^2(E_0) \cap \boldsymbol{H}_0^1(E_0)\right) \times H^1(E_0) \to \boldsymbol{L}^2(E_0) \times \left(\bigtimes_{i=1}^{N} \left(H^1(C_i)\cap L_0^2(C_i)\right)\right) \times \mathbb{R}^N$$

by

$$G\left(t,\boldsymbol{v},p\right) = F\left(t,\boldsymbol{v}+\boldsymbol{G},p\right)$$

and obtain from (16.4)

$$G\left(0,\boldsymbol{u}_0 - \boldsymbol{G}, p_0\right) = F\left(0,\boldsymbol{u}_0,p_0\right) = 0. \tag{16.5}$$

Our goal is to apply the implicit function theorem to G. To this end, we notice that $t \mapsto G\left(t,\boldsymbol{u},p\right)$ is differentiable in a small neighborhood of $t=0$ for all $\boldsymbol{u} \in \boldsymbol{H}^2(E_0)\cap\boldsymbol{H}_0^1(E_0)$ and $p \in H^1(E_0)$, because of the regularity assumptions on the transformation $T \in \overline{\mathcal{T}}_{ad}$. Furthermore, we find for all $(\boldsymbol{u},p) \in \left(\boldsymbol{H}^2(E_0)\cap\boldsymbol{H}_0^1(E_0)\right)\times H^1(E_0)$:

$$\begin{aligned} &\mathrm{D}_{(u,p)} G\left(0,\boldsymbol{u}_0-\boldsymbol{G},p_0\right)(\boldsymbol{u},p) = \\ &= \left(-\mu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}_0 + \boldsymbol{u}_0\cdot\nabla\boldsymbol{u} + \nabla p, \left(\operatorname{div}\boldsymbol{u}|_{C_i}\right)_{i=1}^{N}, \left(\int_{C_i} p\,\mathrm{d}x\right)_{i=1}^{N}\right). \end{aligned}$$

To show, that $\mathrm{D}_{(u,p)}G\left(0,\boldsymbol{u}_0-\boldsymbol{G},p_0\right)$ is an isomorphism, we choose arbitrary

$$\left(\boldsymbol{r},(s_i)_{i=1}^{N},(t_i)_{i=1}^{N}\right) \in \boldsymbol{L}^2(E_0) \times \left(\bigtimes_{i=1}^{N}\left(H^1(C_i)\cap L_0^2(C_i)\right)\right)\times\mathbb{R}^N$$

and find from [Gal11, Lemma 2.3.1] that for any $i = 1,\ldots,N$ there exists some $\boldsymbol{w}_i \in \boldsymbol{H}^2(C_i)\cap\boldsymbol{H}_0^1(C_i)$ such that $\operatorname{div}\boldsymbol{w}_i = s_i$ and $\|\boldsymbol{w}_i\|_{\boldsymbol{H}^2(C_i)} \le c\,\|s_i\|_{H^1(C_i)}$. We can extend those functions by zero, due to the regularity of E_0, to functions $\boldsymbol{w}_i \in \boldsymbol{H}^2(E_0) \cap \boldsymbol{H}_0^1(E_0)$. By Lax-Milgram's theorem A.2 we find a unique $\boldsymbol{u} \in \boldsymbol{H}_0^1(E_0)$ with $\operatorname{div} \boldsymbol{u} = 0$ such that

$$\int_{E_0} \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{v} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x = \int_{E_0} \boldsymbol{r} \cdot \boldsymbol{v} + \\ + \sum_{i=1}^N \left(-\mu \nabla \boldsymbol{w}_i \cdot \nabla \boldsymbol{v} - \boldsymbol{w}_i \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{v} - \boldsymbol{u}_0 \cdot \nabla \boldsymbol{w}_i \cdot \boldsymbol{v} \right) \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(E_0), \operatorname{div} \boldsymbol{v} = 0$$

which fulfills moreover

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(E_0)} \le C \left(\|\boldsymbol{r}\|_{\boldsymbol{L}^2(E_0)} + \sum_{i=1}^N \|\boldsymbol{w}_i\|_{\boldsymbol{H}^2(E_0)} \right) \le C \left(\|\boldsymbol{r}\|_{\boldsymbol{L}^2(E_0)} + \sum_{i=1}^N \|s_i\|_{H^1(C_i)} \right)$$

since it holds by Lemma 13.5

$$\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}$$

and so

$$\left| \int_{E_0} \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{v} \, \mathrm{d}x \right| \le \frac{\mu}{2} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(E_0).$$

Additionally, we get from Lemma 4.4 existence and uniqueness of $p \in L^2(E_0)$ with $\int_{C_i} p \, \mathrm{d}x = t_i$ for all $i = 1, \ldots, N$ such that it holds in the distributional sense

$$-\mu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{r} + \sum_{i=1}^N \left(\mu \Delta \boldsymbol{w}_i - \boldsymbol{w}_i \cdot \nabla \boldsymbol{u}_0 - \boldsymbol{u}_0 \cdot \nabla \boldsymbol{w}_i \right) \quad \text{in } C_i$$

and find that

$$\|p\|_{L^2(E_0)} \le C \left(\sum_{i=1}^N |t_i| + \|\boldsymbol{u}\|_{\boldsymbol{H}^1(E_0)} + \|\boldsymbol{r}\|_{\boldsymbol{L}^2(E_0)} + \sum_{i=1}^N \|s_i\|_{H^1(C_i)} \right).$$

Applying regularity theory for the Stokes equations, see for instance [Tem77, Gal11], we obtain furthermore $\boldsymbol{u} \in \boldsymbol{H}^2(E_0)$ and $p \in H^1(E_0)$ together with

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(E_0)} + \|p\|_{H^1(E_0)} \le C \left(\|\boldsymbol{r}\|_{\boldsymbol{L}^2(E_0)} + \sum_{i=1}^N \left(\|s_i\|_{H^1(C_i)} + |t_i| \right) \right)$$

and thus $\mathrm{D}_{(u,p)} G(0, \boldsymbol{u}_0 - \boldsymbol{G}, p_0)$ is a surjective, open, linear operator. Injectiveness of $\mathrm{D}_{(u,p)} G(0, \boldsymbol{u}_0 - \boldsymbol{G}, p_0)$ follows from the uniqueness of the solution derived above, and we can deduce from the open mapping theorem that $\mathrm{D}_{(u,p)} G(0, \boldsymbol{u}_0 - \boldsymbol{G}, p_0)$ is an isomorphism.

So we can apply the implicit function theorem and obtain that there exists a small interval $I \subset \mathbb{R}$ around 0 and a unique mapping $I \ni t \mapsto (\boldsymbol{u}(t), p(t)) \in \boldsymbol{H}^2(E_0) \cap \boldsymbol{H}_0^1(E_0) \times \boldsymbol{H}^1(E_0)$ such that

$$G(t, \boldsymbol{u}(t), p(t)) = 0 \quad \forall t \in I.$$

We define $\boldsymbol{u}_t := (\boldsymbol{u}(t) + \boldsymbol{G}) \circ T_t^{-1}$ and $p_t := p(t) \circ T_t^{-1} \in \boldsymbol{H}^1(E_0)$ and find that $\boldsymbol{u}_t \in \boldsymbol{H}^1_{\boldsymbol{g}}(E_0) \cap \boldsymbol{H}^2(E_0)$, since $T \in \overline{\mathcal{T}}_{ad}$, cf. Lemma 3.5. By (16.4) we observe furthermore:

$$
\begin{aligned}
0 &= G\left(t, \boldsymbol{u}(t), p(t)\right) = F\left(t, \boldsymbol{u}(t) + \boldsymbol{G}, p(t)\right) = F\left(t, \boldsymbol{u}_t \circ T_t, p_t \circ T_t\right) = \\
&= \left(\left(-\mu\Delta \boldsymbol{u}_t + \boldsymbol{u}_t \cdot \nabla \boldsymbol{u}_t + \nabla p_t - \boldsymbol{f}\right) \circ T_t, \left(\left(\operatorname{div} \boldsymbol{u}_t\right) \circ T_t|_{C_i} - \fint_{C_i} \left(\operatorname{div} \boldsymbol{u}_t\right) \circ T_t \,\mathrm{d}x\right)_{i=1}^N, \right. \\
&\left. \left(\int_{T_t(C_i)} p_t \,\mathrm{d}x\right)_{i=1}^N\right).
\end{aligned}
\tag{16.6}
$$

Considering the second condition, we find for every $i = 1, \ldots, N$:

$$
\operatorname{div} \boldsymbol{u}_t = \left(\fint_{C_i} \left(\operatorname{div} \boldsymbol{u}_t\right) \circ T_t \,\mathrm{d}x\right) \circ T_t^{-1} = \fint_{C_i} \left(\operatorname{div} \boldsymbol{u}_t\right) \circ T_t \,\mathrm{d}x \quad \text{in } T_t(C_i).
$$

If we use, that for t small enough $T_t(C_i)$, $i = 1, \ldots, N$, will be the connected components of $T_t(E_0)$ we find therefrom by $\boldsymbol{u}_t \in \boldsymbol{H}^1_{\boldsymbol{g}}(E_t)$:

$$
0 = \int_{T_t(C_i)} \operatorname{div} \boldsymbol{u}_t \,\mathrm{d}x = \int_{T_t(C_i)} \left(\fint_{C_i} \left(\operatorname{div} \boldsymbol{u}_t\right) \circ T_t \,\mathrm{d}x\right) \mathrm{d}x
$$

and so $\fint_{C_i} \left(\operatorname{div} \boldsymbol{u}_t\right) \circ T_t \,\mathrm{d}x = 0$, wherefrom we can deduce $\operatorname{div} \boldsymbol{u}_t = 0$.
And so we can conclude from (16.6) that $(\boldsymbol{u}_t, p_t)$ solve the following system:

$$
\begin{aligned}
-\mu\Delta \boldsymbol{u}_t + \boldsymbol{u}_t \cdot \nabla \boldsymbol{u}_t + \nabla p_t &= \boldsymbol{f} && \text{in } E_t, \\
\operatorname{div} \boldsymbol{u}_t &= 0 && \text{in } E_t, \\
\int_{T_t(C_i)} p_t \,\mathrm{d}x &= 0 && \forall i = 1, \ldots, N.
\end{aligned}
$$

Hence, $\boldsymbol{u}_t = \boldsymbol{S}_0^N\left(2\chi_{E_t} - 1\right)$, and p_t is the unique pressure associated to the state equations in the sense of (16.3) such that $\int_{T_t(C_i)} p_t \,\mathrm{d}x = 0$ for $i = 1, \ldots, N$ and this implies the first part of the statement.

The implicit function theorem gives more, namely that $I \ni t \mapsto (\boldsymbol{u}(t), p(t)) \in \boldsymbol{H}^2(E_0) \times H^1(E_0)$ are differentiable at $t = 0$. Hence,

$$
I \ni t \mapsto (\boldsymbol{u}_t \circ T_t, p_t \circ T_t) = (\boldsymbol{u}(t) + \boldsymbol{G}, p(t)) \in \boldsymbol{H}^2(E_0) \times H^1(E_0)
$$

is differentiable at $t = 0$ and we can finish the proof.

$\square$

Using this result, we can consider minimizing J_0^N as a classical shape optimization problem in the sense of (16.1), since for the minimizer E_0 and local deformations of it the function J_S^N is well-defined. So the rest of this subsection will be devoted to deriving first order optimality conditions for J_0^N by considering minimizing J_S^N as a shape optimization problem in much the same way as in Section 8.
With Lemma 16.2 we have all assumptions fulfilled to use [Sim80, Theorem 3.1, Theorem 3.2] and conclude the formula for the shape derivative $\boldsymbol{u}'[V]$. In particular, we obtain as

in Lemma 8.4 that the $\boldsymbol{u}'[V]$ is given as the unique weak solution of the following system:

$$\begin{aligned} -\mu\Delta\boldsymbol{u}'[V] + \boldsymbol{u}'[V]\cdot\nabla\boldsymbol{u}_0 + \boldsymbol{u}_0\cdot\nabla\boldsymbol{u}'[V] + \nabla p'[V] &= \mathbf{0} && \text{in } E_0,\\ \operatorname{div}\boldsymbol{u}'[V] &= 0 && \text{in } E_0,\\ \boldsymbol{u}'[V] &= -\left(V(0)\cdot\nu\right)\partial_\nu\boldsymbol{u}_0 && \text{on } \partial E_0\cap\Omega,\\ \boldsymbol{u}'[V] &= \mathbf{0} && \text{on } \partial E_0\cap\partial\Omega. \end{aligned}$$

As in the proof of Theorem 8.1 we can find a Lagrange multiplier for the integral constraint. Moreover, similar arguments as in Theorem 8.1 give a formulation of the shape derivative $\mathrm{D}J_S^N(E_0)[V] = \partial_t|_{t=0}J_S^N(T_t(E_0))$ in the adjoint formulation. We summarize these results in the following theorem:

Theorem 16.1. *Since E_0 is assumed to minimize J_S^N, the following necessary optimality conditions are fulfilled*

$$\mathrm{D}J_S^N(E_0)[V] = -\lambda_0\int_\Omega (2\chi_{E_0}-1)\operatorname{div}V(0)\,\mathrm{d}x \quad \forall V\in\overline{\mathcal{V}}_{ad} \tag{16.7}$$

for some Lagrange multiplier $\lambda_0\geq 0$, which fulfills moreover

$$\lambda_0\left(\int_\Omega (2\chi_{E_0}-1) - \beta|\Omega|\right) = 0 \tag{16.8}$$

and the shape derivative is given by

$$\begin{aligned} \mathrm{D}J_S^N(E_0)[V] = &\int_{\partial E_0}\left(\mu\partial_\nu\boldsymbol{q}_0\cdot\partial_\nu\boldsymbol{u}_0 - (\mathrm{D}_3 f)(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0\right)V(0)\cdot\nu\,\mathrm{d}x+\\ &+\int_{E_0}\mathrm{D}\left(f(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\right)V(0)\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\operatorname{div}V(0)\,\mathrm{d}x+\\ &+\gamma c_0\int_{\partial E_0\cap\Omega}\kappa V(0)\cdot\nu\,\mathrm{d}x. \end{aligned} \tag{16.9}$$

Here $\kappa = \operatorname{div}_{\partial E_0\cap\Omega}\nu$ denotes the mean curvature of $\partial E_0\cap\Omega$, ν is the outer unit normal on E_0 and $\boldsymbol{q}_0\in\boldsymbol{H}_0^1(E_0)$ is the adjoint state given as strong solution of

$$-\mu\Delta\boldsymbol{q}_0 + \nabla\boldsymbol{u}_0^T\cdot\boldsymbol{q}_0 - \boldsymbol{u}_0\cdot\nabla\boldsymbol{q}_0 + \nabla\pi_0 = \mathrm{D}_u f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) \quad \text{in } E_0, \tag{16.10a}$$

$$\operatorname{div}\boldsymbol{q}_0 = 0 \quad \text{in } E_0, \tag{16.10b}$$

$$\boldsymbol{q}_0 = \mathbf{0} \quad \text{on } \partial E_0. \tag{16.10c}$$

Remark 16.1. *Notice, that $\lambda_0\geq 0$ is a Lagrange multiplier for the volume constraint $|E_0|\leq 0.5(1+\beta)|\Omega|$ and (16.8), which can be rewritten as*

$$\lambda_0\left(|E_0| - \frac{(1+\beta)}{2}|\Omega|\right) = 0$$

is the corresponding complementarity condition.

Remark 16.2. *If we assume that $f(x,\boldsymbol{u}(x),\mathrm{D}\boldsymbol{u}(x)) = 0$ for a.e. $x\in\Omega\setminus E_0$ if $\boldsymbol{u}=\mathbf{0}$ a.e. in $\Omega\setminus E_0$ then formula (16.9) reads as*

$$\begin{aligned} \mathrm{D}J_S^N(E_0)[V] = &\int_{\partial E_0}\left(\mu\partial_\nu\boldsymbol{q}_0\cdot\partial_\nu\boldsymbol{u}_0 - (\mathrm{D}_3 f)(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0\right)V(0)\cdot\nu\,\mathrm{d}x+\\ &+\int_{\partial E_0} f(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\,V(0)\cdot\nu\,\mathrm{d}x + \gamma c_0\int_{\partial E_0\cap\Omega}\kappa V(0)\cdot\nu\,\mathrm{d}x. \end{aligned}$$

The resulting formula is then in Hadamard form, see discussion in Remark 8.3. Moreover, already known literature (compare for instance the results in [SS10]) yield the same result.

We want to discuss this result briefly by means of the example of minimizing the total potential power:

Example 16.1. Using the total potential power as an objective functional, which is introduced in Example 2.3, thus

$$f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) = \frac{\mu}{2} |\mathrm{D}\boldsymbol{u}|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u}$$

we get from Theorem 16.1 the following formula for the shape derivative:

$$\begin{aligned} \mathrm{D}J_S^N(E_0)[V] = &\int_{\partial E_0} \left(\mu \partial_\nu \boldsymbol{q}_0 \cdot \partial_\nu \boldsymbol{u}_0 - \mu |\partial_\nu \boldsymbol{u}_0|^2\right) V(0) \cdot \nu \,\mathrm{d}x + \\ &+ \int_{\partial E_0} \left(\frac{\mu}{2} |\partial_\nu \boldsymbol{u}_0|^2 - \boldsymbol{f} \cdot \boldsymbol{u}_0\right) V(0) \cdot \nu \,\mathrm{d}x + \gamma c_0 \int_{\partial E_0 \cap \Omega} \kappa V(0) \cdot \nu \,\mathrm{d}x. \end{aligned} \tag{16.11}$$

Comparing this result for instance with [SS10, Section 5], [Sim80, Remark 7.2] and [Pir74], where $\boldsymbol{f} \equiv \boldsymbol{0}$ was chosen, we see that we obtain the same result. Additionally, we can compare those results to Example 8.1, where the same objective functional was minimized in a Stokes flow. We remark that in contrast to Example 8.1 we cannot simplify formula (16.11) to obtain a formulation without adjoint state. This was only possible in a Stokes flow setting.

16.2 Geometric variations

The purpose of this subsection is to derive first order optimality conditions for the limit problem

$$\min_{(\varphi,\boldsymbol{u})\in L^1(\Omega)\times\boldsymbol{H}^1(\Omega)} J_0^N(\varphi,\boldsymbol{u}) \tag{16.12}$$

without stating the unverified assumptions (16.2) we had to make in Section 16.1. Instead, we want to follow the approach of Section 8.2. Thus we will again vary the minimizing set by a suitable transformation and differentiate in this direction. But in contrast to Section 16.1 we do not assume additional regularity on the minimizer. Consequently, we arrive in an optimality system, that can under certain assumptions on the convergence rate of the minimizers be verified to be the limit of optimality conditions for the phase field model as we will show in Section 17.

For this purpose, we fix for the rest of this subsection

$$E_0 := \{x\in\Omega \mid \varphi_0(x) = 1\}$$

where $(\varphi_0,\boldsymbol{u}_0)\in L^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ are still the fixed minimizer of J_0^N, and notice that E_0 is now in general only a Caccioppoli set. We recall, that due to Lemma 13.5 the state equations (13.4), corresponding to φ_0, inherit a unique solution, thus $\boldsymbol{S}_0^N(\varphi_0) = \{\boldsymbol{u}_0\}$ and it holds

$$\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}. \tag{16.13}$$

We define

$$\varphi_0(t) = \varphi_0\circ T_t^{-1},\quad \Omega_t = T_t(\Omega),\quad E_t = T_t(E_0) = E^{\varphi_0(t)}$$

for some given transformation $T\in\overline{\mathcal{T}}_{ad}$ and see that $\varphi_0(t)\in\overline{\Phi}_{ad}^0$, since due to Lemma 3.6 the function $\left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{u}_0\circ T_t^{-1}\in\boldsymbol{U}^{\varphi_0(t)}$ and so $\boldsymbol{U}^{\varphi_0(t)}\neq\emptyset$.

As in Section 16.1, we can a priori neither guarantee the existence of a solution to the state equations (13.4) corresponding to E_t, nor uniqueness, even though this holds true for E_0. And so we start with deducing the existence of a solution to the state equations corresponding to E_t if t is small enough from the minimizing property of E_0 in the next lemma.

Lemma 16.3. *There exists a small interval $I\subset\mathbb{R}$, $0\in I$, such that there exists some $\boldsymbol{u}_t\in\boldsymbol{S}_0^N\left(\varphi_0\circ T_t^{-1}\right)$ for all $t\in I$. Moreover, there exists a constant $C>0$ independent of $t\in I$ such that it holds*

$$\|\nabla\boldsymbol{u}_t\|_{\boldsymbol{L}^2(\Omega)} \le C. \tag{16.14}$$

Proof. We define $\boldsymbol{u}(t) := \left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\boldsymbol{u}_0\circ T_t^{-1}\in\boldsymbol{U}^{\varphi_0(t)}$ as in Lemma 13.5 and let $\boldsymbol{v}\in\boldsymbol{V}$ be arbitrary. Then we have, by following the arguments of [Gal11, Lemma IX.1.1], the estimate

$$b(\boldsymbol{v},\boldsymbol{u}(t),\boldsymbol{v}) = -b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{u}(t)) \le \|\boldsymbol{v}\|_{\boldsymbol{L}^{2d/(d-2)}(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^d(\Omega)}. \tag{16.15}$$

Using change of variables and $\|\mathrm{D}T_t\|_\infty = \sup_{x\in\overline{\Omega}} \|\mathrm{D}T_t(x)\|_\infty \le 1 + C|t|$ and $\|\det \mathrm{D}T_t\|_\infty \le 1 + C|t|$, which holds for $|t| \ll 1$, we find

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^d(\Omega)} \le (1 + C|t|)\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^d(\Omega)}. \tag{16.16}$$

Combining (16.15) and (16.16) we obtain by using again estimates as in [Gal11, Lemma IX.1.1] that

$$\begin{aligned} |b(\boldsymbol{v},\boldsymbol{u}(t),\boldsymbol{v})| &\le \|\boldsymbol{v}\|_{\boldsymbol{L}^{2d/(d-2)}(\Omega)}\,\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^d(\Omega)}\,(1+C|t|) \le \\ &\le K_\Omega\,\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\,\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}\,(1+C|t|) \le \frac{\mu}{2}\,\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\,(1+C|t|) \end{aligned} \tag{16.17}$$

where in the last step we made in particular use of $\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}$, compare Lemma 13.5. We hence can deduce from (16.17) the existence of some interval $0 \in I \subset \mathbb{R}$ and some constant $c > 0$ with $c < \mu$ such that

$$|b(\boldsymbol{v},\boldsymbol{u}(t),\boldsymbol{v})| \le c\,\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in I. \tag{16.18}$$

As by construction $\boldsymbol{u}(t) \in \boldsymbol{U}^{\varphi_0(t)}$ we obtain from (16.18) and Lemma 13.1 the existence of some $\boldsymbol{u}_t \in \boldsymbol{S}_0^N(\varphi_0(t))$ for all $t \in I$.

To deduce the uniform estimate (16.14) on $(\boldsymbol{u}_t)_{t\in I}$ we proceed as in the proof of Lemma 11.6 to find that $\boldsymbol{w}_t := \boldsymbol{u}_t - \boldsymbol{u}(t) \in \boldsymbol{V}^{\varphi_0(t)}$ fulfills

$$\mu\,\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} + b(\boldsymbol{w}_t,\boldsymbol{u}(t),\boldsymbol{w}_t) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}_t - \mu\nabla\boldsymbol{u}(t)\cdot\nabla\boldsymbol{w}_t\,\mathrm{d}x - b(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{w}_t)$$

and so

$$\begin{aligned} \mu\,\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} &\le |b(\boldsymbol{w}_t,\boldsymbol{u}(t),\boldsymbol{w}_t)| + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)} + \mu\,\|\nabla\boldsymbol{u}(t)\|_{\boldsymbol{L}^2(\Omega)}\,\|\nabla\boldsymbol{w}_t\|_{\boldsymbol{L}^2(\Omega)} + \\ &+ C\,\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^1(\Omega)}\,\|\boldsymbol{w}_t\|_{\boldsymbol{H}^1(\Omega)}. \end{aligned}$$

Applying (16.18) implies then

$$\|\nabla\boldsymbol{w}_t\|^2_{\boldsymbol{L}^2(\Omega)} \le C\left(\|\boldsymbol{u}(t)\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}(t)\|^4_{\boldsymbol{H}^1(\Omega)}\right). \tag{16.19}$$

Similar calculations as in (16.16) yield the existence of some $C > 0$ independent of $t \in I$ such that $\sup_{t\in I}\|\boldsymbol{u}(t)\|_{\boldsymbol{H}^1(\Omega)} \le C$. And thus (16.19) implies the uniform bound (16.14) and we can finish the proof. □

In the next lemma, we will show differentiability of $t \mapsto (\boldsymbol{u}_t \circ T_t)$ if $\boldsymbol{u}_t \in \boldsymbol{S}_0^N(\varphi_0(t))$ is a family of solutions to the state equations corresponding to the transformed state $\varphi_0(t)$. A priori, we only now existence of such a family of solutions by Lemma 16.3, but we do not know if this is unique, and hence it is not clear how to choose this family. But we will obtain implicitly by the arguments of the following proof that $\boldsymbol{S}_0^N(\varphi_0(t)) = \{\boldsymbol{u}_t\}$ for $|t| \ll 1$ and so this choice is well-defined. One could also directly show uniqueness of a solution of the state equations corresponding to $\varphi_0(t)$ for $|t| \ll 1$ by using similar arguments as in third step in the next proof, but here we deduce this fact as a consequence of the following considerations, see Corollary 16.1.

Lemma 16.4. *Let $\boldsymbol{u}_t \in \boldsymbol{S}_0^N(\varphi_0(t))$ be a familiy of solutions to the state equations corresponding to $\varphi_0(t)$, whose existence is guaranteed by Lemma 16.3 for $t \in I$, if $0 \in I \subset \mathbb{R}$ is a small interval.*
Then the function $I \ni t \mapsto (\boldsymbol{u}_t \circ T_t) \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$ and $\dot{\boldsymbol{u}}_0[V] := \partial_t|_{t=0}(\boldsymbol{u}_t \circ T_t) \in \boldsymbol{H}^1(\Omega)$ with $\dot{\boldsymbol{u}}_0[V]|_{\Omega \setminus E_0} = \boldsymbol{0}$ is given as the unique weak solution to

$$\begin{aligned}&\int_{E_0} \mu \nabla \dot{\boldsymbol{u}}_0[V] \cdot \nabla \boldsymbol{z}\,\mathrm{d}x + b(\boldsymbol{u}_0, \dot{\boldsymbol{u}}_0[V], \boldsymbol{z}) + b(\dot{\boldsymbol{u}}_0[V], \boldsymbol{u}_0, \boldsymbol{z}) = \int_{E_0} \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z}\,\mathrm{d}x + \\ &+ \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \mathrm{D}V(0)^T \nabla \boldsymbol{z}\,\mathrm{d}x + \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \nabla(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z})\,\mathrm{d}x - \\ &- \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z} \operatorname{div} V(0)\,\mathrm{d}x + b(\mathrm{D}V(0)\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{z}) - b(\boldsymbol{u}_0, \boldsymbol{u}_0, \mathrm{D}V(0)\boldsymbol{z}) + \\ &+ \int_{E_0} (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}\,\mathrm{d}x + \int_{E_0} \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x\end{aligned} \tag{16.20}$$

which has to hold for every $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$, together with

$$\operatorname{div} \dot{\boldsymbol{u}}_0[V] = \nabla \boldsymbol{u}_0 : \mathrm{D}V(0). \tag{16.21}$$

Proof. We want to use an implicit function argument similar to Theorem A.3 to show the statement. Again we cannot apply Theorem A.3, because of the nonlinear state equations, and a definition of a function as in Lemma 16.2 is not possible, since there may be infinitely many connected components of E_0. Even if E_0 would be connected, the mapping $\operatorname{div} : \boldsymbol{H}_0^1(E_0) \to L_0^2(E_0)$ could not shown to be surjective, because of the lack of regularity of ∂E_0. Instead, we use the idea of Lemma 8.5 and generalize therefore the proof of Theorem A.3 to this nonlinear setting here.
We start by defining the function

$$F : I \times \boldsymbol{V}^{E_0} \to \left(\boldsymbol{V}^{E_0}\right)'$$

by

$$\begin{aligned}F(t, \boldsymbol{u})(\boldsymbol{z}) = &\int_{E_0} \mu \nabla \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right)\mathrm{d}x - \\ &- \int_{E_0} \mu \nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right) \cdot \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t\,\mathrm{d}x + \\ &+ \int_{E_0} \det \mathrm{D}T_t^{-1} (\mathrm{D}T_t)\boldsymbol{u} \cdot \nabla \boldsymbol{u}\left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right)\mathrm{d}x - \\ &- \int_{E_0} \mathrm{D}T_t \boldsymbol{u} \cdot \nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right) \cdot \boldsymbol{u} \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right)\mathrm{d}x + \\ &+ \int_{E_0} \mathrm{D}T_t \boldsymbol{u} \cdot \mathrm{D}T_t^{-T} \nabla \boldsymbol{G} \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right)\mathrm{d}x + \\ &+ \int_{E_0} \boldsymbol{G} \cdot \nabla \boldsymbol{u} \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) - \boldsymbol{G} \cdot \nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right) \boldsymbol{u} \cdot (\mathrm{D}T_t \boldsymbol{z})\,\mathrm{d}x - \\ &- \int_{E_0} \boldsymbol{f} \circ T_t \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t\,\mathrm{d}x,\end{aligned}$$

where $\boldsymbol{G} \in \boldsymbol{U}^{E_0}$ is some fixed chosen function. Roughly speaking, this means that $F(t, \boldsymbol{u})$ describes the state equations on $T_t(E_0)$, but transformed back to the reference region E_0 and reduced to homogeneous boundary data be using the function $\boldsymbol{G}$. We will consider the state equations that are solved for the divergence-free transformation $(\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_t \circ T_t$ of $\boldsymbol{u}_t$ onto $T_t(E_0)$ and so there are some additional terms appearing in the definition of F that correspond to $(\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)$.

Additionally, let

$$f : I \to \left(\boldsymbol{V}^{E_0}\right)'$$

be defined as

$$\begin{aligned} f(t)(\boldsymbol{z}) = &- \int_{E_0} \mu \mathrm{D}T_t^{-T} \nabla \boldsymbol{G} : \mathrm{D}T_t^{-T} \nabla \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x - \\ &- \int_{E_0} \boldsymbol{G} \cdot \mathrm{D}T_t^{-T} \nabla \boldsymbol{G} \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x + \\ &+ \int_{E_0} \boldsymbol{f} \circ T_t \cdot \left(\det \mathrm{D}T_t^{-1} \mathrm{D}T_t \boldsymbol{z}\right) \cdot \det \mathrm{D}T_t \,\mathrm{d}x. \end{aligned}$$

Direct calculations give then for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$ the identity

$$\begin{aligned} &F\left(t, \det \mathrm{D}T_t (\mathrm{D}T_t^{-1}) \boldsymbol{u} \circ T_t - \boldsymbol{G}\right)(\boldsymbol{z}) + f(t)(\boldsymbol{z}) = \\ &= \int_{E_t} \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{z}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{z}_t - \boldsymbol{f} \cdot \boldsymbol{z}_t \,\mathrm{d}x, \end{aligned} \tag{16.22}$$

where we used $\boldsymbol{z}_t := (\det \mathrm{D}T_t)\, \mathrm{D}T_t^{-1} \boldsymbol{z} \circ T_t^{-1} \in \boldsymbol{V}^{E_t}$, see Lemma 3.6. And so in particular, this yields

$$F\left(t, \det \mathrm{D}T_t (\mathrm{D}T_t^{-1}) \boldsymbol{u}_t \circ T_t - \boldsymbol{G}\right) = f(t) \quad \forall t \in I. \tag{16.23}$$

We observe that the differentiability of $t \mapsto F(t, \boldsymbol{u})$ for all $\boldsymbol{u} \in \boldsymbol{V}^{E_0}$ in a small interval around $t = 0$ can be deduced directly by the regularity of the transformation $T \in \overline{\mathcal{T}}_{ad}$. Moreover, we get for arbitrary $\boldsymbol{u} \in \boldsymbol{V}^{E_0}$ and $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$:

$$\mathrm{D}_u F(0, \boldsymbol{u}_0 - \boldsymbol{G})(\boldsymbol{u})\, \boldsymbol{z} = \int_{E_0} \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{z} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{z} \,\mathrm{d}x. \tag{16.24}$$

Now we divide the proof into several steps:

- *1st step:* We first show that there exists some $c > 0$ such that

  $$\left\| F(0, \boldsymbol{v} - \boldsymbol{G}) - F(0, \boldsymbol{u}_0 - \boldsymbol{G}) \right\|_{(\boldsymbol{V}^{E_0})'} \geq c \left\| \boldsymbol{v} - \boldsymbol{u}_0 \right\|_{\boldsymbol{H}^1(\Omega)} \tag{16.25}$$

  which has to hold for all $\boldsymbol{v} \in \boldsymbol{U}^{E_0}$.
  To this end, we notice first that we have

  $$\begin{aligned} (F(0, \boldsymbol{v} - \boldsymbol{G}) - F(0, \boldsymbol{u}_0 - \boldsymbol{G}))\, \boldsymbol{z} = \int_{E_0} &\mu (\nabla \boldsymbol{v} - \nabla \boldsymbol{u}_0) \cdot \nabla \boldsymbol{z} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{z} - \\ &- \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 \cdot \boldsymbol{z} \,\mathrm{d}x \end{aligned} \tag{16.26}$$

  for all $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$. Using

  $$\begin{aligned} &b(\boldsymbol{v} - \boldsymbol{u}_0, \boldsymbol{v} - \boldsymbol{u}_0, \boldsymbol{z}) + b(\boldsymbol{v} - \boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{z}) + b(\boldsymbol{u}_0, \boldsymbol{v} - \boldsymbol{u}_0, \boldsymbol{z}) = \\ &= b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{z}) - b(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{z}) \end{aligned} \tag{16.27}$$

we obtain from (16.26)

$$
\begin{aligned}
&\|F(0,\boldsymbol{v}-\boldsymbol{G})-F_1(0,\boldsymbol{u}_0-\boldsymbol{G})\|_{(\boldsymbol{V}^{E_0})'} = \\
&= \sup_{\boldsymbol{0}\neq\boldsymbol{z}\in\boldsymbol{V}^{E_0}} \frac{\left|\int_\Omega \mu\nabla\left(\boldsymbol{v}-\boldsymbol{u}_0\right)\cdot\nabla\boldsymbol{z}\,\mathrm{d}x+b\left(\boldsymbol{v},\boldsymbol{v},\boldsymbol{z}\right)-b\left(\boldsymbol{u}_0,\boldsymbol{u}_0,\boldsymbol{z}\right)\right|}{\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)}} \geq \\
&\geq \frac{\left|\int_\Omega \mu\left|\nabla\left(\boldsymbol{v}-\boldsymbol{u}_0\right)\right|^2\,\mathrm{d}x+b\left(\boldsymbol{v}-\boldsymbol{u}_0,\boldsymbol{u}_0,\boldsymbol{v}-\boldsymbol{u}_0\right)\right|}{\|\boldsymbol{v}-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}} \geq \\
&\geq \frac{\left|\int_\Omega \mu\left|\nabla\left(\boldsymbol{v}-\boldsymbol{u}_0\right)\right|^2\,\mathrm{d}x\right|-\left|b\left(\boldsymbol{v}-\boldsymbol{u}_0,\boldsymbol{u}_0,\boldsymbol{v}-\boldsymbol{u}_0\right)\right|}{\|\boldsymbol{v}-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}} \geq \\
&\geq \frac{\mu\left\|\nabla\left(\boldsymbol{v}-\boldsymbol{u}_0\right)\right\|^2_{\boldsymbol{L}^2(\Omega)}-K_\Omega\left\|\nabla\boldsymbol{u}_0\right\|_{\boldsymbol{L}^2(\Omega)}\left\|\nabla\left(\boldsymbol{v}-\boldsymbol{u}_0\right)\right\|^2_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{v}-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}}.
\end{aligned}
$$

As $\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{\mu}{2K_\Omega}$, see (16.13), this implies the existence of a constant $c>0$ such that

$$
\|F(0,\boldsymbol{v}-\boldsymbol{G})-F(0,\boldsymbol{u}_0-\boldsymbol{G})\|_{(\boldsymbol{V}^{E_0})'} \geq c\frac{\left\|\nabla\left(\boldsymbol{v}-\boldsymbol{u}_0\right)\right\|^2_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{v}-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}} \geq c\|\boldsymbol{v}-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)}
$$

where we applied in the last step Poincaré's inequality. This proves (16.25).

- *2nd step:* Now we want to derive a similar estimate as in the first step for the derivative of F. More precisely we want to show that there exists some $C>0$ such that

$$
\|\mathrm{D}_u F\left(0,\boldsymbol{u}_0-\boldsymbol{G}\right)\boldsymbol{u}\|_{(\boldsymbol{V}^{E_0})'} \geq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \quad \forall\boldsymbol{u}\in\boldsymbol{V}^{E_0}. \tag{16.28}
$$

Therefore, we use the form of the derivative $\mathrm{D}_u F$ given by (16.24) and $\|\nabla\boldsymbol{u}_0\|_{\boldsymbol{L}^1(\Omega)} \leq \frac{\mu}{2K_\Omega}$, which follows from (16.13), and obtain similar to the first step

$$
\begin{aligned}
&\|\mathrm{D}_u F\left(0,\boldsymbol{u}_0-\boldsymbol{G}\right)\boldsymbol{u}\|_{(\boldsymbol{V}^{E_0})'} = \sup_{\boldsymbol{0}\neq\boldsymbol{z}\in\boldsymbol{V}^{E_0}} \frac{\left|\int_{E_0}\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{z}+\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{z}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}_0\cdot\boldsymbol{z}\,\mathrm{d}x\right|}{\|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)}} \geq \\
&\geq \frac{\left|\int_{E_0}\mu\left|\nabla\boldsymbol{u}\right|^2+\boldsymbol{u}_0\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}_0\cdot\boldsymbol{u}\,\mathrm{d}x\right|}{\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}} \geq \\
&\geq \frac{\mu\left\|\nabla\boldsymbol{u}\right\|^2_{\boldsymbol{L}^2(\Omega)}-K_\Omega\left\|\nabla\boldsymbol{u}\right\|^2_{\boldsymbol{L}^2(\Omega)}\left\|\nabla\boldsymbol{u}_0\right\|_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}} \geq c\frac{\left\|\nabla\boldsymbol{u}\right\|^2_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}} \geq c\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.
\end{aligned}
$$

For the following considerations we will use the notation

$$
m(t) := \left(\det \mathrm{D}T_t\right)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_t\circ T_t \quad \forall|t|\ll 1.
$$

- *3rd step:* Next we want to prove Lipschitz continuity of the mapping $I\ni t\mapsto m(t)\in \boldsymbol{H}^1(\Omega)$ if the interval I is chosen small enough.

We observe that the differentiability of F and f together with the quadratic form of F imply

$$\|f(t) - f(0)\|_{(\boldsymbol{V}^{E_0})'} \le C|t| \quad \forall |t| \ll 1 \tag{16.29}$$

and

$$\begin{aligned} &\|F(t)(\boldsymbol{v} - \boldsymbol{G}) - F(0)(\boldsymbol{v} - \boldsymbol{G})\|_{(\boldsymbol{V}^{E_0})'} \le \\ &\le C|t| \left(\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} \right) \quad \forall |t| \ll 1 \end{aligned} \tag{16.30}$$

which holds for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v}|_{\Omega \setminus E_0} = \boldsymbol{0}$ and $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}$. Moreover, it follows directly from (16.23) that

$$\begin{aligned} F(0, m(t) - \boldsymbol{G}) = F(0, m(t) - \boldsymbol{G}) - F(t, m(t) - \boldsymbol{G}) + \\ + (f(t) - f(0)) + F(0, \boldsymbol{u}_0). \end{aligned} \tag{16.31}$$

Applying (16.25) to this identity yields

$$\begin{aligned} c\|m(t) - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} &\le \|F(0, m(t) - \boldsymbol{G}) - F(0, \boldsymbol{u}_0 - \boldsymbol{G})\|_{(\boldsymbol{V}^{E_0})'} = \\ &= \|F(0, m(t) - \boldsymbol{G}) - F(t, m(t) - \boldsymbol{G}) + f(t) - f(0)\|_{(\boldsymbol{V}^{E_0})'} \le \\ &\le C|t| \left(\|m(t)\|_{\boldsymbol{H}^1(\Omega)} + \|m(t)\|^2_{\boldsymbol{H}^1(\Omega)} + 1 \right) \end{aligned} \tag{16.32}$$

where we made in particular use of (16.29) and (16.30). By using Lemma 16.3 we can deduce that $\|\boldsymbol{u}_t \circ T_t\|_{\boldsymbol{H}^1(\Omega)}$ is bounded uniformly in t for $|t| \ll 1$ and so we can deduce from (16.32) the existence of some $L > 0$ such that it holds for $|t| \ll 1$ small enough

$$\|m(t) - m(0)\|_{\boldsymbol{H}^1(\Omega)} = \left\| (\det \mathrm{D}T_t) \left(\mathrm{D}T_t^{-1} \right) \boldsymbol{u}_t \circ T_t - \boldsymbol{u}_0 \right\|_{\boldsymbol{H}^1(\Omega)} \le L|t|. \tag{16.33}$$

- *4th step:* In this step we want to show the weak differentiability of $I \ni t \mapsto m(t) \in \boldsymbol{H}^1(\Omega)$ at $t = 0$. For this purpose, we start by deducing from (16.33) that

$$\frac{1}{|t|} \|m(t) - m(0)\|_{\boldsymbol{H}^1(\Omega)} \le L \quad \forall |t| \ll 1.$$

And so there exists a subsequence $(t_k)_{k\in\mathbb{N}}$ and some element $\widetilde{m} \in \boldsymbol{V}^{E_0}$ such that $\left(\frac{1}{t_k} (m(t_k) - m(0)) \right)_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\widetilde{m}$. Using the differentiability assumptions on the transformation $T_t \in \overline{\mathcal{T}}_{ad}$, this implies additionally, that $\left(\frac{1}{t_k} (\boldsymbol{u}_{t_k} \circ T_{t_k} - \boldsymbol{u}_0) \right)_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}^1_0(\Omega)$ where $\widetilde{\boldsymbol{u}}|_{\Omega \setminus E_0} = \boldsymbol{0}$.
As $F(0, \cdot) : \boldsymbol{V}^{E_0} \to \left(\boldsymbol{V}^{E_0} \right)'$ is Fréchet differentiable we find that there exists some r_F such that it holds for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{H}^1_0(\Omega)$

$$\lim_{\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{H}^1(\Omega)} \to 0} \frac{\|r_F(\boldsymbol{v}_1)\|_{(\boldsymbol{V}^{E_0})'}}{\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{H}^1(\Omega)}} = 0 \tag{16.34}$$

and

$$\mathrm{D}_u F(0,\boldsymbol{v}_2)(\boldsymbol{v}_1-\boldsymbol{v}_2) = F(0,\boldsymbol{v}_1) - F(0,\boldsymbol{v}_2) + r_F(\boldsymbol{v}_1).$$

From this, we find that

$$\begin{aligned}
&\mathrm{D}_u F(0,\boldsymbol{u}_0-\boldsymbol{G})(m(t_k)-m(0)) = F(0,m(t_k)-\boldsymbol{G}) - F(0,m(0)-\boldsymbol{G}) + \\
&+ r_F(m(t_k)-\boldsymbol{G}) + (-F(t_k,m(t_k)-\boldsymbol{G}) + f(t_k)) + \\
&+ (F(t_k,m(0)-\boldsymbol{G}) - F(t_k,m(0)-\boldsymbol{G})) + (F(0,m(0)-\boldsymbol{G}) - f(0)) + \\
&+ \left(f'(0)t_k - f'(0)t\right) + (\mathrm{D}_t F(0,m(0)-\boldsymbol{G})\,t_k - \mathrm{D}_t F(0,m(0)-\boldsymbol{G})\,t_k) = \\
&= (F(t_k,m(0)-\boldsymbol{G}) - F(t_k,m(t_k)-\boldsymbol{G})) - \\
&- (F(0,m(0)-\boldsymbol{G}) - F(0,m(t_k)-\boldsymbol{G})) + \\
&+ (F(0,m(0)-\boldsymbol{G}) - F(t_k,m(0)-\boldsymbol{G}) + \mathrm{D}_t F(0,m(0)-\boldsymbol{G})\,t_k) + \\
&+ \left(f(t_k) - f(0) - f'(0)t_k\right) + f'(0)t_k - \mathrm{D}_t F(0,\boldsymbol{u}_0-\boldsymbol{G})\,t_k + r_F(m(t_k)-\boldsymbol{G}).
\end{aligned} \tag{16.35}$$

Using (16.22) and (16.27) while making in particular use of the quadratic form of F we can establish similar to (16.30)

$$\begin{aligned}
&\|(F(0,m(t_k)-\boldsymbol{G}) - F(0,m(0)-\boldsymbol{G})) - \\
&- (F(t_k,m(t_k)-\boldsymbol{G}) - F(t_k,m(0)-\boldsymbol{G}))\|_{(\boldsymbol{V}^{E_0})'} \le \\
&\le C|t_k|\left(\|m(t_k)-m(0)\|_{\boldsymbol{H}^1(\Omega)} + \|m(t_k)-m(0)\|^2_{\boldsymbol{H}^1(\Omega)}\right) \le C|t_k|^2 \quad \forall k \gg 1.
\end{aligned}$$

where the last inequality follows from the Lipschitz continuity (16.33). This leads to

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{|t_k|} \|(F(0,m(t_k)-\boldsymbol{G}) - F(0,m(0)-\boldsymbol{G})) - \\
&- (F(t_k,m(t_k)-\boldsymbol{G}) - F(t_k,m(0)-\boldsymbol{G}))\|_{(\boldsymbol{V}^{E_0})'} = 0.
\end{aligned} \tag{16.36}$$

Since $F(\cdot,\boldsymbol{u}_0-\boldsymbol{G}) : I \to \left(\boldsymbol{V}^{E_0}\right)'$ is Fréchet differentiable at $t=0$ we find moreover

$$\|F(0,\boldsymbol{u}_0-\boldsymbol{G}) - F(t_k,\boldsymbol{u}_0-\boldsymbol{G}) + \mathrm{D}_t F(0,\boldsymbol{u}_0-\boldsymbol{G})\,t_k\|_{(\boldsymbol{V}^{E_0})'} = \mathrm{o}(|t_k|)$$

and hence

$$\lim_{k\to\infty} \left\| \frac{1}{t_k} (F(0,\boldsymbol{u}_0-\boldsymbol{G}) - F(t_k,\boldsymbol{u}_0-\boldsymbol{G}) + \mathrm{D}_t F(0,\boldsymbol{u}_0-\boldsymbol{G})\,t_k) \right\|_{(\boldsymbol{V}^{E_0})'} = 0. \tag{16.37}$$

Similarly, we derive from the Fréchet differentiability of f at $t=0$ that it holds

$$\lim_{k\to\infty} \left\| \frac{1}{t_k} \left(f(t_k) - f(0) - f'(0)t_k\right) \right\|_{(\boldsymbol{V}^{E_0})'} = 0. \tag{16.38}$$

Now we combine (16.33), (16.34) with the estimates (16.36), (16.37) and (16.38) to deduce from (16.35) that the weak limit $\widetilde{m}$ of $\left(\frac{1}{t_k}(m(t_k)-m(0))\right)_{k\in\mathbb{N}}$ fulfills

$$\mathrm{D}_u F(0,\boldsymbol{u}_0-\boldsymbol{G})\,\widetilde{m} = f'(0) - \mathrm{D}_t F(0,\boldsymbol{u}_0-\boldsymbol{G}). \tag{16.39}$$

Direct calculations imply hence that $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}_0^1(\Omega)$ with $\widetilde{\boldsymbol{u}}|_{\Omega \setminus E_0} = \mathbf{0}$ solves (16.20)-(16.21) and hereby we guarantee in particular solvability of (16.20)-(16.21).

In view of the result from the second step in this proof, we find that there exists at most one solution to (16.20)-(16.21), and hence $\widetilde{\boldsymbol{u}}$ is the unique solution of (16.20)-(16.21) as stated in the claim of this lemma.
By carrying out the same arguments for any subsequence $(t_k)_{k\in\mathbb{N}}$ we can conclude that $\left(\frac{1}{t}\left(m(t) - m(0)\right)\right)_t$ itself converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\widetilde{m}$.

- *5th step:* We now want to conclude the differentiability of $I \ni t \mapsto \boldsymbol{u}_t \circ T_t \in \boldsymbol{H}^1(\Omega)$ at $t = 0$, which is equivalent to the differentiability of $I \ni t \mapsto m(t) \in \boldsymbol{H}^1(\Omega)$ at $t = 0$. Therefore, we have to show the strong convergence

$$\lim_{t\to 0} \left\| \frac{1}{t}\left(m(t) - m(0)\right) - \widetilde{m} \right\|_{\boldsymbol{H}^1(\Omega)} = 0. \tag{16.40}$$

For this purpose, we start by applying estimate (16.28), which was established in the second step of this proof, and see

$$\begin{aligned} &\|m(t) - m(0) - t\tilde{m}\|_{\boldsymbol{H}^1(\Omega)} \le \\ &\le C\left\|\mathrm{D}_u F\left(0, \boldsymbol{u}_0 - \boldsymbol{G}\right)\left(m(t) - m(0) - t\tilde{m}\right)\right\|_{\left(\boldsymbol{V}^{E_0}\right)'} = \\ &= C\left\|\mathrm{D}_u F\left(0, \boldsymbol{u}_0 - \boldsymbol{G}\right)\left(m(t) - m(0)\right) - t\left(f'(0) - \mathrm{D}_t F\left(0, \boldsymbol{u}_0 - \boldsymbol{G}\right)\right)\right\|_{\left(\boldsymbol{V}^{E_0}\right)'} \end{aligned} \tag{16.41}$$

where we made in the last step use of (16.39). The considerations of the fourth step of this proof give us

$$\left\|\mathrm{D}_u F\left(0, \boldsymbol{u}_0 - \boldsymbol{G}\right)\left(m(t) - m(0)\right) - t\left(f'(0) - \mathrm{D}_t F\left(0, \boldsymbol{u}_0 - \boldsymbol{G}\right)\right)\right\|_{\left(\boldsymbol{V}^{E_0}\right)'} = \mathrm{o}(|t|)$$

and hence we find from (16.41) directly (16.40). This finally proves the statement of the lemma.

□

From the previous lemma we obtain directly the following result concerning uniqueness of the state equations:

Corollary 16.1. *There exists a small interval $I \subset \mathbb{R}$, $0 \in I$, such that $\boldsymbol{S}_0^N\left(\varphi_0 \circ T_t^{-1}\right) = \{\boldsymbol{u}_t\}$ for all $t \in I$, thus there exists a unique solution to the state equations* (13.4) *corresponding to small deformations $\varphi_0 \circ T_t^{-1}$, $|t| \ll 1$, of the minimizer φ_0.*

Proof. By Lemma 16.3 we have for every $t \in I$, if $I \subset \mathbb{R}$ is chosen small enough, a solution $\boldsymbol{u}_t \in \boldsymbol{S}_0^N\left(\varphi_0 \circ T_t^{-1}\right)$ for the state equations (13.4) corresponding to $\varphi_0 \circ T_t^{-1}$. Lemma 16.4 guarantees additionally that $t \mapsto (\boldsymbol{u}_t \circ T_t) \in \boldsymbol{H}^1(\Omega)$ is continuous. Hence there exists some $t' > 0$ such that

$$\left\|\nabla\left(\boldsymbol{u}_t \circ T_t\right) - \nabla \boldsymbol{u}_0\right\|_{\boldsymbol{H}^1(\Omega)} \le \frac{\mu}{4K_\Omega} \quad \forall |t| \le t'$$

which implies

$$\left\|\nabla\left(\boldsymbol{u}_t \circ T_t\right)\right\|_{\boldsymbol{H}^1(\Omega)} \le \frac{\mu}{4K_\Omega} + \left\|\nabla \boldsymbol{u}_0\right\|_{\boldsymbol{L}^2(\Omega)} \overset{(16.13)}{\le} \frac{3\mu}{4K_\Omega} \quad \forall |t| \le t'.$$

Using as in the proof of Lemma 16.3 that $\|\mathrm{D}T_t\|_\infty \le 1 + C|t|$ and $\|\det \mathrm{D}T_t\|_\infty \le 1 + C|t|$ for $|t| \ll 1$ we can deduce therefrom the existence of some $c > 0$ such that $c < \frac{\mu}{K_\Omega}$ and

$$\|\nabla \boldsymbol{u}_t\|_{\boldsymbol{H}^1(\Omega)} \le c < \frac{\mu}{K_\Omega} \quad \forall |t| \ll 1.$$

Now the statement follows from Lemma 13.2. □

We thus have proved that local deformations $\varphi_0(t) = \varphi_0 \circ T_t^{-1}$ along suitable transformations $T \in \overline{\mathcal{T}}_{ad}$ of the minimizer φ_0 still inherit a unique solution of the state equations, thus $\boldsymbol{S}_0^N(\varphi_0(t)) = \{\boldsymbol{u}_t\}$. Moreover, we know that $t \mapsto \boldsymbol{u}_t \circ T_t$ is differentiable at $t = 0$ as a mapping into $\boldsymbol{H}^1(\Omega)$ and have derived a system that defines the derivative $\partial_t|_{t=0}(\boldsymbol{u}_t \circ T_t)$. And so we can for the remainder of this section focus on the calculation of first order optimality conditions for J_0^N at $(\varphi_0, \boldsymbol{u}_0)$.

Due to $\varphi_0(t) \in \Phi_{ad}^0$ for $T \in \mathcal{T}_{ad}$, see Remark 3.4, and $\boldsymbol{u}_t \in \boldsymbol{S}_0^N(\varphi_0(t))$, we have by $(\varphi_0(t), \boldsymbol{u}_t)$ admissible comparison functions for J_0^N and get for all $T \in \mathcal{T}_{ad}$:

$$J_0^N(\varphi_0, \boldsymbol{u}_0) \le J_0^N(\varphi_0(t), \boldsymbol{u}_t) \qquad \forall |t| \ll 1. \tag{16.42}$$

Since Corollary 16.1 implies $\boldsymbol{S}_0^N(\varphi_0(t)) = \{\boldsymbol{u}_t\}$ for t small enough we can define

$$j_0^N(\varphi_0(t)) := J_0^N(\varphi_0(t), \boldsymbol{u}_t).$$

Hence, (16.42) implies

$$j_0^N(\varphi_0) \le j_0^N(\varphi_0 \circ T_t^{-1}) \qquad \forall |t| \ll 1, T \in \mathcal{T}_{ad}$$

and therefrom we obtain

$$\partial_t|_{t=0} j_0^N\left(\varphi_0 \circ T_t^{-1}\right) = 0 \quad \forall T \in \mathcal{T}_{ad}.$$

By following the arguments of Lemma 7.5 we find the existence of a Lagrange multiplier $\lambda_0 \in \mathbb{R}^+$ for the integral constraint. We can now state the main result of this section:

Theorem 16.2. *For any minimizer $\varphi_0 \in \Phi_{ad}^0$ with $\{\boldsymbol{u}_0\} = \boldsymbol{S}_0^N(\varphi_0)$ of (16.12) we have the following necessary optimality condition:*

$$\partial_t|_{t=0} j_0^N\left(\varphi_0 \circ T_t^{-1}\right) = -\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x, \tag{16.43}$$

$$\lambda_0 \left(\int_\Omega \varphi_0 \,\mathrm{d}x - \beta\,|\Omega|\right) = 0 \tag{16.44}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ with velocity $V \in \overline{\mathcal{V}}_{ad}$. Here $\lambda_0 \ge 0$ is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:

$$\begin{aligned}\partial_t|_{t=0} j_0^N\left(\varphi_0 \circ T_t^{-1}\right) = &\int_\Omega \left[\mathrm{D}f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right)\left(V(0), \dot{\boldsymbol{u}}_0\left[V\right], \mathrm{D}\dot{\boldsymbol{u}}_0\left[V\right] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)\right) + \right.\\ &+ \left. f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \operatorname{div} V(0)\right] \mathrm{d}x +\\ &+ \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right) \mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|\end{aligned} \tag{16.45}$$

with ν being the generalised unit normal on $E_0 = \{\varphi_0 = 1\}$. Moreover $\dot{\boldsymbol{u}}_0\left[V\right] \in \boldsymbol{H}_0^1(\Omega)$ with $\dot{\boldsymbol{u}}_0\left[V\right] = \mathbf{0}$ a.e. in $\Omega \setminus E_0$ is given as solution of (16.20) – (16.21).

Proof. This follows by using the same calculations as done in Lemma 15.5. □

16.3 Linking the optimality criteria

This section was so far concerned with the study of optimality conditions for the optimization problem (13.16) – (13.17), which is equivalent to minimizing J_0^N. For this purpose, we have devoted Section 16.1 to the classical shape calculus and have obtained first order optimality conditions that can be stated in the classical Hadamard form and coincide with well-known results from literature. However, additional unverified regularity on the minimizing set had to be assumed. And so we started in Section 16.2 discussing optimality conditions arising from geometric variations that can be verified in our general setting. We thus have established optimality conditions without additional assumptions.

The natural question arises whether it is possible to prove equivalence of both systems under certain assumptions. We can answer this positively in the following sense:

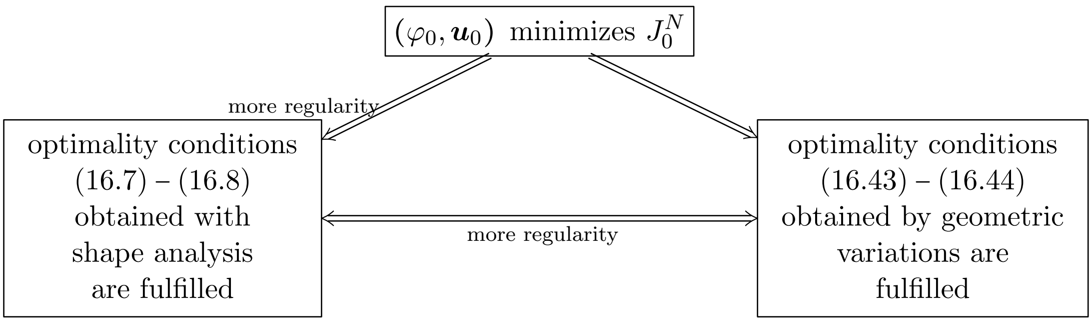
Lemma 16.5. *Let $T \in \overline{\mathcal{T}}_{ad}$ be chosen and $V \in \overline{\mathcal{V}}_{ad}$ be its velocity field.*
Assume that $E_0 := \operatorname{int}(\{x \in \Omega \mid \varphi_0(x) = 1\})$ is a well-defined open subset of Ω and that the regularity assumptions (16.2) *are fulfilled.*
Then we have

$$\partial_t|_{t=0} j_0^N \left(\varphi_0 \circ T_t^{-1}\right) = \mathrm{D}J_S^N(E_0)[V] \tag{16.46}$$

where $\partial_t|_{t=0} j_0^N \left(\varphi_0 \circ T_t^{-1}\right)$ is given by (16.45) *and $\mathrm{D}J_S^N(E_0)[V]$ by* (16.9).
This means that the optimality conditions of Theorem 16.1 and of Theorem 16.2 are equivalent.

Proof. We proceed as in the proof of Lemma 8.6, while using calculations as in Lemma 15.6. □

We summarize our results again in the following diagram, which we have shown to be true:



17 Convergence of the optimality system

Here we want to show the equivalent statement derived in Section 9 while having the stationary Navier-Stokes equations instead of the Stokes equations as a state constraint. For this purpose we assume differentiability of the body force and the objective functional given by Assumptions **(A6)** and **(A7)** for the remainder of this section.
We have already seen in Theorem 14.1 that a subsequence of any sequence of minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$, denoted by $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$, converges in $L^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ to a minimizer $(\varphi_0,\boldsymbol{u}_0)$ of J_0^N if $(\varphi_\varepsilon)_{\varepsilon>0}$ fulfills the convergence rate (14.1). Now we want to investigate what happens to the optimality systems corresponding to $\left(J_\varepsilon^N\right)_{\varepsilon>0}$ as $\varepsilon\searrow 0$. And in fact we will see that, if stated in the right manner, the optimality systems converge to an optimality system of J_0^N as $\varepsilon\searrow 0$.

Theorem 17.1. *Let $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ be minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$. Then there exists a subsequence, which is denoted by the same, such that $(\varphi_\varepsilon)_{\varepsilon>0}$ converges in $L^1(\Omega)$ to some limit element φ_0 and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\boldsymbol{u}_0$. If it holds*

$$\|\varphi_\varepsilon-\varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})}=\mathcal{O}(\varepsilon) \tag{17.1}$$

then $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ and we obtain that $(\varphi_0,\boldsymbol{u}_0)$ are a minimizer of J_0^N. Moreover, it holds then

$$\lim_{\varepsilon\searrow 0}\partial_t|_{t=0}j_\varepsilon^N\left(\varphi_\varepsilon\circ T_t^{-1}\right)=\partial_t|_{t=0}j_0^N\left(\varphi_0\circ T_t^{-1}\right)\quad\forall T\in\overline{\mathcal{T}}_{ad}. \tag{17.2}$$

If additionally

$$|\{\varphi_0=1\}|>0 \tag{17.3}$$

then we have the following convergence results:

$$\begin{aligned}
\varphi_\varepsilon &\xrightarrow{\varepsilon\searrow 0}\varphi_0 && \text{in } L^1(\Omega), && (17.4a)\\
\boldsymbol{u}_\varepsilon &\xrightarrow{\varepsilon\searrow 0}\boldsymbol{u}_0 && \text{in } \boldsymbol{H}^1(\Omega), && (17.4b)\\
\dot{\boldsymbol{u}}_\varepsilon[V] &\xrightarrow{\varepsilon\searrow 0}\dot{\boldsymbol{u}}_0[V] && \text{in } \boldsymbol{H}^1(\Omega), && (17.4c)\\
\lambda_\varepsilon &\xrightarrow{\varepsilon\searrow 0}\lambda_0 && \text{in } \mathbb{R}, && (17.4d)\\
J_\varepsilon^N(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon) &\xrightarrow{\varepsilon\searrow 0}J_0^N(\varphi_0,\boldsymbol{u}_0) && \text{in } \mathbb{R}, && (17.4e)
\end{aligned}$$

where $\{\boldsymbol{u}_\varepsilon\}=\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon)$ for ε small enough and $\{\boldsymbol{u}_0\}=\boldsymbol{S}_0^N(\varphi_0)$. Moreover $(\lambda_\varepsilon)_{\varepsilon>0}\subseteq\mathbb{R}^+$ are Lagrange multipliers for the integral constraint defined due to Lemma 15.5, and $\lambda_0\in\mathbb{R}^+$ is a Lagrange multiplier such that it holds (16.43)–(16.44)*, and thus is a Lagrange multiplier for the integral constraint in the sharp interface according to Theorem 16.2.*

Remark 17.1. *We remark that* (17.2) *and* (17.4a)-(17.4c), (17.4e) *would be true for any $T\in\overline{\mathcal{T}}_{ad}$ even if* (17.3) *would not be fulfilled. Only for the convergence of the Lagrange multipliers $(\lambda_\varepsilon)_{\varepsilon>0}$ we need* (17.3). *But as already discussed in Remark 9.1, condition* (17.3) *is not very restrictive.*

Proof. Convergence of a subsequence of a sequence of minimizers of $\left(J_\varepsilon^N\right)_{\varepsilon>0}$, which will be denoted by $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$, to some limit element $(\varphi_0, \boldsymbol{u}_0)$ follows from Theorem 14.1. From now on we assume (17.1). Then we can again apply Theorem 14.1 to obtain the strong convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ and the fact that the limit elements are minimizer of J_0^N together with (17.4e). Besides, we obtain from Corollary 14.1 for ε small enough $\boldsymbol{S}_\varepsilon^N(\varphi_\varepsilon) = \{\boldsymbol{u}_\varepsilon\}$ and Lemma 13.5 gives $\boldsymbol{S}_0^N(\varphi_0) = \{\boldsymbol{u}_0\}$. The proof of Theorem 14.1 gives more, namely by (14.16)

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = 0.$$

Using the convergence rate (17.1), we can establish as in the second step of Lemma 6.3 that

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = 0 \quad \forall \boldsymbol{v}\in\boldsymbol{H}^1(\Omega), \boldsymbol{v}|_{\Omega\setminus E^{\varphi_0}} = \boldsymbol{0}. \tag{17.5}$$

The remainder of this proof follows closely the arguments of Theorem 9.1 and we only point out the main steps here and focus on the differences that occur in the stationary Navier-Stokes setting.

Thus we define the functions $(FP_\varepsilon)_{\varepsilon>0}$ by

$$\begin{aligned} FP_\varepsilon(\boldsymbol{v}) := \int_\Omega\left(\frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 + \boldsymbol{u}_\varepsilon\cdot\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\cdot\boldsymbol{v} + \dot{\boldsymbol{u}}_\varepsilon[V]\cdot\nabla\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\right)\mathrm{d}x - R_\varepsilon(\boldsymbol{v}) + \\ + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot \mathrm{D}V(0)\boldsymbol{v}\,\mathrm{d}x - D_\varepsilon(\boldsymbol{w}_\varepsilon)(\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\boldsymbol{V}\end{aligned}$$

where $R_\varepsilon\in\boldsymbol{H}^{-1}(\Omega)$ and $D_\varepsilon(\boldsymbol{w}_\varepsilon)\in\boldsymbol{H}^{-1}(\Omega)$ are defined by

$$\begin{aligned} R_\varepsilon(\boldsymbol{z}) := &\int_\Omega \mu \mathrm{D}V(0)^T\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z}\,\mathrm{d}x + \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T\nabla\boldsymbol{z}\,\mathrm{d}x + \\ &+ \mu\int_\Omega \nabla\boldsymbol{u}_\varepsilon : \nabla\left(\operatorname{div}V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right)\mathrm{d}x - \\ &- \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z}\operatorname{div}V(0)\,\mathrm{d}x - b\left(\mathrm{D}V(0)\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right) - \\ &- b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon, \mathrm{D}V(0)\boldsymbol{z}\right) + \int_\Omega\left(\nabla\boldsymbol{f}\cdot V(0)\right)\cdot\boldsymbol{z}\,\mathrm{d}x + \int_\Omega \boldsymbol{f}\cdot\mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x\end{aligned}$$

and

$$D_\varepsilon(\boldsymbol{w}_\varepsilon)(\boldsymbol{z}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{w}_\varepsilon\cdot\boldsymbol{z} + \mu\nabla\boldsymbol{w}_\varepsilon\cdot\nabla\boldsymbol{z}\,\mathrm{d}x + b\left(\boldsymbol{u}_\varepsilon, \boldsymbol{w}_\varepsilon, \boldsymbol{z}\right) + b\left(\boldsymbol{w}_\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{z}\right)$$

with

$$\boldsymbol{w}_\varepsilon := \left(-\operatorname{div}V(0) + \mathrm{D}V(0)\right)\boldsymbol{u}_\varepsilon.$$

Furthermore, let

$$\begin{aligned} FP_0(\boldsymbol{v}) := \int_\Omega\left(\frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 + \boldsymbol{u}_0\cdot\nabla\dot{\boldsymbol{u}}_0[V]\cdot\boldsymbol{v} + \dot{\boldsymbol{u}}_0[V]\cdot\nabla\boldsymbol{u}_0\cdot\boldsymbol{v}\right)\mathrm{d}x - \\ - R_0(\boldsymbol{v}) - D_0(\boldsymbol{w}_0)(\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\boldsymbol{V}\end{aligned}$$

where

$$\begin{aligned}R_0(\boldsymbol{z}) := &\int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z}\,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_0 : \mathrm{D}V(0)^T \nabla \boldsymbol{z}\,\mathrm{d}x + \\ &+ \int_\Omega \mu \nabla \boldsymbol{u}_0 : \nabla \left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right)\,\mathrm{d}x - \\ &- \int_\Omega \mu \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z} \operatorname{div} V(0)\,\mathrm{d}x - b\left(\mathrm{D}V(0)\boldsymbol{w}_0, \boldsymbol{u}_0, \boldsymbol{z}\right) - \\ &- b\left(\boldsymbol{u}_0, \boldsymbol{u}_0, \mathrm{D}V(0)\boldsymbol{z}\right) + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}\,\mathrm{d}x + \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x\end{aligned}$$

for $\boldsymbol{z} \in \boldsymbol{V}$ and

$$D_0(\boldsymbol{w}_0)(\boldsymbol{z}) = \int_\Omega \alpha_0(\varphi_0)\boldsymbol{w}_0 \cdot \boldsymbol{z} + \mu \nabla \boldsymbol{w}_0 \cdot \nabla \boldsymbol{z}\,\mathrm{d}x + b\left(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{z}\right) + b\left(\boldsymbol{w}_0, \boldsymbol{u}_0, \boldsymbol{z}\right) \quad \forall \boldsymbol{z} \in \boldsymbol{V}.$$

Here we use the notation $\dot{\boldsymbol{u}}_0[V]$ for the solution to (16.20) – (16.21), $\dot{\boldsymbol{u}}_\varepsilon[V]$ is defined by (15.17) – (15.18) for every $\varepsilon > 0$ and $\boldsymbol{w}_0 := (-\operatorname{div} V(0) + \mathrm{D}V(0))\,\boldsymbol{u}_0$.

For the next estimate we conclude from Lemma 13.5:

$$\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \le \frac{\mu}{2K_\Omega}.$$

Consequently, the convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{u}_0$ in $\boldsymbol{H}^1(\Omega)$ implies the existence of a constant $0 < c < \frac{\mu}{K_\Omega}$ independent of ε such that it holds for $\varepsilon > 0$ small enough:

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \le c < \frac{\mu}{K_\Omega}.$$

Hence, (11.1) gives the estimate

$$|b\left(\dot{\boldsymbol{u}}_\varepsilon, \boldsymbol{u}_\varepsilon, \dot{\boldsymbol{u}}_\varepsilon\right)| \le K_\Omega \|\nabla \boldsymbol{u}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \dot{\boldsymbol{u}}_\varepsilon[V]\|^2_{\boldsymbol{L}^2(\Omega)} \le cK_\Omega \|\nabla \dot{\boldsymbol{u}}_\varepsilon[V]\|^2_{\boldsymbol{L}^2(\Omega)} \tag{17.6}$$

with $cK_\Omega < \mu$.

Combining this estimate and the calculations carried out in (9.9) – (9.11) we find that there exists a constant $C > 0$ independent of ε such that

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x + \|\nabla \dot{\boldsymbol{u}}_\varepsilon[V]\|^2_{\boldsymbol{L}^2(\Omega)} \le C.$$

From this we obtain the existence of a subsequence of $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$, denoted by the same, that converges weakly in $\boldsymbol{H}^1(\Omega)$. Making use of the continuity properties of the trilinear form b, see Lemma 11.1, we can use the same arguments as in Theorem 9.1 to show that $(FP_\varepsilon)_{\varepsilon>0}$ Γ-converges to FP_0 in $\boldsymbol{V}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology. This implies that the limit of the sequence of solutions $(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ of

$$\min_{\boldsymbol{v}\in\boldsymbol{V}} FP_\varepsilon(\boldsymbol{v})$$

equals the unique solution $(\dot{\boldsymbol{u}}_0[V] - \boldsymbol{w}_0)$ of

$$\min_{\boldsymbol{v}\in\boldsymbol{V}} FP_0(\boldsymbol{v}).$$

Since $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ converges to $\boldsymbol{w}_0$, which follows from the already shown convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$, this yields that $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\dot{\boldsymbol{u}}_0[V]$. Besides, we obtain from this Γ-convergence result that

$$\lim_{\varepsilon\searrow 0} FP_\varepsilon\left(\dot{\boldsymbol{u}}_\varepsilon[V]-\boldsymbol{w}_\varepsilon\right)=FP_0\left(\dot{\boldsymbol{u}}_0[V]-\boldsymbol{w}_0\right).$$

As in the proof of Theorem 9.1, this gives

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)\left|\dot{\boldsymbol{u}}_\varepsilon[V]\right|^2\mathrm{d}x=0,\quad \lim_{\varepsilon\searrow 0}\left\|\dot{\boldsymbol{u}}_\varepsilon[V]-\dot{\boldsymbol{u}}_0[V]\right\|_{\boldsymbol{H}^1(\Omega)}=0.$$

The terms in the expression $\partial_t|_{t=0}j_\varepsilon^N\left(\varphi_\varepsilon\circ T_t^{-1}\right)$ given by (15.21) can be considered as in Theorem 9.1, and thus we can deduce (17.2).

Moreover, we use this convergence result, the special choice of some $V\in\overline{\mathcal{V}}_{ad}$ such that $\int_\Omega\varphi_0\operatorname{div}V(0)\,\mathrm{d}x>0$ to deduce from (15.19) and (17.2) the convergence of $(\lambda_\varepsilon)_{\varepsilon>0}\subseteq\mathbb{R}^+$ to some $\lambda_0\in\mathbb{R}^+$. It follows directly that λ_0 then fulfills (16.43) – (16.44). For more details we refer to the proof of Theorem 9.1.

□

Summarizing the results of this part, we have shown that the phase field approach, which was proposed and discussed in Section 12, approximates the sharp interface model (13.16)-(13.17) describing topology optimization problems in a stationary Navier-Stokes flow in a sharp interface setting, in the following sense: We know, that for any sequence of minimizers of the phase field problems, there exists a subsequence that converges to some limit element as the thickness of the interface tends to zero. If this sequence fulfills a certain convergence rate we find, that it actually converges in the strong $L^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ topology and that the limit element is a minimizer of the sharp interface model. Moreover, we can show in this setting as in Part I that certain optimality conditions of the phase field model approximate an optimality system of the sharp interface model. As we have proven that those optimality conditions of the sharp interface are, under suitable assumptions, equivalent to classical shape derivatives, this gives that the optimality conditions of the phase field model are for small $\varepsilon>0$ also an approximation of shape derivatives. This implies, that the phase field ansatz is a good approximation for the shape topology optimization problem in a sharp interface formulation and is consistent with existing models.

Part III
Pressure functionals in a Stokes flow

18 Introduction

18.1 Problems using a general objective functional

One natural extension of the problem described in Part I and II seems to be considering a general objective functional considering not only the velocity but also the pressure of some fluid. Under appropriate assumptions one could carry out the analysis of the phase field model while considering a pressure in the objective functional. But as soon as we want to analyze the sharp interface model in this BV-setting that we consider here, we encounter directly a problem: assume we solve the Stokes equations in some general Caccioppoli set E in the sense of (6.4), and assume there exists some pressure $p \in \mathcal{D}'(E)$ such that $\nabla p = \mu\Delta\boldsymbol{u} + \boldsymbol{f}$ in the distributional sense. Then this pressure p would only be defined up to constants in every connected component of E. To have a unique pressure, one would need as many additional conditions as there are connected components of E, which may even be infinitely many. Besides, it is not even clear how to understand "connected components" of some Caccioppoli set, cf. Example 3.2. But in the phase field model we only have one degree of freedom, since ∇p_ε is given by

$$\nabla p_\varepsilon = -\alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon + \mu\Delta\boldsymbol{u}_\varepsilon + \boldsymbol{f} \quad \text{in } \boldsymbol{H}^{-1}(\Omega) \tag{18.1}$$

and Ω is connected. Thus it is not apparent how to connect both systems in a consistent way.

Besides, we find that numerical simulations indicate that the limit element of $(p_\varepsilon)_{\varepsilon>0}$, if defined by (18.1), depends on the choice of the interpolation function α_ε. We briefly discuss this on the example of minimizing the total potential power described in Section 10, in particular the setting where the double pipe occurs, see Figure 10 and 11. In Figure 12 we show the values of the pressure along the green curve sketched in Figure 11 using the interpolation function α_ε defined in Example 2.1 for different values of ε. In contrast, we show in Figure 13 the pressure along the green curve in Figure 11 for different values of ε using an interpolation function α_ε as in [BP03]. We see that for both cases, the pressure seems to converge for $\varepsilon \searrow 0$ to a certain distribution. Moreover, in both situation there develop two flat regions, which correspond to the fluid regions. But in those regions they will differ by constants, which reflects the analytic non-uniqueness problem described above. Besides, we see that the behaviour outside the fluid is completely different. In particular the pressure will not tend to zero outside the fluid region. This suggests that the phase field model gives in general not a good approximation for the pressure associated to the sharp interface limit.
We briefly remark that the different scales of the pressure in Figure 12 and 13 are a result of a different scaling of $\overline{\alpha}_\varepsilon$. The numerical calculations have been carried out by Christian Kahle from University of Hamburg.

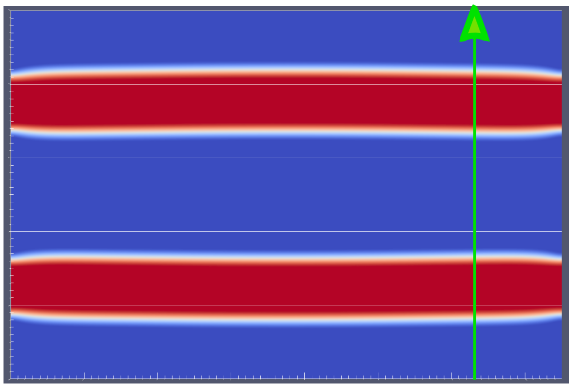


Figure 11: The graphs in Figures 12-13 illustrate the pressure along the green line in this picture.

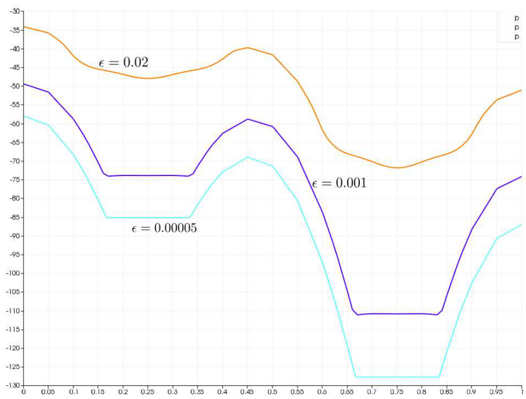


Figure 12: Pressure along the green curve sketched in Figure 11 for different values of ε using the interpolation function α_ε of Example 2.1.

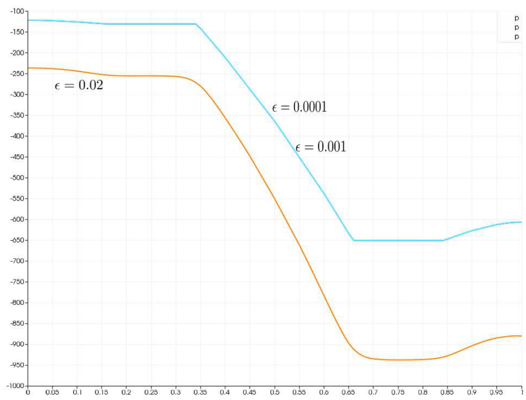


Figure 13: Pressure along the green curve sketched in Figure 11 for different values of ε using an interpolation function α_ε as in [BP03].

So this model seems not to be appropriate for a general objective functional considering the pressure. Instead, a sharp interface model defining the pressure in the whole domain Ω by some equations would be a possible way, for instance by enforcing the pressure to be zero outside the fluid region, see for instance [AHH11]. Then we would have to vary the phase field model in an appropriate way to approximate this sharp interface model. One possible way to do this could be to soften the incompressibility condition, see for instance [Kim96, Tem68]. A similar problem concerning the pressure has also been discussed in [KM12] and included references, where numerics showed that the our diffuse setting can not prevent the pressure diffusion through solid material.

For this reason we restrict ourselves in the following on specific types of functionals involving the pressure, as described in the next subsection.

18.2 Possible choices of objective functionals

As described in the previous subsection, we have to restrict ourselves to specific situations that can be modelled with our phase field approximation, instead of considering a general objective functional.

Our objective functional for the phase field model will be defined similar to (2.5) and is therefore given by

$$\begin{aligned} J_\varepsilon^P(\varphi, \boldsymbol{u}, p) := \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \\ + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x. \end{aligned} \tag{18.2}$$

We indicate here two possible choices of objective functionals that are of interest in applications and are also point of research in different fields. Possible choices of $h(p)$ could be

$$h(p) = (\chi_M p)^2$$

for some fixed domain $M \subset \Omega$, where we assume to have fluid, or

$$h(p) = (\chi_{M_1} p - \chi_{M_2} p)^2$$

for two disjoint domains M_1 and M_2, where we assume to have fluid. The latter functional could be used for instance to minimize the pressure difference between in-and outflow or at two sides of an obstacle.

We generalize those examples by choosing h as a functional fulfilling the following assumptions:

(A9) Assume to have finitely many fixed disjoint Lipschitz domains $(M_i)_{i=1}^m$, $M_i \subset \Omega$. Let $h_M : \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, that means here h_M is assumed to be continuous, such that

$$|h_M(v)| \le C|v|^2 \quad \forall v \in \mathbb{R}^m \tag{18.3}$$

for some constant $C > 0$. Additionally, assume that

$$H : L^2(\Omega) \ni q \mapsto \int_\Omega h_M(q\chi_{M_1}, \dots, q\chi_{M_m})\,\mathrm{d}x \tag{18.4}$$

is weakly lower semicontinuous and bounded from below. We use the following the notation:

$$\int_\Omega h(p)\,\mathrm{d}x = \int_\Omega h_M\left(p\chi_{M_1}, \dots, p\chi_{M_m}\right)\,\mathrm{d}x \quad \forall p \in L^2(\Omega).$$

Moreover, we have to assume some compatibility condition such that the admissible set is not empty:

$$\sum_{i=1}^m |M_i| < \beta\,|\Omega|\,.$$

Remark 18.1. *Of course we could generalise the objective functional considering the pressure, given by Assumption **(A9)**, without much effort to a functional depending on the spatial variable $x \in \Omega$. Hence, we would replace*

$$\int_\Omega h(p)\,\mathrm{d}x$$

by

$$\int_\Omega \widetilde{h}\,(x, p)\,\mathrm{d}x = \int_\Omega \widetilde{h}_M\left(x, p\chi_{M_1}, \dots, p\chi_{M_m}\right)\,\mathrm{d}x$$

in (18.2)*. Then, $\widetilde{h}\,(x, \cdot)$ has to be chosen such that $\widetilde{h}_M : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carthéodory function, i.e. $\widetilde{h}_M\,(\cdot, v)$ is measurable for each $v \in \mathbb{R}^m$ and $\widetilde{h}_M\,(x, \cdot)$ is continuous for a.e. $x \in \Omega$. Moreover, there have to be functions $a \in L^1(\Omega)$, $b \in L^\infty(\Omega)$ such that for almost every $x \in \Omega$ it holds*

$$\left|\widetilde{h}_M(x, v)\right| \le a(x) + b(x)\,|v|^2 \quad \forall v \in \mathbb{R}^m.$$

Finally, we assume that

$$L^2(\Omega) \ni q \mapsto \int_\Omega \widetilde{h}_M\left(x, p\chi_{M_1}, \dots, p\chi_{M_m}\right)\,\mathrm{d}x$$

is weakly lower semicontinuous and bounded from below.

Another possible generalization would be replacing

$$\int_\Omega f\,(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) + h(p)\,\mathrm{d}x$$

by

$$\int_\Omega \widetilde{f}\,(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}, p)\,\mathrm{d}x$$

in the objective functional with an appropriate chosen functional $\widetilde{f}$.

To simplify notations and clearify the used techniques, we focus in the following considerations to a form as outlined in (18.2) *and Assumption **(A9)**.*

Remark 18.2. *If the objective functional h is chosen such that Assumption **(A9)** is fulfilled, we obtain from* (18.3) *that*

$$L^2(\Omega) \ni q \mapsto \int_\Omega h_M(q\chi_{M_1}, \dots, q\chi_{M_m})\,\mathrm{d}x \tag{18.5}$$

is a well-defined and continuous Nemytskii operator. It even holds, that (18.5) *is a well-defined operator if and only if* (18.3) *is fulfilled, see [AZ90, Sho97].*

Applications of objective functionals involving the pressure can be found for instance in [AHH11] and in [AHL08], where a L^2-tracking-type functional for the pressure is considered.

Assumption **(A9)** implies in particular that the objective functional only depends on values of the pressure in the domains M_i. Thus it makes sense to define the following space, which will be our solution space later on:

$$L^2_M(\Omega) := \left\{ q \in L^2(\Omega) \mid \int_{M_i} q\,\mathrm{d}x = 0,\ \forall i = 1,\dots,m,\ q|_{\Omega \setminus \cup_{i=1}^m M_i} = 0 \right\}.$$

Remark 18.3. *Assume that $H|_{L^2_M(\Omega)}$ is bounded from below and continuous, where $H : L^2(\Omega) \to \mathbb{R}$ is defined in* (18.4)*. Then the convexity of*

$$\mathbb{R} \ni p \mapsto h_M\left(p\chi_{M_1}, \dots, p\chi_{M_m}\right) \in \mathbb{R}$$

is sufficient for the weakly lower semicontinuity of H, see for instance [Vis96, XI.4].

Remark 18.4.
- *The two examples discussed above are a special case of h. Another possible choice could be $h(p) = p\chi_M$ for some Lipschitz domain $M \subset \Omega$.*
- *We remark specifically that no radially unboundedness of the objective functional with respect to the pressure is needed, as it is explicitly necessary for the velocity, see Assumption **(A5)**.*

Moreover, we obviously need more regularity on the objective functional when considering first order optimality conditions. Thus the following assumption has to be assumed for Sections 19.3, 20.3 and 21.2:

(A10) Assume that $h_M : \mathbb{R}^m \to \mathbb{R}$, given by Assumption **(A9)**, is differentiable. Besides, let there be a constant $C > 0$ such that

$$|\mathrm{D}h_M(v)| \le C\,|v| \quad \forall v \in \mathbb{R}^m. \tag{18.6}$$

Remark 18.5. *If Assumption **(A10)** is fulfilled for the objective functional, we obtain that*

$$H : L^2(\Omega) \ni q \mapsto \int_\Omega h(p)\,\mathrm{d}x$$

is continuously Fréchet differentiable and the directional derivatives are given as

$$\mathrm{D}H\left(p\right)\left(q\right) = \int_\Omega \mathrm{D}h(p) q\,\mathrm{d}x \quad \forall p, q \in L^2(\Omega).$$

This follows by similar considerations as in Remark 2.5.

We have to impose an additional assumption to state the variational inequality for the phase field problem in an adjoint formulation. This assumption is analytically necessary to have a well-posed adjoint state in the first order necessary optimality condition, see Remark 19.3. Any other result in the following than Theorem 19.1 will hold true even without Assumption **(A11)**.

(A11) For every $p \in L^2_M(\Omega)$ ther exists some $\vartheta \in \mathbb{R}^m$ such that

$$\int_{M_i} (\vartheta_i - \mathrm{D}h(p))\,\mathrm{d}x = 0 \quad \forall i = 1,\dots,m. \tag{18.7}$$

Remark 18.6 (Remarks on Assumption **(A11)**)**.** *We briefly want to discuss Assumption **(A11)** and interpret this condition from a different point of view.*

- *Since the Stokes equations describe the pressure only up to constants in every domain M_i, one could want the objective functional to be invariant with respect to perturbations of p by constants. This implies*

$$\partial_k|_{k=0} \int_{M_i} h(p+k)\,\mathrm{d}x = 0 \quad \forall i = 1,\dots,m$$

  and thus

$$\int_{M_i} \mathrm{D}h(p)\,\mathrm{d}x = 0 \quad \forall i = 1,\dots,m$$

  *which implies for instance Assumption **(A11)** with $\vartheta \equiv 0$.*

- *The problem stated in this part implies implicitly that the objective functional is invariant with respect to perturbations of the pressure by constants, since we always choose $p \in L^2_M(\Omega)$, and thus $\int_{M_i} p\,\mathrm{d}x = 0$. Hence, the argument inserted into $h(\cdot)$ has always mean value zero in every M_i. But assume we want to extend h to the whole space $L^2(\Omega)$, such that it is invariant with respect to addition of constants in M_i. Then a natural choice would be*

$$h^e(p) := h_M\left(p\chi_{M_1} - \fint_{M_1} p\,\mathrm{d}x,\dots,p\chi_{M_m} - \fint_{M_m} p\,\mathrm{d}x\right) \quad \forall p \in L^2(\Omega)$$

  wherefrom we obtain

$$h^e(p) = h(p) \quad \forall p \in L^2_M(\Omega)$$

  and for all $p \in L^2_M(\Omega)$, $\tilde{p} \in L^2(\Omega)$ we see

$$\mathrm{D}h^e(p)\tilde{p} = \mathrm{D}h(p)\left(\tilde{p} - \fint_{M_1} \tilde{p}\,\mathrm{d}x,\dots,\tilde{p} - \fint_{M_m} \tilde{p}\,\mathrm{d}x\right).$$

  This means, that the nonlocal terms $\fint_{M_i} \tilde{p}$ would enter the adjoint equation (see calculations in Section 19.3), and we would have to find an adjoint state $\boldsymbol{q} \in \boldsymbol{H}^1_0(\Omega)$ such that $\operatorname{div}\boldsymbol{q} = \mathrm{D}h^e(p)$, and thus

$$\int_{M_i} \mathrm{D}h^e(p)\tilde{p}\,\mathrm{d}x = 0 \quad \forall \tilde{p} \in L^2(M_i), \fint_{M_i} \tilde{p}\,\mathrm{d}x = 0.$$

  This conditions implies

$$\mathrm{D}h^e(p) \equiv m_i \text{ in } M_i$$

for some constants $m_i \in \mathbb{R}$, $i = 1, \ldots, m$ *and so we have*

$$\int_{M_i} \mathrm{D}h^e(p)\,\mathrm{d}x = |M_i|\, m_i =: \vartheta_i.$$

This implies again Assumption ***(A11)****.*

To complete the definition of

$$J_\varepsilon^P : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \to \overline{\mathbb{R}}$$

we define $J_\varepsilon^P(\varphi, \boldsymbol{u}, p)$ by (18.2) if $\varphi \in \Phi_p$ where

$$\Phi_p := \{\varphi \in \Phi_{ad} \mid \varphi|_{M_i} = 1,\ \forall i = 1, \ldots, m\}$$

and $(\boldsymbol{u}, p) = \boldsymbol{S}_\varepsilon^P(\varphi)$ and $J_\varepsilon^P(\varphi, \boldsymbol{u}, p) = +\infty$ otherwise. The solution operator $\boldsymbol{S}_\varepsilon^P$ for the penalized Stokes equations is defined in Lemma 19.1.
We point out that in this setting a design variable φ will only be admissible if it ensures fluid in the domains where the pressure is taken into account. This imposes an additional constraint, namely $\varphi|_{\cup_{i=1}^m M_i} = 1$, and so this model is not appropriate if $\cup_{i=1}^m M_i$ is too large.

Besides, we will use the following notation for the extended admissible set:

$$\overline{\Phi}_p := \left\{\varphi \in \overline{\Phi}_{ad} \mid \varphi|_{M_i} = 1,\ \forall i = 1, \ldots, m\right\}.$$

We notice, that both Φ_p and $\overline{\Phi}_p$ are convex sets, which will be important in particular for having well-posed first order optimality conditions, see Section 19.3.

Similarly, we have the following admissible sets for the sharp interface model:

$$\Phi_p^0 := \left\{\varphi \in \Phi_{ad}^0 \mid \varphi|_{M_i} = 1,\ \forall i = 1, \ldots, m\right\}$$

and

$$\overline{\Phi}_p^0 := \left\{\varphi \in \overline{\Phi}_{ad}^0 \mid \varphi|_{M_i} = 1,\ \forall i = 1, \ldots, m\right\}.$$

The objective functional in the sharp interface is correspondingly given by

$$J_0^P(\varphi, \boldsymbol{u}, p) := \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \tag{18.8}$$

if $\varphi \in \Phi_p^0$ and $(\boldsymbol{u}, p) = \boldsymbol{S}_0^P(\varphi)$ and $J_0^P(\varphi, \boldsymbol{u}, p) = +\infty$ otherwise. The solution operator $\boldsymbol{S}_0^P$ of the Stokes equations is defined in Lemma 20.2. Besides, we will use the reduced objective functional, which will be defined by

$$j_0^P(\varphi) := J_0^P\left(\varphi, \boldsymbol{S}_0^P(\varphi)\right).$$

Well-posedness of the phase field model is discussed in Section 19.2 and in Section 20.2 we will discuss the corresponding sharp interface model. In Section 21 we will then consider convergence of minimizers of the reduced functionals

$$j_\varepsilon^P(\varphi) := J_\varepsilon^P\left(\varphi, \boldsymbol{S}_\varepsilon^P(\varphi)\right)$$

and state a convergence result for the optimality systems.
Finally, we discuss in Section 22 how to apply these results to a setting where the Stokes equations are replaced by the stationary Navier-Stokes equations in the constraints.

For calculating derivatives by geometric variations in Sections 19.3 and 20.3 we will have to modify the admissible transformations and velocities, too, since in M_i the domain is assumed to be fixed. And so we arrive in the following definitions:

Definition 18.1 ($\mathcal{V}^p_{ad}$, $\mathcal{T}^p_{ad}$, $\overline{\mathcal{V}}^p_{ad}$, $\overline{\mathcal{T}}^p_{ad}$)**.** We say that a velocity field $V \in \mathcal{V}_{ad}$ (resp. $V \in \overline{\mathcal{V}}_{ad}$) is in $\mathcal{V}^p_{ad}$ (resp. in $\overline{\mathcal{V}}^p_{ad}$) if it holds

> **(V5)** $V(t,x) = 0$ for every $x \in M_i$, $i = 1, \dots, m$.

We say then that a transformation $T \in \mathcal{T}_{ad}$ (resp. $T \in \overline{\mathcal{T}}_{ad}$) is in $\mathcal{T}^p_{ad}$ (resp. in $\overline{\mathcal{T}}^p_{ad}$) if its velocity field is in $\mathcal{V}^p_{ad}$ (resp. $\overline{\mathcal{V}}^p_{ad}$). Here we associated transformations $T \in \overline{\mathcal{T}}^p_{ad}$ to velocity fields $V \in \overline{\mathcal{V}}^p_{ad}$ by (2.9), which means it holds

$$\partial_t T_t(x) = V(t, T_t(x)), \quad T_0(x) = x.$$

Remark 18.7.

- *We see directly that $\mathcal{V}^p_{ad} \subset \overline{\mathcal{V}}^p_{ad}$ and $\mathcal{T}^p_{ad} \subset \overline{\mathcal{T}}^p_{ad}$.*
- *It follows as in Lemma 3.5 that for any $T \in \overline{\mathcal{T}}^p_{ad}$ we have $T_t(x) = x$ if $x \in M_i$ for some $i \in \{1, \dots, m\}$.*

19 Phase field model

We start by introducing the phase field model describing a shape and topology optimization problem in a Stokes flow with a general objective functional taking the velocity and the pressure of the fluid into account. After a brief description of the model, we focus on discussing the state constraints and well-posedness of the overall optimization problem. After that we come to deriving first order necessary optimality conditions both by parametric and geometric variations. We will carry out the discussions briefly, and mainly point out the differences to Part I.

19.1 Problem formulation

The general problem in the phase field setting will be minimizing J_ε^P as given in (18.2), which means we want to solve

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u},p)} J_\varepsilon^P(\varphi,\boldsymbol{u},p) = \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \\ + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x\end{aligned} \tag{19.1}$$

with

$$(\varphi,\boldsymbol{u},p) \in \Phi_p \times \boldsymbol{U} \times L^2_M(\Omega)$$

such that

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{V} \tag{19.2}$$

and

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \begin{aligned}&\forall \boldsymbol{v}\in\boldsymbol{H}^1_0(M_i),\\ &i=1,\ldots,m.\end{aligned} \tag{19.3}$$

This definition implies that (19.3) defines a pressure by

$$\nabla p = -\alpha_\varepsilon(\varphi)\,\boldsymbol{u} + \mu\Delta\boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{H}^{-1}(M_i)$$

which is well-defined due to Lemma 19.1.
On the other hand, we see that by (19.2) and Lemma 4.4 we get some $p_\Omega \in L^2(\Omega)$, which is defined up to a constant, such that

$$\nabla p_\Omega = -\alpha_\varepsilon(\varphi)\,\boldsymbol{u} + \mu\Delta\boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega).$$

Consequently, there exist constants $m_i \in \mathbb{R}$, $i = 1,\ldots,m$, such that

$$p_\Omega|_{M_i} = p|_{M_i} + m_i.$$

From this definition, the pressure p defined by (19.3) is a priori only defined on M_i, and arbitrary outside of M_i and only corresponds to p_Ω in M_i up to constants m_i. We remark that we chose our functional J_ε^P such that it only depends on $p|_{M_i}$, so it is enough to define p on M_i. Therefore, the choice of $p|_{\Omega\setminus\cup_{i=1}^m M_i} = 0$ is just an arbitrary extension of p and does not influence the overall problem.

Remark 19.1. *In the following we will often have a functional $\boldsymbol{F} \in \boldsymbol{H}^{-1}(\Omega)$ such that $\boldsymbol{F}(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ and then define a pressure $p \in L^2_M(\Omega)$ by the relation*

$$\nabla p = \boldsymbol{F}|_{\boldsymbol{H}_0^1(M_i)} \in \boldsymbol{H}^{-1}(M_i) \quad i = 1, \ldots, m.$$

This pressure then is uniquely defined by the choice of $p \in L^2_M(\Omega)$, see Lemma 4.4.

In this model our design variable will again be the phase field variable φ, where $\{x \in \Omega \mid \varphi(x) = 1\}$ models the presence of fluid, and $\{x \in \Omega \mid \varphi(x) = -1\}$ the non-presence of fluid. The phase field ansatz yields additionally a small transition area between both region with thickness $\mathcal{O}(\varepsilon)$, $\varepsilon > 0$. If we are in the fluid region, thus $\varphi(x) = 1$, the penalization term α_ε vanishes, and the state equations reduce to the classical Stokes equations. If we are outside the fluid region, thus $\varphi(x) = -1$, we can consider the state equations as a Darcy flow through some medium with permeability $\alpha_\varepsilon(-1)^{-1}$, thus the non-presence of fluid is merely approximated by a fluid through porous medium. For details we refer to the discussion in Section 5.1. We just point out, that we restrict our admissible sets in the optimization problem by enforcing to have fluid in the part where the pressure is considered in the objective functional.

19.2 Existence results

The aim of this section is to establish existence results for the phase field model (19.1) – (19.3), first by considering the state equations and afterwards by showing existence of a minimizer for the optimization problem.

We start by defining a solution operator corresponding to the state equations (19.2) – (19.3):

Lemma 19.1. *For every $\varphi \in L^1(\Omega)$ with $|\varphi(x)| \le 1$ a.e. in Ω there exist unique $\boldsymbol{u} \in \boldsymbol{U}$ and $p \in L^2_M(\Omega)$ such that it holds (19.2) – (19.3). Moreover, the solution $(\boldsymbol{u}, p)$ fulfills*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \le c\left(\Omega, \overline{\alpha}_\varepsilon, \mu, M_i\right)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}\right).$$

This defines a solution operator, which will be denoted by

$$\boldsymbol{S}_\varepsilon^P : \overline{\Phi}_p \to \boldsymbol{U} \times L^2_M(\Omega),$$

$$\boldsymbol{S}_\varepsilon^P(\varphi) := (\boldsymbol{u}, p), \quad \text{if } (\boldsymbol{u}, p) \text{ fulfill (19.2) – (19.3)}.$$

Proof. Existence, uniqueness and the a priori estimate for the velocity $\boldsymbol{u} \in \boldsymbol{U}$ follow from Lemma 5.1. Then we apply Lemma 4.4 to every M_i, $i = 1, \ldots, m$, to get a unique solution $p_i \in L^2(M_i)$ with $\int_{M_i} p_i \,\mathrm{d}x = 0$ of (19.3) together with an estimate of the form

$$\|p_i\|_{L^2(M_i)} \le c(M_i)\left\|\alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} - \mu\Delta\boldsymbol{u} - \boldsymbol{f}\right\|_{\boldsymbol{H}^{-1}(M_i)}$$

which implies by (5.4)

$$\|p_i\|_{L^2(M_i)} \le c(M_i, \Omega, \overline{\alpha}_\varepsilon, \mu)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}\right).$$

By defining $p = \sum_{i=1}^m p_i \chi_{M_i}$ we deduce the statement. □

Thus we can now directly show the existence of a minimizer for (19.1)-(19.3):

Lemma 19.2. *There exists at least one minimizer of* (19.1)-(19.3).

Proof. This result can be shown as in the proof of Theorem 5.1 by the direct method in the calculus of variations. For this purpose, we make in particular use of the weakly lower semicontinuity in $L^2(\Omega)$ of the pressure term in the objective functional given by Assumption **(A9)**. To this end let $(\varphi_k, \boldsymbol{u}_k, p_k)_{k\in\mathbb{N}}$ be a minimizing sequence of (19.1)-(19.3). Then we get due to Assumption **(A5)** a uniform bound on $\left(\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}\right)_{k\in\mathbb{N}}$, and thus by $p_k \in L^2_M(\Omega)$, Lemma 4.1 and (19.3) also a uniform bound on $\left(\|p_k\|_{L^2(\Omega)}\right)_{k\in\mathbb{N}}$. The objective functional together with the pointwise condition $|\varphi_k| \leq 1$ a.e. provide moreover directly a bound on $\left(\|\varphi_k\|_{H^1(\Omega)}\right)_{k\in\mathbb{N}}$. And so we find subsequences, still indexed by $k \in \mathbb{N}$, such that $(\varphi_k)_{k\in\mathbb{N}}$ converges weakly in $H^1(\Omega)$ to $\varphi \in \Phi_p$, $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{u} \in \boldsymbol{U}$ and $(p_k)_{k\in\mathbb{N}}$ converges weakly in $L^2(\Omega)$ to $p \in L^2_M(\Omega)$. We can deduce as in Theorem 5.1 that $\boldsymbol{u} = \boldsymbol{S}_\varepsilon(\varphi)$ and using (19.3) we find therefrom $(\boldsymbol{u}, p) = \boldsymbol{S}^P_\varepsilon(\varphi)$. And so by the weakly lower semicontinuity of the objective functional we can conclude the statement. For details we refer to the proof of Theorem 5.1. $\square$

In particular, due to those considerations, we can now define the reduced objective functional as follows:

$$j^P_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$$

$$j^P_\varepsilon(\varphi) := \begin{cases} J^P_\varepsilon\left(\varphi, \boldsymbol{S}^P_\varepsilon(\varphi)\right), & \text{if } \varphi \in \Phi_p, \\ +\infty, & \text{otherwise.} \end{cases} \tag{19.4}$$

19.3 Optimality conditions

The goal of this subsection is to derive necessary optimality conditions of first order for problem (19.1)-(19.3). One viewpoint of (19.1)-(19.3) is the optimal control theory. So we will start by developing first order optimality conditions arising from this approach, which lead to a variational inequality, see Theorem 19.1. But as we want to approximate with the phase field formulation a shape and topology optimization problem we also consider geometric variations along suitable transformation which leads to a second formulation for optimality conditions, see Theorem 19.2. At the end of this subsection we will then discuss the relation between the two optimality conditions. We remark that we will not be able to prove the same equivalence that we were able to show in the setting of the first part.

We have to assume in this section additionally Assumptions **(A6)**, **(A7)** and **(A10)** to ensure differentiability of the objective functional and of the external force.
We fix in the following $\varepsilon > 0$ and $\varphi_\varepsilon \in \Phi_p$ as a minimizer of (19.1)-(19.3), whose existence is guaranteed by Lemma 19.2. Additionally, we introduce the notation $(\boldsymbol{u}_\varepsilon, p_\varepsilon) = \boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)$.

As we want to deduce a variational inequality for the reduced objective functional j_ε^P, we see that we have to differentiate the solution operator. So the first step is showing differentiability of the solution operator $\boldsymbol{S}_\varepsilon^P$.

Lemma 19.3. *Let $\varphi \in \overline{\Phi}_p$ be given. Then the directional derivative*

$$\partial_t|_{t=0}\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon)) = \mathrm{D}\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) =: (\boldsymbol{u}, p) \in \boldsymbol{V} \times L_M^2(\Omega)$$

exists in $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$, is well-defined and is given as the unique weak solution to

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} \cdot \boldsymbol{v} + \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{19.5}$$

together with

$$\int_\Omega p \operatorname{div} \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} \cdot \boldsymbol{v} + \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\,\mathrm{d}x \tag{19.6}$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0^1(M_i)$ and for all $i = 1, \ldots, m$, where $(\boldsymbol{u}_\varepsilon, p_\varepsilon) := \boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)$. Here we denote by $\partial_t|_{t=0}\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon)) = \lim_{t\to 0, t>0} \frac{1}{t}\left(\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon + t(\varphi - \varphi_\varepsilon)) - \boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)\right)$ the one-sided directional derivative.

Proof. First we apply Lemma 7.2 to get the existence of the directional derivative of $pr_1\boldsymbol{S}_\varepsilon^P : \Phi_p \to \boldsymbol{H}^1(\Omega)$ in the sense of the statement, where pr_1 denotes the projection of $\boldsymbol{S}_\varepsilon^P$ onto the first component, thus onto $\boldsymbol{H}^1(\Omega)$, which implies $pr_1\boldsymbol{S}_\varepsilon^P = \boldsymbol{S}_\varepsilon|_{\Phi_p}$. Moreover, we get from this lemma, that

$$\mathrm{D}pr_1\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) = pr_1\mathrm{D}\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) = \boldsymbol{u}$$

where $\boldsymbol{u} \in \boldsymbol{V}$ is the unique solution to (19.5).
We define the mapping

$$P : \left\{\boldsymbol{F} \in \boldsymbol{H}^{-1}(\Omega) \mid \boldsymbol{F}(\boldsymbol{v}) = 0\,\forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \operatorname{div} \boldsymbol{v} = 0\right\} \to L_M^2(\Omega)$$

where $P(\boldsymbol{F}) = p$ is defined as the unique $p \in L^2_M(\Omega)$ such that $\nabla p = \boldsymbol{F}$ in $\boldsymbol{H}^{-1}(M_i)$ for every $i = 1, \dots, m$, compare Remark 19.1. Using Lemma 4.4, we see that P is a linear, continuous operator and due to (19.3) we get

$$p_t = P\left(\boldsymbol{f} - \alpha_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right)\boldsymbol{u}_t + \mu\Delta\boldsymbol{u}_t\right)$$

where

$$\left(\boldsymbol{u}_t, p_t\right) \coloneqq \boldsymbol{S}^P_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right).$$

If we denote by pr_2 the projection of $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ onto $L^2(\Omega)$, we can find that

$$\begin{aligned} p &= \mathrm{D}pr_2\boldsymbol{S}^P_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right) = pr_2\mathrm{D}\boldsymbol{S}^P_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right) = \\ &= \partial_t|_{t=0}p_t = P\left(\partial_t|_{t=0}\left(\boldsymbol{f} - \alpha_\varepsilon\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right)\boldsymbol{u}_t + \mu\Delta\boldsymbol{u}_t\right)\right) = \\ &= P\left(-\alpha'_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon - \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u} + \mu\Delta\boldsymbol{u}\right) \end{aligned}$$

which implies that $p \in L^2_M(\Omega)$ is given as the solution to (19.6) and proves the statement. □

Remark 19.2. *If we extend $\alpha_\varepsilon : [-1, 1] \to [0, \overline{\alpha}_\varepsilon]$ suitably to a function $\widetilde{\alpha}_\varepsilon \in C^2(\mathbb{R})$, we can extend the solution operator $\boldsymbol{S}^P_\varepsilon : \overline{\Phi}_p \to \boldsymbol{U}$ to an operator $\widetilde{\boldsymbol{S}}^P_\varepsilon : L^6(\Omega) \to \boldsymbol{U}$ and show as in Remark 7.1 by an application of the implicit function theorem that $\widetilde{\boldsymbol{S}}^P_\varepsilon$ is Fréchet differentiable. Hence, we also find that the reduced objective functional $j^P_\varepsilon : \overline{\Phi}_p \to \mathbb{R}$ can be extended to a Fréchet differentiable functional $\widetilde{j}^P_\varepsilon : H^1(\Omega) \to \mathbb{R}$.*

Using this differentiability result we are already in a position deducing the announced variational inequality, which arises if considering (19.1)-(19.3) as an optimal control problem. We obtain a result similar to that in Theorem 7.1 but with a different adjoint state $\boldsymbol{q}_\varepsilon$, as the following theorem shows.

Theorem 19.1. *Assume now additionally Assumption **(A11)** and*

$$\exists \varphi \in \overline{\Phi}_p : \quad \int_\Omega \varphi \,\mathrm{d}x < 0. \tag{19.7}$$

Then the following optimality system is fulfilled for any minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \Phi_p \times \boldsymbol{U} \times L^2_M(\Omega)$ of (19.1)-(19.3):

$$\begin{aligned} &\left(\frac{1}{2}\alpha'_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2 + \frac{\gamma}{\varepsilon}\psi'_0\left(\varphi_\varepsilon\right) - \alpha'_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon + \lambda_\varepsilon, \varphi - \varphi_\varepsilon\right)_{L^2(\Omega)} + \\ &\qquad + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla\left(\varphi - \varphi_\varepsilon\right)\right)_{\boldsymbol{L}^2(\Omega)} \geq 0 \quad \forall \varphi \in \overline{\Phi}_p, \end{aligned} \tag{19.8}$$

$$\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{19.9a}$$

$$\begin{aligned} \int_\Omega \alpha_\varepsilon\left(\varphi\right)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{v} \,\mathrm{d}x - \int_\Omega p_\varepsilon \operatorname{div}\boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \\ \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(M_i),\ i = 1, \dots, m, \end{aligned} \tag{19.9b}$$

$$\alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{q}_\varepsilon - \mu\Delta\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon = \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon + \mathrm{D}_u f(\cdot,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon) \qquad \textit{in } \Omega, \tag{19.9c}$$

$$\operatorname{div}\boldsymbol{q}_\varepsilon = \sum_{i=1}^m \left(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon)\right)\chi_{M_i} \qquad \textit{in } \Omega, \tag{19.9d}$$

$$\boldsymbol{q}_\varepsilon = \boldsymbol{0} \qquad \textit{on } \partial\Omega, \tag{19.9e}$$

$$\lambda_\varepsilon\left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x - \beta|\Omega|\right) = 0,\ \lambda_\varepsilon \geq 0, \tag{19.9f}$$

$$\int_\Omega \varphi_\varepsilon\,\mathrm{d}x \leq \beta|\Omega|, \quad |\varphi_\varepsilon| \leq 1 \ \textit{a.e. in } \Omega, \tag{19.9g}$$

*where $\lambda_\varepsilon \in \mathbb{R}^+$ is a Lagrange multiplier for the integral constraint and $\vartheta_\varepsilon = (\vartheta_{\varepsilon,i})_{i=1}^m \in \mathbb{R}^m$ is the variable from Assumption **(A11)** associated to $p_\varepsilon \in L^2_M(\Omega)$. Here, $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \boldsymbol{U} \times L^2_M(\Omega)$ and $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}^1_0(\Omega)$ are weak solutions of the state equations and adjoint system, respectively.*

Remark 19.3. *Existence and uniqueness of a solution $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}^1_0(\Omega)$ defined by system* (19.9c) – (19.9e) *follows with Lax-Milgram's theorem A.2 after homogenization of the incompressibility condition* (19.9d). *This homogenization can be done by using a field*

$$\boldsymbol{w}_\varepsilon \in \boldsymbol{H}^1_0(\Omega), \quad \operatorname{div}\boldsymbol{w}_\varepsilon = \sum_{i=1}^m \left(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon)\right)\chi_{M_i},$$

$$\|\boldsymbol{w}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} \leq c(\Omega)\left\|\sum_{i=1}^m \left(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon)\right)\chi_{M_i}\right\|_{L^2(\Omega)}$$

*which exists due to Lemma 4.2. To apply Lemma 4.2 we make use of Assumption **(A11)** and see*

$$\int_\Omega \sum_{i=1}^m \left(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon)\right)\chi_{M_i}\,\mathrm{d}x = 0.$$

Remark 19.4. *Assumption* (19.7) *is in particular necessary for proving the regularity assumption of [KZ79]. If* (19.7) *is not fulfilled, the variational inequality*

$$\mathrm{D}j^P_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \geq 0, \quad \forall \varphi \in \Phi_p$$

would still be fulfilled and could be reformulated with the help of the adjoint variable as below, but we could not prove the existence of a Lagrange multiplier λ_ε for the integral constraint by using the results of [KZ79].
Assumption (19.7) *may not be fulfilled, if for instance the domains M_i, where φ is assumed to have value 1, is too large compared to Ω.*

Proof. The existence of a Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}^+$ corresponding to the integral constraint follows exactly as in Theorem 7.1 by using in particular (19.7). It follows therefrom, see (7.21), that it holds

$$\mathrm{D}j^P_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) + \lambda_\varepsilon\int_\Omega (\varphi - \varphi_\varepsilon)\,\mathrm{d}x \geq 0 \quad \forall \varphi \in \overline{\Phi}_p$$

where j^P_ε is due to (19.4) given by

$$j^P_\varepsilon(\varphi) = J^P_\varepsilon\left(\varphi, \boldsymbol{S}^P_\varepsilon(\varphi)\right) \quad \forall \varphi \in \overline{\Phi}_p.$$

Using the notation $(\boldsymbol{u},p) = \mathrm{D}\boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)$ we calculate for any $\varphi \in \overline{\Phi}_p$:

$$\begin{aligned}\mathrm{D}j_\varepsilon^P(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) = \frac{1}{2}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{u}_\varepsilon\,\mathrm{d}x +\\ + \int_\Omega \mathrm{D}_{(2,3)}f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega \mathrm{D}h(p_\varepsilon)p\,\mathrm{d}x +\\ + \gamma\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_\varepsilon)\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

From the linearized equation (19.5) and Lemma 4.4 we get a unique $p_\Omega \in L^2(\Omega)$ with $\int_\Omega p_\Omega\,\mathrm{d}x = 0$ such that

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v} + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - \int_\Omega p_\Omega \operatorname{div}\boldsymbol{v}\,\mathrm{d}x = 0$$

holds for all $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$. This leads to $p_\Omega|_{M_i} = p|_{M_i} + m_i$ for constants some $m_i \in \mathbb{R}$ and all $i = 1,\dots,m$.

Inserting the adjoint state $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}_0^1(\Omega)$ as a test function in the linearized equation (19.5) we obtain

$$\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{q}_\varepsilon\,\mathrm{d}x - \int_\Omega p_\Omega \operatorname{div}\boldsymbol{q}_\varepsilon\,\mathrm{d}x = 0$$

since $\boldsymbol{q}_\varepsilon$ is not solenoidal.
Similar we use the linearized state $\boldsymbol{u} \in \boldsymbol{V}$ as a test function in the adjoint system given by (19.9c) – (19.9e) to get

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{q}_\varepsilon\cdot\nabla\boldsymbol{u}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u}\,\mathrm{d}x +\\ + \int_\Omega \mathrm{D}_{(2,3)}f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x.\end{aligned}$$

Comparing these two equations and using (19.9d) we have

$$\begin{aligned}\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\,\mathrm{d}x - \sum_{i=1}^m\int_{M_i} p_\Omega(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon))\,\mathrm{d}x = -\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u}\,\mathrm{d}x -\\ - \int_\Omega \mathrm{D}_{(2,3)}f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x.\end{aligned}$$

Invoking Assumption **(A11)** yields

$$\begin{aligned}&\sum_{i=1}^m\int_{M_i} p_\Omega(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon))\,\mathrm{d}x = \sum_{i=1}^m\int_{M_i} p(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon))\,\mathrm{d}x +\\ &+ \sum_{i=1}^m m_i\underbrace{\int_{M_i}(\vartheta_{\varepsilon,i} - \mathrm{D}h(p_\varepsilon))\,\mathrm{d}x}_{=0} =\\ &= \sum_{i=1}^m \vartheta_{\varepsilon,i}\underbrace{\int_{M_i} p\,\mathrm{d}x}_{=0} - \sum_{i=1}^m\int_{M_i}\mathrm{D}h(p_\varepsilon)p\,\mathrm{d}x = -\sum_{i=1}^m\int_{M_i}\mathrm{D}h(p_\varepsilon)p\,\mathrm{d}x = -\int_\Omega \mathrm{D}h(p_\varepsilon)p\,\mathrm{d}x\end{aligned} \tag{19.10}$$

and thus

$$\int_\Omega \alpha_\varepsilon' (\varphi_\varepsilon) (\varphi - \varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon \, \mathrm{d}x = - \int_\Omega \alpha_\varepsilon (\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{u} \, \mathrm{d}x - \\ - \int_\Omega \mathrm{D}_{(2,3)} f (x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) (\boldsymbol{u}, \mathrm{D}\boldsymbol{u}) \, \mathrm{d}x - \int_\Omega \mathrm{D}h(p_\varepsilon) p \, \mathrm{d}x.$$

Plugging these results together we end up with

$$\mathrm{D}j_\varepsilon^P (\varphi_\varepsilon) (\varphi - \varphi_\varepsilon) = \frac{1}{2} \int_\Omega \alpha_\varepsilon' (\varphi_\varepsilon) (\varphi - \varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}x - \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon) (\varphi - \varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon \, \mathrm{d}x + \\ + \gamma\varepsilon \int_\Omega \nabla\varphi_\varepsilon \cdot \nabla (\varphi - \varphi_\varepsilon) \, \mathrm{d}x + \frac{\gamma}{\varepsilon} \int_\Omega \psi' (\varphi_\varepsilon) (\varphi - \varphi_\varepsilon) \, \mathrm{d}x$$

and can deduce the statement.

□

We thus have derived first order optimality conditions by considering the phase field model (19.1)-(19.3) as an optimal control problem. Since we want to approximate a sharp interface shape and topology optimization problem we also want to calculate optimality conditions by geometric variations as in Section 7.2. Applying the calculations of Section 7.2 we arrive in the following result:

Theorem 19.2. *For any minimizer* $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \Phi_p \times \boldsymbol{U} \times L^2_M(\Omega)$ *of* (19.1)-(19.3) *we have the following necessary optimality conditions:*

$$\partial_t|_{t=0} j_\varepsilon^P \left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0) \, \mathrm{d}x, \tag{19.11}$$

$$\lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon \, \mathrm{d}x - \beta |\Omega| \right) = 0 \tag{19.12}$$

for all $T \in \overline{\mathcal{T}}^p_{ad}$ *with velocity* $V \in \overline{\mathcal{V}}^p_{ad}$*, where* $\lambda_\varepsilon \in \mathbb{R}^+$ *is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:*

$$\begin{aligned} \partial_t|_{t=0} j_\varepsilon^P \left(\varphi_\varepsilon \circ T_t^{-1}\right) &= \int_\Omega \alpha_\varepsilon (\varphi_\varepsilon) \left(\boldsymbol{u}_\varepsilon \cdot \dot{\boldsymbol{u}}_\varepsilon [V] + \frac{1}{2} |\boldsymbol{u}_\varepsilon|^2 \operatorname{div} V(0) \right) \mathrm{d}x + \\ &+ \int_\Omega [\mathrm{D}f (x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) (V(0), \dot{\boldsymbol{u}}_\varepsilon [V], \mathrm{D}\dot{\boldsymbol{u}}_\varepsilon [V] - \mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0)) + \\ &+ f (x, \boldsymbol{u}_\varepsilon, \mathrm{D}\boldsymbol{u}_\varepsilon) \operatorname{div} V(0)] \, \mathrm{d}x + \\ &+ \int_\Omega \mathrm{D}h (p_\varepsilon) \dot{p}_\varepsilon [V] + h (p_\varepsilon) \operatorname{div} V(0) \, \mathrm{d}x + \\ &+ \int_\Omega \left(\frac{\gamma\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon} \psi (\varphi_\varepsilon) \right) \operatorname{div} V(0) - \gamma\varepsilon \nabla\varphi_\varepsilon \cdot \nabla V(0) \nabla\varphi_\varepsilon \, \mathrm{d}x. \end{aligned} \tag{19.13}$$

Here $\dot{\boldsymbol{u}}_\varepsilon [V] \in \boldsymbol{H}^1_0(\Omega)$ *is given as the solution of* (7.26)-(7.27) *and* $\dot{p}_\varepsilon [V] \in L^2(\Omega)$ *with* $\dot{p}_\varepsilon [V] = 0$ *in* $\Omega \setminus \cup_{i=1}^m M_i$ *is the pressure associated to* $\dot{\boldsymbol{u}}_\varepsilon [V]$ *in the following sense:*

$$\begin{aligned}
&\int_\Omega \dot{p}_\varepsilon [V] \operatorname{div} \boldsymbol{z}_i \, \mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) \dot{\boldsymbol{u}}_\varepsilon [V] \cdot \boldsymbol{z}_i + \mu \nabla \dot{\boldsymbol{u}}_\varepsilon [V] \cdot \nabla \boldsymbol{z}_i \, \mathrm{d}x + \\
&+ \int_\Omega \alpha_\varepsilon (\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{z}_i \operatorname{div} V(0) \, \mathrm{d}x - \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z}_i \, \mathrm{d}x - \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T \nabla \boldsymbol{z}_i \, \mathrm{d}x + \\
&+ \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z}_i \operatorname{div} V(0) \, \mathrm{d}x - \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}_i \, \mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{z}_i \operatorname{div} V(0) \, \mathrm{d}x + \\
&+ \int_\Omega p_\varepsilon \mathrm{D}\boldsymbol{z}_i : \mathrm{D}V(0) - p_\varepsilon \operatorname{div} \boldsymbol{z}_i \operatorname{div} V(0) \, \mathrm{d}x
\end{aligned} \tag{19.14}$$

for all $\boldsymbol{z} = (\boldsymbol{z}_i)_{i=1}^m \in \times_{i=1}^m \boldsymbol{H}_0^1(M_i)$ *together with*

$$\int_{M_i} \dot{p}_\varepsilon [V] \, \mathrm{d}x = -\int_{M_i} p_\varepsilon \operatorname{div} V(0) \, \mathrm{d}x \quad \forall i = 1, \ldots, m. \tag{19.15}$$

Proof. We choose some $T \in \overline{\mathcal{T}}_{ad}^p$ with velocity $V \in \overline{\mathcal{V}}_{ad}^p$ and introduce the notation

$$(\boldsymbol{u}_\varepsilon(t), p_\varepsilon(t)) = \boldsymbol{S}_\varepsilon^P \left(\varphi_\varepsilon \circ T_t^{-1} \right), \quad \varphi_\varepsilon(t) = \varphi_\varepsilon \circ T_t^{-1}.$$

As soon as we have shown that $I \ni t \mapsto (p_\varepsilon(t) \circ T_t)$ is differentiable at $t = 0$ in $L^2(\Omega)$ and that $\dot{p}_\varepsilon [V]$ is given by (19.14)-(19.15) we can proceed as in Theorem 7.5 to get the stated result. Here I is again a small open interval in $\mathbb{R}$ such that $0 \in I$.

To this end, we want to apply Theorem A.3 to the following setting: we define

$$(F_1^p, F_2^p) = F^p : I \times L_0 \to \bigtimes_{i=1}^m \boldsymbol{H}^{-1} (M_i) \times \mathbb{R}^m$$

by

$$F_1^p(t,p)(\boldsymbol{z}) := \left(\int_\Omega p \left(\mathrm{D}\boldsymbol{z}_i : \mathrm{D}T_t^{-1} \right) \det \mathrm{D}T_t \, \mathrm{d}x \right)_{i=1}^m \quad \forall \boldsymbol{z} = (\boldsymbol{z}_i)_{i=1}^m \in \bigtimes_{i=1}^m \boldsymbol{H}_0^1(M_i)$$

and

$$F_2^p(t,p) = \left(\int_{M_i} p \det \mathrm{D}T_t \, \mathrm{d}x \right)_{i=1}^m$$

where

$$L_0 := \left\{ p \in L^2(\Omega) \mid p(x) = 0 \text{ for a.e. } x \in \Omega \setminus \cup_{i=1}^m M_i \right\}.$$

Introducing $(f_1^p, f_2^p) = f^p : I \to \times_{i=1}^m \boldsymbol{H}^{-1}(M_i) \times \mathbb{R}^m$ by

$$f_1^p(t)(\boldsymbol{z}) := \left(F_1 \left(t, \boldsymbol{u}_\varepsilon(t) \circ T_t \right) (\boldsymbol{z}_i) \right)_{i=1}^m \quad \forall \boldsymbol{z} = (\boldsymbol{z}_i)_{i=1}^m \in \bigtimes_{i=1}^m \boldsymbol{H}_0^1(M_i)$$

and

$$f_2^p(t) = (0, \ldots, 0) \in \mathbb{R}^m$$

where $F_1 : I \times \boldsymbol{H}_{\boldsymbol{g}}^1(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is defined by

$$\begin{aligned}
F_1 (t, \boldsymbol{u}) \boldsymbol{z} = &\int_\Omega \alpha_\varepsilon (\varphi_\varepsilon) \boldsymbol{u} \cdot \boldsymbol{z} \det \mathrm{D}T_t \, \mathrm{d}x + \int_\Omega \mu \mathrm{D}T_t^{-T} \nabla \boldsymbol{u} : \mathrm{D}T_t^{-T} \nabla \boldsymbol{z} \det \mathrm{D}T_t \, \mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f} \circ T_t \cdot \boldsymbol{z} \det \mathrm{D}T_t \, \mathrm{d}x
\end{aligned}$$

for all $t \in I$, $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g}}(\Omega)$ and $\boldsymbol{z} \in \boldsymbol{H}^1_0(\Omega)$ we see that

$$f_1^p(t)(\boldsymbol{z}) - F_1^p(t, p_\varepsilon(t) \circ T_t)(\boldsymbol{z}) = \left(F_1 (t, \boldsymbol{u}_\varepsilon(t) \circ T_t)(\boldsymbol{z}_i) - \int_\Omega p_\varepsilon(t) \operatorname{div} \left(\boldsymbol{z} \circ T_t^{-1} \right) \mathrm{d}x \right)_{i=1}^m =$$
$$= \left(\int_\Omega \alpha_\varepsilon (\varphi_\varepsilon(t)) \boldsymbol{u}_\varepsilon(t) \cdot \left(\boldsymbol{z}_i \circ T_t^{-1} \right) \mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon(t) \cdot \nabla \left(\boldsymbol{z}_i \circ T_t^{-1} \right) \mathrm{d}x - \right.$$
$$\left. - \int_\Omega \boldsymbol{f} \cdot \left(\boldsymbol{z}_i \circ T_t^{-1} \right) \mathrm{d}x - \int_\Omega p_\varepsilon(t) \operatorname{div} \left(\boldsymbol{z}_i \circ T_t^{-1} \right) \mathrm{d}x \right)_{i=1}^m = 0.$$

Here, we have used that for $\boldsymbol{z} \in \boldsymbol{H}^1_0(M_i)$ it holds $\boldsymbol{z} \circ T_t^{-1} \in \boldsymbol{H}^1_0(M_i)$ because of our choice of $T \in \overline{\mathcal{T}}^p_{ad}$, which implies due to Remark 18.7 that $T_t|_{M_i} = Id$ for all M_i.

Since $T_t = Id$ on M_i implies $T_t(M_i) = M_i$ we get additionally

$$F_2^p (t, p_\varepsilon(t) \circ T_t) = \left(\int_{M_i} p_\varepsilon(t) \circ T_t \det \mathrm{D}T_t \,\mathrm{d}x \right)_{i=1}^m = \left(\int_{M_i} p_\varepsilon(t) \,\mathrm{d}x \right)_{i=1}^m = (0, \ldots, 0) = f_2^p(t).$$

Next we see that $I \ni t \mapsto F^p(t, \cdot) \in \mathcal{L}\left(L_0, \times_{i=1}^m \boldsymbol{H}^{-1}(M_i) \times \mathbb{R}^m \right)$ is differentiable at $t = 0$ and it holds due to (4.1)

$$\begin{aligned} \|F^p(0,p)\|_{\times_{i=1}^m \boldsymbol{H}^{-1}(M_i) \times \mathbb{R}^m} &= \left(\|\nabla p\|_{\boldsymbol{H}^{-1}(M_i)} \right)_{i=1}^m + \left(\left| \int_{M_i} p \,\mathrm{d}x \right| \right)_{i=1}^m \geq \left(c(M_i) \|p\|_{L^2(M_i)} \right)_{i=1}^m \geq \\ &\geq c(m, M_1, \ldots, M_m, \Omega) \|p\|_{L^2(\Omega)} \end{aligned}$$

which implies (A.1). From Lemma 7.4 we deduce moreover the differentiability of f^p at $t = 0$.
Thus we can apply Theorem A.3 to get differentiability of $I \ni t \mapsto (p_\varepsilon(t) \circ T_t)$ at $t = 0$ and one obtains that it holds for $\dot{p}_\varepsilon[V] := \partial_t|_{t=0} (p_\varepsilon(t) \circ T_t)$:

$$\begin{aligned} \int_\Omega \dot{p}_\varepsilon[V] \operatorname{div} \boldsymbol{z}_i \,\mathrm{d}x &= \partial_t|_{t=0} \left(f_1^p(t) \right) - \partial_t|_{t=0} F_1^p(0, p_\varepsilon)(\boldsymbol{z}) = (\partial_t|_{t=0} F_1)(0, \boldsymbol{u}_\varepsilon)(\boldsymbol{z}_i) + \\ &+ (\partial_u F_1)(0, \boldsymbol{u}_\varepsilon) \dot{\boldsymbol{u}}_\varepsilon[V](\boldsymbol{z}_i) + \int_\Omega p_\varepsilon \mathrm{D}\boldsymbol{z}_i : \mathrm{D}V(0) \,\mathrm{d}x - \int_\Omega p_\varepsilon \operatorname{div} \boldsymbol{z}_i \operatorname{div} V(0) \,\mathrm{d}x \end{aligned}$$

for all $\boldsymbol{z} = (\boldsymbol{z}_i)_{i=1}^m \in \times_{i=1}^m \boldsymbol{H}^1_0(M_i)$. This shows, that $\dot{p}_\varepsilon[V]$ is the pressures associated to $\dot{\boldsymbol{u}}_\varepsilon[V]$ by (19.14). Additionally, we have

$$\int_{M_i} \dot{p}_\varepsilon[V] \,\mathrm{d}x = -\partial_t|_{t=0} F_2^p(0, p_\varepsilon) = -\int_{M_i} p_\varepsilon \operatorname{div} V(0) \,\mathrm{d}x.$$

Consequently, we can now show the result by calculating as in Theorem 7.5.

□

We point out that we cannot reformulate (19.13) as in Section 7.2, even if we assume more regularity on the data. The reason for that is that for those calculations we would have to define an adjoint state $\boldsymbol{q}$ similar as in Theorem 19.1 and in particular $\boldsymbol{q}$ would have to fulfill

$$\operatorname{div} \boldsymbol{q} = \sum_{i=1}^m (\vartheta_i - \mathrm{D}h(p_\varepsilon)) \chi_{M_i}.$$

This is only an L^2–function, and will not obtain H^1-regularity because of the jumps at the boundary of M_i. So we cannot apply regularity theory to the adjoint system and get $\boldsymbol{H}^2$-regularity for the adjoint variable. But this was necessary for the calculations done in Lemma 7.6.

Nevertheless we can still show the following result which is comparable to the discussion carried out in Section 7.3:

Lemma 19.4. *Assume* $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \Phi_p \times \boldsymbol{U} \times L^2_M(\Omega)$ *with* $(\boldsymbol{u}_\varepsilon, p_\varepsilon) = \boldsymbol{S}^P_\varepsilon(\varphi_\varepsilon)$ *fulfill the variational inequality* (19.8)*. Then it holds*

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon) \operatorname{div}\left(|\boldsymbol{u}_\varepsilon|^2 V(0)\right) \mathrm{d}x + \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon \mathrm{D}\varphi_\varepsilon V(0) \,\mathrm{d}x + \\
&+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right) \operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon \,\mathrm{d}x = -\lambda_\varepsilon \int_\Omega \operatorname{div} V(0)\varphi_\varepsilon \,\mathrm{d}x
\end{aligned} \tag{19.16}$$

for all $V \in \overline{\mathcal{V}}^p_{ad}$*, where the adjoint state* $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}^1_0(\Omega)$*, the Lagrange multiplier* $\lambda_\varepsilon \in \mathbb{R}^+$ *and* ϑ_ε *are given as in Theorem 19.1.*

Remark 19.5. *In particular every minimizer* $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \Phi_p \times \boldsymbol{U} \times L^2_M(\Omega)$ *of* (19.1)-(19.3) *fulfills the variational inequality* (19.8) *if* ***(A11)*** *and* (19.7) *are fulfilled and therefore we obtain due to Lemma 19.4 also* (19.16)*.*

Proof. This can be shown by carrying out the same calculations as in Section 7.3. □

20 Sharp interface model

This section contains the introduction and discussion of the sharp interface model that describes the shape and topology optimization problem with an objective functional involving the pressure and the velocity of a fluid which is described by the Stokes equations. Again the state equations are considered in view of solvability before deriving first order optimality conditions for minimizers of the overall optimization problem. The existence of minimizers for this problem is not guaranteed in general, but will be a consequence of the sharp interface convergence result provided in Section 21.1 under suitable assumptions. We only discuss briefly the main results and refer to the first part for details.

Before introducing the sharp interface model, we want to discuss the pressure in a Stokes flow if the equations are solved in a general measurable set.

20.1 Considering the pressure in measurable sets

In the sharp interface limit we will result in a special form of the Stokes equations that have to be fulfilled in some Caccioppoli set. The existence of the associated velocity field follows from Lax-Milgram's theorem, see Section 6.1, whereas the existence of the associated pressure is not so obvious. Standard results, see for instance [Tem77, Proposition 1.1], define the gradient of a distribution in an open set Ω, which is only in $L^2(\Omega)$ if Ω has Lipschitz boundary. But since Caccioppoli sets are in general not open, and in particular not Lipschitz, we want to extend this result to our situation, which is the topic of this subsection.

Lemma 20.1. *Let $E \subset \Omega$ be a measurable set and $\boldsymbol{u} \in \boldsymbol{U}^E$ such that*

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}^E. \tag{20.1}$$

Then there exists some $p \in L^2(E)$ such that

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x - \int_E p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v}|_{\Omega \setminus E} = \boldsymbol{0}. \tag{20.2}$$

Proof. We denote by $\varphi := 2\chi_E - 1 \in L^1(\Omega, \{\pm 1\})$ the function associated to the measurable set E. For $\varepsilon > 0$ we define $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$ as a solution to

$$\int_\Omega \alpha_\varepsilon(\varphi)\, \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{u}_\varepsilon \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_E \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{20.3}$$

which exists for example due to Lemma 5.1 and means that $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi)$. Defining $\varphi_\varepsilon := \varphi$ for all $\varepsilon > 0$ we see as in the proof of Lemma 6.3 that (after possibly choosing a subsequence) $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon \searrow 0$ and $\lim_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi) |\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega \alpha_0(\varphi) |\boldsymbol{u}|^2 \,\mathrm{d}x = 0$. Now from (20.3) and using the convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ we see that $(\alpha_\varepsilon(\varphi)\, \boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $\boldsymbol{V}'$ and thus there exists some $A \in \boldsymbol{V}'$ such that

$$\lim_{\varepsilon \searrow 0} \int_\Omega \alpha_\varepsilon(\varphi)\, \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x = A(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

and so passing to the limit in (20.3) gives

$$A(\boldsymbol{v}) + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_E \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \operatorname{div} \boldsymbol{v} = 0.$$

For some $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ with $\boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$ we obtain

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v}\,\mathrm{d}x = \int_{\{\varphi=-1\}} \alpha_\varepsilon(\varphi)\,\boldsymbol{u}_\varepsilon \cdot \underbrace{\boldsymbol{v}}_{=\mathbf{0}}\,\mathrm{d}x + \int_{\{\varphi=1\}} \underbrace{\alpha_\varepsilon(\varphi)}_{=0}\,\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v}\,\mathrm{d}x = 0. \tag{20.4}$$

So we know that we can extend A to a linear, continuous functional on

$$\left(\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}\right).$$

Since $\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}$ is a linear and closed subspace of $\boldsymbol{H}_0^1(\Omega)$ we can extend A to a linear and continuous functional on $\boldsymbol{H}_0^1(\Omega)$ by defining

$$A(\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \left(\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}\right)^\perp$$

where $\left(\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}\right)^\perp$ denotes the orthogonal complement of $\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}$ in $\boldsymbol{H}_0^1(\Omega)$.
Using Lemma 4.4 we thus can conclude that there exists some $p \in L^2(\Omega)$ such that

$$A(\boldsymbol{v}) + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\,\mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{20.5}$$

Since due to (20.4) it holds $A(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that $\boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$, this implies in particular

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\,\mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$$

and so $p|_E$ is a pressure associated to $\boldsymbol{u}$ fulfilling (20.2).

□

One question that arises during these considerations is, if the set $\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}\right\}$ can be identified with $\boldsymbol{H}_0^1(\mathrm{int}(E))$, because then Lemma 20.1 would define a pressure $p \in L^2(\mathrm{int}(E))$ associated to the Stokes equations that are fulfilled in $\mathrm{int}(E)$, whereas $\mathrm{int}(E)$ is not a Lipschitz set as it is necessary for the classical results (see for instance [Soh01, Tem77, Gal11]). In those results the lack of boundary regularity implies that the pressure can only be found in L^2_{loc} of the corresponding subset.
These sets actually can be identified in some sense, as the following discussion shows:
Lemma 3.2 together with Remarks 3.1 and 3.2 imply, that every Caccioppoli set E has a crack free representative E_c for which it holds in particular

$$\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \mathbf{0} \text{ a.e. in } \Omega \setminus E_c\right\} = \boldsymbol{H}_0^1(\mathrm{int}E_c) = \boldsymbol{H}_0^1(\mathrm{int}\overline{E}_c).$$

Now fixing this representative E_c, we can solve the Stokes equation in $\mathrm{int}E_c$ in the sense of (20.1) and obtain due to Lemma 20.1 an associated pressure $p \in L^2(\mathrm{int}E_c)$, which is more than the standard results stated above have proven so far.

20.2 Statement of the sharp interface model

Even though we could define one pressure in the usual way for the sharp interface equations, as stated in Lemma 20.1, this is not the situation we want to consider. The reason for this is, as already discussed in Section 18.1, that it is not clear which conditions to state to get uniqueness of this pressure, since the Caccioppoli set considered there may have varying, or even infinitely many, connected components. In particular we cannot fix the connected components, since topological changes are allowed during the optimization process and are one key ingredient of the model.
Instead of that, we want to define the pressure only in the domains M_i, which are fixed and assumed to contain fluid, in the way we've done it in Section 19. Thus our overall problem in the sharp interface limit is given by

$$\min_{(\varphi,\boldsymbol{u},p)} J_0^P(\varphi,\boldsymbol{u},p) = \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \tag{20.6}$$

with

$$(\varphi,\boldsymbol{u},p) \in \Phi_p^0 \times \boldsymbol{U}^\varphi \times L_M^2(\Omega)$$

such that

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{V}^\varphi, \tag{20.7}$$

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{H}_0^1(M_i),\ i=1,\dots,m. \tag{20.8}$$

We see, that (20.8) defines a pressure by

$$\nabla p = \mu\Delta\boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{H}^{-1}(M_i)$$

which is well-defined due to Lemma 20.2. On the other hand, we find by using (20.7) and Lemma 20.1 the existence of some $p_E \in L^2(E^\varphi)$, such that

$$\nabla p_E = \mu\Delta\boldsymbol{u} + \boldsymbol{f} \in (\boldsymbol{V}^\varphi)'.$$

And so, there exist constants $m_i \in \mathbb{R}$, $i=1,\dots,m$, such that

$$p_E|_{M_i} = p|_{M_i} + m_i.$$

For further discussion of this sharp interface model we refer to Part I and in particular to Section 6.1. Roughly speaking, we optimize over all Caccioppoli sets, while a perimeter penalization approach is used to regularize the problem and handle the general ill-posedness of the problem. We restrict in this particular situation the admissible design sets to those having fluid in M_i, thus on those parts where the objective functional depends on the pressure.

In the next lemma, we define a solution operator for the state equations (20.7)-(20.8).

Lemma 20.2. *For every* $\varphi \in L^1(\Omega)$ *such that* $\boldsymbol{U}^{\varphi} \neq \emptyset$ *there exists a unique* $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ *and* $p \in L^2_M(\Omega)$ *such that it holds* (20.7)-(20.8). *This defines a solution operator denoted by*

$$\boldsymbol{S}_0^P : \overline{\Phi}_p^0 \to \boldsymbol{U} \times L^2_M(\Omega),$$

$$\boldsymbol{S}_0^P(\varphi) := (\boldsymbol{u}, p), \quad \textit{with } (\boldsymbol{u}, p) \textit{ fulfill } (20.7) - (20.8).$$

Proof. Existence and uniqueness of $\boldsymbol{u} \in \boldsymbol{U}^{\varphi}$ follow from Lemma 6.1, and the existence and uniqueness of p in $L^2(M_i)$ follows with Lemma 4.4. $\square$

And so we end up in defining the reduced objective functional for the sharp interface model by

$$j_0^P : L^1(\Omega) \to \overline{\mathbb{R}},$$

$$j_0^P(\varphi) := \begin{cases} J_0^P\left(\varphi, \boldsymbol{S}_0^P(\varphi)\right), & \text{if } \varphi \in \Phi_p^0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{20.9}$$

In Section 21 we will show that a minimizer of j_0^P can be obtained as limit of a subsequence of a sequence of minimizers of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$, which were defined in (19.4), if a certain convergence rate is fulfilled. In this setting, we thus obtain in particular existence of a minimizer for the overall optimization problem (20.6)-(20.8).

20.3 Optimality conditions

The aim of this section is deriving first order necessary optimality conditions for the sharp interface problem (20.6) – (20.8). Therefore, we have to assume in this section additionally the differentiability Assumptions **(A6)**, **(A7)** and **(A10)** on the objective functional and the body force.
We point out, that we cannot derive classical shape derivatives in the adjoint formulation to arrive in a Hadamard representation formula as for instance done in Section 8, even if assuming more regularity on the data and on the minimizing set E_0. The reason for that is that we would need an adjoint variable $\boldsymbol{q}_0$, which would then be defined as solution to the following system:

$$\begin{aligned} -\mu\Delta\boldsymbol{q}_0 + \nabla\pi_0 &= \mathrm{D}_u f\left(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}\right) && \text{in } E_0,\\ \operatorname{div}\boldsymbol{q}_0 &= \sum_{i=1}^{m}\left(\vartheta_{0,i} - \mathrm{D}h(p_0)\right)\chi_{M_i} && \text{in } E_0,\\ \boldsymbol{q}_0 &= \boldsymbol{0} && \text{on } \partial E_0. \end{aligned}$$

But for this system, we cannot apply standard regularity results to get $\boldsymbol{q}_0 \in \boldsymbol{H}^2(M_i)$, or even in $\boldsymbol{H}^2(E_0)$, because $\sum_{i=1}^{m}\left(\vartheta_{0,i} - \mathrm{D}h(p_0)\right)\chi_{M_i} \notin H^1(E_0)$.
And so the calculations that have been carried out in Section 8.1, in particular in Theorem 8.1, cannot be applied.

Thus we start directly by calculating first order optimality conditions that don't need additional regularity, similar to Section 8.2.
We recall, that we can rewrite (20.6) – (20.8) as

$$\min_{\varphi\in L^1(\Omega)} j_0^P(\varphi). \tag{20.10}$$

Assume that φ_0 is a fixed minimizer of j_0^P. We introduce for the minimizing set $E_0 = \{\varphi_0 = 1\}$ and some arbitrary $T \in \overline{\mathcal{T}}_{ad}^p$ the notation

$$\varphi_0(t) := \varphi_0 \circ T_t^{-1}, \quad (\boldsymbol{u}_0(t), p_0(t)) := \boldsymbol{S}_0^P\left(\varphi_0(t)\right).$$

We see from Remark 3.4 and Remark 18.7 that $\varphi_0(t) \in \overline{\Phi}_p^0$, since $\varphi_0 \in \overline{\Phi}_p^0$, and T is chosen such that $T_t(x) = x$ for all $x \in M_i$, $i = 1,\dots,m$.

Moreover, we know from Lemma 8.5 that $\mathbb{R} \supset I \ni t \mapsto (\boldsymbol{u}_0(t)\circ T_t) \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$, if I is a suitable small interval around $t = 0$, and applying the idea of the proof of Theorem 19.2 to the proof of Lemma 8.5 we can deduce that $I \ni t \mapsto (p_0(t)\circ T_t) \in L^2(M_i)$ is differentiable at $t = 0$ for all $i = 1,\dots,m$. Moreover $\dot{\boldsymbol{u}}_0[V] := \partial_t|_{t=0}(\boldsymbol{u}_0(t)\circ T_t)$ solves (8.13) – (8.14), and $\dot{p}_0[V] := \partial_t|_{t=0}(p_0(t)\circ T_t) \in L^2_M(\Omega)$ is the pressure associated to $\dot{\boldsymbol{u}}_0[V]$ in the sense of (20.14).

Then we get by direct calculations and by using the arguments of Theorem 8.2 the following necessary optimality conditions:

Theorem 20.1. *Assume $\varphi_0 \in \Phi_p^0$ with $(\boldsymbol{u}_0, p_0) = \boldsymbol{S}_0^P(\varphi_0) \in \boldsymbol{U}^{\varphi_0} \times L_M^2(\Omega)$ is a minimizer of* (20.10)*. Then we have the following necessary optimality condition:*

$$\partial_t|_{t=0} j_0^P \left(\varphi_0 \circ T_t^{-1}\right) = -\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0) \,\mathrm{d}x, \tag{20.11}$$

$$\lambda_0 \left(\int_\Omega \varphi_0 \,\mathrm{d}x - \beta |\Omega| \right) = 0 \tag{20.12}$$

for every $T \in \overline{\mathcal{T}}_{ad}^p$ with velocity field $V \in \overline{\mathcal{V}}_{ad}^p$ with some Lagrange multiplier $\lambda_0 \geq 0$ for the integral constraint. The derivative is given by the following formula:

$$\begin{aligned}\partial_t|_{t=0} j_0^P \left(\varphi_0 \circ T_t^{-1}\right) = &\int_\Omega [\mathrm{D}f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0)(V(0), \dot{\boldsymbol{u}}_0[V], \mathrm{D}\dot{\boldsymbol{u}}_0[V] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)) + \\ &+ f(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0) \operatorname{div} V(0)] \,\mathrm{d}x + \int_\Omega \mathrm{D}h(p_0) \dot{p}_0[V] + h(p_0) \operatorname{div} V(0) \,\mathrm{d}x + \\ &+ \gamma c_0 \int_\Omega (\operatorname{div} V(0) - \nu \cdot \nabla V(0) \nu) \,\mathrm{d}|\mathrm{D}\chi_{E_0}|\end{aligned} \tag{20.13}$$

with ν being the generalised unit normal on $E_0 = \{\varphi_0 = 1\}$. Moreover $\dot{\boldsymbol{u}}_0[V] \in \boldsymbol{H}_0^1(\Omega)$ with $\dot{\boldsymbol{u}}_0[V] = \boldsymbol{0}$ a.e. in $\Omega \setminus E_0$ fulfills (8.13)-(8.14) *and $\dot{p}_0[V] \in L^2(\Omega)$ with $\dot{p}_0[V] = 0$ in $\Omega \setminus \cup_{i=1}^m M_i$ is the pressure associated to $\dot{\boldsymbol{u}}_0[V]$ by*

$$\begin{aligned}&\int_{E_0} \dot{p}_0[V] \operatorname{div} \boldsymbol{z}_i \,\mathrm{d}x = \int_{E_0} \mu \nabla \dot{\boldsymbol{u}}_0[V] : \nabla \boldsymbol{z}_i \,\mathrm{d}x - \int_{E_0} \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z}_i \,\mathrm{d}x - \\ &- \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \mathrm{D}V(0)^T \nabla \boldsymbol{z}_i \,\mathrm{d}x + \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z}_i \operatorname{div} V(0) \,\mathrm{d}x - \int_{E_0} (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}_i \,\mathrm{d}x - \\ &- \int_{E_0} \boldsymbol{f} \cdot \boldsymbol{z}_i \operatorname{div} V(0) \,\mathrm{d}x + \int_{E_0} p_0 \mathrm{D}\boldsymbol{z}_i : \mathrm{D}V(0) - p_0 \operatorname{div} \boldsymbol{z}_i \operatorname{div} V(0) \,\mathrm{d}x\end{aligned} \tag{20.14}$$

for all $\boldsymbol{z} = (\boldsymbol{z}_i)_{i=1}^m \in \times_{i=1}^m \boldsymbol{H}_0^1(M_i)$, fulfilling additionally

$$\int_{M_i} \dot{p}_0[V] \,\mathrm{d}x = -\int_{M_i} p_0 \operatorname{div} V(0) \,\mathrm{d}x \quad \forall i = 1, \ldots, m.$$

To summarize this section, we have derived a sharp interface model considering shape and topology optimization in a Stokes flow that can handle pressure functionals. We have shown solvability of the state equations. Finally, we can also state first order optimality conditions for this problem. The existence of a minimizer is still an open problem, but is guaranteed under certain assumptions, see Theorem 21.1.

21 Sharp interface limit

After having introduced and discussed the phase field model in Section 19 and the sharp interface model in Section 20, we wish to investigate in this section the relation between those two approaches. We will see similar to the first part that the phase field model approximates the sharp interface model in the following sense: we will find that any sequence of minimizers of the reduced objective functionals $\left(j_\varepsilon^P\right)_{\varepsilon>0}$ has a subsequence that converges in $L^1(\Omega)$. If the latter fulfills some convergence rate, we can deduce that the limit element is a minimizer of the reduced objective functional of the sharp interface problem j_0^P and that the minimal functional values then converge, too. In particular, this will yield existence of minimizers for j_0^P if the converging subsequence of the sequence of minimizers of the phase field problems fulfills the stated convergence rate. It is also shown that the optimality conditions of the sharp interface description deduced in Section 20.3 can then be obtained as a limit system of optimality conditions for the phase field model. We will touch only the main aspects and point out the differences to the first part and refer for a detailed analysis to Section 6.2 and Section 9.

21.1 Convergence of minimizers

In this section we extend the result of Theorem 6.1 to the setting including pressure functionals. To be more precise, it is shown that any sequence of minimizers of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$, defined by (19.4), has a subsequence that converges in $L^1(\Omega)$. If the converging subsequence fulfills a certain convergence rate depending on ε, then we find that the limit element is a minimizer of j_0^P, which is defined by (20.9). We want to use the construction of Theorem 6.1 to get a recovery sequence. But therefore we have to ensure that the smooth sets given by Lemma 3.1 approximating our Caccioppoli set still contain M_i for all $i = 1, \ldots, m$, such that those sets are admissible. Thus, we need the following adapted version of Lemma 3.1:

Lemma 21.1. *Let E be a measurable subset of Ω. If $\left(E \setminus \bigcup_{i=1}^m M_i\right)$ and $\Omega \setminus E$ both contain a non-empty open ball and $\bigcup_{i=1}^m M_i \subset E$, then there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of open subset of Ω such that*

1. *$\partial E_n \cap \Omega \in C^2$ for n large enough,*
2. *$\lim_{n\to\infty} |E_n \Delta E| = 0, \qquad \lim_{n\to\infty} P_\Omega(E_n) = P_\Omega(E)$,*
3. *$|E_n| = |E|$ for n large enough,*
4. *$\cup_{i=1}^m M_i \subseteq E_n$ for all n large enough,*
5. *$d(\partial M_i \cap \Omega, \partial E_n \cap \Omega) > 0$ for n large enough and all $i = 1, \ldots, m$.*

Moreover, we get the following convergence rate:

$$|E_n \Delta E| = \mathcal{O}\left(n^{-1}\right). \tag{21.1}$$

To prove this lemma we first of all consider the simplified situation where $\partial M_i \cap \Omega$ has a positive distance to $\partial E \cap \Omega$ for all $i = 1, \ldots, m$:

Lemma 21.2. *Let E be a measurable subset of Ω such that $d(\partial M_i \cap \Omega, \partial E \cap \Omega) > 0$ for all $i = 1, \dots, m$. If $E \setminus \bigcup_{i=1}^m M_i$ and $\Omega \setminus E$ both contain a non-empty open ball, then there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of open subset of Ω such that Properties 1.-5. of Lemma 21.1 together with the convergence rate (21.1) are fulfilled for $(E_n)_{n\in\mathbb{N}}$.*

Proof. We adapt the construction of [Mod87, proof of Lemma 1] to our desired result. Let us therefore give the main ideas of this proof and outline its modifications.

Choosing $\varphi := \chi_E$ and standard mollifiers $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\phi_\varepsilon \subseteq B_\varepsilon(0)$, $0 \le \phi_\varepsilon$, $\int_{\mathbb{R}^d} \phi_\varepsilon \,\mathrm{d}x = 1$ we can define

$$\varphi_\varepsilon := \varphi * \phi_\varepsilon$$

and see that

$$\lim_{\varepsilon\searrow 0} \int_\Omega |\varphi - \varphi_\varepsilon| \,\mathrm{d}x = 0,$$

$$\lim_{\varepsilon\searrow 0} |\{x \in \Omega \mid |\varphi_\varepsilon(x) - \varphi(x)| \ge \eta\}| = 0 \qquad \forall \eta > 0$$

and

$$\lim_{\varepsilon\searrow 0} \int_\Omega |\mathrm{D}\varphi_\varepsilon| \,\mathrm{d}x = \int_\Omega \mathrm{d}|\mathrm{D}\varphi| = P_\Omega(E).$$

Now let $x_1 \in E \setminus \bigcup_{i=1}^m M_i$, $x_2 \in \Omega \setminus E$ and $\delta_0 > 0$ be chosen such that

$$B_1 := B_{\delta_0}(x_1) \subset E \setminus \bigcup_{i=1}^m M_i, \qquad B_2 := B_{\delta_0}(x_2) \subset \Omega \setminus E$$

which implies

$$\varphi_\varepsilon = \varphi \quad \text{on } B_1 \cup B_2 \qquad \forall \varepsilon < \frac{\delta_0}{2}. \tag{21.2}$$

For every $n \in \mathbb{N}$ let $\varepsilon_n < \min\left\{\frac{1}{n}, \frac{\delta_0}{2}\right\}$ be such that

$$\left|\left\{x \in \Omega \mid |\varphi_{\varepsilon_n}(x) - \varphi(x)| \ge \frac{1}{n}\right\}\right| \le \frac{1}{n}. \tag{21.3}$$

Using the notation

$$\nu_n = \operatorname{essinf}_{\frac{1}{n} \le t \le 1-\frac{1}{n}} P_\Omega(\{x \in \Omega \mid \varphi_{\varepsilon_n}(x) > t\})$$

we obtain with the help of Sard's Lemma the existence of $t_n \in \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ such that

$$P_\Omega(\{x \in \Omega \mid \varphi_{\varepsilon_n}(x) > t_n\}) \le \nu_n + \frac{1}{n} \tag{21.4}$$

and

$$\mathrm{D}\varphi_{\varepsilon_n}(x) \ne 0 \qquad \forall x \in \Omega : \varphi_{\varepsilon_n}(x) = t_n.$$

Denoting

$$\widetilde{E}_n := \{x \in \Omega \mid \varphi_{\varepsilon_n}(x) > t_n\}, \quad \widehat{E}_n := \left\{x \in \mathbb{R}^d \mid \varphi_{\varepsilon_n}(x) > t_n\right\} \tag{21.5}$$

and

$$\lambda_n := \left|\widetilde{E}_n\right| - |E|$$

we come to defining E_n by

$$E_n := \begin{cases} \widetilde{E}_n \setminus B_{r_n}(x_1) & \text{if } \lambda_n > 0, \\ \widetilde{E}_n & \text{if } \lambda_n = 0, \\ \widetilde{E}_n \cup B_{r_n}(x_2) & \text{if } \lambda_n < 0, \end{cases} \tag{21.6}$$

with r_n such that $|B_{r_n}(x_1)| = |B_{r_n}(x_2)| = |\lambda_n|$.
In addition we define the following sets:

$$F_n := \begin{cases} \widehat{E}_n \setminus B_{r_n}(x_1) & \text{if } \lambda_n > 0, \\ \widehat{E}_n & \text{if } \lambda_n = 0, \\ \widehat{E}_n \cup B_{r_n}(x_2) & \text{if } \lambda_n < 0, \end{cases}$$

and observe that $E_n = F_n \cap \Omega$. We note that

$$x \in \widetilde{E}_n \setminus E \implies \varphi_{\varepsilon_n}(x) > t_n > \frac{1}{n}, \qquad \varphi(x) = 0$$

and

$$x \in E \setminus \widetilde{E}_n \implies \varphi_{\varepsilon_n}(x) \le t_n < 1 - \frac{1}{n}, \qquad \varphi(x) = 1.$$

Thus using (21.3) we obtain:

$$|\lambda_n| \le \left|\widetilde{E}_n \Delta E\right| \le \left|\left\{x \in \Omega \mid |\varphi_{\varepsilon_n}(x) - \varphi(x)| \ge \frac{1}{n}\right\}\right| \le \frac{1}{n} \tag{21.7}$$

which leads to

$$\lim_{n\to\infty} r_n = 0$$

and since for n large enough it holds $\delta_0 > r_n$ we end up with

$$\overline{B_{r_n}(x_1)} \subset B_1, \qquad \overline{B_{r_n}(x_2)} \subset B_2.$$

From $\varepsilon_n < \frac{\delta_0}{2}$ and (21.2) we know that

$$B_1 \subseteq \widetilde{E}_n \qquad B_2 \subseteq \Omega \setminus \widetilde{E}_n$$

and so

$$|E_n| = \left|\widetilde{E}_n\right| - |B_{r_n}(x_1)| = |E| \qquad \text{if } \lambda_n > 0,$$
$$|E_n| = \left|\widetilde{E}_n\right| + |B_{r_n}(x_2)| = |E| \qquad \text{if } \lambda_n < 0$$

wherefrom the third statement follows.

Denoting $M := \bigcup_{i=1}^m M_i$ we obtain that for almost every $x \in \overline{M}$, where $\overline{M} \subseteq \overline{\Omega}$ is the closure of M in $\mathbb{R}^d$, there exists some $n(x)$ such that $x \in \operatorname{int} F_n$ for all $n \ge n(x)$ and so

$$\overline{M} \subseteq \bigcup_{x\in\overline{M}} \operatorname{int} F_{n(x)}.$$

Since $\overline{M}$ is compact, we can choose finitely many $\left\{F_{n(x_i)} \mid x_i \in \overline{M}, i = 1, \dots, N\right\}$ such that

$$\overline{M} \subseteq \bigcup_{i=1}^N \operatorname{int} F_{n(x_i)}.$$

Defining $\overline{n} := \max_{i=1,\ldots,N} n(x_i)$ we see that

$$\overline{M} \subseteq \mathrm{int} F_n \quad \forall n \geq \overline{n}$$

and so the last two statements of the lemma follow from the fact that $E_n = F_n \cap \Omega$. The second property than can be shown exactly as in the proof of [Mod87, Lemma 1]. Finally, we conclude the convergence rate (21.1) from (21.7) and (21.6).

□

Due to the fact that in general we are not in the setting of Lemma 21.2, where the sets $\overline{M}_i$ have a positive distance to E in Ω, we now turn to the general case:

Proof of Lemma 21.1. Since we assume $\sum_{i=1}^{M} |M_i| < \beta |\Omega|$, $\beta \in (-1, 1)$ and $|E| < |\Omega|$ we see that there exists some $\varepsilon > 0$ such that $B_\varepsilon(E) \cap \Omega \subset \Omega$, $\overline{E} \subsetneq B_\varepsilon(E) \cap \Omega$, and we define

$$F_\varepsilon := B_\varepsilon(E) \cap \Omega.$$

We see that $d(\partial M_i \cap \Omega, \partial F_\varepsilon \cap \Omega) > 0$ for all $i = 1, \ldots, m$, and are now in the situation of Lemma 21.2. So we can choose for every $\varepsilon > 0$ a sequence of sets $(E_n^\varepsilon)_{n\in\mathbb{N}}$ such that

$$\partial E_n^\varepsilon \cap \Omega \in C^2, \quad \lim_{n\to\infty} |E_n^\varepsilon \Delta F_\varepsilon| = 0, \quad \lim_{n\to\infty} P_\Omega(E_n^\varepsilon) = P_\Omega(F_\varepsilon), \quad \bigcup_{i=1}^{m} M_i \subseteq E_n^\varepsilon,$$

$$d(\partial M_i \cap \Omega, \partial E_n^\varepsilon \cap \Omega) > 0,\ \forall i = 1, \ldots, m, \quad |E_n^\varepsilon| = |F_\varepsilon|, \qquad \forall n \gg 1$$

and

$$|E_n^\varepsilon \Delta F_\varepsilon| = \mathcal{O}\left(n^{-1}\right).$$

Having a closer look at the volume of those sets, we remark that we don't want the volume of E_n^ε to be equal to the volume of F_ε, but merely to the volume of E, which is smaller. Thus we take in much the same way as in Lemma 3.1 some $x_1 \in E \setminus \bigcup_{i=1}^{m} M_i$ and $\delta_0 > 0$ such that

$$B_{\delta_0}(x_1) \subset E \setminus \bigcup_{i=1}^{m} M_i.$$

Then we define

$$\lambda_\varepsilon := |E_n^\varepsilon| - |E| = |F_\varepsilon| - |E| = \mathcal{O}(\varepsilon), \quad \forall n \gg 1$$

and let

$$\widetilde{E}_n^\varepsilon := E_n^\varepsilon \setminus B_{r_\varepsilon}(x_1)$$

with r_ε such that $|B_{r_\varepsilon}(x_1)| = |\lambda_\varepsilon|$, and $\varepsilon > 0$ small enough, such that $r_\varepsilon < \delta_0$. From that

$$\left|E \Delta \widetilde{E}_n^\varepsilon\right| \leq \underbrace{|E \Delta F_\varepsilon|}_{=\mathcal{O}(\varepsilon)} + \left|F_\varepsilon \Delta \widetilde{E}_n^\varepsilon\right|$$

and

$$\begin{aligned}\left|F_\varepsilon \Delta \widetilde{E}_n^\varepsilon\right| &= |F_\varepsilon \Delta E_n^\varepsilon \cup (B_\varepsilon(E) \cap B_{r_\varepsilon}(x_1))| \leq |F_\varepsilon \Delta E_n^\varepsilon| + |B_{r_\varepsilon}(x_1)| = \\ &= \underbrace{|F_\varepsilon \Delta E_n^\varepsilon|}_{=\mathcal{O}(n^{-1})} + \underbrace{\lambda_\varepsilon}_{=\mathcal{O}(\varepsilon)}.\end{aligned}$$

Consequently, we obtain for a diagonal sequence $\left(\widetilde{E}^{\varepsilon}_{n_\varepsilon}\right)_{\varepsilon>0}$ that

$$\left|\widetilde{E}^{\varepsilon}_{n_\varepsilon}\Delta E\right| = \mathcal{O}(\varepsilon), \quad \lim_{\varepsilon\searrow 0}\left|\widetilde{E}^{\varepsilon}_{n_\varepsilon}\Delta E\right| = 0$$

together with

$$\left|\widetilde{E}^{\varepsilon}_{n_\varepsilon}\right| = |E|, \quad \bigcup_{i=1}^{m} M_i \subseteq \widetilde{E}^{\varepsilon}_{n_\varepsilon} \qquad \forall\varepsilon \ll 1$$

$$d\left(\partial M_i \cap \Omega, \partial\widetilde{E}^{\varepsilon}_{n_\varepsilon} \cap \Omega\right) > 0,\ \forall i = 1,\dots,m, \qquad \partial\widetilde{E}^{\varepsilon}_{n_\varepsilon} \cap \Omega \in C^2.$$

It remains to consider the perimeter. As we know

$$\lim_{\varepsilon\searrow 0}|\Omega \cap (F_\varepsilon \Delta E)| = 0$$

we obtain thanks to the lower semicontinuity of the perimeter

$$P_\Omega(E) \le \liminf_{\varepsilon\searrow 0} P_\Omega(F_\varepsilon).$$

Applying integration by substitution we can calculate

$$P_\Omega(F_\varepsilon) = \int_{\mathbb{R}^d} \mathrm{d}\left|\mathrm{D}\chi_{B_\varepsilon(E)\cap\Omega}\right| = \int_{\mathbb{R}^d} \mathrm{d}\left|\mathrm{D}\chi_{(1+\varepsilon)E\cap\Omega}\right| \le (1+\varepsilon)^{d-1}\mathrm{d}\int_{\mathbb{R}^d}\left|\mathrm{D}\chi_{E\cap\Omega}\right|$$

which gives

$$\limsup_{\varepsilon\searrow 0} P_\Omega(F_\varepsilon) \le P_\Omega(E)$$

and we can conclude

$$\lim_{\varepsilon\searrow 0} P_\Omega(F_\varepsilon) = P_\Omega(E).$$

Moreover, we have due to Lemma 3.1

$$\lim_{n\to\infty} P_\Omega(E^\varepsilon_n) = P_\Omega(F_\varepsilon)$$

and so

$$\lim_{n\to\infty} P_\Omega\left(\widetilde{E}^\varepsilon_n\right) = \lim_{n\to\infty}\left(P_\Omega(E^\varepsilon_n) + \mathcal{H}^{d-1}(\partial B_{r_\varepsilon}(x_1))\right) = P_\Omega(F_\varepsilon) + \underbrace{\mathcal{H}^{d-1}(\partial B_{r_\varepsilon}(x_1))}_{\xrightarrow{\varepsilon\searrow 0} 0}.$$

Plugging these results together we see, that for the diagonal sequence $\left(\widetilde{E}^{\varepsilon}_{n_\varepsilon}\right)_{\varepsilon>0}$ it holds

$$\lim_{\varepsilon\searrow 0} P_\Omega\left(\widetilde{E}^{\varepsilon}_{n_\varepsilon}\right) = P_\Omega(E)$$

which finishes the proof.

□

We are now able to prove the main result of this subsection:

Theorem 21.1. *Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be minimizers of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$. Then there exists a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$, which is denoted by the same, and an element $\varphi_0 \in L^1(\Omega)$ such that*

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0. \tag{21.8}$$

If it holds

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{21.9}$$

then we obtain moreover

$$\lim_{\varepsilon \searrow 0} j_\varepsilon^P(\varphi_\varepsilon) = j_0^P(\varphi_0) \tag{21.10}$$

and φ_0 is a minimizer of j_0^P.

Remark 21.1. *Thus in the special situation of Theorem 21.1 the existence of minimizers of j_0^P is guaranteed, whereas this is still an open problem in general.*
The existence of minimizers φ_ε of j_ε^P for $\varepsilon > 0$ follows from Lemma 19.2.

Proof. We proceed similarly as in the proof of Theorem 6.1. Our main concern will be the behaviour of the pressure and for some detailed analysis concerning the remaining aspects we refer to Theorem 6.1.
We split the proof into several steps:

- *1st step:* Let $\varphi \in L^1(\Omega)$ be chosen arbitrary such that $j_0(\varphi) < \infty$. We start by constructing a recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ by using the ideas of Theorem 6.1. Thus we approximate E^φ with Lemma 21.1 by more regular sets $(E_n)_{n\in\mathbb{N}}$ with $\partial E_n \cap \Omega \in C^2$, $|E_n| = |E^\varphi|$, $d(\partial M_i \cap \Omega, \partial E_n \cap \Omega) > 0$ for all $i = 1, \ldots, m$ and

  $$\lim_{n\to\infty} P_\Omega(E_n) = P_\Omega(E^\varphi), \quad \lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^1(\Omega)} = 0$$

  where we defined $\varphi_n := 2\chi_{E_n} - 1$. Additionally we obtain the following convergence rate:

  $$\|\varphi_n - \varphi\|_{L^1(\Omega)} = \mathcal{O}\left(n^{-1}\right). \tag{21.11}$$

  Let us introduce the abbreviation:

  $$d_n := d\left(\bigcup_{i=1}^m \partial M_i \cap \Omega, \partial E_n \cap \Omega\right) > 0.$$

  An analogous construction as in Theorem 6.1 gives for every $n \gg 1$ sequences $(\varphi_\varepsilon^n)_{\varepsilon>0} \subseteq H^1(\Omega)$ such that

  $$\limsup_{\varepsilon \searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2} |\nabla \varphi_\varepsilon^n|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon^n)\right) \mathrm{d}x \le \gamma c_0 P_\Omega(E_n).$$

  We observe from this construction in particular that

  $$\{\varphi_\varepsilon^n = 1\} \subseteq E_n, \quad d_\varepsilon^n := d(\{\varphi_\varepsilon^n = 1\}, \partial E_n \cap \Omega) \le 2\eta_\varepsilon = \mathcal{O}(\varepsilon).$$

And so if we choose $\varepsilon_n^0 > 0$ small enough, such that $d^n_{\varepsilon_0^n} < d_n$, which implies $d^n_\varepsilon < d_n$ for all $\varepsilon < \varepsilon_n^0$, it holds that $M_i \subset \{\varphi_\varepsilon^n = 1\}$ for all $\varepsilon < \varepsilon_n^0$ and all $i = 1, \dots, m$.

Then we can choose a diagonal sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n := \varphi^n_{\varepsilon_n}$, such that ε_n fulfills $\varepsilon_n < \varepsilon_n^0$.
Hence we obtain that $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $L^1(\Omega)$ as $n \to \infty$,

$$\limsup_{n\to\infty} \int_\Omega \left(\frac{\gamma\varepsilon_n}{2} |\nabla\varphi_n|^2 + \frac{\gamma}{\varepsilon_n}\psi(\varphi_n) \right) \mathrm{d}x \le \gamma c_0 P_\Omega(E^\varphi),$$

and $\varphi_n|_{M_i} = 1$ for all $i = 1, \dots, m$. Moreover, we find in view of (6.26) and (21.11)

$$\|\varphi_n - \varphi\|_{L^1(\Omega)} = \mathcal{O}\left(n^{-1}\right).$$

- *2nd step:* Let $\varphi \in L^1(\Omega)$ be chosen such that $j_0(\varphi) < \infty$. In the first step we have shown that we can construct admissible $(\varphi_\varepsilon)_{\varepsilon>0}$ converging to $\varphi \in L^1(\Omega)$ with the rate
$$\|\varphi_\varepsilon - \varphi\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon)$$
such that
$$\limsup_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon) \right) \mathrm{d}x \le \gamma c_0 P_\Omega(E^\varphi).$$

  We define $(\boldsymbol{u}_\varepsilon, p_\varepsilon) := \boldsymbol{S}^P_\varepsilon(\varphi_\varepsilon)$ and $(\boldsymbol{u}, p) := \boldsymbol{S}^P_0(\varphi)$. According to Lemma 6.3 this yields, after possible choosing a subsequence, that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ and $\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega \alpha_0(\varphi)|\boldsymbol{u}|^2 \,\mathrm{d}x=0$.

  Let $i \in \{1, \dots, m\}$ be fixed. Then we use (4.1) to obtain from the state equations (19.3) and (20.8)

$$\begin{aligned}\|p_\varepsilon - p\|_{L^2(M_i)} &\le c(M_i) \|\nabla p_\varepsilon - \nabla p\|_{\boldsymbol{H}^{-1}(M_i)} \le \\ &\le c(M_i) \|\mu(\Delta\boldsymbol{u}_\varepsilon - \Delta\boldsymbol{u}) - \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^{-1}(M_i)}.\end{aligned} \tag{21.12}$$

  Since $\varphi_\varepsilon \in \Phi_p$ per construction, it follows $\varphi_\varepsilon|_{M_i} = 1$ and so $\alpha_\varepsilon(\varphi_\varepsilon(x)) = 0$ for a.e. $x \in M_i$. Therefore we can estimate

$$\|p_\varepsilon - p\|_{L^2(M_i)} \le c(M_i)\mu \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \xrightarrow{\varepsilon\searrow 0} 0. \tag{21.13}$$

  Knowing that this holds true for any $i = 1, \dots, m$, and $p_\varepsilon|_{\Omega\setminus\cup_{i=1}^m M_i} = p|_{\Omega\setminus\cup_{i=1}^m M_i} = 0$ this gives
$$\lim_{\varepsilon\searrow 0} \|p_\varepsilon - p\|_{L^2(\Omega)} = 0.$$

  Using the continuity of the objective functional we end up with
$$\limsup_{\varepsilon\searrow 0} j^P_\varepsilon(\varphi_\varepsilon) \le j^P_0(\varphi).$$

- *3rd step:* Now let $\varphi \in L^1(\Omega)$ be arbitrary and $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)$ any sequence converging to φ in $L^1(\Omega)$ as $\varepsilon \searrow 0$ such that

$$\|\varphi_\varepsilon - \varphi\|_{L^1(\{x\in\Omega|\varphi(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{21.14}$$

is fulfilled. We may assume without loss of generality

$$\liminf_{\varepsilon\searrow 0} j_\varepsilon^P(\varphi_\varepsilon) < \infty$$

and thus choose a subsequence $\left(j_{\varepsilon_k}^P(\varphi_{\varepsilon_k})\right)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} j_{\varepsilon_k}^P(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon^P(\varphi_\varepsilon).$$

Then we get as in the proof of Theorem 6.1 a subsequence $\left(\varphi_{\varepsilon_{k(l)}}\right)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty}\left\|\boldsymbol{u}_{\varepsilon_{k(l)}} - \boldsymbol{u}\right\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{l\to\infty}\int_\Omega \alpha_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right)\left|\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 \mathrm{d}x = 0$$

and get

$$\gamma c_0 P_\Omega(E^\varphi) \le \liminf_{l\to\infty}\int_\Omega\left(\frac{\gamma\varepsilon_{k(l)}}{2}\left|\nabla\varphi_{\varepsilon_{k(l)}}\right|^2 + \frac{\gamma}{\varepsilon_{k(l)}}\psi\left(\varphi_{\varepsilon_{k(l)}}\right)\right)\mathrm{d}x. \tag{21.15}$$

Since $\lim_{l\to\infty} j_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right) < \infty$, we see that $\varphi_{\varepsilon_{k(l)}} \in \Phi_p$ for all $l \gg 1$ and so $\varphi_{\varepsilon_{k(l)}}|_{M_i} = 1$ for all $i = 1,\dots,m$ and $l \gg 1$.
So we can use the convergence of $\left(\boldsymbol{u}_{\varepsilon_{k(l)}}\right)_{l\in\mathbb{N}}$ to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ and the same calculations as in (21.12)-(21.13) to deduce

$$\lim_{l\to\infty}\left\|p_{\varepsilon_{k(l)}} - p\right\|_{L^2(\Omega)} = 0.$$

Using the continuity of the objective functional this gives

$$\begin{aligned} j_0^P(\varphi) &= \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) + h(p)\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \le \liminf_{l\to\infty} j_{\varepsilon_{k(l)}}^P\left(\varphi_{\varepsilon_{k(l)}}\right) = \\ &= \lim_{k\to\infty} j_{\varepsilon_k}^P(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon^P(\varphi_\varepsilon). \end{aligned}$$

- *4th step:* We can now follow the arguments of the fifth step of the proof of Theorem 6.1 to deduce the statement.

□

21.2 Convergence of the optimality system

In this subsection we assume again additionally Assumptions **(A6)**, **(A7)** and **(A10)** to ensure differentiability of the objective functional and enough regularity of the external force.

Similar to Section 9 we show here that not only minimizers of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$ converge under certain assumptions to a minimizer of j_0^P, but also that the corresponding optimality systems of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$ converge then as $\varepsilon \searrow 0$ to the optimality system of j_0^P derived in Section 20.3. We directly state the main result of this subsection.

Theorem 21.2. *Let $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq \Phi_p$ be minimizers of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$. Then there exists a subsequence, which is denoted by the same, that converges in $L^1(\Omega)$ to some $\varphi_0 \in L^1(\Omega)$. Assume moreover that*

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon). \tag{21.16}$$

Then the limit element φ_0 is a minimizer of j_0^P. Moreover, it holds

$$\lim_{\varepsilon\searrow 0} \partial_t|_{t=0} j_\varepsilon^P\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0^P\left(\varphi_0 \circ T_t^{-1}\right) \quad \forall T \in \overline{\mathcal{T}}_{ad}^p. \tag{21.17}$$

If

$$|\{\varphi_0 = 1\}| > 0 \tag{21.18}$$

then we have additionally the following convergence results:

$$\begin{aligned}
\varphi_\varepsilon &\xrightarrow{\varepsilon\searrow 0} \varphi_0 && \text{in } L^1(\Omega), && (21.19a)\\
\boldsymbol{u}_\varepsilon &\xrightarrow{\varepsilon\searrow 0} \boldsymbol{u}_0 && \text{in } \boldsymbol{H}^1(\Omega), && (21.19b)\\
p_\varepsilon &\xrightarrow{\varepsilon\searrow 0} p_0 && \text{in } L^2(\Omega), && (21.19c)\\
\dot{p}_\varepsilon[V] &\xrightarrow{\varepsilon\searrow 0} \dot{p}_0[V] && \text{in } L^2(\Omega), && (21.19d)\\
\dot{\boldsymbol{u}}_\varepsilon[V] &\xrightarrow{\varepsilon\searrow 0} \dot{\boldsymbol{u}}_0[V] && \text{in } \boldsymbol{H}^1(\Omega), && (21.19e)\\
\lambda_\varepsilon &\xrightarrow{\varepsilon\searrow 0} \lambda_0 && \text{in } \mathbb{R}, && (21.19f)\\
j_\varepsilon^P(\varphi_\varepsilon) &\xrightarrow{\varepsilon\searrow 0} j_0^P(\varphi_0) && \text{in } \mathbb{R}, && (21.19g)
\end{aligned}$$

where $(\boldsymbol{u}_\varepsilon, p_\varepsilon) = \boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)$, $(\boldsymbol{u}_0, p_0) = \boldsymbol{S}_0^P(\varphi_0)$, $(\lambda_\varepsilon)_\varepsilon \subseteq \mathbb{R}^+$ are Lagrange multipliers for the integral constraint defined due to Theorem 19.2, $\lambda_0 \in \mathbb{R}^+$ is a Lagrange multiplier such that it holds (20.11) – (20.12)*, and thus is a Lagrange multiplier for the integral constraint in the sharp interface according to Theorem 20.1.*

Proof. First we use the result of Theorem 21.1 to derive that a subsequence of the sequence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0}$ of $\left(j_\varepsilon^P\right)_{\varepsilon>0}$, which will still be indexed by ε, converges in $L^1(\Omega)$ to some element $\varphi_0 \in L^1(\Omega)$. Thus from now on assume that (21.16) is fulfilled. Then we obtain by Theorem 21.1 that φ_0 is a minimizer of j_0^P and (21.19g). As in the proof of Theorem 9.1 we thus apply Lemma 6.3 to get a subsequence such that it holds

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = 0,\quad \lim_{\varepsilon\searrow 0}\|\boldsymbol{u}_\varepsilon-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0.$$

Then we can apply the pressure inequality (4.1) and the state equations (19.14) and (20.14) to derive therefrom for every $i = 1,\dots,m$:

$$\|p_\varepsilon - p_0\|_{L^2(M_i)} \le c(M_i)\,\|\boldsymbol{u}_\varepsilon-\boldsymbol{u}_0\|_{\boldsymbol{H}^1(M_i)} \xrightarrow{\varepsilon\searrow 0} 0 \tag{21.20}$$

where we used

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x = 0\quad \forall \boldsymbol{v}\in\boldsymbol{H}_0^1(M_i) \tag{21.21}$$

which follows from $\varphi_\varepsilon = 1$ a.e. in M_i for all $i = 1,\dots,m$.
Due to $p_\varepsilon, p_0 \in L^2_M(\Omega)$ we obtain $p_\varepsilon|_{\Omega\setminus\cup_{i=1}^m M_i} = p_0|_{\Omega\setminus\cup_{i=1}^m M_i} = 0$ for all $\varepsilon > 0$ and so we can deduce

$$\lim_{\varepsilon\searrow 0}\|p_\varepsilon - p_0\|_{L^2(\Omega)} = 0. \tag{21.22}$$

Using arguments as in Theorem 9.1 one can establish the convergence of $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ to $\dot{\boldsymbol{u}}_0[V]$ in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon\searrow 0$ and find from the state equations (19.14), (20.14) then, that

$$\lim_{\varepsilon\searrow 0}\|\dot{p}_\varepsilon[V] - \dot{p}_0[V]\|_{L^2(\Omega)} = 0$$

while applying estimates as in (21.20)-(21.22). Thanks to the Reshetnyak continuity theorem, see Theorem 3.2, we can show as in the proof of Theorem 9.1 that

$$\lim_{\varepsilon\searrow 0}\int_\Omega\left(\frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div}V(0)\,\mathrm{d}x = c_0\int_\Omega \operatorname{div}V(0)\,\mathrm{d}|\mathrm{D}\chi_{E_0}|$$

and

$$\lim_{\varepsilon\searrow 0}\varepsilon\int_\Omega \nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x = c_0\int_\Omega \nu\cdot\nabla V(0)\nu\,\mathrm{d}|\mathrm{D}\chi_{E_0}|$$

if we use the notation $E_0 = \{\varphi_0 = 1\}$ and denote by ν the generalised unit normal on E_0. Thus, we finally deduce (21.17) from the already shown convergence results. The convergence of the Lagrange multipliers $(\lambda_\varepsilon)_{\varepsilon>0}$ follows then again as in Theorem 9.1 by using (21.18). □

Remark 21.2. *If* (21.18) *is not fulfilled, we could still show almost the same convergence results, but without the convergence of the Lagrange multipliers* (21.19f)*, compare also discussion in Remark 9.1. But, as already mentioned in Remark 9.1,* (21.18) *is not very restrictive and will be fulfilled for a wide class of problems.*

Summarizing the results of this part, we have shown that the phase field model is even a good approximation of the shape and topology optimization problem that is described by the sharp interface model (20.6)-(20.8) if a pressure functional is involved. And so we have by our phase field ansatz a consistent formulation, where the minimizers of the diffuse interface approximation converge to a sharp interface minimizer if the sequence of minimizers fulfills a certain convergence rate and simultaneously first order necessary optimality conditions can be shown to be an approximation of optimality conditions for the sharp interface setting.

22 Pressure functionals in a stationary Navier-Stokes flow

In this section we want to discuss briefly what happens if we use the stationary Navier-Stokes equations instead of the Stokes equations as state constraints when considering a pressure term in the objective functional. We will point out how the results of the previous sections can then be applied to this setting, too.

Let us start by examining the corresponding phase field model. We define the pressure $p_\varepsilon \in L^2_M(\Omega)$ associated to some $\varphi_\varepsilon \in \Phi_p$ and $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon)$ in the diffuse interface formulation analogously as in Section 19 as solution to

$$\begin{aligned}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u}_\varepsilon,\boldsymbol{u}_\varepsilon,\boldsymbol{v}) - \int_\Omega p_\varepsilon \operatorname{div}\boldsymbol{v}\,\mathrm{d}x = \\ = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in \boldsymbol{H}^1_0(M_i)\end{aligned} \tag{22.1}$$

for all $i = 1,\dots,m$, if $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon)$ is a velocity for the penalized stationary Navier-Stokes equation given by

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{v}\,\mathrm{d}x + b(\boldsymbol{u}_\varepsilon,\boldsymbol{u}_\varepsilon,\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{V}. \tag{22.2}$$

For any $\varphi_\varepsilon \in \Phi_p$ and $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}^N_\varepsilon(\varphi_\varepsilon)$, existence and uniqueness of some pressure $p_\varepsilon \in L^2_M(\Omega)$ fulfilling (22.1) is given by Lemma 4.4, compare also Remark 19.1.
At the same time, Lemma 4.4 implies the following estimate:

$$\|p_\varepsilon\|_{L^2(M_i)} \le c(M_i)\,\|\mu\Delta\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u}_\varepsilon + \boldsymbol{f}\|_{\boldsymbol{H}^{-1}(M_i)} \quad \forall i = 1,\dots,m$$

since $\varphi_\varepsilon|_{M_i} = 1$ and thus $\alpha_\varepsilon(\varphi_\varepsilon)|_{M_i} = 0$ a.e., which gives

$$\|p_\varepsilon\|_{L^2(\Omega)} \le c(M_1,\dots,M_m)\,\|\mu\Delta\boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{u}_\varepsilon + \boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}\,.$$

This is the most important estimate to deduce as in Section 19.2 the existence of a minimizer to the phase field model from the results of Section 12.2. This overall optimization problem is given by:

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u},p)} \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \\ + \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi(\varphi)\,\mathrm{d}x \\ \text{s.t. } \varphi\in\Phi_p \text{ and } (\varphi,\boldsymbol{u},p) \text{ fulfill (22.1)} - \text{(22.2)}.\end{aligned} \tag{22.3}$$

We can now formulate the sharp interface model. Therefore, we state from now on additionally Assumption **(A8)**, see Section 11.2 for a further discussion concerning this condition. Again Lemma 4.4 implies for every $\varphi \in \overline{\Phi}^0_p$ and $\boldsymbol{u} \in \boldsymbol{S}^N_0(\varphi)$ existence and uniqueness of $p \in L^2_M(\Omega)$ such that

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \int_\Omega p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(M_i),\ i = 1,\dots,m. \tag{22.4}$$

Moreover, we can establish as above an estimate of the form

$$\|p\|_{L^2(\Omega)} \leq c(M_1,\dots,M_m) \|\mu\Delta \boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

Consequently, we introduce the sharp interface problem as

$$\begin{aligned} &\min_{(\varphi,\boldsymbol{u},p)} \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \\ &\text{s.t. } \varphi \in \Phi_p^0 \text{ and } (\varphi,\boldsymbol{u},p) \text{ fulfill (22.4)}, \boldsymbol{u} \in \boldsymbol{S}_0^N(\varphi). \end{aligned} \tag{22.5}$$

By Lemma 13.5 we also have the estimate

$$\|\nabla \boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{\mu}{2K_\Omega}$$

if $(\varphi_0,\boldsymbol{u}_0,p_0)$ are a minimizer for problem (22.5). This estimate is one key ingredient for the sharp interface convergence result.

We then obtain that the sequence of minimizers of the phase field problem (22.3) have converging subsequence, which is denoted by $(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$. If $(\varphi_\varepsilon)_{\varepsilon>0}$ fulfills a certain convergence rate we find that this convergence takes place in the strong topology and that the limit element $(\varphi_0,\boldsymbol{u}_0,\varphi_0)$ is a minimizer of the sharp interface problem (22.5). This can be established as in Theorem 14.1 by applying the ideas of Section 21.1. If this convergence rate is fulfilled, we can deduce therefrom in particular the existence of minimizers for (22.5).
Thus we fix for the remainder of this discussion a sequence of minimizers of the phase field problem such that $(\varphi_\varepsilon)_{\varepsilon>0}$ fulfills

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega | \varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon). \tag{22.6}$$

This implies as in Corollary 14.1 for $\varepsilon > 0$ small enough, that the state equations corresponding to the minimizer φ_ε have a unique solution $\boldsymbol{u}_\varepsilon$.
With this result we can then establish necessary first order optimality conditions as in Theorem 15.1 and Lemma 15.5 by applying the ideas of Section 19.3 to (22.3).
Additionally, we obtain first order optimality conditions for the sharp interface problem (22.5) in the form of Theorem 16.2 by applying the ideas of Section 20.3.
And finally, we even can show convergence of the optimality systems by combining the results of Theorem 17.1 and Theorem 21.2.

Hence, also in this setting we can establish the desired results, namely that the phase field approach is a good approximation of the shape and topology optimization problem in the sharp interface formulation. This is shown in the sense that minimizers of the phase field model converge in the strong topology to a minimizer of the sharp interface model as the interface thickness tends to zero if the minimizers converge with a certain convergence rate. Simultaneously, the optimality system of the sharp interface setting is approximated by optimality conditions for the diffuse interface setting.

Part IV
Application in structural optimization

We want to apply the approach and analytical techniques developed in the first part to the shape and topology optimization problem of minimizing a general objective functional depending on the displacement fields of several elastic materials. To be precise, we assume to have inside of the fixed holdall container Ω a mixture of two homogeneous elastic materials, and we want to find the optimal configuration such that a certain objective functional is minimal. The displacement fields will be defined by the equations of linearized elasticity. In particular, the elasticity tensor and the eigenstrain, describing the different material properties, will depend on the design variable. Again we want to use a perimeter penalization to ensure well-posedness of the problem. Moreover, we consider a diffuse interface setting, namely a phase field approximation, which was already formulated and discussed in [BGS⁺12, BFSGS13]. One goal is to show that the reduced objective functionals corresponding to the phase field formulation Γ-converge in $L^1(\Omega)$ to the reduced objective functional corresponding to the perimeter penalized sharp interface problem as the thickness of the interface tends to zero. Therefrom, we can deduce directly the convergence of the corresponding minimizers in $L^1(\Omega)$. Additionally, we derive first order optimality conditions for the phase field model and for the sharp interface approach. We can show that simultaneously to the minimizers, the corresponding first order optimality conditions of the diffuse interface problem converge to a necessary optimality system in the sharp interface setting as the thickness of the interface tends to zero. We remark in particular, that those convergence results are valid without additional assumptions on the convergence rate of the minimizers, which were in general necessary in the previous parts. Besides we can prove the stronger Γ-convergence of the reduced objective functionals instead of convergence of minimizers.

23 Introduction and assumptions

Before formulating the phase field approach to the shape optimization problem mentioned above, we want to give a brief introduction into the most important quantities and equations in linearized elasticity and fix some notation. We only give a brief introduction and refer the reader for instance to [EGK08, Bra97, Cia88] and included references for details. We first assume to have in the holdall container Ω two open subsets Ω_1 and Ω_2 which are separated by a hypersurface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The two subsets should correspond to two different elastic materials whose displacement fields are described by one variable $\boldsymbol{u} : \Omega \to \mathbb{R}^d$. To be precise, $\boldsymbol{u}|_{\Omega_i}$ corresponds to the displacement field of the i-th material where $i \in \{1,2\}$. By Cauchy's theorem, compare [Bra97, Cia88], we find that for elastic materials the following equilibrium constraints hold in Ω_i, $i \in \{1,2\}$:

$$
\begin{aligned}
-\nabla \cdot \left(\mathrm{D}_2 W_i\left(x, \mathcal{E}\left(\boldsymbol{u}\right)\right)\right) &= \boldsymbol{f} && \text{in } \Omega_i, && (23.1\text{a})\\
\mathrm{D}_2 W_i\left(x, \mathcal{E}\left(\boldsymbol{u}\right)\right) \cdot \boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_g \cap \partial\Omega_i, && (23.1\text{b})\\
\boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \Gamma_D, \cap \partial\Omega_i, && (23.1\text{c})
\end{aligned}
$$

where $\Gamma_D, \Gamma_g \subset \partial\Omega$, $\Gamma_D \dot\cup \Gamma_g = \partial\Omega$, $\boldsymbol{g}$ is the applied surface load and $\boldsymbol{f}$ the applied body force. Moreover, Γ_D is a part of the boundary of the container Ω on which the displacement is prescribed by $\boldsymbol{u}_D$. We will see, that at the interface $\partial\Omega_1 \cap \partial\Omega_2$ the boundary conditions are given by certain transmission properties, see for example Remark 25.1. Besides, $W_i : \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}$ denotes the elastic free energy density of the i-th material, and

$$\mathcal{E}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)$$

is the so-called linearized strain, whereon the linear theory is based. In addition to this geometric linearization we introduce a linear constitutive law by following Hooke's law and using a quadratic form of W_i. Here we use

$$W_i(x, \mathcal{E}) := \frac{1}{2}\left(\mathcal{E} - \overline{\mathcal{E}}_i\right) : \mathcal{C}_i\left(\mathcal{E} - \overline{\mathcal{E}}_i\right) \quad \forall \mathcal{E} \in \mathbb{R}^{d\times d}, x \in \mathbb{R}^d \tag{23.2}$$

where $\mathcal{C}_i : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is the elasticity tensor which reflects the material properties for material $i = 1$ and $i = 2$, respectively. Further, $\overline{\mathcal{E}}_i \in \mathbb{R}^{d\times d}$ is the eigenstrain and represents the value of the strain when the i-th material is unstressed.

As already mentioned above, we have two different elastic materials inside the domain Ω. The design variable will again be a measurable function $\varphi : \Omega \to \mathbb{R}$, where $\{x \in \Omega \mid \varphi(x) = 1\} = \Omega_1$ describes the region where the first material is present, and $\{x \in \Omega \mid \varphi(x) = -1\} = \Omega_2$ the region which is filled with the second material. The elasticity tensor and the eigenstrain are functions of the phase field variable φ and interpolate between two different values for the two different materials.

(A12) Elasticity tensor:
Let $\mathcal{C}(\varphi) = \left(\mathcal{C}_{ijkl}(\varphi)\right)_{i,j,k,l=1}^d$ be such that $\mathcal{C}_{ijkl} \in C^{1,1}([-1,1])$ fulfills pointwise the following symmetry properties

$$\mathcal{C}_{ijkl}(\varphi) = \mathcal{C}_{jikl}(\varphi) = \mathcal{C}_{klij}(\varphi) \quad \forall \varphi \in [-1,1],\, i,j,k,l \in \{1,\dots,d\}.$$

Moreover, we assume that there exist constants $C_C, c_C > 0$ such that

$$|\mathcal{C}(\varphi)A : B| \le C_C\,|A|\,|B|\,, \quad \mathcal{C}(\varphi)A : A \ge c_C\,|A|^2 \tag{23.3}$$

holds for all symmetric matrices $A, B \in \mathbb{R}^{d\times d}$ and $\varphi \in [-1,1]$.

Remark 23.1. *We remark that* (23.3) *implies that the elasticity tensor interpolates between two finite positive values, thus in particular no "void", i.e. regions without material, are allowed in this formulation. Anyhow, the possibility of modelling "void" is given by using the so-called ersatz material approach, where a very soft material approximates the non-presence of material, cf. [BC03, BFSGS13]. Moreover, the elasticity tensor is not depending on the phase field variable $\varepsilon > 0$ introduced later on. And so an ersatz material approach with the ersatz parameter modelling the very soft material depending on the phase field parameter $\varepsilon > 0$ cannot be used in this setting. Such a formulation has been part of the studies in [BFSGS13].*

(A13) Eigenstrain:
We choose the eigenstrain $\overline{\mathcal{E}} \in C^{1,1}\left([-1,1], \mathbb{R}^{d\times d}\right)$ as a function with symmetric values, thus $\overline{\mathcal{E}}(\varphi)^T = \overline{\mathcal{E}}(\varphi)$ for all $\varphi \in [-1,1]$.

Remark 23.2. *Following Vegard's law, a commonly used assumption is that the eigenstrain interpolates linear between the two values corresponding to the two materials, thus* $\overline{\mathcal{E}}(x) = \frac{1}{2}(x+1)\overline{\mathcal{E}}_1 - \frac{1}{2}(x-1)\overline{\mathcal{E}}_2 = \left(\frac{1}{2}\overline{\mathcal{E}}_1 - \frac{1}{2}\overline{\mathcal{E}}_2\right)x + \left(\frac{1}{2}\overline{\mathcal{E}}_1 + \frac{1}{2}\overline{\mathcal{E}}_2\right)$, *if* $\overline{\mathcal{E}}_1 = \overline{\mathcal{E}}(1)$ *and* $\overline{\mathcal{E}}_2 = \overline{\mathcal{E}}(-1)$. *This case is of course included in our setting.*

As already mentioned above, we will describe the state by a single variable $\boldsymbol{u}:\Omega\to\mathbb{R}^d$ where $\boldsymbol{u}|_{\{\varphi=\pm1\}}$ will be the displacement field of the two different materials, and thus fulfill the equilibrium constraint (23.1) in any of these subsets separately. Now we divide the boundary of Ω into two parts, one Dirichlet part where we can prescribe the displacement field, and a Neumann part where the applied boundary forces are acting.

(A14) $\Omega\subset\mathbb{R}^d$ is a bounded Lipschitz domain with outer unit normal $\boldsymbol{n}$ and $d\in\{2,3\}$. Moreover, assume $\partial\Omega = \Gamma_D\cup\Gamma_g$ with $\mathcal{H}^{d-1}(\Gamma_D)>0$ and $\Gamma_D\cap\Gamma_g=\varnothing$.

To simplify notations, we will assume for the following considerations homogeneous Dirichlet boundary conditions on Γ_D, thus $\boldsymbol{u}_D\equiv\boldsymbol{0}$.

We remark that, as before, we denote $\mathbb{R}^d$-valued functions and spaces consisting of $\mathbb{R}^d$-valued functions in boldface.

Then we fix for the remainder of this part the external forces:

(A15) Let $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g}\in\boldsymbol{L}^2(\Gamma_g)$ be given.

Using these conventions, we can now give a reasonable weak formulation of the state equations on the whole of Ω, if the design variable is $\varphi\in L^1(\Omega)$ with $|\varphi|\le 1$ a.e. in Ω: Find

$$\boldsymbol{u}\in\boldsymbol{H}^1_D(\Omega) := \left\{\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)\mid \boldsymbol{u}|_{\Gamma_D}=\boldsymbol{0}\right\} \tag{23.4}$$

such that

$$\int_\Omega \mathcal{C}(\varphi)\left(\mathcal{E}(\boldsymbol{u})-\overline{\mathcal{E}}(\varphi)\right):\mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_g}\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{H}^1_D(\Omega). \tag{23.5}$$

We notice, that in any subregion $\{\varphi=\pm1\}$ this yields exactly the weak formulation of (23.1), since we obtain by (23.2) that $\mathrm{D}_2W_i(x,\mathcal{E}(\boldsymbol{u})) = \mathcal{C}_i\left(\mathcal{E}(\boldsymbol{u})-\overline{\mathcal{E}}_i\right)$ if $\overline{\mathcal{E}}_1=\overline{\mathcal{E}}(1)$, $\overline{\mathcal{E}}_2=\overline{\mathcal{E}}(-1)$, $\mathcal{C}_1=\mathcal{C}(1)$ and $\mathcal{C}_2=\mathcal{C}(-1)$.

Thus the state equations will in both the phase field and the sharp interface formulation be given by (23.5), cf. Sections 24 and 25.

Again we will consider a general objective functional of the form

$$\int_\Omega f(x,\boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_g} g(x,\boldsymbol{u})\,\mathrm{d}x$$

for our shape and topology optimization problem. This has to be chosen, such that the following assumptions are satisfied:

(A16) Objective functional:
We choose $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ and $g : \Gamma_g \times \mathbb{R}^d \to \mathbb{R}$ as Carathéodory functions, thus fulfilling

i) $f(\cdot, v) : \Omega \to \mathbb{R}$ and $g(\cdot, v) : \Gamma_g \to \mathbb{R}$ are measurable for each $v \in \mathbb{R}^d$, and
ii) $f(x, \cdot), g(x, \cdot) : \mathbb{R}^d \to \mathbb{R}$ are continuous for almost every $x \in \Omega$ and $x \in \Gamma_g$, respectively.

Moreover, assume that there exist functions $a_1 \in L^1(\Omega)$, $a_2 \in L^1(\Gamma_g)$ and $b_1 \in L^\infty(\Omega)$, $b_2 \in L^\infty(\Gamma_g)$ such that it holds

$$|f(x,v)| \le a_1(x) + b_1(x)|v|^2 \quad \forall v \in \mathbb{R}^d, \text{ a.e. } x \in \Omega, \tag{23.6}$$

and

$$|g(x,v)| \le a_2(x) + b_2(x)|v|^2 \quad \forall v \in \mathbb{R}^d, \text{ a.e. } x \in \Gamma_g, . \tag{23.7}$$

Moreover, we will assume that the objective functional is bounded from below in a certain sense. To be precise, we assume that the set

$$\begin{aligned} \{ &\int_\Omega f\left(x, \boldsymbol{u}(x)\right) \mathrm{d}x + \int_{\Gamma_g} g\left(x, \boldsymbol{u}(x)\right) \mathrm{d}x \mid \\ &\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega) \text{ solves (23.5) for some } \varphi \in L^1(\Omega), |\varphi| \le 1 \text{ a.e. in } \Omega \Big\} \end{aligned} \tag{23.8}$$

is bounded from below.

Remark 23.3. *We give some remarks on the specific assumption that* (23.8) *is bounded from below. We prove in Lemma 24.1 existence and uniqueness of some* $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ *solving* (23.5) *if* $\varphi \in L^1(\Omega)$ *with* $|\varphi| \le 1$ *a.e. in* Ω*. By stating the boundedness from below of the objective functional only for solutions of the state equations, we include in this setting the special case of minimizing the mean compliance, see Example 23.1. There, we minimize* $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \,\mathrm{d}x$ *and we see directly that*

$$\boldsymbol{H}^1_D(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \,\mathrm{d}x$$

is not bounded from below. But if we choose $\boldsymbol{u} = \boldsymbol{S}_E(\varphi)$*, where the solution operator* $\boldsymbol{S}_E$ *will be defined in* (24.5)*, we have*

$$\begin{aligned} &\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi) \left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi)\right) : \mathcal{E}(\boldsymbol{u}) \,\mathrm{d}x \ge \\ &\ge \int_\Omega c_C \left|\mathcal{E}(\boldsymbol{u})\right|^2 - C_c \left|\overline{\mathcal{E}}(\varphi)\right| \left|\mathcal{E}(\boldsymbol{u})\right| \mathrm{d}x \ge c \int_\Omega \left|\mathcal{E}(\boldsymbol{u})\right|^2 - C(\overline{\mathcal{E}}, \mathcal{C}) \,\mathrm{d}x \ge -C(\mathcal{C}, \overline{\mathcal{E}}) \end{aligned}$$

where we made use of (23.3), (23.5) *and Young's inequality. Thus the set* (23.8) *is bounded from below, which is sufficient for our considerations.*

Remark 23.4. *Due to [AZ90, Trö09, Sho97], the Nemytskii operators*

$$L^2(\Omega)^d \ni v \mapsto f(\cdot, v) \in L^1(\Omega), \quad L^2(\Gamma_g)^d \ni v \mapsto g(\cdot, v) \in L^1(\Gamma_g)$$

are well-defined if and only if (23.6) *and* (23.7) *are fulfilled. If this is the case, we obtain directly, that the operators are continuous.*

Remark 23.5. *We could generalise the results to objective functionals satisfying*

$$|f(x,v)| \le a_1(x) + b_1(x)|v|^p, \quad \forall v \in \mathbb{R}^d, \text{ a.e. } x \in \Omega \tag{23.9}$$

for some functions $a_1 \in L^1(\Omega)$ and $b_1 \in L^\infty(\Omega)$, instead of requiring (23.6)*. Here, $p \ge 2$ has to be chosen such that $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ is a compact imbedding, hence $p \ge 2$ for $d = 2$ and $2 \le p < 2d/d-2$ for $d = 3$. We then obtain that $L^p(\Omega)^d \ni v \mapsto f(\cdot, v) \in L^1(\Omega)$ is well-defined and continuous and all proofs can be adapted.*

These are the basic assumptions for the following considerations. As we do not only consider convergence of minimizers but also derive first order optimality conditions for both approaches we have to impose additionally the following differentiability assumptions for Sections 24.3, 25.3 and 26.2:

(A17) Assume that for every fixed $v \in \mathbb{R}^d$ the functions $\Omega \ni x \mapsto f(x,v) \in \mathbb{R}$ and $\Gamma_g \ni x \mapsto g(x,v) \in \mathbb{R}$ are in $W^{1,1}(\Omega)$ and $W^{1,1}(\Gamma_g)$, respectively. Let the partial derivatives $\mathrm{D}_2 f(x,\cdot), \mathrm{D}_2 g(x,\cdot)$ exist for almost every $x \in \Omega$ and $x \in \Gamma_g$, respectively. Besides assume that there exist $\hat{a}_1 \in L^1(\Omega)$, $\hat{a}_2 \in L^1(\Gamma_g)$ and $\hat{b}_1 \in L^\infty(\Omega)$, $\hat{b}_2 \in L^\infty(\Gamma_g)$ such that

$$|\mathrm{D}_2 f(x,v)| \le \hat{a}_1(x) + \hat{b}_1(x)|v| \quad \forall v \in \mathbb{R}^d, \text{ a.e. } x \in \Omega \tag{23.10}$$

and

$$|\mathrm{D}_2 g(x,v)| \le \hat{a}_2(x) + \hat{b}_2(x)|v| \quad \forall v \in \mathbb{R}^d, \text{ a.e. } x \in \Gamma_g. \tag{23.11}$$

Remark 23.6. *We note, that under the assumptions stated in Assumption **(A17)** the operators*

$$F : \boldsymbol{L}^2(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega f(x, \boldsymbol{u}(x))\,\mathrm{d}x$$

and

$$G : \boldsymbol{L}^2(\Gamma_g) \ni \boldsymbol{u} \mapsto \int_{\Gamma_g} g(x, \boldsymbol{u}(x))\,\mathrm{d}x$$

are continuously Fréchet differentiable and that the directional derivatives are given as

$$\mathrm{D}F(\boldsymbol{u})(\boldsymbol{v}) = \int_\Omega \mathrm{D}_2 f(x,\boldsymbol{u})\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),$$

$$\mathrm{D}G(\boldsymbol{u})(\boldsymbol{v}) = \int_{\Gamma_g} \mathrm{D}_2 g(x,\boldsymbol{u})\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{L}^2(\Gamma_g).$$

This follows by similar considerations as in Remark 2.5.

Remark 23.7. *If* (23.6) *is replaced by* (23.9) *for some $p > 0$ such that $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ is a compact imbedding, we have to replace* (23.10) *in Assumption **(A17)** by*

$$|\mathrm{D}_2 f(x,v)| \le \hat{a}_1(x) + \hat{b}_1(x)|v|^{p-1} \quad \forall v \in \mathbb{R}^d, \text{ a.e. } x \in \Omega \tag{23.12}$$

for $\hat{a}_1 \in L^1(\Omega)$, $\hat{b}_1 \in L^\infty(\Omega)$ to obtain that

$$\boldsymbol{L}^p(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega f(x, \boldsymbol{u}(x))\,\mathrm{d}x$$

is continuously Fréchet differentiable.

As in the previous parts, we describe the sharp interface model by a design variable $\varphi : \Omega \to \{\pm 1\}$, where $\{\varphi = \pm 1\}$ describe the two different materials. Since we use a perimeter penalization to ensure the existence of minimizers, we need that $\{\varphi = \pm 1\}$ are sets of finite perimeter, thus our admissible design variables are chosen in

$$\Phi_E^0 := \left\{ \varphi \in BV\left(\Omega, \{\pm 1\}\right) \mid \fint_\Omega \varphi \,\mathrm{d}x \le \beta \right\}.$$

Notice, that we include an integral constraint, where $\beta \in (-1,1)$ is a fixed constant. This yields, that there is a maximal amount of the material corresponding to $\{\varphi = 1\}$ that can be used during the optimization process. We could also add additionally a constraint on the maximal amount of the material corresponding to $\{\varphi = -1\}$ by replacing this constraint with

$$-\beta_1 \le \fint_\Omega \varphi \,\mathrm{d}x \le \beta$$

for a suitable $\beta_1 \in (-1,1)$. In this setting, the same analysis could be carried out.
Moreover, we introduce the extended admissible set, which will be used for first order optimality conditions after introducing a Lagrange multiplier for the integral constraint:

$$\overline{\Phi}_E^0 := BV\left(\Omega, \{\pm 1\}\right).$$

As already mentioned above, we will discuss a phase field formulation of the shape optimization problem. In particular we approximate the perimeter by the Ginzburg-Landau energy and use therefore, as in the previous parts, a double obstacle potential, thus $\psi : \mathbb{R} \to \overline{\mathbb{R}}$ is given by

$$\psi(\varphi) := \begin{cases} \psi_0(\varphi), & \text{if } |\varphi| \le 1 \\ +\infty, & \text{if } |\varphi| > 1 \end{cases}, \qquad \psi_0(\varphi) := \frac{1}{2}\left(1 - \varphi^2\right).$$

The design variable $\varphi : \Omega \to [-1,1]$ is then allowed to have values between minus one and one and thus there may be a transition area between the areas $\{\varphi = -1\}$ and $\{\varphi = 1\}$. The admissible set for the optimization problem will then be given by

$$\Phi_E := \left\{ \varphi \in H^1(\Omega) \mid \fint_\Omega \varphi \,\mathrm{d}x \le \beta, \, |\varphi| \le 1 \text{ a.e. in } \Omega \right\}$$

and correspondingly the extended admissible set by

$$\overline{\Phi}_E := \left\{ \varphi \in H^1(\Omega) \mid |\varphi| \le 1 \text{ a.e. in } \Omega \right\}.$$

For geometric variations we will use again transformations $T \in \mathcal{T}_{ad}$ and $T \in \overline{\mathcal{T}}_{ad}$ associated to velocity fields $V \in \mathcal{V}_{ad}$ and $V \in \overline{\mathcal{V}}_{ad}$, respectively, by (2.9), thus by the ordinary differential equation

$$\partial_t T_t(x) = V\left(t, T_t(x)\right), \quad T_0(x) = x.$$

The spaces $\mathcal{T}_{ad}$, $\overline{\mathcal{T}}_{ad}$, $\mathcal{V}_{ad}$ and $\overline{\mathcal{V}}_{ad}$ are given by Definition 2.1 where condition **(V4)** is replaced by

(V4') $V(t,x) = \mathbf{0}$ for every $x \in \Gamma_D$.

We finish this introduction by two typical examples which are commonly used as objective functionals in structural optimization. For a deeper discussion on those problems and some further applicatons we refer for instance to [Ben03].

Example 23.1 (Mean compliance). One commonly used objective in structural optimization is the minimization of the mean compliance, which is for a structure in ints equilibrium configuration given by

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}x.$$

We notice, that this is equivalent to minimizing

$$\int_\Omega \mathcal{C}(\varphi) \left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi) \right) : \mathcal{E}(\boldsymbol{u}) \, \mathrm{d}x$$

if $\boldsymbol{u}$ solves the state equations (23.5). The aim of minimizing this objective functional can also be interpreted as maximizing the stiffness under the given forces or, in case of $\overline{\mathcal{E}} \equiv 0$, as minimizing the stored mechanical energy. Using this objective functional, we arrive in a self adjoint optimization problem in the sense that the adjoint equations of the first order optimality system equal the state equations if $\overline{\mathcal{E}} \equiv 0$, cf. Section 24.3.

Example 23.2 (Compliant mechanism). The typical compliant mechanism objective functional used in topology optimization is given by the tracking type functional

$$\frac{1}{2} \int_\Omega c(x) \left| \boldsymbol{u} - \boldsymbol{u}_\Omega \right|^2 \mathrm{d}x$$

where $\boldsymbol{u}_\Omega \in \boldsymbol{L}^2(\Omega)$ is some desired structure, which may for instance have certain advantages or desired properties. Moreover, $c \in L^\infty(\Omega)$, $c \geq 0$, is for example a characteristic function or a weighting factor to emphasize certain regions in Ω.

24 Phase field model

We start our considerations by introducing a diffuse interface setting in terms of a phase field formulation for the general shape and topology optimization problem of finding the optimal material distribution of two given materials. For this purpose, we will start by discussing the problem formulation, which will be introduced in Section 24.1. This will then be considered in view of solvability of the state equations and existence of optimal designs in Section 24.2. Finally, we derive first order necessary optimality conditions for this problem in Section 24.3.

24.1 Problem formulation

The overall optimization problem is given by

$$\min_{(\varphi, \boldsymbol{u})} J_\varepsilon^E(\varphi, \boldsymbol{u}) := \int_\Omega f(x, \boldsymbol{u}) \, \mathrm{d}x + \int_{\Gamma_g} g(x, \boldsymbol{u}) \, \mathrm{d}x + \frac{\gamma\varepsilon}{2} \int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \frac{\gamma}{\varepsilon} \int_\Omega \psi(\varphi) \, \mathrm{d}x \quad (24.1)$$

with

$$(\varphi, \boldsymbol{u}) \in \Phi_E \times \boldsymbol{H}_D^1(\Omega)$$

s.t.

$$\int_\Omega \mathcal{C}(\varphi) \left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi) \right) : \mathcal{E}(\boldsymbol{v}) \, \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega). \quad (24.2)$$

Hence the design variable in the shape and topology optimization problem is given by $\varphi \in \Phi_E = \left\{ \varphi \in H^1(\Omega) \mid \fint_\Omega \varphi \, \mathrm{d}x \leq \beta, |\varphi| \leq 1 \text{ a.e. in } \Omega \right\}$. The aim is to minimize the objective functional given by J_ε^E by finding the optimal material composition of the two materials. The regions filled with material one or two are represented by $\{x \in \Omega \mid \varphi(x) = 1\}$ and $\{x \in \Omega \mid \varphi(x) = -1\}$, respectively. The design variable φ is also allowed to take values between minus one and one, which leads to a small transitional area whose thickness is proportional to a small parameter $\varepsilon > 0$. Thus, as ε tends to zero, we will arrive in a sharp interface problem and the interfacial layer vanishes.

The last two terms of the objective functional, namely

$$\frac{\gamma\varepsilon}{2} \int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \frac{\gamma}{\varepsilon} \int_\Omega \psi(\varphi) \, \mathrm{d}x$$

are a multiple of the Ginzburg-Landau energy. This term is essential for the existence of a minimizer of the problem and Γ-converges to a multiple of the perimeter functional as ε tends to zero, cf. [Mod87, MM77]. The parameter $\gamma > 0$ is an arbitrary fixed constant and can be considered as a weighting parameter for the perimeter penalization. For a more detailed discussion of the Ginzburg Landau energy and the phase field approach we refer to Section 5.1.

The state equations are given by (24.2). In the two regions $\{\varphi = 1\}$ and $\{\varphi = -1\}$, representing the two materials, the elasticity tensor $\mathcal{C}$ and the eigenstrain $\overline{\mathcal{E}}$ are constant and thus (24.2) yields for those regions the usual weak formulation of the equilibrium constraints of linearized elasticity, see (23.1). In the interfacial layer $\{-1 < \varphi < 1\}$, the elasticity tensor $\mathcal{C}$ interpolates between the two material properties, which are represented by $\mathcal{C}(-1)$ and $\mathcal{C}(1)$, respectively. Similarly, the eigenstrain $\overline{\mathcal{E}}$ is a function interpolating between the eigenstrains of the two materials, namely $\overline{\mathcal{E}}(-1)$ and $\overline{\mathcal{E}}(1)$.

We remark in particular, that in contrast to the previous parts, the state equations do not depend on the phase field parameter $\varepsilon > 0$ any more, but only on the phase field variable φ.

24.2 Existence results

We start by discussing the optimization problem (24.1) – (24.2) in view of well-posedness for fixed $\varepsilon > 0$. For this purpose, we first have a closer look at the state equations. We find that the state equations (24.2) are a weak formulation of

$$-\nabla \cdot \left(\mathcal{C}(\varphi) \left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi) \right) \right) = \boldsymbol{f} \qquad \text{in } \Omega, \tag{24.3a}$$

$$\left(\mathcal{C}(\varphi) \left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi) \right) \right) \cdot \boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \Gamma_g, \tag{24.3b}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_D. \tag{24.3c}$$

We can establish the following result concerning solvability of this system:

Lemma 24.1. *For every $\varphi \in L^1(\Omega)$ with $|\varphi| \leq 1$ a.e. in Ω there exists a unique $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ such that* (24.3) *is fulfilled in the sense of* (24.2). *Moreover, the solution $\boldsymbol{u}$ fulfills*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\left(\Omega, \mathcal{C}, \overline{\mathcal{E}}\right) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Gamma_g)} + 1 \right). \tag{24.4}$$

Proof. We apply Lax Milgram's theorem A.2 and define for this purpose the bilinear form

$$a : \boldsymbol{H}^1_D(\Omega) \times \boldsymbol{H}^1_D(\Omega) \to \mathbb{R}$$

by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{C}(\varphi)\, \mathcal{E}(\boldsymbol{u}) : \mathcal{E}(\boldsymbol{v})\, \mathrm{d}x.$$

We use Korn's inequality A.6 and the coercivity of the elasticity tensor $\mathcal{C}$, see (23.3), to obtain for any $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ the following estimate:

$$a(\boldsymbol{u}, \boldsymbol{u}) = \int_\Omega \mathcal{C}(\varphi)\, \mathcal{E}(\boldsymbol{u}) : \mathcal{E}(\boldsymbol{u})\, \mathrm{d}x \geq c_C \int_\Omega |\mathcal{E}(\boldsymbol{u})|^2\, \mathrm{d}x \geq C \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}.$$

Thus, a is a coercive bilinear form, which is obviously also continuous by (23.3). Defining $F \in \left(\boldsymbol{H}^1_D(\Omega)\right)'$ by

$$F(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{v}\, \mathrm{d}x + \int_\Omega \mathcal{C}(\varphi)\overline{\mathcal{E}}(\varphi) : \mathcal{E}(\boldsymbol{v})\, \mathrm{d}x$$

we obtain by Lax-Milgram's theorem A.2 a unique $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega),$$

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\mathcal{C}, \Omega) \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Gamma_g)} + \left\|\overline{\mathcal{E}}\right\|_{L^\infty([-1,1])} \right).$$

Thus $\boldsymbol{u}$ solves (24.2) and fulfills (24.4). □

This result implies that there is a well-defined solution operator for the constraints (24.3), which will be denoted by

$$\boldsymbol{S}_E : \left\{ \varphi \in L^1(\Omega) \mid |\varphi| \leq 1 \text{ a.e. in } \Omega \right\} \to \boldsymbol{H}^1_D(\Omega) \tag{24.5}$$

where $\boldsymbol{S}_E(\varphi)$ is defined as the unique solution $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ of (24.3).
Besides, we can now define the reduced objective functional $j^E_\varepsilon : \Phi_E \to \mathbb{R}$ by

$$j^E_\varepsilon(\varphi) := J^E_\varepsilon(\varphi, \boldsymbol{S}_E(\varphi)). \tag{24.6}$$

The next result yields that the overall optimization problem stated above is well-posed.

Theorem 24.1. *There exists at least one minimizer of* (24.1) – (24.2).

Proof. This follows by the direct method in the calculus of variations. To this end, we use that

$$\left\{ J^E_\varepsilon(\varphi, \boldsymbol{S}_E(\varphi)) \mid \varphi \in \Phi_E \right\}$$

is according to Assumption **(A16)** bounded from below. Consequently, we can choose a minimizing sequence $(\varphi_k, \boldsymbol{u}_k)_{k\in\mathbb{N}} \subset \Phi_E \times \boldsymbol{H}^1_D(\Omega)$ for (24.1) – (24.2), in particular $\boldsymbol{u}_k = \boldsymbol{S}_E(\varphi_k)$. We obtain therefrom and the fact that $|\varphi_k| \leq 1$ a.e. in Ω for all $k \in \mathbb{N}$ that

$\sup_{k\in\mathbb{N}} \|\varphi_k\|_{H^1(\Omega)} < \infty$. Inserting $\boldsymbol{u}_k$ as a test function in the state equations (24.2) corresponding to φ_k we obtain moreover

$$\begin{aligned}
c\,\|\boldsymbol{u}_k\|^2_{\boldsymbol{H}^1(\Omega)} - \tilde{c}\,\left\|\overline{\mathcal{E}}\right\|^2_{L^\infty([-1,1])} &\le \int_\Omega c_C\,|\mathcal{E}(\boldsymbol{u}_k)|^2 - C_c\left|\overline{\mathcal{E}}(\varphi_k)\right||\mathcal{E}(\boldsymbol{u}_k)|\,\mathrm{d}x \le \\
&\le \int_\Omega \mathcal{C}(\varphi_k)\left(\mathcal{E}(\boldsymbol{u}_k) - \overline{\mathcal{E}}(\varphi_k)\right) : \mathcal{E}(\boldsymbol{u}_k)\,\mathrm{d}x = \\
&= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_k\,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g}\cdot\boldsymbol{u}_k\,\mathrm{d}x \le \\
&\le \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Gamma_g)}\|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Gamma_g)} \le \\
&\le \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)} + c\,\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Gamma_g)}\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}
\end{aligned} \tag{24.7}$$

where we made in particular use of Young's inequality, Korn's inequality A.6 and the uniform ellipticity of the elasticity tensor stated in (23.3). Thus, we find from (24.7) that

$$\sup_{k\in\mathbb{N}} \|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)} < \infty.$$

And so we deduce the existence of a subsequence of $(\varphi_k, \boldsymbol{u}_k)_{k\in\mathbb{N}}$, which will be denoted by the same, such that $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ and strongly in $\boldsymbol{L}^2(\Omega)$ to some limit element $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$, and $(\varphi_k)_{k\in\mathbb{N}}$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some limit element $\varphi \in \Phi_E$. Here we use that $\boldsymbol{H}^1_D(\Omega)$ and Φ_E are as closed, convex subsets also weakly closed in $\boldsymbol{H}^1(\Omega)$ and $H^1(\Omega)$, respectively. Additionally, we obtain from the compact embedding $\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \hookrightarrow \boldsymbol{L}^2(\partial\Omega)$ that $(\boldsymbol{u}_k|_{\partial\Omega})_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{L}^2(\Gamma_g)$. Because $\mathcal{C}$ is uniformly bounded, we find for every $\boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$ the pointwise estimate

$$|\mathcal{C}(\varphi_k)\mathcal{E}(\boldsymbol{v})|(x) \le C|\mathcal{E}(\boldsymbol{v})(x)| \quad \text{for a.e. } x \in \Omega.$$

By the continuity of $\mathcal{C}$ and Lebesgue's convergence theorem, this implies that $(|\mathcal{C}(\varphi_k)\,\mathcal{E}(\boldsymbol{v})|)_{k\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $|\mathcal{C}(\varphi)\,\mathcal{E}(\boldsymbol{v})|$. Besides, $(\mathcal{E}(\boldsymbol{u}_k))_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{L}^2(\Omega)^d$ and hence

$$\lim_{k\to\infty} \int_\Omega \mathcal{C}(\varphi_k)\,\mathcal{E}(\boldsymbol{u}_k) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi)\,\mathcal{E}(\boldsymbol{u}) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x.$$

Similarly, we find from the uniform boundedness of $\overline{\mathcal{E}}$ that

$$\lim_{k\to\infty} \int_\Omega \mathcal{C}(\varphi_k)\,\overline{\mathcal{E}}(\varphi_k) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi)\,\overline{\mathcal{E}}(\varphi) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x$$

and can deduce therefrom

$$\int_\Omega \mathcal{C}(\varphi)\left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi)\right) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega) \tag{24.8}$$

and thus $\boldsymbol{u} = \boldsymbol{S}_E(\varphi)$. Now we use the weakly lower semicontinuity of

$$H^1(\Omega) \ni \varphi \mapsto \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x$$

and the continuity of the objective functional, compare Remark 23.4, to observe

$$J^E_\varepsilon(\varphi, \boldsymbol{u}) \le \liminf_{k\to\infty} J^E_\varepsilon(\varphi_k, \boldsymbol{u}_k).$$

This gives us that $(\varphi, \boldsymbol{u})$ is a minimizer of (24.1) – (24.2). □

24.3 Optimality conditions

For the following subsection we assume that $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ is a minimizer of (24.1) – (24.2). We want to derive in this subsection first order necessary optimality conditions that have to be fulfilled for a minimizer of the phase field problem (24.1) – (24.2). For this purpose, we consider the optimization problem first of all as a classical optimal control problem and derive optimality conditions in form of a variational inequality. After that, we also vary geometrically and derive optimality conditions therefrom. In particular, the latter will turn out to be an approximation of first order optimality conditions for a sharp interface formulation of the shape optimization problem, see Section 26.2.

We assume for the remainder of this subsection additionally Assumption **(A17)** to ensure differentiability of the objective functional.

We start by stating a differentiability result concerning the solution operator:

Lemma 24.2. *Let $\varphi \in \overline{\Phi}_E$. Then the directional derivative*

$$\partial_t|_{t=0}\boldsymbol{S}_E\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) = \mathrm{D}\boldsymbol{S}_E\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right) =: \boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$$

exists in $\boldsymbol{H}^1(\Omega)$ and is given as the unique solution of

$$\begin{aligned}\int_\Omega &\mathcal{C}'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)\left(\mathcal{E}\left(\boldsymbol{u}_\varepsilon\right) - \overline{\mathcal{E}}\left(\varphi_\varepsilon\right)\right) : \mathcal{E}\left(\boldsymbol{v}\right) + \\ &+ \mathcal{C}\left(\varphi_\varepsilon\right)\left(\mathcal{E}\left(\boldsymbol{u}\right) - \overline{\mathcal{E}}'\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right)\right) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = 0,\end{aligned} \tag{24.9}$$

which has to hold for all $\boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$, where $\boldsymbol{u}_\varepsilon = \boldsymbol{S}^E_\varepsilon(\varphi_\varepsilon)$. Here we denote by $\partial_t|_{t=0}\boldsymbol{S}_E\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) = \lim_{t\to 0, t>0}\frac{1}{t}\left(\boldsymbol{S}_E\left(\varphi_\varepsilon + t\left(\varphi - \varphi_\varepsilon\right)\right) - \boldsymbol{S}_E\left(\varphi_\varepsilon\right)\right)$ the one-sided directional derivative.

Proof. This follows for example as in [BFSGS13, Theorem 3.3], where the case $\overline{\mathcal{E}} \equiv 0$ is treated, and can also be shown by direct calculations similar to those in the proof of Lemma 7.2 or by an application of the implicit function theorem, compare Remark 7.1. □

And so, after introducing an adjoint variable $\boldsymbol{q}_\varepsilon$ and a Lagrange multiplier λ_ε for the integral constraint, we obtain a first version of first order optimality conditions for the optimal control problem (24.1) – (24.2) in form of a variational inequality:

Theorem 24.2. *The following optimality system is fulfilled for any minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ of (24.1) – (24.2):*

$$\begin{aligned}\left(\frac{\gamma}{\varepsilon}\psi_0'\left(\varphi_\varepsilon\right) + \left(\mathcal{C}(\varphi_\varepsilon)\overline{\mathcal{E}}'\left(\varphi_\varepsilon\right) - \mathcal{C}'\left(\varphi_\varepsilon\right)\left(\mathcal{E}\left(\boldsymbol{u}_\varepsilon\right) - \overline{\mathcal{E}}\left(\varphi_\varepsilon\right)\right)\right) : \mathcal{E}\left(\boldsymbol{q}_\varepsilon\right) + \lambda_\varepsilon, \varphi - \varphi_\varepsilon\right)_{L^2(\Omega)} + \\ + \left(\gamma\varepsilon\nabla\varphi_\varepsilon, \nabla\left(\varphi - \varphi_\varepsilon\right)\right)_{\boldsymbol{L}^2(\Omega)} \geq 0 \qquad \forall \varphi \in \overline{\Phi}_E\end{aligned} \tag{24.10}$$

$$
\left.
\begin{aligned}
-\nabla\cdot\left(\mathcal{C}\left(\varphi_\varepsilon\right)\left(\mathcal{E}\left(\boldsymbol{u}_\varepsilon\right)-\overline{\mathcal{E}}\left(\varphi_\varepsilon\right)\right)\right) &= \boldsymbol{f} && \textit{in } \Omega,\\
\left(\mathcal{C}\left(\varphi_\varepsilon\right)\left(\mathcal{E}\left(\boldsymbol{u}_\varepsilon\right)-\overline{\mathcal{E}}\left(\varphi_\varepsilon\right)\right)\right)\cdot\boldsymbol{n} &= \boldsymbol{g} && \textit{on } \Gamma_g,\\
\boldsymbol{u}_\varepsilon &= \boldsymbol{0} && \textit{on } \Gamma_D,\\[1ex]
-\nabla\cdot\left(\mathcal{C}\left(\varphi_\varepsilon\right)\mathcal{E}\left(\boldsymbol{q}_\varepsilon\right)\right) &= \mathrm{D}_u f\left(\cdot,\boldsymbol{u}_\varepsilon\right) && \textit{in } \Omega,\\
\left(\mathcal{C}\left(\varphi_\varepsilon\right)\mathcal{E}\left(\boldsymbol{q}_\varepsilon\right)\right)\cdot\boldsymbol{n} &= \mathrm{D}_u g\left(\cdot,\boldsymbol{u}_\varepsilon\right) && \textit{on } \Gamma_g,\\
\boldsymbol{q}_\varepsilon &= \boldsymbol{0} && \textit{on } \Gamma_D,\\[1ex]
\lambda_\varepsilon\left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x-\beta\left|\Omega\right|\right) &= 0,\ \lambda_\varepsilon\geq 0,\\
\int_\Omega \varphi_\varepsilon\,\mathrm{d}x \leq \beta\left|\Omega\right|, &\quad \left|\varphi_\varepsilon\right|\leq 1 \textit{ a.e. in } \Omega,
\end{aligned}
\right\}
\tag{24.11}
$$

where $\lambda_\varepsilon \in \mathbb{R}^+$ *denotes a Lagrange multiplier for the integral constraint. Moreover,* $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_E\left(\varphi_\varepsilon\right)$ *is the weak solution for the state equations and* $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}^1_D(\Omega)$ *is the adjoint variable being the weak solution of the adjoint state system and thus fulfills*

$$
\int_\Omega \mathcal{C}\left(\varphi_\varepsilon\right)\mathcal{E}\left(\boldsymbol{q}_\varepsilon\right):\mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \mathrm{D}_u f\left(x,\boldsymbol{u}_\varepsilon\right)\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_g}\mathrm{D}_u g\left(x,\boldsymbol{u}_\varepsilon\right)\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{H}^1_D(\Omega).
\tag{24.12}
$$

Proof. Since $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon)$ is a minimizer of (24.1)–(24.2), we see that φ_ε minimizes the reduced objective functional defined by (24.6). And so we find

$$
\mathrm{D}j^E_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi-\varphi_\varepsilon\right)\geq 0 \quad \forall\varphi\in\Phi_E
$$

where we use that this directional derivative exists due to Lemma 24.2 and Remark 23.6. By using the arguments of the proof of Theorem 7.1 we deduce from [KZ79] the existence of some Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}^+$ for the integral constraint such that

$$
\mathrm{D}j^E_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi-\varphi_\varepsilon\right)+\lambda_\varepsilon\int_\Omega\left(\varphi-\varphi_\varepsilon\right)\,\mathrm{d}x\geq 0 \quad \forall\varphi\in\overline{\Phi}_E
\tag{24.13}
$$

and

$$
\lambda_\varepsilon\left(\int_\Omega\varphi_\varepsilon\,\mathrm{d}x-\beta\left|\Omega\right|\right)=0.
\tag{24.14}
$$

Now we want to reformulate the directional derivative

$$
\begin{aligned}
\mathrm{D}j^E_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi-\varphi_\varepsilon\right) = {}& \int_\Omega \mathrm{D}_u f\left(x,\boldsymbol{u}_\varepsilon\right)\boldsymbol{u}\,\mathrm{d}x + \int_{\Gamma_g}\mathrm{D}_u g\left(x,\boldsymbol{u}_\varepsilon\right)\boldsymbol{u}\,\mathrm{d}x +\\
&+\gamma\int_\Omega \varepsilon\nabla\varphi_\varepsilon\cdot\nabla\left(\varphi-\varphi_\varepsilon\right)+\frac{1}{\varepsilon}\psi_0'\left(\varphi_\varepsilon\right)\left(\varphi-\varphi_\varepsilon\right)\,\mathrm{d}x
\end{aligned}
\tag{24.15}
$$

for any $\varphi\in\overline{\Phi}_E$, where we use $\boldsymbol{u} := \mathrm{D}\boldsymbol{S}_E\left(\varphi_\varepsilon\right)\left(\varphi-\varphi_\varepsilon\right)$. For this purpose, we insert $\boldsymbol{u}\in\boldsymbol{H}^1_D(\Omega)$ as a test function in the weak formulation for the adjoint system (24.12) and see

$$
\int_\Omega \mathcal{C}\left(\varphi_\varepsilon\right)\mathcal{E}\left(\boldsymbol{q}_\varepsilon\right):\mathcal{E}\left(\boldsymbol{u}\right)\,\mathrm{d}x = \int_\Omega \mathrm{D}_u f\left(x,\boldsymbol{u}_\varepsilon\right)\boldsymbol{u}\,\mathrm{d}x + \int_{\Gamma_g}\mathrm{D}_u g\left(x,\boldsymbol{u}_\varepsilon\right)\boldsymbol{u}\,\mathrm{d}x.
\tag{24.16}
$$

On the other hand, we see, by testing the linearized system (24.9) with $\boldsymbol{q}_\varepsilon \in \boldsymbol{H}^1_D(\Omega)$, that

$$\begin{aligned}\int_\Omega &\mathcal{C}'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon)-\overline{\mathcal{E}}(\varphi_\varepsilon)\right):\mathcal{E}(\boldsymbol{q}_\varepsilon)+\\ &+\mathcal{C}(\varphi_\varepsilon)\Big(\mathcal{E}(\boldsymbol{u})-\overline{\mathcal{E}}'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\Big):\mathcal{E}(\boldsymbol{q}_\varepsilon)\,\mathrm{d}x=0.\end{aligned}\tag{24.17}$$

Comparing (24.16) and (24.17) yields

$$\begin{aligned}&\int_\Omega -\mathcal{C}'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon)-\overline{\mathcal{E}}(\varphi_\varepsilon)\right):\mathcal{E}(\boldsymbol{q}_\varepsilon)+\mathcal{C}(\varphi_\varepsilon)\overline{\mathcal{E}}'(\varphi_\varepsilon):\mathcal{E}(\boldsymbol{q}_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x=\\ &=\int_\Omega \mathrm{D}_u f(x,\boldsymbol{u}_\varepsilon)\,\boldsymbol{u}\,\mathrm{d}x+\int_{\Gamma_g}\mathrm{D}_u g(x,\boldsymbol{u}_\varepsilon)\,\boldsymbol{u}\,\mathrm{d}x\end{aligned}\tag{24.18}$$

and so we can reformulate (24.15) to

$$\begin{aligned}\mathrm{D}j^E_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)=&-\int_\Omega \mathcal{C}'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon)-\overline{\mathcal{E}}(\varphi_\varepsilon)\right):\mathcal{E}(\boldsymbol{q}_\varepsilon)\,\mathrm{d}x+\\ &+\int_\Omega \mathcal{C}(\varphi_\varepsilon)\overline{\mathcal{E}}'(\varphi_\varepsilon):\mathcal{E}(\boldsymbol{q}_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x+\\ &+\gamma\int_\Omega \varepsilon\nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_\varepsilon)+\frac{1}{\varepsilon}\psi_0'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon)\,\mathrm{d}x.\end{aligned}\tag{24.19}$$

Combining (24.13), (24.14) and (24.19) we obtain the statement. □

This gives a first version of necessary optimality conditions. But as already mentioned above, we also want to obtain optimality conditions by geometric variations. In particular, those optimality conditions will then be the correct formulation for considering the limit process $\varepsilon \searrow 0$ and to obtain a necessary optimality system for the sharp interface problem, cf. Section 26.2.

Theorem 24.3. *For any minimizer $(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon)\in\Phi_E\times\boldsymbol{H}^1_D(\Omega)$ of (24.1) – (24.2) we have the following necessary optimality conditions:*

$$\partial_t|_{t=0}j^E_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=-\lambda_\varepsilon\int_\Omega\varphi_\varepsilon\operatorname{div}V(0)\,\mathrm{d}x,\tag{24.20}$$

$$\lambda_\varepsilon\left(\int_\Omega\varphi_\varepsilon\,\mathrm{d}x-\beta|\Omega|\right)=0\tag{24.21}$$

for all $T\in\overline{\mathcal{T}}_{ad}$ with velocity $V\in\overline{\mathcal{V}}_{ad}$, where $\lambda_\varepsilon\in\mathbb{R}^+$ is a Lagrange multiplier for the integral constraint. The derivative is given by the following formula:

$$\begin{aligned}\partial_t|_{t=0}j^E_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=&\int_\Omega\left[\mathrm{D}f(x,\boldsymbol{u}_\varepsilon)(V(0),\dot{\boldsymbol{u}}_\varepsilon[V])+f(x,\boldsymbol{u}_\varepsilon)\operatorname{div}V(0)\right]\mathrm{d}x+\\ +&\int_{\Gamma_g}\left[\mathrm{D}g(x,\boldsymbol{u}_\varepsilon)(V(0),\dot{\boldsymbol{u}}_\varepsilon[V])+g(x,\boldsymbol{u}_\varepsilon)\left(\operatorname{div}V(0)-\boldsymbol{n}\cdot\nabla V(0)\boldsymbol{n}\right)\right]\mathrm{d}x+\\ +&\int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2+\frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div}V(0)-\gamma\varepsilon\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x\end{aligned}\tag{24.22}$$

where $\dot{\boldsymbol{u}}_\varepsilon[V]\in\boldsymbol{H}^1_D(\Omega)$ is given as the solution of

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_\varepsilon)\mathcal{E}(\dot{\boldsymbol{u}}_\varepsilon[V]) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi_\varepsilon)\frac{1}{2}\left(\mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0) + \nabla V(0)\nabla \boldsymbol{u}_\varepsilon\right) : \mathcal{E}(\boldsymbol{v}) + \\
&+ \mathcal{C}(\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon) - \overline{\mathcal{E}}\left(\varphi_\varepsilon\right)\right) : \frac{1}{2}(\nabla V(0)\nabla \boldsymbol{v} + \mathrm{D}\boldsymbol{v}\mathrm{D}V(0)) - \\
&- \mathcal{C}(\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon) - \overline{\mathcal{E}}\left(\varphi_\varepsilon\right)\right) : \mathcal{E}(\boldsymbol{v})\,\mathrm{div}\,V(0)\,\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot \mathrm{D}\boldsymbol{v}V(0)\,\mathrm{d}x - \int_{\Gamma_g} \boldsymbol{g}\cdot \mathrm{D}\boldsymbol{v}V(0)\,\mathrm{d}x
\end{aligned} \tag{24.23}$$

which has to hold for all $\boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$.

Remark 24.1. *If* Γ_g *is a* C^2*-submanifold, we see that*

$$\mathrm{div}\,V(0) - \boldsymbol{n}\cdot\nabla V(0)\boldsymbol{n} = \mathrm{div}_{\Gamma_g} V(0) \qquad \text{on } \Gamma_g$$

for all $V \in \overline{\mathcal{V}}_{ad}$, *where* div_{Γ_g} *denotes the surface divergence of* Γ_g. *Thus, we can rewrite* (24.22) *as*

$$\begin{aligned}
&\partial_t|_{t=0} j^E_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \int_\Omega \left[\mathrm{D}f\left(x, \boldsymbol{u}_\varepsilon\right)\left(V(0), \dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right) + f\left(x,\boldsymbol{u}_\varepsilon\right)\,\mathrm{div}\,V(0)\right]\mathrm{d}x + \\
&+ \int_{\Gamma_g}\left[\mathrm{D}g\left(x,\boldsymbol{u}_\varepsilon\right)\left(V(0), \dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right) + g\left(x,\boldsymbol{u}_\varepsilon\right)\,\mathrm{div}_{\Gamma_g} V(0)\right]\mathrm{d}x + \\
&+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right)\mathrm{div}\,V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x.
\end{aligned} \tag{24.24}$$

Proof. We concentrate mainly on deriving (24.23), as the remaining calculations can be carried out as in Lemma 7.5. In particular, the existence of a Lagrange multiplier follows by using the arguments of the corresponding parts in the proof of Lemma 7.5, which reduces to an explicit construction.

Let $T \in \overline{\mathcal{T}}_{ad}$ with velocity field $V \in \overline{\mathcal{V}}_{ad}$ be chosen. We introduce the notation

$$\varphi_\varepsilon(t) = \varphi_\varepsilon \circ T_t^{-1}, \qquad \boldsymbol{u}_\varepsilon(t) = \boldsymbol{S}_E\left(\varphi_\varepsilon(t)\right) \quad \forall t \in (-\tau_0, \tau_0)$$

for $\tau_0 > 0$ small enough. To prove that $t \mapsto (\boldsymbol{u}_\varepsilon(t)\circ T_t) \in \boldsymbol{H}^1_D(\Omega)$ is differentiable at $t = 0$, we want to apply the implicit function theorem and define

$$F : (-\tau_0, \tau_0) \times \boldsymbol{H}^1_D(\Omega) \to \left(\boldsymbol{H}^1_D\left(\Omega\right)\right)'$$

by

$$\begin{aligned}
F(t,\boldsymbol{u})(\boldsymbol{v}) = &\int_\Omega \mathcal{C}(\varphi_\varepsilon)\frac{1}{2}\left(\nabla T_t^{-1}\nabla\boldsymbol{u} + \mathrm{D}\boldsymbol{u}\mathrm{D}T_t^{-1}\right) : \frac{1}{2}(\nabla T_t^{-1}\nabla\boldsymbol{v} + \mathrm{D}\boldsymbol{v}\mathrm{D}T_t^{-1})\det \mathrm{D}T_t\,\mathrm{d}x - \\
&- \int_\Omega \mathcal{C}\left(\varphi_\varepsilon\right)\overline{\mathcal{E}}\left(\varphi_\varepsilon\right) : \frac{1}{2}\left(\nabla T_t^{-1}\nabla\boldsymbol{v} + \mathrm{D}\boldsymbol{v}\mathrm{D}T_t^{-1}\right)\det\mathrm{D}T_t\,\mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\mathrm{d}x - \int_{\Gamma_g}\boldsymbol{g}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\mathrm{d}x.
\end{aligned}$$

Using

$$\nabla\left(\boldsymbol{v}\circ T_t\right) = \nabla T_t\left(\nabla\boldsymbol{v}\right)\circ T_t, \quad \mathrm{D}\left(\boldsymbol{v}\circ T_t\right) = \left(\mathrm{D}\boldsymbol{v}\right)\circ T_t \mathrm{D}T_t \quad \forall \boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$$

we can establish

$$\begin{aligned}
&F(t, \boldsymbol{u}_\varepsilon(t)\circ T_t)(\boldsymbol{v}) = \\
&= \int_\Omega \mathcal{C}(\varphi_\varepsilon(t)\circ T_t)\frac{1}{2}\left(\nabla T_t^{-1}\nabla(\boldsymbol{u}_\varepsilon(t)\circ T_t) + \mathrm{D}(\boldsymbol{u}_\varepsilon(t)\circ T_t)\mathrm{D}T_t^{-1}\right) : \frac{1}{2}\left(\nabla T_t^{-1}\nabla \boldsymbol{v} + \right.\\
&\left. + \mathrm{D}\boldsymbol{v}\mathrm{D}T_t^{-1}\right)\det \mathrm{D}T_t\,\mathrm{d}x - \\
&- \int_\Omega \mathcal{C}\left(\varphi_\varepsilon(t)\circ T_t\right)\overline{\mathcal{E}}\left(\varphi_\varepsilon(t)\circ T_t\right) : \frac{1}{2}\left(\nabla T_t^{-1}\nabla\boldsymbol{v} + \mathrm{D}\boldsymbol{v}\mathrm{D}T_t^{-1}\right)\det\mathrm{D}T_t\,\mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\,\mathrm{d}x - \int_{\Gamma_g}\boldsymbol{g}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\,\mathrm{d}x = \\
&= \int_\Omega \mathcal{C}(\varphi_\varepsilon(t)\circ T_t)\frac{1}{2}\left((\nabla\boldsymbol{u}_\varepsilon(t))\circ T_t + (\mathrm{D}\boldsymbol{u}_\varepsilon(t))\circ T_t\right) : \mathcal{E}(\boldsymbol{v}\circ T_t^{-1})\circ T_t\det\mathrm{D}T_t\,\mathrm{d}x - \\
&- \int_\Omega \mathcal{C}\left(\varphi_\varepsilon(t)\circ T_t\right)\overline{\mathcal{E}}\left(\varphi_\varepsilon(t)\circ T_t\right) : \mathcal{E}\left(\boldsymbol{v}\circ T_t^{-1}\right)\circ T_t\det\mathrm{D}T_t\,\mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\,\mathrm{d}x + \int_{\Gamma_g}\boldsymbol{g}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\,\mathrm{d}x = \\
&= \int_\Omega \mathcal{C}(\varphi_\varepsilon(t))\left(\mathcal{E}(\boldsymbol{u}_\varepsilon(t)) - \overline{\mathcal{E}}\left(\varphi_\varepsilon(t)\right)\right) : \mathcal{E}(\boldsymbol{v}\circ T_t^{-1})\,\mathrm{d}x - \\
&- \int_\Omega \boldsymbol{f}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\,\mathrm{d}x - \int_{\Gamma_g}\boldsymbol{g}\cdot\left(\boldsymbol{v}\circ T_t^{-1}\right)\,\mathrm{d}x = 0
\end{aligned}$$

where we made use of $\boldsymbol{v}\circ T_t^{-1}\in\boldsymbol{H}_D^1(\Omega)$ if $\boldsymbol{v}\in\boldsymbol{H}_D^1(\Omega)$ by the particular choice of $T\in\overline{\mathcal{T}}_{ad}$. Besides, $\mathrm{D}_uF\left(0,\boldsymbol{u}_\varepsilon\right):\boldsymbol{H}_D^1(\Omega)\to\left(\boldsymbol{H}_D^1(\Omega)\right)'$, given by

$$\mathrm{D}_uF\left(0,\boldsymbol{u}_\varepsilon\right)(\boldsymbol{u})(\boldsymbol{v}) = \int_\Omega\mathcal{C}\left(\varphi_\varepsilon\right)\mathcal{E}\left(\boldsymbol{u}\right):\mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x\quad\forall\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}_D^1(\Omega)$$

is by Lax-Milgram's theorem A.2 an isomorphism. This follows by analysis similar to that in the proof of Lemma 24.1. And so we can apply the implicit function theorem to obtain differentiability of $(-\tau_0,\tau_0)\ni t\mapsto(\boldsymbol{u}_\varepsilon(t)\circ T_t)\in\boldsymbol{H}_D^1(\Omega)$ at $t=0$ together with

$$\dot{\boldsymbol{u}}_\varepsilon\left[V\right] := \partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t)\circ T_t\right),\qquad \mathrm{D}_uF(0,\boldsymbol{u}_\varepsilon)\dot{\boldsymbol{u}}_\varepsilon[V] = -\partial_tF(0,\boldsymbol{u}_\varepsilon)$$

and obtain therefrom (24.23).

To derive (24.22), we find that the volume integrals appearing in (24.22) as well as the terms resulting from the Ginzburg-Landau energy can be treated as in the proof of Lemma 7.5.
To handle the boundary integrals, we use the calculation rules derived in [DZ11, Chapter 9, Section 4.2] to see

$$\partial_t|_{t=0}\int_{T_t(\Gamma_g)}g\left(x,\boldsymbol{u}_\varepsilon(t)\right)\,\mathrm{d}x = \partial_t|_{t=0}\int_{\Gamma_g}g\left(T_t(x),\boldsymbol{u}_\varepsilon(t)\circ T_t\right)\omega_t\,\mathrm{d}x$$

where $\omega_t = \left|\det\mathrm{D}T_t\mathrm{D}T_t^{-T}\boldsymbol{n}\right|$. The derivative of ω_t with respect to t at can be calculated by

$$\partial_t|_{t=0}\omega_t = \operatorname{div}V(0) - \boldsymbol{n}\cdot\nabla V(0)\boldsymbol{n}.$$

For more details we refer to [DZ11]. And so we arrive in

$$\partial_t|_{t=0} \int_{T_t(\Gamma_g)} g\left(\boldsymbol{u}_\varepsilon(t)\right)\,\mathrm{d}x = \int_{\Gamma_g} \mathrm{D}g\left(x, \boldsymbol{u}_\varepsilon\right)\left(V(0), \dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right) + \\ + g\left(x, \boldsymbol{u}_\varepsilon\right)\left(\operatorname{div} V(0) - \boldsymbol{n} \cdot \nabla V(0)\boldsymbol{n}\right)\,\mathrm{d}x$$

and can finally deduce the statement.

□

This finishes the treatment of the phase field problem. The next section will now be concerned with the formulation and discussion of the sharp interface problem that turns out to be the limit of the phase field problems as $\varepsilon \searrow 0$.

25 Sharp interface model

After we have introduced and discussed the phase field model describing the problem of finding an optimal material distribution for two given materials on a diffuse interface level in Section 24 we now consider a sharp interface formulation, where the boundary between the two materials is given by a free hypersurface. Therefore, we start by formulating the problem in Section 25.1. One significant difference to the previous parts is here, that the state equations do not change in comparison to the phase field setting as there is no explicit ε-dependency. But as the design variable φ does only have two discrete values for the corresponding materials and does not interpolate any more between those two values, we find that the elasticity tensor and the eigenstrain also have only two different values in the whole region Ω. And so the state system here can be seen as a coupling of the equations for linearized elasticity that have to be fulfilled for every subset Ω_i, $i = 1, 2$, which is filled with the corresponding material.
We will find that the sharp interface formulation of the shape optimization problem is equivalent to minimizing a certain reduced objective functional. In Section 26.1 we then prove, that the latter is the Γ-limit of the reduced objective functionals corresponding to the phase field problems. And so in this sense, the problem discussed in the following is the limit problem of the phase field problems as the thickness of the interface tends to zero.

25.1 Problem formulation

The sharp interface problem that we are considering in this section is given by

$$\min_{(\varphi,\boldsymbol{u})} J_0^E(\varphi,\boldsymbol{u}) := \int_\Omega f(x,\boldsymbol{u})\,\mathrm{d}x + \int_{\Gamma_g} g(x,\boldsymbol{u})\,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi) \tag{25.1}$$

with

$$(\varphi,\boldsymbol{u}) \in \Phi_E^0 \times \boldsymbol{H}_D^1(\Omega)$$

s.t.

$$\int_\Omega \mathcal{C}(\varphi)\left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi)\right) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega), \tag{25.2}$$

where we recall that

$$E^\varphi := \{x \in \Omega \mid \varphi(x) = 1\}.$$

This problem is a general shape optimization problem, where the aim is to find an optimal material distribution of two fixed materials, which are represented by $\{x \in \Omega \mid \varphi(x) = 1\}$ and $\{x \in \Omega \mid \varphi(x) = -1\}$, respectively. Thus, $\varphi \in \Phi_E^0 = \left\{\varphi \in BV(\Omega, \{\pm 1\}) \mid \fint_\Omega \varphi\,\mathrm{d}x \le \beta\right\}$ plays the role of the design variable, and can now in contrast to the previous section only have the discrete values ± 1. Besides, by adding a multiple of the perimeter to the objective functional (25.1), we ensure the existence of a minimizer for the overall optimization problem. Hereby, $\gamma > 0$ is an arbitrary given parameter which can be considered as weighting parameter for the perimeter. Besides, $c_0 = \int_{-1}^1 \sqrt{2\Psi_0(s)}\,\mathrm{d}s = \frac{\pi}{2}$ is a constant arising due to technical reasons, since the perimeter functional times this constant is the Γ-limit of the Ginzburg-Landau energy, see for instance [Mod87, MM77, Ste88].

Remark 25.1. *Assume that* $\varphi = \pm 1$ *a.e. in* Ω *and* $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_g)$ *and let* $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ *be a solution of* (25.2)*. Then we see, that* $\mathcal{C}(\varphi) = \mathcal{C}(\pm 1)$ *and* $\overline{\mathcal{E}}(\varphi) = \overline{\mathcal{E}}(\pm 1)$ *a.e. in* Ω*. Now assume that* $\{\varphi = \pm 1\}$ *are smooth open sets and denote* $\Gamma := \partial\{\varphi = 1\} \cap \partial\{\varphi = -1\}$*. Additionally, let the support of the applied surface load* $\boldsymbol{g}$ *have a positive distance to the part of the boundary with Dirichlet boundary conditions* Γ_D *and* Γ *should not intersect with* $\partial\Omega$*.*
We then can apply regularity theory for the equations in linearized elasticity, see [Cia88, CDN10] and included references, to obtain $\boldsymbol{u} \in \boldsymbol{H}^2(\{\varphi = \pm 1\})$*. Testing now the state equation* (25.2) *with* $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\{\varphi = \pm 1\})$ *we arrive in the pointwise relation*

$$-\nabla \cdot (\mathcal{C}(\pm 1)\,\mathcal{E}(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \{\varphi = \pm 1\}. \tag{25.3}$$

Multipliying this identity with some $\boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$ *and integrating by parts leads to*

$$\begin{aligned} &\int_{\{\varphi=\pm 1\}} \mathcal{C}(\pm 1)\,(\mathcal{E}(\boldsymbol{u})) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x - \int_{\partial\{\varphi=\pm 1\}} (\mathcal{C}(\pm 1)\mathcal{E}(\boldsymbol{u})\,(\pm\nu)) \cdot \boldsymbol{v}\,\mathrm{d}x = \\ &= \int_{\{\varphi=\pm 1\}} \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_g \cap \{\varphi=\pm 1\}} (\mathcal{C}(\pm 1)\mathcal{E}(\boldsymbol{u}) \cdot \boldsymbol{n}) \cdot \boldsymbol{v}\,\mathrm{d}x. \end{aligned}$$

Adding up those two terms for $\{\varphi = 1\}$ *and* $\{\varphi = -1\}$*, respectively, and comparison with* (25.2) *yields then*

$$\int_\Gamma [\mathcal{C}(\varphi)\left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi)\right)\nu]_\Gamma \cdot \boldsymbol{v}\,\mathrm{d}x = 0,$$

$$\int_{\Gamma_g \cap \{\varphi=\pm 1\}} \left(\mathcal{C}(\pm 1)\left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\pm 1)\right) \cdot \boldsymbol{n} - \boldsymbol{g}\right) \cdot \boldsymbol{v}\,\mathrm{d}x = 0$$

where we made in particular use of the fact, that the outer unit normal ν *on* $\{\varphi = 1\}$ *is the negative of the one on* $\{\varphi = -1\}$*. Here we use the notation* $[\boldsymbol{w}]_\Gamma = \boldsymbol{w}|_{\{\varphi=1\}} - \boldsymbol{w}|_{\{\varphi=-1\}}$ *for the jump of* $\boldsymbol{w}$ *across the interface* Γ*. Besides,* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ *implies directly* $[\boldsymbol{u}]_\Gamma = \boldsymbol{0}$*. Altogether we find, that in this case* (25.2) *is a weak formulation of*

$$\begin{aligned} -\nabla \cdot (\mathcal{C}(\pm 1)\,\mathcal{E}(\boldsymbol{u})) &= \boldsymbol{f} && \text{in } \{\varphi = \pm 1\}, \\ \left(\mathcal{C}(\pm 1)\left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\pm 1)\right)\right) \cdot \boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_g \cap \partial\{\varphi = \pm 1\}, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D, \\ \left[\mathcal{C}(\varphi)\left(\mathcal{E}(\boldsymbol{u}) - \overline{\mathcal{E}}(\varphi)\right)\nu\right]_\Gamma &= \boldsymbol{0} && \text{on } \Gamma, \\ [\boldsymbol{u}]_\Gamma &= \boldsymbol{0} && \text{on } \Gamma. \end{aligned}$$

Thus in every set filled with material one or two, the equations for linearized elasticity are fulfilled independently and we have a transmission condition on the interface Γ*.*

As already discussed above, the state equations given by the weak formulation (25.2) are the same as in the diffuse interface setting and the only difference is, that the design variable φ now does only take discrete values. This is one main difference to the preceding parts, where the state equations were depending on ε and not only on the design variable φ.

25.2 Existence results

As already mentioned, the state constraints (25.2) are the same as in the phase field model. Therefore, Lemma 24.1 yields well-posedness of (25.2) in the sense of existence and uniqueness of a solution $\boldsymbol{u} \in \boldsymbol{H}_D^1(\Omega)$ for every $\varphi \in L^1(\Omega)$ with $|\varphi| \leq 1$ a.e. in Ω. And so we can come directly to prove well-posedness of the optimization problem.

Theorem 25.1. *There exists at least one minimizer of* (25.1) – (25.2).

Proof. This can be established by using the arguments of the proof of Theorem 24.1, thus by the direct method in the calculus of variations. To this end, we use in particular that $BV(\Omega, \{\pm 1\})$ compactly imbeds into $L^1(\Omega)$, and that the state equations (25.2) imply for fixed boundary data $\boldsymbol{g}$ and force term $\boldsymbol{f}$ by Korn's inequality a uniform bound on $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ if $\boldsymbol{u}$ solves (25.2) for some $\varphi \in L^1(\Omega)$, cf. (24.4). □

Again we can introduce a reduced objective functional

$$j_0^E : \Phi_E^0 \to \mathbb{R}$$

which is then given by

$$j_0^E(\varphi) := J_0^E\left(\varphi, \boldsymbol{S}_E(\varphi)\right).$$

Before proving that j_0^E actually is the $L^1(\Omega)$-Γ-limit of $\left(j_\varepsilon^E\right)_{\varepsilon>0}$, we will discuss optimality conditions for the sharp interface problem in the next subsection.

25.3 Optimality conditions

For this subsection we assume that $(\varphi_0, \boldsymbol{u}_0)$ is a minimizer of (25.1) – (25.2). We want to derive first order necessary optimality conditions. We will use the idea of Theorem 24.3 by varying the geometry and obtain therefrom a rather general formulation of optimality conditions. We will then see, that under suitable regularity assumptions on the minimizer φ_0, those optimality conditions are equivalent to results obtained by classical shape analysis which can be found in literature.

We assume for the remainder of this subsection additionally Assumption **(A17)** to ensure differentiability of the objective functional.

Theorem 25.2. *For any minimizer* $(\varphi_0, \boldsymbol{u}_0) \in \Phi_E^0 \times \boldsymbol{H}_D^1(\Omega)$ *of* (25.1) – (25.2) *we have the following necessary optimality conditions:*

$$\partial_t|_{t=0} j_0^E\left(\varphi_0 \circ T_t^{-1}\right) = -\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x, \tag{25.4}$$

$$\lambda_0 \left(\int_\Omega \varphi_0\,\mathrm{d}x - \beta|\Omega|\right) = 0 \tag{25.5}$$

for all $T \in \overline{\mathcal{T}}_{ad}$ *with velocity* $V \in \overline{\mathcal{V}}_{ad}$*, where* $\lambda_0 \in \mathbb{R}^+$ *is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:*

$$\begin{aligned}\partial_t|_{t=0} j_0^E\left(\varphi_0 \circ T_t^{-1}\right) &= \int_\Omega \left[\mathrm{D}f\left(x, \boldsymbol{u}_0\right)\left(V(0), \dot{\boldsymbol{u}}_0[V]\right) + f\left(x, \boldsymbol{u}_0\right) \operatorname{div} V(0)\right] \mathrm{d}x + \\ &+ \int_{\Gamma_g} \left[\mathrm{D}g\left(x, \boldsymbol{u}_0\right)\left(V(0), \dot{\boldsymbol{u}}_0[V]\right) + g\left(x, \boldsymbol{u}_0\right)\left(\operatorname{div} V(0) - \boldsymbol{n} \cdot \nabla V(0)\boldsymbol{n}\right)\right] \mathrm{d}x + \\ &+ \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right) \mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|\end{aligned} \tag{25.6}$$

with ν being the generalised unit normal on $E_0 := \{\varphi_0 = 1\}$. Moreover, $\dot{\boldsymbol{u}}_0[V] \in \boldsymbol{H}^1_D(\Omega)$ is given as the solution of (24.23) *with φ_ε replaced by φ_0 and $\boldsymbol{u}_\varepsilon$ replaced by $\boldsymbol{u}_0$.*

Proof. For the proof we refer the reader to the proof of Theorem 24.3. The first variation of the perimeter term is given by Lemma 3.4. □

Remark 25.2. *If we assume that Γ_g has C^2-regularity, we can rewrite* (25.6) *into the more convenient form*

$$\begin{aligned}\partial_t|_{t=0} j_0^E\left(\varphi_0 \circ T_t^{-1}\right) = \int_\Omega \left[\mathrm{D}f(x, \boldsymbol{u}_0)(V(0), \dot{\boldsymbol{u}}_0[V]) + f(x, \boldsymbol{u}_0)\operatorname{div} V(0)\right] \mathrm{d}x + \\ + \int_{\Gamma_g}\left[\mathrm{D}g(x, \boldsymbol{u}_0)(V(0), \dot{\boldsymbol{u}}_0[V]) + g(x, \boldsymbol{u}_0)\operatorname{div}_{\Gamma_g} V(0)\right] \mathrm{d}x + \\ + \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right) \mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|\end{aligned} \tag{25.7}$$

by using the identify

$$\operatorname{div}_{\Gamma_g} V(0) = \operatorname{div} V(0) - \boldsymbol{n} \cdot \nabla V(0)\boldsymbol{n} \qquad on\ \Gamma_g.$$

We can now reformulate those optimality conditions under more regularity assumptions on the minimizing set $E_0 = \{\varphi_0 = 1\}$ and the given data. In particular, we then can compare our results to those obtained in literature.

Theorem 25.3. *Let $(\varphi_0, \boldsymbol{u}_0) \in \Phi^0_E \times \boldsymbol{H}^1_D(\Omega)$ be minimizers of* (25.1) – (25.2) *such that $\{\varphi_0 = 1\}$ and $\{\varphi_0 = -1\}$ represent fixed open sets. Let $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ and the objective functional is assumed to be chosen in such a way that $\mathrm{D}_u f(\cdot, \boldsymbol{u}) \in \boldsymbol{L}^2(\Omega)$ and $\mathrm{D}_u g(\cdot, \boldsymbol{u}) \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma_g)$ if $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Additionally, let the boundary data be chosen such that*

$$d\left(\overline{\{x \in \Gamma_g \mid \boldsymbol{g}(x) \neq \boldsymbol{0}\}}, \overline{\Gamma_D}\right), \quad d\left(\overline{\{x \in \Gamma_g \mid \mathrm{D}_u g(x, \boldsymbol{u}_0) \neq \boldsymbol{0}\}}, \overline{\Gamma_D}\right) > 0. \tag{25.8}$$

Let $\Gamma := \partial\{\varphi_0 = 1\} \cap \Omega$ denote the interface between the two phases and assume that $\Gamma \in C^2$ and $d(\Gamma, \partial\Omega) > 0$. By

$$[\boldsymbol{w}]_\Gamma(x) := \boldsymbol{w}|_{\{\varphi_0 = 1\}}(x) - \boldsymbol{w}|_{\{\varphi_0 = -1\}}(x)$$

we denote the jump of $\boldsymbol{w}$ along the interface Γ, and ν is again the outer unit normal on $\{\varphi_0 = 1\}$. Besides, let $\kappa = \operatorname{div}_\Gamma \nu$ be the mean curvature of Γ. If $g(\cdot, \boldsymbol{u}_0(\cdot)) \not\equiv 0$, we assume additionally that Γ_g has C^2-regularity. Then the optimality conditions derived in Theorem 25.2 are equivalent to the following system:

$$\left.\begin{aligned}\gamma c_0 \kappa - \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)\right]_\Gamma + \\ + \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu \cdot \partial_\nu \boldsymbol{q}_0\right]_\Gamma + \\ + \left[\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu \cdot \partial_\nu \boldsymbol{u}_0\right]_\Gamma + 2\lambda_0 + \left[f(x, \boldsymbol{u}_0)\right]_\Gamma = 0 \quad on\ \Gamma\end{aligned}\right\} \tag{25.9}$$

$$
\left.\begin{aligned}
-\nabla\cdot\left(\mathcal{C}\left(\pm 1\right)\mathcal{E}\left(\boldsymbol{u}_0\right)\right) &= \boldsymbol{f} && \textit{in } \{\varphi_0=\pm 1\},\\
\left(\mathcal{C}\left(\pm 1\right)\left(\mathcal{E}\left(\boldsymbol{u}_0\right)-\overline{\mathcal{E}}\left(\pm 1\right)\right)\right)\cdot\boldsymbol{n} &= \boldsymbol{g} && \textit{on } \Gamma_g\cap\partial\{\varphi_0=\pm 1\},\\
\boldsymbol{u}_0 &= \boldsymbol{0} && \textit{on } \Gamma_D,\\
\left[\mathcal{C}(\varphi_0)\left(\mathcal{E}\left(\boldsymbol{u}_0\right)-\overline{\mathcal{E}}\left(\varphi_0\right)\right)\nu\right]_\Gamma &= \boldsymbol{0} && \textit{on } \Gamma,\\
\left[\boldsymbol{u}_0\right]_\Gamma &= \boldsymbol{0} && \textit{on } \Gamma,\\
-\nabla\cdot\left(\mathcal{C}\left(\pm 1\right)\mathcal{E}\left(\boldsymbol{q}_0\right)\right) &= \mathrm{D}_u f\left(\cdot,\boldsymbol{u}_0\right) && \textit{in } \{\varphi_0=\pm 1\},\\
\left(\mathcal{C}\left(\pm 1\right)\mathcal{E}\left(\boldsymbol{q}_0\right)\right)\cdot\boldsymbol{n} &= \mathrm{D}_u g\left(\cdot,\boldsymbol{u}_0\right) && \textit{on } \Gamma_g\cap\partial\{\varphi_0=\pm 1\},\\
\boldsymbol{q}_0 &= \boldsymbol{0} && \textit{on } \Gamma_D,\\
\left[\mathcal{C}(\varphi_0)\mathcal{E}\left(\boldsymbol{q}_0\right)\nu\right]_\Gamma &= \boldsymbol{0} && \textit{on } \Gamma,\\
\left[\boldsymbol{q}_0\right]_\Gamma &= \boldsymbol{0} && \textit{on } \Gamma,\\
\lambda_0\left(\int_\Omega \varphi_0\,\mathrm{d}x-\beta\left|\Omega\right|\right) = 0,\ \lambda_0\geq 0,\ &\int_\Omega\varphi_0\,\mathrm{d}x\leq\beta\left|\Omega\right|.
\end{aligned}\right\}
\tag{25.10}
$$

Hence, $\boldsymbol{u}_0\in\boldsymbol{H}^2(\{\varphi_0=\pm 1\})$ and $\boldsymbol{q}_0\in\boldsymbol{H}^2\left(\{\varphi_0=\pm 1\}\right)$ are strong solutions of the state equations and adjoint system, respectively.

Remark 25.3. *Condition* (25.8) *is necessary to obtain that the state variable $\boldsymbol{u}_0$ and the adjoint variable $\boldsymbol{q}_0$ are in $\boldsymbol{H}^2(\{\varphi_0=\pm 1\})$. Since we assume that Γ has a positive distance to $\partial\Omega$, we could generalize the stated result by dropping* (25.8). *Then we only obtain $\boldsymbol{u}_0,\boldsymbol{q}_0\in\boldsymbol{H}^2(U)$ if $U\subset\{\varphi_0=\pm 1\}$ is an open subset such that $d\left(\partial U,\partial\Omega\right)>0$, see [Cia88, CDN10]. But we could smoothen $\boldsymbol{u}_0,\boldsymbol{q}_0$ and carry out the same calculations with the smoothed functions. Since Γ has a positive distance to $\partial\Omega$, we find that $\boldsymbol{u}_0,\boldsymbol{q}_0\in\boldsymbol{H}^2(U)$ for a neighborhood U of Γ, and hence we find that* (25.9) *still holds true for $\boldsymbol{u}_0$ and $\boldsymbol{q}_0$. Of course, the state and adjoint equations then have to be understood in the usual weak sense.*

Proof. We start by noticing that Lemma 3.4 implies directly

$$\int_\Omega\left(\operatorname{div}V(0)-\nu\cdot\nabla V(0)\nu\right)\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right| = \int_\Gamma \kappa V(0)\cdot\nu\,\mathrm{d}x.$$

If $g\left(\cdot,\boldsymbol{u}_0\left(\cdot\right)\right)\not\equiv 0$, we use the stated C^2–regularity of Γ_g to deduce

$$\int_{\Gamma_g} g\left(x,\boldsymbol{u}_0\right)\left(\operatorname{div}V(0)-\boldsymbol{n}\cdot\nabla V(0)\boldsymbol{n}\right)\,\mathrm{d}x = \int_{\Gamma_g} g\left(x,\boldsymbol{u}_0\right)\,\operatorname{div}_{\Gamma_g}V(0)\,\mathrm{d}x$$

see also considerations in Remark 25.2.
Using the stated regularity assumptions on the data and (25.8), we can apply regularity theory for the equations of linearized elasticity, compare [Cia88, Section 6.3] and included references, to obtain moreover that

$$\boldsymbol{u}_0\in\boldsymbol{H}^2\left(\{\varphi_0\pm 1\}\right).$$

We define the adjoint variable $\boldsymbol{q}_0\in\boldsymbol{H}^1_D(\Omega)$ as weak solution of

$$\int_\Omega \mathcal{C}\left(\varphi_0\right)\mathcal{E}\left(\boldsymbol{q}_0\right):\mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x = \int_\Omega \mathrm{D}_u f\left(x,\boldsymbol{u}_0\right)\boldsymbol{v}\,\mathrm{d}x+\int_{\Gamma_g}\mathrm{D}_u g\left(x,\boldsymbol{u}_0\right)\boldsymbol{v}\,\mathrm{d}x$$

which has to holds for all $\boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$. We notice as above from the stated regularity on the objective functional that $\boldsymbol{q}_0 \in \boldsymbol{H}^2(\{\varphi_0 = \pm 1\})$. Using the regularity of $\boldsymbol{q}_0$ and $\boldsymbol{u}_0$ we obtain as in Remark 25.1 that

$$[\boldsymbol{u}_0]_\Gamma = [\boldsymbol{q}_0]_\Gamma = \mathbf{0} \quad \text{on } \Gamma \tag{25.11}$$

and by making use of the state equations we also have

$$\left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu\right]_\Gamma = \left[\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu\right]_\Gamma = \mathbf{0} \quad \text{on } \Gamma. \tag{25.12}$$

If we choose $\dot{\boldsymbol{u}}_0[V] \in \boldsymbol{H}^1_D(\Omega)$ as a test function in the adjoint state system we obtain

$$\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0) : \mathcal{E}(\dot{\boldsymbol{u}}_0[V])\,\mathrm{d}x = \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_0)(\dot{\boldsymbol{u}}_0[V])\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g(x, \boldsymbol{u}_0)\,\dot{\boldsymbol{u}}_0[V]\,\mathrm{d}x. \tag{25.13}$$

On the other hand, inserting $\boldsymbol{q}_0 \in \boldsymbol{H}^1_D(\Omega)$ as test function into the linearized equation (24.23) for $\dot{\boldsymbol{u}}_0[V]$ we see

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\dot{\boldsymbol{u}}_0[V]) : \mathcal{E}(\boldsymbol{q}_0)\,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi_0)\frac{1}{2}\left(\mathrm{D}\boldsymbol{u}_0\mathrm{D}V(0) + \nabla V(0)\nabla\boldsymbol{u}_0\right) : \mathcal{E}(\boldsymbol{q}_0) + \\
&+ \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \frac{1}{2}(\nabla V(0)\nabla\boldsymbol{q}_0 + \mathrm{D}\boldsymbol{q}_0\mathrm{D}V(0)) - \\
&- \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)\,\mathrm{div}\,V(0)\,\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\,\mathrm{d}x - \int_{\Gamma_g}\boldsymbol{g}\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\,\mathrm{d}x.
\end{aligned} \tag{25.14}$$

We proceed by testing the state equation for $\boldsymbol{u}_0$ in the strong formulation with $\mathrm{D}\boldsymbol{q}_0 V(0) \in \boldsymbol{H}^2(\{\varphi_0 = \pm 1\})$ and find

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \frac{1}{2}\left(\nabla(\mathrm{D}\boldsymbol{q}_0)V(0) + \nabla V(0)\nabla\boldsymbol{q}_0 + \mathrm{D}(\mathrm{D}\boldsymbol{q}_0)V(0) + \right. \\
&\left. + \mathrm{D}\boldsymbol{q}_0\mathrm{D}V(0)\right)\mathrm{d}x - \int_\Gamma \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\right]_\Gamma \mathrm{d}x = \\
&= \int_\Omega \boldsymbol{f}\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\,\mathrm{d}x + \int_{\Gamma_g}\boldsymbol{g}\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\,\mathrm{d}x.
\end{aligned}$$

Hence, (25.14) can be reformulated to

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\dot{\boldsymbol{u}}_0[V]) : \mathcal{E}(\boldsymbol{q}_0)\,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi_0)\frac{1}{2}\left(\mathrm{D}\boldsymbol{u}_0\mathrm{D}V(0) + \nabla V(0)\nabla\boldsymbol{u}_0\right) : \mathcal{E}(\boldsymbol{q}_0) - \\
&- \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \frac{1}{2}(\nabla(\mathrm{D}\boldsymbol{q}_0)V(0) + \mathrm{D}(\mathrm{D}\boldsymbol{q}_0)V(0)) - \\
&- \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)\,\mathrm{div}\,V(0)\,\mathrm{d}x + \\
&+ \int_\Gamma \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\right]_\Gamma \mathrm{d}x.
\end{aligned} \tag{25.15}$$

Similarly, we insert $\mathrm{D}\boldsymbol{u}_0 V(0) \in \boldsymbol{H}^2(\{\varphi_0 = \pm 1\})$ as test function in the adjoint equation to obtain

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0) : \frac{1}{2}\left(\nabla(\mathrm{D}\boldsymbol{u}_0)V(0) + \nabla V(0)\nabla \boldsymbol{u}_0 + \mathrm{D}(\mathrm{D}\boldsymbol{u}_0)V(0) + \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)\right)\,\mathrm{d}x - \\
&- \int_\Gamma [\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu \cdot \mathrm{D}\boldsymbol{u}_0 V(0)]_\Gamma = \\
&= \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x
\end{aligned}$$

and so (25.15) can be rewritten as

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\dot{\boldsymbol{u}}_0[V]) : \mathcal{E}(\boldsymbol{q}_0)\,\mathrm{d}x = \int_\Omega -\mathcal{C}(\varphi_0)\frac{1}{2}\left(\nabla(\mathrm{D}\boldsymbol{u}_0)V(0) + \mathrm{D}(\mathrm{D}\boldsymbol{u}_0)V(0)\right) : \mathcal{E}(\boldsymbol{q}_0) - \\
&- \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \frac{1}{2}(\nabla(\mathrm{D}\boldsymbol{q}_0)V(0) + \mathrm{D}(\mathrm{D}\boldsymbol{q}_0)V(0)) - \\
&- \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)\operatorname{div} V(0)\,\mathrm{d}x + \\
&+ \int_\Gamma \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu \cdot \mathrm{D}\boldsymbol{q}_0 V(0)\right]_\Gamma + [\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu \cdot \mathrm{D}\boldsymbol{u}_0 V(0)]_\Gamma\,\mathrm{d}x + \\
&+ \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x.
\end{aligned} \tag{25.16}$$

Substituting

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)\operatorname{div} V(0)\,\mathrm{d}x = \\
&= -\int_\Omega \nabla(\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)) \cdot V(0)\,\mathrm{d}x + \\
&+ \int_\Gamma \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)(V(0)\cdot\nu)\right]_\Gamma\,\mathrm{d}x
\end{aligned}$$

into (25.16) we have

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\dot{\boldsymbol{u}}_0[V]) : \mathcal{E}(\boldsymbol{q}_0)\,\mathrm{d}x = -\int_\Gamma [\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)]_\Gamma (V(0)\cdot\nu)\,\mathrm{d}x + \\
&+ \int_\Gamma \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu \cdot \mathrm{D}\boldsymbol{q}_0 V(0)\right]_\Gamma + [\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu \cdot \mathrm{D}\boldsymbol{u}_0 V(0)]_\Gamma\,\mathrm{d}x + \\
&+ \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x.
\end{aligned} \tag{25.17}$$

Thus, combining (25.17) and (25.13) we obtain

$$\begin{aligned}
&\int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_0)\dot{\boldsymbol{u}}_0[V]\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g(x, \boldsymbol{u}_0)\,\dot{\boldsymbol{u}}_0[V]\,\mathrm{d}x = \\
&= -\int_\Gamma [\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0)]_\Gamma (V(0)\cdot\nu)\,\mathrm{d}x + \\
&+ \int_\Gamma \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right)\nu \cdot \mathrm{D}\boldsymbol{q}_0 V(0)\right]_\Gamma + [\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu \cdot \mathrm{D}\boldsymbol{u}_0 V(0)]_\Gamma\,\mathrm{d}x + \\
&+ \int_\Omega \mathrm{D}_u f(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g(x, \boldsymbol{u}_0)\left(\mathrm{D}\boldsymbol{u}_0 V(0)\right)\,\mathrm{d}x
\end{aligned}$$

which can be reformulated to

$$
\begin{aligned}
&\int_\Omega \mathrm{D}_u f(x,\boldsymbol{u}_0)\dot{\boldsymbol{u}}_0[V]\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g\,(x,\boldsymbol{u}_0)\,\dot{\boldsymbol{u}}_0\,[V]\,\mathrm{d}x = \\
&= \int_\Gamma \big[-\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0) + \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right)\nu\cdot\partial_\nu\boldsymbol{q}_0 + \\
&+ \mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu\cdot\partial_\nu\boldsymbol{u}_0\big]_\Gamma\,(V(0)\cdot\nu)\,\mathrm{d}x + \\
&+ \int_\Omega \mathrm{D}_u f(x,\boldsymbol{u}_0)\,(\mathrm{D}\boldsymbol{u}_0 V(0))\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}_u g\,(x,\boldsymbol{u}_0)\,(\mathrm{D}\boldsymbol{u}_0 V(0))\,\mathrm{d}x.
\end{aligned}
$$

Here, we made in particular use of

$$
\begin{aligned}
&\left[\mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\mathrm{D}\boldsymbol{u}_0 V(0)\right]_\Gamma = \mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\left[\mathrm{D}\boldsymbol{u}_0 V(0)\right]_\Gamma = \\
&= \mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\left[\partial_\nu\boldsymbol{u}_0(V(0)\cdot\nu)\right]_\Gamma = \left[\mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0\right]_\Gamma(V(0)\cdot\nu)
\end{aligned}
$$

which follows from (25.11) – (25.12), and analogously we find

$$
\left[\mathcal{C}(\varphi_0)\left(\mathcal{E}\,(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right)\nu\cdot\mathrm{D}\boldsymbol{q}_0 V(0)\right]_\Gamma = \left[\mathcal{C}(\varphi_0)\left(\mathcal{E}\,(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right)\nu\cdot\partial_\nu\boldsymbol{q}_0\right]_\Gamma(V(0)\cdot\nu)\,.
$$

Combining these results and using

$$
\mathrm{D}\left(g\left(x,\boldsymbol{u}_0(x)\right)\right) = \mathrm{D}_{\Gamma_g}\left(g\left(x,\boldsymbol{u}_0(x)\right)\right) + \partial_{\boldsymbol{n}}\left(g\left(x,\boldsymbol{u}_0(x)\right)\right)\cdot\boldsymbol{n}
$$

we end up with

$$
\begin{aligned}
&\partial_t|_{t=0} j_0^E\left(\varphi_0\circ T_t^{-1}\right) = \int_\Gamma \big[-\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0) + \\
&+ \mathcal{C}(\varphi_0)\left(\mathcal{E}\,(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right)\nu\cdot\partial_\nu\boldsymbol{q}_0 + \mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0\big]_\Gamma\,(V(0)\cdot\nu)\,\mathrm{d}x + \\
&+ \int_\Omega \mathrm{D}f(x,\boldsymbol{u}_0)\,(V(0),\mathrm{D}\boldsymbol{u}_0 V(0))\,\mathrm{d}x + \int_{\Gamma_g} \mathrm{D}g\,(x,\boldsymbol{u}_0)\,(V(0),\mathrm{D}\boldsymbol{u}_0 V(0))\,\mathrm{d}x + \\
&+ \int_\Omega f\,(x,\boldsymbol{u}_0)\,\operatorname{div} V(0)\,\mathrm{d}x + \int_{\Gamma_g} g\,(x,\boldsymbol{u}_0)\,\operatorname{div}_{\partial\Omega} V(0)\,\mathrm{d}x + \gamma c_0 \int_\Gamma \kappa(V(0)\cdot\nu) = \\
&= \int_\Gamma \big[-\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0) + \mathcal{C}(\varphi_0)\left(\mathcal{E}\,(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right)\nu\cdot\partial_\nu\boldsymbol{q}_0 + \\
&+ \mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0\big]_\Gamma\,(V(0)\cdot\nu)\,\mathrm{d}x + \\
&+ \int_\Omega \operatorname{div}\left(f\,(x,\boldsymbol{u}_0)\,V(0)\right)\mathrm{d}x + \int_{\partial\Omega} \operatorname{div}_{\partial\Omega}\left(g\,(x,\boldsymbol{u}_0)\,V(0)\right)\mathrm{d}x + \\
&+ \int_{\Gamma_g} \mathrm{D}g(x,\boldsymbol{u}_0)\,(\boldsymbol{n},\partial_{\boldsymbol{n}}\boldsymbol{u}_0)\,\underbrace{(V(0)\cdot\boldsymbol{n})}_{=0}\,\mathrm{d}x + \gamma c_0 \int_\Gamma \kappa(V(0)\cdot\nu) = \\
&= \int_\Gamma \big[-\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right) : \mathcal{E}(\boldsymbol{q}_0) + \mathcal{C}(\varphi_0)\left(\mathcal{E}\,(\boldsymbol{u}_0) - \overline{\mathcal{E}}\,(\varphi_0)\right)\nu\cdot\partial_\nu\boldsymbol{q}_0 + \\
&+ \mathcal{C}(\varphi_0)\mathcal{E}\,(\boldsymbol{q}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0 + f\,(x,\boldsymbol{u}_0)\big]_\Gamma\,(V(0)\cdot\nu) + \gamma c_0\kappa(V(0)\cdot\nu)\,\mathrm{d}x.
\end{aligned}
\tag{25.18}
$$

For the last step we made use of the tangential Stokes formula, see [DZ11, Chapter 9, Section 5.5], which yields

$$
\int_{\Gamma_g} \operatorname{div}_{\partial\Omega}\left(g\,(x,\boldsymbol{u}_0)\,V(0)\right)\mathrm{d}x = \int_{\Gamma_g} g(x,\boldsymbol{u}_0)\kappa_{\Gamma_g} V(0)\cdot\boldsymbol{n}\,\mathrm{d}x
$$

if $\kappa_{\Gamma_g} = \operatorname{div}_{\Gamma_g} \boldsymbol{n}$ denotes the mean curvature of Γ_g, and the fact that $V(0)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ for any $V \in \overline{\mathcal{V}}_{ad}$.
Thus, using (25.4) and (25.18) we find

$$\begin{aligned}&\int_\Gamma \left[-\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0)-\overline{\mathcal{E}}(\varphi_0)\right):\mathcal{E}(\boldsymbol{q}_0)+\mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0)-\overline{\mathcal{E}}(\varphi_0)\right)\nu\cdot\partial_\nu\boldsymbol{q}_0+\right.\\ &\left.+\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{q}_0)\nu\cdot\partial_\nu\boldsymbol{u}_0+f(x,\boldsymbol{u}_0)\right]_\Gamma(V(0)\cdot\nu)+\gamma c_0\kappa(V(0)\cdot\nu)\,\mathrm{d}x=\\ &=-\lambda_0\int_\Omega\varphi_0\operatorname{div}V(0)\,\mathrm{d}x=-2\lambda_0\int_\Gamma V(0)\cdot\nu\,\mathrm{d}x.\end{aligned} \tag{25.19}$$

Since (25.19) is fulfilled for any $V \in \overline{\mathcal{V}}_{ad}$ we obtain the pointwise relation (25.9) on Γ.

□

We want to compare the stated results in this smooth setting to those derived in literature. We first of all remark that in [BFSGS13] the same optimality system for the sharp interface setting has been derived from the phase field model by formally matched asymptotics for the mean compliance and compliant mechanism problems, see Examples 23.1 and 23.2. In contrast to our work, no eigenstrain has been taken into account in [BFSGS13].

Besides, applying directly shape sensitivity analysis yields the same result, where we refer for instance to [ADDM13, AJVG11, HHS13].

We finally want to discuss the result of Theorem 25.3 on the basis of the examples introduced in Example 23.1 and 23.2, namely the mean compliance and the compliant mechanism.

Example 25.1 (Mean compliance)**.** Assume the objective functional is given as in Example 23.1, thus we want to minimize the mean compliance. This corresponds to the following choices

$$f(x,\boldsymbol{u}) = \boldsymbol{f}(x)\cdot\boldsymbol{u}, \quad g(x,\boldsymbol{u}) = \boldsymbol{g}(x)\cdot\boldsymbol{u}.$$

Besides, we consider the case where no eigenstrain is taken into account, thus $\overline{\mathcal{E}} \equiv 0$. One can see easily that the adjoint state equation for $\boldsymbol{q}_0$ equals the state equation, and due to the unique solvability we obtain $\boldsymbol{u}_0 = \boldsymbol{q}_0$. Moreover, for $\boldsymbol{f}$ smooth enough, we obtain from $[\boldsymbol{u}_0]_\Gamma = \boldsymbol{0}$, that $[f(x,\boldsymbol{u}_0)]_\Gamma = 0$. And so in this case, the first order optimality conditions (25.9)-(25.10) reduce to

$$\gamma c_0\kappa - \left[\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{u}_0):\mathcal{E}(\boldsymbol{u}_0)\right]_\Gamma + 2\left[\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{u}_0)\nu\cdot\partial_\nu\boldsymbol{u}_0\right]_\Gamma + 2\lambda_0 = 0 \quad \text{on } \Gamma, \tag{25.20}$$

$$\begin{aligned}-\nabla\cdot\left(\mathcal{C}(\pm1)\mathcal{E}(\boldsymbol{u}_0)\right) &= \boldsymbol{f} && \text{in } \{\varphi_0=\pm1\}\\ \left(\mathcal{C}(\pm1)\mathcal{E}(\boldsymbol{u}_0)\right)\cdot\boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_g\cap\partial\{\varphi_0=\pm1\},\\ \boldsymbol{u}_0 &= \boldsymbol{0} && \text{on } \Gamma_D,\\ \left[\mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{u}_0)\nu\right]_\Gamma &= \boldsymbol{0} && \text{on } \Gamma,\\ [\boldsymbol{u}_0]_\Gamma &= \boldsymbol{0} && \text{on } \Gamma,\end{aligned} \tag{25.21}$$

$$\lambda_0\left(\int_\Omega\varphi_0\,\mathrm{d}x - \beta\,|\Omega|\right) = 0,\ \ \lambda_0 \geq 0,\ \ \int_\Omega\varphi_0\,\mathrm{d}x \leq \beta\,|\Omega|\,.$$

Example 25.2 (Compliant mechanism)**.** Considering the compliant mechanism problem, thus

$$f(x, \boldsymbol{u}) = \frac{1}{2} c(x) \left| \boldsymbol{u} - \boldsymbol{u}_\Omega(x) \right|^2, \quad g(x, \boldsymbol{u}) = 0$$

with $\boldsymbol{u}_\Omega \in \boldsymbol{H}^1(\Omega)$, $c \in W^{1,\infty}(\Omega)$, $c \geq 0$, we find that $[f(x, \boldsymbol{u}_0)]_\Gamma = 0$. And so (25.9) reduces to

$$\left. \begin{aligned} \gamma c_0 \kappa - \left[\mathcal{C}(\varphi_0) \left(\mathcal{E}\left(\boldsymbol{u}_0\right) - \overline{\mathcal{E}}\left(\varphi_0\right) \right) : \mathcal{E}\left(\boldsymbol{q}_0\right) \right]_\Gamma + & \\ + \left[\mathcal{C}(\varphi_0) \left(\mathcal{E}\left(\boldsymbol{u}_0\right) - \overline{\mathcal{E}}\left(\varphi_0\right) \right) \nu \cdot \partial_\nu \boldsymbol{q}_0 \right]_\Gamma + & \\ + \left[\mathcal{C}(\varphi_0) \mathcal{E}\left(\boldsymbol{q}_0\right) \nu \cdot \partial_\nu \boldsymbol{u}_0 \right]_\Gamma + 2\lambda_0 \qquad & = 0 \quad \text{on } \Gamma. \end{aligned} \right\} \tag{25.22}$$

We remark that in this case the problem is not self-adjoint anymore, i.e. $\boldsymbol{u}_0 \neq \boldsymbol{q}_0$ in general, even if $\overline{\mathcal{E}} \equiv 0$, and so we need an adjoint state variable.

26 Sharp interface limit

The aim of this section is to relate the phase field problems introduced in Section 24 to the sharp interface formulation, which was discussed in Section 25. For this purpose, we will on the one hand show that the reduced objective functionals $\left(j_\varepsilon^E\right)_{\varepsilon>0}$ Γ-converge in $L^1(\Omega)$ to the reduced objective functional j_0^E describing the sharp interface optimization problem. In addition, we will show that the optimality systems of the phase field model, which were obtained by geometric variations, approximate an optimality system of the sharp interface problem, too.

We point out, that this is in particular a stronger result than we were able to show in the previous parts, where in general only convergence of minimizers could be shown. A Γ-convergence result there was only possible when minimizing the dissipated power in a Stokes flow, see Section 6.3.

26.1 Γ-convergence of the objective functionals

We start by extending the reduced objective functionals, introduced in (24.6) to the whole space $L^1(\Omega)$ by defining $j_\varepsilon^E : L^1(\Omega) \to \overline{\mathbb{R}}$ as follows:

$$j_\varepsilon^E(\varphi) := \begin{cases} J_\varepsilon^E\left(\varphi, \boldsymbol{S}_E\left(\varphi\right)\right) & \text{if } \varphi \in \Phi_E, \\ +\infty & \text{otherwise.} \end{cases}$$

Similarly, we can define $j_0^E : L^1(\Omega) \to \overline{\mathbb{R}}$ by

$$j_0^E(\varphi) := \begin{cases} J_0^E\left(\varphi, \boldsymbol{S}_E\left(\varphi\right)\right) & \text{if } \varphi \in \Phi_E^0, \\ +\infty & \text{otherwise.} \end{cases}$$

We then obtain the following main result:

Theorem 26.1. *The functionals $\left(j_\varepsilon^E\right)_{\varepsilon>0}$ Γ-converge in $L^1(\Omega)$ to j_0^E as $\varepsilon \searrow 0$.*

As a preparation of this theorem, we prove the following lemma:

Lemma 26.1. *The function*

$$F_E : \left\{\varphi \in L^1(\Omega) \mid |\varphi| \le 1 \text{ a.e. in } \Omega\right\} \ni \varphi \mapsto \int_\Omega f\left(x, \boldsymbol{S}_E(\varphi)\right) \mathrm{d}x + \int_{\Gamma_g} g\left(x, \boldsymbol{S}_E\left(\varphi\right)\right) \mathrm{d}x$$

is continuous in $L^1(\Omega)$. Besides we find, that $\boldsymbol{S}_E : \{\varphi \in L^1(\Omega) \mid |\varphi| \le 1 \text{ a.e.}\} \to \boldsymbol{H}_D^1(\Omega)$ is demicontinuous, i.e.,

$$\left(\varphi_k\right)_{k\in\mathbb{N}} \subset L^1(\Omega), |\varphi_k| \le 1 \text{ a.e.}, \lim_{k\to\infty} \|\varphi_k - \varphi\|_{L^1(\Omega)} = 0 \implies \boldsymbol{S}_E(\varphi_k) \overset{k\to\infty}{\rightharpoonup} \boldsymbol{S}_E(\varphi) \text{ in } \boldsymbol{H}^1(\Omega).$$

Proof. Let $\left(\varphi_n\right)_{n\in\mathbb{N}} \subset L^1(\Omega)$ be a sequence such that $|\varphi_n| \le 1$ a.e. in Ω for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^1(\Omega)} = 0$. In particular, this gives directly $|\varphi| \le 1$ a.e. in Ω. Now let $\left(\varphi_{n_k}\right)_{k\in\mathbb{N}}$ be any subsequence of $\left(\varphi_n\right)_{n\in\mathbb{N}}$. Defining $\boldsymbol{u}_{n_k} := \boldsymbol{S}_E(\varphi_{n_k})$ we see, by using (24.2), that it holds

$$\int_\Omega \mathcal{C}\left(\varphi_{n_k}\right)\left(\mathcal{E}\left(\boldsymbol{u}_{n_k}\right) - \overline{\mathcal{E}}\left(\varphi_{n_k}\right)\right) : \mathcal{E}\left(\boldsymbol{u}_{n_k}\right) \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_{n_k} \mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u}_{n_k} \mathrm{d}x \quad \forall k \in \mathbb{N}.$$

Thus, by applying the inequalities of Korn, Young and Hölder and the uniform estimate on the elasticity tensor $\mathcal{C}$ (23.3), we obtain with similar estimates as in Lemma 24.1 that

$$\sup_{k\in\mathbb{N}} \|\boldsymbol{u}_{n_k}\|_{\boldsymbol{H}^1(\Omega)} < \infty.$$

And so we find a subsequence $\left(\boldsymbol{u}_{n_{k(l)}}\right)_{l\in\mathbb{N}}$ such that $\left(\boldsymbol{u}_{n_{k(l)}}\right)_{l\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{u} \in \boldsymbol{H}^1_D(\Omega)$ as $l \to \infty$. Using the uniform boundedness of the tensor-valued function $\mathcal{C} \in C^{1,1}\left([-1,1], \mathbb{R}^{d^2\times d^2}\right)$, see Assumption **(A12)**, we obtain for any $\boldsymbol{v} \in H^1_D(\Omega)$ the uniform estimate

$$\left|\mathcal{C}\left(\varphi_{n_{k(l)}}(x)\right)\mathcal{E}\left(\boldsymbol{v}\right)(x)\right| \le C\left|\mathcal{E}(\boldsymbol{v})(x)\right| \qquad \text{for a.e. } x \in \Omega.$$

Hence, Lebesgue's convergence theorem implies that $\left(\left|\mathcal{C}\left(\varphi_{n_{k(l)}}\right)\mathcal{E}\left(\boldsymbol{v}\right)\right|\right)_{l\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ to $\left|\mathcal{C}\left(\varphi\right)\mathcal{E}\left(\boldsymbol{v}\right)\right|$. Since $\left(\mathcal{E}\left(\boldsymbol{u}_{n_{k(l)}}\right)\right)_{l\in\mathbb{N}}$ converges additionally weakly in $\boldsymbol{L}^2(\Omega)^d$, we obtain that

$$\lim_{l\to\infty}\left|\int_\Omega \mathcal{C}\left(\varphi_{n_{k(l)}}\right)\mathcal{E}\left(\boldsymbol{u}_{n_{k(l)}}\right) : \mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x - \int_\Omega \mathcal{C}\left(\varphi\right)\mathcal{E}\left(\boldsymbol{u}\right) : \mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x\right| = 0.$$

Similarly, we can deduce from the uniform boundedness of $\overline{\mathcal{E}}$ that

$$\lim_{l\to\infty}\left|\int_\Omega \mathcal{C}\left(\varphi_{n_{k(l)}}\right)\overline{\mathcal{E}}\left(\varphi_{n_{k(l)}}\right) : \mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x - \int_\Omega \mathcal{C}\left(\varphi\right)\overline{\mathcal{E}}\left(\varphi\right) : \mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x\right| = 0.$$

This leads to

$$\int_\Omega \mathcal{C}\left(\varphi\right)\left(\mathcal{E}\left(\boldsymbol{u}\right) - \overline{\mathcal{E}}\left(\varphi\right)\right) : \mathcal{E}\left(\boldsymbol{v}\right)\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$$

which yields $\boldsymbol{u} = \boldsymbol{S}_E(\varphi)$.
By applying the same arguments as above for any subsequence of $(\boldsymbol{S}_E(\varphi_n))_{n\in\mathbb{N}}$, we obtain that every subsequence of $(\boldsymbol{S}_E(\varphi_n))_{n\in\mathbb{N}}$ has a subsequence $\left(\boldsymbol{S}_E\left(\varphi_{\hat{n}(k)}\right)\right)_{k\in\mathbb{N}}$ such that $\left(\boldsymbol{S}_E\left(\varphi_{\hat{n}(k)}\right)\right)_{k\in\mathbb{N}}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{S}_E(\varphi) = \boldsymbol{u}$. This implies then the demicontinuity of $\boldsymbol{S}_E$ as stated in the lemma.

We are left with proving the continuity of F_E. For this purpose, we take again a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ such that $|\varphi_k| \le 1$ a.e. in Ω and $\lim_{k\to\infty}\|\varphi_k - \varphi\|_{L^1(\Omega)} = 0$. We have already established, that this implies the weak convergence of $(\boldsymbol{S}_E(\varphi_k))_{k\in\mathbb{N}}$ to $\boldsymbol{S}_E(\varphi)$ in $\boldsymbol{H}^1(\Omega)$. Using the compact imbeddings $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^{\frac{1}{2}}(\Gamma_g) \hookrightarrow \boldsymbol{L}^2(\Gamma_g)$ we moreover find, that $(\boldsymbol{S}_E(\varphi_k))_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{L}^2(\Omega)$ and $\left(\boldsymbol{S}_E(\varphi_k)|_{\Gamma_g}\right)_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{L}^2(\Gamma_g)$. We can now use the continuity of the objective functional stated in Assumption **(A16)**, see Remark 23.4, to obtain

$$\lim_{k\to\infty} F_E(\varphi_k) = F_E(\varphi)$$

and have shown the statement.

□

Using this lemma, we can show the stated Theorem 26.1 by applying known results concerning Γ-convergence of the Ginzburg-Landau energy.

Proof of Theorem 26.1: By [Ste88, Mod87, BE91] we obtain, that the Ginzburg-Landau energy $E_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$, which is given by

$$E_\varepsilon(\varphi) := \begin{cases} \int_\Omega \frac{1}{\varepsilon}\psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2 \,\mathrm{d}x & \text{if } \varphi \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

Γ-converges as $\varepsilon \searrow 0$ in $L^1(\Omega)$ to

$$E_0 : L^1(\Omega) \ni \varphi \mapsto \begin{cases} c_0 P_\Omega(\{\varphi = 1\}) & \text{if } \varphi \in BV(\Omega, \{\pm 1\}), \\ +\infty & \text{else.} \end{cases}$$

We rewrite the reduced objective functional in the following form: $j_\varepsilon^E = \gamma E_\varepsilon + F_E + I_K$, where

$$I_K(\varphi) := \begin{cases} 0, & \text{if } \varphi \in K \\ +\infty, & \text{otherwise} \end{cases}$$

with

$$K := \left\{ \varphi \in L^1(\Omega) \mid \int_\Omega \varphi \,\mathrm{d}x \le \beta |\Omega| \right\}.$$

Making use of Lemma 26.1, we find that $F_E + I_K$ is a continuous function in $L^1(\Omega)$, and so j_ε is the Ginzburg-Landau energy E_ε plus some function which is continuous in $L^1(\Omega)$. Consequently, by standard results for Γ-convergence, see for instance Section 3.4, we find that $\left(j_\varepsilon^E\right)_{\varepsilon>0}$ Γ-converges in $L^1(\Omega)$ to j_0^E, since

$$j_0^E(\varphi) = \gamma E_0(\varphi) + (F_E + I_K)(\varphi).$$

This proves the statement. □

As a consequence, we obtain directly:

Corollary 26.1. *Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be minimizers of $\left(j_\varepsilon^E\right)_{\varepsilon>0}$. Then there exists a subsequence, denoted by the same, and an element $\varphi \in L^1(\Omega)$ such that* $\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi\|_{L^1(\Omega)} = 0$. *Besides, φ is a minimizer of j_0^E and it holds*

$$\lim_{\varepsilon\searrow 0} j_\varepsilon^E(\varphi_\varepsilon) = j_0^E(\varphi).$$

Proof. We apply the compactness argument of the fifth step of the proof of Theorem 6.1 to find a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges in $L^1(\Omega)$ to some element φ as $\varepsilon \searrow 0$. Then the previous theorem and standard results for Γ-convergence, compare Section 3.4, yield the assertion. □

26.2 Convergence of the optimality system

As we have done before in the previous parts, we want to show that in the setting of Corollary 26.1 we can even show that the optimality systems of the phase field model obtained by geometric variations are an approximation of optimality criteria for the sharp interface description. This is the topic of the next theorem.

But as we are considering first order optimality conditions, we assume again for the remainder of this subsection Assumption **(A17)** to ensure differentiability of the objective functional.

Theorem 26.2. *Let* $(\varphi_\varepsilon)_{\varepsilon>0}$ *be minimizers of* $\left(j_\varepsilon^E\right)_{\varepsilon>0}$. *Then there exists a subsequence, which is denoted by the same, that converges in* $L^1(\Omega)$ *to a minimizer* φ_0 *of* j_0^E. *Moreover, it holds*

$$\lim_{\varepsilon\searrow 0} \partial_t|_{t=0} j_\varepsilon^E\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0^E\left(\varphi_0 \circ T_t^{-1}\right) \quad \forall T \in \overline{\mathcal{T}}_{ad}. \tag{26.1}$$

If

$$|\{\varphi_0 = 1\}| > 0 \tag{26.2}$$

then we have additionally the following convergence results:

$$\varphi_\varepsilon \xrightarrow{\varepsilon\searrow 0} \varphi_0 \qquad \text{in } L^1(\Omega), \tag{26.3a}$$

$$\boldsymbol{u}_\varepsilon \overset{\varepsilon\searrow 0}{\rightharpoonup} \boldsymbol{u}_0 \qquad \text{in } \boldsymbol{H}^1(\Omega), \tag{26.3b}$$

$$\dot{\boldsymbol{u}}_\varepsilon[V] \overset{\varepsilon\searrow 0}{\rightharpoonup} \dot{\boldsymbol{u}}_0[V] \qquad \text{in } \boldsymbol{H}^1(\Omega), \tag{26.3c}$$

$$\lambda_\varepsilon \xrightarrow{\varepsilon\searrow 0} \lambda_0 \qquad \text{in } \mathbb{R}, \tag{26.3d}$$

$$j_\varepsilon^E(\varphi_\varepsilon) \xrightarrow{\varepsilon\searrow 0} j_0^E(\varphi_0) \qquad \text{in } \mathbb{R}, \tag{26.3e}$$

where $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_E(\varphi_\varepsilon)$, $\boldsymbol{u}_0 = \boldsymbol{S}_E(\varphi_0)$, $(\lambda_\varepsilon)_{\varepsilon>0} \subseteq \mathbb{R}^+$ *are Lagrange multipliers for the integral constraint defined due to Theorem 24.3,* $\lambda_0 \in \mathbb{R}^+$ *is a Lagrange multiplier such that it holds* (25.4) – (25.5), *and thus is a Lagrange multiplier for the integral constraint in the sharp interface setting according to Theorem 25.2.*

Proof. The result of Corollary 26.1 yields directly the existence of a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ such (26.3a) and (26.3e) are fulfilled. By Lemma 26.1, this implies the weak convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{u}_0 = \boldsymbol{S}_E(\varphi_0)$ in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon \searrow 0$.
Now we recall, that $\dot{\boldsymbol{u}}_\varepsilon[V] \in \boldsymbol{H}_D^1(\Omega)$ is given as the solution of

$$\int_\Omega \mathcal{C}(\varphi_\varepsilon)\mathcal{E}(\dot{\boldsymbol{u}}_\varepsilon[V]) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x = \boldsymbol{R}_\varepsilon(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega) \tag{26.4}$$

where $\boldsymbol{R}_\varepsilon \in \left(\boldsymbol{H}_D^1(\Omega)\right)'$ is given by

$$\begin{aligned}\boldsymbol{R}_\varepsilon(\boldsymbol{v}) := &\int_\Omega \mathcal{C}(\varphi_\varepsilon)\frac{1}{2}\left(\mathrm{D}\boldsymbol{u}_\varepsilon \mathrm{D}V(0) + \nabla V(0)\nabla\boldsymbol{u}_\varepsilon\right) : \mathcal{E}(\boldsymbol{v}) + \\ &+ \mathcal{C}(\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon) - \overline{\mathcal{E}}(\varphi_\varepsilon)\right) : \frac{1}{2}\left(\nabla V(0)\nabla\boldsymbol{v} + \mathrm{D}\boldsymbol{v}\mathrm{D}V(0)\right) - \\ &- \mathcal{C}(\varphi_\varepsilon)\left(\mathcal{E}(\boldsymbol{u}_\varepsilon) - \overline{\mathcal{E}}(\varphi_\varepsilon)\right) : \mathcal{E}(\boldsymbol{v})\operatorname{div} V(0)\,\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot \mathrm{D}\boldsymbol{v}V(0)\,\mathrm{d}x - \int_{\Gamma_g} \boldsymbol{g}\cdot\mathrm{D}\boldsymbol{v}V(0)\,\mathrm{d}x.\end{aligned}$$

Since $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $\boldsymbol{H}^1(\Omega)$, $\|\varphi_\varepsilon\|_{L^\infty(\Omega)} \le 1$ and using the uniform estimate on the elasticity tensor and the eigenstrain given by Assumptions **(A12)** and **(A13)** we can deduce that

$$\sup_{\varepsilon>0} \|\boldsymbol{R}_\varepsilon\|_{(\boldsymbol{H}_D^1(\Omega))'} < \infty.$$

And so we find by Korn's inequality A.6 from (26.4) that

$$C \left\| \dot{\boldsymbol{u}}_\varepsilon [V] \right\|^2_{\boldsymbol{H}^1(\Omega)} \leq \int_\Omega \mathcal{C}(\varphi_\varepsilon) \mathcal{E}(\dot{\boldsymbol{u}}_\varepsilon [V]) : \mathcal{E}(\dot{\boldsymbol{u}}_\varepsilon [V]) \,\mathrm{d}x = \boldsymbol{R}_\varepsilon (\dot{\boldsymbol{u}}_\varepsilon [V]) \leq$$
$$\leq \left\| \boldsymbol{R}_\varepsilon \right\|_{(\boldsymbol{H}^1_D(\Omega))'} \left\| \dot{\boldsymbol{u}}_\varepsilon [V] \right\|_{\boldsymbol{H}^1(\Omega)}$$

which yields

$$\sup_{\varepsilon>0} \left\| \dot{\boldsymbol{u}}_\varepsilon [V] \right\|_{\boldsymbol{H}^1(\Omega)} \leq C \sup_{\varepsilon>0} \left\| \boldsymbol{R}_\varepsilon \right\|_{(\boldsymbol{H}^1_D(\Omega))'} < \infty.$$

This yields the existence of a subsequence, which will be denoted by the same, such that $(\dot{\boldsymbol{u}}_\varepsilon [V])_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{w} \in \boldsymbol{H}^1_D(\Omega)$. Following the arguments of the proof of Lemma 26.1 we see that the limit element $\boldsymbol{w}$ of $(\dot{\boldsymbol{u}}_\varepsilon [V])_{\varepsilon>0}$ is given as the solution to

$$\begin{aligned}
&\int_\Omega \mathcal{C}(\varphi_0)\mathcal{E}(\boldsymbol{w}) : \mathcal{E}(\boldsymbol{v}) \,\mathrm{d}x = \int_\Omega \mathcal{C}(\varphi_0)\frac{1}{2}\left(\mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0) + \nabla V(0)\nabla \boldsymbol{u}_0\right) : \mathcal{E}(\boldsymbol{v}) \\
&+ \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \frac{1}{2}(\nabla V(0)\nabla \boldsymbol{v} + \mathrm{D}\boldsymbol{v}\mathrm{D}V(0)) - \\
&- \mathcal{C}(\varphi_0)\left(\mathcal{E}(\boldsymbol{u}_0) - \overline{\mathcal{E}}(\varphi_0)\right) : \mathcal{E}(\boldsymbol{v}) \operatorname{div} V(0) \,\mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \mathrm{D}\boldsymbol{v} V(0) \,\mathrm{d}x - \int_{\Gamma_g} \boldsymbol{g} \cdot \mathrm{D}\boldsymbol{v} V(0) \,\mathrm{d}x
\end{aligned} \tag{26.5}$$

which has to hold for all $\boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega)$. Hence, by definition of $\dot{\boldsymbol{u}}_0 [V]$, see Theorem 25.2, we get $\boldsymbol{w} = \dot{\boldsymbol{u}}_0 [V]$. In particular, we can deduce by the imbedding theorems that both $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ and $(\dot{\boldsymbol{u}}_\varepsilon [V])_{\varepsilon>0}$ converge strongly in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{L}^2(\Gamma_g)$. And so we obtain by the continuous differentiability of the objective functional, see Remark 23.6, that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\Bigg[\int_\Omega \left[\mathrm{D}f(x,\boldsymbol{u}_\varepsilon)(V(0),\dot{\boldsymbol{u}}_\varepsilon [V]) + f(x,\boldsymbol{u}_\varepsilon) \operatorname{div} V(0)\right] \mathrm{d}x + \\
&+ \int_{\Gamma_g} \left[\mathrm{D}g(x,\boldsymbol{u}_\varepsilon)(V(0),\dot{\boldsymbol{u}}_\varepsilon [V]) + g(x,\boldsymbol{u}_\varepsilon)(\operatorname{div} V(0) - \boldsymbol{n}\cdot\nabla V(0)\boldsymbol{n})\right] \mathrm{d}x\Bigg] = \\
&= \int_\Omega \left[\mathrm{D}f(x,\boldsymbol{u}_0)(V(0),\dot{\boldsymbol{u}}_0 [V]) + f(x,\boldsymbol{u}_0) \operatorname{div} V(0)\right] \mathrm{d}x + \\
&+ \int_{\Gamma_g} \left[\mathrm{D}g(x,\boldsymbol{u}_0)(V(0),\dot{\boldsymbol{u}}_0 [V]) + g(x,\boldsymbol{u}_0)(\operatorname{div} V(0) - \boldsymbol{n}\cdot\nabla V(0)\boldsymbol{n})\right] \mathrm{d}x.
\end{aligned} \tag{26.6}$$

Analogously as in Theorem 9.1 we can apply the Reshetnyak continuity theorem 3.2 to deduce

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\left[\int_\Omega \left(\frac{\gamma\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right) \operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon \,\mathrm{d}x\right] = \\
&\qquad = \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu\cdot\nabla V(0)\nu\right) \mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|.
\end{aligned} \tag{26.7}$$

Plugging those results together we end up with (26.1). As in the proof of Theorem 9.1, we can find some $V \in \overline{\mathcal{V}}_{ad}$ such that $\int_\Omega \varphi_0 \operatorname{div} V(0) \,\mathrm{d}x > 0$ if we assume (26.2). Thus, (26.1) and (24.20) lead to

$$\lim_{\varepsilon\searrow 0} -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0) \,\mathrm{d}x = \lim_{\varepsilon\searrow 0} \partial_t|_{t=0} j^E_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j^E_0\left(\varphi_0 \circ T_t^{-1}\right)$$

wherefrom we obtain that $(\lambda_\varepsilon)_{\varepsilon>0}$ converges to some $\lambda_0 \in \mathbb{R}^+$. Besides, this directly yields that $\lambda_0 \in \mathbb{R}^+$ fulfills

$$-\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x = \partial_t|_{t=0} j_0^E \left(\varphi_0 \circ T_t^{-1}\right) \quad \forall T \in \overline{\mathcal{T}}_{ad}$$

and thus is a Lagrange multiplier associated to the integral constraint by (25.4) – (25.5). This finally proves the statement.

□

To summarize the results of this part, we have shown that by the phase field approach we have found a well-posed optimal control problem, which can be reformulated to minimizing a reduced objective functional. The latter Γ-converges in $L^1(\Omega)$ as the thickness of the interface tends to zero to a functional describing a sharp interface formulation of the problem. Besides, we have shown that first order optimality conditions of the phase field problem can be deduced by geometric variations. As the minimizers converge, also the obtained optimality conditions converge to a system, which is a necessary optimality condition for the sharp interface problem. Besides, this optimality system for the sharp interface problem can be derived in the general setting of functions of bounded variations. But assuming additional regularity assumptions on the minimizing set and the data, it can be shown that the obtained conditions are equivalent to results that were already obtained in literature by either pure shape derivatives and also by formal asymptotics from the phase field model. Thus we have delivered a rigorous proof for the convergence results that were already predicted by formal asymptotics in [BFSGS13]. In contrast to [BFSGS13], we even use a general objective functional. We remark that in [BFSGS13] the state constraints can be ε-dependent. To be precise, an ersatz material approach is used, where the stiffness of the ersatz material scales like ε^2, and thus vanishes as $\varepsilon \searrow 0$. This is not done in our work, but possible generalizations for reasonable objective functionals in the spirit of the first parts may be possible. This means that convergence of minimizers could possibly be shown, but we expect that again certain growth conditions on the convergence of the minimizers play a role, where this rate has to be consistent with the ε-scaling of the ersatz material.

Summary and Conclusions

At the end of this work, we want to summarize the results we have discussed above. As reviewed in the introduction, shape and topology optimization in fluid dynamics is still a young research field where only a few topics have been examined so far. We introduced a rather general topology optimization model for both the Stokes and the stationary Navier-Stokes flow and discussed how to include pressure functionals in this setting. This approach is based on a perimeter penalization. Additionally, we approximated this sharp interface problem by a phase field approach, while parallel weakening the condition of non-permeability through the non-fluid region. This resulted in a phase field formulation, which can be stated as a typical optimal control problem. We showed, that this diffuse interface problem is well-posed and that a subsequence of the sequence of minimizers converge. If the convergent subsequence satisfies a certain convergence rate, we can show that the limit is a minimizer of the sharp interface topology optimization problem. In the specific setting of minimizing the total potential power in a Stokes flow we can even establish the stronger result of Γ-convergence of the corresponding reduced objective functionals. In particular, we obtain therefrom directly even the existence of a minimizer for the sharp interface problem if either the growth condition is fulfilled or the total potential power is used as an objective functional in a Stokes flow. This is not a trivial fact and is an open problem in the general setting. Moreover, we derived first order necessary optimality conditions for both approaches and showed that the optimality system of the phase field model is an approximation of the optimality system of the sharp interface model, as it converges simultaneously to the minimizers to this system if the phase field parameter modelling the interface thickness tends to zero in the above-mentioned settings. Altogether, we arrive in the following diagram:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{minimizer of the}\\ \text{phase field model}\end{array}} \arrow[r, "\varepsilon\searrow 0", "\text{converges to}"'] \arrow[d, Rightarrow, "\text{fulfill}"'] & \boxed{\begin{array}{c}\text{minimizer of the}\\ \text{sharp interface model}\end{array}} \arrow[d, Rightarrow, "\text{fulfill}"'] \\
\boxed{\begin{array}{c}\text{optimality conditions for}\\ \text{the phase field model}\end{array}} \arrow[r, "\varepsilon\searrow 0", "\text{converges to}"'] & \boxed{\begin{array}{c}\text{optimality conditions for}\\ \text{the sharp interface model}\end{array}}
\end{tikzcd}$$

Further, the first order necessary optimality conditions that we derived for the sharp interface model of the shape and topology optimization problem are veritable without additional assumptions on the regularity of the minimizing set. But stating certain regularity assumptions on the minimizer, we can show that those optimality conditions are equivalent to the results stated in known literature about shape sensitivity analysis and can be rewritten in the normal form of Hadamard.

Finally, we applied the ideas and methods developed for optimization of fluids also for structural optimization, i.e. finding optimal material distributions of two given elastic materials. Again the sharp interface problem is formulated in a setting of Caccioppoli sets and by using a perimeter penalization we can ensure the existence of a minimizer for this problem. Moreover, the sharp interface can be replaced by a diffuse interface where the perimeter is approximated by the Ginzburg-Landau energy. The obtained problem can then be reformulated to minimizing a reduced objective functional, and the latter Γ-converges as the thickness of the interface tends to zero to the reduced objective functional corresponding to the original sharp interface problem. Besides, we derived independently first order necessary optimality conditions for both the phase field and sharp interface model. It was shown, that the latter can be approximated by the optimality conditions obtained by geometric variations in the phase field setting. In particular, by these considerations we justify the results from [BFSGS13], where the sharp interface limit has been carried out by formal asymptotics.

This implies, that we have derived a consistent approach that can be used for future researches on shape and topology optimization in fluid dynamics and in structural optimization. Due to the phase field structure, it moreover gives rise to good analytic results and numerical tools and may therefore be a serious alternative to rigorous and systematic investigations in this field.

Appendix

The following results are used in this thesis frequently, and thus we state them briefly.

A.1 Lemma. *Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be sequences that are bounded from below with*

$$\lim_{k\to\infty} (a_k + b_k) = (a + b)$$

where $a, b \in \mathbb{R}$, such that

$$a \le \liminf_{k\to\infty} a_k, \quad b \le \liminf_{k\to\infty} b_k.$$

Then it holds

$$\lim_{k\to\infty} a_k = a, \qquad \lim_{k\to\infty} b_k = b.$$

Proof. We can estimate directly

$$\begin{aligned} a \le \liminf_{k\to\infty} a_k \le \limsup_{k\to\infty} a_k \le \limsup_{k\to\infty} (a_k + b_k - b_k) \le \\ \le \underbrace{\limsup_{k\to\infty} (a_k + b_k)}_{=a+b} + \underbrace{\limsup_{k\to\infty} (-b_k)}_{=-\underbrace{\liminf_{k\to\infty} b_k}_{\ge b}} \le a + b - b = a \end{aligned}$$

wherefrom we obtain $\lim_{k\to\infty} a_k = a$ and so $\lim_{k\to\infty} b_k = b$. This proves the statement. □

The following version of Lax-Milgram's theorem is taken from [Alt06, 4.2, 4.3]:

A.2 Lemma (Lax-Milgram's theorem)**.** *Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ a bilinear form. Assume there exist constants $c, C \in \mathbb{R}$ with $0 < c \le C < \infty$ such that for all $x, y \in H$:*

$$a(x,y) \le C \|x\|_H \|y\|_H, \quad a(x,x) \ge c \|x\|_H^2.$$

Then there exists for every $x' \in H'$ exactly one solution $x \in H$ to

$$a(y,x) = x'(y) \quad \forall y \in H$$

and it holds

$$\|x\|_H \le c^{-1} \|x'\|_{H'}.$$

The state equations in fluid dynamics will often not be compatible with the standard version of the implicit function theorem, thus we use the following adapted version, which is taken from [Sim91].

A.3 Theorem. *Let U be an open set in a Banach space X, $u_0 \in U$, and Y and Z two reflexive Banach spaces. Moreover, assume $F : U \times Y \to Z$ be such that $F(u,\cdot) \in \mathcal{L}(Y,Z)$ for all $u \in U$. Let $m : U \to Y$, $f : U \to Z$ be functions such that*

$$F(u, m(u)) = f(u) \quad \forall u \in U.$$

If $u \mapsto F(u,\cdot)$ is differentiable at u_0 into $\mathcal{L}(Y,Z)$, f is differentiable at u_0 and

$$\|F(u_0, x)\|_Z \ge \alpha \|x\|_Y \quad \forall x \in Y \tag{A.1}$$

for some $\alpha > 0$, then $u \mapsto m(u)$ is differentiable at u_0 and $m'(u_0)v$ is for any $v \in U$ given as the unique solution of

$$F(u_0, m'(u_0)v) = \mathrm{D}f(u_0)v - \mathrm{D}_u f\left(u_0, m(u_0)\right)v.$$

For solving the nonlinear state equations we use the following main theorem on pseudo-monotone operators, which can be found in [Zei90, 27.3]:

A.4 Theorem. *Assume $A : X \to X'$ is a pseudo-monotone, bounded and coercive operator on a real, separable and reflexive Banach space X with $\dim X = \infty$. Then for each $b \in X'$ there exists a solution $u \in X$ to $Au = b$.*

A.5 Remark. *Let $A, B : X \to X'$ be operators and X be a real, reflexive Banach space. We will use the following properties of pseudo-monotone operators to apply Theorem A.4:*

- *If A is linear and monotone, then it is pseudo-monotone.*
- *If A is pseudo-monotone and B is strongly continuous, then $A + B$ is pseudo-monotone.*

Those statements can be found in [Zei90, Section 27.2].

An important inequality in the theory of linearized elasticity is Korn's inequality:

A.6 Lemma (Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Moreover, let $\Gamma_D \subset \partial\Omega$ be an open subset of $\partial\Omega$ with $\mathcal{H}^{d-1}(\Gamma_D) > 0$. Then there exists a constant $c_k > 0$ such that for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}$ it holds*

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} \le c_k \int_\Omega \mathcal{E}\left(\boldsymbol{v}\right) : \mathcal{E}\left(\boldsymbol{v}\right) \,\mathrm{d}x.$$

Proof. See [Cia88, Theorem 6.3-4] or [Zei97, 62.15] and included references. □

Remark A.1. *Let the assumptions of Lemma A.6 be satisfied. Then we even obtain that the semi-norm $\boldsymbol{v} \mapsto \|\mathcal{E}\left(\boldsymbol{v}\right)\|_{\boldsymbol{L}^2(\Omega)}$ is a norm which is equivalent to $\|\cdot\|_{\boldsymbol{H}^1(\Omega)}$ on $\{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}\}$.*

Symbols

$(\cdot,\cdot)_H$	scalar product in some Hilbert space H
$\lvert A\rvert$	$=\sqrt{A:A}=\sqrt{\operatorname{tr}(A^TA)}\ \forall A\in\mathbb{R}^{d\times d}$, standard norm in $\mathbb{R}^{d\times d}$
$A:B$	$=\operatorname{tr}\left(A^TB\right)=\sum_{i,j=1}^d a_{ij}b_{ij}\ \forall A,B\in\mathbb{R}^{d\times d}$ standard scalar product in $\mathbb{R}^{d\times d}$
$\langle\cdot,\cdot\rangle_{X'}$	dual pairing of X and its dual X' for some Banach space X
α_ε	inverse permeability/interpolation function, $\alpha_\varepsilon:[-1,1]\to[0,\overline{\alpha}_\varepsilon]$, p. 20
β	$\in(-1,1)$, given constant for the volume constraint
$\mathcal{C}$	elasticity tensor, p. 225
c_0	$=\int_{-1}^1\sqrt{2\psi(x)}\,\mathrm{d}x$, here: $c_0=\frac{\pi}{2}$, p. 55
$C_0(\Omega)$	continuous functions in $C(\Omega)$ with compact support in Ω
$\boldsymbol{C}_0^\infty(\Omega)$	smooth functions in $(C^\infty(\Omega))^d$ with compact support in Ω
χ_M	characteristic function of some set M
d	space dimension, $d\in\{2,3\}$, p. 19, p. 226
$\operatorname{div}_\Gamma$	surface divergence on a C^2-manifold Γ, see for instance [AFP00, DZ11] for details
$\mathrm{D}_uf(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\boldsymbol{v}$	$=\mathrm{D}_{(2,3)}f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})(\boldsymbol{v},\mathrm{D}\boldsymbol{v})$, p. 22
$\mathcal{E}(\boldsymbol{u})$	linearized strain, $\mathcal{E}(\boldsymbol{u})=\frac{1}{2}\left(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T\right)$, p. 225
$\overline{\mathcal{E}}$	eigenstrain, p. 225
E^φ	$=\{x\in\Omega\mid\varphi(x)=1\}$ for $\varphi\in L^1(\Omega)$, p. 23
$\boldsymbol{f}$	$\in\boldsymbol{L}^2(\Omega)$, applied body force, p. 19, p. 226
f	objective functional, p. 21, p. 227
$\boldsymbol{g}$	given boundary data, p. 19, p. 226
g	objective functional for the boundary terms, only in structural optimization, p. 227
γ	>0, arbitrary fixed parameter used in the model, p. 43
Γ_D,Γ_g	Dirichlet part and Neumann part of $\partial\Omega$ in linearized elasticity, p. 226
$\mathcal{H}^{d-1}$	$(d-1)$-dimensional Hausdorff measure, see for instance [EG92]
h	$h:\mathbb{R}\to\mathbb{R}$, objective functional for the pressure, p. 189
$\boldsymbol{H}^{-1}(\Omega)$	dual space of $\boldsymbol{H}_0^1(\Omega)$

$\boldsymbol{H}_0^1(\Omega)$	the closure of $\boldsymbol{C}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}^1(\Omega)$ norm
$\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$	Sobolev space of order $\frac{1}{2}$, defined for instance in [RR00, Section 7]
$\boldsymbol{H}_D^1(\Omega)$	$= \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{u}\vert_{\Gamma_D} = \boldsymbol{0}\right\}$, p. 226
$\boldsymbol{H}_{\boldsymbol{g}}^1(\Omega)$	$= \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega\right\}$, p. 20
$H^m(U)$	$= \left\{v \in L^2(U) \vert \text{derivatives of order less than or equal to } m \text{ are in } L^2(U)\right\}$, for some open set $U \subset \mathbb{R}^d$, $m \in \mathbb{N}$
$\boldsymbol{H}^m(U)$	$= \left\{\boldsymbol{v} \in (H^m(U))^d\right\}$, for some open set $U \subset \mathbb{R}^d$, $m \in \mathbb{N}$
J_0	objective functional in the sharp interface model in a Stokes flow, $J_0 : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$, p. 24
j_0	reduced objective functional, $j_0(\varphi) = J_0(\varphi, \boldsymbol{S}_0(\varphi))$, p. 24
J_0^E	objective functional for structural optimization in the sharp interface formulation, p. 240
j_0^E	reduced objective functional for structural optimization, $j_0^E(\varphi) = J_0^E(\varphi, \boldsymbol{S}_E(\varphi))$, p. 242
J_0^N	objective functional in the sharp interface model in a stat. Navier-Stokes flow, $J_0^N : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$, p. 25
J_0^P	objective functional in the sharp interface model in a Stokes flow including the pressure, $J_0^P : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \to \overline{\mathbb{R}}$, p. 192
j_0^P	reduced objective functional, $j_0^P(\varphi) = J_0^P(\varphi, \boldsymbol{S}_0^P(\varphi))$, p. 192
J_ε	objective functional in the phase field model in a Stokes flow, $J_\varepsilon : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$, p. 24
j_ε	reduced objective functional, $j_\varepsilon(\varphi) = J_\varepsilon(\varphi, \boldsymbol{S}_\varepsilon(\varphi))$, p. 24
J_ε^E	objective functional for structural optimization in the phase field formulation, p. 231
j_ε^E	reduced objective functional for structural optimization, $j_\varepsilon^E(\varphi) = J_\varepsilon^E(\varphi, \boldsymbol{S}_E(\varphi))$, p. 232
J_ε^N	objective functional in the phase field model in a stat. Navier-Stokes flow, $J_\varepsilon^N : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \overline{\mathbb{R}}$, p. 25
J_ε^P	objective functional in the phase field model in a Stokes flow including the pressure, $J_\varepsilon^P : L^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \to \overline{\mathbb{R}}$, p. 188
j_ε^P	reduced objective functional, $j_\varepsilon^P(\varphi) = J_\varepsilon^P(\varphi, \boldsymbol{S}_\varepsilon^P(\varphi))$, p. 193
κ	$= \mathrm{div}_\Gamma \nu$ is the mean curvature of a C^2-hypersurface Γ if ν is the outer unit normal on Γ

$L^2_0(\Omega)$	$= \left\{ q \in L^2(\Omega) \mid \int_\Omega q \,\mathrm{d}x = 0 \right\}$, p. 20
$L^2_M(\Omega)$	$= \left\{ q \in L^2(\Omega) \mid \int_{M_i} q \,\mathrm{d}x = 0,\ \forall i = 1, \ldots, m,\ q\vert_{\Omega \setminus \cup_{i=1}^m M_i} = 0 \right\}$, p. 190
$\boldsymbol{L}^2(\Omega)$	$= \left\{ \boldsymbol{v} \in L^2(\Omega)^d \right\}$
μ	viscosity
$\boldsymbol{n}$	outer unit normal on Ω, p. 19, p. 226
Ω	bounded Lipschitz domain in $\mathbb{R}^d$, p. 19, p. 226
Φ_{ad}	$= \left\{ \varphi \in H^1(\Omega) \mid ⨍_\Omega \varphi \,\mathrm{d}x \le \beta, \vert\varphi\vert \le 1 \text{ a.e. in } \Omega \right\}$, p. 23
Φ^0_{ad}	$= \left\{ \varphi \in BV(\Omega, \{\pm 1\}) \mid ⨍_\Omega \varphi \,\mathrm{d}x \le \beta,\ \boldsymbol{U}^\varphi \neq \emptyset \right\}$, p. 24
$\overline{\Phi}^0_{ad}$	$= \{ \varphi \in BV(\Omega, \{\pm 1\}) \mid \boldsymbol{U}^\varphi \neq \emptyset \}$, p. 24
$\overline{\Phi}_{ad}$	$= \left\{ \varphi \in H^1(\Omega) \mid \vert\varphi\vert \le 1 \text{ a.e. in } \Omega \right\}$, p. 23
Φ_E	$= \left\{ \varphi \in H^1(\Omega) \mid ⨍_\Omega \varphi \,\mathrm{d}x \le \beta, \vert\varphi\vert \le 1 \text{ a.e. in } \Omega \right\}$, p. 229
Φ^0_E	$= \left\{ \varphi \in BV(\Omega, \{\pm 1\}) \mid ⨍_\Omega \varphi \le \beta \right\}$, p. 229
$\overline{\Phi}_E$	$= \left\{ \varphi \in H^1(\Omega) \mid \vert\varphi\vert \le 1 \text{ a.e. in } \Omega \right\}$, p. 229
$\overline{\Phi}^0_E$	$= BV(\Omega, \{\pm 1\})$, p. 229
Φ_p	$= \{ \varphi \in \Phi_{ad} \mid \varphi\vert_{M_i} = 1,\ \forall i = 1, \ldots, m \}$, p. 192
Φ^0_p	$= \left\{ \varphi \in \Phi^0_{ad} \mid \varphi\vert_{M_i} = 1,\ \forall i = 1, \ldots, m \right\}$, p. 192
$\overline{\Phi}^0_p$	$= \left\{ \varphi \in \overline{\Phi}^0_{ad} \mid \varphi\vert_{M_i} = 1,\ \forall i = 1, \ldots, m \right\}$, p. 192
$\overline{\Phi}_p$	$= \left\{ \varphi \in \overline{\Phi}_{ad} \mid \varphi\vert_{M_i} = 1,\ \forall i = 1, \ldots, m \right\}$, p. 192
$P_\Omega(E)$	perimeter of E in Ω, p. 29
ψ	potential, $\psi : \mathbb{R} \to \overline{\mathbb{R}}$, $\psi(\varphi) = \psi_0(\varphi) = \frac{1}{2}(1 - \varphi^2)$ if $\vert\varphi\vert \le 1$, $\psi(\varphi) = +\infty$ otherwise, p. 19, p. 229
$\mathbb{R}^+$	$= \{ r \in \mathbb{R} \mid r \ge 0 \}$
$\mathbb{R}^-$	$= \{ r \in \mathbb{R} \mid r \le 0 \}$
$\overline{\mathbb{R}}$	$= \mathbb{R} \cup \{\pm\infty\}$
A^T	for some matrix $A \in \mathbb{R}^{d \times d}$ denotes the transpose of A
$\mathcal{T}_{ad}, \overline{\mathcal{T}}_{ad}$	admissible transformations for deformations of the domain, p. 25, p. 229
$\mathcal{T}^p_{ad}, \overline{\mathcal{T}}^p_{ad}$	admissible transformations for deformations of the domain in Part III, p. 193

$\boldsymbol{U}$	$= \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega, \operatorname{div} \boldsymbol{u} = 0\right\}$, p. 20
$\boldsymbol{U}^E$	$= \{\boldsymbol{u} \in \boldsymbol{U} \mid \boldsymbol{u} = \boldsymbol{0}$ a.e. in $\Omega \setminus E\}$ for some measurable set $E \subseteq \Omega$, p. 23
$\boldsymbol{U}^\varphi$	$= \boldsymbol{U}^{E^\varphi}$ for $\varphi \in L^1(\Omega)$, p. 23
$\boldsymbol{V}$	$= \left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0\right\}$, p. 20
$\mathcal{V}_{ad}, \overline{\mathcal{V}}_{ad}$	admissible vector fields for deformations of the domain, p. 25, p. 229
$\mathcal{V}_{ad}^p, \overline{\mathcal{V}}_{ad}^p$	admissible vector fields for deformations of the domain in Part III, p. 193
$\boldsymbol{V}^E$	$= \{\boldsymbol{v} \in \boldsymbol{V} \mid \boldsymbol{v} = \boldsymbol{0}$ a.e. in $\Omega \setminus E\}$ for some measurable set $E \subseteq \Omega$, p. 23
$\boldsymbol{V}^\varphi$	$= \boldsymbol{V}^{E^\varphi}$ for $\varphi \in L^1(\Omega)$, p. 23
$W^{k,p}(U)$	$= \{v \in L^p(U) \vert$derivatives of order less than or equal to k are in $L^p(U)\}$, for some open set $U \subset \mathbb{R}^d$, $1 \leq k, p \leq \infty$

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