Local Cohomology Sheaves on Algebraic Stacks

Dissertation zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der Fakultät für Mathematik der
Universität Regensburg

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im

April 2014

Promotionsgesuch wurde eingereicht am: 08.04.2014

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Introduction

The concept of *local cohomology* was introduced by Grothendieck at the occasion of a seminar at Harvard university in the beginning of the 1960s. The first published account on the subject was a section in Hartshorne's "Residues and Duality" ([Har66]), followed by the publication of his lecture notes "Local cohomology" of Grothendieck's seminar ([Har67]). There is also a treatment of this topic in [SGAII]. It is worth mentioning that the ideas for local cohomology were already present in Serre's "Faisceaux algébriques cohérents" ([Ser55]).

In advance to some results, we recall the definition: Let X be a topological space, $Z \subset X$ a closed subset and $\mathscr{F}$ a sheaf of abelian groups on X. Define $\Gamma_Z(X;\mathscr{F})$ to be the group consisting of global sections of $\mathscr{F}$ having support in Z. Furthermore, define the *local cohomology groups* $\mathrm{H}^\bullet_Z(X;\mathscr{F})$ to be the right derived functors of $\Gamma_Z(X;-)$ in the category of abelian sheaves on X. If we start with a ringed space and a module sheaf $\mathscr{F}$, then we obtain local cohomology modules over its ring of global sections.

There are (at least) two aspects why one should be interested in studying local cohomology groups: There is a long exact sequence relating the cohomology $\mathrm{H}^\bullet(X;-)$ of X, the local cohomology $\mathrm{H}^\bullet_Z(X;-)$ and the cohomology $\mathrm{H}^\bullet(U;-)$ of the open complement $U = X \setminus Z$. There is also a sheaf version $\underline{\mathrm{H}}^\bullet_Z(-)$ and a spectral sequence linking the local to the global version.

Moreover, local cohomology allows a translation of global results on a projective space to purely algebraic results on the corresponding polynomial ring (and hence to its local ring at the origin).

Let us state two typical results for a noetherian local ring $(A,\mathfrak{m})$ and an A-module M:

- The depth of M (i.e. the maximal length of a regular sequence) is equal to the least integer k such that $\mathrm{H}^k_{V(\mathfrak{m})}\big(\mathrm{Spec}(A),\widetilde{M}\big)$ does not vanish.
- The local cohomology modules $\mathrm{H}^k_{V(\mathfrak{m})}\big(\mathrm{Spec}(A),\widetilde{M}\big)$ are dual to $\mathrm{Ext}^{n-k}_A(M,\omega)$, where $n = \dim(A)$ is the dimension of A and ω is a dualizing module. This is a local version of Serre duality for a projective variety.

There are several (newer) textbooks on local cohomology. Yet, most books restrict to the case of affine schemes $\mathrm{Spec}(A)$ and – even worse – often assume that the ring A is noetherian. We prefer living in a non-noetherian world since this is the place where we meet our motivating example, the stack of formal groups. Hence, we stick to the classic and established literature mentioned above and the paper [AJL97] by Alonso, Jeremías and Lipman.

Instead of working on schemes, we deal with algebraic stacks. Stacks can be seen as generalizations of schemes, in a similar vein to schemes generalizing the concept of a projective variety. They enjoy great popularity since the mid-sixties because they are, among other applications, used to solve moduli problems – the first paper on stacks was Mumford's "Picard groups of moduli problems" ([Mum65]). He never uses the term "stack" but the concept is implicit in the paper. His joint work with Deligne [DM69] was revolutionary and they used the language of stacks to solve a long standing open problem. At least since the publication of Artin's [Art74] there cannot be any doubt about the beauty and significance of stacks in modern mathematics.

Since one encounters various flavours of stacks, we should clarify in which kind of objects we are interested. We restrict our attention to algebraic stacks in the sense of Goerss, Naumann,...: a stack is algebraic if it is quasi-compact and has affine diagonal. As a warning, note that these stacks are *not* the same as algebraic stacks in the sense of the book [LMB00] of Laumon and Moret-Bailly. Vistoli's well-written introduction [Vis05] gives basic notions concerning stacks we are going to need in the following.

The stack we always have in mind is the (p-local) stack of formal groups $\mathcal{M}_{\mathrm{FG}}$ – it is an algebraic stack and of vital importance for homotopy theorists. It is represented by the flat Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ for the Brown-Peterson spectrum BP (for some fixed prime p). The BP_*-version of the Adams-Novikov spectral sequence converges to the p-local stable homotopy groups $\pi_*(\mathbb{S}) \otimes \mathbb{Z}_{(p)}$ of the sphere $\mathbb{S}$ and has E^2-page $\mathrm{Ext}^{\bullet;\bullet}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$, where the Ext is understood in the category of $\mathrm{BP}_*\mathrm{BP}$-comodules. Since there is essentially no difference between knowing $\pi_*(\mathbb{S}) \otimes \mathbb{Z}_{(p)}$ for all primes p (together with $\pi_*(\mathbb{S}) \otimes \mathbb{Q} = \mathbb{Q}[0]$ and finite generation in each degree) and knowing $\pi_*(\mathbb{S})$, this should be motivation enough to understand the stack $\mathcal{M}_{\mathrm{FG}}$ and its geometry.

So how does local cohomology enter the stage in this setup? The stack of formal groups has a height filtration

$$\mathcal{M}_{\mathrm{FG}} = \mathfrak{Z}^0 \supsetneq \mathfrak{Z}^1 \supsetneq \mathfrak{Z}^2 \supsetneq \cdots$$

given by closed, reduced substacks such that the closed immersion $\mathfrak{Z}^n \hookrightarrow \mathcal{M}_{\mathrm{FG}}$ is regular for every n. The closed substack $\mathfrak{Z}^n$ corresponds to the Hopf algebroid $(\mathrm{BP}_*/I_n, \mathrm{BP}_*\mathrm{BP}/I_n\mathrm{BP}_*\mathrm{BP})$ where I_n is an invariant regular prime ideal of BP_*.

Hovey has put tremendous effort in understanding the theory of comodules, for instance in his paper [Hov04]. Together with Strickland he developed in [HS05a] and [HS05b] the theory of local cohomology on the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ and we generalize their ideas. Franke's unpublished paper [Fra96] is a treasure of brilliant ideas and great inspiration. We should not forget to also mention Goerss' treatment [Goe].

Let us briefly present some results of this thesis:

For an ordinary scheme X, we can interpret local cohomology sheaves as a colimit of certain Ext sheaves.

Theorem A (2.3.6) *Let X be a scheme, $Z \subset X$ a closed subscheme such that the inclusion of the open complement $U \hookrightarrow X$ is quasi-compact. If $\mathscr{F}$ is a quasi-coherent sheaf, then we have a homomorphism of module sheaves*

$$\mathrm{colim}_n\, \underline{\mathcal{E}\mathrm{xt}}^k_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \longrightarrow \underline{\mathrm{H}}^k_Z(X; \mathscr{F})\,,$$

where $\mathscr{I} \subset \mathscr{O}_X$ denotes the quasi-coherent ideal sheaf corresponding to Z.

This morphism is an isomorphism if

- *(i) $Z \hookrightarrow X$ is a regular closed immersion, or*
- *(ii) X is locally coherent and $Z \hookrightarrow X$ is a weakly proregular closed immersion, or*
- *(iii) X is locally noetherian.*

Here (iii) is a special case of (ii) since any (locally) noetherian scheme is (locally) coherent and any closed immersion in a locally noetherian scheme is weakly proregular. Coherent (resp. locally coherent) schemes are natural generalizations of noetherian (resp. locally noetherian) schemes and weakly proregularity seems to be a rather mild assumption one can put on a closed immersion.

If $\mathfrak{X}$ is an algebraic stack with presentation $P\colon \mathrm{Spec}(A) \to \mathfrak{X}$, then we have the following diagram (of horizontal adjoint functors):

$$\begin{tikzcd}
\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}) \arrow[r, shift right, "Q_{\mathfrak{X}}"'] \arrow[r, phantom, "\perp"] & \mathbf{QCoh}(\mathfrak{X}) \arrow[l, shift right, "\iota_{\mathfrak{X}}"'] \arrow[r, shift left, "P^*"] \arrow[r, phantom, "\perp"] \arrow[d, leftrightarrow, "\simeq"] & \mathbf{QCoh}\big(\mathrm{Spec}(A)\big) \arrow[l, shift left, "P_*"] \arrow[r, shift left, "\iota_A"] \arrow[r, phantom, "\perp"] \arrow[d, leftrightarrow, "\simeq"] & \mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(A)}) \arrow[l, shift left, "Q_A"] \\
& \Gamma{-}\mathbf{Comod} \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & A{-}\mathbf{Mod} \arrow[l, shift left, "\Gamma\otimes-"] &
\end{tikzcd} \tag{$\clubsuit$}$$

Here $\iota_\bullet$ denotes the inclusion of the appropriate full subcategory. Note that modules on the very left are defined in the flat topology whereas we use the Zariski topology on the right hand side. We discuss these functors and their properties, in particular regarding preservation of injective objects.

As indicated in (♣) we will see that the category $\mathbf{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on an algebraic stack $\mathfrak{X}$ is a coreflective subcategory of the category $\mathbf{Mod}(\mathscr{O}_\mathfrak{X})$ of all module sheaves (3.5.2), i.e. the inclusion $\iota_\mathfrak{X}\colon \mathbf{QCoh}(\mathfrak{X}) \hookrightarrow \mathbf{Mod}(\mathscr{O}_\mathfrak{X})$ admits a right adjoint $Q_\mathfrak{X}$. In general, the inclusion $\iota_\mathfrak{X}$ does not preserve injective objects. A priori, this may have the unpleasant feature that the derived functors of the global section functor

$$\Gamma(\mathfrak{X};-)\colon \mathbf{QCoh}(\mathfrak{X}) \longrightarrow \Gamma(\mathfrak{X};\mathscr{O}_\mathfrak{X})\mathbf{-Mod}$$

might have different values when calculated in $\mathbf{QCoh}(\mathfrak{X})$ resp. $\mathbf{Mod}(\mathscr{O}_\mathfrak{X})$. Fortunately, this is not the case:

Theorem B (3.7.1) *If $\mathscr{F}$ is a quasi-coherent sheaf on an algebraic stack $\mathfrak{X}$, then the canonical morphism*

$$\mathbb{R}^\bullet_{\mathbf{QCoh}(\mathfrak{X})}\Gamma(\mathfrak{X};\mathscr{F}) \longrightarrow \mathbb{R}^\bullet_{\mathbf{Mod}(\mathscr{O}_\mathfrak{X})}\Gamma(\mathfrak{X};\mathscr{F})$$

of δ-functors is an isomorphism.

Note that the right hand side is sheaf cohomology of a quasi-coherent sheaf $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$.

We can compare local cohomology sheaves defined on an algebraic stack $\mathfrak{X}$ and "classical" local cohomoloy on the presentation $\operatorname{Spec}(A)$ via

Theorem C (4.4.2) *Let $\mathfrak{X}$ be an algebraic stack and $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ a weakly proregularly embedded closed substack. Then there is a natural equivalence*

$$\underline{\mathrm{H}^\bullet_\mathfrak{Z}}(-) := \mathbb{R}^\bullet_{\mathbf{QCoh}(\mathfrak{X})}\underline{\Gamma_\mathfrak{Z}}(\mathscr{F}) \longrightarrow \mathbb{R}^\bullet_{\mathbf{Mod}(\mathscr{O}_{\operatorname{Spec}(A)})}\underline{\Gamma_Z}(P^*-)\,.$$

of δ-functors on $\mathbf{QCoh}(\mathfrak{X})$ (under one additional technical assumption). Here Z denotes the closed subscheme $Z := \mathfrak{Z} \times_\mathfrak{X} \operatorname{Spec}(A) \subset \operatorname{Spec}(A)$.

Content

We start by introducing some possibly little known concepts from category theory that will be used in this thesis. The category of quasi-coherent sheaves on the (p-local) stack of formal groups is a locally coherent Grothendieck category and therefore locally finitely presentable. Moreover it has the structure of a closed symmetric monoidal category and the dualizable objects form a generating set. Key of this thesis is to elaborate the ideas of localizing subcategories developed in section 1.2.

In section 2 we explain how Grothendieck's definition of local cohomology on schemes fits in the framework of localization. We briefly focus on the case of affine schemes and come across *weakly proregular systems* and *ideals* which are of crucial importance when dealing with non-noetherian rings. If the closed subscheme is defined by a weakly proregular ideal I, then we can use the Čech complex associated to the ideal I to compute the local cohomology groups. Furthermore, we generalize a result of Grothendieck, giving a description of local cohomology sheaves in terms of a colimit of certain Ext-sheafs.

We then give a short introduction to algebraic stacks and flat Hopf algebroids and state their correlation. Sections 3.4.2 and 3.5 illuminate the structure of the category of comodules on a flat Hopf algebroid. The Adams condition on Hopf algebroids is useful at several points, as well as being noetherian resp. coherent. We end this section with a discussion of sheaf cohomology on algebraic stacks.

Section 4 gives the definition of local cohomology sheaves via a localizing pair. We show that under certain assumptions local cohomology commutes with filtered colimits, a very useful observation. We then compare our version of local cohomology for stacks with the classic one and see that they are closely related.

Finally, we give a possible application of the theory developed so far – we consider the stack of formal groups $\mathcal{M}_{\mathrm{FG}}$ together with its height filtration. A proof of chromatic convergence for coherent sheaves on $\mathcal{M}_{\mathrm{FG}}$ using the techniques of section 4.4 completes this thesis.

Acknowledgements

First of all and most importantly, I would like to thank my advisor Niko Naumann for introducing me to the world of stacks and bringing my attention to the concept of local cohomology. His ideas helped shape the outline of this project.

I am deeply grateful to Leo Alonso and Ana Jeremías for helpful e-mail exchanges and their hospitality during my visit in Santiago de Compostela. I have benefited a lot from working with both of them.

It is a pleasure to thank Martin Ruderer for hours of mathematical and non-mathematical discussions and pleasant company. I owe special thanks to Florian Strunk and Georg Tamme for their (sometimes involuntary) interest in my work and useful ideas and suggestions. Moreover, I would like to express my gratitude to Peter Arndt, Martin Brandenburg, Michael Völkl and Marcel Wid for helpful conversations.

This project was supported by the GRK 1692 "Curvature, Cycles, and Cohomology".

Conventions and notation

Throughout we are working within a fixed universe $\boldsymbol{\mathcal{U}}$ containing an infinite set. Morphisms in a category $\boldsymbol{\mathcal{A}}$ are either denoted $\boldsymbol{\mathcal{A}}(-,-)$ or $\mathrm{Hom}_{\boldsymbol{\mathcal{A}}}(-,-)$. All categories $\boldsymbol{\mathcal{C}}$ are assumed to be $\boldsymbol{\mathcal{U}}$-categories in the sense that for each pair of objects $A, B \in \boldsymbol{\mathcal{C}}$ the set $\boldsymbol{\mathcal{C}}(A,B)$ is small, i.e. in bijection with a set in $\boldsymbol{\mathcal{U}}$. A category $\boldsymbol{\mathcal{C}}$ is essentially small provided that the isomorphism classes of objects in $\boldsymbol{\mathcal{C}}$ form a small set. The category of sets resp. abelian groups is denoted by **Set** resp. **Ab**.

The inclusion of a (full) subcategory is denoted by $\hookrightarrow$. A monomorphism (resp. epimorphism) in a category $\boldsymbol{\mathcal{C}}$ is denoted by $\rightarrowtail$ (resp. $\twoheadrightarrow$); subobjects are denoted by $\leq$. Functors between abelian categories are always assumed to be additive. If $\boldsymbol{\mathcal{C}}$ is a category and $X \in \boldsymbol{\mathcal{C}}$, then $\boldsymbol{\mathcal{C}}_{/X}$ denotes the slice category of $\boldsymbol{\mathcal{C}}$ over X. Its objects are all morphisms $Y \to X$ with target X in $\boldsymbol{\mathcal{C}}$ and its morphisms are given by commutative triangles over X.

If $\boldsymbol{\mathcal{D}}$ is a small category and $D\colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{C}}$ is a $\boldsymbol{\mathcal{D}}$-shaped diagram with values in $\boldsymbol{\mathcal{C}}$, we usually write D_d for the object $D(d) \in \boldsymbol{\mathcal{C}}$, $d \in \boldsymbol{\mathcal{D}}$. We use the categorical terms colimits and limits and write $\mathrm{colim}_{d\in\boldsymbol{\mathcal{D}}}$ (instead of $\varinjlim$) resp. $\lim_{d\in\boldsymbol{\mathcal{D}}}$ (instead of $\varprojlim$) and assume that every diagram is small. A filtered (resp. finite) colimit is a colimit over a filtered (resp. finite) diagram; a category is filtered if every finite diagram has a cocone. A functor between categories is said to be continuous (resp. cocontinuous) if it commutes with all (existing) limits (resp. colimits). A category $\boldsymbol{\mathcal{C}}$ is complete (resp. cocomplete) if it has all small limits (resp. colimits).

To stress the commutativity of certain diagrams we use $\circlearrowleft$; pullback squares are indicated by writing $\lrcorner$.

When we have a pair of functors $\big(F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}, G\colon \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}\big)$ such that F is left adjoint to G, we indicate this by writing $F \dashv G$ or

$$F\colon \boldsymbol{\mathcal{A}} \rightleftarrows \boldsymbol{\mathcal{B}} : G \qquad \text{or} \qquad \boldsymbol{\mathcal{A}} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{\mathcal{B}}$$

(the left adjoint is always the upper arrow). If $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ is a left (resp. right) exact functor between abelian categories such that $\boldsymbol{\mathcal{A}}$ has enough F-acyclic objects, then we denote $\mathbb{R}^{\bullet}F$ (resp. $\mathbb{L}^{\bullet}F$) the right (resp. left) derived functor. The unbounded derived category of an abelian category $\boldsymbol{\mathcal{A}}$ is denoted $\mathbf{D}(\boldsymbol{\mathcal{A}})$. If total right (resp. left) derived functors exist, then they are denoted $\mathbf{R}F$ (resp. $\mathbf{L}F$).

Every ring R is commutative and unital, ideals I in a ring R are denoted $I \lhd R$. The category of (left) modules over a ring R is denoted by $R{-}\mathbf{Mod}$.

If $(\boldsymbol{\mathcal{C}}, \mathscr{O})$ is a ringed site, we denote by $\boldsymbol{\mathcal{S}h}(\boldsymbol{\mathcal{C}})$ the category of abelian sheaves on X and by $\mathbf{Mod}(\mathscr{O})$ the category of $\mathscr{O}$-module sheaves.

References to *The Stacks Project* are uniquely defined by their tags, one uses the 4-symbol code at `http://stacks.math.columbia.edu/tag`.

1 Categorical preliminaries

The statements are not formulated in full generality.

1.1 Locally ... categories

We give the definition of locally noetherian (resp. finitely generated resp. finitely presentable resp. coherent) categories and study their relations and properties.

Definition 1.1.1 (Grothendieck category) A *Grothendieck category* is an (AB5)-category having a generator, i.e. a cocomplete abelian category with exact filtered colimits that admits a generating object.

Reminder (generator) An object G in an abelian category $\boldsymbol{\mathcal{C}}$ is a *generator* (sometimes called *separator*) if the covariant hom functor $\boldsymbol{\mathcal{C}}(G,-)\colon \boldsymbol{\mathcal{C}} \to \mathbf{Set}$ is faithful. If $\boldsymbol{\mathcal{C}}$ is cocomplete, then an object G is a generator if and only if for every $X \in \boldsymbol{\mathcal{C}}$ there exists a set I and an epimorphism $\coprod_I G \twoheadrightarrow X$ ([KS06, Proposition 5.2.4]). A set of objects $\{G_i\}_{i\in I}$ is said to be a *generating set* if $\coprod_{i\in I} G_i$ is a generator of $\boldsymbol{\mathcal{C}}$.

Example 1.1.2 (i) If $\boldsymbol{\mathcal{C}}$ is an essentially small preadditive category, then the presheaf category $\operatorname{Fun}(\boldsymbol{\mathcal{C}}^{\mathrm{op}}, \mathbf{Ab})$ is Grothendieck. The set of all $\boldsymbol{\mathcal{C}}(-,C)$, $C \in \boldsymbol{\mathcal{C}}$, is generating.

(ii) The category $R-\mathbf{Mod}$ is Grothendieck for any ring R. R itself can be taken as a generator.

(iii) If $(\boldsymbol{\mathcal{C}}, \mathscr{O})$ is a ringed site, the categories $\boldsymbol{\mathcal{Sh}}(\boldsymbol{\mathcal{C}})$ of abelian sheaves on $\boldsymbol{\mathcal{C}}$ and $\mathscr{O}-\mathbf{Mod}$ of $\mathscr{O}$-modules are Grothendieck categories (cf. [KS06, Theorem 18.1.6]).

(iv) If X is a scheme, then the category of quasi-coherent sheaves $\mathbf{QCoh}(X)$ on X is Grothendieck ([SPA, Proposition 077P]).

Remark If $\boldsymbol{\mathcal{A}}$ is a Grothendieck category, then it is already (AB3$*$), i.e. complete (cf. [SPA, Lemma 07D8]). This can also be deduced from the Gabriel-Popescu-Theorem (cf. [KS06, Theorem 8.5.8]).

Moreover, a Grothendieck category $\boldsymbol{\mathcal{A}}$ has functorial injective embeddings (cf. [SPA, Theorem 079H]) and we can check easily whether a functor $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ between Grothendieck categories admits an adjoint: F has a right (resp. left) adjoint if and only if it is cocontinuous (resp. continuous), cf. [KS06, Proposition 8.3.27].

Definition 1.1.3 Let $\boldsymbol{\mathcal{A}}$ be a Grothendieck category.

(i) An object $A \in \boldsymbol{\mathcal{A}}$ is *noetherian* if each ascending chain of subobjects of A is stationary. Let us write $\boldsymbol{\mathcal{A}}_{\mathrm{noe}}$ for the full subcategory of $\boldsymbol{\mathcal{A}}$ of noetherian objects.

(ii) An object $A \in \boldsymbol{\mathcal{A}}$ is *finitely presentable* if the corepresentable functor

$$\boldsymbol{\mathcal{A}}(A,-)\colon \boldsymbol{\mathcal{A}} \to \mathbf{Set}$$

preserves filtered colimits, i.e. for every filtered category $\boldsymbol{\mathcal{D}}$ and every functor $D\colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{A}}$, the canonical morphism

$$\operatorname{colim}_{d\in\boldsymbol{\mathcal{D}}} \boldsymbol{\mathcal{A}}\big(A, D_d\big) \longrightarrow \boldsymbol{\mathcal{A}}\big(A, \operatorname{colim}_{d\in\boldsymbol{\mathcal{D}}} D_d\big)$$

is an isomorphism. We write $\boldsymbol{\mathcal{A}}_{\mathrm{fp}}$ for the full subcategory of $\boldsymbol{\mathcal{A}}$ consisting of finitely presentable objects.

(iii) An object $A \in \boldsymbol{\mathcal{A}}$ is *finitely generated* if $\boldsymbol{\mathcal{A}}(A,-)$ preserves filtered colimits of monomorphisms. We write $\boldsymbol{\mathcal{A}}_{\mathrm{fg}}$ for the full subcategory spanned by finitely generated objects.

(iv) An object $A \in \boldsymbol{\mathcal{A}}$ is *coherent* if $A \in \boldsymbol{\mathcal{A}}_{\text{fp}}$ and every finitely generated subobject of X is also finitely presented. Denote $\boldsymbol{\mathcal{A}}_{\text{coh}}$ the full subcategory of coherent objects.

Example 1.1.4 (i) A ring R is noetherian if and only if R is a noetherian object in $R-\mathbf{Mod}$.

(ii) An R-Module M is finitely generated if and only if it is finitely generated in the sense of commutative algebra, i.e. if there is an epimorphism from a finite free R-module.

(iii) An R-module M is finitely presentable if and only if it is finitely presented in the sense of commutative algebra, i.e. if it can be written as the cokernel of a morphism of finite free R-modules.
On a scheme X, a locally finitely presented quasi-coherent sheaf is a finitely presentable object in $\mathbf{QCoh}(X)$. If X is *concentrated* (quasi-compact and quasi-separated), then the converse is also true ([Mur06, Proposition 75]). The same holds for finitely generated objects and locally finitely generated quasi-coherent sheaves.

(iv) A ring R is coherent if and only if it is coherent in the sense of commutative algebra, i.e. if every finitely generated ideal $I \lhd R$ is finitely presented.
Note that a coherent sheaf $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$ on a scheme X is not the same as a coherent object in $\mathbf{Mod}(\mathscr{O}_X)$.

Remark (i) An object $A \in \boldsymbol{\mathcal{A}}$ is noetherian if and only if every subobject of A is finitely generated (cf. [JVV95, Proposition 1.33]). In particular, subobjects of noetherian objects are noetherian. The category $\boldsymbol{\mathcal{A}}_{\text{noe}}$ is abelian and the inclusion $\boldsymbol{\mathcal{A}}_{\text{noe}} \hookrightarrow \boldsymbol{\mathcal{A}}$ is exact, i.e. $\boldsymbol{\mathcal{A}}_{\text{noe}}$ is an *exact subcategory* of $\boldsymbol{\mathcal{A}}$.

(ii) An object $A \in \boldsymbol{\mathcal{A}}$ is finitely presentable if and only if every morphism $A \to \operatorname{colim}_{d \in \boldsymbol{\mathcal{D}}} F_d$ with $\boldsymbol{\mathcal{D}}$ filtered factors through some F_d. Similarly for finitely generated.

(iii) The categories $\boldsymbol{\mathcal{A}}_{\text{fp}}$ and $\boldsymbol{\mathcal{A}}_{\text{fg}}$ are usually not abelian. Indeed, the kernel of a morphism between finitely presentable (resp. finitely generated) objects is not necessarily finitely presentable (resp. finitely generated).

(iv) The category $\boldsymbol{\mathcal{A}}_{\text{coh}}$ is an exact subcategory of $\boldsymbol{\mathcal{A}}$ ([Her97, Proposition 1.5]).

(v) The subcategories $\boldsymbol{\mathcal{A}}_{\text{noe}}$, $\boldsymbol{\mathcal{A}}_{\text{fg}}$, $\boldsymbol{\mathcal{A}}_{\text{fp}}$ and $\boldsymbol{\mathcal{A}}_{\text{coh}}$ are closed under finite colimits.

(vi) We have the following implications for an object $A \in \boldsymbol{\mathcal{A}}$:

$$\begin{tikzcd}
 & & \text{noetherian} \arrow[d, Rightarrow] \\
\text{coherent} \arrow[r, Rightarrow] & \text{finitely presentable} \arrow[r, Rightarrow] & \text{finitely generated}
\end{tikzcd}$$

Definition 1.1.5 (locally ... category) Let $\boldsymbol{\mathcal{A}}$ be a Grothendieck category and let P be one of the properties of Definition 1.1.3. We call $\boldsymbol{\mathcal{A}}$ *locally* P if it admits a generating set of P-objects.

Example 1.1.6 (i) If $\boldsymbol{\mathcal{C}}$ is an essentially small preadditive category, then the presheaf category $\operatorname{Fun}(\boldsymbol{\mathcal{C}}^{\text{op}}, \mathbf{Ab})$ is locally finitely generated ([Her97, 1.2]); every functor of the form $\boldsymbol{\mathcal{C}}(-, C)$ is finitely generated. Furthermore, [Her97, Proposition 1.3] shows that this category is even locally finitely presentable. If $\boldsymbol{\mathcal{C}}$ is moreover abelian, then $\operatorname{Fun}(\boldsymbol{\mathcal{C}}^{\text{op}}, \mathbf{Ab})$ is locally coherent ([Her97, Proposition 2.1]).

(ii) $R-\mathbf{Mod}$ is locally noetherian if and only if R is noetherian. This follows from [Rot09, Theorem 3.39 (Bass-Papp)] and [Roo69, p. 198].
If X is a locally noetherian scheme, then the category $\mathbf{Mod}(\mathscr{O}_X)$ is locally noetherian ([Har66, Theorem II.7.8]).
If X is a noetherian scheme, then $\mathbf{QCoh}(X)$ is locally noetherian ([Gab62, Théorème VI.1]). For a noetherian scheme X, $\mathbf{QCoh}(X)_{\text{noe}}$ is the subcategory of coherent sheaves

on X. If X is only locally noetherian, $\mathbf{QCoh}(X)$ may not be locally noetherian ([Har66, p. 135]).

(iii) $R-\mathbf{Mod}$ is locally finitely presentable (and hence locally finitely generated) for any ring R since the generator R of $R-\mathbf{Mod}$ is finitely presentable.
If X is a concentrated scheme, then $\mathbf{QCoh}(X)$ is locally finitely presentable ([EGAI$_{\text{new}}$, Corollaire 6.9.12]).

(iv) $R-\mathbf{Mod}$ is locally coherent if and only if R is a coherent ring. ([Gla89, Theorem 2.3.2]).
If X is a coherent scheme, then $\mathbf{QCoh}(X)$ is locally coherent ([Gar10, Proposition 40]). The converse holds if X is concentrated. We give the definition of a coherent scheme in Definition 2.3.4.

Remark (i) Not every Grothendieck category is locally finitely generated. A nice example is given in [PR10, Example 4.9]: The ringed space $\mathbf{Mod}(\mathscr{O}_X)$ with $X := [0,1] \subset \mathbb{R}$ and $\mathscr{O}_X$ the sheaf of $\mathbb{R}$-valued continuous functions on $[0,1]$ is not locally finitely generated.

(ii) In a locally noetherian category $\boldsymbol{\mathcal{A}}$ every finitely generated object is noetherian.

(iii) If $\boldsymbol{\mathcal{A}}$ is locally coherent, then every finitely presentable object of $\boldsymbol{\mathcal{A}}$ is coherent, [Her97, Theorem 1.6].

(iv) A locally finitely presentable category $\boldsymbol{\mathcal{A}}$ is locally coherent if and only if $\boldsymbol{\mathcal{A}}_{\text{fp}} \subset \boldsymbol{\mathcal{A}}$ is an exact subcategory ([Roo69, Proposition 2.2]).

Proposition 1.1.7 *Let $\boldsymbol{\mathcal{A}}$ be a Grothendieck category and P one of the properties of Definition 1.1.3. Then $\boldsymbol{\mathcal{A}}$ is locally P if any only if every object in $\boldsymbol{\mathcal{A}}$ is a filtered colimit of objects with property P.*

If P stands for finitely presentable (resp. noetherian), then every object can even be written as a filtered union over its finitely generated (resp. noetherian) subobjects.

Proof. We extend the proof of [Bre70, Satz 1.5] for finitely generated and finitely presentable to the other two cases.

Let us first show that the conditions are sufficient:

Let G be a generator of $\boldsymbol{\mathcal{A}}$. If $G = \operatorname{colim}_{d\in\boldsymbol{\mathcal{D}}} D_d$ for a filtered category $\boldsymbol{\mathcal{D}}$ such that $D_d \in \boldsymbol{\mathcal{A}}_P$ ($P =$ fg, fp, noe, coh), then $\{D_d \,|\, d \in \boldsymbol{\mathcal{D}}\}$ is a generating set of $\boldsymbol{\mathcal{A}}$.

The conditions are also necessary:

Assume that $\boldsymbol{\mathcal{A}}$ is locally P. Since the category $\boldsymbol{\mathcal{A}}_P$ is closed under finite colimits, there exists an essentially small subcategory $\boldsymbol{\mathcal{B}} \subset \boldsymbol{\mathcal{A}}_P$ such that $\boldsymbol{\mathcal{B}}$ is closed under finite colimits and the objects of $\boldsymbol{\mathcal{B}}$ form a generating set. Let $A \in \boldsymbol{\mathcal{A}}$ be an arbitrary object. Consider the slice category $\boldsymbol{\mathcal{B}}_{/A}$ together with the functor

$$\boldsymbol{\mathcal{B}}_{/A} \to \boldsymbol{\mathcal{B}} \qquad (b\colon B \to A) \mapsto s(b) = B\,.$$

We obtain a morphism $p\colon \operatorname{colim}_{b\in\boldsymbol{\mathcal{B}}_{/A}} s(b) \to A$ and want to prove that this morphism is an isomorphism.

`Epimorphism:` Consider

$$\operatorname*{colim}_{b\in\boldsymbol{\mathcal{B}}_{/A}} s(b) \xrightarrow{\ p\ } A \xrightarrow{\text{can}} \operatorname{coker}(p) \longrightarrow 0\,.$$

Then $\text{can} \circ b = 0$ for every $b \in \boldsymbol{\mathcal{B}}_{/A}$. Since the objects of $\boldsymbol{\mathcal{B}}$ are supposed to be a generating set, we see that $p = 0$, i.e. $p\colon \operatorname{colim}_b s(b) \to A$ is an epimorphism.

`Monomorphism:` Let first P be either finitely generated or noetherian. Then the image of a finitely generated (resp. noetherian) object is again finitely generated (resp. noetherian). If

$A \in \mathcal{A}$, then the full subcategory $\mathcal{B}^{\rightarrow}_{P_{/A}}$ of $\mathcal{B}_{P_{/A}}$ where we only consider monomorphisms $B \rightarrowtail A$ over B is cofinal in $\mathcal{B}_{P_{/A}}$. In particular, the morphism

$$\operatorname*{colim}_{b \in \mathcal{B}_{P_{/A}}} = \operatorname*{colim}_{b \in \mathcal{B}^{\rightarrow}_{P_{/A}}} s(b) \xrightarrow{p} A$$

is a monomorphism.

We cannot argue the same way for finitely presentable resp. coherent. Let $G \in \mathcal{A}_P$ be an element of the generating set of $\mathcal{A}$ and $g\colon G \to \operatorname{colim}_{b \in \mathcal{B}_{/A}} s(b)$ an arbitrary morphism such that $p \circ g = 0$. If we can show $g = 0$, then $\ker(p) = 0$ by the definition of a generator. Since $\mathcal{B}_{/A}$ is filtered and $G \in \mathcal{A}_P$ is finitely presentable, there exists $(B', b') \in \mathcal{B}_{/A}$ such that

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[dr, dashed, "g'"'] & \operatorname*{colim}\limits_{b \in \mathcal{B}_{/A}} s(b) \arrow[r, "p"] & A \\
& B' \arrow[u, "i_{b'}"] \arrow[ur, "b'"'] &
\end{tikzcd}$$

commutes, i.e. $i_{b'} \circ g' = g$. Hence $b' \circ g' = p \circ i_{b'} \circ g' = p \circ g = 0$ by assumption. Thus b' factorizes over the cokernel of g':

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[dr, "g'"'] & \operatorname*{colim}\limits_{b \in \mathcal{B}_{/A}} s(b) \arrow[r, "p"] & A & \\
& B' \arrow[u, "i_{b'}"] \arrow[ur, "b'"] \arrow[r, dashed, "\text{can}"] & \operatorname{coker}(g') \arrow[u, dashed, "c"'] \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{B}$ is closed under cokernels, $B'' := \operatorname{coker}(g')$ is an object of $\mathcal{B}$ and $c\colon B'' \to A$ is an object of $\mathcal{B}_{/A}$ with canonical morphism $i_{b''}\colon B'' \to \operatorname{colim}_{b \in \mathcal{B}_{/A}} s(b)$ and $i_{b'} = i_{b''} \circ \text{can}$. In particular, we have

$$g = i_{b'} \circ g' = i_{b''} \circ \text{can} \circ g' = 0$$

and we conclude that p is a monomorphism. □

In locally finitely generated categories one has the following characterization of finitely generated objects:

Proposition 1.1.8 ([Bre70, Satz 1.6]) *Let $\mathcal{A}$ be locally finitely generated and $\mathcal{G}$ be a generating set consisting of finitely generated objects. For an object $A \in \mathcal{A}$, the following are equivalent:*

(i) A is finitely generated.

(ii) For every filtered diagram $D\colon \mathcal{D} \to \mathcal{A}$ the canonical morphism

$$\operatorname*{colim}_{d \in \mathcal{D}} \mathcal{A}(A, D_d) \to \mathcal{A}\big(A, \operatorname*{colim}_{d \in \mathcal{D}} D_d\big)$$

is injective.

(iii) There exists $n \in \mathbb{N}$ and an exact sequence

$$\coprod_{1 \le i \le n} G_i \longrightarrow A \longrightarrow 0$$

with $G_i \in \mathcal{G}$.

A similar result holds for locally finitely presentable categories:

Proposition 1.1.9 ([Bre70, Satz 1.11]) *Let $\boldsymbol{\mathcal{A}}$ be a locally finitely presentable category and $\mathcal{G}$ be a generating set consisting of finitely presentable objects. For an object $A \in \boldsymbol{\mathcal{A}}$, the following are equivalent:*

(i) $A \in \boldsymbol{\mathcal{A}}_{\mathrm{fp}}$ *is finitely presentable.*

(ii) $A \in \boldsymbol{\mathcal{A}}_{\mathrm{fg}}$ *and for every epimorphism* $p\colon Y \twoheadrightarrow A$ *with* $Y \in \boldsymbol{\mathcal{A}}_{\mathrm{fg}}$ *we have* $\ker(p) \in \boldsymbol{\mathcal{A}}_{\mathrm{fg}}$.

(iii) We have an exact sequence of the form

$$\coprod_{1\leq i\leq m} G_i^1 \longrightarrow \coprod_{1\leq j\leq n} G_j^0 \longrightarrow A \longrightarrow 0$$

with $G_\bullet^\bullet \in \mathcal{G}$ *and* $m, n \in \mathbb{N}$.

Remark If $\boldsymbol{\mathcal{A}}$ is a locally finitely presentable category, then $\boldsymbol{\mathcal{A}}$ is completely determined by its full subcategory $\boldsymbol{\mathcal{A}}_{\mathrm{fp}}$:

The category $\boldsymbol{\mathcal{A}}_{\mathrm{fp}}$ is additive, essentially small, has cokernels and the functor

$$\boldsymbol{\mathcal{A}} \to \mathrm{Lex}\big(\boldsymbol{\mathcal{A}}_{\mathrm{fp}}^{\mathrm{op}}, \mathbf{Ab}\big)\,, \qquad\qquad X \mapsto \mathrm{Hom}(-,X)\big|_{\boldsymbol{\mathcal{A}}_{\mathrm{fp}}}$$

from $\boldsymbol{\mathcal{A}}$ into the category of additive left exact functors from $\boldsymbol{\mathcal{A}}_{\mathrm{fp}}^{\mathrm{op}}$ to $\mathbf{Ab}$ is an equivalence ([Bre70, Satz 2.4]). Moreover, a functor $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ into a Grothendieck category $\boldsymbol{\mathcal{B}}$ is exact if and only if the restriction $F\big|_{\boldsymbol{\mathcal{A}}_{\mathrm{fp}}}$ to the full subcategory of finitely presentable objects is exact ([Kra98, Proposition 5.10 (3)]). E.g. one can use this observation to extend Landweber's original proof of the Landweber exact functor theorem from finitely presentable comodules to all comodules.

Similar results hold for $\boldsymbol{\mathcal{A}}_{\mathrm{noe}}$ resp. $\boldsymbol{\mathcal{A}}_{\mathrm{coh}}$ in place of $\boldsymbol{\mathcal{A}}_{\mathrm{fp}}$, cf. [Roo69, p. 203f].

The class of finitely generated resp. presentable objects is in general not preserved by functors.

Lemma 1.1.10 *Let $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}$ be Grothendieck categories, $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a faithful exact functor and $A \in \boldsymbol{\mathcal{A}}$.*

(i) If $\boldsymbol{\mathcal{A}}$ is locally finitely generated and $F(A)$ is a finitely generated object in $\boldsymbol{\mathcal{B}}$, then A is finitely generated in $\boldsymbol{\mathcal{A}}$.

(ii) Assume that F has a right adjoint G preserving filtered colimits. If A is a finitely presentable (resp. finitely generated) in $\boldsymbol{\mathcal{A}}$, so is $F(A)$ in $\boldsymbol{\mathcal{B}}$.

Proof. To show (i), we use Proposition 1.1.8. Let $\mathrm{colim}_d\, D_d$ be a colimit over a filtered diagram $D\colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{A}}$. Since F is faithful and filtered colimits commute with finite limits in $\mathbf{Set}$, the horizontal morphisms in the commutative diagram

$$\begin{tikzcd}
\mathrm{colim}_d\, \boldsymbol{\mathcal{A}}(A, D_d) \arrow[r, tail] \arrow[d, "\phi"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{colim}_d\, \boldsymbol{\mathcal{B}}\big(F(A), F(D_d)\big) \arrow[d, "\psi"] \\
\boldsymbol{\mathcal{A}}(A, \mathrm{colim}_d\, D_d) \arrow[r, tail] & \boldsymbol{\mathcal{B}}\big(F(A), \mathrm{colim}_d\, F(D_d)\big)
\end{tikzcd}$$

are injective. If $F(A)$ is finitely generated, then ψ is injective and we conclude that ϕ is also injective. Hence A is finitely generated.

For the second claim, we have natural isomorphisms

$$\begin{aligned}
\boldsymbol{\mathcal{B}}\big(F(A), \operatorname*{colim}_d B_d\big) &\cong \boldsymbol{\mathcal{A}}\big(A, G(\operatorname*{colim}_d B_d)\big) \cong \boldsymbol{\mathcal{A}}\big(A, \operatorname*{colim}_d G(B_d)\big) \\
&\cong \operatorname*{colim}_d \boldsymbol{\mathcal{A}}\big(A, G(B_d)\big) \cong \operatorname*{colim}_d \boldsymbol{\mathcal{B}}\big(F(A), B_d\big)
\end{aligned}$$

for every diagram $B\colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{B}}$. $\square$

In general, a filtered colimit of injective objects in a Grothendieck category $\boldsymbol{\mathcal{A}}$ may not be injective again. As an example, the category $R-$**Mod** satisfies this property if and only if $R-$**Mod** is locally noetherian (cf. [Roo69, Theorem 1]) and we have already stated that this is equivalent to R being noetherian.

Theorem 1.1.11 ([Roo69, Theorem 2]) *Let $\boldsymbol{\mathcal{A}}$ be a locally noetherian Grothendieck category. Then every filtered colimit of injectives is injective.*

In particular, if $\boldsymbol{\mathcal{A}}$ is a locally noetherian Grothendieck category, the full subcategory $\boldsymbol{\mathcal{A}}_{\text{inj}}$ of injective objects is closed under taking products *and* coproducts (in $\boldsymbol{\mathcal{A}}$).

Proposition 1.1.12 ([Kra01, Prop A.11]) *Let $\boldsymbol{\mathcal{A}}$ be a locally finitely presentable Grothendieck category. Then the following are equivalent:*

(i) $\boldsymbol{\mathcal{A}}$ is locally noetherian.
(ii) Every finitely presentable object is noetherian.
(iii) Every fp-injective object is injective.
(iv) Every direct limit of injective objects is injective.

Here, an object $E \in \boldsymbol{\mathcal{A}}$ is *fp-injective* if $\operatorname{Ext}^1_{\boldsymbol{\mathcal{A}}}(A, E) = 0$ for every finitely presentable object $A \in \boldsymbol{\mathcal{A}}_{\text{fp}}$.

The following result will be used to show that the category of comodules over a Hopf algebroid (A, Γ) is locally noetherian if A is a noetherian ring.

Lemma 1.1.13 ([Has09, Lemma 11.1]) *Let $\boldsymbol{\mathcal{A}}$ be a Grothendieck abelian category and $\boldsymbol{\mathcal{B}}$ a locally noetherian category. Let $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a faithful exact functor and G its right adjoint. If G preserves filtered colimits, then*

(i) $\boldsymbol{\mathcal{A}}$ is locally noetherian, and
(ii) $A \in \boldsymbol{\mathcal{A}}$ is a noetherian object if and only if $F(A) \in \boldsymbol{\mathcal{B}}$ is.

1.2 Localizing subcategories

The concept of localizing subcategories is the core for our definition of local cohomology.

Definition 1.2.1 (Serre subcategory) A full subcategory $\boldsymbol{\mathcal{S}}$ of an abelian category $\boldsymbol{\mathcal{A}}$ is a *Serre subcategory* provided that for every exact sequence $0 \to A'' \to A \to A' \to 0$ in $\boldsymbol{\mathcal{A}}$ the object A is in $\boldsymbol{\mathcal{S}}$ if and only if A' and A'' are in $\boldsymbol{\mathcal{S}}$.

Example 1.2.2 (i) If $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ is an exact functor between abelian categories, the full subcategory $\mathbf{Ker}(F)$ consisting of objects $A \in \boldsymbol{\mathcal{A}}$ with $F(A) = 0$ is called the *kernel of F* and is a Serre subcategory, cf. [SPA, Lemma 02MQ].

(ii) If X is a scheme, the full subcategory $\mathbf{QCoh}(X) \subset \mathbf{Mod}(\mathscr{O}_X)$ is not a Serre subcategory. Indeed, a subsheaf of a quasi-coherent sheaf might not be quasi-coherent.

Remark (i) Gabriel ([Gab62, p. 365]) uses the term *épaisse* (french for thick) for a Serre subcategory.

(ii) A Serre subcategory is an abelian category and the inclusion functor $\boldsymbol{\mathcal{S}} \hookrightarrow \boldsymbol{\mathcal{A}}$ is exact ([SPA, Lemma 02MP]).

We fix a Serre subcategory $\boldsymbol{\mathcal{S}}$ of $\boldsymbol{\mathcal{A}}$.

Definition 1.2.3 (quotient category) The *quotient category $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$* of $\boldsymbol{\mathcal{A}}$ relative to $\boldsymbol{\mathcal{S}}$ is defined as follows: The objects of $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$ are those of $\boldsymbol{\mathcal{A}}$ and

$$\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}(A, B) = \operatorname{colim} \boldsymbol{\mathcal{A}}(A', B/B')$$

with $A' \leq A$, $B' \leq B$ and $A/A', B' \in \boldsymbol{\mathcal{S}}$.

Remark (i) The category $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$ is abelian and the canonical quotient functor $q\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$ with $q(X) = X$ is exact, essentially surjective and its kernel is $\boldsymbol{\mathcal{S}}$ ([SPA, Lemma 02MS] and [Gab62, Proposition III.1]).

(ii) $(\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}, q)$ satisfies a universal property: For any exact functor $G\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ such that $\boldsymbol{\mathcal{S}} \subset \mathbf{Ker}(G)$ there exists a factorization $G = H \circ F$ for a unique exact functor $H\colon \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{B}}$:

$$\begin{tikzcd}
\boldsymbol{\mathcal{A}} \arrow[rr, "G"] \arrow[dr, "q"'] & & \boldsymbol{\mathcal{B}} \\
& \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}} \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

(iii) $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$ is the categorical localization of $\boldsymbol{\mathcal{A}}$ with respect to morphisms $f \in \boldsymbol{\mathcal{A}}$ such that $\ker(f), \operatorname{coker}(f) \in \boldsymbol{\mathcal{S}}$.

Definition 1.2.4 (localizing subcategory) A Serre subcategory $\boldsymbol{\mathcal{S}}$ is called *localizing* provided that q admits a right adjoint $s\colon \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{A}}$ which is then called *section functor*.

Lemma 1.2.5 ([Kra97, Lemma 2.1]) *Let $\boldsymbol{\mathcal{A}}$ be an abelian category with enough injectives and $\boldsymbol{\mathcal{S}}$ be a Serre subcategory of $\boldsymbol{\mathcal{A}}$. Then the following conditions are equivalent:*

(i) The inclusion $\iota\colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{A}}$ admits a right adjoint $t\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{S}}$.

(ii) $\boldsymbol{\mathcal{S}}$ is a localizing subcategory.

The right adjoint is then given by

$$t\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{S}}\,, \qquad A \mapsto \ker\bigl(A \xrightarrow{\eta_A} s \circ q(A)\bigr)\,.$$

Remark (i) [Kra97, Lemma 2.1] actually gives more: If $\boldsymbol{\mathcal{A}}$ has enough injectives, then a localizing subcategory $\boldsymbol{\mathcal{S}}$ defines a hereditary torsion theory $\bigl(\boldsymbol{\mathcal{S}}, \{A \in \boldsymbol{\mathcal{A}} \,|\, \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{S}}, A) = 0\}\bigr)$ and vice versa. This gives the connection to [HS05a, Section 1].

(ii) Let $\boldsymbol{\mathcal{S}} \hookrightarrow \boldsymbol{\mathcal{A}}$ be a localizing subcategory. If $\boldsymbol{\mathcal{A}}$ is complete, so is $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$. If $\boldsymbol{\mathcal{A}}$ is Grothendieck, so is $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{S}}$.

(iii) The Gabriel-Popescu Theorem can be formulated as follows: Every Grothendieck category is given as a quotient of the category of modules over a ring by a localizing subcategory. From (ii) we see that this implies that every Grothendieck category is complete.

Example 1.2.6 (localizing pair) Let $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}$ be Grothendieck categories and $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ an exact functor with fully faithful right adjoint G. Then $\iota\colon \mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ is a localizing subcategory and the pair (F, G) is called a *localization pair*. Moreover, we then have an equivalence of categories $\boldsymbol{\mathcal{A}}/\mathbf{Ker}(F) \cong \boldsymbol{\mathcal{B}}$. The diagram with adjoint morphisms looks like

$$\begin{tikzcd}
\mathbf{Ker}(F) \arrow[r, hook, shift left, "\iota"] & \boldsymbol{\mathcal{A}} \arrow[l, shift left, "t"] \arrow[rr, shift left, "F"] \arrow[dr, shift left, "q"] & & \boldsymbol{\mathcal{B}} \arrow[ll, shift left, "G"] \\
& & \boldsymbol{\mathcal{A}}/\mathbf{Ker}(F) \arrow[ul, shift left, "s"] \arrow[ur, "H"'] &
\end{tikzcd}$$

and H is the induced functor obtained by the universal property of the quotient $\boldsymbol{\mathcal{A}}/\mathbf{Ker}(F)$ establishing the equivalence of categories. Furthermore, $L := GF\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ is a localization functor in the categorical sense (cf. [HS05a, Definition 1.2 and Proposition 1.4]).

Definition 1.2.7 (coreflective subcategory) A *coreflective subcategory* of an abelian category is a full subcategory whose inclusion functor has a right adjoint.

Remark Since the inclusion of a full subcategory is always fully faithful, a coreflective subcategory $\iota\colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{C}}$ with right adjoint t always satisfies $\mathrm{id}_{\boldsymbol{\mathcal{S}}} \overset{\sim}{\to} t \circ \iota$ for the unit η of the adjunction $\iota \dashv t$.

Example 1.2.8 (i) If $\boldsymbol{\mathcal{S}} \hookrightarrow \boldsymbol{\mathcal{A}}$ is a localizing subcategory of a Grothendieck category $\boldsymbol{\mathcal{A}}$, then $\boldsymbol{\mathcal{S}}$ is a coreflective subcategory.

(ii) If X is a scheme, then the inclusion $\mathbf{QCoh}(X) \to \mathbf{Mod}(\mathscr{O}_X)$ admits a right adjoint, the *coherator* $Q\colon \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{QCoh}(X)$. Indeed, $\mathbf{QCoh}(X)$ is Grothendieck as mentioned in 1.1.2 (iv) and $\mathbf{QCoh}(X) \to \mathbf{Mod}(\mathscr{O}_X)$ is cocontinuous. Thus $\mathbf{QCoh}(X) \subset \mathbf{Mod}(\mathscr{O}_X)$ is a coreflective subcategory. We give the construction for $X = \mathrm{Spec}(A)$ affine and X quasi-compact and semi-separated in Section 3.5.

Proposition 1.2.9 *Let $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}$ be Grothendieck categories and $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ an exact functor with fully faithful right adjoint G. Let $t\colon \boldsymbol{\mathcal{A}} \to \mathbf{Ker}(F)$ be the right adjoint to the inclusion $\iota\colon \mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ as in Lemma 1.2.5. Assume further that ι preserves injective objects.*

If E is an injective object of $\boldsymbol{\mathcal{A}}$, then the objects $F(E)$ of $\boldsymbol{\mathcal{B}}$ and $GF(E)$, $\iota t(E)$ of $\boldsymbol{\mathcal{A}}$ are also injective. Moreover, we have a (split) short exact sequence of the form

$$0 \longrightarrow \iota t(E) \longrightarrow E \longrightarrow GF(E) \longrightarrow 0$$

in $\boldsymbol{\mathcal{A}}$.

Proof. $t(E)$ is an injective object of $\mathbf{Ker}(F)$ since t has the exact left adjoint $\mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$. By assumption, $\iota t(E)$ is an injective object of $\boldsymbol{\mathcal{A}}$. Hence, the monomorphism $\iota t(E) \rightarrowtail E$ has a section $E \to \iota t(E)$. Moreover, G preserves injective objects due to its exact left adjoint F.

Let us show that $\eta_E\colon E \to GF(E)$ is an epimorphism. To do so, let $F(E) \rightarrowtail J$ be an embedding of $F(E)$ into an injective object of $\boldsymbol{\mathcal{B}}$. Since G is left exact and preserves injective objects, we have an embedding $GF(E) \rightarrowtail G(J)$ into an injective object of $\boldsymbol{\mathcal{A}}$. Combining the section $E \to \iota t(E)$ with $E \overset{\eta_E}{\to} GF(E) \rightarrowtail G(J)$ we get a monomorphism

$$E \rightarrowtail t(E) \oplus G(J)$$

since $E = \iota t(E) \oplus \ker\big(E \to \iota t(E)\big)$. As E is an injective object of $\boldsymbol{\mathcal{A}}$, we can write E as a direct summand of $\iota t(E) \oplus G(J)$, e.g.,

$$\iota t(E) \oplus G(J) = E \oplus \widetilde{E}$$

for some $\widetilde{E} \in \boldsymbol{\mathcal{A}}$. The unit η applied to $\iota t(E) \oplus G(E)$ is an epimorphism since

$$GF\big(\iota t(E) \oplus G(J)\big) \cong 0 \oplus G(J) \cong G(J)\,.$$

Thus $0 = \mathrm{coker}(\eta_{\iota t(E) \oplus G(E)}) = \mathrm{coker}(\eta_E) \oplus \mathrm{coker}(\eta_{\widetilde{E}})$ and we conclude that η_E is an epimorphism, as desired.

Since $\iota t(E)$ is injective, we have a splitting $GF(E) \to E$ forcing $GF(E)$ to be an injective object of $\boldsymbol{\mathcal{A}}$. It remains to show that $F(E)$ is an injective object of $\boldsymbol{\mathcal{B}}$. Since G is a fully faithful right adjoint, it reflects injective objects. As $GF(E)$ is injective, it follows that $F(E)$ is injective. □

Corollary 1.2.10 *Let the assumptions be as before. For any object $A \in \boldsymbol{\mathcal{A}}$, we have an exact sequence*

$$0 \longrightarrow \iota t(A) \longrightarrow A \longrightarrow GF(A) \longrightarrow \mathbb{R}^1(\iota t)(A) \longrightarrow 0 \tag{1.2.1}$$

and isomorphisms

$$\mathbb{R}^{k+1}(\iota t)(A) \cong \mathbb{R}^{k}(GF)(A)$$

for $k \geq 1$.

Proof. Let $A \rightarrowtail E^\bullet$ be an injective resolution of $A \in \boldsymbol{\mathcal{A}}$. By Proposition 1.2.9 we have a natural short exact sequence

$$0 \longrightarrow \iota t(E^\bullet) \longrightarrow E^\bullet \longrightarrow GF(E^\bullet) \longrightarrow 0$$

of complexes and the claim follows after taking the long exact sequence in cohomology. □

The crucial assumption "$\iota\colon \mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ preserves injective objects" can be characterized in the following way:

Lemma 1.2.11 *The following are equivalent:*

(i) The inclusion $\iota\colon \mathbf{Ker}(F) \to \boldsymbol{\mathcal{A}}$ *preserves injective objects.*

(ii) Every object $A \in \mathbf{Ker}(F)$ *admits an embedding* $\iota(A) \rightarrowtail E$ *into an injective object* $E \in \boldsymbol{\mathcal{A}}$ *such that* $E \in \mathbf{Ker}(F)$.

(iii) If $A \in \mathbf{Ker}(F)$, *then* $\mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(\iota A) \in \mathbf{Ker}(F)$. *In other words: the full subcategory* $\mathbf{Ker}(F)$ *is closed under injective hulls (in* $\boldsymbol{\mathcal{A}}$*).*

Proof. Let ι preserve injective objects. As ι is faithful, the unit $A \rightarrowtail t(\iota A)$ is a monomorphism. For every $A \in \mathbf{Ker}(F)$ we can find an embedding $\iota A \rightarrowtail E$ into an injective object of $\boldsymbol{\mathcal{A}}$. Applying the right adjoint t gives a monomorphism $t(\iota A) \rightarrowtail t(E)$ in $\mathbf{Ker}(F)$ and $t(E)$ is an injective object of $\mathbf{Ker}(F)$ since t has an exact left adjoint. Consider the composition $A \rightarrowtail t(\iota A) \rightarrowtail t(E)$. If (i) holds, then $t(E)$ is an injective object of $\boldsymbol{\mathcal{A}}$ and we have found the desired monomorphism in (ii).

If $A \rightarrowtail E$ is an embedding as in (ii), consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, hook] & \mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(A) \arrow[dl, dashed] \\
& E &
\end{tikzcd}$$

in $\boldsymbol{\mathcal{A}}$. Since $A \rightarrowtail \mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(A)$ is an essential extension, the dotted arrow $\mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(A) \to E$ is a monomorphism. In particular, $\mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(A) \in \mathbf{Ker}(F)$, as $\mathbf{Ker}(F)$ is closed under subobjects. Hence (iii) holds.

Let E be an injective object of $\mathbf{Ker}(F)$. Then we have an embedding $\iota(E) \rightarrowtail \mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(E)$ in $\boldsymbol{\mathcal{A}}$. If (iii) holds, then $\mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(E) \in \mathbf{Ker}(F)$ and we have a splitting $\mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(\iota E) \to \iota(E)$ in $\mathbf{Ker}(F)$. We see that $\mathrm{inj.hull}_{\boldsymbol{\mathcal{A}}}(\iota E) \cong \iota E$. In particular, ιE is an injective object of $\boldsymbol{\mathcal{A}}$ and (i) holds. □

Lemma 1.2.12 *Every object of* $\mathbf{Ker}(t) = \big\{A \in \boldsymbol{\mathcal{A}} \,\big|\, t(A) = 0\big\}$ *can be embedded in an injective object* $E \in \boldsymbol{\mathcal{A}}$ *with* $E \in \mathbf{Ker}(t)$.

Proof. If $A \in \boldsymbol{\mathcal{A}}$, let $F(A) \rightarrowtail E$ be an embedding into an injective object E of $\boldsymbol{\mathcal{B}}$. Applying the right adjoint G we obtain an embedding $GF(A) \rightarrowtail G(E)$ and $G(E)$ is injective since the left adjoint F is exact.

If $t(A) = 0$, the unit $\eta_A \colon A \to GF(A)$ is a monomorphism since its kernel vanishes and we can compose to get an embedding $A \rightarrowtail GF(A) \rightarrowtail G(E)$. It remains to show that $tG(E) = 0$. Let $X \in \mathbf{Ker}(F)$ be an arbitrary object of $\mathbf{Ker}(F)$. Then

$$\mathbf{Ker}(F)\big(X, tG(E)\big) \cong \boldsymbol{\mathcal{A}}\big(\iota X, G(E)\big) \cong \boldsymbol{\mathcal{B}}\big(F\iota X, E\big) = 0$$

by using the adjunctions $\iota \dashv t$ and $F \dashv G$ and $X \in \mathbf{Ker}(F)$. Hence, $tG(E) = 0$. □

Corollary 1.2.13 *Let $\iota \colon \mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ preserve injective objects. For every $A \in \boldsymbol{\mathcal{A}}$, $GF(A) \cong GF\big(GF(A)\big)$.*

Proof. If A in $\boldsymbol{\mathcal{A}}$ is an arbitrary object, then $\eta_{GF(A)} \colon GF(A) \to GF\big(GF(A)\big)$ can be identified with $\mathrm{id}_{GF(A)}$. □

Remark If $A \in \mathbf{Ker}(t)$, then we have a monomorphism $A \rightarrowtail GF(A)$ which is in general *not* an isomorphism. Indeed, if $A \rightarrowtail E$ is an embedding in an injective object with cokernel C, then $C \in \mathbf{Ker}(t)$ is equivalent to $\mathbb{R}^1(\iota t)(A) = 0$, which is equivalent to $A \cong GF(A)$. Since t is only left exact, there is no reason for C to be in the kernel of t in general. Clearly, this holds if A is already injective itself.

Lemma 1.2.14 *Assume that $\iota \colon \mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ preserves injective objects and let $A \in \boldsymbol{\mathcal{A}}$. Then $\mathbb{R}^\bullet(\iota t)(A) \in \mathbf{Ker}(F)$ for every object $A \in \boldsymbol{\mathcal{A}}$. Moreover, $\mathbb{R}^k(GF)(A) \in \mathbf{Ker}(F)$ if $k \geq 1$.*

Proof. Let $A \rightarrowtail E^\bullet$ be an injective resolution of A. By definition

$$\mathbb{R}^k(\iota t)(A) = \mathrm{H}^k\big(\iota t(E^\bullet)\big)\,.$$

Since F is an exact functor, it commutes with cohomology and we see that

$$F\big(\mathbb{R}^k(\iota t)(A)\big) \cong \mathrm{H}^k\big(F\iota t(E^\bullet)\big) = 0\,.$$

Thus $\mathbb{R}^\bullet(\iota t)(A) \in \mathbf{Ker}(F)$.

The second claim follows from the first statement and Corollary 1.2.10 since

$$\mathbb{R}^k(GF)(A) \cong \mathbb{R}^{k+1}(\iota t)(A)$$

for $k \geq 1$. □

Lemma 1.2.15 *Let $\iota \colon \mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ preserve injective objects and let $A \in \boldsymbol{\mathcal{A}}$. Then we have natural isomorphisms*

$$\mathbb{R}^\bullet G(FA) \cong \mathbb{R}^\bullet(GF)(A)\,.$$

Proof. Let $E^\bullet$ be an injective resolution of A. Since F is exact and preserves injective objects by Proposition 1.2.9, $F(E^\bullet)$ is an injective resolution of $F(A)$. Hence

$$\mathbb{R}^k G(FA) = \mathrm{H}^k\big(G(FE^\bullet)\big) = \mathrm{H}^k\big((GF)E^\bullet\big) = \mathbb{R}^k(GF)(A)\,.$$ □

Corollary 1.2.16 *Let $\iota \colon \mathbf{Ker}(F) \to \boldsymbol{\mathcal{A}}$ preserve injective objects.*

(i) If $A \in \mathbf{Ker}(F)$, then $\mathbb{R}^\bullet GF(A) = 0$.

(ii) The unit $\eta_A \colon A \to GF(A)$ induces an isomorphism

$$\mathbb{R}^\bullet GF(A) \xrightarrow{\simeq} \mathbb{R}^\bullet GF\big(GF(A)\big)$$

for any $A \in \boldsymbol{\mathcal{A}}$.

Proof. (i) Under the assumption on ι, we can find an injective resolution $A \rightarrowtail E^\bullet$ such that all $E^i \in \mathbf{Ker}(F)$. Hence

$$\mathbb{R}^\bullet GF(A) = \mathrm{H}^\bullet\big(GF(E^\bullet)\big) = \mathrm{H}^\bullet\big(G(FE^\bullet)\big) = 0\,.$$

(ii) From the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \iota t(A) \arrow[r] & A \arrow[rr, "\eta_A"] \arrow[dr] & & GF(A) \arrow[r] & \mathbb{R}^1(\iota t)(A) \arrow[r] & 0 \\
& & & \mathrm{im}(\eta_A) \arrow[ur] & & &
\end{tikzcd}$$

we obtain the two short exact sequences

$$0 \longrightarrow \iota t(A) \longrightarrow A \longrightarrow \mathrm{im}(\eta_A) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im}(\eta_A) \longrightarrow GF(A) \longrightarrow \mathbb{R}^1(\iota t)(A) \longrightarrow 0$$

Applying $\mathbb{R}^\bullet GF$ yields long exact sequences

$$\ldots \longrightarrow \mathbb{R}^k GF\big(\iota t(A)\big) \longrightarrow \mathbb{R}^k GF(A) \longrightarrow \mathbb{R}^k GF\big(\mathrm{im}(\eta_A)\big) \longrightarrow \mathbb{R}^{k+1} GF\big(\iota t(A)\big) \longrightarrow \ldots$$

resp. (where we write A' instead of $\mathbb{R}^1(\iota t)(A)$ for short)

$$\ldots \longrightarrow \mathbb{R}^{k-1} GF(A') \longrightarrow \mathbb{R}^k GF\big(\mathrm{im}(\eta_A)\big) \longrightarrow \mathbb{R}^k GF\big(GF(A)\big) \longrightarrow \mathbb{R}^k GF(A') \longrightarrow \ldots$$

Since $\iota t(A)$ and A' are objects in $\mathbf{Ker}(F)$, we have

$$\mathbb{R}^\bullet GF\big(\iota t(A)\big) = \mathbb{R}^\bullet GF(A') = 0$$

by part (i). Hence, we obtain isomorphisms

$$\mathbb{R}^k GF(A) \xrightarrow{\simeq} \mathbb{R}^k GF\big(\mathrm{im}(\eta_A)\big) \xrightarrow{\simeq} \mathbb{R}^k GF\big(GF(A)\big)$$

and the claim follows. ☐

Remark To be more general, we actually do not have to work with injective objects. It would be enough to have a class of objects $\boldsymbol{\mathcal{K}} \subset \boldsymbol{\mathcal{A}}$ with the following properties:

$$(1.2.2)\qquad \begin{cases} \eta_K \colon K \to GF(K) \text{ is an epimorphism for every } K \in \boldsymbol{\mathcal{K}}. \\ \boldsymbol{\mathcal{K}} \text{ is preserved by } \iota t \text{ and } GF. \\ \text{Every object } A \in \boldsymbol{\mathcal{A}} \text{ admits a monomorphism } A \rightarrowtail K \text{ with } K \in \boldsymbol{\mathcal{K}}. \\ \text{Objects in } \boldsymbol{\mathcal{K}} \text{ are acyclic for } \iota t \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}} \text{ and } GF \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}. \end{cases}$$

These assumptions are fulfilled if $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{A}}_{\mathrm{inj}}$ and $\mathbf{Ker}(F) \hookrightarrow \boldsymbol{\mathcal{A}}$ preserves injective objects.

As an example (cf. Proposition 2.1.1), let X be a scheme and $j \colon U \hookrightarrow X$ be the inclusion of an open subscheme. The immersion j induces an adjunction

$$j^* \colon \mathbf{Mod}(\mathscr{O}_X) \rightleftarrows \mathbf{Mod}(\mathscr{O}_U) : j_*$$

and we write $\underline{\Gamma_Z}(-)$ for $\iota t \colon \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$. Let $\boldsymbol{\mathcal{K}} \subset \mathbf{Mod}(\mathscr{O}_X)$ be the subcategory spanned by flasque sheaves. Then

- $\mathscr{K} \to j_*j^*\mathscr{K}$ is an epimorphism for every flasque sheaf $\mathscr{K}$.
- Flasque sheaves are preserved by push-forward along morphism of sheaves. If $\mathscr{K}$ is flasque, so is $\mathscr{K}\big|_U$ for every open $U \subset X$. The functor $\underline{\Gamma_Z}(-)$ preserves flasque sheaves.
- There are enough injective objects in $\mathbf{Mod}(\mathscr{O}_X)$ and every injective $\mathscr{O}_X$-module is flasque.
- Flasque sheaves are acyclic for $\underline{\Gamma_Z}(-)$ and the direct image functor j_*.

1.3 Dualizable objects in closed symmetric monoidal categories

Definition 1.3.1 (closed symmetric monoidal category) A symmetric monoidal category $\boldsymbol{\mathcal{A}} = (\boldsymbol{\mathcal{A}}, \otimes, \mathbb{1})$ is *closed* if for all objects $B \in \boldsymbol{\mathcal{A}}$ the functor $(- \otimes B) \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ admits a right adjoint functor $[B, -] \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$, i.e. we have natural bijections

$$\boldsymbol{\mathcal{A}}(A \otimes B, C) \simeq \boldsymbol{\mathcal{A}}(A, [B, C])$$

in $A, C \in \boldsymbol{\mathcal{A}}$. The object $[B, C]$ is then called the *internal hom* of B and C. We sometimes write $\underline{\mathrm{hom}}_{\boldsymbol{\mathcal{C}}}(B, C)$.

Example 1.3.2 (i) For a commutative ring R, the category $R{-}\mathbf{Mod}$ of R-modules is closed symmetric monoidal under the functors $- \otimes_R -$ and $\mathrm{Hom}_R(-,-)$.

(ii) If X is a scheme, the category $\mathbf{Mod}(\mathscr{O}_X)$ is closed symmetric monoidal under the tensor product $- \otimes_{\mathscr{O}_X} -$ and the hom sheaf $\underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_X}(-,-)$.

(iii) If X is a scheme, the category $\mathbf{QCoh}(X)$ is closed symmetric monoidal under $- \otimes_{\mathscr{O}_X} -$ and $Q\underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_X}(-,-)$, where Q denotes the coherator.

For any closed symmetric monoidal category $\boldsymbol{\mathcal{A}}$, we have a natural map, called *evaluation*,

$$\mathrm{ev}_{A,B} \colon [A, \mathbb{1}] \otimes B \to [A, B]$$

for objects $A, B \in \boldsymbol{\mathcal{A}}$.

Definition 1.3.3 (dualizable object) An object $A \in \boldsymbol{\mathcal{A}}$ is *strongly dualizable*, or simply *dualizable*, if the evaluation map $[A, \mathbb{1}] \otimes B \to [A, B]$ is an isomorphism for all $B \in \boldsymbol{\mathcal{A}}$. If A is dualizable, we write DA instead of $[A, \mathbb{1}]$.

Theorem 1.3.4 ([HPS97, Theorem A.2.5]) *(i) If A is dualizable, so is DA.*

(ii) If A is dualizable, the natural map $A \to D^2A$ adjoint to the evaluation map $A \otimes DA \to \mathbb{1}$ is an isomorphism.

(iii) If A and B are dualizable, so is $A \otimes B$.

(iv) If A is dualizable and B, C are arbitrary objects of $\boldsymbol{\mathcal{A}}$, there is a natural isomorphism

$$\left[B \otimes A, C\right] \to \left[B, DA \otimes C\right].$$

Example 1.3.5 In $R{-}\mathbf{Mod}$, the dualizable objects are precisely the flat and finitely presented R-modules, or, equivalently, the finitely generated projective R-modules (for the equivalence see [SPA, Lemma 00NX]).

Definition 1.3.6 (flat object) An object $A \in \boldsymbol{\mathcal{A}}$ is *flat* if the functor $(- \otimes A) \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ is exact.

Since the notion of "flatness" might have another meaning in a different context, we sometimes refer to this property as *globally flat* following [Lur05].

Example 1.3.7 (i) An R-module M is globally flat if and only if it is flat in the usual algebraic sense.

(ii) An $\mathscr{O}_X$-module $\mathscr{F}$ on a scheme X is globally flat if and only if it is flat in the algebro-geometric sense, i.e if $\mathscr{F}_x$ is a flat $\mathscr{O}_{X,x}$-module for all points $x \in X$, cf. [SPA, Lemma 05NE].

(iii) On a quasi-separated scheme X, a quasi-coherent sheaf $\mathscr{F}$ is globally flat if and only if it is flat in the usual geometric sense. Indeed, any flat quasi-coherent sheaf is clearly globally flat. On the other hand, let $\mathscr{F}$ be a globally flat quasi-coherent sheaf. It is a standard fact that $\mathscr{F}$ is flat if and only if $\mathscr{F}(U)$ is a flat $\Gamma(U, \mathscr{O}_X)$-module for every affine open $U \subset X$. Let $j\colon U \hookrightarrow X$ be the inclusion of an affine open subset of X and $\mathscr{G} \rightarrowtail \mathscr{H}$ a monomorphism in $\mathbf{QCoh}(U)$. Since X is quasi-separated, j is quasi-compact (and quasi-separated), j_* preserves quasi-coherence and $j^* \dashv j_*$ on the level of quasi-coherent sheaves. Applying j_*, tensoring with $\mathscr{F}$ and then applying j^* we obtain a monomorphism $j^*\mathscr{F} \otimes \mathscr{G} \rightarrowtail j^*\mathscr{F} \otimes \mathscr{H}$ in $\mathbf{QCoh}(U)$. Since U is affine, this is enough to conclude that $\mathscr{F}(U) = \Gamma(U, j^*\mathscr{F})$ is flat.

Lemma 1.3.8 *(i) Any dualizable object is flat.*

(ii) If A and B are flat, so is $A \otimes B$.

(iii) If $\mathbb{1}$ is finitely presentable, then any dualizable object is finitely presentable.

Proof. (i) For any dualizable object A, we have adjunctions

$$(- \otimes A) \dashv (- \otimes DA) \qquad \text{and} \qquad (- \otimes DA) \dashv (- \otimes A)\,.$$

Since $- \otimes A\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}$ is both a left and a right adjoint, it is an exact functor.

(ii) Straightforward.

(iii) Let A be dualizable. For a filtered diagram $B\colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{C}}$ we have natural morphisms

$$\begin{aligned}\operatorname*{colim}_d \boldsymbol{\mathcal{A}}(A, B_d) &\cong \operatorname*{colim}_d \boldsymbol{\mathcal{A}}\big(D(DA), B_d\big) \cong \operatorname*{colim}_d \boldsymbol{\mathcal{A}}(\mathbb{1}, B_d \otimes DA) \\ &\xrightarrow{\varphi} \boldsymbol{\mathcal{A}}\big(\mathbb{1}, \operatorname*{colim}_d (B_d \otimes DA)\big) \cong \boldsymbol{\mathcal{A}}\big(\mathbb{1}, (\operatorname*{colim}_d B_d) \otimes DA\big) \\ &\cong \boldsymbol{\mathcal{A}}\big(\mathbb{1} \otimes D(DA), \operatorname*{colim}_d B_d\big) \cong \boldsymbol{\mathcal{A}}(A, \operatorname*{colim}_d B_d)\end{aligned}$$

and φ is an isomorphism if $\mathbb{1}$ is finitely presentable. □

1.4 Unbounded derived categories and derived functors

We recall some techniques to ensure the existence of derived functors on the whole unbounded derived category of a Grothendieck category $\boldsymbol{\mathcal{A}}$. If $\boldsymbol{\mathcal{A}}$ is Grothendieck, then the derived category $\mathbf{D}(\boldsymbol{\mathcal{A}})$ exists and has “small hom-sets”, cf. [AJS00, Corollary 5.6]. In particular, there is no need of changing the universe $\boldsymbol{\mathcal{U}}$ when constructing the derived category via calculus of fractions.

Right-derived functors

Let $\boldsymbol{\mathcal{A}}$ be an abelian category with enough injectives, e.g. a Grothendieck category. It is a classical fact that then one can define the right derived functor of a (left exact) functor $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{B}}$ abelian, on the bounded below derived categories $\mathbf{D}^+(\boldsymbol{\mathcal{A}}) \to \mathbf{D}^+(\boldsymbol{\mathcal{B}})$, cf. [SPA, Lemma 05TI]. In general, the existence of enough injective objects is not sufficient to ensure the existence of the right derived functor on the unbounded derived category $\mathbf{D}(\boldsymbol{\mathcal{A}})$.

Spaltenstein ([Spa88]) showed the existence of K-injective (sometimes also called q-injective or ho-injective) resolutions for the derived category of abelian sheaves over a ringed space. This result was later generalized by Serpé ([Ser03]) to any Grothendieck category. There is also a section about the existence of enough K-injectives in Grothendieck categories in The Stacks Project, [SPA, Section 079I].

Proposition 1.4.1 ([Ser03, Corollary 3.14]) *Let $F\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ be a left exact functor between abelian categories and let $\boldsymbol{\mathcal{A}}$ be Grothendieck. Then the derived functor $\mathbf{R}F\colon \mathbf{D}(\boldsymbol{\mathcal{A}}) \to \mathbf{D}(\boldsymbol{\mathcal{B}})$ exists.*

Left derived functors

Similarly to the above, one may use K-projective objects to obtain left derived functors on unbounded derived categories. Certainly, we cannot expect the existence of enough K-projective objects for arbitrary Grothendieck categories. Under the assumption that $\boldsymbol{\mathcal{A}}$ has a projective generator (e.g. $\boldsymbol{\mathcal{A}} = A{-}\mathbf{Mod}$), a positive result for this can be found in [AJS00, Proposition 4.3].

We want to define the left derived functors of the tensor product on the category of quasi-coherent sheaves over an algebraic stack $\mathfrak{X}$. A possible approach could be K-flat resolutions. If X is a quasi-compact and semi-separated scheme, then every complex of quasi-coherent sheaves has a K-flat resolution made up of quasi-coherent sheaves, [Alo+08, Lemma 3.3]. The author is not aware of any written acount on the existence of K-flat resolutions for the category of quasi-coherent sheaves on arbitrary algebraic stacks. The proof of [Alo+08, Lemma 3.3] uses the equivalence

$$\mathbf{D}(\mathbf{QCoh}(X)) \xrightarrow{\sim} \mathbf{D}_{\mathbf{QCoh}(X)}(\mathbf{Mod}(\mathscr{O}_X))$$

for semi-separated and quasi-compact schemes X, given in [BN93, Corollary 5.5]. Note that the Corollary in [BN93] is formulated for separated schemes, but the proof immediately generalizes to the semi-separated case. Recently, Hall, Neeman and Rydh have shown that this question is closely related to the question whether $\mathbf{D}_{\mathbf{QCoh}(\mathfrak{X})}(\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}))$ resp. $\mathbf{D}(\mathbf{QCoh}(\mathfrak{X}))$ are compactly generated, [HNR13, Theorem 1.1].

At least if the stack $\mathfrak{X}$ is an Adams stack, then we expect the derived tensor product $\otimes^{\mathbf{L}}$ in $\mathbf{D}\big(\mathbf{QCoh}(\mathfrak{X})\big)$ to exist. This should follow from the approach via weakly flat descent structures of Cisinski and Déglise ([CD09]); the dualizable quasi-coherent sheaves form a flat generating family.

2 Local cohomology for schemes

2.1 Definition via a localizing pair

We define local cohomology sheaves on schemes via a localizing pair and compare this definition to Grothendieck's classic one.

Let X be a scheme, $Z \subset X$ a closed subscheme and $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$. Let $j\colon U \hookrightarrow X$ be the open complement of Z. In the adjoint pair

$$j^*\colon \mathbf{Mod}(\mathscr{O}_X) \rightleftarrows \mathbf{Mod}(\mathscr{O}_U) : j_*\,,$$

the counit $j^*j_* \to \mathrm{id}_{\mathbf{Mod}(\mathscr{O}_U)}$ is an equivalence and the functor j^* is exact. Hence, we have a localizing pair (j^*, j_*) as in Example 1.2.6. In particular, the category of $\mathscr{O}_U$-modules $\mathbf{Mod}(\mathscr{O}_U)$ is equivalent to the quotient $\mathbf{Mod}(\mathscr{O}_X)/\mathbf{Mod}_Z(\mathscr{O}_X)$, where

$$\mathbf{Mod}_Z(\mathscr{O}_X) := \mathbf{Ker}(j^*) = \bigl\{\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X) \bigm| j^*(\mathscr{F}) = 0\bigr\}$$

is a localizing subcategory of $\mathbf{Mod}(\mathscr{O}_X)$. A module sheaf $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$ belongs to $\mathbf{Mod}_Z(\mathscr{O}_X)$ if and only if

$$\mathrm{supp}_X(\mathscr{F}) := \bigl\{x \in X \bigm| \mathscr{F}_x \neq 0\bigr\} \subset Z\,.$$

Thus, the localizing subcategory $\mathbf{Mod}_Z(\mathscr{O}_X) \subset \mathbf{Mod}(\mathscr{O}_X)$ consists of module sheaves which are supported on Z.

The *sheaf of sections of $\mathscr{F}$ supported on Z*, $\underline{\Gamma_Z}(\mathscr{F})$, is given by exactness of

$$0 \longrightarrow \underline{\Gamma_Z}(\mathscr{F}) \longrightarrow \mathscr{F} \xrightarrow{\eta_{\mathscr{F}}} j_*j^*\mathscr{F}\,. \tag{2.1.1}$$

The name is motivated by the observation that $\underline{\Gamma_Z}(\mathscr{F})$ is the biggest subsheaf of $\mathscr{F}$ supported on Z. Let $\Gamma_Z(X;\mathscr{F})$ be the $\Gamma(X,\mathscr{O}_X)$-module of its global sections,

$$\Gamma_Z(X;\mathscr{F}) := \Gamma\bigl(X;\underline{\Gamma_Z}(\mathscr{F})\bigr)\,.$$

The functors

$$\underline{\Gamma_Z}(-)\colon \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)\,, \qquad\qquad \Gamma_Z(X;-)\colon \mathbf{Mod}(\mathscr{O}_X) \to \Gamma(X,\mathscr{O}_X)\mathbf{-Mod}$$

are left exact. We denote their right derived functors in the category of module sheaves $\mathbf{Mod}(\mathscr{O}_X)$ on X by

$$\underline{\mathrm{H}^\bullet_Z}(-) := \mathbb{R}^\bullet_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(-) \qquad \text{resp.} \qquad \mathrm{H}^\bullet_Z(X;-) := \mathbb{R}^\bullet_{\mathbf{Mod}(\mathscr{O}_X)}\Gamma_Z(-)\,.$$

Remark To be precise, we have to be careful when comparing this definition of $\underline{\mathrm{H}^k_Z}(-)$ with the one given in [Har67]. In the latter everything is defined for the category of abelian sheaves $\boldsymbol{\mathcal{S}h}(X)$ on the scheme X. Yet, any injective $\mathscr{O}_X$-module $\mathscr{E}$ is a flasque abelian sheaf (cf. [SPA, Lemma 09SX]) and flasque sheaves are acyclic for the functor $\underline{\Gamma_Z}(-)\colon \boldsymbol{\mathcal{S}h}(X) \to \boldsymbol{\mathcal{S}h}(X)$ (cf. [Har67, Proposition 1.10]). Hence, we can use injective resolutions in $\mathbf{Mod}(\mathscr{O}_X)$ to prove that the canonical morphism

$$\underline{\mathrm{H}^\bullet_Z}(-) := \mathbb{R}^\bullet_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(-) \longrightarrow \mathbb{R}^\bullet_{\boldsymbol{\mathcal{S}h}(X)}\underline{\Gamma_Z}(-)$$

is an isomorphism of (universal) δ-functors $\mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$. Similarly, the higher images of the push-forward of a morphism of schemes $f\colon Y \to X$ can be calculated either in $\mathbf{Mod}(\mathscr{O}_Y)$ or $\boldsymbol{\mathcal{Sh}}(Y)$ by the same argument,

$$\mathbb{R}^{\bullet}_{\mathbf{Mod}(\mathscr{O}_Y)} f_*(-) \xrightarrow{\sim} \mathbb{R}^{\bullet}_{\boldsymbol{\mathcal{Sh}}(Y)} f_*(-)\,,$$

since flasque sheaves are acyclic for $f_*\colon \boldsymbol{\mathcal{Sh}}(X) \to \boldsymbol{\mathcal{Sh}}(Y)$ (cf. [SPA, Lemma 0950]). Hence we convince ourselves that we consider exactly the same functor as Grothendieck.

Proposition 2.1.1 (cf. [Har67, Corollary 1.9]) *If $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$, then we have an exact sequence*

$$0 \longrightarrow \underline{\Gamma_Z}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*j^*\mathscr{F} \longrightarrow \underline{\mathrm{H}^1_Z}(\mathscr{F}) \longrightarrow 0 \tag{2.1.2}$$

and isomorphisms

$$\underline{\mathrm{H}^{k+1}_Z}(\mathscr{F}) \cong \mathbb{R}^k j_*(\mathscr{F}\big|_U) \tag{2.1.3}$$

for $k > 0$. Furthermore, we have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_Z(X;\mathscr{F}) \arrow[r] & \Gamma(X;\mathscr{F}) \arrow[r] & \Gamma(U,j^*\mathscr{F}) \arrow[dlll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\mathrm{H}^1_Z(X;\mathscr{F}) \arrow[r] & \mathrm{H}^1(X;\mathscr{F}) \arrow[r] & \mathrm{H}^1(U;j^*\mathscr{F}) \arrow[r] & \ldots\,.
\end{tikzcd}$$

Proof. Similar to the proof of 1.2.9 – we replace the term “injective object” by “flasque sheaf”, cf. (1.2.2). Indeed, if $\mathscr{E}$ is a flasque sheaf, so is $\underline{\Gamma_Z}(\mathscr{E})$ (this is [Har67, Lemma 1.6]) and we have a short exact sequence

$$0 \longrightarrow \underline{\Gamma_Z}(\mathscr{E}) \longrightarrow \mathscr{E} \longrightarrow j_*j^*\mathscr{E} \longrightarrow 0$$

where the exactness on the right hand side follows from $j_*j^*\mathscr{E}(V) = \mathscr{E}(V \cap U)$ for $V \subset X$ open and $\mathscr{E}$ being flasque. Note that the restriction $j^*\mathscr{E} = \mathscr{E}\big|_U$ of a flasque sheaf $\mathscr{E}$ to an open $U \subset X$ is again flasque. Besides, any flasque sheaf is acyclic for the functor $\underline{\Gamma_Z}(-)$, cf. [Har67, Proposition 1.10]. $\square$

Remark (spectral sequence for $\mathrm{H}^{\bullet}_Z(-)$) Since $\underline{\Gamma_Z}(-)$ preserves flasque sheaves, flasque sheaves are acyclic for $\Gamma(X,-)$ and $\Gamma_Z(X;-)$ is defined by $\Gamma\big(X;\underline{\Gamma_Z}(-)\big)$, we have a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{H}^p\big(X;\underline{\mathrm{H}^q_Z}(-)\big) \Rightarrow E^{p+q} = \mathrm{H}^{p+q}_Z(X;\mathscr{F})\,. \tag{2.1.4}$$

We are mainly interested in quasi-coherent sheaves. To ensure that $\underline{\Gamma_Z}(-)$ and its derived functors preserve quasi-coherence, we need to impose a finiteness condition on the inclusion $U \hookrightarrow X$.

Corollary 2.1.2 ([SGAII, Exposé II, Corollaire 2]) *Let $Z \subset X$ be a closed subscheme such that the inclusion $j\colon U \hookrightarrow X$ is quasi-compact. Then all module sheaves $\underline{\mathrm{H}^k_Z}(\mathscr{F})$, $k \geq 0$, are quasi-coherent.*

Proof. If j is quasi-compact, then $\mathbb{R}^{\bullet}j_*$ preserves quasi-coherence ([SPA, Lemma 01XJ]) and the claim follows from the exact sequence (2.1.2) and the isomorphisms (2.1.3) in Proposition 2.1.1. $\square$

Note that the condition on j is always fulfilled if the scheme X is locally noetherian, cf. [SPA, Lemma 01OX].

Lemma 2.1.3 (flat base change I) *Let $f\colon Y \to X$ be a flat morphism of schemes, $Z \subset X$ a closed subscheme with $j\colon U \hookrightarrow X$ quasi-compact and $\mathscr{F} \in \mathbf{QCoh}(X)$. Then we have a canonical identification*

$$\underline{\Gamma_{f^{-1}(Z)}}(f^*\mathscr{F}) \cong f^*\underline{\Gamma_Z}(\mathscr{F})$$

in $\mathbf{QCoh}(Y)$, where $f^{-1}(Z) = Z \times_X Y$ is the closed subscheme $f^{-1}(Z) \subset Y$ obtained by base change.

Proof. Consider the pullback diagram

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, hook, "j'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
U \arrow[r, hook, "j"] & X
\end{tikzcd}$$

Since f is flat, we can apply the exact functor $f^*\colon \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ to the defining sequence (2.1.1) of $\underline{\Gamma_Z}(\mathscr{F}) \in \mathbf{QCoh}(X)$ and we obtain the exact sequence

$$0 \longrightarrow f^*\underline{\Gamma_Z}(\mathscr{F}) \longrightarrow f^*\mathscr{F} \longrightarrow f^*j_*j^*\mathscr{F}$$

in $\mathbf{QCoh}(Y)$. By usual base change arguments, we can write the sheaf on the very right as

$$f^*j_*j^*\mathscr{F} \cong j'_*j'^*f^*\mathscr{F}.$$

The evident diagram commutes and the identification follows from the universal property of the kernel. □

Note that the last lemma also holds more generally for $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$.

Proposition 2.1.4 (flat base change II) *Let $f\colon Y \to X$ be a flat morphism of schemes, $Z \subset X$ a closed subscheme such that $U \hookrightarrow X$ is quasi-compact and $\mathscr{F} \in \mathbf{QCoh}(X)$. Then there are canonical base change isomorphisms*

$$f^*\underline{\mathrm{H}^k_Z}(\mathscr{F}) \cong \underline{\mathrm{H}^k_{f^{-1}(Z)}}(f^*\mathscr{F})$$

in $\mathbf{QCoh}(Y)$ for every $k \geq 0$.

Proof. Let us again consider the diagram

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, hook, "j'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
U \arrow[r, hook, "j"] & X
\end{tikzcd}$$

with flat f and quasi-compact, quasi-separated j. By [SPA, Lemma 02KH] we have canonical isomorphisms

$$\mathbb{R}^\ell j'_*(f'^*\mathscr{G}) \cong f^*\mathbb{R}^\ell j_*\mathscr{G}$$

for every $\mathscr{G} \in \mathbf{QCoh}(U)$ and $\ell \geq 0$. Applying these isomorphisms to $\mathscr{G} = j^*\mathscr{F}$ for $\mathscr{F} \in \mathbf{QCoh}(X)$ and using $f \circ j' = j \circ f'$, we obtain isomorphisms

$$\mathbb{R}^\ell j'_* j'^*(f^*\mathscr{F}) \cong f^*(\mathbb{R}^\ell j_* j^*\mathscr{F})\,.$$

If $\ell \geq 1$, the left hand side is isomorphic to $\underline{\mathrm{H}^{\ell+1}_{f^{-1}(Z)}}(f^*\mathscr{F})$ and the right hand side is isomorphic to $f^*\underline{\mathrm{H}^{\ell+1}_Z}(\mathscr{F})$ by Proposition 2.1.1, hence the claim for $k \geq 2$.

The remaining cases $k = 0$ and $k = 1$ follow from Lemma 2.1.3 above and the exact sequence (2.1.2) in Proposition 2.1.1. □

2.2 Special case: affine schemes

We now want to restrict ourselves to the case of an affine scheme $X = \mathrm{Spec}(A)$.

Remark The assumption "$U \hookrightarrow X$ is quasi-compact" translates to "the ideal I defining Z is finitely generated" in the affine case $X = \mathrm{Spec}(A)$.

Proposition 2.2.1 ([Har67, Proposition 2.2]) *Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let $Z \subset X$ be a closed subscheme such that $U \hookrightarrow X$ is quasi-compact. If $\mathscr{F} \in \mathbf{QCoh}(X)$ and $k \geq 0$, then $\underline{\mathrm{H}^k_Z}(\mathscr{F})$ is the sheaf associated to the A-module $\mathrm{H}^k_Z(X;\mathscr{F})$, and we have an exact sequence*

$$0 \longrightarrow \Gamma_Z(X;\mathscr{F}) \longrightarrow \Gamma(X;\mathscr{F}) \longrightarrow \Gamma(U;j^*\mathscr{F}) \longrightarrow \mathrm{H}^1_Z(X;\mathscr{F}) \longrightarrow 0$$

and isomorphisms

$$\mathrm{H}^k(U;j^*\mathscr{F}) \cong \mathrm{H}^{k+1}_Z(X;\mathscr{F})$$

for $k > 0$.

Proof. Remember the spectral sequence (2.1.4),

$$\mathrm{H}^p\big(X;\underline{\mathrm{H}^q_Z}(\mathscr{F})\big) \Rightarrow \mathrm{H}^{p+q}_Z(X;\mathscr{F})\,.$$

From Corollary 2.1.2 we know that the $\underline{\mathrm{H}^q_Z}(\mathscr{F})$ are quasi-coherent. Since X is affine, the spectral sequence degenerates and we see that

$$\mathrm{H}^k_Z(X;\mathscr{F}) \cong \Gamma\big(X;\underline{\mathrm{H}^k_Z}(\mathscr{F})\big)$$

which proves the first statement. The remaining two statements follow from the long exact sequence in cohomology. □

Definition 2.2.2 ($\Gamma_I(-)$) If A is a ring, $I \lhd A$ a finitely generated ideal and M an A-module, we define

$$\Gamma_I(M) := \Gamma\big(\mathrm{Spec}(A);\Gamma_{V(I)}(\widetilde{M})\big)\,,$$

where $V(I)$ is the closed subscheme $V(I) \subset \mathrm{Spec}(A)$ defined by I. This gives a left exact functor

$$\Gamma_I(-)\colon A\text{--}\mathbf{Mod} \to A\text{--}\mathbf{Mod}\,.$$

Lemma 2.2.3 *If $M \in A\text{--}\mathbf{Mod}$ and $I \lhd A$ is finitely generated, then*

$$\Gamma_I(M) = \big\{m \in M \,\big|\, \exists k \in \mathbb{N} : I^k m = 0\big\}\,.$$

Proof. Let $u_1, \dots, u_r$ be generators of the ideal $I \lhd A$ and let $j\colon U \hookrightarrow \operatorname{Spec}(A)$ the inclusion of the open complement of $\operatorname{Spec}(A/I)$. Then U can be covered by the open affines $\operatorname{Spec}\bigl(A[u_i^{-1}]\bigr)$, $1 \le i \le n$, and $\Gamma\bigl(\operatorname{Spec}(A); j_* j^* \widetilde{M}\bigr)$ is the kernel of

$$\bigoplus_i M[u_i^{-1}] \to \bigoplus_{i<j} M[u_i^{-1}, u_j^{-1}]\,, \qquad (m_1, \dots, m_r) \mapsto \Bigl(\frac{m_i}{1} - \frac{m_j}{1}\Bigr)_{i<j}\,.$$

One checks that the unit $\eta_{\widetilde{M}}$ corresponds to the morphism

$$M \to \Gamma\bigl(\operatorname{Spec}(A); j_* j^* \widetilde{M}\bigr) \subset \bigoplus_i M[u_i^{-1}]\,, \qquad m \mapsto \Bigl(\frac{m}{1}, \dots, \frac{m}{1}\Bigr)$$

and hence

$$\begin{aligned}\Gamma_I(M) &= \Bigl\{ m \in M \Bigm| m = 0 \text{ in } \bigoplus_i M[u_i^{-1}] \Bigr\} \\ &= \bigl\{ m \in M \bigm| \exists k \in \mathbb{N} : I^k m = 0 \bigr\}\,. \end{aligned}$$ □

Remark If $M \in A\text{-}\mathbf{Mod}$ satisfies $\Gamma_I(M) = M$, then one says that M is *I-torsion.*

We want to calculate the right derived functors of $\Gamma_I(-)$. Since

$$\iota_A\colon \mathbf{QCoh}\bigl(\operatorname{Spec}(A)\bigr) \hookrightarrow \mathbf{Mod}\bigl(\mathscr{O}_{\operatorname{Spec}(A)}\bigr)$$

does not preserve injective objects in general, the resulting A-modules

$$\mathrm{H}^k_I(M) := \Gamma\bigl(\operatorname{Spec}(A), \underline{\mathrm{H}^k_I}(M)\bigr) \qquad \text{and} \qquad \mathbb{R}^k_{A-\mathbf{Mod}}\Gamma_I(M)$$

might be different.

Lemma 2.2.4 ([SGAVI, Exposé II, Appendice I, 0.1], due to Verdier) *There exists an affine scheme $X = \operatorname{Spec}(A)$ and an injective A-module E such that $\widetilde{E}$ is not a flasque $\mathscr{O}_X$-module. In particular, $\widetilde{E}$ is not an injective $\mathscr{O}_X$-module.*

We study the obstruction to the existence of such an A-module in Proposition 2.2.12. An explicit example of an injective A-module E with $\widetilde{\mathscr{E}}$ not flasque is given in [SPA, Section 0273]).

At least if A is noetherian, such an example cannot exist:

Lemma 2.2.5 ([Har66, Corollary II.7.14]) *If A is a noetherian ring, then*

$$\iota_A\colon \mathbf{QCoh}\bigl(\operatorname{Spec}(A)\bigr) \hookrightarrow \mathbf{Mod}\bigl(\mathscr{O}_{\operatorname{Spec}(A)}\bigr)$$

preserves injective objects.

Remark Although it is not explicitly stated in [Har66], the last Lemma also holds more generally for any locally noetherian scheme X. Indeed, let $\mathscr{F} \in \mathbf{QCoh}(X)$ be an injective object in $\mathbf{QCoh}(X)$. By [Har66, Theorem 7.18] there exists a monomorphism $\mathscr{F} \rightarrowtail \mathscr{E}$ in $\mathbf{QCoh}(X)$ such that $\mathscr{E}$ is an injective object in $\mathbf{Mod}(\mathscr{O}_X)$. Since $\mathscr{F}$ is injective, this monomorphism splits (in $\mathbf{QCoh}(X)$) and hence $\mathscr{F}$ is also injective in $\mathbf{Mod}(\mathscr{O}_X)$.

Corollary 2.2.6 *If A is a noetherian ring, the natural morphism*

$$\mathbb{R}^\bullet_{A-\mathbf{Mod}}\Gamma_I(M) \to \mathrm{H}^\bullet_I(M)$$

of δ-functors $A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ is an isomorphism for every $M \in A\text{-}\mathbf{Mod}$.

Instead of imposing a finiteness condition on the ring, we also can put an assumption on the regularity of the ideal $I \lhd A$.

Weakly proregular sequences

Let A be a commutative ring and $\underline{\boldsymbol{u}} = (u_i)_{1\leq i\leq r}$ a system of r elements of A. We write $\underline{\boldsymbol{u}}^m$ for the system $(u_i^m)_{1\leq i\leq r}$ and denote by $(\underline{\boldsymbol{u}})$ the ideal generated by $u_1, \dots, u_r$. For every i we consider the chain complex

$$\begin{tikzcd} \mathrm{K}_\bullet(u_i) : & \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\cdot u_i"] & A \arrow[r] & 0 \arrow[r] & \dots \\ \text{degree} & & 2 & 1 & 0 & -1 & \end{tikzcd}$$

and define

$$\mathrm{K}_\bullet(\underline{\boldsymbol{u}}) := \mathrm{K}_\bullet(u_1) \otimes_A \mathrm{K}_\bullet(u_2) \otimes_A \cdots \otimes_A \mathrm{K}_\bullet(u_r)$$

to be the tensor product of chain complexes. For $M \in A-\mathbf{Mod}$, we define the Koszul chain complex

$$\mathrm{K}_\bullet(\underline{\boldsymbol{u}}; M) := \mathrm{K}_\bullet(\underline{\boldsymbol{u}}) \otimes_A M$$

as well as the Koszul cochain complex

$$\mathrm{K}^\bullet(\underline{\boldsymbol{u}}; M) := \mathrm{Hom}_A\big(\mathrm{K}_\bullet(\underline{\boldsymbol{u}}), M\big)\,.$$

Example 2.2.7 Let $\underline{\boldsymbol{u}} = u_1, u_2$ and $M \in A-\mathbf{Mod}$. Then $K_\bullet(\underline{\boldsymbol{u}}; M)$ is given by the complex

$$\begin{tikzcd} K_\bullet(\underline{\boldsymbol{u}}; M) : & \dots \arrow[r] & 0 \arrow[r] & M \arrow[r, "\begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}"] & M^2 \arrow[r, "(u_1\ u_2)"] & M \arrow[r] & 0 \arrow[r] & \dots \\ \text{degree} & & 3 & 2 & 1 & 0 & -1 & \end{tikzcd}$$

Remark (properties of the Koszul chain complex) Let $\underline{\boldsymbol{u}} = u_1, \dots, u_r$ be a system of elements in A and $M \in A-\mathbf{Mod}$.

(i) $\mathrm{H}_0\big(K_\bullet(\underline{\boldsymbol{u}}; M)\big) = M/\underline{\boldsymbol{u}}M$ by identifying $K_1(\underline{\boldsymbol{u}}; M) \to K_0(\underline{\boldsymbol{u}}; M)$ with

$$M^r \to M, \qquad (m_1, \dots, m_r) \mapsto u_1m_1 + \cdots + u_rm_r\,.$$

(ii) $\mathrm{H}_r\big(K_\bullet(\underline{\boldsymbol{u}}; M)\big) = \big\{m \in M \,\big|\, u_im = 0 \text{ for all } i\big\}$ by identifying $K_r(\underline{\boldsymbol{u}}; M) \to K_{r-1}(\underline{\boldsymbol{u}}; M)$ with

$$M \to M^r, \qquad m \mapsto \big((-1)^{r-1}u_rm, \dots, u_1m\big)\,.$$

(iii) If $u_1, \dots, u_r$ is an M-regular sequence, then $\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}; M)\big) = 0$ for $k \neq 0$ ([SPA, Lemma 062F]). In particular, $K_\bullet(\underline{\boldsymbol{u}}; A)$ is a finite free resolution of A/I if I is generated by a regular sequence $\underline{\boldsymbol{u}}$.

(iv) $\mathrm{H}^k\big(K^\bullet(\underline{\boldsymbol{u}}; M)\big) \cong \mathrm{H}_{r-k}\big(K_\bullet(\underline{\boldsymbol{u}}; M)\big)$ for every $0 \leq k \leq r$.

If $m \leq m'$, then we have a homomorphism of complexes

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "u_i^{m'}"] \arrow[d, "\cdot u_i^{m'-m}"'] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \arrow[r] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & A \arrow[r, "u_i^m"] & A \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

inducing morphisms

$$K_\bullet(\underline{\boldsymbol{u}}^{m'};M) \longrightarrow K_\bullet(\underline{\boldsymbol{u}}^{m};M) \qquad\qquad K^\bullet(\underline{\boldsymbol{u}}^{m};M) \longrightarrow K^\bullet(\underline{\boldsymbol{u}}^{m'};M)\,.$$

With these morphisms, we have inverse resp. direct systems

$$\big\{K_\bullet(\underline{\boldsymbol{u}}^{m};M)\big\}_{m\geq 0} \qquad\qquad \big\{K^\bullet(\underline{\boldsymbol{u}}^{m};M)\big\}_{m\geq 0}$$

of chain resp. cochain complexes. Taking (co-)homology, we obtain inverse resp. direct systems

$$\big\{\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^{m};M)\big)\big\}_{m\geq 0} \qquad\qquad \big\{\mathrm{H}^k\big(K^\bullet(\underline{\boldsymbol{u}}^{m};M)\big)\big\}_{m\geq 0}\,.$$

of A-modules for every $k \geq 0$.

Definition 2.2.8 (Čech complex) We define

$$\check{C}^\bullet_{\underline{\boldsymbol{u}}}(M) := \operatorname*{colim}_m K^\bullet(\underline{\boldsymbol{u}}^m;M)$$

to be the *Čech complex with respect to* $\underline{\boldsymbol{u}}$.

We stick to the term Čech complex rather than Čech cocomplex. This complex is sometimes also called *stable Koszul complex.*

Remark One can show that

$$\check{C}^\bullet_{\underline{\boldsymbol{u}}}(M) \cong \check{C}^\bullet_{u_1}(M) \otimes_A \check{C}^\bullet_{u_2}(M) \otimes_A \cdots \otimes_A \check{C}^\bullet_{u_r}(M)\,,$$

where

$$\begin{array}{lcccccccccc}
\check{C}^\bullet_{u_i}(M) & \qquad \dots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\text{can}} & M[u_i^{-1}] & \longrightarrow & 0 & \longrightarrow \dots \\
\text{degree} & & & -1 & & 0 & & 1 & & 2 &
\end{array}$$

Definition 2.2.9 Following [EGAIII$_1$, (1.1.6)] we set

$$\mathrm{H}^\bullet\big(\underline{\boldsymbol{u}};M\big) := \operatorname*{colim}_{m\in\mathbb{N}} \mathrm{H}^\bullet\big(K^\bullet(\underline{\boldsymbol{u}}^m;M)\big)\,.$$

Since filtered colimits are exact and colimits in $\mathrm{CoCh}\big(A\text{−}\mathbf{Mod}\big)$ are defined degreewise (note that $\mathrm{CoCh}(\boldsymbol{\mathcal{A}})$ is a Grothendieck category if $\boldsymbol{\mathcal{A}}$ is), $\mathrm{H}^\bullet(\underline{\boldsymbol{u}};M)$ is the cohomology of the Čech complex $\check{C}^\bullet_{\underline{\boldsymbol{u}}}(M)$.

The connection to local cohomology is given by

Proposition 2.2.10 ([SGAII, Exposé II, Proposition 5]) *We have an isomorphism*

$$\mathrm{H}^\bullet\big(\underline{\boldsymbol{u}};M\big) \cong \mathrm{H}^\bullet_{V(\underline{\boldsymbol{u}})}\big(\mathrm{Spec}(A);\widetilde{M}\big)$$

of δ-functors A−$\mathbf{Mod} \to A$−$\mathbf{Mod}$, where $V(\underline{\boldsymbol{u}})$ denotes the closed subscheme defined by the ideal $(\underline{\boldsymbol{u}})$ generated of the elements of $\underline{\boldsymbol{u}}$.

Definition 2.2.11 (weakly proregular system) Let $\underline{\boldsymbol{u}}$ be a finite system of elements in A. Then we call the system *weakly proregular* if the inverse system

$$\Big\{\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^m)\big)\Big\}_{m\in\mathbb{N}}$$

is *essentially zero* for every $k > 0$, i.e. if for every $m \geq 0$ there exists an $m' \geq m$ such that the comparison map

$$\rho^{\underline{\boldsymbol{u}}}_{m',m} \colon \mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^{m'})\big) \to \mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^{m})\big)$$

is the zero map. An ideal $I \lhd A$ is a *weakly proregular ideal* if it can be generated by a weakly proregular system.

Remark We use to prefer weakly proregular system over weakly proregular sequence to indicate the independence of the order, cf. Lemma 2.2.13.

Proposition 2.2.12 *The following are equivalent:*

(i) $\underline{\boldsymbol{u}}$ *is a weakly proregular system, i.e.* $\big\{\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^m)\big)\big\}_{m\in\mathbb{N}}$ *is essentially zero for all* $k \neq 0$.
(ii) $\mathrm{H}^k(\underline{\boldsymbol{u}}; E) = 0$ *for any injective* A*-module* E *and any* $k \neq 0$.
(iii) $\big\{\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^m; F)\big)\big\}_{m\in\mathbb{N}}$ *is essentially zero for all* $k \neq 0$ *and each flat* A*-module* F.
(iv) The $\mathrm{H}^k(\underline{\boldsymbol{u}}; -) \colon A-\mathbf{Mod} \to A-\mathbf{Mod}$, $k \geq 0$, *form a universal* δ*-functor.*
(v) For every A*-module* $M \in A-\mathbf{Mod}$ *and every* $k \geq 0$ *we have an isomorphism*

$$\mathbb{R}^k_{A-\mathbf{Mod}}\Gamma_I(M) \cong \mathrm{H}^k(\underline{\boldsymbol{u}}; M)\,.$$

(vi) For every A*-module* $M \in A-\mathbf{Mod}$ *and every* $k \geq 0$ *we have an isomorphism*

$$\mathbb{R}^\bullet_{A-\mathbf{Mod}}\Gamma_I(M) \cong \mathrm{H}^\bullet_I(M)\,.$$

Proof. The equivalence (i)⇔(ii) is shown in [SGAII, Exposé 2, Lemme 9]. (i)⇔(iii) follows from

$$\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}; F)\big) \cong \mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}; A)\big) \otimes_A F$$

for flat $F \in A-\mathbf{Mod}$. The statements (iv) and (v) are equivalent to (ii) since the $\big\{\mathrm{H}^k(\underline{\boldsymbol{u}}; -)\big\}_{k\geq 0}$ define a δ-functor with $\mathrm{H}^0(\underline{\boldsymbol{u}}; M) = \Gamma_I(M)$ for $M \in A-\mathbf{Mod}$. Finally, (vi) is equivalent to (v) by Proposition 2.2.10. □

Remark The existence of an injective A-module E such that $\widetilde{E}$ is not flasque mentioned in Example 2.2.4 uses the equivalence (i)⇔(ii) of the previous proposition. Verdier gives an example of a system $\underline{\boldsymbol{u}} \subset A$ in a ring A such that $\underline{\boldsymbol{u}}$ is not weakly proregular. Hence, there exists an injective A-module E such that $\mathrm{H}^k_{V(\underline{\boldsymbol{u}})}(\widetilde{E}) \neq 0$ for some $k \neq 0$. Since flasque sheaves are acylic for $\Gamma_{(\underline{\boldsymbol{u}})}$, the quasi-coherent sheaf $\widetilde{E}$ cannot be flasque.

Lemma 2.2.13 *Let* $\underline{\boldsymbol{u}} = u_1, \ldots, u_r$ *be a system of elements of* A*. Then the following conditions are equivalent:*

(i) $\underline{\boldsymbol{u}}$ *is a weakly proregular system.*
(ii) There exists an $n > 0$ *such that* $\underline{\boldsymbol{u}}^n = u_1^n, \ldots, u_r^n$ *is a weakly proregular system.*
(iii) For any permutation $\sigma \in S_r$*, the system* $(u_{\sigma(1)}, \ldots u_{\sigma(r)})$ *is weakly proregular.*

Proof. Since the construction of the Koszul complex is symmetric, we have (i)⇔(iii). For (i)⇔(ii) note that

$$\mathrm{H}^k(\underline{\boldsymbol{u}}; M) = \operatorname*{colim}_m \mathrm{H}^k(\underline{\boldsymbol{u}}^m; M) = \operatorname*{colim}_m \mathrm{H}^k(\underline{\boldsymbol{u}}^{mn}; M) = \operatorname*{colim}_m \mathrm{H}^k\big((\underline{\boldsymbol{u}}^n)^m; M)\big) = \mathrm{H}^k\big((\underline{\boldsymbol{u}}^n); M\big)$$

for $n > 0$ and use Proposition 2.2.12, (i)⇔(ii). □

Lemma 2.2.14 (weaky proregular systems and (faithfully) flat base change) *Let* $f\colon A \to B$ *be a flat ring homomorphism and* $\underline{\boldsymbol{u}}$ *a system of elements in* A.

(i) If $\underline{\boldsymbol{u}}$ *is a weakly proregular system in* A*, then* $f(\underline{\boldsymbol{u}})$ *is a weakly proregular system in* B.

(ii) If f *is faithfully flat, then also the converse holds.*

Proof. If f is flat, then $K_\bullet(\underline{\boldsymbol{u}}; A) \otimes B \cong K_\bullet\big(f(\underline{\boldsymbol{u}}), B\big)$. Using flatness again, we see that

$$\mathrm{H}_k\Big(K_\bullet\big(f(\underline{\boldsymbol{u}}); B\big)\Big) \cong \mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}; A)\big) \otimes B$$

and $\rho^{f(\underline{\boldsymbol{u}})}_{m',m} = \rho^{\underline{\boldsymbol{u}}}_{m',m} \otimes \mathrm{id}_B$. □

Proposition 2.2.15 *(i) Every regular sequence* $\underline{\boldsymbol{u}} = u_1, \dots, u_r$ *is weakly proregular.*

(ii) In a noetherian ring A*, every finite sequence* $\underline{\boldsymbol{u}} = u_1, \dots, u_r$ *is weakly proregular.*

Proof. (i) This follows from [Mat86, Theorem 16.5]: We have a short exact sequence

$$0 \longrightarrow K_\bullet(A; u_2, \dots, u_r) \xrightarrow{\cdot u_1} K_\bullet(A; u_2, \dots, u_r) \longrightarrow K_\bullet(A/(u_1), \overline{u_2}, \dots, \overline{u_r}) \longrightarrow 0$$

of complexes. The complex $K_\bullet(A; u_1, \dots, u_r)$ is isomorphic to the cone of the first non-trivial map, hence we obtain a quasi-isomorphism

$$K_\bullet(A/(u_1); \overline{u_2}, \dots, \overline{u_r}) \overset{\text{qiso}}{\sim} K_\bullet(A; u_1, \dots, u_r)\,.$$

Since $\overline{u_2}, \dots, \overline{u_r}$ is a regular sequence in $A/(u_1)$, we can use induction and the long exact sequence in homology to reduce to the case of a single non-zero divisor $u \in A$. But the multiplication with u is injective, hence $\mathrm{H}_k\big(K_\bullet(u)\big) = 0$ for every $k \neq 0$. Since $\underline{\boldsymbol{u}}^m$ is also a regular sequence, $\mathrm{H}_k\big(K_\bullet(\underline{\boldsymbol{u}}^m)\big) = 0$ for every $k \neq 0$.

(ii) A proof is given in [Har67, Lemma 2.5]. □

2.3 Local cohomology sheaves and Ext sheaves

We generalize [Har67, Theorem 2.8], relating local cohomology sheaves to a direct limit of Ext sheaves. We get rid of Grothendieck's noetherian assumption (for completeness always stated as (iii)) by imposing a regularity assumption on the closed subscheme resp. by weakening noetherian to coherent and adding a weakly proregularity assumption.

Let us first show the connection between the sheaf of sections supported on Z and the internal hom in $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$, i.e. $[\mathscr{F}, \mathscr{G}] = \underline{\mathcal{H}om}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ for $\mathscr{F}, \mathscr{G} \in \mathbf{Mod}(\mathscr{O}_X)$. Recall that a closed subscheme $Z \subset X$ corresponds to a quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$ and vice versa.

Lemma 2.3.1 *Let* X *be a scheme and* $Z \subset X$ *a closed subscheme with* $U \hookrightarrow X$ *quasi-compact. If* $\mathscr{F} \in \mathbf{QCoh}(X)$*, then we have an isomorphism*

$$\operatorname*{colim}_n \underline{\mathcal{H}om}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \cong \underline{\Gamma_Z}(\mathscr{F})\,.$$

Proof. We may assume $X = \mathrm{Spec}(A)$ affine, $\mathscr{I} = \widetilde{I}$ and $\mathscr{F} = \widetilde{M}$. Then both sides are given by

$$\big\{m \in M \,\big|\, \exists k \in \mathbb{N} : I^k m = 0\big\}\,.$$

This is clear for the left hand side and for the right hand side this was shown in Lemma 2.2.3. □

Lemma 2.3.2 *Let A be a ring, $I \lhd A$ an ideal and $M \in A-\mathbf{Mod}$ an A-module. Then we have a natural morphism*

$$\mathrm{Ext}^\bullet_A(A/I, M)^\sim \longrightarrow \underline{\mathscr{E}\mathrm{xt}}^\bullet_{\mathscr{O}_{\mathrm{Spec}(A)}}(\widetilde{A/I}, \widetilde{M})$$

of δ-functors $A-\mathbf{Mod} \to \mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(A)})$. It is an isomorphism if

(i) I is generated by a finite regular sequence, or

(ii) A is coherent and I is finitely generated, or

(iii) A is noetherian.

Proof. We use the fact that we can calculate $\underline{\mathscr{E}\mathrm{xt}}^\bullet_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ for $\mathscr{F}, \mathscr{G} \in \mathbf{Mod}(\mathscr{O}_X)$ via locally free resolutions of $\mathscr{F}$. Indeed, if

$$\ldots \longrightarrow \mathscr{L}_1 \longrightarrow \mathscr{L}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

is a resolution of $\mathscr{F}$ by locally free sheaves of finite rank, then

$$\underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathrm{H}^k\big(\underline{\mathscr{H}\mathrm{om}}_{\mathscr{O}_X}(\mathscr{L}_\bullet, \mathscr{G})\big)$$

is a morphism of universal δ-functors $\mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$, hence an isomorphism.

(i) If $I = (u_1, \ldots, u_r) \lhd A$ is regular, then A/I has a finite free resolution of finite length

$$0 \longrightarrow A^{b_r} \longrightarrow \ldots \longrightarrow A^{b_1} \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

with $b_i = \binom{r}{i}$ given by the Koszul complex, cf. the remark on p. 20. Thus we are reduced to show $\underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_{\mathrm{Spec}(A)}}(\widetilde{A}^\ell, M) = 0$ for $k > 0$. Since $\underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_{\mathrm{Spec}(A)}}$ commutes with finite direct sums, it is sufficient to prove that $\underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_{\mathrm{Spec}(A)}}(\widetilde{A}, \widetilde{M}) = 0$ for $k > 0$. But $\underline{\mathscr{H}\mathrm{om}}_{\mathscr{O}_{\mathrm{Spec}(A)}}(\widetilde{A}, \mathscr{F}) = \mathscr{F}$ is an exact functor in $\mathscr{F}$ for any $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$.

(ii) Over a coherent ring A every finitely presented A-module M admits a finite free resolution ([Gla89, Corollary 2.5.2]). Since I is finitely generated, A/I is finitely presented and we can argue as in (i).

(iii) Since every noetherian ring is coherent, this follows from (ii). Let us nevertheless give a another proof using a different technique: Since A/I is finitely presented, we have

$$\mathrm{Hom}_A(A/I, M)^\sim \cong \underline{\mathscr{H}\mathrm{om}}_{\mathscr{O}_{\mathrm{Spec}(A)}}(\widetilde{A/I}, \widetilde{M})\,.$$

Let $I^\bullet$ be an injective resolution of M in $A-\mathbf{Mod}$. Then $\widetilde{I^\bullet}$ is an injective resolution of $\widetilde{M}$ in $\mathbf{Mod}(\mathscr{O}_X)$ by Lemma 2.2.5 and the claim follows as $\widetilde{\cdot}\colon A-\mathbf{Mod} \to \mathbf{QCoh}(\mathrm{Spec}(A))$ is exact. □

Definition 2.3.3 (weakly proregular immersion) We call a closed immersion $Z \hookrightarrow X$

(i) *regular* if Z is locally defined by a regular ideal.

(ii) *weakly proregular* if Z is locally defined by a weakly proregular ideal.

Following [Gar10] we also define

Definition 2.3.4 ((locally) coherent scheme) (i) A scheme X is *locally coherent* if it can be covered by open affine subsets $\mathrm{Spec}(R_i) \subset X$, where each R_i is a coherent ring.

(ii) A scheme X is *coherent* if it is locally coherent, quasi-compact, and quasi-separated.

Analogously to locally noetherian schemes, we can give the following characterization.

Lemma 2.3.5 *For a scheme X the following are equivalent:*

- *(i) The scheme X is locally coherent.*
- *(ii) For every affine open $U = \mathrm{Spec}(A) \subset X$ the ring A is coherent.*
- *(iii) For every $x \in X$ there exists an affine open neighbourhood $U = \mathrm{Spec}(A) \subset X$ such that R is coherent.*
- *(iv) There exists an open covering $X = \bigcup_i X_i$ such that each open subscheme X_i is locally coherent.*

Moreover, if X is locally coherent, then every open subscheme is locally coherent.

Proof. By [SPA, Lemma 01OR] it is enough to check that "coherent" is a local property of rings ([SPA, Definition 01OP]), i.e. we have to check

- (1) If R is coherent and $f \in R$, then R_f is coherent.
- (2) If $f_i \in R$, $1 \leq i \leq r$, is a finite sequence of elements such that $(f_1, \dots, f_r) = 1$ and each R_{f_i} a coherent ring, then R is coherent.

But (1) is proven in [Gla89, Theorem 2.4.2] and (2) follows from [Gla89, Theorem 2.4.3 & Corollary 2.4.5]. Indeed, if each R_{f_i} is a coherent ring, so is $\prod_{i=1}^r R_{f_i}$. Since $R \to \prod R_{f_i}$ is faithfully flat we conclude that R is coherent. □

Theorem 2.3.6 *Let X be a scheme, $Z \subset X$ a closed subscheme with $U \hookrightarrow X$ quasi-compact and $\mathscr{F} \in \mathbf{QCoh}(X)$. Then we have a homomorphism of module sheaves*

$$\mathrm{colim}_n \, \underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \longrightarrow \underline{\mathrm{H}^k_Z}(X; \mathscr{F}),$$

where $\mathscr{I} \subset \mathscr{O}_X$ denotes the quasi-coherent ideal sheaf corresponding to Z.

It is an isomorphism if

- *(i) $Z \hookrightarrow X$ is a regular closed immersion, or*
- *(ii) X is locally coherent and $Z \hookrightarrow X$ is a weakly proregular closed immersion, or*
- *(iii) X is locally noetherian.*

Proof. For every $n \in \mathbb{N}$ and $\mathscr{F} \in \mathbf{QCoh}(X)$, we have morphisms $\underline{\mathscr{H}\mathrm{om}}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \to \underline{\Gamma_Z}(\mathscr{F})$. By the universal properties of derived functor $\mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$ we obtain morphisms

$$\underline{\mathscr{E}\mathrm{xt}}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \longrightarrow \underline{\mathrm{H}^k_Z}(\mathscr{F}).$$

The Ext's form a direct system mapping into $\underline{\mathrm{H}^k_Z}(\mathscr{F})$ and thus we get

$$\mathrm{colim}_n \, \underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \longrightarrow \underline{\mathrm{H}^k_Z}(X; \mathscr{F}).$$

(i) The question is local and we may assume $X = \mathrm{Spec}(A)$, $\mathscr{I} = \widetilde{I}$ and $\mathscr{F} = \widetilde{M}$. We have to show that

$$\mathrm{colim}_n \, \underline{\mathscr{E}\mathrm{xt}}^k_{\mathscr{O}_X}\big((A/I^n)^\sim, \widetilde{M}\big) \xrightarrow{\simeq} \underline{\mathrm{H}^k_{V(I)}}(\widetilde{M}). \tag{2.3.1}$$

We want to rewrite the left hand side via the natural morphism

$$\mathrm{colim}_n \big(\mathrm{Ext}^\bullet_A(A/I^n, M)\big)^\sim \longrightarrow \mathrm{colim}_n \, \underline{\mathscr{E}\mathrm{xt}}^\bullet_{\mathscr{O}_{\mathrm{Spec}(A)}}(\widetilde{A/I^n}, \widetilde{M}) \tag{2.3.2}$$

as a colimit over Ext sheafs. Note that the sheafification functor is a left adjoint and hence preserves colimits. The desired isomorphism does not follow directly from Lemma 2.3.2, since we cannot assume I^n to be generated by a regular sequence in general. We overcome this difficulty by a little trick.

Let $\underline{\boldsymbol{u}} = u_1, \dots, u_r$ be a regular sequence generating I. By [SPA, Lemma 07DV] $\underline{\boldsymbol{u}}^n = u_1^n \dots, u_r^n$ is again a regular sequence for every $n > 0$. Denote by I_n the ideal generated by the sequence $\underline{\boldsymbol{u}}^n$, i.e. $I_m = (u_1^n, \dots, u_r^n)$. Clearly, $I_n \subset I^n$. On the other hand, for every n one can choose $\ell = \ell(n) \in \mathbb{N}$, $\ell \gg 0$, such that $(I^n)^\ell \subset I_n$. Let $\boldsymbol{\mathcal{D}}$ be the category with objects the ideals I^n, I_n and inclusions as morphisms. Then both $\{I^n\}_n$ and $\{I_n\}_n$ are cofinal in $\boldsymbol{\mathcal{D}}$. From the existence of morphisms $\big(\mathrm{Ext}_A^\bullet(A/J, M)\big)^\sim \to \underline{\mathcal{E}\mathrm{xt}}_{\mathcal{O}_{\mathrm{Spec}(A)}}^\bullet(\widetilde{A/J}, \widetilde{M})$ for every $J \in \boldsymbol{\mathcal{D}}$ we conclude that in (2.3.2) we might as well take the colimit over the category $\boldsymbol{\mathcal{D}}$ instead of $\mathbb{N}$. In particular, we see that (2.3.2) is indeed an isomorphism of $\mathcal{O}_X$-modules.

For the right hand side of (2.3.1), one can use Proposition 2.2.1 to show that

$$\underline{\mathrm{H}_{V(I)}^k}(\widetilde{M}) \cong \mathrm{H}_{V(I)}^k\big(\mathrm{Spec}(A); M\big)^\sim .$$

Hence, we are left to show that

$$\mathrm{colim}_n \, \mathrm{Ext}_A^k(A/I^n, M) \longrightarrow \mathrm{H}_Z^k\big(\mathrm{Spec}(A), M\big) \tag{2.3.3}$$

is an isomorphism. But this is a morphism of universal δ-functors $A-\mathbf{Mod} \to A-\mathbf{Mod}$: For $k > 0$ and $M = E$ injective, the left hand side vanishes since the Ext's are derived functors; the right hand side vanishes since any regular sequence is weakly proregular by Proposition 2.2.15.

(ii) We use the same arguments as in (i). Since $U \hookrightarrow X$ is quasi-compact, $\mathcal{I}$ is locally finitely generated. If X is locally coherent, (2.3.2) is an isomorphism by Lemma 2.3.2. Hence, we are again reduced to show that (2.3.3) is an isomorphism of A-modules. Since we assume I to be weakly proregular, this is again an isomorphism of universal δ-functors.

(iii) Follows from the proof of (ii) and the fact that every finite system of elements in a noetherian ring is weakly proregular, cf. Proposition 2.2.15 again. □

There is also a version for local cohomology and Ext *groups*:

Proposition 2.3.7 *Let the situation be as before. Then we have a morphism*

$$\mathrm{colim}_n \, \mathrm{Ext}_{\mathcal{O}_X}^k(\mathcal{O}_X/\mathcal{I}^n, \mathcal{F}) \longrightarrow \mathrm{H}_Z^k(X, \mathcal{F}) .$$

It is an isomorphism if

(i) X is quasi-compact and quasi-separated and $Z \hookrightarrow X$ is a regular closed immersion, or

(ii) X is coherent and $Z \hookrightarrow X$ is a weakly proregular closed immersion, or

(iii) X is noetherian.

Proof. If X is a quasi-compact and quasi-separated scheme and $\mathcal{F} : \boldsymbol{\mathcal{D}} \to \mathbf{Mod}(\mathcal{O}_X)$ is a filtered diagram of module sheaves on X, then we have

$$\operatorname*{colim}_d \mathrm{H}^k(X; \mathcal{F}_d) \xrightarrow{\simeq} \mathrm{H}^k(X; \operatorname*{colim}_d \mathcal{F}_d)$$

by [SPA, Lemma 01FF] and [SPA, Lemma 054D]. In particular, this holds for any noetherian scheme since any locally noetherian scheme is quasi-separated (cf. [SPA, Lemma 01OY]) and for coherent schemes since they are quasi-compact and quasi-separated by definition.

Let us now consider the Grothendieck spectral sequence

$$\mathrm{H}^p\big(X; \underline{\mathscr{E}\mathrm{xt}}^q_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F})\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n)$$

(if $\mathscr{E}$ is an injective $\mathscr{O}_X$-module, then $\underline{\mathscr{H}om}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{E})$ is a flasque sheaf for every $\mathscr{O}_X$-module $\mathscr{G}$, cf. [Har67, Lemma 2.9]) as well as the Grothendieck spectral sequence

$$\mathrm{H}^p\big(X; \underline{\mathrm{H}^q_Z}(\mathscr{F})\big) \Rightarrow \mathrm{H}^{p+q}_Z(X; \mathscr{F})$$

already mentioned in (2.1.4).

From the fact that a filtered colimit of spectral sequences is a spectral sequence and since we have functorial homomorphisms of spectral sequences, we obtain the following diagram

$$\begin{tikzcd}
\operatorname{colim}_n \mathrm{H}^p\big(X; \underline{\mathscr{E}\mathrm{xt}}^q_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F})\big) \arrow[r] \arrow[d, Rightarrow] & \mathrm{H}^p\big(X; \underline{\mathrm{H}^q_Z}(\mathscr{F})\big) \arrow[d, Rightarrow] \\
\operatorname{colim}_n \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}) \arrow[r] & \mathrm{H}^k_Z(X; \mathscr{F})
\end{tikzcd}$$

To show the isomorphism of the abutment terms, it is enough to show that the upper horizontal morphism is an isomorphism. But this follows for both situations from the proposition before and the little remark at the beginning of the proof. □

3 Algebraic stacks and Hopf algebroids

3.1 Short introduction to algebraic stacks

Let S be an affine scheme and denote the category of affine S-schemes by $\mathbf{Aff}_{/S}$. We sometimes simply write $\mathbf{Aff}$ for $\mathbf{Aff}_{/S}$ if S is understood. We equip the category $\mathbf{Aff}_{/S}$ with the *fpqc* topology and consider stacks $\mathfrak{X} \to \mathbf{Aff}_{/S}$.

Remark (*fpqc* topology) A family of morphisms $\{f_i \colon T_i \to T\}_i \in \mathbf{Aff}_{/S}$ is an *fpqc covering* of T if and only if the induced morphism $\coprod_i T_i \to T$ is *fpqc*. A morphism of schemes $X \to Y$ is *fpqc* if it is faithfully flat and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X (*fpqc* is an abbreviation for *fidèlement plat quasi-compact*). For characterizations of *fpqc* covering families $\{f_i \colon T_i \to T\}$ consult [SPA, Lemma 03L7]. The *fpqc* topology is subcanonical, i.e. all representable presheaves are sheaves ([SPA, Lemma 023Q]).

For us, a 1-morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ of stacks is *representable* if for any $U \in \mathbf{Aff}_{/S}$, $U \to \mathfrak{Y}$, the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} U$ is a scheme (one may more generally take an algebraic space). If $\mathcal{P}$ is a property of schemes that is preserved under arbitrary base change and *fpqc* local on the base, then a representable morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ is said to have property $\mathcal{P}$, if the resulting morphism of schemes $f' \colon \mathfrak{X} \times_{\mathfrak{Y}} U \to U$ has property $\mathcal{P}$ for all $U \to \mathfrak{Y}$. For a list of properties being stable under base change and being *fpqc* local on the base, see [SPA, Remark 02WH]. E.g. we can talk about affine, quasi-compact, flat, surjective (hence faithfully flat), (quasi-)separated, open and closed immersions,. . . .

Definition 3.1.1 (algebraic stack) We call a stack $\mathfrak{X} \to \mathbf{Aff}_{/S}$ *algebraic* if

(i) the diagonal morphism $\Delta_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is affine, and

(ii) there exists an affine scheme $X = \operatorname{Spec}(A)$ and a faithfully flat, quasi-compact morphism $P \colon X \to \mathfrak{X}$. We call P a *presentation* of $\mathfrak{X}$.

Example 3.1.2 (i) Any quasi-compact and semi-separated scheme defines an algebraic stack via its functor of points. Recall that a scheme X is called *semi-separated* if its diagonal Δ_X is affine. This is equivalent to the existence of an affine open cover $\{U_i\}_{i \in I}$ of X such that $U_i \cap U_j$ is affine for every pair $(i,j) \in I^2$. Clearly, every separated scheme is semi-separated and semi-separated implies quasi-separated.

(ii) If X is a quasi-compact and semi-separated scheme and G an affine group scheme acting (on the right) on X, then the quotient stack $[X/G]$ is an algebraic stack. Recall that the quotient stack $[X/G]$ has as objects principal G-bundles $E \to T$ together with a G-equivariant morphism $E \to X$.
Indeed, a presentation is given by $\coprod U_i \to X \to [X/G]$, where U_i is a finite affine open cover of X. Moreover, since $X \to [X/G]$ is faithfully flat and being affine is *fpqc* local, it remains to show that the upper arrow in the pullback diagram

$$\begin{tikzcd}
{[X/G] \times_{[X/G]\times[X/G]} X \times X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times X \arrow[d] \\
{[X/G]} \arrow[r, "\Delta_{[X/G]}"] & {[X/G] \times [X/G]}
\end{tikzcd}$$

is affine. But we can identify this morphism with

$$G \times X \to X \times X \qquad (g,x) \mapsto (x, gx)\,,$$

which is the composition of

$$\begin{array}{ccccc} G\times X & \longrightarrow & G\times X\times X & \xrightarrow{\mathrm{pr}} & X\times X \\ (g,x) & \mapsto & (g,x,gx) & & \\ & & (g,x_1,x_2) & \mapsto & (x_1,x_2) \end{array}$$

The first morphism in the composition is given by the pullback diagram

$$\begin{tikzcd} G\times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G\times X\times X \arrow[d, "f"] \\ X \arrow[r, "\Delta_X"] & X\times X \end{tikzcd}$$

with $f(g,x_1,x_2)=(gx_1,x_2)$ and is therefore affine. The second morphism is affine since G is affine.

Hence any affine scheme gives an example of an algebraic stack. Furthermore, we can consider quotients like $[\mathbb{A}^n/\mathbb{G}_m]$, which cannot be represented by a scheme.

Remark The category of algebraic stacks $\mathfrak{X}$ with a fixed presentation $P\colon X\to\mathfrak{X}$ (Naumann calls these stacks *rigidified* in [Nau07]) carries the structure of a 2-category: A 1-morphism f from $P\colon X\to\mathfrak{X}$ to $Q\colon Y\to\mathfrak{Y}$ is a pair $(f_0\colon X\to Y,\ f_1\colon\mathfrak{X}\to\mathfrak{Y})$ consisting of a morphism of affine schemes $f_0\colon X\to Y$ and a 1-morphism of stacks $f_1\colon\mathfrak{X}\to\mathfrak{Y}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "P"'] & Y \arrow[d, "Q"] \\ \mathfrak{X} \arrow[r, "f_1"] & \mathfrak{Y} \end{tikzcd}$$

is 2-commutative; composition is defined componentwise. For the definition of 2-morphisms cf. [SPA, Definition 02XS].

Lemma 3.1.3 *Any morphism from an affine scheme to an algebraic stack is affine.*

Proof. Let $v\colon V\to\mathfrak{X}$ be a morphism with V affine and let $u\colon U\to\mathfrak{X}$ be a test map with U affine. In the pullback diagram

$$\begin{tikzcd} \mathfrak{X}\times_{\mathfrak{X}\times\mathfrak{X}}(U\times_S V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U\times_S V \arrow[d, "{(u,v)}"] \\ \mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}}"] & \mathfrak{X}\times_S\mathfrak{X} \end{tikzcd}$$

the left upper corner is affine since $\Delta_{\mathfrak{X}}$ and $U\times_S V$ are affine. Moreover, it is isomorphic to $U\times_{\mathfrak{X}} V$, cf. [Alo+13, 3.1]. Hence, the induced morphism $v'\colon U\times_{\mathfrak{X}} V\to U$ is affine. □

In particular, the presentation $P\colon X\to\mathfrak{X}$ is an affine morphism.

Lemma 3.1.4 *Any morphism $\mathfrak{X}\to\mathfrak{Y}$ of algebraic stacks is quasi-compact and quasi-separated.*

Proof. Let $f\colon\mathfrak{X}\to\mathfrak{Y}$ be a morphism of algebraic stacks and $U\to\mathfrak{Y}$ any test map from an affine scheme U. Consider the pullback diagrams

$$\begin{tikzcd} X\times_{\mathfrak{Y}} U \arrow[r, "P'"] \arrow[d] & \mathfrak{X}\times_{\mathfrak{Y}} U \arrow[r, "f'"] \arrow[d] & U \arrow[d] \\ X \arrow[r, "P"] & \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

where $P\colon X \to \mathfrak{X}$ is a presentation of $\mathfrak{X}$.

Since X is affine, $X \times_{\mathfrak{Y}} U$ is affine and thus $f' \circ P'$ is affine, hence quasi-compact (and quasi-separated). Since P' is surjective, f' is quasi-compact by [EGAIV$_1$, Proposition 1.1.3].

Since $f' \circ P'$ is quasi-separated and P' is quasi-compact and surjective, f' is quasi-separated by [GW10, Proposition 10.25]. □

The 2-category of categories fibred in groupoids over any Grothendieck site has 2-fibre products ([SPA, Lemma 0041]). Since stackification commutes with 2-fibre products ([SPA, Lemma 04Y1]), the 2-subcategory of stacks is closed under 2-fibre products.

Lemma 3.1.5 ([Alo+13, Proposition 3.3]) *Algebraic stacks are closed under 2-fibre products. I.e. if $\mathfrak{X}$ and $\mathfrak{Y}$ are algebraic stacks over an algebraic stack $\mathfrak{Z}$, then $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is also an algebraic stack.*

Definition 3.1.6 (open and closed substack) An *open substack* $\mathfrak{U}$ of an algebraic stack $\mathfrak{X}$ is a substack such that $\mathfrak{U} \hookrightarrow \mathfrak{X}$ is an open immersion. Similarly, one defines a *closed substack* $\mathfrak{Z} \hookrightarrow \mathfrak{X}$.

Example 3.1.7 Let R be a ring. Then the projective space of dimension n can be defined by setting $\mathbb{P}^n := \left[\mathbb{A}^{n+1} \setminus \{0\}/\mathbb{G}_m\right]$ (where we omit writing R). Since the action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} \setminus \{0\}$ is free, this quotient is representable by a scheme. This does not hold for $[\mathbb{A}^{n+1}/\mathbb{G}_m]$. We can consider $\mathbb{P}^n$ as an open substack of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$; the complement $[\{0\}/\mathbb{G}_m]$ is a closed substack of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Note that a closed substack $\mathfrak{Z}$ of an algebraic stack $\mathfrak{X}$ is automatically algebraic again and that an open substack $\mathfrak{U}$ is algebraic if and only if the inclusion $\mathfrak{U} \hookrightarrow \mathfrak{X}$ is quasi-compact. Indeed, $\mathfrak{Z}$ and $\mathfrak{U}$ have both affine diagonal and $\mathfrak{Z} \times_{\mathfrak{X}} \operatorname{Spec}(A) \to \mathfrak{Z}$ is a presentation of $\mathfrak{Z}$ by an affine scheme since any closed immersion is affine. The scheme $U := \mathfrak{U} \times_{\mathfrak{X}} \operatorname{Spec}(A)$ is not affine, but if j is quasi-compact we can cover U by finitely many affine schemes U_i and obtain a presentation $\coprod_i U_i \to U \to \mathfrak{U}$ by the affine scheme $\coprod_i U_i$.

3.2 The topos $\mathfrak{X}_{\mathrm{fpqc}}$ and quasi-coherent sheaves

Let $S = \operatorname{Spec}(\Bbbk)$ be an affine scheme and $\mathfrak{X}$ an algebraic stack over $\mathbf{Aff}_{/S}$ with the fpqc topology.

Definition 3.2.1 ($\mathfrak{X}_{\mathrm{fpqc}}$) Let $\mathfrak{X}_{\mathrm{fpqc}}$ be the topos associated to the *small* fpqc site of $\mathfrak{X}$, i.e. the site build by

- (objects) flat S-morphisms $t\colon T \to \mathfrak{X}$ with $T \in \mathbf{Aff}_{/S}$.
- (morphisms) 2-commutative triangles over S

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & \Leftarrow & T \arrow[dl] \\ & \mathfrak{X} & \end{tikzcd}$$

- (coverings) A collection of morphisms $\{T_i \to T\}$ is a *covering* if the underlying collection of maps in $\mathbf{Aff}_{/S}$ is an fpqc covering.

Remark (smallness) If $R \neq 0$ is a ring, any faithfully flat ring map $R \to R'$ gives an *fpqc* covering $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$. There does *not* exist a set $\mathcal{S}$ of *fpqc*-coverings of $\operatorname{Spec}(R)$ such that every *fpqc*-covering can be refined by an element of $\mathcal{S}$. This implies that the collection of *fpqc* covers of a scheme X does not admit a cofinal set. Thus, sheaves on the *fpqc*-site do not form a Grothendieck topos. This unpleasant feature of the *fpqc*-topology has the following consequence: there exists a presheaf with no associated sheaf (cf. [Wat75]).

We circumvent this problem by setting an appropriate bound on the generation of the objects in $\mathfrak{X}_{\mathrm{fpqc}}$ in the following way: We choose the cardinality of a (minimal) system of generators of a flat cover $X \to \mathfrak{X}$. Now we take its successor (hence regular) cardinal α and allow only affine schemes over $\mathfrak{X}$ that are at most α-generated. This guarantees us to work with a small site.

The topos $\mathfrak{X}_{\mathrm{fpqc}}$ is ringed with structure sheaf $\mathscr{O}_{\mathfrak{X}}$ given by

$$\mathscr{O}_{\mathfrak{X}}(T \to \mathfrak{X}) := \Gamma(T; \mathscr{O}_T)\,.$$

Definition 3.2.2 ($\mathscr{O}_{\mathfrak{X}}$-module) An $\mathscr{O}_{\mathfrak{X}}$*-module* is an $\mathscr{O}$-module in the category of abelian sheaves on the ringed topos $(\mathfrak{X}_{\mathrm{fpqc}}, \mathscr{O}_{\mathfrak{X}})$. Denote the category of $\mathscr{O}_{\mathfrak{X}}$-modules by $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$.

Remark Almost by definition, one sees that the category $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$ is equivalent to the category of collections of data $\{(\mathscr{F}_{(T,t)}, \varphi)\}$ as follows:

(i) For every flat morphism $t\colon T \to \mathfrak{X}$ in $\mathfrak{X}_{\mathrm{fpqc}}$ we have a sheaf of $\mathscr{O}_T$-modules $\mathscr{F}_{(T,t)}$.

(ii) For every morphism $f\colon (T', t') \to (T, t)$ in $\mathfrak{X}_{\mathrm{fpqc}}$ we have a morphism of $\mathscr{O}_{T'}$-modules

$$\varphi\colon f^*\mathscr{F}_{(T,t)} \to \mathscr{F}_{(T',t')}\,.$$

These morphisms are required to satisfy a cocycle condition for compositions.

If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of algebraic stacks, then there is a functor

$$f_*\colon \mathfrak{X}_{\mathrm{fpqc}} \to \mathfrak{Y}_{\mathrm{fpqc}}$$

sending $\mathscr{F} \in \boldsymbol{Sh}(\mathfrak{X})$ to the sheaf

$$(T \to \mathscr{Y}) \mapsto \Gamma\big((T \times_{\mathfrak{Y}} \mathfrak{X})_{\mathrm{fpqc}}; \mathscr{F}\big)\,.$$

This functor has a left adjoint f^{-1} (see [Alo+13, p. 6] for a definition), but this functor is in general *not* exact. If f^{-1} is exact, then f induces a morphism of ringed topoi (cf. [SGAIV$_1$, Exposé IV, 13.1]) or [SPA, Section 01D2])

$$\begin{tikzcd} \mathfrak{Y}_{\mathrm{fpqc}} \arrow[r, shift left, "f^{-1}"] & \mathfrak{X}_{\mathrm{fpqc}} \arrow[l, shift left, "f_*"] \end{tikzcd}$$

and we obtain an adjunction

$$\begin{tikzcd} \mathbf{Mod}(\mathscr{O}_{\mathfrak{Y}}) \arrow[r, shift left, "f^*"] & \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}) \arrow[l, shift left, "f_*"] \end{tikzcd}$$

by defining

$$f^*\colon \mathbf{Mod}(\mathscr{O}_{\mathfrak{Y}}) \to \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})\,, \qquad \mathscr{F} \mapsto \mathscr{O}_{\mathfrak{X}} \otimes_{f^{-1}\mathscr{O}_{\mathfrak{Y}}} f^{-1}\mathscr{F}\,.$$

If f^{-1} is not exact, then one has to spend a little bit more effort to define an adjoint pair, cf. [Alo+13, Section 6]. Nevertheless, every 1-morphism of algebraic stacks induces an adjunction on the level of module sheaves, compatible with 2-morphisms. For the sake of clarity we often omit writing down the 2-morphisms.

3.3 Connection with Hopf algebroids

Let $\Bbbk$ be a ring.

Definition 3.3.1 (Hopf algebroid) A *Hopf algebroid (over* $\Bbbk$*)* is a cogroupoid object in the category of commutative $\Bbbk$-algebras. A morphism of Hopf algebroids (over $\Bbbk$) corepresents a morphism of groupoids.

Remark Explicitly, a Hopf algebroid is given by a pair (A, Γ) of commutative $\Bbbk$-algebras together with structure morphisms

$$\begin{tikzcd}
\Gamma \arrow[r, "\epsilon"] & A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \Gamma \arrow[loop above, "c"] \arrow[r, "\Delta"] & \Gamma \otimes_A \Gamma
\end{tikzcd}$$

and plenty of commuting diagrams, cf. [Rav04, Definition A1.1.1]. Γ is given the structure of an A-bimodule via s and t. We use the convention to write $\Gamma \otimes -$ to indicate the tensor product $\Gamma_t \otimes -$ with Γ as right-module via t. Hence $\Gamma \otimes_A \Gamma$ should be read as $\Gamma_t \otimes_s \Gamma$. We identify $s\colon A \to \Gamma$ with $A \to \Gamma \otimes A$, $a \mapsto 1 \otimes a$.

Being a groupoid object is the same as corepresenting a groupoid valued functor. In the pair (A, Γ) of $\Bbbk$-algebras A corepresents the objects and Γ corepresents the morphisms. This gives a good insight in the meaning of the structure morphisms and their relations: The identity morphism is corepresented by ϵ, the source (resp. target) is corepresented by s (resp. t), composition is corepresented by Δ and forming the inverse (every morphism is invertible in a groupoid by definition) is corepresented by c. E.g. the source of the inverse of an isomorphism is the target of the original one, and vice versa. Hence one relation is given by $c \circ s = t$ resp. $c \circ t = s$.

A morphism $(A, \Gamma) \to (B, \Sigma)$ of Hopf algebroids is a pair of $\Bbbk$-algebra morphisms ($A \to B$, $\Gamma \to \Sigma$) respecting the structure.

Definition 3.3.2 (flat Hopf algebroid) A Hopf algebroid (A, Γ) is *flat* if s is a flat ring morphism.

Since $c^2 = \mathrm{id}_\Gamma$ (inverting twice gives the identity) and $c \circ s = t$ this assertion is equivalent to t being flat.

Example 3.3.3 (i) Every $\Bbbk$-Algebra A defines a trivial Hopf algebroid (A, A).

(ii) If $\Bbbk = A$ and $s = t$, then we obtain a Hopf algebra.

(iii) If $A \to B$ is a morphism of $\Bbbk$-algebras, we can form the Hopf algebroid $(B, B \otimes_A B)$.

(iv) If (A, Γ) is a Hopf algebroid and $A \to B$ is a morphism of $\Bbbk$-algebras, then we can form the Hopf algebroid

$$(B, B \otimes_A \Gamma \otimes_A B)\,.$$

Even if we start with a flat Hopf algebroid then the induced Hopf algebroid $(B, B \otimes \Gamma \otimes B)$ may not be flat. Sufficient for flatness of $(B, B \otimes \Gamma \otimes B)$ is *Landweber exactness* of B over A (cf. [HS05a, Corollary 2.3]). Recall that B is called Landweber exact over (A, Γ) if the functor $\Gamma-\mathbf{Comod} \to B-\mathbf{Mod}$, $M \mapsto B \otimes_A M$, is exact.

(v) (Hopf algebroids from ring spectra) If E is a commutative ring spectrum such that $E_*E = \pi_*(E \wedge E)$ is flat over $E_* = \pi_*(E)$, then (E_*, E_*E) is a Hopf algebroid (over $\pi_0(E)$). The condition "E_*E is flat over E_*" is known to be satisfied in many important cases like MU, BP, … and ensures that we have an isomorphism

$$E_*E \otimes_{E_*} E_*(X) \xrightarrow{\simeq} \pi_*(E \wedge E \wedge X)$$

for every spectrum X. This isomorphism is needed to define $\Delta\colon \Gamma \to \Gamma \otimes \Gamma$.

Remark (i) If (A, Γ) is a flat Hopf algebroid, then Γ is a faithfully flat A-module since $\epsilon \circ s = \epsilon \circ t = \mathrm{id}_A$.

(ii) The category of flat Hopf algebroids carries the structure of a 2-category, cf. [Nau07, 3.1].

If $\mathrm{Spec}(A) \to \mathfrak{X}$ is a presentation of an algebraic stack $\mathfrak{X}$, we can form the (2-categorical) pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}(A) \times_{\mathfrak{X}} \mathrm{Spec}(A) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}(A) \arrow[d, "P"] \\
\mathrm{Spec}(A) \arrow[r, "P"] & \mathfrak{X}
\end{tikzcd}$$

and $\mathrm{Spec}(A) \times_{\mathfrak{X}} \mathrm{Spec}(A) \cong \mathrm{Spec}(\Gamma)$ is affine by Lemma 3.1.3 and the pair (A, Γ) is a Hopf algebroid. On the other hand, any flat Hopf algebroid (A, Γ) defines a prestack and $\mathfrak{X} = \big[(A, \Gamma)\big]$ is the stackification of (A, Γ). One checks that $\mathfrak{X}$ is algebraic (cf. [Nau07, 3.3] for details).

Theorem 3.3.4 ([Nau07, Theorem 8]) *We have an equivalence of 2-categories*

$$\big\{\textit{algebraic stacks with fixed presentation } \mathrm{Spec}(A) \xrightarrow{P} \mathfrak{X}\big\} \rightleftarrows \big\{\textit{flat Hopf algebroids } (A, \Gamma)\big\}\,.$$

3.4 The categories of quasi-coherent sheaves and comodules

3.4.1 Quasi-coherent sheaves on algebraic stacks

Definition 3.4.1 (quasi-coherent sheaf) An $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ is said to be *quasi-coherent* if it has a local presentation. I.e. for every object $T \in \mathfrak{X}_{\mathrm{fpqc}}$ there exists a covering $\{T_i \to T\}_i$ such that each restriction $\mathscr{F}\big|_{\mathfrak{X}/T_i}$ has a presentation

$$\mathscr{O}_{T_i}^{(J)} \longrightarrow \mathscr{O}_{T_i}^{(J')} \longrightarrow \mathscr{F}\big|_{\mathfrak{X}/T_i} \longrightarrow 0\,.$$

Here $\mathfrak{X}/T_i$ denotes the slice category $\mathfrak{X}_{\mathrm{fpqc}_{/T_i}}$ over T_i. Denote the full subcategory of $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$ spanned by quasi-coherent modules by $\mathbf{QCoh}(\mathfrak{X})$.

Remark (quasi-coherent and cartesian sheaves) One calls a module sheaf $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$ *cartesian* if every $\mathscr{F}_{T,t}$ is a quasi-coherent sheaf on T and if every transition morphism $\varphi\colon f^*\mathscr{F}_{T,t} \to \mathscr{F}_{T',t'}$ is an isomorphism of quasi-coherent sheaves on T' for all possible choices of $f, T, T' \in \mathfrak{X}_{\mathrm{fpqc}}$. Using mild modifications of the arguments in [Alo+13] (they work in the *fppf* category) one can show that the category of cartesian $\mathscr{O}_{\mathfrak{X}}$-modules is equivalent to the category of quasi-coherent sheaves.

We have the natural inclusion

$$\iota_{\mathfrak{X}}\colon \mathbf{QCoh}(\mathfrak{X}) \hookrightarrow \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})\,.$$

Lemma 3.4.2 ([Ols11, Lemma C.5]) *The essential image of $\iota_{\mathfrak{X}}$ is closed under kernels, cokernels, and extensions.*

3.4.2 Comodules on Hopf algebroids

Let (A, Γ) be a flat Hopf algebroid.

Definition 3.4.3 (Γ-comodule) A (left) Γ-comodule is a (left) A-module M together with a map $\psi_M \colon M \to \Gamma \otimes M$ of (left) A-modules which is counitary and coassociative, i.e. the diagrams

$$\begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[dr, "\simeq"'] & \Gamma \otimes M \arrow[d, "\epsilon \otimes \mathrm{id}_M"] \\ & A \otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[d, "\psi_M"] & \Gamma \otimes M \arrow[d, "\mathrm{id}_\Gamma \otimes \psi_M"] \\ \Gamma \otimes M \arrow[r, "\Delta \otimes \mathrm{id}_M"] & \Gamma \otimes \Gamma \otimes M \end{tikzcd}$$

commute. A morphism of comodules $f \colon M \to N$ is a morphism of A-modules such that

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\psi_M"] & N \arrow[d, "\psi_N"] \\ \Gamma \otimes M \arrow[r, "\mathrm{id}_\Gamma \otimes f"] & \Gamma \otimes N \end{tikzcd}$$

commutes. Denote the category of Γ-comodules by $\Gamma-$**Comod**. Let

$$U \colon \Gamma-\mathbf{Comod} \longrightarrow A-\mathbf{Mod}\,, \qquad (M, \psi_M) \mapsto M$$

be the forgetful functor from $\Gamma-$**Comod** to $A-$**Mod**. We write $\mathrm{Hom}_\Gamma(-,-)$ for the set of morphisms.

Reminder Analogously, there is the notion of a right Γ-comodule.

Example 3.4.4 (i) A itself is a (left) Γ-comodule via $\psi_A = s$.

(ii) Γ is a Γ-comodule via $\psi_\Gamma = \Delta$.

(iii) If E is a ring spectrum as in example 3.3.3 (v), then E_*X is a comodule over (E_*, E_*E) for every X; the structure morphism is induced by

$$\mathrm{id}_E \wedge \eta \wedge \mathrm{id}_X \colon E \wedge X = E \wedge S^0 \wedge X \to E \wedge E \wedge X\,.$$

Remark Since Γ is supposed to be flat over A, the category $\Gamma-$**Comod** is abelian (cf. [Rav04, Theorem A.1.1.3]) The forgetful functor $U \colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ is exact, faithful and admits a right adjoint

$$\Gamma \otimes - \colon A-\mathbf{Mod} \to \Gamma-\mathbf{Comod}\,, \qquad M \mapsto \big(\Gamma \otimes M, \psi_{\Gamma \otimes M}\big)\,,$$

where $\psi_{\Gamma \otimes M} = \Delta \otimes \mathrm{id}_M$. If $M \in A-\mathbf{Mod}$, we call $(\Gamma \otimes M, \psi_{\Gamma \otimes M})$ the *cofree comodule*. One immediately concludes that $\Gamma-$**Comod** is cocomplete and has finite limits. The functor $\Gamma \otimes -$ also admits a right adjoint, cf. [Hov04, p. 5].

Lemma 3.4.5 *In the category* $\Gamma-$**Comod** *of* Γ*-comodules, finite limits commute with filtered colimits. In other words, filtered colimits are exact in* $\Gamma-$**Comod**.

Proof. The forgetful functor $U \colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ preserves filtered limits, since it is left adjoint to the cofree comodule functor $\Gamma \otimes -$. Moreover, it preserves finite limits since it is exact. The canonical comparison morphism

$$\operatorname*{colim}_{\mathrm{filt}}{}^{\Gamma} \lim_{\mathrm{fin}}{}^{\Gamma} \to \lim_{\mathrm{fin}}{}^{\Gamma} \operatorname*{colim}_{\mathrm{filt}}{}^{\Gamma}$$

is an isomorphism after applying U, since filtered colimits commute with finite limits in $A-$**Mod**. Now U is faithful and exact, hence conservative, i.e. it reflects isomorphisms. □

The following observation is rather obvious.

Lemma 3.4.6 *If M is a comodule and $N \subset M$ a submodule, then N carries at most one comodule structure such that N is a subcomodule of M.*

Proof. Let ψ, ψ' two structure morphisms of N such that $i\colon N \rightarrowtail M$ is a morphism of comodules. Then $(\mathrm{id}_\Gamma \otimes i)(\psi - \psi') = 0$ and $\mathrm{id}_\Gamma \otimes i$ is a monomorphism of A-modules, hence $\psi = \psi'$. □

The next proposition gives a new proof of the known statement that the category of Γ-comodules over a flat Hopf algebroid is Grothendieck, cf. [Fra96, Proposition 3.3.1] or [Alo+13, Theorem 5.3] for different proofs.

Proposition 3.4.7 *The category $\Gamma-$**Comod** of Γ-comodules is Grothendieck.*

Proof. We have already seen that $\Gamma-$**Comod** is cocomplete and that filtered colimits are exact. It remains to show the existence of a generating object.

We have an adjoint pair of functors

$$U\colon \Gamma-\mathbf{Comod} \rightleftarrows A-\mathbf{Mod} : \Gamma \otimes - \,.$$

where $A-$**Mod** is Grothendieck, $\Gamma-$**Comod** is cocomplete abelian, U is faithful exact and $\Gamma \otimes -$ preserves all (co-)limits. The preservation of colimits follows from the fact that colimits commute with tensor products. Let $M \in \Gamma-$**Comod** be an arbitrary comodule. Then $U(M)$ can be written as a filtered colimit over finitely presentable A-modules, i.e.

$$U(M) \cong \operatorname*{colim}_{d \in \boldsymbol{\mathcal{D}}} N_d$$

for a filtered category $\boldsymbol{\mathcal{D}}$ and a diagram $N\colon \boldsymbol{\mathcal{D}} \to A-\mathbf{Mod}_{\mathrm{fp}}$, cf. Theorem 1.1.7. Note that the collection of all finitely presentable A-modules is essentially small by [AR04, Remark 1.9 (2)]. Applying $\Gamma \otimes -$ and using the unit of the adjunction $U \dashv (\Gamma \otimes -)$, we form the pullback diagram

$$\begin{tikzcd} M_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \otimes N_d \arrow[d] \\ M \arrow[r, tail] & \Gamma \otimes UM \end{tikzcd}$$

in $\Gamma-$**Comod**. Recall that monomorphisms are stable under pullback, hence $M_d \rightarrowtail \Gamma \otimes N_d$. Since filtered colimits commute with pullbacks by Lemma 3.4.5 we see that

$$\operatorname*{colim}_{d \in \boldsymbol{\mathcal{D}}} M_d \cong M$$

in the category $\Gamma-$**Comod**. For fixed $i \in \boldsymbol{\mathcal{I}}$, the collection of all subobjects of $\Gamma \otimes N_d$ is small. This follows from the valid statement for the underlying A-module and Lemma 3.4.6. Hence, we have an essentially small collection of objects forming a colimit-dense generating set ([Shu, Definition 3.5]). Since any colimit-dense generating set is a generating set, we are done. Indeed, by the description of a colimit as cokernel between coproducts indexed over objects and morphisms of the diagram category, we see that for any comodule M we have an epimorphism from a coproduct of objects of the colimit dense generating set. □

Every submodule of a Γ-comodule has a biggest subcomodule in the following sense.

Lemma 3.4.8 *Let $M \in \Gamma-$**Comod** and let $M' \leq U(M)$ be a submodule of $U(M)$. Define*

$$N = \left\{ m \in M \,\middle|\, \psi_M(m) \in \Gamma \otimes M' \right\}.$$

Then N is a subcomodule of M with $N \leq U(M')$. If $\widetilde{N}$ is another subcomodule of M with $\widetilde{N} \leq U(M)$, then $\widetilde{N} \leq N$.

Proof. We mimic the proof given in [Ser68, Proposition 1] for comodules over Hopf *algebras.*

Let $i\colon M' \rightarrowtail M$ denote the inclusion of M' into M. By definition, N is given by the pullback diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \otimes M' \arrow[d, "\mathrm{id}_\Gamma \otimes i"] \\ M \arrow[r, tail, "\psi_M"] & \Gamma \otimes M \end{tikzcd}$$

in A-modules. Since all modules (resp. morphisms) involved are comodules (resp. morphisms of comodules), we see that N can be given the structure of a comodule and $N \rightarrowtail M$ turns N into a subcomodule of M.

As ψ_M is counitary (i.e. $(\epsilon \otimes \mathrm{id}_M) \circ \psi_M = \mathrm{id}_M$), N is contained in $(\epsilon \otimes \mathrm{id}_M)(\Gamma \otimes M')$ and hence in M'.

If $\widetilde{N}$ is another subcomodule of M contained in M', then we have $\psi_M(\widetilde{N}) \subset \Gamma \otimes \tilde{N} \subset \Gamma \otimes M'$ and hence $\widetilde{N} \subset N$. $\square$

Remark Since $\Gamma-$**Comod** has pullbacks, every subset S of a comodule $M \in \Gamma-$**Comod** sits in a smallest subcomodule of M, the Γ-*comodule generated by S*. Indeed, the set

$$\left\{ N \in \Gamma-\mathbf{Comod} \,\middle|\, N \leq U(M) \text{ is a submodule and } S \subset N \right\}$$

is partially ordered, non-empty and every chain has a lower bound. Then this set contains a minimal element by Zorn's lemma.

Lemma 3.4.9 *Let $M \in \Gamma-$**Comod** and $M' \leq U(M)$ a finitely generated A-submodule. Then there exists a subcomodule $\widetilde{N} \leq M$ such that $M' \leq U(\widetilde{N})$ and $U(\widetilde{N})$ is finitely generated as A-module.*

Proof. It is enough to show the claim for $M' = Am$. Indeed, if $M' = \sum_{j=1}^{s} Am_j$, then $\sum_{j=1}^{s} \widetilde{N_j}$ is again a subcomodule of M by Lemma 3.4.6 and the underlying A-module of $\sum_{j=1}^{s} \widetilde{N_j}$ is still finitely generated.

Thus let $M' = Am$ for an element $m \in M$. Define $\widetilde{N}$ to be the Γ-subcomodule generated by $\{m\}$. W have to show that $\widetilde{N}$ is finitely generated as A-module.

Let us write $\Gamma \otimes M \ni \psi(m) = \sum_{i=1}^{r} \gamma_i \otimes n_i$ with $n_i \in \widetilde{N}$ and $\gamma_i \in \Gamma$. Let $N' = \sum_{i=1}^{r} An_i$ be the A-module generated by the n_i. We claim that $N' = \widetilde{N}$.

Using the same arguments as in the proof we can construct a Γ-comodule E such that $m \in E$ and $E \subset N'$ by defining $E := \psi^{-1}(\Gamma \otimes N')$. By minimality of $\widetilde{N}$ we see that $\widetilde{N} \subset E$ and therefore $\widetilde{N} \subset N'$.

On the other hand, we also have $N' \subset \widetilde{N}$: since $\widetilde{N}$ is supposed to be a Γ-comodule, we have $\psi(\widetilde{N}) \subset \Gamma \otimes \widetilde{N}$ and hence $n_i \in \widetilde{N}$ for each $i = 1, \ldots, n$. $\square$

Let us now investigate the categorical property of a Γ-comodule M to be "finitely presentable" resp. "finitely generated" resp. "coherent", cf. Definition 1.1.3. We adapt the proof for the case "finitely presentable" treated in [Hov04, Proposition 1.3.3.].

Lemma 3.4.10 *A Γ-comodule M is finitely presentable (resp. finitely generated) if and only if the underlying A-module $U(M)$ is finitely presentable (resp. finitely generated). A Γ-comodule is coherent if $U(M)$ is coherent.*

Proof. If $M \in \Gamma-$**Comod** is finitely presentable (resp. finitely generated), then $U(M) \in A-$**Mod** is finitely presented (resp. finitely generated) by Lemma 1.1.10 (ii). Let us prove the converse. To do so, let $M \in \Gamma-$**Comod** such that $U(M)$ is finitely presented (resp. finitely generated) and $D\colon \boldsymbol{\mathcal{D}} \to \Gamma-\mathbf{Comod}$ a filtered diagram (of monomorphisms for the finitely generated case). The same argument as in the proof of Lemma 1.1.10 (i) shows that the canonical morphism

$$\underset{d}{\operatorname{colim}} \operatorname{Hom}_\Gamma(M, D_d) \longrightarrow \operatorname{Hom}_\Gamma(M, \underset{d}{\operatorname{colim}} D_d)$$

is injective in both cases. We have to show that this map is also surjective. For this, let $f \in \operatorname{Hom}_\Gamma\big(M, \operatorname{colim}_d D_d\big)$. Applying the forgetful functor $U\colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ yields a factorization

$$\begin{tikzcd}
UM \arrow[rr, "Uf"] \arrow[dr, "g"'] & & \underset{d}{\operatorname{colim}} UD_d \\
& UD_k \arrow[ur, "\text{can}"'] &
\end{tikzcd}$$

since UM is supposed to be finitely presented (resp. finitely generated) in $A-\mathbf{Mod}$. In general, g is not a map of comodules since the left square of the diagram

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "\psi_M"'] & D_k \arrow[r, "\text{can}"] \arrow[d, "\psi_{D_k}"'] \arrow[dr, phantom, "\circlearrowleft"] & \underset{d}{\operatorname{colim}} D_d \arrow[d, "\psi_{\operatorname{colim}_d D_d}"] \\
\Gamma \otimes M \arrow[r, "\text{id}\otimes g"] & \Gamma \otimes D_k \arrow[r, "\text{id}\otimes\text{can}"] & \Gamma \otimes \underset{d}{\operatorname{colim}} D_d
\end{tikzcd}$$

may not commute. Yet, we have

$$(\text{id} \otimes \text{can}) \circ \psi_{D_k} \circ g = (\text{id} \otimes \text{can}) \circ (\text{id} \otimes g) \circ \psi_M$$

in $A-\mathbf{Mod}$. Since the target can be written as $\operatorname{colim}_d \Gamma \otimes D_d$ we have a factorization

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, "g'"'] & & \underset{d}{\operatorname{colim}} \Gamma \otimes D_d \\
& \Gamma \otimes D_\ell \arrow[ur, "\text{can}"'] &
\end{tikzcd}$$

in $A-\mathbf{Mod}$ and we get a commuting diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\psi_M"'] \arrow[dr, phantom, "\circlearrowleft"] & D_\ell \arrow[r, "\text{can}"] \arrow[d, "\psi_{D_\ell}"'] \arrow[dr, phantom, "\circlearrowleft"] & \underset{d}{\operatorname{colim}} D_d \arrow[d] \\
\Gamma \otimes M \arrow[r] & \Gamma \otimes D_\ell \arrow[r, "\text{id}\otimes\text{can}"] & \Gamma \otimes \underset{d}{\operatorname{colim}} D_d
\end{tikzcd}$$

as desired.

If M is a Γ-comodule such that $U(M)$ is a coherent A-module, then M is finitely presentable. Every subobject $M' \leq M$ gives a subobject $U(M') \leq U(M)$. If M' is finitely generated, so is $U(M')$ and since $U(M)$ is supposed to be coherent, $U(M')$ is finitely presentable. Hence M' is finitely presentable. $\square$

Corollary 3.4.11 *The category* $\Gamma-\mathbf{Comod}$ *is locally finitely generated for any flat Hopf algebroid* (A, Γ).

We end this section with the definition of an invariant ideal.

Definition 3.4.12 (invariant ideal) Let (A, Γ) be a Hopf algebroid. An ideal $I \lhd A$ is said to be *invariant* if I is a Γ-subcomodule of A.

Lemma 3.4.13 *Let (A, Γ) be a Hopf algebroid and $I \lhd A$ an ideal. Then the following are equivalent:*

(i) I is invariant,
(ii) $s(I) \subset t(I)\Gamma$,
(iii) $\big(A/I, \Gamma/s(I)\big)$ is a Hopf algebroid, and
*(iv) A/I is a quotient in the category $\Gamma-$**Comod**.*

Proof. Omitted. These are basically just restatements of the definition of an invariant ideal; one has to pay attention with the left resp. right module structures. □

Definition 3.4.14 (primitive element) An element $m \in M$ of a Γ-comodule M is called *primitive* if $\psi_M(m) = 1 \otimes m$.

Remark If $M \in \Gamma-\mathbf{Comod}$, then

$$\begin{aligned} \mathrm{Hom}_\Gamma(A, M) &\cong \big\{m \in M \,\big|\, \psi_M(m) = 1 \otimes m\big\} \\ \mathrm{Hom}_\Gamma(A, A) &\cong \big\{a \in A \,\big|\, s(a) = t(a)\big\}\,. \end{aligned}$$

In particular, primitive elements in A are invariant.

3.4.3 Equivalence of quasi-coherent sheaves and comodules

Proposition 3.4.15 ([Nau07, Section 3.4]) *We have an equivalence of categories*

$$\mathbf{QCoh}(\mathfrak{X}) \rightleftarrows \Gamma-\mathbf{Comod}\,.$$

The proof relies on the observation that the presentation $P\colon \mathrm{Spec}(A) \to \mathfrak{X}$ is affine and induces an equivalence

$$\mathbf{QCoh}(\mathfrak{X}) \rightleftarrows \big\{\mathscr{F} \in \mathbf{QCoh}\big(\mathrm{Spec}(A)\big) \text{ plus descent data}\big\}\,.$$

Descent data on a quasi-coherent sheaf $\mathscr{F} \in \mathbf{QCoh}\big(\mathrm{Spec}(A)\big)$ is given by an isomorphism $s^*\mathscr{F} \to t^*\mathscr{F}$ in $\mathbf{QCoh}\big(\mathrm{Spec}(\Gamma)\big)$ satisfying a cocycle condition.

Remark Under this equivalence, we have the following identifications:

(i) The morphisms in the adjunction

$$P^*\colon \mathbf{QCoh}(\mathfrak{X}) \rightleftarrows \mathbf{QCoh}\big(\mathrm{Spec}(A)\big) : P_*$$

correspond to the forgetful functor $U\colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ resp. to the cofree comodule functor $\Gamma \otimes -\colon A-\mathbf{Mod} \to \Gamma-\mathbf{Comod}$.

(ii) The structure sheaf $\mathscr{O}_\mathfrak{X}$ corresponds to the trivial Γ-comodule A.

(iii) The functor $\mathrm{Hom}_\Gamma(A, -)\colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ corresponds to

$$\mathrm{Hom}_{\mathbf{QCoh}(\mathfrak{X})}(\mathscr{O}_\mathfrak{X}, -) = \mathrm{Hom}_{\mathbf{Mod}(\mathscr{O}_\mathfrak{X})}(\mathscr{O}_\mathfrak{X}, -) = \mathrm{H}^0(\mathfrak{X}, -)\,.$$

In particular, we may interpret the primitive elements of a comodule M as its global sections.

Corollary 3.4.16 *Let* $\operatorname{Spec}(A) \to \mathfrak{X}$ *be an algebraic stack corresponding to a flat Hopf algebroid* (A, Γ). *We have a 1-1 correspondence between*

(i) closed substacks $\mathfrak{Z} \subset \mathfrak{X}$,

(ii) invariant ideals $I \lhd A$.

Proof. Let $i\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ be a closed immersion. Just like in the scheme case, $\mathfrak{Z}$ defines a quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_{\mathfrak{X}}$ and vice versa. The corresponding comodule I is then a subcomodule of A, hence invariant by Lemma 3.4.13. □

Corollary 3.4.17 *A quasi-coherent sheaf* $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ *is finitely presentable (resp. finitely generated) if and only if* $\mathscr{F}_{T,t}$ *is a finitely presentable (resp. finitely generated) object in* $\mathbf{QCoh}(T)$ *for every* $t\colon T \to \mathfrak{X}$ *in* $\mathfrak{X}_{\text{fpqc}}$.

Proof. If $\mathscr{F}_{T,t}$ is finitely presentable (resp. finitely generated) for all $t\colon T \to \mathfrak{X}$, then in particular for the presentation $P\colon \operatorname{Spec}(A) \to \mathfrak{X}$ and the claim follows from Lemma 3.4.10.

On the other hand, assume $\mathscr{F}_{\operatorname{Spec}(A),P}$ is a finitely presented (resp. finitely generated) object in $\mathbf{QCoh}\big(\operatorname{Spec}(A)\big)$.

Let $t\colon T = \operatorname{Spec}(B) \to \mathfrak{X}$ be an object in $\mathfrak{X}_{\text{fpqc}}$ and form the pullback diagram

$$\begin{tikzcd} \operatorname{Spec}(C) \arrow[r, "t'"] \arrow[d, "P'"] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(A) \arrow[d, "P"] \\ \operatorname{Spec}(B) \arrow[r, "t"] & \mathfrak{X} \end{tikzcd}$$

Then $\mathscr{F}\big(\operatorname{Spec}(C) \to \mathfrak{X}\big)$ is finitely presentable (resp. finitely generated) and the claim follows from P' being faithfully flat,

$$C \otimes \mathscr{F}\big(\operatorname{Spec}(B) \to \mathfrak{X}\big) \cong \mathscr{F}\big(\operatorname{Spec}(C) \to \mathfrak{X}\big)\,. \qquad \square$$

and [SPA, Lemma 03C4].

3.5 The closed symmetric monoidal structure

We want to give the closed monoidal structure on $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$.

Via quasi-coherent sheaves

Lemma 3.5.1 *The category* $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$ *is closed symmetric monoidal with respect to the tensor product* $- \otimes_{\mathscr{O}_{\mathfrak{X}}} -$ *and the internal hom functor* $\underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(-,-)$.

Proof. We briefly recall the construction of tensor product and the internal hom for ringed sites, cf. [SPA, Section 03EK] resp. [SPA, Section 04TT]. If $\mathscr{F}, \mathscr{G} \in \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$, then $\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}$ is defined to be the sheafification of the rule

$$\big(T \to \mathfrak{X} \text{ in } \mathfrak{X}_{\text{fpqc}}\big) \longmapsto \mathscr{F}(T) \otimes_{\mathscr{O}(T)} \mathscr{G}(T)\,.$$

The internal hom of two module sheaves $\mathscr{F}, \mathscr{G}$ is defined via the rule

$$\big(T \to \mathfrak{X} \text{ in } \mathfrak{X}_{\text{fpqc}}\big) \longmapsto \operatorname{Hom}_{\mathscr{O}_T}\big(\mathscr{F}\big|_T, \mathscr{G}\big|_T\big)$$

and is already a sheaf. If $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ are $\mathscr{O}_{\mathfrak{X}}$-modules, then there is a canonical isomorphism

$$\underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}\big(\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}, \mathscr{H}\big) \xrightarrow{\simeq} \underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}\big(\mathscr{F}, \underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{G}, \mathscr{H})\big),$$

functorial in all three entries. In particular,

$$\mathrm{Hom}_{\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})}\big(\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}, \mathscr{H}\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})}\big(\mathscr{F}, \underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{G}, \mathscr{H})\big). \qquad \square$$

If $\mathscr{F}, \mathscr{G} \in \mathbf{QCoh}(\mathfrak{X})$, then the module sheaf $\underline{\mathcal{H}om}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ is not quasi-coherent in general, cf. the scheme case. We fix this by composing with the coherator $Q\colon \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}) \to \mathbf{QCoh}(\mathfrak{X})$, which is right adjoint to the inclusion of quasi-coherent sheaves in module sheaves.

Let us first recall the coherator for affine resp. quasi-compact and semi-separated schemes ([TT90, B.14]).

Reminder (the coherator for affine schemes) If $X = \mathrm{Spec}(A)$ is an affine scheme, the inclusion $\mathbf{QCoh}(X) \hookrightarrow \mathbf{Mod}(\mathscr{O}_X)$ has a right adjoint Q_A given by

$$Q_A(\mathscr{G}) = \Gamma(X, \mathscr{G})^{\sim}.$$

The unit $\eta\colon \mathrm{id}_{\mathbf{QCoh}(X)} \to Q_A \circ \iota$ is an isomorphism and the counit $\epsilon\colon \iota \circ Q_A \to \mathrm{id}_{\mathbf{Mod}(\mathscr{O}_X)}$ gives a morphism

$$\iota\Gamma(X, \mathscr{G})^{\sim} \to \mathscr{G} \tag{3.5.1}$$

for every $\mathscr{G} \in \mathbf{Mod}(\mathscr{O}_X)$. From the counit-unit equation, ι being fully faithful and η being an isomorphism, we see that (3.5.1) is an isomorphism if $\mathscr{G}$ is quasi-coherent.

Remark (the coherator for quasi-compact and semi-separated schemes) Let X be a quasi-compact and semi-separated scheme with affine open semi-separating cover $\{U_i\}_{i \in I}$ and I finite. By definition, each $U_i \times_X U_k = U_i \cap U_k$ is affine. Writing $U_{ik} := U_i \cap U_k$ for short and $j_i\colon U_i \hookrightarrow X$, $j_{ik}\colon U_{ik} \hookrightarrow X$ for the inclusions, we see that the sheaf axioms give an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_{i \in I} j_{i*}\big(\mathscr{F}|_{U_i}\big) \rightrightarrows \bigoplus_{i,k \in I^2} j_{ik*}\big(\mathscr{F}|_{U_{ik}}\big) \tag{3.5.2}$$

for every $\mathscr{F} \in \mathbf{Mod}(\mathscr{O}_X)$. Hence, one has to define $Q_X(\mathscr{F})$ by exactness of

$$0 \longrightarrow Q_X(\mathscr{F}) \longrightarrow \bigoplus_i j_{i*}\big(Q_{U_i}(\mathscr{F}|_{U_i})\big) \rightrightarrows \bigoplus_{i,k} j_{ik*}\big(Q_{U_{ik}}(\mathscr{F}|_{U_{ik}})\big),$$

where $Q_\bullet$ is the coherator for the particular affine scheme.

Note that the construction generalizes to quasi-compact and quasi-separated schemes by taking tripe intersetions $U_i \cap U_k \cap U_\ell$.

If X is quasi-compact and semi-separated, the (1-categorical) pullback diagram

$$\begin{tikzcd}
\coprod_{i,k \in I^2} U_i \times_X U_k \arrow[r, "t"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner"] & \coprod_{i \in I} U_i \arrow[d, "P"] \\
\coprod_{i \in I} U_i \arrow[r, "P"] & X
\end{tikzcd}$$

gives a presentation of X by a flat Hopf algebroid and we can rewrite (3.5.2) as

$$0 \longrightarrow \mathscr{F} \longrightarrow P_*P^*\mathscr{F} \rightrightarrows r_*r^*\mathscr{F}$$

where $r = P \circ s = P \circ t$. The construction of the coherator for X given above motivates

Theorem 3.5.2 (the coherator for algebraic stacks, cf. [Ols11, Lemma C.6]) *The inclusion*

$$\iota_{\mathfrak{X}} \colon \mathbf{QCoh}(\mathfrak{X}) \hookrightarrow \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$$

of the full subcategory of quasi-coherent sheaves on $\mathfrak{X}$ *has a right adjoint* Q*. The unit of this adjunction,* $\eta\colon \mathrm{id}_{\mathbf{QCoh}(\mathfrak{X})} \to Q_{\mathfrak{X}} \circ \iota_{\mathfrak{X}}$*, is an isomorphism turning* $\mathbf{QCoh}(\mathfrak{X})$ *into a coreflective subcategory of* $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$*.*

Proof. As usual, our situation looks like

$$\begin{tikzcd}
\mathrm{Spec}(\Gamma) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[rr, bend right, "r"'] & \mathrm{Spec}(A) \arrow[r, "P"] & \mathfrak{X} .
\end{tikzcd}$$

If $\mathscr{G} \in \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$, consider the diagram

$$\begin{tikzcd}
P_*P^*\mathscr{G} \arrow[r] & (P\circ s)_*(P\circ s)^*\mathscr{G} \\
P_*\Gamma\big(\mathrm{Spec}(A), P^*\mathscr{G}\big)^{\sim} \arrow[u] \arrow[r] & (P\circ s)_*\Gamma\big(\mathrm{Spec}(\Gamma), (P\circ s)^*\mathscr{G}\big)^{\sim} . \arrow[u]
\end{tikzcd}$$

The vertical arrows are given by applying P_* to (3.5.1) for the sheaf $P^*\mathscr{G} \in \mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(A)})$. For the horizontal ones, note that

$$\begin{aligned}
s_*\Gamma\big(\mathrm{Spec}(\Gamma), (P\circ s)^*\mathscr{G}\big)^{\sim} &\cong \Gamma\big(\mathrm{Spec}(A), s_*(P\circ s)^*\mathscr{G}\big)^{\sim} \\
&\cong \Gamma\big(\mathrm{Spec}(A), s_*s^*(P^*\mathscr{G})\big)^{\sim} .
\end{aligned}$$

Then the horizontal arrows are induced by the unit of $s^* \dashv s_*$ applied to $P^*\mathscr{G} \in \mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(A)})$.

Mimicking this with t instead of s and writing $r = P\circ s \Rightarrow P\circ t$ as indicated above, we get commutative diagrams of the form

$$\begin{tikzcd}
P_*P^*\mathscr{G} \arrow[r, shift left] \arrow[r, shift right] & r_*r^*\mathscr{G} \\
P_*\Gamma\big(\mathrm{Spec}(A), P^*\mathscr{G}\big)^{\sim} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & r_*\Gamma\big(\mathrm{Spec}(\Gamma), r^*\mathscr{G}\big)^{\sim} . \arrow[u]
\end{tikzcd}$$

Since $\mathscr{G}$ is supposed to be a sheaf, the equalizer of the upper arrows is given by $\mathscr{G}$. We define the equalizer of the lower arrows to be $Q(\mathscr{G})$ together with the induced morphism $Q(\mathscr{G}) \to \mathscr{G}$. Since $Q(\mathscr{G})$ is an equalizer of morphisms of quasi-coherent sheaves, we have $Q(\mathscr{G}) \in \mathbf{QCoh}(\mathfrak{X})$ and sometimes emphasize this by writing $\iota Q(\mathscr{G}) \to \mathscr{G}$.

Putting everything together, we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{G} \arrow[r] & P_*P^*\mathscr{G} \arrow[r, shift left] \arrow[r, shift right] & r_*r^*\mathscr{G} \\
0 \arrow[r] & Q(\mathscr{G}) \arrow[u] \arrow[r] & P_*\Gamma\big(\mathrm{Spec}(A), P^*\mathscr{G}\big)^{\sim} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & r_*\Gamma\big(\mathrm{Spec}(\Gamma), r^*\mathscr{G}\big)^{\sim} \arrow[u]
\end{tikzcd}$$

We want to prove that $\iota \dashv Q$. Note that the first vertical morphism,

$$\epsilon\colon \iota\circ Q \longrightarrow \mathrm{id}_{\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})} ,$$

is a reasonable candidate for the counit of the adjunction. Let us seek for a possible unit, i.e. a natural transformation

$$\eta\colon \mathrm{id}_{\mathbf{QCoh}(\mathfrak{X})} \longrightarrow Q \circ \iota$$

satisfying certain properties. To do so, let $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ be quasi-coherent. Then $P^*\mathscr{F} \in \mathbf{QCoh}\big(\mathrm{Spec}(A)\big)$, $r^*\mathscr{F} \in \mathbf{QCoh}\big(\mathrm{Spec}(\Gamma)\big)$ and we obtain an isomorphism $\iota Q(\iota\mathscr{F}) \xrightarrow{\sim} \iota\mathscr{F}$ coming from an isomorphism $Q\iota\mathscr{F} \xrightarrow{\sim} \mathscr{F}$ in $\mathbf{QCoh}(\mathfrak{X})$. Let $\eta_{\mathscr{F}} : \mathscr{F} \to Q\iota\mathscr{F}$ be its inverse.

It remains to show that ϵ and η satisfy the counit-unit equations, i.e.

$$\mathrm{id}_{\iota\mathscr{F}} = \epsilon_{\iota\mathscr{F}} \circ \iota(\eta_{\mathscr{F}}) \qquad \text{and} \qquad \mathrm{id}_{Q\mathscr{G}} = Q(\epsilon_{\mathscr{G}}) \circ \eta_{Q\mathscr{G}} \, .$$

If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$, then

$$\begin{tikzcd} \iota\mathscr{F} \arrow[r, "\iota(\eta_{\mathscr{F}})", "\simeq"'] \arrow[rr, bend right, "\mathrm{id}_{\iota\mathscr{F}}"'] & \iota Q\iota\mathscr{F} \arrow[r, "\epsilon_{\iota\mathscr{F}}"] & \iota\mathscr{F} \end{tikzcd}$$

commutes, since by definition $\iota(\eta_{\mathscr{F}}) = (\epsilon_{\iota\mathscr{F}})^{-1}$.

On the other hand, $\eta_{Q\mathscr{G}} = \epsilon^{-1}_{\iota Q\mathscr{G}}$ and $Q(\epsilon_{\mathscr{G}}) = \epsilon_{\iota Q\mathscr{G}}$. Thus, the diagram

$$\begin{tikzcd} Q\mathscr{G} \arrow[r, "\eta_{Q\mathscr{G}}"] \arrow[rr, bend right, "\mathrm{id}_{Q\mathscr{G}}"'] & Q\iota Q\mathscr{G} \arrow[r, "Q(\epsilon_{\mathscr{G}})"] & Q\mathscr{G} \end{tikzcd}$$

commutes and the pair (η, ϵ) defines an adjunction

$$\iota\colon \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}) \rightleftarrows \mathbf{QCoh}(\mathfrak{X}) : Q$$

as desired.

If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$, then we have already seen that $\eta_{\mathscr{F}} : \mathscr{F} \to Q\iota\mathscr{F}$ is an isomorphism. Therefore $\mathbf{QCoh}(\mathfrak{X})$ is a coreflective subcategory of $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$. □

Corollary 3.5.3 *The essential image of ι is closed under colimits.*

Corollary 3.5.4 *The category $\mathbf{QCoh}(\mathfrak{X})$ is closed symmetric monoidal with respect to the tensor product $- \otimes_{\mathscr{O}_{\mathfrak{X}}} -$ of $\mathscr{O}_{\mathfrak{X}}$-modules and the internal hom functor $Q\underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(-,-)$.*

Proof. It is a standard procedure to prove that the tensor product of two quasi-coherent module sheaves is again quasi-coherent, cf. [Goe, Lemma 6.2].

Moreover, we have adjunctions

$$\begin{aligned} \mathrm{Hom}_{\mathbf{QCoh}(\mathfrak{X})}(\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}, \mathscr{H}) &\cong \mathrm{Hom}_{\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})}(\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G}, \mathscr{H}) \\ &\cong \mathrm{Hom}_{\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})}\big(\mathscr{F}, \underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{G}, \mathscr{H})\big) \\ &\cong \mathrm{Hom}_{\mathbf{QCoh}(\mathfrak{X})}\big(\mathscr{F}, Q\underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{G}, \mathscr{H})\big) \, . \end{aligned}$$

for $\mathscr{F}, \mathscr{G}, \mathscr{H} \in \mathbf{QCoh}(\mathfrak{X})$. □

Via comodules

Throughout this section (A, Γ) denotes a flat Hopf algebroid.

Lemma 3.5.5 *The symmetric monoidal structure $\wedge$ in $\Gamma-$**Comod** is given as follows: If $M, N \in \Gamma-$**Comod**, then the usual tensor product $M \otimes_A N$ is given a structure map via the composition*

$$M \otimes_A N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes_A M \otimes_A \Gamma \otimes_A N \xrightarrow{\text{switch}} \Gamma \otimes_A \Gamma \otimes_A M \otimes_A N \xrightarrow{\text{mult}} \Gamma \otimes_A M \otimes_A N\,.$$

This comodule is denoted $M \wedge N$.

Proof. Straightforward. ☐

Clearly, $U(M \wedge N) = M \otimes_A N$.

Lemma 3.5.6 ([Hov04, Lemma 1.1.5]) *Let $M \in A-$**Mod** and $N \in \Gamma-$**Comod**. Then we have a natural isomorphism*

$$(\Gamma \otimes M) \wedge N \xrightarrow{\simeq} \Gamma \otimes \big(M \otimes U(N)\big)$$

of Γ-comodules.

We want to characterize the internal hom $\underline{\hom}_\Gamma(M, N)$ of comodules M, N. Let us first consider the case that N is a cofree comodule.

Lemma 3.5.7 *Let $M \in \Gamma-$**Comod** and $N \in A-$**Mod**. Then*

$$\underline{\hom}_\Gamma(M, \Gamma \otimes N) = \Gamma \otimes \underline{\hom}_A(M, N)$$

Proof. If $P \in \Gamma-$**Comod**, then we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_\Gamma(P \wedge M, \Gamma \otimes N) &\cong \mathrm{Hom}_A\big(U(P \wedge M), N\big) \\ &\cong \mathrm{Hom}_A\big(U(P), \underline{\hom}_A(U(M), N)\big) \\ &\cong \mathrm{Hom}_\Gamma\big(P, \Gamma \otimes \underline{\hom}_A(U(M), N)\big)\,. \end{aligned}$$ ☐

For the general case, we use the following useful observation due to Hovey, [Hov04, p. 6].

Lemma 3.5.8 *Any comodule can be written as the kernel of a map of cofree comodules.*

Proof. Let $M \in \Gamma-$**Comod**. Then $\psi\colon M \rightarrowtail \Gamma \otimes M$ is a map of comodules, where we consider $\Gamma \otimes M$ as cofree comodule on M since

$$\begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[d, "\psi_M"] & \Gamma \otimes M \arrow[d, "\Delta \otimes \mathrm{id}_M"] \\ \Gamma \otimes M \arrow[r, "\mathrm{id}_\Gamma \otimes \psi_M"] & \Gamma \otimes \Gamma \otimes M \end{tikzcd}$$

commutes by definition. Let $M' := \mathrm{coker}(\psi)$ be the cokernel of ψ in $\Gamma-$**Comod**. As $M' \rightarrowtail \Gamma \otimes M'$, we see that

$$M \cong \ker\big(\Gamma \otimes M \overset{\text{can}}{\to} M' \overset{\psi_{M'}}{\rightarrowtail} \Gamma \otimes M'\big)$$

and all morphisms are morphisms of comodules. Note that the composition $\psi_{M'} \circ \text{can}$ is in general not a morphism of cofree comodules. ☐

The last two lemmas give the internal hom in Γ-comodules. We want to compare this with the internal hom in A-modules.

Proposition 3.5.9 *Let $M, N \in \Gamma-\mathbf{Comod}$. Then we have a natural comparison map*

$$U\big(\underline{\hom}_\Gamma(M,N)\big) \longrightarrow \underline{\hom}_A\big(U(M),U(N)\big)$$

of A-modules. It is

- *(i) injective if M is finitely generated as A-module.*
- *(ii) an isomorphism if M is finitely presented as A-module.*

Proof. Consider the natural diagram in $A-\mathbf{Mod}$:

$$\begin{tikzcd}
0 \arrow[r] & \underline{\hom}_\Gamma(M,N) \arrow[r] & \Gamma \otimes \underline{\hom}_A(M,N) \arrow[r] \arrow[d] & \Gamma \otimes \underline{\hom}_A(M,N') \arrow[d] \\
0 \arrow[r] & \underline{\hom}_A(M,N) \arrow[r] & \underline{\hom}_A(M,\Gamma \otimes N) \arrow[r] & \underline{\hom}_A(M,\Gamma \otimes N')\ ,
\end{tikzcd}$$

where N' is given as in the proof of Lemma 3.5.8. According to Hovey, a careful diagram chase shows that it is commutative and we get a natural induced map $\underline{\hom}_\Gamma(M,N) \to \underline{\hom}_A(M,N)$.

Since Γ is a flat A-module, we can write $\Gamma = \operatorname{colim}_d \Gamma_d$ over a filtered diagram with Γ_d finite free by Lazard's theorem ([SPA, Theorem 058G]). Thus, the natural map $\Gamma_d \otimes \underline{\hom}_A(M,N) \to \underline{\hom}_A(M,\Gamma_d \otimes N)$ is an isomorphism and so is the upper row in

$$\begin{tikzcd}
\Gamma \otimes \underline{\hom}_A(M,N) \arrow[r, "\simeq"] & \operatorname*{colim}_d \underline{\hom}_A(M,\Gamma_d \otimes N) \arrow[d, "\varphi"] & \\
& \underline{\hom}_A\left(M, \operatorname*{colim}_d \Gamma_d \otimes N\right) \arrow[r, "\simeq"] & \underline{\hom}_A(M,\Gamma \otimes N)\ .
\end{tikzcd}$$

Now the vertical morphism φ is a monomorphim (resp. isomorphism) if M is finitely generated (resp. finitely presented) as A-module, cf. Proposition 1.1.8. The claim follows. □

Lemma 3.5.10 *A Γ-comodule M is flat as a Γ-comodule if and only if it is flat as an A-module.*

Proof. Let M be a flat comodule and $0 \to N' \to N \to N'' \to 0$ a short exact sequence of A-modules. From the exactness of the cofree comodule functor $\Gamma \otimes - \colon A-\mathbf{Mod} \to \Gamma-\mathbf{Comod}$, Lemma 3.5.6 and the exactness of the forgetful functor $U \colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ we see that the sequence

$$0 \longrightarrow \Gamma \otimes N' \otimes M \longrightarrow \Gamma \otimes N \otimes M \longrightarrow \Gamma \otimes N'' \otimes M \longrightarrow 0$$

is exact in $A-\mathbf{Mod}$ and thus M is a flat A-module since Γ is faithfully flat over A.

The other direction follows from U being exact and faithful. Thus the forgetful functor $U \colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$ reflects exact sequences. □

Corollary 3.5.11 *The following are equivalent for a Γ-comodule M:*

- *(i) M is dualizable in $\Gamma-\mathbf{Comod}$.*
- *(ii) M is finitely presentable and flat in $\Gamma-\mathbf{Comod}$.*
- *(iii) $U(M)$ is dualizable in $A-\mathbf{Mod}$.*
- *(iv) $U(M)$ is finitely presentable and flat in $A-\mathbf{Mod}$.*
- *(v) $U(M)$ is finitely generated and projective in $A-\mathbf{Mod}$.*

Proof. (i) ⇔ (v) is shown in [Hov04, Proposition 1.3.4], (v) ⇔ (iv) ⇔ (iii) is Example 1.3.5 and (ii) ⇔ (iv) follows from Lemma 3.5.10 and Lemma 3.4.10. □

Products in $\Gamma-$Comod

Since the category $\Gamma-$**Comod** is Grothendieck, it is complete. The forgetful functor

$$U\colon \Gamma-\mathbf{Comod} \to A-\mathbf{Mod}$$

does not preserve products in general. It may be tempting to think that the product $\prod^{\Gamma} M_i$ of an arbitrary family $\{M_i\}_i$ of comodules is the biggest subcomodule of $\prod^{A} U(M_i)$. This is made precise in the following

Lemma 3.5.12 *Let (A,Γ) be flat Hopf algebroid and $\big\{(M_i,\psi_{M_i})\big\}_{i\in I}$ a family of Γ-comodules. Assume further that the canonical morphisms*

$$\mathrm{can}\colon \Gamma\otimes\prod_i{}^{A} M_i \to \prod_i{}^{A}\Gamma\otimes M_i\,,\qquad \gamma\otimes(m_i)_i\mapsto(\gamma\otimes m_i)_i$$

and

$$\widetilde{\mathrm{can}}\colon \Gamma\otimes\prod_i{}^{A}(\Gamma\otimes M_i) \to \prod_i{}^{A}\Gamma\otimes\Gamma\otimes M_i\,,\qquad \gamma\otimes(\gamma_i\otimes m_i)_i\mapsto(\gamma\otimes\gamma_i\otimes m_i)_i$$

*induced by the universal property of the product in $A-$**Mod** are monomorphisms. Then the product $\prod_i^{\Gamma} M_i$ in the category $\Gamma-$**Comod** is given by*

$$\prod_i{}^{\Gamma} M_i = \Big\{(m_i)_i\in\prod_i{}^{A} M_i\,\Big|\,\prod_i{}^{A}\psi_{M_i}(m_i)\in\Gamma\otimes\prod_i{}^{A} M_i\Big\}$$

with structure morphism $\psi_{\prod_i^{\Gamma}}$ given by restriction of $\prod_i^{A}\psi_{M_i}$.

Proof. We omit writing the index i and the superscript A to indicate the product in $A-$**Mod**. Let us consider the pullback diagram

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma\otimes\prod M_i \arrow[d, tail, "\mathrm{can}"] \\
\prod\limits_i M_i \arrow[r, tail, "\prod\psi_{M_i}"] & \prod\limits_i\Gamma\otimes M_i
\end{tikzcd}\tag{3.5.3}$$

in $A-$**Mod**. Since monomorphisms are stable under pullback, we consider P via j as a submodule of $\prod_i M_i$ and p as the restriction of $\prod\psi_{M_i}$ to P. Then P is the A-module given in the formulation of the lemma.

`Step 1:` The morphism p factors through $\Gamma\otimes P$.

Consider the diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "(\Delta\otimes\mathrm{id}_M)\circ p"] \arrow[dr, bend right, "p"'] & & \\
& \Gamma\otimes P \arrow[r, "\mathrm{id}_\Gamma\otimes p"] \arrow[d, "\mathrm{id}_\Gamma\otimes j"'] & \Gamma\otimes\Gamma\otimes\prod\limits_i M_i \arrow[d, "\mathrm{id}_\Gamma\otimes\mathrm{can}"] \\
& \Gamma\otimes\prod\limits_i M_i \arrow[r, "\mathrm{id}_\Gamma\otimes\prod\psi_{M_i}"] & \Gamma\otimes\prod\limits_i\Gamma\otimes M_i
\end{tikzcd}$$

If the two compositions

$$(\dagger) \qquad P \xrightarrow{p} \Gamma \otimes \prod_i M_i \xrightarrow{\Delta \otimes \mathrm{id}_{M_i}} \Gamma \otimes \Gamma \otimes \prod_i \xrightarrow{\mathrm{id}_\Gamma \otimes \mathrm{can}} \Gamma \otimes \prod_i \Gamma \otimes M_i$$

and

$$(\ddagger) \qquad P \xrightarrow{p} \Gamma \otimes \prod_i M_i \xrightarrow{\mathrm{id}_\Gamma \otimes \prod \psi_{M_i}} \Gamma \otimes \prod_i \Gamma \otimes M_i$$

are equal, then by the universal property of the pullback $\Gamma \otimes P$ (note the $\Gamma \otimes -$ is an exact functor and hence preserves finite limits like pullbacks) we obtain a morphism $\rho\colon P \to \Gamma \otimes P$ such that

$$(3.5.4) \qquad \begin{tikzcd} P \arrow[r, dashed, "\rho"] \arrow[dr, "p"'] & \Gamma \otimes P \arrow[d, "\mathrm{id}_\Gamma \otimes j"] \\ & \Gamma \otimes \prod_i M_i \end{tikzcd}$$

commutes. This is the desired factorization.

To show the equality of (†) and (‡) we further compose with the monomorphism

$$\Gamma \otimes \prod_i \Gamma \otimes M_i \xrightarrow{\widetilde{\mathrm{can}}} \prod_i \Gamma \otimes \Gamma \otimes M_i$$

to obtain

$$\begin{tikzcd} & \Gamma \otimes P \arrow[r, "\mathrm{id}_\Gamma \otimes p"] \arrow[d, "\mathrm{id}_\Gamma \otimes j"'] & \Gamma \otimes \Gamma \otimes \prod_i M_i \arrow[d, "\mathrm{id}_\Gamma \otimes \mathrm{can}"] \\ P \arrow[r, "p"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \otimes \prod_i M_i \arrow[ur, "\Delta \otimes \mathrm{id}_{\prod M_i}"] \arrow[r, "\mathrm{id}_\Gamma \otimes \prod \psi_{M_i}"] \arrow[d, tail, "\mathrm{can}"] & \Gamma \otimes \prod_i \Gamma \otimes M_i \arrow[d, "\widetilde{\mathrm{can}}"] \\ \prod_i M_i \arrow[r, tail, "\prod \psi_{M_i}"] & \prod_i \Gamma \otimes M_i \arrow[r, shift left, "\prod \mathrm{id}_\Gamma \otimes \psi_{M_i}"] \arrow[r, shift right, "\prod \Delta \otimes \mathrm{id}_{M_i}"'] & \prod_i \Gamma \otimes \Gamma \otimes M_i \end{tikzcd}$$

On simple tensors one checks that

$$(3.5.5) \qquad \widetilde{\mathrm{can}} \circ (\mathrm{id}_\Gamma \otimes \prod \psi_{M_i}) = \big(\prod \mathrm{id}_\Gamma \otimes \psi_{M_i}\big) \circ \mathrm{can}$$

and

$$(3.5.6) \qquad \widetilde{\mathrm{can}} \circ (\mathrm{id}_\Gamma \otimes \mathrm{can}) \circ (\Delta \otimes \mathrm{id}_{\prod M_i}) = \prod (\Delta \otimes \mathrm{id}_{M_i}) \circ \mathrm{can}\,.$$

Then

$$\begin{aligned} \widetilde{\mathrm{can}} \circ (\mathrm{id}_\Gamma \otimes \prod \psi_{M_i}) \circ p &\overset{(3.5.5)}{=} \prod (\mathrm{id}_\Gamma \otimes \psi_{M_i}) \circ \mathrm{can} \circ p \overset{(3.5.3)}{=} \prod (\mathrm{id}_\Gamma \otimes \psi_{M_i}) \circ \prod \psi_{M_i} \circ j \\ &= \prod \big((\mathrm{id}_\Gamma \otimes \psi_{M_i}) \circ \psi_{M_i}\big) \circ j = \prod \big((\Delta \otimes \mathrm{id}_{M_i}) \circ \psi_{M_i}\big) \circ j \\ &= \prod (\Delta \otimes \mathrm{id}_{M_i}) \circ \prod \psi_{M_i} \circ j \overset{(3.5.3)}{=} \prod (\Delta \otimes \mathrm{id}_{M_i}) \circ \mathrm{can} \circ p \\ &\overset{(3.5.6)}{=} \widetilde{\mathrm{can}} \circ (\mathrm{id}_\Gamma \otimes \mathrm{can}) \circ (\Delta \otimes \mathrm{id}_{\prod M_i}) \circ p \end{aligned}$$

and we have shown the desired equality of the compositions (†) and (‡).

`Step 2:` The morphism $\rho\colon P \to \Gamma \otimes P$ defines the structure of a Γ-comodule on P.

ρ is counitary: We have to show that $(\epsilon \otimes \mathrm{id}_P) \circ \rho = \mathrm{id}_P$. To do so, let us compose both sides with the monomorphism j as indicated in

$$\begin{tikzcd}
P \arrow[r, "\rho"] \arrow[d, "j"'] & \Gamma \otimes P \arrow[r, "\epsilon\otimes\mathrm{id}_P"] \arrow[d, "\mathrm{can}\circ(\mathrm{id}_\Gamma\otimes j)"] & P \arrow[d, tail, "j"] \\
\prod M_i \arrow[r, "\prod \psi_{M_i}"] \arrow[rr, bend right, "\prod \mathrm{id}_{M_i}"'] & \prod\limits_i \Gamma \otimes M_i \arrow[r, "\prod \epsilon\otimes\mathrm{id}_{M_i}"] & \prod\limits_i M_i
\end{tikzcd}$$

The left square commutes since

$$\mathrm{can} \circ \mathrm{id}_\Gamma \otimes j \circ \rho \overset{(3.5.4)}{=} \mathrm{can} \circ p \overset{(3.5.3)}{=} \prod \psi_{M_i} \circ j$$

and the commutativity of the right square can be checked on simple tensors. The claim follows from $\prod \mathrm{id}_{M_i} \circ j = j \circ \mathrm{id}_P$.

ψ_M is coassociative: We have to show that $(\Delta \otimes \mathrm{id}_P) \circ \rho = (\mathrm{id}_\Gamma \otimes \rho) \circ \rho$. Consider the diagram

$$\begin{tikzcd}
P \arrow[r, "\rho"] \arrow[d, "j"'] & \Gamma \otimes P \arrow[r, shift left, "\mathrm{id}_\Gamma\otimes\rho"] \arrow[r, shift right, "\Delta\otimes\mathrm{id}_P"'] \arrow[d] & \Gamma \otimes \Gamma \otimes P \arrow[d, tail] \\
\prod\limits_i M_i \arrow[r, "\prod \psi_{M_i}"] & \prod\limits_i \Gamma \otimes M_i \arrow[r, shift left, "\prod \mathrm{id}_\Gamma\otimes\psi_{M_i}"] \arrow[r, shift right, "\prod \Delta\otimes\mathrm{id}_{M_i}"'] & \prod \Gamma \otimes \Gamma \otimes M_i
\end{tikzcd}$$

where the unlabeled vertical morphisms are the canonical inclusions. Each square commutes and the claim follows just like before.

`Step 3:` If we define $q_i := \mathrm{pr}_i^{\prod M_i} \circ j$, then the q_i are morphisms of Γ-comodules.

We define the projections q_i by

$$q_i\colon P \xrightarrow{j} \prod_i M_i \xrightarrow{\mathrm{pr}_i^{\prod M_i}} M_i$$

Since the diagrams

$$\begin{tikzcd}
\prod\limits_i M_i \arrow[r, "\prod \psi_{M_i}"] \arrow[d, "\mathrm{pr}_i^{\prod M_i}"'] & \prod\limits_i \Gamma \otimes M_i \arrow[d, "\mathrm{pr}_i^{\prod \Gamma\otimes M_i}"] \\
M_i \arrow[r, "\psi_{M_i}"] & \Gamma \otimes M_i
\end{tikzcd} \tag{3.5.7}$$

and

$$\begin{tikzcd}
\Gamma \otimes \prod\limits_i M_i \arrow[d, "\mathrm{can}"'] \arrow[dr, bend left, "\mathrm{id}_\Gamma\otimes\mathrm{pr}_i^{\prod M_i}"] & \\
\prod\limits_i \Gamma \otimes M_i \arrow[r, "\mathrm{pr}_i^{\prod \Gamma\otimes M_i}"] & \Gamma \otimes M_i
\end{tikzcd} \tag{3.5.8}$$

commute, we see that

$$\begin{aligned}(\mathrm{id}_\Gamma \otimes q_i) \circ \rho &\overset{\text{def}}{=} \big(\mathrm{id}_\Gamma \otimes (\mathrm{pr}_i^{\prod M_i} \circ j)\big) \circ \rho = (\mathrm{id}_\Gamma \otimes \mathrm{pr}_i^{\prod M_i}) \circ (\mathrm{id}_\Gamma \otimes j) \circ \rho \\ &\overset{(3.5.8)}{=} \mathrm{pr}_i^{\prod \Gamma\otimes M_i} \circ \mathrm{can} \circ p \overset{(3.5.3)}{=} \mathrm{pr}_i^{\prod \Gamma\otimes M_i} \circ \prod \psi_{M_i} \circ j \\ &\overset{(3.5.7)}{=} \psi_{M_i} \circ \mathrm{pr}_i^{\prod M_i} \circ j \overset{\text{def}}{=} \psi_{M_i} \circ q_i\,.\end{aligned}$$

`Step 4:` The comodule (P, ρ) together with the projections $(q_i)_i$ satisfies the universal property of the product in $\Gamma-$**Comod**.

Let $N \in \Gamma-$**Comod** and $f_i \colon N \to M_i$ be a family of comodule morphisms. By the universal property of the product in $A-$**Mod** we obtain a morphism $f \colon N \to \prod_i M_i$ of A-modules such that

$$\mathrm{pr}_i^{\prod M_i} \circ f = f_i \tag{3.5.9}$$

for every i. We consider the diagram

$$\begin{tikzcd}
N \arrow[r, "\psi_N"] \arrow[ddr, bend right, "f"'] & \Gamma \otimes N \arrow[dr, bend left, "\mathrm{id}_\Gamma \otimes f"] & & \\
& P \arrow[r, "p"] \arrow[d, "j"'] & \Gamma \otimes \prod\limits_i M_i \arrow[d, "\mathrm{can}"] \arrow[dr, bend left, "\mathrm{id}_\Gamma \otimes \mathrm{pr}_i^{\prod M_i}"] & \\
& \prod\limits_i M_i \arrow[r, "\prod \psi_{M_i}"] & \prod\limits_i \Gamma \otimes M_i \arrow[r, "\mathrm{pr}_i^{\prod \Gamma\otimes M_i}"] & \Gamma \otimes M_i
\end{tikzcd}$$

For fixed i we have

$$\begin{aligned}\mathrm{pr}_i^{\prod \Gamma\otimes M_i} \circ \mathrm{can} \circ (\mathrm{id}_\Gamma \otimes f) \circ \psi_N &\overset{(3.5.8)}{=} (\mathrm{id}_\Gamma \otimes \mathrm{pr}_i^{\prod M_i}) \circ (\mathrm{id}_\Gamma \otimes f) \circ \psi_N \\ &= \big(\mathrm{id} \otimes (\mathrm{pr}_i^{\prod M_i} \circ f)\big) \circ \psi_N \\ &\overset{(3.5.9)}{=} (\mathrm{id} \otimes f_i) \circ \psi_N = \psi_{M_i} \circ f_i\end{aligned}$$

and on the other hand

$$\mathrm{pr}_i^{\prod \Gamma\otimes M_i} \circ \prod \psi_{M_i} \circ f \overset{(3.5.7)}{=} \psi_{M_i} \circ \mathrm{pr}_i^{\prod M_i} \circ f \overset{(3.5.9)}{=} \psi_{M_i} \circ f_i\,.$$

Hence we have an equality

$$\mathrm{can} \circ (\mathrm{id}_\Gamma \otimes f) \circ \psi_N = \prod \psi_{M_i} \circ f$$

and obtain an induced morphism $g \colon N \to P$ of A-modules such that the diagram

$$\begin{tikzcd}
N \arrow[drr, bend left, "(\mathrm{id}_\Gamma \otimes f) \circ \psi_N"] \arrow[dr, dashed, "g"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r, "p"] \arrow[d, "j"'] & \Gamma \otimes \prod\limits_i M_i \\
& \prod\limits_i M_i &
\end{tikzcd} \tag{3.5.10}$$

commute. Moreover we have

$$q_i \circ g \overset{\text{def}}{=} \mathrm{pr}_i^{\prod M_i} \circ j \circ g \overset{(3.5.10)}{=} \mathrm{pr}_i^{\prod M_i} \circ f \overset{(3.5.9)}{=} f_i \,.$$

Let us show the uniqueness of g: If we have another morphism $h\colon N \to P$ of Γ-comodules with $q_i \circ g = q_i \circ h$ for every i, then we have $\mathrm{pr}_i \circ j \circ g = \mathrm{pr}_i \circ j \circ h$ and thus $j \circ g = j \circ h$. Since j is a monomorphism, we see that $g = h$.

It remains to show that g is a morphism of Γ-comodules, i.e. the commutativity of

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "\psi_N"'] & P \arrow[d, "\rho"] \\ \Gamma \otimes N \arrow[r, "\mathrm{id}_\Gamma \otimes g"] & \Gamma \otimes P \,. \end{tikzcd}$$

We compose with the monomorphisms $\mathrm{id}_\Gamma \otimes j\colon \Gamma \otimes P \rightarrowtail \Gamma \otimes \prod_i M_i$ and see that

$$(\mathrm{id}_\Gamma \otimes j) \circ (\mathrm{id}_\Gamma \otimes g) \circ \psi_N = \big(\mathrm{id}_\Gamma \otimes (j \circ g)\big) \circ \psi_N \overset{(3.5.10)}{=} (\mathrm{id}_\Gamma \otimes f) \circ \psi_N$$

resp.

$$(\mathrm{id}_\Gamma \otimes j) \circ \rho \circ g \overset{(3.5.4)}{=} p \circ g \overset{(3.5.10)}{=} (\mathrm{id}_\Gamma \otimes f) \circ \psi_N \,.$$

Therefore we obtain $(\mathrm{id}_\Gamma \otimes g) \circ \psi_N = \rho \circ g$. □

Remark The assumption holds if Γ is a Mittag-Leffler module, i.e. if the comparison morphism

$$\varphi\colon \Gamma \otimes \prod_i{}^{A} U(M_i) \longrightarrow \prod_i{}^{A} \big(\Gamma \otimes U(M_i)\big)$$

is a monomorphism. We refer to [SPA, Proposition 059M] and [SPA, Section 0599] for a precise definition and discussion of this property. Let us just note some examples of Mittag-Leffler modules ([SPA, Example 059R]):

(i) Projective modules are Mittag-Leffler.
(ii) Any finitely presentable module is Mittag-Leffler (then φ is even an isomorphism).

In this case, this gives (in theory) a characterization of all limits in $\Gamma-$**Comod** since kernels are computed in the same way in $\Gamma-$**Comod** and $A-$**Mod**.

3.6 Properties of algebraic stacks

3.6.1 The Adams condition and the strong resolution property

Definition 3.6.1 (Adams Hopf algebroids & Adams stacks) A Hopf algebroid (A, Γ) is an *Adams Hopf algebroid* if there is a filtered system $\{\Gamma_\alpha\}$ of (left) Γ-comodules such that each Γ_α is dualizable and there is an isomorphism

$$\operatorname*{colim}_\alpha \Gamma_\alpha \cong \Gamma$$

of Γ-comodules.

An algebraic stack associated to an Adams Hopf algebroid is called an *Adams stack*.

Example 3.6.2 (i) If E is a commutative ring spectrum that is *topologically flat*, then the pair (E_*, E_*E) is an Adams Hopf algebroid. We refer to [Hov04, Definition 1.4.5 and Theorem 1.4.7] for the definition and examples of topologically flat spectra and just state that MU and BP are topologically flat. Note that every Landweber exact spectrum over a topologically flat ring spectrum is again topologically flat ([Hov04, Theorem 1.4.9]).

(ii) If (A, Γ) is an Adams Hopf algebroid, $I \lhd A$ is an invariant ideal in A and v is a primitive element in A, then $(A/I, \Gamma/I\Gamma)$ and $(v^{-1}A, v^{-1}\Gamma)$ are Adams Hopf algebroids ([Hov04, Proposition 1.4.11]).

(iii) If $f \colon \mathfrak{X} \to \mathfrak{Y}$ is an affine morphism of algebraic stacks and $\mathfrak{Y}$ is an Adams stack, so is $\mathfrak{X}$ ([Sch12b, Proposition 4.6]).

(iv) Adams stacks are closed under 2-fibre products ([Sch12b, Corollary 4.7]).

Definition 3.6.3 (strong resolution property) An algebraic stack $\mathfrak{X}$ has the *strong resolution property* if the dualizable quasi-coherent sheaves form a generating family of the category of quasi-coherent sheaves.

Hovey and Schäppi have proven that Adams stacks are precisely the algebraic stacks with the strong resolution property:

Theorem 3.6.4 ([Hov04, Proposition 1.4.4] & [Sch12a, Theorem 1.3.1]) *An algebraic stack $\mathfrak{X}$ has the strong resolution property if and only if it is an Adams stack. In other words: A Hopf algebroid (A, Γ) is Adams if and only if the dualizable comodules form a generating family of* $\Gamma-$**Comod**.

3.6.2 Coherent and noetherian algebraic stacks

Definition 3.6.5 (coherent and noetherian algebraic stacks) An algebraic stack $\mathfrak{X}$ is called *coherent* (resp. *noetherian*) if there exists a presentation $P \colon X \to \operatorname{Spec}(\mathfrak{X})$ with a coherent (resp. noetherian) affine scheme X.

Clearly, every noetherian algebraic stack is coherent.

Lemma 3.6.6 *Let (A, Γ) be a flat Adams Hopf algebroid with A coherent. Then* $\Gamma-$**Comod** *is locally coherent.*

Proof. Since (A, Γ) is Adams, the category $\Gamma-$**Comod** is locally finitely presentable and the claim is equivalent to $\Gamma-\mathbf{Comod}_{\mathrm{coh}} = \Gamma-\mathbf{Comod}_{\mathrm{fp}}$ by [Her97, Theorem 1.6]. Hence, we have to show that every finitely presentable object is already coherent. Let $M \in \Gamma-\mathbf{Comod}$ be finitely presentable and $M' \leq M$ a finitely generated subobject. By Lemma 3.4.10 $U(M) \in A-\mathbf{Mod}$ is finitely presentable and $U(M') \leq U(M)$ is a finitely generated subobject. Since A is supposed to be coherent, $U(M')$ is finitely presentable, hence M' is finitely presentable. Thus, M is coherent. □

Remark The Adams assumption in the last lemma is used to ensure that $\Gamma-$**Comod** is locally finitely presentable.

Lemma 3.6.7 *Let (A, Γ) be a flat Hopf algebroid with A noetherian. Then* $\Gamma-$**Comod** *is locally noetherian.*

Proof. This follows from Lemma 1.1.13. □

The following two propositions show that injective objects behave well for noetherian algebraic stacks.

Proposition 3.6.8 *Let (A, Γ) be a flat Hopf algebroid with A noetherian. Then the forgetful functor*

$$U\colon \Gamma{-}\mathbf{Comod} \to A{-}\mathbf{Mod}$$

preserves injective objects.

Proof. Let $E \in \Gamma{-}\mathbf{Comod}$ be an injective comodule. We can embed $U(E) \in A{-}\mathbf{Mod}$ into an injective A-module J, $U(E) \rightarrowtail J$. By adjointness, this is equivalent to a map $E \to \Gamma \otimes J$ of comodules, which is also a monomorphism. Since E is an injective comodule, we have a splitting $\Gamma \otimes J \to E$. Now $U(\Gamma \otimes J)$ is an injective A-module by the next lemma an thus $U(E)$ is a split subobject of an injective A-module, hence itself injective. □

Lemma 3.6.9 *Let A be a noetherian ring and $F, E \in A{-}\mathbf{Mod}$ with F flat and E injective. Then the tensor product $F \otimes_A E$ is injective again.*

Proof. By Lazard's theorem ([SPA, Theorem 058G]) we can write the flat A-module F as a filtered colimit over finite free A-modules $F_d \cong A^{n_d}$ $(n_d \in \mathbb{N})$. Since $F \cong \operatorname{colim}_{d \in \boldsymbol{\mathcal{D}}} F_d$ we have isomorphisms

$$F \otimes_A E \cong (\operatorname*{colim}_{d \in \boldsymbol{\mathcal{D}}} F_d) \otimes_A E \cong \operatorname*{colim}_{d \in \boldsymbol{\mathcal{D}}} \bigl(F_d \otimes_A E\bigr) \cong \operatorname*{colim}_{\boldsymbol{\mathcal{I}}} E^{n_d} .$$

The E^{n_d}, $d \in \boldsymbol{\mathcal{D}}$, are injective A-modules. Since A is noetherian, any filtered colimit of injectives is injectives and we are done. □

Proposition 3.6.10 *Let $\mathfrak{X}$ be a noetherian algebraic stack. Then the inclusion*

$$\mathbf{QCoh}(\mathfrak{X}) \hookrightarrow \mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$$

preserves injective objects.

Proof. Let $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ be an injective object. It is enough to show that we can embed $\mathscr{F}$ into an object which is injective in $\mathbf{QCoh}(\mathfrak{X})$ *and* $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$.

Since $\mathfrak{X}$ is supposed to be noetherian, we can find a presentation $P\colon X \to \mathfrak{X}$ with a noetherian affine scheme X. By Lemma 2.2.5 there is an inclusion $P^*\mathscr{F} \rightarrowtail \mathscr{E}$ in an injective quasi-coherent sheaf $\mathscr{E}$, which is also injective as an object in $\mathbf{Mod}(\mathscr{O}_X)$. Applying P_* and combining with the unit of the adjunction $P^* \dashv P_*$ we get inclusions $\mathscr{F} \rightarrowtail P_*P^*\mathscr{F} \rightarrowtail P_*\mathscr{E}$ in $\mathbf{QCoh}(\mathfrak{X})$. Since P_* has the exact left adjoint P^*, the object $P_*\mathscr{E}$ is injective in both $\mathbf{QCoh}(\mathfrak{X})$ and $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$. □

Lemma 3.6.11 *Let $\mathfrak{X}$ be an algebraic stack and $\mathfrak{Z}$ a closed substack*

(i) If $\mathfrak{X}$ is coherent and $\mathfrak{Z}$ is defined by a finitely generated ideal, then $\mathfrak{Z}$ is coherent.

(ii) If $\mathfrak{X}$ is noetherian, so is $\mathfrak{Z}$.

Proof. Let $X = \operatorname{Spec}(A) \to \mathfrak{X}$ be a presentation of $\mathfrak{X}$ and let $I \lhd A$ be the invariant ideal defining $\mathfrak{Z}$. Then $\operatorname{Spec}(A/I) \to \mathfrak{Z}$ is a presentation of $\mathfrak{Z}$.

(i) This is [Gla89, Theorem 2.4.1].

(ii) A quotient of a noetherian ring is noetherian again. □

Lemma 3.6.12 *Let $\mathfrak{X}$ be an algebraic stack, $\mathfrak{Z} \subset \mathfrak{X}$ a closed substack and $\mathfrak{U} \subset \mathfrak{X}$ the open complement such that $\mathfrak{U} \hookrightarrow \mathfrak{X}$ is quasi-compact.*

(i) If $\mathfrak{X}$ is noetherian, so is $\mathfrak{U}$.

(ii) If $\mathfrak{X}$ is coherent, so is $\mathfrak{U}$.

Proof. (i) Every open subscheme of a (locally) noetherian scheme is locally noetherian.

(ii) Using Lemma 2.3.5, the argument is the same as for the noetherian case. □

Remark Being coherent (resp. noetherian) descends along faithfully flat ring morphisms (see [Gla89, Corollary 2.4.5] resp. [SPA, Lemma 033E]), i.e. if $A \to B$ is faithfully flat and B is coherent (resp. noetherian), so is A. These properties do not ascend (see [Gla89, Example 7.3.13] for an example of a coherent ring A with $A[X]$ not coherent), one has to impose an extra finiteness condition.

Corollary 3.6.13 *If $\mathfrak{X}$ is a noetherian algebraic stack, then we have an equivalence*

$$\mathbf{D}^+(\mathbf{QCoh}(\mathfrak{X})) \xrightarrow{\sim} \mathbf{D}^+_{\mathbf{QCoh}(\mathfrak{X})}(\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}))$$

induced by the natural embedding $\mathbf{D}(\mathbf{QCoh}(\mathfrak{X})) \to \mathbf{D}(\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}}))$

Proof. Clear from Proposition 3.6.10. □

3.7 Sheaf cohomology

From 3.6.10 we immediately see that sheaf cohomology of a quasi-coherent sheaf computed via resolutions in $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$ resp. $\mathbf{QCoh}(\mathfrak{X})$ is the same for a noetherian algebraic stack – any injective quasi-coherent sheaf is also injective as module sheaf. Yet, we don't need the noetherian assumption to prove this statement. The proof is motivated by [TT90, Proposition B.8].

Theorem 3.7.1 *If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$, then the canonical morphism*

$$\mathbb{R}^{\bullet}_{\mathbf{QCoh}(\mathfrak{X})}\Gamma(\mathfrak{X};\mathscr{F}) \xrightarrow{\cong} \mathbb{R}^{\bullet}_{\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})}\Gamma(\mathfrak{X};\mathscr{F})$$

of δ-functors is an isomorphism.

Proof. As for quasi-compact and semi-separated schemes (cf. [TT90, Proposition B.8]) we use Čech cohomology to compute the cohomology groups.

`Step 1:` $\mathbb{R}^{\ell}_{\mathrm{Mod}}\Gamma(\mathfrak{X};\mathscr{F}) \cong \check{\mathrm{H}}\big(X \to \mathfrak{X};\mathscr{F}\big)$.

By [SPA, Theorem 03OW] we have the *Čech-to-cohomology spectral sequence*

$$\check{\mathrm{H}}^p\big(X \to \mathfrak{X};\mathcal{H}^q(\mathscr{F})\big) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{X};\mathscr{F}),$$

for the ringed site $(\mathfrak{X},\mathscr{O}_{\mathfrak{X}})$ and the covering $P\colon X \to \mathfrak{X}$. Here $\mathscr{F}$ is an arbitrary abelian sheaf and $\mathcal{H}^q(\mathscr{F})$ is the abelian presheaf $V \mapsto \mathrm{H}^q(V,\mathscr{F})$. Recall that $\check{\mathrm{H}}^p(X \to \mathfrak{X};-)$ is given by the p-th cohomology of the Čech complex w.r.t. the covering $X \to \mathfrak{X}$.

Since all $X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$ are affine and $\mathscr{F}$ is quasi-coherent, we have $\mathrm{H}^q\big(X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X;\mathscr{F}\big) = 0$ for all $q > 0$. This implies the degeneration of the spectral sequence at E_2 and thus

$$\check{\mathrm{H}}^p(X \to \mathfrak{X};\mathscr{F}) \cong \mathrm{H}^p(\mathfrak{X},\mathscr{F}).$$

Eventually, note that the cohomology groups of a module sheaf are the same computed in the category of abelian sheaves $\boldsymbol{\mathcal{Sh}}(\mathfrak{X})$ or the category of module sheaves $\mathbf{Mod}(\mathscr{O}_{\mathfrak{X}})$, cf. [SPA, Lemma 03FD].

`Step 2:` $\check{\mathrm{H}}\big(X \to \mathfrak{X};\mathscr{F}\big) \cong \mathbb{R}^{\ell}_{\mathrm{QCoh}}\Gamma(\mathfrak{X};\mathscr{F})$.

We show that the Čech complex of a quasi-coherent sheaf gives a resolution by acyclic ones.

Let us define $X^{\times k} := X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$ to be the k-fold pullback of the presentation $P\colon X \to \mathfrak{X}$ with morphisms $P_k\colon X^k \to \mathfrak{X}$. The X^k and P_k are affine and we get adjoint pairs of functors

$$P_k^*\colon \mathbf{QCoh}(\mathfrak{X}) \rightleftarrows \mathbf{QCoh}_{\mathrm{fpqc}}(X^{\times k}) : P_{k_*}$$

for every k. Both functors are exact and P_{k_*} preserves injective objects by adjointness.

From $\Gamma(\mathfrak{X};-) \circ P_{k_*} = \Gamma(X^k,-)$ we get a Grothendieck spectral sequence

$$\mathbb{R}^p_{\mathrm{QCoh}}\big(\Gamma(\mathfrak{X};-)\big) \circ \mathbb{R}^q_{\mathrm{QCoh}} P_{k_*}(-) \Rightarrow \mathbb{R}^{p+q}_{\mathrm{QCoh}}\Gamma(X^{\times k};-)\,.$$

Since P_{k_*} is exact, the spectral sequence collapses and we obtain isomorphisms

$$\mathbb{R}^\ell_{\mathrm{QCoh}}\Gamma\big(\mathfrak{X};P_{k_*}\mathscr{G}\big) \cong \mathbb{R}^\ell_{\mathrm{QCoh}}\Gamma\big(X^{\times k};\mathscr{G}\big)$$

for any $\mathscr{G} \in \mathrm{QCoh}_{fpqc}(X^{\times k})$ and $\ell \geq 0$. From the comparison of the *fpqc* and the Zariski site and the fact that all $X^{\times k}$ are affine we get

$$\mathbb{R}^\ell_{\mathrm{QCoh}}\Gamma(X^{\times k};\mathscr{G}) = 0 \qquad \text{for all } \ell > 0$$

and so

$$\mathbb{R}^\ell_{\mathrm{QCoh}}\Gamma\big(\mathfrak{X};P_{k_*}\mathscr{G}\big) = 0 \qquad \text{for all } \ell > 0\,.$$

Hence, every quasi-coherent sheaf of the form $P_{k_*}\mathscr{G}$ is acyclic for $\Gamma(\mathfrak{X};-)\colon \mathbf{QCoh}(\mathfrak{X}) \to \mathbf{Ab}$.

Therefore, we can use the exact Čech complex

$$0 \to P_{1_*}P_1^*\mathscr{F} \to P_{2_*}P_2^*\mathscr{F} \to P_{3_*}P_3^*\mathscr{F} \to \dots$$

to compute

$$\begin{aligned}\mathbb{R}^\ell_{\mathrm{QCoh}}\Gamma\big(\mathfrak{X};\mathscr{F}\big) &\cong \mathrm{H}^\ell\Big(\Gamma\big(\mathfrak{X};P_{1_*}P_1^*\mathscr{F}\big) \to \Gamma\big(\mathfrak{X};P_{2_*}P_2^*\mathscr{F}\big) \to \dots\Big)\\ &\cong \mathrm{H}^\ell\Big(\Gamma\big(X^{\times 1};P_1^*\mathscr{F}\big) \to \Gamma\big(X^{\times 2};P_2^*\mathscr{F}\big) \to \dots\Big)\\ &= \check{\mathrm{H}}^\ell\big(X \to \mathfrak{X};\mathscr{F}\big)\end{aligned}$$

□

Corollary 3.7.2 *If $X \to \mathfrak{X}$ is an algebraic stack with associated Hopf algebroid (A,Γ) and $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ is a quasi-coherent sheaf with associated comodule $M \in \Gamma-\mathbf{Comod}$, then*

$$\mathrm{H}^\bullet(\mathfrak{X},\mathscr{F}) \cong \mathrm{Ext}^\bullet_\Gamma(A,M)\,.$$

Proof. Under the usual identifications, $\mathrm{Ext}^k_\Gamma(A,M) \cong \mathbb{R}^k_{\mathbf{QCoh}(\mathfrak{X})}\Gamma\big(\mathfrak{X};\mathscr{F}\big)$. □

4 Local cohomology sheaves for algebraic stacks

4.1 Definition and first properties

We fix the following setting:

> Let $\mathfrak{X}$ be an algebraic stack (over an affine scheme $S = \operatorname{Spec}(\Bbbk)$) and
>
> $$\begin{tikzcd}\mathfrak{Z} \arrow[r, hook, "i"] & \mathfrak{X} & \mathfrak{U} \arrow[l, hook', "j"']\end{tikzcd}$$
>
> a diagram of stacks where i is a closed immersion of a closed substack and $\mathfrak{U}$ is the open complement. We assume j to be a quasi-compact morphism. This guarantees that all stacks involved are algebraic.

Hence, we have an adjoint pair of functors

$$j^*\colon \mathbf{QCoh}(\mathfrak{X}) \rightleftarrows \mathbf{QCoh}(\mathfrak{U}) : j_*$$

between the Grothendieck categories of quasi-coherent sheaves on $\mathfrak{X}$ resp. $\mathfrak{U}$. The functor j^* is exact and its right adjoint j_* is fully faithful. As mentioned in Example 1.2.6, the full subcategory

$$\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) := \mathbf{Ker}(j^*) = \left\{\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X}) \,\middle|\, j^*\mathscr{F} = 0\right\}$$

is a localizing subcategory of $\mathbf{QCoh}(\mathfrak{X})$ and (j^*, j_*) is a localizing pair. The corresponding diagram reads as

$$\begin{tikzcd}\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \arrow[r, hook, shift left, "\iota"] & \mathbf{QCoh}(\mathfrak{X}) \arrow[l, shift left, "t"] \arrow[rr, shift left, "j^*"] \arrow[dr, shift left, "q"] & & \mathbf{QCoh}(\mathfrak{U}) \arrow[ll, hook', shift left, "j_*"] \\ & & \mathbf{QCoh}(\mathfrak{X})/\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \arrow[ul, hook', shift left, "s"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

and we give the following

Definition 4.1.1 (section with support and localization functor) We define the *section with support functor (with respect to* $\mathfrak{Z}$*)* by

$$\underline{\Gamma_{\mathfrak{Z}}} := \iota \circ t\colon \mathbf{QCoh}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})\,.$$

Moreover, we define the *localization functor (with respect to* $\mathfrak{U}$*)* by

$$\underline{L_{\mathfrak{U}}} := j_* j^*\colon \mathbf{QCoh}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})\,.$$

If a quasi-coherent sheaf $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ lies in $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X})$, then we say that $\mathscr{F}$ is *supported on* $\mathfrak{Z}$.

Recall that $\underline{\Gamma_{\mathfrak{Z}}}$ is a left exact functor.

Definition 4.1.2 (local cohomology sheaves) We define

$$\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}}}(-) := \mathbb{R}^{\bullet}_{\mathbf{QCoh}(\mathfrak{X})}\underline{\Gamma_{\mathfrak{Z}}}(-)\,.$$

If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$, we call $\underline{\mathrm{H}^{k}_{\mathfrak{Z}}}(\mathscr{F})$ the *k-th local cohomology sheaf of $\mathscr{F}$ with respect to* $\mathfrak{Z}$.

The following corollaries are reformulations of the results in Section 1.2.

Corollary 4.1.3 (cf. Proposition 1.2.9) *Let us assume that* $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserves injective objects. If* $\mathscr{E} \in \mathbf{QCoh}(\mathfrak{X})$ *is an injective object, then so are* $j^*\mathscr{E}$ *in* $\mathbf{QCoh}(\mathfrak{U})$ *and* $\underline{L_{\mathfrak{U}}}(\mathscr{E})$, $\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{E})$ *in* $\mathbf{QCoh}(\mathfrak{X})$ *and we have a split short exact sequence*

$$0 \longrightarrow \underline{\Gamma_{\mathfrak{Z}}}(\mathscr{E}) \longrightarrow \mathscr{E} \longrightarrow \underline{L_{\mathfrak{U}}}(\mathscr{E}) \longrightarrow 0$$

in $\mathbf{QCoh}(\mathfrak{X})$.

Corollary 4.1.4 (cf. Corollary 1.2.10) *Let* $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserve injective objects. If* $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$*, then we have an exact sequence*

$$0 \longrightarrow \underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \underline{L_{\mathfrak{U}}}(\mathscr{F}) \longrightarrow \mathbb{R}^1\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F}) \longrightarrow 0$$

in $\mathbf{QCoh}(\mathfrak{X})$ *and we have canonical isomorphisms*

$$\mathbb{R}^k\underline{L_{\mathfrak{U}}}(\mathscr{F}) \xrightarrow{\simeq} \underline{\mathrm{H}^{k+1}_{\mathfrak{Z}}}(\mathscr{F})$$

for $k \geq 1$.

Corollary 4.1.5 (cf. Lemma 1.2.12) *Assume that* $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserves injective objects and let* $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$*. If* $\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F}) = 0$*, then we can embed* $\mathscr{F}$ *into an injective quasi-coherent sheaf* $\mathscr{E} \in \mathbf{QCoh}(\mathfrak{X})$ *with* $\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{E}) = 0$.

Corollary 4.1.6 (cf. Lemma 1.2.14) *Assume that the inclusion* $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserves injective objects. If* $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$*, then* $\mathbb{R}^k\underline{L_{\mathfrak{U}}}(\mathscr{F})$ *is supported on* $\mathfrak{Z}$ *for* $k > 0$.

Corollary 4.1.7 (cf. Lemma 1.2.15) *Assume that* $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserves injective objects. If* $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$*, then we have natural isomorphisms*

$$\mathbb{R}^\bullet j_*(j^*\mathscr{F}) \cong \mathbb{R}^\bullet(j_*j^*)(\mathscr{F}) =: \mathbb{R}^\bullet\underline{L_{\mathfrak{U}}}(\mathscr{F})\,.$$

Corollary 4.1.8 (cf. Corollary 1.2.16) *Let* $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserve injective objects.*

(i) If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ *is supported on* $\mathfrak{Z}$*, then* $\mathbb{R}^\bullet\underline{L_{\mathfrak{U}}}(\mathscr{F}) = 0$.

(ii) The unit $\eta_{\mathscr{F}} : \mathscr{F} \to \underline{L_{\mathfrak{U}}}(\mathscr{F})$ *of the adjunction* $j^* \dashv j_*$ *induces isomorphisms*

$$\mathbb{R}^\bullet\underline{L_{\mathfrak{U}}}(\mathscr{F}) \xrightarrow{\simeq} \mathbb{R}^\bullet\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F})\big)$$

for any $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$.

Remark If one defines the *local cohomology groups* $\mathrm{H}^\bullet_{\mathfrak{Z}}(-)$ via the right derived functors of

$$\Gamma_{\mathfrak{Z}}(-) := \Gamma\big(\mathfrak{X}; \underline{\Gamma_{\mathfrak{Z}}}(-)\big) \colon \mathbf{QCoh}(\mathfrak{X}) \to \Gamma(\mathfrak{X}; \mathscr{O}_{\mathfrak{X}})\text{–}\mathbf{Mod}\,,$$

then we have a spectral sequence

$$\mathrm{H}^p\big(\mathfrak{X}; \underline{\mathrm{H}^q_{\mathfrak{Z}}}(-)\big) \Rightarrow \mathrm{H}^{p+q}_{\mathfrak{Z}}(-)\,.$$

by Corollary 4.1.3.

So far we did not state any criterion when $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ actually preserves injective objects.

Lemma 4.1.9 *If $\mathfrak{X}$ is a noetherian algebraic stack and $\mathfrak{Z}$ is a closed substack, then ι preserves injective objects.*

Proof. Let $X = \operatorname{Spec}(A) \to \mathfrak{X}$ be a presentation of $\mathfrak{X}$ by a noetherian affine scheme X and form the two cartesian diagrams

$$\begin{tikzcd}
U \arrow[r, "j'"] \arrow[d, "P'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "P"] & Z \arrow[l, "i'"'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathfrak{U} \arrow[r, "j"] & \mathfrak{X} & \mathfrak{Z} \arrow[l, "i"']
\end{tikzcd}$$

Let $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ be supported on $\mathfrak{Z}$, i.e. $\mathscr{F} \in \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X})$. By Lemma 1.2.11 it is enough to find an embedding $\mathscr{F} \rightarrowtail \mathscr{E}$ into an injective object of $\mathbf{QCoh}(\mathfrak{X})$ such that $\mathscr{E}$ is supported on $\mathfrak{Z}$. Since X is noetherian, the injective hull $\text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(P^*\mathscr{F})$ of $P^*\mathscr{F} \in \mathbf{QCoh}(\operatorname{Spec}(X))$ in $\mathbf{Mod}(\mathscr{O}_X)$ is again quasi-coherent.
`Claim:` $\mathscr{J} := \text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(P^*\mathscr{F}) \in \mathbf{QCoh}_Z(X)$, i.e. $j'^*\mathscr{J} = 0$.
Let us assume the claim for a moment. Then $P_*\mathscr{J} \in \mathbf{QCoh}(\mathfrak{X})$ and we have a morphism $\mathscr{F} \to P_*\mathscr{J}$ in $\mathbf{QCoh}(\mathfrak{X})$ given by adjointness, i.e. by composition of

$$\mathscr{F} \overset{\eta_{\mathscr{F}}}{\rightarrowtail} P_*P^*\mathscr{F} \rightarrowtail P_*\mathscr{J} \,.$$

The first arrow is a monomorphism since P^* is faithful. The second arrow is a monomorphism since $P^*\mathscr{F} \rightarrowtail \mathscr{J}$ and P_* is left exact. Since P_* has an exact left adjoint, $P_*\mathscr{J}$ is an injective object of $\mathbf{QCoh}(\mathfrak{X})$. Finally, we see that

$$j^*(P_*\mathscr{J}) \cong P'^*j'^*\mathscr{J} = 0$$

and $P_*\mathscr{J}$ is supported on $\mathfrak{Z}$.

It remains to prove the claim. Let us write $\mathscr{G} := P^*\mathscr{F} \in \mathbf{QCoh}(X)$ and consider the diagram

$$\begin{tikzcd}
\mathscr{G} \arrow[r, tail] \arrow[d, tail] & \text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(\mathscr{G}) \\
\prod\limits_{x \in X} j_{x_*}E(\mathscr{G}_x)\,, &
\end{tikzcd}$$

where $j_x \colon \{x\} \hookrightarrow X$ is the inclusion of a point $\{x\} \subset X$ and $\mathscr{G}_x \rightarrowtail E(\mathscr{G}_x)$ is an embedding of $\mathscr{G}_x$ in an injective $\mathscr{O}_{X,x}$-module. The sheaf $\prod_x j_{x_*}E(\mathscr{G}_x)$ is an injective $\mathscr{O}_X$-module and $\mathscr{G} \rightarrowtail \prod_{x \in X} j_{x_*}E(\mathscr{G}_x)$ is a monomorphism (cf. [SPA, Lemma 01DI]). By injectivity of $\prod_{x \in X} j_{x_*}E(\mathscr{G}_x)$ we obtain a morphism

$$\varphi \colon \text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(\mathscr{G}) \longrightarrow \prod_{x \in X} j_{x_*}E(\mathscr{G}_x)$$

making the diagram commute. Since $\mathscr{G} \rightarrowtail \text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(\mathscr{G})$ is an essential monomorphism, φ is a monomorphism. As $j \colon U := X \setminus Z \hookrightarrow X$ is an open immersion, the functor $j^* \colon \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_U)$ admits a right adjoint $j_!$ (cf. [SPA, Lemma 00A7]) and thus preserves limits. By exactness of j^* we obtain a monomorphism

$$j^*\big(\text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(\mathscr{G})\big) \overset{j^*\varphi}{\rightarrowtail} j^* \prod_{x \in X} j_{x_*}E(\mathscr{G}_x) \xrightarrow{\simeq} \prod_{x \in X} j^*j_{x_*}E(\mathscr{G}_x)\,.$$

Note that the product restricts to a product over all points $z \in Z$ since $\mathscr{G}_x = 0$ for all $x \in U$. Hence we have $j^*j_{z_*}E(\mathscr{G}_z) = 0$ for all $z \in Z$ and thus $j^*\big(\text{inj.hull}_{\mathbf{Mod}(\mathscr{O}_X)}(\mathscr{G})\big) = 0$ as desired. $\square$

4.2 Preservation of filtered colimits

From now on, we assume $\mathfrak{X}$ to be an Adams stack. In particular, the category $\mathbf{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on $\mathfrak{X}$ is generated by dualizable objects. Hence it is locally finitely presentable, i.e. every quasi-coherent sheaf $\mathscr{F}$ can be written as a filtered colimit of dualizable (hence finitely presentable) objects.

In this section we see that it is enough to know the right derived functors of $\underline{L_{\mathfrak{U}}}$ and $\underline{\Gamma_{\mathfrak{Z}}}$ on the subcategory of dualizable quasi-coherent sheaves.

Proposition 4.2.1 *Let $\mathfrak{X}$ be an Adams stack and assume that $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ preserves injective objects. Then the functors*

$$\underline{L_{\mathfrak{U}}}, \underline{\Gamma_{\mathfrak{Z}}}\colon \mathbf{QCoh}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$$

commute with finite limits and filtered colimits.

Proof. Since $\underline{L_{\mathfrak{U}}}$ and $\underline{\Gamma_{\mathfrak{Z}}}$ are left exact, they preserve finite limits. By definition, $\underline{L_{\mathfrak{U}}} = j_*j^*$ and j^* is a left adjoint and hence preserves all colimits. Thus to show the claim for $\underline{L_{\mathfrak{U}}}$ it will be enough to prove that j_* preserves filtered colimits.

Therefore, let $\mathscr{G}\colon \boldsymbol{\mathcal{D}} \to \mathbf{QCoh}(\mathfrak{U})$ be a filtered diagram of quasi-coherent sheaves on $\mathfrak{U}$. By the universal property of the colimit, we have a natural morphism

$$\varphi\colon \operatorname*{colim}_{d\in\boldsymbol{\mathcal{D}}} j_*\mathscr{G}_d \to j_*\big(\operatorname*{colim}_{d\in\boldsymbol{\mathcal{D}}} \mathscr{G}_d\big)$$

in $\mathbf{QCoh}(\mathfrak{X})$. We show that

$$\varphi_*\colon \operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{X})}\big(-, \operatorname*{colim}_d j_*\mathscr{G}_d\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{X})}\big(-, j_* \operatorname*{colim}_d \mathscr{G}_d\big)$$

and then apply Yoneda to see that φ is an isomorphism. Since $\mathfrak{X}$ is supposed to be an Adams stack, we only have to check this for all dualizable quasi-coherent sheaves sitting in $-$. Indeed, if $G \in \boldsymbol{\mathcal{C}}$ is a generator of an (abelian) category $\boldsymbol{\mathcal{A}}$, then the hom-functor $\boldsymbol{\mathcal{A}}(G, -)\colon \boldsymbol{\mathcal{A}} \to \mathbf{Set}$ is faithful. Since faithful functors reflect monomorphisms resp. epimorphisms and an abelian category is balanced, $\boldsymbol{\mathcal{A}}(G, -)$ reflects isomorphisms.

If $\mathscr{D} \in \mathbf{QCoh}(\mathfrak{X})$ is dualizable (and hence finitely presentable), then

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{X})}\big(\mathscr{D}, \operatorname*{colim}_d j_*\mathscr{G}_d\big) &\cong \operatorname*{colim}_d \operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{X})}\big(\mathscr{D}, j_*\mathscr{G}_d\big)\\
&\cong \operatorname*{colim}_d \operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{U})}\big(j^*\mathscr{D}, \mathscr{G}_d\big)\\
&\cong \operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{U})}\big(j^*\mathscr{D}, \operatorname*{colim}_d \mathscr{G}_d\big)\\
&\cong \operatorname{Hom}_{\mathbf{QCoh}(\mathfrak{X})}\big(\mathscr{D}, j_* \operatorname*{colim}_d \mathscr{G}_d\big),
\end{aligned}$$

where we have used that j^* preserves dualizable objects (the functor j^* is a functor of closed symmetric monoidal categories).

The claim for $\underline{\Gamma_{\mathfrak{Z}}}$ follows from the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname*{colim}_d \underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F}_d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname*{colim}_d \mathscr{F}_d \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d) \arrow[d, "\simeq"', "\varphi_*"] \\
0 \arrow[r] & \underline{\Gamma_{\mathfrak{Z}}}(\operatorname*{colim}_d \mathscr{F}_d) \arrow[r] & \operatorname*{colim}_d \mathscr{F}_d \arrow[r] & \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d)
\end{tikzcd}$$

and the universal property of the kernel. □

Remark The proof of the last proposition shows the following: Let $\boldsymbol{\mathcal{A}}$ be a locally finitely presentable category, $\boldsymbol{\mathcal{B}}$ an abelian category and $G\colon \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}$ be part of an adjoint pair $F \dashv G$. Then G preserves filtered colimits if and only if its left adjoint F preserves finitely presentable objects.

Even on an Adams stack, filtered colimits of injective quasi-coherent sheaves may not be injective again. Indeed, this is equivalent to $\mathbf{QCoh}(\mathfrak{X})$ being locally noetherian. Nevertheless, we see that the subcategory of injective sheaves $\mathscr{E}$ with $\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{E}) = 0$ is closed under filtered colimits if the open complement $\mathfrak{U}$ is noetherian.

Proposition 4.2.2 *Assume that* $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserves injective objects and that* $\mathfrak{U}$ *is a noetherian stack. Let* $\mathscr{E} : \boldsymbol{\mathcal{D}} \to \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X})_{\mathrm{inj}}$ *be a filtered diagram of injective quasi-coherent* $\mathscr{O}_{\mathfrak{X}}$*-modules with* $\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{E}_d) = 0$ *for all* $d \in \boldsymbol{\mathcal{D}}$*. Then* $\operatorname{colim}_d \mathscr{E}_d$ *is also injective.*

Proof. We have $\mathscr{E}_d \cong \underline{L_{\mathfrak{U}}}(\mathscr{E}_d)$ by the remark after Corollary 1.2.13 and hence

$$\operatorname*{colim}_d \mathscr{E}_d \cong \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}\big(\mathscr{E}_d\big) = \operatorname*{colim}_d j_* j^* \mathscr{E}_d \cong j_* \operatorname*{colim}_d j^* \mathscr{E}_d \,.$$

Each $j^*\mathscr{E}_d$ is injective by Corollary 4.1.3 and since $\mathbf{QCoh}(\mathfrak{U})$ is locally noetherian, $\operatorname{colim}_d j^*\mathscr{E}_d$ is an injective object of $\mathbf{QCoh}(\mathfrak{U})$. Finally, j_* preserves injective objects and we see that $\operatorname{colim}_d \mathscr{E}_d$ is injective.

Note that $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X})_{\mathrm{inj}} = \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \cap \mathbf{QCoh}(\mathfrak{X})_{\mathrm{inj}}$ by the assumption on ι. □

We can use the last result to prove

Proposition 4.2.3 *Let* $\mathfrak{X}$ *be an Adams stack such that* $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ *preserves injective objects and let* $\mathfrak{U}$ *be a noetherian stack. Then filtered colimits are preserved by the right derived functors* $\mathbb{R}^\bullet \underline{L_{\mathfrak{U}}}(-)$ *of* $\underline{L_{\mathfrak{U}}}(-)$*.*

Proof. Let $\mathscr{F} : \boldsymbol{\mathcal{D}} \to \mathbf{QCoh}(\mathfrak{X})$ be a filtered diagram. We have to show that

$$\operatorname*{colim}_d \mathbb{R}^k \underline{L_{\mathfrak{U}}}(\mathscr{F}_d) \xrightarrow{\sim} \mathbb{R}^k \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d)\,.$$

for all $k \geq 0$.

`Case` $k = 0$: This was shown in Proposition 4.2.1.

`Case` $k = 1$: Let $\mathscr{F} : \boldsymbol{\mathcal{D}} \to \mathbf{QCoh}(\mathfrak{X})$ be a filtered diagram of quasi-coherent sheaves on $\mathfrak{X}$. Then $\big\{\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big\}_{d \in \boldsymbol{\mathcal{D}}}$ is a filtered diagram of quasi-coherent sheaves with $\underline{\Gamma_{\mathfrak{Z}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) = 0$ for every d by Corollary 4.1.8.

Using Corollary 4.1.5 we can find a filtered diagram $\mathscr{E} \colon \boldsymbol{\mathcal{D}} \to \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X})^{\mathrm{inj}}$ of injectives $\mathscr{E}_d$ which are supported on $\mathfrak{Z}$ and a short exact sequence of filtered diagrams

$$\{0\}_d \longrightarrow \big\{\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big\}_d \longrightarrow \{\mathscr{E}_d\}_d \longrightarrow \{\mathscr{G}_d\}_d \longrightarrow \{0\}_d\,.$$

Applying the exact functor colim_i we get a short exact sequence

$$0 \longrightarrow \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d) \longrightarrow \operatorname*{colim}_d \mathscr{E}_d \longrightarrow \operatorname*{colim}_d \mathscr{G}_d \longrightarrow 0$$

and $\operatorname{colim}_d \mathscr{E}_d$ is injective by Proposition 4.2.2. Applying $\mathbb{R}^\bullet \underline{L_{\mathfrak{U}}}(-)$ to this short exact sequence gives a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \arrow[r] & \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{E}_d) \arrow[r] & \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{G}_d) \arrow[dll, rounded corners] \\
& \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \arrow[r] & \mathbb{R}^1\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{E}_d) \arrow[r] & \dots\,.
\end{tikzcd}$$

and since $\operatorname{colim}_d \mathscr{E}_d$ is injective, $\mathbb{R}^k\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{E}_d) = 0$ for $k \geq 1$. Thus, we get an exact sequence

$$0 \longrightarrow \underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \longrightarrow \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{E}_d) \longrightarrow \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{G}_d) \longrightarrow \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \longrightarrow 0\,.$$

On the other hand, we can start with the exact sequences ($d \in \boldsymbol{\mathcal{D}}$)

$$0 \longrightarrow \underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \longrightarrow \underline{L_{\mathfrak{U}}}(\mathscr{E}_d) \longrightarrow \underline{L_{\mathfrak{U}}}(\mathscr{G}_d) \longrightarrow \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \longrightarrow 0$$

to obtain an exact sequence of the form

$$0 \longrightarrow \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \longrightarrow \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{E}_d) \longrightarrow \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{G}_i d) \longrightarrow \operatorname*{colim}_d \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \longrightarrow 0$$

These two exact sequences fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \arrow[r] \arrow[d] & \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{E}_d) \arrow[r] \arrow[d] & \operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{G}_d) \arrow[r] \arrow[d] & \operatorname*{colim}_d \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \arrow[r] & \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{E}_d) \arrow[r] & \underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{G}_d) \arrow[r] & \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \arrow[r] & 0
\end{tikzcd}$$

and we get an isomorphism

$$\operatorname*{colim}_d \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \xrightarrow{\cong} \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big)\,.$$

We obtain isomorphisms

$$\mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \cong \mathbb{R}^1\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d) \cong \mathbb{R}^1\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d)\,.$$

Putting these together we see that

$$\operatorname*{colim}_d \mathbb{R}^1\underline{L_{\mathfrak{U}}}(\mathscr{F}_d) \cong \mathbb{R}^1\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d)\,,$$

i.e. $\mathbf{R}^1\underline{L_{\mathfrak{U}}}(-)$ commutes with filtered colimits.
`Case` $k > 1$: By induction, assume that the claim is shown for some fixed k. Then

$$\begin{aligned}
\operatorname*{colim}_d \mathbb{R}^{k+1}\underline{L_{\mathfrak{U}}}(\mathscr{F}_d) &\cong \operatorname*{colim}_d \mathbb{R}^{k+1}\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \\
&\cong \operatorname*{colim}_d \mathbb{R}^{k}\underline{L_{\mathfrak{U}}}(\mathscr{G}_d) \\
&\cong \mathbb{R}^{k}\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{G}_d) \\
&\cong \mathbb{R}^{k+1}\underline{L_{\mathfrak{U}}}\big(\operatorname*{colim}_d \underline{L_{\mathfrak{U}}}(\mathscr{F}_d)\big) \\
&\cong \mathbb{R}^{k+1}\underline{L_{\mathfrak{U}}}\big(\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d)\big) \\
&\cong \mathbb{R}^{k+1}\underline{L_{\mathfrak{U}}}(\operatorname*{colim}_d \mathscr{F}_d)
\end{aligned}$$

□

Corollary 4.2.4 *Let $\mathfrak{X}$ be an Adams stack such that $\mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathbf{QCoh}(\mathfrak{X})$ preserves injective objects and let $\mathfrak{U}$ be a noetherian stack. Then filtered colimits are preserved by the right derived functors $\underline{\mathrm{H}^\bullet_{\mathfrak{Z}}}(-)$ of $\underline{\Gamma_{\mathfrak{Z}}}(-)$.*

4.3 Translation to Hopf algebroids

We use the following dictionary

stacky language	groupoid language	reference
algebraic stack $P\colon \mathrm{Spec}(A) \to \mathfrak{X}$	(flat) Hopf algebroid (A, Γ)	Theorem 3.3.4
quasi-coherent sheaf $\mathscr{F}$	comodule M	Proposition 3.4.15
closed substack $\mathfrak{Z} \subset \mathfrak{X}$	invariant ideal $I \lhd A$	Corollary 3.4.16

The assumption that $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$ is quasi-compact translates to I being finitely generated. We assume in this section that $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}}(\mathfrak{X}) \hookrightarrow \mathbf{QCoh}(\mathfrak{X})$ preserves injective objects.

Lemma 4.3.1 *Under the equivalences stated above, we have*

$$\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F}) \approx \left\{ m \in M \,\middle|\, \exists n \in \mathbb{N} :\ I^n m = 0 \right\}$$

where $M \in \Gamma{-}\mathbf{Comod}$ is the comodule corresponding to $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$.

Proof. Similar to Lemma 2.2.3. □

Definition 4.3.2 ($\underline{\Gamma_I}(-)$) If M is a comodule and $I \lhd A$ is a finitely generated invariant ideal, we define the comodule

$$\underline{\Gamma_I}(M) := \left\{ m \in M \,\middle|\, \exists n \in \mathbb{N} :\ I^n m = 0 \right\} \in \Gamma{-}\mathbf{Comod}\,.$$

Similar to Lemma 2.3.1 we have a characterization of $\underline{\Gamma_I}(-)$ in terms of the internal hom of the closed symmetric monoidal category $\Gamma{-}\mathbf{Comod}$.

Lemma 4.3.3 *The natural morphism*

$$\operatorname*{colim}_n \underline{\mathrm{hom}}_\Gamma(A/I^n, M) = \underline{\Gamma_I}(M)$$

is an isomorphism of Γ-comodules.

Proof. Since I is supposed to be finitely generated invariant, A/I^n is a finitely presentable comodule and Proposition 3.5.9 gives an isomorphism

$$U\big(\operatorname*{colim}_n \underline{\mathrm{hom}}_\Gamma\,(A/I^n, M)\big) \cong \operatorname*{colim}_n \underline{\mathrm{hom}}_A\,(A/I^n, M) = \operatorname*{colim}_n \mathrm{Hom}_A(A/I^n, M)\,.$$

of A-modules and $\mathrm{colim}_n\, \underline{\mathrm{hom}}_\Gamma\,(A/I^n, M)$ is a subcomodule of M. Now $\mathrm{colim}_n\, \underline{\mathrm{hom}}_\Gamma\,(A/I^n, M)$ and $\underline{\Gamma_I}(M)$ are two subcomodules of M whose underlying A-modules agree. The claim now follows from Lemma 3.4.6. □

Corollary 4.3.4 *Let $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$. Then we have a natural isomorphism*

$$\mathrm{colim}_n\, \underline{\mathcal{H}\mathrm{om}}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{O}_{\mathfrak{X}}/\mathscr{I}^n, \mathscr{F}) \xrightarrow{\simeq} \underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F})\,.$$

4.4 Comparison with the scheme situation

Let $P\colon X \to \mathfrak{X}$ be a presentation of an algebraic stack $\mathfrak{X}$. Then we have commuting diagrams of the form

$$\begin{tikzcd}
\mathbf{QCoh}(\mathfrak{X}) \arrow[r, "P^*"] \arrow[d, "\underline{\Gamma_{\mathfrak{Z}}}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbf{QCoh}(X) \arrow[r, "\iota_X"] \arrow[d, "\underline{\Gamma_{Z}}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbf{Mod}(\mathcal{O}_X) \arrow[d, "\underline{\Gamma_{Z}}"] \\
\mathbf{QCoh}(\mathfrak{X}) \arrow[r, "P^*"] & \mathbf{QCoh}(X) \arrow[r, "\iota_X"] & \mathbf{Mod}(\mathcal{O}_X)
\end{tikzcd}$$

where $Z = \mathfrak{Z} \times_{\mathfrak{X}} X \hookrightarrow X$ is the closed (invariant) subscheme defining $\mathfrak{Z}$. The commutativity of the left square follows from P^* being exact and $P^* j_* j^* \mathscr{F} \cong j'_* j'^* P^* \mathscr{F}$ for every $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ where j' is defined by base change of j along $P\colon X \to \mathfrak{X}$,

$$\begin{tikzcd}
U \arrow[r, "j'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "P"] \\
\mathfrak{U} \arrow[r, "j"] & \mathfrak{X} .
\end{tikzcd}$$

Proposition 4.4.1 *If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$, we can give $\mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_X)}\underline{\Gamma_Z}(P^*\mathscr{F})$ the structure of an object of $\mathbf{QCoh}(\mathfrak{X})$, i.e. $\mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_X)}\underline{\Gamma_Z}(P^*\mathscr{F})$ is a quasi-coherent sheaf on X together with descent data for every $k \geq 0$.*

Proof. Recall Proposition 2.1.4: If $f\colon Y \to X$ is a flat morphism of schemes and $\mathscr{G} \in \mathbf{QCoh}(X)$, then we have canonical base change morphisms

$$f^*\underline{\mathrm{H}^\bullet_Z}(\mathscr{G}) \cong \underline{\mathrm{H}^\bullet_{f^{-1}(Z)}}(f^*\mathscr{G})$$

in $\mathbf{QCoh}(Y)$. We can apply this result to the flat morphisms $s, t\colon \operatorname{Spec}(\Gamma) \to \operatorname{Spec}(A)$ and the quasi-coherent sheaf $P^*\mathscr{F} \in \mathbf{QCoh}(X)$ with $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$. Thus,

$$\begin{aligned}
s^*\mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_X)}\underline{\Gamma_Z}(P^*\mathscr{F}) &\cong \mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_Y)}\underline{\Gamma_{s^{-1}(Z)}}(s^*P^*\mathscr{F}) \\
&= \mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_Y)}\underline{\Gamma_{t^{-1}(Z)}}(s^*P^*\mathscr{F}) \\
&\xrightarrow{\varphi_{\mathscr{F}}} \mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_Y)}\underline{\Gamma_{t^{-1}(Z)}}(t^*P^*\mathscr{F}) \\
&\cong t^*\mathbb{R}^k_{\mathbf{Mod}(\mathcal{O}_X)}\underline{\Gamma_Z}(P^*\mathscr{F})
\end{aligned}$$

of quasi-coherent $\mathcal{O}_X$-modules, where we used the invariance of Z (i.e. $s^{-1}(Z) = t^{-1}(Z)$) and $\varphi_{\mathscr{F}}$ is the isomorphism induced by $s^*(P^*\mathscr{F}) \xrightarrow{\sim} t^*(P^*\mathscr{F})$. We leave out checking that this morphism satisfies the cocycle conditions. □

Theorem 4.4.2 *There is a unique natural transformation of cohomological δ-functors*

$$\underline{\mathrm{H}^\bullet_{\mathfrak{Z}}}(\mathscr{F}) := \mathbb{R}^\bullet_{\mathbf{QCoh}(\mathfrak{X})}\underline{\Gamma_{\mathfrak{Z}}}(\mathscr{F}) \longrightarrow \mathbb{R}^\bullet_{\mathbf{Mod}(\mathcal{O}_X)}\underline{\Gamma_Z}(P^*\mathscr{F}) .$$

on $\mathbf{QCoh}(\mathfrak{X})$. This natural transformation is an isomorphism if the following condition is satisfied:

> *The canonical natural transformation*
>
> (4.4.1) $$\mathbb{R}^\bullet_{\mathbf{QCoh}(X)}\underline{\Gamma_Z}(-) \longrightarrow \mathbb{R}^\bullet_{\mathbf{Mod}(\mathcal{O}_X)}\underline{\Gamma_Z}(-)$$
>
> *of δ-functors on $\mathbf{QCoh}(X)$ is an isomorphism.*

Proof. The existence of the natural transformation follows from the universal property of a derived functor.

To prove that it is an isomorphism it is sufficient to show that every object $\mathscr{F}$ of $\mathbf{QCoh}(\mathfrak{X})$ can be embedded in an object $\mathscr{E}$ of $\mathbf{QCoh}(\mathfrak{X})$ with $\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(P^*\mathscr{E}) = 0$ for any $k > 0$.

Thus, let $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ be an arbitrary quasi-coherent sheaf on $\mathfrak{X}$. Embed $\mathbf{QCoh}(X) \ni P^*\mathscr{F}$ in an injective quasi-coherent sheaf $\mathscr{J}$ on X. By adjunction $P^* \dashv P_*$, this corresponds to a *mono*morphism $\mathscr{F} \rightarrowtail P_*\mathscr{J}$ in $\mathbf{QCoh}(\mathfrak{X})$. We want to show that $P^*P_*\mathscr{J}$ is acyclic for the functor $\underline{\Gamma_Z}\colon \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$. This amounts to prove

$$\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(P^*P_*\mathscr{J}) = 0 \qquad \text{for } k > 0\,.$$

`Step 1:` $\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(P^*P_*\mathscr{J}) \cong \mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(s_*t^*\mathscr{J})$

This follows from the cartesian diagram

$$\begin{tikzcd}
Y := X \times_{\mathfrak{X}} X \arrow[r, "t"] \arrow[d, "s"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "P"] \\
X \arrow[r, "P"] & \mathfrak{X}
\end{tikzcd}$$

and P being flat.

`Step 2:` $\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(s_*t^*\mathscr{J}) \cong s_*\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_Y)}\underline{\Gamma_{s^{-1}(Z)}}(t^*\mathscr{J})$

We have $\underline{\Gamma_Z} \circ s_* = s_* \circ \underline{\Gamma_{s^{-1}(Z)}}$ and since s_* is affine and preserves injective objects (has exact left adjoint s^*), we obtain isomorphisms

$$\big(\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}\big)(s_*\mathscr{G}) \cong R^k_{\mathbf{Mod}(\mathscr{O}_X)}s_*\big(\underline{\Gamma_{s^{-1}(Z)}}(\mathscr{G})\big)$$

for every $\mathscr{G} \in \mathbf{QCoh}(Y)$ and $k \geq 0$. Using the same arguments again we see that

$$\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}s_*\big(\underline{\Gamma_{s^{-1}(Z)}}(\mathscr{G})\big) \cong s_*\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_Y)}\underline{\Gamma_{s^{-1}(Z)}}(\mathscr{G})$$

Now choose $\mathscr{G} = t^*\mathscr{J}$.

`Step 3:` $s_*\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_Y)}\underline{\Gamma_{s^{-1}(Z)}}(t^*\mathscr{J}) \cong s_*t^*\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(\mathscr{J})$

Z is an invariant subscheme and hence we have canonical isomorphisms

$$\begin{aligned}\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_Y)}\underline{\Gamma_{s^{-1}(Z)}}(t^*\mathscr{J}) &\cong \mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_Y)}\underline{\Gamma_{t^{-1}(Z)}}(t^*\mathscr{J}) \\ &\cong t^*\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}\mathscr{J}\,.\end{aligned}$$

by Proposition 2.1.4.

`Step 4:` $s_*t^*\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\Gamma_Z(\mathscr{J}) = 0$ for $k > 0$.

If the natural transformation (4.4.1) is an isomorphism, we see that

$$\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_X)}\underline{\Gamma_Z}(\mathscr{J}) \cong \mathbb{R}^k_{\mathbf{QCoh}(X)}\underline{\Gamma_Z}(\mathscr{J}) = 0$$

for $k > 0$, since $\mathscr{J}$ was chosen to be an injective object in $\mathbf{QCoh}(X)$. □

The proposition is of little use if we don't state sufficient criteria for (4.4.1) being an isomorphism. The next Corollary follows immediately from Proposition 2.2.10 and Proposition 2.2.12.

Corollary 4.4.3 *Let $Z = \operatorname{Spec}(A/I)$ be defined by a weakly proregular ideal $I \lhd A$. Then the natural transformation* (4.4.1) *is an isomorphism in* $\mathbf{QCoh}(X)$.

Example 4.4.4 In Example 3.1.7 we have seen how to interpret $[\{0\}/\mathbb{G}_m]$ as a closed substack of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$. The stack $[\mathbb{A}^{n+1}/\mathbb{G}_m]$ has a presentation P by the affine scheme $\mathbb{A}^{n+1} = \operatorname{Spec}\big(R[X_0, \dots, X_n]\big)$ and $[\{0\}/\mathbb{G}_m]$ is given by the regular maximal ideal $\mathfrak{m} = (X_0, \dots, X_n)$. If $\mathscr{F} \in \mathbf{QCoh}\big([\mathbb{A}^{n+1}/\mathbb{G}_m]\big)$ is a quasi-coherent sheaf on $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, then the local cohomology sheaves of $\mathscr{F}$ with respect to the closed substack $[\{0\}/\mathbb{G}_m]$ can be calculated as

$$\mathbf{R}^k_{\mathbf{Mod}(\mathscr{O}_{\mathbb{A}^{n+1}})}\underline{\Gamma_{\{0\}}}(P^*\mathscr{F})\,.$$

If we choose $\mathscr{F}$ to be the structure sheaf on $\mathbb{A}^{n+1}/\mathbb{G}_m$, then

$$\underline{\mathrm{H}^k_{[\{0\}/\mathbb{G}_m]}}\big(\mathscr{O}_{[\mathbb{A}^{n+1}/\mathbb{G}_m]}\big) = \begin{cases} 0 & k \neq n+1 \\ R[X_0, \dots, X_n]/(X_0^\infty, \dots, X_n^\infty) & k = n+1\,. \end{cases}$$

by regularity of $\mathfrak{m}$.

Note that the Čech complex $\check{C}^\bullet_{\underline{\boldsymbol{u}}}\big(U(M)\big)$ (given in Definition 2.2.8) of a comodule M is *not* a complex of comodules. Indeed, we only want the ideal generated by $\underline{\boldsymbol{u}}$ to be invariant, not the elements u_i of $\underline{\boldsymbol{u}}$ itself. But multiplication with u_i on A is a map of comodules if and only if u_i is invariant. Nevertheless we will see that its cohomology modules $\mathrm{H}^k(\underline{\boldsymbol{u}}; M)$ are the underlying A-modules of the Γ-comodules $\mathbb{R}^k_{\Gamma-\mathbf{Comod}}\underline{\Gamma_I}(M)$.

Proposition 4.4.5 *If the sequence $\underline{\boldsymbol{u}}$ is weakly proregular, the functor*

$$\mathrm{H}^k(\underline{\boldsymbol{u}}; -) \circ U \colon \Gamma{-}\mathbf{Comod} \to A{-}\mathbf{Mod}$$

is a universal δ-functor.

Proof. δ`-functor:` The forgetful functor $U \colon \Gamma{-}\mathbf{Comod} \to A{-}\mathbf{Mod}$ is exact and with Proposition 2.2.12 we conclude that $\mathrm{H}^k(\underline{\boldsymbol{u}}; -) \colon A{-}\mathbf{Mod} \to A{-}\mathbf{Mod}$ is a universal δ-functor.
`effaceable:` If $M \in \Gamma{-}\mathbf{Comod}$ is a comodule, then we have to show the existence of a monomorphism $i \colon M \rightarrowtail N$ in $\Gamma{-}\mathbf{Comod}$ with $\mathrm{H}^k(\underline{\boldsymbol{u}}; U(i)) = 0$ for $k > 0$. This holds in particular if we can choose N in such a manner that $\mathrm{H}^k(\underline{\boldsymbol{u}}; U(N)) = 0$ for $k > 0$.

As usual, we take $N = \Gamma \otimes_A E$ for some embedding $U(M) \rightarrowtail E$ in an injective object E in $A{-}\mathbf{Mod}$. Since Γ is a flat A-module, we can write $\Gamma \cong \operatorname{colim}_{d \in \boldsymbol{\mathcal{D}}} \Gamma_d$ for finite free A-modules Γ_d over a filtered category $\boldsymbol{\mathcal{D}}$ by Lazard's theorem (cf. [SPA, Theorem 058G]). On the level of A-modules one has

$$\begin{aligned} \mathrm{H}^k(\underline{\boldsymbol{u}}; \Gamma \otimes E) &= \mathrm{H}^k\big(\mathrm{K}^\bullet(\underline{\boldsymbol{u}}; \Gamma \otimes E)\big) \\ &\cong \mathrm{H}^k\big(K^\bullet(\underline{\boldsymbol{u}}) \otimes (\Gamma \otimes E)\big) \\ &\cong \mathrm{H}^k\big(\operatorname*{colim}_{d \in \boldsymbol{\mathcal{D}}} \mathrm{K}^\bullet(\underline{\boldsymbol{u}}; \Gamma_d \otimes E)\big) \\ &\cong \operatorname*{colim}_{d \in \boldsymbol{\mathcal{D}}} \mathrm{H}^k(\underline{\boldsymbol{u}}; \Gamma_d \otimes E)\,, \end{aligned}$$

where we have used the exactness of filtered colimits in $A{-}\mathbf{Mod}$. Each $\Gamma_d \otimes_A E$ is an injective A-module and thus $\mathrm{H}^k(\underline{\boldsymbol{u}}; \Gamma_d \otimes_A E) = 0$ for every $k > 0$. □

Corollary 4.4.6 *Let $\underline{\boldsymbol{u}}$ be a weakly proregular sequence in A and assume that the ideal $I \lhd A$ generated by $\underline{\boldsymbol{u}}$ is invariant. Then $\mathrm{H}^k(\underline{\boldsymbol{u}}; -) \circ U$ is the k-th derived functor of*

$$U \circ \underline{\Gamma_I}(-) \colon \Gamma{-}\mathbf{Comod} \to A{-}\mathbf{Mod}\,.$$

Definition 4.4.7 Let $\mathfrak{X}$ be an algebraic stack and $\mathfrak{Z} \subset \mathfrak{X}$ a closed substack. Then we call $i\colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ *a weakly proregular closed immersion* (or *weakly proregularly embedded*) if there exists a presentation $P\colon \mathrm{Spec}(A) \to \mathfrak{X}$ such that the corresponding invariant ideal $I \lhd A$ is weakly proregular.

From Lemma 2.2.14 we conclude that this definition does not depend of the actual choice of the presentation P.

Corollary 4.4.8 *Let $\mathfrak{X}$ be an algebraic stack and $\mathfrak{Z} \subset \mathfrak{X}$ a closed substack such that the inclusion of the open complement $\mathfrak{U} \hookrightarrow \mathfrak{X}$ is quasi-compact. Assume that $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ is a weakly proregular closed immersion and let* $\mathrm{Spec}(A) \to \mathfrak{X}$ *be an arbitrary presentation of $\mathfrak{X}$. If $\mathscr{F} \in \mathbf{QCoh}(\mathfrak{X})$ with corresponding Γ-comodule M, then one can calculate $\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}}}(\mathscr{F})$ via the Čech complex $C^{\bullet}_{I}(M)$ associated to the invariant ideal $I \lhd A$ defining $\mathfrak{Z}$.*

5 Application: The stack of formal groups

5.1 The Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$

Let BP be the Brown-Peterson spectrum for a fixed prime p ([Rav04, Theorem 4.1.12]). As in Example 3.3.3 (v), the pair of homotopy groups $\mathrm{BP}_* = \pi_*(\mathrm{BP})$ and BP_*-homology $\mathrm{BP}_*\mathrm{BP} = \pi_*(\mathrm{BP} \wedge \mathrm{BP})$ forms a flat Hopf algebroid (over $\pi_0(\mathrm{BP}_*) = \mathbb{Z}_{(p)}$). It is well known ([Rav04, Theorem 4.1.19]) that $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ is isomorphic to the Hopf algebroid (V, VT) where

$$\begin{aligned} V &= \mathbb{Z}_{(p)}[v_1, v_2, \dots] \qquad & |v_n| &= 2(p^n - 1) \\ VT &= V[t_1, t_2, \dots] \qquad & |t_n| &= 2(p^n - 1) \end{aligned}$$

and structure morphisms given in [Rav04, Theorem 4.1.18]. There may be (at least) two possibile choices for the elements v_i, namely the Hazewinkel or Araki generators. Nevertheless, all interesting properties are independent of the choice of the generators. Let us freely identify (V, VT) with $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$.

Define $I_n := (p, v_1, v_2, \dots, v_{n-1}) \lhd \mathrm{BP}_*$ and set $I_0 := (0)$, $I_\infty = (p, v_1, v_2, \dots)$. The $I_n \lhd \mathrm{BP}_*$ are invariant prime ideals ([Rav04, Theorem 4.3.2]) and are the only invariant prime ideals in BP_* ([Rav04, Theorem 4.3.1]). Moreover,

$$s(v_n) \equiv t(v_n) \bmod I_n$$

for every $n \geq 0$ and $(p, v_1, v_2, \dots)$ is a regular sequence in BP_*.

Since the I_n are invariant, we can form the Hopf algebroid

$$(\mathrm{BP}_*/I_n, \mathrm{BP}_*\mathrm{BP}/I_n\mathrm{BP}_*\mathrm{BP}) \cong (\mathrm{BP}_*/I_n, \mathrm{BP}_*/I_n \otimes \mathrm{BP}_*\mathrm{BP} \otimes \mathrm{BP}_*/I_n)\,.$$

Definition 5.1.1 (v_n-torsion) Let $m \in M$ be an element of a BP_*-comodule and $v_n \in \mathrm{BP}_*$. Then m is *v_n-torsion* if $v_n^k m = 0$ for some k. If all elements of M are v_n-torsion, then M is said the be a v_n-torsion. If no non-zero element of M is v_n-torsion, then we say M is *v_n-torsion free.*

Proposition 5.1.2 ([JY80, Theorem 0.1]) *Let $M \in \mathrm{BP}_*-$**Comod**. If $m \in M$ is v_n-torsion, it is a v_{n-1}-torsion element. Consequently, if M is a v_n-torsion module, then it is a v_{n-1}-torsion module.*

We conclude:

Corollary 5.1.3 *A $\mathrm{BP}_*\mathrm{BP}$-comodule is v_n-torsion if and only if it is I_{n+1}-torsion.*

Definition 5.1.4 (The stack $\mathcal{M}_{\mathrm{FG}}$) We let $\mathcal{M}_{\mathrm{FG}}$ be the stack associated to the flat Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ and define $\mathfrak{Z}^n$ to be the algebraic stack associated to the flat Hopf algebroid $(\mathrm{BP}_*/I_n, \mathrm{BP}_*\mathrm{BP}/I_n\mathrm{BP}_*\mathrm{BP})$. Moreover, we define $\mathfrak{U}_n$ to be the open complement of $\mathfrak{Z}^{n+1}$. Denote by $j_n\colon \mathfrak{U}_n \hookrightarrow \mathcal{M}_{\mathrm{FG}}$ the quasi-compact open immersion.

Lemma 5.1.5 *The stacks $\mathcal{M}_{\mathrm{FG}}$, $\mathfrak{Z}^n$, $\mathfrak{U}_n$ and $\mathfrak{Z}^n \times_{\mathcal{M}_{\mathrm{FG}}} \mathfrak{U}_n$ are Adams stacks.*

Proof. This follows from Example 3.6.2, cf. [Goe, Proposition 6.9]. Presentations are given by

$$\begin{aligned} \operatorname{Spec}\bigl(\mathbb{Z}_{(p)}[v_1, v_2, \dots]\bigr) &\to \mathcal{M}_{\mathrm{FG}} \\ \operatorname{Spec}\bigl(\mathbb{F}_p[v_n, v_{n+1}, \dots]\bigr) &\to \mathfrak{Z}^n \\ \operatorname{Spec}\bigl(\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]\bigr) &\to \mathfrak{U}_n \\ \operatorname{Spec}\bigl(\mathbb{F}_p[v_n^{\pm 1}, v_{n+1}, \dots]\bigr) &\to \mathfrak{Z}^n \times_{\mathcal{M}_{\mathrm{FG}}} \mathfrak{U}_n\,. \end{aligned}$$

□

We want to define local cohomology sheaves on the stack $\mathcal{M}_{\mathrm{FG}}$ with respect to the closed substacks $\mathfrak{Z}^n$, $n \geq 0$, as before. In order to use the results of Section 4.1, we need to verify that the inclusion of the localizing subcategory of quasi-coherent sheaves supported on a fixed closed substack $\mathfrak{Z}^n$ into the category $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ preserves injective objects.

Proposition 5.1.6 ([HS05b, Proposition 2.2]) *Let $\mathscr{F} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ be supported on $\mathfrak{Z}^n$. Then the injective hull of $\mathscr{F}$ is also supported on $\mathfrak{Z}^n$.*

From Lemma 1.2.11 we immediately conclude:

Corollary 5.1.7 *The inclusion $\iota\colon \mathbf{QCoh}_{\mathfrak{Z}^n}(\mathcal{M}_{\mathrm{FG}}) \hookrightarrow \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ preserves injective objects.*

Remark ($\mathcal{M}_{\mathrm{FG}}$ and formal groups) Our ad hoc definition $\mathcal{M}_{\mathrm{FG}}$ and its closed (resp. open) substacks $\mathfrak{Z}^n$ (resp. $\mathfrak{U}_n$) does not make clear how they are connected to formal groups resp. formal group laws. Let us enlight the situation very briefly, the reader is referred to [Rav04, Appendix A2] and Lurie's lecture notes [Lur10] on chromatic homotopy theory for details.

A (commutative, one-dimensional) formal group law f over a ring R is a power series $f(x, y) \in R[\![x, y]\!]$ over R satisfying

(i) (commutativity) $f(x, y) = f(y, x)$,
(ii) (associativity) $f\big(f(x, y), z\big) = f\big(x, f(y, z)\big)$, and
(iii) (existence of unit) $f(x, 0) = x$.

This is enough to conclude the existence of a power series $i(x) \in R[\![x]\!]$ over R such that $f\big(x, i(x)\big) = 0$. Two common examples are the *additive formal group law* $f_a(x, y) = x + y$ and the *multiplicative formal group law* $f_m(x, y) = x + y + xy$. Let us denote the set of formal groups over a ring R by $\mathrm{FGL}(R)$. A homomorphism of formal group laws $f(x, y)$ and $g(x, y)$ over R is a power series $\varphi(x) \in xR[\![x]\!]$ such that $\varphi\big(f(x, y)\big) = g\big(\varphi(x), \varphi(y)\big)$. It is an isomorphism if $\varphi'(0) \in R^\times$ is invertible and we call φ a *strict isomorphism* if even $\varphi'(0) = 1$. Moreover, we define the *n-series*, $n \in \mathbb{N}$, of a formal group f by

$$[n]_f(x) := \underbrace{x +_f \cdots +_f x}_{n\text{-times}} := f\big(x, [n-1]_f(x)\big)\,,$$

where we set $[-1]_f(x) := 0$. We have $[n]_{f_a}(x) = nx$ and $[n]_{f_m}(x) = (1 + x)^n - 1$.

The functor $\mathbf{Rng} \to \mathbf{Set}$, $R \mapsto \mathrm{FGL}(R)$, assigning each ring its set of formal group laws is corepresentable, i.e. there exists a universal ring L together with a formal group law $f_{\mathrm{univ}}(X, Y)$ on L, such that any formal group law f on another ring R determines a unique ring morphism $L \to R$ sending f_{univ} to f. This ring is called *Lazard ring* and is isomorphic to a polynomial ring over $\mathbb{Z}$ in infinitely many variables. Moreover, there is a ring LB corepresenting strict isomorphisms between two formal group laws. The pair (L, LB) defines a Hopf algebroid. One can show that (L, LB) is isomorphic to the Hopf algebroid associated to complex cobordism MU.

If $f = f(x, y) \in R[\![x, y]\!]$ is a formal group over a $\mathbb{Z}_{(p)}$-algebra R, then consider the algebra $R[\xi_q]$ obtained by adjoining a q-th root of unity of R and define $f_q(x)$ to be the series

$$f_q(x) := [\tfrac{1}{q}]\Big(\sum_{1 \leq i \leq q}^{f} \xi_q^i \cdot x\Big)\,,$$

where the summation is understood via the formal group law f and $[\frac{1}{q}](x)$ denotes the inverse of the q-series of x. By symmetry we see that f_q is already defined over R (and not only over $R[\xi_q]$). The formal group law $f = f(x, y)$ is said to be *p-typical* if $f_q(x) = 0$ for every prime $q \neq p$. A homomorphism of p-typical formal group laws is a homomorphism of formal group laws. Due to a theorem of Cartier, every formal group law over a $\mathbb{Z}_{(p)}$-algebra is naturally strictly isomorphic to a p-typical one. As before, there exists a Hopf algebroid (V, VT) corepresenting p-typical formal group laws and strict isomorphisms between them. This is the pair of $\mathbb{Z}_{(p)}$-algebras mentioned on page 67 and we denote by $\mathcal{M}_{\mathrm{FG}}$ its associated stack.

Let $f(x, y)$ be a formal group law over a ring R and fix a prime p. We let v_n denote the coefficient of x^{p^n} in the p-series $[p]_f$ of f. Then we say that f has *height* $\geq n$ if $v_i \equiv 0 \bmod p$ for all $i < n$ and f has *height exactly* n if it has height $\geq n$ and $v_n \in R^\times$ is invertible. Since $v_0 =$ coefficient of $x = p$, we see that f has height ≥ 1 if and only if R is of characteristic p and f has height exactly zero if and only if $p \in R^\times$ is invertible. If $p = 0$ in R, then the additive formal group law f_a has height ∞ and the multiplicative formal group law f_m has height exactly 1 since

$$\begin{aligned}[p]_{f_a}(x) &= px = 0 \cdot x^{p^0} + 0 \cdot x^{p^1} + \dots \\ [p]_{f_m}(x) &= (x+1)^p - 1 \overset{p=0}{=} x^p = 0 \cdot x^{p^0} + 1 \cdot x^{p^1} + 0 \cdot x^{p^2} + \dots .\end{aligned}$$

The closed substack $\mathfrak{Z}^n$ of $\mathcal{M}_{\mathrm{FG}}$ is the stack associated to the Hopf algebroid corepresenting p-typical formal group laws of height at least n and strict isomorphisms; and there is a similar interpretation for $\mathfrak{U}_n$ – it corepresents formal group laws of height at most n. The presentation of $\mathfrak{U}_n$ in Lemma 5.1.5 shows that the Hopf algebroid representing $\mathfrak{U}_n$ is given by the Hopf algebroid associated to the n-th Johnson-Wilson theory $\mathrm{E}(n)$.

Formal groups are, roughly said, coordinate-free versions of formal groups. We refer the reader to [Goe], in particular to [Goe, Proposition 2.43], for a precise definition and discussion. We stick to formal group laws.

5.2 Local cohomology sheaves on $\mathcal{M}_{\mathrm{FG}}$ and chromatic convergence

Definition 5.2.1 (cf. Definition 4.1.1) Let $\underline{L_{\mathfrak{U}_n}}(-) := j_{n*}j_n^*$ and define $\underline{\Gamma_{\mathfrak{Z}^n}}(\mathscr{F})$ via exactness of

$$0 \longrightarrow \underline{\Gamma_{\mathfrak{Z}^n}}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \underline{L_{\mathfrak{U}_{n-1}}}(\mathscr{F}) \,,$$

where the morphism $\mathscr{F} \to \underline{L_{\mathfrak{U}_{n-1}}}(\mathscr{F})$ is given by the unit of $j_{n-1}^* \dashv j_{n-1_*}$.

Remark (Correlation with [HS05b]) The functor $\underline{L_{\mathfrak{U}_n}}$ corresponds to the functor L_n, the functor $\underline{\Gamma_{\mathfrak{Z}^{n+1}}}$ corresponds to T_n, note the shift. Since the calculations are not easier than in the comodule situation, it seems to be of little use to reformulate the statements of [HS05b] in the language of stacks.

Since $\mathcal{M}_{\mathrm{FG}}$ is an Adams stack and the closed substacks $\mathfrak{Z}^n$ are noetherian, every result developed in section 4 can be used. In particular, we have

Proposition 5.2.2 *If $\mathscr{F} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$, then we have isomorphisms*

$$\underline{\mathrm{H}^\bullet_{\mathfrak{Z}^n}}(\mathscr{F}) \cong \mathbb{R}^\bullet_{\mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(\mathrm{BP}_*)})}\underline{\Gamma_{V(I_n)}}(P^*\mathscr{F}) \,.$$

Proof. The invariant ideal $I_n = (p, v_1, \dots, v_{n-1})$ defining the closed substack $\mathfrak{Z}^n$ is regular, in particular weakly proregular. The claim is just a reformulation of Theorem 4.4.2. □

Example 5.2.3 Like in Example 4.4.4 we see that

$$\underline{\mathrm{H}^k_{\mathfrak{Z}^n}}(\mathscr{O}_{\mathcal{M}_{\mathrm{FG}}}) \cong \begin{cases} 0 & k \neq n \\ \mathscr{O}_{\mathcal{M}_{\mathrm{FG}}}/\mathscr{I}_n^\infty & k = n \,. \end{cases}$$

From the last proposition we see that we can use the Čech complex to calculate the comodules corresponding to the local cohomology sheaves of a quasi-coherent sheaf on $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$. We refer to [HS05b, Theorem B & Theorem C] for explicit calculations of the comodules BP_*/I_k corresponding to the quasi-coherent sheaves $\mathscr{O}_{\mathcal{M}_{\mathrm{FG}}}/\mathscr{I}_k$. Note that this suffices to know the value of $\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}^n}}(-)$ on the full subcategory of finitely presentable comodules by the Landweber filtration theorem – every finitely presentable $\mathrm{BP}_*\mathrm{BP}$-comodule has a filtration

$$0 = M_0 \leq M_1 \leq \cdots \leq M_s = M$$

where $M_i \in \Gamma-\mathbf{Comod}$ and $M_{i+1}/M_i \cong \mathrm{BP}_*/I_{s_i}$. Since every comodule is the filtered colimit of finitely presentable ones and $\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}^n}}(-)$ commutes with filtered colimits by Corollary 4.2.4, one can "calculate" all local cohomology sheaves.

The substack $\mathfrak{Z}^n \cap \mathfrak{U}_{n+1} := \mathfrak{Z}^n \times_{\mathcal{M}_{\mathrm{FG}}} \mathfrak{U}_n$ is a closed substack of $\mathfrak{U}_n$ (and hence algebraic) with open complement $\mathfrak{U}_{n-1}$ (in $\mathfrak{U}_n$). It corepresents formal groups of height exactly n.

Proposition 5.2.4 *If $\mathscr{F} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$, then we have natural isomorphisms*

$$j_n^*\big(\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}^n}}(\mathscr{F})\big) \cong \underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(j_n^*\mathscr{F})\,.$$

Proof. We show that both sides define universal δ-functors $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}}) \to \mathbf{QCoh}(\mathfrak{U}_n)$ with the same values for $k = 0$. We have a commuting diagram of the form

$$\begin{tikzcd}
\mathfrak{U}_{n-1} \arrow[rr, hook, "j^n_{n-1}"] \arrow[dr, hook, "j_{n-1}"'] & \circlearrowleft & \mathfrak{U}_n \arrow[dl, hook', "j_n"] \\
& \mathcal{M}_{\mathrm{FG}} &
\end{tikzcd}$$

and one checks that $j_n^* \circ j_{n-1_*} = j^n_{n-1_*}$. Indeed, since j_{n*} is fully faithful, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{QCoh}(\mathfrak{U}_n)}\big(j_n^* \circ j_{n-1_*}, -\big) &\cong \mathrm{Hom}_{\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})}\big(j_{n*} \circ j^n_{n-1_*}, j_{n*}\big)\\
&\cong \mathrm{Hom}_{\mathbf{QCoh}(\mathfrak{U}_n)}\big(j^n_{n-1_*}, -\big)\,,
\end{aligned}$$

where we have omitted arguments.
`Values for` $k = 0$: The left hand side is then defined by exactness of

$$0 \longrightarrow j_n^*\big(\underline{\Gamma_{\mathfrak{Z}^n}}(\mathscr{F})\big) \longrightarrow j_n^*\mathscr{F} \xrightarrow{j_n^*(\eta_{\mathscr{F}})} j_n^*\big(\underline{L_{\mathfrak{U}_{n-1}}}(\mathscr{F})\big)\,.$$

The right hand side is given by

$$0 \longrightarrow \underline{\Gamma_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(j_n^*\mathscr{F}) \longrightarrow j_n^*\mathscr{F} \xrightarrow{\eta'_{j_n^*\mathscr{F}}} j^n_{n-1_*}j^{n^*}_{n-1}(j_n^*\mathscr{F})$$

The evident diagram commutes and the equality follows from the universal property of the kernel.
δ`-functors:` The functor $j_n^*\colon \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}}) \to \mathbf{QCoh}(\mathfrak{U}_n)$ is exact.
`effaceable:` Since $\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}^n}}(-)$ and $\underline{\mathrm{H}^{\bullet}_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(-)$ are right derived functors it is enough to note that j_n^* preserves injective objects and this was shown in Corollary 4.1.3. ☐

There is a similar statement for the stacks $\mathfrak{Z}^n \cap \mathfrak{U}_m$, $n \geq m$.

Remark Since BP_* is a coherent ring and $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ is Adams, the category $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ is locally coherent (Lemma 3.6.6) and the notions "coherent" and "finitely presentable" coincide.

If $\mathscr{F} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ is a quasi-coherent sheaf, we will see that we have an inverse system $\big\{\mathbf{R}\underline{L_{\mathfrak{U}_n}}(\mathscr{F})\big\}_{n\in\mathbb{N}}$. The open substacks $\mathfrak{U}_n$, $n \in \mathbb{N}$, do not cover $\mathcal{M}_{\mathrm{FG}}$, since we miss formal groups of height ∞. Nevertheless, it is enough to know the data of a quasi-coherent sheaf on the $\mathfrak{U}_n$'s. This is made precise in

Proposition 5.2.5 (chromatic convergence) *If* $\mathscr{F} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ *is finitely presentable, then the canonical morphism*

$$\mathscr{F} \longrightarrow \operatorname*{holim}_{n} \mathbf{R}\underline{L_{\mathfrak{U}_n}}(\mathscr{F})$$

is an isomorphism in $\mathbf{D}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$.

Here the countable homotopy limit is understood in the sense of Bökstedt and Neeman, cf. [SPA, Definition 08TC]. Before giving the proof, we explain how $\big\{\underline{L_{\mathfrak{U}_n}}(\mathscr{F})\big\}_n$ is made into an inverse system and that the homotopy limit may be interpreted as a honest limit.

Lemma 5.2.6 *Let* $\mathscr{E} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ *be an injective object. In the system* $\big(\underline{L_{\mathfrak{U}_n}}(\mathscr{E}), \mu_n\big)_{n\geq 0}$ *every*

$$\mu_n \colon \underline{L_{\mathfrak{U}_n}}(\mathscr{E}) \to \underline{L_{\mathfrak{U}_{n-1}}}(\mathscr{E})$$

is a split epimorphism.

Proof. Let us consider the commuting triangle

$$\begin{tikzcd}
\mathfrak{U}_{n-1} \arrow[rr, hook, "j^n_{n-1}"] \arrow[dr, hook, "j_{n-1}"'] & & \mathfrak{U}_n \arrow[dl, hook', "j_n"] \\
& \mathcal{M}_{\mathrm{FG}} &
\end{tikzcd}$$

If $\mathscr{J}$ is an arbitrary injective object of $\mathbf{QCoh}(\mathfrak{U}_n)$, then we have a short exact sequence

$$0 \longrightarrow \underline{\Gamma_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(\mathscr{J}) \longrightarrow \mathscr{J} \xrightarrow{\eta_{\mathscr{J}}} j^n_{n-1_*} j^{n^*}_{n-1}(\mathscr{J}) \longrightarrow 0$$

where η is the unit of $j^{n^*}_{n-1} \dashv j^n_{n-1_*}$. Since $\mathscr{J} = j_n^*\mathscr{E}$ for an injective object $\mathscr{E} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ (the functor j_{n_*} is fully faithful and reflects injectives), this gives a short exact sequence

$$0 \longrightarrow j_{n_*}\big(\underline{\Gamma_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(j_n^*\mathscr{E})\big) \longrightarrow j_{n*}j_n^*\mathscr{E} \longrightarrow j_{n-1_*}j_{n-1}^*\mathscr{E} \longrightarrow 0$$

after applying j_{n_*}. Note that $\underline{\Gamma_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(j_n^*\mathscr{E})$ is an injective object of $\mathbf{QCoh}(\mathfrak{U}_n)$ by Corollary 4.1.3 and $j_{n-1} = j_n \circ j^n_{n-1}$. Now $j_{n_*}\big(\underline{\Gamma_{\mathfrak{Z}^n\cap\mathfrak{U}_n}}(j_n^*\mathscr{E})\big)$ is an injective object in $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ and the short exact sequence splits. ☐

Corollary 5.2.7 *If* $\mathscr{F}^\bullet \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ *is a quasi-coherent sheaf, then the canonical morphism*

$$\lim_n \underline{L_{\mathfrak{U}_n}}(\mathscr{F}) \longrightarrow \operatorname*{holim}_n \underline{L_{\mathfrak{U}_n}}(\mathscr{F})$$

is a quasi-isomorphism.

Proof. Let $\mathscr{F} \rightarrowtail \mathscr{E}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$. Then $\mathbf{R}\underline{L_{\mathfrak{U}_n}}(\mathscr{F}) \in \mathbf{D}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$ is represented by the complex $\underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet)$. For every j we have a split short exact sequence of the form

$$0 \longrightarrow \lim_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^j) \longrightarrow \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^j) \xrightarrow{\mathrm{id}-\nu} \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^j) \longrightarrow 0$$

in $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ by the last lemma. Thus, we have an exact sequence

$$0 \longrightarrow \lim_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \longrightarrow \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \xrightarrow{\mathrm{id}-\nu} \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \longrightarrow 0$$

in $\mathrm{CoCh}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$, which is split exact. Hence, the sequence of complexes defines a triangle in the derived category $\mathbf{D}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$. The canonical morphism $\lim_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \to \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet)$ composes with $\mathrm{id} - \nu$ to give zero, so we deduce a canonical morphism

$$\lim\nolimits_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \longrightarrow \operatorname{holim}\nolimits_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet)$$

of complexes such that

$$\begin{tikzcd}
\operatorname*{holim}_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \arrow[r] & \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \\
\lim_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

commutes. From the discussion above we see that this morphism is a quasi-isomorphism. □

Let us now come to the

Proof of Proposition 5.2.5. Let us fix n for a moment. If $\mathscr{E} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ is injective, then we have a short exact sequence

$$0 \longrightarrow \underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{E}) \longrightarrow \mathscr{E} \longrightarrow \underline{L_{\mathfrak{U}_n}}(\mathscr{E}) \longrightarrow 0 \tag{5.2.1}$$

in $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ by Corollary 4.1.3. Hence, we have a distinguished triangle

$$\mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathbf{R}\underline{L_{\mathfrak{U}_n}}(\mathscr{F})$$

for every $\mathscr{F} \in \mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ in $\mathbf{D}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$ and we obtain a distinguished (!) triangle

$$\operatorname*{holim}_n \mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \operatorname*{holim}_n \mathbf{R}\underline{L_{\mathfrak{U}_n}}(\mathscr{F})\,.$$

In a general triangulated category, the homotopy limit of distinguished triangles may not be distinguished again. Nevertheless this holds in our situation. One possible approach is the injective model structure on the category of (unbounded) chain complexes over $\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})$ and the equivalence of its homotopy category (with respect to the injective model structure) with the derived category $\mathbf{D}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$. Then one can use a "homotopy limits commute with homotopy limits"-argument.

Yet, we can also give a direct proof. Let $\mathscr{E}^\bullet$ be an injective resolution of $\mathscr{F}$ and consider the diagram

$$\begin{tikzcd}
\operatorname*{holim}_n \mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{F}) \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \operatorname*{holim}_n \mathbf{R}\underline{L_{\mathfrak{U}_n}}(\mathscr{F}) \arrow[d] \\
\prod_n \underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{E}^\bullet) \arrow[r] \arrow[d, "\mathrm{id}-\nu"] & \prod_n \mathscr{E}^\bullet \arrow[r] \arrow[d, "\mathrm{id}-\nu"] & \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet) \arrow[d, "\mathrm{id}-\nu"] \\
\prod_n \underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{E}^\bullet) \arrow[r] & \prod_n \mathscr{E}^\bullet \arrow[r] & \prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet)
\end{tikzcd}$$

Since $\underline{\Gamma_{\mathfrak{Z}^{n+1}}}(-)$ and $\underline{L_{\mathfrak{U}_n}}(-)$ both preserve injective objects by Corollary 4.1.3, the products $\prod_n \underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{E}^\bullet)$ resp. $\prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{E}^\bullet)$ of complexes represent the derived product $\prod_n \underline{\Gamma_{\mathfrak{Z}^{n+1}}}(\mathscr{F})$ resp. $\prod_n \underline{L_{\mathfrak{U}_n}}(\mathscr{F})$ in $\mathbf{D}\big(\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})\big)$. The upper row is then given as the mapping cylinder of the vertical morphisms $\mathrm{id}-\nu$. This also shows that the homotopy limit over the constant diagram indexed over $\mathbb{N}$ is given by its constant value (cf. [Nee01, Lemma 1.6.6]).

We are left to show that

$$\operatorname*{holim}_n \mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{n}}}(\mathscr{F}) = 0\,.$$

`Claim:` For every $k \geq 0$ there is an $\ell(k)$ such that

$$\mathrm{H}^k\big(\mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{m}}}(\mathscr{F})\big) = 0$$

for every $m > \ell(k)$.

Then certainly $\operatorname{holim}_n \mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{n}}}(\mathscr{F}) = 0$.

`Proof of the claim:` Since the defining sequence I_n of $\mathfrak{Z}^n$ is regular, we can apply Theorem 4.4.2 to see that the natural morphism

$$\mathbb{R}^k_{\mathbf{QCoh}(\mathcal{M}_{\mathrm{FG}})}\underline{\Gamma_{\mathfrak{Z}^{n}}}(\mathscr{F}) \longrightarrow \mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(\mathrm{BP}_*)})}\underline{\Gamma_{Z^{n}}}(P^*\mathscr{F})$$

is an isomorphism. Here, $Z^n = \mathfrak{Z}^n \times_{\mathcal{M}_{\mathrm{FG}}} \mathrm{Spec}(\mathrm{BP}_*) \simeq \mathrm{Spec}(\mathrm{BP}_*/I_n)$.

The ring $A := \mathrm{BP}_*$ is coherent and hence every finitely presented module M has a finite free resolution ([Gla89, Corollary 2.5.2]). We can even assume that we have a resolution of finite length: Write $A = \bigcup_j A_j$, where $A_j = \mathbb{Z}_{(p)}[v_1, \ldots, v_j]$ (cf. p. 67) are polynomial rings over the regular local ring $\mathbb{Z}_{(p)}$; note that A is flat over each A_j. Since $M := P^*\mathscr{F}$ is assumed to be finitely presented, we may assume that M is a finite A_j-module, $j \gg 0$. By [Lan02, XXI, Theorem 2.8] (together with [SPA, Proposition 00O7]) every finite A_j-module admits a finite free resolution of *finite length*. Since A is flat over A_j, the A-module M has a finite free resolution of finite length.

Hence, we can reduce to the case $P^*\mathscr{F} = \mathscr{O}_{\mathrm{Spec}(A)}$. But then

$$\mathbb{R}^k_{\mathbf{Mod}(\mathscr{O}_{\mathrm{Spec}(A)})}\underline{\Gamma_{Z^{n}}}(\mathscr{O}_{\mathrm{Spec}(A)}) = \mathrm{H}^k_{I_n}(A) = 0$$

for all $k \neq n$, cf. Example 4.4.4. If we fix k, then this shows that

$$\mathrm{H}^k\big(\mathbf{R}\underline{\Gamma_{\mathfrak{Z}^{m}}}(\mathscr{F})\big) = 0$$

for $m > k$ and proves the claim. □

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