Nonperturbative physics, fractional instantons and matter fields

Dissertation

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To my daughter
May she always seek out knowledge

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Chapter 1

Introduction

The Yang-Mills (YM) theory – the theory of gluons – and quantum chromodynamics (QCD) – the theory of quarks and gluons – has had great success in the past 40+ years in describing the physics of strong interactions. These theories drew even more appeal because they are asymptotically free, a feature which allowed, in contrast to quantum electrodynamics (QED), to define the theory[1] at arbitrarily short scales and to make lattice simulations with continuum limit extrapolations.

Although seemingly perfectly tame and well behaved at short distances where perturbation theory is applicable[2], the theory has a very nontrivial infrared behavior which still precludes analytic treatment, exhibiting dynamical mass gap generation and confinement at large distances, phenomena not accessible in the perturbation theory. Nevertheless, our knowledge of nonlinear theories has significantly increased in the past decades. Asymptotically free nonlinear sigma models such as $O(N)$ and $CP(N)$ were

[1]The theory needs to be defined in a continuum and a standard way to do this is to put the theory on the lattice and let the lattice spacing run to zero. Until today this has not been done self-consistently for QED as there is a phase transition in the lattice theory when the bare coupling exceeds a certain value. Nevertheless the theory is perfectly tame when the lattice is coarse enough, i.e. it is a low energy effective theory. On the other hand YM and QCD are perfectly well defined in the ultraviolet where the coupling flows to zero. Many take the view that because of this fact QCD is *fundamental* while QED is not. This author takes the view that neither are fundamental (in the sense that they are the theories of nature), as we simply do not have access to experiments at sufficiently large energies and that the entire standard model can come from a fundamental theory which is quite different from either QCD or QED.

[2]This is perhaps a misnomer which is widely accepted. Although indeed the perturbation theory has definite applicability at short distances because of the small coupling in this regime, the perturbation series is not well defined because of the large number of diagrams which make the series non-Borel summable due to the singularities (so called 't Hooft renormalons in Yang-Mils theory) in the Borel plane. There are examples, however, where the perturbation theory is cured by systematic resummation of the non-perturbative objects, (e.g. instanton-anti-instanton in quantum mechanics) but instantons in Yang-Mills theories did not cure the problem of these singularities. For a long time it was believed that these singularities are a sickness of the perturbation theory and cannot be cured by semi-classical objects. This view, however, is being challenged by recent progress in QCD-like theories. For recent development see [48, 19, 34, 50, 47, 49]

solved in the large N expansion, where a mass gap is computable and is of the form $m \sim e^{-\cdots/g^2}$, with g^2 being the bare coupling of the theory. Such a mass gap is obviously non-perturbative in the coupling and vanishes to any order in the perturbation theory.

It is very appealing to think that such a mass gap is generated by semi-classical objects with an action $S \propto 1/g^2$. Indeed instantons, classical solutions of YM equations in Euclidean space-time, were thought to be such objects. Further, because of their topological character, the Atiyah-Singer (AS) *index theorem* [16] guarantees that instantons have fermionic bound-states referred to as *zero modes*. It was discovered by 't Hooft [106] that these bound states can be viewed as effective $2N_f$ fermion interactions, much like the interactions of the Nambu-Jona-Lasinio (NJL) model [76, 77]. This interaction was a welcomed surprise, as it solved the long standing $U(1)$ problem: why there is no ninth Goldstone boson, or why is the η' meson so massive. Instantons simply revealed that the $U(1)$ axial symmetry is *not* spontaneously broken but is *explicitly* broken by instanton events. In addition it provided the desired fermion self-interaction of the appropriate symmetry which could induce spontaneous chiral symmetry breaking pattern observed in nature (i.e. light Goldstone pions).

The instantons, however, although successful in solving the $U(1)$ anomaly problem and generating the fermion interactions required for chiral symmetry breaking, suffer from a severe problem when quantum effects are taken into account. In fact, it was shown by 't Hooft in [106] that an instanton, although classically a scale invariant object, has an action which develops dependence on its size, reflecting the running of the coupling as a function of the typical background field size: the instanton size. The problem is that the integration over the size of the instanton is dominated by a large instanton-size contribution in the one loop approximation. This is, however, precisely the region where one-loop is not to be trusted: the elusive infrared region. As a result, a reliable semi-classical treatment of instantons, although possible under certain circumstances (e.g. in supersymmetric QCD (sQCD) [5] in the Higgs phase, and finite temperature YM/QCD theory[3] [57]), is impossible in QCD and YM at zero temperature. Nevertheless, instanton phenomenology had a noted success in explaining the origin and mechanism of chiral symmetry breaking and phenomena connected to it by fixing the instanton size phenomenologically to $\rho \approx 1/3$ fm (see [93] and references therein), but these models could never explain the area law and confinement. Presumably this is because they ignore the large instanton contributions which are believed, for some time now, to have something to do with confinement. Indeed it was argued a long time ago by Callan, Dashen and Gross (CDG) [31] that an instanton of large size breaks into 4D, point-like objects with fractional topological charge, which they dubbed *merons*. Merons interact logarithmically, much like vortices in two dimensional theories, and can

[3]In the case of pure YM/QCD at finite temperature although the instanton calculus is reliable because of the one loop suppression of large instantons, the reader will note that the theory is far from solvable as there are observables which are non-calculable in perturbation theory such as the magnetic mass.

condense if their action cost is smaller than their entropy gain. Upon condensation of merons, CDG argued that Wilson loops will have an area law behavior. These considerations remain, however, purely at a phenomenological level and no rigorously calculable scenario in four dimensions based on the meron picture was found.

In the past several years non-supersymmetric YM-like theories which are under complete theoretical control emerged, most particularly deformed YM and QCD with quarks in the adjoint representation (QCD(adj)) [109, 98]. These theories are generically center-symmetry-preserving (confining) compactifications (as opposed to thermal compactifications) of their four-dimensional versions. Such a compactification enforces the adjoint Higgs mechanism, where, for compactification of the 4-direction, A_4 plays the role of a (compact) Higgs field and the YM theory abelianizes at scales larger than the radius of compactification L. Nonetheless even though the theory is abelian at low energies, nontrivial topological objects emerge on the scene with monopole-like fields and fractional topological charge. These are instanton-monopoles discovered as constituents of instantons over a decade ago[4] [66, 67]. They dramatically change the behavior of the theories in question, dynamically generating a mass gap and confinement. Further, instanton-monopoles have been connected to dyons, object responsible for confinement in $\mathcal{N}=2$ Super Yang-Mills (SYM) theory on $\mathbb{R}^4$, by Poisson resummation [90]. In the works of Poppitz, Schäfer and Ünsal (PSU) [87, 86] the microscopic picture of $\mathcal{N}=1$ SYM gauge theories was analyzed on $\mathbb{R}^3 \times S^1$. The partition function is well known to be the *Witten index* [116], which is independent of the compact radius L, so no phase transition can occur in the decompactification limit. On the other hand, PSU have shown that by breaking supersymmetry (SUSY) softly, giving an explicit, but small mass m to gauginos, a confinement/deconfinement phase transition occurs of the correct order (i.e. second for $SU(2)$ and first for $SU(N \geq 3)$) at some small compact radius $L_c \ll 1/\Lambda$, where Λ is the strong scale of the theory. They have conjectured that this analytically tractable scenario is continuously connected to the case when $m \to \infty$, i.e. when gauginos are completely decoupled and the theory is pure YM. If this is indeed the case it would be a strong evidence that the same mechanism is responsible for confinement both in SYM and the pure YM theory.

In pure YM the confinement/deconfinement phase transition happens at a large radius of order $L = 1/T_c \sim 1/\Lambda$, where T_c is the transition temperature, so that the fluctuations of the fields are large, and semiclassical analysis is not strictly applicable. The physical interpretation of the confinement/deconfinement scenario is that in the confined phase, quarks are connected with a narrow electric flux tube, so that the energy of

[4]The credit of discovering the instanton-monopoles is often attributed to these references. However the reader will note that similar objects were discussed previously, for example in [57]. Their contribution was mostly ignored in the community as the they were thought to be irrelevant in the high-temperature phase of QCD. In [118] an attempt to compute their contribution was made, but little effort was made to understand the physics of them.

a quark–anti-quark pair is linearly growing with their distance. At high temperatures, however, thermal gluons can be excited and are able to screen the electric flux tube if the temperature is high enough. In supersymmetric theories this screening is protected by supersymmetry, where fermionic super-partners are kept periodic in the compact direction, and do not have a thermal interpretation. Because of that the would-be thermal fermions cancel the contribution of the thermal gluons. In pure YM theory this does not happen and the Polyakov loop – an order parameter of the confinement/deconfinement transition – is nonzero: the theory is in the deconfined phase. As a result the gauge field in the temporal direction is zero up to a gauge transformation, and no adjoint Higgs mechanism, which is crucial for analytical control, takes place.

Nevertheless the question "what causes confinement" in YM/QCD has been tackled by the lattice and the phenomenology community in the past years from the viewpoint of the instanton-monopoles. In the works by Ilgenfritz et. al. [59] and Bornyakov et. al. [22] the instanton-monopoles (there referred to as dyons) were identified in lattice simulations above and below T_c. This did stir some phenomenological interest in the community [43, 40, 44] where suggestions were made that confinement is driven by the moduli space metric of instanton-monopoles[5]. A metric for the instanton-monopoles was proposed (for a review see [41] and references therein), which, apart from having a deficit of including only the self-dual sector[6], suffered from problems at short distances of monopoles (i.e. large monopole densities) [26], so it is unclear to what extent the moduli-space metric describes the correct interactions of instanton-monopoles. On the other hand, Bruckmann et. al. [25] have shown that a random gas of instanton-monopoles is confining, while in the works [100, 51] chiral symmetry breaking was connected with randomization of instanton-monopoles, giving a potential connection between confinement and chiral symmetry breaking. Until recently this development paralleled the development in SUSY theories with little or no overlap. In our work [101] a crude model was built based on the language used in controllable theories and some comparison to lattice data was made with decent agreement.

Although a lot of work has been done in YM and its supersymmetric incarnations, a systematic understanding of the interplay between the instanton-monopoles and the fundamental fermions is still in its early infancy. Like in the case of instantons the instanton-monopoles can carry fermionic zero modes which generate 't Hooft interactions

[5]The classical interactions were ignored in this argument, because the authors focused on the self-dual sector only in which instanton-monopoles do not interact. They also ignored loop effects which would generate Debye screening and make even self-dual monopoles interacting (see however next footnote).

[6]A suggestion on how to include the anti–self-dual monopoles was briefly discussed in [41] where a non-interaction assumption between the two sectors was made invoking electric Debye screening. It is unclear to this author why this is justified, as this would a) not eliminate the interactions between the self-dual and anti–self-dual monopoles and b) would induce interactions between the self-dual monopoles, which was absent from their analysis and the guiding reason why the authors in these works consider moduli space metric to be the leading contribution to the confinement.

and in principle should break chiral symmetry via a similar mechanism. In [100, 51] initial steps were taken in this directions, attempting to reconstruct the chirally broken picture of the Instanton Liquid Model. In these works, however, the detailed structure of the fermionic zero modes and interactions of monopoles via fermionic zero modes was not analyzed and only naive interactions were introduced. In addition phenomena related to finite density systems, which have eluded lattice simulations because of the infamous sign problem, must necessarily go through the fermions, so it is worthwhile to explore how the spectrum (and most importantly zero modes) changes when chemical potential is introduced. In our work [28] a detailed analysis was made of the fermionic zero modes and the so-called hopping matrix element, which is responsible for the fermion facilitated interactions between topological objects which carry zero modes.

Additionally, phenomena related to strong magnetic fields are of considerable interests in recent years due to their possible relevance for Heavy Ion Collisions (HIC), as well as to the physics of magnetars and the early universe. Interesting effects such as Chiral Magnetic Effect [53, 63] and Chiral Separation Effect [104, 72] were found when fermions move in the background with topological charge and magnetic field. In our work [24] it was shown that a novel phenomenon happens in the background of the instanton-monopoles in magnetic field. This phenomenon was dubbed *charge catalysis* as it exhibits imaginary charge[7] accumulation in between a monopole–anti-monopole pair. This phenomenon transcends QCD and has direct application in strained graphene.

It is of great interest to see how fundamental matter influences a theory where instanton-monopoles are important and where their computation is reliable. The most natural candidates are SUSY theories. Although extensive work has been done in the super YM case, the theories with matter where instanton monopoles appear have only gotten side (but important) remarks in generic analysis of 3D supersymmetric theories in both old and recent literature [7, 8]. Even though the addition of massless flavor multiplets renders the theory gapless and non-semiclassical, adding massive flavors allows semiclassical treatment and analytical control. This was the main interest of our work [88] where the one loop calculation around instanton-monopoles in supersymmetric QCD (sQCD) on $\mathbb{R}^3 \times S^1$ with heavy flavors was performed and the microscopic picture of the Polyakov loop screening and string breaking was analyzed.

Unfortunately supersymmetric theories, in addition to not being realized in nature, are much more difficult to handle at finite quark density, because the presence of the baryon charge would violate supersymmetry[8]. An $O(N)$ model in 2D however – a common toy model of non-abelian gauge theories – has a rigid $SO(N)$ continuous symmetry and a chemical potential μ for some $SO(2) \in SO(N)$ can be added. For sufficiently

[7]The charge being imaginary is, of course, not observable, but its effect appears in the charge-charge correlation functions. For details see Chapter 3.

[8]Adding an equal chemical potential to squarks does not seem to help much, because bosons and fermions have different statistics.

large μ, $SO(N)$ symmetry breaks to $SO(2)$ and vortex solutions appear [4, 29] which, in the case of the $O(3)$ model, have fractional topological charge. They can be identified as instanton constituents, i.e. instanton-monopoles, which have the ability to generate a dynamical mass gap by disordering themselves.

In this thesis we will review our works [28, 24, 88] mentioned above. We have organized the thesis as follows: In Chapter 2 we give a motivation why monopoles are important for expected infrared (IR) behavior of YM-like theories, as well as introduce notations and important results from the literature relevant for our discussions in the chapters to follow. We also review some basic facts about the $O(2)$ model in two dimensions, vortices and the physics of the Kosterlitz-Thouless transition, and show how to introduce chemical potential in the $O(N)$ and $CP(N-1)$ models. In Chapter 3 we discuss the phenomenon of charge catalysis in magnetic fields, a novel phenomenon of instanton-monopoles in the magnetic field, as well as its manifestation in strained graphene. In the same chapter we discuss the explicit zero mode solutions for instanton-monopoles and calorons at finite (complex) chemical potential. Finally in Chapter 4 we discuss two dynamical models where monopole-like objects with fractional topology have a crucial influence on the IR dynamics: supersymmertic QCD on $\mathbb{R}^3 \times S^1$ and the $O(N)$ nonlinear sigma model in two dimensional with chemical potential. In Chapter 5 we give a summary and speculate about the potential significance of instanton-monopoles in QCD and beyond.

Chapter 2

Preliminaries

This chapter deals with preliminaries which are important for the recent developments that the thesis focuses on. The chapter is organized as follows: In Section 2.1 we discuss in detail the Georgi-Glashow model which is well known to have monopoles and confinement. In Section 2.2 we review the construction of instanton-monopoles on $\mathbb{R}^3 \times S^1$ and their connection with calorons. These objects are the backbone of this thesis. In Section 2.4 the index theorem is discussed, which guarantees the existence of localized fermionic states on top of topological objects, in particular the instanton-monopoles. In Section 2.5 some basic language, terminology and relations of supersymmetries are reviewed, which we will require later in Section 4.1.

2.1 Motivation: Georgi-Glashow model and Polyakov's confinement

Before we consider instanton-monopoles it is instructive to give some motivation why monopoles are important in non-abelian gauge theories. The simplest example is the SU(2) Georgi-Glashow model in 3D, i.e. an $SU(2)$ YM + adjoint Higgs theory. We will see that this model has monopole solutions that influence the IR behavior heavily.

The Georgi-Glashow model is given by a Lagrangian density

$$\mathcal{L} = \frac{1}{g_3^2}\mathrm{Tr}\left[\frac{1}{2}F_{ij}^2 + (D_i\phi)^2\right] + V(|\phi|)\ , \tag{2.1}$$

where $\phi = \phi^a \frac{\tau^a}{2}$, $D_i\phi = \partial_i\phi - i[A_i, \phi]$ and $|\phi| = \sqrt{\sum_{a=1,2,3}(\phi^a)^2}$ and g_3^2 is the 3D coupling with the dimension of energy. If the potential $V(|\phi|^2)$ has a minimum at $|\phi| = v$, and if one makes a gauge transformation so that ϕ is in the 3rd color direction, the scalar field can be written as $\phi = v\frac{\tau^3}{2} +$ (fluctuations). Such a decomposition makes it transparent that the $A_\mu^{1,2}$ terms in the above Lagrangian obtains the mass from the Higgs field, while the A^3_μ gauge field remains massless. In other words the gauge symmetry breaks spontaneously from $SU(2)$ to $U(1)$.

A priori it seems that the low energy effective theory is gap-less and it is just a free $U(1)$ theory. However, the theory breaks to $U(1)$ at large distances, while at short distances $\lesssim 1/v$ the full $SU(2)$ theory can be restored. This fact hints at the possibility that the theory can have fluctuations which are highly localized defects (i.e. appear to be singular) in the $U(1)$ theory, but are perfectly smooth in the core because of the underlying $SU(2)$ structure. Indeed such objects exist as solutions to the equations of motion of the Yang-Mills + adjoint Higgs theory. They were found independently by 't Hoof and Polyakov [105, 81] and are referred to as 't Hoof-Polyakov monopole because of their monopole character. Here we review the arguments for their existence, while we postpone their explicit solution for the next section.

To see that such objects indeed exist in the theory, it is simplest not to fix the gauge to the third color direction for the moment and just demand that the Higgs field at space-time infinity takes its expectation value, i.e. that $|\phi|^2 \to v^2$ when $|\boldsymbol{r}| \to \infty$. The space-time at infinity has a topology of a S^2 sphere. On the other hand, since the length of the Higgs field is constrained to be v, which is invariant under the $SU(2)$ gauge rotations, the Higgs field at spatial infinity is a 3-vector of fixed length, and therefore also takes values on a S^2 sphere. The space-time at infinity to target-space Higgs field mapping is then a map $S^2_{|r|\to\infty} \to S^2_{|\phi|=v}$. It is well known that such maps are characterized by an integer number $Q \in \mathbb{Z}$, where Q is the so-called winding number, i.e. a number which tells us how many times does the target space sphere $S^2_{|\phi|=v}$ get covered when we cover a sphere at space-time infinity[1] $S^2_{|\boldsymbol{r}|=\infty}$.

Since the nonzero Q configuration cannot be contracted continuously to trivial vacuum, such configurations must have nontrivial properties in the bulk. For our purposes it will suffice to consider the $Q = 1$ sector. The task now would be to find the finite action solutions of the equations of motion in this sector with $Q = 1$. This is however not possible analytically for arbitrary potential $V(|\phi|)$ nor is it possible for the mexican hat potential[2] $\lambda(|\phi|^2 - v^2)^2$. Nevertheless the most important properties of the solution can be extracted without explicit computations. The necessary condition that the action is finite is that $(D_i\phi)^2 \to 0$ faster then $1/r^3$. This translates into

$$(D_i\phi)^a = \partial_i\phi^a + \epsilon_{abc}A^b_i\hat{\phi}^c \to v\partial_i\hat{r}^a + v\epsilon_{abc}A^b_i\hat{r}^c \ , \qquad (r\to\infty) \tag{2.2}$$

[1] To get an intuitive feel for the winding number it helps to think about the map from $\varphi : S^1 \to S^1$, i.e. $\varphi(\alpha)$ where both φ and α are angular variables. Since we demand that $\alpha \equiv \alpha + 2\pi$ and $\varphi \equiv \varphi + 2\pi$, the maps are constrained so that $\varphi(2\pi) - \varphi(0) = 2\pi Q$, where Q is an integer. The maps of fixed Q cannot be continuously deformed to each other without violating the boundary conditions that $\alpha \equiv \alpha + 2\pi$ and $\varphi \equiv \varphi + 2\pi$.

[2] It is, however, possible for the limit $\lambda \to 0$, i.e. the vanishing potential limit, which we discuss in the next section.

where we have assumed that $\phi^a \sim v\hat{r}^a$ at spatial infinity (i.e. we assumed spherical symmetry). Asymptotically the covariant derivative of the Higgs field is given by

$$(D_i\phi)^a \sim v\frac{\delta^{ai} - \hat{r}^a\hat{r}^i}{r} + v\epsilon_{abc}A_i^b\hat{r}^c \, . \tag{2.3}$$

The above combination must vanish at infinity because the first term above decays too slowly for the action to be finite (i.e. as $1/r$ so that its square, which is $1/r^2$, would give a linear IR divergence of the action). It is easy to see that this asymptotic equation is solved by $A_i^b = \frac{1}{r}\epsilon_{bik}\hat{r}^k$. Computing the asymptotic field strength gives that $B_i^a = \frac{1}{2}\epsilon^{ijk}F_{jk}^a \sim -\frac{\hat{r}^i\hat{r}^a}{r^2}$. Notice that the asymptotic field is in the same color direction as the Higgs field, i.e. that $B_i^a\phi^a \sim -\frac{\hat{r}^i}{r^2}$. This is as it should be as we said that the long range theory is a $U(1)$ theory, where the $U(1)$ subgroup was selected by the Higgs vacuum expectations value (VEV). So we have a field of a (anti-)monopole[3]!

Since the coupling g_3^2 has dimensions of energy, and since the only scale in the problem is v, we must have that the classical action of the monopole solution is of the form

$$S = c\frac{v}{g_3^2} \, . \tag{2.4}$$

where c is some dimensionless constant. This will suffice to illustrate the important points.

Now that we have found the asymptotics of the solution, we must find the way how to incorporate them into the effective theory. Firstly we must understand how to integrate over such configurations. Let us say that we want to include a monopole at position $\boldsymbol{x}_0$ in the path integral. Then we would write the gauge field as $A_i = A_i^{mon}(\boldsymbol{x}_0) + a_i$ where a_i are the fluctuations around the monopole solution. Heuristically we expect that the integration measure should change as follows

$$\int \mathcal{D}A_i \to \int \mathcal{D}a_i \int d^3x_0 \tag{2.5}$$

so that integration breaks into distinct parts, one of fluctuation around the monopole at point x_0 and the other the integration over all monopole positions.

Indeed this is what will happen. The details of how this decomposition occurs are given in Appendix C.1. However the integration over the so called collective coordinates $\boldsymbol{x}_0$ clearly must be accompanied by a dimensionfull metric g_{ij} which will render the integration measure $\sqrt{g}d^3x_0$ dimensionless. We will not worry about this at the moment, instead we leave a more precise treatment for the case of four dimensional YM theory, where similar objects appear. It will suffice to say that this metric is constant for a

[3] Whether it is a monopole or an anti-monopole will depend on the sign of v.

single monopole and does not depend on[4] $\boldsymbol{x}_0$ because of translational invariance. For now we will simply include a constant C with any integration over d^3x_0.

The obstacle now is to account for the interactions between multiple monopoles. These interactions will, unsurprisingly, be coulomb-like because of the long range abelian fields. It is not difficult to show[5] that the interaction action between a monopole located at $\boldsymbol{r}_1$ and an (anti-)monopole located at $\boldsymbol{r}_2$ is

$$S_{int}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = \pm \frac{1}{g_3^2} \frac{4\pi}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \tag{2.6}$$

where the upper sign refers to the like charges (i.e. two monopoles, or two anti-monopoles) and the lower sign refers to the unlike charges (i.e. monopole and anti-monopole). The partition function of the monopole gas would then have to include these long range interactions.

In 1976 Polyakov [83] devised an ingenious duality by noticing that the interactions can be accounted for by introducing a scalar field σ with the kinetic term $S[\sigma] = \int d^3x \, \frac{g_3^2}{2(2\pi)^2}(\partial_i\sigma)^2$. The (anti-)monopole at position $\boldsymbol{x}_0$ would then couple to the σ field in the following fashion

$$\text{monopole operator} \propto e^{\pm i\sigma(\boldsymbol{x}_0)} \; . \tag{2.7}$$

Indeed it takes little effort to show that

$$\int \mathcal{D}\sigma \; e^{i\sigma(\boldsymbol{r}_1)} e^{\pm i\sigma(\boldsymbol{r}_2)} e^{-S[\sigma]} \propto e^{\mp \frac{4\pi}{g_3^2 |\boldsymbol{r}_1 - \boldsymbol{r}_2|}} = e^{-S_{int}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)} \tag{2.8}$$

The effective U(1) theory we want to describe would then have a dual description in terms of the free Lagrangian

$$\mathcal{L}_{dual}^{free} = \frac{g_3^2}{2(4\pi)^2}(\partial_i\sigma)^2 \; . \tag{2.9}$$

There is a more formal way of showing the above duality. Indeed because of the Higgs

[4]For multiple monopoles the metric gets corrections which depend on the distances between monopoles. These corrections, however, can be neglected for dilute regimes, which is what we will discuss. The reader is however advised that in pure YM theory in 4D these "moduli space interactions" might be important, as was argued in some models based on instanton-monopoles [40] (see also [41])

[5]The total asymptotic chromo-magnetic field of the two monopoles is given by $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2 = \frac{\hat{r}_1}{r_1^2}\frac{\tau^3}{2} + \frac{\hat{r}_2}{r_2^2}\frac{\tau^3}{2}$, where r_1, r_2 are distances from the monopole centers to the observation point and $\hat{r}_1, \hat{r}_2$ are unit vectors pointing from the monopoles to the observation point. Since the action is $g^2S = \int d^3x \; \mathrm{tr}\, \boldsymbol{B}_1^2 + \int d^3x \; \mathrm{tr}\, \boldsymbol{B}_2^2 + 2\int d^3x \; \mathrm{tr}\, \boldsymbol{B}_1 \cdot \boldsymbol{B}_2 + \dots$, where the dots stand for Higgs field action which, because of the Higgs mass does not contribute to the interaction between monopoles. The first two terms do not depend on the distance between monopoles and they combine with the Higgs action to make $2S_0$, while the third term yields (neglecting irrelevant core-contribution) $g_3^2 S_{int} = 2\int d^3x \mathrm{tr}\, \boldsymbol{B}_1 \cdot \boldsymbol{B}_2 = \frac{4\pi}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}$

mechanism which gives two out of three gluons a mass (i.e. they become heavy W bosons) the YM part of the $SU(2)$ action becomes $\frac{1}{2g_3^2}\mathrm{tr}\, F_{ij}^2 \to \frac{1}{4g_3^2}\mathcal{F}_{ij}$, where $\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i$ is the abelian field strength, and $\mathcal{A}_i$ is the gauge field which remained mass-less (e.g. $\mathcal{A}_i = A_i^3$ in the gauge where $\langle\phi\rangle = v\frac{\tau^3}{2}$). In writing this we have simply integrated over the massive W-bosons which are short ranged (of order $\sim 1/v$) and will not contribute to the long-distance dynamics.

The partition function is then an integral over the mass-less gauge field $\mathcal{A}_i$ only. However, this is equivalent to integrating over the field-strength $\mathcal{F}_{ij}$ and imposing the Bianchi identity $\epsilon^{ijk}\partial_i \mathcal{F}_{jk} = 0$. We therefore have

$$\begin{aligned} &\int \mathcal{D}\mathcal{A}_i \; e^{-\frac{1}{4g_3^2}\int d^3x \mathcal{F}_{ij}^2[A]} = \int \mathcal{D}\mathcal{F}_{ij} \; e^{-\frac{1}{4g_3^2}\int d^3x \mathcal{F}_{ij}^2} \delta(\epsilon^{ijk}\partial_i \mathcal{F}_{jk}) = \\ &= \int \mathcal{D}\mathcal{F}_{ij} \int \mathcal{D}\sigma \; e^{-\frac{1}{4g_3^2}\int d^3x \mathcal{F}_{ij}^2 + i\frac{1}{8\pi}\int d^3x \; \epsilon^{ijk}\partial_i \sigma \mathcal{F}_{jk}} = \int \mathcal{D}\sigma \; e^{-\frac{g_3^2}{2(4\pi)^2}\int d^3x \; (\partial_i \sigma)^2} \; . \end{aligned} \tag{2.10}$$

In the above we introduced a term $\frac{i}{8\pi}\sigma\epsilon^{ijk}\partial_i F_{jk}$ in the Lagrangian, which upon integration over the Lagrange multiplier σ imposes the Bianchi identity. This allowed us to integrate over the field strength $\mathcal{F}_{ij}$ generating a theory which depends only on the σ field. The theory we ended up with is a theory of the scalar field σ only. The Bianchi constraint is precisely the condition that there are no monopoles. However, inserting a (anti-)monopole at position $\boldsymbol{x}_0$ is as simple as inserting an operator $e^{\pm i\sigma(\boldsymbol{x}_0)}$. Indeed by integrating over the σ field with the insertion of this operator it is easy to see that the constraint is now $\frac{1}{2}\epsilon^{ijk}\partial_i \mathcal{F}_{jk} = \boldsymbol{\nabla}\cdot\boldsymbol{B} = 4\pi\delta(\boldsymbol{x} - \boldsymbol{x}_0)$, which corresponds to a monopole at $\boldsymbol{x}_0$.

We are ready now to include many monopoles and anti-monopoles into the partition function. Firstly we go to a dual description in terms of the σ field with the kinetic Lagrangian $\frac{g_3^2}{2(4\pi)^2}(\partial_i\sigma)^2$. Including monopoles means integrating over monopole-like configurations and so each monopole should be weighted by a factor $e^{-S_0 \pm \sigma(x_0)}$, where $\sigma(\boldsymbol{x}_0)$ accounts for the magnetic charges, and then integrated over the moduli-space metric $\int d^3x_0 C$. The partition function in the dual description accounting for the monopoles can then be written as

$$\begin{aligned} Z = &\int \mathcal{D}\sigma \; e^{-\frac{g_3^2}{2(4\pi)}\int d^3x \; \frac{1}{2}(\partial_i\sigma)^2} \\ &\times \sum_{M=0}^{\infty} \frac{1}{M!}\left(\int d^3x_0 \; Ce^{-S_0 + i\sigma(\boldsymbol{x}_0)}\right)^M \sum_{\bar{M}=0}^{\infty} \frac{1}{\bar{M}!}\left(\int d^3x_0 \; Ce^{-S_0 - i\sigma(\boldsymbol{x}_0)}\right)^{\bar{M}} = \\ &= \int \mathcal{D}\sigma \; e^{-\frac{g_3^2}{2(4\pi)}\int d^3x \; \left(\frac{1}{2}(\partial_i\sigma)^2 - m^2\cos\sigma\right)} \end{aligned} \tag{2.11}$$

where the second line accounts for monopoles (sum over M) and anti-monopoles (sum over $\bar{M}$) which exponentiated and formed a $\cos(\sigma)$ potential for the σ field with $m^2 = \frac{2(4\pi)}{g_3^2}C$.

So what happened here? An effective U(1) theory dual to a free scalar σ-field theory, which without monopoles had no potential for the scalar field σ, has, upon monopole resummation, obtained a potential, and, therefore, the mass m for the σ field. That this happened is not horribly surprising once we think about the physics of this mechanism. Because of its three-dimensional nature, the effective theory was a free theory of U(1) "magnetic field" only[6] (i.e. $\mathcal{L}_{eff} \propto \boldsymbol{B}^2$). Without monopoles the magnetic interactions would be long ranged and decay non-exponentially. Introduction of monopoles in the theory changes this qualitative picture completely. Since monopoles are able to screen the magnetic field by redistributing themselves appropriately in the vacuum, the correlation functions become exponential. This is known as the *magnetic Debye screening*[7].

To make this more explicit let us consider a Wilson loop observable i.e.

$$W[C] = \left\langle \mathrm{tr}\,\exp(i\frac{\tau^3}{2}\oint_C dx^i \mathcal{A}_i) \right\rangle \tag{2.12}$$

where C is some contour. Notice the factor of $\tau^3/2$ in the exponent. This is due to the fact that we consider quarks in fundamental representation. The above observable is diagonal in color and can be computed by first evaluating

$$\left\langle \exp(i\frac{1}{2}\oint_C dx^i \mathcal{A}_i) \right\rangle = \left\langle \exp(i\frac{1}{2}\int_S d\boldsymbol{S}_i\, \mathcal{B}^i \right\rangle \tag{2.13}$$

where $d\boldsymbol{S}_i$ is a surface element and the integration is over a surface S such that its boundary is C, i.e. $C = \partial S$ and $\mathcal{B}^i = \frac{1}{2}\epsilon^{ijk}\mathcal{F}_{jk}$ is the abelian magnetic field. Let us take a contour C lying entirely in the x_1, x_2 plane, and let us take the surface S also in the (x_1, x_2) plane. Then inserting the above operator in the Lagrangian (2.10) containing both σ and $\mathcal{F}_{ij}$ fields, integrating over $\mathcal{F}_{ij}$ and summing over (anti-)monopoles we get the following action

$$\mathcal{L}_{dual}^{W-loop} = \frac{g_3^2}{2(4\pi)^2}\left[\Big(\partial_k\sigma + 2\pi\delta(x_3)\theta(x_1,x_2)\delta_{k,3}\Big)^2 - m^2\cos\sigma\right] \tag{2.14}$$

where $\theta(x_1, x_2) = 1$ for $x_1, x_2 \in S$ and zero otherwise. The above action is infinite unless

[6] What we call here "magnetic field" is perhaps misleading. Namely we think of the three-dimensional problem at hand as a static four-dimensional one. In that sense the F_{ij} are simply magnetic field components. Note however that this analogy is incomplete as static electric charges should also exist in a truly four-dimensional problem. This will indeed be the case in four-dimensional theories which we discuss later on.

[7] This is not to be confused to the *electric* Debye screening, responsible for deconfinement.

the term in the parenthesis is a smooth function across the surface S, i.e. we demand that

$$\partial_3\sigma + 2\pi\delta(x_3)\theta(x_1,x_2) = (\text{smooth function}) \ . \tag{2.15}$$

The above condition translates into

$$\sigma(x_3 = 0^+) - \sigma(x_3 = 0^-) = -2\pi \tag{2.16}$$

i.e. the σ field has a jump across the surface whose boundary is the contour C. This means that σ field is a periodic field with a period 2π. This is the crucial difference between a SU(2) theory and a SO(3) theory, where integer charges would be allowed, and σ field would have a 4π period. (For more details discussing this difference see e.g. [10].)

Let us now solve the classical equations of motion in the presence of the Wilson loop. The equations of motion from (2.14) are

$$-2\partial_k\left(\partial_k\sigma + 2\pi\delta(x_3)\theta(x_1,x_2)\delta_{k3}\right) + m^2\sin\sigma = 0 \ . \tag{2.17}$$

For a large surface S we can look for a solution which only depends on x_3. Since we need to make a jump by 2π at $x_3 = 0$ we take the following solution[8]

$$\sigma = \begin{cases} 4\arctan(e^{-mx_3/\sqrt{2}}) & x_3 > 0 \\ -4\arctan(e^{mx_3/\sqrt{2}}) & x_3 < 0 \end{cases} \tag{2.18}$$

The form above has $\sigma \to 0$ for $x \to \pm\infty$ and has the appropriate jump of 2π across the surface. The action of the above configuration can be computed and is

$$S \approx \frac{g_3^2 m}{2\sqrt{2}\pi^2} A + S_{vac} \tag{2.19}$$

where A is the area and where we ignored potential edge corrections and S_{vac} is the action of the vacuum (i.e. $\cos\sigma = 1$, $\partial_i\sigma = 0$). The average expectation value of the Wilson loop is then

$$\left\langle e^{i\frac{\tau^3}{2}\oint A_i dx^i} \right\rangle \approx 2e^{-\frac{g_3^2 m}{2\pi^2}A} \tag{2.20}$$

which is the famous area law.

We conclude this section by summarizing the important points:

[8]The solution is found for $x_3 \neq 0$ by multiplying the equation of motion $-2\partial_3^2\sigma + m^2\sin\sigma = 0$ by $\partial_3\sigma$ and integrating over sigma we get that $(\partial_3\sigma)^2 + m^2\cos\sigma = const$. Since we must have that for $x_3 \to \pm\infty$ the Lagrangian (2.14) goes to the vacuum $\cos\sigma = 1$ and $\partial_3\sigma = 0$, then we have $\sigma = 4\arctan(e^{\pm m(x_3 - x_3^0)/\sqrt{2}}) + 2\pi k = \pm\arctan(e^{m(x_3-x_3^0)/\sqrt{2}}) + 2\pi k'$, where k, k' are integers, and x_3^0 is an integration constant.

- The Georgi-Glashow model in the color broken phase is analytically tractable
- The low energy effective theory is a $U(1)$ theory with monopoles
- It exhibits nontrivial IR behavior such as the mass the gap generation and the area law

The crucial component in the non-trivial IR behavior of the theory were magnetic monopoles. The natural question now whether similar objects exist 4D theories. Although there are no such solutions known on[9] $\mathbb{R}^4$, we shall see in the next section that similar objects do appear in theories defined on $\mathbb{R}^3 \times S^1$.

2.2 Monopoles and calorons on $\mathbb{R}^3 \times S^1$

In this section we describe in detail the construction of instanton-monopoles on $\mathbb{R}^3 \times S^1$ and explain their connection to instantons. These objects will be the backbone of the chapters to come and around which the entire thesis revolves. Various discussions in this section can be found in [41, 99, 66], but the reader will keep in mind that there are powerful D-brane arguments of their existence which are well worth exploring [68] (for a pedagogical introduction into D-branes and their connection with self-dual solutions see [108]).

2.2.1 The BPS monopole

We start with the dimensional reduction of the 4D Yang-Mills theory, which has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\, F_{ij}^2 + \mathrm{tr}\,(D_i A_4)^2 \tag{2.21}$$

where we have ignored the field dependence on the x_4-component and where the covariant derivative acts in the adjoint representation, i.e. as[10] $D_i = \partial_i - i[A_i, \quad]$. The above Lagrangian is reminiscent of the Georgy-Glashow Lagrangian (2.1) with a vanishing Higgs potential. The equations of motion are

$$D_i F_{ij} = 0 \,, \qquad\qquad D_i^2 A_4 = 0 \,. \tag{2.22}$$

[9]A popular mechanism of confinement on $\mathbb{R}^4$ is 't Hooft's abelian projection [107]. The monopoles there, however, appear as gauge-dependent objects. Although they have been very important in phenomenology of QCD and pure YM, we do not discuss these here.

[10]We always assume, unless otherwise specified, that the generators are in the fundamental representation.

These are second order nonlinear differential equations which are difficult to solve. However it is possible to simplify the problem by considering a positive definite quantity

$$0 \le \frac{1}{g^2}\int d^4x \,\mathrm{tr}\,(D_iA_4 \pm \frac{1}{2}\epsilon^{ijk}F_{jk})^2 =$$
$$= \frac{1}{g^2}\int d^4x \,\mathrm{tr}\left((D_iA_4)^2 + \frac{1}{2}F_{ij}^2 \pm \epsilon^{ijk}D_iA_4F_{jk}\right) = S \pm \frac{8\pi^2}{g^2}Q \quad (2.23)$$

where S is the action, and $Q = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \,\mathrm{tr}\,(F_{\mu\nu}F_{\nu\rho}) = \frac{1}{8\pi^2}\epsilon^{ijk}\int d^4x \,\mathrm{tr}\,(D_iA_4F_{jk})$ is the topological charge (see Appendix B). The above equation implies that

$$S \ge \frac{8\pi^2}{g^2}|Q| \,. \quad (2.24)$$

In other words the lower bound of the action is not zero, but depends on its topological charge Q. We can look for the solutions which saturate the lower bound. These solutions were found by Prasad and Sommerfield [91] and independently by Bogomolny [21] and are known as Bogomol'nyi-Prasad-Sommerfield (BPS) solutions and their equations are often referred to as self-duality equations

$$D_iA_4 = \pm\frac{1}{2}\epsilon_{ijk}F^{jk} \quad (2.25)$$

The equations of motion (e.o.m.) can be solved by the following spherically symmetric ansatz

$$A_i^a = \mathcal{A}(r)\epsilon_{aij}\hat{r}^j \,, \qquad\qquad A_4^a = \mathcal{H}(r)\,\hat{r}^a \quad (2.26)$$

where $\mathcal{A}(r), \mathcal{H}(r)$ are functions of the radial component r only. The above ansatz is motivated by the same reasoning as in the previous section (see eqs. (2.2-2.3) and the text below them). Since we are interested in the solutions in the abelian vacuum, we take $\mathcal{H} \to v = \pm|v|$, $(r \to \infty)$ where, as we shall see, the sign will depend on whether the solution is self-dual or anti–self-dual.

We have

$$(D_iA_4)^a = \delta^{ia}\left(\frac{\mathcal{H}}{r} - \mathcal{H}\mathcal{A}\right) + \hat{r}^i\hat{r}^a\left(\mathcal{H}' + \mathcal{H}\mathcal{A} - \frac{\mathcal{H}}{r}\right) \quad (2.27)$$
$$\partial_iA_j^a = \left(\mathcal{A}' - \frac{\mathcal{A}}{r}\right)\hat{r}^i\epsilon_{ajk}\hat{r}^k + \frac{\mathcal{A}}{r}\epsilon_{aji} \quad (2.28)$$
$$- i[A_i, A_j]^a = \epsilon_{ijk}\hat{r}^k\hat{r}^a\mathcal{A}^2 \quad (2.29)$$
$$\epsilon^{ijk}F_{jk}^a = 2\left(\mathcal{A}' - \frac{\mathcal{A}}{r} + \mathcal{A}^2\right)\hat{r}^i\hat{r}^a - 2\delta^{ia}\left(\mathcal{A}' + \frac{\mathcal{A}}{r}\right) \quad (2.30)$$

or, taking $\mathcal{A} = (1 - A(r))/r$

$$E_i^a = (D_i A_4)^a = \left(\hat{r}^i \hat{r}^a - \delta^{ia}\right)\left(\frac{-\mathcal{H}A}{r}\right) + \hat{r}^i \hat{r}^a \mathcal{H}' \tag{2.31a}$$

$$B_i^a = \frac{1}{2}\epsilon_{ijk} F_{jk}^a = \left(\hat{r}^i \hat{r}^a - \delta^{ia}\right)\left(-\frac{A'}{r}\right) + \hat{r}^a \hat{r}^i \left(\frac{A^2 - 1}{r^2}\right) \tag{2.31b}$$

The tensors $(\delta^{ia} - \hat{r}^i \hat{r}^a)$ and $\hat{r}^i \hat{r}^a$ are orthogonal to each other, therefore demanding (anti–)self-duality $E_i^a = \pm B_i^a$ and identifying the corresponding tensor factors we must have

$$A' = \pm \mathcal{H} A \qquad \mathcal{H}' = \pm \frac{A^2 - 1}{r^2} \tag{2.32}$$

Since we demand that $\mathcal{H}(r \to \infty) = v$, from the first equation we have that the $A \sim e^{\pm vr}$ asymptotically, so that we must take $v = -|v|$ for self-dual and $v = |v|$ for anti–self-dual solution in order to avoid A blowing up at infinity. Plugging into the second equation, we see that the Higgs field must asymptote as $\mathcal{H} \sim v \mp \frac{1}{r}$, so we write $\mathcal{H}(r) = h(r) \mp \frac{1}{r}$ and also $a(r) = \frac{A(r)}{r}$. The above equations simplify to

$$a'(r) = \pm h a \tag{2.33}$$

$$h'(r) = \pm a^2 \tag{2.34}$$

From the ratio of these two equations we can easily get that $h^2 = a^2 + const.$ Since a is zero asymptotically we must have $h^2 = a^2 + v^2$ or that $a^2 = h^2 - v^2$. Substituting in the second equation we obtain

$$h'(r) = \pm(h^2(h) - v^2) \to h(r) = \pm v \coth(vr) \tag{2.35}$$

Where the integration constant was chosen so that $\mathcal{H} = h \pm \frac{1}{r} = \pm\left(\frac{1}{r} - v\coth(vr)\right)$ is regular at $r = 0$. Since $a(r)^2 = h(r)^2 - v^2$ we get

$$a = \frac{v}{\sinh(rv)} \tag{2.36}$$

Finally we have that

$$\mathcal{H} = \pm\left(\frac{1 - vr\coth(vr)}{r}\right), \qquad \mathcal{A} = \frac{1}{r} - \frac{v}{\sinh(rv)} \tag{2.37}$$

The positive and negative signs correspond to the self-dual and anti–self-dual solutions.

Now let us compute the asymptotic fields. From (2.31) we have that

$$E_i^a \approx \mp \hat{r}^i \hat{r}^a \frac{1}{r^2} \qquad\qquad B_i^a \approx -\hat{r}^a \hat{r}^i \frac{1}{r^2} \,. \tag{2.38a}$$

The upper solution is obviously self-dual and the lower anti–self-dual. Projecting onto the Higgs field $\hat{\phi}^a = \mp \hat{r}^a$ we see that the self-dual solution has monopole-like asymptotics and anti-self–dual anti-monopole-like asymptotics.

2.2.2 The KK monopole

In the previous section we constructed the BPS monopole in Yang-Mills theory on $\mathbb{R}^3 \times S^1$. This monopole is a three-dimensional object, i.e. completely static in "time" x_4, and it exists in the Georgi-Glashow model as well.

However due to the fact that the theory is locally four-dimensional, there exists another solution which carries a twist in the 4-direction. Because of this twist the monopole is referred to as the Kaluza-Klein (KK) monopole.

Before we explain how to construct this monopole, let us first discuss our choice of gauge. Namely in the previous section we constructed the monopole in the radial gauge, where the Higgs field had a hedgehog r dependence, i.e. $A_4^a \propto \hat{r}^a$. This gauge is convenient for constructing the solution, but it is utterly bad for considering an ensemble of many (anti-)monopoles, the reason being is that in this gauge the monopole center is a special point. It is much more convenient to rotate the Higgs field so that it takes one color direction, for example the 3-direction, so that $A_4 \propto \frac{\tau^3}{2}$. This is always possible by an appropriate x_4-independent gauge transformation, at the cost of introducing Dirac strings[11] where the gauge field is not defined[12]. The multi-monopole solution would then consist of (anti-)monopoles which have the same asymptotics of the compact Higgs field A_4.

So far we have seen that there are BPS monopoles and BPS anti-monopoles which, in the radial gauge, differed by whether the "Higgs" field winds in the positive or negative direction, i.e. taking $v > 0$ whether $A_4^a \propto v\hat{r}^a$ or $\phi^a \propto -v\hat{r}^a$ asymptotically. To classify objects which exist with the same holonomy asymptotics (up to a gauge transformation) we must go to a gauge where the asymptotic Higgs field A_4 does not depend on the position of the monopoles. To that end let us apply a x_4-independent gauge transformation

[11] These Dirac strings will appear in the caloron solution which is comprised out of a monopole and an anti-monopole, as we shall see.

[12] Strictly speaking one would have to have two patches of gauge field in order to describe a $U(1)$ Dirac monopole field [46].

which rotates the color direction $\hat{r}\cdot\boldsymbol{\tau}$ into τ^3 and $-\tau^3$, i.e.

$$U_+^\dagger \hat{r}\cdot\boldsymbol{\tau} U_+ = \tau^3 \tag{2.39}$$

$$U_-^\dagger \hat{r}\cdot\boldsymbol{\tau} U_- = -\tau^3 \tag{2.40}$$

The explicit forms of matrices $U_\pm$ can be found in e.g. [41], but we omit them since they are irrelevant for our discussion. If we apply the U_+ gauge transformation above to the solutions with the $A_4^a \propto \hat{r}^a$ and U_- transformation to the solution with $A_4^a \propto -\hat{r}^a$ we have that the asymptotic fields of both monopole and anti-monopole is the same, i.e. we have that asymptotically

$$A_4^a \sim v\delta^{a3} \tag{2.41}$$

This gauge is known as *the abelian gauge* or sometimes *the stringy gauge* because it introduces Dirac string singularities in the gauge field.

Now we ask a question: could we construct other solutions which have the same asymptotic behavior of the Higgs field but are different solutions? The answer is affirmative, as is seen by doing the following. Take a BPS solution of the previous section and replace the parameter v by

$$\bar{v} = 2\pi/L - v\ , \tag{2.42}$$

so that the asymptotic Higgs field A_4 has the behavior $A_4^a \sim \bar{v}\hat{r}^a$. Such a solution has topological charge $Q = \frac{\bar{v}L}{2\pi}$.

Now take the gauge transformation U_- discussed above. The asymptotic Higgs field becomes

$$A_4 \sim \bar{v}\frac{\hat{r}\cdot\boldsymbol{\tau}}{2} \to A_4' = U_-^\dagger A_4 U_- \sim -\bar{v}\frac{\tau^3}{2}\ . \tag{2.43}$$

Next we can take the x_4-dependent gauge transformation $V(x_4) = e^{-i\frac{\pi x_4}{L}\tau^3}$. Notice that this gauge transformation is anti-periodic, i.e. that $V(0) = -V(L)$. Since the gauge fields are in the adjoint representation, they will remain periodic and this gauge transformation is allowed[13]. The gauge transformation acts on the Higgs as

$$A_4' \sim -\bar{v}\frac{\tau^3}{2} \to A_4'' = V^\dagger A_4' V + V^\dagger \partial_4 V \sim v\frac{\tau^3}{2}\ . \tag{2.44}$$

We have found a new solution which has the same asymptotic in the abelian gauge as our two original BPS solutions! The cost of constructing such a solution was to introduce a x_4-dependent gauge twist in a solution with the parameter $\bar{v} = 2\pi/L - v$. This gauge twist does not affect the asymptotics of the gauge fields, as it is solely in the unbroken

[13] The same is not true for fermions in the fundamental representation, which has a great consequence on whether or not the monopole solution has a fermionic zero mode. We will discuss these issues in Section 2.4

$U(1)$ sector. However the monopole has a non-abelian core, and the gauge twist affects the gauge fields in this core. For this reason this monopole is often referred to as the *Kaluza-Klein monopole* or KK monopole[14]. This explains the different topological charge $Q_{KK} = \frac{\bar{v}L}{2\pi}$ and a different action $S_{KK} = \frac{4\pi\bar{v}L}{g^2}$.

Another curious thing is that the topological charge of the BPS monopole and the KK monopole add to unity: $Q_{BPS} + Q_{KK} = \frac{(v+\bar{v})L}{2\pi} = 1$. One could therefore be tempted to conclude that the superposition of these objects makes an instanton, or a caloron[15]. In fact this was shown by an explicit construction of the caloron with nontrivial asymptotic Polyakov loop in the papers [66, 67]. The construction used the ADHM construction [12] to obtain an array of instantons on $\mathbb{R}^4$ along the x_4 direction, which differ by a gauge transformation $e^{i\frac{vL}{2}\tau^3}$. In this construction the gauge fields go to zero at infinity, so in particular $A_4 \to 0$,$(r \to \infty)$. By gauge transforming the construction with $U(x_4) = e^{-i\frac{v}{2}\tau^3 x_4}$ however the asymptotic value of the Higgs field becomes $A_4^\infty = v\frac{\tau^3}{2}$ and the gauge fields become periodic with the period L. The Nahm transformation [75] was then used to find the solution of the ADHM constraint. The solution turned out to be comprised out of two monopoles, one time-independent, and the other time dependent. These precisely correspond to the KK and BPS monopoles discussed previously.

Since it would take us too far off track and since the details are not very illuminating, we do not perform this construction which can be found in the original papers [66, 67]. We will however give some explicit formulas for the caloron configuration in the next section and check that they correspond to a system of a BPS and KK monopole by looking at some limits.

2.3 The caloron

Here we give an overview of the caloron solution with nontrivial holonomy which was constructed in [66, 67], and show its connection to the BPS and KK monopoles discussed earlier. Our notation and conventions are mostly from [66].

[14]The *winding modes* in the compact direction are usually referred to as the Kaluza-Klein modes, as opposed to the Matsubara modes.

[15]The term caloron might not be entirely appropriate here as these are not the objects which appear in the high-temperature phase of pure YM or QCD. They are however generalization of such objects to $|A_4| \neq 0 \mod 4\pi/L$. Further we have made no restriction on the boundary conditions of other fields, i.e. matter fields and fermions. In fact it is possible to compactify the theory non-thermally, i.e. endowing fermions (and scalars) with periodicity up to a phase, as we will discuss in the next Chapter. Nonetheless the gauge field is always kept periodic and therefore one can still use the term caloron for these objects.

The gauge field of the caloron[16] is given by

$$A_\mu = -\frac{1}{2}\bar{\eta}^3_{\mu\nu}\tau_3\partial_\nu \ln\phi - \frac{1}{2}\phi\,\mathrm{Re}\left((\bar{\eta}^1_{\mu\nu} - i\eta^2_{\mu\nu})(\tau_1 + i\tau_2)((\partial_\nu + v\delta_{40})\tilde{\chi})\right) \tag{2.45}$$

where τ_i are the Pauli matrices, $\mathrm{Re}(M) = \frac{1}{2}(M + M^\dagger)$ and where $\phi, \tilde{\chi}$ are given by, at fist site, horribly looking expressions of mostly hyperbolic functions

$$\phi = \frac{\psi}{\hat{\psi}} \tag{2.46a}$$

$$\tilde{\chi} = \frac{\pi\rho^2}{\psi}\left\{e^{-2\pi i x_4}\frac{\sinh(4\pi s\omega)}{s} + \frac{\sinh(4\pi r\bar{\omega})}{r}\right\} \tag{2.46b}$$

$$\begin{aligned}\psi = &-\cos(2\pi x_4) + \cosh(4\pi r\bar{\omega})\cosh(4\pi s\omega) + \frac{r^2 + s^2 + \pi^2\rho^4}{2rs}\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega) \\ &+ \pi\rho^2\left(\frac{\sinh(4\pi s\omega)\cosh(4\pi r\bar{\omega})}{s} + \frac{\sinh(4\pi r\bar{\omega})\cosh(4\pi s\omega)}{r}\right)\end{aligned} \tag{2.46c}$$

$$\begin{aligned}\hat{\psi} = &-\cos(2\pi x_4) + \cosh(4\pi r\bar{\omega})\cosh(4\pi s\omega) \\ &+ \frac{r^2 + s^2 - \pi^2\rho^4}{2rs}\sinh(4\pi r\bar{\omega})\sinh(4\pi s\omega)\end{aligned} \tag{2.46d}$$

It is important to note that the above expression is for a caloron with its constituents lying along the x_3-axis. In the above equations ρ is the caloron size parameter, while, as we shall see, r and s can be interpreted as the distances from the observation point to the constituent KK and BPS monopole centers respectively. They are related to the size parameter ρ by noting that the distance between the constituent monopoles is given by $d = \pi\rho^2$, so that by placing one of the monopoles at the center of the coordinate system, r becomes the length of the observation point vector and $s = |\boldsymbol{r} - \hat{x}^3 d| = \sqrt{r^2 + d^2 - 2rd\cos\theta}$ where θ is the azimuthal angle.

Let us see that this intimidating expression indeed reduces to the monopole when we send one of the constituents to infinity, i.e. $d \to \infty$. It is not very difficult to show that in this limit

$$\phi \approx \frac{2d}{r}\frac{1}{\coth(\bar{v}r) - \cos\theta}\,, \qquad \psi \approx \frac{d}{r}\sinh(r\bar{v})e^{vd}\,, \qquad \tilde{\chi} \approx \frac{r}{d}\frac{e^{-i\bar{v}x_4}}{\sinh(r\bar{v})}\,, \tag{2.47}$$

Further since

$$\frac{\partial r}{\partial x^i} = \hat{r}^i\,, \qquad \frac{\partial\theta}{\partial x^i} = \frac{1}{r}\hat{\theta}^i \tag{2.48}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are the unit vectors in the radial direction, and along the θ-direction, we

[16] The radius of the compact circle L is set to unity in this section. It can always be reinstated on dimensional grounds.

have that

$$\partial_i \ln\phi = -\left(\frac{1}{r} + \frac{1}{\sinh^2(\bar{v}r)}\frac{\bar{v}}{\coth(\bar{v}r)-\cos\theta}\right)\hat{r}^i - \frac{1}{r}\frac{\sin\theta}{\coth(\bar{v}r)-\cos\theta}\hat{\theta}^i \tag{2.49}$$

Note that $\tilde{\chi}$ field vanishes exponentially fast for $r\bar{v} \ll 1$, so that the $\tilde{\chi}$ dependent part in (2.45) is irrelevant at large distances. Then the gauge field reduces to

$$A_i \approx \frac{1}{2r}\epsilon_{ij3}\tau_3(\hat{r}^j + \cot\frac{\theta}{2}\hat{\theta}^j)\ , \qquad A_4 \approx \left(v + \frac{1}{r}\right)\frac{\tau^3}{2} \tag{2.50}$$

or

$$A_r = A_\theta = 0\ , \qquad A_\varphi = -\frac{1}{2r}\cot\frac{\theta}{2}\tau^3\ , \qquad A_4 = \left(v + \frac{1}{r}\right)\frac{\tau^3}{2}\ . \tag{2.51a}$$

The field is clearly (quasi-)abelian and has the structure of the Dirac monopole with the Dirac-string singularity along the $\theta = 0$ half-axis with electric and magnetic fields given by

$$E_i \approx -\frac{\hat{r}^i}{r^2}\frac{\tau^3}{2}\ , \qquad B_i \approx -\frac{\hat{r}^i}{r^2}\frac{\tau^3}{2} \tag{2.52}$$

So the field is (asymptotically) self-dual and has an anti-monopole-like character. Notice that the monopole has x_4 dependence for $r \lesssim \bar{v}$ which comes solely from the function $\tilde{\chi}$ which is periodic in time. Therefore this is the field of the twisted KK monopole.

If we had sent the other constituent to infinity, i.e. if $r \to \infty$ and s is kept fixed, the electric and magnetic fields would both reverse the signs, and the $\tilde{\chi}$ function would be independent of x_4. We can then claim that the caloron is composed out of a *monopole* and an anti-monopole. The long range fields of a caloron have a dipole character.

So we see that the limit of the infinite-size caloron is a monopole with expected asymptotics from our analysis of the previous section. However since the caloron satisfies the first-order self-duality equations, by uniqueness these have to be the same solution even inside the core. Therefore the caloron is an object which comprises the BPS and the KK instanton-monopole, i.e. the pair $BPS + KK$ monopole and the caloron are the same object!

It is well known that instantons have highly localized fermionic zero modes. The question is then if an $SU(2)$ instanton breaks into two monopoles, how do the zero modes distribute themselves? This question is answered by the *index theorem* on $\mathbb{R}^3 \times S^1$, which is the topic of the next section.

2.4 Index theorem and fractional topology

The index theorem is a powerful theorem discovered by Atiyah and Singer (AS) on compact manifolds without boundary in their seminal works [16] and later extended to manifolds with boundary by Atiyah, Patodi and Singer (APS) [15, 14, 13]. Specifically, the index theorem on $\mathbb{R}^3 \times S^1$ was considered in [79]. Here, however, we derive the index theorem mostly following [112, 89] (see also [78]).

The main statement of the AS index theorem for the Dirac operator on four dimensional (compact) manifolds is that the following equation holds

$$Q = N_L - N_R \tag{2.53}$$

where Q is the topological charge of the background field and $N_{R,L}$ are the number of right (R) and left (L) zero-modes of the Dirac operator. The quantity $N_R - N_L$ is usually referred to as *the index*[17] of the Dirac operator $\not{D}$. More precisely for a Dirac operator (see Appendix A for our conventions)

$$\not{D} = \begin{pmatrix} 0 & \mathcal{D} \\ \bar{\mathcal{D}} & 0 \end{pmatrix} \tag{2.54}$$

where $\mathcal{D} = \sigma^\mu D_\mu$ and $\bar{\mathcal{D}} = \bar{\sigma}^\mu D_\mu$ with $\mathcal{D} = -\bar{\mathcal{D}}^\dagger$. The index is defined as

$$Index = \dim\ker\mathcal{D} - \dim\ker\bar{\mathcal{D}} = N_R - N_L \tag{2.55}$$

where "ker" denotes the *kernel* of an operator (i.e. the set of states annihilated by the operator).

To quantify the index, consider the expression

$$I(m^2) = \mathrm{Tr}\left(\frac{m^2}{-\not{D}^2 + m^2}\gamma_5\right) . \tag{2.56}$$

Since

$$\not{D}^2 = \begin{pmatrix} \mathcal{D}\bar{\mathcal{D}} & 0 \\ 0 & \bar{\mathcal{D}}\mathcal{D} \end{pmatrix} \tag{2.57}$$

the operator $\not{D}^2$ is chiral (i.e. it commutes with γ_5), and eigenstates of $\not{D}^2$ separate into

[17] The reader will be careful with the overall sign which is purely a matter of convention and is not fixed across the literature.

left and right eigenstates. Therefore

$$I(m^2) = \mathrm{Tr}\left(\frac{m^2}{-\not{D}^2+m^2}\gamma_5\right) = \mathrm{Tr}\left(\frac{m^2}{-\mathcal{D}\bar{\mathcal{D}}+m^2}\right) - \mathrm{Tr}\left(\frac{m^2}{-\bar{\mathcal{D}}\mathcal{D}+m^2}\right) = \\ = \sum_n \frac{m^2}{\lambda_n^{R2}+m^2} - \sum_n \frac{m^2}{\lambda_n^{L2}+m^2}\ , \quad (2.58)$$

where λ_n^{L,R^2} are eigenvalues[18] of operators $-\mathcal{D}\bar{\mathcal{D}}$ and $-\bar{\mathcal{D}}\mathcal{D}$ respectively. If we take the limit $m^2 \to 0$, it is clear that nonzero eigenvalues will not contribute to the quantity (2.56), while the zero eigenvalues will contribute unity exactly for all values of m^2. Moreover we have that $\ker\mathcal{D} \subset \ker\bar{\mathcal{D}}\mathcal{D}$ and $\ker\bar{\mathcal{D}} \subset \ker\mathcal{D}\bar{\mathcal{D}}$. This is because a zero mode of $\mathcal{D}$ and $\bar{\mathcal{D}}$ is also the zero mode of $\bar{\mathcal{D}}\mathcal{D}$ and $\mathcal{D}\bar{\mathcal{D}}$, but the converse is also true, and here is why: Assume that ψ is a zero mode of $\bar{\mathcal{D}}\mathcal{D}$, i.e. $\bar{\mathcal{D}}\mathcal{D}\psi = 0$. Multiplying this equation from the left by $\psi^\dagger$ and integrating we get that the norm $||\mathcal{D}\psi||^2 = 0$, so that $\mathcal{D}\psi = 0$. In other words a zero mode of $\bar{\mathcal{D}}\mathcal{D}$ is also a zero mode of $\mathcal{D}$. In the same way one can show that a zero mode of $\mathcal{D}\bar{\mathcal{D}}$ implies a zero mode of $\bar{\mathcal{D}}$.

This allows us to write the index as

$$N_R - N_L = \lim_{m^2\to 0} \mathrm{Tr}\left(\frac{m^2}{-\not{D}^2+m^2}\gamma_5\right) = \lim_{m^2\to 0} I(m^2)\ . \quad (2.59)$$

because the zero modes of $\mathcal{D}\bar{\mathcal{D}}$ has the same amount of zero modes as the number of left-handed zero modes of $\not{D}$ and $\bar{\mathcal{D}}\mathcal{D}$ has the same amount of zero modes as the number of right-handed zero modes of $\not{D}$.

The limit of $m^2 \to 0$ in the expression (2.59) is difficult to take. The usual argument is that the above expression is actually independent of m^2. This is argued by showing that the non-zero-mode spectra of the operators $\mathcal{D}\bar{\mathcal{D}}$ and $\bar{\mathcal{D}}\mathcal{D}$ are the same. This is in general not the case (see discussion in [110, 111]). In Appendix F.1 we show that there is an m^2 dependent part coming from the boundary of the space and obtain formulas (F.12) and (F.13), which we repeat here for convenience

$$Index = I_S(0) + I_B \quad (2.60)$$

where we have defined

$$I_S(m^2) = \frac{1}{2}\oint dS_\mu j_5^\mu \quad (2.61a)$$

$$I_B = \lim_{M^2\to\infty} \mathrm{Tr}\left(\gamma_5 \frac{M^2}{-\not{D}^2+M^2}\right) \quad (2.61b)$$

[18] Note that these eigenvalues are positive definite because the operators $\not{D}^2$ is a square of an anti-Hermitian operator, so it has strictly negative eigenvalues.

where j_5^μ is the chiral current (F.2). The index function $I(m^2)$ separates into the m-independent bulk part I_B and into an m-dependent surface contribution.

Note that all the m^2 dependence of the index function is in the surface term, as the j_5^μ depends on m. The formula (F.11) is much easier to compute than the form (2.59) as it consists of a surface contribution of the chiral current where the field-strength falls off to zero and the bulk contribution with infinite mass M, which can be computed in the gradient expansion. We start with the latter, *bulk* contribution and examine

$$I(M^2) = \mathrm{Tr}\,\gamma_5 \frac{M^2}{-\not{D}^2 + M^2}\ . \tag{2.62}$$

for M large. Since

$$\not{D}^2 = D_\mu^2 - \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \tag{2.63}$$

we have that the index function

$$\begin{aligned} I(M^2) &= \mathrm{Tr}\,\gamma_5 \frac{M^2}{-D_\mu^2 + \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} = \\ &= \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\,\gamma_5 \frac{M^2}{-(D_\mu + ik_\mu)^2 + \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + M^2} \end{aligned} \tag{2.64}$$

where we have taken the trace in the plane-wave basis, and where "tr" stands for the trace in color and Dirac indices. We now want to make a large M expansion. As we will see in a moment this is particularly simple because of the γ_5 matrix which ensures that the only term which contributes in this limit is the term with 4 gamma matrices, i.e. the term $\mathrm{tr}\,[F_{\mu\nu}F_{\rho\sigma}\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \propto \mathrm{tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu})$, which is precisely the topological density. More precisely, we can rewrite the above expression as

$$I(M^2) = \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\,\gamma_5 \frac{M^2}{k^2 + M^2 + \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \cdots} \tag{2.65}$$

where the dots include terms which will not contribute under the γ_5 trace. Now we expand for large M

$$\begin{aligned} &\int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\,\gamma_5 \frac{M^2}{k^2+M^2} \sum_n \left(\frac{1}{k^2+M^2}\right)^n \left(-\frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + \cdots\right)^n = \\ &\qquad = -\int d^4x \int \frac{d^4k}{(2\pi)^4} \frac{M^2}{(k^2+M^2)^3} \frac{1}{4} \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho F_{\mu\nu}F_{\sigma\rho} + \cdots = \\ &= -\frac{1}{(2\pi)^4} \int d^4x \int \frac{d^4\xi}{(1+\xi^2)^3} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma} = -\frac{1}{16\pi^2} \int d^4x\ \mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} = -2T(\mathcal{R})Q \end{aligned} \tag{2.66}$$

where $T(\mathcal{R})$ is defined as $\operatorname{tr} T^a T^b = T(\mathcal{R})\delta^{ab}$ for arbitrary representation $\mathcal{R}$ of the generators T^a. When the topological charge is integer (e.g. when the manifold has no boundary), this is all, and the above expression is the usual statement of the index theorem. For fractional topology, however, this cannot be true as the expression on the RHS is non-integral. As we will soon see, the surface contribution will conspire with the above term to make the index an integer.

Now we turn to the evaluation of the surface contribution, which can be written as

$$\begin{aligned} 2I_S(m) = \oint dS^\mu j^5_\mu = \oint dS^\mu \int \frac{d^4k}{(2\pi)^4}\, i\operatorname{tr}\gamma^\mu\gamma_5 \frac{1}{i(\not{D}+i\not{k})+im} = \\ = \oint dS^\mu \int \frac{d^4k}{(2\pi)^4}\, \operatorname{tr}\gamma^\mu\gamma_5(\not{D}+i\not{k})\frac{1}{-(\not{D}+i\not{k})^2+m^2} \end{aligned} \tag{2.67}$$

Since

$$-(\not{D}+i\not{k})^2 = -(D_\mu+ik_\mu)^2 + \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \tag{2.68}$$

and assuming that $F_{\mu\nu}\to 0$ at large distances, to leading order we have

$$\begin{aligned} 2I_S(m^2) = -\frac{i}{2}\oint dS^\mu \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\ \times \operatorname{tr}(D_\nu+ik_\nu)\frac{1}{-(D_\mu+ik_\mu)^2+m^2}F_{\rho\sigma}\frac{1}{-(D_\mu+ik_\mu)^2+m^2}\ . \end{aligned} \tag{2.69}$$

where “Tr” is the trace over the gamma matrices and “tr” the trace over the color matrices. Taking the trace over the gamma matrices we have

$$\begin{aligned} 2I_S(m^2) = -2i\epsilon^{\mu\nu\rho\sigma}\int dS^\mu \int \frac{d^4k}{(2\pi)^4} \\ \times \operatorname{tr}\left[(D_\nu+ik_\nu)\frac{1}{-(D_\mu+ik_\mu)^2+m^2}F_{\rho\sigma}\frac{1}{-(D_\mu+ik_\mu)^2+m^2}\right]\ . \end{aligned} \tag{2.70}$$

Since we are particularly interested in the index theorem for the instanton-monopoles, we specialize to the manifold $\mathbb{R}^3\times S^1$, and let the 4-component be in the compact direction. Then we must replace $\int\frac{dk_4}{2\pi}\to\frac{1}{L}\sum_n$ and $k_4\to 2\pi n/L$, where L is the compact radius. Further we assume that the asymptotic fields A_μ do not depend on the 4-coordinate and that the asymptotic A_4^∞ field is constant, while all spatial derivatives acting on F_{ij} and A_4 we drop as these are sub-leading at infinity. The above expression then takes the form

$$2I_S(m^2) = -2\int dS^i\epsilon^{ijk}\int\sum_n\frac{d^3k}{(2\pi)^3}\operatorname{tr}\left(F_{jk}\frac{A_4^\infty-\frac{2\pi n}{L}}{\left((A_4^\infty-\frac{2\pi n}{L})^2+k_i^2+m^2\right)^2}\right)\ . \tag{2.71}$$

Integrating over k_i we have

$$I_S(m^2) = -\frac{1}{8\pi}\sum_n \int dS^i \epsilon^{ijk}\mathrm{tr}\left(F_{jk}\frac{A_4^\infty - \frac{2\pi n}{L}}{\sqrt{(A_4^\infty - \frac{2\pi n}{L})^2 + m^2}}\right) \tag{2.72}$$

From the above expression it is clear that the surface contribution is non-vanishing only if asymptotically we have $F_{ij} \sim 1/r^2$, i.e. if the system contains monopole-like charges. So far the only assumptions about the background we made is that it is asymptotically static and that its field strength decays to zero at large distances. Notice that under these very limited assumptions we can extract a very direct consequence for the background fields with fractional topological charge. For such fields the above expression must contribute to make the index an integer. So the objects with fractional topological charges have a monopole character, but notice that the converse need not be true.

Now let us compute the bulk contribution to the index (see Appendix B)

$$I_B = -2T(\mathcal{R})Q = -\frac{1}{16\pi^2}\int dS^\mu\, K_\mu = -\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\int dS^\mu\,\mathrm{tr}\left(A_\nu\partial_\rho A_\sigma - i\frac{2}{3}A_\nu A_\rho A_\sigma\right). \tag{2.73}$$

where K_μ is referred to as the Chern-Simons current. Again, for asymptotically x_4 independent gauge fields we have

$$I_B = -\frac{L}{8\pi^2}\epsilon^{ijk}\int dS^i \mathrm{tr}\,(A_4^\infty F_{jk}) = -\mathrm{tr}\,\frac{LA_4^\infty \hat{B}}{\pi} \tag{2.74}$$

while the surface contribution can formally be written as

$$I_S(0) = -\mathrm{tr}\left(\hat{B}\sum_n \mathrm{sign}\left(\frac{A_4^\infty L}{2\pi} - n\right)\right) = 2\mathrm{tr}\left(\hat{B}\left(\frac{A_4^\infty L}{2\pi} - \hat{B}\left\lfloor\frac{A_4^\infty L}{2\pi}\right\rfloor\right)\right) \tag{2.75}$$

where $\lfloor x\rfloor$ denotes the integer part of x, we defined $\lim_{r\to\infty}\hat{r}^i\epsilon^{ijk}F_{jk}r^2 = \hat{B}$ and where we have regulated the sum as (see Appendix H)

$$\sum_n \mathrm{sign}\,(a-n) = \lim_{s\to 0}\sum_n \frac{\mathrm{sign}(a-n)}{|a-n|^s} = 1 - 2(a - \lfloor a\rfloor) \tag{2.76}$$

Finally, the index is given by

$$Index = I_B + I_S(0) = -2\mathrm{tr}\left(\hat{B}\left\lfloor\frac{A_4^\infty L}{2\pi}\right\rfloor\right) \tag{2.77}$$

Notice that so far we have not used abelianization of any sort, and only assumed that the background configuration has monopole-like asymptotics, with $\hat{B}$ being a non-abelian monopole charge matrix. In the case of an instanton-monopole the charge matrix $\hat{B}$ has unit eigenvalues and commutes with the holonomy A_4^∞, so the above expression is an integer.

Let us now specialize to the case of the SU(2) abelian BPS monopole in the stringy gauge where

$$B^{i,\infty} = \frac{1}{2}\epsilon^{ijk}F_{jk}^\infty = \frac{\hat{r}^i}{r^2}T^3 \ , \tag{2.78}$$

$$A_4^\infty = vT^3 \ . \tag{2.79}$$

T^3 is the generator in the representation $\mathcal{R}$. The index is

$$I_{BPS} = -2\mathrm{tr}\left(T^3\left\lfloor\frac{vL}{2\pi}T^3\right\rfloor\right) \tag{2.80}$$

If we had considered fermions which are periodic up to a phase $\psi(L) = e^{i\varphi}\psi(0)$ we only need to change the formula (2.80) so that $vT^3 \to vT^3 + i\varphi/L$, i.e.

$$I_{BPS} = -2\mathrm{tr}\left(T^3\left\lfloor\frac{vL}{2\pi}T^3 + \frac{\varphi}{2\pi}\right\rfloor\right) \tag{2.81}$$

For the fundamental representation this reduces to

$$I_{BPS}^f = -\left\lfloor\frac{vL+2\varphi}{4\pi}\right\rfloor + \left\lfloor\frac{-vL+2\varphi}{4\pi}\right\rfloor \tag{2.82}$$

while for the adjoint representation, the result is

$$I_{BPS}^{adj} = -2\left\lfloor\frac{vL+\varphi}{2\pi}\right\rfloor + 2\left\lfloor\frac{-vL+\varphi}{2\pi}\right\rfloor \ . \tag{2.83}$$

For definiteness we consider a BPS monopole with topological charge between $Q \in [0, 1/2]$, which translates into[19] $vL \in [0, \pi]$.

Notice that for a fixed vL the index for the fundamental fermions changes from -1 to 0 when φ goes from the value less then $vL/2$ to the value greater then $vL/2$. For the adjoint fermions when $\varphi < vL$ the index is -2, while it changes to 0 for $\varphi > vL$.

Now consider the KK monopole. As we discussed in the introduction the BPS and KK monopole constitute an instanton with integer topological charge. We would then expect that the sum of the two indices is just the BPST instanton index which is $I(inst) = -2T(\mathcal{R})Q$. Indeed as we have seen the surface contribution vanishes if there are no monopole-like fields at infinity, and since the BPS and KK monopole have opposite magnetic charges, the field at infinity will be dipole-like and will not contribute

[19] This choice of region for v is motivated by taking the positive Polyakov loop branch $\mathrm{tr}\, L > 0$.

to the index. Keeping this in mind we can simply write

$$I_{BPS} + I_{KK} = -2T(\mathcal{R}) \,. \tag{2.84}$$

so that

$$I_{KK} = -2T(\mathcal{R}) - I_{BPS} \,. \tag{2.85}$$

We will, however, compute this index for the fundamental and the adjoint representation and use the above formulas as a check. As we discussed in the introduction the KK monopole is obtained from the BPS solution by substituting $v \to \bar{v} = 2\pi/L - v$ and then gauge transforming appropriately with an anti-periodic gauge transformation. The periodicity of the adjoint fermions is unaffected by the anti-periodic gauge transformation, and the formulas are the same as before, except that we replace $vL \to \bar{v}L = 2\pi - vL$, so that

$$I^{adj}_{KK} = -4 + 2\left\lfloor \frac{vL+\varphi}{2\pi} \right\rfloor - 2\left\lfloor \frac{-vL+\varphi}{2\pi} \right\rfloor = -4 - I^{adj}_{BPS} \,. \tag{2.86}$$

We see that the index changes from -2 to -4 as φ increases from $\varphi < vL$ to $\varphi > vL$. The sum of the index of the BPS and that of the KK monopole is -4, which is just the BPST instanton index for the adjoint representation and (2.84) is satisfied.

Consider now the KK monopole index for the fundamental representations. This time the gauge twist affects the periodicity of fermions in that it gives them a $\pi\tau^3$ phase. Therefore we must replace $v\frac{\tau^3}{2} \to \bar{v}\frac{\tau^3}{2} - \pi\tau^3$, i.e. $vL \to \bar{v}L - 2\pi = -vL$ in the surface contribution to account for this phase shift by a gauge twist. However the bulk contribution remains unchanged and is given by $I^{KK}_B = \frac{\bar{v}L}{2\pi} = 1 - \frac{vL}{2\pi}$. We can then simply replace $vL \to -vL$ in (2.82) and add unity to account for the bulk contribution in order to get the KK index in fundamental representation, i.e.

$$I^f_{KK} = 1 + \left\lfloor \frac{vL+2\varphi}{4\pi} \right\rfloor - \left\lfloor \frac{-vL+2\varphi}{4\pi} \right\rfloor = 1 - I^f_{BPS} \tag{2.87}$$

Again the sum of BPS and KK index is unity, which is just the BPST instanton index for the fundamental representation and again (2.84) is satisfied.

An analogous theorem in 2D abelian gauge theory exists, which we discuss in details in the Appendix F.2. The main observation that, up to boundary terms[20], the index is

$$I = \frac{\Phi_B}{2\pi} + \text{(boundary terms)} \tag{2.88}$$

where Φ_B is the magnetic flux. The boundary terms above serve so as to make the index an integer.

[20]The boundary terms will not be of much interest to us in the following chapters.

2.5 Supersymmetry and superspace

Here we review some of the formalism of supersymmetry and superspace. We will need this formalism in Section 4.1 when we consider instanton-monopoles in supersymmetric QCD (sQCD) with heavy flavors. This exposition is textbook material, so we will not dwell on ti much. The interested reader is referred to the standard references [114, 55].

To introduce notations we will consider the so-called chiral multiplet of the supersymmetric theory, i.e. a SUSY theory with a single scalar field and one Weyl fermion[21]. The claim is that the Lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 + i\psi_\alpha (\sigma^\mu)^{\alpha\dot{\beta}} \partial_\mu \bar{\psi}_{\dot{\beta}} \tag{2.89}$$

enjoys an internal symmetry[22] which takes fermions into bosons and vice versa. The above Lagrangian requires some explanation. Since we work in Euclidean space the dotted and undotted indices refer to the different $SU(2)$s of the Euclidean "Lorentz" group $SO(4) = SU(2) \times SU(2)$. The fields $\psi_\alpha, \bar{\psi}_{\dot{\alpha}}$ are therefore unrelated fields which transform under different irreducible $SU(2)$ representations of $SO(4)$. This is unlike Minkowski space where the two spinors $\psi, \bar{\psi}$ are related by complex conjugations.

The position of spinor indices is important to keep track of how the objects ψ_α, ψ^α transform, i.e.

$$\psi'_\alpha = U_\alpha{}^\beta \psi_\beta \; , \qquad\qquad \psi'^\alpha = U^*{}_\beta{}^\alpha \psi^\beta \tag{2.90}$$

with $U \in SU(2)$ so that $\psi^\alpha \chi_\alpha$ is and $SO(4)$ invariant. Due to the pseudo-reality of $SU(2)$ we can take $\psi^\beta = \epsilon^{\beta\alpha}\psi_\alpha$ (the sign is conventional) with $\epsilon^{\beta\alpha} = -\epsilon^{\alpha\beta}$. We define $\epsilon^{12} = 1 = -\epsilon_{12}$ so that $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$ and define $\psi_\beta = \epsilon_{\beta\alpha}\psi^\alpha$. Similar relations for the dotted index which transforms under the other $SU(2)$ of $SO(4)$ hold where we take $\bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$ and $\bar{\psi}_{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}$.

Further an object $(\sigma^\mu)^{\alpha\dot{\beta}}$ transforms as a vector under $SO(4)$ transformations. More particularly taking U, V matrices to be in the two different $SU(2)$ groups of $SO(4)$ this object would transform as

$$(\sigma^\mu)^{\alpha\dot{\beta}} \to U^*{}_\gamma{}^\alpha V^*{}_{\dot{\delta}}{}^{\dot{\beta}} (\sigma^\mu)^{\gamma\dot{\delta}} \; . \tag{2.91}$$

[21] Apart from the fact that this is the simplest SUSY multiplet, we will see that it is the only thing we will need to construct the low energy effective theories of supersymmetric Yang-Mills and QCD on $\mathbb{R}^3 \times S^1$ in Section 4.1

[22] A modern view of SUSY is that the supersymmetry is actually a generalization of the Lorentz symmetry, i.e. that it is a translational symmetry in some fermionic coordinates. These can be integrated out from the action exactly and what is left is the remnant of this translational symmetry which looks like an internal symmetry. We will discuss this in the next section on superspace.

Consider now the transformation $V = U^*$. Then the above transformation becomes

$$(\sigma^\mu)^{\alpha\dot{\beta}} \to U^\dagger \sigma^\mu U \ . \tag{2.92}$$

The $\mu = 4$ component remains unchanged, and the spatial components $i = 1, 2, 3$ transform under the $SO(3)$.

On the other hand if we take $V = U$ we have

$$(\sigma^\mu)^{\alpha\dot{\beta}} \to U^\dagger \sigma^\mu U^* \ . \tag{2.93}$$

One can check[23] that the above expression reduces to "boosts", i.e. rotations between the spatial components and "temporal"[24] 4-component.

Similar statements hold for the object $(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}$ (note the position of indices). Now consider the following transformation of the Lagrangian (2.89)

$$\delta\psi_\alpha = i\sqrt{2}\bar{\xi}^{\dot{\beta}}(\bar{\sigma}^\mu)_{\dot{\beta}\alpha}\partial_\mu\phi \tag{2.94a}$$

$$\delta\phi = \sqrt{2}\psi_\alpha\xi^\alpha \tag{2.94b}$$

$$\delta\bar{\psi}_{\dot{\alpha}} = -i\sqrt{2}(\bar{\sigma}^\mu)_{\dot{\alpha}\beta}\xi^\beta\partial_\mu\bar{\phi} \tag{2.94c}$$

$$\delta\bar{\phi} = \sqrt{2}\bar{\xi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \tag{2.94d}$$

where $\xi, \bar{\xi}$ are infinitesimal Grassmann transformation parameters. We now check explicitly that the Lagrangian (2.89) is invariant under the above transformation. Varying the Lagrangian we get (up to partial integration)

$$\delta\mathcal{L} = -(\partial_\mu^2\bar{\phi})\delta\phi - \delta\bar{\phi}(\partial_\mu^2\phi) - i(\partial_\mu\delta\psi_\alpha)(\sigma^\mu)^{\alpha\dot{\beta}} + i\psi_\alpha(\sigma^\mu)^{\alpha\dot{\beta}}(\partial_\mu\delta\bar{\psi}_{\dot{\beta}}) + \partial_\mu(\dots) \tag{2.95}$$

---

[23]Take for example $U = e^{\alpha i\tau^3/2}$. Then one can write for infinitesimal α

$$U^\dagger\sigma^\mu U^* \approx (1 - i\alpha\frac{\tau^3}{2})\sigma^\mu(1 - i\alpha\frac{\tau^3}{2}) = \sigma^\mu - i\frac{\alpha}{2}\{\tau^3, \sigma^\mu\} = \sigma^\mu + \alpha\delta^{\mu 4}\sigma^3 - \alpha\delta^{\mu 3}\mathbb{I}$$

or

$$\sigma^4 \to \sigma^4 + \alpha\sigma^3$$
$$\sigma^3 \to -\alpha\sigma^4 + \sigma^3$$

which are the infinitesimal rotations in the (x_4, x_3) plane.

[24]We will often use the term temporal component for the 4-component, although when we compactify the theory on a circle we will endow the fermions with the same boundary conditions as bosons to preserve supersymmetry. Therefore this compactification is sometimes referred to as "spatial" compactification.

We have that

$$-(\partial_\mu^2\bar{\phi})\delta\phi = -\sqrt{2}(\psi_\alpha\xi^\alpha)(\partial_\mu^2\bar{\phi}) \tag{2.96a}$$
$$-(\partial_\mu^2\phi)\delta\bar{\phi} = -\sqrt{2}(\bar{\xi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}})(\partial_\mu^2\phi) \tag{2.96b}$$
$$-i(\partial_\mu\delta\psi_\alpha)(\sigma^\mu)^{\alpha\dot{\beta}}\bar{\psi}_{\dot{\beta}} = \sqrt{2}\bar{\xi}^{\dot{\beta}}(\bar{\sigma}^\mu)_{\dot{\beta}\alpha}(\sigma^\nu)^{\alpha\dot{\beta}}\bar{\psi}_{\dot{\beta}}\partial_\mu\partial_\nu\phi = \sqrt{2}(\bar{\xi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}})\partial_\mu^2\phi \tag{2.96c}$$
$$i\psi_\alpha(\sigma^\mu)^{\alpha\dot{\beta}}\partial_\mu\delta\bar{\psi}_{\dot{\beta}} = \sqrt{2}\psi_\alpha(\sigma^\mu)^{\alpha\dot{\beta}}(\sigma^\nu)_{\dot{\beta}\gamma}\xi^\gamma\partial_\mu\partial_\nu\bar{\phi} = \sqrt{2}(\psi_\alpha\xi^\alpha)\partial_\mu^2\bar{\phi} \tag{2.96d}$$

The sum of these terms on the LHS is the variation of the action, while the right hand side adds to zero, so the action is invariant under the SUSY transformations (2.94).

Perhaps we should comment on these transformations. They are indeed awkward, as in Euclidean space ψ_α and $\bar{\psi}_{\dot{\alpha}}$ are not complex conjugates of each other. However the second and fourth equation in (2.94) would be expected to be each-others complex conjugates, as they are the transformation of the complex scalar fields. The transformations (2.94) should therefore be thought as an analytical continuation of the Minkowski space transformations where such problems disappear. In the next section where we discuss superspace and superfields we will again be faced with a similar problem. There we will be forced to go to Minkowski space and analytically continue at the end of the day.

2.5.1 Superfields

There is a very beautiful alternative to considering SUSY theories: the formalism of superspace. The formalism introduces, in addition to the bosonic coordinates x^μ, the fermionic coordinates $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$. The supersymmetry transformation is then seen as translations in the supersymmetric coordinates (see [114]), i.e. as $\theta \to \theta + \xi, \bar{\theta} \to \bar{\theta} + \bar{\xi}$. A superfield $\mathcal{F}$ is defined as a function of bosonic and fermionic coordinates $x, \theta, \bar{\theta}$, i.e. $\mathcal{F} = \mathcal{F}(x, \theta, \bar{\theta})$. Any Lagrangian which is built of such superfields and integrated over the Grassmannian coordinates $\int d^2\theta \int d^2\bar{\theta}$ is manifestly supersymmetric because the integrals over Grassmann numbers are invariant under the shifts[25] (see Appendix G).

We will not go into details of this derivation as it is textbook material (see e.g. [114]), and just state that the theory (2.89) can be described by a Lagrangian

$$\mathcal{L}_{chiral} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}(x, \theta, \bar{\theta})\Phi(x, \theta, \bar{\theta}) \tag{2.97}$$

with $\Phi(x, \theta, \bar{\theta})$ and $\bar{\Phi}(x, \theta, \bar{\theta})$ are the so called *chiral superfields*, i.e. fields obeying the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0\ , \qquad D_\alpha\bar{\Phi} = 0\ , \tag{2.98}$$

[25] This is not the entire story, however, as the SUSY transformations also induce translations in the bosonic space (see [114]). This however is also of no consequence because the theory is also translationally invariant.

where $\bar{D}_{\dot{\alpha}}, D_\alpha$ are the supercovariant derivatives defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma_M^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{2.99}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\theta^{\dot{\alpha}}} - i\theta^\alpha(\sigma_M^\mu)_{\alpha\dot{\alpha}}\partial_\mu \tag{2.100}$$

where σ_M^μ are the Minkowski space matrices from [114]. Note that both $\Phi, \bar{\Phi}$ and $D_\alpha, \bar{D}_{\dot{\alpha}}$ are each others complex conjugates.

The above covariant derivatives are chosen to commute with the SUSY generators. The details can be found in [114]. One can easily check that

$$\bar{D}_{\dot{\alpha}}(x^\mu + i\theta_\alpha(\sigma_M^\mu)^{\alpha\dot{\beta}}\bar{\theta}_{\dot{\beta}}) = 0\ . \tag{2.101}$$

So by defining $y = x^\mu + i\theta_\alpha(\sigma_M^\mu)^{\alpha\dot{\beta}}\bar{\theta}_{\dot{\beta}}$ we can satisfy the constraint (2.98) by requiring that the superfield is a function of y and θ only, i.e. that

$$\Phi = \Phi(y,\theta) = \phi(y) + \sqrt{2}\theta_\alpha\psi^\alpha(y) + \theta_\alpha\theta^\alpha F(y)\ , \tag{2.102}$$

Here ϕ, F are scalar fields and ψ_α is the Weyl fermion so the above superfield is a good candidate to reproduce (2.89) as it has a scalar and a Weyl spinor. In addition we have a scalar field F. We will see in a moment that this is a non-propagating, auxiliary field, which will not interest us and we will always integrate it out. It does have a very important role, however, of closing the SUSY algebra off mass-shell (see e.g. [114]).

Now consider the explicitly SUSY invariant Lagrangian

$$\int d^2\theta \int d^2\bar{\theta}\ \bar{\Phi}(\bar{y},\bar{\theta})\Phi(y,\theta)\ , \tag{2.103}$$

built from the chiral superfield $\Phi(y,\theta)$ and anti-chiral superfield $\bar{\Phi}(\bar{y},\bar{\theta})$ where

$$\bar{y}^\mu = x^\mu - i\theta_\alpha(\sigma^\mu)^{\alpha\dot{\beta}}\bar{\theta}_{\dot{\beta}}\ . \tag{2.104}$$

is the complex conjugate of y^μ and the field $\bar{\Phi}(\bar{y},\bar{\theta})$ satisfies the constraint

$$D_\alpha\bar{\Phi}(y,\bar{\theta}) = 0\ . \tag{2.105}$$

Since the integration over the Grassmann coordinates[26] picks up only the terms proportional to $\theta\theta\bar{\theta}\bar{\theta}$, only these terms are of interest to us. The product of the chiral

[26] The derivatives and integrals with respect to fermionic variables are defined in Appendix G

and anti-chiral fields gives

$$\begin{aligned}\bar{\Phi}(\bar{y},\bar{\theta})\Phi(y,\theta) &= \\ &= \theta\theta\bar{\theta}\bar{\theta}\left(\frac{1}{4}\phi\partial_\mu^2\bar{\phi} + \frac{1}{4}\bar{\phi}\partial_\mu^2\phi - \frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}(\partial_\mu\bar{\psi})\bar{\sigma}_M^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}_M^\mu\partial_\mu\psi + F\bar{F}\right) + \dots\end{aligned} \tag{2.106}$$

where the dots are terms which do not contribute after the integration over the Grassmann coordinates. Further [27]

$$\begin{aligned}\mathcal{L} = \int d^2\theta \int d^2\bar{\theta}\ \bar{\Phi}(\bar{y},\bar{\theta})\Phi(y,\theta) &= \\ &= -\frac{1}{4}\phi\partial_\mu^2\bar{\phi} - \frac{1}{4}\bar{\phi}\partial_\mu^2\phi + \frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}(\partial_\mu\bar{\psi})\bar{\sigma}_M^\mu\psi - \frac{i}{2}\bar{\psi}\bar{\sigma}_M^\mu\partial_\mu\psi + F\bar{F}\end{aligned} \tag{2.107}$$

which is equivalent to (2.89) upon analytical continuation to Euclidean space and up to the complex scalar field F which is non-dynamical and can be integrated out.

The superfield language, however, is a powerful tool to construct any supersymmetric field theory. In fact we can generalize the above Lagrangian to

$$\int d^2\theta \int d^2\bar{\theta} K(\Phi,\bar{\Phi}) \tag{2.108}$$

where K is an arbitrary function of $\Phi,\bar{\Phi}$. Note that terms such as

$$\int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}) \tag{2.109}$$

are also allowed in the Lagrangian. The function W is commonly called the superpotential, and K is referred as the Kähler potential due to its connection with Kähler geometry. An interested reader is again referred to [114]. What we have written down is in fact the most general Lagrangian consistent with supersymmetry containing the complex scalar ϕ and a Weyl fermion ψ_α. By the power of supersymmetry the problem boiled down to finding analytical functions of K, W of superfield Φ. What is more the Kähler potential $K(\Phi,\bar{\Phi})$ generates kinetic terms, while the superpotential W generates potential terms for the scalar field ϕ, determining the vacuum structure of the low energy theory. Even better, W depends only on Φ and not $\bar{\Phi}$ and by the power of holomorphy the problem of finding the terms which determine the condensates of the theory has been reduced to finding the function W of a single variable.

There is a remarkably simple formula for computing the scalar potential, which we sketch briefly. After integrating out the super-coordinate $\theta,\bar{\theta}$ we get that the scalar

[27] Note that $\int d^2\theta\ \theta\theta = 1$ and $\int d^2\bar{\theta}\ \bar{\theta}\bar{\theta} = 1$ by convention.

components are

$$\mathcal{L} = \left(\frac{\partial^2 K}{\partial \Phi \partial \bar{\Phi}} F \bar{F} + F \frac{\partial W}{\partial \Phi} + \bar{F} \frac{\partial \bar{W}}{\partial \bar{\Phi}} \right) \Big|_{\Phi=\phi, \bar{\Phi}=\bar{\phi}} + \dots \tag{2.110}$$

where the dots represent terms which depend on fermions[28] ψ. The above Lagrangian is Gaussian in the auxiliary field F and we can integrate it out, which is equivalent to imposing the equations of motion for F. The result is

$$\mathcal{L}_{bos} = -\frac{1}{G} \left| \frac{\partial W}{\partial \Phi} \right|^2 \Bigg|_{\Phi=\phi} \tag{2.111}$$

where

$$G = \frac{\partial^2 K}{\partial \Phi \partial \bar{\Phi}} \Big|_{\Phi=\phi, \bar{\Phi}=\bar{\phi}} . \tag{2.112}$$

This means that the bosonic potential (recall we are in Minkowski space) is given by

$$V_{bos} = \frac{1}{G} \left| \frac{\partial W}{\partial \Phi} \right|^2 \Bigg|_{\Phi=\phi} . \tag{2.113}$$

Finding the bosonic potential for the scalar fields ϕ then reduced to finding the superpotential $W(\Phi)$.

2.6 Spin models, duality and vortices

In this section we review models in two dimensions which are very popular as toy models for the YM theory because of their similarities with non-abelian gauge theories. The models in question are the so-called $O(N)$ *nonlinear sigma model* (or *principal chiral model*) and $CP(N-1)$ model in two dimensions. In particular the $CP(1)$ model and the $O(3)$ model are equivalent, an it is the simplest 2D nonlinear sigma model which has the following features in common with pure YM:

- it is asymptotically free,
- it has a dynamically generated mass gap,
- it has instanton solutions.

The continuum Lagrangian of the $O(3)$ model is nothing but a low energy effective Hamiltonian for the Heisenberg ferromagnet.

[28] These terms also depend on $F, \bar{F}$ but will not contribute to the result that we will now state.

The $O(N)$ generalizations, however, although still gapped dynamically, no longer have a topological charge, and, therefore, no instantons. As a result no θ-angle[29] can be defined in this theory, a feature not shared by $SU(N)$ YM theory.

The $O(3) \equiv CP(1)$ can nevertheless be generalized to $CP(N-1)$ nonlinear sigma model, which has a rigid $SU(N)$ symmetry. The $O(3)$ instanton solution can be embedded in such a model, and topological charge, and therefore the θ-angle, can be defined.

The $O(2)$ model is not asymptotically free, and needs to be defined with some regularization in the UV. Depending on the UV regularization, the model may exhibit vortex and anti-vortex solutions which are able to distort the system.

We will see in Chapter 4 that $O(N)$ models with large chemical potential reduce to an $O(2)$ model, which have vortex solutions. Vortices are able to disorder the system and generate a nonzero correlation length, much like monopoles in the $U(1)$ gauge theory.

This section is divided into two parts: Sec. 2.6.2 and Sec. 2.6.1. In Sec. 2.6.1 we discuss the $O(2)$ model, its dual description and vortices, while in Sec. 2.6.2 we show how chemical potential enters in the $O(N)$ and $CP(N-1)$ model.

2.6.1 The $O(2)$ model

Consider an $O(2)$ sigma model in 2 dimensions, defined with the Lagrangian

$$\mathcal{L} = \frac{1}{2g^2}(\partial_\mu \boldsymbol{n})^2 \tag{2.114}$$

where $\boldsymbol{n} = (n_1, n_2)$ is a two component vector with a constraint $\boldsymbol{n}^2 = 1$. This Lagrangian is nothing but the continuum limit of the classical lattice XY-model Hamiltonian, where 2D rotors live on lattice sites, parametrized by the vector $\boldsymbol{n}_i$ at site i and interact via a term $\boldsymbol{n}_i \cdot \boldsymbol{n}_j$ for neighboring sites i and j (see e.g. textbooks [32, 9]).

We can parametrize the $\boldsymbol{n}$ field by an angle ϕ. Taking $n_1 = \cos\phi, n_2 = \sin\phi$, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2g^2}(\partial_\mu \phi)^2 \; . \tag{2.115}$$

The above Lagrangian has a duality very similar to the $U(1)$ gauge field duality discussed in Section 2.1. As before we wish to write the Lagrangian in terms of a dual variable σ. To do this note that writing $v_\mu = \partial_\mu \phi$, a constraint $\epsilon^{\mu\nu}\partial_\mu v_\nu = 0$ is automatically imposed. We can however introduce a term in the action which imposes this constraint

[29] the θ-angle is a topological term added to the action $\Delta S_\theta = i\theta Q$, where Q is the topological charge. This term is a boundary term, and does not influence the classical equations of motion, but it can influence the quantum theory heavily via quantum interference.

and integrate over v_μ instead, i.e.

$$\begin{aligned} Z = & \int \mathcal{D}\phi \, e^{-\frac{1}{2g^2}\int d^2x \, (\partial_\mu \phi)^2} = \\ = & \int \mathcal{D}v_\mu \mathcal{D}\sigma \, e^{-\int d^2x \left[\frac{1}{2g^2} v_\mu^2 - i\frac{1}{2\pi}\sigma \partial_\mu v_\nu \epsilon^{\mu\nu}\right]} = \\ = & \int \mathcal{D}\sigma e^{-\frac{g^2}{2(2\pi)^2}\int d^2x \, (\partial_\mu \sigma)^2} \end{aligned} \tag{2.116}$$

where in the last step we integrated out the field v_μ. What we have obtained is a dual theory of field σ. Similarly to the $U(1)$ duality, the σ field must be a compact field, as can be seen from considering a correlator of the form[30]

$$\left\langle e^{i\phi(\boldsymbol{x})} e^{-i\phi(\boldsymbol{y})} \right\rangle = \left\langle e^{i\int_{\boldsymbol{y}}^{\boldsymbol{x}} dx^\mu \, v_\mu} \right\rangle \,, \tag{2.117}$$

where we have written the operator as a contour integral from the point $\boldsymbol{y}$ to the point $\boldsymbol{x}$. If we again repeat the steps of the eq. (2.116) we get that the action of the above correlator is infinite unless the field σ has a 2π jump across the interface of the contour, so $\sigma \equiv \sigma + 2\pi$ and σ is an angular variable (see Fig. 2.1).

We have so far assumed that the above theory is defined on a smooth $\mathbb{R}^2$ manifold. In practice, however, a theory is usually defined on the lattice, or at length scales larger then some typical microscopic length a. It happens often that microscopic theories allow for vortex solutions: defects for which the angle-valued field ϕ goes from 0 to 2π as we transverse a circle around some point $\boldsymbol{x}_0$. These objects are characterized by a singularity at $\boldsymbol{x}_0$ in the continuum version:

$$\epsilon^{\mu\nu}\partial_\mu \partial_\nu \phi(x) = 2\pi \delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_0) \,. \tag{2.118}$$

It is easy to see that by integrating the above expression we get

$$\oint dx^\mu \partial_\mu \phi = 2\pi \,, \tag{2.119}$$

where the contour is taken around the point $\boldsymbol{x}_0$.

To account for these "monopoles", we do the same thing that we did in the $U(1)$ gauge theory duality. Namely, notice that an operator insertion $e^{\pm i\sigma(\boldsymbol{x}_0)}$ corresponds to an (anti-)vortex at position $\boldsymbol{x}_0$, just like before. So accounting for monopoles is as simple as introducing a term $-m^2 \cos\sigma$ in the dual Lagrangian[31].

However, as opposed to the $U(1)$ gauge theory, the theory at hand is a 2D theory,

[30]this correlator appears when computing the $\langle \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{y}) \rangle$ correlator, i.e. a spin-spin correlation function.

[31]m is given dimensions by the cutoff of the theory.

and vortices have infrared divergent actions. In fact it is easy to see that $\phi = \varphi$ where φ is the polar angle is a solution of the classical equations of motion $-\partial_\mu^2 \phi = 0$ and represents a vortex. The action is

$$S_{vortex} = \frac{1}{2g^2} \int_a^R r dr \int_0^{2\pi} d\phi \, \frac{1}{r^2} \sim \frac{2\pi}{2g^2} \ln \frac{R}{a} \tag{2.120}$$

where R is the IR cutoff (i.e. the size of the system), and a is the UV cutoff (i.e. lattice spacing, for example). So the vortices cost infinite action in an infinite system.

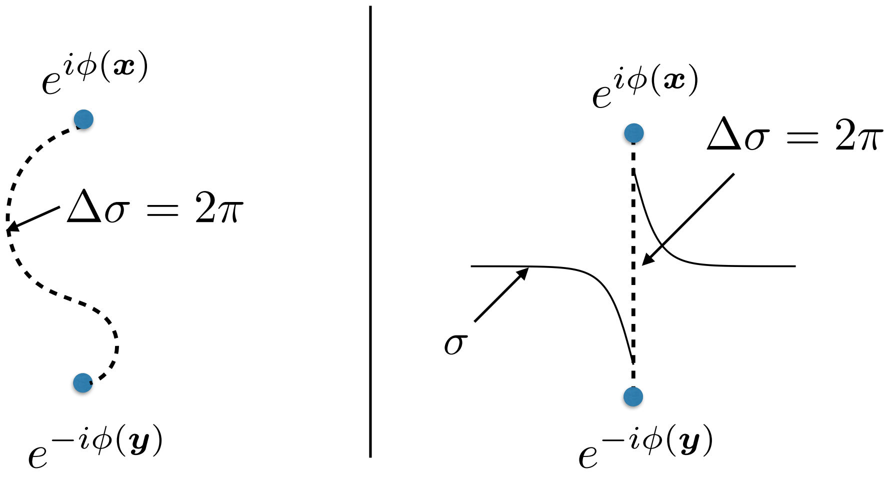

Figure 2.1: A pictorial representation of the correlation function (2.117). Upon integrating out the ϕ field and obtaining the dual theory of σ field, a cut (dashed line) in between the source at $\boldsymbol{x}$ and a source at $\boldsymbol{y}$ is created which demands that the jump of the sigma field across this cut is $\Delta\sigma = 2\pi$. On the right a straight contour was chosen, which has a kink solution with a 2π jump across this controur. The action of the kink is proportional to the length where the jump is required to occur, inducing an exponential decaying correlation, just like in the case of monopoles in the $U(1)$ gauge theory.

On the other hand, the gain in entropy of having a vortex is proportional to the logarithm of the number of ways a vortex can be placed in the system, which is $\ln \frac{A}{a^2} \propto \ln \frac{R}{a}$ where $A \propto R$ is the area of the system. So although a cost in action is infinite in the thermodynamic $R \to \infty$ limit, so is the gain in entropy. What is different is that the vortex action depends on the coupling g^2, while the entropy does not. Therefore if the bare coupling g^2 is sufficiently large, the ionized vortices and anti-vortices become entropically favored and percolate, disordering the system and causing it to undergo a phase transition called the *Kosterlitz-Thouless (KT) transition [65]* also attributed to Berezinsky [20]. The transition is signaled by an algebraic decay of the spin-spin correlators (2.117) for subcritical couplings $g^2 < g_{KT}^2$ and by an exponential decay in the supercritical phase $g^2 > g_{KT}^2$, where g_{KT}^2 is the critical coupling of the KT transition.

If vortices do condense and the $\cos\sigma$ term survives in the IR, an analogue to the area law in $U(1)$ theory with monopoles can be made by looking at the correlator (2.117), and, for simplicity, taking the contour there to be a straight line from $\boldsymbol{x}$ to $\boldsymbol{y}$, we have a scenario which is depicted on the right of Fig. 2.1. For very large separations $|\boldsymbol{x}-\boldsymbol{y}|$ the problem becomes quasi-one dimensional and dependent on one coordinate perpendicular to the line connecting points $\boldsymbol{x}$ and $\boldsymbol{y}$. Just like in the case of $U(1)$ gauge theory, a kink solution exist with the same profile as (2.18) and with an approximate action[32]

$$S \approx \sqrt{2}\frac{g^2 m}{\pi^2}|\boldsymbol{x}-\boldsymbol{y}| + S_{vac} \tag{2.121}$$

generating a linear confining potential between operators $e^{i\phi}$ and $e^{-i\phi}$.

Actually this correlator behavior can be interpreted as the mass of the fields $e^{\pm i\phi}$. As is well known from basic quantum field theory, temporal correlators[33] for massive fields are asymptotically exponential, behaving as $\sim e^{-E_0 t}$ where E_0 is the first excited state, or mass gap. Therefore we can identify $E_0 = \frac{g^2 m}{2\pi^2}$ with the mass gap of the theory.

2.6.2 $CP(N-1)$ and $O(N)$ model with chemical potential

We wish to consider adding chemical potential to the $O(N)$ model defined by the Lagrangian

$$\mathcal{L}^{O(N)} = \frac{1}{2g^2}(\partial_\mu \boldsymbol{n})^2 \tag{2.122}$$

where $\boldsymbol{n}$ is a N dimensional real vector with $\boldsymbol{n}^2 = 1$, and the $CP(N-1)$ model defined by the Lagrangian

$$\mathcal{L}^{CP(N-1)} = \frac{1}{g^2}(D_\mu u)^\dagger (D_\mu u) \tag{2.123}$$

where u is an N dimensional *complex* vector with $u^\dagger u = 1$ and $D_\mu = \partial_\mu - iA_\mu$ where A_μ is a non-dynamical gauge field which can be integrated out. The $O(N)$ model enjoys the global[34] $SO(N)$ symmetry which rotates the unit $\boldsymbol{n}$ vector, while the $CP(N-1)$ enjoys a local $U(1)$ symmetry, $u \to e^{i\alpha(x)}u$ and a global $SU(N)$ symmetry $u \to Uu\ ,\ \ U \in SU(N)$.

To find the conserved current[35] we simply vary the action by a *local* variation $\delta\boldsymbol{n} = O(x)\boldsymbol{n}$, $O(x) \in SO(N)$ for the $O(N)$ case and with $\delta u = U(x)u$, $U(x) \in SU(N)$. Writing these transformations as $O(x) = e^{t^a\phi^a}$ and $U = e^{iT^a\phi^a}$ where t^a are the anti-Hermitian

[32] The coupling in this expression is not the bare coupling, but the renormalized coupling in the IR.

[33] Because of the Euclidean symmetry, whether it is a temporal or spatial correlator does not matter.

[34] although the model enjoys the $O(N)$ symmetry, only the $SO(N)$ part has conserved currents as this corresponds to the continuous rotations of the $\boldsymbol{n}$ vector, while the rest are reflections.

[35] when considering conserved currents and conserved charges it is best to think of the Lagrangian as being the Minkowski space Lagrangian, where charge is conserved in *real* time.

generators of the $SO(N)$ transformation and T^a are the Hermitian generators of the $SU(N)$ transformation, we get

$$\delta S^{O(N)} = \frac{1}{g^2}\int d^2x\, \boldsymbol{n}\cdot(t^a\partial_\mu \boldsymbol{n})\partial_\mu\phi^a = \int d^2x j^\mu_a\partial_\mu\phi^a\ , \tag{2.124}$$

$$\delta S^{CP(N-1)} = \frac{i}{g^2}\int d^2x\, \left(u^\dagger T^a D_\mu u - (D_\mu u)^\dagger T^a u\right)\partial_\mu\phi^a = \int d^2x\, j^\mu_a\partial_\mu\phi^a\ , \tag{2.125}$$

where

$$j^\mu_a = \frac{1}{g^2}\boldsymbol{n}\cdot(t^a\partial_\mu\boldsymbol{n}) \qquad \text{for the } O(N) \text{ model} \tag{2.126}$$

$$j^\mu_a = \frac{-2}{g^2}\mathrm{Im}\left(u^\dagger T^a D_\mu u\right)\ , \qquad \text{for the } CP(N-1) \text{ model.} \tag{2.127}$$

If $\boldsymbol{n}$ and u obey the equations of motion, then the actions are invariant under any transformation, and since ϕ^a is arbitrary it follows that the currents j^a_μ are conserved, so there are conserved charges[36]

$$Q^a = \int dx\, j^a_0\ . \tag{2.128}$$

Therefore one can consider charged thermal systems with the partition function

$$\mathcal{Z} = \mathrm{tr}\, e^{-\beta H + Q^a\mu^a\beta}\ . \tag{2.129}$$

By the standard methods, the above partition functions can be written in terms of the path integrals with Euclidean actions given by[37]

$$S^{O(N)} = \frac{1}{2g^2}[(\partial_\nu - i\delta_{\nu 0}\mu^a t^a)\boldsymbol{n}]^2\ , \tag{2.130}$$

$$S^{CP(N-1)} = \frac{1}{2g^2}[(D_\mu + \delta_{\nu 0}\mu^a T^a)u]^\dagger[(D_\mu - \delta_{\nu 0}\mu^a T^a)u]. \tag{2.131}$$

Notice that the above Lagrangians both have imaginary parts proportional to μ^a. This is a generic feature of theories with chemical potential which cause the infamous *sign problem* when attempting to simulate them on the lattice.

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[36]we will use the 0 component to denote both the Euclidean and Minkowski time in 2D models.

[37]note that in the case of the $CP(N-1)$ model, there is a subtlety with the canonical quantization which is present in all gauge theories. Namely the Gauss law has to be imposed in the partition function. In this case since the gauge fields are non-dynamical, the Gauss law is a statement that $U(1)$ charges are exactly zero for all physical states. This is why $U(1)$ chemical potential does not change the partition function at all, and there is no sense in adding it.

Chapter 3

Zero modes, charged systems and magnetic field(s)

In this Chapter we will discuss the fermionic zero modes which appear in the backgrounds with non-trivial topology, and is based on our works [24, 28]. The Chapter is organized as follows: In Section 3.1 we consider abelian gauge theories in 2+1 dimensions and show how the nontrivial interplay between iso-spin chemical potential, and magnetic and pseudo-magnetic fields affects the system. This setup has direct application to strained graphene with which we conclude the section. In Section 3.2 we explain a similar phenomenon in 3+1 dimensions occurring in the instanton-monopole and anti-monopole system in abelian magnetic field. In Section 3.3 we discuss the Dirac operator zero modes for finite chemical potential in the instanton-monopole as well as in caloron background and the constriction of the *hopping matrix element* important for simulations of instanton-monopole ensembles. We conclude the chapter with the summary of the results in Section 3.4.

3.1 2+1 dimensional system and charge catalysis

3.1.1 (2+1)D and electric charge catalysis

In this section we describe the system which will help us understand the case of the caloron in the magnetic field. The setup we will consider here is a system with $U(1)\times U(1)$ gauge fields in 2+1 dimensions. The one particle Dirac Hamiltonian in 2+1 dimensions is given by

$$H = i\not{D} \ , \qquad \not{D} = \sigma_1 D_1 + \sigma_2 D_2 \ , \tag{3.1}$$

where $\sigma^{1,2}$ are the usual Pauli matrices and where D_i is

$$D_i = \partial_i - iA_i - i\tilde{A}_i\tau^3 = \partial_i - i\mathcal{A}_i \ . \tag{3.2}$$

Here $i = 1, 2$ and $A_i, \tilde{A}_i$ are the gauge fields of $U(1) \times U(1)$ gauge group, while the τ^3 matrix is a Pauli matrix acting on an internal index which we refer to as iso-spin[1]. Let us now consider a background of the gauge fields for which the magnetic field is

$$\mathcal{B} = B + F\tau^3 \ , \qquad B = \epsilon^{ij}\partial_i A_j \ , F = \epsilon^{ij}\partial_i \tilde{A}_j \ , \tag{3.3}$$

where $\epsilon^{ij} = -\epsilon^{ji}$ is the totally antisymmetric tensor with $\epsilon^{12} = 1$ and $\mathcal{B} = \epsilon^{ij}\partial_i \mathcal{A}_j$ is the total magnetic field.

If B, F are constant magnetic fields it is well known that the spectrum of the Hamiltonian is given by

$$H\psi_n^{\pm} = E_n^{\pm}\psi_n^{\pm} \ , \qquad \tau^3\psi^{\pm} = \pm\psi^{\pm} \ . \tag{3.4}$$

with

$$E_n^{\pm} = \pm\sqrt{2n|B \pm F|} \ . \tag{3.5}$$

The sign above comes from the fact that the Hamiltonian anti-commutes with the matrix σ_3, i.e. $\{H, \sigma^3\} = 0$ and therefore the energy eigenstates always come in $\pm E_n$ pairs (except for $n = 0$ states) which, upon second quantization, will become degenerate particle-antiparticle states[2]. We however consider these states as degenerate spin-up and spin-down states (although they should really be viewed as spin-up particle and spin down-hole, i.e. absence of spin up particle) and when we consider particle/hole we will implicitly assume that there is another quantum number with spin-down particles and spin-up holes, so that the Hamiltonian acts on 4-spinors. Incidentally this is precisely what happens in the case of graphene where another index, called valley index exists.

Therefore, introducing the chemical potential μ, as well as the iso-spin chemical potential μ_3 these fermions in arbitrary background would have a partition function

$$\begin{aligned}\mathcal{Z} = \sum_n \Big[& g_n^+ \ln(1 + e^{(\mu+\mu_3-E_n^+)\beta}) + g_n^- \ln(1 + e^{(\mu-\mu_3-E_n^-)\beta}) \\ & + \bar{g}_n^+ \ln(1 + e^{(-\mu-\mu_3-E_n^+)\beta}) + \bar{g}_n^- \ln(1 + e^{(-\mu+\mu_3-E_n^-)\beta}) \Big]\end{aligned} \tag{3.6}$$

The terms in the first line correspond to the particle states with iso-spin $\tau^3 = 1$ and $\tau^3 = -1$, and degeneracies $g_n^{\pm} = g_n^{\tau^3=\pm1}$, while the second line corresponds to anti-particle states with degeneracies $\bar{g}_n^{\pm} = \bar{g}_n^{\tau^3=\pm1}$. In the case of magnetic fields (3.3) with constant F and B the energy levels are given by $E_n^{\pm} = \sqrt{2n(B \pm F)}$ and the degeneracies are $g_n^{\pm} = s_n \frac{|\Phi_B \pm \Phi_F|}{2\pi}$, where $\Phi_{F,B}$ are the fluxes of the fields defined as $\Phi_F = FA$, $\Phi_B = BA$ with $A-$the area of the 2D system, while $s_0 = 1, s_{n\neq0} = 2$ is the

[1]This index will be the color index when we discuss the KvBLL caloron in the magnetic field, as this objects breaks the $SU(2) \to U(1)$ spontaneously. The same index will have a more direct interpretation as the *valley* index in graphene.

[2]For zero energy states this degeneracy is not there, and therefore the zero modes are half as degenerate as excited states.

spin degeneracy. Taking $\mu_3 > 0$ (but smaller than the first excited level) and the zero temperature limit $\beta \to \infty$ for simplicity[3], taking $\Phi_F > 0, \Phi_B > 0$ for simplicity, the charge in the system is

$$\langle Q \rangle = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln \mathcal{Z} = \frac{|\Phi_B + \Phi_F| - |\Phi_B - \Phi_F|}{2\pi} = \frac{A\min(|\Phi_B|, |\Phi_F|)}{\pi} \tag{3.7}$$

or that the charge per area σ is given by

$$\sigma = \frac{\min(|B|, |F|)}{\pi} \tag{3.8}$$

In other words the different degeneracies between particle states of opposite iso-spins have induced the charge in the system. This situation is depicted in Fig. 3.1, where the filling of differently degenerate particle and anti-particle states is made more transparent.

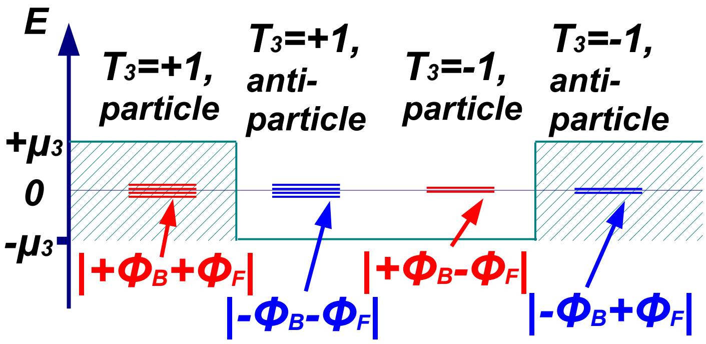


Figure 3.1: The schematic explanation of charge catalysis at finite iso-spin chemical potential μ_3. The states which are filled due to chemical potential μ_3 are particle $\tau_3 = +1$ states and anti-particle $\tau_3 = -1$ states, while the $\tau_3 = 1$ anti-particle and $\tau_3 = -1$ particle states are depleted. The degeneracies are indicated as being proportional to the $|\Phi_B \pm \Phi_F|$, where $\Phi_{B,F}$ are the fluxes of the B and F magnetic fields. Due to the different degeneracy of the filled particle and anti-particle states there is an overall induced charge (3.7) in the system. Figure taken from [24].

3.1.2 Charge halos and charge separation by magnetic fields

Now we want to ask the question what happens if instead of constant fields we consider fields which are localized within some region? Again we will consider only zero modes

[3] Apart from simplifying the expressions, the zero temperature limit takes into account only the zero mode states, which are protected by the index theorem discussed at the end of Sec. 2.4. Therefore the zero temperature limit is valid for *any* magnetic field profiles, and the result depends only on the magnetic fluxes.

and effects associated with them. This is justified at low enough temperature.

To begin let us review the construction of zero modes for arbitrary magnetic field $B(x_1, x_2)$ which was presented in [6]. The gauge field can be chosen to satisfy $\partial_1 A_1 + \partial_2 A_2 = 0$ (2D Lorenz gauge), so that we can take

$$A_1 = -\partial_2 \phi\ , \qquad\qquad A_2 = \partial_1 \phi \tag{3.9}$$

which immediately satisfies the gauge condition. On the other hand we have

$$\partial_1 A_2 - \partial_2 A_1 = B(x, y) = (\partial_1^2 + \partial_2^2)\phi \tag{3.10}$$

i.e. the function ϕ must satisfy the Laplace equation with the source $B(x_1, x_2)$. The equation on 2-spinors $i\not{D}\Psi = 0$ is equivalent to the following system of equations

$$\mathcal{D}\psi^{\downarrow} = 0 \qquad\qquad \bar{\mathcal{D}}\psi^{\uparrow} = 0 \tag{3.11}$$

where $\mathcal{D} = D_1 - iD_2$ and $\bar{\mathcal{D}} = D_1 + iD_2$ and where $\psi^{\uparrow}$ is the upper (spin-up) and $\psi^{\downarrow}$ the lower (spin-down) component of the two component spinor Ψ. Since $\mathcal{D} = \partial - (\partial\phi)$ where $\partial = \partial_z$ with $z = x_1 + ix_2$ and $\bar{D} = \bar{\partial} + \bar{\partial}\phi$, with $\bar{\partial} = \partial_{\bar{z}}$, we can easily solve the above equations by setting $\psi^{\uparrow,\downarrow} = f^{\uparrow,\downarrow}(z, \bar{z})e^{\mp\phi}$ with

$$\partial f^{\uparrow} = 0\ , \qquad\qquad \bar{\partial} f^{\downarrow} = 0\ . \tag{3.12}$$

so $f^{\uparrow} = f^{\uparrow}(\bar{z})$ is anti-holomorphic and $f^{\downarrow} = f^{\downarrow}(z)$ holomorphic. However note that at spatial infinity we must have (up to an irrelevant constant) $\phi = \frac{\Phi}{2\pi}\ln|r|$, where Φ is the flux of the magnetic field. This is because the magnetic field vanishes at infinity, but we still must have $\int dx^i\ A_i = \Phi$, and $\phi = \frac{\Phi}{2\pi}\ln|r|$ follows from the solution of the Laplace equation. Since the exponential factor behaves as $e^{\mp\phi} = |r|^{\mp\frac{\Phi}{2\pi}}$, in order to have normalizable zero modes, we must have that $\lim_{r\to\infty} r^{2+\epsilon}|\psi^{\uparrow,\downarrow}|^2 = 0$ where ϵ is a small positive number. It is then clear that $\pm\frac{\Phi}{2\pi} > 1$, so that only $\psi^{\uparrow}$ can have a zero mode for $\Phi/(2\pi) > 1$ and only $\psi^{\downarrow}$ has a zero mode for $\Phi/(2\pi) < -1$ [4]. However, taking for definiteness $\Phi > 0$, we can also take $f^{\uparrow}(z) = z^n$ with integer n and as long as $n < \Phi/(2\pi) - 1$ the solution is normalizable. Therefore we explicitly constructed $\lfloor \Phi/(2\pi) \rfloor$ zero modes of the form

$$\psi_n^{\uparrow} = N z^n e^{-\phi} \tag{3.13}$$

where ϕ is the solution of (3.10). For $\Phi < 0$ similar equation for $\psi^{\downarrow}$ with $z \to \bar{z}$ and

[4] Notice that in the case of $\mathbb{R} \times S^1$ for distances much large then the compact radius $r \ll L$, the laplace equation would be solved by $\phi \sim |r|$. Then we would have that the zero mode solution is $e^{\mp\phi} \sim e^{-\Phi/(2\pi)|r|}$ asymptotically so that the solution is always normalizable and the zero mode always exists, thus confirming the index theorem result discussed in the Appendix F.2.

$\phi \to -\phi$ holds. Notice that the various zero modes labeled by n, although linearly independent, are in general not orthogonal to each other. However for cylindrically symmetric magnetic fields the function ϕ is also cylindrically symmetric and since $\int d\varphi\, \bar{z}^n z^m \propto \delta_{nm}$ where φ is the polar angle, the above zero modes are orthogonal to each other.

Let us now see what happens in the presence of the $U(1) \times U(1)$ fields, i.e. for $\mathcal{B}_{tot}(x_1, x_2) = B(x_1, x_2) + \tau^3 F(x_1, x_2)$. We have two functions $\phi^\pm$ which solve

$$\partial_i^2 \phi^\pm = B(x_1, x_2) \pm F(x_1, x_2) \tag{3.14}$$

for the two sectors $\tau^3 = \pm 1$. Assuming both B and F field fluxes to be positive and dropping the $\uparrow$ in the superscript for simplicity, this yields two solutions for zero modes, for the $\tau^3 = \pm 1$ sectors, i.e.

$$\psi^\pm = N z^n e^{-\phi^\pm} \,. \tag{3.15}$$

Notice that when magnetic fields B, F are constant the Laplace equation is solved by $\phi^\pm = (F \pm B) r^2$, so that zero modes behave as $\psi^\pm \propto e^{-|B \pm F| r^2}$. This means that zero modes in $\tau^3 = +1$ and $\tau^3 = -1$ sector decay with different Gaussian widths. It is because of this reason that a local *charge separation* can be achieved.

To make this idea more explicit, let us consider the cylindrically symmetric distribution with B, F being constant and positive for $r \le R$ and zero for $r > R$, i.e.

$$F(r) = \begin{cases} F & r \le R \\ 0 & r > R \end{cases} \qquad B(r) = \begin{cases} B & r \le R \\ 0 & r > R \end{cases} \tag{3.16}$$

The equation (3.14) can be solved by

$$\phi^\pm(r) = \begin{cases} (B \pm F)(r^2 - R^2)/4\,, & r \le R \\ (B \pm F) R^2/2 \ln(r/R)\,, & r > R \end{cases} \tag{3.17}$$

The wave-functions are

$$\psi_n^\pm(x_1, x_2) = c_n^\pm z^n \begin{cases} e^{\frac{1}{4}|F \pm B|(R^2 - r^2)}\,, & r \le R \\ \left(\frac{R}{r}\right)^{\frac{1}{2}|F \pm B| R^2}\,, & r > R \end{cases} \tag{3.18}$$

In order for the solutions to be normalizable we must have that $n < \frac{1}{2}|F \pm B| R^2 - 1$ so the number of zero modes $\lim_{\epsilon \to 0} \left\lfloor \frac{|\Phi_F + \Phi_B|}{2\pi} - \epsilon \right\rfloor$. The normalization constants $c_n^\pm$ can be found analytically. They are

$$c_n^{\pm} = \frac{1}{\sqrt{\pi}} \left(\frac{|B \pm F|}{2} \right)^{\frac{n+1}{2}} \left\{ e^{\frac{\Phi^{\pm}}{2\pi}} \left[n! - \Gamma\left(n+1, \frac{\Phi^{\pm}}{2\pi}\right) \right] + \frac{2\left(\frac{\Phi^{\pm}}{2\pi}\right)^{2(n+1)}}{-2(n+1) + \frac{\Phi^{\pm}}{\pi}} \right\}^{-1/2} \tag{3.19}$$

Assuming that we introduce a small iso-spin chemical potential μ_3 all zero modes states

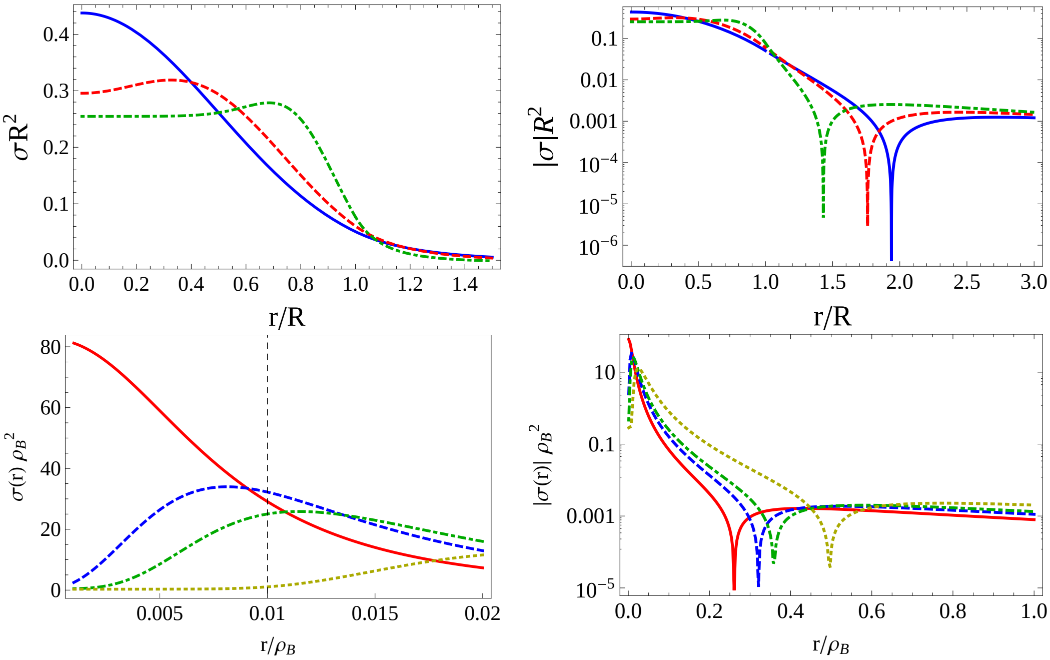


Figure 3.2: Left: The charge per area σ given by (3.20) for the field configuration (3.16) (top) and (3.23) (bottom), for fixed magnetic fluxe $\Phi_B/(2\pi) = 0.4$. The values for pseudo-magnetic field lines are $\Phi_F = 1.5, 2.5, 10.5$ (solid-blue, dashed-red, and dot-dashed-green lines) in the top panel and $\Phi_F = 1.5, 2.5, 3.5, 10.5$ (solid-red,dashed-blue, dot-dashed-green and dotted-yellow lines) in the bottom bottom panel. Taking the flux of $\Phi_B/(2\pi) < 1/2$ makes the degeneracies of the particles and anti-particles the same, so there no global charge induced (see Fig. 3.1). Right: the absolute value of the same charge density on a logarithmic scale. Notice that the charge density becomes negative between $r/R \gtrsim 1$. Figures taken from [24].

will be filled and the charge density (charge per area) is

$$\sigma = \sum_{n=0}^{N^+} |\psi_n^+|^2 - \sum_{n=0}^{N^-} |\psi_n^-|^2 \tag{3.20}$$

where $N^{\pm}$ are the number of $\tau^3 = \pm 1$ zero modes and where the negative sign in the second term is due to the fact that iso-spin $\tau^3 = -1$ states are anti-particle (hole) states. To compare to (3.8) we take the limit of large fluxes $\Phi^{\pm}/(2\pi) \gg 1$ and the coefficients simplify drastically

$$c_n^{\pm} \approx \frac{1}{\sqrt{\pi}\sqrt{n!}} \left(\frac{|B \pm F|}{2}\right)^{\frac{n+1}{2}} e^{-\frac{\Phi^{\pm}}{4\pi}} \tag{3.21}$$

The charge density near the center $r \ll R$ is

$$\begin{aligned}\sigma(r \ll R) \approx \sum_{n=0}^{N^+} \frac{1}{\pi\, n!} r^{2n} \left(\frac{|B+F|}{2}\right)^{n+1} e^{-\frac{|B+F|}{2} r^2} \\ - \sum_{n=0}^{N^-} \frac{1}{\pi\, n!} r^{2n} \left(\frac{|B-F|}{2}\right)^{n+1} e^{-\frac{|B-F|}{2} r^2} \approx \\ \approx \frac{|B+F| - |B-F|}{2\pi}\end{aligned} \tag{3.22}$$

where in the last step we extended the sum to infinity, which is justified for small r/R and large fluxes $\Phi^{\pm}$. This is precisely the same result as (3.8) obtained for uniform fields. In the top-left panel of Fig. 3.2 we show the plot of local charge density for fluxes $\Phi^{\pm}$ such that the number of $\tau^3 = \pm$ zero modes are the same, i.e. $N^+ = N^-$, so that there is no overall charge in the system. It is clearly seen that as the fluxes $\Phi^{\pm}$ become strong, a plateau forms in the region $r \ll R$. Since there is no induced charge in the system, this charge is compensated by an infinitely long negative tail which can be seen in the log-plot of the absolute value of charge density σ in the top-right panel of Fig. 3.2, i.e. charge is *separated*.

Next let us consider another case with the profiles of the B and F fields given by

$$B(r) = \frac{\Phi_B \rho_B^2}{\pi(r^2 + \rho_B^2)^2}\,, \qquad F(r) = \frac{2\Phi_F \rho_F^2}{\pi(r^2 + \rho_F^2)^2} \tag{3.23}$$

where $\Phi_{B,F}$ are the total flux parameters of field B and F and $\rho_{F,B}$ is the parameter which determines the size over which the fields are spread.

The zero modes in this background are again given by (3.15) with

$$\phi^{\pm}(r) = \frac{\Phi_B}{4\pi} \ln\left(1 + \frac{r^2}{\rho_B^2}\right) \pm \frac{\Phi_F}{4\pi} \ln\left(1 + \frac{r^2}{\rho_F^2}\right)\,. \tag{3.24}$$

Assuming $\Phi_F > \Phi_B > 0$, the charge density in this case reads

$$\sigma(r) = \sum_{n=0}^{N_+} |c_n^+|^2 r^{2n} \left(1 + \frac{r^2}{\rho_F^2}\right)^{-\frac{\Phi_F}{2\pi}} \left(1 + \frac{r^2}{\rho_B^2}\right)^{-\frac{\Phi_B}{2\pi}} \\ - \sum_{n=0}^{N_-} |c_n^+|^2 r^{2n} \left(1 + \frac{r^2}{\rho_F^2}\right)^{-\frac{\Phi_F}{2\pi}} \left(1 + \frac{r^2}{\rho_B^2}\right)^{\frac{\Phi_B}{2\pi}} \tag{3.25}$$

This spatial distribution is shown in bottom-left panel of Fig. 3.2, along with its logarithmic counterpart in the bottom-right panel of the same figure. Again we see that compensating charge is pushed to infinity.

3.1.3 Charge catalysis in graphene

The effect of magnetic charge catalysis described in the previous sections has a potential physical realization in graphene, a two dimensional lattice of carbon atoms with hexagonal structure (see Fig. 3.3). There has been enormous interest in graphene in recent years because of its unique low energy properties, where electrons behave like massless Dirac particles. For this reason graphene has become a table-top high energy physics experiment where interesting phenomena of relativistic physics can be observed directly. Further, graphene upon straining exhibits a low energy Hamiltonian with minimally coupled gauge field acting with a different sign on what are known as K-points or *valleys* in the condensed matter literature (see below). By straining the graphene, pseudo-magnetic fields of the order of 300T have been experimentally observed [69]. It is precisely this field which will play the role of the pseudo-magnetic field described earlier in this chapter.

Firstly let us discuss the low energy limit of graphene starting from the tight-binding Hamiltonian of the hexagonal lattice

$$H = \sum_{\xi,i} t_i \Big[|\boldsymbol{\xi}, A\rangle\langle\boldsymbol{\xi} + \boldsymbol{\eta}_i, B| + |\boldsymbol{\xi} + \boldsymbol{\eta}_i, B\rangle\langle\xi, A| \Big] \,. \tag{3.26}$$

Here the pair $\boldsymbol{\xi}$ and A, B label the position on the lattice. Namely ξ labels the spatial location, and A, B label whether the state is on even or odd sub-lattice. t_i are tunneling amplitudes to three nearest neighbors and $\boldsymbol{\eta}_i$ are vectors in the direction corresponding to t_i (see Fig. 3.3).

The remarkable thing about this Hamiltonian is that by going to the lattice momentum space and considering only the low energy limit (i.e. low momentum limit) one can show that for $t_1 = t_2 = t_3 = t$ the Hamiltonian has two quasi-momentum points where the energy vanishes, sometimes referred to as K-points and denoted by $K^{\pm}$. Expanding

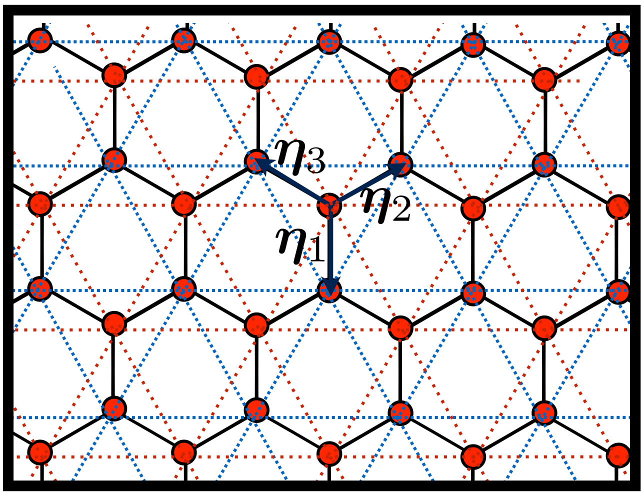


Figure 3.3: The structure of the graphene plane. Hexagonal lattice decomposes into two Bravis sub-lattices, here depicted by dashed red and blue lines. The hopping amplitude from one site to the nearest neighbor site is given by $t_{1,2,3}$ in the directions $\boldsymbol{\eta}_{1,2,3}$.

the Hamiltonian around these K-points in momentum space, we have[5]

$$H^{\pm} \approx v_f \begin{pmatrix} 0 & (p_x \pm i p_y) \\ (p_x \mp i p_y) & 0 \end{pmatrix} \tag{3.27}$$

where $v_f = \frac{3at}{2}$ is the Fermi velocity at these points. Here a is the lattice spacing (i.e. $a = |\boldsymbol{\eta}_i|$). The two Hamiltonians H^+, H^- correspond to the doubling due to the expansion around two K-points[6] $K^{\pm}$.

So although we assumed only one type of spin-less electron, what we have found is that the hexagonal lattice really has 4 types of electrons propagating on top of it. Firstly there is the 2×2 matrix structure of the Hamiltonian (3.27) which comes from the fact that the hexagonal lattice is composed out of two Bravis lattices. Additionally the Hamiltonian has two points where the energy is zero, i.e. two Dirac points. The low energy (i.e. small momentum) expansion around these two points doubles the femions further. The index which distinguishes these two valleys is often referred to as the valley index.

If instead of taking all t_i to be the same, we take instead

$$t_1 = e^{i\theta_1} t \,, \qquad t_2 = t e^{i\theta_2}(1+\epsilon_1) \,, \qquad t_3 = t e^{i\theta_3}(1+\epsilon_2) \,, \tag{3.28}$$

[5]For details see Katsnelson [62] and references therein.

[6]A similar phenomenon occurs in the cubic lattice and the additional states are known as *doublers*. In lattice QCD simulations these are artifacts of the discretezation, but here they are physically different excitations.

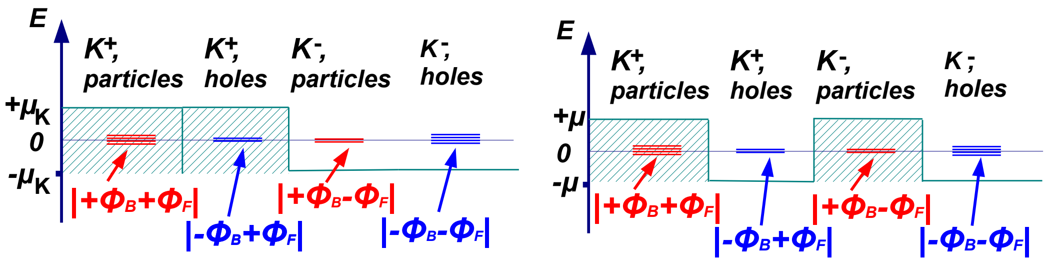


Figure 3.4: Left: The schematic view of charge catalysis for graphene at zero temperature and finite valley chemical potential μ_K. Right: the valley population at finite charge chemical potential μ. Figure taken from [24].

we obtain the same Hamiltonian with the replacement

$$p_x \to p_x \pm \frac{1}{v_F}\tilde{A}_x + A_x \ , \tag{3.29}$$

$$p_y \to p_y \pm \frac{1}{v_F}\tilde{A}_y + A_y \ . \tag{3.30}$$

with A_i being defined by $\theta_i = \boldsymbol{\xi}_i \cdot \boldsymbol{A}$ and

$$\tilde{A}_x = \frac{1}{2}(2t_2 - t_1 - t_3) \ , \qquad \tilde{A}_y = \frac{\sqrt{3}}{2}(t_3 - t_1) \tag{3.31}$$

Therefore introducing different complex hopping coefficients of the hexagonal crystal introduces gauge fields in the Hamiltonian. The phases naturally correspond to the usual gauge fields, but the lengths, in the approximation that their differences are small, introduce an additional gauge field into the picture, which couples with a different sign to the electrons depending on whether they are in the the first or the second valley. This is precisely the scenario discussed previously in the context of a 2+1 dimensional Dirac equation.

As we discussed in the previous section, charge generation can take place if we arrange for nontrivial fluxes $\Phi_{B,F}$ and have a valley chemical potential (i.e. an excess of electrons in one of the valleys compared to the other). For zero temperature this is guaranteed by the index theorem of the two gauge sectors. On the other hand, we can invert the mechanism and populate the valleys by adding a charge density, which is easier to do, for example by connecting the sample to a charge reservoir.

This mechanism is shown in Fig. 3.4 where the lowest Landau level populations are taken into account and small chemical potentials (i.e. smaller then the first excited level) are assumed.

We have setup the idea behind what we call charge catalysis on the simple 2D

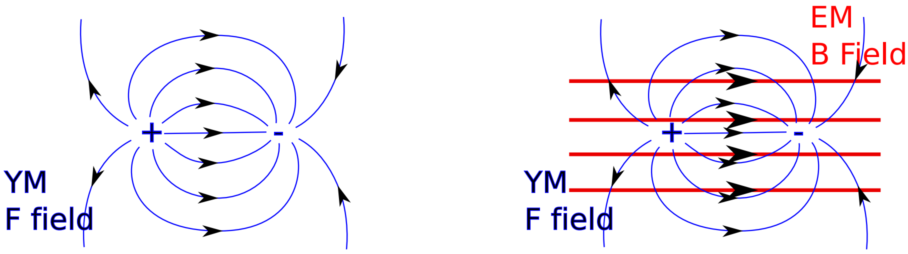


Figure 3.5: Left: schematic chromomagnetic field-lines of a monopole–anti-monopole pair Right: the same in the external magnetic field.

example. We have seen that the phenomenon is directly relevant for the low energy graphene physics. The phenomenon however is also relevant for instanton-monopoles in magnetic field which is what we discuss next.

3.2 Instanton-monopoles and charge catalysis

Let us consider a monopole–anti-monopole aligned along the z-axis. Their magnetic field lines are quasi-abelian, so we take them to be in the τ^3 color direction. The magnetic field-lines are those of a dipole and are depicted in Fig. 3.5. Now imagine that we put this system into an abelian magnetic field. Then there is a region in between the two monopoles where the quasi-abelian YM field and the abelian field are parallel, so the situation in this region is similar to that discussed in 2+1 dimensions.

In this case, however, we have no iso-spin (or in this case, color) chemical potential. Indeed such a chemical potential would be gauge non-invariant. Remember, however, that instanton-monopoles are objects which exist in the presence of a gauge holonomy, i.e. $A_4 \neq 0$. In fact this is precisely what acts in place of the chemical potential. Keep in mind that gauge invariance is not violated here, as the color direction of A_4 is arbitrary, i.e. it is merely a gauge choice.

In order to simplify the problem, let us approximate the magnetic and chromomagnetic fields in the region between the monopole–anti-monopole pair as before $\mathcal{B}_{tot} = B + F\tau^3$, and with $A_4 = v\frac{\tau^3}{2}$ the partition function can be written as

$$\begin{aligned}\frac{\ln \mathcal{Z}(v,\mu)}{V} &= \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \sum_{n=0}^{\infty} s_n \Bigg\{ \frac{|B+F|}{2\pi} \ln\left(1 + e^{-\beta E^+_{n,k_3} + i\beta v/2 + \mu\beta}\right) \\ &+ \frac{|-B-F|}{2\pi} \ln\left(1 + e^{-\beta E^+_{n,k_3} - i\beta v/2 - \mu\beta}\right) + \frac{|B-F|}{2\pi} \ln\left(1 + e^{-\beta E^-_{n,k_3} - i\beta v/2 + \mu\beta}\right) \\ &+ \frac{|-B+F|}{2\pi} \ln\left(1 + e^{-\beta E^-_{n,k_3} + i\beta v/2 - \mu\beta}\right) \Bigg\} \qquad (3.32)\end{aligned}$$

where

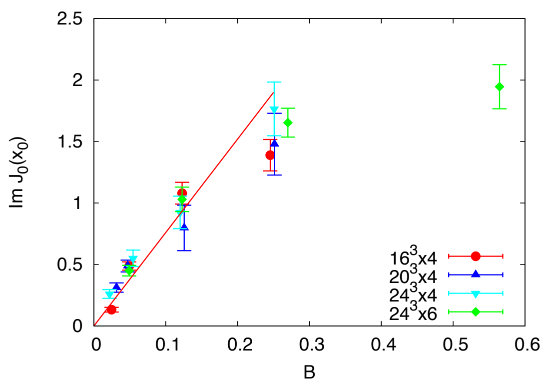
$$E^{\pm}_{n,k_3} = \sqrt{2n|B \pm F| + k_3^2} \tag{3.33}$$

are the energy levels for $\tau^3 = \pm 1$. It is then easy to compute the induced charge, assuming low enough temperature and taking into account only the $n = 0$ level, i.e. assuming $T^2 \ll \min(|B+F|, |B-F|)$. Performing the k_3 integral we get for the charge density

$$\rho = \frac{\langle Q \rangle}{V} = \frac{1}{\beta V} \frac{\partial}{\partial \mu} \ln \mathcal{Z}(v, \mu) = \frac{iv}{(2\pi)^2} (|B+F| - |B-F|) \tag{3.34}$$

Note that v is a compact variable with period $4\pi/\beta$ and that the above expression is only valid for $v\beta \in [-2\pi, 2\pi]$. For other values periodic continuation on other branches is implied.

Since the caloron is an object comprising a monopole and an anti-monopole (i.e. BPS and KK monopole) this behavior should be observed on calorons with the magnetic field oriented along the symmetry axis of the caloron. In fact this is precisely observed on the lattice as is shown in Fig. 3.6 (for details see our work [24]). The lattice result also shows a peak at the center of one monopole and a dip at the center of another. The existence of this peak and dip at the monopole centers can be argued as follows. Because of self-duality, monopoles have color-electric field around them as well. Since in the absence of the magnetic field there is absolute democracy between the color directions $\tau^3 = \pm 1$, the screening with quarks would not show itself in the electric charge as there would be an equal amount of particles with $\tau^3 = +1$ and antiparticles with $\tau^3 = -1$. However



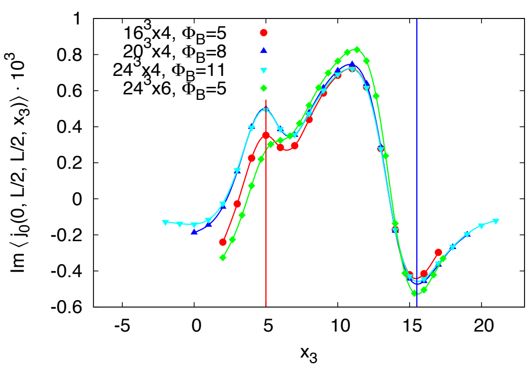


Figure 3.6: Left: Imaginary part of the net electric charge $J_0 \propto \int d^3x \mathrm{Tr}\, \not{D}^{-1}\gamma_4$ averaged over the time component labeled here by x_0 as a function of magnetic field B in lattice units. Right: The charge distribution along the symmetry axis of the caloron. The red and blue vertical lines are positions of the instanton-monopoles. There is a pronounced peak in the middle which is precisely where the two fields are parallel and the mechanism described in the text is at play. Figures taken from [24].

since we have broken the $\tau^3 = \pm 1$ symmetry by introducing the longitudinal magnetic field so that the two color sectors see different magnetic fields, these screening clouds become visible.

What is worrying, however, is that the charge we obtained is imaginary. This seems quite peculiar. On the other hand, the charge operator $\langle \bar{\Psi}\gamma_4\Psi \rangle = \mathrm{Tr}\left(\frac{1}{\not{D}+\mu\gamma_4}\gamma_4\right)$ must be either zero or purely imaginary when $\mu = 0$ because of the anti-Hermiticity of $\not{D}$. What is then going on? The resolution is that the holonomy A_4 is not a physical field and configurations corresponding to fixed A_4 configurations are not physical configurations. Instead, A_4 is introduced in the thermal theory as a Lagrange multiplier imposing the Gauss constraint, and should be integrated over. Further the integration over the configurations would then also include dipole pairs oriented oppositely which would give the same contribution with the opposite sign. The induced charge is then identically zero, as is expected.

However, such configurations could contribute to the charge fluctuations in the vacuum, as the square of the charge operator need not vanish. Naively the charge fluctuations would then have a somewhat unnatural volume scaling and negative charge $\langle Q^2 \rangle \propto -V^2 v^2$. To further explore the phenomenon let us consider a simple situation of the fermion back-reaction on v in uniform fields and consider charge fluctuations in a system with the partition function

$$Z(\mu) = \int_{-2\pi/\beta}^{2\pi/\beta} dv\ \mathcal{Z}(v,\mu)\ . \tag{3.35}$$

where $\mathcal{Z}$ is given by the expression (3.32). We could in principle simply compute the above partition function by integration over v and differentiate the logarithmic derivative twice to get the charge fluctuations. However in order to separate the contribution due to the magnetic catalysis, let us first think of the charge fluctuations as coming from the system with v fixed, i.e. from a system with the partition function $\mathcal{Z}(v,\mu)$. To compute the averages over vacua with different v we will have to integrate over v with the weight $\mathcal{Z}(v,\mu)$. We will denote averages at fixed v as $\langle \ldots \rangle_v$. Assuming $F, B > 0$ for simplicity, we have $\langle Q \rangle_v = \frac{i2v\min(B,F)}{(2\pi)^2}$. The total charge fluctuation is however given by

$$\langle Q^2 \rangle_v = \frac{1}{\beta^2}\frac{\frac{\partial^2}{\partial\mu^2}\mathcal{Z}(v,\mu)}{\mathcal{Z}} = \frac{1}{\beta^2}\left(\frac{\frac{\partial}{\partial\mu}\mathcal{Z}(v,\mu)}{\mathcal{Z}}\right)^2 + \frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\ln\mathcal{Z}(v,\mu)\ . \tag{3.36}$$

First term is simply $\langle Q \rangle_v^2$ and is referred to as the disconnected contribution, while the second is the charge susceptibility, and is simply the charge fluctuations[7] $\langle Q^2 \rangle_v - \langle Q \rangle_v^2$.

[7] Remember, this is charge fluctuation in the v-vacuum, which is not the physical vacuum. The disconnected part in the physical vacuum is zero.

The charge susceptibility is

$$\frac{\partial^2}{\partial\mu^2}\ln\mathcal{Z}(v,\mu) = \frac{V\max(F,B)}{\beta\pi^2} \tag{3.37}$$

This contribution exists for any holonomy and any magnetic field, even when $F = 0$. It is the contribution to the fluctuations due to the magnetic field. Using this formula and eqs. (3.36) and (3.34) we get that

$$\left\langle Q^2\right\rangle_v = \frac{V\max(F,B)}{\beta\pi^2} - \frac{4v^2V^2\min(B^2,F^2)}{(2\pi)^4}\ . \tag{3.38}$$

The negative charge above is due to the fact that the induced charge in the v-vacuum is imaginary. Next we must multiply the above result with $\mathcal{Z}(v,\mu)$, integrate over v and divide by $\int dv\ \mathcal{Z}(v,\mu)$. The first term will be unchanged, since it does not depend on holonomy, while the second term we now compute.

The integral over k_3 in expression (3.32) can be done analytically when we keep only the $n = 0$ Landau level[8]

$$\mathcal{Z}(v,\mu=0) = -4\frac{V}{\beta}\frac{\max(|B|,|F|)}{\pi^2}\sum_{m=1}^{\infty}(-1)^m\frac{\cos\left(\frac{vm}{2}\right)}{m^2} \tag{3.39}$$

The above sum can be re-expressed by noting that the expansion around $v = 0$ terminates at second order, i.e.

$$f(x) = \sum_{m=1}^{\infty}\frac{(-1)^m\cos(xm)}{m^2} = f(0) + \frac{1}{2}f''(0)x^2\ . \tag{3.40}$$

To see this note that the first derivative $f'(0) = 0$ and that the second derivative is $f''(x) = -\sum_{m=1}^{\infty}(-1)^m\cos(xm)$. Regulating this sum we get

$$f''(x) = -\lim_{\epsilon\to 0}\sum_{m=1}(-1)^m\cos(xm)e^{-\epsilon m} = -\frac{1}{2}\ , \tag{3.41}$$

so $f'''(x) = 0$ and the Taylor expansion terminates. Since $f(0) = \sum_{m=1}^{\infty}\frac{(-1)^m}{m^2} = -\frac{\pi^2}{12}$ we can write

$$f(x) = -\frac{\pi^2}{12} + \frac{1}{4}x^2\ . \tag{3.42}$$

[8]To get this formula we expand the expressions $\ln(1+e^{-\beta|k_3|\pm iv\beta/2}) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{e^{-m|k_3|\beta\pm imv\beta/2}}{m}$, which upon integration over k_3 becomes

$$\int_{-\infty}^{\infty}dk_3\ \ln(1+e^{-\beta|k_3|\pm iv\beta/2}) = \frac{2}{\beta}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{e^{imv\beta/2}}{m^2}$$

Note however that due to the periodicity of the function $f(x+2\pi) = f(x)$ this expansion cannot possibly be valid for all values of x. Indeed our regularization of the sum has ruined this periodicity. Nonetheless the expression is correct for $x \in [-\pi, \pi]$ and the series $\sum_{m=1}^{\infty}(-1)^m \frac{\cos(xm)}{m^2}$ is just the Fourier expansion of the function $-\frac{\pi^2}{12} + \frac{1}{4}x^2$ defined on the interval $x \in [-\pi, \pi]$.

This means that when $v\beta \in [-2\pi, 2\pi]$ we can write

$$\ln \mathcal{Z}(v, \mu = 0) = -4\frac{V}{\beta}\frac{\max(|B|, |F|)}{2\pi^2}\left(-\frac{\pi^2}{12} + \frac{v^2\beta^2}{16}\right) . \tag{3.43}$$

The partition function is then a simple Gaussian in v. If we take one of the fields B, F to be very strong then we can ignore the periodicity in v. Then the normalized partition function is

$$\mathcal{Z}(v, \mu = 0) = \sqrt{\frac{V}{\beta}\frac{\max(|B|, |F|)v^2\beta^2}{8\pi^3}} e^{-V\frac{\max(|B|,|F|)v^2\beta}{8\pi^2}} \tag{3.44}$$

and we have that

$$\langle v^2 \rangle \approx \int_{-\infty}^{\infty} dv\ \mathcal{Z}(v, \mu = 0)v^2 = \frac{4\pi^2}{V\max(|B|, |F|)\beta} . \tag{3.45}$$

Averaging over v-vacua we finally have

$$\langle Q^2 \rangle = \frac{V\max(|B|^2, |F|^2)}{\beta\pi^2}\left(1 - \frac{\min(B^2, F^2)}{\max(B^2, F^2)}\right) . \tag{3.46}$$

The above formula shows that the charge fluctuations at fixed B gets increasingly suppressed as we turn on the field F. It would seem that the charge fluctuation is exactly zero when $|F| = |B|$, but we must remember that the approximation was made that $|F \pm B| >> 1/\beta^2$ in taking only the zero Landau level which breaks down in this region.

Let us estimate the charge fluctuations in Heavy Ion Collisions using the formula (3.46). Taking $F \sim 0.1$ GeV^2 from the lattice measurements of the gluon condensate [73], and $(2e/3)B \sim m_\pi^2 \approx 0.02$ GeV^2 (here $2e/3$ is the physical charge of the u quark) which corresponds to Gold-Gold collision with impact parameter $b = 4$ fm at $\sqrt{s} = 200$GeV [102, 70], the charge fluctuations would seem to be suppressed by several percents only. The formula (3.46), however, was made under some very unrealistic simplifications and in the conditions where the holonomy parameter v fluctuates around zero. Close to T_c these fluctuations may shift to a non-zero value and suppression would be greater. A model of the instanton monopoles which develops monopole-anti-monopole strings along the magnetic field is suggested in Chapter 5, which might increase the phenomenon significantly.

3.3 Zero modes with chemical potential

An important development in the instanton phenomenology was the generalization to the finite baryon chemical potential. The introduction of the chemical potential in the quark action changes the profiles of zero modes, and this change can influence the underlying structure of the vacuum. In [92, 94] this is precisely what was done over a decade ago where instantons were used to construct the model of QCD. The crucial ingredient is weighting the instanton configurations with the Dirac determinant. Since by index theorem the topological objects have fermionic zero mode bound states[9] which will facilitate their interactions, while the higher states have little to do with the ensemble of the topological objects and they are not much different from the perturbative states which are present in the absence of topology.

The analysis of the influence of fermions (at zero or finite density) would then boil down to the study of the zero mode profiles and the interactions they induce between topological objects. The purpose of this section is precisely to do that for instanton monopoles and calorons. It is actually only recently in [36] that the zero modes with non-zero chemical potential were generalized from zero temperature instantons to Harrington-Shephard calorons [58] (i.e. calorons with $A_4^\infty = 0$). We will use this result as a check on our calculations in these limits.

3.3.1 The index theorem revisited

Here we briefly discuss the index theorem for the Dirac operator with chemical potential, i.e.

$$\not{D}(\mu) = \gamma^\mu D_\mu - \mu\gamma_4 = \begin{pmatrix} 0 & \mathcal{D}(\mu) \\ \bar{\mathcal{D}}(\mu) & 0 \end{pmatrix} . \tag{3.47}$$

where we have defined $\mathcal{D}(\mu) = \sigma^\mu \mathcal{D}_\mu - \mu$ and $\bar{\mathcal{D}}(\mu) = \bar{\sigma}^\mu \mathcal{D}_\mu - \mu$. Note that the above operator does not have definite Hermiticity. We would like to state the index theorem for the above operator. We may be tempted to write the index function as before

$$I(M^2) = \operatorname{tr} \frac{M^2}{-\not{D}(\mu)^2 + M^2}\gamma_5 = \operatorname{tr} \frac{M^2}{-\mathcal{D}(\mu)\bar{\mathcal{D}}(\mu) + M^2} - \operatorname{tr} \frac{M^2}{-\bar{\mathcal{D}}(\mu)\mathcal{D}(\mu) + M^2} , \tag{3.48}$$

and argue that the zero modes of $\bar{\mathcal{D}}(\mu)$ are the same as the zero modes of $\mathcal{D}(\mu)\bar{\mathcal{D}}(\mu)$, which is clearly still true. The trouble comes when we want to show that zero modes of $\mathcal{D}(\mu)\bar{\mathcal{D}}(\mu)$ are also zero modes of $\bar{\mathcal{D}}(\mu)$. Indeed if we try to repeat the steps made after eq. (2.58) we run into troubles because $\mathcal{D}(\mu) \neq -\bar{\mathcal{D}}(\mu)^\dagger$. In other words there could be a state such that $\mathcal{D}(\mu)\bar{\mathcal{D}}(\mu)\psi = 0$ but $\bar{\mathcal{D}}(\mu)\psi = \chi \neq 0$ and that $\mathcal{D}(\mu)\chi = 0$.

[9] In the instanton anti-instanton ensemble these would cease to be zero modes and would be quasi zero modes instead.

A more careful calculation would involve using Hermitian operators $\not{D}(\mu)\not{D}(\mu)^\dagger$ and $\not{D}(\mu)^\dagger\not{D}(\mu)$. This was done in [60] there it was shown that

$$\dim\ker\mathcal{D}(\mu) - \dim\ker\bar{\mathcal{D}}(-\mu) = Q \;, \tag{3.49}$$

$$\dim\ker\mathcal{D}(-\mu) - \dim\ker\bar{\mathcal{D}}(\mu) = Q \;. \tag{3.50}$$

At $\mu = 0$ the above two equations reduce to the one and the same which is just $N_L - N_R = Q$. However in [60] the authors argue that unless the gauge fields are fine-tuned $\dim\ker\mathcal{D}(\mu) = \dim\ker\mathcal{D}(-\mu)$ and $\dim\ker\bar{\mathcal{D}}(\mu) = \dim\ker\bar{\mathcal{D}}(-\mu)$, so that both of the above relation reduce to the usual $N_L - N_R = Q$ index relation[10].

3.3.2 Bi-orthogonal normalization

The key problem in dealing with non-Hermitian operators $\not{D}(\mu)$ is that eigenstates are not orthogonal. However consider the eigenvalue equations

$$\not{D}(\mu)\psi_n(\mu) = \lambda_n(\mu)\psi_n(\mu) \;, \tag{3.51}$$

$$\chi_n(\mu)^\dagger\not{D}(\mu) = \chi_n(\mu)^\dagger\tilde{\lambda}_n(\mu) \tag{3.52}$$

The two are called the *left* and *right* eigenvalue equation, respectively. However since $\not{D}(\mu)^\dagger = -\not{D}(-\mu)$, by Hermitian conjugation, the second equation becomes

$$\not{D}(-\mu)\chi_n(\mu) = -\tilde{\lambda}_n^*(\mu)\chi_n(\mu) \tag{3.53}$$

Clearly we can take $\chi_n(\mu) = \psi_n(-\mu)$ and then $\tilde{\lambda}_n^*(\mu) = -\lambda_n(-\mu)$. Now since

$$\left\langle\psi_n(-\mu)\left|\not{D}(\mu)\right|\psi_m(\mu)\right\rangle = \lambda_m\left\langle\psi_n(-\mu)|\psi_m(\mu)\right\rangle = \tilde{\lambda}_n\left\langle\psi_n(-\mu)|\psi_m(\mu)\right\rangle \;, \tag{3.54}$$

it follows that

$$\left(\lambda_m(\mu) + \lambda_n^*(-\mu)\right)\left\langle\psi_n(-\mu)|\psi_m(\mu)\right\rangle = 0 \tag{3.55}$$

so that either $\lambda_m(\mu) = -\lambda_n^*(-\mu)$ or $\langle\psi_n(-\mu)|\psi_m(\mu)\rangle = 0$. Then we can orthogonalize the left and right eigenmodes to each other as

$$\left\langle\psi_n(-\mu)|\psi_m(\mu)\right\rangle = \delta_{nm} \tag{3.56}$$

In fact this is nothing more than analytical continuation of the orthogonalization when $\mu = i2\pi z$ is purely imaginary (i.e. z is real) and $\not{D}(i2\pi z)$ is Hermitian. There the

[10] Of course for fractional topology the boundary terms are expected to make the index integer. We will not discuss this for chemical potential, as we will explicitly construct the zero mode.

eigenbasis ψ_n is orthogonal, i.e.

$$\langle \psi_n(-2\pi iz) | \psi_m(2\pi iz) \rangle = \delta_{nm} \tag{3.57}$$

Identifying $\mu = 2\pi iz$ reduces precisely to (3.56).

We will use this concept of analytical continuation to construct the zero modes. We will also examine the pseudo-probability density

$$\rho(x) = \psi^\dagger(-\mu, x)\psi(\mu, x) \tag{3.58}$$

as an analytical continuation of the probability density at imaginary μ.

3.3.3 Monopole in the radial gauge

We want to construct the fundamental fermionic zero mode with chemical potential in the background of a BPS monopole. This section follows the work [28] closely, although the original derivation of zero modes for purely complex μ (i.e. periodicity up to a phase) was done in [100].

The left-handed zero mode equation is

$$(\sigma^\mu D_\mu - \mu)^{\alpha\beta}_{AB} (\Psi_L)^B_\beta = 0 \tag{3.59}$$

where α, β are the spinor and A, B are the color indices. We can make the usual radial ansatz

$$\Psi^A_\alpha = \Big[(\alpha_1(r)\mathbf{1} + \alpha_2(r)\hat{\boldsymbol{r}} \cdot \boldsymbol{\tau})\epsilon \Big]_{A\alpha} \tag{3.60}$$

where $\epsilon = i\tau^2$ is a totally antisymmetric 2×2 matrix. Notice that in writing the Weyl fermion this way, we can consider a 2×2 matrix

$$\eta(\boldsymbol{r}) = \alpha_1(r)\mathbf{1} + \alpha_2(r)\hat{\boldsymbol{r}} \cdot \boldsymbol{\tau} \; . \tag{3.61}$$

The color matrices act on the above matrix from the left, while the spatial spin matrices σ^i act on the spinor Ψ as

$$(\sigma^i)^{\alpha\beta} (\eta\epsilon)_{A\beta} = \big(\eta\epsilon(\sigma^i)^T \big)_{A\alpha} = -\eta\sigma^i\epsilon \; , \tag{3.62}$$

i.e. the spatial spin matrices σ^i act on η from the right with an additional minus sign.

The equation for η is then

$$-\partial_i \eta \sigma^i - iA_4\eta + iA_i\eta\sigma^i - \mu\eta = 0 \tag{3.63}$$

which, with the gauge fields given by (2.26), after some straightforward algebra, gives

$$\left(\alpha_1'(r) - \frac{\mathcal{H}(r)}{2}\alpha_1(r) + \alpha_1(r)\mathcal{A}(r) + i\mu\alpha_2(r)\right) i\hat{\boldsymbol{r}}\cdot\boldsymbol{\tau} \\ + i\left(\alpha_2'(r) + \frac{2}{r}\alpha_2(r) - \mathcal{A}\alpha_2(r) - \frac{\mathcal{H}(r)}{2}\alpha_2(r) + i\mu\alpha_1(r)\right) = 0 \quad (3.64)$$

Clearly this boils down to two equations

$$\alpha_1'(r) + \frac{-\mathcal{H}(r) + 2\mathcal{A}(r)}{2}\alpha_1(r) + i\mu\alpha_2(r) = 0 \quad (3.65)$$

$$\alpha_2'(r) + \left(\frac{-\mathcal{H}(r) - \mathcal{A}(r)}{2} + \frac{2}{r}\right)\alpha_2(r) + i\mu\alpha_2(r) = 0 \quad (3.66)$$

For the right-handed zero mode we would simply need to change the sign of $\mathcal{A}$ and the kinetic terms $\alpha_1'(r), \alpha_2'(r), \frac{2}{r}\alpha_2(r)$. This is equivalent to changing the sign of $\mathcal{H}$ and μ.

These equations are solved in the Appendix E with the result (E.26). We repeat it here for convenience

$$\alpha_1(r) = \frac{c}{\sqrt{rv\sinh(rv)}}\left(2\frac{\mu}{v}\sin(\mu r) + \tanh\left(\frac{rv}{2}\right)\cos(\mu r)\right) \quad (3.67a)$$

$$\alpha_2(r) = \frac{c}{\sqrt{rv\sinh(rv)}}\left(2i\frac{\mu}{v}\cos(\mu r) - i\coth\left(\frac{rv}{2}\right)\sin(\mu r)\right) \quad (3.67b)$$

The above equations are correct for μ both real and imaginary. Notice that the pseudo-density (3.58) is given by

$$\rho_0(r) = \alpha_1^*(r, -\mu^*)\alpha_1(r,\mu) + \alpha_2^*(r,-\mu^*)\alpha_2(r,\mu)\ . \quad (3.68)$$

By integrating the above expression we get that the normalization constant (in the sense of $\int d^4x$ normalization) is given by $c = \sqrt{\frac{v^3}{4\pi\beta}}$.

In Fig. 3.7 we plot the density (3.68) for Im $\mu = 0$. This corresponds to the periodic zero mode. To account for periodicity up to a phase we simply need to make μ complex[11], with its imaginary part identified with the phase as Im $\mu = \frac{\varphi}{\beta}$. However if we try to make the zero mode anti-periodic the sine and cosine terms in (3.67) make the zero mode non-normalizable if Im $\mu > v/2$. Since we consider $v\beta \in [0,\pi]$, the anti-periodic fermion will not have a normalizable zero mode, in accordance with the index theorem we discussed at zero μ. However the KK monopole might still have a zero mode. Indeed the KK monopole zero mode can be constructed in the same way by replacing $v \to \bar{v}$

[11] A spinor ψ which is periodic in x_4 up to a phase $\psi(x_4+L) = e^{i\varphi}\psi(x_4)$ satisfying the Dirac eigenequation $\not{D}\psi = \lambda\psi$ can be written as $\psi = e^{i\varphi\frac{x_4}{L}}\tilde{\psi}$, where $\tilde{\psi}$ satisfies a Dirac eigenequation with purely *imaginary* chemical potential, i.e. $(\not{D} + \frac{\varphi}{L}\gamma_4)\tilde{\psi}$. See also the discussion after (3.72).

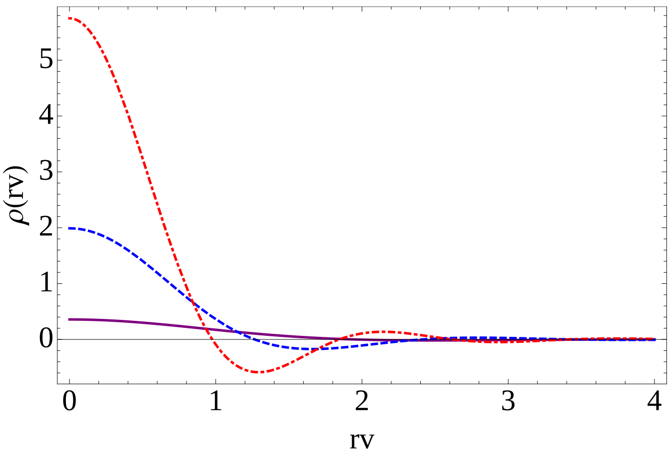


Figure 3.7: The density of the instanton-monopole zero mode given by (3.68) for $\mu = 0.9v$ (solid blue), $\mu = 1.5v$ (dashed red) and $\mu = 2v$ (dot-dashed purple).

and then applying the same gauge transformations we made in Sec. 2.2.2. This way, due to the time-dependent anti-periodic gauge twist the fermions become anti-periodic. Therefore just like in the case of $\mu = 0$ the BPS monopole loses the zero mode and the KK monopole gains it when we go from periodic to anti-periodic fermions.

3.3.4 Caloron zero modes

In Chapter 2 we have briefly discussed the Caloron solution, which was obtained by the ADHM construction and Nahm transformation. This powerful method can also give fermionic zero modes once the solution is known [80], but more specifically for calorons it was given in [54, 27] and in our notation it reads

$$\tilde{\psi}^A_\alpha = \rho\frac{\sqrt{\phi}}{2\pi}\left\{e^{-ivx_4\frac{\tau^3}{2}}\left[\partial_\nu \hat{f}\left(\frac{v}{4\pi}, z\right)\frac{\mathbf{1}+\tau^3}{2}\right.\right.$$
$$\left.\left.+\,\partial_\nu \hat{f}\left(\frac{1}{2\beta}+\frac{\bar{v}}{4\pi}, z\right)\frac{\mathbf{1}-\tau^3}{2}\right]\bar{\sigma}^\nu\epsilon\right\}_{A\alpha} \qquad (3.69)$$

where ρ is the caloron size parameter while $\hat{f}(z,z')$ and[12] the function ϕ is given by

$$\begin{aligned}\hat{f}(z,z') =& e^{2\pi i x_4(z-z')}\frac{\pi}{\beta}(rs\psi)^{-1}\Bigg\{ e^{-\frac{2\pi i x_4}{\beta}\mathrm{sign}(z-z')}\sinh(2\pi r|z-z'|)s \\ & - r^{-1}\cosh\left(2\pi r\left(z+z'-\frac{1}{\beta}\right)\right)\left[s\,d\cosh(vs)+\frac{1}{2}(s^2-r^2+d^2)\sinh(vs)\right] \\ & + r^{-1}\cosh(r(\bar{v}-2\pi|z-z'|))\left[s\,d\cosh(vs)+\frac{1}{2}(r^2+s^2+d^2)\sinh(vs)\right] \\ & + \sinh(r(v-2\pi|z-z'|))\left[s\cosh(sv)+d\sinh(sv)\right] \end{aligned} \quad (3.70a)$$

$$\phi = \frac{\psi}{\hat{\psi}}\ , \quad (3.70b)$$

$$\begin{aligned}\hat{\psi} =& -\cosh(2\pi x_4/\beta)+\cosh(vr)\cosh(\bar{v}s) \\ & + \frac{r^2+s^2-d^2}{2rs}\sinh(vr)\sinh(vs)\ , \end{aligned} \quad (3.70c)$$

$$\begin{aligned}\psi =& -\cosh(2\pi x_4/\beta)+\cosh(vr)\cosh(\bar{v}s) \\ & + \frac{r^2+s^2+d^2}{2rs}\sinh(vr)\sinh(vs) \\ & + d(s^{-1}\sinh(vs)\cosh(\bar{v}r)+r^{-1}\sinh(vr)\cosh(vs)) \end{aligned} \quad (3.70d)$$

where $d=\frac{\pi\rho^2}{\beta}$ is the distance between constituent monopoles. The parameter z is the phase twist of the solution. Note that since the solution of [54, 27] is in the algebraic gauge, we have gauge transformed it by a gauge transformation $e^{-i\frac{vx_4}{2}\tau^3}$ so that the solution is periodic up to a phase, i.e.

$$\tilde{\psi}(x_4=\beta)=e^{-2\pi i z\beta}\tilde{\psi}(x_4=0)\ . \quad (3.71)$$

Also note that the function $f(z,z')$ has the above form only when $z,z'\in[\frac{v}{4\pi},\frac{\bar{v}}{4\pi}]$ which is the appropriate range for the anti-periodic solution. For the complementary range the function f is similar with the replacement $r\leftrightarrow s$ and $v\leftrightarrow\bar{v}$. Since the above is a solution of the Weyl equation

$$\sigma^\mu D_\mu\tilde{\psi}=0 \quad (3.72)$$

we can construct an anti-periodic solution (up to the gauge twist) $\psi=e^{(2\pi i z-i\pi/\beta)x_4}\tilde{\psi}$ which solves

$$(\sigma^\mu D_\mu-(2\pi i z-i\pi/\beta))\psi=0\ . \quad (3.73)$$

[12] Although ϕ is given in the introduction, we repeat it here for convenience with the dimensions restored by β.

To get a solution for chemical potential μ we must then replace $(2\pi iz - i\pi/\beta) \rightarrow \mu$ or $z \rightarrow -\frac{i\mu}{2\pi} + \frac{1}{2\beta}$. Therefore the anti-periodic caloron zero mode is given by

$$\psi^A_\alpha = \rho\frac{\sqrt{\phi}}{2\pi}e^{\mu x_4}\Bigg\{e^{-ivx_4\frac{\tau^3}{2}}\Big[\partial_\nu \hat{f}\left(\frac{v}{4\pi},\frac{1}{2\beta}-\frac{i\mu}{2\pi}\right)\frac{\mathbf{1}+\tau^3}{2} + \partial_\nu \hat{f}\left(\frac{1}{2\beta}+\frac{\bar{v}}{4\pi},\frac{1}{2\beta}-\frac{i\mu}{2\pi}\right)\frac{\mathbf{1}-\tau^3}{2}\Big]\bar{\sigma}^\nu\epsilon\Bigg\}_{A\alpha} \tag{3.74}$$

where

$$\hat{f}^+ \equiv \hat{f}\left(\frac{v}{4\pi},\frac{1}{2\beta}-\frac{i\mu}{2\pi}\right) = e^{-\frac{\pi i x_4}{\beta}}e^{i\frac{vx_4}{2}}e^{-\mu x_4}\frac{\pi}{rs\pi\beta}\Bigg\{e^{\frac{2\pi i t}{\beta}}\sinh\left(\frac{\pi r}{\beta}-\frac{vr}{2}-i\mu r\right)s + \cosh\left(\frac{\pi r}{\beta}-\frac{vr}{2}+i\mu r\right)r\sinh(vs) + \sinh\left(\frac{\pi r}{\beta}-\frac{vr}{2}+i\mu r\right)\Big[s\cosh(vs)+d\sinh(vs)\Big]\Bigg\} \tag{3.75}$$

$$\hat{f}^- \equiv \hat{f}\left(\frac{1}{\beta}+\frac{\bar{v}}{4\pi},\frac{1}{2\beta}-\frac{i\mu}{2\pi}\right) = (\hat{f}^+)^* \tag{3.76}$$

Further, the formalism has a wonderfully simple expression for the probability density

$$\psi(x)^\dagger\psi(x) = -\frac{1}{2\pi^2}\partial^2_\nu\hat{f}(z,z)\ . \tag{3.77}$$

Upon analytical continuation we can write that the pseudo-density is given by

$$\psi(-\mu)^\dagger\psi(\mu) = -\frac{1}{2\pi^2}\partial^2_\nu\hat{f} \tag{3.78}$$

where

$$\hat{f} \equiv \hat{f}\left(\frac{1}{2\beta}-\frac{i\mu}{2\pi},\frac{1}{2\beta}-\frac{i\mu}{2\pi}\right) = \\ = \frac{\pi}{\beta rs\psi}\Bigg\{-\frac{\cos(2\mu r)}{r}\left[s\,d\cosh(vs)+\frac{1}{2}(s^2-r^2+d^2)\sinh(vs)\right] + \frac{\cosh(\bar{v}r)}{r}\left[s\,d\cosh(vs)+\frac{1}{2}(r^2+s^2+d^2)\sinh(vs)\right] + \sinh(\bar{v}r)\Big[s\cosh(vs)+d\sinh(vs)\Big]\Bigg\} \tag{3.79}$$

We will now take several limits of the above solution and show that it reduces to the known results.

The trivial holonomy $\upsilon \to 0$ limit

In this limit we should recover the result of the Harrington-Shephard instanton at finite chemical potential. This solution was discussed in [36] and is given by

$$\psi^A_\alpha = \frac{1}{2\pi\rho} e^{\mu x_4} \sqrt{\Pi} \partial_\nu \left(\frac{\Phi}{\Pi} e^{\mu x_4} \right) [\bar{\sigma}^\nu \epsilon]_{A\alpha} \tag{3.80}$$

where

$$\Pi = 1 + \frac{\pi\rho^2}{r\beta} \frac{\sinh(2\pi r/\beta)}{\cosh(2\pi r/\beta) - \cosh(2\pi x_4/\beta)} \tag{3.81}$$

$$\Phi = (\Pi - 1) \left\{ \frac{\cos(\mu r)\cos(\pi t/\beta)}{\cosh(\pi r/\beta)} + \frac{\sin(\mu r)\sin(\pi t/\beta)}{\sinh(\pi r/\beta)} \right\} \tag{3.82}$$

Now if we take the $\upsilon \to 0$, notice that all dependence on s in equations (3.70) disappears, as it should because the Harrington-Shephard caloron is spherically symmetric. Firstly we have

$$\hat{\psi} = -\cos(2\pi x_4/\beta) + \cosh(2\pi r/\beta) \ , \tag{3.83}$$

$$\psi = -\cos(2\pi x_4/\beta) + \cosh(2\pi r/\beta) + \frac{\pi\rho^2}{r\beta} \sinh(2\pi r/\beta) \tag{3.84}$$

$$\phi = 1 + \frac{\pi\rho^2}{r\beta} \frac{\sinh(2\pi r/\beta)}{\cosh(2\pi r/\beta) - \cos(2\pi x_4/\beta)} = \Pi \tag{3.85}$$

Further for $\hat{f}$ function we get

$$\begin{aligned} \hat{f}^+ = \hat{f}^- &= \frac{\pi}{r\beta\psi} e^{-\mu x_4} \left\{ e^{i\pi x_4/\beta} \sinh\left(\frac{\pi r}{\beta} - i\mu r\right) + e^{-i\pi x_4/\beta} \sinh\left(\frac{\pi r}{\beta} + i\mu r\right) \right\} \\ &= \frac{2\pi}{r\beta\psi} e^{-\mu x_4} \left\{ \cos(\mu r)\cos\left(\frac{\pi x_4}{\beta}\right)\sinh\left(\frac{\pi r}{\beta}\right) \right. \\ &\qquad \left. + \sin(\mu r)\sin\left(\frac{\pi x_4}{\beta}\right)\cosh\left(\frac{\pi r}{\beta}\right) \right\} = \frac{e^{-\mu x_4}}{\rho^2}\frac{\Phi}{\Pi} \end{aligned} \tag{3.86}$$

so that the projection matrices $\frac{1+\tau^3}{2}$ and $\frac{1-\tau^3}{2}$ in (3.74) combine to unity. Combining everything we recover (3.80).

For plotting the pseudo-density of the zero mode we also give

$$\hat{f} = \frac{\pi^2\rho^2}{r^2\beta^2}\frac{-\cos(2\mu r) + \cosh(2\pi r/\beta) + \frac{r\beta}{\pi^2\rho^2}\sinh(2\pi r/\beta)}{-\cos(2\pi x_4/\beta) + \cosh(2\pi r/\beta) + \frac{\rho^2\pi^2}{r\beta}\sinh(2\pi r/\beta)} \tag{3.87}$$

Instanton limit $\beta \to \infty$

The zero temperature limit of the caloron reduces to that of the instanton on $\mathbb{R}^4$. Indeed all of the dependence on $\bar{v}$ and v carries the β dependence. Taking $\beta \to \infty$ and keeping ρ fixed we get the same solution (3.80) with $\beta \to \infty$, i.e.

$$\Pi = 1 + \frac{\rho^2}{r^2 + x_4^2}\ , \tag{3.88}$$

$$\Phi = \frac{\rho^2}{r^2 + x_4^2}\left(\cos(\mu r) + \frac{t}{r}\sin(\mu r)\right) \tag{3.89}$$

which, of course, agrees with references [36, 11, 3].

Again we give $\hat{f}$ in this limit

$$\hat{f} = \frac{1}{r^2 + t^2 + \rho^2}\left(1 + \frac{\sin^2(\mu r)\rho^2}{r^2}\right) \tag{3.90}$$

Single monopole $d \to \infty$ limit

We finally take the single monopole limit, i.e. infinite caloron size $d \to \infty$. In this limit $\rho = \sqrt{d/\pi} \to \infty$, $s = \sqrt{d^2 + r^2 - 2dr\cos\theta} \to \infty$. The $\hat{f}$ function becomes

$$\hat{f} = \frac{\pi}{r\beta}\frac{\cosh(r\bar{v}) - \cos(2\mu r)}{\sinh(r\bar{v})}\ . \tag{3.91}$$

The function ϕ becomes

$$\phi \approx \frac{2d}{r}\frac{1}{\coth(\bar{v}r) - \cos\theta}\ , \tag{3.92}$$

so it is linear in $d = \pi\rho^2$. Therefore we take

$$\lim_{\rho\to\infty}\rho^2\hat{f}^{\pm} = e^{\mp i\frac{\bar{v}x_4}{2} - \mu x_4}\frac{\sinh\left(\frac{\bar{v}r}{2} \pm i\mu r\right)}{\sinh(\bar{v}r)} \tag{3.93}$$

$$\tag{3.94}$$

or

$$\lim_{\rho\to\infty}\rho^2\hat{f}^{\pm} = \frac{1}{2}e^{\mp i\frac{\bar{v}x_4}{2} - \mu x_4}\left(\frac{\cos(\mu r)}{\cosh\left(\frac{\bar{v}r}{2}\right)} \pm i\frac{\sin(\mu r)}{\sinh\left(\frac{\bar{v}r}{2}\right)}\right) \tag{3.95}$$

and the zero mode becomes

$$\psi_\alpha^A = \frac{1}{2\sqrt{2\pi r\beta}} \frac{1}{\sqrt{\coth(\bar{v}r) - \cos\theta}} e^{\mu x_4}$$
$$\times \left\{ e^{-i\frac{vx_4}{2}\tau^3} \partial_\nu \left[e^{-\mu x_4} \left(\frac{\cos(\mu r)}{\cosh\left(\frac{\bar{v}r}{2}\right)} + i\tau_3 \frac{\sin(\mu r)}{\sinh\left(\frac{\bar{v}r}{2}\right)} \right) e^{-i\frac{\bar{v}x_4}{2}\tau^3} \right] \bar{\sigma}^\nu \epsilon \right\}_{A\alpha} . \quad (3.96)$$

The above solution clearly has a quasi-Dirac string-like singularity[13] by the appearance of the factor $1/\sqrt{\coth(\bar{v}r) - \cos\theta}$, which is completely in the core, but exponentially approaches the Dirac string behavior.

Doing the space-time derivative we obtain the form

$$\psi_\alpha^A = \frac{\bar{v}T^{3/2}}{8\sqrt{2\pi r}} \left[e^{-\pi i x_4/\beta\tau_3} Q \left(\frac{\sin(\mu r)}{\sinh\left(\frac{\bar{v}r}{2}\right)} + \frac{2\mu}{\bar{v}} \frac{\cos(\mu r)}{\cosh\left(\frac{\bar{v}r}{2}\right)} \right) \epsilon \right.$$
$$\left. - i \left(\frac{\cos(\mu r)}{\cosh\left(\frac{\bar{v}r}{2}\right)} + \frac{2\mu}{\bar{v}} \frac{\sin(\mu r)}{\sinh\left(\frac{\bar{v}r}{2}\right)} \right) \hat{\boldsymbol{r}} \cdot \boldsymbol{\tau}\epsilon \right]_{A\alpha} \quad (3.97)$$

where

$$Q = \frac{\tau_3 \hat{\boldsymbol{r}} \cdot \boldsymbol{\tau} + \tanh\left(\frac{\bar{v}r}{2}\right)}{\sqrt{\coth(\bar{v}r) - \cos\theta}} . \quad (3.98)$$

Note that all of the dependence on θ has been put into Q. By computing $Q^\dagger Q$ one easily sees that

$$Q = 2\frac{\sinh\left(\frac{\bar{v}r}{2}\right)}{\sqrt{\sinh(\bar{v}r)}} \mathcal{U}(r,\theta) \quad (3.99)$$

where $\mathcal{U}(r,\theta)$ is a space-dependent unitary matrix. So finally

$$(i\hat{\boldsymbol{r}} \cdot \boldsymbol{\tau})\mathcal{U}^\dagger e^{\pi i x_4 \tau_3/\beta} \psi =$$
$$= \frac{\bar{v}T^{3/2}}{4\sqrt{2\pi r \sinh(\bar{v}r)}} T^{3/2} \left[\left(\cos(\mu r) \tanh\left(\frac{\bar{v}r}{2}\right) + \frac{2\mu}{\bar{v}} \sin(\mu r) \right) \right.$$
$$\left. - i \left(-\sin(\mu r) \coth\left(\frac{\bar{v}r}{2}\right) + \frac{2\mu}{\bar{v}} \cos(\mu r) \right) \hat{\boldsymbol{r}} \cdot \boldsymbol{\tau} \right] \quad (3.100)$$

where on the left we have performed the "untwisting" by $e^{\pi i x_4 \tau_3/\beta}$ and the space-dependent gauge transform $i\hat{\boldsymbol{r}} \cdot \boldsymbol{\tau}\mathcal{U}^\dagger$. The solution on the RHS is precisely the radial-gauge solution we had for the instanton-monopole (see eqs. (3.60) and (3.67)).

In our work [28] the fermionic zero modes at finite chemical potential were confirmed

[13] In this gauge the fields approach the singularity exponentially fast, but since $\coth(\bar{v}r) - \cos\theta > 0$ there is no true singularity except at $r \to \infty$.

on the lattice using staggered fermions[14]. In addition the spectrum in the presence of a caloron and in the vacuum was compared with striking similarities, except for the presence of the extra zero mode. This will justify the factorization of the Dirac determinant into a (quasi-)zero mode part and the perturbative part, which was used in the Instanton Liquid Model simulations and which we argue in the next section for the instanton-monopoles as well.

3.3.5 The hopping matrix element

In this section we discuss the so-called hopping matrix element. It is an important ingredient in constructing models a-la Instanton Liquid Model, where it was extensively used. It incorporates the soliton interactions via fermionic zero mode exchange[15]. To understand the interactions due to fermions, consider the Dirac operator in the presence of a KK monopole and a KK anti-monopole, because it is these that carry a zero mode for anti-periodic fermion boundary conditions. We have seen that such objects carry a left and the right handed zero mode respectively. Their gauge field can approximately be written as

$$A_\mu^{tot} \approx A_\mu^I + A_\mu^{\bar{J}} \tag{3.101}$$

where the index I labels a monopole and the index $\bar{J}$ labels an anti-monopole.

We want to discuss the effect of fermions on this configuration. What is then the usual assumption is that the Dirac determinant decomposes into the determinant in the quasi-zero mode[16] basis and the determinant over the plane-wave scattering states. This assumption is reasonable as the solitons are localized objects and should not affect much the plane waves of the theory (except to correctly renormalize the coupling and etc.)

Keeping that in mind we can write that the Dirac determinant of the soliton–anti-soliton system is given by the 2×2 matrix of their would-be zero modes

$$T_{I\bar{J}} = \int d^4x \; \Psi_I^\dagger(-\mu) i \not{D}(\mu) \Psi_{\bar{J}}(\mu) \tag{3.102}$$

where $\Psi_I, \Psi_{\bar{J}}$ are the would-be zero modes of the monopole and anti-monopole. However because $(\not{\partial} - i\not{A}^{I,\bar{J}} - \mu\gamma_4)\Psi_{I,\bar{J}} = 0$ our approximation (3.101) simplifies the above

[14] Although staggered fermions do not have strict zero modes, their eigenvalues become zero in the continuum limit up to machine precision.

[15] Note that the alternative way to include the zero mode exchange is via the 't Hooft vertex discussed in the next chapter on supersymmetric theories. The method described here, however, is simpler for simulations of the solitonic ensembles.

[16] They are no longer zero modes as the configuration is neither self-dual nor anti–self-dual, and the topological charge is zero, so the zero modes get lifted.

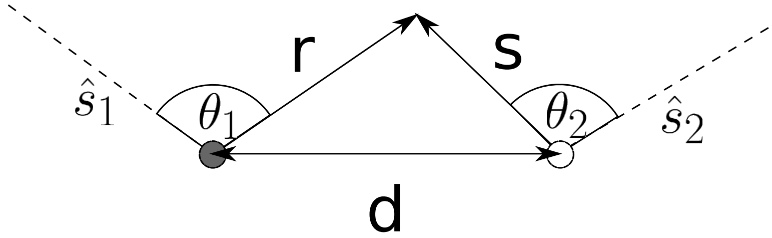


Figure 3.8: The sketch of the monopole–anti-monopole system and their string orientation labeled by the $\hat{\boldsymbol{s}}_{1,2}$ respectively.

expression considerably (see e.g. [92])

$$T_{I\bar{J}} = \int d^4x \; \Psi_I^\dagger(-\mu)(-i\partial\!\!\!/ + i\mu\gamma_4)\Psi_{\bar{J}}(\mu) \; . \tag{3.103}$$

So far we haven not assumed any particular structure of solutions, except that their configurations can be approximated by (3.101) and that they carry topological zero modes. The above expression clearly depends on the relative distance between the solutions, as the zero modes themselves depend on their positions. In the case of an instanton the above matrix element should be modified to account for the rigid $SU(2)$ rotations of the instanton, so an $SU(2)$ space-time independent matrix U should be inserted in between the fermionic states. This is not important for a single instanton, as nothing depends on its color orientation, but for multi-instanton ensembles their relative color orientation matters.

On the other hand, for monopoles the unbroken gauge symmetry is $U(1)$ and the matrix should be a $U(1)$ matrix

$$U = e^{i\alpha\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}} \; , \tag{3.104}$$

in the gauge where $A_4^\infty \propto v\frac{\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}}{2}$, i.e.

$$T_{I\bar{J}} = \int d^4x \; \Psi_I^\dagger(-\mu)(-i\partial\!\!\!/ + i\mu\gamma_4)U\Psi_{\bar{J}}(\mu) \; . \tag{3.105}$$

We want to use the single monopole zero mode solutions of the previous section to study this matrix element. There are several things we must keep in mind

- The radial gauge solution given by (3.60) and (3.67) is not appropriate for the superposition of monpoles, as their asymptotic fields do not match, i.e. they go as $A_4^I \propto \hat{\boldsymbol{r}}\cdot\boldsymbol{\tau}$ and $A^{\bar{J}} \propto \hat{\boldsymbol{s}}\cdot\boldsymbol{\tau}$ where $\hat{r}$ and $\hat{\boldsymbol{s}}$ are the direction vectors from the centers of the monopole and anti-monopole to the observation point respectively. The solution (3.96) is much more convenient as it is in a gauge where the holonomy is in a single color direction.

- In order to be able to write the approximate configuration of the monopole and anti-monopole, we must go to the algebraic gauge where $A_4^\infty = 0$, so the solution (3.96) should be multiplied by $e^{i\frac{vx_4}{2}\tau^3}$.

- The caloron solution, as well as the monopole solution (3.96) derived from it is in the gauge where both the holonomy and the quasi-string direction is the 3-direction of color and space respectively, i.e. they are locked to each other. However we can orient the strings of the monopole and the anti-monopole to arbitrary directions parametrized by unit vectors $\hat{\boldsymbol{s}}_1$ and $\hat{\boldsymbol{s}}_2$ respectively (see Fig. 3.8).

To address the third point we can simply generalize the solution (3.96) to have a string along $\hat{\boldsymbol{s}}_1$ by replacing $\tau_3 \to \pm\hat{\boldsymbol{s}}_1 \cdot \boldsymbol{\tau}$, and similarly for the anti-monopole[17] solution i.e. we have

$$\psi_{I\,\alpha}^{\;A} = \frac{1}{2\sqrt{2\pi r\beta}}\frac{1}{\sqrt{\coth(\bar{v}r) - \cos\theta_1}}e^{\mu x_4} \times \left\{\partial_\nu\left[e^{-\mu x_4}\left(\frac{\cos(\mu r)}{\cosh\left(\frac{\bar{v}r}{2}\right)} - i\hat{\boldsymbol{s}}_1\cdot\boldsymbol{\tau}\frac{\sin(\mu r)}{\sinh\left(\frac{\bar{v}r}{2}\right)}\right)e^{i\frac{\bar{v}x_4}{2}\hat{\boldsymbol{s}}_1\cdot\boldsymbol{\tau}}\right]\bar{\sigma}^\nu\epsilon\right\}_{A\alpha} . \tag{3.106a}$$

$$\psi_{\bar{J}\,\alpha}^{\;A} = \frac{1}{2\sqrt{2\pi r\beta}}\frac{1}{\sqrt{\coth(\bar{v}r) - \cos\theta_2}}e^{\mu x_4} \times \left\{\partial_\nu\left[e^{-\mu x_4}\left(\frac{\cos(\mu r)}{\cosh\left(\frac{\bar{v}r}{2}\right)} + i\hat{\boldsymbol{s}}_2\cdot\boldsymbol{\tau}\frac{\sin(\mu r)}{\sinh\left(\frac{\bar{v}r}{2}\right)}\right)e^{-i\frac{\bar{v}x_4}{2}\hat{\boldsymbol{s}}_2\cdot\boldsymbol{\tau}}\right]\sigma^\nu\epsilon\right\}_{A\alpha} . \tag{3.106b}$$

where the angles $\theta_{1,2}$ are defined in Fig. 3.8. Note that the holonomies of the solutions for I monopole and the $\bar{J}$ anti-monopole have holonomies pointing in $-\hat{\boldsymbol{s}}_1$ and $\hat{\boldsymbol{s}}_2$ respectively[18]. To put both monopoles into the gauge where they have the same holonomy (i.e. same boundary twist, as we are in the algebraic gauge) we make use of the transformations

$$\mathcal{U}_1\hat{\boldsymbol{s}}_1\cdot\boldsymbol{\tau}\,\mathcal{U}_1 = -\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}\ , \qquad \mathcal{U}_2\hat{\boldsymbol{s}}_2\cdot\boldsymbol{\tau}\,\mathcal{U}_2 = \hat{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}\ , \tag{3.107}$$

where $\hat{\boldsymbol{\omega}}$ is an arbitrary vector parametrizing the direction of the asymptotic Polyakov

[17] To get the anti-monopole (i.e. anti–self-dual monopole) we replace $A_\mu = (A_i(\boldsymbol{x}, x_4), A_4(\boldsymbol{x}, x_4)) \to \tilde{A}_\mu = (A_i(\boldsymbol{x}, x_4), A_4(\boldsymbol{x}, x_4))$. Now since $\not{D}(A)\psi(\boldsymbol{x}, x_4) = 0$ then $\not{D}(\tilde{A})\gamma_4\psi(-\boldsymbol{x}, x_4) = 0$ so one can construct an anti-monopole zero mode by flipping $\boldsymbol{x} \to -\boldsymbol{x}$ and multiplying by γ_4 (i.e. changing left to right). Note that the Dirac string must also flip.

[18] The caloron solution of [66] in the form we quoted in Chapter 2, as well as the zero mode solutions of the previous sections is in a gauge where the line connecting the BPS monopole and the KK monopole is in the positive 3-direction

loop. So finally we need to compute the hopping matrix element

$$T_{I\bar{J}} = \int d^4x \; \psi_I^\dagger(-\mu)\mathcal{U}_2^\dagger U \mathcal{U}_1(-i\partial\!\!\!/ + i\mu\gamma_4)\psi_{\bar{J}}(\mu) \tag{3.108}$$

Note that for the instanton the relative (rigid) $SU(2)$ transformation was relevant. Here the relative quasi-abelian transformation U is relevant, whereas the rest of the moduli $SU(2)/U(1)$ is parametrized by string orientations, i.e. transformations $\mathcal{U}_{1,2}$.

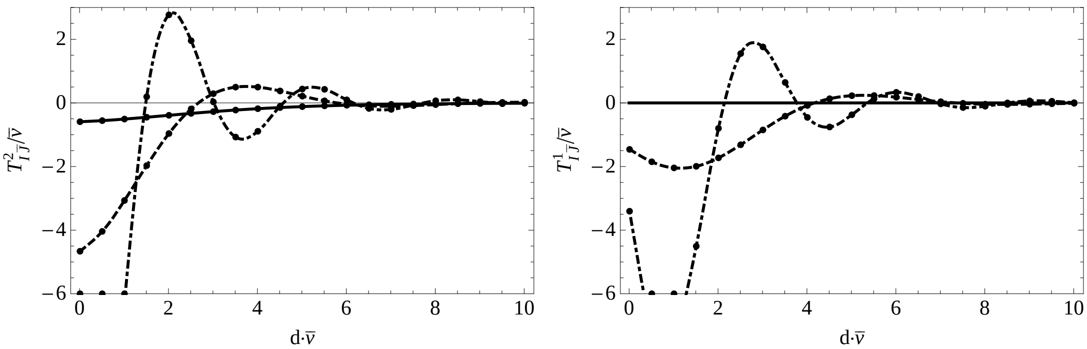


Figure 3.9: Plots of numerical integration of functions $T_{I\bar{J}}$ function of distance d between the KK monopole and the KK anti-monopole in units of $\bar{v}$ the plot shows the results for $\mu = 0$ (solid), $\mu = \bar{v}$ (dashed) and $\mu = 2\bar{v}$ (dot-dashed). Note that the behavior is similar to that of the behavior of two spatially separated instantons (see Fig. 1 in [92]). Figures taken from [28].

Since we can expand $U = \cos\alpha + i\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\tau}\sin\alpha$ we can write the matrix element as

$$T_{I\bar{J}} = i(T^1_{I\bar{J}}\cos\alpha + T^2_{I\bar{J}}\sin\alpha)\ . \tag{3.109}$$

We take the string so that they point oppositely to each other and to lie on the 3-axis. Then $\hat{\boldsymbol{s}}_{1,2} = (0,0,\mp 1)$ and $\mathcal{U}_{1,2} = \mathbf{1}$.

For $\mu = 0$ it can be shown analytically that $T^1_{I\bar{J}}$ vanishes. This is purely because of the color and spin index structure of the expressions. However for $\mu \neq 0$ the contribution of the second term in parenthesis of the expressions (3.106) is no longer vanishing, and $T^1_{I\bar{J}} \neq 0$.

The integral needed to compute $T_{I\bar{J}}$ has to be done numerically, which is given by real functions $T^{1,2}_{I\bar{J}}$ as a function of distance. The results are shown in Fig. 3.9 and the corresponding logarithmic plots are shown in Fig. 3.10. Notice that the asymptotic behavior of the envelope of the matrix element is $T_{IJ} \sim e^{-vd/2}$ where d is the distance between the monopole and the anti-monopole. What is clearly visible is that the chemical potential introduces an oscillatory correlation between monopole and anti-monopole. This effect is very similar to what happens in the case of instantons [92].

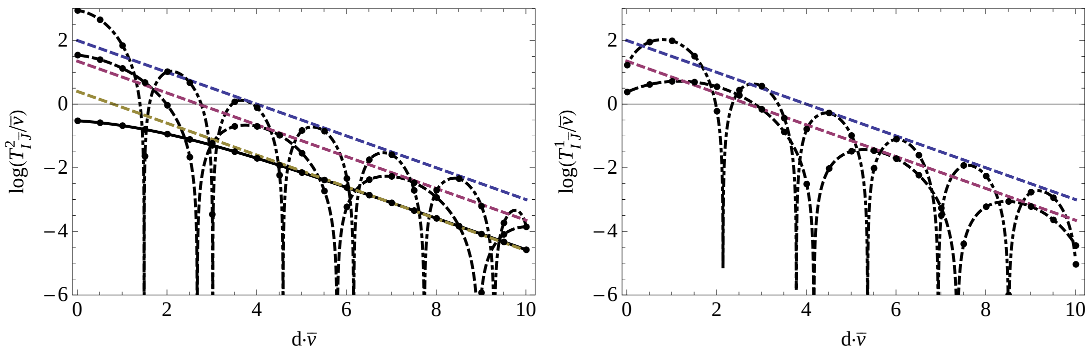


Figure 3.10: The logarithmic plots of $\log(|T_{I\bar{J}}|)$ as function of distance d between the monopole and anti-monopole. The curves are for the same values of μ as in Fig. 3.9. The straight lines are linear functions with slopes $-1/2$ to guide the eye. Figures taken from [28].

3.4 Summary

In this chapter we have extensively studied the zero modes in topological backgrounds. We have seen that Dirac fermions in $2+1$ dimension have interesting behavior when they move in the $U(1) \times U(1)$ magnetic field backgrounds and where there is an excess of one of the $U(1)$ charges over the other. Due to the non-matching degeneracies of the zero modes (and for uniform fields, excited states as well) the charge is generated whenever an iso-spin chemical potential is present. Further, non-uniform field show a localized charge distribution in the region where the field is localized. By using different magnetic field profiles the charge can be manipulated and we have shown that for generic, algebraically decaying (and cylindrically symmetric) profiles of magnetic fields, halo charge distribution may form.

These phenomena in $2+1$ could have direct application in graphene and the field of *valleytronics*, where valley-biased currents are used for transport. There the role of the pseudo-magnetic field is played by non-uniform in-plane strain of the graphene. In [69] the strain magnetic field was measured to produce a pseudo-magnetic field equivalent to 300T, so it is feasible that the method of charge catalysis can be used to manipulate valley-charge with chemical potential, and charge with valley chemical potential. Further, by localized fields, one may be able to do high-precision manipulation of this charge.

We have also shown that the charge catalysis exists in a more intricate scenario of instanton-monopoles in magnetic field. Although the configuration has no chemical potential, we have seen that because of the nontrivial holonomy an imaginary charge gets generated. This, we explained, is an artifact of the theory in fixed gauge backgrounds as the A_4 gauge field, which acts like an imaginary chemical potential, serves as a Lagrange multiplier imposing Gauss constraint, and not a physical field, so the result is meaningless without integration over the holonomy A_4. The effect, however, does have an influence on operators which are quadratic in the charge density. We have illustrated this explicitly for a simple setup of non-interacting fermions in $U(1) \times U(1)$ magnetic fields and have shown that the presence of two magnetic fields decreases the charge *fluctuations*. This might be relevant for the Heavy Ion Collision experiments where charge fluctuation of finite hot and dense QCD systems can be measured.

Finally, the chapter was concluded by deriving the zero modes of instanton-monopoles, as well as of calorons in the presence of the chemical potential. The solution is qualitatively very similar to the equivalent instanton solution with chemical potential, exhibiting the so-called *Friedel oscillations* [61, 52] in space. These are the consequence of non-participation of fermions below the Fermi surface and are well known from experimental observations of Fermi gases at finite density[19]. The zero modes were used to construct the hopping matrix element which was key in Instanton Liquid Model simulations at finite baryon density [92, 94]. This will hopefully lead to a phenomenological microscopic model of instanton-monopoles which has both chiral symmetry breaking and confinement, two of the most important phenomena of QCD.

[19] One should warn that in QCD the Fermi surface is unstable because of the attractive channels in the quark-quark interactions, so that di-quark condensates can form in dynamical models. Since we have not discussed the dynamics here, the asymptotic zero modes show the qualitative behavior of the free fermionic propagator in the Fermi sea background and, therefore, the Friedel oscillations.

Chapter 4

Dynamical Models

4.1 Supersymmetry and instanton-monopoles

In this section we will discuss the instanton-monopoles in supersymmetric theories. We will restrict ourselves to $SU(2)$ gauge theories. As we will see in the next section, the instanton-monopoles will be crucial for understanding the infrared dynamics of supersymmetric gauge theories on $\mathbb{R}^3 \times S^1$. One might wonder, however, why not try to apply the calculation directly to Yang-Mills theory. Well firstly, a pure Yang-Mills theory compactified on $\mathbb{R}^3 \times S^1$ is a thermal theory, and it is well known that a thermal theory at large temperature (i.e. small compact radius) is deconfined. In other words a Polyakov loop potential which has a minimum at $A_0 = 0$ forms [57, 113] and no Higgs phenomenon takes place.

It is not very difficult to compute why this happens, if we again take the gauge $A_4 = vT^3$, where T^3 is the Cartan generator of $SU(2)$, and note that it appears as a color chemical potential in the covariant Laplace operator $-D_\mu^2$. It is well known that the finite part of the free energy[1] of bosons with chemical potential μ is proportional to $\sum_n g_n \ln(1 - e^{(\mu - E_n)\beta})$ so that

$$F_{gauge}\beta = 2V \int \frac{d^3k}{(2\pi)^3} \mathrm{tr} \ln(1 - e^{-k\beta + iA_4\beta}) \tag{4.1}$$

where the factor of 2 is due to the fact that there are two physical polarizations of gluons. Writing $A_4\beta = v\beta T^3$ where T^3 is the generator in the adjoint representation,

[1] The free energy, of course, has an UV divergent part, which, however, does not depend on the field v nor on the temperature, and therefore we ignore it.

and expanding the logarithm we easily get

$$F_{gauge}\beta = -2V\frac{1}{\pi^2}\sum_{n=1}^{\infty} 2^n \frac{\cos(nv\beta)}{n^4} \; . \tag{4.2}$$

Quite clearly all the terms in the sum[2] prefer $v = 0 \mod 2\pi/\beta$.

If we introduce adjoint fermions, and compactify them with periodic boundary conditions, they will have the same free energy with the opposite sign and exactly cancel the gauge field contribution. This is a supersymmetric YM theory, and it is the first step in our discussions of supersymmetry. This theory was discussed in detail in the works[3] [37, 87] as well as [38, 86] for higher gauge groups. We will review this work in the next section, because the results will be necessary for the introduction of the massive fundamental multiplet in Sec. 4.1.2.

4.1.1 The instanton-monopoles in SU(2) and the superpotential

In the supersymmetric theory on $\mathbb{R}^3 \times S^1$ due to the presence of gauginos, the perturbative potential for the holonomy A_4 cancels exactly. Before we show this, however, it will serve us well to define the integration over the Weyl fermions in Euclidean space. Namely the Dirac action reads

$$\mathcal{L}_{dirac} = \bar{\lambda} i \bar{\sigma}^{\mu} D_{\mu} \lambda \; . \tag{4.3}$$

Since $\bar{\mathcal{D}} = \bar{\sigma}^{\mu} D_{\mu}$ has no definite Hermiticity property, and indeed cannot have any as it maps spinors which transform under one irreducible representation $SU(2)$ of $SO(4)$ to the other one, and we should be careful in defining what we mean by integration over the Weyl fermions λ. Instead of dealing with an awkward Dirac operator, consider operators $\mathcal{D}\mathcal{D}^{\dagger}$, $\mathcal{D}^{\dagger}\mathcal{D}$, where $\mathcal{D} = \sigma^{\mu} D_{\mu}$. Notice that since $\bar{\mathcal{D}} = -\mathcal{D}^{\dagger}$, these operators are Hermitian and positive definite[4], so that we can construct an eigenbasis ψ_n of the operator $\mathcal{D}^{\dagger}\mathcal{D}$ and $\tilde{\psi}_n$ of the operator $\mathcal{D}\mathcal{D}^{\dagger}$ i.e.

$$\mathcal{D}\mathcal{D}^{\dagger}\psi_n = \epsilon_n \psi_n \; , \qquad\qquad \mathcal{D}^{\dagger}\mathcal{D}\tilde{\psi}_n = \epsilon_n \tilde{\psi}_n \; . \tag{4.4}$$

[2]The higher n terms have other minima, but they are not global minima of the sum.

[3]The works by Davies et. al, although deriving the effective action exactly, have done so relying highly on supersymmetries, and not having the microscopic picture in mind. On the other hand Poppitz, Schafer and Unsal have understood the terms in the effective action microscopically which allowed them to break SUSY softly and induce a phase transition much like that in the thermal YM theory.

[4]The positive definiteness follows from the fact that if $D^{\dagger}D|\psi\rangle = \lambda|\psi\rangle$, then it follows that $\langle\psi\left|\mathcal{D}^{\dagger}\mathcal{D}\right|\psi\rangle = \lambda \Rightarrow ||\mathcal{D}\psi|| = \lambda$ where $||\ldots||$ is the norm. Since the norm is always larger then zero, positivity of eigenvalues follows.

We have made explicit that the spectrum of the two operators is identical, which can be checked by taking[5] $\psi_n \propto \mathcal{D}\tilde{\psi}_n$. Normalizing we have $\psi_n = \frac{1}{\sqrt{\epsilon_n}}\tilde{\psi}_n$.

Then we can expand $\lambda = \sum_n \xi_n \psi_n(x)$ and $\bar{\lambda} = \sum_n \bar{\xi}_n \tilde{\psi}_n(x)$ where $\xi_n, \bar{\xi}_n$ are Grassmann numbers. Then the Dirac action becomes

$$\int d^4x \bar{\lambda} i \sigma^\mu D_\mu \lambda = \sum_n \bar{\xi}_n \xi_n (-i)\sqrt{\epsilon_n} \ . \tag{4.5}$$

Now *defining* the measure of integration as

$$\int \mathcal{D}\lambda \mathcal{D}\bar{\lambda} = \prod_n \int d^2\xi_n \tag{4.6}$$

we get that the integral over the Grassmann numbers ξ_n

$$\prod_n \sqrt{\epsilon_n} = \sqrt{\det(-\bar{\mathcal{D}}\mathcal{D})} \ , \tag{4.7}$$

where we dropped the i factor as it is simply a constant in front of the partition function on which nothing depends.

An SU(2) $\mathcal{N} = 1$ super QCD Lagrangian contains bosonic gauge fields A_μ, their super-partner gauginos λ, as well as N_f Dirac quarks Ψ and their super-partners squaks (complex scalars) ϕ, with canonical kinetic terms $F_{\mu\nu}^2, \bar{\lambda}\sigma^\mu\partial_\mu\lambda, \bar{\Psi}\not{D}\Psi, |D_\mu\phi|^2$. By integrating out these fields to one loop, the effective action of a supersymmetric QCD is given by

$$V_{1-loop}(A_4) = -\ln\left(\frac{[\det_{ff}(\not{D}+M)]^{N_f}\det_{gh}(-D^2)[\det_{af}(-\bar{\mathcal{D}}\mathcal{D})]^{1/2}}{[\det_{fs}(-D^2+M^2)]^{2N_f}[\det_{gf}(-D^2\delta_{\mu\nu}-F_{\mu\nu})]^{1/2}}\right) \tag{4.8}$$

where $D^2 = D_\mu^2$, the subscripts ff, fs refer to massive *fundamental fermions* (N_f Dirac fermions) and massive *fundamental scalars* ($2N_f$ of them[6]) both with mass M, while gf, gh, af refers to the gauge field determinant, ghost determinant and the adjoint fermion (gaugino) determinant.

Since we want to compute the perturbative contribution to the A_4 potential, we take the background field to be $A_4 = v(x)\frac{\tau^3}{2}$ and $A_i = 0$. Then $F_{\mu\nu} = 0$ and the above one loop determinants cancel exactly[7]. The cancellation is just a balance of degrees of freedom, i.e. N_f Dirac fermions have $4N_f$ polarizations (2 spin and particle-

[5] This is true only for $\epsilon_n \neq 0$. We will treat zero modes separately later.

[6] One for each Weyl flavor

[7] The operators in the fundamental representation $\det_{ff}\not{D} = [\det(-D_{fund}^2 + M^2)]^4$ which cancels the scalar contribution exactly, while the ghost and the gaugino determinant combine into a scalar operator in the adjoint representation $[\det(-D_{adjoint}^2)]^2$ which cancels exactly the gauge field which is $\det_{gf}(-D^2\delta_{\mu\nu})^{1/2} = [\det(-D_{adjoint}^2)]^2$.

antiparticles d.o.f.) while $2N_f$ scalars have $4N_f$ polarization, so the same. Similarly the gauge fields have two physical polarization (4 minus 2 from the ghosts) which cancel the two polarizations of a single Weyl gaugino.

Since the perturbative contribution to the holonomy $\langle A_4 \rangle$ cancels in SUSY theories, it gives us a clean setup to explore the contributions of non-perturbative objects. Before we do so, let us determine what the low energy degrees of freedom are. Assuming that $A_4 = v\frac{\tau^3}{2}$ such that $\langle v \rangle \neq 0$ the Higgs mechanism gives the mass to the 2 out of 3 SU(2) gauge fields, just like in the Georgi-Glashow model. Similarly 2 out of 3 gauginos get a mass through the same mechanism, i.e. due to the presence of the commutator term $[A_4, \lambda]$ in the action, the adjoint gauginos λ which are not in the color direction of A_4 are massive. The low energy effective theory is therefore a 3D[8] $U(1)$ theory with a single flavor of free Weyl fermion λ^3, i.e.

$$\mathcal{S}_{eff} = \frac{L}{g^2} \int d^3x \, \left(\frac{1}{4}\mathcal{F}_{ij}^2 + \frac{1}{2}(\partial_i v)^2 \right) + \int d^3x \, \frac{L}{g^2} \bar{\lambda}^3 i\sigma^i \partial_i \bar{\lambda}^3 + \dots \tag{4.9}$$

where L is the radius of the S^1 and $\mathcal{F}_{ij} = F_{ij}^3$ is the abelian field-strength tensor of the unbroken U(1) gauge group. The dots above indicate the potential terms which can (and, as we shall see, will) be generated by non-perturbative effects. The $U(1)$ theory can be dualized like in the Georgi-Glashow model and the action in the dual description becomes

$$S_{eff}^{dual} = L \int d^3x \left(\frac{g^2}{2(4\pi)^2} \left((\partial_i \sigma)^2 + (\partial_i b)^2 \right) + \frac{i}{g^2} \bar{\lambda} \sigma^i \partial_i \lambda \right) \tag{4.10}$$

where we have defined $b = \frac{4\pi v}{g^2}$. The real scalar fields can be combined into a complex scalar $b + i\sigma$ with a canonical kinetic term. The low energy effective theory is then a theory of a single complex scalar and a single Weyl fermion[9] λ in three dimensions. Keeping in mind that this theory has to be supersymmetric, the theory should be describable by a single chiral superfield. In fact, up to field redefinitions, the action (4.10) is the same as the action of the Lagrangian (2.89), up to compactification in the 4-direction (i.e. ignoring field dependence on the x_4 coordinate).

However apart from the kinetic terms in the theory, a superpotential is also allowed. The question is then whether a superpotential is somehow generated non-perturbatively? To that end let us consider a BPS instanton-monopole configuration, and for the moment ignore the fundamental matter. According to the index theorem discussed in Section 2.4 an instanton-monopole background has two zero modes of the operator $\sigma^\mu D_\mu$ in the adjoint representation on $\mathbb{R}^3 \times S^1$. In fact we can explicitly construct these solutions by noticing that

$$\sigma^\mu D_\mu (\bar{\sigma}^{\nu\rho} F_{\nu\rho}) = 0 \tag{4.11}$$

[8] Recall that we are on $\mathbb{R}^3 \times S^1$ and the long distance theory is effectively $\mathbb{R}^3$.

[9] Not to clutter notation we drop the color superscript 3 from now on

where $\bar{\sigma}^{\mu\nu} = \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$. To show that the above equation is correct we use the identity (A.16)

$$\sigma^\mu\bar{\sigma}^{\nu\rho} = \delta^{\mu\nu}\sigma^\rho - \delta^{\mu\rho}\sigma^\nu + \epsilon^{\mu\nu\rho\sigma}\sigma^\sigma \tag{4.12}$$

then (4.11) becomes

$$\sigma^\mu D_\mu(\bar{\sigma}^{\mu\nu})F_{\mu\nu} = 2D^\mu F_{\mu\rho} + \epsilon^{\mu\nu\rho\sigma}D_\mu F_{\nu\rho}\sigma^\sigma \ . \tag{4.13}$$

The first term on the right vanishes because of the equations of motion satisfied by self-dual fields, and the second term vanishes because of the Bianchi identity, so that (4.11) is satisfied. Then we can construct a zero mode[10]

$$\lambda_{0,\alpha} = (\bar{\sigma}^{\mu\nu})_{\alpha\dot{\beta}}F^a_{\mu\nu}\xi^{\dot{\beta}} \tag{4.14}$$

where ξ is an arbitrary (anti-commuting) two spinor, which we take as normalized to one.

Furthermore since $\bar{\sigma}^{ij} = -\epsilon^{ijk}\sigma^k$, $\bar{\sigma}^{i4} = -\sigma^i$, and since $F_{i4} = \frac{1}{2}\epsilon_{ijk}F_{jk}$, one obtains

$$\bar{\sigma}^{\mu\nu}F_{\mu\nu} = \bar{\sigma}^{ij}F_{ij} + 2\bar{\sigma}^{i4}F_{i4} = -2F^{ij}\epsilon^{ijk}\sigma^k \approx 4\frac{\hat{r}^i\sigma^i}{r^2}\frac{\tau^3}{2} \tag{4.15}$$

where in the last step we have used the asymptotics of the instanton monopole (in stringy gauge) $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk} \approx \frac{\hat{r}^i}{r^2}\frac{\tau^3}{2}$, where τ^3 is the color matrix. Now

$$\Delta(\boldsymbol{x}) = \frac{r^i\sigma^i}{4\pi r^3} \tag{4.16}$$

is the fermionic propagator of the effective theory (4.10), so we can write the zero modes asymptotically as

$$\lambda_0^I(\boldsymbol{x} - \boldsymbol{x}_0) \approx 8\pi\Delta(\boldsymbol{x} - \boldsymbol{x}_0)\xi\tau^3 \tag{4.17}$$

where $\boldsymbol{x}_0$ is the position of the instanton-monopole.

Because of the zero mode, in the presence of the BPS instanton-monopole (or indeed any background with zero modes) the quantum weight of the configuration is exactly zero. The question is now how does this term contribute when fermionic sources into the Lagrangian. To that end let us introduce a source term in the Dirac Lagrangian of the form[106] $-\frac{1}{g^2}\bar{\lambda}\bar{J}\bar{\lambda} = -\frac{1}{g^2}\bar{\lambda}_{\dot{\alpha}}J^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}}$ where the source $\bar{J}(x)$ has some spinor structure. The non-zero modes, as before, produce determinants which can in principle be computed, but now $\bar{\lambda}$ contains the zero modes $\lambda_0(\boldsymbol{x} - \boldsymbol{x}_0)$ with the fermionic collective coordinate ξ, i.e.

$$\mathcal{S}_{Dirac} = \int d^4x\frac{1}{g^2}(i\bar{\lambda}\bar{\mathcal{D}}\lambda - i\bar{\lambda}\bar{J}\bar{\lambda}) = -\frac{1}{g^2}\int d^4x\ \lambda_0\bar{J}\lambda_0\ , \tag{4.18}$$

[10]The zero mode here is nothing but the SUSY partner of the translational bosonic zero mode.

In other words we have

$$\int d^2\xi e^{-S_{Dirac}} = \frac{1}{g^2}\int d^2\xi \int d^4x \; \lambda_0(\boldsymbol{x}-\boldsymbol{x}_0)J\lambda_0(\boldsymbol{x}-\boldsymbol{x}_0) \; , \tag{4.19}$$

which, after the integration over Grassmanns, becomes

$$\int d^2\xi e^{-S_{Dirac}} = \frac{(8\pi)^2 L}{g^2}\int d^3x \; \epsilon^{\beta\delta}J^{\gamma\alpha}\Delta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}_0)\Delta_{\gamma\delta}(\boldsymbol{x}-\boldsymbol{x}_0) \tag{4.20}$$

Consider now an insertion $C\lambda\lambda(\boldsymbol{x}_0) = C\lambda_\alpha(\boldsymbol{x}_0)\epsilon^{\alpha\beta}\lambda_\beta(\boldsymbol{x}_0)$ in the effective theory (4.10). Since

$$\left\langle \lambda_\alpha(\boldsymbol{x})\bar{\lambda}_\beta(\boldsymbol{y})\right\rangle = \frac{g^2}{L}\Delta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y}) \tag{4.21}$$

we have

$$\langle C\lambda\lambda(\boldsymbol{x}_0)\rangle = -2C\frac{1}{g^2L^2}g^4\int d^3x \; \epsilon^{\beta\delta}J^{\gamma\alpha}\Delta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}_0)\Delta_{\gamma\delta}(\boldsymbol{x}-\boldsymbol{x}_0) \tag{4.22}$$

which is precisely the term generated by the monopole zero mode. So if a monopole has a weight $\mu_B e^{-S_0}$, where μ_B is the bosonic measure (C.14), then the monopole vertex is

$$-\frac{(8\pi)^2L^3}{2g^4}\mu_B \; e^{-S} \; \lambda\lambda \; . \tag{4.23}$$

However, we have to additionally regulate the fermionic determinant. We do this by dividing by the zero mode measure (4.19) with the insertion[11] $J^{\alpha\beta} = \Lambda_{PV}\epsilon^{\alpha\beta}$, i.e. we must divide by

$$\frac{1}{g^2}\Lambda_{PV}\int d^2\xi \int d^4x \; \lambda_0(\boldsymbol{x}-\boldsymbol{x}_0)\lambda_0(\boldsymbol{x}-\boldsymbol{x}_0) \tag{4.24}$$

Computing the above integral we have

$$\int d^2\xi \int d^4x \; \lambda_0(\boldsymbol{x}-\boldsymbol{x}_0)\lambda_0(\boldsymbol{x}-\boldsymbol{x}_0) = \frac{1}{4}\mathrm{Tr}\left(\{\bar{\sigma}^{\mu\nu},\bar{\sigma}^{\rho\sigma}\}\right)\int d^4x \; \mathrm{tr}\left(F_{\mu\nu}F_{\rho\sigma}\right) \tag{4.25}$$

where Tr is the trace over spinor and tr over color indices and where we have replaced $\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma}$ with the part symmetric in the interchange of $\mu\nu \leftrightarrow \rho\sigma$ because it multiplies the $\mathrm{tr}\left(F_{\mu\nu}F_{\rho\sigma}\right)$ which is symmetric in these indices. Then because

$$\frac{1}{4}\mathrm{Tr}\left\{\bar{\sigma}^{\mu\nu},\bar{\sigma}^{\rho\sigma}\right\} = -\epsilon^{\mu\nu\rho\sigma} \tag{4.26}$$

so that

[11]This corresponds to adding the Pauli-Villars mass term Λ_{PV} to the gauginos.

$$\int d^2\xi \int d^4x\ \lambda_0(\boldsymbol{x}-\boldsymbol{x}_0)\lambda_0(\boldsymbol{x}-\boldsymbol{x}_0) = \epsilon^{\mu\nu\rho\sigma}\int d^4x\ \mathrm{tr}\,(F_{\mu\nu}F_{\rho\sigma}) = \\ = 2\int d^4x\ \mathrm{tr}\,(F_{\mu\nu}^2) = 4g^2 S_{BPS} \quad (4.27)$$

where in we used the self-duality $F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and where S_{BPS} is the action of the BPS monopole. Since $g^2 S_{BPS} = 4\pi v$ we have that

$$\frac{1}{g^2}\Lambda_{PV}\int d^2\xi \int d^4x\ \lambda_0(\boldsymbol{x}-\boldsymbol{x}_0)\lambda_0(\boldsymbol{x}-\boldsymbol{x}_0) = -\frac{\Lambda_{PV}}{g^2}16\pi v L \quad (4.28)$$

so that the measure finally becomes

$$-\frac{2\pi L^2}{g^2 v L \Lambda_{PV}}\mu_B\ e^{-S}\ \lambda\lambda = -\ \frac{16\pi^2 L^3}{g^6}\Lambda_{PV}^3\ e^{-S}\ \lambda\lambda \quad (4.29)$$

where we have plugged in the bosonic measure from (C.14) (and integrated over the collective coordinate α on which nothing depends)

$$\mu_B = \frac{8\pi L^2 v}{g^4}\Lambda_{PV}^4 \quad (4.30)$$

The monopole vertex (4.29) has to be resummed in the same fashion that we resummed monopoles in Sec. 2.1. This procedure introduces a term

$$\mathcal{L}_{BPS} = -\frac{16\pi^2 L^3}{g^6}\Lambda_{PV}^3 e^{-S}\lambda\lambda e^{i\sigma} \quad (4.31)$$

in the effective Lagrangian. We have added the operator $e^{i\sigma}$ to account for the monopole charge in the effective theory.

The action of the monopole is $S = \frac{4\pi v}{g^2}$. We want to use the fields b, σ of the dual action (4.10). It is more convenient, however, to shift the b field by a constant and re-express the vertex in terms of a shifted field $b' = \frac{4\pi v}{g^2} - \frac{4\pi^2}{g^2} = b - \frac{4\pi^2}{g^2}$. This has an advantage that $b' = 0$ is the center symmetric (confining) point. Then the BPS monopole contribution becomes

$$\mathcal{L}_{BPS} = \frac{16\pi^2 L^3}{g^6}\Lambda_{PV}^3 e^{-S_0 - b' + i\sigma}\lambda\lambda \quad (4.32)$$

where $S_0 = \frac{4\pi^2}{g^2}$. The BPS anti-monopole would simply give the hermitian conjugate contribution, so that the combined contribution is

$$\mathcal{L}_{BPS+\overline{BPS}} = \frac{m_\lambda L}{g^2} e^{-b'+i\sigma} \lambda\lambda + \frac{m_\lambda L}{g^2} e^{-b'-i\sigma} \bar{\lambda}\bar{\lambda} \tag{4.33}$$

where $m_\lambda = \frac{16\pi^2 L^2 \Lambda_{PV}^3}{g^4}$. Proceeding in a similar way we can construct the KK monopole contribution. It corresponds to adding a term[12]

$$\mathcal{L}_{KK+\overline{KK}} = \frac{m_\lambda L}{g^2} e^{b'-i\sigma} \lambda\lambda + \frac{m_\lambda L}{g^2} e^{b'+i\sigma} \bar{\lambda}\bar{\lambda} \tag{4.34}$$

so that the effective $3D$ Lagrangian takes the form

$$\begin{aligned} \frac{g^2}{2(4\pi)^2 L} \Big((\partial_i \sigma)^2 + (\partial_i b')^2 + \frac{2(4\pi)^2 L^2}{g^4} \bar{\lambda} i \sigma^i \partial_i \lambda \\ + \frac{2(4\pi)^2 L^2}{g^4} \Big[m_\lambda \lambda\lambda e^{-b'+i\sigma} + m_\lambda \lambda\lambda e^{b'-i\sigma} + h.c \Big] \Big) = \\ = \frac{g^2}{2(4\pi)^2 L} \left(|\partial_i \phi|^2 + i\bar{\psi}\sigma^i \partial_i \psi \right) + \frac{g^2}{2(4\pi)^2 L} \Big[m_\lambda \psi\psi e^{\phi} + m_\lambda \psi\psi e^{-\phi} + h.c. \Big] \end{aligned} \tag{4.35}$$

where

$$\phi = -b' + i\sigma \ , \qquad \psi = \frac{4\pi\sqrt{2}L}{g^2} \lambda \ . \tag{4.36}$$

Notice that the potential in (4.35) is generated by a superpotential

$$W(\Phi) = \frac{g^2}{2(4\pi)^2 L} m_\lambda \left(e^{\Phi} + e^{-\Phi} \right) \tag{4.37}$$

with $\Phi = \phi(y, \theta, \bar{\theta}) + \sqrt{2}\theta\psi(y, \theta, \bar{\theta}) + \theta\theta F(y, \theta, \bar{\theta})$ and $y^i = x^i + \bar{\theta}\sigma^i\theta$. In turn, as discussed in the Section 2.5.1, this superpotential gives a bosonic potential

$$\begin{aligned} V_{bos} = \frac{1}{\partial_\Phi \partial_{\bar{\Phi}} K} \left| \frac{\partial W}{\partial \Phi} \right|^2 \Bigg|_{\Phi=\phi} = \frac{g^2}{2(4\pi)^2 L} m_\lambda^2 |e^{-b'+i\sigma} - e^{b'-i\sigma}|^2 = \\ = \frac{g^2 m_\lambda^2}{2(4\pi)^2 L} (e^{2b'} + e^{-2b'} - e^{i\sigma} - e^{-i\sigma}) \end{aligned} \tag{4.38}$$

What is remarkable is that, by supersymmetry, we get new terms in the Lagrangian, which are required by supersymmetry. In fact these terms have an interpretation. In [87] the term $-\cos(2\sigma)$ was interpreted as *magnetic bion* contribution. These are contribu-

[12]The signs in front of the $i\sigma$ field have changed because the KK monopole has *opposite* magnetic charge to that of the BPS monopole.

tions of a doubly magnetically charged objects, bound by fermion exchange, of correlated $BPS - \overline{KK}$ pair, yielding a contribution $-e^{2i\sigma}$ in the effective action, and similarly of a $KK - \overline{BPS}$ pair yields a term $-e^{2i\sigma}$ in the effective Lagrangian (see Fig. 4.1). The term $\cosh(2b')$ is due to the correlated pair of a BPS (KK) and $\overline{BPS}$ ($\overline{KK}$) giving $e^{2b'}$ and $e^{-2b'}$ which was dubbed the *neutral bion* (see Fig. 4.1). There is a subtlety here however. The magnetic bions can be treated semi-classically because the interaction of the magnetic bion is the fermion zero mode exchange, which is attractive and the b', σ field exchanges, which are repulsive[13]. The dominant contribution of the integral over the distance between the monopole pair comes from the region L/g^2 which is (for weak coupling) much greater then their core sizes $\sim L$, so that the low energy theory can be used there. However for neutral bions both fermion exchange and the σ, b' exchange are attractive[14] the integral over the separation between the two monopoles is dominated by the region where the monopole–anti-monopole heavily overlap, region where all of our assumptions are invalid. Via the analytical continuation in the coupling in the complex plane, Poppitz, Schäfer and Ünsal [87][15] have managed to compute this contribution. Taking the coupling back the amplitude gets a minus sign and the potential $\cosh(2b')$ emerges, in complete agreement with SUSY. This sign is crucial, because it attains a minimum at $b' = 0$, which makes the Polyakov loop in the compact direction confining $\left\langle \mathrm{tr}\, e^{iv\tau^3/2} \right\rangle = 0$, i.e. center symmetric.

With this we conclude our review of the work in pure Yang-Mills. This was important as it sets the groundwork for adding heavy fundamental multiplets [88], which will require computing the one loop determinants around BPS and KK (anti-)monopoles.

4.1.2 sQCD: introducing the fundamental matter multiplet

We want to see how adding fundamental multiplet (quarks and squarks) would change in the SU(2) supersymmetric gauge theory and review our work [88]. Perhaps the most curious thing would be to add a mass-less multiplet and hope to construct a low energy effective $U(1)$ theory of the gauge multiplet and the matter multiplet. There are however several problems with this.

- Because the matter fields are in the fundamental representation they will be given a by the holonomy condensate A_4. This means that if we can construct the theory

[13]As we have already seen in the Georgi-Glashow model the σ exchange amounts to the attraction between a monopole and anti-monopole and repulsion between the monopoles. The instanton-monopoles couple to the σ field as $e^{i\sigma}$ depending on their charge. However they couple to the b' field as $e^{\pm b'}$, with no i. Because of this reason the exchange of "like b charges" is attractive rather then repulsive, and the exchange of unlike charges is repulsive rather then attractive. One can also see that for this reason the BPS monopole which couples as $e^{-b'+i\sigma}$ and KK monopole which couples like $e^{b'-i\sigma}$ do not interact.

[14]Their attractiveness is nothing else then saying that this pair belongs to the perturbative vacuum (i.e. topological charge zero) and therefore its action can be minimized by annihilating them.

[15]See [86] for higher gauge groups

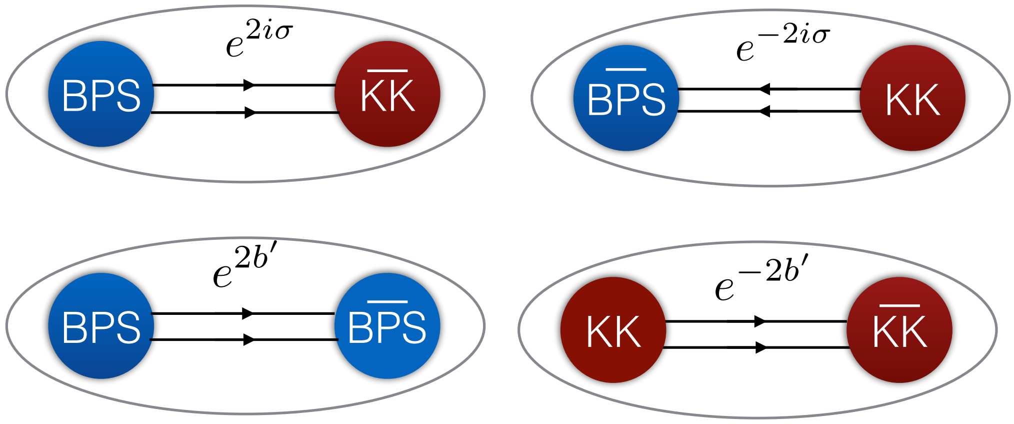


Figure 4.1: Schematic representation of bion molecules in Super YM theory. In the upper part of the figure the gaugino exchange (arrows) binds the $(KK)BPS$ and anti-$(BPS)KK$ monopole to generate a molecule with ± 2 magnetic charges, while the BPS and anti BPS (KK and anti-KK) monopoles bind to form terms $e^{\pm 2b'}$ which stabilize the center symmetry.

in the Coulomb phase where A_4 gets an expectation value, all the matter multiplets are heavy and should be integrated out.

- The fundamental matter is charged under the $U(1)$ and therefore is a magnetic object of the σ field. It is unclear how to take into account this interaction.
- The fundamental matter does not obey the center symmetry and it is possible (and indeed true, unless certain esoteric conditions are satisfied) that the theory will completely collapse to a center broken vacuum where abelianization does not happen and the coulomb phase is lifted.

Despite these problems let us for the moment think about the instanton-monopole vertex and how it would be affected by fundamental fermions. It is important to keep in mind that we can set the fundamental multiplet to obey any periodicity conditions, even periodicity up to a phase, as long as we make both quarks and squarks obey the same condition to preserve supersymmetry.

So what would a vertex of an instanton-monopole look like if we introduced N_f Dirac fermions into the theory? Firstly the topological objects would have more zero modes then just the gaugino zero modes.

To consider the zero modes recall that the index of a BPS monopole in the fundamental representation is given by (2.82)

$$I_{BPS}(f) = -\left\lfloor \frac{vL + 2\varphi}{4\pi} \right\rfloor + \left\lfloor \frac{-vL + 2\varphi}{4\pi} \right\rfloor \tag{4.39}$$

which means that for periodic boundary conditions (i.e. $\phi = 0$) and under the assumption that $vL \in [0, \pi]$ there is a single left-handed zero modes on the BPS instanton-monopole.

On the other hand the the KK monopole has the index

$$I_{KK}(f) = -\left\lfloor \frac{2\pi - vL + 2\varphi}{4\pi} \right\rfloor + \left\lfloor \frac{-2\pi + vL + 2\varphi}{4\pi} \right\rfloor \tag{4.40}$$

which has a zero mode for anti-periodic, but not for the periodic boundary conditions.

The monopole vertices, therefore, would have the following structure (for anti-periodic boundary conditions of the fundamental multiplet)

$$[BPS] = \lambda\lambda e^{b'+i\sigma}\ , \qquad [KK] = \lambda\lambda\psi^{2N_f} e^{-b'-i\sigma} \tag{4.41}$$

$$[\overline{BPS}] = \bar\lambda\bar\lambda e^{b'-i\sigma}\ , \qquad [\overline{KK}] = \bar\lambda\bar\lambda\bar\psi^{2N_f} e^{b'+i\sigma} \tag{4.42}$$

where $\psi^{2N_f} = \epsilon^{i_1 i_2 \ldots i_{2N_f}} \psi_{i_1}\psi_{i_2}\ldots\psi_{i_{2N_f}}$ with $i_1 \ldots i_{2N_f}$ being the (Weyl) flavor index, is the flavor symmetric 't Hooft vertex. Note that the 't Hooft vertex is symmetric under $SU(2N_f)$ flavor transformations[16], i.e. that it transforms as $\psi^{2N_f} \to \det U \psi^{2N_f}$ where U is the flavor transformation matrix. Clearly if $U \in SU(2N_f)$ the vertex is invariant, while the $U(1)$ part is anomalous.

Now, a shift in $\sigma \to \sigma + \delta$ can always be compensated by a corresponding shift in $\lambda \to e^{-i\delta/2}\lambda, \psi \to e^{i\frac{\delta}{N_f}}\psi$. Therefore the potential for the σ field cannot be generated by these monopole vertices, the σ field remains ungapped and Wilson lines remain unconfined even after the monopole resummation (see Sec. 2.1).

Another way of seeing this is to consider the microscopic picture of the magnetic bion formation, discussed in the previous section. In Fig. 4.2 we show schematically the monopole operator. Whereas before we could have formed magnetic bions which gap the theory, this can no longer be done once massless fundamentals are introduced, because there will always be fundamental zero modes attached to the molecule and the contribution vanishes unless there are external fermion sources. In fact there is no way to contract all the fermion zero modes and generate a potential for the σ field. This follows from the symmetry arguments above.

[16]Because of the pseudo-reality of the $SU(2)$ color group the classical flavor symmetry is $U(2N_f)$ instead of $U(1) \times SU_L(N_f) \times SU_R(N_f)$.

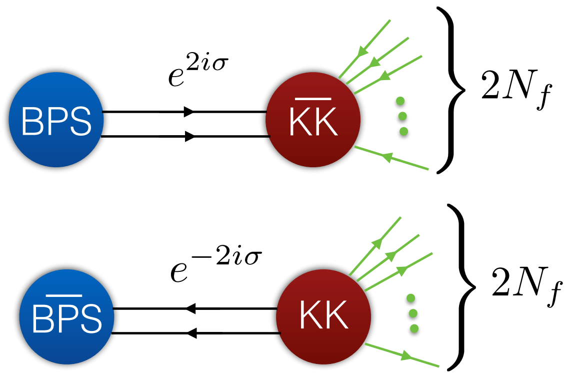


Figure 4.2: The magnetic bion (non-)formation in sQCD. The extra fundamental zero modes prohibit mass gap generation of the σ field.

In lieu of these problems we add a mass to the fundamental flavors ψ (and their scalar super-partners). With fundamental fermions massive, the presence of fundamental zero modes does not make the monopole vertex vanish. Since they are heavy they can be integrated out of the theory and the effective theory has the same basic fields as that of the super YM theory (4.10), provided that the coulomb phase is not lifted, i.e. that the expectation value of $v \neq 0 \mod 2\pi/L$.

Qualitative expectation

As we have seen in the SYM case, the microscopic picture of the theory is a theory of monopoles. However, since monopoles could not appear in the vacuum without having gaugino zero modes, they could not generate the bosonic potential for the dual σ field, and therefore cannot by themselves be responsible for the mass gap generation and area law. The objects responsible for these effects are magnetic bions. The center symmetry, on the other hand, was preserved by the formation of *neutral bions*. Upon addition of massive multiplets, the center symmetry is no longer a symmetry of the theory because the fundamental fermions are not invariant under it. We are interested in understanding how this picture changes in sQCD with heavy flavors.

Recall that the neutral bions, objects responsible for the center symmetry, were bound states of $BPS-\overline{BPS}$ and $KK-\overline{KK}$ monopoles. Because of complete symmetry between the gaugino zero mode exchange, these objects contributed $e^{2b'}$ and $e^{-2b'}$ with the *same* coefficient. In the case of sQCD, however, we know that fundamental zero modes will rest on the BPS type monopole or KK type monopole depending on their boundary conditions. This means that the coefficients in front of neutral bion factors $e^{2b'}$ and $e^{-2b'}$ need no longer be the same, i.e. depending on the boundary conditions of the fundamental multiplet, the fundamental fermion zero mode exchange can bind either $BPS - \overline{BPS}$ or $KK - \overline{KK}$ more tightly (see Fig. 4.3), and the minimum of b' field is shifted from zero to some finite nonzero value δ. Additionally we can have contributions to the Polyakov loop screening from the perturbative dynamical quarks and squarks, which would also contribute to the δ shift of the minimum of b'.

Whatever the mechanism, we expect that the effective potential of b' field to be of the form

$$\sim \cosh(2(b' - \delta)) - c_1 \cos(2\sigma) \ . \tag{4.43}$$

which attains its minimum at $b' = \delta$.

We would like to compute δ by similar methods used for SYM. This time, however, we cannot keep the fundamental matter in the low energy effective theory, as they are explicitly massive, and we should integrate out both the quarks and squarks altogether to obtain the theory of the dual field σ and b'. To do this we need to compute the one loop correction to the instanton-monopoles arising from the fundamental multiplet as well as the vector multiplet.

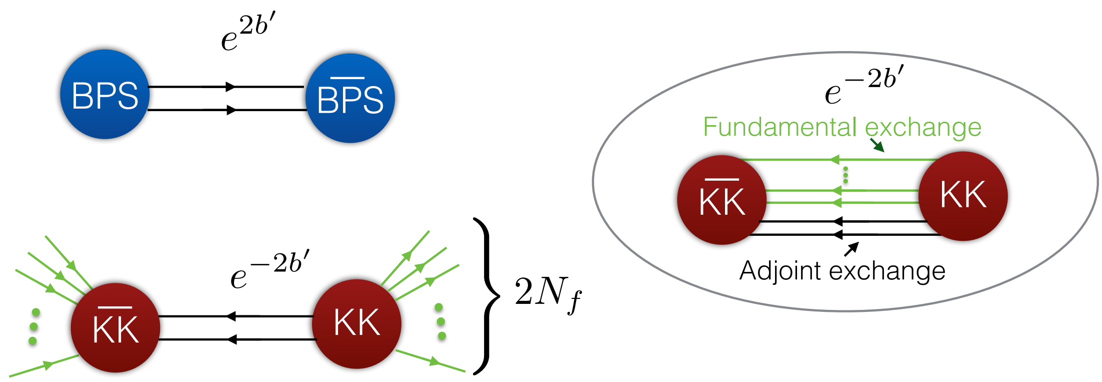


Figure 4.3: A schematic depiction of neutral bion operators in sQCD. The green lines represent the fundamental zero modes (for anti-periodic boundary conditions of the fermions), while the black lines represent the adjoint zero modes. On the right the potential contribution to the neutral bion contribution $e^{-2b'}$ is shown, where an exchange of (massive) fundamentals is possible.

One loop determinants

The one loop fundamental determinants can be written as

$$\frac{\det(\not{D} + M)^{N_f}}{\det(-D_\mu^2 + M^2)^{2N_f}} \tag{4.44}$$

where the determinants in the numerator is that of a 4D Dirac operator, and the determinants in the denominator are those of $2N_f$ complex scalars. We can rewrite the determinants as

$$\frac{\det(-D_\mu^2\mathbf{1} + \frac{i}{2}\bar{\sigma}^{\mu\nu}F_{\mu\nu} + M^2\mathbf{1})^{N_f/2}\det(-D_\mu^2\mathbf{1} + M^2\mathbf{1})^{N_f/2}}{\det(-D_\mu^2\mathbf{1} + M^2\mathbf{1})^{N_f}} =$$
$$= \left(\frac{\det(-D_\mu^2\mathbf{1} + \frac{i}{2}\bar{\sigma}^{\mu\nu}F_{\mu\nu} + M^2\mathbf{1})}{\det(-D_\mu^2\mathbf{1} + M^2\mathbf{1})}\right)^{N_f/2} \tag{4.45}$$

where $\mathbf{1}$ is a two by two spin matrix. This follows from rewriting $\det(\not{D} + M) = \det(-\not{D}^2 + M^2)^{1/2} = \left(\det(-D_\mu^2\mathbf{1} + M^2\mathbf{1})\det(-D_\mu^2\mathbf{1} + \frac{i}{2}\bar{\sigma}^{\mu\nu}F_{\mu\nu} + M^2)\right)^{1/2}$, which is true in the self-dual background. Further we must regulate these determinants. Using the Pauli-Villars regulators we rewrite the right hand side as

$$R = \left(\frac{\det(\Delta_- + M^2)}{\det(\Delta_+ + M^2)}\right)^{N_f/2}\left(\frac{\det(\Delta_+ + \Lambda_{PV}^2)}{\det(\Delta_- + \Lambda_{PV}^2)}\right)^{N_f/2} \tag{4.46}$$

where we denoted $\Delta_+ = -\mathcal{D}\bar{\mathcal{D}} = -D_\mu^2$ and $\Delta_- = -\bar{\mathcal{D}}\mathcal{D} = -D_\mu^2 + \frac{i}{2}\bar{\sigma}^{\mu\nu}F_{\mu\nu}$, and where we dropped the explicit writing of the 2×2 spinor structure of the operator. This expression can further be written as

$$R_f = \exp\left[\frac{N_f}{2}\int_{\Lambda_{PV}^2}^{M^2} d\mu^2 \mathrm{Tr}\left(\frac{1}{\Delta_- + \mu^2} - \frac{1}{\Delta_+ + \mu^2}\right)\right] = \exp\left[\frac{N_f}{2}\int_{M^2}^{\Lambda_{PV}^2}\frac{d\mu^2}{\mu^2}I_f(\mu)\right] \tag{4.47}$$

where subscript f stands for *fundamental* and $I_f(\mu^2)$ is the index function (2.58) in the fundamental representation. This is a remarkable result which takes such an elegant form because of the supersymmetry.

In a similar fashion we can construct the adjoint determinant ratio [87] (see also [110])

$$R_{adj} = \left(\frac{\det\Delta_+}{\det'\Delta_-}\right)^{3/4}\left(\frac{\det(\Delta_- + \Lambda_{PV}^2)}{\det(\Delta_+ + \Lambda_{PV}^2)}\right)^{3/4} \tag{4.48}$$

where the prime indicates that the two zero modes are excluded[17] which can be written as

$$\det{}'(\Delta_-) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^4}\det(\Delta_- + \mathbf{1}\epsilon^2) \tag{4.49}$$

[17] The reader will keep in mind that this expression already includes the Λ_{PV}^3 that we explicitly wrote in (4.32), so we will have to be careful not to include it when we use this expression to correct the monopole vertex.

where the RHS determinant includes all modes now. Then we can write

$$R_{adj} = \lim_{\epsilon \to 0} \exp\left(-\frac{3}{4} \int_{\epsilon^2}^{\Lambda_{PV}^2} \frac{d\mu^2}{\mu^2} I_{ajd}(\mu^2) + 3 \ln \epsilon \right) \tag{4.50}$$

We have already discussed the index function in Sec. 2.4. It consists of an μ independent bulk part, which is proportional to the topological charge

$$I_B = -2T(\mathcal{R})Q \tag{4.51}$$

and a surface term given by the expression (2.72) which is

$$\begin{aligned} I_S(\mu) = -\frac{1}{8\pi} \sum_{n=-\infty}^{\infty} \int dS^i \epsilon^{ijk} \mathrm{tr} \left(F_{jk} \frac{A_4^\infty - \frac{2\pi n}{L}}{\sqrt{(A_4^\infty - \frac{2\pi n}{L})^2 + \mu^2}} \right) = \\ = - \sum_{n=-\infty}^{\infty} \mathrm{tr} \left(T^3 \frac{vT^3 - \frac{2\pi n}{L}}{\sqrt{(vT^3 - \frac{2\pi n}{L})^2 + \mu^2}} \right) \end{aligned} \tag{4.52}$$

where T^3 is the Cartan generator in the representation at hand, and the trace is over the color indices.

Let us first consider the bulk contribution. This contribution is UV divergent and gives the following contribution in the exponent of (4.47) and (4.50)

$$\frac{N_f}{2} Q \ln \frac{\Lambda_{PV}^2}{M^2} - 3Q \ln \frac{\Lambda_{PV}^2}{\epsilon^2} \ . \tag{4.53}$$

Since the ratios $R_f R_{adj}$ will multiply the monopole vertex which is $e^{-\frac{8\pi^2}{g_0^2} Q}$ where g_0 is the bare coupling at scale Λ_{PV}, the exponent of the vertex will then contain the term

$$-\left(\frac{8\pi^2}{g^2(\Lambda_{PV})} - 3 \ln \frac{\Lambda_{PV}^2}{\epsilon^2} + \frac{N_f}{2} \ln \frac{\Lambda_{PV}^2}{M^2} \right) \frac{vL}{2\pi} \tag{4.54}$$

where we used that $Q = \frac{vL}{2\pi}$ for the BPS monopole. Rewriting the above expression in the following way

$$-\left(\frac{8\pi^2}{g^2(\Lambda_{PV})} - 6 \ln \frac{\Lambda_{PV}}{M} + N_f \ln \frac{\Lambda_{PV}}{M} - 6 \ln(ML) + 6 \ln(\epsilon L) \right) \frac{vL}{2\pi} \tag{4.55}$$

by using a 1-loop beta function for the N_f-Dirac flavor sQCD, we see that $\frac{8\pi^2}{g^2(\Lambda_{PV})} -$

$6\ln\frac{\Lambda_{PV}}{M} + N_f \ln\frac{\Lambda_{PV}}{M} = \frac{8\pi^2}{g^2(M^2)}$, so that the above expression can be written as

$$-\left(\frac{8\pi^2}{g^2(M)} - 6\ln(ML) + 6\ln(L\epsilon)\right)\frac{vL}{2\pi} \tag{4.56}$$

The above coupling is a sQCD coupling evolved with the sQCD beta function to the mass scale M. The other terms in the above expression we rewrote anticipating that the scale L will appear with the appropriate coefficients and that the last ϵ-dependent term will be canceled. Indeed we see easily from (4.50) that at the center symmetric point $vL = \pi$ the ϵ-dependence is canceled, while the L dependence comes from the L^3 dependence of the monopole vertex (4.32). However off center $vL \neq \pi$, additional terms will come from the surface contribution of the index which will completely cancel the last term in the expression in the parenthesis above for any v.

Disregarding the last term, the above expression is a statement that once the coupling has been evolved with the sQCD beta function all the way to the scale M, the fermions decouple and the evolution from the scale M to $1/L$ is only through the SYM beta function (assuming that $1/L < M$). This hierarchy has an interpretation of the massive fundamental multiplet decoupling at the scale M. It would be slightly awkward to combine the first two terms in the above expression to the term only involving the effective coupling $1/g^2(1/L)$ as this cannot be written with the strong scale of the pure SYM theory nor sQCD theory. To write the theory entirely in terms of the effective coupling of the sQCD theory we can write the exponent as

$$\begin{aligned}
-\left(\frac{8\pi^2}{g^2(M)} - 6\ln(ML) + N_f\ln(ML) - N_f\ln(ML) + 6\ln(L\epsilon)\right)\frac{vL}{2\pi} = \\
= -\left(\frac{8\pi^2}{g^2(1/L)} - N_f\ln(ML) + 6\ln(L\epsilon)\right)\frac{vL}{2\pi}
\end{aligned} \tag{4.57}$$

For consistency we demand that the combination $\frac{8\pi^2}{g^2(1/L)} - \frac{N_f}{2}\ln(M^2L^2) = \frac{8\pi^2}{g^2_{SYM}(1/L)}$ in the (s)quark decoupling limit $M \to \infty$, where $g^2_{SYM}(1/L)$ is the effective coupling of the pure SYM at scale $1/L$. This means that

$$(\Lambda_{SQCD})^{6-N_f} M^{N_f} = \Lambda^3_{SYM} \tag{4.58}$$

which is the one loop scale-matching condition.

Now we turn to the evaluation of I_S contribution to the one loop determinants. We start with the fundamental multiplet

$$I_S^f(\mu) = -\sum_{n=-\infty}^{\infty}\left(\frac{\frac{v}{2} - \frac{2\pi n}{L}}{\sqrt{(\frac{v}{2} - \frac{2\pi n}{L})^2 + \mu^2}} - (v \to -v)\right) \tag{4.59}$$

The above sum can be rewritten as

$$I_S^f(\mu) = -\sum_{n=-\infty}^{\infty} \frac{\frac{vL}{4\pi}+n}{\sqrt{(\frac{vL}{4\pi}+n)^2+\left(\frac{\mu L}{2\pi}\right)^2}} - (v\to -v) =$$
$$= -\partial_a F(-\frac{1}{2};a,c) - (a\to -a) \tag{4.60}$$

with $a=\frac{vL}{4\pi}$ and $c=\frac{\mu L}{2\pi}$ and where $F(s;a,c)$ is given by (H.6). From (H.12) we have that

$$I_S^f(\mu) = 4\frac{\mu L}{\pi}\sum_{p=1}^{\infty}\sin\left(\frac{pvL}{2}\right)K_1(p\mu L) \tag{4.61}$$

The integral over μ, which, following notation of [88] we label by a function $2X(vL,ML)$, gives

$$2X(vL,ML) = \int_M^\infty \frac{d\mu}{\mu} I_S^f(\mu) =$$
$$= \frac{4L}{\pi}\sum_{p=1}^{\infty}\sin\left(\frac{pvL}{2}\right)\int_M^\infty d\mu\; K_1(p\mu L) = -\frac{4}{\pi}\sum_{p=1}^{\infty}\frac{\sin\left(\frac{pvL}{2}\right)}{p}K_0(pML) \tag{4.62}$$

Consider now the surface contribution from the index function of the vector multiplet, i.e. consider

$$I_S^{adj}(\mu) = -\sum_{n=-\infty}^{\infty}\left(\frac{v+\frac{2\pi n}{L}}{\sqrt{(v+\frac{2\pi n}{L})^2+\mu^2}} - (v\to -v)\right) \tag{4.63}$$

and, since

$$\int_{\epsilon^2}^\infty \frac{d(\mu^2)}{\mu^2}\frac{1}{\sqrt{A^2+\mu^2}} = \frac{2\sinh^{-1}\left(\frac{|A|}{\epsilon}\right)}{|A|} \approx \frac{\ln(\frac{(2A)^2}{\epsilon^2})}{|A|} + o(\epsilon^2) \tag{4.64}$$

we have that

$$-\int_{\epsilon^2}^\infty \frac{d\mu^2}{\mu^2} I_S^{ajd}(\mu) = 4\sum_n \mathrm{sign}\left(n+\frac{vL}{2\pi}\right)\ln\left|n+\frac{vL}{2\pi}\right|$$
$$-4\sum_n \mathrm{sign}\left(n+\frac{vL}{2\pi}\right)\ln\frac{\epsilon L}{4\pi} \tag{4.65}$$

The sums appearing above are discussed in the Appendix H.2. In particular using (H.17) we have that

$$-\int_{\epsilon^2}^{\infty}\frac{d\mu^2}{\mu^2}I_S^{ajd}(\mu) = -4\left(1-\frac{vL}{\pi}\right)\ln\frac{\epsilon L}{4\pi} + 4\ln\frac{\Gamma\left(\frac{vL}{2\pi}\right)}{\Gamma\left(1-\frac{vL}{2\pi}\right)} \tag{4.66}$$

Combining equations (4.47), (4.50), (4.62) and (4.66) we get that the monopole vertex, up to pre-exponential moduli-space factors, is

$$\begin{aligned} R_{adj}^{BPS}R_f^{BPS}e^{-S_{BPS}} = \left(\frac{4\pi}{L}\right)^3 \exp\Bigg\{ &- \left[\frac{8\pi^2}{g^2\left(\frac{4\pi}{L}\right)} + N_f \ln\left(\frac{4\pi}{ML}\right)\right]\frac{vL}{2\pi} \\ &+ 3\ln\frac{\Gamma\left(\frac{vL}{2\pi}\right)}{\Gamma\left(1-\frac{vL}{2\pi}\right)} + N_f X(vL, ML)\Bigg\} \end{aligned} \tag{4.67}$$

where we have explicitly indicated that the result holds for the BPS (anti-)monopole.

So far we assumed the fundamental multiplet has periodic boundary conditions. To take a multiplet with a twist (i.e. that $\Phi(L) = e^{i\varphi}\Phi(0)$) all we need to do is change the index function in (4.59) by

$$I_S^f(\mu) = -\sum_{n=-\infty}^{\infty}\left(\frac{\frac{v}{2}+\frac{\varphi}{L}-\frac{2\pi n}{L}}{\sqrt{(\frac{v}{2}+\frac{\varphi}{L}-\frac{2\pi n}{L})^2+\mu^2}} - (v\to -v)\right) \tag{4.68}$$

This changes equation (4.61) to

$$I_S^f(\mu) = 4\frac{\mu L}{\pi}\sum_{p=1}^{\infty}\sin\left(\frac{pvL}{2}\right)\cos(p\varphi)K_1(p\mu L) \tag{4.69}$$

and therefore (4.62) becomes

$$X_\varphi(vL, ML) = -\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\sin\left(\frac{pvL}{2}\right)\cos(p\varphi)K_0(pML) \tag{4.70}$$

The KK monopole determinants can also be calculated by using the index functions which we discussed in Section 2.4. Recall that in the case of adjoint fermions the modification $v \to \bar{v}$ was all that needed to be changed. In the case of fundamental fermions an additional shift to the Matsubara sum needed to be implemented, which contributed a negative sign to the surface contribution, so that

$$R_{adj}^{KK} R_f^{KK} e^{-S_{KK}} = \left(\frac{4\pi}{L}\right)^3 \exp\left\{ - \left[\frac{8\pi^2}{g^2\left(\frac{4\pi}{L}\right)} + N_f \ln\left(\frac{4\pi}{ML}\right)\right]\left(1 - \frac{vL}{2\pi}\right) \right.$$
$$\left. - 3\ln\frac{\Gamma\left(\frac{vL}{2\pi}\right)}{\Gamma\left(1-\frac{vL}{2\pi}\right)} - N_f X(vL, ML)\right\} \quad (4.71)$$

We recognize that the anti-BPS and anti-KK monopoles have a very similar structure, with the only difference being an additive constant which is the instanton action $\frac{8\pi^2}{g^2(4\pi/L)}$ and a negative sign in front of the other terms.

Now we come to the main point. The BPS and KK monopole, in addition to the above dependence on v (and therefore b') should be endowed with monopole charges by inserting $e^{\pm i\sigma}$ as well as the 't Hooft vertex $\lambda\lambda$ (and $\bar\lambda\bar\lambda$ for the anti–self-dual pair). The above contribution (along with moduli space metric in front of the exponent) must be writable in terms of the superpotential $W(\boldsymbol{B})$. Indeed one can easily check that such terms are reproduced if we take

$$W(\boldsymbol{B}) = \frac{g^2}{2(4\pi)^2 L} m_\lambda \left(e^{\boldsymbol{B}} + e^{-\boldsymbol{B}}\right) \quad (4.72)$$

where $\boldsymbol{B}$ is a superfield in which for which the $\theta, \bar\theta$ independent part (commonly denoted by $\boldsymbol{B}\big|$) is

$$\boldsymbol{B}\Big| = -b'\left(1 + \frac{g^2 N_f}{8\pi^2}\ln\left(\frac{4\pi}{ML}\right) + 3\ln\frac{\Gamma\left(\frac{1}{2} - \frac{g^2}{8\pi^2}b'\right)}{\frac{1}{2} + \frac{g^2}{8\pi^2}b'}\right) + N_f X_\varphi(\pi + \frac{g^2}{4\pi}b', ML) + i\sigma\ . \quad (4.73)$$

The coupling above is the coupling at the scale $4\pi/L$ evolved with the sQCD beta function.

It might appear strange that the lowest component of the chiral superfield B is highly nonlinear in b'. This in fact happens because of the moduli space metric (i.e. the coefficient in front of the kinetic term $(\partial_i b')^2$) is modified by the presence of matter (and in fact of adjoint fermions when $b' \neq 0$) as was discussed in [87] for the pure Yang-Mills. We will ignore this modification, as it is a correction in the coupling $g^2(4\pi/L)$ and comment on this in the conclusion of this section, but for now let us see what scalar potential we obtain from this superpotential.

Since $g^2 \ll 1$ (assuming $L \ll \Lambda$), the above identification of the lowest component of the superfield can be approximated by

$$\boldsymbol{B}\Big| = -b' + N_f X_\varphi(\pi, ML) + i\sigma\ . \quad (4.74)$$

The N_f dependent part was, of course, absent in the SYM case, so, to leading order in the coupling, the addition of fundamental fermions introduces a shift in the b' field. Therefore

$$V_{bos} = \frac{1}{\partial_{\boldsymbol{B}}\partial_{\bar{\boldsymbol{B}}}K}\left|\frac{\partial W}{\partial \Phi}\right|\Bigg|_{\Phi=\phi} \propto \cosh\Big[2(b' - N_f X_\varphi(\pi, ML))\Big] - \cos(2\sigma)\ . \tag{4.75}$$

The minimum is attained when $b' = N_f X_\varphi(\pi, ML)$, i.e.

$$\left\langle b' \right\rangle \simeq N_f X_\varphi(\pi, ML) = -\frac{2N_f}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\sin\frac{n\pi}{2}\cos(p\varphi)K_0(pML) \tag{4.76}$$

rendering the average Polyakov loop

$$\langle \mathrm{tr}\, L\rangle \approx \frac{g^2}{4\pi}\left\langle b'\right\rangle \simeq N_f\frac{g^2}{2\pi^2}\sum_{p=1}^{\infty}\sin\frac{n\pi}{2}\cos(p\varphi)K_0(pML)\ . \tag{4.77}$$

If we take $ML \gtrsim 1$ we obtain

$$\langle \mathrm{tr}\, L\rangle \approx N_f\sqrt{\frac{2}{\pi}}\frac{e^{-ML}}{\sqrt{ML}}\ . \tag{4.78}$$

This result has a simple interpretation as the screening of the heavy quark by the fundamental matter. Indeed we see that the free energy of the quark is $F = -\frac{1}{L}\ln\langle \mathrm{tr}\, L\rangle \propto M$, i.e. the cost of pulling a single (s)quark from the vacuum.

Further, looking at the expression (4.62) in the effective action of the single monopole, the contribution comes from the Kaluza-Klein sums of heavy electric charges, i.e. objects $\propto e^{\pm p\frac{i}{2}vL}e^{-pML}$. Clearly such objects carry fundamental electric charge $\pm p$ because a static worldline of such a charge would give a contribution $e^{\pm\frac{p}{2}i\int dx^4 A_4^3} = e^{\frac{\pm pivL}{2}}$.

Therefore the fundamental multiplets screen heavy quarks, as would be expected. Notice, however, that supersymmetry is not violated, and that the superpotential did not change its qualitative form. What did change is the identification of the lowest component of the superfield $\boldsymbol{B}$, which gets corrections from the Matsubara modes. In fact this change affects the moduli space metric of the b' field. We do not discuss this here, for brevity. The discussion is given in our original paper [88] (see also the discussion in [87]). This is not very surprising as having dynamical quarks, charged under the unbroken $U(1)$, are sure to modify the coupling of the effective U(1) theory.

Let us now discuss the microscopic picture a bit more. Namely one can ask a question what causes this shift from a potential $\cosh(2b')$ to $\cosh(2(b'-\delta))$? One mechanism may be simply the quark screening. Indeed we have seen that this is certainly part of the story, as the X-function corresponds to a sum over charged particles.

As we discussed in Section 4.1.2 there is another mechanism one can envision, where the fundamental fermion exchange can affect the neutral bion factors $e^{2b'}$ and $e^{-2b'}$. We would like to see if our results indicates such a fundamental zero mode exchange.

To answer this question, recall from Section 3.3.5 that the hopping matrix element had the form[18]

$$T_{I\bar{J}} \sim e^{-\frac{vR}{2}} \tag{4.79}$$

where R is the distance between the monopole and anti-monopole. The above behavior is their result of the zero modes decaying as $e^{-vr/2}$ with the distance r from the monopole center. This matrix element is simply the eigenvalue shift of the Dirac operator from zero eigenvalue of infinitely separated monopoles, to $\sim \pm e^{-vR/2}$ at a finite separation distance R.

If we now weight the monopole–anti-monopole system with the determinant of the massive fundamental fermions, schematically, we would have

$$\det(\not{D} + M) \sim e^{\mathrm{tr}\,\ln(1+\frac{\not{D}}{M})+\ldots} \approx 1 - \frac{1}{2}\mathrm{tr}\,\frac{\not{D}^2}{M^2} \sim 1 - \frac{C}{M^2}e^{-vR} \tag{4.80}$$

where we have neglected the constant factors common to the perturbative vacuum which will cancel upon proper normalization, and taken the trace in the quasi-zero mode states only. C above is some numerical constant and we expanded the determinant for large[19] M.

If we put different boundary conditions from strictly (anti-)periodic ones, i.e. if we make them periodic up to a phase, the zero modes will decay as $e^{-\left(\frac{v}{2}-\frac{\varphi}{L}\right)}$. This change in behavior would change the decay of the hopping matrix element appropriately, and the above weight of the pair at the distance R is instead

$$\det(\not{D} + M) \sim 1 - \frac{C}{M^2}e^{-(v-2\varphi/L)R} \tag{4.81}$$

To compute the coefficient in front of the neutral bion term in the effective action, i.e. the amplitude of the neutral bions, we would have to integrate over all separations R, which, as we discussed earlier is dominated by the $R \ll v^{-1}$ even in the case of SYM where we cannot talk about a monopole and an anti-monopole anymore, and where all of our approximations fail (see however [101] for a phenomenological resolution in pure YM theory).

Supersymmetry, however, has already provided us with the result, and we are not interested in the computation of this amplitude, but only in whether there is such a

[18] This form is for the periodic boundary conditions, as the monopoles in question are the BPS and $\overline{BPS}$ monopoles, while for the anti-periodic boundary conditions we would have to change $v \to \bar{v}$.

[19] This is justified because we assumed higher eigenvalue states are similar to the vacuum states, so they cancel upon proper normalization and coupling redefinition.

contribution and since the zero mode profiles explicitly depend on the periodicity phase φ, it is quite clear that the amplitude will depend on it. So if there is such a contribution in the one loop computation we did, it must necessarily be φ dependent. The only φ dependent part of the effective one loop monopole action is in the function $X_\varphi(vL, ML)$.

To see that there is no such contribution to the amplitude of the neutral bions, let us consider a case where N_f is even and half of the (s)quarks are periodic up to a phase φ while the other half is anti-periodic up to a phase φ, i.e. they are periodic up to a phase $\pi+\varphi$. If there is an exchange of fundamental fermions in this case, it will be completely symmetric between the $BPS - \overline{BPS}$ and $KK - \overline{KK}$ monopoles, and the amplitudes in front of $e^{2b'}$ and $e^{-2b'}$ will be the same but *dependent on* φ. However, the Polyakov loop screening contribution in this case is exactly zero, because

$$\frac{N_f}{2}\left(X_\varphi(vL, ML) + X_{\varphi+\pi}(vL, ML)\right) = 0 \ . \tag{4.82}$$

The neutral bion amplitudes are completely independent on φ! But as we said, we expect that if there is a fundamental fermion exchange contribution to the neutral bion amplitude that it is necessarily φ dependent, although symmetrical between $BPS-\overline{BPS}$ and $KK - \overline{KK}$ neutral bions. We are therefore forced to conclude that the fundamental zero mode exchange such as the one on the right of Fig. 4.3 does not exist and the screening effects are only due to the perturbative (s)quarks.

4.2 $O(3)$ model and chemical potential

In this section we will discuss another model where instanton-monopoles will appear: the $O(3)$ non-linear sigma model in two dimensions. As we discussed in the introduction, this model (and its generalizations to higher groups) has been a toy model of YM theory for quite some time because of their similar features (see e.g. [84]).

Recently very important steps in understanding the mass gap generation via so-called *resurgence* expansion in $CP(N)$ models[20] were made [49, 47]. These works rely on compactifying the theory to $\mathbb{R}^1 \times S^1$ with a small radius, and imposing twisted boundary conditions in S^1, such that the fields differ by a global symmetry group transformation upon transversing the compact circle. By doing this appropriately the theory abelianizes, and has instanton-monopole–like solutions [30, 23] (referred to as instanton-kinks in the works of Ünsal et al.).

[20] $CP(N)$ model is a natural generalization of $O(3)$. In fact $CP(1) \equiv O(3)$. While it may seem that $O(N)$ models are more natural generalizations of $O(3)$ model, they do not have instanton solutions, but do have dynamical mass gap generation. This is one of the arguments why instantons were not believed to be crucial for dynamical mass gap generation. Recently, however, there are works suggesting that it is not only the topologically stable solutions, but also the unstable saddles which are crucial for IR dynamics of the models like $O(N)$ [34].

There is another deformation of the O(3) model which was suggested long ago by I. Affleck with the aim of understanding the IR dynamics [4]. In this work an explicit mass term m was introduced, breaking the model to $O(2)$ at length scales larger then $1/m$ with the effective coupling depending on m: small at large m and growing as m is decreased. Further the model has vortex solutions which carry a half-integer topological charge: i.e. they are instanton-monopole–like solutions referred to as *merons* in[21] [4]. It is well known that if the $O(2)$ models in 2D have vortex solutions, the model has a phase transition as the function of the coupling: at weak coupling, vortices interact strongly and form tightly bound pairs, while at strong coupling the gain in entropy makes it favorable for vortex–anti-vortex pairs to ionize and percolate through the system. Affleck then conjectured that as we take m to zero, a phase transition occurs from the massless phase to a gapped phase precisely due to this vortex condensation.

We will see that something similar happens upon introduction of the chemical potential[22] μ for the $O(3)$ model[23]. The theory with chemical potential breaks to $O(2)$ at length scales larger than $1/\mu$ and meron solutions with half-integer topological charge appear[24]. The theory can be dualized much like the $U(1)$ gauge theory with monopoles. We will see that due to the presence of the chemical potential there is a condensation of kinks in the dual theory and that there is an asymmetry between the temporal correlation functions and spatial correlation functions. This asymmetry is, of course, expected, because the chemical potential explicitly breaks the (Euclidean) Lorentz invariance.

4.2.1 $O(2)$ model, vortices and chemical potential

In Section 2.6.2 we have seen how to put a chemical potential in the $O(N)$ model, which we now specialize to the $O(2)$ model. The action is

$$g_0^2\mathcal{L} = \frac{1}{2}(\partial_\nu\phi)^2 + i\mu\dot{\phi} - \frac{1}{2}\mu^2 \tag{4.83}$$

where the dot represents the temporal derivative and where g_0 is the bare coupling, which does not run perturbatively [82]. One thing we notice is that the second term is a topological term, and does not depend on the details of the field ϕ, but only on

[21] As discussed in the introduction, the meron solutions were also suggested for YM [31]. These are very different from merons in $O(3)$ model. In the $O(3)$ model, merons interact logarithmically, which is a coulomb interaction in two dimensions. Merons in YM theory also interact logarithmically, but this is *not* a coulomb interaction in four dimensions.

[22] Since there is global symmetry $SO(3)$, there are conserved charges.

[23] The twists discussed in [30, 23, 49, 47] can be viewed as imaginary chemical potentials for the rigid $SO(3)$ symmetries.

[24] Here a difference appears compared to the twists of [30, 23, 49, 47]. In these works, the topological charge of instanton-kinks (i.e. instanton-monopoles or merons) depend on the twist introduced, while in a theory with a large (real) chemical potential, the merons always carry half-integer topological charge. It is an interesting question if a theory with the imaginary chemical potential (twists), can be continuously connected to the theory with the real chemical potential.)

the winding of ϕ around the temporal direction, while the third term is just an additive constant not depending on the fields of the effective theory.

We can now go to a dual picture by introducing a term $-i\frac{1}{2\pi}\partial_\mu\partial_\nu\epsilon^{\mu\nu}\phi$ and integrating out $\partial_\nu\phi$, obtaining the dual Lagrangian

$$\mathcal{L}_{dual} = \frac{g_0^2}{2(2\pi)^2}(\partial_\mu\sigma)^2 - \mu\frac{\partial_x\sigma}{2\pi} \ . \tag{4.84}$$

Notice a remarkable thing: the dual theory has no imaginary part of the action, and is instead completely real. The μ-dependence is again entirely in the topological term, but now a topological term in the compact σ field, coupling to the total winding in the *x-direction.* Finally, differentiating with respect to μ shows that the total charge is proportional to the winding number of the σ field in the x-direction, i.e. to the number of kinks in the spatial direction of the dual field[25].

If we assume that the original theory had vortices and that the bare coupling was sufficiently large for them to percolate[26], the dual effective Lagrangian is given by

$$\mathcal{L}_{dual}^{eff} = \frac{g^2}{2(2\pi)^2}\left[(\partial_\mu\sigma)^2 - m^2\cos\sigma\right] - \mu\frac{\partial_x\sigma}{2\pi} \ . \tag{4.85}$$

To compute the correlation function between $e^{i\phi}$ and $e^{-i\phi}$ like in Section 2.6, we first notice that there is a difference between spatial and temporal correlators. Namely if we look at a spatial correlator $\left\langle e^{-i\phi(x_1)}e^{i\phi(x_2)}\right\rangle$, we notice no change at all, because the kink solution across the line connecting the two insertion points does not couple to the chemical potential (see left panel of Fig. 4.4). On the other hand a temporal correlator $\left\langle e^{-i\phi(t_1)}e^{i\phi(t_2)}\right\rangle$ does feel this kink, and it contributes a term to the action $\Delta S_\mu = -\mu(t_2 - t_1)$ (see right panel of Fig. 4.4). This contribution changes sign when ordering of t_1, t_2 is changed, enhancing (for $\mu > 0$) a forward correlation in time, but suppressing the backward one. This is typical of theories with chemical potential, but here we see quite a nontrivial manifestation of it via topology.

The kink action is given by $\frac{g^2 m}{2\pi^2}|t_2 - t_1| = E_0|t_2 - t_1|$, where E_0 is the mass gap of the theory (see Section 2.6), and we expect the temporal correlation functions to cease being exponential in the regime when $\mu > E_0$, because we start populating the the charged excited levels, which have continuum energy levels spaced as $\sim 1/L$ where L is the spatial extent of the system[27].

A phase transition, therefore, must occur at $\mu_c = E_0$. This is not surprising, as this

[25] The kink number of the field σ in the x-direction is just $Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ \partial_x\sigma = \frac{\sigma(+\infty)-\sigma(-\infty)}{2\pi}$, i.e. it is the net winding of the field σ in the spatial direction.

[26] Notice that because the chemical potential term couples to boundary terms only, it does not affect the interactions between vortices.

[27] That these are always continuum states is clear because it is always possible to boost a particle with mass m to a particle with energy $\sqrt{m^2 + \boldsymbol{p}^2}$ where $\boldsymbol{p}$ is the momentum vector.

is the critical value of the chemical potential where it becomes favorable to populate the system with charge and it is often referred to as the *Silver-Blaze phenomenon* in the high energy community. It is nothing but the statement that the partition function cannot change (at zero temperature) until the chemical potential reaches the value of the mass of the first charged state (although correlators may still be affected). Since we have shown that the charge is directly related to the average number of kinks in the spatial direction, a chemical potential $\mu > \mu_c$ will cause kinks to condense.

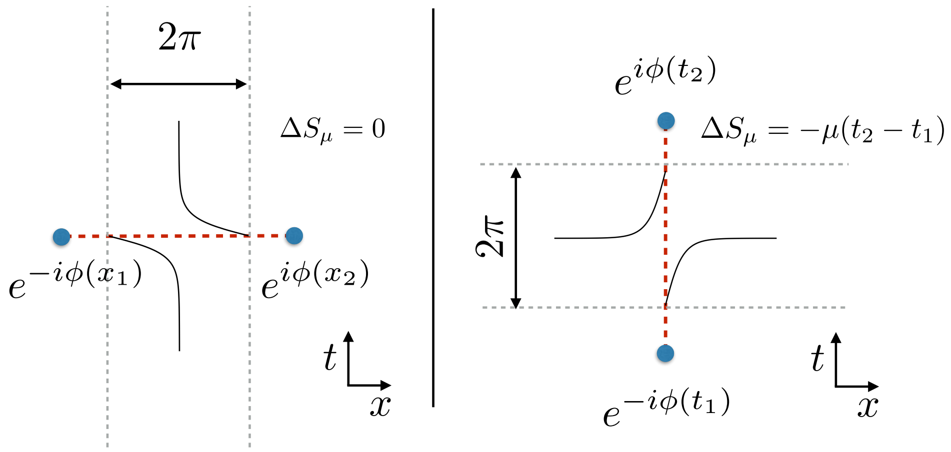


Figure 4.4: A schematic depiction of the spatial and temporal correlators in the presence of the chemical potential. The spatial correlator (left) does not change due to the presence of the chemical potential, as the chemical potential couples to kinks in the spatial direction. On the other hand, the temporal correlator (right) has an extra contribution to the action $\Delta S_\mu = -\mu(t_2 - t_1)$ due to a required jump at the interface of the line connecting the sources.

In the extreme case that $\mu \gg E_0$, the kink density will be so large that the $\cos\sigma$ potential in the Lagrangian (4.85) will be highly oscillating and negligible, so $\sigma = \frac{2\pi x Q}{L}$ will be an approximate solution for the Q kinks in the x direction. The action of Q kinks is then approximately

$$S_{\mu \gg E_0} \approx \frac{g^2}{2(2\pi)^2} \left(\frac{Q}{L}\right)^2 L\beta - \mu Q\beta \tag{4.86}$$

where $\beta = \frac{1}{T}$ is the temporal extent and L is the spatial size of the system. Quite clearly the action is minimized for

$$\frac{Q}{L} = \mu \frac{(2\pi)^2}{g^2} \,, \tag{4.87}$$

In other words the charge density $\rho = Q/L$ directly proportional to μ is induced in the system.

4.2.2 $O(3)$ and $O(N)$ with chemical potential and the effective action

Now we specialize to the $O(3)$ model. Because of the $O(3)$ symmetry, we can add a chemical potential for the Q^3 charge (2.128) without loss of generality. The Lagrangian is

$$g_0^2\mathcal{L} = \frac{1}{2}(\partial_\nu \boldsymbol{n})^2 + i\mu(n_1\dot{n}_2 - n_2\dot{n}_1) - \frac{1}{2}\mu^2(n_1^2 + n_2^2) \ . \tag{4.88}$$

where we used that $[t^3]_{ab} = \epsilon_{ab}$ is the generator of rotations around the 3-axis of the $\boldsymbol{n}$-field. Using the constraint $\boldsymbol{n}^2 = 1$, the above Lagrangian can be rewritten as

$$g_0^2\mathcal{L} = \frac{1}{2}(\partial_\nu \boldsymbol{n})^2 + i\mu(n_1\dot{n}_2 - n_2\dot{n}_1) + \frac{1}{2}\mu^2 n_3^2 - \frac{1}{2}\mu^2 \ . \tag{4.89}$$

Clearly μ acts as a mass for the 3rd component of the $\boldsymbol{n}$ field. In the asymptotic limit $\mu^2 \to \infty$, the last term completely suppress n_3 fluctuations, and imposes $n_3 = 0$ at length scales larger then $1/|\mu|$, so that $n_1^2 + n_2^2 = 1$. The model, therefore, becomes an effective $O(2)$ model. Parametrizing $n_1 = \cos\phi, n_2 = \sin\phi$, the effective Lagrangian becomes

$$g^2\mathcal{L}_{eff} = \frac{1}{2}(\partial_\nu\phi)^2 + i\mu\dot{\phi} - \frac{1}{2}\mu^2 \ . \tag{4.90}$$

Several comments about the above Lagrangian are in order

- It is a low energy effective Lagrangian valid for $\mu \gg \Lambda$ where Λ is the strong scale of the theory
- As we integrate out the modes from the UV scale $\tilde{\Lambda} \gg \mu$, the coupling g_0^2 runs with the full $O(3)$ beta function to the scale μ. At this point the coupling freezes as fluctuations of the n_3 field are suppressed and the theory at length scales $\gtrsim 1/\mu$ is an $O(2)$ theory for which the coupling does not run. Therefore the coupling $g^2 = g^2(\mu)$ in (4.90) is the $O(3)$ coupling at scale μ.
- The model (4.89) has vortex solutions[28] where ϕ winds around some point $\boldsymbol{x}_0$ and the $\boldsymbol{n}$ field goes to the north or south pole at $\boldsymbol{x}_0$, i.e. $n_3(\boldsymbol{x}_0) = \pm 1$.
- These vortex solutions have half-integer topological charge.

A similar effective Lagrangian was in fact considered a long time ago by Affleck [4], where an explicit deformation was introduced breaking $O(3)$ to $O(2)$. Here however the

[28] There are no vortex solutions in the $O(3)$ model without any deformation, because the mapping $S^1 \to S^2$ is topologically trivial.

deformation has the meaning of chemical potential, instead of being an ad hoc deformation, and has, in addition, an imaginary part which was absent in the consideration by Affleck.

To explore the theory further, let us do a duality transformation of the above Lagrangian. Just like in the previous section we have that

$$\mathcal{L}_{dual} = \frac{g^2}{2(2\pi)^2}(\partial_\mu\sigma)^2 - \mu\frac{\partial_x\sigma}{2\pi} \, . \tag{4.91}$$

where g^2 is the coupling at scale μ. For large μ the coupling is weak, so we expect no vortex formation there (i.e. algebraic spatial correlation functions). Nevertheless the theory has a potential to form vortices, but their IR behavior will be irrelevant for large μ as they will prefer to form tightly bound vortex–anti-vortex pairs. It is important to stress that there is no longer only one vortex (and anti-vortex) solution, but two distinct vortices now, as a vortex can attain values $n_3 = \pm 1$ at the center. The difference in these vortices will be in their winding number and they will have different weights if the θ angle is introduced [4], so that they contribute

$$-m^2 \cos\sigma \cos\frac{\theta}{2} \tag{4.92}$$

to the effective action, where m is some constant with the dimension of mass.

As we go down in chemical potential μ we expect that the vortex solutions will become important at some critical μ_{KT}, where the subscript stands for Kosterlitz-Thouless. In addition there is another transition which has to take place, namely the transition where the charge starts condensing and the $O(3)$ breaks to $O(2)$. This value of the chemical potential we call μ_c.

One can wonder which phase transition comes first. In fact if $\mu_c > \mu_{KT}$, then as we lower the chemical potential from above we restore the $O(3)$ symmetry gets restored, and there is no reason to expect a KT transition anymore. On the other hand we expect that the spatial correlators are exponential at $\mu = \mu_c = E_0$ where E_0 is the mass gap of the theory, with the correlation length $\xi \propto E_0^{-1}$ as in the $O(2)$ model. The KT transition should therefore happen at $\mu_{KT} > \mu_c$, so that there is a window in values of μ for which the model is effectively an $O(2)$ model and vortices condense, gapping the spatial correlation functions (see the phase diagram in Fig. 4.5).

We could have repeated all the arguments so far for the $O(N)$ model by adding a chemical potential for some $O(2)$ subgroup. The action would be identical to (4.88), but with $\boldsymbol{n}$ being an N-dimensional unit vector instead. The chemical potential would tend to maximize the value of $n_1^2+n_2^2$ and suppress all other components. The same breaking $O(N) \to O(2)$ would occur, and an effective model would look identical to (4.91). Again vortices can exist, by demanding that the value of the $n_1^2+n_2^2$ becomes zero at the vortex

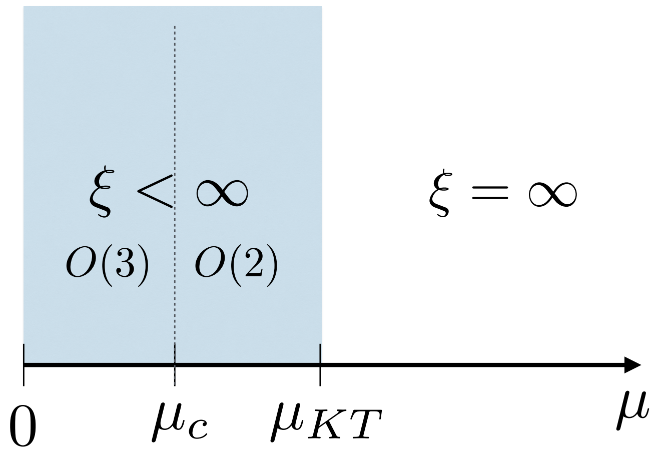


Figure 4.5: A phase diagram of the $O(N)$ model with chemical potential. The shaded region $\mu < \mu_{KT}$ represents the phase where the spatial correlation length $\xi \neq 0$. The dashed line at $\mu = \mu_c$ indicated where the breaking from $O(N)$ to $O(2)$ takes place and where charge condensation occurs. For the $O(3)$ model the θ-angle dependence is expected to change at the KT transition point μ_{KT} from $\cos\frac{\theta}{2}$ at $\mu < \mu_{KT}$ to $\cos\theta$, which would show itself as the jump in the topological susceptibility.

center.

An additional tool for studying the $O(N)$ model is the large N expansion (see e.g. [39, 115]) which can be used to study the $O(N) \to O(2)$ transition [29]. In the large N expansion the low energy effective theory is a theory of N particles with a dynamically generated mass M, and the Lagrangian containing the term $(\partial_\mu \boldsymbol{n})^2 + M^2\boldsymbol{n}^2$. Upon the introduction of the chemical potential, a term $-\mu^2(n_1^2 + n_2^2) = -\mu^2\boldsymbol{n}_\parallel^2$ starts favoring a condensate of $\boldsymbol{n}_\parallel^2$, which would start to generate when $\mu > M$, breaking the symmetry to $O(2)$. Parametrizing $\boldsymbol{n}_\parallel = |\boldsymbol{n}_\parallel|(\cos\phi, \sin\phi)$, the kinetic term becomes $(\partial_\mu \boldsymbol{n}_\parallel)^2 = \boldsymbol{n}_\parallel^2(\partial_\mu\phi)^2 + (\partial_\mu|\boldsymbol{n}_\parallel|)^2$. In the mean field approximation, the condensate of $\boldsymbol{n}_\parallel^2$ acts like an inverse coupling in the effective $O(2)$ model, and would allow vortices to condense when $\mu \gtrsim M$, corroborating the phase diagram 4.5.

In the end, let us briefly discuss the introducing a chemical potential for the $CP(N-1)$ model. Similarly as before, this would induce the breaking of $SU(N)$ to its $U(1)$ subgroup, and similar vortex solutions with fractional topological charge seem to appear. The model has an added similarity with the $SU(N)$ gauge theory in that the θ-angle can be introduced for any N, as opposed to the $O(N)$ model where only $O(3)$ can have the topological *theta*-angle. If the naive lattice formulation of these models can be successfully reformulated in terms of the dual variables on the lattice, avoiding the sign problem, a study of topological susceptibility transition can be made which for $N > 2$ has a well defined continuum extrapolation. Again one can study the large N expansion and hope to understand the $SU(N) \to U(1)$ phase transition better. We expect the phase diagram of the $CP(N-1)$ model to be similar to the one in Fig. 4.5, with an appropriate jump in topological susceptibility at μ_{KT}, which happens when the vortices pair with anti-vortices, leaving instantons as the sole contributions to the θ angle dependence of the vacuum energy, but this still remains to be analyzed in detail.

4.2.3 Summary and conclusion

We have seen that the $O(N)$ and, in particular, the $O(3)$ and the $O(2)$ models have dual descriptions when a large enough chemical potential in the $O(2)$ subgroup is added. The $O(2)$ model with the appropriate UV definition (e.g. lattice) allows for vortex solutions which, if the coupling is strong enough, generate a mass gap in the system and exhibits the *Silver-Blaze* phenomenon at some value of the chemical potential μ_c where $O(2)$ charges condense. What is fascinating is that these charges have a direct interpretation in the dual picture of the model, as kinks in the spatial direction. Upon kink condensation the spatial correlators still exhibit exponential decay.

In the case of $O(3)$ model, however, the coupling is not a parameter of the model, but is set by μ according to the renormalization group flow. Although the theory has vortex solutions, for asymptotically large μ the coupling is weak and the vortices do not condense. We however conjecture that a Kosterlitz-Thouless transition takes place at some μ_{KT} as we lower the chemical potential from above, and that $\mu_{KT} > \mu_c$, where μ_c is the chemical potential at which the $O(3)$ symmetry is restored. As both transitions are expected to happen at $\mu \sim \Lambda$, where Λ is the strong scale, the fluctuations are strong and semiclassical analysis is unreliable.

In spite of the fact that naive Lagrangian formulations of these theories have a sign problem, tremendous progress has been made in scalar field theories where an action is rewritten in term of the so-called dual variables [33] which was extensively used to compute finite chemical potential phenomena on the lattice [18, 56, 64, 71, 95]. If these methods can be adopted for the $O(N)$ and $CP(N-1)$ models, our conjectured phase diagram in Fig. 4.5 can be tested.

If it is indeed true that μ_{KT} is larger then μ_c, we expect that spatial correlation length will not be affected much as μ_c is transversed. If this is shown to be the case it would be a strong evidence that the physics of the mass gap (i.e. spatial correlation length) generation can be understood as the condensation of vortices with half-integer topological charge: instanton-monopoles or merons. Further a change in behavior in the θ-angle dependence must occur if the KT transition takes place, since above μ_{KT} vortices will form pairs and only instantons will couple to the θ-angle as $\cos\theta$. The instanton calculus, however, has a well known UV divergence in the $O(3)$ model (see e.g. [97])

and continuum extrapolations would make instanton $\cos\theta$ contributions dominant over the completely regular vortex–anti-vortex contributions. This would not be a problem in the $CP(N-1)$ generalizations of the model. These will be studied in closer detail in the forthcoming publication. Nevertheless, even for the $O(3)$ model and for sufficiently coarse lattices, a distinct phase from $\cos\theta$ to $\cos\frac{\theta}{2}$ vacuum energy dependence should occur as we go down in chemical potential.

4.3 Future prospects: pure Yang-Mills

As a conclusion to this chapter we would like to comment on the possibility of using similar methods of this presented here in pure YM theory. Although this is a problem without matter, it is nonetheless a crucial theory to understand if one hopes to gain insights into QCD. As we have said at the beginning of Section 4.1, the applicability to the high-temperature YM theory is very limited due to the deconfinement and non-abelianization of the theory at high temperature. However steps to understand pure Yang-Mills systematically have already been made some time ago [98] by adding terms to the action to preserve the center, often referred to as *the double trace deformation*. Although this does not directly give access to the low temperature theory, a conjecture of continuity in the compact radius L has often been made in the works of Ünsal and collaborators, both in relations to the double trace deformed theory (often denoted as YM*) and to QCD(adj). This conjecture still remains to be proven, although there is some evidence to the contrary from lattice simulations[29] [35], studying these theories at small spatial compactification allows one to gain insights on the richness and complexity of YM-like theories.

Can we, however, hope to use similar methods to understand the theory when the temperature $T \ll \Lambda$ where Λ is the strong scale? Although this is a topic for future considerations, we want to connect the discussions of instanton-monopoles to the low energy Polyakov loop effective actions at distances much larger then the inverse temperature, i.e. for energy scales $\ll T \ll \Lambda_{YM}$ where Λ_{YM} is the strong scale of the YM theory. Although effective Lagrangians of the Polyakov loop such as (4.93) below are fairly standard, we believe that sufficient emphasis has not been made in the past on the connection of the low energy Polyakov loop actions and the instanton monopoles[30].

A standard way of constructing a low energy effective theory is to identify the fields

[29] Note that this evidence is for QCD(adj) only and not for strictly chiral adjoint fermions. It is unclear to what extent the finite mass of fermions would change the picture. On the other hand the QCD(adj) on $\mathbb{R}^3 \times S^1$ does not have non-abelian chiral symmetry breaking which is expected at large radius L, invalidating the continuity conjecture for strictly massless adjoint fermions. Whether continuity can be achieved for some fine tuned value of the fermion mass still remains to be seen.

[30] Arguments to the smooth connectedness of the YM-like theories at large compact circle were made with the small compact circle theories, where instanton-monopoles are crucial for the IR physics (see e.g. [98]), but little consideration was given to the effective Polyakov loop models at large compact radius.

and symmetries which appear in it. Since we are interested in a theory at length scales much larger then the inverse temperature, the effective theory is a theory of spatial gauge fields and Polyakov loops Ω, which transform as $U^\dagger \Omega U$ under the 3D gauge transforms. To preserve gauge invariance, therefore, the minimal kinetic terms of the low energy effective action is

$$\frac{1}{g_{eff}^2}\mathrm{tr}\, F_{ij}^2 + \Lambda^2 \mathrm{tr}\left[\left(D_i\Omega(x)\right)^\dagger D_i\Omega(x)\right] + \dots \tag{4.93}$$

where the coupling g_{eff} is the effective $3D$ gauge coupling, Λ is some dynamically generated scale, and where $D_i = \partial_i - i[A_i, \dots]$. Further, since we expect confinement we expect terms which stabilize the center in some way, i.e terms $\mathrm{tr}\left(\Omega^n + (\Omega^\dagger)^n\right)$ with some coefficients, or for $SU(2)$, written in terms of the usual parametrization $\Omega = \exp\left[i\theta\frac{\hat{n}\cdot T}{2}\right]$, where $\hat{n}$ is a unit vector, the Polyakov loop potential terms would be $\cos(n\theta)$, with some unknown coefficients, but constrained to have a minimum at the center symmetric point $\theta = \pi \mod 4\pi$. We will comment on the potential physical meaning of these terms later, but coefficients in front of these terms are in principle not known. However one can also argue that the potential for the Polyakov loop is flat, and that the average is zero due to the fluctuations of the Polyakov loop field [31]. This, however, would not account for the exponential correlation of the Polyakov loop at large distances, which is expected if the theory is in the confined phase, so center symmetric potential terms are expected to be present.

If such a potential exists, then the theory abelianizes[32] by the presence of the commutator term $|[A_i, \Omega]|^2$. The theory can therefore be described by a single abelian component $\mathcal{A}_i = A_i^3$ for example, and then dualized to the abelian dual σ field just like before. We know that the UV theory has regular monopole-like solutions with the appropriate Polyakov loop IR asymptotics. These can generate terms like $\cos(n\sigma)$ in the effective action where n gives the n-monopole contribution. The theory is then gapped completely.

Let us now briefly comment on terms $\cos(n\theta)$ in the effective action. Naively one might think that they come from the resummation of electric charges[33] $e^{\pm ni\frac{\theta}{2}}$, where n is the fundamental charge of the object. This unfortunately would induce terms $-\cos\left(\frac{n\theta}{2}\right)$ in the effective action, breaking center symmetry. Indeed this is how the high-temperature confinement/deconfinement transition can be understood, as being due to the thermal charged gluons (and quarks in the fundamental representation in the case of

[31] In lattice works [45, 42] this was precisely the claim, which was challenged in [103]

[32] This statement has to be made with care. In fact it is very unlikely that the low energy theory is an abelian theory. However the would-be abelian sector will be further gapped by the presence of monopoles, which most probably kick in at the same scale as the abelianization scale. The abelianization is then simply a way of systematically writing the terms in the effective Lagrangian.

[33] as this is a pure gauge theory and we do not expect fundamental charges, but only adjoint charges, for $SU(2)$ we should take n to be even.

QCD). Some mechanism must, therefore, change the sign of the fugacities of a sufficient number of these terms in order to stabilize the center. Although there have been some suggestions how this works (see [87, 101]) there is no conclusive understanding of it which goes further than the phenomenological. This mechanism is crucial for understanding confinement, and as we have seen the rest of the phenomenon such as mass gap and area law follow immediately.

Chapter 5

Summary, future prospects and speculations

In this thesis an overview of the instanton-monopoles and their interplay with fundamental matter was given based on publications [24, 88, 28].

Due to their monopole-like character, instanton-monopoles are a very appealing ingredient in any model of QCD-like theories which hopes to emulate confinement and mass gap generation. The index theorem shows that qualitative features of instanton-monopoles in connection with fundamental matter are similar to that of instantons, in that they exhibit fermionic bound states and generate 't Hooft vertices in the IR effective theories. We have in fact explicitly constructed these zero modes, both for arbitrary periodicity conditions and, by analytical continuation, for finite chemical potential. We have also computed the hopping matrix element, a vital ingredient of the instanton-monopole based models of QCD, numerically which can be used to improve models like [100, 51].

Further, the fermionic spectrum was shown to exhibit a very nontrivial behavior when a monopole–anti-monopole system is immersed into a magnetic field, generating an "imaginary charge" which vanishes upon integration over the holonomy. Nevertheless, the effect can still show itself in observables quadratic in the charge density and generically has a tendency to suppress charge fluctuations in such backgrounds. This suppression could be a measurable effect in lattice simulations and may have significance for the Heavy Ion Collisions. The effect can also have consequences for finite density QCD in magnetic field. In this setup chromo-magnetic field parallel to the abelian-magnetic field would generate color quark charges. If confinement is assumed, such charges would be heavily suppressed, which would manifest itself in a strong suppression of the chromo-magnetic field in the direction of the abelian magnetic field. Whether this happens can be tested in lattice simulations of the $SU(2)$ theories with chemical potential where no sign problem occurs for even number of flavors. On the other hand, a fundamental zero mode exchange is expected between the KK monopoles[1]. Since in strong enough magnetic field fermions can propagate only along the magnetic field B, a KK monopole–anti-monopole pair creation along the magnetic field will be more probable[2] (see Fig. 5.1), possibly making strings of monopoles along the magnetic field, and inducing strong chromo-magnetic fields there allowing for charge catalysis and the suppression of charge fluctuations. On the other hand, chiral condensation along the magnetic field of some sort might be possible, which would allow pions to form and propagate along the magnetic field[3].

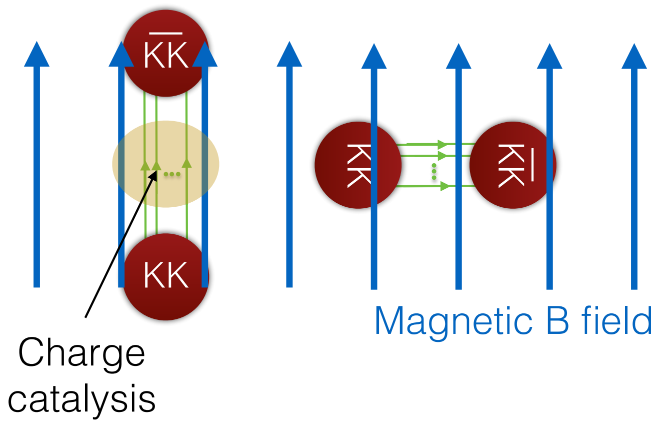


Figure 5.1: KK monopole–anti-monopole in the magnetic field. The monopole–anti-monopole pair creation on the left is more probable than the one on the right, because the fermions, which bind them together, get an effective mass $\sqrt{B}$ in the transverse direction of the magnetic field. This implies that the proper picture of KK monopoles in the vacuum would be strings of monopole–anti-monopole pairs where charge catalysis can occur. Preference of the configurations on the left would increase the probability of charge catalysis effect and, therefore, stronger suppression of charge fluctuations.

We have also discussed a first order effect of charge-generation in 2+1D systems

[1] Although we have found no contributions of the fundamental zero mode exchange in sQCD, we believe that this is due to supersymmetry, rather then a generic feature (see below).

[2] It is important to stress that such a scenario is not expected in the chirally broken phase, which would be a disordered phase of monopoles and where fermions would not be strong enough to fight this disorder. Instead this scenario, if it exists, is expected at a temperature slightly above the transition temperature T_c.

[3] This "chiral symmetry breaking" transition might not be (and probably is not) a usual phase transition, as it is unclear whether it is characterized by any order parameter, but is instead characterized by the possibility to form pions which can only propagate along the magnetic field. Further, since we are arguing a dimensional reduction to 1+1D, Mermin-Wagner theorem prohibits spontaneous symmetry breaking anyway (see however [74] and references therein), and the effective Lagrangian of pions would be a two-dimensional nonlinear sigma model. These models are generically gapped and would imply pion formation with a dynamical mass generation.

with the $U(1) \times U(1)$ magnetic background fields, where the chemical potential of one $U(1)$ sector results in the generation of charge in the other $U(1)$ sector. By localizing the magnetic fields the charge can be generated locally and halo-shaped charge density can appear by appropriate magnetic field profiles. The setup has a direct correspondence with graphene and potentially important use in the field of "valleytronics", i.e. manipulation of valley charges in graphene.

In Chapter 4, we discussed the appearance of the instanton-monopoles in two dynamical models: sQCD and 2D $O(N)$ nonlinear sigma models. While we have argued that for massless fundamental multiplets sQCD is ungapped (a feature not shared with QCD) and non-semiclassical, sQCD with heavy (s)quarks is perfectly computable and exhibits Polyakov loop screening and string breaking phenomenon which are also present in QCD. We have shown that the shift of the Polyakov loop average from the center symmetric value is caused by the resummation of electrically charged massive (s)quark winding modes. Surprisingly, however, we find no fundamental zero mode exchange contribution to the neutral bion amplitude. A similar feature is found in the $\mathcal{N} = 2$ SYM on $\mathbb{R}^3 \times S^1$, a theory with two Weyl fermions, where no exchange occurs to form magnetic and neutral bions [90]. The microscopic picture of why this is not happening has not yet been elucidated, although speculations that scalars are responsible for this behavior have been made in [90]. Understanding the zero mode exchange (or lack thereof) in SUSY theories is bound to give insights into understanding their non-supersymmetric counterparts.

As a toy model of a QCD-like theory at finite densities, we also considered the two dimensional $O(N)$ model. We have found that at finite density of some $SO(2) \in SO(N)$ charge, the model abelianizes to the $O(2)$ subgroup and has monopole-like solutions, i.e. vortices. If they percolate in the system, the vortices gap the spatial correlators. Unfortunately, strict analytical control is only possible for $\mu \gg \Lambda$, where Λ is the strong scale of the theory. In this regime vortices form tightly bound pairs and the spatial correlation functions are algebraically (i.e. non-exponentially) decaying. Nevertheless, as we reduce the value of the chemical potential two things are bound to occur: 1. the spatial correlators must become exponential and 2. the full $O(N)$ symmetry must be restored. The latter is expected to happen at the point $\mu = \mu_c$ where the chemical potential is equal to the mass of the first exited charged state E_0. The former should occur because of the disorder in the system. We have conjectured that the disordering is due to the vortex percolation at chemical potentials larger then μ_c where the symmetry is still broken to $O(2)$, and is therefore the famous Berezinsky-Kosterlitz-Thouless transition, characteristic to two dimensional systems. This is plausible because at μ_c the temporal correlation functions also become exponential, but are expected to decay as $e^{-(E_0-\mu)t}$ because of the chemical potential enhancement in the temporal direction. On the other hand we expect that the spatial correlation functions decays as $e^{-E_0|x|}$, so some mechanism should generate the nonzero correlation length $\xi = 1/E_0$ before μ_c is reached from above. The window between the value of μ where abelianization occurs and where spatial correlators become cease to be exponential may be adjustable by using the deformation like the one in [5] allowing easier lattice analysis.

Although our interest in the $O(N)$ models was mainly because of its similarity to YM theory, these models also appear in condensed matter systems such as models of topological supefluids [1], and quantum simulators [96]. The latter work is especially interesting for us as in this proposal the magnetic field plays the role of the chemical potential, which would make our discussion easy to implement in this setup and test experimentally.

Finally let us speculate how instanton monopoles can be used to construct a model of QCD vacuum with fundamental matter. One way would be to simulate the ensemble a-la Instanton Liquid Model, taking into account all the classical interactions between monopoles, as well as those due to fermions. This approach has been applied in [51], although this work relied on the view of Diakonov et al. [43, 44, 41] where moduli space, rather then classical, interactions were assumed to be dominant, which is not the case according to our understanding. Another approach would be to construct an effective theory of fermions and instanton monopoles with terms[4] $\sim \det \Psi\Psi e^{\pm i\sigma}$ which should be present due to the zero modes of the KK (anti-)monopoles, and terms $e^{\pm i\sigma}$ due to the BPS (anti-)monopoles. As opposed to sQCD we discussed in Section 4.1, in QCD with massless fermions BPS monopoles can generate a mass gap and yield a term $\cos(\sigma)$, while the KK monopoles generate the $2N_f$ Fermi interaction needed for spontaneous chiral symmetry breaking[5]. However a question how can electrically charged objects such as fermion, exist in the effective theory of monopoles still remains, and it is not clear to this author whether the above, although naively confining Wilson loops, would confine dynamical quarks properly. Regardless, an ad hoc model such as the one containing the terms discussed above may be worth exploring as it has both the Wilson loop area law and chiral symmetry breaking. Of course chemical potential and magnetic field can coupled in a usual way to fermions, allowing one to study a confining, chirally broken theory at finite density and magnetic field.

[4]The determinant here is in flavor space

[5]These fermion interactions would couple to the dual field σ and it is an interesting question how this would affect chiral symmetry breaking.

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I am grateful to Edward Shuryak for teaching me that details are sometimes not important, to Erich Poppitz for teaching me that details are sometimes important, to Mithat Ünsal for teaching me optimism and to Falk Bruckmann for teaching me skepticism. These four virtues of science I shall cherish forever. I addition I am very thankful to Pavel Buidovich for many interesting discussions.

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Appendix A

Abbreviations and notation

A.1 Abbreviations

- LHS = Left Hand Side
- RHS = Right Hand Side
- VEV = Vacuum Expectation Value
- IR = Infrared
- UV = Ultraviolet
- YM = Yang-Mills
- QCD = Quantum Chromodynamics
- QCD(adj) = QCD with quarks in the adjoint representation
- QED = Quantum Electrodynamics
- e.o.m = equation(s) of motion

A.2 Pauli and Dirac matrices and gauge fields

- T^a , $a = 1, \ldots, N(N-1)/2$ = Generators of $SU(N)$ (in arbitrary representation)
- τ^a , $a = 1, 2, 3$ = Pauli matrices defined as

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.1}$$

- $\sigma^\mu = (-i\tau^i, \mathbb{I}), \qquad \bar{\sigma}^\mu = (i\tau^i, \mathbb{I})$
- $A_\mu = A_\mu^a T^a$ (unless otherwise specified)
- $D_\mu = \partial_\mu - iA_\mu$

We take the Dirac matrices in Euclidean space in the chiral basis to be

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.2}$$

and

$$\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{A.3}$$

so that the Dirac operator is

$$\not{D} = \gamma_\mu D_\mu = \begin{pmatrix} 0 & \mathcal{D} \\ \bar{\mathcal{D}} & 0 \end{pmatrix} \tag{A.4}$$

with

$$\mathcal{D} = \sigma^\mu D_\mu \ , \qquad\qquad \bar{\mathcal{D}} = \bar{\sigma}^\mu D_\mu \tag{A.5}$$

and we define the left-handed (ψ_L) and the right-handed (ψ_R) Dirac component to be

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{A.6}$$

The identities

$$\epsilon\sigma^\mu\epsilon = -(\sigma^\mu)^* = -(\bar{\sigma}^\mu)^T \ , \qquad\qquad \epsilon\bar{\sigma}^\mu\epsilon = -(\sigma^\mu)^* = -(\bar{\sigma}^\mu)^T \tag{A.7}$$

are easily shown to hold, which is equivalent to

$$\sigma^\mu\epsilon = \epsilon(\sigma^\mu)^* = \epsilon(\bar{\sigma}^\mu)^T \ , \qquad\qquad \bar{\sigma}^\mu\epsilon = \epsilon(\bar{\sigma}^\mu)^* = \epsilon(\sigma^\mu)^T \tag{A.8}$$

If we construct matrices

$$\sigma^{\mu\nu} = \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \tag{A.9}$$

$$\bar{\sigma}^{\mu\nu} = \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) \tag{A.10}$$

it is easily checked that $\bar{\sigma}^{\mu\nu}$ is self-dual, and that $\sigma^{\mu\nu}$ is anti–self-dual i.e. that

$$\sigma^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu} \tag{A.11}$$

$$\bar{\sigma}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\mu\nu} \tag{A.12}$$

From the identity for the gamma matrices

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} = \delta^{\mu\nu}\gamma^{\rho} + \delta^{\nu\rho}\gamma^{\mu} - \delta^{\mu\rho}\gamma^{\nu} - \epsilon^{\mu\nu\rho\sigma}\gamma^{\sigma}\gamma_{5} \tag{A.13}$$

which follows by inspection, we get that the $\sigma^{\mu}, \sigma^{\nu}$ matrices obey

$$\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho} = \delta^{\mu\nu}\sigma^{\rho} + \delta^{\nu\rho}\sigma^{\mu} - \delta^{\mu\rho}\sigma^{\nu} + \epsilon^{\mu\nu\rho\sigma}\sigma^{\sigma} \tag{A.14}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} = \delta^{\mu\nu}\bar{\sigma}^{\rho} + \delta^{\nu\rho}\bar{\sigma}^{\mu} - \delta^{\mu\rho}\bar{\sigma}^{\nu} - \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}^{\sigma} \tag{A.15}$$

from which it follows that

$$\sigma^{\mu}\bar{\sigma}^{\nu\rho} = \delta^{\mu\nu}\sigma^{\rho} - \delta^{\mu\rho}\sigma^{\nu} + \epsilon^{\mu\nu\rho\sigma}\sigma^{\sigma} \qquad \sigma^{\mu\nu}\sigma^{\rho} = \delta^{\nu\rho}\sigma^{\mu} - \delta^{\mu\rho}\sigma^{\nu} + \epsilon^{\mu\nu\rho\sigma}\sigma^{\sigma} \tag{A.16a}$$

$$\bar{\sigma}^{\mu}\sigma^{\nu\rho} = \delta^{\mu\nu}\bar{\sigma}^{\rho} - \delta^{\mu\rho}\bar{\sigma}^{\nu} - \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}^{\sigma} \qquad \bar{\sigma}^{\mu\nu}\sigma^{\rho} = \delta^{\nu\rho}\bar{\sigma}^{\mu} - \delta^{\mu\rho}\bar{\sigma}^{\nu} + \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}^{\sigma} \tag{A.16b}$$

They also obey

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = -2\left(\delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\nu\rho}\sigma_{\mu\sigma}\right) \tag{A.17a}$$

$$\{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} = -2\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}\right) \tag{A.17b}$$

$$[\bar{\sigma}_{\mu\nu}, \bar{\sigma}_{\rho\sigma}] = -2\left(\delta_{\mu\rho}\bar{\sigma}_{\nu\sigma} + \delta_{\nu\sigma}\sigma_{\mu\rho} - \delta_{\mu\sigma}\bar{\sigma}_{\nu\rho} - \delta_{\nu\rho}\bar{\sigma}_{\mu\sigma}\right) \tag{A.17c}$$

$$\{\bar{\sigma}_{\mu\nu}, \bar{\sigma}_{\rho\sigma}\} = -2\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}\right) \tag{A.17d}$$

Further the 't Hooft symbols are defined as

$$\eta^{a}_{\mu\nu} = \frac{1}{2i}\mathrm{tr}\left(\tau^{a}\bar{\sigma}_{\mu\nu}\right), \qquad \bar{\eta}^{a}_{\mu\nu} = \frac{1}{2i}\mathrm{tr}\left(\tau^{a}\sigma_{\mu\nu}\right) \tag{A.18}$$

from where we get

$$\eta^{a}_{4i} = -\bar{\eta}^{a}_{4i} = \delta_{ia}, \qquad \eta^{a}_{ij} = \bar{\eta}^{a}_{ij} = \epsilon_{ija} \tag{A.19}$$

Appendix B

Topological charge and winding

Here we want to show that the quantity

$$Q = \frac{1}{16\pi^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{B.1}$$

is the winding number, where $F_{\mu\nu} = F^a_{\mu\nu}\frac{\tau^a}{2}$ and $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. We mostly follow [110]. First note that

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \\ &= 2\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,(\partial_\mu A_\nu\partial_\rho A_\sigma + iA_\mu A_\nu\partial_\rho A_\sigma + i\partial_\mu A_\nu A_\rho A_\sigma - A_\mu A_\nu A_\rho A_\sigma) = \\ &= 2\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,(\partial_\mu A_\nu\partial_\rho A_\sigma - 2i\partial_\mu A_\nu A_\rho A_\sigma) = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu\,\mathrm{Tr}\,(A_\nu\partial_\rho A_\sigma - i\frac{2}{3}A_\nu A_\rho A_\sigma) = \\ &= \partial_\mu K^\mu \end{aligned} \tag{B.2}$$

with

$$K^\mu = 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(A_\nu\partial_\rho A_\sigma - i\frac{2}{3}A_\nu A_\rho A_\sigma) \tag{B.3}$$

Then

$$\int_{\mathcal{M}} d^4x \,\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \oint_{S^3=\partial\mathcal{M}} dS_\mu K^\mu \tag{B.4}$$

where S_μ is a vector on perpendicular to the S^3 sphere at infinity. Assuming that $F_{\mu\nu} = 0$ at infinity, we have that $\epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma = i\epsilon^{\mu\nu\rho\sigma}A_\rho A_\sigma$, and taking that $A_\mu = iU^\dagger\partial_\mu U$, i.e. that A_μ is a pure gauge, we have

$$\int_{\mathcal{M}} d^4x\; F_{\mu\nu}\tilde{F}^{\mu\nu} = \oint_{S^3} dS_\mu \frac{2}{3}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,\left(U^\dagger\partial_\nu U\; U^\dagger\partial_\rho U\; U^\dagger\partial_\sigma U\right) \tag{B.5}$$

Let $\xi_i, i = 1,2,3$ be parameters parametrizing $SU(2)$ group element. We can write $U^\dagger\frac{\partial}{\partial\xi_i}U = e^a_i i\frac{\tau_a}{2}$, where $\tau_a/2$ are the $SU(2)$ the group generators, and e^a_i is the $SU(2)$

group vielbein. The integral then becomes

$$\int_{\mathcal{M}} d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{-i}{12}\oint_{S^3} d^3x\, \epsilon^{ijk}\frac{\partial\xi^l}{\partial x^i}\frac{\partial\xi^m}{\partial x^j}\frac{\partial\xi^s}{\partial x^k}\,\mathrm{Tr}\,(\tau_a\tau_b\tau_c)\, e_i^a e_j^b e_k^c = \int_{group} d^3\xi\,\det(e(\xi)) \tag{B.6}$$

where we have used[1] $\epsilon^{ijk}\frac{\partial\xi^l}{\partial x^i}\frac{\partial\xi^m}{\partial x^j}\frac{\partial\xi^s}{\partial x^k} = \epsilon^{lms}\det\frac{\partial\xi_i}{\partial x_j}$.

The integral

$$\int d^3\xi\,\det(e(\xi)) \tag{B.7}$$

is an integral over the group manifold. The measure $\det(e(\xi))$ is clearly parametrization invariant, and all we have to do is determine the norm. This can be checked by considering the parametrization $U = \bar{\sigma}^\mu\xi_\mu$ with $\xi_\mu\xi^\mu = 1$. ξ_μ are then coordinates on a 3-sphere. Computing $\det(e(\xi))$ at $\xi_0 = 1$ and $\xi_i = 0$ we have

$$U^\dagger(\xi)\frac{\partial}{\partial\xi_i}U(\xi)\bigg|_{\xi_i=0} = i\tau^i \tag{B.8}$$

so $e_i^a(\xi_i = 0) = 2\delta_i^a$ so that $\det(e_i^a(\xi_i = 0)) = 8$. Due to isotropy of the parametrization, we have that this is valid for arbitrary ξ_i. Then we have

$$\int d^4x\mathrm{Tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} = \int_{group} d^3\xi\,\det(e(\xi)) = 8(2\pi^2)Q \tag{B.9}$$

where $2\pi^2$ is the volume of the S^3 sphere and Q is the number of times the group is covered. From above it follows that

$$Q = \frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{B.10}$$

[1] This holds because of the anti-symmetry with respect to lms and the norm is found by, for example, setting $l, m, s = 1, 2, 3$.

Appendix C

Monopole measure

C.1 Moduli space metric

Before considering gauge fields let us first consider a problem in the case of a scalar field ϕ instead. The problem is that of doing a semi-classical saddle point approximation around a classical solution $\phi_0(\{\alpha_i\})$, where α_i are parameters of the solution which leave the action invariant. These parameters are associated with the global symmetries of the theory (e.g. translational symmetry). If we now want to do a saddle point around this background ϕ_0, we may write the field as $\phi = \phi_0 + \delta\phi$, and then expand the action to second order in the quantum fluctuations $\delta\phi$. However as we already said, the fluctuations of $\delta\phi$ along the symmetry directions of ϕ_0 will produce no change in the action and will not be small, so we must do this integral exactly.

The task now is to construct an integration measure. To that end let us separate an arbitrary variation into variations along the symmetry coordinates and those perpendicular to it, i.e. $\delta\phi = \delta\phi_\perp + \partial_{\alpha_i}\phi_0 d\alpha_i$, where $\delta\phi_\perp$ is orthogonal to the variations ∂_{α_i}. To find the integration measure we simply look at the invariant "line element"

$$(\delta\phi, \delta\phi) = (\delta\phi_\perp, \delta\phi_\perp) + (\partial_{\alpha_i}\phi_0, \partial_{\alpha_j}\phi_0) d\alpha_i d\alpha_j \tag{C.1}$$

We can now read off the metric for the α_i integration as

$$G_{\alpha_i,\alpha_j} = (\partial_{\alpha_i}\phi_0, \partial_{\alpha_j}\phi_0) \ , \tag{C.2}$$

and the integration measure becomes

$$\int \mathcal{D}\phi \to \int \mathcal{D}\delta_\perp\phi \int d^K\alpha_i \sqrt{\det G_{\alpha_i,\alpha_j}} \ . \tag{C.3}$$

In the case of the gauge background field A^0_μ, the situation is only slightly more subtle. Again we must consider a variation $\delta A_\mu^\perp$ which is orthogonal to the variation along the symmetry moduli $\partial_{\alpha_i} A_\mu^0$. However $\partial_{\alpha_i} A_\mu^0$ can also have components which are pure gauge $D_\mu \Lambda$. Therefore we must insist that $\delta_{\alpha_i} A_\mu = \partial_{\alpha_i} A_\mu^0 + D_\mu \Lambda_1$ is orthogonal to the pure gauge $D_\mu \Lambda_2$ for any gauge transformation Λ_2. In other words we must find Λ_1 such that $(\delta_{\alpha_i} A_\mu, D_\mu \Lambda_2) \Rightarrow -(D_\mu \delta_{\alpha_i} A_\mu, \Lambda_2), \forall \Lambda_2 \Rightarrow D_\mu \delta_{\alpha_i} A_\mu = 0$, i.e. $\delta_{\alpha_i} A_\mu$ must be in a background field gauge.

To compute the metric (C.2) of the monopole we take the monopole solution in the stringy gauge. Let us first consider the spatial translation moduli and write

$$\delta_j A_\mu = \partial_j A_\mu^{mon} + D_\mu \Lambda \ . \tag{C.4}$$

If we choose $\Lambda = -A_j^{mon}$ we have $\delta_j A_i = F_{j\mu}$ and, indeed $D_\mu \delta_j A_\mu = 0$ by the e.o.m. Further the monopole has another symmetry associated with global $U(1)$ transformations (i.e. transformations which leave the Higgs unchanged). To compute these moduli we take

$$A_\mu^\alpha = e^{-i\alpha\tau^3/2} A_\mu e^{i\alpha\tau^3/2} \tag{C.5}$$

and so

$$\delta_\alpha A_\mu = \partial_\alpha A_\mu^\alpha \Big|_{\alpha=0} + D_\mu \Lambda = \frac{i}{2}[A_\mu, \tau^3] + D_\mu \Lambda \tag{C.6}$$

Taking $\Lambda = -A_4/v + \tau^3/2$ we have that $\delta_\alpha A_\mu = -\frac{1}{v} D_\mu A_4$ and, again, $D_\mu \delta_\alpha A_\mu = 0$ by the e.o.m.

The metric is then given by

$$G_{ij} = \int d^4x \ 2\mathrm{tr}\,(F_{i\mu} F_{j\mu}) = \frac{\delta_{ij}}{3} \int d^4x \ 2\mathrm{tr}\, F_{kl}^2 + \frac{\delta_{ij}}{3} \int d^4x \ 2\mathrm{tr}\, F_{k4}^2 = 4\pi v L \delta_{ij} \ , \tag{C.7}$$

$$G_{\alpha\alpha} = \frac{1}{v^2} \int d^4x \ 2\mathrm{tr}\,(D_i A_4)^2 = \frac{1}{v^2} \int d^4x \ \mathrm{tr}\, F_{i4}^2 = \frac{4\pi L}{v} \tag{C.8}$$

so that the integration over monopole position is

$$\int_0^{2\pi} d\alpha \int d^3x_0 \sqrt{\det(G)} = \int_0^{2\pi} d\alpha \int d^3x_0 \ (4\pi L)^2 v \ . \tag{C.9}$$

C.2 The one loop determinants

The one loop determinants around the monopole have to be done with care. As we already mentioned the bosonic determinant over the gauge field fluctuations has to be restricted to non-zero mode subspace, as the zero mode fluctuations are taken into account by the moduli space metric. Therefore the gauge field determinant structure is of the form (see (D.20))

$$\frac{\det\left(-D^2\right)}{\det'\left(\frac{2\pi}{g^2}\left(-D^2\delta_{\mu\nu}-2F_{\mu\nu}\right)\right)^{1/2}} \tag{C.10}$$

where the prime on the determinant indicated stat the zero modes must be omitted. We have also inserted the gauge coupling in front of the operators as well as the factor of 2π coming from the Gaussian integration[1]. These are usually ignored because they factor in front of the partition function. However because of the absence of zero modes in the determinant, they will be crucial for the one-loop measure of monopoles (and instantons, which we don't discuss here. See [106]).

The above determinants are divergent and must be regularized. In principle the regulator is common for the vacuum and one (and indeed multi-)monopole contributions and is given by

$$\frac{\det\left(\frac{1}{g^2\sqrt{2\pi}}\left(-\partial^2+\Lambda_{PV}^2\right)\delta_{\mu\nu}\right)^{1/2}}{\det\left(-\partial^2+\Lambda_{PV}^2\right)} \propto \det(-\partial^2+\Lambda_{PV}^2) \tag{C.11}$$

However the ratio between this regulator and the one where the free vacuum operators are replaced by the background operators vanishes in the large Λ_{PV} limit. Therefore we will use the following regularization for the one loop determinants:

$$\frac{\det\left(\frac{2\pi}{g^2}\left((-D^2-2F_{\mu\nu}+\Lambda_{PV}^2)\delta_{\mu\nu}\right)\right)^{1/2}}{\det\left(-D^2+\Lambda_{PV}^2\right)}\frac{\det\left(-D^2\right)}{\det'\left(\frac{2\pi}{g^2}\left(-D^2\delta_{\mu\nu}-2F_{\mu\nu}\right)\right)^{1/2}} \tag{C.12}$$

Note that the regulator determinant is not the zero-mode amputated one, but the full one. As a result the one loop measure of the topological background is given by

$$\left(\frac{1}{g\sqrt{2\pi}}\Lambda_{PV}\right)^{N_{zm}}\frac{\det\left(-D^2\right)}{\det\left(-D^2+\Lambda_{PV}^2\right)}\frac{\det'\left(\frac{2\pi}{g^2}\left((-D^2-2F_{\mu\nu}+\Lambda_{PV}^2)\delta_{\mu\nu}\right)\right)^{1/2}}{\det'\left(\frac{2\pi}{g^2}\left(-D^2\delta_{\mu\nu}-2F_{\mu\nu}\right)\right)^{1/2}} \tag{C.13}$$

where the N_{zm} is the number of bosonic zero modes in the background and now all determinants are without zero modes[2].

[1] More specifically if we have an integral over the measure

$$\left(\prod_i\int d\phi_i\right)e^{-\sum_{i,j}\phi_i O_{ij}\phi_j}=\prod_i\int d\phi_i' e^{-\lambda_i\phi_i'^2}=\prod_i\left(\sqrt{\frac{2\pi}{\lambda_i}}\right)$$

Note that such factors of 2π do not occur in the Grassmann integrals.

[2] It is perhaps a misnomer to say that the regulator determinant has a zero mode. What we mean is

Combining (C.9) and (C.13) we finally get for the monopole measure

$$\int d^3x\ \mu_B = \int_0^{2\pi} d\alpha \int d^3x_0\ \frac{4L^2 v}{g^4}\Lambda_{PV}^4 \times (\text{non-zero mode determinants}) \tag{C.14}$$

that its Λ_{PV} independent part has a zero mode.

Appendix D

The background field quantization

D.1 YM action to quadratic order

Writing $A_\mu \to A_\mu + \delta A_\mu$ we have

$$F_{\mu\nu} \to F_{\mu\nu} + D_\mu \delta A_\nu - D_\nu \delta A_\mu - i[\delta A_\mu, \delta A_\nu] \tag{D.1}$$

where $D_\mu = \partial_\mu - i[A_\mu, \dots]$ is the background field covariant derivative in the adjoint representation.

The action density then is

$$\begin{aligned} F_{\mu\nu}^2 \to &+ F_{\mu\nu}^2 && \text{(D.2a)} \\ &+ 2F_{\mu\nu}(D_\mu \delta A_\nu - D_\nu \delta A_\mu) && \text{(D.2b)} \\ &+ (D_\mu \delta A_\nu - D_\nu \delta A_\mu)^2 - 4iF_{\mu\nu}\delta A_\mu \delta A_\nu && \text{(D.2c)} \\ &- 2i(D_\mu \delta A_\nu - D_\nu \delta A_\mu)[\delta A_\mu, \delta A_\nu] - [\delta A_\mu, \delta A_\nu]^2 && \text{(D.2d)} \end{aligned}$$

Since

$$\text{tr } (F_{\mu\nu}\delta A_\mu \delta A_\nu) = \text{tr } ([F_{\mu\nu}, \delta A_\mu]\delta A_\nu - F_{\mu\nu}\delta A_\mu \delta A_\nu) \tag{D.3}$$

$$\Downarrow$$

$$\text{tr } (F_{\mu\nu}\delta A_\mu \delta A_\nu) = \frac{1}{2}\text{tr } [F_{\mu\nu}, \delta A_\mu] \tag{D.4}$$

and since we can write

$$(D_\mu \delta A_\nu - D_\nu \delta A_\mu)^2 = 2\delta A_\mu(-D^2 \delta_{\mu\nu} + D_\nu D_\mu)\delta A_\nu \tag{D.5}$$

we can write the quadratic term in δA_μ in the YM action is

$$\frac{1}{g^2}\mathrm{tr}\left[\delta A_\mu\left(-D^2\delta_{\mu\nu}+D_\nu D_\mu+i[F_{\mu\nu},\,\cdot\,]\right)\delta A_\nu\right] \tag{D.6}$$

Notice that the operator $-i[F_{\mu\nu},\,\cdot\,]$ is really $F_{\mu\nu}$ in the adjoint representation. So the quadratic term can be written as

$$\frac{1}{g^2}\mathrm{tr}\left[\delta A_\mu\left(-D^2\delta_{\mu\nu}+D_\nu D_\mu-F_{\mu\nu}\right)\delta A_\nu\right] \tag{D.7}$$

where all fields are in the adjoint representation.

Notice that the field δA_μ has a local gauge transformation transforming as

$$\delta A_\mu \to U^\dagger \delta A_\mu U + U^\dagger A_\mu U - A_\mu + iU^\dagger\partial_\mu U \tag{D.8a}$$

or infinitesimally as

$$\delta A_\mu \to \delta A_\mu + (D_\mu - i\delta A_\mu)\Lambda\ . \tag{D.8b}$$

D.2 The FaddeevPopov ghosts

We want to impose the gauge fixing condition on the background field. In particular we will use the following modified background Lorenz gauge

$$(D_\mu \delta A_\mu)^a = \omega^a \tag{D.9}$$

where a is the color index in the adjoint representation. As discussed in the previous section δA_μ can be gauge transformed with (D.8b). We denote the gauge transformed field as δA_μ^Λ, where Λ is the parameter of the gauge transformation. Now we must insert unity in the path integral in the form

$$\int \mathcal{D}\Lambda\ \det\left(\frac{\delta G^a(\delta A_\mu^\Lambda)}{\delta\Lambda^b}\right)\delta(G^a(\delta A_\mu^\Lambda)) = 1 \tag{D.10}$$

where $G^a(A_\mu^\Lambda) = \omega^a - (D_\mu\delta A_\mu)^a$ is the gauge fixing function.

Since the delta function imposes the background gauge (D.9), it is sufficient to use infinitesimal gauge transformations in the vicinity of this gauge to evaluate the determinant[1]. In other words

$$G^a(\delta A_\mu^\Lambda) \approx G^a(\delta A_\mu + (D_\mu - i\delta A_\mu)\Lambda) = \omega^a - (D_\mu\delta A_\mu)^a - (D_\mu(D_\mu - i\delta A_\mu)\Lambda)^a \tag{D.11}$$

[1]This is not exactly correct, as there are Gribov copies. We do not deal with such subtleties here.

so that

$$\det\left(\frac{\delta G^a(\delta A^\Lambda_\mu)}{\delta\Lambda^b}\right) = \det(-(D_\mu(D_\mu - i\delta A_\mu))^{ab}) \tag{D.12}$$

The partition function can then be written as

$$Z = \int \mathcal{D}\Lambda^a \int \mathcal{D}\delta A_\mu \; e^{-\mathcal{S}(\delta A^\Lambda)} \delta^{(\infty)}(G(\delta A^\Lambda_\mu)) \det\left(\frac{\delta G^a(\delta A^\Lambda_\mu)}{\delta\Lambda^b}\right) , \tag{D.13}$$

Since $\delta G^a(\delta A^\Lambda_\mu)/\delta\Lambda^b$ is independent of Λ we can do a coordinate change $A^\Lambda_\mu \to A_\mu$. Notice that the measure is invariant under the local gauge transformation of the fluctuating field δA_μ defined by (D.2), i.e. $\mathcal{D}A_\mu \to \mathcal{D}A^\Lambda_\mu$. This is because is a combination of a shift independent of the fluctuating field δA_μ and a unitary rotation in color[2]

The FP determinant can then be exponentiated with the help of Grassmann fields $\bar{c}$ and c, which then read in the Lagrangian

$$\mathcal{L}_{ghost} = \bar{c}\,[-D_\mu(D_\mu - i\delta A_\mu)]\,c \tag{D.14}$$

The total partition function is then

$$Z = \int \mathcal{D}\Lambda^a \int \mathcal{D}\delta A_\mu \; e^{-\mathcal{S}[\delta A]-\mathcal{S}_{ghost}} \delta^{(\infty)}(\omega^a - (D_\mu \delta A_\mu)^a) , \tag{D.15}$$

Since the integrand does not depend on Λ^a, the integral $\int \mathcal{D}\Lambda^a$ is just an infinite constant which can be dropped as it doesn't contribute to correlation functions. In principle one could say that we are done. We have successfully managed to restrict the integration over the fields which have a background gauge condition imposed on them, i.e. $(D_\mu\delta A_\mu)^a = \omega^a$. However having constraints in the Lagrangian can be difficult to work with, to that end we do our final trick which will eliminate constraints on the field δA_μ.

Since the equation (D.15) holds for any ω^a we can integrate with an arbitrary weight over ω^a as long as it is normalized to unity. Taking Gaussian weight $e^{-\omega^2/\xi}$ we have

$$Z = N(\xi)\int \mathcal{D}\omega^a \int \mathcal{D}\delta A_\mu \; e^{-\mathcal{S}[\delta A]-\mathcal{S}_{ghost}} \delta^{(\infty)}(\omega^a - (D_\mu\delta A_\mu)^a) e^{-\int d^4x \, \frac{\omega^2}{g^2\xi}} , \tag{D.16}$$

where we dropped the infinity integral over Λ and where $N(\xi)$ is the Gaussian integral norm (on which physics does not depend). Integrating over the ω^a finally produces the Lagrangian

$$S_{GF} = S[\delta A] + S_{ghost} + \int d^4x \frac{(D_\mu\delta A_\mu)^2}{\xi g^2} \tag{D.17}$$

The ξ is the parameter which we can choose at will.

[2]More specifically we have a transformation of the type $v' = Uv + f(U)$. Quite clearly the Jacobian $\det \frac{\partial v^a}{\partial v^b} = \det U = 1$ for U belonging to the special orthogonal or unitary groups.

A comment about (D.17) is in order. In the literature it is often said that the action (D.17) is gauge fixed to the gauge $D_\mu \delta A_\mu = 0$. However this is not true for the off-shell fields but only true in the sense that the expectation value[3] $\langle D_\mu \delta A_\mu \rangle = 0$. The fields δA_μ over which we integrate, however, need not obey this condition.

D.2.1 The one loop determinants

Now we are ready to compute the one loop determinats in the background of a classical solution, i.e. a solution for which the linear term (D.2b) vanishes[4] or, more specifically where the background is self-dual[5]. Then the effective gauge-fixed action to quadratic order in fluctuating fields δA_μ takes the form

$$\mathcal{S}_{GF} = S_0 + \frac{1}{g^2} \int d^4x \delta A_\mu (-D^2 \delta_\mu \nu - F_{\mu\nu} + D_\nu D_\mu - \frac{1}{\xi} D_\mu D_\nu) \delta A_\nu + \int d^4x\ \bar{c}(-D_\mu^2)c + \int d^4x\ (D_\mu \bar{c}) i \delta A_\mu c \quad \text{(D.18)}$$

Taking $\xi = 1$ we have a simple form of the quadratic operator

$$W_{\mu\nu} = -D^2 \delta_{\mu\nu} - 2F_{\mu\nu} \quad \text{(D.19)}$$

Integrating over the quantum fluctuations δA_μ and the ghost fields $\bar{c}, c$ we have the determinants[6]

$$\frac{\det(-D_\mu^2)}{\sqrt{\det(-D^2 \delta_{\mu\nu} - 2F_{\mu\nu})}} \quad \text{(D.20)}$$

A comment about this ration is in order. The gauge field determinant has a μ, ν index structure, which accounts for 4 polarizations of the fluctuating gauge field, i.e. $\mu = 0, 1, 2, 3$. However the ghost determinants is due to the two ghost field $\bar{c}, c$, which exactly cancels two unphysical polarizations of the gauge field fluctuations in the vacuum.

[3] This is true because we have integrated with a Gaussian weight with peak around $\omega = 0$. Since $\langle \omega \rangle = \langle D_\mu \delta A_\mu \rangle = 0$

[4] To get the effective action, the linear term need not vanish, as one can always put an external source which cancels it. This is the standard method of the effective action, and would also be required for the monopole–anti-monopole system in a similar spirit of [17, 117] where instantons are discussed. We will not deal with such backgrounds here.

[5] The self-duality allows a simple rewriting of the

[6] One might be worried by the appearance of the linear term in δA_μ. However this term is only important for higher loops (see e.g. [106]).

Appendix E

Fundamental zero mode and chemical potential

E.1 The single monopole background

I think the convention of the covariant derivative here is $D_\mu = \partial_\mu + iA_\mu$. I must check this.

Here we find the solutions of the equations

$$\frac{d\alpha_1(r)}{dr} + \frac{\pm\mathcal{H} + 2\mathcal{A}}{2}\alpha_1(r) \pm i\mu\alpha_2 = 0 \tag{E.1a}$$

$$\frac{d\alpha_2(r)}{dr} + \left(\frac{\pm\mathcal{H} - 2\mathcal{A}}{2} + \frac{2}{r}\right)\alpha_1(r) \pm i\mu\alpha_2 = 0 \tag{E.1b}$$

This corresponds to the Dirac equation with

$$((\sigma^\mu)_{\alpha\beta}(D_\nu)_{AB} - \mu\delta_{AB}\delta_{\alpha\beta})(\Psi_L)^B_\beta = 0 \tag{E.2a}$$

$$((\bar{\sigma}^\mu)_{\alpha\beta}(D_\nu)_{AB} - \mu\delta_{AB}\delta_{\alpha\beta})(\Psi_R)^B_\beta = 0 \tag{E.2b}$$

where $\sigma^\mu = (\mathbb{I}, -i\tau^i), \bar{\sigma}^\mu = (\mathbb{I}, i\tau^i)$ and with the ansatz

$$\Psi^A_\alpha = \left[(\alpha_1(r)\mathbb{I} + \alpha_2(r)\hat{\boldsymbol{r}}\cdot\boldsymbol{\tau})\,\epsilon\right]_{A\alpha} \tag{E.3}$$

in the background

$$A^a_4 = \mathcal{H}(r)\hat{r}^a \qquad\qquad A^a_i = \mathcal{A}(r)\epsilon_{aik}\hat{r}^k\ . \tag{E.4}$$

The upper sign in (E.1) refers to the right-handed solution and the lower sign to the left-handed solution, i.e. it refers to eqs. (E.2a) and (E.2b), respectively. The *BPS* monopole solution is given by

$$\mathcal{H} = \mp \frac{1 - vr\coth(vr)}{r} \,, \qquad \mathcal{A} = \frac{1}{r}\left(1 - \frac{vr}{\sinh(vr)}\right) \tag{E.5}$$

where the upper sign refers to the self-dual solution of the YM equations and the lower sign to the anti-self-dual solution. Note that μ can be real or imaginary.

To solve the above equations let us write them in the following form

$$\frac{d}{dr}\boldsymbol{\alpha}(r) = -M\boldsymbol{\alpha} \tag{E.6}$$

where

$$\boldsymbol{\alpha}(r) = \begin{pmatrix} \alpha_1(r) \\ \alpha_2(r) \end{pmatrix} \qquad M = \begin{pmatrix} \frac{\pm\mathcal{H}+2\mathcal{A}}{2} & -i\mu \\ -i\mu & \frac{\pm\mathcal{H}-2\mathcal{A}}{2} + \frac{2}{r} \end{pmatrix} \tag{E.7}$$

It is well known that the formal solution of (E.6) is

$$\boldsymbol{\alpha}(r) = \mathcal{P}\, e^{-\int_0^r dr'\; M(r')}\boldsymbol{\alpha}(0) \tag{E.8}$$

where $\mathcal{P}$ is the path-ordering.

Let us, however, split the matrix $M(r)$ into its non-zero trace component and traceless component

$$M(r) = M_0(r) + M_1(r) \tag{E.9}$$

with

$$M_0(r) = \left(\frac{\pm\mathcal{H}}{2} + \frac{1}{r}\right)\mathbb{I} \tag{E.10}$$

$$M_1(r) = -i\mu\tau_1 + \left(\mathcal{A}(r) - \frac{1}{r}\right)\tau^3 \tag{E.11}$$

where $\tau^{1,3}$ are usual Pauli matrices. The solution can then be written as

$$\boldsymbol{\alpha}(r) = e^{-\int dr'\; M_0(r')}\boldsymbol{\chi}(r) \tag{E.12}$$

with the equation for $\boldsymbol{\chi}(r)$ being

$$\frac{d}{dr}\boldsymbol{\chi}(r) = -M_1(r)\boldsymbol{\chi}(r) \tag{E.13}$$

or

$$\frac{d\chi_1}{d\xi} = \frac{1}{\sinh\xi}\chi_1 + i\zeta\chi_2 \tag{E.14a}$$

$$\frac{d\chi_2}{d\xi} = i\zeta\chi_1 - \frac{1}{\sinh\xi}\chi_2 \tag{E.14b}$$

where we have labeled

$$\xi = vr, \zeta = \mu/v\ . \tag{E.15}$$

In order for the solution to be normalizable from (E.12) we see that $M_0(r \to \infty) > 0$ which means that for self-dual solutions, left-handed mode is normalizable and for anti-self-dual solutions the right-handed mode is normalizable, as it should be. We can specialize to self-dual solutions with $\mathcal{H} \to v$, so that[1]

$$e^{-\int dr\ M_0(r)} = \frac{c}{\sqrt{\xi\sinh(\xi)}} \tag{E.16}$$

Now by expressing χ_2 through χ_1 from (E.14) we get a second order equation for χ_1

$$-\left(\frac{d}{d\xi} + \frac{1}{\sinh\xi}\right)\left(\frac{d}{d\xi} - \frac{1}{\sinh\xi}\right)\chi_1 = -\frac{d^2}{d\xi}\chi_1 - \frac{1}{2\cosh^2\frac{\xi}{2}}\chi_1 = \zeta^2\chi_1 \tag{E.17}$$

This is the Schödigner equation for a particle moving in a $1/\cosh^2(r)$ potential. This potential is solvable by noting that the above equation is a Schrödinger equation of a supersymmetric Hamiltonian with the superpotential $W(\xi) = -1/\sinh\xi$, i.e.

$$H = \left(-\frac{d}{d\xi} + W(\xi)\right)\left(\frac{d}{d\xi} + W(\xi)\right) = \left(-\frac{d}{d\xi} - \frac{1}{\sinh\xi}\right)\left(\frac{d}{d\xi} - \frac{1}{\sinh\xi}\right)\ . \tag{E.18}$$

However we can just as well write another Hamiltonian with the superpotential $W_1 = \frac{1}{2}\tanh(\xi/2)$

$$H_1 = \left(-\frac{d}{d\xi} + W_1(\xi)\right)\left(\frac{d}{d\xi} + W_1(\xi)\right) = H + \frac{1}{4} \tag{E.19}$$

But this Hamiltonian has a super-partner Hamiltonian which is just the free particle Hamiltonian, i.e.

$$\tilde{H}_1 = \left(\frac{d}{d\xi} + W_1(\xi)\right)\left(-\frac{d}{d\xi} + W_1(\xi)\right) = -\frac{d^2}{d\xi^2} + \frac{1}{4} \tag{E.20}$$

Since the eigenstates of $\tilde{H}_1$ are simply plane waves $\tilde{\psi}_\zeta(\xi) = e^{\pm i\zeta\xi}$ with eigenvalues

$$\tilde{E}_1(\zeta) = \zeta^2 + \frac{1}{4} \tag{E.21}$$

[1]The constant of integration is irrelevant as it can be absorbed into the normalization.

To get the eigenstates of H_1 (and therefore of H) we simply take

$$\psi_\zeta(\xi) = \left(-\frac{d}{d\xi} + \frac{1}{2}\tanh\frac{\xi}{2}\right)\tilde{\psi}_\zeta(\xi) = \left(\mp i\zeta + \frac{1}{2}\tanh\frac{\xi}{2}\right)e^{\pm i\zeta\xi} \tag{E.22}$$

Therefore the general solution of (E.17) is

$$\chi_1(\xi) = c_+\left(-2i\zeta + \tanh\frac{\xi}{2}\right)e^{i\zeta\xi} + c_-\left(2i\zeta + \tanh\frac{\xi}{2}\right)e^{-i\zeta\xi} \tag{E.23}$$

Using the first order equations (E.14) we get that

$$\chi_2(\xi) = c_+\left(2i\zeta - \coth\frac{\xi}{2}\right)e^{i\zeta\xi} + c_-\left(2i\zeta + \coth\frac{\xi}{2}\right)e^{-i\zeta\xi} \tag{E.24}$$

The function χ_2 is divergent in the limit $\xi \to 0$ unless $c_+ = c_-$. So we must set $c_+ = c_- = c/2$. The solution is then

$$\chi_1 = c\left(2\zeta\sin(\zeta\xi) + \tanh\left(\frac{\xi}{2}\right)\cos(\zeta\xi)\right) \tag{E.25a}$$

$$\chi_2 = c\left(2i\zeta\cos(\zeta\xi) - i\coth\left(\frac{\xi}{2}\right)\sin(\zeta\xi)\right) \tag{E.25b}$$

Finally we can combine with (E.16) and obtain

$$\alpha_1(r) = \frac{c}{\sqrt{rv\sinh(rv)}}\left(2\frac{\mu}{v}\sin(\mu r) + \tanh\left(\frac{rv}{2}\right)\cos(\mu r)\right) \tag{E.26a}$$

$$\alpha_2(r) = \frac{c}{\sqrt{rv\sinh(rv)}}\left(2i\frac{\mu}{v}\cos(\mu r) - i\coth\left(\frac{rv}{2}\right)\sin(\mu r)\right) \tag{E.26b}$$

Appendix F

The index theorem

F.1 The index formula

To see this consider a massive Dirac propagator $\Delta(x, y; m)$ obeying

$$(i\not{D} + im)\Delta(x, y; m) = \mathbb{I}\delta(x - y) \tag{F.1}$$

where $\mathbb{I}$ is the identity in spinor and color indices. We want to consider the divergence of the chiral current defined as

$$j_5^\mu(x) = \lim_{y\to x} \mathrm{Tr}\, i\gamma^\mu\gamma_5\Delta(x, y; m) \ . \tag{F.2}$$

The propagator is UV divergent, so it must be regulated. Taking Pauli-Villars (PV) regularization we can write

$$\Delta_{reg}(x, y; m, M) = \Delta(x, y; m) - \Delta(x, y; M) \tag{F.3}$$

where M is the PV mass which will be sent to infinity. The above expression then defines the regulated chiral current

$$j_5^\mu(x) = \lim_{y\to x} \mathrm{Tr}\, i\gamma^\mu\gamma_5\Delta_{reg}(x, y; m) \ . \tag{F.4}$$

Consider the point-split divergence

$$\begin{aligned}\frac{1}{2}(\partial_\mu^x + \partial_\mu^y)j_5^\mu(x, y) &= -\frac{1}{2}\mathrm{Tr}\,\gamma_5(i\not{\partial}_x + i\not{\partial}_y)\Delta_{reg}(x, y; m, M) = \\ &= -\mathrm{Tr}\,\gamma_5(i\not{\partial}_x\Delta_{reg}(x, y; m, M) + \Delta_{reg}(x, y; m, M)(-i\overleftarrow{\not{\partial}_y}))\end{aligned} \tag{F.5}$$

where we have used the cyclicity of the trace and where

$$j_5^\mu(x,y) = \mathrm{Tr}\, i\gamma^\mu\gamma_5\Delta_{reg}(x,y;m,M)\ . \tag{F.6}$$

Further

$$i\partial\!\!\!/_x\Delta(x,y;m) = \mathbb{I}\delta(x-y) - (A\!\!\!/(x)+im)\Delta(x,y;m)\ . \tag{F.7}$$

Hermitian conjugating[1] the above expression and replacing $x \leftrightarrow y$ as well as $m \to -m$

$$\Delta(x,y;m)(-i\overleftarrow{\partial\!\!\!/}_y) = \mathbb{I}\delta(x-y) - \Delta(x,y;m)(A\!\!\!/(y)+im) \tag{F.8}$$

so that (F.5) becomes

$$\begin{aligned}\frac{1}{2}(\partial_\mu^x + \partial_\mu^y)j_5^\mu(x,y) &= im\mathrm{tr}\,(\gamma_5\Delta(x,y;m)) - iM\mathrm{tr}\,(\gamma_5\Delta(x,y;M)) \\ &\quad + \frac{1}{2}\mathrm{Tr}\,\gamma_5((A\!\!\!/(x) - A\!\!\!/(y))\Delta_{reg}(x,y;m,M))\end{aligned} \tag{F.9}$$

If we take the limit $y \to 0$ the last expression vanishes as $\Delta_{reg}(x,y;m,M)$ is regular in this limit. Upon taking the integral over the divergence of the chiral current we have

$$\begin{aligned}\frac{1}{2}\int d^4x\ \partial_\mu j_5^\mu &= \int d^4x\ \mathrm{tr}\left\langle x\left|\gamma_5\frac{im}{iD\!\!\!/ + im}\right|x\right\rangle - \mathrm{tr}\int d^4x\ \left\langle x\left|\gamma_5\frac{iM}{iD\!\!\!/ + iM}\right|x\right\rangle = \\ &= \mathrm{Tr}\left(\gamma_5\frac{m^2}{-D\!\!\!/^2 + m^2}\right) - \mathrm{Tr}\left(\gamma_5\frac{M^2}{-D\!\!\!/^2 + M^2}\right)\end{aligned} \tag{F.10}$$

We recognize the two expressions as the index function $I(m^2) - I(M^2)$. Putting the fermion mass to zero and the regulator to infinity we get the following expression for the index

$$\begin{aligned}Index &= \lim_{m^2\to 0}\mathrm{Tr}\left(\gamma_5\frac{m^2}{-D\!\!\!/^2 + m^2}\right) = \\ &= \lim_{m^2\to 0}\frac{1}{2}\oint dS_\mu j_5^\mu + \lim_{M^2\to\infty}\mathrm{Tr}\left(\gamma_5\frac{M^2}{-D\!\!\!/^2 + M^2}\right)\end{aligned} \tag{F.11}$$

or, more concisely

$$Index = I_S(0) + I_B \tag{F.12}$$

[1]Note that $\Delta(x,y;m)^\dagger = \Delta(y,x;-m)$

where we have defined

$$I_S(m^2) = \frac{1}{2} \oint dS_\mu j_5^\mu \tag{F.13a}$$

$$I_B = \lim_{M^2 \to \infty} \operatorname{Tr} \left(\gamma_5 \frac{M^2}{-\not{D}^2 + M^2} \right) \tag{F.13b}$$

F.2 The index theorem for 2D

An analog of the four dimensional index theorem exists in two dimensions as well. Let us repeat the arguments for the 2D Dirac operator[2]

$$\not{D} = D_1 \sigma^1 + D_2 \sigma^2 \tag{F.14}$$

where $\sigma^{1,2}$ are the usual Pauli matrices, i.e. $\{\sigma^i, \sigma^j\} = 2\delta_{ij}$, and $D_i = \partial_i - iA_i$ where A_i is the $U(1)$ gauge connection. The square of the Dirac operator is

$$\not{D}^2 = D_i^2 + \sigma^3 B \tag{F.15}$$

where $B = F_{12} = \partial_1 A_2 - \partial_2 A_1$.

Let us now assume that there is a nonzero magnetic flux going through our 2D plane, i.e.

$$\Phi = \int d^2 x B = \oint dx^\mu A_\mu \neq 0 \; . \tag{F.16}$$

Quite clearly the gauge connection A_μ cannot vanish at infinity[3].

We can consider an index function

$$I(m^2) = \operatorname{Tr} \sigma^3 \frac{m^2}{-\not{D}^2 + m^2} \; . \tag{F.17}$$

By the same arguments as before we have that

$$I(m^2) = \frac{1}{2} \oint dx^\mu j_{3,\mu} + \lim_{M^2 \to \infty} \operatorname{Tr} \left(\sigma^3 \frac{M^2}{-\not{D}^2 + M^2} \right) \tag{F.18}$$

where

$$j_3^i = \operatorname{tr} \left(i \sigma^i \sigma^3 \Delta(x, x; m) \right) \; . \tag{F.19}$$

[2]The Dirac operator in 2D is the same as the Dirac-Hamiltonian in (2+1)D. As we will see in the Chapter 3, we will be concerned with the energy eigenstates of the Dirac-Hamiltonian with the same structure.

[3]The gauge field can be gauged away at infinity at the cost of introducing the singularity in the bulk, which would require another boundary. We therefore assume the gauge field is well defined at all points in the bulk.

with

$$\Delta(x,y;m) = \left\langle x \left| \frac{1}{i\not{D} + im} \right| y \right\rangle \tag{F.20}$$

The $M^2 \to \infty$ contribution is related to the axial anomaly, as before, and is easy to evaluate in this limit. Similarly as before we write

$$I_B(M^2) = Tr\left(\sigma^3 \frac{M^2}{-\not{D}^2 + M^2}\right) = \int d^2x \int \frac{d^2k}{(2\pi)^2} \mathrm{tr}\left(\sigma^3 \left(\frac{M^2}{k^2 + M^2 - \sigma^3 B + \dots}\right)\right) \tag{F.21}$$

where again the dots represent terms not contributing under the trace in the large M^2 limit. Expanding in large M^2 we have

$$I_B(M^2) = \int d^2x \int \frac{d^2k}{(2\pi)^2} \frac{M^2}{(k^2+M^2)^2} 2B = \frac{1}{2\pi}\int d^2x\, B = \frac{\Phi}{2\pi} \tag{F.22}$$

Therefore the surface contribution can then be written as

$$\begin{aligned} 2I_S(m^2) &= \oint dx^i \epsilon_{ij} j_3^i = \\ &= \oint dx^i \epsilon_{ij} \int \frac{d^2k}{(2\pi)^2} \mathrm{tr}\left(\sigma^j \sigma^3 \sigma^i (D_i + ik_i) \frac{1}{-(D_i+ik_i)^2 + \frac{i}{2} B\sigma^3 + m^2}\right) . \end{aligned} \tag{F.23}$$

At the boundary we assume $B \to 0$. However, as opposed to the 4D case, the leading contribution is given by the zeroth term in the expansion of B at infinity, i.e.

$$\oint dx^i \epsilon_{ij} j_3^i = 2i \oint dx^i \int \frac{d^2k}{(2\pi)^2} (D_i + ik_i) \frac{1}{-(D_i+ik_i)^2 + m^2} . \tag{F.24}$$

Let us now assume a simple rectangular geometry at infinity to evaluate the above expression. Further let us assume that the space is compactified in the 1-direction and endow the fermions with the twisted boundary conditions, i.e. $\psi(x_1 + L) = e^{i\phi}\psi(x_1)$, so $\int \frac{dk_1}{2\pi} \to \frac{1}{L}\sum_n$ and $k_1 \to (2\pi n + \phi)/L$ where L is the size of the compact direction. The gauge fields we take as strictly periodic, i.e. that $A_i(x_1 = 0) = A_i(x_1 = L)$, so that the contribution to the magnetic flux in eq. (F.16) cancels along the $x_1 = 0$ and $x_1 = L$ directions. So the only contributions of the magnetic flux can come from the $x_2 = \pm\infty$, i.e.

$$\Phi = \int_0^L dx^1 A_1(x_2 = \infty) - \int_0^L dx^1 A_1(x_2 = -\infty) . \tag{F.25}$$

In fact let us choose the gauge such that A_1 is constant along these edges[4] and equal

[4] We can always do this because if this is not true and $A_1^{\pm\infty} = A_1(x_2 = \pm\infty)$ is a function of x_1 then we can do a gauge transformation $A_1^{\pm\infty\prime} = A_1^{\pm\infty} - \partial_1\Lambda$. Demanding that $\partial_1^2\Lambda = \partial_1 A_1^{\pm\infty}$ we obtain precisely that $A_1^{\pm\infty\prime}$ is x_1 independent.

to $A_1^{\pm\infty} = A_1(x_2 = \pm\infty) = \pm\frac{A}{2} = const$. Clearly $AL = \Phi$. Now we go back to the expression (F.24) and write

$$\int dx^i \epsilon_{ij} j_3^j = -2\int \frac{dk_2}{(2\pi)} \sum_n \left(\frac{2\pi n + \phi}{L} - \frac{A}{2}\right) \frac{1}{\left(\frac{2\pi n+\phi}{L} - \frac{A}{2}\right)^2 + k_2^2 + m^2} - (A \to -A) =$$

$$= -\sum_n \frac{\frac{2\pi n+\phi}{L} - \frac{A}{2}}{\sqrt{\left(\frac{2\pi n+\phi}{L} - \frac{A}{2}\right)^2 + m^2}} - (A \to -A) \quad \text{(F.26)}$$

Taking the limit $m^2 \to 0$ we get

$$2I_S(0) = \sum_n \frac{\frac{AL}{4\pi} + \frac{\phi}{2\pi} - n}{\sqrt{\left(\frac{AL}{4\pi} + \frac{\phi}{2\pi} - n\right)^2}} - (A \to -A) = \sum_n (\mathrm{sign}\,(a + \nu - n) - (a \to -a)\ . \quad \text{(F.27)}$$

where $a = AL/(4\pi)$ and $\nu = \phi/(2\pi)$. Standard regularization gives (see Appendix H)

$$2I_S(0, a) = \lim_{s \to 0} \sum_n \frac{\mathrm{sign}(a + \nu - n)}{|a + \nu - n|^s} + (\nu \to -\nu) \quad \text{(F.28)}$$

Let us write $a + \nu = \lfloor a \pm \nu \rfloor + \epsilon_\pm$, so that $\epsilon_\pm$ is the non-integer part. We then have that

$$2I_S(0) = \zeta(0, \epsilon_+) - \zeta(0, 1 - \epsilon_+) + (\epsilon_+ \to \epsilon_-) =$$
$$= 2 - 2\epsilon_+ - 2\epsilon_- = 2 + 2\lfloor a + \nu \rfloor + 2\lfloor a - \nu \rfloor - 4a \quad \text{(F.29)}$$

where $\zeta(x, y)$ is the incomplete zeta function and where we used $\zeta(0, x) = \frac{1}{2} - x$. The total index is

$$I = I_B + I_S(0) = 1 + \left\lfloor \frac{\Phi + 2\phi}{4\pi} \right\rfloor + \left\lfloor \frac{\Phi - 2\phi}{4\pi} \right\rfloor \quad \text{(F.30)}$$

Notice a curious thing that if $\phi = 0$, i.e. strictly periodic fermions, the index is exactly equal $+1$ for arbitrary positive magnetic flux, even when this flux goes to zero. This implies the existence of a zero modes on $\mathbb{R} \times S^1$ for arbitrary fluxes.

Appendix G

Grassmann algebra and relations

Consider an anti-commuting number θ, η, i.e. $\{\theta, \eta\} = 0$. Any function of anti-commuting numbers has a finite Taylor expansion because $\theta^2 = \eta^2 = 0$, so

$$f(\theta) = A + B\theta \ . \tag{G.1}$$

where A, B are ordinary, commuting, numbers.The derivative respect to an anti-commuting number is defined

$$\frac{\partial}{\partial\theta}\theta = 1 \ , \left\{\frac{\partial}{\partial\theta}, \eta\right\} = 0 \text{ for } \eta \neq \theta \tag{G.2}$$

To define the integral $\int d\theta \ f(\theta)$ we require that the integral of the total derivative vanishes, i.e.

$$\int d\theta \ \frac{\partial}{\partial\theta} f(\theta) = \int d\theta B = 0 \tag{G.3}$$

which defines the action of the integral on the commuting number to be zero.

On the other hand the integral over an arbitrary function $f(\theta)$ should be defined so that it is shift invariant, i.e.

$$\int d\theta \ f(\theta) = \int d\theta \ f(\theta + \eta) \tag{G.4}$$

This is a direct generalization of the ordinary bosonic integrals $\int_{-\infty}^{\infty} dx$. From the above requirement it follows that

$$\int d\theta (A + B\theta) = \int d\theta (A + B(\theta + \eta)) \ . \tag{G.5}$$

from the above (requiring linearity in integration) we get that

$$\int d\theta \ \eta = 0 \ . \tag{G.6}$$

So all its left to define is

$$\int d\theta \; \theta = 1 \; , \tag{G.7}$$

by convention. Therefore the integration over Grassmann numbers is the same as differentiation.

$$\int d\theta \equiv \frac{\partial}{\partial\theta} \; . \tag{G.8}$$

Appendix H

Sums

H.1 Poisson resummation

Consider a sum

$$F(s;a,\gamma;c) = \sum_{n=-\infty}^{\infty} \frac{1}{[(n+a+i\gamma)^2+c^2]^s} \tag{H.1}$$

We can write an arbitrary sum over n as follows

$$\sum_n g(n+a+i\gamma) = \oint \frac{dz}{2\pi i}\, g(z)w(z) \tag{H.2}$$

where $w(z)$ is a function which has poles with unit residues at $z = n + a + i\gamma$ and the integral is taken around these poles. Such a function is given by $w(z) = \pi\cot(\pi(z-a-i\gamma))$. By taking $g(z) = \frac{1}{(z^2+c^2)^s}$ (see Fig. H.1) we obtain the following expression for the sum

$$\oint \frac{dz}{2\pi i}\pi\cot\left(\pi(z-a-i\gamma)\right)\frac{1}{(z^2+c^2)^s}\ . \tag{H.3}$$

where the contour is given in Fig. H.1.

Now we can deform the contour to envelope the upper and lower branch-cut, obtain-

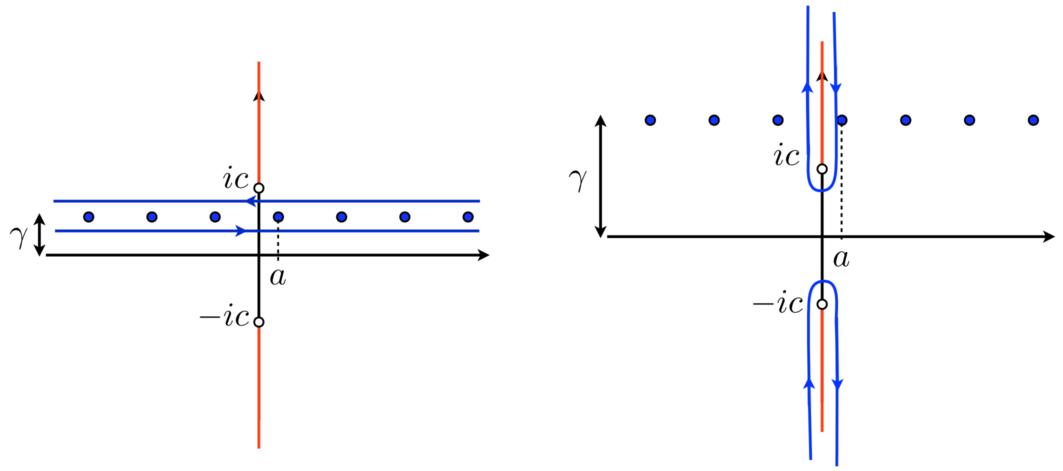

Figure H.1: (left) The pole structure and the contour of integration in (H.2). The blue points represent poles of $w(z) = \pi \cot(\pi(z - a - i\gamma))$, while the red lines represent the branch cuts of $g(z) = \frac{1}{(z^2+c^2)^s}$. (right) Contour of the expression (H.4)

ing for the sum (see right panel of Fig. H.1)

$$\begin{aligned} F(s;a,\gamma;c) = &\int_{-i\infty-\epsilon}^{-ic-\epsilon} dz \frac{1}{2i} \cot(\pi(z-a-i\gamma)) \frac{1}{(z^2+c^2)^s} \\ &+ \int_{-ic+\epsilon}^{-i\infty+\epsilon} dz \frac{1}{2i} \cot(\pi(z-a-i\gamma)) \frac{1}{(z^2+c^2)^s} \\ &+ \int_{i\infty+\epsilon}^{ic+\epsilon} dz \frac{1}{2i} \cot(\pi(z-a-i\gamma)) \frac{1}{(z^2+c^2)^s} \\ &+ \int_{ic-\epsilon}^{i\infty-\epsilon} dz \frac{1}{2i} \cot(\pi(z-a-i\gamma)) \frac{1}{(z^2+c^2)^s} = \\ = i \int_c^\infty dy &\left[\cot(i\pi(y-\gamma) - \pi a) - \cot(\pi(i(-y-\gamma) - \pi a)) \right] \frac{\sin(s\pi)}{(y^2-c^2)^s} \end{aligned} \tag{H.4}$$

where we have taken into account the branch cut discontinuity. Taking the substitution $y = ct$ we can rewrite the above equation as

$$F(s;a,\gamma;c) = 2\text{Re}\, c^{1-2s} \int_1^\infty dt \;\coth(\pi(tc-\gamma) + i\pi a) \frac{\sin(s\pi)}{(t^2-1)^s} \tag{H.5}$$

Take now $\gamma = 0$, and write

$$F(s;a,c) \equiv F(s;a,0;c) = \sum_{n=-\infty}^{\infty} \frac{1}{[(n+a)^2 + c^2]^s} \tag{H.6}$$

We can expand

$$\coth(\pi tc + i\pi a) = 1 + 2\sum_{p=1}^{\infty} e^{-(2\pi tc + 2\pi i a)} \tag{H.7}$$

so that the sum becomes

$$F(s;a,c) = 2\, c^{(1-2s)} \int_1^{\infty} dt\, \left(1 + 2\sum_{p=1}^{\infty} e^{-2\pi tpc} \cos(2\pi a)\right) \frac{1}{(t^2-1)^s} \sin(\pi s) \tag{H.8}$$

Since the modified Bessel functions $K_\nu(x)$ are given by [2]

$$K_\nu(z) = \frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu + 1/2)} \int_1^{\infty} e^{-zt}(t^2-1)^{\nu-1/2} \tag{H.9}$$

and

$$\int_0^1 du\, u^{x-1}(1-u)^{y-1} = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{H.10}$$

where $B(x,y)$ is the beta function, and $\Gamma(x)$ is the gamma function, then

$$\int_1^{\infty} dt\, (t^2-1)^{-s} = \frac{1}{2}\int_0^1 du\, u^{-\frac{1}{2}+s-1}(1-u)^{-s+1-1} = \frac{\Gamma(s-1/2)\Gamma(1-s)}{2\Gamma(1/2)} \tag{H.11}$$

and the sum becomes

$$F(s;a,c) = \frac{c^{1-2s}\sqrt{\pi}}{\Gamma(s)} \left(\Gamma(s-\frac{1}{2}) + 4\sum_{p=1}^{\infty} (p\pi c)^{s-1/2}\cos(2\pi pa) K_{\frac{1}{2}-s}(2\pi pc)\right) \tag{H.12}$$

where we used the Euler reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{H.13}$$

The sum (H.12) is the same as that obtained in ref. [85], up to using the fact that $K_\nu(x) = K_{-\nu}(x)$.

H.2 Hurwitz zeta sum

Consider the sum

$$S(s;a) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{sign}(n+a)}{|n+a|^s} \tag{H.14}$$

where a is non integer. The sum is obviously periodic in a with unity period. Defining $\hat{a} = a - \lfloor a \rfloor$ we can write, so that $\hat{a} \in [0,1)$, we have that

$$S(s;a) = \sum_{n=0}^{\infty} \frac{1}{(n+\hat{a})^s} - \sum_{n=1}^{\infty} \frac{1}{(n-\hat{a})^s} = \zeta(s,\hat{a}) - \zeta(s,1-\hat{a}) \tag{H.15}$$

where $\zeta(s,x)$ is the Hurwitz zeta function.

The sum

$$S_1(s;a) = \sum_n \frac{\mathrm{sign}(n+a)}{|n+a|^s} \ln|n+a| = -\partial_s S(s;a) = -\partial_s\big[\zeta(s,\hat{a}) - \zeta(s,1-\hat{a})\big] \tag{H.16}$$

In particular we have that for $s=0$

$$S(0;a) = 1 - 2\hat{a}\ , \qquad\qquad S_1(0,a) = -\ln\frac{\Gamma(\hat{a})}{\Gamma(1-\hat{a})} \tag{H.17}$$

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