Wild ramification of schemes via curves, the rank 1 case (with applications to higher class field theory)

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Ivan Barrientos aus Mexiko-Stadt, Mexiko

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Die Arbeit wurde angeleitet von: Prof. Dr. Moritz Kerz

Prüfungsausschuss:
Vorsitzender: Prof. Dr. Abels
Erst-Gutachter: Prof. Dr. Kerz
Zweit-Gutachter: Prof. Dr. Jannsen
weiterer Prüfer: Prof. Dr. Künnemann

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Chapter 1

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Chapter 2

Introduction

Let R be a discrete valuation ring with residue field κ, fraction field $K = \mathrm{Frac}(R)$, and L a finite Galois extension of K. If κ is perfect, there is a satisfactory ramification theory for L/K, see for example [Ser 68]. On the other hand, if κ is not perfect this theory is not as well-established; for a comprehensive survey on ramification see [Xia-Zhu 13].

In the 1980's, Brylinski and Kato defined conductors when κ is not necessarily perfect and L/K is attached to a character of rank 1, see [Bry 83] and [Kat 89]. Recently, Abbes and Saito succeeded in providing a general definition of higher ramification groups in an ℓ-adic context that agrees with the classical case of a perfect residue field; see [Abb-Sai 02] and [Abb-Sai 11]. Consequently, they also define a conductor in [Abb-Sai 11]. Moreover, their conductor agrees with that of Brylinski and Kato's in rank 1, see [Abb-Sai 09].

In another direction, in the 1970's Deligne had initiated a program of measuring ramification of sheaves along a divisor in terms of transversal curves, see [Del 76], which was further developed by Laumon in [Lau 81] and [Lau 82]. We remark that one of the principal aims of these works was to achieve new Euler-Poincaré formulas for surfaces (over algebraically closed fields). Let us also mention work by Brylinski [Bry 83] and Zhukov [Zhu 02].

A natural question then became if one could follow Deligne's program and express Abbes-Saito's conductor in terms of curves. In this direction, Matsuda established results in the so-called 'non-log' case of Brylinski-Kato's conductor, see [Mat 97] and [Ker-Sai 13, Coro 2.7]. There are also stronger results for the 'non-log' case recently obtained by T. Saito, see [Sai 13]. For a modern survey on wild ramification in the sheaf-theoretic context, see [Sai 10].

The main result of this thesis is the description of Brylinski-Kato's 'log' conductor in terms of curves, see Theorem 6.1.2. In other words, we establish that Abbes-Saito's 'log' conductor in rank 1 is given in terms of curves.

Our approach is a slight generalization of Deligne's original idea of analyzing ramification via transversal curves and instead considering *all* curves on a scheme with special attention to *tangent* curves. We conjecture that our result also holds for ℓ-adic sheaves of finite rank, see Conjecture A as well as Conjecture B below. We hope our conjectures can be used for non-abelian higher class field theory.

As applications of our main result, we confirm an expectation of Esnault and Kerz [Esn-Ker 12, §3] in the smooth rank 1 case, see Theorem 7.2.1, and we also obtain a reformulation of the recent 'non-log' Existence Theorem of higher class field theory obtained by Kerz and S. Saito, [Ker-Sai 13, Coro II] in the 'log' case, see §8.3. In a slightly different direction, we also generalize an Isomorphism Theorem of Kerz and Schmidt [Ker-Sch 09, Thm. 8.2] by relaxing an assumption on being generically proper over a curve, see Thm. 8.2.4.

Chapter 3

Notation and definitions

3.1 Notation and definitions

Unless otherwise mentioned:

- By the *dimension* of a ring or a scheme we mean the Krull dimension.
- The group of *effective divisors* on a scheme X is denoted by $\mathrm{Div}^{+}(X)$.
- An effective divisor D is a *simple normal crossing divisor* on X if Zariski locally[1] there is a finite family of sections $(f_i)_{i \in I}$, $f_i \in \mathcal{O}_X$ such that the following two conditions hold:

  (i) $D = \sum_{i \in I} div(f_i)$

  (ii) for each $x \in \mathrm{Supp}(D)$, the restrictions $(f_i)_x$ that satisfy $(f_i)_x \in \mathfrak{m}_{X,x}$ form a part of a regular system of parameters in $\mathcal{O}_{X,x}$.
- We will routinely abbreviate 'simple normal crossing divisor' by *sncd*.
- An effective divisor D is a *normal crossings divisor* (ncd) if (i) and (ii) hold étale locally.
- A *variety* is a scheme that is reduced, separated, and of finite type over a field k.
- Sm_k is the set of connected, smooth varieties over $\mathrm{Spec}(k)$.
- A *curve* C is an integral scheme of dimension 1. A *curve* C *on* a scheme X is a closed curve $C \subset X$.
- A *smooth sheaf* is a locally constant constructible sheaf.

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[1] Our definition follows [Gro-Mur 71], Def 1.8.2 and is more flexible than that of SGA 1 [XIII, 2.1] and SGA 5 [3.1.5] where the f_i are required to be *global* sections $f_i \in \Gamma(X, \mathcal{O}_X)$.

- The set of closed points of a scheme X is denoted by $|X|$. The set of codimension r points is denoted by $X^{(r)}$.
- $\mathbb{N}$ denotes the natural numbers; this set includes 0 (the cardinality of the empty set).
- For a noetherian local ring A, denote by A^h the henselization of A and $\hat{A}$ its completion. We have $A \subset A^h \subset \hat{A}$.

Chapter 4

Background on Swan conductors

4.1 Overview

In this chapter, we begin by defining the Swan conductor at the level of finite extensions of discrete valuation fields with *separable* residue field extensions in Sec. 4.2, see Def. 4.2. In Sec 4.3 we give the definition of the conductor for a curve defined over a perfect field of finite characteristic, see Def. 4.3. We provide an example calculation of this conductor for an Artin-Schreier extension in Ex. 4.3.4. Basic lemmas for Artin-Schreier extensions are provided in Lemmas 4.3.1 and 4.3.2.

The Brylinski-Kato filtration is introduced in Sec. 4.4 from which we derive a generalized conductor for finite extensions of discrete valuation fields whose residue field extension is not necessarily separable, see Def. 4.4.4. Finally, in Sec. 4.5 we define the log conductor of an $\mathbb{F}_\ell$-sheaf of rank 1 at a given generic point of a divisor on the boundary in Def. 4.5.1. As a prelude to the next chapter, where we state our two main expectations, we include in subsection 4.5.1 an example that gives evidence for computing the log conductor in terms of a limit of conductors of curves tangent to a given divisor.

4.2 Conductor for discrete valuation fields with perfect residue field

We briefly review here the classical case of Swan conductors which we later relate to the Brylinski-Kato conductor. Classically, the Swan conductor is defined for finite extensions of discrete valuation rings with *separable* residue field extensions, which we begin with here. Let R be a henselian discrete valuation ring of characteristic $p > 0$. Let k be the residue field of R. Put $K = \mathrm{Frac}(R)$. Let $\mathcal{F}$ be a constructible $\mathbb{F}_\ell$- sheaf of finite rank on $S = \mathrm{Spec}\,(R)$ and $\bar{\eta} = \mathrm{Spec}\,(K^{\mathrm{sep}})$ a geometric point of S. Then the stalk $M = \mathcal{F}_{\bar{\eta}}$ is an

$\mathbb{F}_\ell[\mathrm{Gal}(K^{\mathrm{sep}}/K)]$-module and $\dim_{\mathbb{F}_\ell}(\mathcal{F}_{\bar{\eta}})$ is finite. The action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\mathcal{F}_{\bar{\eta}}$ factors through a quotient of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ by a closed normal subgroup Γ of finite index. Fix such a quotient and write it as

$$G = \mathrm{Gal}(K^{\mathrm{sep}}/K)/\,\Gamma.$$

Let L be the fixed field of Γ, i.e.

$$L = (K^{\mathrm{sep}})^{\Gamma};$$

then $G = \mathrm{Gal}(L/K)$. Let R' be the normalization of R in L and let k' be the residue field of R'.

For $i \geq 0$, let G_i denote the i-th (lower) ramification group of G, i.e.

$$G_i = \ker[G \to \mathrm{Aut}(R'/{\mathfrak{m}_L}^{i+1})],$$

where $\mathfrak{m}_L$ is the maximal ideal of R'. Clearly, each G_i is a normal subgroup of G.

We now state the main assumption of this section: assume that

$$\text{the extension of residue fields } k'/k \text{ of } R'/R \text{ is } \textbf{separable.} \tag{4.1}$$

Lemma 4.2.1. For $i \gg 0$, G_i is trivial.

Proof. Let v_L denote the discrete valuation of L. First observe that since the extension of residue fields k'/k is finite and separable, the theorem of the primitive element implies that there exists $\bar{x}' \in k^{\mathrm{sep}}$ such that $k' = k[\bar{x}']$. There is a lifting $x' \in R'$ of $\bar{x}'$ such that $R' = R[x']$ (see e.g. [Neu 99, II.10.4]).

We have then

$$G_i = \{\sigma \in G : v_L(\sigma(x') - x') \geq i + 1\}.$$

Since G is a finite group, there are only finitely many values $v_L(\sigma(x') - x')$ as σ ranges over all elements of G and therefore

$$i' > \max_{\sigma \in G}\{v_L(\sigma(x') - x')\} \Rightarrow G_{i'} \text{ is trivial.}$$

□

Since G acts on M, then so does each G_i. Let M^{G_i} denote submodule of M fixed by G_i. Let $s = \mathrm{Spec}\,(k)$ be the closed point of $S = \mathrm{Spec}\,(R)$. Then we define

$$\mathrm{Sw}_s(M) = \sum_{i=1}^{\infty} \frac{1}{[G_0 : G_i]} . \dim_{\mathbb{F}_\ell}(M/M^{G_i}). \tag{4.2}$$

Notice that this sum starts at $i = 1$ and not at $i = 0$ and this sum is finite by Lemma 4.2.1.

Lemma 4.2.2. The definition (4.2) does not depend on the choice of Γ.

Proof. This result depends on a compatibility between the lower and upper higher ramification groups of G_0, which itself depends on our assumption that k'/k is separable; see [Ser 68, Ch IV, §3]. $\square$

4.3 The case of a curve

Let k be a perfect field and X/k a smooth curve. Fix a codimension 1 point $x \in X^{(1)}$ and fix a geometric point $\bar{x}$ above x. Now consider the strict henselization of X at x,

$$R = \mathcal{O}^{sh}_{X,\bar{x}},$$

and let

$$S = \mathrm{Spec}\,(R).$$

Having fixed a geometric point $\bar{x}$, a choice of a separable closure k_s of the residue field $\mathbf{k}(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is fixed. The scheme S is a strictly henselian local scheme; let $s = \mathrm{Spec}\,(k_s)$ be its closed point and $\eta = \mathrm{Spec}\,(K)$ the generic point of S. For a fixed separable closure K^{sep} of K, let $\overline{\eta}$ be the corresponding geometric point of S. Let $\mathcal{F}$ be a constructible $\mathbb{F}_\ell$-sheaf of finite rank on X. The action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\mathcal{F}_{\overline{\eta}}$ factors through a finite quotient $G = \mathrm{Gal}(K'/K)$ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Since X is of finite-type over k and $x \in |X|$ (as $\dim(X) = 1$), then $\mathbf{k}(x)$ is a finite extension of k. Therefore $\mathbf{k}(x)$ is perfect and so $k_s = \mathbf{k}(x)^{\mathrm{sep}}$ is algebraically closed. So for $R = \mathcal{O}^{sh}_{X,\bar{x}}$ and R' the normalization of R in K', we have that the extension of residue fields from R'/R is trivial and therefore the finite extension R'/R satisfies (4.1). We then define

$$\mathrm{Sw}_x(\mathcal{F}) = \mathrm{Sw}_s(\mathcal{F}_{\overline{\eta}}). \tag{4.3}$$

We include the following Example 4.3.4 below, which is [Lau 81, Exemple 1.1.7], since it will fit in nicely with subsequent examples. Before doing so, we prove the following well-known results about Artin-Schreier extensions, which will be used in the sequel.

Lemma 4.3.1. Let K be a field of characteristic $p > 0$. Let $a \in K$ and consider the polynomial

$$f(t) = t^p - t - a \in K[t].$$

Then the finite extension of K

$$L = K[t]/(f(t)),$$

is Galois with $\mathrm{Gal}(L/K)$ either trivial or isomorphic $\mathbb{Z}/p$.

Proof. First observe that the formal derivative of $f(t)$ is equal to -1 which is coprime to $f(t)$ and therefore L/K is separable. Next, since

$$t^p - t = \prod_{i=0}^{p-1}(t-i) \in K[t],$$

then if $\alpha \in K^{\mathrm{sep}}$ is a root of $f(t)$, so is $\alpha + i$ for $i = 0, \dots p-1$. Therefore, L is the splitting field of $f(t)$ and since L/K is separable, then L/K is normal. We conclude that if $f(t)$ admits no roots in K, the extension L/K is non-trivial and Galois with $\mathrm{Gal}(L/K) = \mathbb{Z}/p$. □

Lemma 4.3.2. Let K be a discrete valuation field of characteristic $p > 0$ with uniformizer π and normalized valuation $v : K^\times \to \mathbb{Z}$. For a fixed $a \in K$, consider the polynomial

$$f(t) = t^p - t - a \in K[t].$$

Then if $L = K[t]/(f(t))$ is a non-trivial extension of K, we have

(i) L/K is unramified if $\exists y \in K$ such that $v(a - (y^p - y)) \geq 0$.

(ii) L/K is totally ramified if $\exists y \in K$ and $m > 0$ such that $v(a - (y^p - y)) = -m < 0$, and $(p, m) = 1$.

Proof. Let π' be a uniformizer of L. Let A, resp. B, denote the ring of integers of K, resp. L. Recall the different $\mathfrak{D}_{B/A}$ from [Ser 68, Ch III]. Let $\alpha \in K^{\mathrm{sep}}$ be a root of $f(t)$ so that $L = K[\alpha]$.

(i). Suppose $y \in K$ is such that $v(a - (y^p - y)) \geq 0$. Then for $w = \alpha - y$ and $b = a - (y^p - y)$ we have $L = K[w]$ and the minimal polynomial of w is $g(t) = t^p - t - b$. Note that w is integral over A since $v(b) \geq 0$. Now by [Ser 68, Ch. III, §6, Coro 2], $\mathfrak{D}_{B/A}$ divides the principal ideal $(g'(w))$. Since $g'(t) = -1$, then $g'(w) = -1$ and so (π') does not divide $\mathfrak{D}_{B/A}$ and hence B/A is unramified by [Ser 68, Ch. III, §5, Thm. 1].

(ii). Let w and b be as in (i), so

$$w^p - w = b.$$

Let $e = e(B/A)$ be the ramification index of B/A and v' the normalized discrete valuation on L. We have

$$v'(b) = -m.e,$$

and

$$v'(b) = v'(w^p - w) = p.v'(w).$$

Since $e \leq [L : K] = p$, then $e = p$ and $v'(w) = -m$. □

Remark 4.3.3. In fact, in both (i) and (ii) of Lemma 4.3.2 we also have the converse statements if K is perfect. See [Sti 09, Prop. 3.7.8]; our proof above is only a slight refinement of the proof in loc. cit.

We now turn to [Lau 81, Example 1.1.7]. For a general discussion of Artin-Schreier-Witt extensions, see Section 4.4 below. We point out that this example, while elementary, is slightly long. We will see a significant reduction in its required computations in Example 4.4.11 below.

Example 4.3.4. (cf. [Lau 81, Exemple 1.1.7]) Let k be an algebraically closed field of characteristic $p > 0$, and let $X = \mathbb{A}^1_k = \operatorname{Spec}(k[x])$. Let U be the complement of the closed subvariety $V(x)$, that is

$$U = \operatorname{Spec}(k[x, 1/x]).$$

Fix an integer $m \geq 1$ and let $U' \to U$ by the Artin-Schreier cover of U given by

$$t^p - t = x^{-m},$$

i.e.

$$U' = \operatorname{Spec}(k[x, 1/x][t]/(t^p - t - x^{-m})).$$

To calculate ramification induced by $U' \to U$ along the boundary $V(x)$, we define the following (henselian) discrete valuation rings R and R' as follows. Let R be the henselization of $k[x]$ at the the prime ideal (x) and let $K = \operatorname{Frac}(R)$. Set

$$L = K[t]/(t^p - t - x^{-m}),$$

and define R' as the integral closure of R in L. [1]

By Lemma 4.3.2 (ii), the prime ideal (x) of R is totally ramified in R'/R and by Lemma 4.3.1 $\operatorname{Gal}(U'/U) = \mathbb{Z}/p = \mathbb{F}_p$. An element $\sigma \in \operatorname{Gal}(U'/U)$ acts on t as $\sigma(t) = t + \sigma$. A fixed non-trivial homomorphism

$$\rho : \mathbb{F}_p \to \mathbb{F}_\ell^\times,$$

and the morphism $U' \to U$ yields a smooth $\mathbb{F}_\ell$-sheaf $\mathcal{F}$ on U. For $j : U \to X$ the open immersion of U into X and $0 \in X$ the closed point of X that corresponds to the prime ideal $(x) \leq k[x]$, we show that the Swan conductor

$$\operatorname{Sw}_0(j_! \mathcal{F}) = m',$$

where $m = m'.p^e$ with $(m', p) = 1$ and $e \geq 0$. There are two methods to see that we can reduce to the case that $U' \to U$ is given by solving the equation

$$T^p - T = x^{-m'}. \tag{4.4}$$

---

[1] Recall that the integral closure of a henselian dvr in a finite extension of its fraction field is again a henselian dvr.

The first is by the Artin-Schreier-Witt isomorphism, provided in Section 4.4.1 below, which in this case reads

$$W_1(R)/(F-1)W_1(R) \cong H^1(\mathrm{Spec}\,(R), \mathbb{Z}/p),$$

where $F : R \to R$ is the Frobenius morphism $a \mapsto a^p$ and $(F-1)R \leq W_1(R)$ denotes the subgroup generated by $\{(F-1)(b) = b^p - b;\ b \in R\}$. According to this isomorphism, R' is given by solving

$$T^p - T = \overline{x^{-m'.p^e}},$$

where $\overline{x^{-m'.p^e}}$ is the image of $x^{-m'.p^e}$ in the quotient $W_1(R)/(F-1)$. Since

$$(F-1)^e\,(x^{-m'}) = x^{-m'.p^e} - x^{-m'},$$

we have

$$x^{-m'.p^e} \equiv x^{-m'} \mod (F-1)R.$$

Therefore, the extension of R'/R obtained from $t^p - t = x^{-m}$ is equivalent to the extension obtained from $T^p - T = x^{-m'}$. The second method of seeing this isomorphism is to consider the polynomial $f(t) = t^p - t - x^{-m} \in R[t]$ and set

$$u = x^{-m'.p^{e-1}} + x^{-m'.p^{e-2}} + \cdots + x^{-m'}.$$

Then

$$f(t+u) = t^p - t + u^p - u - x^{-m} = t^p - t - x^d,$$

and since $u \in R$, we have an isomorphism of R-algebras

$$R[t]/(f(t)) \cong R[t]/(f(t+u)),$$

which gives the desired reduction.

In order to compute $\mathrm{Sw}_0(j_!\mathcal{F})$ by (4.2), we compute the ramification groups G_i of $G = \mathrm{Gal}(U'/U)$. Since the order of G is prime, then either $G_i = G$ or $G_i = \langle 0 \rangle$. Therefore, we just have to compute the number of subgroups G_i that are non-trivial. To do so, observe that on denoting the discrete valuation of R' of by v' (so $v'(x) = p$) and π' a uniformizer of R', we have $G_i = \{\sigma \in G : v'(\sigma(\pi') - \pi') \geq i+1\}$. We show that the value $v'(\sigma(\pi') - \pi')$ is constant for $\sigma \in G \backslash \{0\}$; namely, we claim that

$$v'(\sigma(\pi') - \pi') = m' + 1,\ \forall \sigma \neq 0.$$

It will then follow that the number of non-trivial G_i is equal to $m' + 1$ and so G_j is non-trivial for m' many indices $j \geq 1$. Let $\pi' = x^a.t^b$ where $a, b \geq 1$ are such that $ap - bm' = 1$. Then,

$$v'(t) = -m' < 0.$$

Observe that

$$\begin{aligned}
\sigma(\pi') &= \sigma(x^a.t^b) \\
&= x^a(\sigma+t)^b \\
&= x^a(t(1+t^{-1}.\sigma))^b \\
&= \pi'(1+t^{-1}.\sigma)^b
\end{aligned}$$

Thus,

$$\begin{aligned}
v'(\sigma(\pi')-\pi') &= v'(\pi'[(1+t^{-1}.\sigma)^b-1]) \\
&= v'(\sigma.\pi'[b.t^{-1}+\sigma.\frac{b(b-1)}{2}.t^{-2}+\cdots+\sigma^{b-1}.t^{-b}]) \\
&= v'(\pi')+\min\{m',2m',\ldots,bm'\} \\
&= v'(\pi')+m' \\
&= 1+m'.
\end{aligned}$$

Therefore, $G_i = G$ for $0 \leq i \leq m'$ and $G_j = \langle 0 \rangle$ for $j \geq m'+1$ and since the module M is $M = \mathbb{F}_\ell$ in this case we have

$$\mathrm{Sw}_0(M) = \sum_{i=1}^{\infty} \frac{1}{[G_0 : G_i]}.\dim_{\mathbb{F}_\ell}(M/M^{G_i}) = \sum_{i=1}^{m'} 1 = m' \,,$$

as desired. □

Luckily, the computation of such a Swan conductor can be significantly simplified, as we will see in the next section. See in particular Example (4.4.11) where the above computation is essentially reduced to one line.

4.3.1 Remarks on separable residue field extensions

4.4 Conductor for general discrete valuation fields

We mostly follow the notation of [Abb-Sai 09, §9]. Let R be a discrete valuation ring of characteristic $p > 0$.

Let K be the fraction field of R, fix a separable closure K^{sep} of K. First we recall the Brylinski-Kato filtration on truncated Witt vectors over K. Let v denote the normalized discrete valuation on K. Let $n \geq 0$ and denote by $W_{n+1}(K)$ the ring of Witt vectors of length $n+1$ over K.

We define an increasing filtration on the additive group of $W_{n+1}(K)$, which was studied by Brylinski [Bry 83] and Kato [Kat 89].

Definition 4.4.1. For $m \geq 0$ define the subgroup $\mathrm{fil}_m W_{n+1}(K)$ as

$$\mathrm{fil}_m W_{n+1}(K) = \Big\{(x_0, \dots, x_n) : p^{n-i}.v(x_i) \geq -m \ (0 \leq i \leq n)\Big\}.$$

We have that for $\mathbf{x} = (x_0, \dots, x_n) \in W_{n+1}(K)$, $\mathbf{x} \in \mathrm{fil}_m W_{n+1}(K)$ if and only if

$$m \geq \sup\{-p^n.v(x_0), \dots, -p.v(x_{n-1}), -v(x_n), 0\}. \tag{4.5}$$

This filtration yields a conductor. Being in rank 1, we choose to define conductors at the level of characters. See Appendix 9.2 for the precise correspondence between characters, extensions of K, and sheaves for $\mathbb{Z}/p^n$-extensions. Set

$$H^1(K) := \varinjlim_{r \geq 1} H^1(K, \mathbb{Z}/r\mathbb{Z}) = H^1(K, \mathbb{Q}/\mathbb{Z}).$$

Remark 4.4.2. Note that since $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is a compact topological group (under the usual Krull topology) and $\mathbb{Q}/\mathbb{Z}$ is a discrete topological group, then an element

$$\chi \in H^1(K) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{ab}/K), \mathbb{Q}/\mathbb{Z}),$$

must have finite image in $\mathbb{Q}/\mathbb{Z}$ and this image is therefore a finite cyclic group.

We continue to assume that K is the fraction field of a dvr R of characteristic $p > 0$. The relationship between truncated Witt vectors over K and cyclic p^n-extensions of K is provided by the Artin-Schreier-Witt isomorphism which we now recall. Put $\eta = \mathrm{Spec}\,(K)$ and recall that for $n \geq 1$ there is a short exact sequence of sheaves over $\eta_{\text{ét}}$:

$$0 \longrightarrow \mathbb{Z}/p^n \longrightarrow W_n \xrightarrow{F-1} W_n \longrightarrow 0. \tag{4.6}$$

Let us show that $H^1(K, W_n)$ vanishes.

We first briefly recall the argument establishing the additive version of Hilbert '90 [Ser 68, Ch. X, Prop. 1], which says that $H^1(G_K, K^{\mathrm{sep}}) = 0$. Let L/K be a Galois extension with Galois group G and write $G = \{\sigma_1, \dots, \sigma_r\}$. The normal basis theorem yields an element $x \in L$ such that $\{\sigma_i(x)\}_{1 \leq i \leq r}$ is a basis for L/K. Then we have an isomorphism of G-modules: $K \cong K \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, implying that K is an induced G-module (in the sense of [Ser 68, Ch. VII, §2]) and thus $H^i(G_K, K) = 0$ for each $i > 0$. Note that since G is finite, then in fact $\mathbb{Z}[G] \otimes_{\mathbb{Z}} K = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], K)$ and so K is also a co-induced G-module. Taking colimits one immediately obtains $H^1(G_K, K^{\mathrm{sep}}) = 0$, or in our notation: $H^1(K, W_1) = 0$.

Next, we show that for $n > 1, H^1(K, W_n) = 0$. Consider the short exact sequence

$$0 \longrightarrow W_n \xrightarrow{V} W_{n+1} \longrightarrow W_{n+1}/V(W_n) \longrightarrow 0,$$

where $V : W_n(K) \to W_{n+1}(K)$ is the Verschiebung, i.e. it is the additive map that sends $(x_0, \dots, x_{n-1})$ to $(0, x_0, \dots, x_{n-1})$. This gives a filtration on $H^1(K, W_{n+1})$ whose quotients are each trivial by the case $n = 1$, and so $H^1(K, W_{n+1}) = 0$ as well.

Therefore, the beginning of the long exact sequence obtained from the short exact sequence (4.6) is

$$0 \longrightarrow \mathbb{Z}/p^n \longrightarrow W_n(K) \xrightarrow{F-1} W_n(K) \xrightarrow{\delta} H^1(K, \mathbb{Z}/p^n) \to H^1(K, W_n) = 0. \tag{4.7}$$

Hence, we have a canonical isomorphism of groups

$$W_n(K)/(F-1) \cong H^1(K, \mathbb{Z}/p^n), \tag{4.8}$$

and this isomorphism classifies finite cyclic extensions of K of order $p^{n'}$, for $n' \leq n$. Moreover, there is a commutative diagram

$$\begin{tikzcd} W_n(K) \arrow[r, "\delta_n"] \arrow[d, "V"] & H^1(K, \mathbb{Z}/p^n) \arrow[d, "\cdot p"] \\ W_{n+1}(K) \arrow[r, "\delta_{n+1}"] & H^1(K, \mathbb{Z}/p^{n+1}) \end{tikzcd} . \tag{4.9}$$

Now write

$$\mathrm{fil}_m H^1(K, \mathbb{Z}/p^n) := \delta_n(\mathrm{fil}_m W_n(K)). \tag{4.10}$$

The above filtration induces a filtration on $H^1(K)$ via

$$\mathrm{fil}_m H^1(K) := H^1(K)[p'] + \mathrm{fil}_m H^1(K)[p^\infty], \tag{4.11}$$

where

$$H^1(K)[p'] = \varinjlim_{l \geq 1, (l,p)=1} H^1(K, \mathbb{Z}/l),$$

and

$$H^1(K)[p^\infty] = \varinjlim_{r \geq 1} H^1(K, \mathbb{Z}/p^r).$$

Definition 4.4.3. Let $\chi \in H^1(K)$. The Swan conductor of χ, denoted by $\mathrm{Sw}(\chi)$, is the minimal integer $m \geq 0$ for which $\chi \in \mathrm{fil}_m H^1(K)$.

Remark 4.4.4. That this definition agrees with the classical definition appearing in (4.3) when K is complete and the residue field of K is finite [2] is a theorem of Brylinski [Bry 83, Coro. to Thm. 1] and the case where K is complete and has

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[2] such fields are traditionally called *local fields*, [Neu 99, Ch. II, §2]

perfect residue field is due to Kato, [Kat 89]. Note that in [Bry 83], the proof is for the Artin conductor, which differs from the Swan conductor by a potential difference of +1, cf. [Bor 02, §2.8].

We will use the following results in the sequel to justify explicit calculations of conductors using Witt vectors. Recall that v denotes the normalized discrete valuation on K. To begin, the Brylinski filtration inspires the following map:

$$\Gamma : W_{n+1}(K) \to \mathbb{Z},$$

defined by $(x_0, \ldots, x_n) \mapsto \sup\{-p^n v(x_0), -p^{n-1}v(x_1), \ldots, -v(x_n), 0\}$.

Proposition 4.4.5. Let $\chi \in H^1(K, \mathbb{Z}/p^{n+1})$ with $\mathrm{Sw}(\chi) = N \geq 0$. Let $\bar{\mathbf{x}} \in W_{n+1}(K)/(F-1)$ correspond to χ. Suppose $\mathbf{x} = (x_0, \ldots, x_n) \in W_{n+1}(K)$ satisfies $\mathbf{x} \equiv \bar{\mathbf{x}} \mod (F-1)$ and $\Gamma(\mathbf{x}) = N$. Furthermore, suppose $N = -v(x_n) > -p^{n-m}.v(x_m)$ for $m = 0, \ldots, n-1$. Then for each $\mathbf{y} = (y_0, \ldots, y_n) \in W_{n+1}(K)$ we have the inequality

$$\Gamma(\mathbf{x} + (F-1)\mathbf{y}) \geq \Gamma(\mathbf{x}). \tag{4.12}$$

We begin with a basic lemma used to prove Prop. 4.4.5.

Lemma 4.4.6. Suppose x and y are elements of a discrete valuation field with uniformizer π and normalized discrete valuation v. Then

(i) $v(x) + v(y) \neq 0 \Rightarrow v(x+y) = \min\{v(x), v(y)\}$

(ii) $x + y = 0 \Rightarrow v(x+y) = +\infty$.

Proof. It suffices to prove (i). We may assume $v(x) < v(y)$. Set $m = v(x)$. There is a $k \in \mathbb{N}$ such that $v(y) = m + k$. We have $v(x+y) = v(\pi^m(1+\pi^k)) = mv(\pi) + v(1+\pi^k) = m = v(x)$, as desired. $\square$

Next, observe that for $\mathbf{y} = (y_0, \ldots, y_n) \in W_{n+1}(K)$ we have $\mathbf{y} = (y_0, \ldots, y_{n-1}, 0) + (0, \ldots, 0, y_n)$ and therefore

$$(F-1)\mathbf{y} = (F-1)(y_0, \ldots, y_{n-1}, 0) + (F-1)(0, \ldots, 0, y_n).$$

Hence, it suffices to prove Prop. 4.4.5 for $\mathbf{y} = (y_0, \ldots, y_{n-1}, 0)$ and $\mathbf{y} = (0, \ldots, 0, y_n)$. We begin with the first case.

Lemma 4.4.7. Prop. 4.4.5 holds for $\mathbf{y} = (y_0, \ldots, y_{n-1}, 0)$.

Proof. The proof is by induction on $n \geq 1$. The base case is $n = 1$ which is verified as follows. Setting $z_0 = y_0 - y_0^p$ (recall $y_1 = 0$ here) we have

$$\vec{w} = (x_0 + z_0,\ x_1 + p^{-1}(x_0^p + z_0^p - (x_0 + z_0)^p)).$$

We divide the argument into two cases: case i) $-p.v(z_0) \leq -v(x_1)$ and case ii) $-p.v(z_0) > -v(x_1)$.

For case (i),we show that $v(w_1) = v(x_1)$. Observe that

$$v(w_1 - x_1) = v\left(\sum_{k=1}^{p-1} x_0^{p-k} z_0^k\right),$$

and this sum consists of $p-1$ monomials of degree p. Hence, the conditions $-p.v(x_0) < -v(x_1)$ and $-p.v(z_0) \leq -v(x_1)$ yield that

$$-v(w_1 - x_1) < -v(x_1),$$

i.e. $v(x_1) < v(w_1 - x_1)$ and so $v(w_1) = v(x_1)$.

Next, for case (ii) we have

$$-p.v(x_0) < -v(x_1) < -p.v(z_0),$$

hence $v(w_0) = v(z_0)$, which gives $\Gamma(\vec{w}) \geq -p.v(z_0) > -v(x_1)$ and the case $n = 1$ is complete. [3]

Now assume the lemma is true for $W_n(K)$ for a fixed $n > 1$. Given $\vec{y} \in W_{n+1}(K)$, let $z_i = y_i - y_i^p$ for $1 \leq i \leq n$ (recall $y_n = 0$ by hypothesis). Again, the argument is divided into two cases: Case i) $-p.v(z_{i-1}) \leq -v(x_i), \forall i$ and Case ii) $-p.v(z_{i-1}) > -v(x_i), \exists i$, where $1 \leq i \leq n$.

For Case (i) we show that $v(w_n) = v(x_n)$. To compare $v(w_n - x_n)$ and $v(x_n)$ we make use of the following observation. Use the polynomials S_j as in [Ser 68, II.6] to express the addition law in $W_{n+1}(K)$ as

$$\vec{a} + \vec{b} = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \ldots, S_n(\vec{a}, \vec{b})).$$

Providing the ring $\mathbb{Z}[a_0, \ldots, a_n, b_0, \ldots, b_n]$ with the grading where a_i and b_i have weight p^i, then $S_j(a_0, \ldots, a_j, b_0, \ldots, b_j)$ is a homogeneous polynomial of degree p^j, for $j = 0, \ldots, n$.

Thus, $w_n - x_n$ is comprised of the following sums: for each $m \in [0, n-1]$, there are $p^{n-m} - 1$ monomials in x_m, z_m of polynomial degree p^{n-m}. Our assumptions in this case yield that $v(x_n)$ is strictly less than each valuation $v(\sum_{k=1}^{p^{n-m}-1} x_m^{p^{n-m}-k} z_m^k)$ for $m \in [0, n-1]$, from which we conclude that $v(x_n) < v(x_n - w_n)$. Hence, $v(w_n) = v(x_n)$.

For Case (ii), we divide the argument into two subcases. The first is if the inequality $-p.v(z_{n-1}) > -v(x_n)$ holds. Then, we have

$$-p.(z_{n-1}) > -v(x_n) > -p.v(x_{n-1}),$$

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[3] NB: we only needed $-p.v(z_0) \geq -v(x_1)$ for case (ii).

hence, $v(z_{n-1}) < v(x_{n-1})$. Induction applied to $(w_0, \ldots, w_{n-1})$ gives $v(w_{n-1}) = v(z_{n-1})$ and therefore

$$\Gamma(\vec{w} = (w_0, \ldots, w_n)) \geq -p.v(z_{n-1}) > -v(x_n),$$

as desired. On the other hand, suppose $-p.v(z_{n-1}) \leq -v(x_n)$. Here, we need to compare the remaining z_k with x_n. If for some k' with $0 \leq k' \leq n-2$ the inequality $-p^{n-k'}.v(z_{k'}) > -v(x_n)$ holds, then induction gives $v(w_{k'}) = v(z_{k'})$ and then

$$\Gamma(\vec{w} = (w_0, \ldots, w_n)) \geq -p^{n-k'}.v(z_{k'}).$$

Finally, if $-p^{n-k'}.v(z_{k'}) < -v(x_n)$ for all such k', then applying a similar argument as in Case (i) above gives $v(w_n) = v(x_n)$. □

The following lemma completes the proof of Prop. 4.4.5. Namely, replacing $\mathbf{x}$ by $\mathbf{x} + (F-1)(y_0, \ldots, y_{n-1}, 0)$ we show that $\mathbf{x} + (F-1)(0, \ldots, 0, y_n)$ has the desired property.

Lemma 4.4.8. Suppose $\mathbf{x} \in W_{n+1}(K)$ satisfies $\Gamma(\mathbf{x}) \geq N$ and $-v(x_n) \geq -p^{n-m}v(x_m)$ for $m = 0, \ldots, n-1$. Then

$$\Gamma(\mathbf{x} + (F-1)(0, \ldots, 0, y_n)) \geq -v(x_n).$$

Proof. We have

$$(x_0, \ldots, x_n) + (F-1)(0, \ldots, 0, y_n) = (x_0, \ldots, x_n + y_n^p - y_n).$$

By Lemma 4.4.6, $v(x_n + y_n^p - y_n) = \min\{v(x_n), v(y_n^p - y_n)\}$, if $v(x_n) \neq v(y_n^p - y_n)$. Finally, observe that $v(x_n) \geq \min\{v(x_n), v(y_n^p - y_n)\}$, i.e.

$$-\min\{v(x_n), v(y_n^p - y_n)\} \geq -v(x_n),$$

which completes the proof. □

We make some additional remarks regarding ramification.

Remark 4.4.9. If $\chi' \in H^1(K)[p']$, then $\mathrm{Sw}(\chi') = 0$, by (4.11) above. Therefore, in the rank 1 case, our study of wild ramification is reduced to that of characters in $H^1(K)[p^\infty]$.

4.4.1 Witt vectors and extensions; computation of conductors

Lemma 4.4.10. Let $\chi \in H^1(K, \mathbb{Z}/p^n)$ and suppose $n' \leq n$. Then χ corresponds to a $\mathbb{Z}/p^{n'}$ extension of K if and only if its corresponding Witt vector is in the image of the iterated Verschiebung modulo $(F-1)$,

$$V^{n-n'} : W_{n'}(K)/(F-1) \to W_n(K)/(F-1).$$

Proof. This follows by the the isomorphism (4.8) and the commutative diagram (4.9). □

Using the isomorphism (4.8), we can explicate p-cyclic extensions of K as follows. If L/K is Galois and satisfies $\mathrm{Gal}(L/K) = \mathbb{Z}/p^{n+1}$, then there are Witt vectors

$$\mathbf{x} = (x_0, \dots, x_n) \in W_{n+1}(K), \ \ (\alpha_0, \dots, \alpha_n) \in W_{n+1}(K^{\mathrm{sep}})$$

such that

$$L = K[\alpha_0, \dots, \alpha_n] \tag{4.13}$$

where $(\alpha_0, \dots, \alpha_n) \in W_{n+1}(K^{\mathrm{sep}})$ satisfies

$$(F-1)(\alpha_0, \dots, \alpha_n) = (x_0, \dots, x_n).$$

The Swan conductor of L/K can be computed as follows.

By (4.8) there is an attached character $\chi_{\mathbf{x}} \in H^1(K)$. Then $\mathrm{Sw}(\chi_{\mathbf{x}})$ is the minimal integer $m \geq 0$ satisfying

$$\mathbf{x} \in (\mathrm{fil}_m W_{n+1}(K))/(F-1).$$

Moreover, we may also illustrate the calculation of this conductor by choosing a lift of the image of $\mathbf{x}$ in $W_{n+1}(K)/(F-1)$ as follows. Choose an element $\mathbf{y} = (y_0, \dots, y_n) \in W_{n+1}(K)$ such that

(i) $\mathbf{y} \equiv \mathbf{x} \mod (F-1)W_{n+1}(K)$.

(ii) There is a $k \in [0, n]$ such that $-p^{n-k}v(y_k) = \mathrm{Sw}(\chi_{\mathbf{x}})$.

Clearly such a $\mathbf{y}$ exists simply because $W_{n+1}(K) \to W_{n+1}(K)/(F-1)$ is surjective.

Then with such a $\mathbf{y}$ satisfying (i) and (ii) above we have by (4.5) that

$$\mathrm{Sw}(\chi_{\mathbf{x}}) = \sup\{-p^n.v(y_0), \dots, -p.v(y_{n-1}), -v(y_n), 0\} \tag{4.14}$$

Note that we consider the integer 0 in the right-hand side of (4.14) since it is possible that each y_i has $v(y_i) > 0$; in this case there is no wild ramification and the conductor is zero.

The isomorphism (4.8) for $n+1 = 1$ gives us a relatively explicit method of exhibiting examples of Artin-Schreier extensions of any field of characteristic p. Namely, if K is such a field, then for a fixed $a \in K$, the extension

$$K[T]/(T^p - T - a),$$

is a $\mathbb{Z}/p$-extension of K if and only if $a \neq (F-1)(y) = y^p - y, \forall y \in K$. We now provide examples of Artin-Schreier-Witt extensions and the computation of their conductors.

Example 4.4.11. Let K be a discrete valuation field of characteristic $p > 0$. Let $m \geq 1$ and write $m = p^r.m'$ where $m' \geq 1$ is prime to p and $r \geq 0$. Denote by π a uniformizer of K. Consider the polynomial

$$f(T) = T^p - T - \frac{1}{\pi^m} \in K[T].$$

By Lemma 4.3.1, the splitting field of this polynomial is a $\mathbb{Z}/p$-extension of K. We now compute the Swan conductor of this extension using (4.14) in $W_1(K)$; let $\chi \in H^1(K, \mathbb{Z}/p)$ be the character attached to this extension.

We have $1/\pi^m \equiv 1/\pi^{m'} \mod (F-1)$. Now, $\mathbf{y} = 1/\pi^{m'} \in W_1(K)$ is a 'minimal lift' of the image of $1/\pi^m$ in $W_1(K)/(F-1)$ in the sense of conditions (i) and (ii) above (4.14).

Therefore,

$$\mathrm{Sw}(\chi) = \sup\{-(-m'), 0\} = m'.$$

Cf. the computations in Example 4.3.4 above. $\square$

Example 4.4.12. Let K be as in Example (4.4.11) and let

$$\bar{x} = (x_0, x_1) = \left(\frac{1}{\pi}, 0\right) \in W_2(K).$$

Since $\bar{x}$ is not in the image of the Verschiebung

$$V : W_1(K) \to W_2(K),$$

then by Lemma 4.4.10 $\bar{x}$ defines a $\mathbb{Z}/p^2$-extension of K. Let $\chi \in H^1(K, \mathbb{Z}/p^2)$ be the corresponding character obtained from the Artin-Schreier-Witt isomorphism (4.8) above. The conductor is computed via (4.14) and we have

$$\mathrm{Sw}(\chi) = \sup(-p.v(x_0), -v(x_1), 0) = \sup(-p.(-1), -\infty, 0) = p.$$

We can further describe this extension as follows. Let the extension be L/K; then L is expressed as

$$L = K[\alpha_0, \alpha_1],$$

where $(\alpha_0, \alpha_1) \in W_2(K^{\mathrm{sep}})$ satisfy

$$(F-1)(\alpha_0, \alpha_1) = (x_0, x_1).$$

To go further, we write down the coordinates of $(F-1)(\alpha_0, \alpha_1)$. In order to keep the calculations brief in this particular example, we now assume that $p = 2$. Then the additive inverse of an element $(a_0, b_0) \in W_2(K)$ is given as

$$-(a_0, a_1) = (-a_0, -a_0^2 - a_1) = (a_0, a_0^2 + a_1),$$

where the last equality is valid since $\mathrm{char}(K) = 2$. We have then

$$\begin{aligned}(F-1)(\alpha_0, \alpha_1) &= (\alpha_0^p, \alpha_1^p) - (\alpha_0, \alpha_1) \\ &= (\alpha_0^2, \alpha_1^2) + (\alpha_0,\ \alpha_0^2 + \alpha_1) \\ &= (\alpha_0^2 + \alpha_0,\ \alpha_0^2 + \alpha_1^2 + \alpha_1 + \frac{1}{2}(\alpha_0^4 + \alpha_0^2 - (\alpha_0^2 + \alpha_0)^2)) \\ &= (\alpha_0^2 + \alpha_0,\ \alpha_0^2 + \alpha_1^2 + \alpha_1 + \alpha_0^3).\end{aligned}$$

Therefore, the ordered pair $(\alpha_0, \alpha_1) \in K^{\mathrm{sep}} \times K^{\mathrm{sep}}$ gives a solution to the system of equations in $K[T_0, T_1]$:

$$T_0^2 + T_0 = \frac{1}{\pi}\ ,\ T_0^3 + T_0^2 + T_1^2 + T_1 = 0.$$

$\square$

Combining Remark 4.4.2 and Remark 4.4.9 narrows our focus to p-cyclic extensions of K, but for further applications, such as higher class field theory, we define a conductor for any finite abelian extension of K as follows. Given a homomorphism

$$\chi : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \bigoplus_j \mathbb{Z}/p^{e_j},$$

with finite image, χ then corresponds to a finite extension L/K with $\mathrm{Gal}(L/K) \leq \oplus_j \mathbb{Z}/p^{e_j}$. There are natural projections

$$p_j : \bigoplus_j \mathbb{Z}/p^{e_j} \to \mathbb{Z}/p^{e_j}.$$

Put

$$\chi_j = p_j \circ \chi \in H^1(K, \mathbb{Z}/p^{e_j}) \subset H^1(K, \mathbb{Q}/\mathbb{Z}).$$

We have then

Definition 4.4.13. The conductor of $\chi = \oplus_j \chi_j$ is defined as

$$\mathrm{Sw}(\chi) = \max_j \{Sw(\chi_j)\},$$

where the $\mathrm{Sw}(\chi_j)$ are defined in Definition 4.4.3.

We can check that Def 4.4.13 is well-defined by showing that it is invariant under the automorphisms of a given Galois extension of K. Namely, suppose that L/K is Galois with degree $[L : K] = m > 0$.

Lemma 4.4.14. Let (A, v) be a valuation ring (not necessarily discrete) with fraction field K and value group Γ. Define a function

$$\mathbf{v} : K^m \to \Gamma$$

by

$$\mathbf{v}(b_1, \ldots, b_m) = \min_i(v(b_i)).$$

Then $\mathbf{v}$ is invariant under the action of $GL_m(A)$.

Proof. Given $\mathbf{b} = (b_1, \ldots, b_m) \in K^m$, suppose the index j_0 is such that $v(b_{j_0}) = \mathbf{v}(\mathbf{b})$. Then each $b_{j_0}.b_i$ is in the ring A and by the choice of j_0, we have $\mathbf{v}(b_{j_0}^{-1}.\mathbf{b}) = 0$. Now for any $M \in GL_n(A)$ we have

$$\mathbf{v}(A\mathbf{b}) = \mathbf{v}(b_{j_0}^{-1}.A\mathbf{b}) + v(b_{j_0}) = \mathbf{v}(Ab_{j_0}^{-1}.\mathbf{b}) + v(b_{j_0}).$$

Therefore, it suffices to show shat if $\mathbf{b}$ satisfies $\mathbf{v}(\mathbf{b}) = 0$, then $\mathbf{v}(M\mathbf{b}) = 0$, for arbitrary $M \in GL_m(A)$. In fact, this assertion is equivalent to: if $\mathbf{b}$ has a non-zero coordinate modulo $\mathfrak{m}$, the maximal ideal of A, then $A\mathbf{b}$ also has a non-zero coordinate modulo $\mathfrak{m}$. Since $M \in GL_m(A)$, we have $\det(M) \in R \backslash \mathfrak{m}$ and hence $det(M)$ is non-zero modulo $\mathfrak{m}$. The assertion is then reduced to: for $k = A/\mathfrak{m}$, if $\bar{\mathbf{b}} \in k^m$ is non-zero, then for any $\bar{M} \in GL_m(k)$, we have $\bar{M}\bar{\mathbf{b}} \neq 0 \in k^m$, which is clear. $\square$

4.5 Conductor of a variety

Let X/k be a normal variety and $U \subset X$ a k-smooth open subscheme such that the closed complement $X \backslash U = E$ is the support of an effective Cartier divisor and let $D \in \mathrm{Div}^+(X)$ be an effective Cartier divisor with $\mathrm{Supp}(D) \subset E$.

Denote by I the set of generic points of E; then $I \subset X^{(1)}$. For $\lambda \in I$, let K_λ be the fraction field of the henselian discrete valuation ring $\mathcal{O}^h_{X,\lambda}$. Note that the residue field of K_λ is the function field of the divisor $\overline{\{\lambda\}}$ on X.

Recall that in Def. 4.4.3 we defined the Swan conductor for characters over discrete valuation fields.

Let $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$.

Definition 4.5.1. We define

$$\chi|_\lambda \in H^1(K_\lambda, \mathbb{Q}/\mathbb{Z}),$$

to be the pullback of χ to $\mathrm{Spec}(K_\lambda)$ induced by the canonical composition

$$\mathrm{Spec}(K_\lambda) \to \mathrm{Spec}(\mathcal{O}^h_{X,\lambda}) \to X.$$

The log Swan conductor of $\chi|_\lambda$, as defined by Def. 4.4.3, is denoted by

$$\mathrm{Sw}_\lambda(\chi)_{\log}.$$

Remark 4.5.2. Observe that since k is perfect and X/k is assumed to be of finite type, the residue field of K_λ has transcendence degree equal to $\dim(X) - 1$ and therefore this residue field is perfect if and only if $\dim(X) = 1$.

Definition 4.5.3. A *curve* C is a one-dimensional integral scheme and a *curve* C *on* a scheme is a closed subscheme that is a curve.

Definition 4.5.4. We let $Z_1(X, E)$ denote the set of curves $\bar{C}$ on X such that $\bar{C}$ is not contained in E, i.e. such that $\bar{C} \not\subset \mathrm{Supp}(E)$.

Definition 4.5.5. For $\bar{C} \in Z_1(X, E)$ let

$$\Psi_{\bar{C}} : \bar{C}^N \to \bar{C},$$

denote the normalization of $\bar{C}$. We put

$$\bar{C}_\infty = \{z \in |\bar{C}^N| : \Psi_{\bar{C}}(z) \in E\}. \tag{4.15}$$

When we want to specify a divisor $D' \subset \mathrm{Supp}(E)$, we write $Z_1(X, D')$, resp. $\bar{C}_\infty(D')$.

We can think of $\bar{C}_\infty$ as the set of places of the global field $K(\bar{C}) = K(\bar{C}^N)$ "on the boundary" E. Given $\bar{C} \in Z_1(X, E)$ and $z \in \bar{C}_\infty$, we write $K(\bar{C})_z$ for the henselization of $K(\bar{C})$ at the place corresponding to z.

Definition 4.5.6. Given $\bar{C} \in Z_1(X, E)$ and $z \in \bar{C}_\infty$, we define

$$\chi|_{\bar{C},z} \in H^1(K(\bar{C})_z, \mathbb{Q}/\mathbb{Z}),$$

to be the restriction of χ via

$$\mathrm{Spec}(K(\bar{C})_z) \to \bar{C} \hookrightarrow X.$$

Since $\dim \bar{C} = 1$, we do not need to emphasize a 'log' conductor and we unambiguously write

$$\mathrm{Sw}_z(\chi|_{\bar{C}}),$$

for the conductor of $\chi|_{\bar{C},z}$, which coincides with the classical Swan conductor (see Remark 4.4.4). More generally, suppose $\tilde{C}$ is a curve with a given finite morphism

$$\phi : \tilde{C} \to X,$$

and that $\tilde{z} \in |\tilde{C}|$ is a closed point satisfying $\phi(\tilde{z}) \in E$. We will often emphasize the morphism ϕ by writing

$$\mathrm{Sw}_z(\phi^*\chi),$$

for the conductor of χ restricted to $\tilde{C}$.

4.5.1 Example of computing by tangent curves

After giving examples of computing conductors by restriction to curves, we conjecture a general formula of this phenomena in 5.1 and prove this conjecture in rank 1 for an sncd in Sec. 6.1.

Example 4.5.7. (Artin-Schreier for a surface.) Let $k = \mathbb{F}_p$ and

$$U = \operatorname{Spec}(k[x,y][1/y]).$$

The surface U is an open subscheme of the affine plane $X = \operatorname{Spec}(k[x,y])$ and $X \backslash U = V(y)$. Take $D = V(y)$; then D is indeed an sncd ($D \cong \mathbb{A}^1_k$) and D has unique generic point (y). We will consider both "fierce" and "non-fierce" wild ramification (cf. [Lau 82, §2]) to give evidence that conductors obtained from curves (locally) tangent to (y) handle both phenomena.

Let A be the henselization of $k[x,y]$ at (y) and $K = \operatorname{Frac}(A)$. We will consider Galois covers of U with group $\mathbb{F}_p$. Given a non-trivial morphism

$$\rho : \mathbb{F}_p \to GL_1(\mathbb{F}_\ell) = \mathbb{F}_\ell^\times$$

consider an Artin-Schreier equation over A:

$$t^p - t = x^a/y^b,$$

with $a \geq 0, b \geq 1$. We can further assume that x^a/y^b is not a p^{th}-root in K, i.e. that p is prime to a or b. This equation and the character ρ yield a rank 1 sheaf $\mathcal{F}$ of $\mathbb{F}_\ell$-modules on U. Let

$$L = K[t]/(t^p - t - x^a/y^b),$$

and let B be the integral closure of A in L; then B is also a discrete valuation ring. We will restrict $\mathcal{F}$ to $\operatorname{Spec}(K)$ via $\operatorname{Spec}(K) \to X$. Let $\mathbf{k}(A) = k(x)$ denote the residue field of A and $\mathbf{k}(B)$ the residue field of B. By definition B/A is fierce if the extension $\mathbf{k}(B)/\mathbf{k}(A)$ is inseparable, i.e. (in this case) if $\mathbf{k}(B)$ contains a p-th root of x. For both cases, consider curves of the form $C_e \subset X$ defined by the equation $y = x^e$, for $e > 0$. Each C_e is a 1-dimensional closed, irreducible, subscheme of X since $C_e \cong \mathbb{A}^1_k$ and, geometrically, as e increases, the C_e progressively become more tangent to the x-axis near the origin in the plane, i.e. to the divisor defined by $D = V(y)$. Let $j : U \to X$ be the open immersion of U. Then $j_!\mathcal{F}$ is a constructible $\mathbb{F}_\ell$-sheaf on X and its restriction $j_!\mathcal{F}|_{C_e}$ is also constructible and is associated to the Artin-Schreier equation given by

$$t^p - t = x^a/x^{be} = \frac{1}{x^{be-a}}.$$

Let $\phi : C_e \hookrightarrow X$ denote the closed immersion of C_e into X and fix a closed point $z \in |C|$ such that $\phi(z) \in D$

(i) Non-fierce case. In this case, $p|a$ or $(b,p)=1$. For simplicity, assume $a=1$ and that b is prime to p. The residue field extension is separable because $(a,p)=1$, so

$$\mathrm{Sw}_{(y)}(j_!\mathcal{F})_{\log} = \mathrm{Sw}_{(y)}(j_!\mathcal{F}) = b,$$

where the middle term is the classical Swan conductor as defined in [Lau 81]. The Artin-Schreier equation over C_e is then

$$t^p - t = \frac{1}{x^{be-1}}.$$

To calculate $\mathrm{Sw}_z(j_!\mathcal{F}|_{C_e})$, observe that $(b,p)=1$ does not necessarily give $(be-1,p)=1$, $\forall e>0$. Using (4.14) we have that if $p|(be-1)$, then (by $(F-1)$ equivalence) $\mathrm{Sw}_z(j_!\mathcal{F}|_{C_e}) < be-1$ and if $(be-1,p)=1$ then $\mathrm{Sw}_z(j_!\mathcal{F}|_{C_e}) = be-1$. Therefore,

$$\limsup_{e>0} \frac{\mathrm{Sw}_z(j_!\mathcal{F}|_{C_e})}{e} = \lim_{e\to+\infty} b - \frac{1}{e} = b,$$

as expected.

(ii) Fierce case. In this case $(a,p)=1$ and $p|b$. For simplicity, take now $a=1$ and $b=p$ so that the extension over A is given by

$$t^p - t = x/y^p.$$

Let us observe that the extension of residue fields is purely inseparable. Let $K=\mathrm{Frac}(A)$ so that the corresponding extension B of A is the normalization of A in

$$L = K[t]/(t^p - t - x/y^p).$$

Let $k(A)$ denote the residue field of A and $k(B)$ the residue field of B. Clearly, $x \notin k(A)^p$; we show that $x \in k(B)^p$.

Let $v_B : L^\times \to \mathbb{Q}$ be a valuation on L normalized by $v_B(y)=1$. Then $v_B(t) = -p/v_B(y^p) = -1$ since $t^p - t = x/y^p$. Noting that in L we also have the relation

$$(ty)^p - t.y^p = x, \tag{4.16}$$

then we see that $v_B(t.y^p) = -1+p > 0$ and so $t.y^p$ is an element of the maximal ideal $\mathfrak{m}_B$ of B. Therefore,

$$(ty)^p \equiv x \pmod{\mathfrak{m}_B},$$

which means that $x \in k(B)^p$, i.e. that x admits a p-th root in $k(B)$. So since the extension $k(B)/k(A)$ is (purely) inseparable, we cannot appeal to the classical Swan to compute the conductor over the surface, so we use $\mathrm{Sw}_{(y)}(j_!\mathcal{F})_{\log}$ only. Namely, we explicitly compute this conductor by the Brylinski-Kato filtration. We consider the class of x/y^p in $W_1(K)/(F-1)$. Since x is not a p-th root in K, x/y^p is irreducible with respect to $F-1$ equivalence and so it suffices to find the least integer $m > 0$ such that $v_y(x/y^p) \geq -m$, where v_y is the (normalized) valuation on K. Indeed,

$$v_y(x/y^p) = -p,$$

and $m = p$, so

$$\mathrm{Sw}_{(y)}(j_!\mathcal{F})_{\log} = p.$$

Finally, observe that restricting to *any* curve yields *perfect* residue fields. The restricted equation is now

$$t^p - t = x/x^{pe} = \frac{1}{x^{pe-1}}$$

and since $(pe-1, p) = 1, \forall e > 0$, we have

$$\lim_{e\to+\infty} \frac{\mathrm{Sw}_z(j_!\mathcal{F}|_{C_e})}{e} = \lim_{e\to+\infty} p - \frac{1}{e} = p,$$

as desired.

□

Example 4.5.8. Same setup as above, but now let

$$D = V(x) \cup V(y).$$

Consider the Artin-Schreier extension of U given by

$$t^p - t = \frac{1}{y^{m.p^2}} + \frac{1}{y^{p-1}.x^p}, \tag{4.17}$$

with $m > 0$ and $(m, p) = 1$. On setting

$$z = t - \frac{1}{y^{mp}} - \frac{1}{y^m},$$

we see that (4.17) is Artin-Schreier equivalent to

$$z^p - z = f := \frac{1}{y^m} + \frac{1}{y^{p-1}.x^p}. \tag{4.18}$$

We first take the Swan conductor on X at the codimension 1 point that corresponds to the generic point of $V(y)$, which we write as $\xi_y \in X^{(1)}$.

From (4.18) we see that in the fraction field of the henselian ring $\mathcal{O}^h_{X,\xi_y}$ with discrete valuation v_y we have

$$\begin{array}{rcl} m \geq p-1 \Rightarrow v_y(f) & = & m \\ m < p-1 \Rightarrow v_y(f) & = & p-1 \end{array}$$

so in either case we have $(v_y(f), p) = 1$. We have, for $\mathcal{F}$ an Artin-Schreier sheaf associated to (4.17),

$$\mathrm{Sw}_{\xi_y}(\mathcal{F})_{\log} = \mathrm{Sw}_{\xi_y}(\mathcal{F}) = max(m, p-1),$$

as desired.

Again, consider the curves C^y_e on X given by $y = x^e$ and let $\mathfrak{m} \in |C^y_e|$ be a closed point that meets $V(y)$. For $e > p$ and $(e,p) = 1$ we have

$$\begin{array}{rcl} m > p \Rightarrow \mathrm{Sw}_{\mathfrak{m}}(\mathcal{F}|_{C^y_e}) & = & me \\ m < p \Rightarrow \mathrm{Sw}_{\mathfrak{m}}(\mathcal{F}|_{C^y_e}) & = & p + e(p-1) \end{array}$$

and therefore

$$\limsup_{e>0,(e,p)=1} \frac{\mathrm{Sw}_{\mathfrak{m}}(\mathcal{F}|_{C^y_e})}{e} = \mathrm{Sw}_{\xi_y}(\mathcal{F})_{\log},$$

as expected.

Now consider the generic point of $V(x)$, written $\xi_x \in X^{(1)}$. In this case consider the curves C^x_e on X given by $(x = y^e)$. We have, for $\mathfrak{m} \in |C|$ meeting $V(x)$,

$$\lim_{e\to+\infty} \frac{\mathrm{Sw}_{\mathfrak{m}}(\mathcal{F}|_{C^x_e})}{e} = \mathrm{Sw}_{\xi_x}(\mathcal{F})_{\log} = p,$$

as desired.

$\square$

Chapter 5

Conjectures on log conductors and curves

Definition 5.0.9. For a scheme X, we let $\mathrm{Cu}(X)$ denote the set of normalizations of closed integral one-dimensional subschemes of X.

For $\bar{C} \in \mathrm{Cu}(X)$ let

$$\bar{\phi} : \bar{C} \to X,$$

denote the induced morphism to X.

Definition 5.0.10. Given a Cartier divisor D and a codimension one point y on a locally Noetherian scheme, we denote the multiplicity of y in D by $m_y(D)$; see e.g. [Liu 02, 7.1.2, Eq. (1.2)].

Fix a perfect field k. Let X/k be a normal variety and $j : U \hookrightarrow X$ a k-smooth open subscheme such that the closed complement $E := X \backslash U$ is the support of an effective Cartier divisor and let $D \in \mathrm{Div}^+(X)$ be an effective Cartier divisor with $\mathrm{Supp}(D) \subset E$. Let $\mathcal{F}$ be a smooth $\mathbb{F}_\ell$- sheaf of finite rank on U.

Denote by $\mathrm{Sw}_{\cdot}(-)_{\log}$ the Abbes-Saito log conductor of a constructible sheaf at a codimension 1 point of a scheme, see [Abb-Sai 11, Definition 8.10 (i)]. We omit the subscript 'log' when the scheme has dimension one. Note that the Abbes-Saito log conductor simplifies in rank 1, see Prop. 6.1.1.f

5.1 Conjecture A

Conjecture A. Suppose $\mathrm{Supp}(D) = E$ and that D is a smooth irreducible divisor. Then

$$\limsup_{\bar{\phi},z} \frac{\mathrm{Sw}_z(\bar{\phi}^* j_! \mathcal{F})}{m_z(\bar{\phi}^* D)} = \mathrm{Sw}_\xi (j_! \mathcal{F})_{\log}, \tag{5.1}$$

where the supremum ranges over $\bar{C} \in \mathrm{Cu}(X)$ with $\bar{\phi}(\bar{C}) \not\subset D$ and closed points z of $\bar{C}$ that satisfy $m_z(\bar{\phi}^* D) > 0$.

Remark 5.1.1. In [Zhu 02, Remark 2.5.3], a limit similar to that in (5.1) is mentioned for a smooth surface defined over an algebraically closed field.

5.2 Conjecture B

Instances of the following ideas are related to previous work by Deligne [Del 76], Matsuda [Mat 97], and Russel [Rus 10, Lemma 3.31].

Definition 5.2.1. ([Esn-Ker 12, Def. 3.6]) We say that *the ramification of $\mathcal{F}$ is bounded by D* if for each $C \in \mathrm{Cu}(U)$ we have the inequality of divisors in $\mathrm{Div}^+(\bar{C})$:

$$\mathrm{Sw}(\phi^* \mathcal{F}) \leq \bar{\phi}^*(D),$$

where $\phi : C \to U$ is the induced morphism from C to U and $\bar{\phi} : \bar{C} \to X$ is the normalization of the closure of C in X and

$$\mathrm{Sw}(\phi^* \mathcal{F}) := \sum_{z \in |\bar{C}|} \mathrm{Sw}_z(\bar{\phi}^* j_! \mathcal{F}).[z] \in \mathrm{Div}^+(\bar{C}).$$

In this case, we formally write

$$\mathrm{Sw}(\mathcal{F}) \leq D.$$

Conjecture B. As in the first paragraph of this chapter, suppose $D \in \mathrm{Div}^+(X)$ is an effective Cartier divisor with $\mathrm{Supp}(D) \subset E$. We conjecture that the following are equivalent (cf. [Esn-Ker 12, §3]).

(i) $\mathrm{Sw}(\mathcal{F}) \leq D$.

(ii) for each open immersion $j' : U \subset X'$ over k such that $X' \backslash U$ is an sncd and for a morphism $h : X' \to X$, *that extends* $j : U \to X$,

$$\mathrm{Sw}_{\xi'}(h^* j_! \mathcal{F})_{\log} \leq m_{\xi'}(h^*(D)),$$

for each generic point $\xi' \in D' := X' \backslash U$.

Our main result is that if X/k is smooth and D is an sncd on X, then if $\mathcal{F}$ has rank 1 Conjecture A is true (see Theorem 6.1.2) and implies Conjecture B (Theorem 7.2.1). We do not require resolution of singularities; e.g. we do not require X/k to be proper in Theorem 6.1.2. On the other hand, if one insists that we begin with a smooth variety U/k and a *proper*, but not-necessarily smooth, variety X/k that contains U as an open dense subscheme, then resolution of singularities seems to be required to apply our techniques.

In the case $D = 0$ (and arbitrary rank), the equivalence in Conjecture B is shown by Kerz and Schmidt in [Ker-Sch 10] relying on work of Wiesend in [Wie 06]. Moreover, in the 'non-log' rank 1 setting Conjecture B follows by work of Matsuda on Kato's Artin conductor, see [Mat 97] and [Ker-Sai 13, Coro 2.7].

5.2.1 Remarks on the higher rank case.

We make some brief remarks relating recent work by T. Saito to both of the above conjectures for general rank. First, the inequality $\leq$ in Conjecture A (a form of 'semi-continuity') is evidenced by [Sai 09, Lemma 2.31]. Furthermore, the implication (ii) $\Rightarrow$ (i) in Conjecture B is also evidenced by [Sai 09, Lemma 2.22]. Finally, results from [Sai 13] seem to shed light on the converse of these statements as well. In particular, a careful analysis of the vanishing of the residue of the refined Swan conductor seems to be indispensable here and is part of future work.

Chapter 6

Proof of Expectation 1 in rank 1

6.1 Theorem 6.1.2

Throughout, k is a perfect field of $\mathrm{char}(k) = p > 0$. Recalling the definitions from §4.5, we will state our main theorem in this section.

First, we begin with the observation that in rank 1 the Abbes-Saito log conductor agrees with our previously defined log conductor in Def. 4.4.3. This result is due to Abbes and T. Saito.

Proposition 6.1.1. The conductor defined above in Def. 4.4.3 is equal to the log conductor of Abbes and Saito in rank 1.

Proof. The Abbes-Saito log conductor is defined in [Abb-Sai 11, Def. 8.10(i)] and that this definition agrees with ours is given by [Abb-Sai 09, Coro 9.12 and Def. 10.16]. □

Theorem 6.1.2. (Conjecture A in the smooth rank 1 case, D irreducible) Let X/k be a smooth variety and $U \subset X$ an open subvariety such that $E := X \backslash U$ is the support of a smooth effective Cartier divisor on X. Let $D \in \mathrm{Div}^+(X)$ be an irreducible smooth divisor on X with $\mathrm{Supp}(D) \subset E$. Let $\xi \in D$ be its generic point. Let $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$. Then,

$$\sup_{\bar{C},z} \frac{\mathrm{Sw}_z(\chi|_{\bar{C}})}{m_z(\bar{\phi}^* D)} = \mathrm{Sw}_\xi(\chi)_{\log}, \tag{6.1}$$

where the supremum ranges over all $\bar{C} \in Z_1(X, D)$ and $z \in \bar{C}_\infty$ satisfying $m_z(\bar{\phi}^* D) > 0$.

Let us illustrate the necessity of the irreducibility hypothesis on D in the statement of Thm. 6.1.2 by showing that if D is an sncd with more than one component, the left-hand side can be strictly larger than the right-hand side in (6.1) . Essentially, the problem is that too much ramification may occur along other components if D is not irreducible.

Example 6.1.3. Let $X = \operatorname{Spec} k[x,y], D = V(x) \cup V(y)$, and $U = X \backslash D$. Let $r > 0$ and consider the Artin-Schreier extension of U defined by

$$T^p - T = \frac{1}{y.x^r}. \tag{6.2}$$

Let $\chi \in H^1(U, \mathbb{Z}/p)$ be the character corresponding to this extension.

We first compute the log conductor along $V(y)$. So let ξ_y be the generic point of $V(y)$. Then

$$\mathrm{Sw}_{\xi_y}(\chi)_{\log} = 1.$$

Next, we compute the conductor restricted to the following curves on X. For $e > 0$, define the curve C_e on X by $y = x^e$; hence for each $e > 0$, $C_e \in Z_1(X, D)$ and C_e is normal. Let z be the closed point of C_e corresponding to the ideal (x); then $m_z(\bar{\phi}^* V(y)) = e$. The restriction $\chi|_{C_e}$ corresponds to the equation (in the fraction field of the henselization of C_e localized at z) :

$$T^p - T = \frac{1}{x^{e+r}}.$$

Let $e > 0$ be such that $\gcd(r + e, p) = 1$, then

$$\mathrm{Sw}_z(\chi|_{C_e}) = r + e,$$

which is a problem since

$$\mathrm{Sw}_z(\chi|_{C_e})/e = (r+e)/e > \mathrm{Sw}_{\xi_y}(\chi)_{\log} = 1. \tag{6.3}$$

That is, since the left-hand side is strictly larger than the right-hand side in (6.3), we do not have a form of 'semi-continuity' saying that the conductor restricted to curves (divided by multiplicity) is always less than or equal to the log conductor. Thus, this example illustrates that the formula (6.1) does not hold in general when D has more than one component. $\square$

Remark 6.1.4. Keeping the notation of Example 6.1.3, note that we have in fact

$$\limsup_{e'} \mathrm{Sw}_z(\chi|_{C_{e'}}) \,/\, e' = \lim_{e \to +\infty} \frac{r+e}{e} = 1,$$

where the lim sup ranges over those $e' > 0$ with the property that $\gcd(r + e', p) = 1$. This limit suggests that a modification to the formula in (6.1) should make it possible to compute the log conductor in terms of curves when D is an sncd with more than one component. Namely, we should take sums over the generic points of D on both sides of (6.1).

We first prove the theorem locally around a closed point in §6.1.1 and proceed to the general case in §6.1.4.

6.1.1 Local setting

Let us give the basic outline of the proof. We begin by formulating the problem around a closed point. The main reason we localize at a closed point is to apply Cohen's structure theorem which enables us to explicate Artin-Schreier-Witt extensions. With relatively explicit polynomials at hand, we define our 'tangent' curves to the divisor which capture the necessary ramification. We then transfer back to the henselian world by simply noting that these curves are defined algebraically and therefore the relations defining them imply their definition at the level of henselian local rings (not just over completions).

Let $x \in |X| \cap \mathrm{Supp}(D)$. Write the completion of the localization of X at x as

$$X_x := \mathrm{Spec}\,(\widehat{\mathcal{O}}_{X,x}).$$

Then $\widehat{\mathcal{O}}_{X,x}$ is a Noetherian and complete regular local ring and by Cohen's structure theorem (see e.g. [Liu 02, 4.2.28]) there is a finite extension k'/k such that

$$\widehat{\mathcal{O}}_{X,x} \cong k'[[t_1, \ldots, t_d]].$$

Because D is irreducible and regular around any point of X we can assume without loss of generality that locally around x, this divisor D is defined, reordering regular parameters at x if necessary, by the regular parameter t_1, i.e. locally around x, it is defined by $(t_1 = 0)$. Now let $U_x = X_x \backslash V(t_1)$. We prove the following

Theorem 6.1.5. (Local Tangent theorem) Let $\chi \in H^1(U_x) = H^1(U_x, \mathbb{Q}/\mathbb{Z})$. For ξ the generic point of D, there are irreducible, *regular* curves $\phi_e : C_e \to X_x$ with $C_e \in Z_1(X_x, D)$ indexed by integers $e > 0$ such that

$$\limsup_{e\to\infty} \frac{\mathrm{Sw}_x(\chi|_{C_e})}{m_x(\phi_e^* D)} = \mathrm{Sw}_\xi(\chi)_{\log}. \tag{6.4}$$

Remark 6.1.6. Clearly, the C_e depend on the generic point ξ of $X_x \backslash U_x$ (cf. Example 6.1.3 above). Also note that since X_x is a local scheme, then so is C_e and $|C_e| = \{x\}$.

Let us summarize the strategy of the proof in this local setting. First we make some immediate reductions.

Theorem 6.1.5 is obvious if $\dim(U_x) = 1$, for then we simply take $C_e = X_x$, so from now on we assume $\dim(U_x) > 1$. By the previous remarks 4.4.2 and 4.4.9, we are reduced to the case where $\chi \in H^1(U_x, \mathbb{Z}/p^{n+1})$ for some $n \geq 0$ (recall $\mathrm{char}(k) = p > 0$).

Now, for the Artin-Schreier case, i.e. $n = 0$, there is a corresponding polynomial f satisfying $T^p - T = f$; see (6.7). We extract a certain 'leading term' of f and construct 'tangent' curves on X_x from this term. In particular, these curves are defined by algebraic relations (in $t_1, \dots, t_d$) depending on this leading term. These relations are found by means of a combinatorial argument similar to that in the proof of Noether's normalization lemma; see Lemma 6.1.8. To take care of $(F-1)$-equivalence, as required by the Artin-Schreier-Witt isomorphism (4.8) and the Brylinski-Kato conductor (Def. 4.4.3), we must ensure that certain properties of our curves do not 'shrink' with respect to $(F-1)$-equivalence. This task is performed by means of so-called 'B-good' vectors (Def. 6.1.9) via Lemma 6.13. We then show that our curves have sufficiently good ramification properties in order to achieve the desired lim sup in (6.4); see §6.1.2. For the Artin-Schreier-Witt case, i.e. $n \geq 1$, essentially the same strategy as in the $n = 0$ case is applied by selecting a 'leading term' from a certain'maximal' coordinate of the Witt vector corresponding to the given p-cyclic extension; see §6.1.3.

We start by proving the Local Tangent theorem (6.1.5) first for $n = 0$ in §6.1.1 and then proceed to the general case in §6.1.3. We fix the following notation. Let

$$A_x := k'[[t_1, \dots, t_d]], \tag{6.5}$$

and

$$R := \Gamma(U_x, \mathcal{O}_{U_x}) = A_x[1/t_1]. \tag{6.6}$$

Set

$$N = \mathrm{Sw}_{\xi}(\chi)_{\log}.$$

If $N = 0$ (i.e. if χ is tame at ξ), then for *any* subscheme $Z \subset X_x$, the restriction $\mathrm{Sw}_x(\chi|_Z) = 0$. Hence we assume that $N > 0$ in what follows. Furthermore, by the Artin-Schreier-Witt isomorphism (4.8), χ corresponds to a unique element $\overline{(f_0, \dots, f_n)} \in W_{n+1}(R)/(F-1)$.

The Artin-Schreier case

In this case, χ corresponds to a unique element $\bar{f} \in R/(F-1)$. We choose a lift $f \in R$ satisfying

$$-v_{t_1}(f) = N.$$

Note that $f \notin R^p$. Write

$$f = \frac{B}{t_1^N} + \frac{B'}{t_1^{N-1}}, \tag{6.7}$$

where B is a non-zero element of $A_x/(t_1)$ and $B' \in A_x$. We regard B as an element of A_x via a section $\iota : A_x/(t_1) \to A_x$, which exists since the residue fields of $A_x/(t_1)$ and A_x are both k'.

Remark 6.1.7. If $p|N$, we modify f so that each term of B is not a pth power in A_x as follows. Suppose that for our initial choice of f, the term B has elements of the form h^p for some (non-zero) $h \in A_x$. Write $N = p.N'$ for some $0 < N' < N$. Then in $R/(F-1)$ we have

$$\frac{h^p}{t_1^{pN'}} \equiv \frac{h}{t_1^{N'}},$$

and

$$\frac{B}{t_1^N} \equiv \frac{B - h^p}{t_1^N} + \frac{h}{t_1^{N'}}.$$

Therefore, we may choose $f \in R$ so that h is a term of B' in the expression $f = B/t_1^N + B'/t^{N-1}$. With this choice of f, each term in B is not a pth power in A_x.

We begin by defining a surjective morphism

$$\Phi_e : k'[[t_1, \ldots, t_d]] \to k'[[w]],$$

that sends each parameter t_i to either a (non-negative integral) power of w or zero and satisfies $\Phi_e(B) \neq 0$. Write B as a sum of monomials:

$$B = \sum_{(i_2,\ldots,i_d)} u_{(i_2,\ldots,i_d)} t_2^{i_2} \cdots t_d^{i_d},$$

where $u_{(i_2,\ldots,i_d)} \in k'$. From this sum, choose a multi-index $(j_2, \ldots, j_d)$ that has at least one non-zero entry and that has a *minimal* number of $j_q \neq 0$. By a change of coordinates, we may assume such a term is in the $r-1$ variables $t_2, \ldots, t_r$ for some r with $2 \leq r \leq d$. Denote by $\mathbb{B}_r$ the set of monomials of B of the form $t_2^{a_2} \cdots t_r^{a_r}$ with each $a_i > 0$ $(1 \leq i \leq r)$.

Lemma 6.1.8. Given $\mathfrak{m}' = (m_2', \ldots, m_r') \in \mathbb{N}^{r-1}$, define

$$\Psi^{\mathbf{m}'} : \mathbb{B}_r \to \mathbb{N},$$

by

$$\Psi^{\mathbf{m}'}(t_2^{a_2} \cdots t_r^{a_r}) = m_2' a_2 + \cdots + m_r' a_r.$$

Let $g = t_2^{b_2} \cdots t_r^{b_r} \in \mathbb{B}_r$ be the monomial that is minimal in $\mathbb{B}_r$ with respect to degree in lexicographical ordering in $\mathbb{N}^{r-1}$. Then there is an element $\mathfrak{m} = (m_2, \ldots, m_r) \in \mathbb{N}^{r-1}$ depending on the minimal term $g \in \mathbb{B}_r$ satisfying

$$m_r = 1 \text{ and } \Psi^{\mathfrak{m}}(g) < \Psi^{\mathfrak{m}}(h), \forall h \in \mathbb{B}_r \backslash \{g\}. \tag{6.8}$$

The method employed in the following construction of $\mathfrak{m}$ is very similar in spirit to the classical proof of Noether normalization (see e.g. [Liu 02, Prop. 2.1.9]).

Proof. Write $\mathbf{b} = (b_2, \ldots, b_r)$. Define

$$\mathbf{m} = (m_2, \ldots, m_r),$$

via

$$m_r = 1,\ m_j = \sum_{i=j+1}^{r} m_i b_i,\ (2 \leq j < r).$$

Now assume $\Psi^{\mathbf{m}}(h) \leq \Psi^{\mathbf{m}}(g)$ for some $h = t_2^{a_2} \cdots t_r^{a_r} \in \mathbb{B}_r$. We claim that $h = g$. First suppose $r = 2$, so $m_2 = 1$. Since g is minimal with respect to lexicographical ordering of degree in $\mathbb{B}_r$, we have $a_2 \geq b_2$. Moreover, the hypothesis $\Psi^{\mathbf{m}}(h) \leq \Psi^{\mathbf{m}}(g)$ in this case means $a_2 \leq b_2$; therefore, $a_2 = b_2$ and the claim is true for $r = 2$. Now we prove the claim for $r \geq 3$. We have

$$\sum_{i=2}^{r} m_i a_i \leq \sum_{i=2}^{r} m_i b_i \tag{6.9}$$

$$a_2 + \frac{\sum_{i=3}^{r} m_i a_i}{m_2} \leq b_2 + 1 \tag{6.10}$$

$$(a_2 - b_2) + \frac{\sum_{i=3}^{r} m_i a_i}{m_2} \leq 1 \tag{6.11}$$

where the second inequality is obtained by dividing by m_2 and using the definition $m_2 = \sum_{i=3}^{r} m_i b_i$. Since $\mathbf{b}$ is minimal in the lexicographical ordering, we have $a_2 \geq b_2$, and so the inequality (6.11) implies that either

(i) $a_2 = b_2 + 1$ and $\sum_{i=3}^{r} m_i a_i = 0$, or

(ii) $a_2 = b_2$ and $\sum_{i=3}^{r} m_i a_i = \sum_{i=3}^{r} m_i b_i$.

Case (i) is impossible since the choice of $\mathbf{b}$ and induction on r gives $\sum_{i=3}^{r} m_i a_i > 0$. Only case (ii) is possible. In this case induction on r gives $a_i = b_i$. Therefore $h = g$.

□

Definition 6.1.9. We call an element $\mathfrak{m} \in \mathbb{N}^{r-1}$ satisfying (6.8) in Lemma 6.1.8 a *B-good vector*. Such a vector always has last coordinate $m_r = 1$.

Proposition 6.1.10. There are regular curves C_e on X_x indexed by integers $e > 0$ such that if the morphism defining C_e is $\phi_e : C_e \hookrightarrow X_x$, then $\phi_e^{\#}(B) \neq 0$.

Proof. Let $\mathfrak{m} = (m_2, \ldots, m_r)$ be a B-good vector. Given $e > 0$, our curves C_e are defined by the the following surjective morphism of rings over k'

$$\Phi_e : k'[[t_1, \ldots, t_d]] \to k'[[w]],$$

given by

$$\begin{aligned} \Phi_e(t_1) &= w^e \\ \Phi_e(t_2) &= w^{m_2} \\ \Phi_e(t_3) &= w^{m_3} \\ &\vdots \\ \Phi_e(t_r) &= w^{m_r} \\ \Phi_e(t_s) &= 0, \; (s > r). \end{aligned}$$

Since $\mathfrak{m}$ is a B-good vector, we have $\Phi_e(B) \neq 0$. The morphism

$$\phi_e : C_e \to X_x,$$

is defined as the morphism of affine schemes that corresponds to Φ_e. Note that in terms of ideals (recall $m_r = 1$), for

$$I = (t_1 - t_r^e, t_2 - t_r^{m_2}, \ldots, t_{r-1} - t_r^{m_{r-1}}, \; t_{r+1}, \ldots, t_d) \leq A_x,$$

we have the equality of closed subschemes

$$C_e = V(I) \subset X_x.$$

□

By construction of Φ_e, we may assume B consists only of monomials in $t_2^{i_2} \cdots t_r^{i_r}$ with $i_j > 0$ (since $\Phi_e(t_s) = 0$ for $s > r$). That is, we now write (up to units in A_x):

$$B = \sum_{(i_2, \ldots, i_r)} t_2^{i_2} \cdots t_r^{i_r},$$

with each $i_j > 0$ for $j = 2, \ldots, r$. [1]

We record the following remark and lemmas that will be used in the sequel.

Remark 6.1.11. Observe that the image $\Phi_e(B)$ is independent of $e > 0$ since $\mathfrak{m}$ is independent of e. If $e > 0$ is understood from the context, we write the image of B under the morphism ϕ_e constructed above as

$$\Phi_{\mathfrak{m}}(B) := \Phi_e(B).$$

If we want to emphasize both $\mathfrak{m}$ and e, then we write $\Phi_{e,\mathfrak{m}}$.

---

[1] As the previous sentences asserts, by definition of Φ_e, we can (and do) assume from now on that B is a sum of monomials in only r-terms. We can make this assumption precisely because all of the other $d - r$ terms get sent to zero by Φ_e; i.e. $\Phi_e(t_s) = 0$ for $s > r$.

Lemma 6.1.12. Suppose that $r > 2$. Again write $g = t_2^{b_2} \cdots t_r^{b_r}$ for the minimal element in $\mathbb{B}_r$ with respect to lexicographic ordering in degree. Suppose that $\mathfrak{m} = (m_2, \ldots, m_r = 1)$ is a B-good vector. Then if $Q_i \in \mathbb{N}$ $(i = 2, \ldots, r-1)$ are integers satisfying

$$Q_i \geq \sum_{j=i+1}^{r-1} b_j.Q_j, \tag{6.12}$$

then

$$\mathfrak{m}' = (m_2 + Q_2, \ldots, m_{r-1} + Q_{r-1}, m_r = 1),$$

is a B-good vector.

Proof. Same method as in the proof of Lemma 6.1.8.

□

Although $f = B/t_1^N + B'/t_1^{N-1}$ is assumed to not be a p-power in $R = A_x[1/t_1]$, it is possible that $p|N$. In this case, we modify $\mathfrak{m}$ as follows. Denote by v the normalized discrete valuation on $k'[[w]]$.

Lemma 6.1.13. Recall that $N := \mathrm{Sw}_{\xi}(\chi)_{\log}$. Suppose that $p|N$. Given a B-good vector $\mathfrak{m}$, there is a B-good vector $\mathfrak{m}'$ satisfying the following condition

$$p \nmid Ne - v(\Phi_{\mathfrak{m}'}(B)), \text{for infinitely many } e > 0. \tag{6.13}$$

Proof. If $p \nmid v(\Phi_{\mathfrak{m}}(B))$, take $\mathfrak{m}' = \mathfrak{m}$. So now assume $p|v(\Phi_{\mathfrak{m}}(B))$.

Observe that the hypothesis $p|N$ implies that B is not a p-power in A_x by our choice of $f \in R$ because otherwise $-v_{t_1}(f) < N$, which contradicts our hypothesis that f is a lift of $\bar{f}$ with $-v_{t_1}(f) = N$.

Recall that by Remark 6.1.7, each term of B is not a pth power. Also recall that $g = t_2^{b_2} \cdots t_r^{b_r} \in \mathbb{B}_r$ denotes the minimal lexicographic element of $\mathbb{B}_r$. Now, if $p \nmid v(\Phi_{e,\mathfrak{m}}(B))$ or $r = 2$, no modification is necessary and we put $\mathfrak{m}' = \mathfrak{m}$. So now assume $r > 2$ and $p|v(\Phi_{e,\mathfrak{m}}(B))$..

It is then clear that there are integers $Q_j \geq 0$ such that $\mathfrak{m}' = (m_2 + Q_2, \ldots, m_{r-1} + Q_r, 1)$ is a B-good vector and such that

$$p \nmid v(\Phi'_{\mathfrak{m}}(B)) = (m_2 + Q_2).b_2 + \cdots + (m_{r-1} + Q_{r-1}).b_{r-1} + b_r.$$

□

6.1.2 The limsup in the Artin-Schreier case

We now turn to calculating the Swan conductor over the curves $\phi_e : C_e \to X_x$. Put

$$\mathbb{K} = K(C_e) \cong \mathrm{Frac}\left(k'[[w]]\right),$$

the function field of C_e. Then $\mathbb{K}$ is a complete discrete valuation field and we write v for the discrete valuation on $\mathbb{K}$ normalized with respect to the uniformizer w. Recall that $R = A_x[1/t_1]$ and denote again by

$$\Phi_{e,\mathfrak{m}} : R \to \mathbb{K},$$

the morphism of rings for a B-good vector $\mathfrak{m}$. Write

$$c := v(\Phi_{e,\mathfrak{m}}(B)).$$

Since $-v(\Phi_{e,\mathfrak{m}}(B/t_1^N)) > -v(\Phi_{e,\mathfrak{m}}(B'/t_1^{N'})$ for all $e \gg 0$, the term B/t_1^N of f determines the non-zero rational number

$$\frac{\mathrm{Sw}_x(\chi|_{C_e})}{m_x(\phi_e^* D)} = \frac{\mathrm{Sw}_x(\chi|_{C_e})}{e}. \tag{6.14}$$

We can further assume that $e > 0$ is sufficiently large so that $Ne - c > 0$ (recall that $N > 1$ and $c \geq 0$ are independent of e). We now calculate $\mathrm{Sw}_x(\chi|_{C_e})$ using the Brylinski-Kato filtration (4.4.1) on $W_1(\mathbb{K})/(F-1)$. The image by $\Phi_{e,\mathfrak{m}}$ of $B/t_1^N \in R$ is, up to a unit, equal to $w^{-(Ne-c)}$ in $\mathbb{K}$. There are two cases to consider

(i) The first case is if $p|(Ne-c)$. Then $w^{-(Ne-c)} = w^{-p^a.N'}$ for some $a > 1$ and $N' \geq 1$ with $(N', p) = 1$. Then, $w^{-(Ne-c)}$ is equal to $w^{-N'}$ in $\mathbb{K}/(F-1)$. In this case $\mathrm{Sw}_x(\chi|_{C_e}) < Ne - c$.

(ii) The second case is if $p \nmid (Ne-c)$. Then there is no reduction modulo $(F-1)$, and hence $\mathrm{Sw}_x(\chi|_{C_e}) = -v(w^{-(Ne-c)}) = Ne - c$.

Clearly, if $p \nmid N$, there are infinitely-many $e > 0$ such that case (ii) above holds. If $p|N$, then we modify ϕ_e by Lemma 6.1.13 to achieve case (ii) above for infinitely-many $e > 0$. Therefore, to verify the asserted limsup in (6.4) it is sufficient to take the limit over those $e > 0$ in case (ii) and we have

$$\limsup_{e\to\infty} \frac{\mathrm{Sw}_x(\chi|_{C_e})}{m_x(\phi_e^* D)} = \lim_{e\to\infty} \frac{Ne-c}{e} = N,$$

as desired. The proof of Theorem 6.1.5 for $\chi \in H^1(U_x, \mathbb{Z}/p)$ is complete.

□

6.1.3 The general case: Artin-Schreier-Witt

Now given $n \geq 1$, we prove that Theorem 6.1.5 is true for cyclic $\mathbb{Z}/p^{n+1}$-extensions assuming it is true for all cyclic $\mathbb{Z}/p^{n'}$-extensions with $n' < n+1$.

So fix a character $\chi \in H^1(K, \mathbb{Z}/p^{n+1})$ and put

$$N := \mathrm{Sw}_{\xi}(\chi) > 0.$$

Suppose $\bar{\mathbf{f}} \in W_{n+1}(R)/(F-1)$ corresponds to χ. Choose elements $f_i \in R$, for $i = 0, \ldots, n$, such that on writing

$$\mathbf{f} = (f_0, \ldots, f_n) \in W_{n+1}(R),$$

we have

(i) $\mathbf{f} \equiv \bar{\mathbf{f}} \mod (F-1)W_{n+1}(R)$.

(ii) There is a $k \in [0, n]$ such that

$$-p^{n-k}.v(f_k) = N. \tag{6.15}$$

The existence of such an $\mathbf{f}$ follows directly from (4.14). We proceed by analyzing the case where $k < n$ and then with $k = n$.

If $k < n$, we apply the induction hypothesis as follows. Denote by $V^{k+1} : W_{n-k}(K) \to W_{n+1}(K)$ the $(k+1)$-iterated Verschiebung. The diagram (4.9) extends to a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
W_{n-k}(K) \arrow[r, "\delta_{n-k}"] \arrow[d, "V^{k+1}"] & H^1(K, \mathbb{Z}/p^{n-k}) \arrow[d, "\cdot p^{k+1}"] \\
W_{n+1}(K) \arrow[r, "\delta_{n+1}"] \arrow[d, "\mathrm{mod}\ V^{k+1}"] & H^1(K, \mathbb{Z}/p^{n+1}) \arrow[d, "\mathrm{mod}\ p^{k+1}"] \\
W_{k+1}(K) \arrow[r, "\delta_{k+1}"] \arrow[d] & H^1(K, \mathbb{Z}/p^{k+1}) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{6.16}$$

Denote by Ψ^{k+1} the morphism corresponding to mod p^{k+1} in the above diagram , so

$$\Psi^{k+1} : H^1(K, \mathbb{Z}/p^{n+1}) \to H^1(K, \mathbb{Z}/p^{k+1}).$$

Let $\chi' = \Psi^{k+1}(\chi)$. Note that in terms of Witt vectors, if χ corresponds to $(z_0, \ldots, z_n) \in W_{n+1}(K)/(F-1)$, then χ' corresponds to $(z_0, \ldots, z_k) \in W_{k+1}(K)/(F-1)$. Observing that

$$\mathrm{Sw}_{\xi}(\chi)_{\log} = p^{k+1}.\mathrm{Sw}_{\xi}(\chi')_{\log},$$

then the commutativity of the diagram (6.16) and our induction hypothesis applied to χ' verifies the assertion of Theorem 6.1.5 in the case $k \in [0, n-1]$. It therefore remains to analyze the case $k = n$, i.e.

$$N = -v(f_n) \text{ and } -v(f_n) > -p^{n-i}.v(f_i), \forall i \in [0, n-1],$$

which we now place ourselves in. We first construct the morphism $\Phi_e : R \to \mathbb{K}$ and then proceed to the calculation of $\limsup_{e>0} \mathrm{Sw}_x(\chi|_{C_e})/m_x(\phi_e^* D)$, where $\phi_e : C_e \to X_x$ is the morphism of schemes defined by Φ_e. For each $f_i \neq 0$, let

$$N_i = -v(f_i),$$

and as in (6.7) write

$$f_i = \frac{B_i}{t_1^{N_i}} + \frac{B_i'}{t_1^{N_i-1}}, \text{with } 0 \neq B_i \in A_x/(t_1), B_i' \in A_x. \tag{6.17}$$

In order to define Φ_e, consider the term B_n of f_n and fix an integer $e > 0$. Then we define $\Phi_e : R \to \mathbb{K}$ with respect to B_n by setting $B = B_n$ so that Φ_e is defined by a B-good vector as in Prop. 6.1.10. If $p | N := \mathrm{Sw}_\xi(\chi)_{\log}$, we further choose this B-good vector to satisfy (6.13) in Lemma 6.1.13. Then,

$$\Phi_e(B_n) \neq 0, \forall e > 0.$$

Note that for $i \in [0, n]$:

$$\Phi_e(f_i) = 0 \Leftrightarrow \Phi_e(B_i) = -t_2^e.\Phi(B_i'). \tag{6.18}$$

Therefore, for each $i \in [0, n]$ with $f_i \neq 0$, we can only have $\Phi_e(f_i) = 0$ for at most one $e > 0$ and so

$$\text{if } f_i \neq 0, \text{ then } \Phi_e(f_i) = 0 \text{ for at most } n+1-\text{many } e > 0.$$

Set

$$\Phi_e(\mathbf{f}) = (\Phi_e(f_0), \ldots, \Phi_e(f_n)) \in W_{n+1}(\mathbb{K}).$$

For each $e \gg 0$ we claim that $\Phi_e(f_n)$ will determine the conductor corresponding to the image of $\Phi_e(\mathbf{f})$ in $W_{n+1}(\mathbb{K})/(F-1)$ upon dividing by the multiplicity $m_x(\phi_e^* D) = e$.

More specifically, we first show that for $m(e)$ the smallest integer for which $\Phi_e(\mathbf{f}) \in \mathrm{fil}_{m(e)} W_{n+1}(\mathbb{K})$, the limit of the ratio $m(e)/e$ as $e \to \infty$ is N (see Lemma 6.1.14). Then we show that the lim sup of the ratio $s(e)/e$, for $s(e)$ is the smallest integer for which $\Phi_e(\mathbf{f}) \in (\mathrm{fil}_{s(e)} W_{n+1}(\mathbb{K}))/(F-1)$, is also N by means of Lemmas 4.4.7 and 6.1.15.

Lemma 6.1.14. Keep the above notation and hypothesis. Again write v for the normalized discrete valuation on $\mathbb{K}$. For a fixed $e > 0$, let $m(e) \geq 1$ denote the *minimal* integer for which

$$\Phi_e(\mathbf{f}) \in \mathrm{fil}_{m(e)} W_{n+1}(\mathbb{K}).$$

Then $m(e)/e \to N$ as $e \to \infty$.

Proof. This is clear since for each $i \in [0, n]$, we have

$$\Phi_e(f_i) = \frac{\Phi_e(B_i)}{t_2^{N_i.e}} + \frac{\Phi_e(B_i')}{t_2^{(N_i-1).e}}.$$

□

It remains to compare the valuation of each coordinate in $\Phi_e(\mathbf{f})$ *relative to* $(F-1)$ equivalence in $W_{n+1}(\mathbb{K})$. We claim that modulo $(F-1)$ in $W_{n+1}(\mathbb{K})$, the image of the $n+1$-th coordinate of $\phi(\mathbf{f}) \in W_{n+1}(\mathbb{K})$, i.e. $\Phi_e(f_n)$, determines the conductor $\mathrm{Sw}(\chi|_{C_e})$. In fact, the truth of this assertion is provided by Prop. 4.4.5 above.

We now proceed by following the principle of Lemma 6.1.14 to compute $\limsup_e \mathrm{Sw}(\chi|_{C_e})/e$ as $e \to \infty$.

Lemma 6.1.15. For $e > 0$, consider the composite

$$\overline{\Phi_e} : W_{n+1}(R) \to W_{n+1}(\mathbb{K}) \to W_{n+1}(\mathbb{K})/(F-1).$$

For an integer $h \geq 1$, let $s(h) \geq 0$ denote the *minimal* integer for which

$$\overline{\Phi_h}(\mathbf{f}) \in \mathrm{fil}_{s(h)}(W_{n+1}(\mathbb{K})/(F-1)).$$

Then

$$\limsup_{e \to \infty} s(e)/e = \sup_{e'} s(e')/e' = N,$$

where e' ranges over all $e' > 0$ for which $p \nmid v(\Phi_{e'}(f_n))$.

Proof. The proof is as in Lemma 6.1.14 with the $\lim_{e\to\infty}$ replaced by $\limsup_{e\to\infty}$. □

End of proof of Theorem 6.1.5

We now finish the proof of Thm. 6.1.5. By induction we've reduced to the case where $\mathrm{Sw}_\xi(\chi)_{\log} = -v(f_n)$. Combining Lemmas 4.4.7 and 6.1.15 enables us to use the technique in §6.1.2 to compute $N := \mathrm{Sw}_\xi(\chi)_{\log}$ in terms of the C_e. Recall that the curves $\phi_e : C_e \to X_x$ on X_x are based on the term B_n of the coordinate f_n in $\overline{\mathbf{f}}$. Let

$$c = v(\Phi_e(B_n)).$$

We have

$$\limsup_{e\to\infty} \frac{\mathrm{Sw}_x(\chi|_{C_e})}{m_x(\phi_e^* D)} = \lim_{e\to\infty} \frac{Ne-c}{e} = N.$$

This completes the proof of the Local Tangent Theorem, Thm. 6.1.5.

□

The following corollary expresses a form of (upper) semi-continuity of restriction to curves with respect to the conductor on the boundary $\mathrm{Sw}_{\log}$. [2]

Corollary 6.1.16. If $\phi : C_x \to X_x$ is a curve on X_x, then we have the inequality

$$\mathrm{Sw}_x(\chi|_{C_x}) \leq m_x(\phi^* D) \cdot \mathrm{Sw}_\xi(\chi)_{\log},$$

Proof. From the above proof, this assertion is clear on pulling-back Witt vectors over discrete valuation rings from $\mathcal{O}^h_{X_x,\xi}$ to $\mathcal{O}^h_{C_x,x}$.

In particular, suppose as above that t_1 is the local parameter defining the irreducible Cartier divisor $D \subset X_x$. Let $C \subset X_x$ be a curve on X_x with $C \not\subset \mathrm{Supp}(D)$ where C is defined by a morphism of rings

$$f_C : k'[[t_1, \ldots, t_d]] \to k'[[w]].$$

Let

$$N = \mathrm{Sw}_\xi(\chi)_{\log},$$

and fix a character $\chi \in H^1(K, \mathbb{Z}/p^{n+1})$. In the Artin-Schreier case ($n = 0$), the character χ corresponds to an equation of the form

$$T^p - T = \frac{B}{t_1^N} + H,$$

where H consists of higher-order terms in the parameter t_1. Let v denote the (normalized) discrete valuation on $k'[[w]]$, fix a point $z \in C_\infty(D)$, and let $D|_C$ denote the pullback of D to the curve C. Then the image $f_C(t_1) \in k'[[w]]$ has the property that

$$0 < \mathrm{Sw}_z(\chi|_C) \leq -v(f_C(t_1)) \leq N.m_z(D|_C),$$

so it follows that

$$\frac{\mathrm{Sw}_z(\chi|_C)}{m_z(D|_C)} \leq N = \mathrm{Sw}_\xi(\chi)_{\log}.$$

---

[2] Recall that on a metric space, a function f is upper semi-continuous near a point x_0 if $\limsup_{x\to x_0} f(x) \leq f(x_0)$. For $x < x_0$ we think of $f(x)$ as $f(x; C, z) = \mathrm{Sw}_x(\chi|_C)/m_z(\phi^* D)$ and $f(x_0)$ as the number $\mathrm{Sw}_\xi(\chi)_{\log}$.

The general Artin-Schreier-Witt case follows similarly by the methods in Sec. 6.1.3.

□

6.1.4 Global setting

We proceed to the global setting, i.e. to Thm. 6.1.2, via the following two propositions. Recall that k is a perfect field, X/k is a smooth variety and $U \subset X$ is an open subscheme such that $X \backslash U$ is the support of an effective Cartier divisor on X and D is an sncd on X with $\mathrm{Supp}(D) \subset E$.

Proposition 6.1.17. Suppose that D is a smooth irreducible divisor on X with generic point ξ and let $x \in |X| \cap \mathrm{Supp}(D)$. Since D and X are regular, there is a unique point $\lambda \in \mathrm{Spec}\, \mathcal{O}^h_{X,x}$ lying over $\xi \in \mathrm{Spec}\, \mathcal{O}_{X,x}$. Consider the henselian discrete valuation rings $A = \mathcal{O}^h_{X,\xi}$ and $B = ((\mathcal{O}_{X,x})^h{}_\lambda)^h$. Denote by ω the generic point of Spec B. Let $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$. Then

$$\mathrm{Sw}_\xi(\chi)_{\log} = \mathrm{Sw}_\lambda(\chi|_\omega)_{\log}. \tag{6.19}$$

Proof. This is clear since B/A is unramified (note that the extension of residue fields is separably generated). □

We will need the above proposition for complete local rings. Having proven the assertion for B/A, it follows for the completions $\hat{B}/\hat{A}$ by standard arguments. In particular,

Proposition 6.1.18. Keep the hypotheses and notation in Prop. 6.1.17. Let $\hat{\lambda}$ in Spec $\hat{\mathcal{O}}_{X,x}$ be the unique point of Spec $\hat{\mathcal{O}}_{X,x}$ lying over $\xi \in \mathrm{Spec}\, \mathcal{O}_{X,x}$. Consider the completions $\hat{A}$ and $\hat{B}$ of A and B, respectively. Denote by $\hat{\omega}$ the generic point of Spec $\hat{B}$. Let $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$. Then

$$\mathrm{Sw}_\xi(\chi)_{\log} = \mathrm{Sw}_{\hat{\lambda}}(\chi|_{\hat{\omega}})_{\log}. \tag{6.20}$$

Having proven the assertion for B/A, it follows for the completions $\hat{B}/\hat{A}$ by standard arguments; namely,

Proof. This follows from Prop. 6.1.17 above and [EGA 4, Part. 4, Prop. 17.4.4.].

□

In order to ensure that we get 1-dimensional subschemes on X from the local setting above, we use henselian local rings of X since they are unions of finitely-generated k-algebras, as opposed to complete local rings. To do so we simply note that the curves constructed above (see Prop. 6.1.10) are defined by *algebraic* relations in the regular parameters of X_x and therefore these curves are *defined* in the henselian localization $\mathcal{O}^h_{X,x}$.

In particular, keep the notation from Thm. 6.1.5 and let A_x^h be the henselization of X at x and $R^h = A_x^h[1/t_1]$. Because the curves C_e constructed in the proof of Prop. 6.1.10 are defined by algebraic relations, we thus have curves C_e^h on $\mathrm{Spec}(A_x^h)$ for the following

Proposition 6.1.19. There are regular curves $\phi_e : C_e^h \to \mathrm{Spec}(A_x^h)$ with $C_e^h \in Z_1(\mathrm{Spec}(A_x^h), D)$ indexed by $e > 0$ such that

$$\limsup_{e\to\infty} \frac{\mathrm{Sw}_x(\chi|_{C_e^h})}{m_x(\phi_e^* D)} = \mathrm{Sw}_\xi(\chi)_{\log}. \tag{6.21}$$

□

We are now in a position to prove Theorem 6.1.2.

6.1.5 Proof of Theorem 6.1.2

Proof. We may assume that D is irreducible with generic point ξ. Let $x \in |X| \cap \mathrm{Supp}(D)$. Let $X_x^h = \mathrm{Spec}(A_x^h)$ and write

$$C_e' \subset X_x^h,$$

for the curves given in Prop. 6.1.19 above and consider the image $f_x(C_e')$ where $f_x : X_x^h \to X$ is the canonical morphism. Since x is a closed point of X and X_x^h is the henselization of X at x, it follows that that the *closure* of $f_x(C_e')$ is a curve on X. In fact, the claim that the Zariski closure of $f_x(C_e')$ has dimension 1 follows by noting that the function field of $f_x(C_e')$ has transcendence degree 1 over the base field k. Then the $f_x(C_e')$ are the desired set of curves $\bar{C}$ on X. Recall that by definition of $C_\infty(D)$, it is their normalizations $\bar{C}^N$ over which the conductor $\mathrm{Sw}_z(\chi|_C)$ is computed. Since $\dim(\bar{C}) = 1$, then $\bar{C}^N$ is a *regular* scheme.

The theorem then follows immediately on combining Proposition 6.1.18 and the Local Tangent Theorem 6.1.5. □

Corollary 6.1.20. We have that for each curve $\bar{\phi} : \bar{C} \to X$ on X and $z \in \bar{C}_\infty(D_i)$,

$$\mathrm{Sw}_z(\bar{\phi}^* \chi) \leq m_z(\bar{\phi}^* D_i). \mathrm{Sw}_\xi(\chi)_{\log}.$$

Proof. Follows by combining Coro. 6.1.16 and Prop. 6.1.18. □

6.2 Discussion of the curves C_e

We make some remarks regarding the above construction of the curves C_e. First, we give an illustrative example of our construction.

Example 6.2.1. Suppose that

$$B = t_2.t_3^3.t_4 - t_2^2.t_3.t_4 + t_1.t_2.t_3.t_4.t_5.$$

Then, $r = 4$, our minimal monomial is $g = t_2.t_3^3.t_4, \mathbf{b} = (1,3,1), m_4 = 1, m_3 = 1 \cdot 1 = 1$, and $m_2 = 1 \cdot 3 + 1 \cdot 1 = 4$ so $\mathbf{m} = (4,1,1)$. Then the image of B is non-zero:

$$\phi_e(B) = t^8 - t^{10} \neq 0.$$

□

Remark 6.2.2. If we want each of the images $\phi_e(f_i) \neq 0$ for $f_i \neq 0$, $i = 0,\ldots,n$ for the proof in Section 6.1.3, we slightly modify ϕ_e as follows. Observe that in the construction of ϕ_e in Lemma 6.1.10, if $\mathfrak{m} = (m_2,\ldots,m_d)$ has the desired property that $\phi_e(B_i) \neq 0$, then $(m_2 + Q,\ldots,m_r + Q)$ also satisfies this property for each integer $Q > 0$. Each series $B_0,\ldots,B_n$ obtained from the 'leading numerators' of the coordinates of $\mathbf{f}$ determines an $\mathfrak{m}_i$ for $i = 0,\ldots,n$. Arbitrarily fixing $\mathfrak{m}_1$, then an $\mathfrak{m}$ that we want is given by setting Q equal to the sum of all coordinates appearing in each of the $\mathfrak{m}_i$ and then setting

$$\mathfrak{m} = (m_{11} + Q,\ldots,m_{d1} + Q).$$

It follows that the morphism $\phi_e : R \to \mathbb{K}$ defined by this $\mathfrak{m}$ has the property that $\phi_e(f_i) \neq 0$ whenever $f_i \neq 0$, for $i = 0,\ldots n$.

Remark 6.2.3. We expect that this method, perhaps combined with the method in the next subsection, yields an $\mathbf{m}$ such that the image of B is non-zero and is minimal with respect to sum of coordinates. For our present applications, this minimality is not necessary.

Note that our construction of B-good vectors could be phrased in a more geometric context. The exponents of a monomial in B forms a point in $\mathbb{N}^{d-1}$. A B-good vector then corresponds to an appropriate corner from the convex hull of these points. So in fact, a similar and more explicit proof can give a *minimal* $\mathbf{m}$ with respect to the sum of its coordinates.

Chapter 7

Proof of Expectation 2 in rank 1

7.1 Proof of Expectation 2 in rank 1

7.2 Theorem 7.2.1

Conjecture B in our setting is the following theorem. Recall that for D an effective Cartier divisor, $Z_1(X, D)$ denotes the set of curves on X that are not contained in the support of D and given $\bar{C} \in Z_1(X, D)$,

$$\bar{C}_\infty(D) = \{z \in |\bar{C}^N| : \Psi_{\bar{C}}(z) \in D\},$$

see Def. 4.5.4.

Theorem 7.2.1. (Conjecture B in the smooth rank 1 case) Let X/k be a smooth variety and $U \subset X$ an open subvariety such that $E := X \backslash U$ is the support of a smooth effective Cartier divisor on X. Let $\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$. Let $D \in \mathrm{Div}^+(X)$ be a a regular sncd with $\mathrm{Supp}(D) = E$. The following are equivalent.

(i) For each curve $\bar{\phi} : \bar{C} \to X$ and each $z \in \bar{C}_\infty(D)$,

$$\mathrm{Sw}_z(\bar{\phi}^*\chi) \leq m_z(\bar{\phi}^* D).$$

(ii) For each generic point $\xi \in D$,

$$\mathrm{Sw}_\xi(\chi)_{\log} \leq m_\xi(D).$$

Remark 7.2.2. We've made the simplifying assumption that $\mathrm{Supp(D)} = E$, instead of $\mathrm{Supp}(D) \subset E$, so that $\bar{\phi}^*\chi$ makes sense.

7.2.1 Proof of Theorem 7.2.1

Proof. For ease of notation,

$$m_\xi := m_\xi(D).$$

(i)⇒ (ii): Given ξ, let D' be the component of D that contains ξ with its multiplicity so that in $\mathrm{Div}^+(X)$ we have

$$D' = m_\xi . D'_{\mathrm{red}}.$$

By hypothesis, for each curve $\bar{C} \in Z_1(X, D')$ and each point $z \in \bar{C}_\infty(D')$, we have

$$\mathrm{Sw}_z(\bar{\phi}^*\chi) \le m_z(\bar{\phi}^* D') \tag{7.1}$$

By Thm. 6.1.2, we know that

$$\mathrm{Sw}_\xi(\chi)_{\log} = \sup_{z',\bar{C}} \frac{\mathrm{Sw}_{z'}(\bar{\phi}^*\chi)}{m_{z'}(\bar{\phi}^*(D'_{\mathrm{red}}))}, \tag{7.2}$$

where the supremum ranges over $\bar{C} \in Z_1(X, D'^{\mathrm{sm}})$ and points $z' \in \bar{C}_\infty(D'^{\mathrm{sm}})$ satisfying the property that the image of z' in X is contained in D' but not any other component of D.

For such points z', we further have

$$m_{z'}(\bar{\phi}^* D') \le m_{z'}(\bar{\phi}^*(D'_{\mathrm{red}})).m_\xi,$$

Then by (7.1) we have

$$\mathrm{Sw}_{z'}(\bar{\phi}^*\chi) \le m_{z'}(\bar{\phi}^*(D'_{\mathrm{red}})).m_\xi$$

and then

$$\frac{\mathrm{Sw}_{z'}(\bar{\phi}^*\chi)}{m_{z'}(\bar{\phi}^*(D'_{\mathrm{red}}))} \le m_\xi.$$

Thus, by (7.2) we have

$$\mathrm{Sw}_\xi(\chi)_{\log} \le m_\xi.$$

(ii)⇒ (i): We may assume $\chi \in H^1(U, \mathbb{Z}/p^n)$ for some $n \ge 1$.

First suppose that D is irreducible with generic point λ.

By Coro. 6.1.20, we may assume that the multiplicity $m_z(\bar{\phi}^* D)$ is non-zero, i.e. that $m_z(\bar{\phi}^* D) > 0$. There is an integer $M_{\bar{C}} \ge 1$ such that

$$m_z(\bar{\phi}^* D) = M_{\bar{C}}. m_\lambda(D).$$

The proof of Thm. 6.1.5 and Coro. 6.1.20 yields that

$$\mathrm{Sw}_z(\bar{\phi}^*\chi) \leq M_{\bar{C}}.\,\mathrm{Sw}_\lambda(\chi)_{\log}.$$

By hypothesis, $\mathrm{Sw}_\lambda(\chi)_{\log} \leq m_\lambda(D)$ and so

$$\mathrm{Sw}_z(\bar{\phi}^*\chi) \leq M_{\bar{C}}.\,m_\lambda(D) = m_z(\bar{\phi}^*D),$$

as desired.

These arguments and purity of the branch locus complete the proof for the case where D has more than one component. □

Chapter 8

Higher class field theory

In this chapter we discuss the higher class field theory for schemes that was developed in the original papers of G. Wiesend [Wie 06], [Wie 07] and completed in [Ker-Sch 09].

After recalling some of the basic definitions and fundamental results in Sec. 8.1.2, we focus on the case of varieties in positive characteristic in Sec. 8.2. In subsection 8.2.1 we prove a result on space filling curves, Thm. 8.2.1, that is used in the proof of an isomorphism theorem for varieties, Thm. 8.2.4.

In the last section, Sec. 8.3, we place ourselves in the context of ray class groups and define both a class group and a fundamental group relative to a modulus. In particular, we use our results for the log filtration, that is Expectation 2 (in rank 1), to define a fundamental group. Note that this filtration does not coincide with the non-log filtration employed in [Ker-Sai 13]; see Lemma 8.3.1 below.

8.1 Wiesend class field theory

One of the fundamental ideas in Wiesend's version of class field theory for schemes is to construct a class group for a scheme based on its zero and one-dimensional data; specifically from the closed points and curves on the given scheme. At present, it is an open question if there is a formulation in which Wiesend's framework can be made to satisfy a local-global principle in the sense of classical class field theory (see e.g. [Neu 99, Ch. VI]).

8.1.1 Wiesend's class group

We follow Wiesend's construction of the class group as in [Wie 06], as presented in [Ker-Sch 09, §7].

Let $\mathrm{Sch}(\mathbb{Z})$ denote the category of schemes that are separated and finite type over $\mathrm{Spec}\,(\mathbb{Z})$.

For $C \in \mathrm{Sch}(\mathbb{Z})$ a curve, denote by $\tilde{C}$ the normalization of C and $\bar{C}$ the regular compactification of $\tilde{C}$, i.e. $\bar{C}$ is (up to isomorphism) the regular proper curve $\tilde{C} \in \mathrm{Sch}(\mathbb{Z})$ that contains $\tilde{C}$ as a dense open subscheme. Let C_∞ denote the set of normalized discrete valuations of the global field $\mathbf{k}(C)$ that correspond to the points of the closed complement $\bar{C} \backslash \tilde{C}$ together with the set of archimedean places of $\mathbf{k}(C)$.

Given $v \in C_\infty$, denote by $\mathbf{k}(C)_v$ the completion of $\mathbf{k}(C)$ with respect to v.

Definition 8.1.1. The idéle group $I(X)$ is defined to be the group

$$I(X) = \bigoplus_{x \in |X|} \mathbb{Z} \oplus \bigoplus_{C \subset X} \bigoplus_{v \in C_\infty} \mathbf{k}(C)_v^\times,$$

where $C \subset X$ ranges through curves on X. This group is regarded as a topological group by endowing it with the direct sum topology.

For a morphism of schemes $f : X \to Y$ in $\mathrm{Sch}(\mathbb{Z})$, there is a continuous push-forward map of idéle class groups which is covariant functorial in global schemes: $f_* : I(X) \to I(Y)$; see [Ker-Sch 09, §7].

For a curve C on X, define a canonical map $\mathbf{k}(C)^\times \to I(X)$ as follows. Define $\mathbf{k}(C)^\times \to I(\tilde{C})$ as the sum of all embeddings $\mathbf{k}(C)^\times \hookrightarrow \mathbf{k}(C)^\times \subset I(\tilde{C})$ for $v \in C_\infty$ and all discrete valuations $\mathbf{k}(C)^\times \to \mathbb{Z} \subset I(\tilde{C})$ that correspond to closed points of $\tilde{C}$. Composing this map with $I(\tilde{C}) \to I(X)$ gives the desired canonical map $\mathbf{k}(C)^\times \to I(X)$.

Definition 8.1.2. The idéle class group $C(X)$ is defined to be the cokernel of the morphism of groups

$$\bigoplus_{C \subset X} \mathbf{k}(C)^\times \to I(X),$$

and $C(X)$ is endowed with the quotient topology.

Using 1-dimensional local and global class field theory, one defines the *reciprocity* map

$$\rho_X : C(X) \to \pi_1^{ab}(X).$$

For details, see [Ker-Sch 09, §7].

8.1.2 Basic results

We recall basic results that will be used in the proof of our Isomorphism Theorem, Thm. 8.2.4, below.

Let $X \in \mathrm{Sch}(\mathbb{Z})$ be an integral scheme of dimension d and let M be a subset of $|X|$. For $x \in |X|$ write $N(x) = \#\mathbf{k}(x)$. Recall that M has Dirichlet density

$$\delta(M) = \lim_{s\to d+0}\left(\sum_{x\in M}\frac{1}{N(x)^s}\right) / \log\left(\frac{1}{s-d}\right),$$

if this limit exists.

Proposition 8.1.3. (Chebotarev density; [Ser 63, Thm. 7]) Let $Y \to X$ be a Galois covering of connected normal schemes in $\mathrm{Sch}(\mathbb{Z})$. Let R be a subset of $G = \mathrm{Gal}(Y|X)$ with $gRg^{-1} = R, \forall g \in G$. Set

$$M = \{x \in |X| : \mathrm{Frob}_x \in R\}.$$

Then the density $\delta(M)$ is defined and equal to $\#R/\#G$.

The following assertions, Lemma 8.1.5 and Prop. 8.1.6, may be found in [Ker 11] where they are stated and proved for *arithmetic* schemes $X \in \mathrm{Sch}(\mathbb{Z})$, i.e. $X \in \mathrm{Sch}(\mathbb{Z})$ where X is an integral, regular, separated scheme *flat* and of finite-type over $\mathrm{Spec}\,(\mathbb{Z})$. However, the schemes of principal interest in this chapter are schemes $X \in Sm_k$ for k a *finite field*. Then, $X \in Sm_k$ is *not* flat over $\mathrm{Spec}\,(\mathbb{Z})$. (One way to see this is that since $\mathrm{Spec}\,(\mathbb{Z})$ is connected, then $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(\mathbb{Z})$ is not flat which shows that X cannot be flat over $\mathrm{Spec}\,(\mathbb{Z})$.)

Nevertheless, these three assertions given in [Ker 11] remain true without their flatness assumption over $\mathrm{Spec}\,(\mathbb{Z})$ and so apply to $X \in Sm_k$. Note that $Sm_k \subset \mathrm{Sch}(\mathbb{Z})$.

Definition 8.1.4. We say that $X \in \mathrm{Sch}(\mathbb{Z})$ is a *global* scheme if X is either an arithmetic scheme or if $X \in Sm_k$ for some finite field k.

Lemma 8.1.5. ([Ker 11, Lemma 4.5]) Suppose X is a global scheme. The image of $\bigoplus_{x\in|X|}\mathbb{Z} \to C(X)$ is dense in $C(X)$. If moreover X is regular and $U \subset X$ is a dense open subscheme, then the image of $\bigoplus_{x\in|U|}\mathrm{Spec}\,(\mathbb{Z}) \to C(X)$ is dense in $C(X)$.

Proposition 8.1.6. ([Ker 11], Proposition 4.10) Suppose X is a global scheme. The image of ρ is dense in $\pi_1^{ab}(X)$.

On the other hand, there is one result we will need to translate from [Ker 11] whose proof therein only holds for arithmetic schemes. This result is on space filling curves which we translate and prove in the next section in Thm. 8.2.1. Our result is the analogue in positive characteristic of that in [Ker 11, Prop. 2.2] (Bloch approximation) and in addition, we include a prescription on tangent spaces; see Thm. 8.2.1 below.

8.2 Varieties in positive characteristic

8.2.1 Space filling curves

As an additional tool for class field theory in finite characteristic we include the following result, which was presented in the Forschungsseminar in Essen during the Summer 2011 semester.

The following theorem is an alternate proof to [Drin 12, Thm. 2.15]; we use a Bertini theorem as alluded to in [Drin 12, §2.5]. We note that the methods of our proof are similar in spirit to those in [Katz 99, Lemma 3]. See also [Ker-Sch 09, Prop. 1.5] for a characteristic zero case.

If a base is not specified, we always mean a given finite field $\mathbb{F}$.

Theorem 8.2.1. Let X be an irreducible, smooth *quasi-projective variety* over a finite field $\mathbb{F}$. Let $Y \to X$ be a Galois cover of X and $S = \{p_1, \ldots, p_n\} \subset X$ a finite set of closed points in X. Then, there is a curve $C \subset X$ such that

(i) C contains S as regular points, i.e. $S \subset C^{reg}$.

(ii) Given, at each $s \in S$, a 1-dimensional subspace $l_s \subset T_s X$, then

$$T_s C = l_s.$$

(iii) C has the property that

$$\pi_1(C) \to \mathrm{Gal}(Y/X)$$

is surjective.

We make use of the following Bertini theorem in positive characteristic:

Proposition 8.2.2. (Gabber-Poonen) Suppose that X is a quasi-projective subscheme of $\mathbb{P}^N$ and $M \subset X$ is a finite set of closed points. For $m \gg 0$, there exists $f \in H^0(\mathbb{P}^N, \mathcal{O}(m))$ such that the hypersurface H_f defined by f intersects X properly, $M \subset H_f$, and $X \cap H_f$ is smooth.

The plan of the proof of Thm. 8.2.1 is as follows. For $U \subset X$ an affine open neighborhood of S of dimension d, construct a morphism $\phi : U \to \mathbb{A}^d$ étale at S "pointed along" $l_s, s \in S$; find a suitable curve $C' \subset \mathbb{A}^d$ via Prop. 8.2.2; pull back C' and verify that closure of this pullback, $C := \overline{\phi^{-1}(C')}$, satisfies the properties (i)-(iii).

Step 1. Let $A = \varinjlim_{V \supset S} \mathcal{O}_X(V)$ where V ranges over affine opens of X containing S; A is then a semi-local ring with $max(A) = S$. For $p_i \in S$ let $\mathfrak{m}_i$ denote its corresponding maximal ideal. Set $d = \dim(A)$. As A is regular, choose d-elements $e_{1,i}, \ldots, e_{d,i}$ that form a $\mathbf{k}(p_i)$-basis of $\mathfrak{m}_i/\mathfrak{m}_i^2$; do so for $1 \leq$

$i \leq n = \#S$. Let $I = \cap_i \mathfrak{m}_i \subset A$, the Jacobson radical of A. Since $A/I^2 = \prod_i A_{\mathfrak{m}_i}/\mathfrak{m}_i^2$, choose $e_1, \ldots, e_d \in A$ such that each e_j has image equal to $e_{j,i}$ in $A_{\mathfrak{m}_i}/\mathfrak{m}_i^2$. Then, the map $t_j \mapsto e_j$ gives an étale-at-S morphism $\operatorname{Spec} A \to \mathbb{A}^d$. We now make the e_j satisfy an additional condition.

Write $l_{p_i} = \langle \gamma_{p_i} \rangle_{\mathbf{k}(p_i)}$, $1 \leq i \leq n$. Because $\{e_{j,i}\}_{1 \leq j \leq d}$ forms a basis in $\mathfrak{m}_i/\mathfrak{m}_i^2$ we further impose the following $d-1$ conditions on the $e_j \in A$:

$$e_{ji} \in ker(\gamma_{p_i}) \subset \mathfrak{m}_{p_i}/\mathfrak{m}_{p_i}^2, \ (1 \leq j \leq d-1; 1 \leq i \leq n).$$

Necessarily, $\gamma_{p_i}(e_{i,d}) \neq 0, \forall 1 \leq i \leq n$. Define ϕ via the morphism $\phi^{\#}$ of rings that maps $t_j \mapsto e_j$. Observe that

$$\gamma_{p_i}(\phi^{\#}_{p_i}(t_j)) = 0, \ 1 \leq i \leq d-1$$

where $\phi^{\#}_{p_i} : \mathcal{O}_{\mathbb{A}^d, \phi(p_i)} \to A_{\mathfrak{m}_i}$. This last observation will be crucial in proving that condition (ii) holds.

Step 2.

We now add to the set $\phi(S) \subset \mathbb{A}^d$ certain conjugacy class coming from $\operatorname{Gal}(Y/X)$ as follows. Let $U = \operatorname{Spec} A$. Applying Zariski's main theorem to the morphism $\phi : U \to \mathbb{A}^d$, there is a non-empty open $W \subset \mathbb{A}^d$ such that $\phi^{-1}(W) \to W$ is an étale covering of W. Restricting Y to the open subset $\phi^{-1}(W)$ gives an étale covering of W whose Galois closure we denote by $Y_W \to W$. Now let $\{w_1, \ldots, w_k\} \subset |W| \subset \mathbb{A}^d$ be a finite set of closed points such that each conjugacy class in $\operatorname{Gal}(Y_W/W)$ contains at least one of the $\operatorname{Frob}_{w_i}$.

Let

$$\Gamma = \phi(S) \cup \{w_i\} \subset \mathbb{A}^d.$$

Write $j : \mathbb{A}^d \subset \mathbb{P}^d$ for a fixed open immersion of $\mathbb{A}^d$ into $\mathbb{P}^d$ and define

$$S' = j(\Gamma) \subset \mathbb{P}^d.$$

Applying Gabber-Poonen to the pair $\{\mathbb{P}^d, S'\}$ produces a hypersurface section f_1 such that H_{f_1} is smooth and contains S'. Let $\hat{f} \in \mathbb{F}[t_1, \ldots, t_d]$ denote a chosen dehomogenization of a homogeneous f in the coordinate ring of $\mathbb{P}^d$. Let $\mathfrak{m}_0 = (t_1, \ldots, t_d)$. Now, [Poonen 04, Thm. 1.2] shows that we may further assume that f_1 satisfies the property that $\hat{f}_1 \equiv \overline{t_1}$ in $\mathfrak{m}_0/\mathfrak{m}_0^2$. Note that the closed subscheme $V(\hat{f}_1) \subset \mathbb{A}^d$ has dimension $d-1$, is irreducible, smooth, and contains Γ. Applying this procedure to $H_{f_1} \subset \mathbb{P}^d$ produces an f_2 such that $V(\hat{f}_2)$ is irreducible, smooth, $\dim(V(\hat{f}_2)) = d-2$, $V(f_2)$ contains Γ, and $\hat{f}_2 \equiv \overline{t_2}$ in $\mathfrak{m}_0/\mathfrak{m}_0^2$. Continuing in this way, we obtain $\hat{f}_1, \ldots, \hat{f}_{d-1} \in \mathbb{F}[t_1, \ldots, t_d]$ such that

$$C' := V(\hat{f}_1) \cap \ldots \cap V(\hat{f}_{d-1}) \subset \mathbb{A}^d,$$

is irreducible, smooth, contains Γ, has $\dim(C') = 1$, and $\hat{f}_i \equiv \overline{t_i}$ in $\mathfrak{m}_0/\mathfrak{m}_0^2$ $(1 \leq i \leq d-1)$.

Step 3.

Without loss of generality, replace U by the étale locus of ϕ so that $\phi : U \to \mathbb{A}^d$ is étale *everywhere* on U.

Let $D = \phi^{-1}(C')$ and note that

$$D = V(\phi^{\#}(\hat{f}_1)) \cap \cdots \cap V(\phi^{\#}(\hat{f}_{d-1})).$$

Being a finite intersection of closed sets, D is closed in U, D contains Γ as smooth (hence regular) points, and $\dim(D) = 1$. By construction of ϕ and the $\hat{f}_1, \ldots, \hat{f}_{d-1}$ observe that

$$l_p \subset T_p V(\phi^{\#}(\hat{f}_i)), \forall p \in S,\ 1 \leq i \leq d-1.$$

Hence,

$$l_p = \bigcap_{1 \leq i \leq d-1} T_p V(\phi^{\#}(\hat{f}_i)) = T_p D, \ \forall p \in S$$

because $l_p \subset T_p D$ and $\dim_{\mathbf{k}(p)} l_p = \dim_{\mathbf{k}(p)} T_p D$.Thus, $D = \phi^{-1}(C')$ verifies property (ii).

We now verify (iii) for the Galois covering $Y \to X$ and complete the proof the theorem. We follow the proof of [Ker 11, Prop. 2.2].

First of all, the composite

$$\pi_1(C' \cap W) \to \pi_1(W) \to \mathrm{Gal}(Y_W/W)$$

is surjective since $C' \cap W \supset \{w_i\}$. It follows by [SGA 1, Exposé 5, Prop 6.9] that $Y_W \times_{\mathbb{A}^d} C'$ is irreducible; hence, $U \times_{\mathbb{A}^d} C' = \phi^{-1}(C')$ is also irreducible. The latter follows since the w_i are regular points of C' (recall that if a variety is connected and regular, then it is irreducible).[1]

Finally, enlarging S if necessary (throughout), assume S contains closed points $x_j \in X$ such that each conjugacy class in $\mathrm{Gal}(Y/X)$ contains one of the Frob_{x_j}. It follows that the Zariski closure in X of $\phi^{-1}(C')$:

$$C := \overline{\phi^{-1}(C')} \subset X,$$

satisfies (i)-(iii). □

8.2.2 Isomorphism theorem for varieties

We record the following theorem. Fix a finite field k.

---

[1] For a more explicit argument on the irreducibility of $\phi^{-1}(C')$, see [Ker-Sch 09, Prop. 1.5].

Theorem 8.2.3. ([Ker-Sch 09, Thm. 8.2] Isomorphism theorem) Let X/k be a separated, connected smooth variety. Assume that there exists an étale morphism $X' \to X$ and a proper, generically smooth morphism $X \to Z$, where Z/k is a smooth curve. Let $\phi : Y \to X$ be a Galois covering and let $Y' \to X$ be its maximal abelian subcovering. Then the reciprocity map induces an isomorphism of finite abelian groups

$$C(X)/\phi_* C(Y) \to \mathrm{Gal}(Y'/X).$$

We generalize this result as follows.

Theorem 8.2.4. (Isomorphism theorem for varieties) Let $\phi : Y \to X$ be a finite dominant morphism of irreducible schemes of finite type over Spec (k). Further assume X to be regular, Y normal, and that the normalization of X in the separable closure of $K(X)$ in $K(Y)$ is finite Galois over X. Then the reciprocity map induces an isomorphism of finite abelian groups

$$\rho_{Y/X} : C(X)/\phi_* C(Y) \to \mathrm{Aut}(Y/X)^{ab}.$$

Proof. In the arithmetic case, the proof is given in [Ker 11, Thm. 6.1] and we prove the assertion here in the variety case following the same method *ibid.*, with the only exception being that we use our Thm. 8.2.1 instead Bloch approximation in the arithmetic case.

Suppose that $X \in Sm_k$. Denote the normalization of X in the separable closure of $K(X)$ in $K(Y)$ by $\phi' : Y' \to X$. First we prove that there is an isomorphism

$$\rho_{Y'/X} : C(X)/\phi'_* C(Y') \cong \mathrm{Gal}(Y'/X)^{ab}. \tag{8.1}$$

The isomorphism for $\phi' : Y' \to X$ now follows by the proof of [Ker-Sch 09, Thm. 6.1]. We briefly review these steps. The surjectivity of ϕ' follows directly from Prop. 8.1.6 on noting that $\mathrm{Gal}(Y'/X)^{ab} = \pi_1^{ab}(X)/\phi'_*\pi_1^{ab}(Y')$.

To prove that ϕ' is injective, let $U \subset X$ be an affine open dense subscheme that is *smooth over* k. Let $W \subset |U|$ be a finite set and let

$$im_W := im[\oplus_W \mathbb{Z} \to C(X)/\phi_* C(Y)].$$

Theorem 8.2.1 yields a curve D with $W \subset D^{reg}$ and such that the base change $D_{Y'} \times D \times_X Y'$ is irreducible. Now because $Y' \to X$ is *Galois* and D_Y is irreducible, we have

$$\mathrm{Gal}(K(D_{Y'})/K(D)) \xrightarrow{\sim} \mathrm{Gal}(Y'/X).$$

(One uses that if B and C are finite A-algebras, $B \otimes_A C$ has no non-trivial minimal primes, and C/A is Galois, then $Aut_B(B \otimes_A C) = Aut_A(C)$.)

Then, the injectivity of ϕ' follows exactly as in *ibid.* since the remaining arguments rely on the isomorphism theorem in global class field theory for function fields and Lemma 8.1.5.

We now turn to $\phi : Y \to X$. First,

$$\mathrm{Aut}(Y/X) = \mathrm{Gal}(Y'/X),$$

as $\mathrm{Gal}(K(Y)/K(Y')) = 1$ since $K(Y)/K(Y')$ is a purely inseparable extension. Let $\Psi : Y \to Y'$ be the morphism induced by $K(Y') \hookrightarrow K(Y)$. Then we have $\Psi_* : C(Y) \to C(Y')$ which in turn induces the left vertical arrow in the following commutative diagram:

$$\begin{tikzcd}
C(X)/\phi_* C(Y) \arrow[r] \arrow[d] & Aut(Y/X)^{ab} \arrow[d, "\cong"] \\
C(X)/\phi'_* C(Y') \arrow[r, "\cong"] & \mathrm{Gal}(Y'/X)^{ab}
\end{tikzcd}$$

To finish the proof, we must show that the morphism on class groups $\Psi_* : C(Y) \to C(Y')$ is surjective.

Recalling how the pushforward Ψ_* is defined (see [Ker-Sch 09, §7]) observe that for $y \in |Y|, \deg(k(y)/k(\Psi(y))) = 1$ since $K(Y)/K(Y')$ is a finite purely inseparable extension; if $C \subset Y$ is a curve and $v \in C_\infty$, then if $D = \overline{\Psi(C)}$ is a curve on Y' and $v|_{K(D)}$ is a valuation $w \in D_\infty$ we have that the norm $k(C)_v^\times \to k(D)_w^\times$ is a *surjection* by Lemma 8.2.5 below; the other cases for $v \in C_\infty$ show that $\Psi_*^{v \to y'}$ is surjective as well. Thus, we conclude that $\Psi_* : C(Y) \to C(Y')$ is surjective.

□

Lemma 8.2.5. Let K be a complete local field of characteristic $p > 0$ and L a finite purely inseparable extension of K. Then the norm $N : L^* \to K^*$ is surjective.

Proof. Let $\mathbb{F}$ be the residue field of K ($\mathbb{F} = \mathbb{F}_q$ for some $q = p^f$). Choose a uniformizer π_K of K so that $K = \mathbb{F}((\pi_K))$. Let π_L be a uniformizer of L and $p^d := [L : K]$. Then, $\pi_L^{p^d} \in \mathbb{F}[[\pi_K]]$. Denoting by v_L the discrete valuation on L, we get $v_L(\pi_K) = p^d$. Therefore, π_K^{1/p^d} is a uniformizer of L, i.e. $L = \mathbb{F}((\pi_K^{\frac{1}{p^d}}))$. Now, the norm is explicitly given as follows:

$$u.(\pi_K^{\frac{m}{p^d}}) \mapsto u.(\pi_K^{\frac{m}{p^d}})^{p^d} = u^{p^d}.\pi_K^m,$$

for $u \in U_L, m \in \mathbb{Z}$. The claim then follows. □

8.3 Class groups with modulus and Existence theorem

Recently, Kerz and S. Saito proved an existence theorem for varieties using, in particular, extensions with bounded ramification and Artin conductors. One of their results is a generalization of the existence theorem for ray class groups and ray class fields in the classical case. They employ a so-called 'non-log' filtration that is similar to the 'log' filtration in rank 1, the latter being the Brylinski-Kato filtration, see Def. 4.4.1. Since the log and non-log filtrations are nested in rank 1, see Lemma 8.3.1 below, we are able to use our result Thm. 7.2.1 to reformulate one of their main results [Ker-Sai 13, Coro II] to the log case.

We recall the non-log filtration used in [Ker-Sai 13], cf. [Mat 97]. Let K be a henselian discrete valuation field of characteristic $p > 0$. Recall the Brylinski-Kato filtration fil_m on $W_n(K)$ in Def. 4.4.1 and set

$$\mathrm{fil}_m^{\mathrm{nonlog}} W_n(K) := \mathrm{fil}_m W_n(K) + V^{n-n'} \mathrm{fil}_m W_{n'}(K), \tag{8.2}$$

where $n' = \min\{n, \mathrm{ord}_p(m+1)\}$.

As in Def. (4.11) from §4.4, definition (8.2) yields a so-called *non-log* filtration on all of $H^1(K)$, which we write as $\mathrm{fil}_m^{\mathrm{nonlog}}$. The relationship between the log and non-log filtrations is summarized by the following lemma.

Lemma 8.3.1. (see [Mat 97] and [Kat 89])

The following properties are satisfied by the non-log filtration.

(i) $\mathrm{fil}_1^{\mathrm{nonlog}} H^1(K)$ is the subgroup of tamely ramified characters

(ii) $\mathrm{fil}_m^{\mathrm{nonlog}} H^1(K_\lambda) \subset \mathrm{fil}_m H^1(K) \subset \mathrm{fil}_{m+1}^{\mathrm{nonlog}} H^1(K)$.

(iii) $\mathrm{fil}_m^{\mathrm{nonlog}} H^1(K) = \mathrm{fil}_{m-1} H^1(K)$ if $(m,p) = 1$.

For simplicity, let X/k a normal variety and U/k an open subscheme such that the reduced closed complement $X \backslash U$ is a (nontrivial) sncd D on X.

Consider the following property on a character $\chi \in H^1(U) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{ab}}(U), \mathbb{Q}/\mathbb{Z})$: for each generic point $\xi \in D$,

$$\mathrm{Sw}_\xi(\chi)_{\log} \leq m_\xi(D), \tag{8.3}$$

(recall Def. 4.5.1).

Definition 8.3.2. For D as above, define $\mathrm{fil}_D H^1(U)$ to be the subgroup of $H^1(U)$ that consists of those χ satisfying (8.3) above. Also define

$$\pi_1^{\mathrm{ab}}(X, D)_{\log} = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{fil}_D H^1(U), \mathbb{Q}/\mathbb{Z}),$$

with the pro-finite topology of the dual.

The topological group $\pi_1^{\mathrm{ab}}(X,D)_{\log}$ is a quotient of $\pi_1^{\mathrm{ab}}(U)$ which should be thought of as classifying extensions of U with "ramification bounded by D".

Following [Rus 10, §3.4], let $Z_0(X,D)$ be the subgroup of 0-cycles on $X\backslash D$ and set

$$\mathcal{R}(X,D) = \{(C,f) : C \in Z_1(X,E), f \in K(C)^\times, \tilde{f} \equiv 1 \bmod D^\#\},$$

where $\tilde{f}$ is the image of f in $K(C^N)$ and $D^\# = (D - D_{\mathrm{red}}) \cdot C^N + (D \cdot C^N)_{\mathrm{red}}$.

Let $R_0(X,D)$ be the subgroup of $Z_0(X\backslash D)$ generated by $div(f)_C$ for $(C,f) \in \mathcal{R}(X,D)$. We set

$$\mathrm{C}(X,D)_{\log} = Z_0(X\backslash D)/R_0(X,D).$$

This definition of a relative Chow group of zero cycles is tailor-made to fit the Brylinksi-Kato filtration, see [Rus 10].

Example 8.3.3. Let $X = \operatorname{Spec} k[u,v], E = V(v), U = \operatorname{Spec}[u,v][1/v], m \geq 1$ and $D = m.E$. For $e \geq 1$, consider the curves C_e on X given by $v = u^e$. The C_e are normal and only meet D at the origin of the plane. We have

$$D^\# = (m.E - E) \cdot C_e + (m.E \cdot C_e)_{\mathrm{red}} = ((m-1)e+1).V(u) \in \mathrm{Div}^+(C_e).$$

■

Remark 8.3.4. If D is reduced, then so is $D^\#$.

Now consider the degree-zero parts

$$\mathrm{C}(X,D)^0_{\log} = \ker[\mathrm{C}(X,D)_{\log} \xrightarrow{\deg} \mathbb{Z}],$$

and

$$\pi_1^{\mathrm{ab}}(X,D)^0_{\log} = \ker[\pi_1^{\mathrm{ab}}(X,D)_{\log} \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k) \simeq \widehat{\mathbb{Z}})].$$

Using our Theorem 7.2.1 and Lemma 8.3.1 combined with [Ker-Sai 13, Coro II], we have the following log-version of their Existence theorem:

Theorem 8.3.5. (Existence theorem, with log filtration) Suppose k is a finite field with $\mathrm{char}(k) \neq 2$. Let X/k be a *proper and* smooth k-variety and $U \subset X$ an open k-subscheme such that the reduced closed complement $D = X\backslash U$ is an sncd on X. Then

$$\mathrm{C}(X,D)^0_{\log} \to \pi_1^{\mathrm{ab}}(X,D)^0_{\log},$$

is an isomorphism of *finite* abelian groups.

□

Chapter 9

Appendix

9.1 Remarks on compactifications and ramification

9.1.1 Tame ramification

In this section we briefly review tame ramification in the context of Wiesend's class field theory. For further information on tame ramification we refer to [Ker-Sch 10]. At the end of this section we remark that the fundamental group of a proper scheme is isomorphic to its tame quotient, see (9.3).

We begin with Wiesend's intrinsic definition of tameness ([Wie 08, §2.3]). Let X be a global scheme (see Def. 8.1.4). Recall that if v is a discrete valuation of $K(X)$, then v is said to be *tame* in a finite separable extension K' of $K(X)$ if for each extension $w|v$ in K', the ramification index $e_{w|v}$ of v in w is relatively prime to the characteristic of the residue field $\mathbf{k}(v)$ and if the corresponding extension of residue fields $\mathbf{k}(w)/\mathbf{k}(v)$ is separable. If C is a curve, then we say that a finite étale cover $C' \to C$ is tame if the extension of function fields $K(C')/K(C)$ is tame.

Recall that by Def. 5.0.9, $\mathrm{Cu}(X)$ is the set of pairs (ϕ, Z^N) where Z^N is the normalization of a curve Z on X and $\phi : Z^N \to X$ is the composition of the normalization morphism with the closed embedding of Z into X.

Definition 9.1.1. An étale covering $Y \to X$ is said to be *tame* if for each $C \in \mathrm{Cu}(X)$, the induced finite étale cover

$$Y \times_X C \to C,$$

is tame.

Remark 9.1.2. Let $C' = Y \times_X C$ in the above definition and observe that since $C' \to C$ is in particular unramified, then for each closed point $x \in |C|$ and its corresponding discrete valuation v_x, each extension of v_x in $K(C')$ is unramified, and hence tame. On the other hand, if C is not proper, then there are discrete valuations v of $K(C)$ that do not correspond to closed points of C. More precisely, if X is an arithmetic scheme, let $S = \mathrm{Spec}\,(\mathbb{Z})$ and if X is a variety over a field k, let $S = \mathrm{Spec}\,(k)$.

Let $C \in \mathrm{Cu}(X)$. If C is proper over S, it follows that every S-scheme étale over C is tame. If C is not proper over S, then C/S determines a regular proper curve $\bar{C}/S$ and an open immersion $C \hookrightarrow \bar{C}$. Since $K(\bar{C}) = K(C)$ but $\bar{C} \backslash C \neq \emptyset$, there are discrete valuations on $K(C)$ that *do not* correspond to closed points on C.

Definition 9.1.3. We say that an étale covering $Y \to X$ is *tame along* $\bar{C}\backslash C$ if for each discrete valuation v_x that corresponds to a point $x \in \bar{C}\backslash C, v_x$ is tame in $K(C')|K(C)$.

By Rem. 9.1.2, we immediately see the following equivalence.

$$Y \to X \text{ is tame iff } Y \times_X C \to C \text{ is tame along } \bar{C}\backslash C, \forall C \in \mathrm{Cu}(X) \tag{9.1}$$

Tame groups

Let X be a global scheme. For a fixed curve $C \subset X$ on X and a fixed place $v \in C_\infty$, consider the group of principal units $U_v^{(1)} \leq k(C)_v^\times$. Let $U(X)^t \leq I(X)$ denote the subgroup of $I(X)$ generated by all such principal unit subgroups for all curves $C \subset X$ and let

$$I(X)^t = I(X)/U(X)^t.$$

Definition 9.1.4. The tame class group $C(X)^t$ of a global scheme X is defined as the quotient of topological groups

$$C(X)^t = C(X)/I(X)^t.$$

Definition 9.1.5. The tame fundamental group of a global scheme X is defined as the quotient of $\pi_1^{ab}(X)^{\mathrm{tame}}$ that classifies tame étale coverings of X, in the sense of Def. 9.1.1.

In [Ker-Sch 09, §7], a reciprocity map of tame groups is defined:

$$\rho_X^t : C(X)^t \to \pi_1^{ab}(X)^{\mathrm{tame}}.$$

The Isomorphism theorem for tame coverings of varieties is

Theorem 9.1.6. ([Ker-Sch 09, Thm. 8.3]) Let k be a finite field and suppose $X \in Sm_k$. Let $f : Y \to X$ be a connected tame étale covering and let $Y' \to X$ be its maximal abelian subcovering. Then ρ_X^t induces an isomorphism of finite abelian groups

$$C(X)^t/f_*C(Y)^t \cong \mathrm{Gal}(Y'/X).$$

9.1.2 Partial compactifications

Let k be a perfect field and let $X \in Sm_k$. By Nagata's theorem on compactifications (see [Con 07]), there exists a normal, proper k-variety X' that contains X as an open dense subscheme. Let $\{\xi_1, \dots, \xi_n\}$ be the generic points of the closed complement $X' \backslash X$.

Take open subschemes $\{U_i\}_{1 \leq i \leq n}$ of X' that satisfy $\xi_j \in U_i \Leftrightarrow j = i$ and let U_j^{sm} denote the smooth k-locus of U_j. Now take the following unions inside X':

$$U' = \cup_{1 \leq i \leq n} U_i^{\mathrm{sm}}$$

and

$$\bar{X} = X \cup U'.$$

Then $\bar{X}$ is a smooth k-variety that contains X as an open dense subscheme and the closed complement $\bar{X} \backslash X$ is an sncd on $\bar{X}$.

Note that

$$X/k \,\text{not proper} \Rightarrow \bar{X} \backslash X \neq \emptyset. \tag{9.2}$$

From the above discussion on tame ramification, we have

$$\bar{X}/k \,\text{proper} \Rightarrow \pi_1(\bar{X}) = \pi_1(\bar{X})^{\text{tame}}. \tag{9.3}$$

9.2 Basics of Artin-Schreier-Witt sheaves

For completeness we include here the explicit correspondence between cyclic p^m-extensions and locally constant constructible sheaves of $\mathbb{F}_\ell$-modules. For short, we call such sheaves *lisse* sheaves of $\mathbb{F}_\ell$-modules. As a general reference, we mention [Mil 80, Ch II, Exa. 2.18(c) & Ch. Y, §1].

Let X be a connected scheme defined over a finite field k of characteristic $p > 0$, $\ell \neq p$ a prime number, and $m > 0$ an integer.

Recall that an *Artin-Schreier-Witt extension* of X is a Galois cover $Y \to X$ with *cyclic* group $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/p^m$ and a *Artin-Schreier-Witt sheaf lisse sheaf* that is trivialized by an étale cover isomorphic to $Y \to X$. Here, χ will denote a non-trivial homomorphism of groups $\chi : G \to GL(V)$, where V is a finite-dimensional $\mathbb{F}_\ell$-vector space.

Proposition 9.2.1. There is a one-to-one correspondence between

(i) Artin-Schreier-Witt extensions of X with a given morphism χ.

(ii) Lisse sheaves $\mathcal{L}_\chi$ of $\mathbb{F}_\ell$-modules on X with a morphism χ that are trivialized by a $\mathbb{Z}/p^m$- Galois cover of X and whose stalks at geometric points of X are all isomorphic to $\mathbb{F}_{l^n}$, for some $n > 0$.

Proof. Fix $m > 0$ and a geometric point $\overline{x}$ of X.

(i)$\Rightarrow$ (ii): Let $X' \to X$ be an Artin-Schreier-Witt extension of X with $\mathrm{Gal}(X'/X) = \mathbb{Z}/p^m$. The group $\mathrm{Gal}(X'/X)$ is then a finite quotient of $\pi_1(X, \overline{x})$; composing with the given morphism χ induces an action of $\pi_1(X, \overline{x})$ on $GL(V) = GL_n(\mathbb{F}_\ell)$ ($\exists n > 0$) via

$$\pi_1(X, \overline{x}) \twoheadrightarrow \mathrm{Gal}(X'/X) \xrightarrow{\chi} GL_n(\mathbb{F}_\ell).$$

Hence, $GL_n(\mathbb{F}_\ell)$ is a finite $\pi_1(X, \overline{x})$-module. Now define the sheaf $\mathcal{L}_\chi$ on X as follows: for a finite étale morphism $Y \to X$, put

$$\mathcal{L}_\chi(Y) = \mathrm{Hom}_{\pi_1}(\mathrm{Hom}_X(\overline{x}, Y), V).$$

Note that if Y is connected, then $\mathcal{L}_\chi(Y) = V^{\pi_1(Y, \overline{x})}$. It follows that $(\mathcal{L}_\chi)_{\overline{x}} \cong V$.

(ii)$\Rightarrow$(i): The lisse sheaf $\mathcal{L}_\chi$ is trivialized by a single finite Galois cover $X' \to X$ and $(\mathcal{L}_\chi)_{\overline{x}} \cong V$ is a finite $\pi_1(X, \overline{x})$-module.

□

Note that the explicit trivialization $X' \to X$ of an Artin-Schreier-Witt sheaf is defined by an element in the ring of Witt vectors $W_m(\Gamma(X, \mathcal{O}_X))$.

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