Semistable Extension of Families of Curves

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Chapter 1

Introduction

The stable reduction theorem of Deligne-Mumford says that for any smooth projective curve C over the function field K of a discrete valuation ring R, there exists a finite separable extension L of K such that the curve $C \otimes_K L$ has stable reduction over the integral closure of R in L. For the reason why one can choose the field extension to be separable, see, for example [23], Exercise 10.4.2.

One can prove the properness of the moduli space of stable $n-$pointed genus g curves using the above theorem.

Deligne generalised the result above (See [9] or Theorem 3.12 [2]) as follows: for any proper stable curve C of genus $g \geq 2$ over an open dense subscheme of a quasi-compact quasi-separated integral scheme S there exists a proper surjective morphism $S' \to S$ such that $C \times_S S'$ can be extended to a proper stable curve over S'. The theorem is so-called Stable Extension Theorem.

Remark: The curve C need not be smooth.

A. de Jong extended Deligne's result in [5] (See also [7]) showing that for any proper curve C over an integral quasi-compact excellent scheme S, there exists an alteration $S' \to S$ and a modification $C' \to C \times_S S'$ such that C' is a proper semi-stable curve over S'. For the precise statement see Chapter 5, Theorem 5.3.1.

Our Main Concern: Roughly speaking, we would like to give an explicit construction describing "the best possible model" for a given semi-stable curve (Definition 4.2.1) which is defined over a discrete valuation field.

We proceed in two different ways;

I) Using properties of moduli space of stable curves: since the moduli space of stable curves of genus g is proper, we can use the 'extended' version of valuation criterion for properness of morphisms of algebraic stacks given by Deligne and Mumford to show that the reduction remains stable after possibly a finite extension of the base field (See 5.2).

II) Using the semi-stable reduction theorem on every component of the given semi-stable curve C and then gluing the suitable models in a certain way (See 5.3).

In the last chapter we introduce some monodromy conditions which ensure that families of curves, defined over an open dense subset of the base scheme, can be extended to the whole base in a stable manner (i.e., the family extends to a "stable family" over the whole scheme).

Chapter 2

Normality and gluing along sections

2.1 Product

In this chapter we collect the prerequisite results which are needed for the next chapters. They can be found in several books written on algebraic geometry. Our main source for this chapter is Algebraic geometry and arithmetic curves, Q. Liu [23].

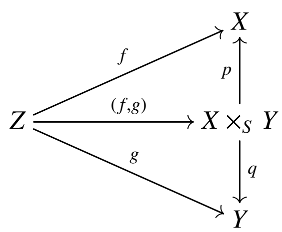
2.1.1 Product along arbitrary base scheme

Definition and convention 2.1.1. If X is a scheme and $x \in X$ is a point, then $k(x)$ denotes the residue field of the local ring $O_{X,x}$. Sometimes we consider a point as a scheme $x = \operatorname{Spec} k(x)$. A scheme S is integral if it is irreducible and reduced. Its function field is denoted $R(S)$.

Definition 2.1.2. Let S be a scheme, and let X and Y be two schemes over S ($S-$schemes). The fibre product of X and Y over S is defined to be an $S-$scheme $X \times_S Y$, together with two morphisms of $S-$schemes $p : X \times_S Y \to X$, $q : X \times_S Y \to Y$ (the projections), verifying the following universal property:

Let $f : Z \to X$, $g : Z \to Y$ be two morphisms of $S-$schemes. Then there exists a unique morphism of $S-$schemes a

$(f,g) : Z \to X \times_S Y$ which makes the following diagram commutative:

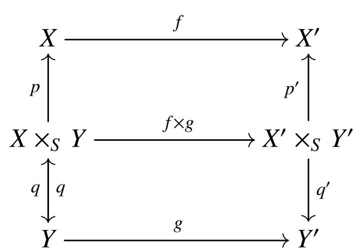


It is well known that the fibered product $(X \times_S Y, p, q)$ exists and is unique up to unique isomorphism.

The following facts easily follow from the universal property of the fibered product.

Theorem 2.1.3. Let S be a scheme and X, Y and Z are $S-$ schemes. The following properties hold.

- $X \times_S S \simeq X$, $Y \times_S X \simeq X \times_S Y$, $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$
- Let Z be a $Y-$scheme, considered as an $S-$scheme via $Z \to Y \to S$. Then we have a canonical isomorphism of $S-$schemes $(X \times_S Y) \times_Y Z \simeq X \times_S Z$. Where $X \times_S Y$ is endowed with the structure of a $Y-$scheme via the second projection.
- Let $f : X \to X'$, $g : Y \to Y'$ be morphisms of $S-$schemes. There exists a unique morphism of $S-$schemes $f \times g : X \times_S Y \to X' \times_S Y'$ which makes the following diagram commutative:



- let $i : U \to X$, $j : V \to Y$ be open subschemes. Then the morphism $i \times j$ includes an isomorphism $U \times_S V \simeq p^{-1}(U) \cap q^{-1}(V) \subseteq X \times_S Y$.

Definition 2.1.4. We say that the morphism of schemes $X \to S$ is projective if it factors into a closed immersion $X \to \mathbb{P}^n_S$ followed by the canonical morphism $\mathbb{P}^n_S \to S$.

2.1.2 Fibres of morphisms

Definition 2.1.5. Let $f : X \to Y$ be a morphism of schemes. For any $y \in Y$

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

This is the fibre of f over y. The second projection $X_y \to \operatorname{Spec} k(y)$ makes X_y into a scheme over $k(y)$. In case Y is irreducible of generic point ξ; we call X_ξ the generic fibre of f.

It can be seen that the first projection

$$p : X_y = X \times_Y \operatorname{Spec} k(y) \to X$$

induces a homeomorphism from X_y onto $f^{-1}(y)$.

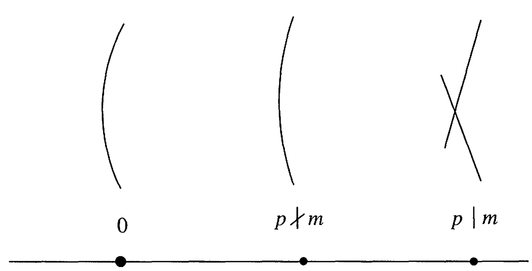
Example 2.1.6. (See [23]) Let m be a non-zero integer. Let

$$f : X = \operatorname{Spec} \mathbb{Z}[Y, Z]/(YZ^2 - m) \to \operatorname{Spec} \mathbb{Z}$$

be the canonical morphism. For any prime number p, we let X_p is the fibre of f over the point $p\mathbb{Z} \in \operatorname{Spec} \mathbb{Z}$. Then the generic fibre of X is

$$\operatorname{Spec} \mathbb{Q}[Y, Z]/(YZ^2 - m) = \operatorname{Spec} \mathbb{Q}[Z, 1/Z].$$

And the fibre X_p is equal to $\operatorname{Spec} \mathbb{F}_p[Y, Z]/(YZ^2 - m)$. Therefore if the prime p does not divide m the fibre X_p is integral, being isomorphic to $\operatorname{Spec} \mathbb{F}_p[Y, 1/Y]$. Otherwise (if $p|m$) it has two irreducible components. Note that the scheme X itself is integral. So, we can 'cut' X into slices, most of these slices X_p (for $p \nmid m$) staying integral, but some become reducible. This phenomenon is called 'degeneration'.



2.2 Base Change

Definition 2.2.1. A morphism of schemes $f : X \to Y$ is of finite type if f is quasi-compact (inverse image of any affine open subset of codomain can be covered by a finite number of open subsets of domain) and if for every affine subset V of Y, and for every affine open subset U of $f^{-1}(V)$, the canonical homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_Y(V)$–algebra.

Definition 2.2.2. Let k be a field. A scheme of finite type over $\operatorname{Spec} k$ is called algebraic variety over k. (We sometimes restrict the definition by imposing additional conditions such as irreducibility.)

Theorem 2.2.3. Let X be an algebraic variety over k, and let K/k be an algebraic extension. Then the following properties are true.

1. We have $\dim X_K = \dim X$.
2. If X is reduced and K/k is separable, then X_K is reduced.

Proof. Since we may assume that X is affine, say $X = \operatorname{Spec} A$.

(1) is obvious because $A \to A_K = A \otimes_k K$ is injective and integral therefore $\dim A = \dim A_K$. In general for if $\phi : A \to B$ is an injective morphism of rings and $q \in \operatorname{Spec} B$ and $p = \phi^{-1}(q) \in \operatorname{Spec} A$ then we have $\dim V(p) = \dim V(q)$.

(2) Since the ring A can be embedded in $\oplus_i A/p_i$ for $p_1, \dots p_n$ the minimal prime ideals of A, one can assume A is an integral ring. Now A_K is a subring of $\operatorname{Frac}(A) \otimes_k K$, therefore it is enough to show that the ring $F \otimes_k K$ is reduced for any field F containing k. Every element of $F \otimes_k K$ is contained $F \otimes_k K'$, with K' finite separable over k, and we therefore we can assume that K is finite over k. It follows that $K \simeq k[T]/(P(T))$, where $P(T) \in k[T]$ is a separable polynomial but $P(T)$ is still separable in $F[T]$, therefore $F \otimes_k K \simeq F[T]/(P)$ is reduced. □

2.2.1 Extending varieties to algebraically closed field

Definition 2.2.4. Let X be an algebraic variety over k. Let $\bar{k}$ be the algebraic closure of k. We say that X is geometrically reduced (resp. geometrically irreducible, geometrically integral, geometrically connected) if $X_{\bar{k}}$ is reduced (resp. irreducible, integral, connected).

Example 2.2.5. Let k be a field of characteristic different from 2, and $a \in k$ which is not a square. Consider the projective variety

$$X = \operatorname{Proj} k[x, y, z]/(x^2 - ay^2)$$

Let $\alpha \in \bar{k}$ be a square root of a and $K = k[\alpha]$. Then we see that X is integral while $X_K = \operatorname{Proj} K[x, y, z]/(x - \alpha y)(x + \alpha y)$ is not.

Definition 2.2.6. Let X be an algebraic variety over a field k, and let K/k be a field extension and $X(K)$ denote the set of morphisms of $K-$schemes from $\operatorname{Spec}(K) \to X$. The elements of $X(K)$ are called $K-$valued points of X. The instant remark is that $X(K)$ is not in general the set of points $x \in X$ such that $k(x) \subseteq K$.

Theorem 2.2.7. If X is an algebraic variety over k, and K/k is a field extension, the following properties hold.

1. There is a canonical bijection $X(K) \to X_K(K)$
2. The data consisting of a point $x \in X$ and a homomorphism of $k-$algebras $k(x) \to K$ uniquely determine an element of $X(K)$
3. For any extension K'/K, we have a natural inclusion $X(K) \subseteq X(K')$.

Proof. (1) If we denote the set of morphisms of $S-$schemes from X to Y by $\mathrm{Mor}_S(X, Y)$ then the projections p and q induce the maps

$$\mathrm{Mor}_S(Z, X \times_S Y) \to \mathrm{Mor}_S(Z, X)$$

$$\mathrm{Mor}_S(Z, X \times_S Y) \to \mathrm{Mor}_S(Z, Y)$$

where Z is an $S-$scheme. This gives a map

$$\mathrm{Mor}_S(Z, X \times_S Y) \to \mathrm{Mor}_S(Z, X) \times \mathrm{Mor}_S(Z, Y),$$

By the universal property of product it is bijective. Now by taking $Z = S$ we have a canonical bijection of sections

$$(X \times_S Y)(S) \simeq X(S) \times Y(S).$$

If $Y = Z$ and we have

$$\mathrm{Mor}_S(Y, X \times_S Y) \simeq \mathrm{Mor}_S(Y, X),$$

in which $X \times_S Y$ is endowed with the structure of a $Y-$scheme via the second projection.

(2) Take $s \in X(K)$ and let $x \in X$ be the image of $s : \operatorname{Spec} K \to X$. We have $s^{\#}_x : \mathcal{O}_{X,x} \to K$ induces a homomorphism $k(x) \to K$ Conversely, if $x \in X$ and $k(x) \to K$ a given homomorphism, define a morphism of $k-$schemes $\operatorname{Spec} K \to \operatorname{Spec} k(x)$. Compose this with the canonical morphism $\operatorname{Spec} k(x) \to X$ we obtain an element of $X(K)$ and these two processes are inverses.

(3) It is obvious because the composition with $\operatorname{Spec} K' \to \operatorname{Spec} K$ induces a map $X(K) \to X(K')$ in (2) we showed that the map is injective. □

Theorem 2.2.8. Let X be an integral algebraic variety over k. Then X is geometrically reduced if and only if $K(X)$ is a finite separable extension of a purely transcendental extension $k(T_1, \ldots, T_d)$.

Proof. It is obvious if $K(X)$ is a finite separable extension of the field $L := k(T_1, \ldots, T_d)$, then $K(X)$ can be written as $L[S]/P(S)$ for an irreducible separable polynomial $P(S) \in L[S]$. Therefore $K(X) \otimes_k \bar{k} = L'[S]/P(S)$, where $L' = L \otimes_k \bar{k} = \bar{k}(T_1, \ldots, T_d)$. But $P(S)$

remains separable in $L'[S]$. Therefore $K(X) \otimes_k \bar{k}$ is reduced. So X is geometrically reduced.

Conversely, If X is geometrically reduced and $K(X)$ is a finite extension of $L := k(T_1, \ldots, T_d)$, then if $K(X)/L$ is separable, there is nothing to prove. So assume that $K(X) \neq L_s$ where L_s is the separable closure of L in $K(X)$ and $f \in K(X) \backslash L_s$ is such that $f^p \in L_s$. $p = \mathrm{Char}(k)$. We show in the following that $L_s[f]$ is finite and separable over a purely transcendental extension. Since $K(X)/L$ is finite extension, this process decomposes the extension $K(X)/L_s$ into a sequence of purely inseparable extensions which proves the theorem.

Let $p(S) = S^r + Q_{r-1}S^{r-1} + \cdots + Q_0 \in L[S]$ be the minimal polynomial of f^p over L. We show that at least one $Q_i \notin k(T_1^p, \ldots, T_d^p)$. If not, $P(S^p) = H(S)^p$ for a polynomial $H(S) \in \bar{k}[S]$. Therefore $L_s[f] \otimes_k \bar{k} = (L_s \otimes_k \bar{k})[S]/P(S^p)$ is not reduced. This is in contradiction with the fact that $K(X) \otimes_k \bar{k}$ is reduced and $L_s[f] \otimes_k \bar{k} \subseteq K(X) \otimes_k \bar{k}$ as algebras. So, a power of say, T_1 prime to p appears in one of the Q_i. Hence T_1 is algebraic and separable over $k(f, T_2, \ldots, T_d)$. Since $L_s[f]$ is finite separable over $k(f, T_1, T_2, \ldots, T_d)$, we have $L_s[f]$ finite and separable over $k(f, T_2, \ldots, T_d)$. This extension is purely transcendental because its transcendental degree over k is equal to that of $L_s[f]$, which is d. □

2.2.2 Rational points of geometrically reduced varieties

Theorem 2.2.9. Let X be a geometrically reduced algebraic variety over a field k. Let k^s be the separable closure of k. Then $X(k^s) \neq \emptyset$.

Proof. (See [23], Proposition 3.2.20): We may assume that $k = k^s$ replacing X by X_{k^s} (X_{k^s} is geometrically reduced). Now we would like to show that $X(k) \neq \emptyset$. Again by replacing X by an irreducible affine open sebset, we may assume that X is affine and integral. Since now X is integral algebraic variety which is geometrically reduced, by theorem above its fraction field $K(X)$ is finite separable over $k(T_1, \ldots, T_d)$ therefore $K(X) = k(T_1, \ldots, T_d)[f]$. Assume that $P(S) \in \mathrm{Frac}(A)[S]$ is the minimal polynomial of f and $A = k[T_1, \ldots, T_d]$, $B = \mathcal{O}_X(X)$. We may assume that $A[f] \subseteq B$ after localising B if necessary. Since $\mathrm{Frac}(B) = \mathrm{Frac}(A)[f]$, there exists a $g \in A$ such that $B \subseteq A_g[f]$ and $P(S) \in A_g[S]$. It follows that $B_g = A_g[f] = A_g[S]/P(S)$. As $P(S)$ is a separable polynomial, its resultant $h := \mathrm{Res}(P(S), P(S')) \in A_g$ is non-zero. The field k is infinite because it is separably closed. Therefore there exists a $t \in k^d$ such that $g(t) \neq 0$ and $h(t) \neq 0$. Let $y \in \mathrm{Spec}\, A_g$ be the point corresponding to t. Then $k(y) = k$ and

$$B_g \otimes_{A_g} k(y) = k[S]/\tilde{P}(S),$$

in which $\tilde{P}(S)$ is the image of $P(S) \in k(y)[S]$. The resultant of $\tilde{P}(S)$ is $h(t) \neq 0$. Therefore $\tilde{P}(S)$ is separable and $B_g \otimes_{A_g} k(y)$ is a direct sum of k's. Hence the points of $\mathrm{Spec}\, B_g$ over y are rational over k which means $X(k) \neq \emptyset$. □

2.3 Some global properties

2.3.1 Separatedness

Definition 2.3.1. Let X be a topological space. Let $\Delta : X \to X \times X$ be the diagonal map sending x to (x, x) in which $X \times X$ is endowed with the product topology. From topology we know that X is separated if and only if the image of diagonal map $\Delta(X)$ is closed in $X \times X$.

Although the underlying topological space of a scheme is almost never separated, we would like to define the separatedness of schemes in accordance to the above equivalence.

Definition 2.3.2. Let $f : X \to Y$ be a morphism of schemes. The morphism $\Delta := (Id_X, Id_x) : X \to X \times_Y X$ is called the diagonal morphism of f. We say X is separated over Y if Δ is a closed immersion of schemes. This is a local property on Y. The scheme X is separated if it is separated over $\mathbb{Z}$.

It is obvious that any morphism of affine schemes $X \to Y$ is separated because in this case $X \times_Y X$ and Δ are respectively $B \otimes_A B$ and $\rho : B \otimes_A B \to B$ defined by $\rho(b_1 \otimes b_2) = b_1 b_2$. Since ρ is surjective, Δ is a closed immersion.

From this we conclude that if $f : X \to Y$ is a morphism of schemes such that $\Delta(X)$ is a closed subset of $X \times_Y X$. Then f is separated.

Theorem 2.3.3. If X is scheme then X is separated if and only if there exists a covering of X by affine open subsets U_i such that for all i, j, $U_i \cap U_j$ is affine and $\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective.

Proof. First of all we have the inverse image of $U \times_{\mathbb{Z}} V$ under the diagonal morphism $\Delta^{-1}(U \times V) = U \cap V$ and also $\Delta : U \cap V \to U \times_{\mathbb{Z}} V$ corresponds to $\mathcal{O}(U) \otimes_{\mathbb{Z}} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Now if X is separated then Δ is a closed immersion and $U \times V$ is affine therefore $U \cap V$ is affine as well.

Conversely, since $\Delta : \Delta^{-1}(U_i \times_{\mathbb{Z}} U_j) \to U_i \times U_j$ is a closed immersion for i and j and $U_i \times_{\mathbb{Z}} U_j$ cover $X \times_{\mathbb{Z}} X$, Δ is a closed immersion. □

Application of the theorem above. 2.3.4. The projective space $\mathbb{P}^n_{\mathbb{Z}}$ is separated because we can cover it with the affine open subsets $D_+(T_i)$ for $0 \leq i \leq n$ and the condition of the theorem above is satisfied.

It is trivial from the definition of separatedness that open and closed immersions are separated morphisms and that the composition of two separated morphisms is a separated morphism. Therefore any projective morphism is separated.

2.3.2 Properness

Definition 2.3.5. We say that a morphism $f : X \to Y$ is closed if f maps any closed subset of X onto a closed subset of Y. We say that f is universally closed if for any base change $Y' \to Y$, morphism $X \times_Y Y' \to Y'$ stays a closed morphism.

We say that a morphism of schemes $f : X \to Y$ is proper if it is of finite type, separated and universally closed. We say that a $Y-$scheme is proper if the structure morphism is proper. Clearly, properness is a local property on Y.

Example 2.3.6. Closed immersions are proper. It is also well-known that if the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is proper then B is finite over A. Moreover if X is proper over $\operatorname{Spec} A$ then $O_X(X)$ is integral over A. From this one can deduce if X is a reduced algebraic variety which is proper over a field k then $O_X(X)$ is a $k-$vector space of finite dimension. Even more generally, if X is a scheme (not necessarily reduced) over an arbitrary Noetherian ring A, one can show that $O_X(X)$ is finite over A, using the finiteness theorem of coherent sheaves.

Definition 2.3.7. Let K be a field. A valuation of K is a map ν from K^* to a totally ordered Abelian group Γ, verifying the following properties:

1. $\nu(\alpha\beta) = \nu(\alpha) + \nu(\beta)$, (i.e., ν is a group homomorphism);
2. $\nu(\alpha + \beta) \geq \min \nu(\alpha), \nu(\beta)$.

Convention: $\nu(0) = +\infty$. The set $O_\nu = \{\alpha \in K \mid \nu(\alpha) \geq 0\}$ is called the valuation ring of ν (or the valuation ring of K). In general, a ring is called a valuation ring if it is the valuation ring of a field for a valuation. The valuation ring is a local ring with the maximal ideal $m_\nu = \{\alpha \in K \mid \nu(\alpha) > 0\}$.

Lemma 2.3.8. Let O_K be a valuation ring, $K = \operatorname{Frac}(O_K)$, and A a local subring of K which dominates O_K, meaning $O_K \subseteq A$ and the morphism $O_K \to A$ is a local homomorphism of local rings. Then $A = O_K$.

Proof. If there exists an $a \in A \backslash O_K$, then $\nu(1/a) > 0$. Hence $1/a \in m_\nu \subseteq m_A$. This implies that $1 = a.(1/a) \in m_A$ which is impossible. □

2.3.3 Extending of rational points

The following result characterizes the properness (See [16], Theorem II.4.7).

Theorem (Valuation criterion for properness) 2.3.9. A morphism of finite type $f : X \to Y$ is proper if and only if for any valuation ring O_K over Y, with fraction field K, the canonical map

$$X(O_K) \to X_K(K)$$

is bijective.

Proof. See [16], Theorem II.4.7. □

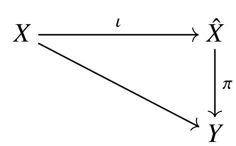
Remark 2.3.10. Concerning the property of properness, it follows from the above criterion that if the morphism of schemes $f : X \to Y$ is proper then the fibre $X_y \to \operatorname{Spec} k(y)$ is proper. One can ask whether the converse is true. If we do not impose any additional condition, then it is trivially false; as an counter-example we can take Y to be a Noetherian scheme and $f : X \to Y$ an open immersion. Nevertheless under some additional assumptions we can see that the converse holds. The most important case for us is as follows;

Let O_K be a discrete valuation ring, X an irreducible scheme, $X \to \operatorname{Spec} O_K$ a surjective morphism and of finite type. If the fibres $X \to \operatorname{Spec} O_K$ are geometrically connected, and if the special fibre $X_s \to \operatorname{Spec} k(s)$ is proper then $X \to \operatorname{Spec} O_K$ is proper. See [15], IV 15.7.10 (the main reference) and [23], Remark 3.3.28

In what follows, we mention an important theorem from Nagata without proof.

2.3.4 Compactifications of schemes and Chow's lemma

Theorem (Nagata) 2.3.11. Let X be a separated scheme of finite type over a Noetherian scheme Y. There exists a proper scheme $\hat{X}$ over Y such that X is embedded in $\hat{X}$ through an open immersion which scheme-theoretically has a dense image; i.e., one has the following commutative diagram



where ι is a dense open immersion and where π is proper.

Proof. For a proof of the theorem see [31]. □

Definition 2.3.12. A proper algebraic variety over a field is called a complete variety.

The most common class of proper morphisms is that of projective morphisms. Projective morphisms enjoy the same good properties as proper morphisms, such as stability under base change and composition. Projective morphisms and proper morphisms are connected with one another by Chow's lemma.

Theorem 2.3.13. (Chow's lemma) Let Y be a Noetherian scheme. For any proper morphism $X \to Y$, there exists a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ & \searrow^{g} & \downarrow \\ & & Y \end{array}$$

with f, g projective, and $f^{-1}(U) \to U$ is an isomorphism for some everywhere dense open subset $U \subseteq X$.

Definition 2.3.14. The morphism $f : X \to Y$ is quasi-projective if it can be decomposed into an open immersion of finite type $X \to Z$ and a projective morphism $Z \to Y$.

2.4 Normality

Definition 2.4.1. A normal Noetherian integral domain of dimension 0 or 1 is called a Dedekind domain. Let X be a scheme. We say $x \in X$ is a normal point of X if the local ring $\mathcal{O}_{X,x}$ is normal (i.e., it is integral and integrally closed in its fraction field). A scheme X is normal if it is normal at every point $x \in X$. A normal locally Noetherian scheme of dimension 0 or 1 is called a Dedekind scheme. We have included dimension 0 to the definition so that the property of being a Dedekind domain is stable by localisation, which in turns means an open subscheme of a Dedekind scheme is – by our definition – a Dedekind scheme as well.

Theorem 2.4.2. Let X be a normal locally Noetherian scheme and F be a closed subset of X of codimension ≥ 2. Then the restriction

$$\mathcal{O}_X(X) \to \mathcal{O}_X(X \backslash F)$$

is an isomorphism. This means that every regular function on $X \backslash F$ extends uniquely to a regular function on X.

Proof. By assuming $X = \operatorname{Spec} A$ is affine, every prime ideal $p \subset A$ of height 1 is in $X \backslash F$. Therefore the theorem immediately follows from the fact that $A = \bigcap_{p \in \operatorname{Spec} A, \operatorname{ht} p = 1} A_p$. for normal rings of dimension ≥ 1. □

Lemma 2.4.3. Let X be a reduced $S-$scheme, Y a separated $S-$scheme and also consider two morphisms of $S-$schemes $f, g : X \to Y$. If $f|_U = g|_U$ for some everywhere dense open subset $U \subseteq X$, then $f = g$.

Proof. Set $\Delta = \Delta_{Y/S}$ and $h = (f, g) : X \to Y \times_S Y$. We have $\Delta \circ f = (f, f)$ due to the universal property of the morphism (f, f). Therefore $\Delta \circ f$ and h coincide on U.

Consequently, $U \subseteq h^{-1}(\Delta(Y))$. Since $\Delta(Y)$ is closed, we have $X = h^{-1}(\Delta(Y))$. Hence $f(x) = g(x)$ for every $x \in X$.

Now for showing $f = g$ we can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and let φ, ψ be ring homomorphisms corresponding to f and g respectively. For $b \in B$ set $a = \varphi(b) - \psi(b)$. Then $a|_U = 0$. It follows that $U \subseteq V(a)$, and hence $V(a) = \operatorname{Spec} A$ since U is dense. This implies that a is nilpotent and since A is reduced, we have $a = 0$ Therefore $\varphi = \psi$, and $f = g$. □

2.4.1 Extending morphism to points of codimension 1

Theorem 2.4.4. Let $Y \to S$ be a proper morphism over a locally Noetherian scheme. Let X be a normal $S-$scheme of finite type and consider a morphism of $S-$schemes $f : U \to Y$ defined on a non-empty open subset U of X. Then f extends uniquely to a morphism $V \to Y$, where V is an open subset of X containing all points of codimension 1.

Proof. (See [23], Proposition 4.1.16) Since $Y \to S$ is separated and X reduced from the lemma above the uniqueness is obvious. Let ξ be the generic point of X f induces a morphism $f_\xi : \operatorname{Spec} K(X) \to Y$. Let $x \in X$ be a point of codimension 1. Then $\mathcal{O}_{X,x}$ is a discrete valuation ring with the field of fractions $K(X)$. From Corollary 2.3.10 f_ξ extends to a morphism $f_x : \operatorname{Spec} \mathcal{O}_{X,x} \to Y$. Since Y is of finite type over S, f_x can be extended to $g : U_x \to Y$ where U_x is an open neighbourhood of x.

Set W to be an affine open neighbourhood of $g(x)$. Consider the restriction of f and g to $\xi \in U' := f^{-1}(W) \cap g^{-1}(W)$. Since $\mathcal{O}_X(U') \subseteq K(X)$, the ring homomorphisms $\mathcal{O}_Y(W) \to \mathcal{O}_X(U')$ corresponding to $f|_{U'}$ and $g|_{U'}$ are identical. Therefore their corresponding morphisms of schemes are identical as well; $f|_{U'} = g|_{U'}$ by the Theorem 2.3.9. Now using the lemma 2.4.3 above f and g coincide on $U \cap U_x$. If we take another point of codimension 1, say $x' \in X$ the same reasoning shows that $g' : U_{x'} \to Y$ coincide with f and g respectively on $U \cap U_{x'}$ and $U_x \cap U_{x'}$ Therefore f can be extended to an open subset $V \subseteq X$ containing all points of codimension 1. □

With the same assumptions as theorem 2.4.4, if $\dim X = 1$ then f extends uniquely to a morphism $X \to Y$. In what follows, we are going to explain a useful lemma for normality.

2.4.2 A criterion for normality

Theorem 2.4.5. ([23], Lemma 4.1.18) Let $\mathcal{O}_K$ be a discrete valuation ring, with field of fractions K and residue field k. Let X be an $\mathcal{O}_K-$scheme such that $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_K$ for every affine open subset U of X. We suppose that X_K is normal and that X_k is reduced. Then X is normal.

Proof. As always, we may assume that $X = \operatorname{Spec} A$ is affine. Since A is flat over $\mathcal{O}_K$ then $A \to A \otimes_{\mathcal{O}_K} K$ is injective. Therefore A is integral domain. If t is a uniformizer for $\mathcal{O}_K$ take $\alpha \in \mathrm{Frac}(A)$ so that α is integral over A. Since $A \otimes_{\mathcal{O}_K} K$ is normal, there exist $a \in A,\ r \in \mathbb{Z}$ such that $\alpha = t^{-r}a$. Now if $a \notin tA$ we are going to show that $r \leq 0$. We have $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$ an integral equation for α over A. If $r > 0$ by multiplying this equation by t^{rn} we see that α is nilpotent in A/tA. Hence $a \in tA$ which is a contradiction. Therefore $r \leq 0$ and $\alpha \in A$. □

There are several other criteria for normality such as Serre's R_k and S_k conditions.

Definition 2.4.6. Let X be a locally Noetherian scheme and $k \geq 0$ be an integer. We say that X has the property (R_k) is X is regular at all of its points of codimension $\leq k$. We say that X verifies property (S_k) if for any $x \in X$ we have

$$\operatorname{depth} \mathcal{O}_{X,x} \geq \inf\{k, \dim \mathcal{O}_{X,x}\}.$$

Then one can show that a locally Noetherian connected scheme X is normal if and only if it verifies R_1 and S_2.

Definition 2.4.7. Let X be an integral scheme. A morphism $\pi : X' \to X$ is called a normalisation morphism if X' is normal and if every dominant morphism $f : Y \to X$ with Y normal factors uniquely through π:

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & X \\ \downarrow & \nearrow_{\pi} & \\ X' & & \end{array}$$

We can extend the definition of normalisation to reducible schemes.

Extended Definition 2.4.8. Let X be a scheme having only finite number of irreducible components $X_1, \ldots, X_n$ (endowed with the reduced closed subscheme structure). The disjoint union $X' = \coprod_{1 \leq i \leq n} X'_i$ where X'_i is the normalisation of the integral scheme X_i (defined in 1.4.6 above) is called the normalisation of X. By construction, X' is endowed with a surjective integral morphism $\pi : X' \to X$. If X_{red} is the reduced scheme associated to X, then $X'_{\mathrm{red}} = X'$.

Definition 2.4.9. Assume X is an integral scheme and L is an algebraic extension of its function field $K(X)$. We define the normalisation of X in L to be an integral morphism $\pi : X' \to X$ with X' normal, $K(X') = L$, and such that π extends the canonical morphism $\operatorname{Spec} L \to X$.

Remarks 2.4.10.

1. If $\pi : X' \to X$ is a normalisation of X then for any open subscheme U of X, the restriction morphism $\pi^{-1}(U) \to U$ is a normalisation of U.

2. If A is an integral domain and A' is the integral closure of A in $\mathrm{Frac}(A)$ then the morphism $\operatorname{Spec} A' \to \operatorname{Spec} A$ induced by the canonical injection $A \to A'$ is a normalisation morphism.

Theorem 2.4.11. The normalisation of an integral scheme X exists and it is unique up to isomorphism. Moreover, a morphism $f : Y \to X$ is the normalisation morphism if and only if Y is normal, and f is birational and integral.

Proof. According to the universal property of normality, the uniqueness is immediate. For the existence it is enough to cover X with affine open subsets U_i and apply the remark above to get normalisation morphisms $U'_i \to U_i$. Now gluing morphisms along with their intersections gives us the desired morphism. The rest of statement is obvious because of the remark above. □

In the same manner, one can show that the normalisation X in $L := K(X')$ exists and is unique. Moreover, for any affine open subset $U \subseteq X$, $\pi^{-1}(U)$ is affine and $\mathcal{O}'_X(\pi^{-1}(U))$ is integral closure of $\mathcal{O}_X(U)$ in L.

Theorem 2.4.12. (See [23], Proposition 4.1.25) Let X be a normal Noetherian scheme and L be a finite separable extension of $K(X)$. Then the normalisation $X' \to X$ of X in L is a finite morphism.

Proof. We may assume that $X = \operatorname{Spec} A$. If B is the integral closure of A in L, we want to show that B is finite as an $A-$module. Since A is Noetherian, one can extend L (By definition, $L = \operatorname{Frac} B$) in such a way that it is Galois over $K := K(X)$.

Now consider the trace form $\operatorname{Tr}_{L/K} : L \times L \to K$ sending $(x, y) \mapsto \operatorname{Tr}_{L/K}(xy)$. From linear algebra, the form is non-degenerate and bilinear because L/K is separable extension. Take $\{e_1, \ldots, e_n\}$ as a base of L/K with the $e_i \in B$. There exists a basis $\{e_1^*, \ldots, e_n^*\} \subset L$ dual to the basis above, meaning $\operatorname{Tr}_{L/K}(e_i e_j^*) = \delta_{ij}$. If we take $b \in B$ we can represent it as $b = \sum_j \lambda_j e_j^*$ with $\lambda_j \in K$. But then we have $\lambda_j = \operatorname{Tr}_{L/K}(be_j) \in B \cap K = A$. Therefore B is a sub-$A-$module of $\sum_j Ae_j^*$ and finite over A. □

Since the integral closure of $k[x_1, \ldots, x_n]$ in L (finite extension of $k(x_1, \ldots x_n)$) is finite over $k[x_1, \ldots, x_n]$, the normalisation in L of an integral algebraic variety over the field k is finite. In particular, the normalisation of an integral algebraic variety over a field k is again an algebraic variety over the same field.

2.4.3 Integral closure of Dedekind rings

Theorem 2.4.13. Let A be a Dedekind ring with field of fractions K. If L is a finite extension of K, and B is the integral closure of A in L, then B is a Dedekind ring and the canonical morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ has finite fibres.

Proof. Consider the field extension L/K we can decompose it into a separable extension and a purely inseparable extension. Due to Theorem 2.4.12, we only have to deal with the inseparable part. So, we can assume that the extension L/K is purely inseparable. Therefore there exists a power p^e of the characteristic $p = \operatorname{Char}(K)$ such that $L^{p^e} \subseteq K$. Hence we have $B^{p^e} \subseteq A$. Let $p \in \operatorname{Spec} A$ then $\sqrt{pB}$ is the unique prime ideal of B lying above p. This shows that f is bijective and since B is integral over A

$\dim B = \dim A = 1$.

Now take $q \in \operatorname{Spec} B$ and set $p := q \cap A$ to be a maximal ideal of A. Since A_p is a discrete valuation ring, assume that $v : K \to \mathbb{Z}$ is a discrete valuation associated to A_p Define $v_L(\beta) = v(\beta^{p^e})$. We see that v_L is also a discrete valuation of L with the valuation ring B_q. Therefore B_q is a discrete valuation ring for all $q \in \operatorname{Spec} B$. All we have to show now is that B is Noetherian. Let I be a non-zero ideal of B. We show that I is finitely generated. Take $0 \neq b \in I$. Consider the ring B/bB. It is integral over $A/bB \cap A$. We have $b^{p^e} \in bB \cap A$, therefore it is non-zero and $\dim B/bB = \dim A/bB \cap A = 0$. Since f is bijective, $V(b)$ is a finite set $q_1, \ldots, q_n$. On the other hand we have $B/bB \simeq \oplus_{1 \leq i \leq n} B_{q_i}/(b)$ is Noetherian (note that all B_{q_i} are Noetherian local rings). Therefore $I/(f)$ is finitely generated ideal and consequently I is finitely generated. □

Definition 2.4.14. Assume that X is a scheme and $x \in X$. Let m_x be the maximal ideal of $O_{X,x}$ and $k(x) = O_{X,x}/m_x$ be the residue field. Then $m_x/m_x^2 = m_x \otimes_{O_{X,x}} k(x)$ as a $k(x)-$vector space. Its dual $(m_x/m_x^2)^\vee$ is called the (Zariski) tangent space to X at x. We denote it by $T_{X,x}$. If $f : X \to Y$ is a morphism of schemes and $x \in X$ and $y = f(x)$, then $f_x^\# : O_{Y,y} \to O_{X,x}$ induces a $k(x)-$linear map $T_{f,x} : T_{X,x} \to T_{Y,y} \otimes_{k(x)} k(x)$, which is called tangent map of f at x.

Let (A, m) be a Noetherian local ring with residue field $k = A/m$. We know that always $\dim_k m/m^2 \geq \dim A$. The Noetherian local ring A is called regular if the equality holds which means if m is generated by $\dim A$ elements.

Now, let X be a locally Noetherian scheme, and $x \in X$ be a point. We say that X is regular at $x \in X$ if $O_{X,x}$ is a regular local ring, which means if $\dim O_{X,x} = \dim_{k(x)} T_{X,x}$ if $x \in X$ is not regular, we call it a singular point of X.

2.4.4 The relation between regularity and normality

Theorem 2.4.15. If X is a Noetherian scheme then X is regular if and only if it is regular at its closed points. Moreover, if X is regular then any connected component of X is normal.

Proof. We may assume that $X = \operatorname{Spec} A$ is affine. Since we have $(A_m)_{pA_m} = A_p$ for $p \in \operatorname{Spec} A$ and m maximal ideal of A such that $p \subset m$, the first part of the theorem is trivial (recall that a ring is regular if its localisations are regular local rings). The rest of theorem follows from the fact that local regular rings are integrally closed in their field of fractions. □

Example 2.4.16. The affine space $\mathbb{A}_k^n$ and the projective space $\mathbb{P}_k^n$ are regular. The affine space is regular because its local rings at its closed points are regular. The projective space is regular because it is a union of open subschemes isomorphic to $\mathbb{A}_k^n$.

2.5 Smoothness

In the study of a family of varieties parameterised by a base scheme, the flatness is a crucial notion which somehow shows the continuity of fibres.

Definition 2.5.1. Let $f : X \to Y$ be a morphism of scheme. We say that f is flat at $x \in X$ is the corresponding homomorphism $f_x^\# : O_{Y,f(x)} \to O_{X,x}$ is flat. From commutative algebra, the property of flatness is stable under base change, composition and fibre product.

Example 2.5.2. Since any algebra over a field k is flat, the morphism $X \to \operatorname{Spec} k$ of an algebraic variety over k is flat. As another class of examples, one can mention open immersions as well, while closed immersions are not in general flat. One can go further and see that a closed immersion is flat if it is an open immersion.

2.5.1 A criterion for flatness

As an important consequence for flatness, we have the following theorem.

Theorem 2.5.3. ([23], Lemma 4.3.7) Let $f : X \to Y$ be a flat morphism with Y is irreducible. Then every non-empty open subset U of X dominates Y (i.e., $f(U)$ is dense in Y). If X has only a finite number of irreducible components, then every one of them dominates Y.

Proof. We may assume that $Y = \operatorname{Spec} A$ and $U = \operatorname{Spec} B$ are affine. Since open immersions are flat, we have $U \to Y$ is flat. Let η be the generic point of Y and N the nilradical of A. By flatness of B as an $A-$module we have

$$B/NB = B \otimes_A (A/N) \subseteq B \otimes_A \operatorname{Frac}(A/N) = B \otimes_A k(\eta) = O(U_\eta)$$

If the fibre $U_\eta = \emptyset$ then $B = NB$ which means the ring B is nilpotent and therefore $U = \emptyset$, a contradiction. Therefore the fibre $U_\eta \neq \emptyset$ and $f(U)$ is dense in Y.

If X has only finite number of irreducible components, then every component has a non-empty open subset. Hence every one of them dominates Y. □

Theorem 2.5.4. ([23], Proposition 4.3.9) Let Y be a Dedekind scheme. Let $f : X \to Y$ be a morphism with X reduced. Then f is flat if and only if every irreducible component of X dominates Y.

Proof. First we suppose every component of X dominates Y. Let $x \in X$ and $y = f(x)$. If y is the generic point of Y, then $O_{X,x}$ is an $O_{Y,y} = K(Y)-$module therefore it is flat. So, let us suppose that $y \in Y$ is a closed point and $\pi \in O_{Y,y}$ is a uniformizer. We are going to show that π is not a zero divisor in $O_{X,x}$. By definition, this shows that $O_{Y,y}$ is flat over $O_{X,x}$. By assumption π is not contained in any of minimal prime ideals of $O_{X,x}$. Since X is reduced this implies that π is not a zero divisor in $O_{X,x}$. The converse is deduced from theorem 2.5.3. □

Corollary 2.5.5. Let Y be a Dedekind scheme, and $f : X \to Y$ be a non-constant morphism with X integral. Then f is flat.

Proof. Since Y is irreducible and of dimension 1, $f(X)$ is dense in Y (because f is not constant). Therefore by theorem 2.5.4 f is flat. □

2.5.2 Dimension of fibres

Theorem 2.5.6. (See [23], Theorem 4.3.12 and [16], III 9.5) If $f : X \to Y$ is a morphism of Noetherian schemes and if $x \in X$ and $y = f(x)$ then

$$\dim \mathcal{O}_{X_y,x} \geq \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y}.$$

Moreover if f is flat then the equality holds.

Proof. We may assume that Y is affine and the spectrum of a Noetherian local ring. Because simply one can change the base via $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$. So, y is a closed point of Y.

We proceed by induction on $\dim Y$. If $\dim Y = 0$ then we have $X_{\text{red}} = (X_y)_{\text{red}}$, therefore the equality holds.

If $\dim Y \geq 1$, we may assume that Y is reduced via the base change $Y_{\text{red}} \to Y$. We have

$$\begin{array}{ccc} X \times_Y Y_{\text{red}} & \longrightarrow & Y_{\text{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

This does not change the dimensions of Y and X_y at x. And since flatness is stable under base change, therefore we may assume that Y is reduced. Take $t \in A$ which is neither a zero divisor nor invertible. We have

$$\dim(A/tA) = \dim A - 1, \quad \dim(B/tB) \geq \dim B - 1$$

in which $B := \mathcal{O}_{X,x}$ B is a flat $A-$module. Therefore tensoring the injective homomorphism $A \xrightarrow{t} A$ by B keeps the injectivity and therefore $t \in B$ (we show the image of t in B by the same letter t) is a non-zero divisor in B. Hence $\dim(B/tB) = \dim B - 1$.

Set $Y' = \operatorname{Spec}(A/tA)$ and $X' = X \times_Y Y'$. Then by the induction hypothesis we have

$$\dim \mathcal{O}_{X'_y,x} \geq \dim \mathcal{O}_{X',x} - \dim \mathcal{O}_{Y',y}$$

and the equality holds if f is flat, because if so, $X' \to Y'$ is also flat.

But now $X'_y = X_y$ therefore we have

$$\dim \mathcal{O}_{X_y,x} \geq (\dim \mathcal{O}_{X,x} - 1) - (\dim \mathcal{O}_{Y,y} - 1) = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y},$$

where the equality holds if f is a flat morphism. □

Corollary 2.5.7. Let $f : X \to Y$ be a flat and surjective morphism of algebraic varieties. If Y is irreducible and X is equidimensional (which means that all of its irreducible components have the same dimension), then for every $y \in Y$, the fibre X_y is equidimensional, and we have

$$\dim X_y = \dim X - \dim Y$$

.

Proof. If $x \in X_y$ is a closed point, then for every irreducible component X_i of X passing through x we have

$$\dim \mathcal{O}_{X_i,x} = \dim X_i - \dim \overline{\{x\}}$$

therefore

$$\dim \mathcal{O}_{X,x} = \dim X - \dim \overline{\{x\}}$$

Since x is the generic point of $\overline{\{x\}}$ and the latter is an algebraic variety we have

$$\dim \overline{\{x\}} = \mathrm{trdeg}_{\mathrm{k}}\, k(x)$$

On the other hand, $x \in X_y$ is a closed point therefore $k(y)/k(x)$ is an algebraic extension. Hence

$$\dim \overline{\{x\}} = \mathrm{trdeg}_k\, k(x) = \mathrm{trdeg}_k\, k(y) = \dim \overline{\{y\}} = \dim Y - \dim \mathcal{O}_{Y,y}.$$

By the above equalities and theorem 2.5.6 we finally have

$$\dim X_y = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,y} = \dim X - \dim Y.$$

□

2.5.3 Étale morphisms

Definition 2.5.8. Let $f : X \to Y$ be a morphism of finite type of locally Noetherian schemes. Let $x \in X$ and $y = f(x)$. We say that f is unramified at x if the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ for which we have $m_y\mathcal{O}_{X,x} = m_x$ (meaning $\mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x} = k(x)$), and if the (finite) extension of residue fields $k(y) \to k(x)$ is separable. We say that f is étale at x is it is uniramified and flat at x.

A local homomorphism of Noetherian local rings $A \to B$ is called étale if it is flat and unramified morphism such that B is a localisation of a finitely generated $A-$algebra.

Example 2.5.9. If L/K is a finite field extension then $\operatorname{Spec} L \to \operatorname{Spec} K$ is unramified (and therefore étale) if and only if the extension L/K is separable.

Definition 2.5.10. Assume X is an algebraic variety over a field k and $\bar{k}$ is the algebraic closure of k. We say that X is smooth at $x \in X$ if the points of $X_{\bar{k}}$ lying above x are regular points of $X_{\bar{k}}$. As the simplest examples, $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are smooth varieties.

2.5.4 Relation between regularity and smoothness

In the following theorem we show that the Jacobian criterion is a criterion for smoothness.

Theorem 2.5.11. ([23], Proposition 4.3.30) Let X is an algebraic variety over a field k, and let $x \in X$ be a closed point. If X is smooth at x, then it is regular at x. Moreover, the converse is true if $k(x)$ is separable over k.

Proof. We may assume that X is affine. Let $x' \in X_{\bar{k}}$ lying above x. The fibers of morphism $X_{\bar{k}} \to X$ are of dimension 0. By theorem 2.5.6, we have $\dim \mathcal{O}_{X_{\bar{k}},x'} = \dim \mathcal{O}_{X,x}$ Let us take $X = V(I) \subseteq \mathbb{A}^n_k$ and $I = (f_1, \ldots, f_r)$ and J_x denote the Jacobian of X at x. Of course, $J_x = J_{x'}$ as matrices in $\bar{k}$. Set $D_xP : k[X_1, \ldots, X_n] \to (k^n)^\vee$ (dual of k^n as $k-$vector space), defined by

$$D_xP : (x_1, \ldots, x_n) \mapsto \sum_{1 \leq i \leq n} \frac{\partial P}{\partial X_i}(y)x_i.$$

Then we have a surjective map $I/I \cap m^2 \to D_xI$. Therefore we have

$$\dim_{k(x)} T_X, x = n - \dim_{k(x)}((I/I \cap m^2)) \leq n - \operatorname{rank} D_xI = n - \operatorname{rank} J_x.$$

Note that so far we have not used the assumption of smoothness. If X is smooth at x then we have

$$\dim_{k(x)} T_{X,x} \leq n - \operatorname{rank} J'_x = \dim \mathcal{O}_{X_{\bar{k}},x'} = \dim \mathcal{O}_{X,x}$$

Hence X is regular at x. For the second part, if $k(x) = k$ (i.e., if x is rational) then the we have

$$\dim_k(I/I \cap m^2) = \operatorname{rank} D_xI.$$

Therefore it is obvious by the Jacobian criterion (applied to $X_{\bar{k}}$) that $X_{\bar{k}}$ is regular at x'. If not, consider the map $X_{\bar{k}} \to X$. Since $\operatorname{Spec} k' \to \operatorname{Spec} k$ is finite and étale according to the assumption, we have $X_{\bar{k}} \to X$ étale and finite as well. Now all the points of X'_k lying above x are rational over k' Therefore we return to the first case. So X is smooth at x.

In general if $f : X \to Y$ is of finite type and étale at $x \in X$ then we have X is regular at x if and only if Y is regular at $f(x)$ because in this case we have $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,f(x)}$ and also $T_{X,x} \simeq T_{Y,f(x)} \otimes_{k(y)} k(x)$

□

Corollary 2.5.12. If X is an algebraic variety over a field k and $x \in X$ is an arbitrary point. If X is smooth at x, then X is regular at x.

Proof. Take $x' \in X_{\bar{k}}$ as a point above $x \in X$. Since X is smooth at x then x' is a regular point. Now consider $\overline{\{x'\}}$. It is a reduced algebraic variety over an algebraically closed field then it is well-known that it contains a closed regular point $y' \in X_{\bar{k}}$. Let $y \in X$ be the image of y'. Then from Theorem 2.5.11 y is regular. Since y is a specialisation of x therefore x is regular as well. □

Smooth morphisms are a generalisation of étale morphisms.

Definition 2.5.13. Let Y be a locally Noetherian scheme and $f : X \to Y$ be a morphism of finite type. We say f is smooth at a point $x \in X$ if f is flat at x and if $X_y \to \operatorname{Spec} k(y)$ (in which $y = f(x)$) is smooth at $x \in X_y$

We say that f is smooth of relative dimension n if it is smooth at all $x \in X$ and all of its non-empty fibres are equidimensional of dimension n. As an example, étale morphisms of finite type are smooth of relative dimension 0.

By the same process as the corollary 2.5.12, one can show that if Y is a locally Noetherian regular scheme and if $f : X \to Y$ is smooth morphism then X is regular too.

2.6 Divisors

2.6.1 Cartier divisors

Definition 2.6.1. Let X be a scheme. The sheaf of algebras associated to the presheaf $\mathcal{K}'_X$ (defined by $\mathcal{K}'_X(U) = \operatorname{Frac}(O_X(U)) := (O_X(U) \setminus Z(O_X(U)))^{-1} O_X(U)$, in which $Z(O_X(U))$ is the set of zero divisors of the ring) is denoted by $\mathcal{K}_X$. We call it the sheaf of stalks of meromorphic functions on X.

The subsheaf of invertible elements of $\mathcal{K}_X$ is denoted by $\mathcal{K}^*_X$. An element of $\mathcal{K}_X(X)$ is called a meromorphic element on X.

Definition 2.6.2. Let X be a scheme. Denote the group $H^0(X, \mathcal{K}^*_X / O^*_X)$ by $\operatorname{Div}(X)$. The elements of $\operatorname{Div}(X)$ are called divisors on X. If $f \in H^0(X, \mathcal{K}^*_X)$; its image in $\operatorname{Div}(X)$ is called a principal Cartier divisor and denoted by $\operatorname{div}(f)$.

Convention: We show the group law in $\operatorname{Div}(X)$ additively.

We say two divisors D_1 and D_2 are linearly equivalent ($D_1 \sim D_2$) if $D_1 - D_2$ is principal.

A Cartier divisor X is called effective if it is in the image of

$$H^0(X, O_X \cap \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/O_X^*)$$

and we denote it by $D \geq 0$.

We can represent a Cartier divisor D by $\{(U_i, f_i)_i\}$ where the U_i are open subsets of X and they cover X. the $f_i \in \mathcal{K}_X^*(U_i)$ the quotient of two regular elements of $O_X(U_i)$, and $f_i|_{U_i \cap U_j} \in f_j|_{U_i \cap U_j} O_X(U_i \cap U_j)^*$ for every i, j.
Two systems $\{(U_i, f_i)_i\}$ and $\{(V_j, g_j)_j\}$ represent the same divisor if on $U_i \cap V_j$, f_i and g_j differ by a multiplicative factor in $O_X(U_i \cap V_j)^*$.

2.6.2 Weil divisors

For more details see for example [23].

Definition 2.6.3. Let X be a Noetherian scheme. A prime cycle on X is an irreducible closed subset of X. A cycle on X is an element of $\mathbb{Z}^{(X)}$ and can be represented in a unique way by

$$Z = \sum_{x \in X} n_x \overline{\{x\}}$$

If all of $n_x = 0,\ Z = 0$ The n_x is called the multiplicity of Z at x. If al of $\mathrm{mult}_x(Z) \geq 0$ for every $x \in X$, we say that Z is positive.

The finite union of $\overline{\{x\}}$ for which $n_x \neq 0$ is called the support of Z. It is a closed subset of X. The support of divisor 0 is set to be the empty set by convention. If all of irreducible components of $\mathrm{Supp}(Z)$ are of codimension 1, we say that Z is of codimension 1. Note that $\overline{\{x\}}$ if and only if $\dim O_{X,x} = 1$.

If X is a Noetherian integral scheme, a cycle of codimension 1 on X is called a Weil divisor on X. They form an abelian group by component-wise addition. If X is a normal Noetherian scheme and $f \in K(X)$ be a non-zero divisor, then for $x \in X$ of codimension 1 we have $O_{X,x}$ normal local ring of dimension 1 (by definition, a discrete valuation ring). Therefore we can define

$$\mathrm{mult}_x : K(X) \to \mathbb{Z} \cup \{\infty\}$$

to be the normalised valuation of $K(X)$ Set

$$(f) := \sum_{x \in X,\ \dim O_{X,x} = 1} \mathrm{mult}_x(f)[\overline{\{x\}}].$$

Such a divisor is called a principal Weil divisor.

The quotient of the group of cycles of codimension 1 on X by the subgroup of principal divisors is denoted by $\mathrm{Cl}(X)$. Two Weil divisors Z_1 and Z_2 are equivalent ($Z_1 \sim Z_2$) if $Z_1 - Z_2$ is a principal Weil divisor.

2.6.3 Relation between Cartier divisors and Weil divisors

Definition 2.6.4. Let A be a Noetherian local ring of dimension 1. We know that if $f \in A$ is a regular element then $\mathrm{length}_A(A/fA)$ is a finite integer and it is additive over a short exact sequence. Therefore the map

$$f \mapsto \mathrm{length}_A(A/fA)$$

extends to a group homomorphism

$$\mathrm{Frac}(A)^* \to \mathbb{Z}.$$

Since the invertible elements of A are contained in the kernel we obtain a homomorphism

$$\mathrm{mult}_A : \mathrm{Frac}(A)^*/A^* \to \mathbb{Z}.$$

Now let X be a Noetherian scheme and $D \in \mathrm{Div}(X)$ a Cartier divisor. For any $x \in X$ of codimension 1. We have

$$(\frac{\mathcal{K}_X^*}{\mathcal{O}_X^*})_x = \frac{\mathrm{Frac}\,\mathcal{O}_{X,x}^*}{\mathcal{O}_{X,x}^*}.$$

Define

$$\mathrm{mult}_x(D) := \mathrm{mult}_{\mathcal{O}_{X,x}}(D_x)$$

.

We can assign to a Cartier divisor a Weil divisor as follows;

If D is a Cartier divisor, we set

$$[D] = \sum_{x \in X,\ \dim \mathcal{O}_{X,x}=1} \mathrm{mult}_x(D)[\overline{\{x\}}].$$

Therefore $[D]$ is a cycle of codimension 1 such that $\mathrm{mult}_x([D]) = \mathrm{mult}_x(D)$ at every point of codimension 1.

One can show that if X is a regular Noetherian (hence it is normal) integral scheme. Then the canonical homomorphisms

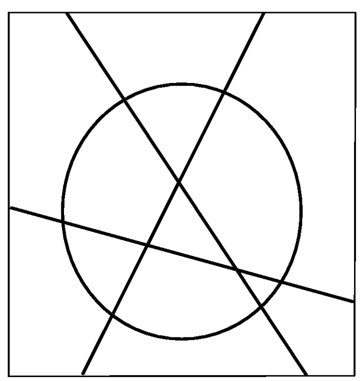
$$\mathrm{Div}(X) \to \mathrm{Z}^1(X), \quad \mathrm{CaCl}(X) \to \mathrm{Cl}(X)$$

are isomorphism. ($\mathrm{CaCl}(X)$ is the quotient group of Cartier divisors on X modulo the principal Cartier divisors and $\mathrm{Z}^1(X)$ is the group of cycles of codimension 1 on X.)

2.6.4 Meeting transversally

Definition 2.6.5. Let (A, m) be an $n-$dimensional Noetherian local ring. It is well known (See for instance [25], Theorem 13.4) that there exists an $m-$primary ideal generated by n elements but not by any fewer number of elements. If $a_1, \dots, a_n \in m$ generate an $m-$primary ideal, then $\{a_1, \dots, a_n\}$ is called a system of parameters of A. A system of parameters which generates the maximal ideal m is called a regular system of parameters. Of course in the latter case, (A, m) by definition is a regular local ring.

Definition 2.6.6. ([23], Definition 9.1.6) Let Y be a regular Noetherian scheme, and D be an effective Cartier divisor on Y. We say that D has normal crossings at a point $y \in Y$ if there exist a regular system of parameters $f_1, \dots, f_n$ of Y at y, an integer $0 \le m \le n$, and integers $r_1, \dots, r_m \ge 1$ such that the ideal $\mathcal{O}_Y(-D)_y \in \mathcal{O}_{Y,y}$ is generated by $f_1^{r_1} \dots f_m^{r_m}$. The divisor D has normal crossings if it has normal crossings at every point $y \in Y$. We say that the prime divisors $D_1, \dots, D_l$ meets transversally at $y \in Y$ if they are pairwise distinct and if the divisor $D_1 + \dots + D_l$ has normal crossings at y.



2.6.5 Intersection with horizontal divisors

Theorem 2.6.7. ([23], Proposition 9.1.30) Suppose $\pi : X \to S$ be an arithmetic surface (i.e., X is regular, integral, projective, flat scheme of dimension 2 over a Dedekind scheme S of dimension 1). Let η be the generic point and s be a closed point. Then for any closed point $P \in X_\eta$, we have

$$\overline{\{P\}} \cdot X_s = [K(P) : K(S)] := \deg_{k(s)} O_X(\overline{\{P\}})|_{X_s},$$

in which $\overline{\{P\}}$ is the Zariski closure of $\{P\}$ in X, endowed with the reduced closed subscheme structure.

Proof. Let denote the horizontal divisor $\overline{\{P\}}$ by D. If $i : D \to X$ is the canonical closed immersion and $h : D \to S$ the finite surjective morphism $\pi \circ i$, then we have $X_s = \pi^* s$ (because $X \to S$ is regular fibered surface) and we have $X_s|_D = i^*(\pi^* s) = h^* s$. Now

$$h_*[X_s|_D] = h_*[h^* s] = d[s], \quad \text{where } \; d = [K(D) : K(S)],$$

and

$$h^* : \mathrm{Div}(S) \to \mathrm{Div}(D),$$
$$h_* : \mathrm{Div}(D) \to \mathrm{Div}(S).$$

are the canonical morphisms of abelian groups of Weil divisors. On the other hand we have

$$h_*[X_s|_D] = \sum_{s \in S} i_s(V, D)[s].$$

where $i_s(V, D) := \deg_{k(s)} O_X(D)|_V$. Therefore $D \cdot X_s = d$ because $K(D) = K(P)$. □

2.6.6 Exceptional divisors and van der Waerden's purity theorem

We conclude this chapter by stating without proof an important theorem by Van dar Waerden which is the so-called the purity theorem. It shows that the notion of divisors is connected to the exceptional locus of a separated birational morphisms of finite type. It is crucial for constructing minimal regular models in the next chapters.

Assume X and Y are Noetherian integral schemes and $f : X \to Y$ is a separated birational morphism of finite type. There exists an open set $W \subseteq Y$ for which we have $x \in f^{-1}(W)$ if and only if $O_{Y,f(x)} \to O_{X,x}$ is an isomorphism (See, for instance, [23], Corollary 4.4.3 (b)). We call the subset $E := X \backslash f^{-1}(W)$ the exceptional locus of f. The definition is local in the sense that if $U \subseteq X$ is open then the exceptional locus of $f|_U : U \to Y$ is $U \cap E$.

Theorem 2.6.8. ([23], Theorem 7.2.22. van der Waerden's purity theorem) Let X, Y be Noetherian integral schemes, and let $f : X \to Y$ be a separated birational morphism of finite type. Suppose that Y is regular. Then the exceptional locus E of f is empty or of pure codimension 1 in X.

There is a nice criterion distinguishing exceptional divisors among vertical divisors.

Theorem 2.6.9. (Castelnuovo's criterion) If X is a regular, integral, projective and flat $S-$curve and $E \subset X_s$ is a vertical prime divisor then E is an exceptional divisor if and only if $E \simeq \mathbb{P}^1_{k'}$ and $E^2 = -[k' : k(s)]$ in which $k' := O_E(E)$.

Proof. See Theorem 9.3.8 [23]. □

In the following theorem, we see that the projective line $\mathbb{P}^1_k$ is the unique (up to isomorphism) geometrically integral, projective curve of genus $g \leq 0$ which has an $k-$rational point.

Theorem 2.6.10. (Curves of small genus) Let X be a geometrically integral projective curve of arithmetic genus $p_a \leq 0$ over a field k. We have $X \simeq \mathbb{P}^1_k$ if and only if $X(k) \neq \emptyset$.

Proof. The curve X is smooth over k because if X' is the normalisation of $X_{\bar{k}}$, then $p_a(X') \geq 0$ $(H^0(X', O_{X'}) = \bar{k})$ and hence $X' = X_{\bar{k}}$. Now take $y \in X(k)$ and consider the $k-$vector space

$$L(y) := \{f \in K(X)^* \mid \text{mult}_y(f) + 1 \geq 0\} \cup \{0\}.$$

From Riemann-Roch Theorem, $\dim_k L(y) = 2$. Therefore $X \simeq \mathbb{P}^1_k$ (In general, a normal projective curve X is isomorphic to the projective line if and only if there exists a Cartier divisor D such that $\deg D = 1$ and $l(D) \geq 2$). □

2.7 Abelian varieties

In this section we state the preliminary definitions and facts about the extensive subject of abelian varieties. We will use the following results in the last chapter, where we take advantage of the Jacobian of families of curves so as to extract properties of the given families (See for instance Chapter 6. Theorem 6.3.1). For more details and proves in this subsection, refer to [30] or [28], or [40].

2.7.1 Group schemes

Definitions and basic properties 2.7.1. A scheme G over S is a group scheme if it is endowed with the following morphisms over S,

$$m : G \times_S G \to G$$

$$\epsilon : S \to G$$

$$\text{inv} : G \to G$$

with the following properties;

1. The diagrams below are commutative

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{m \times Id_G} & G \times_S G \\ \downarrow{\scriptstyle Id_G \times m} & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} G \times_S S & \xrightarrow{Id_G \times \epsilon} & G \times_S G \\ & \searrow^{Id_G} & \downarrow{\scriptstyle m} \\ & & G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\Delta_{G/S}} & G \times_S G & \xrightarrow{Id_G \times \mathrm{inv}} & G \times_S G \\ \downarrow & & & & \downarrow{\scriptstyle m} \\ S & & \xrightarrow{\epsilon} & & G \end{array}$$

2. For any scheme T over S, morphisms m and inv induces maps $m(T) : G(T) \times G(T) \to G(T)$, $\mathrm{inv} : G(T) \to G(T)$ in a canonical manner.

The above data is equivalent to say that $G(T)$ is a group with ϵ_T as its unit for every $S-$scheme T. It is easy to see that the fibered product of two group schemes over the same base scheme is again a group scheme over the common base. If the group $G(T)$ is a commutative group for every $S-$scheme T, then we say that the group scheme G is commutative.

A subgroup scheme H of the group scheme G is a closed subscheme of G such that $H(T)$ is a subgroup of $G(T)$ for all $S-$scheme T. Morphisms of group schemes over S are $S-$morphisms of schemes which are compatible with the two axioms above.

The Kernel of morphism of group schemes $f : G \to G'$ is the group scheme $G \times'_G S$, in which S is considered as a G' scheme via the standard morphism of $\epsilon : S \to G'$.

A group scheme G over a field k is called algebraic group scheme over k if G is of finite type over k.

Example 2.7.2. The first basic example of group schemes is $\mathbb{G}_a = \mathrm{Spec}\,\mathbb{Z}[T]$. We can take morphism $m : \mathbb{G}_a \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{G}_a \to \mathbb{G}_a$ corresponding to morphisms of rings $\mathbb{Z}[T] \to \mathbb{Z}[T_1, T_2]$ which send $T \mapsto T_1 + T_2$, and $\mathrm{inv} : \mathbb{G}_a \to \mathbb{G}_a$ corresponding to $\mathbb{Z}[T] \to \mathbb{Z}[T]$, $T \mapsto -T$. For any scheme S the group scheme $\mathbb{G}_{a,S} := \mathbb{G}_a \times_{\mathrm{Spec}\,\mathbb{Z}} S$ obtained by base change over S is called the additive group over S. The group $\mathbb{G}_a(T)$ is the additive group of $O_T(T)$.

Example 2.7.3. The second basic example of group schemes is $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[T, 1/T]$. We can take morphism $m : \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{G}_m \to \mathbb{G}_m$ corresponding to morpohism of rings $\mathbb{Z}[T, 1/T] \to \mathbb{Z}[T_1, 1/T_1, T_2, 1/T_2]$, $T \mapsto T_1T_2$. In the same way as the example 2.7.2 above, for every scheme T we call the group scheme $\mathbb{G}_{m,S} := \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} S$ obtained by the base change the multiplicative group over S. The reason for this terminology is that the group $\mathbb{G}_m(T) = \mathcal{O}_T^*(T)$.

The following theorem shows the property of G^0; the connected component containing the identity element $e \in G$

Theorem 2.7.4. Let G be a group scheme of finite type over k. The following properties are true for the G^0,

- G^0 is open, closed and is a subgroup scheme of G.
- G^0 is geometrically irreducible.
- G^0 is of finite type over k.

Proof.
- Since G is of finite type over k, it is locally Noetherian. Therefore the connected component is always closed and open. On the other hand, since G^0 is itself connected, the map $G^0 \times G^0 \to G$ factors through G^0. So G^0 is a group.

- Since the identity element $e \in G$ is rational over k (by definition) and since G^0 is connected, it is geometrically connected. After base change to an algebraic closure $\bar{k}$ and considering the reduced induced structure over G^0_{red}, it is regular over $\bar{k}$. Now since G^0 is smooth and geometrically connected, it is geometrically irreducible.
- It is easy to see that for any open affine subset $U \subset G^0$, the map $U \times U \to G^0_{\text{red}}$ is surjective. Therefore G^0_{red} is quasi-compact. Since it is already locally of finite type, it is of finite type.

□

Definition 2.7.5. (For more, see [23]) Let k be a field. An abelian variety over k is defined to be an algebraic group that is geometrically integral and proper over k. An abelian variety is always projective and commutative. See [26], Section 2 and 7.

Notation: Assume G is a commutative algebraic group over field k. For any extension k'/k, $G(k'$ is an abstract commutative group. For any $n \in \mathbb{Z}$, $G[n]$ is the kernel of the multiplication by n morphism $G \to G$. Therefore $G[n](k')$ is the kernel of multiplication by n on $G(k')$.
The following theorem shows the behaviour of the torsion subgroups of a group scheme A ([23]).

Theorem 2.7.6. Let A be an abelian variety of dimension g over a field k, and let $\bar{k}$ be the algebraic closure of k. Fix a non-zero integer n.

- If $(n, \mathrm{char}(k)) = 1$, then $A[n]$ is étale over $\operatorname{Spec} k$, and $A[n](\bar{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.
- If $p = \mathrm{char}(k) > 0$, then there exists an $0 \leq h \leq g$ such that for any $n = p^m$, we have $A[n](\bar{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^h$.

2.7.2 Jacobian of curves

The following theorem guarantees the existence of the Jacobian variety for a smooth, geometrically connected projective curve over a field k.

Theorem 2.7.7. Let X be a smooth, geometrically connected, projective curve of genus g over k (Note that under this conditions geometric genus and arithmetic genus coincide). Then there exists an abelian variety J of dimension g over k such that $J(K) \simeq \mathrm{Pic}^0(X_K)$ (i.e., invertible sheaves of degree 0) for any extension K/k such that $X(K) \neq \emptyset$. The isomorphism is moreover compatible with field extensions.

Proof. For the proof see [27], Theorem 1.1. □

Definition 2.7.8. The abelian variety J in the theorem above is called the Jacobian of curve X.

Combining theorems 2.7.5 and 2.7.6 we have,

Corollary 2.7.9. Let X be a smooth, connected, projective curve over an algebraically closed field k, of genus g. Let $n \in \mathbb{Z}$ be a non-zero.

- If $(n, \mathrm{char}(k)) = 1$, then $\mathrm{Pic}^0(X)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.
- If $p = \mathrm{char}(k) > 0$, then there exists an $0 \leq h \leq g$ such that for any $n = p^m$, we have $\mathrm{Pic}^0(X)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^h$.

In a more canonical way, we can define the Jacobian of a smooth, geometrically connected projective curve X over k using some universal properties as follows,
For any scheme T over k, let us denote by $\mathrm{Pic}^0(X \times_k T)$ the subgroup of $\mathrm{Pic}(X \times T)$ consisting of invertible sheaves whose restriction to each fibre X_t for $t \in T$ has degree 0 (recall that the degree of an invertible sheaf $\mathcal{L}$ over scheme X is defined to be $\chi_k(\mathcal{L}) - \chi_k(\mathcal{O}_X)$.)

Let $p : X \times T \to T$ be the second projection. For any invertible sheaf $\mathcal{N}$ on T, $p^*\mathcal{N} \in \mathrm{Pic}^0(X \times_k T)$ since it is trivial on each fibre.

Put

$$\mathrm{Pic}^0(X/T) := \frac{\mathrm{Pic}^0(X \times_k T)}{p^* \operatorname{Pic} T}$$

Another definition for Jacobian variety 2.7.10. Let X be a curve of genus g over k. The Jacobian variety of X is a scheme J of finite type over k, together with an element $\mathcal{L} \in \mathrm{Pic}^0(X/J)$, such that it admits the following universal property: for any scheme T of finite type over k, and for any $\mathcal{M} \in \mathrm{Pic}^0(X/T)$, there is a unique morphism $f : T \to J$ such that $f^*\mathcal{L} \simeq \mathcal{M}$ (canonical isomorphism in $\mathrm{Pic}^0(X/T)$). In other words, if we define a function from the category of schemes of finite type over k to the category of commutative groups by sending scheme T to $\mathrm{Pic}^0(X/T)$, then the definition above says that this functor is representable with scheme J.

For primary results on this approach see [16], Chapter IV, Section 4 and [30].

Let X is a projective variety over k and $x \in X(k)$ is a rational point. Grothendieck proved the following fact for the Picard functor,

Theorem 2.7.11. We have

1. $\mathrm{Pic}_{X/k}$ is represented by a scheme (hence it is a group scheme), which is locally of finite type over k.
2. The connected component $\mathrm{Pic}^0_{X/k}$ is quasiprojective, and if X is smooth variety, then it is projective.

Proof. See [30]. □

Remark 2.7.12. Assume that A is an abelian variety over k. Define

$$\mathrm{Pic}^0(A) := \mathrm{Ker}(\phi : \mathrm{Pic}(A) \to \mathrm{Hom}(A(\bar{k}), \mathrm{Pic}(A_{\bar{k}})))$$

In other words, $\mathcal{L} \in \mathrm{Pic}^0(A)$ if and only if $T^*\mathcal{L} \simeq \mathcal{L}$ for all $x \in A(\bar{k})$. Then it can be shown that $\mathrm{Pic}^0_{A/k}(k) = \mathrm{Pic}^(A)$.

Since the abelian variety A is projective there is an ample sheaf $\mathcal{L}$ on A. Consider the following map

$$\phi_{\mathcal{L}} : A(\bar{k}) \to \mathrm{Pic}^0_{A/k}(\bar{k}) \to \mathrm{Pic}^0(A_{\bar{k}})$$

The last map is due to the Remark 2.7.12 above.

The following theorem is essential for definition of polarisation.

Theorem 2.7.13. With the notations above, if $\mathcal{L}$ is an ample invertible sheaf on A, then the map

$$\phi_{\mathcal{L}} : A(\bar{k}) \to \mathrm{Pic}^0(A_{\bar{k}})$$
$$x \mapsto T^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

is surjective.

Definition 2.7.14. A polarisation of an abelian variety A is an isogeny $\lambda : A \to \hat{A}$ such that $\lambda \otimes \bar{k} = \phi_{\mathcal{L}}$ for some ample invertible sheaf $\mathcal{L}$ on $A_{\bar{k}}$. λ is called a principally polarisation if λ is an isomorphism (equivalently if $\deg \lambda = 1$).

For example if X is a proper curve over k then it can be shown that the Jacobian of X (Definition 2.7.8) admits a canonical polarisation.

Definition 2.7.15. Let A and B be abelian varieties. A homomorphism $\alpha : A \to B$ is called an isogeny if α is surjective and has finite kernel. For example if A is an abelian variety of dimension g then the morphism $n_A : A \to A$ is an isogeny of degree n^{2g}.

One can extend the construction 2.7.9 to the higher dimensions so as to define the dual of abelian varieties over k. Roughly speaking, the contravariant functor which associates to each $k-$variety T the set of families of degree 0 invertible sheaves on T and to each morphism $f : T \to T'$ over k, the mapping induced by the pullback with f, is representable. The pair $(A^\vee, P)$ which represent this functor is called the dual of abelian variety A.

More generally, let G be a commutative finite (hence affine) group scheme over k. The algebra $H := \Gamma(G, \mathcal{O}_G)$ is a finite dimensional commutative and cocommutative Hopf algebra over k. Set $H^* := \operatorname{Hom}_k(H, k)$ as the dual of algebra H. It has a canonical structure of commutative and cocommutative Hopf algebra over k induced by H. For more details, see [30]

Definition 2.7.16. With the notations above, $\hat{G} := \operatorname{Spec} H^*$ is a commutative finite group scheme over k. It is called the Cartier dual of G. The natural dual functor $\mathbb{D} : G \to \hat{G}$ satisfies $\mathbb{D}^2 = \operatorname{Id}$.

Thoerem 2.7.17. Let G_1 and G_2 be commutative group schemes over S. Define the functor

$$\underline{\operatorname{Hom}}(G_1, G_2) : \mathcal{S}ch/S \to \mathcal{A}b,$$
$$T/S \mapsto \operatorname{Hom}_T(G_{1,T}, G_{2,T}).$$

Then we have $\hat{G} \simeq \underline{\operatorname{Hom}}(G, \mathbb{G}_m)$. Moreover the isomorphism is canonical (For the definition of $\mathbb{G}_m$ see Example 2.7.3).

Proof. See [30] □

Remark 2.7.18. In fact if A is an abelian variety over k then by definitions above, the dual variety $\hat{A}$ is nothing but $\operatorname{Pic}^0_{A/k}$.

Notation 2.7.19. For a finite group scheme A over k, we use the notations $\hat{A}$ and A^D for the dual of A interchangeably.

Example 2.7.20. For $G = \mathbb{Z}/n\mathbb{Z}$, $\hat{G}(R) \simeq \{\phi \in R\backslash 0 : \phi^n = 1\}$. Hence $\hat{G} = \mu_n$.

Theorem 2.7.21. Let $f : A \to B$ be an isogeny of abelian varieties. Then the induced morphism of dual abelian varieties $\hat{f} : \hat{B} \to \hat{A}$ is also an isogeny and $\operatorname{Ker} \hat{f} = \widehat{\operatorname{Ker} f}$ Moreover $\deg f = \deg \hat{f}$.

Proof. See [30] □

2.7.3 Finite group schemes and torsions

Our main reference for this section is [32].

Definition 2.7.22. Let G be a finite group scheme over k. We say

- G is local if $G = G^0$.

- G is étale if $k[G]$ is étale $k-$algebra (a $k-$algebra A is called étale if it is of finite type and if $\Omega_{A/k} = 0$, equivalently if A is of the form $A = \prod_i L_i$, for the L_i being finite separable field extensions of k).

Example 2.7.23. Define $\mu_n := \operatorname{Spec} k[x]/(x^n - 1)$. It is étale if and only if $(n, p) = 1$. It is local if and only if n is a $p-$power.

Example 2.7.24. Define $\alpha_n := \operatorname{Spec} k[x]/x^n$. The group scheme α_{p^n} is local.

The local group schemes and étale group schemes are building blocks of finite group schemes in the following way: the connected component of the identity element G^0 is local $k-$group scheme and the quotient G/G^0 is étale.

We borrow the following theorem from [23], Proposition 10.2.18

Theorem 2.7.25. Let X be a scheme of finite type over k (e.g., an algebraic variety). There exists a unique scheme $\pi_0(X)$, finite étale over k, and a morphism $f : X \to \pi_0(X)$ verifying the following universal property: any $k-$morphism $X \to Z$ of X to a finite étale $k-$scheme Z factors in a unique way as

$$\begin{array}{ccc} X & \longrightarrow & Z \\ {\scriptstyle f}\big\downarrow & \nearrow & \\ \pi_0(X) & & \end{array}$$

Proof. Let $X_i, \ldots, X_n$ be affine open subschemas that cover X. Then $O_X(X)$ is a $k-$subalgebra of $\oplus_{1 \le i \le n} O_X(X_i)$. Therefore we can define $O_X(X)^{et}$. Now since the connected components of X and $\operatorname{Spec} O_X(X)$ are the same, then it is enough to set $\pi_0(X) = \operatorname{Spec} O_X(X)^{et}$. □

The scheme $\pi_0(X)$ is called the group of components of X.

Theorem 2.7.26. For every finite group scheme G over the field k, the following short exact sequence holds,

$$1 \to G_{loc} \to G \to G_{et} \to 1.$$

This means that the morphism of $k-$group schemes $G \to G_{et}$ is faithfully flat and the kernel is G_{loc}. Moreover if k is perfect, this exact sequence canonically splits.

Proof. (Sketch) We take $G_{loc} := G^0$ and $G_{et} := \pi_0(G)$. If k is perfect, G_{red} is a $k-$group scheme because when k is perfect the fibre product of reduced schemes is still reduced scheme. So we obtain a morphism $G_{red} \hookrightarrow G$. All it needs to be checked is that the composition

$$G_{red} \hookrightarrow G \longrightarrow \pi_0(G)$$

is an isomorphism. □

Definition 2.7.27. Let G be a commutative finite $k-$group scheme. We say G is étale-étale if G is étale and its dual $\hat{G}$ is étale. We define similarly the notion of étale-local, local-étale, and local-local.

Theorem 2.7.28. Assume the field k is perfect. Let G be a commutative finite group scheme over k. Then G can be decomposed into a product of these four types of groups

$$G \simeq G_{et,et} \times G_{et,loc} \times G_{loc,et} \times G_{et,et}.$$

Moreover this decomposition is unique.

Proof. It is enough to apply Theorem 2.7.25 above twice. □

Remark 2.7.29. Let k be an algebraically closed field. All étale-étale $k-$group schemes must be a product of étale $k-$group schemes of the form μ_n. Moreover there is a non-canonical isomorphism $\mu_n \sim \mathbb{Z}/n\mathbb{Z}$ depending on the choice of primitive root of unity. In a similar manner, all śtale-local $k-$group schemes must be a product of $\mathbb{Z}/p^n\mathbb{Z}$ and all local-étale $k-$group schemes must be a product of μ_{p^n}. However there are a lot of local-local $k-$group schemes even in this case where we assumes k is algebraically closed.

Remark 2.7.30. Suppose $\text{char}(k) = 0$. Then $G = G_{et,et}$ since there are no nontrivial local $k-$group schemes.

2.7.4 Tate modules

Let A be an abelian variety of dimension g over k. Recall that for any integer $l \neq p = \text{char}(k)$ we defined the $l-$adic Tate module $T_l(A) := \varprojlim_m A[l^m](\bar{k})$ (which is equal to $\varprojlim_m A[l^m](k^s)$ because $A[l^m]$ is étale). As it is well-known the $l-$adic Tate module $T_l(A)$ is a free $\mathbb{Z}_l-$module of rank g with a continuous action of the absolute Galois group $\text{Gal}(\bar{k}/k)$.

Any isogeny $f : A \to B$ induces a continuous map of Tate modules $T_l(f) : T_l(A) \to T_l(B)$. This notion is valid for any commutative group scheme other than abelian varieties. For instance, the $l-$adic Tate module of the multiplicative group $\mathbb{G}_m$ is $\varprojlim_m \mu_{l^m} := \mathbb{Z}_l(1)$ is a free $\mathbb{Z}_l-$module of rank 1 where $\text{Gal}(\bar{k}/k)$ acts via cyclotomic character $\xi : \text{Gal}(\bar{k}/k) \to \mathbb{Z}_l^\times$.

2.7.5 Ordinary abelian variety

Definition 2.7.31. Two invariants of (the $p-$torsion of) an abelian variety A are the $p-$rank and $a-$number.

- The $p-$rank of A is defined to be $f = \dim_{\mathbb{F}_p} \text{Hom}(\mu_p, A[p])$. Then p^f is the cardinality of $A[p](k)$.
- The $a-$number of A is $a = \dim_k \text{Hom}(\alpha_p, A[p])$.

Definition 2.7.32. An abelian variety A of dimension g is ordinary if $A[p]$ has $p-$rank $f = g$. If A is ordinary then it can be shown that $A[p] \simeq L^g$.

At the other extreme, we have the following notion,

Definition 2.7.33. An abelian variety A is called supersingular if $A[p]$ has $a-$number $a = g$. In this case, it can be shown that $A \simeq E^g$ for a supersingular elliptic curve E.

Chapter 3

Fibered surfaces and regular models

3.1 Basic properties of fibered surfaces

Definition 3.1.1. Let S be a Dedekind scheme (Definition 2.4.1). An integral, projective, flat $S-$scheme $\pi : X \to S$ of dimension 2 is called a fibered surface over S. The generic point of S is usually denoted by η and the fibre X_η the generic fibre. If $s \in S$ is a closed point the fibre X_s is called a closed fibre of π. Note that the flatness is equivalent to the surjectivity of π. If X is normal (regular), we call it a normal (regular) fibered surface over S.

A morphism between fibered surfaces is a morphism of schemes which is compatible with the structure of $S-$schemes.

Example 3.1.2. Let $S = \operatorname{Spec}\mathbb{Z}$ and $X = \operatorname{Proj}\mathbb{Z}[x, y, z]/(y^2z + yz^2 - x^3 + xz^2)$. We are going to show that X is a normal fibered surface. Of course, $X \to S$ is projective and flat. So all we have to show is that X is normal. By the Jacobian criterion all the fibres of $X \to S$ are smooth except for the closed point $p = 37$ (See [23]). The fibre X_{37} is reduced, therefore the normality of X follows by Theorem 2.4.5.

Theorem 3.1.3. If S is a Dedekind scheme of dimension 1 and $X \to S$ is a fibered (resp. normal fibered) surface, then X_η is an integral (resp. normal) curve over $K(S)$ and for $s \in S$ closed point X_s is a projective curve over $k(s)$.

Proof. X_η is irreducible because if ξ is the generic point of X it lies on X_η since $X \to S$ is dominant. On the other hand, for $x \in X_\eta$ we have $\mathcal{O}_{X_\eta,x} = \mathcal{O}_{X,x}$ Therefore if X is normal then X_η is normal too. If $x \in X_s$ is a closed point then we know that $\dim \mathcal{O}_{X_s,x} = \dim X_\eta$. Hence all we have to show that $\dim X_\eta = 1$.

Now if $x \in X$ is a point with $\dim \mathcal{O}_{X,x} = 2$ then x has to be a closed point in the closed fibre X_s and then we have from Corollary 2.5.7

$$\dim \mathcal{O}_{X_s,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s} = 1.$$

□

3.1.1 Local properties of fibered surfaces

Let us concentrate on closed fibres.

Theorem 3.1.4. Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme of dimension 1.

1. Let $x \in X_\eta$ be a closed point. Then $\overline{\{x\}}$ is an irreducible closed subset of X finite and surjective to S.
2. Let D be an irreducible closed subset of X. If $\dim D = 1$, then either D is an irreducible component of a closed fibre, or $D = \overline{\{x\}}$ where x is a closed point of X_η.
3. Let $x_0 \in X$ be a closed point. Then $\dim \mathcal{O}_{X,x_0} = 2$.

Proof. (1) Assume $D = \overline{\{x\}}$ then D is obviously irreducible. On the other hand, π is projective and therefore proper. So $\pi(D)$ is a closed subset of S containing the generic point of S. Hence $\pi(D) = S$ Now we are going to show that $\pi|_D : D \to S$ is finite. Of course it is projective therefore all we have to show is that $\pi|_D$ is quasi-finite. For any $s \in S$, $\dim X_s = 1$ If $\dim(D \cap X_s \neq 0$, D would contain an irreducible component of X_s therefore would be equal to it, which is impossible (because D is irreducible in the first place). Hence $D \cap X_s$ is finite and we are done.

(2) Since we have assumed that D is irreducible and closed therefore $\pi(D)$ is irreducible and closed subset of S. If $\pi(D) = \{s\}$ then we have $D \subseteq X_s$. But we have $\dim X_s = \dim D$ therefore D is an irreducible component of X_s. If $\pi(D)$ is not reduced to a point, it is therefore equal to S. Hence it contains a point in X_η Consider the closure of that point say, $\overline{\{x\}}$ we have $\dim \overline{\{x\}} = 1$ and by (1) we have $D = \overline{\{x\}}$.

(3) Since π is proper, $\pi(x_0) \in S$ is closed point s and also we have $\dim \mathcal{O}_{X_s,x_0} = 1$. From Theorem 2.6.6 we have

$$\dim \mathcal{O}_{X,x_0} = \mathcal{O}_{X_s,x_0} + \dim \mathcal{O}_{S,s} = 2.$$

□

Definition 3.1.5. Let $\pi : X \to S$ be a fibered surface and D be an irreducible Weil divisor. We say that D is horizontal if $\dim S = 1$ and if $\pi|_D : D \to S$ is surjective (hence finite). If $\pi(D)$ is a point in S, we say that D is vertical. An arbitrary Weil divisor is called horizontal (resp. vertical) if its components are horizontal (resp. vertical). A Cartier divisor is called horizontal (resp. vertical) if its associated Weil divisor is horizontal (vertical).

3.1.2 Relation between generic fibre and closed fibres

Theorem 3.1.6. Let $\pi : X \to S$ be a fibered surface and $s \in S$. Then we have

1. The fiber X_s is a projective curve over $k(s)$, and the equality of arithmetic genera $p_a(X_s) = p_a(X_\eta)$
2. If X_η is geometrically connected (e.g., if $\mathcal{O}_S \simeq \pi_*\mathcal{O}_X$), then X_s is geometrically connected as well.
3. If X_η is geometrically integral, then the canonical homomorphism $\mathcal{O}_S \to \pi_*\mathcal{O}_X$ is an isomorphism.

Proof. (1) It is results from the projective formula that the Euler-Poincaré characteristic $\mathcal{O}_{X_\eta}$ and $\mathcal{O}_{X_s}$ are equal, i.e.,

$$\chi_{k(s)}(\mathcal{O}_{X_\eta}) = \chi_{k(\eta)}(\mathcal{O}_{X_s})$$

(2) Let $s \in S$ be a close point. We can assume that $S = \operatorname{Spec} A$ is a local scheme (by localising at s if necessary). Now set $B = \mathcal{O}_X(X)$ and $L = \mathcal{O}_X(X_\eta)$. First we want to show that X_s is connected. Consider the canonical morphisms $X \to \operatorname{Spec}(\mathcal{O}_X)$. It is projective and of course all of its fibres are geometrically connected. On the other hand, since X_η is geometrically connected therefore $L/K(S)$ is a finite purely inseparable extension. Hence there is a bijection between their integral rings, i.e., $\operatorname{Spec} B \to \operatorname{Spec} A$ is bijective and we are done.

Take $k'/k(s)$ to be a finite simple extension. There is a finite discrete valuation ring A' such that its residue field is k'. Now X'_A is again flat and projective with the geometrically connected generic fibre. Therefore X'_k is connected (from the discussion above). If k'' is an arbitrary finite extension, then we can decompose it into a sequence of simple extensions. So, X_s is geometrically connected.

(3) Again, we can assume that S is affine. All we have to show is that $\mathcal{O}_X(X) = \mathcal{O}_S(S)$. We have $\mathcal{O}_X(X) \subseteq \mathcal{O}_X(X_\eta)$ and since $X \to S$ is a fibered surface therefore $\mathcal{O}_X(X)$ is integral over $\mathcal{O}_S(S)$. By assumption X_η is geometrically integral therefore we have $\mathcal{O}_X(X_\eta) = K(S)$. Hence we have done. □

3.1.3 Smooth locus of fibered surfaces

Theorem 3.1.7. Let $\pi : X \to S$ be a fibered surface. Suppose that the generic fibre X_η is smooth. Then there exists a non-empty open subset V of S such that $\pi^{-1}(V) \to V$ is smooth. In other words, X_s is smooth over $k(s)$ except for a finite number of closed points s.

Proof. We know that there exists an open subset of X on which π is smooth. Let us denote this open subset by X_{sm}. Take $X \backslash X_{sm}$ and therefore $\pi(X \backslash X_{sm}) \subseteq S$ is closed. Now if we set

$$V = S \backslash \pi(X \backslash X_{sm})$$

then it contains η and its complement in S is finite because $\dim S \leq 1$ and we are done. □

Remark: J.M. Fontaine in [12] showed that the fibered surface $X \to \operatorname{Spec} \mathbb{Z}$ is never smooth if $g(X_\eta) \neq 0$, meaning in this case there always exists a fibre X_s which is not smooth.

Convention: Throughout the thesis an arithmetic surface means a regular fibered surface $X \to S$ over a Dedekind scheme S of dimension 1.

3.2 Desingularisation

Definition 3.2.1. Let X be a reduced and locally Noetherian scheme. A proper birational morphism $\pi : Z \to X$ with Z regular is called a desingularisation of X (or a resolution of singularities of X). If π is an isomorphism above every regular point of X, we say that it is a desingularisation in the strong sense.

Example 3.2.2. If X is a reduced curve over a field k then the normalisation $X' \to X$ is a desingularisation of X. Even more generally, if X is excellent, reduced and Noetherian of dimension 1, then the normalisation is a desingularisation. Because in this case the normalisation is finite and therefore proper.

3.2.1 Hironaka's theorem

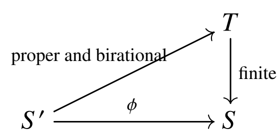
In the case of varieties over fields of characteristic 0, Hironaka proved that resolution of singularities in the strong sense exists.

Theorem (Hironaka) 3.2.3. Let X be reduced algebraic variety over a field of characteristic 0, or more generally X be a reduced scheme which is locally of finite type over a reduced, excellent and locally Noetherian scheme of characteristic 0 (i.e., $\operatorname{char} k(x) = 0$ for every $x \in X$). Then X admits a desingularisation in the strong sense.

3.2.2 Alteration and a theorem by de Jong

It has remained as an open problem for the case of arbitrary characteristic. But still a weaker version of desingularisation – alteration – is enough.

Definition 3.2.4. See [5]. Let S be a Noetherian integral scheme. An alteration S' of S is an integral scheme S', together with a morphism $\phi : S' \to S$, which is dominant, proper and such that for some nonempty open $U \subset S$, the morphism $\phi^{-1}(U) \to U$ is finite. (This condition is equivalent to the condition $\dim S = \dim S'$, when both are finite.) This is in turn equivalent to say that the morphism $S' \to S$ can be decomposed into a proper birational morphism $S' \to T$ and a finite surjective morphism $T \to S$, by using Stein factorization (See [23], Exercise 5.3.11).



Theorem (de Jong) 3.2.5. (See [5]) Let X be a separated integral scheme of finite type over a complete discrete valuation ring (that can be a field). Then there exists an alteration $Y \to X$ with Y regular.

3.2.3 Existence of desingularisation in some certain cases

The following theorem of Lipman is crucial in resolving the singularities of an excellent surface.

Theorem 3.2.6. (Lipman, See [21], [22]) Let X be an excellent, reduced and Noetherian scheme of dimension 2. Then the following sequence if finite;

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X,$$

in which $X_1 \to X$ is the normalisation of X, and for every $i \geq 1$, $X_{i+1} \to X_i$ is the composition of the blowing-up $X'_i \to X_i$ of the singular locus $\mathrm{Sing}(X_i) := X_i \setminus \mathrm{Reg}(X_i)$ (which is closed because X_i is excellent) endowed with the reduced scheme structure, and the normalisation $X_{i+1} \to X'_i$.

In other words, the theorem says that the sequence above stops at $n < \infty$. Of course the sequence stops at n if and only if X_n is regular. In particular, X admits a desingularisation in the strong sense.

Therefore if $X \to S$ is a fibered surface and S is an excellent Dedekind scheme (e.g., $S = \mathrm{Spec}\, A$ in which A is a Dedekind domain of characteristic 0). Then X admits a desingularisation in the strong sense, because X is of finite type over S. Hence it is excellent and the assertion follows from theorem 3.2.6.

We state without proof the following useful theorem. For a proof refer to [23], Theorem 8.3.50.

Theorem 3.2.7. Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme S of dimension 1. Let the generic fibre of π be regular. Then the following conditions are equivalent.

1. The scheme X admits a desingularisation in the strong sense.
2. The set $\mathrm{Sing}(X)$ is contained in a finite union of closed fibres $X_{s_1}, \ldots, X_{s_r}$ and the curve $X \times_S \operatorname{Spec} \operatorname{Frac}(\hat{O}_{S,s_i})$ is regular for every $i \le r$.

Proof. [23], Theorem 8.3.50. □

Corollary 3.2.8. Let $X \to S$ be a fibered surface. Suppose that $\dim S = 1$ and that X has a smooth generic fibre X_η. Then $X \to S$ verifies the conditions of Theorem 3.2.7. In particular, X admits a desingularisation in the strong sense.

Proof. We know from Theorem 3.1.7 that $X \to S$ is smooth above a non-empty open subscheme $V \subseteq S$. Therefore after the base change $V \to S$, X_V is regular. This means that the singular points of X lies in a finite union of closed fibres X_{s_i} in which the s_i are in the finite set $S \setminus V$ and we are done. □

3.2.4 Embedded resolution of singularities (of curves)

Theorem 3.2.9. (For a proof of the theorem, see [23], section 9.2.4). Suppose S is a Dedekind scheme and $X \to S$ is a regular fibered surface. Let us fix an effective Cartier divisor D on X. If D is excellent then there exists a projective birational morphism $f : X' \to X$ with X' regular, such that f^*D is a divisor with normal crossings.

Note that if D is integral, then the strict transform $\tilde{D}$ of D in X' is an irreducible component of f^*D. Therefore $\tilde{D}$ is a regular curve because f^*D is normal crossings.

1. If S is excellent then D is excellent as well, because it is a closed subscheme of an excellent scheme.

2. If D is a vertical divisor then D is excellent, because it is a projective curve (and therefore complete) over a field.

Convention: The arithmetic surface $X \to S$ is called normal crossings if for every closed point $s \in S$, the divisor X_s has normal crossings.

The following corollary is directly deduced from Theorem 3.2.9.

Corollary 3.2.10. Suppose $X \to S$ is an arithmetic surface that has only a finite number of singular fibres (e.g., its generic fibre is smooth, Theorem 3.1.7). Then there exists a projective birational morphism $X' \to X$ such that $X' \to S$ is an arithmetic surface with normal crossings.

Proof. In Theorem 3.2.9, set D to be the sum of all singular fibres. □

Remark (See [23], 9.2.36) It is worth noting that in the higher dimension Theorem 3.2.9 is generalised by Hironaka in case characteristic S is zero. If characteristic S is non-zero, we have a version of embedded resolution with alteration morphisms involved as follows;

Theorem 3.2.11. (See [5], Theorem 6.5 and 8.2) Let $X \to S$ be an integral projective scheme over the spectrum of a complete discrete valuation ring. Let Z be a closed subset of X. Then there exists an alteration $f : X_1 \to X$ such that X_1 is regular and that $f^{-1}(Z)$ is the support of an effective Cartier divisor with normal crossings.

3.3 Regular models

Definition 3.3.1. Let $X \to S$ be a normal fibered surface. We call a regular fibered surface $Y \to S$ together with a birational map $Y \dashrightarrow X$ a regular model of X over S. Note that if $\dim S = 1$, then $Y_\eta \dashrightarrow X_\eta$ is a birational map of projective normal curves. Therefore it is an isomorphism.

A morphism between two regular models Y, Z of X is morphism of fibered $S-$surfaces $Y \to Z$ that is compatible with the birational maps $Y \dashrightarrow X$ and $Z \dashrightarrow X$.

Suppose S is a Dedekind scheme of dimension 1, with the function field K and C is a connected, normal, projective curve over K. We call a normal fibered surface $\mathcal{C} \to S$ together with an isomorphism $f : \mathcal{C}_\eta \simeq C$ a model of C over S. If $\mathcal{C}$ is regular, we say it is a regular model of C. The same as before, a morphism between two models of C say, $\mathcal{C}_1$, and $\mathcal{C}_2$, is a morphism $\mathcal{C}_1\mathcal{C}_2$ of $S-$schemes such that it is compatible with the isomorphism $\mathcal{C}_\eta \simeq C, \mathcal{C}'_\eta \simeq C$. For example, if C is an elliptic curve over K, then the Weierstrass model of C over K is a model of C over S.

Definition 3.3.2. Assume $X \to S$ is regular. A prime divisor E on X is called an exceptional divisor (or $(-1)-$curve) if there exists a regular fibered surface $Y \to S$ and a morphism $f : X \to Y$ of $S-$schemes such that $f(E)$ is reduced to a point, and that $f : X\backslash E \to Y\backslash f(E)$ is an isomorphism. By definition, E is a vertical divisor and $f(E) \in Y$ is a closed regular point and E is an integral curve.

We say that a regular fibered surface $X \to S$ is relatively minimal if it does not contain any exceptional divisor, which is equivalent to say that every birational morphism of regular fibered surfaces $X \to Y$ is an isomorphism.

We say that $X \to S$ is minimal if every birational map of regular fibered surface $Y \dashrightarrow X$ is a birational morphism.

It can be shown that if $X \to S$ is an arithmetic surface with generic fibre of arithmetic genus ≥ 1, then X admits a unique minimal model over S, up to unique isomorphism.

It is well-known that if there is only a finite number of singular fibres (e.g., the generic fibre of $X \to S$ is smooth), then we always have a regular model X' of X which dominates X and has normal crossings, and is minimal for this property as well. In addition if the arithmetic genus of generic fibre is ≥ 1 then there exists a regular model of X that has normal crossings, and is minimal for this property (See for instance [23], Proposition 10.1.8).

Example 3.3.3. Suppose $X \to S$ is a smooth arithmetic surface. Then $X \to S$ is relatively minimal because for any arbitrary vertical divisor V on X we have $V^2 = 0$ (See for instance [23], Proposition 9.1.2). Therefore X contains no exceptional divisor from Castelnuovo's criterion (Theorem 2.6.9). If in addition, $g(X_\eta) \geq 1$, then $X \to S$ is minimal.

3.3.1 Reduction

For more details on this section see [23], 10.1.3.

Definition 3.3.4. Let S be the spectrum of a Henselian discrete valuation ring $\mathcal{O}_K$. Let $\mathcal{X} \to S$ be a surjective proper morphism with generic fibre X. If we denote the set of closed points of X by X^0 then define the map r as follows;

$$r_X : X^0 \to \mathcal{X}_s$$

$$x \mapsto y$$

where $\{y\} = \overline{\{x\}} \cap \mathcal{X}_s$ and $\overline{\{x\}} \cap \mathcal{X}_s$ is a closed point of $\mathcal{X}_s$ because $\overline{\{x\}}$ is irreducible and finite scheme over S, therefore it is a local scheme. When the model $\mathcal{X}$ is fixed, we set $r(x) := \tilde{x}$. Consider that we assume S is the spectrum of a Henselian discrete valuation ring. This guarantees that the $\bar{\{x\}}$ is a local scheme.

Now the question is whether the map above is surjective to the set of closed point. The answer is affirmative.

Theorem 3.3.5. Let S be a Dedekind scheme of dimension 1, $\mathcal{X} \to S$ a dominant morphism of finite type with $\mathcal{X}$ irreducible. Let $\tilde{x} \in \mathcal{X}_s$ be a closed point of the closed fibre. Then there exists a closed point $x \in \mathcal{X}_\eta$ such that $\tilde{x} \in \overline{\{x\}}$.

Proof. We may assume that $\mathcal{X}$ is integral because the question is of topological nature. Therefore we have $\mathcal{X} \to S$ is flat over S. We proceed by induction of $d = \dim \mathcal{O}_{\mathcal{X},\tilde{x}}$.

If $d = 1$, then $\dim \mathcal{O}_{\mathcal{X}_s,\tilde{x}} = 0$. and therefore $\dim \mathcal{X}_\eta = 0$ because it contains at most one point. It follows that $\mathcal{X}_\eta$ is reduced to a point x and we necessarily have $\tilde{x} \in \overline{\{x\}}$ because x is the generic point of $\mathcal{X}$, so we are done in this case.

Suppose that $d \geq 2$. We construct an integral closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ such that $\tilde{x} \in \mathcal{Y}$, and $\mathcal{Y}$ is not contained in $\mathcal{X}_s$ and $\mathcal{Y}_s$ does not contain any irreducible component of $\mathcal{X}_s$. The construction is as follows;
Let m_x (resp. m_s) be the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$ (resp. of $\mathcal{O}_{S,s}$). If $p_1, \ldots, p_n$ are the minimal prime ideals of $\mathcal{O}_{\mathcal{X},x}$ containing the ideal of $m_s\mathcal{O}_{\mathcal{X},x}$. Since $\dim \mathcal{O}_{\mathcal{X},x} = 2$, we have $m_x \neq p_i$. Therefore there exists an element

$$f \in m_x \backslash (\cup_i p_i).$$

Therefore $V(f)$ does not contain any irreducible components of $\operatorname{Spec} \mathcal{O}_{\mathcal{X}_s,x}$. Find an open affine subset $\mathcal{W}$ such that $x \in \mathcal{W} \subseteq \mathcal{X}$ and also a regular element $g \in \mathcal{O}_{\mathcal{X}}(\mathcal{W})$ such that $g = f$, and $V(g) \cap \mathcal{W}_s = \{x\}$. Now $V(g)$ is the desired closed subscheme. Set

$$V(g) := \mathcal{Y}.$$

We have $\mathcal{Y}$ is flat of finite type over S and

$$\dim \mathcal{O}_{\mathcal{Y},\tilde{x}} = \dim \mathcal{O}_{\mathcal{Y}_s,\tilde{x}} + 1 < \dim \mathcal{O}_{\mathcal{X}_s,\tilde{x}} + 1 = \dim \mathcal{O}_{\mathcal{X},\tilde{x}}.$$

By the induction hypothesis, there exists a closed point $x \in \mathcal{Y}_\eta$ such that $\tilde{x} \in \overline{\{x\}}$. Since $\mathcal{Y} \subset \mathcal{X}$ is a closed subset, the point x has the desired property and we are done. □

Chapter 4

Semi-stable curves

4.1 Ordinary double points

There are several definitions for the ordinary double points of curves. The following definition is taken from [23], Definition 7.5.13.

Definition 4.1.1. Let X be a reduced curve over an algebraically closed field k and $\pi : X' \to X$ be the normalisation morphism. We say that a closed point $x \in X$ is an ordinary multiple point if $\delta_x = m_x - 1$, where $m_x := \# \{\text{points of } \pi^{-1}(x)\}$ and $\delta_x := \text{length}_{O_{X,x}} \mathcal{S}_x = [k(x) : k]^{-1} \dim_k \mathcal{S}_x$ in which $\mathcal{S}$ is the quotient coherent sheaf which makes the following short sequence exact.

$$0 \to O_X \to \pi_* O'_X \to \mathcal{S} \to 0.$$

If m_x in the definition of ordinary multiple point is equal 2, then we say x is an ordinary double point or a node.

Remark: Comparing the notion "ordinary double point" with "meeting transversely" which was introduced in 2.6.6, the curve X in an étale neighbourhood of its ordinary double point is isomorphic to $\text{Spec}(k[x, y]/(xy))$. This implies that the residue field at the singular point is separable over the base field k. Yet there are some other equivalent definitions for the notion as well; see for instance [11], Definition V-31: Let $X \to k$ be a curve over an algebraically closed field k. An ordinary double point is defined to be a closed point p of the curve X at which the formal completion $\hat{O}$ of the local ring $O_{X,p}$, with respect to its maximal ideal, is isomorphic to $k[[U, V]]/(U.V)$.

If k is not necessarily algebraically closed, we define the ordinary multiple point $x \in X$ to be the point for which every point $x' \in X_{\bar{k}}$ lying above x are ordinary multiple point. $\bar{k}$ is the algebraic closure of k.

Example: Let X be the curve $\text{Spec}\, k[t, s]/(s^2 - t^2(1 + t))$ over a field k of characteristic $\neq 2$. The point $x := (0, 0)$ is an ordinary double point because we have $\text{Spec}\, k[u]$ with

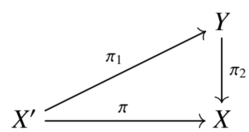
$u = s/t$ and $u^2 = 1 + t$ as the normalisation of X. It follows that $\delta_x = 1$ and $m_x = 2$.

Construction (See [23], the explanation before the Lemma 7.5.12) Suppose X is a reduced curve over an algebraically closed field k and $\pi : X' \to X$ is the normalisation of X. We are going to construct a curve between X and X' having ordinary points as its singularities. Let $x \in X$ be a singular point and $y_1, \ldots, y_m$ denote the points of $\pi^{-1}(x)$. Let V_x be an open affine neighbourhood of x such that x is the only singularity of V_x. Define W_x to be the affine curve corresponding to the $\mathcal{O}_X(V_x)$−algebra

$$\{b \in \mathcal{O}'_X(\pi^{-1}(V_x)) \mid b(y_1) = \cdots = b(y_m)\}.$$

We can decompose the morphism $\pi^{-1}(V_x) \to V_x$ into finite surjective morphism $\pi^{-1}(V_x) \to W_x \to V_x$. Note that $W_x \to V_x$ is isomorphism over $V_x \backslash \{x\}$.

Now by gluing the affine curves W_x as x varies in X_{sing}, we obtain a reduced curve Y, with only ordinary points as singularities making the following diagram commutative;



4.2 Semi-stable curves

The simplest singular curves are those whose singular points are ordinary double points. Their origin goes back to the study of moduli of smooth curves.

Definition 4.2.1. Suppose C is an algebraic curve over an algebraically closed field k. We say that C is semi-stable if it is reduced, and if its singularities are ordinary double points. In general, we say that a curve C over a field k (not necessarily algebraically closed) is semi-stable if its extension $C_{\bar{k}}$ is semi-stable over $\bar{k}$.

We also call a morphism of finite type $f : X \to S$ semi-stable if it is flat and if for any $s \in S$, the fibre X_s is a semi-stable curve over $k(S)$. It is obvious that semi-stable morphisms are stable under base change.

In case S is a Dedekind scheme of dimension 1 and C is a smooth projective curve over $K(S)$, we say that C has semi-stable reduction at $s \in S$ if there exists a model $\mathcal{C}$ of C over $\operatorname{Spec} \mathcal{O}_{S,s}$ which is semi-stable over $\operatorname{Spec} \mathcal{O}_{S,s}$.

One can show if S is an affine Dedekind scheme of dimension 1 and C is of genus ≥ 1 then the minimal regular model $\mathcal{C}_{\min}$ of C over S is a semi-stable model (for a proof see [23], Theorem 10.3.34).

Example A smooth curve over a field k is semi-stable.

4.2.1 Basic facts about semi-stable curves

Theorem 4.2.2. (See [23], Theorem 10.7.3) Suppose C is a semi-stable curve over a field k.

1. If C is regular, then it is smooth over k.
2. Let $\pi : C' \to C$ be the normalisation morphism and $x \in C$ a singular point. Then for $y \in \pi^{-1}(x)$, $k(x)$ and $k(y)$ are separable over k.
3. Keeping the assumption of (2), suppose in addition that the points in $\pi^{-1}(x)$ are rational over k. Then $\pi^{-1}(x)$ contains exactly two points y_1, y_2. Let V be an affine open neighbourhood of x such that $V_{\text{sing}} = \{x\}$. Then

$$\mathcal{O}_C(V) = \{f \in \mathcal{O}'_C(\pi^{-1}(V)) \mid f(y_1) = f(y_2)\}, \quad \hat{\mathcal{O}}_{C,x} \simeq k[[u, v]]/(uv).$$

Proof. (1) We may assume that C is affine and integral. Set $C := \operatorname{Spec} A$. Therefore by assumption A is regular and integral. Since C is regular over k, we know that it is also regular over the separable closure k^s. Hence we can assume that C is affine, integral and regular over a separably closed field k. By definition of semi-stability, C is geometrically reduced. Therefore the function field of $C_{\bar{k}}$ is $K(C_{\bar{k}}) = K(C) \otimes_k \bar{k}$.

We are going to show that the normalisation $(C_{\bar{k}})' \to C_{\bar{k}}$ is isomorphism. We know that the morphism $C_{\bar{k}} \to C$ is a homeomorphism (because we have assumed that k is separably closed) and since $C_{\bar{k}}$ is semi-stable (by definition), which in turns means that all singularities are ordinary double points, it is enough to show that $(C_{\bar{k}})' \to C$ is bijective.

Let B be the integral closure of A in $K(C) \otimes_k \bar{k}$. Since we have assumed that k is separably closed, if we take $b \in B$ there exists a power $q \geq 1$ of char(k) such that $b^q \in K(C)$. But b^q is integral over A. So $b^q \in A$ because A is regular and therefore normal. This means for every prime ideal p of A, $\sqrt{pB}$ is a prime of B. In other words $(C_{\bar{k}})' \to C$ is bijective and we are done.

(2) The proof is the same as (1): We may again suppose that k is algebraically closed field and C is affine, say $\operatorname{Spec} A$. If $x \in C$ is the unique singularity of C, then $\bar{x} \in C_{\bar{k}}$ is the unique point above x and the unique singular point of $C_{\bar{k}}$. Assume $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ and $\operatorname{Spec} D \to \operatorname{Spec}(A \otimes_k \bar{k}$ are the normalisation's respectively corresponding to the morphisms $C' \to C$ and $(C_{\bar{k}})' \to C_{\bar{k}}$. We are going to show that $D = B \otimes_k \bar{k}$.

Since $\bar{x} \in C_{\bar{k}}$ is the unique singularity which is moreover an ordinary double point, we have $\dim_{\bar{k}} D/(A \otimes_k \bar{k} = 1$. Since $A \neq B$, it follows that $\dim_k B/A = 1$. On the other hand, $\operatorname{Spec} D \to \operatorname{Spec} B$ is a homeomorphism. Therefore $\pi^{-1}(x)$ contains exactly two points y_1, y_2. If m is the maximal ideal of A the following homomorphism of $k-$vector spaces is surjective;

$$(B/mB)/(A/m) \to (k(y_1) \oplus k(y_2))/(A/m).$$

Therefore $k(x) = k(y_1) = k(y_2) = k$ and we are done.

(3) Since in this case points in $\pi^{-1}(x)$ are $k-$rational and therefore x is $k-$rational. On the other hand, we have the following commutative diagram;

$$\begin{array}{ccc} C'_{\bar{k}} & \longrightarrow & C_{\bar{k}} \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

We have the projections $C'_{\bar{k}} \to C'$ and $C_{\bar{k}} \to C$ are bijections above $\pi^{-1}(x)$ and x. Therefore there are exactly two points in $\pi^{-1}(x)$ and the rest of (3) is obvious. □

We conclude this chapter with the definition of "split ordinary double points" as follows,

Definition 4.2.3. Let C be a semi-stable curve over a field k and $\pi : C' \to C$ be the normalisation morphism, and $x \in C$ a singular point. We say that x is split if the points of $\pi^{-1}(x)$ are all rational over k. Of course, this means that x is rational over k as well. From Theorem 4.2.2 (2), every singular point becomes split over a finite separable extension of k.

Chapter 5

Semi-stable reduction theorems and some of the main results

5.1 Some generalisations of the stable reduction theorem

Theorem 5.1.1. Let S be Dedekind scheme of dimension 1, C a smooth projective and geometrically connected curve of genus $g \geq 2$ over $K(S)$. Then there exists a Dedekind scheme S' that is finite flat over S such that $C_{K(S')}$ has a unique stable model over S'. Moreover, we can take $K(S')$ separable over $K(S)$ (See for instance [23], Theorem 10.4.3).

Definition 5.1.2. Let S be an integral Noetherian scheme. A modification S' of S is an integral scheme S', together with a proper birational morphism $\varphi : S' \to S$. There is the (largest) closed subset of S over which φ is not an isomorphism; it is called the centre of modification. It is obvious that the composition of two modifications is again a modification.

We recall that an alteration of an integral Noetherian scheme S is an integral scheme S', together with a morphism $\varphi : S' \to S$, which is dominant, proper and such that for some nonempty open $U \subset S$, the morphism $\varphi^{-1}(U) \to U$ is finite. If $\dim S$ and $\dim S'$ are finite, then the finiteness of φ over an open subset of S is equivalent to the condition $\dim S = \dim S'$. The complement of the largest non-empty open subscheme $U \subset S$ over which $\varphi^{-1}(U) \to U$ is finite and flat is called the centre of alteration. Again it is obvious that the composition of two alterations is an alteration.

Historically, this theorem was first proved by Deligne and Mumford using a theorem of Raynaud [8] which links the reduction of a regular model of the given curve to that of the Néron model of its Jacobian. Deligne-Mumford took the Jac(C) over S and they considered its Néron model and compared it to the regular model of C over S. Since the corresponding theorem was already known for abelian varieties (**Theorem:** Let R

be a discrete valuation ring with quotient field K and A be an abelian variety over K. Then there exists a finite separable extension L of K such that if R_L = integral closure of R in L, and if $\mathcal{A}_L$ is the Néron model of $A \times_K L$ over R_L, then the closed fibre $\mathcal{A}_{L,s}$, of $\mathcal{A}_L$ has no unipotent radical), they extended the theorem above to the case of algebraic curves.

Takeshi Saito [37] proved the theorem using the theory of vanishing cycles. He moreover characterised the case when $S' \to S$ is wildly ramified.

This theorem is crucial for proving that moduli space of stable curves of genus g is proper.

Using the fact that the moduli space of stable curves of genus g is proper, Deligne ([9], 1.6) extended the semistable reduction theorem above as follows: Let U be an open (quasi-compact) dense subset of algebraic space S and $u : X \to S$ a family of stable curves of genus $g \geq 2$ parametrised by U. There exists a proper surjective morphism $f : S' \to S$ such that the inverse image X by $f^{-1}(U)$ extends to a family of stable curves parametrised by S'.

In [7] de Jong showed that: if C is a proper curve over a quasi-compact excellent and integral scheme S, there exists an alteration $S' \to S$ and a modification $C' \to C \times_S S'$ such that C' is a proper semi-stable curve over S'.

In the fundamental article [8], Deligne and Mumford succeeded to enlarge the category of schemes – by introducing the object of algebraic stacks – so as to study the family of stable curves of a given genus g. In addition, they extended the notions of separatedness and properness (and also their classic criteria) to the algebraic stacks.

Eventually they showed that the algebraic space $\mathcal{M}_g$ corresponding to the family of stable curves of genus g, which is called moduli space of stable curves, is proper.

This raises the following question:

If one starts with a stable curve of genus $g \geq 2$, which is projective and geometrically connected, to what extent does the Theorem 5.1.1 hold?

In what follows, we are going to scrutinise the question above.

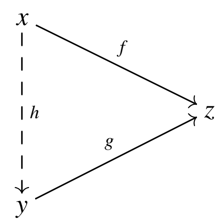
5.2 Algebraic stacks and properness of $\mathcal{M}_g$

Our main reference for this chapter is [8].

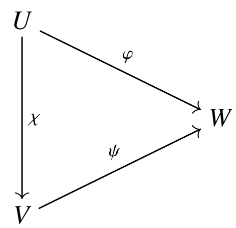
Definition 5.2.1. Suppose $\mathcal{S}$ is the category of schemes and $p : \mathscr{S} \to \mathcal{S}$ is a category over $\mathcal{S}$. For each $U \in \operatorname{Ob}\mathcal{S}$, set $\mathscr{S}_U := p^{-1}(U)$. The category $\mathscr{S}$ is called fibered in groupoids over $\mathcal{S}$ if the following two conditions are verified:

1. For all $\varphi : U \to V$ in $\mathcal{S}$ and $y \in \operatorname{Ob}\mathscr{S}_V$ there is a map $f : x \to y$ in $\mathscr{S}$ with $p(f) = \varphi$.

2. Given a diagram



   in $\mathscr{S}$, let



   be its image in $\mathcal{S}$. Then for all $\chi : U \to V$ such that $\varphi = \psi\chi$, there is a unique $h : x \to y$ such that $f = gh$ and $p(h) = \chi$.

Condition (2) means that the $f : x \to y$ whose existence is guaranteed in the condition (1) is unique. Therefore if we denote the unique x by $x := \chi^* y$. This means that we consider χ^* as a functor from the category $\mathscr{S}_V$ to that of $\mathscr{S}_U$. Obviously, we have $(\varphi\psi)^* = \psi^*\varphi^*$.

Definition 5.2.2. Let $\mathcal{S}$ be the category of schemes with étale topology. A stack in groupoids over $\mathcal{S}$ is a category over $\mathcal{S}$, $p : \mathscr{S} \to \mathcal{S}$ such that:

1. $\mathscr{S}$ is fibered in groupoids over $\mathcal{S}$.

2. For any $U \in \operatorname{Ob} C$ and any objects x, y in $\mathscr{S}_U$ the functor from C/U to sets which to any $\varphi : V \to U$ associates $\operatorname{Hom}_{\varphi_V}(\varphi^* x, \varphi^* y)$ is a sheaf.

3. If we have an étale covering of the object U, say $\varphi_i : V_i \to U$, any decent datum relative to the φ_i for objects in $\mathscr{S}$ is effective.

More explicitly, if $x \in \operatorname{Ob}\mathscr{S}_U$, there are given isomorphisms between the inverse images of $x_i = \varphi_i^* x$ and $x_j = \varphi_j^* x$ over $V_{ij} = V_i \times_U V_j$, and pull-backs of these isomorphisms on $V_{ijk} := V_i \times_U V_j \times_U V_k$ satisfy a cocycle condition and (3) asserts that any such cocycles are defined by some $x \in \mathscr{S}_U$.

Definition 5.2.3. A 1-morphism of stacks over the category of schemes $\mathcal{S}$, $F : \mathscr{S}_1 \to \mathscr{S}_2$, is called representable if for any $X \in \operatorname{Ob} \mathcal{S}$ and any 1-morphism $x : X \to \mathscr{S}_2$, (we have considered X as a stack), the fibre product $X \times_{\mathscr{S}_2} \mathscr{S}_1$, is a representable stack. We say a representable 1-morphism $F : \mathscr{S}_1 \to \mathscr{S}_2$ of stacks over $\mathcal{S}$ has property **P** if for any 1-morphism $x : X \to \mathscr{S}_2$ the morphism in $\mathcal{S}$ obtains by base change $F' : X \times_{\mathscr{S}_2} \mathscr{S}_1 \to X$ has the same property **P**.

Example: One can see that the diagonal map

$$\mathscr{S} \to \mathscr{S} \times_{\mathcal{S}} \mathscr{S}$$

is representable.

Definition 5.2.4. A stack $\mathscr{S}$ is quasi-separated if the diagonal morphisms above is representable, quasi-compact and separated.

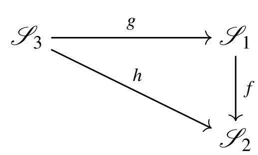
Algebraic stack: A stack $\mathscr{S}$ is an algebraic stack if

1. $\mathscr{S} \to \mathscr{S} \times \mathscr{S}$ is representable.
2. there exists a 1-morphism $x : X \to \mathscr{S}$ such that for all $y : Y \to \mathscr{S}$, the projection morphism $X \times_{\mathscr{S}} Y \to Y$ is surjective and étale (by definition this means that x is étale and surjective).

Definition 5.2.5. An algebraic stack $\mathscr{S}$ is called separated if the diagonal morphism $\mathscr{S} \to \mathscr{S} \times \mathscr{S}$ is proper (note that the diagonal morphism is representable).

Separatedness: A 1-morphism $f : \mathscr{S}_1 \to \mathscr{S}_2$ is separated if for any morphism $x : X \to \mathscr{S}_2$ from a separated scheme X to $\mathscr{S}_2$, the fibre product $\mathscr{S}_1 \times_{\mathscr{S}_2} X$ is separated as an algebraic stack.

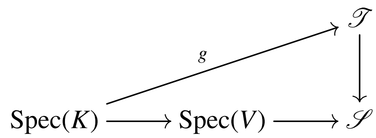
properness: A morphism $f : \mathscr{S}_1 \to \mathscr{S}_2$ is proper if it is separated, of finite type and if, locally over $\mathscr{S}_2$, there exists commutative diagrams



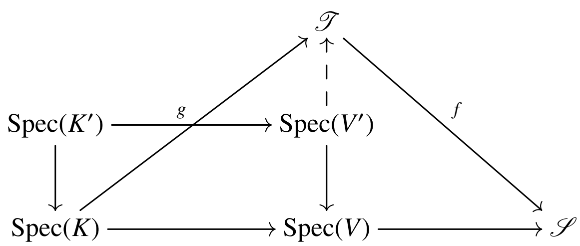
with g surjective and h representable and proper.

For the proof of the following criterion see [8], Theorem 4.19.

Theorem (Valuation criterion for properness) 5.2.6. Let $f : \mathscr{T} \to \mathscr{S}$ be a separated (1-)morphism of (algebraic) stacks. f is proper if and only if for any discrete valuation ring V with field of fractions K and any commutative diagram



there exists a finite extension K' of K such that g extends to Spec(V'), where V' is the integral closure of V in K'



Important remark: To prove a given morphism of stacks is proper, it is enough to verify the above criterion only if V is complete and has algebraically closed residue field. Moreover, if $\mathcal{U} \subset \mathcal{T}$ is an open dense subset, it suffices to test only g's which factor through $\mathcal{U}$.

Using the criterion above, one can show that

Theorem 5.2.7. Let us denote the stack of stable curves of genus $g \geq 2$ defined over a scheme S by $\mathcal{M}_g$. The algebraic stack $\mathcal{M}_g$ is proper and smooth over Spec($\mathbb{Z}$).

The theorem says if one starts with a stable curve, say C/K, then there exists a finite field extension K'/K such that the curve $C_{K'}$ has stable reduction over V' (the integral closure of V in K').

By "stable model of C over $K(S)$ at $s \in S$" (S is a Dedekind scheme of dimension 1), we mean a normal model $\mathcal{C}$ over Spec $\mathcal{O}_{S,s}$ of C such that its closed fibre is a stable curve. A curve C "has stable reduction over S" if the property is true for every $s \in S$. In this case, the model $\mathcal{C}$ over S is called a stable model of the curve. We recall the following definition of stable curves from [23], Definition 10.3.1,

Definition: Let C be an algebraic curve over an algebraically closed field k. We say that C is stable if it is semi-stable (Definition 4.2.1) and if the following conditions are verified:

- C is connected and projective, of arithmetic genus $p_a(C) \geq 2$.
- Let Γ be an irreducible component of C that is isomorphic to $\mathbb{P}^1_k$. Then it intersects the other irreducible components at at least three points.

Now the curve C over an arbitrary field k is stable if its extension $C_{\bar{k}}$ ($\bar{k}$ is the algebraic closure of k) is stable.

5.3 Main results

Through the rest of this thesis $S := \operatorname{Spec} R$ where R is a complete discrete valuation ring.

Our objective is to construct a suitable model of a given semi-stable curve over S. In other words, we start with a semi-stable curve C/K (K is a discrete valuation field) and then glue in a certain way the suitable models of every irreducible components along with their common regular intersections to obtain a model for the reduction of the semi-stable curve C.

Let us first start with the simplest case where the curve C is projective semi-stable with regular irreducible components (and therefore smooth, [23], Theorem 10.3.7), which is defined over the function field $K := K(S)$ of a local scheme $S := \operatorname{Spec} A$ of a complete discrete valuation ring.

Remark 5.3.1. According to the semistable reduction theorem of Deligne-Mumford, each irreducible component has a semi-stable reduction (consequently a semi-stable model) possibly after a ramified extension of K.

We consider the following cases:

- If the arithmetic genus of a component of C is ≤ 0 then the component is rational (because it is geometrically integral and has a rational point as its intersection with other components after possibly a separable extension of the base field, now we can use [23], Proposition 7.4.1). Therefore it has a good reduction (and a unique smooth model). In this case the minimal regular model of the component is smooth.
- If the arithmetic genus of a component of C is 1, then after a finite separable extension of K, we can assume that it has a rational point, hence an elliptic curve. After a finite separable extension, it has a semi-stable reduction ([23], Proposition 10.2.33, Example 10.3.35, Exercise 10.2.3 or 10.4.2).
- If the arithmetic genus of a component of C is > 1 then the component is a stable curve. Therfore, due to the Stable Reduction Theorem of Deligne-Mumford (5.1.1), there exist a semi-stable model (not necessarily unique) for the component, perhaps after a ramified extension of the base.

As already mentioned, our objective here is to find a suitable model for C over S.

Construction 5.3.2. For simplicity, we assume that C has two irreducible components C_1 and C_2. If C is not connected then there will be nothing left to show. Suppose the components intersect at a point p. Since semi-stability is stable under base change, we can assume that the point p is rational possibly after a finite separable extension of the base field K ([23], Theorem 10.3.7 (b) and [23], Definition 10.3.8). On the other hand, we extend the base field even further to guarantee the existence of semi-stable models for C_1 and C_2.

Under our assumptions, each component C_i has a regular model $\mathcal{C}_i$ (it is obvious for the arithmetic genus 0 thanks to Liu [23], Theorem 7.4.1 and for $g \geq 1$, use Theorem 10.1.21 (a)). Denote the closure of p in $\mathcal{C}_i$ as D_i. Since D_i is horizontal divisor in $\mathcal{C}_i$, it is finite and surjective to S. Hence $D_i \simeq S$. From now on, we denote the (canonically) isomorphic divisors D_1 and D_2 by D. We are going to glue the semi-stable S−curves $\mathcal{C}_i$ along the common horizontal divisor D in such a way to obtain a semi-stable curve at the special fibre of the glued surface. We proceed as follows,

Since the situation of gluing along the D is local, by taking an open affine neighbours $U_i = \operatorname{Spec} A_i$ of q_i, the intersection point of D and the corresponding closed fibre $(\mathcal{C}_i)_s$; namely, $D \cap (\mathcal{C}_i)_s$, we have the following morphisms,

$$\begin{array}{ccc} & & A_1 \\ & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & A \end{array}$$

corresponding to the closed immersions $D \hookrightarrow \mathcal{C}_i$. We claim the pull-back of the A_i along A is the desired model, for which the generic and special fibres are both semi-stable.

Remark 5.3.3. Since the point p is rational and the models are regular, the closed points q_1 and q_2 are smooth and $k(s)$−rational points of their corresponding closed fibres (See [23], Corollary 9.1.32). Therefore when we merge them into one point after gluing the models, it will be an ordinary double point again.

Let us denote the pull-back ring by B, we have the following commutative diagram.

$$\begin{array}{ccc} B & \longrightarrow & A_1 \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & A \end{array}$$

in which $B = \{(x, y) : f_1(x) = f_2(y)\}$, as a subring of $A_1 \times A_2$. Suppose that m is the maximal ideal of A and $k = k(s) = A/m$ is its residue field. By definition, we have to show that $B \otimes_A \bar{k}$ is a semi-stable (affine) curve over $\bar{k}$. Tensoring the diagram above, we obtain the following commutative diagram,

$$\begin{array}{ccc} B\otimes\bar{k} & \longrightarrow & A_1\otimes\bar{k} \\ \big\downarrow & & \big\downarrow{\scriptstyle f_1\otimes\bar{k}} \\ A_2\otimes\bar{k} & \xrightarrow{f_2\otimes\bar{k}} & A\otimes\bar{k} \end{array}$$

Let m_i be the corresponding maximal ideal of q_i in $A_i \otimes_A \bar{k}$ and the maximal ideal n be the glued point in $B \otimes_A \bar{k}$. Since $B \otimes_A \bar{k}$ is regular except at the maximal ideal n, using the classification theorem for ordinary double points (Liu, Theorem 7.5.15 (i) and (ii)) it immediately deduces that $B \otimes_A \bar{k}$ is semi-stable. In the same manner, we can show that the generic fibre of $B \to A$ is semi-stable.

Definition 5.3.4. (See [5], 2.4) Let S be a Noetherian scheme. Let $D \subset S$ be a divisor (positive divisor, regularly embedded of codimension 1) and $D_i \subset D$, $i \in I$ be its irreducible components (considered as reduced closed subschemas of D or S). We say that D is a strict normal crossings divisor in S if

- for any $s \in D$ the local ring $O_{S,s}$ is regular,
- D is a reduced scheme, i.e. $D = \cup D_i$ (scheme-theoretically), and
- for any nonempty subset $J \subset I$, the closed subschema $D_J = \cap_{j\in J} D_j$ is a regular scheme of codimension $\#J$ in S.

Remark 5.3.5. (Sketch) The resulted model for the given curve C/K is a strict normal crossings divisor in a smooth ambient space of dimension 3. To see this remember that the resulted model corresponding to each component of C is regular and semi-stable. Using [23], Theorem 9.2.34, we can locally embed the model to a regular (in fact, smooth) scheme Z. Therefore the local description of the constructed model C at every point z is of the form $O_{Z,z}(-C)/(f_1, \dots, f_t)$, in which $f_1, \dots, f_t$ is a part of system of parameters of Z at z.

Now we consider a stable curve C of genus g. Since it might has some rational irreducible components and the latter do not stable model, we instead assume that C is a stable n-pointed curve over $K = K(S)$ with n distinct $K-$rational points $p_1, \dots, p_N$ which lies away from the singular locus of C.

Definition 5.3.6. (See [17], Definition 2.12) A stable $n-$pointed curve (or sometimes $n-$marked) over an algebraically closed field is a complete connected curve C that has only ordinary double points as singularities, together with an ordered collection $p_1, \dots, p_n \in C$ of distinct smooth points of C, such that the $(n+1)-$tuple $(C; p_1, \dots, p_n)$ has only finitely many automorphisms. We know that if the curve C over an algebraically closed field is connected and has no rational component, then the automorphism group is finite. To avoid the other cases, we can assume that:

- the number of marked points plus intersection points of every smooth rational component C is at least 3.

- every rational components of the normalisation of C has at least 3 points lying over singular and/or marked points of C.

In conditions above, if we replace 3 by 2, C is called semi-stable marked curve. As always, a curve C over an arbitrary field K is called (semi-)stable if $C_{\bar{K}}$ obtained by base change is (semi-)stable.

Now, if we start with a marked curve, we have a unique stable model for each component of C after a ramified extension of K. Gluing the stable models of the components along the intersection points p_i on the generic fibres in the same way as we proceed above we obtain a model with stable closed fibre. This is obvious because the procedure of gluing already gives a semi-stable model. On the other hand, the closure of marked (rational) points on the generic fibre intersect exactly one irreducible component of the corresponding closed fibre (this is because we can choose the semi-stable model to be regular). Hence at the intersection point of $\overline{\{p_i\}}$ and the closed fibre of the glued surface, the number of singularities increases by one.

Example 5.3.7. To illustrate the construction 5.3.2, we give an example here.

Let K be a field and $R := K[[t]]$ the power series ring in one variable. Define the curve $C/K((t))$ as follows:

$$C := \operatorname{Spec} \frac{K((t))[X,Y]}{Y(Y - X^2 - t^2)}.$$

Now consider the following model of the curve $C/K((t))$;

$$C := \operatorname{Spec} \frac{R[X,Y]}{Y(Y - X^2 - t^2)}.$$

It is clear that the normalisation of $\frac{R[X,Y]}{Y(Y-X^2-t^2)}$ is the cartesian product $\frac{R[X,Y]}{Y} \times \frac{R[X,Y]}{Y-X^2-t^2} \simeq R[X] \times \frac{R[X,Y]}{Y-X^2-t^2}$.

Denote the image of X and Y in $\frac{R[X,Y]}{Y-X^2-t^2}$ by x and y respectively. We are going to glue the two fibered surfaces along the (canonically) isomorphic sections $\frac{R[X]}{X-t} \simeq \frac{R[x,y]}{x-t} \simeq R$. We have the following surjective $R-$homomorphisms

$$f : R[X] \to R$$
$$X \mapsto t$$

and

$$g : R[x,y] \to R$$
$$x \mapsto t.$$

Now the fibre product $f \times_R g$, which is the kernel of the linear map

$$(f,g) : \frac{R[X,Y]}{Y} \times R[x,y] \to R$$

is the following subalgebra defined by

$$D := \{(P(X,Y), g(x,y)) \ : \ f(t,0) = g(t, 2t^2)\}.$$

The generic (special) fibre of D corresponds to the rings $D \otimes K((t))$ (resp. $D \otimes K$) whose singularities are clearly similar to that of on the generic tiber. Therefore D is semi-stable model of the affine curve C. You may consider that this example is somewhat an special case for which one can glue both smooth models $R[X]$ and $\frac{R[X,Y]}{Y-X^2-t^2}$ along the parallel sections of the normalisation, separately obtained in each model as the closure of the points $(t,0)$ and$(-t,0)$.

Assume $f : X \to S$ is a proper morphism of Noetherian schemes which satisfies the following conditions:

- All fibres of f are nonempty and equidimensional of dimension 1 (the subjectivity here is equivalent to flatness in our case where S is a trait).
- The smooth locus of f is dense in all fibres of f.

Then we have the following fact (See [5], Lemma 5.2):

Theorem 5.3.8. Let S be an excellent integral scheme. Let $f : X \to S$ be a projective morphism satisfying both conditions above and $\sigma_1, \ldots, \sigma_n : S \to X$ be sections of f. There exists a projective alteration $\psi : S' \to S$, and sections $\sigma_1, \ldots, \sigma_n : S' \to X \times_S S'$ such that for any geometric points $\bar{s}$ of S and any irreducible component C of $X_{\bar{s}}$, there exist $i, j, k \in \{1, \ldots, n\}$ such that $\sigma_i(\bar{s}), \sigma_j(\bar{s})$, and $\sigma_k(\bar{s})$ are three distinct points lying on $C \cap \mathrm{sm}(X/S)$.

It indicates that if we start with an $n-$pointed stable curve C over $K = K(S)$, then glue the (regular) semi-stable models of each component as in the Construction 5.3.3, we can obtain a stable model after a finite extension of the base field K. Clearly we must take the sections $\sigma_i : S \to X$ to assign $\eta \mapsto p_i$ and $s \mapsto q_i$ where the points p_i are the marked points in the given stable curve and the q_i are the intersection point of $\overline{\{p_i\}}$ with the closed fibre of the glued model. See the remark right after the following proof.

Proof. (Sketch) The problem is of local nature in the sense if $S = \bigcup U_\alpha$ is a finite covering of S by open affines and $\psi_\alpha : U'_\alpha \to U_\alpha$ are projective alterations, and $\sigma_1^\alpha, \ldots, \sigma_{n_\alpha}^\alpha$ are sections of $X \times_{U_\alpha} U'_\alpha \to U'_\alpha$ for which the assertion holds, then we can construct $S' \to S$ and sections $\sigma_1, \ldots, \sigma_n$ as above.

We choose a closed point $s \in S$ and an affine neighbourhood U of s. Since the based scheme S is Noetherian it can be covered by a finite number of affine opens and since the morphism f is projective there exists a very ample sheaf $\mathcal{L}$ on X over U. Suppose $n \geq 3$ is a large enough integer for which

$$H^0(X, f_*\mathcal{L}^{\otimes n}) \to H^0(X_s, \mathcal{L}^{\otimes n}|_{X_s}).$$

It is plausible because $\mathcal{L}$ is ample and we can choose n so large that H^1 is zero and then use [23], Lemma 5.3.19. Now using Bertini Theorem on the generically smooth curve X_s over s we find a finite separable extension $k(s) \subset k'$ and a section

$$t \in \Gamma(X \otimes k', (\mathcal{L} \otimes k')^{\otimes n}),$$

such that its corresponding divisor $H(t) \subset X \otimes_U k'$ is finite étale over $\mathrm{Spec}\, k'$ and also the divisor $H(t)$ lies in the smooth locus of $f : X \to S$.

There exists a finite étale morphism $\psi : U' \to U$ such that $\psi^{-1}(s) = s'$ and $k(s') \simeq k'$. Shrinking U we can first lift the section t to

$$\tilde{t} \in \Gamma(X'_U, \mathcal{L}^{\otimes n})$$

because it is in the smooth locus of f (Theorem 6.2.3, Liu) and we can also disregard points of U' (by making U small enough) on which $H(\tilde{t}) \to U'$ has fibres of dimension 1 and points of U' on which it is not étale. Hence we get U' such that $H(\tilde{t}) \to U'$ is finite and étale. Considering a geometric point $\bar{u}'$ of U' and C an irreducible component of $X_{\bar{u}}$, we have

$$deg\mathcal{L}|_C \geq 1.$$

Therefore degree $\mathcal{L}^{\otimes n}$ is at least 3 and since $H(\tilde{t}) \to U'$ is finite and étale $H(\tilde{t})$ intersects C at at least three points.

□

Remark 5.3.9. Combining this fact with the Semi-stable Reduction Theorem, we can assume that there exists a stable $n-$pointed curve $(C, \tau_1, \dots, \tau_n)$ over S, a nonempty open subscheme $U \subset S$ and an isomorphism $\beta : C_U \to X_U$ mapping the sections $\tau_i|_U$ to the section $\sigma_i|_U$. In our case, the image of these sections are nothing else but the closure (in the corresponding minimal model of the component) of the marked point in the generic fibre.

To see this (Main references: [19] and [18]. The second reference explains the existence of scheme $M_{g,n}$ when $g = 0$). First recall that there is a projective scheme $\overline{M}$ over the algebraic stack of stable $n-$pointed curves $\overline{\mathcal{M}}_{g,n}$. Suppose the genus $g \geq 0$ and $n \geq 3$. The open substack $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ classifies smooth $n-$pointed curves. Suppose $l \geq 3$ is a prime number and let

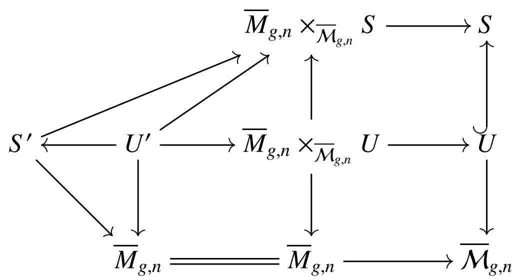
$${}_l\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}[1/l] = \mathcal{M}_{g,n} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[1/l]$$

be the finite étale cover obtained by trivialising the $l-$torsion of the Jacobian of the universal genus g curve over $\mathcal{M}_{g,n}[1/l]$. We can see that ${}_l\mathcal{M}_{g,n} = {}_lM_{g,n}$ is a scheme. Consider the normalisation of $\overline{\mathcal{M}}_{g,n}[1/l]$ in the fundtion field of ${}_l\mathcal{M}_{g,n}$ we have

$${}_l\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}[1/l] = \overline{\mathcal{M}}_{g,n} \times_{\mathrm{spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[1/l]$$

Again one can show that ${}_l\overline{\mathcal{M}}_{g,n} = {}_l\overline{M}_{g,n}$ is a projective scheme over $\mathrm{Spec}\,\mathbb{Z}[1/l]$. Pulling back the universal curve over $\overline{\mathcal{M}}_{g,n}[1/l]$ along the morphism above, we obtain a universal stable $n-$pointed curve of genus g over ${}_l\overline{M}_{g,n}$. Now taking two distinct primes $l_1, l_2 \geq 3$ and putting $\overline{\mathcal{M}}$ equal to the normalisation of $\overline{\mathcal{M}}_{g,n}$ in the function field of ${}_{l_1 l_2}\mathcal{M}_{g,n}$. One can see that $\overline{\mathcal{M}} = \overline{M}$ is a projective scheme over $\mathbb{Z}$ with a finite dominant morphism $\overline{M} \to \overline{\mathcal{M}}_{g,n}$ and a universal curve over $\overline{M}$.

In the situation of theorem above, there is a nonempty subschema $U \subset S$ such that $(X_U, \sigma_1|_U, \dots, \sigma|_U)$ is a smooth stable $n-$pointed curve of genus g (g is the genus of the generic fibre). This gives a 1−morphism $U \to \overline{\mathcal{M}}_{g,n}$. We have the following commutative diagram



Remark that the alteration $S' \to S$ is projective, simply because $\overline{M}$ is projective over $\operatorname{Spec}\mathbb{Z}$. Therefore we are done.

Remark 5.3.10. As you see de Jong's proof is also based on existence and properness of the moduli spaces.

Another question naturally arises is if it is possible to obtain any similar result for an integral, separable, flat and of finite type scheme X (S−variety) over the spectrum of a complete discrete valuation ring (trait) for higher dimensions X and arbitrary characteristic of the base scheme S?

The answer is affirmative. Note that the S−variety X is integral here. First we define some notions.

Definition 5.3.11. (de Jong [5], 2.16) Let S be a trait and X be an S−variety. Let X_i, $i \in I$ be the irreducible components of X. Set $X_J = \bigcap_{j \in J} X_j$ (scheme-theoretic intersection), for a nonempty subset J of I. We say X is strictly semi-stable over S if the following properties hold:

- X_η is smooth over $k(\eta)$,
- X_s is a reduced scheme, i.e. $X_s = \bigcup X_i$ scheme-theoretically,
- for each $i \in I$, X_i is a divisor on X, and
- for each nonempty $J \subset I$, the scheme X_J is smooth over $k(s)$ and has codimension $\#J$ in X.

Remark: The condition above on the scheme $X \to S$ implies that X is regular. (See [5], 2.16)

Theorem 5.3.12. (See [5], Theorem 6.5) Let X be an S−variety (S is a trait). There exist a trait S_1 finite over S, an S_1−variety X_1, an alteration of schemes over S

$$\phi_1 : X_1 \to X$$

and an open immersion

$$j_1 : X_1 \to \overline{X_1}$$

of S_1-varieties, with the following properties:

- $\overline{X_1}$ is projective S_1-variety with geometrically irreducible generic fibre, and
- $\overline{X_1}$ is strictly semi-stable over S.

One can further extend the result above in order to show the behaviour of alteration ϕ_1 on a proper closed subset of $S-$variety X, containing the whole closed fibre.

Let (X, Z) be a pair of an $S-$variety and a closed subset $Z \subset X$ (also considered as a reduced closed subscheme of X). One can write $Z = Z_f \cup Z'$, where $Z_f \to S$ is flat and $Z' \subset f^{-1}(\{s\})$.

Definition: 5.3.13. (See [5], 6.3) We say (X, Z) is strict semi-stable pair if the following conditions are satisfied:

- X is strict semi-stable over S.
- Z is a divisor with strict normal crossings on X-
- Let $Z_f = \bigcup_{i \in I} Z_i$ be the composition of Z_f in its irreducible components, for each $J \subset I$, the scheme $Z_J = \bigcap_{j \in J} Z_j$ is a disjoint union of $S-$varieties which are strict semi-stable over S.

In particular, the third condition implies that Z_J is flat over S.

The main purpose of the following theorem is to show how one can use our Construction 5.3.3 to deal with the higher dimensional case (here $S-$varieties). In order to do that, we first introduced the notion of $S-$varieties and semi-stability of $S-$varieties (resp. Remark 5.3.11 and Definition 5.3.12), then we mentioned the Theorem 5.3.9 which used the sections (corresponding to the pointed curves in our one-dimensional case) and eventually the following theorem shows that after an alteration we can find a suitable semi-stable model for a given $S-$variety.
Considering this concept, the theorem above extends as follows:

Theorem 5.3.14. (See [5], 6.5) Under the hypotheses of the theorem above, if we assume that Z is a proper closed subset of X with $f^{-1}(\{s\}) \subset Z$, there exist a trait S_1 finite over S, an S_1-variety X_1, an alteration of schemes over S

$$\phi_1 : X_1 \to X$$

and an open immersion

$$j_1 : X_1 \to \overline{X_1}$$

of S_1-varieties with the following properties:

- $\overline{X_1}$ is projective S_1-variety with geometrically irreducible generic fibre, and
- The pair $(\overline{X_1},\ {\phi_1}^{-1}(Z)_{\text{red}} \cup \overline{X_1} \backslash j_1(X_1))$ is strict semi-stable.

Remark: Note that the $S-$variety X is integral by definition. If we have more than one irreducible component, considering the disjoint union of the irreducible components $Y := \bigcup_i X_i$, it is still possible to obtain a similar result, since $Y' \to X$ is an isomorphism on a dense subset of Y'.

Chapter 6

A monodromy criterion for extending families of curves

In line with the problem of extending families of smooth curves, we investigate some conditions (in terms of monodromy) sufficient for extending families of curves over an open subset of a normal variety to that of stable curves over the whole base variety of characteristic $p > 0$. To begin with, we introduce some of the known facts,

6.1 Introduction and prerequisite notions

In [6], Oort and de Jong investigate the question whether a given family of stable curves over a dense open subscheme $U \subset S$ (S is an arbitrary scheme) extends to a family of stable curves over whole S. They established the following fact based on stratification of the boundary of moduli space of stable curves:

Theorem 6.1.1. (See [6], Section 5, Main Theorem) Let C_U be a stable curve over U of locally constant topological type. If C_U extends to a stable curve over the generic points of the divisor D (which is a normal crossing divisor), then C_U extends to a stable curve over S.

Remark 6.1.2. The extension theorem above holds for families (C_U, Σ_U) (See [6], Remark 4.9).

The key point of proof is that the extension of the family after blowup comes from its extension in downstairs and the behaviour of extension on the exceptional divisor. We then use the following key lemma. First a definition;

Definition 6.1.3. A stable $r-$pointed curve over a base scheme S is a pair (C, Σ), where $C \to S$ is a flat, projective family of curves and $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ is an ordered set of disjoint sections into the smooth locus of $C \to S$. The geometric fibres of $C \to S$ are reduced, connected and have only nodes as singularities and these fibres have finitely many automorphisms as $r-$pointed curves.

Theorem 6.1.4. ([6], Key Lemma) Let k be any field. Let $(C, \Sigma) \to \mathbb{P}^1_k$ be a stable $r-$pointed curve of genus g, If the topological type of $C, \Sigma)$ is locally constant over $\mathbb{P}^1\backslash\{0, \infty\}$ then (C, Σ) is constant: there exists a stable pair (C_0, Σ_0) over k such that

$$(C, \Sigma) \simeq (C_0 \times \mathbb{P}^1_k, \Sigma_0 \times \mathbb{P}^1_k).$$

Remark 6.1.5. We denote the moduli stack of smooth $n-$pointed curves of genus g by $\mathcal{M}_{g,n}$ and its corresponding coarse moduli space by $M_{g,n}$. Moreover $\overline{\mathcal{M}_{g,n}}$ and $\overline{M_{g,n}}$ are their respective Deligne-Mumford compactifications. In the language of moduli spaces, the Theorem above reads as follows:

Consider a scheme S and a normal crossing divisor $D \subset S$ (i.e., S is regular along D and locally in the étale topology, D is given as the zero set of a product $t_1, \ldots, t_r$ is part of a regular system of parameters). Suppose we are given a morphism $U \to \mathcal{M}_{g,n}$ where $U := S\backslash D$. If the morphism extends to Zariski's neighbourhoods around the generic points of D, then it extends to a morphism $S \to \overline{\mathcal{M}_{g,n}}$.

On the other hand, the condition of extension of morphism to the generic points of D is sufficient to lift the extended morphism $S \to \overline{M_{g,n}}$ to $S \to \overline{\mathcal{M}_{g,n}}$ (See [3] Corollary 4.10). Therefore one can naturally ask: "when does a given morphism $U \to \mathcal{M}_{g,n}$ extend to a morphism of schemes $S \to \overline{M_{g,n}}$?"

Remark 6.1.6. If $\dim S = 1$ one may assume that the base scheme $S = C$ is a curve (because the problem of extending a family is of local nature) and $U = C\backslash\{p\}$, the complement of a closed point.

Historically, Deligne and Mumford showed in [8] that a family of smooth curves over $C\backslash\{p\}$ extends to a family of stable curves over C if and only if the associated Jacobian family extends to a family of semi-abelian varieties. Previously, Grothendieck had shown in [14] that a family of abelian varieties over $C\backslash\{p\}$ extends to a family of semi-abelian varieties over C if and only if the associated monodromy on the H_1 homology of a fibre in a small (analytic) neighbourhood of p is unipotent.

Combining these results one obtains the following fact:

Theorem 6.1.7. (Deligne–Mumford–Grothendieck) A family of smooth curves over $C\backslash\{p\}$ extends to a family of stable curves over C if and only if the induced monodromy on H_1 of the fibres around an (analytic) neighbourhood of p is unipotent.

Example 6.1.8. (See [3]) In case $\dim S > 1$, one may have to blow up the base before being able to extend $U \to \mathcal{M}_g$. As an example, consider the equations $y^2 = x^3 + ax + b$ as a family of elliptic curves over $a, b \in \mathbb{C}^2$. The fibres are smooth over $U = \mathbb{C}^2\backslash\{(a, b) \mid 4a^3 + 27b^2 = 0\}$ and stable over $\mathbb{C}^2\backslash\{(0, 0)\}$. The induced map $\mathbb{C}^2\backslash\{(0, 0)\} \to \overline{M_{1,1}}$ does not extend over $(0, 0)$. To extend the morphism one need to blow up three times to get the surface S'. The map $U \to \mathcal{M}_{1,1}$ now extends to a morphism $S' \to \overline{M_{1,1}}$ to the coarse moduli compactification which collapses the exceptional divisors E_1 and E_2 to the points with $j-$invariants 0 and 1728 and maps E_3 one-to-one onto $\overline{M_{1,1}}$.

The point is that this is not a coincidence. In fact, one has to resolve $D = S\backslash U$ to a normal divisor before being able to extend the given map. So, we can restate Remark 6.1.4 as follows,

Theorem 6.1.9. Let $D = S\backslash U$ be a normal crossing divisor at p. Then a morphism

$$U \to \mathcal{M}_{g,n}$$

extends to a regular map

$$V \to \overline{M_{g,n}}$$

in a Zariski neighbourhood V of p containing U.

S. Cautis showed that the following generalization can be perceived over the complex field $\mathbb{C}$ (See [3]):

Theorem 6.1.10. Let $U \subset S$ be an open subvariety of an irreducible, normal variety S. A morphism $U \to \mathcal{M}_{g,n}$ extends to a regular map $V \to \overline{M_{g,n}}$ in a Zariski neighbourhood $V \supset U$ of $p \in S\backslash U$ if and only if the local monodromy around p is virtually abelian (i.e., it contains an abelian subgroup of finite index).

Remark 6.1.11. Theorem 6.1.10 implies Theorem 6.1.9 over $\mathbb{C}$ because "the local fundamental group" of the complement of a normal crossing divisor is abelian.

For sake of simplicity, he defines a property so-called **AME** (Abelian Monodromy Extension) which is defined as follows:

Definition 6.1.12. ("AME" property for varieties) Given an open embedding of normal varieties $X \subset \overline{X}$ over $\mathbb{C}$, the pair $(X, \overline{X})$ has AME property if given any open subvariety U of a normal variety S such that $D := S\backslash U$ is a normal crossing divisor in S, any given morphism $U \to S$ extends to a morphism $V \to \overline{X}$ in a neighbourhood $V \supset U \ni p$ whenever the local monodromy around p is virtually abelian. If the pair has AME property, then it is shown that $\overline{X}$ is a complete variety which is called an **AME compactification of X.**

Remark 6.1.13. Again, one can analogously extend the AME propety to stacks; more preciesly to the pair of $(\mathcal{M}_{g,n}, \overline{M_{g,n}})$ (i.e., stacks which are integral, separated, normal Deligne-Mumford stacks of finite type over $\mathbb{C}$). In fact, Theorem 6.1.10 shows that the pair $(\mathcal{M}_{g,n}, \overline{M_{g,n}})$ has AME property. We can also check that the AME compactification of product of two stacks is the product of their AME compactifications and that AME property induces to closed normal substacks (More on the properties of AME compactification and its basic properties see [3]).

Remark 6.1.14. There are plenty of varieties over $\mathbb{C}$ which have AME property. More precisely, if $X \subset \overline{X}$ is a dense, open immersion with $\overline{X}$ normal, complete variety. Then there exists an open $X^0 \subset X$ such that $(X^0, \overline{X})$ has the AME property.

Remark 6.1.15. One can analogously define the AME property for Deligne-Mumford stacks. Cautis showed that the pair of $(\mathcal{M}_{g,n}, \overline{M_{g,n}})$ has AME property (See [3], Theorem 4.1).

Naturally, one can ask if the base scheme S is defined over a field of characteristic $p > 0$, does the same monodromy criterion hold for a family of curves over $U \subset S$? We are going to tackle this question in this article.

6.2 Counterexample

As a matter of fact if the base field k is algebraically closed of characteristic 0, then a family of pointed stable curves $C \to S$ over a normal variety S over k is isotrivial if the global monodromy in the sense of [3] (i.e., image of $\pi_1(U) \to \pi_1(\mathcal{M}_{g,n})$) is virtually abelian (See [3], Proposition 4.4).

Remark 6.2.1. Since the image of $\pi(U) \to \pi(\mathcal{M}_g)$ can be identified with the usual notion of monodromy on the fundamental groups of fibres, we terminologically use the same name for the image of $\pi_1(U)$ in the fundamental group of $\pi_1(\mathcal{M}_g)$.

In this section (our main references are L. Moret-Bailly [29] and F. Oort [35]), we construct an example which shows that in general if $\mathrm{char}(k) > 0$ commutativity of the global monodromy does not imply isotriviality of the family.

Let E be a supersingular elliptic curve over an algebraically closed field $k \supset \mathbb{F}_p$. Denote by α_p kernel of the Frobenius morphism $E \xrightarrow{F} E^{(p)}$. It is well-known that $\alpha_p \hookrightarrow E$ is a subgroup scheme of E (in fact, it is equivalent to the supersingularity of E, See [33], Section 4).

For any $a \in k$ define

$$\alpha_p \xrightarrow{\times a} \alpha_p \hookrightarrow E$$

Obviously, the composition morphism is injective if and only if $a \in k^{\times}$. Choose a pair of $t := (a, b) \in \mathbb{P}^1_k$, such that

$$(a, b) : \alpha_p \to E \times E$$

such that

$$(\ \alpha_p \xrightarrow[\sim]{a} {}_F E \xrightarrow[\sim]{b^{-1}} \alpha_p\ ) = \frac{a}{b} \in k \simeq \mathrm{End}_k(\alpha_p)$$

where ${}_F E : \mathrm{Ker}(F : E \to E^{(p)})$ (See [35]).

Now, consider the family of abelian varieties $\mathcal{A}$ over the projective line $\mathbb{P}^1_k$ as follows,

$$\mathcal{A} := \bigcup_{t \in \mathbb{P}^1_k} \frac{E \times E}{t(\alpha_p)} \to \mathbb{P}^1_k.$$

It is known due to Moret-Bailly [29] that there exists a family of curves $C \to \mathbb{P}^1_k$ such that $\mathcal{A}$ is the Jacobian of C. Note that the relative genus of $C \to \mathbb{P}^1_k$ is 2.

Notation If X is an abelian variety over an algebrically closed field k, then we set

$$a(X) := \dim_k \mathrm{Hom}(\alpha_p, X).$$

From [35], we have

$$a(\frac{E \times E}{t(\alpha_p}) = 2$$

if and only if

$$t \in \mathbb{P}^1_k(\mathbb{F}_{p^2}) := \Delta'.$$

Therefore if $x \in \Delta'$ ($\#\Delta' = p^2 + 1$) C_x consists of two components and the fibreC_x is regular otherwise. We have the following diagram,

$$\begin{array}{ccc} \mathcal{A}^o & \hookrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ x, y \in \mathbb{P}^1_k \backslash \Delta' & \hookrightarrow & \mathbb{P}^1_k \end{array}$$

and $X := \mathcal{A}^o_x[p^\infty] \otimes_k \overline{k(x)} \simeq \mathcal{A}^o_y[p^\infty] \otimes_k \overline{k(y)}$.

Since the fundamental group of the projective line, $\pi_1(\mathbb{P}^1_k, *) = 1$, $C \to \mathbb{P}^1_k$ is an example of a non-isotrivial family of curves for which the global monodromy is abelian (in fact, trivial).

6.3 Main theorem

To tackle the problem in case the scheme S is of equi-characteristic $p > 0$, we describe the $p-$adic (and $l-$adic, $l \neq p$) monodromy of the base scheme S to obtain the following proposition.

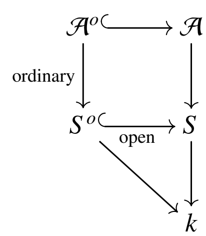
Theorem 6.3.1. Let $C \to S$ be a family of pointed stable curves over a normal variety S (which is defined over algebrically closed field $k \supset \mathbb{F}_p$). We assume the generic fibres ordinary. If the $l-$adic monodromy ρ_l is abelian for some prime $l \neq p$ then the family is isotrivial.

Proof. Recall $l-$adic monodromy of $C \to S$ is defined to be the image of the following homomorphism

$$\pi_1(S, \bar{\eta}) \xrightarrow{\rho_l} \mathrm{Aut}(T_l(Jac(C_\eta)))$$

where $\bar{\eta}$ is the geometric generic point of S. It is a classical result by Grothendieck and Katz that the ordinary locus in $A_{g,n}$ (the moduli scheme of principally polarised abelian varieties of a fixed dimension $g \geq 1$ in characteristic p with level-n-structure ($(n, p) = 1$)) is open and dense. The fact that the ordinary locus is open results from Grothendieck's specialisation theorem for crystals (See [13]).

Taking into account the Jacobian of fibres, we have the following diagram (since the generic fibre C_η is ordinary);



where $\mathcal{A}/S := Jac(C/S)$.

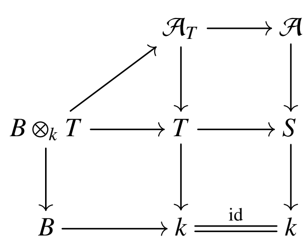
It is enough to show that the family $\mathcal{A}^o \to S$ is isotrivial because $\mathcal{A}^o \subset \mathcal{A}$ is open and dense. As a convention, let us denote $S := S^o$ and $\mathcal{A} := \mathcal{A}^o$.

If the variety S over k is not complete, compactify it using Nagata's theorem. By Chow's lemma we can assume that S is projective over k. Taking a branched cover of this new curve the pullback of C extends to a family of stable curves. By abuse of notation we denote this family by the same $C \to S$. Since any two points in S can be connected by a series of irreducible curves (because projective varieties are path–connected: any two closed points on the variety can be connected by the image of a finite number of nonsingular curves). Therefore it suffices to prove the result when S is a curve. On the other hand, since the image of ρ_l is abelian, there exists an unramified (finite and surjective) $k-$covering $T \to S$, an abelian variety B over k and an isogeny

$$tr : B \otimes_k T \to \mathcal{A} \times_S T.$$

Note that the isogeny is defined over k because the generic fibre $\mathcal{A}_\eta$ is ordinary (See [33], Theorem 2.1).

We have the following commutative diagram,



Remark: Considering the diagram above on the generic point of S, the $k-$variety B is $\mathrm{Tr}_{k(\bar{\eta})/k}(\mathcal{A}_{\bar{\eta}})$ where $\eta \in T$ is its generic point (See the construction of morphism tr in [34]).

Let us return to the proof. We have the following exact sequence on the variety T with $\mathcal{N}$ finite and local.

$$0 \to \mathcal{N} \to B \otimes_k T \to \mathcal{A}_T \to 0.$$

Every geometric fibre of $\mathcal{N} \to T$ is a finite local group scheme over k, because it is the kernel of a purely inseparable isogeny (therefore is connected). Moreover, since the generic fibre of $\mathcal{A}_T \to T$ is ordinary and isogenous to $B \otimes_k k(\eta)$, B is ordinary too. In other words, every geometric fibre of $\mathcal{N} \to T$ sits in an ordinary abelian variety.

On the other hand, any homomorphism $\operatorname{Spec} k[X]/(X^p) \simeq \alpha_p \to B$ is zero —because B is ordinary and obviously any homomorphism $\alpha_p \to \mu_p \simeq \operatorname{Spec} k[X]/(X^p - 1)$ is zero. Hence by the universal property of kernel we have

$$\mathcal{N}_{\bar{t}} \subset (\mu_{p^n})^g \otimes_k k(\bar{t})$$

for some n. By classification of finite group schemes [32] if we take a geometric point $\bar{t} \to T$, this implies $((\mu_{p^n})^g)^D \otimes k(\bar{t}) \to (\mathcal{N}_{\bar{t}})^D$ is surjective. Observing the fact that $(\mu_{p^n})^D \simeq \mathbb{Z}/p^n$ (non-canonically) shows that $(N_{\bar{t}})^D$ is étale. Hence every geometric fibre $N_{\bar{t}}$ is local-étale.

Since $\mathcal{N}$ is a finite group scheme over T, the morphism $\mathcal{N}^D \to T$ becomes constant after a finite étale base change $T' \to T$ therefore the same statement holds for its dual $\mathcal{N}^{DD} = \mathcal{N} \to T$. Pulling back to $T' \to T$, we obtain $\mathcal{N}'$ constant in $B \otimes_k T'$. Hence $\mathcal{A}_{T'}$ is constant and $\mathcal{A}_T \to T$ is isotrivial. Since the composition morphism of schemes $T' \to T \to S$ is finite and surjective, $\mathcal{A} \to S$ is isotrivial and we are done. □

6.4 Extending families of curves

Theorem 6.4.1. Let $C_U \to U$ be a family of smooth curves over U; an open subset of a normal variety S over an algebraically closed field $k \supset \mathbb{F}_p$ such that $D := S \backslash U$ is a normal crossing divisor in S. If the generic fibreC_η is ordinary and if the $l-$adic monodromy ρ_l is commutative for some $l \neq p$, then there exists a morphism of $k-$varieties $S \to \overline{M_g}$ extending $U \to \mathcal{M}_g$.

Before giving the proof, a remark;

Remark: We can always find a blowup $\pi : S' \to S$ such that $D' := \pi^{-1}(D)$ is a normal crossing divisor and then work over the pullback $C'_{U'} \to U'$ where $D' = S' \backslash U'$.

Proof. In this proof all points (fibres) are geometrical points (fibres).
There exists a finite flat morphism from a projective scheme Z to the compactification $\overline{\mathcal{M}_g}$ (See [36], the remark immediately after Theorem 7.4.6 and [1], Remark 2.3.7). Denote by $\tilde{S}$ the normalisation of the closure of $U \times_{\overline{\mathcal{M}_g}} Z$ inside $S \times Z$. The composition $\tilde{S} \to S \times Z \to S$ is projective and generically finite and flat. In fact the morphism $\tilde{S} \longrightarrow \overline{\mathcal{M}_g}$ resolves the rationality of the map $S - - \to \overline{\mathcal{M}_g}$. By definition, this gives a family of stable curves over $\tilde{S}$.

Consider the fibre $\tilde{S}_p$, for $p \in S$ and an irreducible component T of this fibre. We have the following commutative diagram,

$$
\begin{array}{ccc}
\tilde{C} := C \times_S \tilde{S} & \longrightarrow & C \\
\downarrow & & \downarrow \\
\tilde{S} & \longrightarrow & S
\end{array}
$$

Since $\tilde{S} \to S$ is generically finite the generic fibre $\tilde{C}_{\tilde{\eta}}$ is ordinary where $\tilde{\eta}$ is the generic point of $\tilde{S}$. Also the image of corresponding monodromy is commutative, because $\tilde{S} \to S$ is generically flat and we have

$$
\begin{array}{ccc}
\pi_1(\tilde{S}, \eta_{\tilde{s}}) & \xrightarrow{\rho} & \mathrm{Aut}(T_l(\mathrm{Jac}(\tilde{C}_{\eta_{\tilde{s}}})) \\
\downarrow & & \uparrow \\
\pi_1(S, \eta_s) & \longrightarrow & \mathrm{Aut}(T_l(\mathrm{Jac}(C_{\eta_s}))
\end{array}
$$

Also the image of homomorphism ρ coincides with the image of composition of the other three arrows. By Theorem 6.3.1 the family of $\tilde{C} \to \tilde{S}$ is isotrivial and so is the family over the irreducible component T of the fibre $\tilde{S}_p$.

Using the Stein factorisation, we can factorize morphism $\tilde{S} \to S$ as follows,

$$
\begin{array}{ccc}
S'' & & \\
{\scriptstyle f}\uparrow & \searrow^{g} & \\
\tilde{S} & \longrightarrow & S
\end{array}
$$

where f has connected fibres and g is a finite morphism. Let $q \in g^{-1}(p)$ and consider the fibre $\tilde{S}_q$. From the above arguments, the image of each irreducible component of $\tilde{S}_q$ in $\overline{M_g}$ and since $\tilde{S}_q$ is connected the whole fibre maps to a point.

Therefore the morphism $S'' \to \overline{M_g}$ in a neighbourhood of $g^{-1}(p)$ is well defined. On the other hand, since $\tilde{S}$ is normal, S'' is normal too and using the following lemma implies that we have the morphism $S \to \overline{M_g}$ as desired.

□

Remark 6.4.2. In all of the theorems, "ordinary" can be replaced by "almost ordinary". An n-dimensional abelian variety over a field k of positive characteristic p is almost ordinary if the rank of its group of p-torsion points over the algebraic closure of k is equal to $n - 1$.

Lemma 6.4.3. (See [3]) Assume S and S'' are quasi-projective varieties and that S is normal and X is a complete variety such that we have the following commutative diagram;

$$
\begin{array}{ccc}
S'' & & \\
{\scriptstyle \pi}\downarrow & & \\
S & \overset{h}{\dashrightarrow} & X
\end{array}
$$

If π is a finite, surjective morphism then the rational map h is regular if and only if $h \circ \pi$ is a (regular) morphism.

Proof. Denote by S' the closure of the image of $S \dashrightarrow S \times X$. If U is the domain of definition of h then $\pi^{-1}(U)$ is open and dense in S''. Consider the the following diagram

$$\begin{array}{ccc} S'' & \overset{(\pi, h\circ\pi)}{\dashrightarrow} & S \times X \\ {\scriptstyle \pi}\downarrow & \swarrow & \\ S & & \end{array}$$

Now the image of $(\pi, h \circ \pi)$ is S'. Therefore $f : S' \subset S \times X \to S$ is finite (because π is finite) and birational (since S' is the closure of the image of S). Since we assumed that S is normal, f is isomorphism and therefore h is defined over S. □

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