Aspects of uniformly finite homology

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Introduction

Homological methods in coarse geometry arise as a way to understand properties of coarse spaces by using techniques coming from algebraic topology. In particular, coarse (co)homology was introduced by Roe [38] whose main motivations came from index theory but turned out to have other applications in the field of coarse geometry [39, 40]. However, the space of coarse (co)chains is rather large and one might need to find a way to overcome this problem by forcing growth restrictions on the chains. This idea has led to the development of a new coarse homology theory, namely *uniformly finite homology*.

Uniformly finite homology was introduced by Block and Weinberger to study large-scale structures of metric spaces having bounded geometry [7]. It is a coarse invariant in the sense that metric spaces that are *uniformly close* have isomorphic uniformly finite homology [7, Corollary 2.2]. In particular, this is the case when the spaces are quasi-isometric. The uniformly finite chains are the ones considered by Roe for coarse homology with an additional boundedness condition on the coefficients. More precisely, we can summarize the original definition of uniformly finite homology [7, Section 2] as follows:

Let A be a unital ring with norm (e.g. $A = \mathbb{R}$) and let X be a metric space. For each $n \in \mathbb{N}$ let $C_n^{\mathrm{uf}}(X; A)$ be the A-module of functions $c\colon X^{n+1} \longrightarrow A$ satisfying the following conditions:

(i) The map c is bounded.

(ii) In every ball in X^{n+1} there is a uniformly bounded number of simplices on which c is supported.

(iii) The simplices on which c is supported have uniformly bounded diameter.

Together with a suitable boundary operator, this forms a chain complex. The corresponding homology is denoted by $H_*^{\mathrm{uf}}(X; A)$.

Every quasi-isometric embedding $f\colon X \longrightarrow Y$ between two metric spaces (or, more generally, every *effectively proper lipschitz map* [7, Section 2]) induces a well-defined chain map $f_*\colon C^{\mathrm{uf}}_*(X;A) \longrightarrow C^{\mathrm{uf}}_*(Y;A)$. In this way, quasi-isometric embeddings which are at bounded distance between each other (or, more generally, *uniformly close effectively proper lipschitz maps* [7, Section 2]) induce the same map on uniformly finite homology.

Block and Weinberger characterized amenability for metric spaces of coarse bounded geometry in terms of uniformly finite homology. In particular, they proved [7, Theorem 3.1]:

Theorem (Block and Weinberger's characterization of amenability)**.** *Let X be a metric space of coarse bounded geometry. Then X is non-amenable if and only if $H^{\mathrm{uf}}_0(X;\mathbb{R}) = 0$.*

So, for example, for $k \geq 3$ the zero degree uniformly finite homology of the k-valent tree is trivial. The geometric intuition behind this phenomenon can be described as follows: any class in the zero degree uniformly finite homology of a tree is represented by an infinite sum of vertices with uniformly bounded coefficients; any vertex in the tree is the boundary of an infinite "tail" of edges; in the k-valent tree, for $k \geq 3$, there is enough branching to construct infinitely many disjoint tails and bound any infinite sum of vertices. This process is known as the *Eilenberg-swindle construction*.

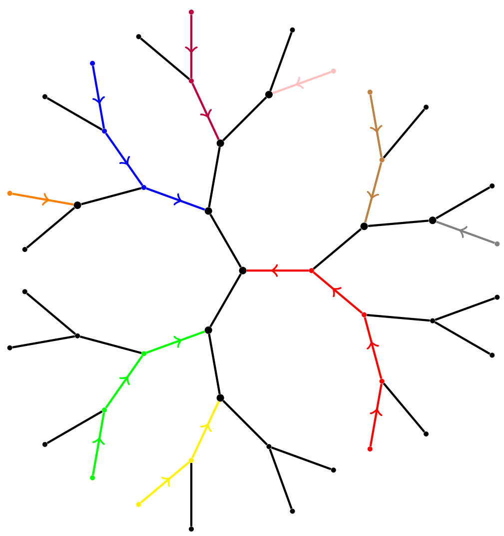

Figure 1: The Eilenberg-swindle construction in the 3-valent tree.

Block and Weinberger also provided some applications following this characterization of amenability. In particular, they constructed aperiodic tilings for certain non-amenable manifolds by using the coefficients of a 1-chain bounding a 0-cycle to "decorate" the tiles and produce an "unbalanced" set of tiles [7, Theorem 4.2]. They also used vanishing classes in zero degree to construct metrics of positive scalar curvature on certain manifolds [7, Proposition 5.1]. Other authors have used uniformly finite homology for a number of different applications. For instance, Whyte studied rigidity problems for uniformly discrete metric spaces of bounded geometry [43]. On the other hand, Dranishnikov developed a technique to detect if certain manifolds are "macroscopically small" using a comparison map between standard homology and uniformly finite homology [19, 20]. Except for the study of large-scale notions of dimension carried on by Dranishnikov who uses the comparison map in every degree, most of the applications are based on the fact that the zero degree uniformly finite homology group is trivial in the non-amenable case. In a survey on large-scale homology theories [8], Block and Weinberger analysed a few cases in higher degree. In particular, they presented a computation of uniformly finite homology for symmetric spaces, whose proof is based on an observation of Gromov [23, Example 0.1.C].

This lack of information on uniformly finite homology in higher degrees has motivated a further analysis in this direction. More precisely, the original questions that have inspired this thesis are:

- What is the geometric information contained in the uniformly finite homology groups in higher degrees?
- What are their inheritance properties and their algebraic structure?
- Do uniformly finite homology groups of higher degree have nice properties concerning the (non-)amenability of metric spaces?

Uniformly finite homology and homology of groups

Uniformly finite homology has also interesting connections with geometric group theory. Indeed, finitely generated groups endowed with the word metric with respect to some finite set of generators are metric spaces of coarse bounded geometry. In particular, Block and Weinberger's characterization of amenability also holds in the case of finitely generated groups, where the classical notion of amenability is considered. Moreover, there is a correspondence between uniformly finite homology and homology of groups. Indeed, for a finitely generated group G, the uniformly finite homology of G with coefficients in a unital ring A with norm is isomorphic to the standard homology of G with coefficients in the module $\ell^\infty(G, A)$ of bounded functions. This follows from an observation of

Brodsky, Niblo and Wright [11] and it represents an important result since it gives the possibility to understand uniformly finite homology using techniques coming from standard homological algebra. In view of this, in the case of amenable groups one can use invariant means to construct transfer maps from uniformly finite homology to standard homology. In joint work with Matthias Blank [6], we have used these transfer maps to detect non-trivial classes in the uniformly finite homology of amenable groups. More precisely, we have (Theorem 2.3.9):

Theorem (Uniformly finite homology of amenable groups)**.** *Let $n \in \mathbb{N}$ and let G be a finitely generated amenable group. Let $H \leq G$ be a subgroup such that $[G : H] = \infty$ and such that the inclusion $i\colon H \hookrightarrow G$ induces a non-trivial map $i_n\colon H_n(H;\mathbb{R}) \longrightarrow H_n(G;\mathbb{R})$. Then* $\dim_{\mathbb{R}}(H_n^{\mathrm{uf}}(G;\mathbb{R})) = \infty$.

The idea is to construct a family of infinitely many invariant means that can be distinguished by H-invariant bounded functions. A series of examples follows from this result and from more classical homological techniques. For instance, we can determine the uniformly finite homology of finitely generated nilpotent groups (Example 2.3.16):

Example (Uniformly finite homology of nilpotent groups)**.** Let G be a finitely generated virtually nilpotent group and let $h \in \mathbb{N}$ be its Hirsch rank. Then

$$H_k^{\mathrm{uf}}(G;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = h \\ \text{infinite dimensional} & \text{if } k \in \{0, \dots, h-1\} \\ 0 & \text{else.} \end{cases}$$

Uniformly finite homology and products

The most natural approach to the study of a new theory consists in comparing it to more classical ones. A natural question that arises when investigating uniformly finite homology is: how does uniformly finite homology behave with respect to products? In particular, is it possible to get information about the uniformly finite homology of a product of spaces or groups knowing the uniformly finite homology of the factors? Following the classical construction of the cross-product in simplicial homology, it is possible to define cross-product maps in the context of uniformly finite homology (Remark 3.1.2). Invariant means on amenable groups turn out to be useful also to examine this type of maps and, in some cases, to deduce their injectivity. As we have explained before, certain classes in uniformly finite homology of a group can be detected by means. On the other hand, there are also non-trivial *mean-invisible* classes, which are sent to zero by any invariant mean. For a finitely generated group G we denote by $\widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})$ the space of mean-invisible classes in $H_0^{\mathrm{uf}}(G;\mathbb{R})$. We have (Theorem 3.1.3):

Theorem (Injectivity of cross-product maps)**.** *Let G, H be finitely generated groups. Suppose H is amenable. Then for any $\alpha \in H_0^{\mathrm{uf}}(H;\mathbb{R})\backslash\widehat{H}_0^{\mathrm{uf}}(H;\mathbb{R})$ the cross-product*

$$-\times\alpha\colon H_*^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(G\times H;\mathbb{R})$$

is injective.

In the case of cross-product maps by mean-invisible classes, a counterexample to the injectivity is provided by Matthias Blank in his Ph.D. thesis [5] (Example 3.1.4). However, in general, deducing properties of cross-products is often a complicated task which cannot be solved with classical approaches. Indeed, in the classical construction of cross-product maps, one defines a bilinear map from the product of the homology groups to the homology group of the product and extends it to a map which takes values on the tensor products of the homology groups. However, since the uniformly finite chains are infinite sums of simplices, it is not clear how to deal with bilinear maps and in this setting one would need a "new" tensor product and its derived version. Using so-called *controlled* and *cocontrolled* tensor products, Hair proved a coarse version of the Künneth theorem for coarse cohomology [24]. In his Ph.D. thesis, Hair described the module of coarse (co)chains in terms of direct and inverse limits of modules having finite geometric properties. However, using the same description for uniformly finite chains by restricting to modules of finite chains, we loose information on the uniformly boundedness of the coefficients. So it is not clear how to use this approach in the case of uniformly finite homology. These difficulties have led us to choose a more geometric approach to the problem. In joint work with Piotr Nowak [16] we develop a geometric method for "killing" homology classes of products of spaces. The main idea is based on a generalization of the Eilenberg-swindle construction in higher dimensions. Indeed, in the original Eilenberg-swindle construction, classes in zero degree are "killed" by tails of 1-simplices attached to vertices of non-amenable simplicial complexes. Using an image conceived by Block and Weinberger, these tails can be seen as many "spaghetti" that "kill" any zero class [8]. In a similar way, using Eilenberg-swindle of 2-simplices ("tagliatelle") we have (Theorem 3.2.1):

Theorem (Vanishing of the uniformly finite homology in degree 1 for non-amenable products)**.** *Let $A \in \{\mathbb{R},\mathbb{Z}\}$ and let $X\times Y$ be the cartesian product of uniformly contractible, non-amenable simplicial complexes X, Y of bounded* *geometry. Then*

$$H_1^{\mathrm{uf}}(X \times Y; A) = 0.$$

By developing this construction further, it is possible to determine completely the uniformly finite homology of the product of three non-amenable trees. In particular, we have (Theorem 3.3.3):

Theorem (Uniformly finite homology of the cartesian product of trees)**.** *Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let $T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}$ be the cartesian product of uniformly locally finite, non-amenable trees $T_{\mathbf{x}}, T_{\mathbf{y}}, T_{\mathbf{z}}$. Then*

$$H_n^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A) = \begin{cases} \text{infinite dimensional} & \text{if } n = 3 \\ 0 & \text{otherwise.} \end{cases}$$

By using 3-dimensional Eilenberg-swindles ("spaghetti quadrati"), one can prove that certain special cycles in $C_2^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A)$ always bound (Lemma 3.3.9). Then, one "kills" every class in $H_2^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A)$ by finding special cycles as representatives (Lemma 3.3.11).

As an important application, we obtain a result for the uniformly finite homology of groups acting on products of trees (Corollary 3.3.2). A class of such groups is given by lattices in products of automorphisms groups of trees whose structure was studied by Burger and Mozes [13].

Corollary (Uniformly finite homology of groups acting on products of trees)**.** *Let G be a group acting by isometries on a product of two uniformly locally finite, non-amenable trees. Suppose the action is proper and cocompact. Then*

$$H_n^{\mathrm{uf}}(G; A) = \begin{cases} \text{infinite dimensional} & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

As another application, we present a characterization of amenability using uniformly finite homology in degree 1 (Theorem 3.2.12):

Theorem (Characterization of amenability in degree 1)**.** *Let G be a finitely generated group acting by isometries on a uniformly contractible simplicial complex of bounded geometry and let F_2 be the free group of rank 2. Suppose the action of G is proper and cocompact. Then G is non-amenable if and only if $H_1^{\mathrm{uf}}(G \times F_2; \mathbb{R}) = 0$.*

Whyte's rigidity criterion

We have already mentioned that uniformly finite homology was used to study rigidity problems for certain metric spaces. In particular, Whyte developed a criterion to distinguish between quasi-isometries and bilipschitz equivalences in the case of uniformly discrete metric spaces of bounded geometry (UDBG-spaces) [43, Theorem 1.1]:

Theorem (Whyte's rigidity criterion)**.** *Let $f\colon X \longrightarrow Y$ be a quasi-isometry between* UDBG*-spaces and let $f_0\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ be the induced map. Let $[X] = \left[\sum_{x\in X} x\right] \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ and $[Y] = \left[\sum_{y\in Y} y\right] \in H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ be the fundamental classes of X and Y respectively. Then there is a bilipschitz map at bounded distance from f if and only if $f_0([X]) = [Y]$.*

This theorem, together with Block and Weinberger's characterization of amenability, provides a proof of the fact that any quasi-isometry between non-amenable UDBG-spaces is at bounded distance from a bilipschitz equivalence [43]. Several authors worked on this problem using other methods [9, 36]. On the other hand, as observed by Dymarz, the inclusion of any proper subgroup of finite index $H \hookrightarrow G$ in a finitely generated amenable group G is not at bounded distance from a bilipschitz equivalence [17].

Rigidity criterion using semi-norms

We derive another criterion to distinguish between quasi-isometries and bilipschitz equivalences using norms on uniformly finite homology. We consider the natural supremum norm on uniformly finite chains and we have (Theorem 4.6.3):

Theorem (Rigidity criterion using semi-norms in uniformly finite homology)**.** *Let $f\colon X \longrightarrow Y$ be a quasi-isometry between* UDBG*-spaces. Then f is at bounded distance from a bilipschitz equivalence if and only if the induced map $f_0\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ is an isometric isomorphism.*

The original motivation to introduce norms on the uniformly finite chain complex was to provide an analytic tool to "measure" homology classes. For instance, as we have already seen, in some cases uniformly finite homology is infinite dimensional but we do not have much information about its "size". It turns out that in many cases uniformly finite homology classes have trivial supremum semi-norm. For example, for metric spaces having no isolated points and for non-amenable UDBG-spaces the supremum semi-norm on uniformly finite homology is always zero (Proposition 4.2.3 and Proposition 4.3.4). Moreover, for amenable groups the vanishing of the supremum semi-norm in higher degrees is a consequence of the vanishing of ℓ^1-homology [29] (Proposition 4.4.2). So, in these cases, one might need to introduce another tool (another norm?) to measure the size of certain classes in uniformly finite homology. On the other hand, we can find key classes in the zero degree uniformly finite homology of amenable UDBG-spaces having non-trivial semi-norm (Lemma 4.2.2) and in the case of amenable groups we can completely classify them using invariant means (Proposition 4.4.1).

Organization of the work

The thesis is structured in the following way: In Chapter 1, we summarize the necessary background in uniformly finite homology, in particular, we introduce the main objects of study, namely the *Block-Weinberger uniformly finite homology* for metric spaces and the *simplicial uniformly finite homology* for simplicial complexes. By endowing simplicial complexes with a suitable metric, we will see that in the uniformly contractible case the Block-Weinberger and the simplicial uniformly finite homology are equivalent. This fact is proved in detail in Appendix B. In Section 1.5, we provide some examples, in particular a complete computation of uniformly finite homology for non-amenable trees. In Section 1.7, we give another definition of uniformly finite homology using Rips complexes (*Rips uniformly finite homology*) and in Section 1.8, we prove that the Rips uniformly finite homology is equivalent to the Block-Weinberger uniformly finite homology. In Section 1.4, we introduce the Eilenberg-swindle construction providing a sketch of the proof of Block and Weinberger's characterization of amenability. In Section 1.6, we present Whyte's rigidity criterion.

In Chapter 2, we recall standard homology of groups and we state that uniformly finite homology for finitely generated groups is isomorphic to standard homology with coefficients in the module of bounded functions (Proposition 2.2.4). A detailed proof of this fact can be found in Appendix A. In Section 2.3, we state and prove the main results concerning uniformly finite homology of amenable groups in degree zero (Section 2.3.1) and in higher degrees (Section 2.3.2). In Section 2.3.3, we provide some examples.

In Chapter 3, we consider uniformly finite homology for products of spaces and groups. In Section 3.1, we prove the injectivity of some cross-product maps. In Section 3.2, we prove the vanishing of the uniformly finite homology in degree 1 for products of non-amenable simplicial complexes. In Section 3.3, we compute the uniformly finite homology of products of two or three non-amenable trees and of groups acting on such products. We also provide an idea for a generalization of our results for products of n-trees, for $n > 3$ (Conjecture 3.3.12). In Section 3.4, we present a conjecture for a vanishing Künneth theorem for uniformly finite homology.

In Chapter 4, we introduce the supremum norm on uniformly finite chains and the corresponding semi-norm in homology. In Section 4.2, 4.3 and 4.4, we compute the supremum semi-norm in many cases and we analyse the behavior of the semi-norm with respect to the various comparison maps seen in Chapter 1. In Section 4.5, we give a proof of the rigidity criterion for finitely generated groups using semi-norms and in

Section 4.6, we give a more general proof for UDBG-spaces which was suggested by Clara Löh. Both results are based on Whyte's rigidity criterion.

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Contents

1 Uniformly finite homology of spaces **1**
- 1.1 Block-Weinberger uniformly finite homology 2
- 1.2 Simplicial uniformly finite homology 4
  - 1.2.1 Simplicial maps of bounded geometry 5
  - 1.2.2 Simplicial uniformly finite chain complex 6
- 1.3 Equivalence between Block-Weinberger and simplicial uniformly finite homology 7
- 1.4 Characterization of amenability 8
  - 1.4.1 Eilenberg-swindle construction 10
- 1.5 Examples 12
  - 1.5.1 Uniformly finite homology of Euclidean space 12
  - 1.5.2 Uniformly finite homology of a tree 15
- 1.6 Whyte's rigidity result 21
  - 1.6.1 Definition of UDBG-spaces 21
  - 1.6.2 Rigidity of UDBG-spaces 22
- 1.7 Rips uniformly finite homology 23
- 1.8 Equivalence of definitions 27

2 Uniformly finite homology of groups **29**
- 2.1 The word metric 29
- 2.2 Definition of homology of groups 30
- 2.3 Uniformly finite homology and amenable groups 32
  - 2.3.1 Zero degree uniformly finite homology 33
  - 2.3.2 Higher degree uniformly finite homology 36
  - 2.3.3 Examples 39

3 Uniformly finite homology of products **43**
- 3.1 Cross-product and transfer maps 44
- 3.2 Products of non-amenable simplicial complexes 46
  - 3.2.1 Simplicial structures on products 47
  - 3.2.2 Proof of Theorem 3.2.1 49
  - 3.2.3 Characterization of amenability in degree 1 54

3.3 Products of trees 55
3.3.1 Uniformly finite homology of the product of two trees 55
3.3.2 Uniformly finite homology of the product of three trees 57
3.4 The Künneth-vanishing-Conjecture 72

4 Semi-norm on uniformly finite homology 73
4.1 Norm on chain complexes 74
4.2 Semi-norms in the case of metric spaces 75
4.3 Semi-norms in the case of UDBG-spaces 79
4.3.1 Isometries for UDBG-spaces 79
4.3.2 Semi-norms for non-amenable UDBG-spaces 81
4.4 Semi-norms for amenable groups 81
4.5 Rigidity for finitely generated groups via semi-norms 83
4.6 Rigidity for UDBG-spaces via semi-norms 96

Appendices 99

A Uniformly finite homology and homology with ℓ^∞-coefficients 101
A.1 Uniformly finite homology defined using small simplices . 101
A.2 Chain isomorphism 102

B Uniformly finite homology and simplicial uniformly finite homology 105

Chapter 1

Uniformly finite homology of spaces

In this chapter, we present the main object of investigation, namely, uniformly finite homology. We start with the uniformly finite chain complex introduced by Block and Weinberger for any metric space [7]. Then we pass to simplicial complexes having bounded geometry and we define a simplicial version of uniformly finite homology, called the *simplicial uniformly finite homology* [1, 33]. In the case of uniformly contractible simplicial complexes, the two chain complexes are chain homotopy equivalent (Proposition 1.3.3). In Section 1.4, we introduce the relation between uniformly finite homology and amenability for metric spaces of coarse bounded geometry and in Section 1.5, we give some examples. In Section 1.6, we give another application of uniformly finite homology due to Whyte [43] regarding rigidity properties of uniformly discrete metric spaces of bounded geometry. In the last section, we define uniformly finite homology using Rips complexes of discrete metric spaces. This second approach is used by Whyte in the case of uniformly discrete metric spaces of bounded geometry [43] and by Mosher for simplicial complexes of bounded geometry [33]. We construct Rips complexes of quasi-lattices inside metric spaces of coarse bounded geometry: in this case the two approaches to uniformly finite homology are equivalent, more precisely, there is an isomorphism between the Block-Weinberger uniformly finite homology and the Rips uniformly finite homology (Corollary 1.8.2). In Chapter 4, we will investigate this equivalence further focusing on its behaviour with respect to semi-norms of homology classes.

In the following chapters, we will often follow the Block and Weinberger approach to homology. When we are not concerned with the choice of chain complex, we will refer to it simply as the uniformly finite homology.

1.1 Block-Weinberger uniformly finite homology

In this section, we define the uniformly finite homology of a metric space following the approach used by Block and Weinberger [7]. We start by recalling the definition of quasi-isometries and bilipschitz equivalences between metric spaces.

Definition 1.1.1. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f\colon X \longrightarrow Y$ be a map between them.

- Let $C, D \in \mathbb{R}_{>0}$. The map f is a (C, D)-*quasi-isometric embedding* if

$$\forall_{x,x' \in X} \quad \frac{1}{C} \cdot d_X(x, x') - D \leq d_Y(f(x), f(x')) \leq C \cdot d_X(x, x') + D.$$

- A map $f'\colon X \longrightarrow Y$ is at *bounded distance* from f if there exists a constant $K \in \mathbb{R}_{\geq 0}$ such that

$$\forall_{x \in X} \quad d_Y(f(x), f'(x)) \leq K.$$

- The map f is a (C, D)-*quasi-isometry* if f is a (C, D)-quasi-isometric embedding for some $C, D \in \mathbb{R}_{>0}$ and if there is a (C', D')-quasi-isometric embedding $g\colon Y \longrightarrow X$ for some $C', D' \in \mathbb{R}_{>0}$ such that $f \circ g$ is at bounded distance from id_Y and $g \circ f$ is at bounded distance from id_X. The map g is called a *quasi-inverse* of f.

When we are not concerned with the value of C and D we say that f is a *quasi-isometric embedding* (resp. *quasi-isometry*).

Definition 1.1.2. Let $f\colon X \longrightarrow Y$ be a map between metric spaces (X, d_X) and (Y, d_Y).

- The map f is a *bilipschitz embedding* if there is a $C \in \mathbb{R}_{>0}$ such that

$$\forall_{x,x' \in X} \quad \frac{1}{C} \cdot d_X(x, x') \leq d_Y(f(x), f(x')) \leq C \cdot d_X(x, x').$$

- The map f is a *bilipschitz equivalence* if f is a bilipschitz embedding and if there is a bilipschitz embedding $g\colon Y \longrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Example 1.1.3. Every bilipschitz equivalence is clearly a quasi-isometry but the converse does not hold in general. Consider $\mathbb{R}$ endowed with the standard metric and $\mathbb{Z}$ with the induced metric. Then the inclusion $i\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometric embedding. One can easily prove that $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric but not bilipschitz equivalent (there cannot be a bijection between the two spaces). On the other hand, multiplication by 2 is a bilipschitz equivalence between $\mathbb{Z}$ and $2\mathbb{Z}$.

Let A be a unital ring with norm

$$|-|\colon A \longrightarrow \mathbb{R}_{\geq 0} \tag{1.1}$$

satisfying the following conditions:

(i) For any $a \in A$ we have $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$.

(ii) For any $a, a' \in A$ we have $|a + a'| \leq |a| + |a'|$.

(iii) For any $a, a' \in A$ we have $|a \cdot a'| = |a| \cdot |a'|$.

For any metric space (X, d) and for any $n \in \mathbb{N}$ consider the cartesian product X^{n+1} endowed with the *maximum metric*:

$$\forall_{\overline{x}=(x_0,\dots,x_n)\in X^{n+1}, \overline{y}=(y_0,\dots,y_n)\in Y^{n+1}} \quad d(\overline{x}, \overline{y}) = \max_{i\in\{0,\dots,n\}} d(x_i, y_i). \tag{1.2}$$

Definition 1.1.4. Let (X, d) be a metric space.

(i) For each $n \in \mathbb{N}$ denote by $C_n^{\mathrm{uf}}(X; A)$ be the A-module of functions $c\colon X^{n+1} \longrightarrow A$ satisfying:

 (a) There exists a constant $K_c \in \mathbb{R}_{>0}$ (depending on c) such that

 $$\forall_{\overline{x}\in X^{n+1}} \quad |c(\overline{x})| \leq K_c.$$

 In other words, the map c is bounded.

 (b) For all $r \in \mathbb{R}_{>0}$ there exists a constant $N_{r,c} \in \mathbb{R}_{>0}$ (depending on r and on c) such that for all $\overline{y} \in X^{n+1}$

 $$\big|\{\overline{x} \in B_r(\overline{y}) \mid c(\overline{x}) \neq 0\}\big| \leq N_{r,c}.$$

 (c) There exists a constant $R_c \in \mathbb{R}_{>0}$ (depending on c) such that

 $$\forall_{\overline{x}=(x_0,\dots,x_n)\in X^{n+1}} \qquad \sup_{i,j\in\{0,\dots,n\}} d(x_i, x_j) > R_c \Longrightarrow c(\overline{x}) = 0.$$

 We will write such functions as formal sums $c = \sum_{\overline{x}\in X^{n+1}} c(\overline{x}) \cdot \overline{x}$ and we will denote by $\operatorname{supp}(c)$ its support, i.e, the set of tuples $\overline{x} \in X^{n+1}$ for which $c(\overline{x}) \neq 0$.

(ii) Define for each $n \in \mathbb{N}_{>0}$ a boundary operator

$$\partial_n \colon C_n^{\mathrm{uf}}(X;A) \longrightarrow C_{n-1}^{\mathrm{uf}}(X;A)$$

by setting for each $\overline{x} \in X^{n+1}$

$$\partial_n(\overline{x}) = \sum_{j=0}^{n} (-1)^j (x_0, \dots, \widehat{x}_j, \dots, x_n).$$

and extending to chains in $C_n^{\mathrm{uf}}(X;A)$ in the obvious way. Clearly, for any $n \in \mathbb{N}$ we have $\partial_n \circ \partial_{n+1} = 0$. An easy computation shows that for any $c \in C_n^{\mathrm{uf}}(X;A)$ the chain $\partial_n(c)$ satisfies conditions (i)-(a),(i)-(b) and (i)-(c). In particular, the boundary map ∂_* is well-defined. In this way we get indeed a chain complex.

(iii) The homology of $(C_n^{\mathrm{uf}}(X;A), \partial_n)_{n\in\mathbb{N}}$ is called the *Block-Weinberger uniformly finite homology of X* and it is denoted by $H_*^{\mathrm{uf}}(X;A)$.

The next proposition is due to Block and Weinberger and it shows that uniformly finite homology is a quasi-isometry invariant.

Proposition 1.1.5 ([7, Proposition 2.1])**.** *Let X and Y be metric spaces. Then*

(i) Any quasi-isometric embedding $f \colon X \longrightarrow Y$ induces a chain map

$$\begin{aligned} C_n^{\mathrm{uf}}(X;A) &\longrightarrow C_n^{\mathrm{uf}}(Y;A) \\ \sum_{\overline{x}\in X^{n+1}} c(\overline{x}) \cdot \overline{x} &\longmapsto \sum_{\overline{x}\in X^{n+1}} c(\overline{x}) \cdot (f(x_0), \dots, f(x_n)). \end{aligned}$$

(ii) If $f, g \colon X \longrightarrow Y$ are quasi-isometric embedding at bounded distance from each other then $f_*, g_* \colon C_*^{\mathrm{uf}}(X;A) \longrightarrow C_*^{\mathrm{uf}}(Y;A)$ are chain homotopic and, consequently, $f_* = g_*$ in homology. In particular, any quasi-isometry induces an isomorphism in uniformly finite homology.

Remark 1.1.6. *Block and Weinberger prove the invariance for a more general class of maps between metric spaces, namely, effectively proper lipschitz maps [7, Section 2]: these are just coarse maps in the sense of Roe [39, Definition 1.8] with the additional property that the inverse image of any bounded set is uniformly bounded.*

1.2 Simplicial uniformly finite homology

In this section, we focus on simplicial complexes and we introduce the simplicial uniformly finite homology. This serves as a useful variant of the Block-Weinberger uniformly finite homology in many situations. As we have seen in Definition 1.1.4, in the Block-Weinberger approach one considers "abstract" simplices, namely tuples of points having uniformly bounded diameter. On the other hand, in the simplicial version one considers only simplices coming from the simplicial structure of a space. The simplicial uniformly finite homology is considered by many authors: Attie [1] calls it the "fine" uniformly finite homology and denotes it by H_*^{uff}, while Mosher [33] refers to it as the simplicial uniformly finite homology. There is also a notion of simplicial uniformly finite cohomology for simplicial complexes of bounded geometry which is defined using bounded cochains or bounded differential forms (de Rham cohomology) [2, 3, 8].

1.2.1 Simplicial maps of bounded geometry

We start by defining simplicial complexes and simplicial maps of bounded geometry.

Definition 1.2.1. A simplicial complex X is of *bounded geometry* if there is a finite uniform bound on the number of simplices contained in the link of any vertex of X.

Let X be a simplicial complex. We denote by V_X the set of its vertices and for any $n \in \mathbb{N}$, we denote by $\Delta_n(X)$ the set of the n-simplices of X.

Definition 1.2.2. Let X, Y be simplicial complexes of bounded geometry.

- A simplicial map $f\colon X \longrightarrow Y$ is of *bounded geometry* if there exists $N \in \mathbb{N}_{>0}$ such that

  $$\forall_{n\in\mathbb{N}} \quad \forall_{\sigma\in\Delta_n(Y)} \quad |f^{-1}(\sigma)| < N.$$

  In other words a simplicial map is of bounded geometry if the inverse image of any simplex contains a uniformly bounded number of simplices.

- Let $f_0, f_1\colon X \longrightarrow Y$ be two simplicial maps of bounded geometry. A *homotopy of bounded geometry between f_0 and f_1* is a simplicial map of bounded geometry $F\colon X \times [0,1] \longrightarrow Y$ such that $F_{|_{X\times\{0\}}} = f_0$ and $F_{|_{X\times\{1\}}} = f_1$. In this case we say that the maps f_0, f_1 are *bg-homotopic*.

- A simplicial map $f\colon X \longrightarrow Y$ is a *homotopy equivalence of bounded geometry* if f has bounded geometry and if there exists a simplicial map of bounded geometry $g\colon Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are bg-homotopic to id_Y and to id_X respectively.

1.2.2 Simplicial uniformly finite chain complex

In this section, we define the simplicial uniformly finite chain complex of a simplicial complex having bounded geometry. We want to consider a simplicial complex X endowed with a binary relation on V_X in such a way that any simplex in X is a totally ordered set of vertices. More precisely:

Definition 1.2.3. A simplicial complex is *ordered* if there is a binary relation $\leq$ on V_X satisfying the following conditions:

(i) If $x \leq x'$ and $x' \leq x$ then $x = x'$.

(ii) Two elements x, x' of V_X are vertices of a given simplex in X if and only if $x \leq x'$ or $x' \leq x$.

(iii) If $x, x', x'' \in V_X$ are vertices of a given simplex in X and if $x \leq x'$ and $x' \leq x''$, then $x \leq x''$.

Let X be an ordered simplicial complex. Then, for any $n \in \mathbb{N}$ the set $\Delta_n(X)$ is just the set of tuples $[x_0, \ldots, x_n]$ such that $x_0 \leq \cdots \leq x_n$ in V_X.

Definition 1.2.4. Let $n \in \mathbb{N}$ and let X be an ordered simplicial complex. An simplex $\sigma = [x_0, \ldots, x_n] \in \Delta_n(X)$ is called *degenerate* if the vertices $x_0 \leq \cdots \leq x_n$ are not all distinct.

We consider A to be a unital ring with norm as in (1.1) on page 3.

Definition 1.2.5. Let X be an ordered simplicial complex of bounded geometry.

(i) For each $n \in \mathbb{N}$ define $C_n^{\mathrm{suf}}(X; A)$ to be the A-module of bounded functions $c\colon \Delta_n(X) \longrightarrow A$ that vanish on degenerate simplices.

We will write such functions as formal sums $c = \sum_{\sigma \in \Delta_n(X)} c(\sigma) \cdot \sigma$ and we will denote by $\mathrm{supp}(c)$ its support, i.e, the set of simplices $\sigma \in \Delta_n(X)$ for which $c(\sigma) \neq 0$.

(ii) Define for each $n \in \mathbb{N}_{>0}$ a boundary operator

$$\partial_n\colon C_n^{\mathrm{suf}}(X; A) \longrightarrow C_{n-1}^{\mathrm{suf}}(X; A)$$

by setting for each $\sigma = [x_0, \ldots, x_n] \in \Delta_n(X)$

$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j [x_0, \ldots, \widehat{x}_j, \ldots, x_n].$$

and extending to any chain in $C_n^{\mathrm{suf}}(X; A)$ in the obvious way as in Definition 1.1.4. In this fashion we get indeed a chain complex.

(iii) The homology of $(C_n^{\mathrm{suf}}(X;A), \partial_n)_{n\in\mathbb{N}}$ is called the *simplicial uniformly finite homology of* X and it is denoted by $H_*^{\mathrm{suf}}(X;A)$.

Homotopy equivalent simplicial complexes of bounded geometry have the same simplicial uniformly finite homology as the next proposition shows.

Proposition 1.2.6 ([1, Proposition 2.4]). *Let X and Y be ordered simplicial complexes with bounded geometry and let $f\colon X \longrightarrow Y$ be a simplicial map of bounded geometry. Then f induces a chain map*

$$\begin{aligned} C_n^{\mathrm{suf}}(X;A) &\longrightarrow C_n^{\mathrm{suf}}(Y;A) \\ \sum_{\sigma\in\Delta_n(X)} c(\sigma)\cdot\sigma &\longmapsto \sum_{\sigma\in\Delta_n(X)} c(\sigma)\cdot f(\sigma). \end{aligned}$$

If f is a homotopy equivalence of bounded geometry then the induced map in homology

$$f_*\colon H_*^{\mathrm{suf}}(X;A) \longrightarrow H_*^{\mathrm{suf}}(Y;A)$$

is an isomorphism.

1.3 Equivalence between Block-Weinberger and simplicial uniformly finite homology

In this section, we state the equivalence between Block-Weinberger uniformly finite homology and simplicial uniformly finite homology for uniformly contractible simplicial complexes of bounded geometry. Following Mosher [33], we will give a detailed proof of this result in Appendix B.

We consider a simplicial complex X endowed with the *ℓ^1-path metric*, i.e. the ℓ^1-metric applied to the barycentric coordinates on every simplex and extended by paths [10, Chapter I.7-A.9]. We rescale the metric on every simplex so that each edge has length 1. For any subset $A \subset X$ and for any $r \in \mathbb{R}_{>0}$ we consider:

$$N_r(A) = \left\{ x \in X \;\middle|\; d(x,A) \leq r \right\}.$$

Definition 1.3.1. A simplicial complex X of bounded geometry is *uniformly contractible* if for any $r \in \mathbb{R}_{>0}$ there exists $S_r \in \mathbb{R}_{>0}$ such that any set $A \subset X$ of diameter $diam(A) < r$ is contractible to a point inside $N_{S_r}(A)$.

Example 1.3.2. A uniformly locally finite tree is a uniformly contractible simplicial complex of bounded geometry. The universal cover of any compact, aspherical simplicial complex has bounded geometry and it is uniformly contractible.

Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let X be a (ordered) simplicial complex of bounded geometry endowed with the ℓ^1-path metric. For any $n \in \mathbb{N}$ a simplex $[x_0, \ldots, x_n] \in \Delta_n(X)$ can be viewed as an element of the cartesian product V_X^n. In particular, there is a natural map $i_*\colon H_*^{\mathrm{suf}}(X;A) \longrightarrow H_*^{\mathrm{uf}}(X;A)$ induced by the inclusion of simplices (Appendix B). We have the following:

Proposition 1.3.3. *Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let X be a uniformly contractible simplicial complex of bounded geometry. Then the map above induces an isomorphism*

$$H_*^{\mathrm{uf}}(X;A) \cong H_*^{\mathrm{suf}}(X;A).$$

Proof. See Appendix B for a proof. $\square$

Remark 1.3.4. *In the case of coarse homology for a metric space X, one can define a coarsening map $c\colon H_*(X) \longrightarrow HX_*(X)$ where H_* is a generalized homology theory for locally compact topological spaces and HX_* is the corresponding coarse homology [35, Section 7.6]. In the case of uniformly contractible metric spaces of bounded geometry the coarsening map is an isomorphism [35, Theorem 7.6.2].*

1.4 Characterization of amenability

One of the most interesting applications of uniformly finite homology is a characterization of amenability for spaces of bounded geometry. The following result is due to Block and Weinberger [7, Theorem 3.1]:

Theorem 1.4.1. *Let X be a metric space of coarse bounded geometry. Then X is non-amenable if and only if $H_0^{\mathrm{uf}}(X;\mathbb{R}) = 0$.*

In this section, we introduce metric spaces of coarse bounded geometry and we define the notion of amenability for metric spaces. We consider a more general result than Theorem 1.4.1: we do not give the detailed proof, but we describe a strategy that allows to "kill" any class in degree zero in any non-amenable space.

Definition 1.4.2. A metric space (X, d) is of *coarse bounded geometry* if it contains a subset $\Gamma \subseteq X$ satisfying the following conditions:

(i) The subset $\Gamma \subseteq X$ is coarsely dense in X, i.e. there exists some $a > 0$ such that

$$\forall_{x \in X} \quad d(x, \Gamma) \leq a.$$

(ii) For all $r \in \mathbb{R}_{>0}$ there exists $K_r > 0$ such that

$$\forall_{x \in X} \quad |B_r(x) \cap \Gamma| \leq K_r$$

where $B_r(x)$ denotes the ball in X having radius r and centered in $x \in X$.

A subset $\Gamma \subseteq X$ with these properties is said to be a *quasi-lattice* in X.

Example 1.4.3. A simplicial complex of bounded geometry X endowed with the ℓ^1-path metric is a metric space of coarse bounded geometry having $V_X \subset X$ as quasi-lattice. A Riemannian manifold is of bounded geometry if and only if it can be triangulated by a simplicial complex of bounded geometry [1, Theorem 1.14].

For any subset $S \subset X$ of a metric space of coarse bounded geometry we consider its boundary:

Definition 1.4.4. Let $S \subset X$ be a subset in a metric space X of coarse bounded geometry. For any $r \in \mathbb{R}_{>0}$, we define the *r-boundary of S* as

$$\partial_r(S) = \left\{ x \in X \;\middle|\; 0 < d(x,S) \leq r \right\}. \tag{1.3}$$

We have the following notion of amenability for metric spaces of coarse bounded geometry:

Definition 1.4.5. A metric space of coarse bounded geometry is *amenable* if it contains a quasi-lattice Γ admitting a sequence $(F_i)_{i\in I}$ of non-empty finite subsets $F_i \subset \Gamma$ such that:

$$\forall_{r\in\mathbb{R}_{>0}} \quad \lim_{i\to\infty} \frac{|\partial_r(F_i)|}{|F_i|} = 0.$$

A sequence $(F_i)_{i\in I}$ satisfying this property is called *Følner sequence*.

Any finitely generated group endowed with the word metric is a metric space of coarse bounded geometry. In this case, Definition 1.4.5 is equivalent to the classical definition of amenability for groups using Følner sequences [14, Definition 4.7.2]. In Chapter 2, we will consider another definition of amenability for groups which uses invariant means on the space of bounded functions on the group (Definition 2.3.1). The two definitions are equivalent [14, Theorem 4.9.2].

The following result is due to Block and Weinberger and it is a more general version of Theorem 1.4.1. In particular, it states that for a metric space of coarse bounded geometry X if one "positive" class in $H_0^{\mathrm{uf}}(X;\mathbb{R})$ is trivial, then the whole group vanishes (and the space is necessary non-amenable).

Proposition 1.4.6 ([7, Proposition 2.3])**.** *Let X be a space of coarse bounded geometry and let $\Gamma \subseteq X$ be a quasi-lattice. The following are equivalent:*

(i) $H_0^{\mathrm{uf}}(X;\mathbb{R}) = 0$.

(ii) $H_0^{\mathrm{uf}}(X;\mathbb{Z}) = 0$.

(iii) *There exists a cycle* $a = \sum_{\gamma\in\Gamma} a(\gamma)\cdot\gamma \in C_0^{\mathrm{uf}}(X;\mathbb{R})$ *such that*

$$\exists_{\epsilon>0} \quad \forall_{\gamma\in\Gamma} \quad a(\gamma) \geq \epsilon$$

and such that $[a] = 0$ *in* $H_0^{\mathrm{uf}}(X;\mathbb{R})$.

(iv) *There exists a cycle* $a = \sum_{\gamma\in\Gamma} a(\gamma)\cdot\gamma \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ *such that*

$$\forall_{\gamma\in\Gamma} \quad a(\gamma) > 0$$

and such that $[a] = 0$ *in* $H_0^{\mathrm{uf}}(X;\mathbb{Z})$.

(v) X *is non-amenable.*

1.4.1 Eilenberg-swindle construction

In Proposition 1.4.6 (in particular, the implication $(iv) \Rightarrow (ii)$) we have seen that, for a space of coarse bounded geometry X, if a certain class is trivial in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$ then the whole group $H_0^{\mathrm{uf}}(X;\mathbb{Z})$ vanishes. In this section, we give the idea behind the proof of this fact, in particular we present a method to "kill" classes in the zero degree uniformly finite homology of a non-amenable space:

Let X be a metric space of coarse bounded geometry and let $\Gamma \subseteq X$ be a quasi-lattice in X. Let $[a] \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ be a class represented by a cycle $\sum_{\gamma\in\Gamma} a(\gamma)\cdot\gamma$ such that for any $\gamma\in\Gamma$ we have $a(\gamma)\in\mathbb{Z}_{>0}$. Suppose $[a] = 0$. Then there exists a $\psi \in C_1^{\mathrm{uf}}(X;\mathbb{Z})$ such that $\partial\psi = a$. It is possible to "rearrange" the elements of $\mathrm{supp}(\psi)$ to get another chain in $C_1^{\mathrm{uf}}(X;\mathbb{Z})$ with which we can bound any cycle in $C_0^{\mathrm{uf}}(X;\mathbb{Z})$ [7, proof of Lemma 2.4]. More precisely, for any $\gamma\in\Gamma$ we can find a sequence of points $\{\gamma_j\}_{j\in\mathbb{Z}_{\leq 0}}$ such that $\gamma_0 = \gamma$ and such that

$$\forall_{j\in\mathbb{Z}_{\leq 0}} \quad (\gamma_{j-1},\gamma_j) \in \mathrm{supp}(\psi).$$

For any $\gamma\in\Gamma$, define a *tail attached to* $\gamma\in\Gamma$ as follows:

$$t_\gamma = \sum_{j\in\mathbb{Z}_{\leq 0}} (\gamma_{j-1},\gamma_j). \tag{1.4}$$

Clearly, $t_\gamma \in C_1^{\mathrm{uf}}(X;\mathbb{Z})$ and $\partial t_\gamma = \gamma$ (Figure 1.1). With an induction argument on the maximum of the coefficients of the chain a, it is possible to construct a tail for any $\gamma\in\Gamma$ in such a way that for any simplex in $\mathrm{supp}(\psi)$ there is a uniformly bounded number of tails passing through it [7, proof of Lemma 2.4]. In other words, it is possible to choose a tail t_γ for any $\gamma\in\Gamma$ such that

$$\exists_{K>0} \quad \forall_{(\gamma',\gamma'')\in\Gamma^2} \quad |E(\gamma',\gamma'')| \leq K \tag{1.5}$$

where $E(\gamma', \gamma'') := \{\gamma \in \Gamma \mid t_\gamma \text{ passes through } (\gamma', \gamma'')\}$. In this way we have that $\sum_{\gamma \in \Gamma} t_\gamma$ is a well-defined chain in $C_1^{\mathrm{uf}}(X; \mathbb{Z})$ and for any $b = \sum_{\gamma \in \Gamma} b(\gamma) \cdot \gamma \in C_0^{\mathrm{uf}}(X; \mathbb{Z})$ we have

$$\partial \left(\sum_{\gamma \in \Gamma} b(\gamma) \cdot t_\gamma \right) = b.$$

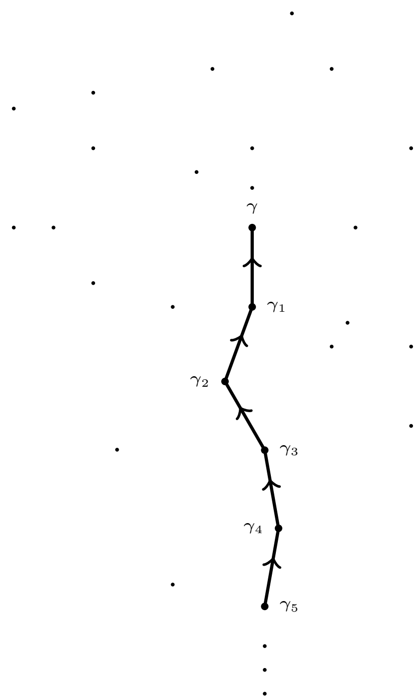


Figure 1.1: A tail attached to the point γ.

It follows that $H_0^{\mathrm{uf}}(X; \mathbb{Z}) = 0$. This procedure is known as the *Eilenberg-swindle construction*. Clearly one can use the tails also to "kill" every class in $H_0^{\mathrm{uf}}(X; \mathbb{R})$. By Proposition 1.4.6, this construction is possible if and only if X is non-amenable.

The idea behind this is that in a non-amenable space there is enough "branching" that allows to construct infinitely many tails which do not intersect too much between each other (Figure 1.2). In the case of non-amenable simplicial complexes of bounded geometry, one can construct tails of 1-simplices attached to the vertices. More precisely, for an ordered simplicial complex of bounded geometry X and for any vertex $x \in V_X$ we can find a tail

$$t_x = \sum_{j \in \mathbb{Z}_{\leq 0}} (-1)^{\epsilon_j} e_j \tag{1.6}$$

where, for any $j \in \mathbb{Z}_{\leq 0}$ we have

$$\epsilon_j = \begin{cases} 0 & \text{if } x_{j-1} < x_j \text{ and } e_j = [x_{j-1}, x_j] \in \Delta_1(X) \\ 1 & \text{if } x_{j-1} > x_j \text{ and } e_j = [x_j, x_{j-1}] \in \Delta_1(X) \end{cases}$$

and for $j = 0$ we have $x_0 = x$.

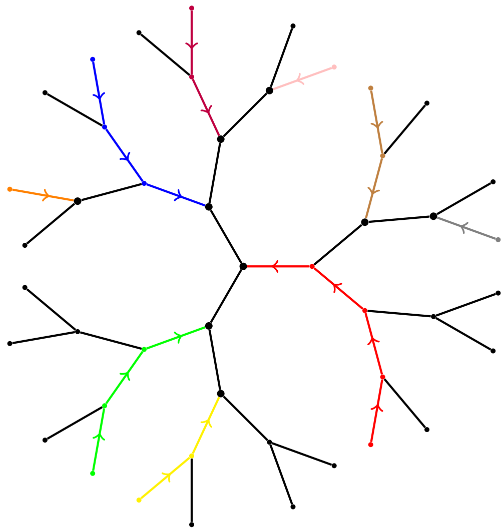


Figure 1.2: The Eilenberg-swindle construction in the 3-valent tree.

As in the case seen above for uniformly finite homology, if X is non-amenable we can use these tails to construct chains in $C_1^{\mathrm{suf}}(X; \mathbb{R})$ which bound cycles in $C_0^{\mathrm{suf}}(X; \mathbb{R})$. In this way, we can prove that any class in $H_0^{\mathrm{suf}}(X; \mathbb{R})$ vanishes.

1.5 Examples

In Section 1.4, we have seen the vanishing of uniformly finite homology in degree zero for non-amenable spaces of coarse bounded geometry. In this section, we want to see some other examples to give a taste of what one could expect from uniformly finite homology. We will see that, in general, uniformly finite homology is very different from standard homology.

1.5.1 Uniformly finite homology of Euclidean space

Consider $\mathbb{R}$ as a metric space of coarse bounded geometry equipped with the standard metric and having $\mathbb{Z} \subset \mathbb{R}$ as a quasi-lattice. Then one could try to "kill" any class in $H_0^{\mathrm{uf}}(\mathbb{R}; \mathbb{R})$ by attaching a tail of edges to any vertex in $\mathbb{Z}$, as we did in the previous section for non-amenable spaces. However, there is not enough space in $\mathbb{Z}$ to ensure the uniform boundedness of the coefficients of these tails (Figure 1.3). On the other hand, in the case of coarse homology, since there is no boundedness condition on the coefficients of the chains, we have $HX_0(\mathbb{Z};\mathbb{R}) = 0$ [39, Example, Chapter 2.2].

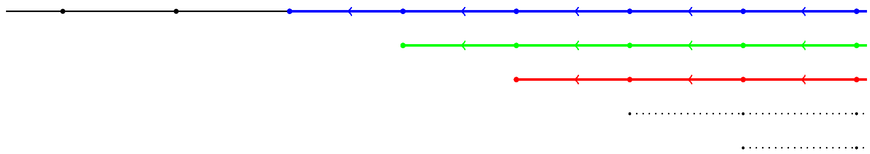


Figure 1.3: The Eilenberg-swindle construction does not work in $\mathbb{R}$.

The following is an explicit formulation of $H_0^{\mathrm{uf}}(\mathbb{Z};A)$ for $A \in \{\mathbb{R},\mathbb{Z}\}$.

Example 1.5.1 ([7]). Let $\|-\|_\infty$ be the supremum norm on $A \in \{\mathbb{R},\mathbb{Z}\}$. We have

$$H_0^{\mathrm{uf}}(\mathbb{Z};A) \cong \frac{\{\varphi\colon \mathbb{Z} \longrightarrow A \mid \|\delta\varphi\|_\infty < \infty\}}{\{\varphi\colon \mathbb{Z} \longrightarrow A \mid \|\varphi\|_\infty < \infty\}}$$

where for any $\varphi\colon \mathbb{Z} \longrightarrow A$ we define $\delta\varphi := (z \mapsto \varphi(z) - \varphi(z-1))$. In particular, $H_0^{\mathrm{uf}}(\mathbb{Z};A)$ is infinite dimensional.

Proof. Let $Q := \frac{\{\varphi\colon \mathbb{Z}\longrightarrow A \mid \|\delta\varphi\|_\infty<\infty\}}{\{\varphi\colon \mathbb{Z}\longrightarrow A \mid \|\varphi\|_\infty<\infty\}}$. We define the following map

$$\begin{aligned} F\colon C_0^{\mathrm{uf}}(\mathbb{Z};A) &\longrightarrow \{\varphi\colon \mathbb{Z} \longrightarrow A \mid \|\delta\varphi\|_\infty < \infty\} \\ c &\longmapsto F(c) := \left(z \mapsto \begin{cases} \sum_{j=1}^{z} c(j) & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ \sum_{j=z+1}^{0} -c(j) & \text{if } z < 0 \end{cases} \right). \end{aligned}$$

It is easy to see that F maps $\operatorname{Im}\partial_1$ to $\{\varphi\colon \mathbb{Z} \longrightarrow A \mid \|\varphi\|_\infty < \infty\}$. In particular, F induces a well-defined map $F\colon H_0^{\mathrm{uf}}(\mathbb{Z};A) \longrightarrow Q$. On the other hand, we have

$$\begin{aligned} \delta\colon \{\varphi\colon \mathbb{Z} \longrightarrow A \mid \|\delta\varphi\|_\infty < \infty\} &\longrightarrow C_0^{\mathrm{uf}}(\mathbb{Z};A) \\ \varphi &\longmapsto \delta\varphi. \end{aligned}$$

An easy computation shows that δ maps $\{\varphi\colon \mathbb{Z} \longrightarrow A \mid \|\varphi\|_\infty < \infty\}$ to $\operatorname{Im}\partial_1$ and it is an inverse for F. Clearly, the space Q is infinite dimensional. □

By the quasi-isometry invariance of uniformly finite homology established in Proposition 1.1.5, we deduce that $H_0^{\mathrm{uf}}(\mathbb{R}; A)$ is also infinite dimensional. More generally, for any $n \in \mathbb{N}$ and for $A \in \{\mathbb{R}, \mathbb{Z}\}$ we have

$$H_k^{\mathrm{uf}}(\mathbb{R}^n; A) = \begin{cases} A & \text{if } k = n \\ \text{infinite dimensional} & \text{if } k \in \{0, \dots, n-1\} \\ 0 & \text{else.} \end{cases}$$

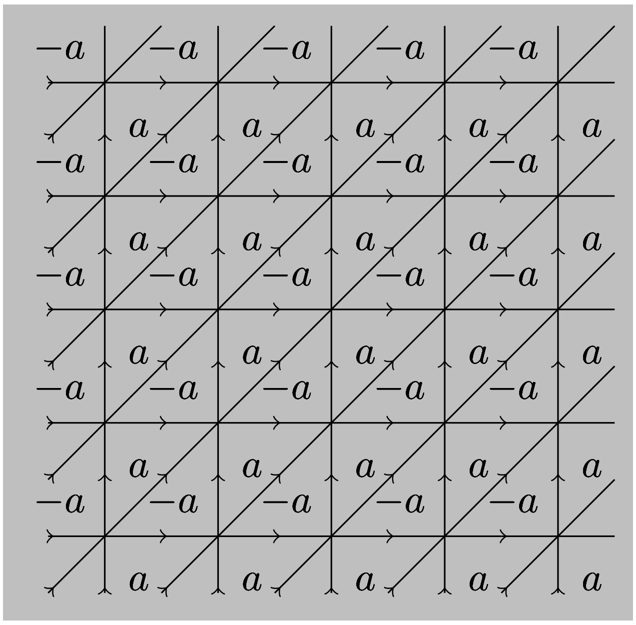


Figure 1.4: For any $a \in A$, we have a cycle in $C_2^{\mathrm{suf}}(\mathbb{R}^2; A)$.

We will see a proof of this in the next chapter (Example 2.3.15) as a consequence of Theorem 2.3.9 where we compute the uniformly finite homology of amenable groups in many cases. The approach used in Chapter 2 is very algebraic and it is based on techniques coming from group homology. However, we can also give a geometric explanation of this example:

For any $n \in \mathbb{N}$ we can take the space $\mathbb{R}^n$ with its standard triangulation so that it is a uniformly contractible simplicial complex of bounded geometry. By Proposition 1.3.3, we can compute $H_*^{\mathrm{suf}}(\mathbb{R}^n; A)$ to deduce the result for $H_*^{\mathrm{uf}}(\mathbb{R}^n; A)$.

Since $\mathbb{R}^n$ is n-dimensional, the cases $k > n$ follow immediately. Consider $k = n$. Any class in $C_n^{\mathrm{suf}}(\mathbb{R}^n; A)$ is an infinite sum of n-simplices in $\mathbb{R}^n$ having uniformly bounded coefficients. Let $c \in C_n^{\mathrm{suf}}(\mathbb{R}^n; A)$ be a cycle and let $\sigma \in \Delta_n(\mathbb{R}^n)$ be a simplex appearing in c with coefficient $c(\sigma) \in A$. By the cycle condition, the coefficients at each face of σ must sum up to zero. In the triangulation of $\mathbb{R}^n$, every $n-1$-simplex is the face of two n- simplices. It follows that the coefficients assigned to any n-simplex in $\mathbb{R}^n$ in c is uniquely determined by $c(\sigma)$ (Figure 1.4). For $k < n$, one can construct infinitely many linearly independent classes in $H_k^{\mathrm{suf}}(\mathbb{R}^n; A)$ represented by infinitely many "parallel tails" of k-simplices in $\mathbb{R}^n$. Indeed, in $\mathbb{R}^n$ there is not enough space to construct a $k+1$-chain that can bound these cycles and whose coefficient stay uniformly bounded (Figure 1.5). On the other hand, the standard coarse homology $HX_k(\mathbb{R}^n; A)$ vanishes for $k < n$ [39, Example, Chapter 2.2].

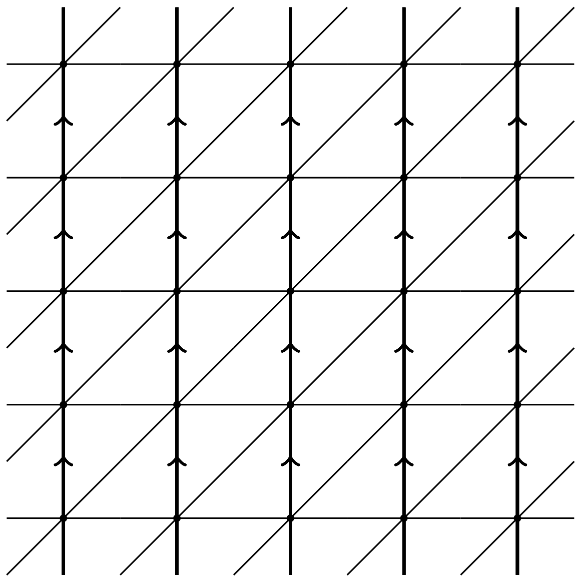


Figure 1.5: The cycle $\sum_{(z,z') \in \mathbb{Z}^2} \left[(z,z'),(z,z'+1)\right]$ in $C_1^{\mathrm{suf}}(\mathbb{R}^2; A)$.

1.5.2 Uniformly finite homology of a tree

In this section we consider the uniformly finite homology of *uniformly locally finite trees*, i.e. trees for which there is a finite uniform bound on the number of edges connected to any vertex. We first focus our attention on uniformly finite homology in degree 1 and then we give a complete computation of uniformly finite homology of non-amenable trees (Theorem 1.5.7). Any uniformly locally finite tree T is a 1-dimensional simplicial complex of bounded geometry and a metric space of coarse bounded geometry with ℓ^1-path metric rescaled so that every edge has length 1. Thus, by Proposition 1.3.3, for $A \in \{\mathbb{R}, \mathbb{Z}\}$, we have $H_*^{\mathrm{uf}}(T; A) \cong H_*^{\mathrm{suf}}(T; A)$. In particular, for any $n \geq 2$ we have $H_n^{\mathrm{uf}}(T;A) = 0$. From Theorem 1.4.1 we know that $H_0^{\mathrm{uf}}(T;A) = 0$ if and only if T is non-amenable. On the other hand, Example 1.5.1 gives an explicit description of uniformly finite homology in degree 0 for the 2-valent tree.

For a tree T, we denote by V_T the set of vertices of T. After choosing an order $\leq$ on V_T following Definition 1.2.3, we have an orientation on the edges in T. We denote by E_T the set of oriented edges. For any vertex $v \in V_T$, we denote by $\deg(v)$ its *degree*, i.e. the number of edges in E_T that connect to v. We always assume trees to be infinite and without leaves (i.e. each vertex has at least degree 2).

Definition 1.5.2. Let T be a tree. A *bi-infinite path in T* is an infinite sequence of distinct vertices $(v_n)_{n\in\mathbb{Z}}$ such that for any $n \in \mathbb{Z}$ we have

$$[v_n, v_{n+1}] \in E_T \text{ or } [v_{n+1}, v_n] \in E_T.$$

In other words any two consecutive vertices $v_n, v_{n+1} \in V_T$ in the sequence are connected by an edge.

Let $A \in \{\mathbb{R}, \mathbb{Z}\}$. Every bi-infinite path $(v_n)_{n\in\mathbb{Z}}$ in a tree T gives a cycle in $C_1^{\mathrm{suf}}(T;A)$:

$$\sum_{n\in\mathbb{Z}} (-1)^{\epsilon_n} e_n \tag{1.7}$$

where, for any $n \in \mathbb{Z}$ we have

$$\epsilon_n = \begin{cases} 0 & \text{if } e_n = [v_n, v_{n+1}] \\ 1 & \text{if } e_n = [v_{n+1}, v_n]. \end{cases}$$

In particular, it gives a cycle in $C_1^{\mathrm{uf}}(T;A)$ (Figure 1.6).

Definition 1.5.3. Let T be a tree and let $A \in \{\mathbb{R}, \mathbb{Z}\}$. A cycle in $C_1^{\mathrm{suf}}(T;A)$ (resp. $C_1^{\mathrm{uf}}(T;A)$) is a *path cycle* if it is of the form $\sum_{n\in\mathbb{Z}}(-1)^{\epsilon_n} e_n$ for a bi-infinite path $(v_n)_{n\in\mathbb{Z}}$ in T. A class in $H_1^{\mathrm{suf}}(T;A)$ (resp. $H_1^{\mathrm{uf}}(T;A)$) is a *path class* if it is represented by a path cycle.

Remark 1.5.4. *Any path class is a non-trivial class in $H_1^{\mathrm{suf}}(T;A)$. Indeed, since T is a 1-dimensional simplicial complex, the space $C_2^{\mathrm{suf}}(T;A)$ is trivial and any non-zero cycle in $C_1^{\mathrm{suf}}(T;A)$ gives a non-trivial class in $H_1^{\mathrm{suf}}(T;A)$. Clearly, $\sum_{n\in\mathbb{Z}}(-1)^{\epsilon_n} e_n$ is a cycle also in $C_1^{\mathrm{uf}}(T;A)$ but in uniformly finite homology the simplices considered are just tuples of points in T which do not come necessarily from the simplicial structure of the space. In particular, $C_2^{\mathrm{uf}}(T;A)$ is non-trivial and so a priori it is not clear that path classes are not zero in $H_1^{\mathrm{uf}}(T;A)$.*

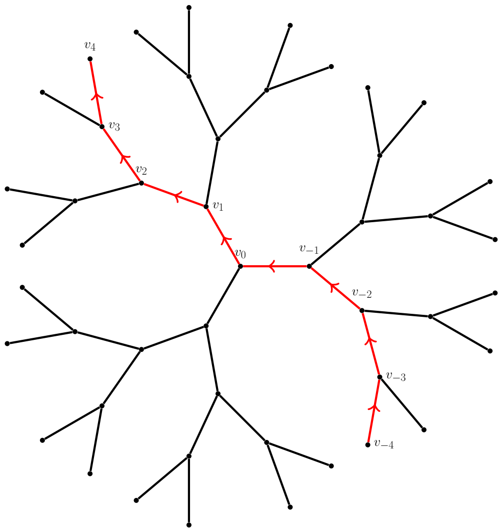


Figure 1.6: A path cycle in a 3-valent tree

For $k \in \mathbb{N}_{>2}$, it is possible to construct infinitely many linearly independent path classes on k-valent trees. Indeed, at each vertex there are enough edges to be able to construct a bi-infinite path and "move away" from it to construct the next one (Figure 1.7). More generally, we have the following result on the uniformly finite homology of uniformly locally finite trees in degree 1:

Proposition 1.5.5. *Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let T be a uniformly locally finite (infinite) tree. Consider the set $D_T = \{v \in V_T \mid \deg(v) \geq 3\}$.*

(i) Suppose $D_T = \emptyset$. Then $H_1^{\mathrm{uf}}(T; A) \cong A$.

(ii) Suppose $D_T \neq \emptyset$ and $|D_T| < \infty$. Then we have $H_1^{\mathrm{uf}}(T; A) \cong A^m$, where $m = \sum_{v \in D_T}(\deg(v) - 1)$.

(iii) Suppose $|D_T| = \infty$. Then $H_1^{\mathrm{uf}}(T; A)$ is infinite dimensional.

Proof. Since T is a uniformly contractible simplicial complex of bounded geometry, we can prove the statement for $H_*^{\mathrm{suf}}(T; A)$ and by Proposition 1.3.3, the claim will also hold for $H_*^{\mathrm{uf}}(T; A)$.

We start with the case (i). If $D_T = \emptyset$, then T is a 2-valent tree and thus it is isometric to the Cayley graph of $\mathbb{Z}$. From Example 2.3.15, we conclude that $H_1^{\mathrm{uf}}(T; A) \cong A$.

Now we consider the case (ii). Suppose $D_T \neq \emptyset$ and $|D_T| < \infty$. We want to construct a basis for the space $H_1^{\mathrm{suf}}(T; A)$. Let $v \in D_T$.

For all $i \in \{1, \dots, \deg(v)\}$, denote by v_i the vertices connected to v by an edge $e_i \in E_T$. For all $i \in \{1, \dots, \deg(v) - 1\}$ choose a bi-infinite path

$$p_{v_i} = (w_n)_{n \in \mathbb{Z}} \tag{1.8}$$

in T such that $w_{-1} = v_i$, $w_0 = v$, $w_1 = v_{i+1}$ and consider its corresponding path cycle γ_{v_i} as given in (1.7). We can repeat the construction for all $v \in D_T$ and by Remark 1.5.4 any of these path cycles gives a non-trivial class in $H_1^{\mathrm{suf}}(T; A)$. Moreover, since every vertex in D_T has degree at least 3, we can choose these bi-infinite paths such that

$$\forall_{v,v' \in D_T} \quad \forall_{i \in \{1,\dots,\deg(v)-1\}} \quad \forall_{j \in \{1,\dots,\deg(v')-1\}} \quad \mathrm{supp}(\gamma_{v_i}) \neq \mathrm{supp}(\gamma_{v'_j}).$$

Let $B = \{[\gamma_{v_i}]\}_{v \in D_T, i \in \{1,\dots,\deg(v)-1\}}$ be a family of path classes in $H_1^{\mathrm{suf}}(T; A)$. Since p_{v_i} are all pairwise distinct, we have $|B| = \sum_{v \in D_T} (\deg(v) - 1)$. We want to prove that B is a basis for the space $H_1^{\mathrm{suf}}(T; A)$. Since D_T is finite and $C_2^{\mathrm{suf}}(T; A) = 0$, every non-trivial linear combination $\sum_{[\gamma_{v_i}] \in B} a_{\gamma_{v_i}} \cdot [\gamma_{v_i}]$ gives a non-trivial class in $H_1^{\mathrm{suf}}(T; A)$. Now for any cycle $c \in C_1^{\mathrm{suf}}(T; A)$, consider the set

$$V_{\mathrm{supp}(c)} = \{v \in V_T \mid v \text{ is a vertex of some edge in } \mathrm{supp}(c)\}.$$

By the cycle condition, for any cycle $c \in C_1^{\mathrm{suf}}(T; A)$ and for any $v \in V_{\mathrm{supp}(c)}$ we can always find some $v_1, v_2 \in V_T$ with $v_1 \neq v_2$ connected to v by edges $e_1 \neq e_2 \in \mathrm{supp}(c)$. Moreover, since T is connected we have that $D_T \cap V_{\mathrm{supp}(c)} \neq \emptyset$. Proceeding by induction on the number of vertices in $D_T \cap V_{\mathrm{supp}(c)}$, we prove that every class in $H_1^{\mathrm{suf}}(T; A)$ can be written as linear combination of elements in B. Let $c \in C_1^{\mathrm{suf}}(T; A)$ be a cycle such that $|D_T \cap V_{\mathrm{supp}(c)}| = 1$ and let $v \in D_T \cap V_{\mathrm{supp}(c)}$. For every $i \in \{1, \dots, \deg(v)\}$, let $c_i := c(e_i) \in A$ be the coefficient associated to the edge e_i that connects to v. By the cycle condition on c and by the fact that there is only a vertex $v \in V_{\mathrm{supp}(c)}$ with $\deg(v) \geq 3$, every edge in $\mathrm{supp}(c)$ has coefficient c_i for some $i \in \{1, \dots, \deg(v)\}$. For simplicity, suppose that all the edges in $\mathrm{supp}(c)$ are oriented towards $v \in V_T$ (if not, we can just change the orientation by changing the sign of the coefficients of c). By the cycle condition on c we have that $\sum_{i=1}^{\deg(v)} c_i = 0$. For any $i \in \{1, \dots, \deg(v) - 1\}$ consider γ_{v_i} to be the path cycle associated to the bi-infinite path defined in (1.8): by construction, for any $i \in \{1, \dots, \deg(v) - 1\}$, the edges e_i, e_{i+1} are contained in $\mathrm{supp}(\gamma_{v_i})$. Thus, it is easy to see that the cycle c can be written as:

$$c = c_1 \cdot \gamma_{v_1} + (c_1 + c_2) \cdot \gamma_{v_2} + \cdots + (c_1 + \cdots + c_{deg(v)-1}) \cdot \gamma_{v_{deg(v)-1}}.$$

In particular, the class $[c] \in H_1^{\text{suf}}(T;A)$ can be written as linear combination of elements in B. Now suppose the claim has been proved for all cycles $c \in C_1^{\text{suf}}(T;A)$ for which $|D_T \cap V_{\text{supp}(c)}| \leq n-1$, for some $n > 1$. Proceeding with the induction step, we consider a cycle $\tilde{c} \in C_1^{\text{suf}}(T;A)$ for which $|D_T \cap V_{\text{supp}(\tilde{c})}| = n$, for some $n > 1$. Let $v \in D_T \cap V_{\text{supp}(\tilde{c})}$ and for every $i \in \{1,\ldots,\deg(v)\}$, let $\widetilde{c}_i := \widetilde{c}(e_i) \in A$ be the coefficient associated to the edge e_i that connects to v. Suppose, again without loss of generality, that all the edges in $\text{supp}(\widetilde{c})$ are oriented towards v. By the cycle condition on $\widetilde{c}$ we have that $\sum_{i=1}^{\deg(v)} \widetilde{c}_i = 0$. Thus

$$b := \widetilde{c}_1 \cdot \gamma_{v_1} + (\widetilde{c}_1 + \widetilde{c}_2) \cdot \gamma_{v_2} + \cdots + (\widetilde{c}_1 + \cdots + \widetilde{c}_{deg(v)-1}) \cdot \gamma_{v_{deg(v)-1}}$$

is a cycle in $C_1^{\text{suf}}(T;A)$. In particular, $c := \widetilde{c} - b$ is also a cycle in $C_1^{\text{suf}}(T;A)$. Moreover, by construction we have that $v \notin V_{\text{supp}(c)}$. It follows that $|D_T \cap V_{\text{supp}(c)}| \leq n-1$. By the induction hypothesis, $[c]$ can be written as linear combination of elements in B. Thus $[\widetilde{c}] = [c+b]$ can also be written as linear combination of elements of B. So B is a basis for $H_1^{\text{suf}}(T;A)$ and $H_1^{\text{suf}}(T;A) \cong A^m$, where $m = \sum_{v \in D_T}(\deg(v)-1)$. Thus (ii) follows.

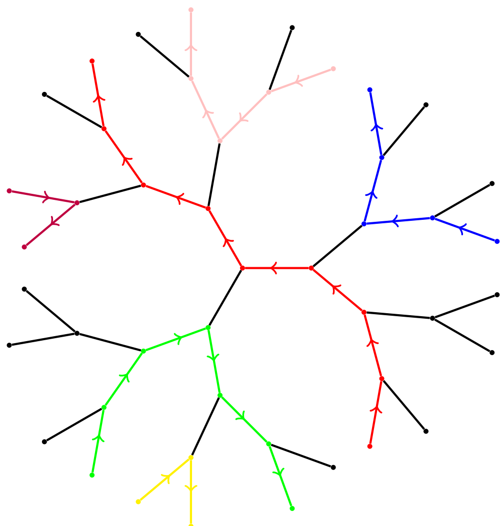


Figure 1.7: Infinitely many disjoint path cycles on the 3-valent tree

We now prove (iii). Suppose $|D_T| = \infty$. Then it is possible to find a bi-infinite path p containing infinitely many vertices in D_T. Indeed starting from a vertex $v \in D_T$ one can always choose edges in a complementary component of v where there are infinitely many vertices in D_T. Let γ be the path cycle corresponding to p as given in (1.7). We can enumerate the vertices in p that are contained in D_T obtaining a subsequence $(v_n)_{n\in\mathbb{Z}} \subseteq p$. Since every vertex v_n of this subsequence has at least degree 3, for any $n \in 2\mathbb{Z}$ we can choose a bi-infinite path p_{v_n} connecting v_n with v_{n+1} such that $\operatorname{supp}(\gamma_{v_n}) \neq \operatorname{supp}(\gamma)$ and such that

$$\forall_{n,m\in 2\mathbb{Z}, n\neq m} \quad \operatorname{supp}(\gamma_{v_n}) \cap \operatorname{supp}(\gamma_{v_m}) = \emptyset.$$

So we have an infinite family of path cycles $(\gamma_{v_n})_{n\in 2\mathbb{Z}}$ whose supports are pairwise disjoint. Since the corresponding path classes $\big([\gamma_{v_n}]\big)_{n\in 2\mathbb{Z}}$ are non-trivial and linearly independent in $H_1^{\mathrm{suf}}(T;A)$, we have that $H_1^{\mathrm{suf}}(T;A)$ must be infinite dimensional. Thus (iii) follows. □

We can consider any uniformly locally finite tree as a metric space of coarse bounded geometry. Definition 1.4.5 gives a notion of amenability for trees.

Lemma 1.5.6. *Let T be a uniformly locally finite (infinite) tree. If the set D_T is finite, then T is amenable.*

Proof. Let T be a uniformly locally finite tree. If $D_T = \emptyset$, then T is 2-valent and it is clearly amenable. Suppose $D_T \neq \emptyset$ and $|D_T| = k < \infty$ for some $k \in \mathbb{N}_{>0}$. Since D_T is a finite set, there exists $N \in \mathbb{N}$ such that $D_T \subseteq B_N(v)$ where $B_N(v)$ is the ball of radius N centered at some $v \in V_T$. For any $n \in \mathbb{N}$, we take $F_n := B_{N+n}(v)$. We want to prove that $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence in T. Notice that for any $n \in \mathbb{N}$ we have $|F_n| \geq N + n$. Moreover, for any $n \in \mathbb{N}$ the set $V_T \backslash F_n$ contains only vertices of degree 2, so for any $r \in \mathbb{R}_{>0}$ the set $\partial_r(F_n)$ contains paths of length r. In particular, for each $v \in D_T$ every complementary component of v gives a path of length r contained in $\partial_r(F_n)$. Thus, for any $r \in \mathbb{R}_{>0}$ we have

$$\lim_{n\to\infty} \frac{|\partial_r(F_n)|}{|F_n|} \leq \lim_{n\to\infty} \frac{\sum_{v\in D_T} \deg(v)\cdot r}{N+n} = 0.$$

It follows that $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence in T. Thus T is amenable. □

We can now give a complete computation of uniformly finite homology of non-amenable trees:

Theorem 1.5.7. *Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let T be a uniformly locally finite non-amenable tree. Then*

$$H_n^{\mathrm{uf}}(T;A) = \begin{cases} \textit{infinite dimensional} & \textit{if } n = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Clearly for any $n \geq 2$ we have $H_n^{\mathrm{uf}}(T;A) \cong H_n^{\mathrm{suf}}(T;A) = 0$. Since T is non-amenable, by Theorem 1.4.1 we have that $H_0^{\mathrm{uf}}(T;A) = 0$. Moreover, by Lemma 1.5.6 the set D_T must contain infinitely many elements. Thus, by Proposition 1.5.5-(ii) it follows that $H_1^{\mathrm{uf}}(T;A)$ is infinite dimensional. □

1.6 Whyte's rigidity result

In this section we introduce a rigidity result due to Whyte [43] which uses the uniformly finite homology of uniformly discrete metric spaces having bounded geometry (UDBG-spaces). In particular, Whyte gives a criterion to distinguish between quasi-isometries and bilipschitz equivalences. We give the definition of UDBG-space and some examples. Then we introduce Whyte's rigidity result. We will use this result in Chapter 4 to study semi-norms of classes in the uniformly finite homology of UDBG-spaces and to establish another rigidity result via semi-norms (Theorem 4.5.1 and Theorem 4.6.3).

1.6.1 Definition of UDBG-spaces

We start with the definition of uniformly discrete metric spaces having bounded geometry.

Definition 1.6.1. (i) A metric space (X,d) is *uniformly discrete* if there exists $\epsilon > 0$ such that

$$\forall_{x,y\in X} \quad d(x,y) < \epsilon \Leftrightarrow x = y.$$

(ii) A uniformly discrete metric space (X,d) has *bounded geometry* if for any $r \in \mathbb{R}_{>0}$ there exists $K_r > 0$ such that

$$\forall_{x\in X} \quad |B_r(x)| < K_r.$$

We call any uniformly discrete metric space with bounded geometry a UDBG-*space*.

Example 1.6.2. Clearly, every finitely generated group endowed with some word metric (Definition 2.1.1) is a UDBG-space. By Definion 1.4.2-(ii), quasi-lattices are metric spaces having bounded geometry but, in general, they are not UDBG-spaces since they are not always uniformly discrete. However, in any metric space of coarse bounded geometry there is a quasi-lattice which is a UDBG-space with the induced metric: indeed, one can always makes a quasi-lattice uniformly discrete by "getting rid" of some points. There are also examples of UDBG-spaces in which the points are not "uniformly distributed". For example, the space

$$\left\{n^2 \;\middle|\; n \in \mathbb{N}\right\}$$

is a UDBG-space with the metric induced by the standard metric in $\mathbb{R}$. However, this is not a quasi-lattice in $\mathbb{R}$ since it is not coarsely dense in $\mathbb{R}$.

Let X be any UDBG-space. By taking X to be a quasi-lattice in itself, we can consider it as a metric space of coarse bounded geometry. Following Definition 1.4.5, we say that a UDBG-space is *amenable* if it admits a Følner sequence.

1.6.2 Rigidity of UDBG-spaces

Let X be a UDBG-space. The class in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$ represented by the cycle $\sum_{x\in X} x \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ which assigns 1 to every point $x \in X$ is called the *fundamental class of* X and it is denoted by $[X]$.

Theorem 1.6.3 ([43, Theorem 1.1]). *Let $f\colon X \longrightarrow Y$ be a quasi-isometry between* UDBG*-spaces and let $f_0\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ be the induced map. Then there is a bilipschitz equivalence at bounded distance from f if and only if $f_0([X]) = [Y]$.*

We can make some immediate observations:

Remark 1.6.4. *(i) From Proposition 1.4.6 and Theorem 1.6.3, it follows immediately that any quasi-isometry between non-amenable* UDBG*-spaces is at bounded distance from a bilipschitz equivalence [43]. This answers a question of Gromov [22]. Other authors worked on this problem independently and using different tools [9, 36].*

(ii) If $H \leq G$ is a proper subgroup of a finitely generated amenable group G with finite index $[G : H] = n > 1$ then the inclusion map $i : H \hookrightarrow G$ is not at bounded distance from a bilipschitz equivalence. Indeed, the fundamental class $[G]$ is equal to the class $n \cdot [H]$ in $H_0^{\mathrm{uf}}(G;\mathbb{Z})$, so

$$[H] = i_0([H]) = [G]$$

only for $n = 1$. Dymarz proved this studying the asymptotic behaviour of non-zero classes in the zero degree uniformly finite homology [17]. As an application of Theorem 1.6.3, Dymarz showed that certain quasi-isometric lamplighter groups cannot be bilipschitz equivalent [18].

(iii) More generally, for any subset $S \subset X$ of a UDBG-space X taken as metric space with the induced metric, one can consider the inclusion $i\colon S \hookrightarrow X$. If S is coarsely dense in X, then this map is a quasi-isometry and one can ask if it is at bounded distance from a bilipschitz equivalence. Contrary to the case of a subgroup in a finitely generated group, in many examples of UDBG*-spaces one can find coarsely dense subsets such that the inclusion is at bounded distance from a bilipschitz equivalence. For example, if X is a finitely generated infinite group (or any infinite* UDBG*-space where the*

points are uniformly distributed in a certain sense), then for $S = X \backslash \{e\}$ *we have*

$$i_0([S]) = [S] = [X \backslash \{e\}],$$

where $i_0 : H_0^{\mathrm{uf}}(S;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(X;\mathbb{Z})$ *is the map induced by the inclusion* $i\colon S \hookrightarrow X$*. Clearly* $[e] = 0$ *in* $H_0^{\mathrm{uf}}(G;\mathbb{R})$*, since we can always construct an infinite tails of* 1*-simplices in* G *which bounds the point* $e \in G$*. So we have that* $[X \backslash \{e\}] = [X]$ *and by Theorem 1.6.3 it follows that* i *must be at bounded distance from a bilipschitz equivalence.*

1.7 Rips uniformly finite homology

We dedicate this section to a further approach to uniformly finite homology which uses the Rips complex on metric spaces. In Section 1.8, we will see that for metric spaces of coarse bounded geometry the Rips uniformly finite homology is equivalent to the Block-Weinberger uniformly finite homology defined in Section 1.1. As first step, following Block and Weinberger [8] we define the Rips chain complex for any quasi-lattice in a metric space of coarse bounded geometry (Definition 1.7.2). Then we consider the direct limit over the set of all quasi-lattices in X (Definition 1.7.4). In this way, we get a more complete definition of Rips uniformly finite homology in which all quasi-lattices are considered together. We start with the construction of Rips complexes for discrete metric spaces.

Definition 1.7.1. Let (X, d) be a discrete metric space and let $r \in \mathbb{R}_{>0}$. The *r-Rips complex of* X is the simplicial complex $R_r(X)$ whose vertices are the points of X. For any $n \in \mathbb{N}$, a tuple $(x_0, \ldots, x_n) \in X^{n+1}$ of vertices forms an n-simplex if and only if $d(x_i, x_j) \leq r$ for all $i, j \in \{0, \ldots, n\}$.

For any $r \in \mathbb{R}_{\geq 0}$ and for any $n \in \mathbb{N}$ we denote by $\Delta_n(R_r(X))$ the set of n-simplices in $R_r(X)$. Now we want to consider the Rips complex of a quasi-lattice Γ in a metric space of coarse bounded geometry X. By Definition 1.4.2-(ii), it follows that the r-Rips complex of any quasi-lattice is a finite dimensional simplicial complex of bounded geometry. We fix a unital ring with norm A and we define:

Definition 1.7.2. Let X be a metric space of coarse bounded geometry and let $\Gamma \subseteq X$ be a quasi-lattice. For any $r \in \mathbb{R}_{>0}$, let $R_r(\Gamma)$ be the r-Rips complex of Γ. The family of Rips complexes $\{R_r(\Gamma) \mid r \in \mathbb{R}_{>0}\}$ is a direct system where for any $r \leq s$, the morphisms $i_{r,s}\colon R_r(\Gamma) \hookrightarrow R_s(\Gamma)$ are given by the inclusion. This gives a direct system $\left(C_*^{\mathrm{suf}}(R_r(\Gamma); A)\right)_{r \in \mathbb{R}_{>0}}$ where the morphisms are the induced maps:

$$i_{r,s*}\colon C_*^{\mathrm{suf}}(R_r(\Gamma); A) \longrightarrow C_*^{\mathrm{suf}}(R_s(\Gamma); A).$$

(i) For each $n \in \mathbb{N}$ define

$$C_n^{\text{R-uf}}(\Gamma; A) := \varinjlim_r C_n^{\text{suf}}(R_r(\Gamma); A)$$

and denote an equivalence class in $C_n^{\text{R-uf}}(\Gamma; A)$ by $[-]_{\text{Rips}}$.

(ii) For each $n \in \mathbb{N}$ define ∂_n to be the boundary operator

$$\partial_n \colon C_n^{\text{R-uf}}(\Gamma; A) \longrightarrow C_{n-1}^{\text{R-uf}}(\Gamma; A)$$

induced by the operator given in Definition 1.2.5-(ii).

(iii) The homology of $\left(C_n^{\text{R-uf}}(\Gamma; A), \partial_n\right)_{n \in \mathbb{N}}$ is denoted by $H_*^{\text{R-uf}}(\Gamma; A)$.

The next proposition shows that any quasi-isometry between quasi-lattices induces an isomorphism in homology.

Proposition 1.7.3. *Let $\Gamma, \Lambda \subseteq X$ be quasi-lattices in X. Then any quasi-isometry $f \colon \Gamma \longrightarrow \Lambda$ induces an isomorphism $H_*^{\text{R-uf}}(\Gamma; A) \cong H_*^{\text{R-uf}}(\Lambda; A)$.*

Proof. Let $r \in \mathbb{R}_{>0}$. Suppose $f \colon \Gamma \longrightarrow \Lambda$ is a (C, D)-quasi-isometry for some $C, D > 0$. Then, for $s = C \cdot r + D$ we have a map

$$f \colon R_r(\Gamma) \longrightarrow R_s(\Lambda)$$

defined for any $n \in \mathbb{N}$ and any simplex $\overline{\gamma} = (\gamma_0, \ldots, \gamma_n) \in \Delta_n(R_r(\Gamma))$ as $f(\overline{\gamma}) = (f(\gamma_0), \ldots, f(\gamma_n))$. Clearly, f is a simplicial map. Moreover, for all $n \in \mathbb{N}$ and all $\overline{\lambda} \in \Delta_n(R_s(\Lambda))$, we have

$$\left| f^{-1}(\overline{\lambda}) \right| \leq \prod_{0 \leq i \leq n} \left| \{ \gamma \in \Gamma \mid f(\gamma) = \lambda_i \} \right|.$$

For any $0 \leq i \leq n$ consider an $\gamma_i \in \Gamma$ such that $\gamma_i \in f^{-1}(\lambda_i)$. Then $f^{-1}(\lambda_i)$ is contained in $B_{C \cdot D}(\gamma_i)$, the ball of radius $C \cdot D$ centered in γ_i. Indeed for any $\gamma' \in f^{-1}(\lambda_i)$ we have

$$d_X(\gamma_i, \gamma') \leq C \cdot d_Y(f(\gamma_i), f(\gamma')) + C \cdot D = C \cdot D.$$

From condition (ii) of Definition 1.4.2, any ball in Γ has uniform bounded cardinality, in particular, there exists a $K_{C \cdot D} > 0$ such that for all $\gamma \in \Gamma$ we have $|B_{C \cdot D}(\gamma) \cap \Gamma| \leq K_{C \cdot D}$. It follows that for all $\overline{\lambda} \in \Delta_n(\Lambda)$, we have

$$\left| f^{-1}(\overline{\lambda}) \right| \leq \prod_{0 \leq i \leq n} |B_{C \cdot D}(\gamma_i)| \leq (K_{C \cdot D})^{n+1}.$$

Since for all $r \in \mathbb{R}_{>0}$ the simplicial complex $R_r(\Gamma)$ is finite dimensional, we can find a constant K such that for all $n \in \mathbb{N}$, we have $K_{C \cdot D}^{n+1} < K$.

So $|f^{-1}(\overline{\lambda})| < K$ and the simplicial map has bounded geometry. Thus, we have a well-defined chain map

$$f_*\colon C_*^{\text{R-uf}}(\Gamma; A) \longrightarrow C_*^{\text{R-uf}}(\Lambda; A).$$

induced by f and, similarly, we have a well-defined chain map

$$g_*\colon C_*^{\text{R-uf}}(\Lambda; A) \longrightarrow C_*^{\text{R-uf}}(\Gamma; A).$$

induced by g. The proof that f_* is an isomorphism in homology is analogous to the one for Proposition 1.1.5. □

For a metric space of bounded geometry X consider the set

$$\mathrm{QL}(X) := \left\{\Gamma \subseteq X \;\middle|\; \Gamma \text{ quasi-lattice in } X\right\}.$$

This is a directed set with respect to the inclusion relation. Indeed every quasi-lattice is included in itself, the relation is transitive and for any two quasi-lattices Γ and Λ there is an upper bound, namely the quasi-lattice given by their union. For any two quasi-lattices $\Gamma \subseteq \Lambda$, the inclusion $i\colon \Gamma \hookrightarrow \Lambda$ is a quasi-isometry.

We can consider the family

$$\left(C_*^{\text{R-uf}}(\Gamma; A)\right)_{\Gamma \in \mathrm{QL}(X)}$$

where for any $\Gamma \subseteq \Lambda$ there is a morphism

$$I_*^{\Gamma,\Lambda} : C_*^{\text{R-uf}}(\Gamma; A) \longrightarrow C_*^{\text{R-uf}}(\Lambda; A)$$

induced by the inclusion $i : \Gamma \hookrightarrow \Lambda$.

Now we can define uniformly finite homology of a metric space X with bounded geometry using Rips complexes.

Definition 1.7.4. Let X be a metric space with coarse bounded geometry.

(i) For each $n \in \mathbb{N}$ define

$$C_n^{\text{R-uf}}(X; A) := \varinjlim_{\Gamma \in \mathrm{QL}(X)} C_n^{\text{R-uf}}(\Gamma; A)$$

and denote an equivalence class in $C_*^{\text{R-uf}}(X; A)$ by $[-]_{\mathrm{QL}}$.

(ii) For each $n \in \mathbb{N}$ define ∂_n to be the boundary operator

$$\partial_n : C_n^{\text{R-uf}}(X; A) \longrightarrow C_{n-1}^{\text{R-uf}}(X; A)$$

induced by the operator given in Definition 1.7.2-(ii).

(iii) The homology of $\big(C_n^{\text{R-uf}}(X;A), \partial_n\big)_{n\in\mathbb{N}}$ is called the *Rips-uniformly finite homology* of X and it is denoted by $H_*^{\text{R-uf}}(X;A)$.

Remark 1.7.5. *This construction follows a more general approach to define coarse homology using anti-Čech systems [39, Chapter 2.2]. Indeed, as we have done here, one can define the coarse homology of a space with respect to a fixed anti-Čech system of covers and then consider the direct limit over all the anti-Čech systems on the space. Notice that for any $r \in \mathbb{R}_{>0}$, the r-Rips complex of a discrete metric space X is the nerve of a cover of balls in X [35, Section 7.5]. These covers form an anti-Čech system for X.*

The next proposition shows that for any quasi-lattice $\Gamma \in \mathrm{QL}(X)$ the canonical map $\Psi_*^\Gamma : C_*^{\text{R-uf}}(\Gamma;A) \longrightarrow C_*^{\text{R-uf}}(X;A)$ which sends an element to its equivalence class in the direct limit induces an isomorphism in homology.

Proposition 1.7.6. *Let X be a metric space of coarse bounded geometry. For every $\Gamma \in \mathrm{QL}(X)$, the canonical map*

$$\begin{aligned} \Psi_*^\Gamma : C_*^{\text{R-uf}}(\Gamma;A) &\longrightarrow C_*^{\text{R-uf}}(X;A) \\ c &\longmapsto [c]_{\mathrm{QL}}. \end{aligned}$$

induces an isomorphism $H_*^{\text{R-uf}}(\Gamma;A) \cong H_*^{\text{R-uf}}(X;A)$

Proof. Let $\Gamma \in \mathrm{QL}(X)$. First we prove the injectivity of Ψ_*^Γ at the level of homology. Let $n \in \mathbb{N}$ and suppose $\Psi_n^\Gamma(\alpha) = 0$ in $H_n^{\text{R-uf}}(X;A)$ for some $\alpha = [c] \in H_n^{\text{R-uf}}(\Gamma;A)$. Then $[c]_{\mathrm{QL}} = \partial_{n+1}[d]_{\mathrm{QL}} \in C_n^{\text{R-uf}}(X;A)$ where $[d]_{\mathrm{QL}} \in C_{n+1}^{\text{R-uf}}(X;A)$ is a class represented by some $d \in C_{n+1}^{\text{R-uf}}(\Lambda;A)$ for some $\Lambda \in \mathrm{QL}(X)$. In particular, $[c]_{\mathrm{QL}} = [\partial_{n+1}d]_{\mathrm{QL}} \in C_n^{\text{R-uf}}(X;A)$. Thus, there exists $\Omega \in \mathrm{QL}(X)$ with $\Lambda, \Gamma \subseteq \Omega$ such that

$$I_n^{\Gamma,\Omega}(c) = I_n^{\Lambda,\Omega}(\partial_{n+1}d) = \partial_{n+1} I_{n+1}^{\Lambda,\Omega}(d) \in C_n^{\text{R-uf}}(\Omega;A).$$

It follows that $\left[I_n^{\Gamma,\Omega}(c)\right] = 0$ in $H_n^{\text{R-uf}}(\Omega;A)$. Since the inclusion $\Gamma \hookrightarrow \Omega$ is a quasi-isometry, by Proposition 1.7.3 the induced map $I_*^{\Gamma,\Omega}$ is an isomorphism in homology. Thus $\alpha = [c] = 0$ in $H_n^{\text{R-uf}}(\Gamma;A)$. So for any $\Gamma \in \mathrm{QL}(X)$ the induced map in homology Ψ_*^Γ is injective.

Now we are left to prove that $\Psi_*^\Gamma\colon H_*^{\text{R-uf}}(\Gamma;A) \longrightarrow H_*^{\text{R-uf}}(X;A)$ is surjective. Let $\Gamma \in \mathrm{QL}(X)$ and let $n \in \mathbb{N}$. Consider $\alpha \in H_n^{\text{R-uf}}(X;A)$ and suppose that $\alpha = [[c]_{\mathrm{QL}}]$ for some $c \in C_n^{\text{R-uf}}(\Lambda;A)$ and some $\Lambda \in \mathrm{QL}(X)$. Consider now an upper bound of Γ and Λ in $\mathrm{QL}(X)$, namely the quasi-lattice $\Gamma \cup \Lambda$. Since the inclusion $\Lambda \hookrightarrow \Gamma \cup \Lambda$ is a quasi-isometry, it induces an isomorphism

$$I_n^{\Lambda,\Gamma\cup\Lambda}\colon H_n^{\text{R-uf}}(\Lambda;A) \longrightarrow H_n^{\text{R-uf}}(\Gamma\cup\Lambda;A)$$

which sends the class $[c]$ to the class $I_n^{\Lambda,\Gamma\cup\Lambda}([c]) \in H_n^{\text{R-uf}}(\Gamma \cup \Lambda; A)$. By Proposition 1.7.3, we also have an isomorphism

$$I_n^{\Gamma,\Gamma\cup\Lambda} \colon H_n^{\text{R-uf}}(\Gamma; A) \longrightarrow H_n^{\text{R-uf}}(\Gamma \cup \Lambda; A).$$

Let $[a] \in H_n^{\text{R-uf}}(\Gamma; A)$ be a class such that $I_n^{\Gamma,\Gamma\cup\Lambda}([a]) = I_n^{\Lambda,\Gamma\cup\Lambda}([c])$. It follows that $[a]_{\text{QL}} = [c]_{\text{QL}}$ in $C_n^{\text{R-uf}}(X; A)$ and $\Psi_n^{\Gamma}([a]) = [[a]_{\text{QL}}] = [[c]_{\text{QL}}]$ in $H_n^{\text{R-uf}}(X; A)$. Thus $\Psi_*^{\Gamma} \colon H_*^{\text{R-uf}}(\Gamma; A) \longrightarrow H_*^{\text{R-uf}}(X; A)$ is surjective. □

1.8 Equivalence of definitions

In this section, we prove the equivalence between the Block-Weinberger uniformly finite homology (Definition 1.1.4) and the Rips uniformly finite homology (Definition 1.7.4). First we restrict to the case of quasi-lattices in some space X viewed as metric spaces with the induced metric. The equivalence for any metric space with coarse bounded geometry will, then, follow as an immediate corollary.

Proposition 1.8.1. *Let X be a metric space with coarse bounded geometry and let $\Gamma \in \text{QL}(X)$. Then the chain complexes $C_*^{\text{uf}}(\Gamma; A)$ and $C_*^{\text{R-uf}}(\Gamma; A)$ are chain isomorphic. In particular $H_*^{\text{uf}}(\Gamma; A) \cong H_*^{\text{R-uf}}(\Gamma; A)$.*

Proof. Let $n \in \mathbb{N}$. By condition (i)-(c) of Definition 1.1.4, it follows that for any chain $a = \sum_{\overline{\gamma}\in\Gamma^{n+1}} a(\overline{\gamma}) \cdot \overline{\gamma} \in C_n^{\text{uf}}(\Gamma; A)$ there is a constant $R_a > 0$ such that every $\overline{\gamma} \in \text{supp}(a)$ must be in $\Delta_n(R_{R_a}(\Gamma))$. So we have a well-defined map

$$\begin{aligned} \phi_n \colon C_n^{\text{uf}}(\Gamma; A) &\longrightarrow C_n^{\text{R-uf}}(\Gamma; A) \\ \sum_{\overline{\gamma}\in\Gamma^{n+1}} a(\overline{\gamma}) \cdot \overline{\gamma} &\longmapsto \left[\sum_{\overline{\gamma}\in\Delta_n(R_{R_a}(\Gamma))} a(\overline{\gamma}) \cdot \overline{\gamma} \right]_{\text{Rips}}. \end{aligned}$$

On the other hand, let $a \in C_n^{\text{suf}}(R_r(\Gamma); A)$ and $b \in C_n^{\text{suf}}(R_s(\Gamma); A)$ for some $r, s \in \mathbb{R}_{>0}$. Suppose there exists $R \geq r, s$ such that $i_{r,R}(a) = i_{s,R}(b)$, then a and b are the same element in $C_n^{\text{uf}}(\Gamma, \mathbb{R})$. In particular the map

$$\begin{aligned} \psi_n \colon C_n^{\text{R-uf}}(\Gamma; A) &\longrightarrow C_n^{\text{uf}}(\Gamma; A) \\ \left[\sum_{\overline{\gamma}\in\Delta_n(R_r(\Gamma))} a(\overline{\gamma}) \cdot \overline{\gamma} \right]_{\text{Rips}} &\longmapsto \sum_{\overline{\gamma}\in\Gamma^{n+1}} a(\overline{\gamma}) \cdot \overline{\gamma} \end{aligned}$$

is well-defined. It is easy to check that ϕ_* and ψ_* are chain maps and that they are inverse to each other. So the claim follows. □

From this we have an easy corollary.

Corollary 1.8.2. *Let X be a metric space of coarse bounded geometry. Then*

$$H_*^{\mathrm{uf}}(X;A) \cong H_*^{\text{R-uf}}(X;A).$$

Proof. Let $\Gamma \in \mathrm{QL}(X)$. Since the inclusion $i : \Gamma \hookrightarrow X$ is a quasi-isometry, by Proposition 1.1.5 we have $H_*^{\mathrm{uf}}(X;A) \cong H_*^{\mathrm{uf}}(\Gamma;A)$. On the other hand, by the isomorphisms established in Proposition 1.8.1 and Proposition 1.7.6 we have $H_*^{\mathrm{uf}}(\Gamma;A) \cong H_*^{\text{R-uf}}(\Gamma;A) \cong H_*^{\text{R-uf}}(X;A)$, so the claim follows. □

Remark 1.8.3. *This last corollary can also be proved directly, by providing a chain isomorphism $C_*^{\mathrm{uf}}(X;A) \longrightarrow C_*^{\text{R-uf}}(X;A)$ for any metric space of coarse bounded geometry X.*

Chapter 2

Uniformly finite homology of groups

This chapter is dedicated to uniformly finite homology of finitely generated groups. We consider groups as metric spaces by endowing them with the word metric (Definition 2.1.1). In Section 2.2, we introduce standard homology of groups and we establish an isomorphism between uniformly finite homology and homology of groups with coefficients in the module of bounded functions (Proposition 2.2.4). In Section 2.3, we present part of a joint work with Matthias Blank [6] where we compute uniformly finite homology for amenable groups in most cases. In particular, we prove that the uniformly finite homology of a finitely generated amenable group is always infinite dimensional in degree zero (Theorem 2.3.7). In Section 2.3.2, we prove that the higher degree uniformly finite homology of amenable groups is infinite dimensional in many cases (Theorem 2.3.9). Any mean on an amenable group induces a transfer map from uniformly finite homology to standard homology (Proposition 2.3.4), so the idea behind the proof of Theorem 2.3.7 and Theorem 2.3.9 is to use different means to distinguish between classes in uniformly finite homology.

2.1 The word metric

Let G be a (discrete) group. We can consider G as a metric space by endowing it with the word metric with respect to some generating set.

Definition 2.1.1. Let G be a group with generating set S. The *word metric on G with respect to S* is the metric defined as

$$d_S(g,h) := \min\{n \in \mathbb{N} \mid \exists_{s_1,\dots,s_n \in S \cup S^{-1}}\ g^{-1} \cdot h = s_1 \cdots s_n\}$$

for any $g, h \in G$.

Remark 2.1.2. *Notice that, by definition, the word metric on a group G with respect to a generating set S is left invariant, i.e. for any $h, g, g' \in G$ we have $d_S(h \cdot g, h \cdot g') = d_S(g, g')$.*

Clearly, the word metric on G depends on the choice of the generating set. However, we have the following well-known fact in geometric group theory:

Proposition 2.1.3 ([27, Proposition 5.2.4]). *Let G be a finitely generated group and let S and S' be two finite generating sets of G. The identity map id_G is a bilipschitz equivalence between (G, d_S) and (G, d'_S).*

Thus, from Proposition 1.1.5 it follows that the uniformly finite homology of a group does not depend on the choice of generating set up to canonical isomorphism.

In the previous chapter, we have seen how uniformly finite homology can be different from ordinary homology theories. However, as we will see in the next section, there is a strong relation between uniformly finite homology and standard homology for finitely generated groups.

2.2 Definition of homology of groups

We first give the general definition of group homology. We consider A to be a unital ring with norm as in (1.1) on page 3.

Definition 2.2.1. Let G be a discrete group and let M be a (left) $A[G]$-module. Let $(C_*(G; A), \partial_*)$ be the chain complex defined as follows:

(i) For each $n \in \mathbb{N}$, let $C_n(G; A) := \bigoplus_{(g_0,\dots,g_n)\in G^{n+1}} A \cdot (g_0, \dots, g_n)$ with the G-action given by $g \cdot (g_0, \dots, g_n) = (gg_0, \dots, gg_n)$.

(ii) For each $n \in \mathbb{N}$, let $\partial_n \colon C_n(G; A) \longrightarrow C_{n-1}(G; A)$ be the boundary map defined as

$$(g_0, \dots, g_n) \longmapsto \sum_{j=0}^{n} (-1)^j (g_0, \dots, \widehat{g}_j, \dots, g_n)$$

for any $(g_0, \dots, g_n) \in C_n(G; A)$.

Let $C_*(G; M)$ be the A-chain complex given by

$$C_*(G; M) := \overline{C}_*(G; A) \otimes_{A[G]} M.$$

where $\overline{C}_*(G; A)$ denotes the right $A[G]$-module obtained by $C_*(G; A)$ via the canonical involution $g \mapsto g^{-1}$. For a group G we define the *group homology of G with coefficients in M* by $H_*(G; M) := H_*(C_*(G; M))$.

Group homology is a functor from the category GrpMod of pairs of groups and modules to the category of graded abelian groups.

Definition 2.2.2. Let GrpMod be the category defined as follows:

(i) The objects of GrpMod are pairs (G, M) where G is a discrete group and M is a (left) $A[G]$-module.

(ii) A morphism between two objects (G, M) and (H, N) in GrpMod is a pair (φ, ψ) where:

- $\varphi \colon G \longrightarrow H$ is a group homomorphism;
- $\psi \colon M \longrightarrow \varphi^* N$ is a $A[G]$-module homomorphism, where $\varphi^* N$ is the $A[G]$-module having N as underlying additive group and $A[G]$-action given by

$$\begin{aligned} G \times N &\longrightarrow N \\ (g, n) &\longmapsto \varphi(g) \cdot n. \end{aligned}$$

Group homology is the functor $H_* : \mathrm{GrpMod} \longrightarrow \mathrm{Ab}_*$ which sends every object (G, M) to $H_*(G; M)$ and every morphism (φ, ψ) to $H_*(C_*(\varphi; \psi))$ where for any $n \in \mathbb{N}$ we have

$$\begin{aligned} C_n(\varphi; \psi) \colon C_n(G; M) &\longrightarrow C_n(H; N) \\ (g_0, \ldots, g_n) \otimes m &\longmapsto (\varphi(g_0), \ldots, \varphi(g_n)) \otimes \psi(m). \end{aligned}$$

As a generating set for the module of chains we can take tuples whose first element is the identity as the next remark shows:

Remark 2.2.3. *Let M be a left $A[G]$-module and let $e \in G$ be the identity element. Let $n \in \mathbb{N}$. As a generating set for $C_n(G; M)$ we can take all the elements of the form*

$$(e, t_1, \ldots, t_n) \otimes m$$

Indeed, for any $(g_0, \ldots, g_n) \otimes m \in C_n(G; M)$ we have

$$\begin{aligned} (g_0, \ldots, g_n) \otimes m &= g_0 \cdot (e, g_0^{-1} g_1, \ldots, g_0^{-1} g_n) \otimes m \\ &= (e, g_0^{-1} g_1, \ldots, g_0^{-1} g_n) \otimes g_0^{-1} \cdot m. \end{aligned}$$

Now we consider a special $A[G]$-module. More precisely, let

$$\ell^\infty(G, A) := \{\varphi : G \longrightarrow A \mid \|\varphi\|_\infty < \infty\}$$

where $\| - \|_\infty$ is the supremum norm defined for any $\varphi \in \ell^\infty(G, A)$ as

$$\|\varphi\|_\infty := \sup_{g \in G} |\varphi(g)|.$$

The space $\ell^\infty(G, A)$ has a natural structure of an $A[G]$-module with respect to the action

$$\begin{aligned} G \times \ell^\infty(G, A) &\longrightarrow \ell^\infty(G, A) \\ (g, \varphi) &\longmapsto \left(g \cdot \varphi \colon g' \longmapsto \varphi(g^{-1}g')\right). \end{aligned} \tag{2.1}$$

The next proposition follows from an observation of Brodzki, Niblo and Wright [11].

Proposition 2.2.4. *Let G be a finitely generated group and let A be a normed unital ring. Then there is a canonical natural isomorphism*

$$H_*^{\mathrm{uf}}(G; A) \cong H_*(G; \ell^\infty(G, A)).$$

Since this result will play an important role in the rest of this chapter, we will give a detailed proof of it in Appendix A.2.

We can, thus, face problems concerning uniformly finite homology with the help of standard techniques coming from group homology. One important consequence of Proposition 2.2.4 is that it gives an upper bound to the uniformly finite homological dimension of a group.

Definition 2.2.5. The *uniformly finite homological dimension* of a finitely generated group G is defined by

$$\mathrm{hd}_{A,\mathrm{uf}}\, G = \sup\{n \in \mathbb{N} \mid H_n^{\mathrm{uf}}(G; A) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

By Proposition 2.2.4, for any finitely generated group G we have

$$\mathrm{hd}_{A,\mathrm{uf}}\, G \leq \mathrm{hd}_A\, G$$

where $\mathrm{hd}_A\, G$ denotes the standard homological dimension of G over A.

2.3 Uniformly finite homology and amenable groups

In this section, we compute the uniformly finite homology for amenable groups in many cases. From Theorem 1.4.1 we know that a finitely generated group G is amenable if and only if $H_0^{\mathrm{uf}}(G; \mathbb{R}) \neq 0$. In this section, we show that the zero degree uniformly finite homology of a finitely generated infinite amenable group is always infinite dimensional. We also consider the higher degree uniformly finite homology of finitely generated amenable groups and we prove that it is infinite dimensional in many cases.

The results contained in this section are part of a joint work with Matthias Blank [6].

2.3.1 Zero degree uniformly finite homology

For a group G we denote by $\ell^\infty(G) := \ell^\infty(G, \mathbb{R})$ the space of real-valued bounded functions. This is a $\mathbb{R}[G]$-module with the action given in (2.1). We use the characterization of amenability for groups using invariant means on the space of real-valued bounded functions. More precisely:

Definition 2.3.1. Let G be a group. A *left invariant mean* is an $\mathbb{R}$-linear map $m\colon \ell^\infty(G) \longrightarrow \mathbb{R}$ satisfying the following conditions:

(i) We have $m(\chi_G) = 1$, where $\chi_G \in \ell^\infty(G)$ is the characteristic function of G.

(ii) For any $\varphi \in \ell^\infty(G)$ such that $\varphi \geq 0$ (i.e., $\varphi(g) \geq 0$ for all $g \in G$) we have $m(\varphi) \geq 0$.

(iii) For any $g \in G$ and any $\varphi \in \ell^\infty(G)$, we have $m(g \cdot \varphi) = m(\varphi)$.

A group is *amenable* if it admits a left invariant mean.

As we have already observed in Section 1.4, this definition is equivalent to Definition 1.4.5 if we consider a finitely generated group as a metric space of coarse bounded geometry [14, Theorem 4.9.2].

Remark 2.3.2. *There is a corresponding definition of right invariant mean where condition (iii) of Definition 2.3.1 is replaced by invariance with respect to a right action of G on $\ell^\infty(G)$. One can pass from left invariant to right invariant means by precomposing with the canonical involution*

$$\begin{aligned} \ell^\infty(G) &\longrightarrow \ell^\infty(G) \\ \varphi &\longmapsto (g \longmapsto \varphi(g^{-1})). \end{aligned}$$

For an amenable group we denote by $M(G)$ the set of all left invariant means on G and by $LM(G)$ its linear span in $\operatorname{Hom}_{\mathbb{R}}(\ell^\infty(G), \mathbb{R})$. For an infinite amenable group, the space $LM(G)$ is infinite dimensional. More precisely:

Theorem 2.3.3 ([15, Theorem 1])**.** *If G is an infinite amenable group, then G has exactly $2^{2^{|G|}}$ left invariant means, where $|G|$ denotes the cardinality of G. Thus $LM(G)$ is infinite dimensional.*

The following proposition is similar to a result of Attie [1, Proposition 2.15].

Proposition 2.3.4. *Let G be an amenable group and let $\chi_G \in \ell^\infty(G)$ be the characteristic function of G. Then every mean $m \in M(G)$ induces a transfer map*

$$m_*\colon H_*(G; \ell^\infty(G)) \longrightarrow H_*(G; \mathbb{R})$$

defined for any chain $c = \sum_{(e,g_1,\dots,g_n)\in G^{n+1}} (e, g_1, \dots, g_n) \otimes \varphi_{(e,g_1,\dots,g_n)}$ *as*

$$m_*(c) = \sum_{(e,g_1,\dots,g_n)\in G^{n+1}} m\left(\varphi_{(e,g_1,\dots,g_n)}\right) \cdot (e, g_1, \dots, g_n).$$

For any $m \in M(G)$*, the induced map* m_* *is a left inverse to the map*

$$i_* \colon H_*(G; \mathbb{R}) \longrightarrow H_*(G; \ell^\infty(G))$$

defined for any chain $c = \sum_{(e,g_1,\dots,g_n)\in G^{n+1}} c(e, g_1, \dots, g_n) \cdot (e, g_1, \dots, g_n)$ *as*

$$i_*(c) = \sum_{(e,g_1,\dots,g_n)\in G^{n+1}} (e, g_1, \dots, g_n) \otimes c(e, g_1, \dots, g_n) \cdot \chi_G$$

and induced by the canonical inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G)$ *as constant functions.*

Proof. Let $m \in M(G)$. Then the map $m_* \colon H_*(G; \ell^\infty(G)) \longrightarrow H_*(G; \mathbb{R})$ is induced by the morphism (id_G, m) in the category GrpMod, where id_G is the identity homomorphism and $m \colon \ell^\infty(G) \longrightarrow \mathbb{R}$ is the left invariant mean. Indeed, if we consider $\mathbb{R}$ as a G-module with the trivial action, then by the left invariance property of means (Definition 2.3.1-(iii)) we have that $m \colon \ell^\infty(G) \longrightarrow \mathbb{R}$ is an $\mathbb{R}[G]$-module homomorphism. On the other hand, the map $i_* \colon H_*(G; \mathbb{R}) \longrightarrow H_*(G; \ell^\infty(G))$ is induced by the morphism (id_G, i) in the category GrpMod, where $i \colon \mathbb{R} \hookrightarrow \ell^\infty(G), 1 \mapsto \chi_G$ is the $\mathbb{R}[G]$-module homomorphism given by the inclusion as constant functions. It is easy to see that (id_G, m) is a left inverse for (id_G, i). Thus the claim follows. □

The next corollary follows immediately from Proposition 2.2.4 and Proposition 2.3.4.

Corollary 2.3.5. *Let* G *be a finitely generated amenable group. Then there is an injection*

$$i_* \colon H_*(G; \mathbb{R}) \hookrightarrow H_*^{\mathrm{uf}}(G; \mathbb{R}).$$

Proof. A left inverse map for i_* can be found by taking any $m \in M(G)$ and considering the map $\overline{m}_* \colon H_*^{\mathrm{uf}}(G; \mathbb{R}) \longrightarrow H_*(G; \mathbb{R})$ obtained by precomposing m_* with the chain isomorphism ρ_* given in Appendix A.2. □

For any $m \in M(G)$, the transfer map in degree zero is of the form

$$\begin{aligned} m_0 : H_0(G; \ell^\infty(G)) &\longrightarrow \mathbb{R} \\ [e \otimes \varphi] &\longrightarrow m(\varphi). \end{aligned}$$

By Remark A.2.1, the corresponding transfer map $\overline{m}_0 : H_0^{\mathrm{uf}}(G; \mathbb{R}) \longrightarrow \mathbb{R}$ is obtained by precomposing m_0 with the canonical involution. Notice that $C_0^{\mathrm{uf}}(G; \mathbb{R}) = \ell^\infty(G)$, thus $\overline{m}_0$ is the map induced by the right invariant mean $\overline{m} : C_0^{\mathrm{uf}}(G; \mathbb{R}) \longrightarrow \mathbb{R}$ obtained by precomposing m with the canonical involution (Remark 2.3.2).

Definition 2.3.6. We call the subspace

$$\widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R}) := \{\alpha \in H_0^{\mathrm{uf}}(G;\mathbb{R}) \mid \forall_{m\in M(G)} \quad \overline{m}_0(\alpha) = 0\}$$

the *mean-invisible part of* $H_0^{\mathrm{uf}}(G;\mathbb{R})$.

We have the following result on the uniformly finite homology of amenable groups in degree zero.

Theorem 2.3.7. *Let G be a finitely generated infinite amenable group. Then*

$$\dim_{\mathbb{R}} H_0^{\mathrm{uf}}(G;\mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R}) = \infty.$$

In particular, $\dim_{\mathbb{R}} H_0^{\mathrm{uf}}(G;\mathbb{R})$ *is infinite dimensional.*

Proof. By Proposition 2.3.4, any $m \in M(G)$ gives a transfer map

$$\begin{aligned} m_0 \colon H_0(G;\ell^\infty(G)) &\longrightarrow \mathbb{R} \\ [e\otimes\varphi] &\longmapsto m(\varphi). \end{aligned}$$

Thus, we have an inclusion

$$\begin{aligned} LM(G) &\hookrightarrow H_0(G;\ell^\infty(G))^* \\ m &\longmapsto m_0 \end{aligned}$$

where $H_0(G;\ell^\infty(G))^* = \mathrm{Hom}_{\mathbb{R}}\big(H_0(G;\ell^\infty(G)),\mathbb{R}\big)$. In particular, by Proposition 2.2.4 and Definition 2.3.6 we have an inclusion

$$LM(G) \hookrightarrow \big(H_0^{\mathrm{uf}}(G;\mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})\big)^*.$$

By Theorem 2.3.3, the space $LM(G)$ is infinite dimensional. Thus the dual space $\big(H_0^{\mathrm{uf}}(G;\mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})\big)^*$ is also infinite dimensional. It follows that $H_0^{\mathrm{uf}}(G;\mathbb{R}) / \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})$ must be infinite dimensional. □

For a finitely generated amenable group G there is a geometric condition to detect classes that belong to the mean-invisible part of $H_0^{\mathrm{uf}}(G;\mathbb{R})$. It is possible to construct infinitely many non-trivial classes which are supported on *asymptotically sparse* subsets of an amenable group G [6, Definition 5.8] and are, therefore, contained in $\widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})$. We have the following:

Theorem 2.3.8 ([6, Theorem 5.1])**.** *Let G be a finitely generated infinite amenable group. Then*

$$\dim_{\mathbb{R}} \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R}) = \infty.$$

2.3.2 Higher degree uniformly finite homology

In this section we focus on higher degree uniformly finite homology for amenable groups. In particular, we prove the following:

Theorem 2.3.9. *Let $n \in \mathbb{N}$ and let G be a finitely generated amenable group. Let $H \leq G$ be a subgroup such that $[G : H] = \infty$ and such that the inclusion $i \colon H \hookrightarrow G$ induces a non-trivial map $i_n \colon H_n(H; \mathbb{R}) \longrightarrow H_n(G; \mathbb{R})$. Then $\dim_{\mathbb{R}}(H_n^{\mathrm{uf}}(G; \mathbb{R})) = \infty$.*

The idea of the proof is to construct infinitely many linearly independent non-trivial classes in $H_*^{\mathrm{uf}}(G; \mathbb{R})$ coming from elements in $\ell^\infty(G)$ which are invariant with respect to the action of an infinite index subgroup $H \leq G$. More precisely, we construct a family of infinitely many left invariant means that can be distinguished by an infinite family of H-invariant functions.

The following theorem is due to Mitchell and gives a condition for a subset $S \subseteq G$ to support a left invariant mean:

Theorem 2.3.10 ([32, Theorem 7])**.** *Let G be an amenable group and let $S \subseteq G$ be a subset of G. Denote by χ_S the characteristic function of S. The following conditions are equivalent:*

(i) The subset S is left-thick, *i.e. for each finite $F \subseteq G$ there exists $g \in G$ such that $F \cdot g \subseteq S$.*

(ii) There exists $m \in M(G)$ such that $m(\chi_S) = 1$.

We construct infinitely many such subsets and we separate them using the following lemma:

Lemma 2.3.11. *Let G be an amenable group and $H \leq G$ such that $[G : H] = \infty$. Let $\pi \colon G \longrightarrow H \backslash G$ be the canonical projection. Then for any pair of finite subsets $T, T' \subseteq G$ there exists $g \in G$ such that $\pi(T \cdot g) \cap \pi(T') = \emptyset$.*

Proof. Suppose, for a contradiction, that there exist finite subsets $T, T' \subseteq G$ such that for any $g \in G$ we have $\pi(T \cdot g) \cap \pi(T') \neq \emptyset$. Then, for any $g \in G$ there exist $t \in T$ and $t' \in T'$ such that $tg \in Ht'$, hence $g \in THT'$. In particular, $G = THT'$. Since G is amenable, it admits a bi-invariant mean m [14, Proposition 4.4.4]. Moreover, by the coset decomposition on G and by Definition 2.3.1 one can easily see that, since H is an infinite index subgroup, we have $m(\chi_H) = 0$. Thus:

$$1 = m(\chi_G) = m(\chi_{THT'}) \leq \sum_{t \in T, t' \in T'} m(\chi_{t \cdot H \cdot t'}) = \sum_{t \in T, t' \in T'} m(\chi_H) = 0.$$

So we have a contradiction and the claim follows. $\square$

Theorem 2.3.12. *Let G be a finitely generated amenable group and $H \leq G$ a subgroup such that $[G : H] = \infty$. Then there exists an infinite family $(m_j)_{j\in J}$ of left invariant means on G and an infinite family $(f_j)_{j\in J}$ of (left) H-invariant functions in $\ell^\infty(G)$, such that $m_k(f_j) = \delta_{k,j}$ for any $k, j \in J$.*

Proof. The idea of the proof is to construct an infinite family $(f_j)_{j\in J}$ of functions in $\ell^\infty(G)$ by taking disjoint thick subsets in the quotient $H\backslash G$. We can use Theorem 2.3.10 to obtain a corresponding infinite family $(m_j)_{j\in J}$ of left invariant means.

We consider G equipped with the word metric with respect to some finite generating set. Let $n \in \mathbb{N}$ and let $\pi\colon G \longrightarrow H\backslash G$ be the canonical projection. By induction we construct a family of subsets $(A_l^k)_{k\in\{1,\ldots,n\},l\in\mathbb{N}}$ such that

- The family $\big(\pi(A_l^k)\big)_{k\in\{1,\ldots,n\},l\in\mathbb{N}}$ is pairwise disjoint.
- For all $k \in \{1,\ldots,n\}$ and for all $l \in \mathbb{N}$ we have $A_l^k = B_l(e)\cdot g$ for some $g \in G$, where $B_l(e)$ denotes the ball of radius $l \in \mathbb{N}$ centered at the identity $e \in G$.

We construct the sets $(A_l^k)_{k\in\{1,\ldots,n\},l\in\mathbb{N}}$ using the lexicographic order on the indices $(k,l) \in \mathbb{N}^2$. Define $A_1^1 := B_1(e)$. We consider $(k,l) \in \mathbb{N}^2$ and, by induction, we suppose the sets $A_{j'}^{k'}$ have been constructed for all $(k',l') \leq (k,l)$. Then

- If $k < n$, by Lemma 2.3.11 there exists $g \in G$ such that
$$\pi\Bigg(\bigcup_{(l',k')\leq(l,k)} A_{l'}^{k'}\Bigg) \cap \pi\big(B_l(e)\cdot g\big) = \emptyset.$$
Define $A_l^{k+1} := B_l(e)\cdot g$
- If $k = n$, by Lemma 2.3.11 there exists $g \in G$ such that
$$\pi\Bigg(\bigcup_{(l',k')\leq(l,k)} A_{l'}^{k'}\Bigg) \cap \pi\big(B_{l+1}(e)\cdot g\big) = \emptyset.$$
Define $A_{l+1}^1 := B_{l+1}(e)\cdot g$.

For all $k \in \{1,\ldots,n\}$ define
$$T^k := \bigcup_{l\in\mathbb{N}} H\cdot A_l^k.$$
Since the sets $T^1,\ldots,T^n$ are H-invariant with respect to multiplication from the left, the corresponding characteristic functions $\chi_{T^1},\ldots,\chi_{T^n}$ are

left invariant with respect to the action given in (2.1). Notice that, by construction, the sets $T^1, \dots, T^n$ are left-thick in the sense of Theorem 2.3.10. Indeed, for every finite subset $F \subseteq G$ there exists some $r \in \mathbb{R}_{\geq 0}$ such that $F \subseteq B_r(e)$. Thus, for all $k \in \{1, \dots, n\}$ there exists some $g \in G$ such that $F \cdot g \subseteq B_r(e) \cdot g \subseteq T^k$. Thus, by Theorem 2.3.10, for all $k \in \{1, \dots, n\}$ there exists $m_k \in M(G)$ such that $m_k(\chi_{T^k}) = 1$. Notice that the sets $T^1, \dots, T^n$ are pairwise disjoint. Thus for any $i, j \in \{1, \dots, n\}$ with $i \neq j$ we have $m_j(\chi_{T^i}) = 0$. In this way we have a family of left invariant means $(m_k)_{k \in \{1,\dots,n\}}$ and a family of H-invariant bounded functions $(\chi_{T^k})_{k \in \{1,\dots,n\}}$ on G such that $m_k(\chi_{T^j}) = \delta_{k,j}$ for any $k, j \in \{1, \dots, n\}$. Notice that we can repeat the same construction for any $n \in \mathbb{N}$. In other words, for any arbitrary large $N \in \mathbb{N}$, we can construct a family $(m_j)_{j \in \{1,\dots,N\}}$ of left invariant means and a family $(f_j)_{j \in \{1,\dots,N\}}$ such that for any $k, j \in \{1, \dots, N\}$ we have $m_k(f_j) = \delta_{k,j}$. Using a slightly different induction argument, one can construct an infinitely family $(A^k_l)_{l,k \in \mathbb{N}}$ of sets as above and directly obtain an infinite family of left invariant means and an infinite family of H-invariant bounded functions satisfying the theorem. Thus the claim follows. □

We are now ready to prove Theorem 2.3.9.

Proof of Theorem 2.3.9. If $H_n(G; \mathbb{R})$ is infinite dimensional, then by Corollary 2.3.5, we have that $H^{\text{uf}}_n(G; \mathbb{R})$ is also infinite dimensional. Suppose that $H_n(G; \mathbb{R})$ is finite dimensional. Consider the space of all H-invariant bounded functions on G, namely

$$\ell^\infty(G)^H := \{f \in \ell^\infty(G) \mid \forall_{h \in H} \quad h \cdot f = f\}.$$

Since the map $i_n \colon H_n(H; \mathbb{R}) \longrightarrow H_n(G; \mathbb{R})$ induced by the inclusion is not trivial, we can find a cycle $c \in C_n(H; \mathbb{R})$ such that $i_n([c]) \neq 0$ in $H_n(G; \mathbb{R})$. Consider the set:

$$S_c := \{i(c) \otimes f \in C_n(G; \ell^\infty(G)) \mid f \in \ell^\infty(G)^H\}.$$

Notice that any $i(c) \otimes f \in S_c$ is a cycle in $C_n(G; \ell^\infty(G))$. Indeed, for any $f \in \ell^\infty(G)^H$, there is a chain map ${\varphi_f}_* \colon C_*(H; \mathbb{R}) \longrightarrow C_*(G; \ell^\infty(G))$ induced by the inclusion $i \colon H \hookrightarrow G$ and by the $\mathbb{R}[H]$-module homomorphism $\mathbb{R} \longrightarrow \ell^\infty(G)$, $1 \mapsto f$. In particular, for any $f \in \ell^\infty(G)^H$, we have ${\varphi_f}_n(c) = i(c) \otimes f$ and since $c \in C_n(H; \mathbb{R})$ is a cycle, $i(c) \otimes f$ must be a cycle.

By Theorem 2.3.12, there exists an infinite family $(m_j)_{j \in J}$ of left invariant means and an infinite family $(f_j)_{j \in J}$ of H-invariant bounded functions such that for any $k, j \in J$ we have $m_k(f_j) = \delta_{k,j}$. By Proposition 2.3.4, for any $j \in J$ the mean $m_j \in M(G)$ induces a transfer map

$$m_{j_*} \colon H_*(G; \ell^\infty(G)) \longrightarrow H_*(G; \mathbb{R}).$$

Since for any $j \in J$ the function f_j is H-invariant, we obtain an infinite family of classes $\big([i(c) \otimes f_j]\big)_{j \in J} \in H_n(G; \ell^\infty(G))$. Thus for any $k, j \in J$ we have

$$m_{k*}\big([i(c) \otimes f_j]\big) = \big[m_k(f_j) \cdot i(c)\big] = \delta_{k,j} \cdot [i(c)]$$

Since $[i(c)] \neq 0$ in $H_n(G; \mathbb{R})$, it follows that the family $\big(m_{j_*}\big)_{j \in J}$ of induced maps is linearly independent in $\mathrm{Hom}_{\mathbb{R}}\big(H_n(G; \ell^\infty(G)), H_n(G; \mathbb{R})\big)$. Thus $H_n(G; \ell^\infty(G))$ is infinite dimensional. By Proposition 2.2.4, we conclude that $H_n^{\mathrm{uf}}(G; \mathbb{R})$ is also infinite dimensional. □

There is another proof of Theorem 2.3.9 which does not need the result of Mitchell given in Theorem 2.3.10 [6]. A more direct proof of Theorem 2.3.9 can be given with the additional assumption that $H \leq G$ is a normal subgroup [6, Second proof of Theorem 3.8)].

2.3.3 Examples

We give some examples for the uniformly finite homology for amenable groups. In the following, beside Theorem 2.3.9, we will also use standard arguments coming from homological algebra and group homology. In particular, we will see that in some cases higher degree uniformly finite homology can be computed using Poincaré duality applied to homology with ℓ^∞-coefficients.

Example 2.3.13 (Uniformly finite homology in degree 1)**.** Let G be a finitely generated infinite amenable group. Assume that $H_1(G; \mathbb{R})$ is non-trivial, i.e., that the abelianization of G is not a torsion group. Then

$$H_1^{\mathrm{uf}}(G; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } G \text{ is virtually } \mathbb{Z} \\ \text{infinite dimensional} & \text{otherwise.} \end{cases}$$

Proof. Suppose G is not virtually $\mathbb{Z}$. Since $H_1(G; \mathbb{R}) \cong G_{ab} \otimes \mathbb{R}$ and it is non-trivial, there exists a non-trivial class of the form $[(e, g)] \in H_1(G; \mathbb{R})$. Let $H = \langle g \rangle$. Then clearly $[G : H] = \infty$ and the map induced by the inclusion $H_1(H; \mathbb{R}) \longrightarrow H_1(G; \mathbb{R})$ is non-trivial. By Theorem 2.3.9 we conclude that $H_1^{\mathrm{uf}}(G; R)$ must be infinite dimensional. On the other hand, if G is virtually $\mathbb{Z}$ then it is quasi-isometric to $\mathbb{Z}$. In particular, by the quasi-isometry invariance of uniformly finite homology, the claim follows from what we have seen in Section 1.5.1 concerning the uniformly finite homology of the Euclidean space. However, using a more algebraic approach one can deduce the result by Poincaré duality as Example 2.3.15 shows. □

Example 2.3.14. Let N, Q be finitely generated amenable groups. Suppose N is infinite. Moreover, suppose Q acts on N and let $N \rtimes Q$ be the semi-direct product. Then, for all $k \in \mathbb{N}$ such that $H_k(Q;\mathbb{R}) \neq 0$ we have $\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(N \rtimes Q;\mathbb{R}) = \infty$.

Proof. By definition, we have a split short exact sequence

$$1 \longrightarrow N \longrightarrow N \rtimes Q \longrightarrow Q \longrightarrow 1.$$

The splitting map $Q \hookrightarrow N \rtimes Q$ is the inclusion on the second component. Let $k \in \mathbb{N}$ such that $H_k(Q;\mathbb{R}) \neq 0$. Then the splitting map induces a non-trivial map in homology $H_k(Q;\mathbb{R}) \longrightarrow H_k(N \rtimes Q;\mathbb{R})$. Thus, by Theorem 2.3.9, we have that $\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(N \rtimes Q;\mathbb{R}) = \infty$. □

We can completely compute the uniformly finite homology of $\mathbb{Z}^n$ using Example 2.3.14 and Poincaré duality. More precisely:

Example 2.3.15. For all $n \in \mathbb{N}$ we have

$$H_k^{\mathrm{uf}}(\mathbb{Z}^n;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = n \\ \text{infinite dimensional} & \text{if } k \in \{0,\dots,n-1\} \\ 0 & \text{else.} \end{cases}$$

Proof. Since $\mathbb{Z}^n = \mathbb{Z} \times \mathbb{Z}^{n-1}$ is a special case of semi-direct product, the statement for $k \in \{0,\dots,n-1\}$ follows from Example 2.3.14. Since $\mathbb{Z}^n$ is a Poincaré duality group [12, Chapter VIII.10] for any k we have

$$H_k(\mathbb{Z}^n;\ell^\infty(\mathbb{Z}^n)) \cong H^{n-k}(\mathbb{Z}^n;\ell^\infty(\mathbb{Z}^n)).$$

In particular, for $k > n$ we have $H_k(\mathbb{Z}^n;\ell^\infty(\mathbb{Z}^n)) = 0$. Since the cohomology functor in degree zero coincides with the functor of taking invariants [28, Proposition 1.3.12], for any group G we have

$$H^0(G;\ell^\infty(G)) \cong \ell^\infty(G)^G \cong \mathbb{R}.$$

Thus $H_n(\mathbb{Z}^n;\ell^\infty(\mathbb{Z}^n)) \cong \mathbb{R}$ and the claim follows. □

We can prove a more general result for nilpotent groups:

Example 2.3.16. Let G be a finitely generated virtually nilpotent group and let $h \in \mathbb{N}$ be its Hirsch rank. Then

$$H_k^{\mathrm{uf}}(G;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = h \\ \text{infinite dimensional} & \text{if } k \in \{0,\dots,h-1\} \\ 0 & \text{else.} \end{cases}$$

Proof. We can assume G to be nilpotent and torsion free. Indeed G contains a nilpotent subgroup $G' \leq G$ of finite index and since G' is finitely generated and nilpotent, it has finite torsion subgroup. Thus, G is quasi-isometric to a torsion free nilpotent group.

For a finitely generated, torsion free nilpotent group G the Hirsch rank coincides with the largest integer n for which $H_n(G;\mathbb{R}) \neq 0$ and with its homological dimension [41]. Since G is a Poincaré duality group [12, VIII.10, Example 1] we can prove the statement for $k \geq h$ using the same argument as in the previous example. On the other hand, we can prove the claim for $k \in \{0, \ldots, h-1\}$ using Theorem 2.3.9: we need a subgroup $H \leq G$ of infinite index such that the inclusion map induces a non-trivial map in homology. For this we follow the computation of homology of nilpotent groups given by Baumslag, Miller and Short [4]. First of all, since G is torsion free, there is a normal subgroup $N \lhd G$ and a split extension of the form

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1.$$

By the Hochschild spectral sequence, we get a short exact sequence

$$0 \longrightarrow H_0(\mathbb{Z}; H_i(N;\mathbb{R})) \longrightarrow H_i(G;\mathbb{R}) \longrightarrow H_1(\mathbb{Z}; H_{i-1}(N;\mathbb{R})) \longrightarrow 0$$

We have $H_0(\mathbb{Z}, H_i(N;\mathbb{R})) \cong H_i(N;\mathbb{R})_{\mathbb{Z}}$. The map on the left hand side is one edge map of the spectral sequence and is induced by the canonical map $H_i(N;\mathbb{R}) \longrightarrow H_i(G;\mathbb{R})$ [42, Chapter 8.6]. Following the computation given by Baumslag, Miller and Short [4, proof of Theorem 16], for all $i \in \{0, \ldots, h-1\}$ we have that $H_i(N;\mathbb{R})_{\mathbb{Z}}$ is non-trivial. Thus the canonical map $H_i(N;\mathbb{R}) \longrightarrow H_i(G;\mathbb{R})$ is also non-trivial. Thus, by Theorem 2.3.9, for $k \in \{0, \ldots, h-1\}$ we have that $H_k^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional. □

Using Theorem 2.3.9 and standard homological techniques other examples can be given in the amenable case [6]. Similarly to Example 2.3.16, one can prove that for semi-direct products of type $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ for $A \in \mathrm{SL}(2,\mathbb{Z})$ the uniformly finite homology is infinite dimensional up to degree 2 [6, Example 4.5]. This includes cocompact lattices in Sol. There are also examples of amenable groups having infinite uniformly finite dimension [6, Examples 4.6-4.7].

Chapter 3

Uniformly finite homology of products

As we have mentioned in the introduction, a method to investigate uniformly finite homology is to compare it to more standard homology theories. We have already seen that, in general, uniformly finite homology behaves very differently from standard homology. In this chapter, we want to analyse further its behavior with respect to products of spaces or groups. In Section 3.1, we investigate some particular cross-product maps and we prove their injectivity in some cases. We do this by providing left inverse maps. It is possible to define more general cross-product maps for uniformly finite homology using standard techniques coming from algebraic topology (Remark 3.1.2). In Section 3.2, we present results from a joint work with Piotr Nowak [16] where we consider the uniformly finite homology of products of non-amenable simplicial complexes. Using an Eilenberg-swindle construction in higher dimension, we prove some vanishing results for uniformly finite homology of non-amenable products. In particular, using *panels of* 2*-simplices* attached to 1-simplices in the product, we prove that the uniformly finite homology of a product of non-amenable simplicial complexes vanishes in degree 1 (Theorem 3.2.1). In a similar way, we prove that the uniformly finite homology of the product of three non-amenable trees vanishes in degree 2 (Theorem 3.3.3). In this case, we use *beams of* 3*-simplices* attached to 2-simplices in the product. We conclude Section 3.3.2 with a possible generalization of our construction for the product of n non-amenable trees (Conjecture 3.3.12-Work in Progress). In Section 3.4, we present a conjecture for a vanishing Künneth theorem for uniformly finite homology.

3.1 Cross-product and transfer maps

In this section, we give a particular cross-product map from the uniformly finite homology of a group to the uniformly finite homology of a product. We prove that in some cases this map is injective. We show an application of this result in Section 3.2.3 where we give a characterization of amenability using uniformly finite homology in degree 1. We first define the cross-product map in the general setting of metric spaces. For two metric spaces $(X, d_X), (Y, d_Y)$ we consider their cartesian product $X \times Y$ with the maximum metric as defined in (1.2) on page 3. Let A be a unital ring with norm as in (1.1) on page 3.

Definition 3.1.1. Let X, Y be metric spaces. For any chain $c \in C_0^{\mathrm{uf}}(Y; A)$, the *cross-product with* c is the map $- \times c \colon C_*^{\mathrm{uf}}(X; A) \longrightarrow C_*^{\mathrm{uf}}(X \times Y; A)$ defined as follows:

For any $n \in \mathbb{N}$, and for any $a = \sum_{\overline{x} \in X^{n+1}} a(\overline{x}) \cdot \overline{x} \in C_n^{\mathrm{uf}}(X; A)$ we have

$$a \times c = \sum_{(\overline{x}, y) = ((x_0, y), \ldots, (x_n, y)) \in (X \times Y)^{n+1}} a(\overline{x}) c(y) \cdot (\overline{x}, y).$$

It is easy to see that the cross-product map with $[c]$ is a well-defined map in homology. We denote this map as $- \times [c]$ for any $[c] \in H_0^{\mathrm{uf}}(Y; A)$.

Remark 3.1.2. *More generally, following the classical construction of the simplicial cross-product, for any $n, m \in \mathbb{N}$ and for any metric spaces X, Y, it is possible to define a map $- \times - \colon H_n^{\mathrm{uf}}(X; A) \times H_m^{\mathrm{uf}}(Y; A) \longrightarrow H_{n+m}^{\mathrm{uf}}(X \times Y; A)$ by shuffling the variables in the tuples $(x_0, \ldots, x_n) \in X^{n+1}, (y_0, \ldots, y_m) \in Y^{m+1}$ to get a well-defined chain in $C_{n+m}^{\mathrm{uf}}(X \times Y; \mathbb{R})$ [25, Chapter 3.B].*

We now consider finitely generated groups as metric spaces. By Proposition 2.2.4, to compute the uniformly finite homology of a group G we can consider the homology with coefficients in the module of bounded functions on G.

Let G be a finitely generated group and let $\ell^\infty(G) = \ell^\infty(G, \mathbb{R})$. By Proposition 2.3.4, we know that if G is amenable, then every left invariant mean $m \in M(G)$ induces a transfer map

$$m_* \colon H_*(G; \ell^\infty(G)) \longrightarrow H_*(G; \mathbb{R}).$$

We, thus, have a corresponding transfer map $\overline{m}_* \colon H_*^{\mathrm{uf}}(G; \mathbb{R}) \longrightarrow H_*(G; \mathbb{R})$ obtained by precomposing m_* with the chain isomorphism ρ_* given in Appendix A.2. In degree zero, we can define the mean-invisible part to be the subspace $\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) \subseteq H_0^{\mathrm{uf}}(G; \mathbb{R})$ of classes α such that $\overline{m}_0(\alpha) = 0$ for any $m \in M(G)$ (Definition 2.3.6).

We use invariant means on amenable groups to construct left inverses for the cross-product maps. In particular we prove the following:

Theorem 3.1.3. *Let G, H be finitely generated groups. Suppose H is amenable. Then for any $\alpha \in H_0^{\mathrm{uf}}(H;\mathbb{R})\backslash\widehat{H}_0^{\mathrm{uf}}(H;\mathbb{R})$ the cross-product*

$$- \times \alpha \colon H_*^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(G \times H;\mathbb{R})$$

is injective.

Proof. By Proposition 2.2.4, for any finitely generated group G we have $H_*^{\mathrm{uf}}(G;\mathbb{R}) \cong H_*(G;\ell^\infty(G))$. In particular, by composing the cross-product map with the chain isomorphisms defined in Appendix A.2, we obtain a cross-product map on the level of homology with coefficients in the module of bounded functions. Let G, H be finitely generated groups. It is easy to see that for any class $[e \otimes \varphi] \in H_0(H;\ell^\infty(H))$, the cross-product map $- \times [e \otimes \varphi] \colon H_*(G;\ell^\infty(G)) \longrightarrow H_*(G \times H;\ell^\infty(G \times H))$ is induced by the group homomorphism

$$\begin{aligned} i \colon G &\longrightarrow G \times H \\ g &\longmapsto (g, e) \end{aligned}$$

and the $\mathbb{R}[G]$-module homomorphism

$$\begin{aligned} P_\varphi \colon \ell^\infty(G) &\longrightarrow \ell^\infty(G \times H) \\ \psi &\longmapsto \big((g,h) \longmapsto \psi(g)\varphi(h)\big). \end{aligned}$$

Suppose H is amenable and let $\alpha = [c] \in H_0^{\mathrm{uf}}(H;\mathbb{R})\backslash\widehat{H}_0^{\mathrm{uf}}(H;\mathbb{R})$ be a class represented by a cycle $c \in C_0^{\mathrm{uf}}(H;\mathbb{R})$. Since α is not mean-invisible, there exists a $m \in M(H)$ such that $\overline{m}_0(c) = m(\rho_0(c)) \neq 0$, where, by Remark A.2.1, the map $\rho_0(c) \in \ell^\infty(H)$ is defined as:

$$\begin{aligned} \rho_0(c) \colon H &\longrightarrow \mathbb{R} \\ h &\longmapsto c(h^{-1}). \end{aligned}$$

Consider

$$\begin{aligned} \tau_m \colon \ell^\infty(G \times H) &\longrightarrow \pi^*\ell^\infty(G) \\ \Psi &\longmapsto \big(g \longmapsto m(h \longmapsto \Psi(g,h))\big). \end{aligned}$$

Here $\pi^*\ell^\infty(G)$ denotes the $\mathbb{R}[G \times H]$-module having $\ell^\infty(G)$ as additive group (Definition 2.2.2). Using the left invariance of m (Definition 2.3.1-(iii)), it is easy to show that τ_m is a $\mathbb{R}[G \times H]$-module homomorphism. Let $\pi \colon G \times H \longrightarrow G$ is the standard projection homomorphism. Then the pair (π, τ_m) is a well-defined morphism in the category GrpMod.

Consider now $\rho_0(c) \in \ell^\infty(H)$ and the corresponding $\mathbb{R}[G]$-module homomorphism $P_{\rho(c)}$. It is easy to see that the composition $\tau_m \circ P_{\rho_0(c)}$ is given for any $\psi \in \ell^\infty(G)$ by:

$$\tau_m \circ P_{\rho_0(c)}(\psi)\colon g \longmapsto m(\rho_0(c)) \cdot \psi(g).$$

So the composition $\tau_m \circ P_{\rho_0(c)}$ is the multiplication by $m(\rho_0(c)) \in \mathbb{R}\backslash\{0\}$. Since $m(\rho_0(c)) \neq 0$, the pair $(\pi, \frac{1}{m(\rho_0(c))} \cdot \tau_m)$ is a well-defined morphism in the category GrpMod. Clearly $(\pi, \frac{1}{m(\rho_0(c))} \cdot \tau_m)$ is a left inverse for $(i, P_{\rho_0(c)})$, so the map induced by $(\pi, \frac{1}{m(\rho_0(c))} \cdot \tau_m)$ in homology is a left inverse for the cross-product $- \times [e \otimes \rho_0(c)]$. It follows that the cross-product map $- \times \alpha\colon H_*^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(G \times H;\mathbb{R})$ is injective. □

If H is amenable, the fundamental class $[H] = [\sum_{h\in H} h] \in H_0^{\mathrm{uf}}(H;\mathbb{R})$ is clearly not mean-invisible. Thus the cross-product $- \times [H]$ gives an injection $H_*^{\mathrm{uf}}(G;\mathbb{R}) \hookrightarrow H_*^{\mathrm{uf}}(G \times H;\mathbb{R})$.

On the other hand, if a class $\alpha \in H_0^{\mathrm{uf}}(H;\mathbb{R})\backslash\{0\}$ is mean-invisible, then the cross-product map is not injective in general.

Example 3.1.4. For any $k \in \mathbb{N}_{>1}$, the class $[n^k] = \left[\sum_{n\in\mathbb{N}} n^k\right] \in H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{R})$ is mean-invisible and non-trivial and the cross-product

$$- \times [n^k]\colon H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{R}) \longrightarrow H_0^{\mathrm{uf}}(\mathbb{Z}^2;\mathbb{R})$$

is not injective.

Sketch of the proof. For any $k \in \mathbb{N}_{>1}$, the class $[n^k] = \left[\sum_{n\in\mathbb{N}} n^k\right]$ is mean-invisible in $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{R})$ [6, Lemma 5.10]. Moreover, it is easy to see that $[n^k]$ is non-trivial in $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{R})$ since it cannot be bounded by a 1-chain having uniformly bounded coefficients and supported on simplices with uniformly bounded diameter. On the other hand, since the points (m^2, n^k) are "sparsely" distributed in $\mathbb{Z}^2$, there is enough space to construct infinite tails of 1-simplices in $\mathbb{Z}^2$ attached to any (m^2, n^k). These tails bound the class $[m^2] \times [n^k] = \left[\sum_{(m,n)\in\mathbb{N}^2} (m^2, n^k)\right]$. Thus, $[m^2] \times [n^k] = 0$ in $H_0^{\mathrm{uf}}(\mathbb{Z}^2;\mathbb{R})$. For a detailed proof of this result, see Matthias Blank's PhD thesis [5]. □

3.2 Products of non-amenable simplicial complexes

In this section we consider the cartesian product of non-amenable simplicial complexes endowed with the maximum metric. In particular, we prove the following:

Theorem 3.2.1. *Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let $X \times Y$ be the cartesian product of uniformly contractible, non-amenable simplicial complexes X, Y of bounded geometry. Then*

$$H_1^{\mathrm{uf}}(X \times Y; A) = 0.$$

We first define a suitable triangulation on the product of two simplicial complexes. Then we prove the theorem for the simplicial uniformly finite homology of the triangulated product.

The results contained in this section are part of a joint work with Piotr Nowak [16].

3.2.1 Simplicial structures on products

We consider ordered simplicial complexes (Definition 1.2.3). Following a classical construction due to Eilenberg and Steenrod [21, Chapter II.8], we give a simplicial structure to the cartesian product of two ordered simplicial complexes. More precisely:

Definition 3.2.2. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be ordered simplicial complexes. Let $\leq_{X\times Y}$ be the partial order on the cartesian product $V_X \times V_Y$ defined as follows:

$$\forall_{(x,y),(x',y') \in V_X \times V_Y} \quad (x, y) \leq_{X\times Y} (x', y') \Leftrightarrow x \leq_X x' \text{ in } V_X \text{ and } y \leq_Y y' \text{ in } V_Y.$$

The *triangulated cartesian product* $X \times_t Y$ of X and Y is the simplicial complex whose set of vertices is $V_X \times V_Y$. For any $n \in \mathbb{N}$, the set of simplices $\Delta_n(X \times_t Y)$ is given by all tuples $[(x_0, y_0), \dots, (x_n, y_n)] \in (V_X \times V_Y)^{n+1}$ such that $(x_0, y_0) \leq_{X\times Y} \cdots \leq_{X\times Y} (x_n, y_n)$.

To simplify the notation, we always use $\leq$ to denote the partial orders on X, Y or on $X \times_t Y$

Remark 3.2.3.
(i) *By Definition 1.2.3 and by the partial order on $X \times Y$ given in Definition 3.2.2, for any $n \in \mathbb{N}$ a simplex in $\Delta_n(X \times_t Y)$ is a tuple $[(x_0, y_0), \dots, (x_n, y_n)]$ with $x_0 \leq \cdots \leq x_n$ in V_X and $y_0 \leq \cdots \leq y_n$ in V_Y. In other words, $[x_0, \dots, x_n]$ is a (possibly degenerate) n-simplex in X and $[y_0, \dots, y_n]$ is a (possibly degenerate) n-simplex in Y.*

(ii) *Clearly $X \times_t Y$ is an ordered simplicial complex with respect to the partial order given in Definition 3.2.2.*

We want to consider uniformly contractible simplicial complexes (Definition 1.3.1).

Lemma 3.2.4. *The triangulated cartesian product of uniformly contractible ordered simplicial complexes of bounded geometry is a uniformly contractible simplicial complex of bounded geometry.*

Proof. If X, Y are of bounded geometry then $X \times_t Y$ is clearly of bounded geometry. We consider $X \times_t Y$ endowed with the ℓ^1-path metric. Suppose that X and Y are uniformly contractible. Let $r \in \mathbb{R}_{>0}$ and let $A \subset X \times_t X$ be a subset of $diam(A) < r$. Consider $\pi_X(A) \subset X$ and $\pi_Y(A) \subset Y$ where $\pi_X \colon X \times Y \longrightarrow X$ and $\pi_Y \colon X \times Y \longrightarrow Y$ are the standard projections. Clearly $diam(\pi_X(A)) < r$ in X and $diam(\pi_Y(A)) < r$ in Y. Since X and Y are both uniformly contractible there exist some $S_r^X, S_r^Y \in \mathbb{R}_{>0}$ such that the set $\pi_X(A) \subset X$ (resp. $\pi_Y(A) \subset Y$) is contractible to a point inside $N_{S_r^X}(\pi_X(A)) \subset X$ (resp. $N_{S_r^Y}(\pi_Y(A)) \subset Y$). By taking the product of the contracting homotopies of $\pi_X(A)$ and of $\pi_Y(A)$, we have that $\pi_X(A) \times \pi_Y(A)$ is contractible to a point in $N_{S_r^X}(\pi_X(A)) \times N_{S_r^Y}(\pi_Y(A))$. Let $S_r := \max\{S_r^X, S_r^Y\}$. It is easy to see that $X \times_t Y$ endowed with the ℓ^1-path metric is quasi-isometric to the cartesian product $X \times Y$ endowed with the maximum metric. Thus, for a suitable $\widetilde{S}_r \in \mathbb{R}_{>0}$ we have

$$N_{S_r^X}(\pi_X(A)) \times N_{S_r^Y}(\pi_Y(A)) \subseteq N_{S_r}(\pi_X(A) \times \pi_Y(A)) \subseteq N_{\widetilde{S}_r}(A).$$

Since $A \subset \pi_X(A) \times \pi_Y(A)$, it follows that A is contractible to a point inside $N_{\widetilde{S}_r}(A)$. Thus $X \times_t Y$ is uniformly contractible.

□

Let $(X, \leq)$ be any ordered simplicial complex. By Definition 1.2.3, for any $x, x' \in V_X$ the following holds

$$[x, x'] \in \Delta_1(X) \Leftrightarrow x \leq x'.$$

In particular, for any $x \in V_X$ there are two sets of vertices in V_X connected with x by a 1-simplex.

Definition 3.2.5. For any $x \in V_X$, let

$$A_x := \{x_A \in V_X \mid x_A > x\}.$$

$$B_x := \{x_B \in V_X \mid x_B < x\}.$$

From Definition 3.2.2, we can easily classify the 1-simplices in the triangulated cartesian product as follows:

Definition 3.2.6. Let $X \times_t Y$ be the triangulated cartesian product of two simplicial complexes X and Y. Any simplex in $\Delta_1(X \times_t Y)$ belongs to one of the following sets:

- The set of *X-simplices*:

$$\Delta_1(X \times_t Y)_X := \left\{ \left[(x,y),(x,y')\right] \mid y < y' \right\}.$$

These simplices are given by the product of a vertex in X and a 1-simplex in Y.

- The set of *Y-simplices*:

$$\Delta_1(X \times_t Y)_Y := \left\{ \left[(x,y),(x',y)\right] \mid x < x' \right\}.$$

These simplices are given by the product of a 1-simplex in X and a vertex in Y.

- The set of *diagonal simplices*:

$$\Delta_1(X \times_t Y)_{\text{diag}} := \left\{ \left[(x,y),(x',y')\right] \mid x < x', y < y' \right\}.$$

3.2.2 Proof of Theorem 3.2.1

In this section we give the proof of Theorem 3.2.1.

Let X, Y be simplicial complexes of bounded geometry and let $X \times_t Y$ be their triangulated cartesian product. Let $A \in \{\mathbb{R}, \mathbb{Z}\}$. We consider the simplicial uniformly finite chain complex $(C_n^{\text{suf}}(X \times_t Y; A), \partial_n)_{n\in\mathbb{N}}$ given in Definition 1.2.5. Following Definition 3.2.6, we can classify the chains in $C_1^{\text{suf}}(X \times_t Y; A)$ as follows:

Definition 3.2.7. We have the following chains in $C_1^{\text{suf}}(X \times_t Y; A)$:

- Any chain of the form $c_X = \sum_{\sigma_X \in \Delta_1(X\times_t Y)_X} c(\sigma_X) \cdot \sigma_X$ is a *X-chain*.
- Any chain of the form $c_Y = \sum_{\sigma_Y \in \Delta_1(X\times_t Y)_Y} c(\sigma_Y) \cdot \sigma_Y$ is a *Y-chain*.
- Any chain of the form $c_{\text{diag}} = \sum_{\sigma_{\text{diag}} \in \Delta_1(X\times_t Y)_{\text{diag}}} c(\sigma_{\text{diag}}) \cdot \sigma_{\text{diag}}$ is a *diagonal chain*.

A class in $H_1^{\text{suf}}(X \times_t Y; A)$ is an *X-class* (resp. a *Y-class*) if it is represented by an X-chain (resp. a Y-chain) that is a cycle in $C_1^{\text{suf}}(X \times_t Y; A)$.

Remark 3.2.8. *Every chain $c \in C_1^{\text{suf}}(X \times_t Y; A)$ can be written as*

$$c = c_X + c_Y + c_{\text{diag}}$$

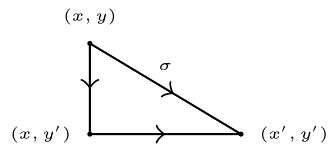
for some X-chain c_X, Y-chain c_Y and some diagonal chain c_{diag}.

We want to reduce any cycle in $C_1^{\text{suf}}(X \times_t Y; A)$ to a cycle suppported only on X and Y-simplices.

Lemma 3.2.9. *Let $X \times_t Y$ be the triangulated cartesian product of two simplicial complexes X, Y of bounded geometry. Let c be a cycle in $C_1^{\mathrm{suf}}(X \times_t Y; A)$. Then there exists a cycle b in $C_1^{\mathrm{suf}}(X \times_t Y; A)$ of the form $b = b_X + b_Y$ such that $[c] = [b]$ in $H_1^{\mathrm{suf}}(X \times_t Y; A)$.*

Proof. Let $c \in C_1^{\mathrm{suf}}(X \times_t Y; A)$ be a cycle. From Remark 3.2.8, we know that c can be written as $c = c_X + c_Y + c_{\mathrm{diag}}$, for some X-chain c_X, Y-chain c_Y and some diagonal chain c_{diag}. Let $\sigma \in \Delta_1(X \times_t Y)_{\mathrm{diag}}$ be a diagonal simplex belonging to the support of c. Then $\sigma = [(x,y),(x',y')]$ for some $[x,x'] \in \Delta_1(X)$ and $[y,y'] \in \Delta_1(Y)$ with $x < x'$ and $y < y'$. Consider:

$$\tau_\sigma = [(x,y),(x,y'),(x',y')].$$



By Definition 3.2.2, it is easy to see that $\tau_\sigma \in \Delta_2(X \times_t Y)$. Moreover

$$\partial \tau_\sigma = [(x,y'),(x',y')] - \sigma + [(x,y),(x,y')]. \tag{3.1}$$

Thus, $\sigma + \partial\tau_\sigma$ is the sum of an X-simplex and a Y-simplex in $\Delta_1(X \times_t Y)$. For any diagonal simplex σ_{diag} in the support of c, construct $\tau_{\sigma_{\mathrm{diag}}}$ as above. Then, define:

$$\varphi = \sum_{\sigma_{\mathrm{diag}} \in \Delta_1(X \times_t Y)_{\mathrm{diag}}} c(\sigma_{\mathrm{diag}}) \cdot \tau_{\sigma_{\mathrm{diag}}}.$$

Notice that this is a (infinite) sum of simplices in $\Delta_2(X \times_t Y)$ having uniformly bounded coefficients. Indeed, any simplex $\tau_{\sigma_{\mathrm{diag}}}$ contained in the support of φ has coefficient $c(\sigma_{\mathrm{diag}}) \in A$ and since c is bounded by Definition 1.1.4 -(a), φ is also bounded. So φ is a well-defined element in $C_2^{\mathrm{suf}}(X \times_t Y; A)$. By (3.1), we have

$$c + \partial\varphi = c_X + c_Y + c_{\mathrm{diag}} + \partial\left(\sum_{\sigma_{\mathrm{diag}} \in \Delta_1(X \times_t Y)_{\mathrm{diag}}} c(\sigma_{\mathrm{diag}}) \cdot \tau_{\sigma_{\mathrm{diag}}}\right) = b_X + b_Y.$$

for some X-chain b_X and some Y-chain b_Y. Then for $b := b_X + b_Y$ we have $[c] = [b] \in H_1^{\mathrm{suf}}(X \times_t Y; A)$ and the claim follows. $\square$

We consider now non-amenable simplicial complexes of bounded geometry. Notice that a simplicial complex of bounded geometry X with the ℓ^1-path metric is a metric space of coarse bounded geometry. We have a notion of amenability for such metric spaces (Definition 1.4.5) and we can define X to be an amenable simplicial complex if it is amenable as metric space of coarse bounded geometry. The next lemma shows that if X is non-amenable (respectively, Y non-amenable), then any cycle supported only on X-simplices (respectively, on Y-simplices) gives a trivial class in $H_1^{\mathrm{suf}}(X \times_t Y; A)$.

Lemma 3.2.10. *Let $X \times_t Y$ be their triangulated cartesian product of two simplicial complexes X, Y of bounded geometry. Then*

(i) If X is non-amenable, any X-class is trivial in $H_1^{\mathrm{suf}}(X \times_t Y; A)$.

(ii) If Y is non-amenable, any Y-class is trivial in $H_1^{\mathrm{suf}}(X \times_t Y; A)$.

Proof. We prove (i) and the same argument can be used to prove (ii). Suppose X is non-amenable and let $\alpha \in H_1^{\mathrm{suf}}(X \times_t Y; A)$ be a X-class. By Definition 3.2.7, α is represented by a cycle $c = \sum_{\sigma_X \in \Delta_1(X \times_t Y)_X} c(\sigma_X) \cdot \sigma_X$. We want to find a chain in $C_2^{\mathrm{suf}}(X \times_t Y; A)$ that bounds c. For any X-simplex σ_X appearing in c_X we construct a "tail" of 2-simplices in $X \times_t Y$ attached to σ_X (Figure 3.1). Since X is non-amenable and $V_X \subset X$ is a quasi-lattice in X, by Proposition 1.4.6, we have $[V_X] = [\sum_{x \in V_X} x] = 0$ in $H_0^{\mathrm{suf}}(X; A)$. Following the construction given in Section 1.4.1, for any vertex $x \in V_X$, we can find a tail t_x of 1-simplices of the form (1.6). For simplicity, we suppose that the simplices in any tail t_x are all oriented in the same direction towards x (if not, we can always invert the orientation and add a sign -1 as in (1.6)). More precisely, for any $x \in V_X$ we have

$$t_x = \sum_{j \in \mathbb{Z}_{\leq 0}} [x_{j-1}, x_j] \in C_1^{\mathrm{suf}}(X; A)$$

such that for any $j \in \mathbb{Z}_{\leq 0}$ we have $x_{j-1} < x_j$ in X and for $j = 0$ we have $x_0 = x$. Clearly, $\partial t_x = x$. Since X is non-amenable, by the Eilenberg-swindle construction given in Section 1.4.1, for any $x \in V_X$ we can find t_x such that $\sum_{x \in X} t_x \in C_1^{\mathrm{suf}}(X; A)$ and $\partial\left(\sum_{x \in X} t_x\right) = \sum_{x \in V_X} x$. Any tail can be considered as a Y-chain in $C_1^{\mathrm{suf}}(X \times_t Y; A)$. More precisely, for any vertex $(x, y) \in V_{X \times_t Y}$, we can define

$$t_{(x,y)} = \sum_{j \in \mathbb{Z}_{\leq 0}} \big[(x_{j-1}, y), (x_j, y)\big] \in C_1^{\mathrm{suf}}(X \times_t Y; A).$$

such that for any $j \in \mathbb{Z}_{\leq 0}$ we have $(x_{j-1}, y) < (x_j, y)$ and $(x_0, y) = (x, y)$.

Now let $\sigma = [(x, y), (x, y')]$ be a simplex in $\Delta_1(X \times_t Y)_X$ appearing in c. Let $t_{(x,y)}$ be a tail as above. Consider:

$$p_\sigma = \sum_{j \in \mathbb{Z}_{\leq 0}} [(x_{j-1}, y), (x_j, y), (x_j, y')] - [(x_{j-1}, y), (x_{j-1}, y'), (x_j, y')]. \quad (3.2)$$

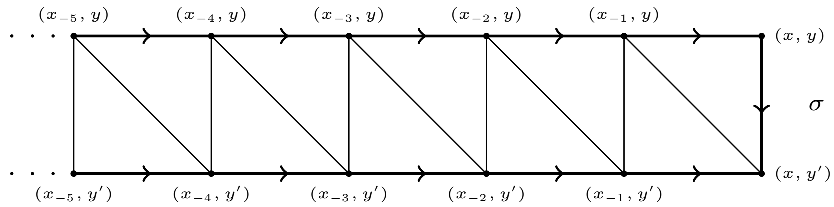


Figure 3.1: A panel of 2-simplices p_σ attached to $\sigma = [(x,y),(x,y')]$.

Notice that p_σ is given by the cross-product between $t_x \in C_1^{\mathrm{suf}}(X;A)$ and the simplex $[y,y'] \in \Delta_1(Y)$. Indeed, as we observed in Remark 3.1.2, this cross-product map is defined following the classical construction of cross-product in simplicial homology. Clearly $p_\sigma \in C_2^{\mathrm{suf}}(X \times_t Y; A)$. Moreover, we have

$$\begin{aligned}\partial p_\sigma &= [(x,y),(x,y')] + \sum_{j \in \mathbb{Z}_{\leq 0}} [(x_{j-1},y),(x_j,y)] - [(x_{j-1},y'),(x_j,y')] \\ &= \sigma + t_{(x,y)} - t_{(x,y')}. \end{aligned} \tag{3.3}$$

We can construct p_{σ_X} as above for any $\sigma_X \in \Delta_1(X \times_t Y)_X$ appearing in c. We call p_{σ_X} a *panel of 2-simplices attached to* σ_X. Then we define:

$$\varphi = \sum_{\sigma_X \in \Delta_1(X \times_t Y)_X} c(\sigma_X) \cdot p_{\sigma_X}. \tag{3.4}$$

Notice that φ is a well-defined element in $C_2^{\mathrm{suf}}(X \times_t Y; A)$. Indeed, the simplices appearing in φ are of one of the following forms:

(a) $[(x,y),(x',y),(x',y')] \in \Delta_2(X \times_t Y)$ for some $x < x'$ in X and some $y < y'$ in Y

(b) $[(x,y),(x,y'),(x',y')] \in \Delta_2(X \times_t Y)$ for some $x < x'$ in X and some $y < y'$ in Y.

Every simplex of the form (a) appears in φ with coefficient

$$\sum_{\widetilde{x} \in E(x,x')} c([(\widetilde{x},y),(\widetilde{x},y')]),$$

while every simplex of the form (b) appears in φ with coefficient

$$-\sum_{\widetilde{x} \in E(x,x')} c([(\widetilde{x},y),(\widetilde{x},y')])$$

where $E(x,x') := \{\tilde{x} \in V_X \mid t_{\tilde{x}} \text{ passes through } [x,x'] \in \Delta_1(X \times_t Y)\}$. Since X is non-amenable, from the construction given in Section 1.4.1 there exists $K > 0$ such that for any $[x,x'] \in \Delta_1(X)$ we have $|E(x,x')| < K$. Thus, the coefficients of φ are uniformly bounded. From (3.3), it follows that

$$\partial\varphi = \sum_{\sigma_X=[(x,y),(x,y')]\in\Delta_1(X\times_t Y)_X} c(\sigma_X)\cdot(\sigma_X + t_{(x,y)} - t_{(x,y')}). \tag{3.5}$$

So to prove that $\partial\varphi = c$, it suffices to show that

$$\sum_{\sigma_X=[(x,y),(x,y')]\in\Delta_1(X\times_t Y)_X} c(\sigma_X)\cdot(t_{(x,y)} - t_{(x,y')}) = 0. \tag{3.6}$$

Now, for any $y \in V_Y$ consider the sets A_y and B_y given in Definition 3.2.5. Since c is a cycle in $C_1^{\mathrm{suf}}(X \times_t Y; A)$, we have $\partial c = 0$. In particular, for any $(x,y) \in V_{X\times_t Y}$, we have

$$\partial c(x,y) = \sum_{y_B\in B_y} c([(x,y_B),(x,y)]) - \sum_{y_A\in A_y} c([(x,y),(x,y_A)]) = 0. \tag{3.7}$$

On the other hand, from (3.5), it is easy to see that for every $(x,y) \in V_{X\times_t Y}$, all the X-simplices in $X \times_t Y$ having (x,y) as a vertex contribute to the coefficient of $t_{(x,y)}$ in $\partial\varphi$ (Figure 3.2). More precisely, for any $(x,y) \in V_{X\times_t Y}$ the coefficient of $t_{(x,y)}$ in $\partial\varphi$ is given by

$$\sum_{y_A\in A_y} c([(x,y),(x,y_A)]) - \sum_{y_B\in B_y} c([(x,y_B),(x,y)])$$

which is zero by (3.7). So (3.6) follows and we have $\partial\varphi = c$. Thus $\alpha = 0$. $\square$

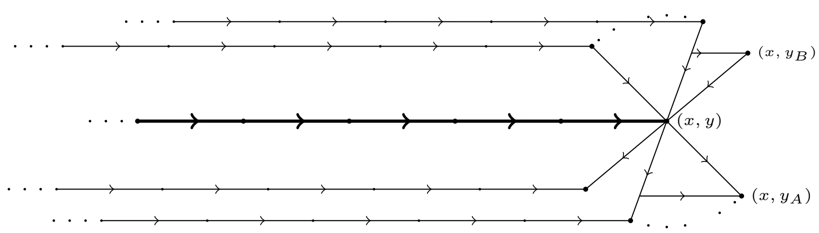


Figure 3.2: Different panels of type p_{σ_X} meeting at the tail $t_{(x,y)}$.

As last step before the proof of Theorem 3.2.1, we reduce any class in $H_1^{\mathrm{suf}}(X \times_t Y; A)$ to a Y-class.

Lemma 3.2.11. *Let $X \times_t Y$ be the triangulated cartesian product of two simplicial complexes X, Y of bounded geometry. Let $c \in C_1^{\mathrm{suf}}(X \times_t Y; A)$ be a cycle. Then* *(i) If X is non-amenable, there exists a Y-chain $b_Y \in C_1^{\mathrm{suf}}(X \times_t Y; A)$ that is a cycle in $C_1^{\mathrm{suf}}(X \times_t Y; A)$ such that $[c] = [b_Y]$ in $H_1^{\mathrm{suf}}(X \times_t Y; A)$.*

(ii) If Y is non-amenable, there exists a X-chain $b_X \in C_1^{\mathrm{suf}}(X \times_t Y; A)$ that is a cycle in $C_1^{\mathrm{suf}}(X \times_t Y; A)$ such that $[c] = [b_X]$ in $H_1^{\mathrm{suf}}(X \times_t Y; A)$.

Proof. We prove (i) and we notice that (ii) can be proved using similar arguments. Suppose X is non-amenable and let $c \in C_1^{\mathrm{suf}}(X \times_t Y)$ be a cycle. By Lemma 3.2.9, we can assume $c = c_X + c_Y$ for some X-chain of the form $c_X = \sum_{\sigma_X \in \Delta_1(X \times_t Y)_X} c(\sigma_X) \cdot \sigma_X$ and some Y-chain of the form $c_Y = \sum_{\sigma_Y \in \Delta_1(X \times_t Y)_Y} c(\sigma_Y) \cdot \sigma_Y$. We want to find a $\varphi \in C_2^{\mathrm{suf}}(X \times_t Y; A)$ such that $c - \partial\varphi = b_Y$ for some Y-chain b_Y. We proceed as in Lemma 3.2.10. In particular, using the Eilenberg-swindle construction on X, for any X-simplex σ_X appearing in c_X we construct a panel p_{σ_X} as in (3.2) and we define φ as in (3.4). By (3.5), we have

$$\partial\varphi = c_X + \sum_{\sigma_X = [(x,y),(x,y')] \in \Delta_1(X \times_t Y)_X} c(\sigma_X) \cdot (t_{(x,y)} - t_{(x,y')}).$$

Notice that for any $(x,y) \in V_{X \times_t Y}$ the tail $t_{(x,y)} = \sum_{j \in \mathbb{Z}_{\leq 0}} [(x_{j-1}, y), (x_j, y)]$ is a Y-chain in $C_1^{\mathrm{suf}}(X \times_t Y; A)$ since it is supported only on Y-simplices in $X \times_t Y$. Thus

$$b_Y := c - \partial\varphi = c_Y - \sum_{\sigma_X = [(x,y),(x,y')] \in \Delta_1(X \times_t Y)_X} c(\sigma_X) \cdot (t_{(x,y)} - t_{(x,y')})$$

is a Y-chain and a cycle in $C_1^{\mathrm{suf}}(X \times_t Y; A)$. Clearly $[b_Y] = [c]$, so the claim follows. □

Then, Theorem 3.2.1 is an immediate consequence of Lemma 3.2.10 and Lemma 3.2.11.

Proof of Theorem 3.2.1. Let X, Y be uniformly contractible, non-amenable simplicial complexes of bounded geometry. From Lemma 3.2.10 and from Lemma 3.2.11 we have that $H_1^{\mathrm{suf}}(X \times_t Y; A) = 0$. Since $X \times_t Y$ is also uniformly contractible and has bounded geometry, from Proposition 1.3.3 it follows that $H_1^{\mathrm{uf}}(X \times_t Y; A) = 0$. It is easy to see that the cartesian product $X \times Y$ endowed with the maximum metric is quasi-isometric to the triangulated cartesian product $X \times_t Y$. Thus, by Proposition 1.1.5, we have that $H_1^{\mathrm{uf}}(X \times Y; A) \cong H_1^{\mathrm{uf}}(X \times_t Y; A) = 0$. □

3.2.3 Characterization of amenability in degree 1

Theorem 1.4.1 gives a characterization of amenability using the uniformly finite homology in degree zero. Using the results seen in the previous sections of this chapter and the computation of uniformly finite homology of a tree given in Section 1.5.2, we give a characterization of amenability for certain finitely generated groups using uniformly finite homology in degree 1.

Theorem 3.2.12. *Let G be a finitely generated group acting by isometries on a uniformly contractible simplicial complex of bounded geometry and let F_2 be the free group of rank 2. Suppose the action of G is proper and cocompact. Then G is non-amenable if and only if $H_1^{\mathrm{uf}}(G \times F_2; \mathbb{R}) = 0$.*

Proof. Since the Cayley graph of F_2 is a uniformly locally finite, non-amenable tree, by Theorem 1.5.7, we have that $H_1^{\mathrm{uf}}(F_2; \mathbb{R})$ is infinite dimensional. Let $[G]$ be the fundamental class of G in $H_0^{\mathrm{uf}}(G; \mathbb{R})$. If G is amenable, then by Theorem 3.1.3 the cross-product map $- \times [G]$ gives an injection $H_1^{\mathrm{uf}}(F_2; \mathbb{R}) \longrightarrow H_1^{\mathrm{uf}}(G \times F_2; \mathbb{R})$. It follows that $H_1^{\mathrm{uf}}(G \times F_2; \mathbb{R}) \neq 0$ (indeed, it is infinite dimensional). Suppose G acts on a uniformly locally finite, uniformly contractible simplicial complex X. Then, by the Švarc-Milnor lemma, G is quasi-isometric to X. Thus, if G is non-amenable, by Theorem 3.2.1 we have that $H_1^{\mathrm{uf}}(G \times F_2; \mathbb{R}) \cong H_1^{\mathrm{uf}}(X \times \mathrm{Cay}(F_2); \mathbb{R}) = 0$. □

3.3 Products of trees

In this section we compute the uniformly finite homology of products of trees. We consider non-amenable trees as 1-dimensional ordered simplicial complexes. We first consider the product of two trees and then, using similar techniques as the ones seen in the precious section, we compute the uniformly finite homology of the product of three trees.

3.3.1 Uniformly finite homology of the product of two trees

Let T be any non-amenable, uniformly locally finite tree. Denote by V_T the set of its vertices and by E_T the set of its edges. We consider T as a 1-dimensional simplicial complex and we consider an order $\leq$ on V_T as in Definition 1.2.3. In particular, for any $x, x' \in V_T$ we have

$$[x, x'] \in E_T \Leftrightarrow x < x'.$$

Theorem 3.3.1. *Let $T_{\mathbf{x}} \times T_{\mathbf{y}}$ be the cartesian product of uniformly locally finite non-amenable trees $T_{\mathbf{x}}, T_{\mathbf{y}}$. Let $A \in \{\mathbb{R}, \mathbb{Z}\}$. Then*

$$H_n^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}}; A) = \begin{cases} \text{infinite dimensional} & \text{if } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

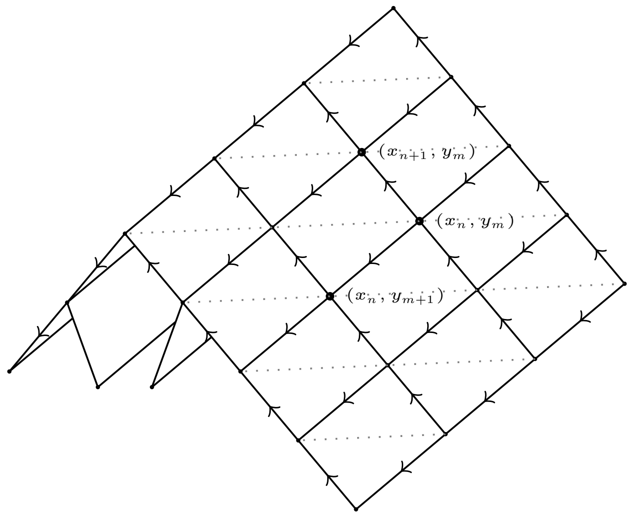


Figure 3.3: Construction of a 2-cycle in the product of two trees using bi-infinite paths in two directions.

Proof. The case $n = 0$ follows from Theorem 1.4.1. Since $T_{\mathbf{x}}$ and $T_{\mathbf{y}}$ are uniformly contractible simplicial complexes of bounded geometry, the case $n = 1$ follows from Theorem 3.2.1. Since the cartesian product $T_{\mathbf{x}} \times T_{\mathbf{y}}$ is quasi-isometric to the triangulated cartesian product, it suffices to prove the statement for $T_{\mathbf{x}} \times_t T_{\mathbf{y}}$. By Lemma 3.2.4, we know that $T_{\mathbf{x}} \times_t T_{\mathbf{y}}$ is a uniformly contractible simplicial complex of bounded geometry. Moreover, since $T_{\mathbf{x}} \times_t T_{\mathbf{y}}$ is 2-dimensional, by Proposition 1.3.3 it follows that $H_n^{\mathrm{uf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A) \cong H_n^{\mathrm{suf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A) = 0$ for $n > 2$.

We consider the case $n = 2$. Let $(x_n)_{n \in \mathbb{Z}}$ be a bi-infinite path in $T_{\mathbf{x}}$ and $(y_m)_{m \in \mathbb{Z}}$ a bi-infinite path in $T_{\mathbf{y}}$ (Definition 1.5.2). For simplicity, suppose that the order on the vertices of $T_{\mathbf{x}}$ and $T_{\mathbf{y}}$ is given in such a way that for any $n, m \in \mathbb{Z}$ we have $x_{n-1} < x_n$ in $T_{\mathbf{x}}$ and $y_{m-1} < y_m$ in $T_{\mathbf{y}}$ (for more general bi-infinite path it suffices to change the orientation of some edges and add a sign -1). Consider

$$\begin{aligned} c = & \sum_{n,m \in \mathbb{Z}} [(x_n, y_{m-1}), (x_n, y_m), (x_{n+1}, y_m)] \\ & - \sum_{n,m \in \mathbb{Z}} [(x_n, y_{m-1}), (x_{n+1}, y_{m-1}), (x_{n+1}, y_m)]. \end{aligned}$$

An easy computation shows that c is a cycle in $C_2^{\mathrm{suf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A)$ (Figure 3.3). Moreover, since $T_{\mathbf{x}} \times_t T_{\mathbf{y}}$ is 2-dimensional for any $n > 2$ we have $C_n^{\mathrm{suf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A) = 0$. It follows that c represents a non-trivial class in $H_2^{\mathrm{suf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A)$. In Theorem 1.5.7 and Proposition 1.5.5, we have seen that in any non-amenable tree T it is possible to construct infinitely many disjoint bi-infinite paths which give rise to infinitely many linearly independent non-trivial classes in $H_1^{\text{suf}}(T;A)$. In the same way, by choosing different bi-infinite paths in $T_{\mathbf{x}}$ and in $T_{\mathbf{y}}$ we can find infinitely many linearly independent cycles like c having disjoint support. It follows that $H_2^{\text{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}}; A) \cong H_2^{\text{uf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A) \cong H_2^{\text{suf}}(T_{\mathbf{x}} \times_t T_{\mathbf{y}}; A)$ is infinite dimensional. □

We have already seen some examples of product of amenable trees. Indeed, from Example 2.3.15 it follows that the uniformly finite homology of $\mathbb{Z} \times \mathbb{Z}$ is infinite dimensional in degree 0 and 1, is isomorphic to A in degree 2 and it is trivial otherwise.

We have the following corollary for groups acting by isometries on product of non-amenable trees:

Corollary 3.3.2. *Let G be a group acting by isometries on a product of two uniformly locally finite, non-amenable trees. Suppose the action is proper and cocompact. Then*

$$H_n^{\text{uf}}(G;A) = \begin{cases} \textit{infinite dimensional} & \textit{if } n = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is an immediate consequence of Theorem 3.3.1 and of the Švarc-Milnor lemma. □

Reiter Ahlin [37] showed that any finitely generated group that is quasi-isometric to a product of bushy or linelike trees contains a finite index subgroup which is a lattice in the isometry group of a new product of trees. Moreover, these trees are quasi-isometric to the original ones. These lattices can be reducible if they are commensurable to a product of lattices in the factors or irreducible otherwise. Burger and Mozes studied lattices in the automorphism group of the product of trees an provided examples of irreducible, cocompact lattices which are not residually finite [13].

3.3.2 Uniformly finite homology of the product of three trees

We now consider the triangulated cartesian product of non-amenable, uniformly locally finite ordered trees $T_{\mathbf{x}}$, $T_{\mathbf{y}}$ and $T_{\mathbf{z}}$. We want to prove the following:

Theorem 3.3.3. *Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let $T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}$ be the cartesian product of uniformly locally finite, non-amenable trees $T_{\mathbf{x}}, T_{\mathbf{y}}, T_{\mathbf{z}}$. Then*

$$H_n^{\text{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A) = \begin{cases} \textit{infinite dimensional} & \textit{if } n = 3 \\ 0 & \textit{otherwise.} \end{cases}$$

We will see that for all $n \neq 2$ the theorem can be proved with similar arguments as the ones used in the proof of Theorem 3.3.1. We focus our attention on the case $n = 2$. Following Section 3.2, we give a simplicial structure to the space $T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}$ and we work on chains supported on 2-simplices.

As in Definition 3.2.2, we can define a partial order on $V_{T_{\mathbf{x}}} \times V_{T_{\mathbf{y}}} \times V_{T_{\mathbf{z}}}$ as follows: for any $(x,y,z),(x',y',z') \in V_{T_{\mathbf{x}}} \times V_{T_{\mathbf{y}}} \times V_{T_{\mathbf{z}}}$ we have

$$(x,y,z) \leq (x',y',z') \Leftrightarrow x \leq x' \text{ in } V_{T_{\mathbf{x}}},\ y \leq y' \text{ in } V_{T_{\mathbf{y}}} \text{ and } z \leq z' \text{ in } V_{T_{\mathbf{z}}}.$$

In this way, the triangulated cartesian product $T_t^3 := T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ is the simplicial complex with vertex set $V_{T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}} = V_{T_{\mathbf{x}}} \times V_{T_{\mathbf{y}}} \times V_{T_{\mathbf{z}}}$ and whose simplices are totally ordered tuples by the binary relation above. In particular, any 2-simplex of $T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ is of the form

$$[(x,y,z),(x',y',z'),(x'',y'',z'')]$$

with $(x,y,z) \leq (x',y',z') \leq (x'',y'',z'')$. Notice that, since $T_{\mathbf{x}}$ is a 1-dimensional simplicial complex, we have $x \leq x' \leq x''$ in $V_{T_{\mathbf{x}}}$ if and only if one of the following situation occurs:

(i) $x = x' = x''$ in $V_{T_{\mathbf{x}}}$.

(ii) $x < x' = x''$ in $V_{T_{\mathbf{x}}}$ (in particular, $[x,x']$ in $E_{T_{\mathbf{x}}}$).

(iii) $x = x' < x''$ in $V_{T_{\mathbf{x}}}$ (in particular, $[x',x'']$ in $E_{T_{\mathbf{x}}}$).

So, either the vertices $x \leq x' \leq x''$ in $V_{T_{\mathbf{x}}}$ are all equal or they change only once. The same holds for $y \leq y' \leq y''$ in $T_{\mathbf{y}}$ and for $z \leq z' \leq z''$ in $T_{\mathbf{z}}$. Following Definition 3.2.6, we can classify all the simplices in $\Delta_2(T_t^3)$ as follows:

Definition 3.3.4. Let $T_t^3 = T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ be the triangulated cartesian product of uniformly locally finite ordered trees $T_{\mathbf{x}}, T_{\mathbf{y}}$ and $T_{\mathbf{z}}$. Any non-degenerate simplex belongs to one of the following sets:

- The simplex σ is an $\mathbf{x}$-*simplex*, i.e. $\sigma = [(x,y,z),(x,y',z'),(x,y'',z'')]$ with
$$y < y' = y'' \text{ or } y = y' < y'' \in V_{T_{\mathbf{y}}},$$
$$z < z' = z'' \text{ or } z = z' < z'' \in V_{T_{\mathbf{z}}}.$$

- The simplex σ is a $\mathbf{y}$-*simplex*, i.e. $\sigma = [(x,y,z),(x',y,z'),(x'',y,z'')]$ with
$$x < x' = x'' \text{ or } x = x' < x'' \in V_{T_{\mathbf{x}}},$$
$$z < z' = z'' \text{ or } z = z' < z'' \in V_{T_{\mathbf{z}}}.$$

- The simplex σ is a $\mathbf{z}$-*simplex*, i.e. $\sigma = [(x,y,z),(x',y',z),(x'',y'',z)]$ with
$$x < x' = x'' \text{ or } x = x' < x'' \in V_{T_{\mathbf{x}}},$$
$$y < y' = y'' \text{ or } y = y' < y'' \in V_{T_{\mathbf{y}}}.$$

We denote by $\Delta_2(T_t^3)_{\mathbf{x}}$ the set of $\mathbf{x}$-simplices, by $\Delta_2(T_t^3)_{\mathbf{y}}$ the set of $\mathbf{y}$-simplices and by $\Delta_2(T_t^3)_{\mathbf{z}}$ the set of $\mathbf{z}$-simplices. If a non-degenerate simplex in $\Delta_2(T_t^3)$ does not belong to any of these sets, then σ is a *diagonal simplex*. The set of diagonal simplices is denoted by $\Delta_2(T_t^3)_{\mathrm{diag}}$.

Notice that T_t^3 has also the structure of a cube complex. In particular, the diagonal simplices are exactly those which do not lie on the boundary of any 3-cube in T_t^3. Let $A \in \{\mathbb{R}, \mathbb{Z}\}$. Since T_t^3 is a simplicial complex of bounded geometry, we can consider its simplicial uniformly finite chain complex $(C_*^{\mathrm{suf}}(T_t^3;A), \partial_*)$. Following Definition 3.2.7 we can classify the elements in $C_2^{\mathrm{suf}}(T_t^3;A)$ as follows:

Definition 3.3.5. Let $T_t^3 = T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ be the triangulated cartesian product of uniformly locally finite trees $T_{\mathbf{x}}, T_{\mathbf{y}}$ and $T_{\mathbf{z}}$. In $C_2^{\mathrm{suf}}(T_t^3;A)$ we can find the following chains:

- Any chain of the form $c_{\mathbf{x}} = \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot \sigma_{\mathbf{x}}$ is an $\mathbf{x}$-*chain*.
- Any chain of the form $c_{\mathbf{y}} = \sum_{\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}}) \cdot \sigma_{\mathbf{y}}$ is a $\mathbf{y}$-*chain*.
- Any chain of the form $c_{\mathbf{z}} = \sum_{\sigma_{\mathbf{z}} \in \Delta_2(T_t^3)_{\mathbf{z}}} c(\sigma_{\mathbf{z}}) \cdot \sigma_{\mathbf{z}}$ is a $\mathbf{z}$-*chain*.
- Any chain of the form $c_{\mathrm{diag}} = \sum_{\sigma_{\mathrm{diag}} \in \Delta_2(T_t^3)_{\mathrm{diag}}} c(\sigma_{\mathrm{diag}}) \cdot \sigma_{\mathrm{diag}}$ is a *diagonal chain*.

A class in $H_2^{\mathrm{suf}}(T_t^3;A)$ is called an $\mathbf{x}$-*class* (resp. $\mathbf{y}$-*class* and $\mathbf{z}$-*class*) if it is represented by an $\mathbf{x}$-chain (resp. $\mathbf{y}$-chain and $\mathbf{z}$-chain) that is a cycle in $C_2^{\mathrm{suf}}(T_t^3;A)$.

Remark 3.3.6. *Every chain $c \in C_2^{\mathrm{suf}}(T^3;A)$ can be written as*
$$c = c_{\mathbf{x}} + c_{\mathbf{y}} + c_{\mathbf{z}} + c_{\mathrm{diag}}$$
for some $\mathbf{x}$-chain $c_{\mathbf{x}}$, $\mathbf{y}$-chain $c_{\mathbf{y}}$, $\mathbf{z}$-chain $c_{\mathbf{z}}$ and some diagonal chain c_{diag}.

Similarly to Lemma 3.2.9, we can restrict to classes in $H_2^{\mathrm{suf}}(T_t^3;A)$ represented by cycles supported on the boundary of any cube in T_t^3. In other words, we can restrict to cycles c with $c_{\mathrm{diag}} = 0$.

Lemma 3.3.7. *Let $T_t^3 = T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ be the triangulated cartesian product of uniformly locally finite trees $T_{\mathbf{x}}, T_{\mathbf{y}}, T_{\mathbf{z}}$. Let $c \in C_2^{\mathrm{suf}}(T_t^3;A)$ be a cycle. Then there is a cycle $b \in C_2^{\mathrm{suf}}(T_t^3;A)$ of the form $b = b_{\mathbf{x}} + b_{\mathbf{y}} + b_{\mathbf{z}}$ such that $[c] = [b]$ in $H_2^{\mathrm{suf}}(T_t^3;A)$.*

Proof. Let $c = c_{\mathbf{x}} + c_{\mathbf{y}} + c_{\mathbf{z}} + c_{\mathrm{diag}} \in C_2^{\mathrm{suf}}(T_t^3; A)$ be a cycle. We prove the lemma by finding a chain $\varphi \in C_3^{\mathrm{suf}}(T_t^3; A)$ such that $c - \partial\varphi$ is of the form $b_{\mathbf{x}} + b_{\mathbf{y}} + b_{\mathbf{z}}$ for some $\mathbf{x}$-chain $b_{\mathbf{x}}$, some $\mathbf{y}$-chain $b_{\mathbf{y}}$ and some $\mathbf{z}$-chain $b_{\mathbf{z}}$ in $C_2^{\mathrm{suf}}(T_t^3; A)$. In other words, we want to find a $\varphi \in C_3^{\mathrm{suf}}(T_t^3; A)$ whose boundary can "kill" all the diagonal simplices in c. For any $(x, y, z) \in V_{T_t^3}$, consider $(x', y', z') \in V_{T_t^3}$ with $x' > x$ in $T_{\mathbf{x}}$, $y' > y$ in $T_{\mathbf{y}}$ and $z' > z$ in $T_{\mathbf{z}}$. It is easy to see that the simplex $e = [(x, y, z), (x', y', z')] \in \Delta_1(T_t^3)$ is the face of six different diagonal 2-simplices contained in a cube in T_t^3 (Figure 3.4). More precisely, all the simplices in $\Delta_2(T_t^3)$ having e as a face are:

- $\sigma_0 = [(x, y, z), (x', y, z), (x', y', z')] \in \Delta_2(T_t^3)_{diag}$;
- $\sigma_1 = [(x, y, z), (x', y', z), (x', y', z')] \in \Delta_2(T_t^3)_{diag}$;
- $\sigma_2 = [(x, y, z), (x, y', z), (x', y', z')] \in \Delta_2(T_t^3)_{diag}$;
- $\sigma_3 = [(x, y, z), (x, y', z'), (x', y', z')] \in \Delta_2(T_t^3)_{diag}$;
- $\sigma_4 = [(x, y, z), (x, y, z'), (x', y', z')] \in \Delta_2(T_t^3)_{diag}$;
- $\sigma_5 = [(x, y, z), (x', y, z'), (x', y', z')] \in \Delta_2(T_t^3)_{diag}$.

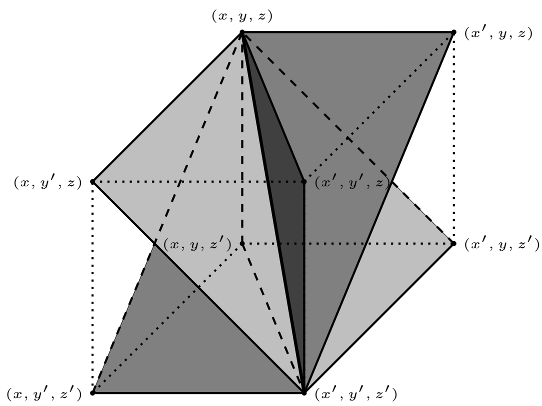


Figure 3.4: The diagonal 2-simplices sharing the face $[(x, y, z), (x', y', z')]$ in a triangulated cube in T_t^3.

Suppose that for any $i \in \{0, \ldots, 5\}$ the simplex σ_i appears in c with coefficient c_i. Then, since c is a cycle, it is easy to see that

$$c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = 0. \tag{3.8}$$

We "connect" the diagonal simplices listed above with 3-simplices in T_t^3. More precisely, consider the following simplices in $\Delta_3(T_t^3)$:

- $\tau_0 = [(x,y,z),(x',y,z),(x',y',z),(x',y',z')] \in \Delta_3(T_t^3)$. This simplex has σ_0 and σ_1 as faces;
- $\tau_1 = [(x,y,z),(x,y',z),(x',y',z),(x',y',z')] \in \Delta_3(T_t^3)$. This simplex has σ_1 and σ_2 as faces;
- $\tau_2 = [(x,y,z),(x,y',z),(x,y',z'),(x',y',z')] \in \Delta_3(T_t^3)$. This simplex has σ_2 and σ_3 as faces;
- $\tau_3 = [(x,y,z),(x,y,z'),(x,y',z'),(x',y',z')] \in \Delta_3(T_t^3)$. This simplex has σ_3 and σ_4 as faces;
- $\tau_4 = [(x,y,z),(x,y,z'),(x',y,z'),(x',y',z')] \in \Delta_3(T_t^3)$. This simplex has σ_4 and σ_5 as faces.

Now define:

$$d_{[(x,y,z),(x',y',z')]} := d_0 \cdot \tau_0 - d_1 \cdot \tau_1 + d_2 \cdot \tau_2 - d_3 \cdot \tau_3 + d_4 \cdot \tau_4$$

for $d_0 := c_0$, $d_1 := c_0 + c_1$, $d_2 := c_0 + c_1 + c_2$, $d_3 := c_0 + c_1 + c_2 + c_3$, $d_4 := c_0 + c_1 + c_2 + c_3 + c_4 \in A$. Notice that, by (3.8), we have $d_4 = -c_5$. An easy computation shows that

$$c'_{[(x,y,z),(x',y',z')]} := \sum_{i=0}^{5} c_i \cdot \sigma_i - \partial d_{[(x,y,z),(x',y',z')]} \tag{3.9}$$

does not contain any simplex in $\Delta_2(T_t^3)_{\mathrm{diag}}$. In particular, $c'_{[(x,y,z),(x',y',z')]}$ is supported only on $\mathbf{x}$-simplices, $\mathbf{y}$-simplices and $\mathbf{z}$-simplices. Now, define

$$\varphi = \sum_{(x,y,z)\in V_{T_t^3}} \sum_{x'>x,y'>y,z'>z} d_{[(x,y,z),(x',y',z')]}.$$

Notice that any diagonal simplex in $\Delta_2(T_t^3)$ is contained in a unique 3-cube in T_t^3. Moreover, for any $(x,y,z) \in V_{T_t^3}$ any choice of $(x',y',z') \in V_{T_t^3}$ with $x' > x$ in $T_{\mathbf{x}}$, $y' > y$ in $T_{\mathbf{y}}$ and $z' > z$ in $T_{\mathbf{z}}$ uniquely determines six simplices in $\Delta_2(T_t^3)_{\mathrm{diag}}$ contained in the same cube in T_t^3. Thus φ is a well-defined element in $C_3^{\mathrm{suf}}(T_t^3; A)$. From (3.9) it follows that $c - \partial\varphi = b_{\mathbf{x}} + b_{\mathbf{y}} + b_{\mathbf{z}}$ for some $\mathbf{x}$-chain $b_{\mathbf{x}}$, $\mathbf{y}$-chain $b_{\mathbf{y}}$ and some $\mathbf{z}$-chain $b_{\mathbf{z}}$ in $C_2^{\mathrm{suf}}(X; A)$. Thus the claim follows. □

Remark 3.3.8. *The cartesian product of three trees $T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}$ can be given the structure of a cube complex. One could define the "cubical uniformly finite homology" where the chains considered are bounded functions supported on cubes coming from the cubical structure of the space. In this way, Lemma 3.3.7 would not be needed since we could directly operate on cubes, instead of simplices.*

In classical homology, one can prove the equivalence between cubical and singular homology using acyclic models [26, Chapter 8.4]. However, chain complexes are usually assumed to be free with natural basis. Since in uniformly finite homology the chains considered are infinite sums of simplices, a priori it is not clear if the acyclic models method could also work in this case. On the other hand, one could avoid the acyclic models theorem and construct an explicit chain isomorphism between the simplicial and the cubical uniformly finite chain complex. Indeed, it is possible to pass from the cubical description to the simplicial one by triangulating each cube inside a cube complex. On the other hand, one can divide any simplex into cubes by performing a suitable subdivision on the simplices. This operation, however, requires a lot of tedious computations and checks and in order to avoid this we prefer to keep working with simplices.

For any $\sigma = [(x,y,z),(x',y',z'),(x'',y'',z'')] \in \Delta_2(T_t^3)$, denote by σ^0, σ^1 and σ^2 the faces of σ, i.e. the simplices

$$\begin{aligned} \sigma^0 &= [(x',y',z'),(x'',y'',z'')] \in \Delta_1(T_t^3) \\ \sigma^1 &= [(x,y,z),(x'',y'',z'')] \in \Delta_1(T_t^3) \\ \sigma^2 &= [(x,y,z),(x',y',z')] \in \Delta_1(T_t^3). \end{aligned} \tag{3.10}$$

Similarly to the case of the product $X \times_t Y$ of two simplicial complexes of bounded geometry (Lemma 3.2.10), the next lemma shows that $\mathbf{x}$-classes, $\mathbf{y}$-classes and $\mathbf{z}$-classes are all trivial in $H_2^{\mathrm{suf}}(T_t^3; A)$.

Lemma 3.3.9. *Let $T_t^3 = T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ be the triangulated cartesian product of uniformly locally finite, non-amenable trees $T_{\mathbf{x}}, T_{\mathbf{y}}, T_{\mathbf{z}}$. Then $\mathbf{x}$-classes, $\mathbf{y}$-classes and $\mathbf{z}$-classes are trivial in $H_2^{\mathrm{suf}}(T_t^3; A)$.*

Proof. The proof of the lemma follows a similar idea given in the proof of Lemma 3.2.10. We prove the statement only for $\mathbf{x}$-classes. The same argument can be used in the case of $\mathbf{y}$-classes and $\mathbf{z}$-classes. Let $\alpha \in H_2^{\mathrm{suf}}(T_t^3; A)$ be an $\mathbf{x}$-class and let $c = \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot \sigma_{\mathbf{x}}$ be a cycle representing α. To prove the lemma, we provide a chain that bounds c using the Eilenberg-swindle construction on $T_{\mathbf{x}}$. We construct this chain by taking the cross-product of $\mathbf{x}$-simplices appearing in c and tails of 1-simplices in $T_{\mathbf{x}}$. More precisely, following the construction given in Section 1.4.1, for any $x \in V_{T_{\mathbf{x}}}$, we find a tail of simplices in $\Delta_1(T_{\mathbf{x}})$ as in (1.6). For simplicity, we suppose that for any $x \in V_{T_{\mathbf{x}}}$ the simplices in the tail t_x are all oriented in the direction of x. More precisely, for any $x \in V_{T_{\mathbf{x}}}$ consider

$$t_x = \sum_{j \in \mathbb{Z}_{\leq 0}} [x_{j-1}, x_j] \in C_1^{\mathrm{suf}}(T_{\mathbf{x}}; A)$$

such that for any $j \in \mathbb{Z}_{\leq 0}$ we have $x_{j-1} < x_j$ in $T_{\mathbf{x}}$ and for $j = 0$ we have $x_0 = x$. Now for any $x \in V_{T_{\mathbf{x}}}$ we consider t_x as a chain in $C_1^{\mathrm{suf}}(T_t^3; A)$, i.e. for any $(x,y,z) \in V_{T_t^3}$ we define

$$t^{\mathbf{x}}_{(x,y,z)} = \sum_{j\in\mathbb{Z}_{\le 0}} [(x_{j-1},y,z),(x_j,y,z)] \in C_1^{\mathrm{suf}}(T_t^3;A). \tag{3.11}$$

From Definition 3.3.4, it follows that any $\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}$ is of one of the following forms:

- $[(x,y,z),(x,y',z),(x,y',z')]$ for some $(x,y,z) \in V_{T^3}$, some $y' > y$ and some $z' > z$;
- $[(x,y,z),(x,y,z'),(x,y',z')]$ for some $(x,y,z) \in V_{T^3}$, some $y' > y$ and some $z' > z$.

For some $(x,y,z) \in V_{T_t^3}$, fix $y' \in A_y$ and $z' \in A_z$ and take $\sigma_x \in \Delta_2(T^3)_{\mathbf{x}}$ of the form $\sigma_x = [(x,y,z),(x,y',z),(x,y',z')]$. Using the tails $t^{\mathbf{x}}_{(x,y,z)}, t^{\mathbf{x}}_{(x,y',z)}$ and $t^{\mathbf{x}}_{(x,y',z')}$ attached to any vertex of σ_x, for any $j \in \mathbb{Z}_{\le 0}$ we consider the following simplices in $\Delta_3(T_t^3)$ (Figure 3.5):

$$\tau_j^0 = [(x_{j-1},y,z),(x_j,y,z),(x_j,y',z),(x_j,y',z')]$$

$$\tau_j^1 = [(x_{j-1},y,z),(x_{j-1},y',z),(x_j,y',z),(x_j,y',z')]$$

$$\tau_j^2 = [(x_{j-1},y,z),(x_{j-1},y',z),(x_{j-1},y',z'),(x_j,y',z')].$$

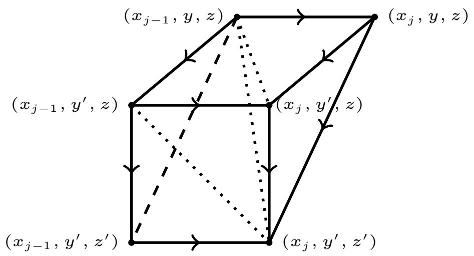


Figure 3.5: The 3-simplices τ_j^0, τ_j^1 and τ_j^2 used to construct the beam of simplices attached to σ_x.

Now consider the following element in $C_3^{\mathrm{suf}}(T_t^3;A)$ (Figure 3.7):

$$p_{\sigma_x} = \sum_{j\in\mathbb{Z}_{\le 0}} \tau_j^0 - \tau_j^1 + \tau_j^2. \tag{3.12}$$

Notice that p_{σ_x} is given by the cross-product of $t_x \in C_1^{\mathrm{suf}}(T_{\mathbf{x}};A)$ with the simplex $[(y,z),(y',z),(y',z')] \in \Delta_2(T_{\mathbf{y}} \times T_{\mathbf{z}})$. Notice that the boundary of p_{σ_x} consists of σ_x and panels of 2-dimensional simplices attached to each face of σ_x (Figure 3.6). More precisely, for any face of σ_x consider:

$$\begin{aligned} p_{\sigma_x^0} &= \sum_{j\in\mathbb{Z}_{\leq 0}} [(x_{j-1},y',z),(x_j,y',z),(x_j,y',z')] \\ &\quad - [(x_{j-1},y',z),(x_{j-1},y',z'),(x_j,y',z')] \\ p_{\sigma_x^1} &= \sum_{j\in\mathbb{Z}_{\leq 0}} [(x_{j-1},y,z),(x_j,y,z),(x_j,y',z')] \\ &\quad - [(x_{j-1},y,z),(x_{j-1},y',z'),(x_j,y',z')] \\ p_{\sigma_x^2} &= \sum_{j\in\mathbb{Z}_{\leq 0}} [(x_{j-1},y,z),(x_j,y,z),(x_j,y',z)] \\ &\quad - [(x_{j-1},y,z),(x_{j-1},y',z),(x_j,y',z)]. \end{aligned} \tag{3.13}$$

An easy calculation shows that

$$\partial p_{\sigma_x} = \sigma_x - p_{\sigma_x^0} + p_{\sigma_x^1} - p_{\sigma_x^2}. \tag{3.14}$$

We can repeat the construction for any $\mathbf{x}$-simplex appearing in the cycle c. More precisely, for any $\sigma_\mathbf{x} \in \Delta_2(T_t^3)_\mathbf{x}$, using tails of the form (3.11) attached to the vertices of $\sigma_\mathbf{x}$ we can construct $p_{\sigma_\mathbf{x}} \in C_3^{\mathrm{suf}}(T_t^3;A)$. For any $\sigma_\mathbf{x} \in \Delta_2(T_t^3)_\mathbf{x}$ we call $p_{\sigma_\mathbf{x}}$ a *beam of* 3*-simplices attached to* $\sigma_\mathbf{x}$. We can, thus, define:

$$\varphi = \sum_{\sigma_\mathbf{x}\in\Delta_2(T^3)_\mathbf{x}} c(\sigma_\mathbf{x}) \cdot p_{\sigma_\mathbf{x}}.$$

By the Eilenberg-swindle construction given in Section 1.4.1, since $T_\mathbf{x}$ is non-amenable, for any $x \in V_\mathbf{x}$ we can choose a tail t_x such that $\sum_{x\in V_{T_\mathbf{x}}} t_x$ is a well-defined element in $C_1^{\mathrm{suf}}(T_\mathbf{x};A)$. It follows that any simplex in $\Delta_2(T_t^3)$ is contained in a uniformly bounded number of beams of type $p_{\sigma_\mathbf{x}}$. Thus, φ is a well-defined chain in $C_3^{\mathrm{suf}}(T_t^3;A)$. By (3.14), we have

$$\partial\varphi = c - \sum_{\sigma_\mathbf{x}\in\Delta_2(T_t^3)_\mathbf{x}} c(\sigma_\mathbf{x}) \cdot (p_{\sigma_\mathbf{x}^0} - p_{\sigma_\mathbf{x}^1} + p_{\sigma_\mathbf{x}^2}). \tag{3.15}$$

So, if we show that $\sum_{\sigma_\mathbf{x}\in\Delta_2(T_t^3)_\mathbf{x}} c(\sigma_\mathbf{x}) \cdot (p_{\sigma_\mathbf{x}^0} - p_{\sigma_\mathbf{x}^1} + p_{\sigma_\mathbf{x}^2}) = 0$, then we will have $\partial\varphi = c$ and the claim will follow. Consider, again, the simplex $\sigma_x = [(x,y,z),(x,y',z)(x,y',z')]$ appearing in c with coefficient $c(\sigma_x)$. Notice that σ_x shares the face σ_x^1 with a unique simplex in $\Delta_2(T_t^3)$ of the form $\widetilde{\sigma}_x = [(x,y,z),(x,y,z'),(x,y',z')]$. Since c is a cycle, we have $\partial c = 0$. In particular, $c(\sigma_x) + c(\widetilde{\sigma}_x) = 0$. Moreover, since $\sigma_x^1 = \widetilde{\sigma}_x^1$, by the construction of the 3-dimensional tails, we have that $p_{\sigma_x^1} = p_{\widetilde{\sigma}_x^1}$. Since for any $\sigma_\mathbf{x} \in \Delta_2(T_t^3)_\mathbf{x}$ there is a unique $\widetilde{\sigma}_\mathbf{x}$ such that $\sigma_\mathbf{x}^1 = \widetilde{\sigma}_\mathbf{x}^1$ and such that $c(\sigma_\mathbf{x}) + c(\widetilde{\sigma}_\mathbf{x}) = 0$, it follows that

$$\sum_{\sigma_\mathbf{x}\in\Delta_2(T_t^3)_\mathbf{x}} c(\sigma_\mathbf{x}) \cdot p_{\sigma_\mathbf{x}^1} = 0.$$

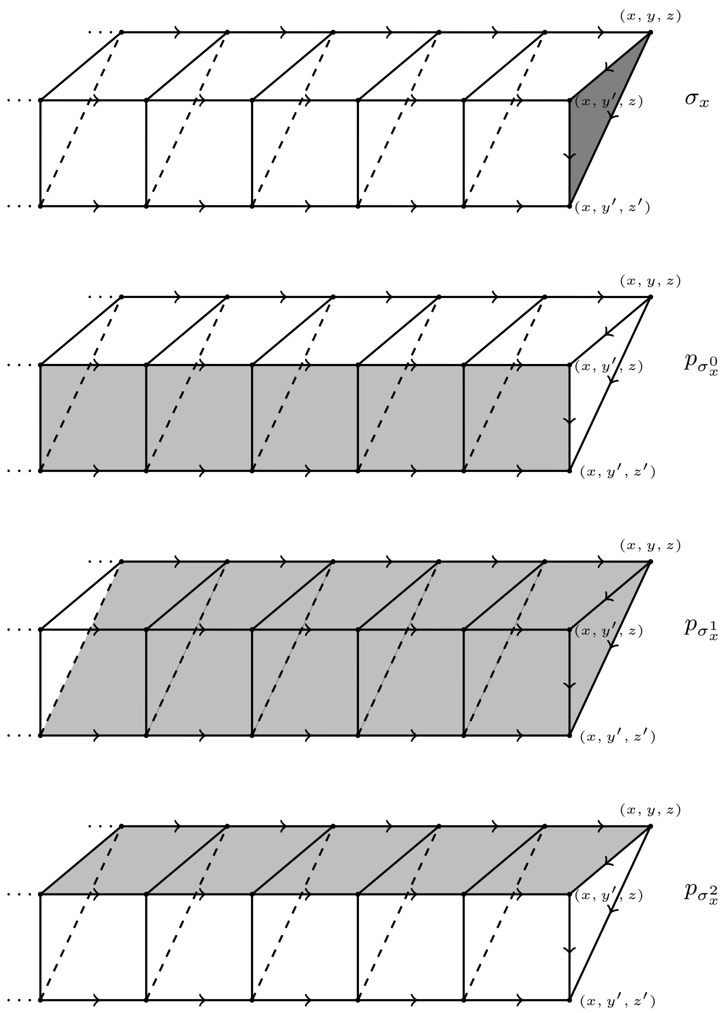


Figure 3.6: The boundary of p_{σ_x} is given by σ, $p_{\sigma_x^0}$, $p_{\sigma_x^1}$ and $p_{\sigma_x^2}$.

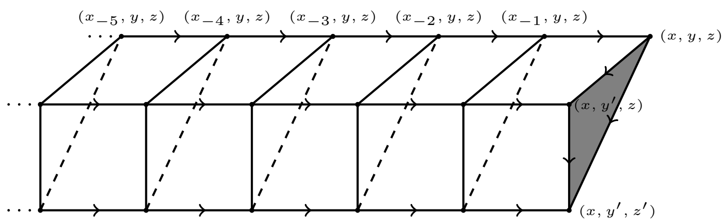


Figure 3.7: Attaching a tail of 1-simplices to each vertex of σ_x, one can construct a beam of 3-simplices p_{σ_x}.

We want to prove that $\sum_{\sigma_{\mathbf{x}}\in\Delta_2(T^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}})\cdot p_{\sigma^0_{\mathbf{x}}} = 0$. Notice that, every beam whose boundary contains a given simplex contributes to the coefficient of this simplex in the sum $\sum_{\sigma_{\mathbf{x}}\in\Delta_2(T^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}})\cdot p_{\sigma^0_{\mathbf{x}}}$. In other words, every simplex appears in this sum with coefficient given by the sum of the coefficients of each $\mathbf{x}$-simplex whose corresponding beams "passes through" it (Figure 3.8). More precisely, let $\sigma^j_{y'} = [(x_{j-1}, y', z), (x_j, y', z), (x_j, y', z')]$ be any $\mathbf{y}$-simplex appearing in $p_{\sigma^0_x}$ for some $\sigma_x \in \Delta_2(T^3)_{\mathbf{x}}$.

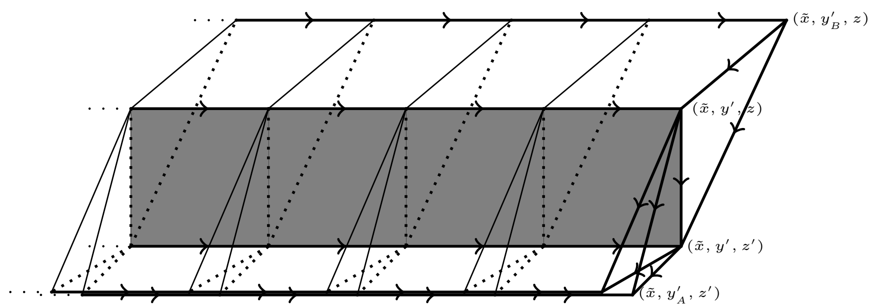


Figure 3.8: Beams attached to $\mathbf{x}$-simplices sharing the edge $[(\tilde{x}, y', z), (\tilde{x}, y', z')]$.

Consider the sets of vertices $B_{y'}$ and $A_{y'}$ in $T_{\mathbf{y}}$ as given in Definition 3.2.5 and the set $E(x_{j-1}, x_j)$ of all vertices in $T_{\mathbf{x}}$ whose tail passes through $[x_{j-1}, x_j] \in \Delta_1(T_{\mathbf{x}})$. For any $\widetilde{x} \in E(x_{j-1}, x_j)$ define

$$\begin{aligned} c_{[(\widetilde{x},y',z),(\widetilde{x},y',z')]} := &\sum_{y'_B\in B_{y'}} c([(\widetilde{x}, y'_B, z), (\widetilde{x}, y', z), (\widetilde{x}, y', z')]) \\ &+ \sum_{y'_A\in A_{y'}} c([(\widetilde{x}, y', z), (\widetilde{x}, y', z'), (\widetilde{x}, y'_A, z')]). \end{aligned} \tag{3.16}$$

It is easy to see that the simplex $\sigma^j_{y'}$ appears in $\sum_{\sigma_{\mathbf{x}}\in\Delta_2(T^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}})\cdot p_{\sigma^0_{\mathbf{x}}}$ with coefficient

$$\sum_{\widetilde{x}\in E(x_{j-1},x_j)} c_{[(\widetilde{x},y',z),(\widetilde{x},y',z')]}. \tag{3.17}$$

So to prove that $\sum_{\sigma_{\mathbf{x}}\in\Delta_2(T^3_t)_{\mathbf{x}}} c(\sigma_{\mathbf{x}})\cdot p_{\sigma^0_{\mathbf{x}}} = 0$, it suffices to prove that (3.17) is trivial. Notice that for any $\sigma^j_{y'} \in \Delta_2(T^3)_{\mathbf{y}}$ and any $\widetilde{x} \in E(x_{j-1}, x_j)$ we have that $c_{[(\widetilde{x},y',z),(\widetilde{x},y',z')]} = 0$. Indeed, $c_{[(\widetilde{x},y',z),(\widetilde{x},y',z')]}$ is just the sum of all the $\mathbf{x}$-simplices in T^3_t having the simplex $[(\widetilde{x}, y', z), (\widetilde{x}, y', z')] \in \Delta_1(T^3_t)$ as a common face. So by the cycle condition on c and the fact that c is only supported on $\mathbf{x}$-simplices, it follows that $c_{[(\widetilde{x},y',z),(\widetilde{x},y',z')]}$=0. In particular, the sum given in (3.17) vanishes. It follows that $\sum_{\sigma_{\mathbf{x}}\in\Delta_2(T^3_t)_{\mathbf{x}}} c(\sigma_{\mathbf{x}})\cdot p_{\sigma^0_{\mathbf{x}}} = 0$.

In a similar way, it is possible to prove that $\sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot p_{\sigma_{\mathbf{x}}^2} = 0$. From (3.15), it follows that $c = \partial\varphi$ and, thus, $\alpha = 0$ in $H_1^{\mathrm{suf}}(T_t^3; A)$. □

As we did in the previous section, we want to reduce any class in $H_2^{\mathrm{suf}}(T_t^3; A)$ to an **x**-class (or **y**-class or **z**-class). We want to prove a similar result as Lemma 3.2.11 but the uniformly finite homology group is now one degree higher than the one considered in the previous section. We, thus, need a step more. First of all we restrict to cycles which take values only on **y**-simplices and **z**-simplices.

Lemma 3.3.10. *Let $T_t^3 = T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ be the triangulated cartesian product of uniformly locally finite, non-amenable trees $T_{\mathbf{x}}, T_{\mathbf{y}}, T_{\mathbf{z}}$. Let $c \in C_2^{\mathrm{suf}}(T_t^3; A)$ be a cycle. Then there is a cycle $b \in C_2^{\mathrm{suf}}(T_t^3; A)$ of the form $b = b_{\mathbf{y}} + b_{\mathbf{z}}$ such that $[b] = [c]$ in $H_2^{\mathrm{suf}}(T_t^3; A)$.*

Proof. Let c be a cycle in $C_2^{\mathrm{suf}}(T_t^3; A)$. By Lemma 3.3.7, we can assume $c = c_{\mathbf{x}} + c_{\mathbf{y}} + c_{\mathbf{z}}$, for some **x**-chain $c_{\mathbf{x}}$, some **y**-chain $c_{\mathbf{y}}$ and some **z**-chain $c_{\mathbf{z}}$. We want to find a $\varphi \in C_3^{\mathrm{suf}}(T_t^3; A)$ such that $c - \partial\varphi = b_{\mathbf{y}} + b_{\mathbf{z}}$ for some **y**-chain $b_{\mathbf{y}}$ and some **z**-chain $b_{\mathbf{z}}$. For any $\sigma_{\mathbf{x}}$ appearing in c we construct a beam $p_{\sigma_{\mathbf{x}}}$ of the form (3.12) as we did in the proof of Lemma 3.3.9 and we define

$$\varphi = \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot p_{\sigma_{\mathbf{x}}} \in C_3^{\mathrm{suf}}(T_t^3; A).$$

As in (3.14), for any $\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}$ we have

$$\partial p_{\sigma_{\mathbf{x}}} = \sigma_{\mathbf{x}} - p_{\sigma_{\mathbf{x}}^0} + p_{\sigma_{\mathbf{x}}^1} - p_{\sigma_{\mathbf{x}}^2}.$$

It follows that

$$\partial\varphi = c_{\mathbf{x}} - \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot (p_{\sigma_{\mathbf{x}}^0} - p_{\sigma_{\mathbf{x}}^1} + p_{\sigma_{\mathbf{x}}^2}).$$

By the same argument used in Lemma 3.3.9, we have

$$\sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot p_{\sigma_{\mathbf{x}}^1} = 0. \tag{3.18}$$

Moreover from (3.13) in the proof of Lemma 3.3.9, one can easily see that for any $\sigma_{\mathbf{x}} \in \Delta_2(T_{\mathbf{x}}^3)$ the chain $p_{\sigma_{\mathbf{x}}^0}$ is a **y**-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$ and $p_{\sigma_{\mathbf{x}}^2}$ is a **z**-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$. In particular, $b_{\mathbf{y}} := c_{\mathbf{y}} + \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot p_{\sigma_{\mathbf{x}}^0}$ is a **y**-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$, while $b_{\mathbf{z}} := c_{\mathbf{z}} + \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot p_{\sigma_{\mathbf{x}}^2}$ is a **z**-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$. From (3.15) and (3.18), we have

$$\partial\varphi = c_{\mathbf{x}} - \sum_{\sigma_{\mathbf{x}} \in \Delta_2(T_t^3)_{\mathbf{x}}} c(\sigma_{\mathbf{x}}) \cdot (p_{\sigma_{\mathbf{x}}^0} + p_{\sigma_{\mathbf{x}}^2}).$$

In particular, $c - \partial\varphi = b_{\mathbf{y}} + b_{\mathbf{z}}$. Thus $[c] = [b_{\mathbf{y}} + b_{\mathbf{z}}]$ in $H_2^{\mathrm{suf}}(T_t^3; A)$ and the claim follows. □

In the last lemma before the proof of Theorem 3.3.3, we reduce every cycle in $C_2^{\mathrm{suf}}(T_t^3; A)$ to a cycle which is supported only on $\mathbf{z}$-simplices.

Lemma 3.3.11. *Let $T_t^3 = T_{\mathbf{x}} \times_t T_{\mathbf{y}} \times_t T_{\mathbf{z}}$ be the triangulated cartesian product of uniformly locally finite, non-amenable trees $T_{\mathbf{x}}, T_{\mathbf{y}}, T_{\mathbf{z}}$. Let $c \in C_2^{\mathrm{suf}}(T_t^3; A)$ be a cycle. Then there is a $\mathbf{z}$-cycle $b_{\mathbf{z}} \in C_2^{\mathrm{suf}}(T_t^3; A)$ such that $[c] = [b_{\mathbf{z}}]$ in $H_2^{\mathrm{suf}}(T_t^3; A)$.*

Proof. From Lemma 3.3.10, we can assume c to be of the form $c = c_{\mathbf{y}} + c_{\mathbf{z}}$, for some $\mathbf{y}$-chain $c_{\mathbf{y}}$ and some $\mathbf{z}$-chain $c_{\mathbf{z}}$ in $C_2^{\mathrm{suf}}(T_t^3; A)$. We proceed, again, by constructing $\varphi \in C_3^{\mathrm{suf}}(T_t^3; A)$ such that $c - \partial\varphi$ is a $\mathbf{z}$-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$. In the rest of the proof we follow the same construction used in Lemma 3.3.9: in particular, we construct beams of 3-dimensional simplices by taking cross-products of $\mathbf{y}$-simplices appearing in c and tails of 1-simplices in $T_{\mathbf{y}}$. We consider tails of 1-simplices as in the Eilenberg-swindle construction given in Section 1.4.1. Again, for simplicity, we assume there is a suitable orientation of the simplices in all tails. More precisely, for any $y \in V_{T_{\mathbf{y}}}$ let

$$t_y = \sum_{j \in \mathbb{Z}_{\leq 0}} [y_{j-1}, y_j] \in C_1^{\mathrm{suf}}(T_{\mathbf{y}}; A)$$

such that for any $j \in \mathbb{Z}_{\leq 0}$, we have $y_{j-1} < y_j$ and for $j = 0$, we have $y_0 = y$. Now consider these tails as elements of $C_1^{\mathrm{suf}}(T_t^3; A)$. More precisely, for any $(x, y, z) \in V_{T_t^3}$ consider

$$t^{\mathbf{y}}_{(x,y,z)} = \sum_{j \in \mathbb{Z}_{\leq 0}} [(x, y_{j-1}, z), (x, y_j, z)] \in C_1^{\mathrm{suf}}(T_t^3; A). \tag{3.19}$$

For some $(x, y, z) \in V_{T_t^3}$, fix $x' \in A_x$ and $z' \in A_z$ and take $\sigma_y \in \Delta_2(T_t^3)_{\mathbf{y}}$ of the form $\sigma_y = [(x, y, z), (x, y, z'), (x', y, z')]$. Using the tails $t^{\mathbf{y}}_{(x,y,z)}, t^{\mathbf{y}}_{(x,y,z')}$ and $t^{\mathbf{y}}_{(x',y,z')}$ attached to any vertex of σ_y, for any $j \in \mathbb{Z}_{\leq 0}$ we consider the following simplices in $\Delta_3(T_t^3)$:

$$\tau_j^0 = [(x, y_{j-1}, z), (x, y_j, z), (x, y_j, z'), (x', y_j, z')]$$

$$\tau_j^1 = [(x, y_{j-1}, z), (x, y_{j-1}, z'), (x, y_j, z'), (x', y_j, z')]$$

$$\tau_j^2 = [(x, y_{j-1}, z), (x, y_{j-1}, z'), (x', y_{j-1}, z'), (x', y_j, z')].$$

Now take the following sums of simplices in $\Delta_3(T_t^3)$

$$p_{\sigma_y} = \sum_{j \in \mathbb{Z}_{\leq 0}} \tau_j^0 - \tau_j^1 + \tau_j^2. \tag{3.20}$$

Let σ_y^0, σ_y^1 and σ_y^2 be the faces of σ_y as in (3.10). Proceeding as in the proof of Lemma 3.3.9, we have

$$\partial p_{\sigma_y} = \sigma_y - p_{\sigma_y^0} + p_{\sigma_y^1} - p_{\sigma_y^2} \tag{3.21}$$

where

$$\begin{aligned} p_{\sigma_y^0} &= \sum_{j\in\mathbb{Z}_{\le 0}} [(x,y_{j-1},z'),(x,y_j,z'),(x',y_j,z')] \\ &\quad - [(x,y_{j-1},z'),(x',y_{j-1},z'),(x',y_j,z')], \\ p_{\sigma_y^1} &= \sum_{j\in\mathbb{Z}_{\le 0}} [(x,y_{j-1},z),(x,y_j,z),(x',y_j,z')] \\ &\quad - [(x,y_{j-1},z),(x',y_{j-1},z'),(x',y_j,z')], \\ p_{\sigma_y^2} &= \sum_{j\in\mathbb{Z}_{\le 0}} [(x,y_{j-1},z),(x,y_j,z),(x,y_j,z')] \\ &\quad - [(x,y_{j-1},z),(x,y_{j-1},z'),(x,y_j,z')]. \end{aligned} \tag{3.22}$$

We can repeat the construction above for any $\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}$ appearing in c. More precisely, for any $\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}$ we attach a tail of type (3.19) to its vertices any we construct $p_{\sigma_{\mathbf{y}}}$ as in (3.20). For any $\sigma_{\mathbf{y}}$ we call $p_{\sigma_{\mathbf{y}}}$ a *beam of* 3-*simplices attached to* $\sigma_{\mathbf{y}}$. We can, thus, define:

$$\varphi = \sum_{\sigma_{\mathbf{y}}\in\Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}})\cdot p_{\sigma_{\mathbf{y}}}.$$

Since $T_{\mathbf{y}}$ is non-amenable, for any $y \in V_{T_{\mathbf{y}}}$ we can choose a tail t_y such that $\sum_{y\in V_{T_{\mathbf{y}}}} t_y$ is a well-defined element in $C_1^{\mathrm{suf}}(T_t^3; A)$. In particular, φ is a well-defined element in $C_3^{\mathrm{suf}}(T_t^3; A)$. By (3.21), it follows that

$$\partial\varphi = c_{\mathbf{y}} - \sum_{\sigma_{\mathbf{y}}\in\Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}})\cdot(p_{\sigma_{\mathbf{y}}^0} - p_{\sigma_{\mathbf{y}}^1} + p_{\sigma_{\mathbf{y}}^2}). \tag{3.23}$$

Notice that $\sigma_y = [(x,y,z),(x,y,z'),(x',y,z')]$ shares the face σ_y^1 with a unique 2-simplex of the form $\widetilde{\sigma}_y = [(x,y,z),(x',y,z),(x',y,z')]$. By the cycle condition on c, it follows that $c(\sigma_y) + c(\widetilde{\sigma}_y) = 0$. Since $\sigma_y^1 = \widetilde{\sigma}_y^1$, by the construction of the panels, we have that $p_{\sigma_y^1} = p_{\widetilde{\sigma}_y^1}$. Since for any $\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}$ there is a unique simplex $\widetilde{\sigma}_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}$ such hat $\sigma_{\mathbf{y}}^1 = \widetilde{\sigma}_y^1$ and such that $c(\sigma_{\mathbf{y}}) + c(\widetilde{\sigma}_{\mathbf{y}}) = 0$, we have

$$\sum_{\sigma_{\mathbf{y}}\in\Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}})\cdot p_{\sigma_{\mathbf{y}}^1} = 0.$$

As in Lemma 3.3.9, the cycle condition on c and the fact that c is supported only on $\mathbf{y}$ and $\mathbf{z}$-simplices can be used to prove that the coefficient of each simplex appearing in $\sum_{\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}}) \cdot p_{\sigma_{\mathbf{y}}^2}$ must be trivial (Figure 3.9). Thus, we have $\sum_{\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}}) \cdot p_{\sigma_{\mathbf{y}}^2} = 0$. By (3.23), it follows that

$$c - \partial\varphi = c_{\mathbf{z}} + \sum_{\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}}) \cdot p_{\sigma_{\mathbf{y}}^0}. \tag{3.24}$$

From (3.22), it is easy to see that for any $\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}$ the chain $p_{\sigma_{\mathbf{y}}^0}$ is a **z**-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$. It follows that $b_{\mathbf{z}} := c_{\mathbf{z}} + \sum_{\sigma_{\mathbf{y}} \in \Delta_2(T_t^3)_{\mathbf{y}}} c(\sigma_{\mathbf{y}}) \cdot p_{\sigma_{\mathbf{y}}^0}$ is a **z**-chain in $C_2^{\mathrm{suf}}(T_t^3; A)$. By (3.24), we have $[c] = [b_{\mathbf{z}}]$ and the claim follows. □

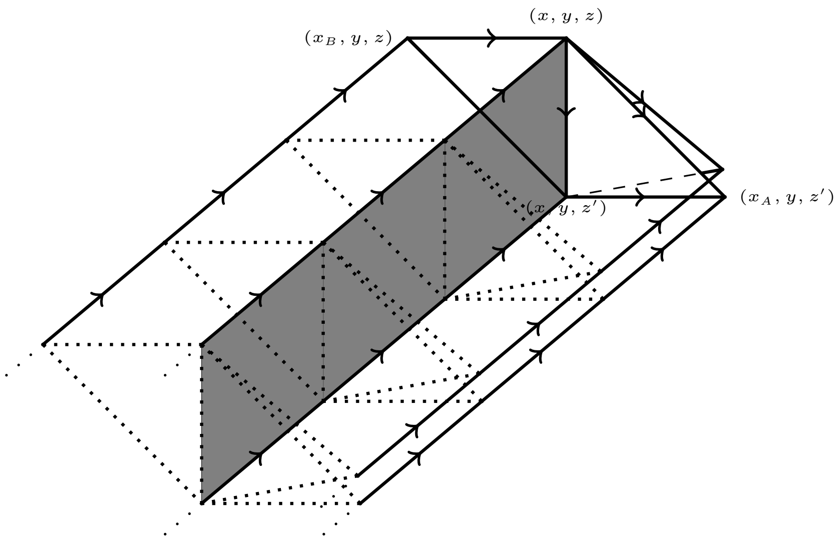


Figure 3.9: Beams attached to **y**-simplices sharing the edge $[(x, y, z), (x, y, z')]$ will have a common panel.

A similar argument shows that any class in $H_2^{\mathrm{suf}}(T_t^3; A)$ is equivalent to an **x**-class and to a **y**-class. We are now ready to prove Theorem 3.3.3.

Proof of Theorem 3.3.3. The case $n = 0$ follows from Theorem 1.4.1. Since the product $T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}$ is quasi isometric to $(T_{\mathbf{x}} \times_t T_{\mathbf{y}}) \times T_{\mathbf{z}}$, by Theorem 3.2.1 we have that $H_1^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A) = 0$. Moreover, $T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}$ is quasi-isometric to the triangulated cartesian product T_t^3. Since T_t^3 is a uniformly contractible simplicial complex of bounded geometry, we can prove the statement for $H_*^{\mathrm{suf}}(T_t^3; A)$ and by Proposition 1.3.3 the theorem will follow for $H_*^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A)$. In particular, since T_t^3 is a 3-dimensional simplicial complex, for any $n > 3$ we have $H_n^{\mathrm{suf}}(T_t^3; A) = 0$.

From Lemma 3.3.11, we know that every class in $H_2^{\mathrm{suf}}(T_t^3; A)$ is equivalent to a $\mathbf{z}$-class and from Lemma 3.3.9, we know that every $\mathbf{z}$-class in $H_2^{\mathrm{suf}}(T_t^3; A)$ is trivial, so the case $n = 2$ follows.

The proof of the case $n = 3$ is analogous to the one seen in Theorem 3.3.1: we can use bi-infinite paths on $T_{\mathbf{x}}$, $T_{\mathbf{y}}$ and $T_{\mathbf{z}}$ to construct infinitely many cycles having disjoint support and giving rise to infinitely many linearly independent non-trivial classes in $H_3^{\mathrm{suf}}(T_t^3; A)$. More precisely, let $(x_n)_{n\in\mathbb{Z}}$ be a bi-infinite path in $T_{\mathbf{x}}$, $(y_m)_{m\in\mathbb{Z}}$ a bi-infinite path in $T_{\mathbf{y}}$ and $(z_l)_{l\in\mathbb{Z}}$ a bi-infinite path in $T_{\mathbf{z}}$. For any $m \in \mathbb{Z}, n \in \mathbb{Z}$ and $l \in \mathbb{Z}$ we consider the 3-simplices contained in the cube in T_t^3 having the edge $[(x_m, y_n, z_l), (x_{m+1}, y_{n+1}, z_{l+1})]$ as diagonal (these simplices are exactly the τ_i's simplices listed in the proof of Lemma 3.3.7 for $(x, y, z) = (x_m, y_n, z_l)$ and $(x', y', z') = (x_{m+1}, y_{n+1}, z_{l+1})$). Similarly to the cycle c given in the proof of Theorem 3.3.1, summing over the indices m, n and l, we obtain a cycle in $C_3^{\mathrm{suf}}(T_t^3; \mathbb{Z})$. For any $n > 0$ we have $C_n^{\mathrm{suf}}(T_t^3; A) = 0$, so this cycle represents a non-trivial class in $H_3^{\mathrm{suf}}(T_t^3; A)$. By choosing infinitely many disjoint bi-infinite paths in $T_{\mathbf{x}}$, $T_{\mathbf{y}}$ and $T_{\mathbf{z}}$, we obtain infinitely many linearly independent non-trivial classes in $H_3^{\mathrm{suf}}(T_t^3; A)$. It follows that $H_3^{\mathrm{uf}}(T_{\mathbf{x}} \times T_{\mathbf{y}} \times T_{\mathbf{z}}; A) \cong H_3^{\mathrm{suf}}(T_t^3; A)$ is infinite dimensional. □

At this point, a natural way to proceed would be to investigate uniformly finite homology of higher dimensional products of non-amenable trees. We state the following conjecture as a possible development for the joint work with Piotr Nowak [16]:

Conjecture 3.3.12 (Work in Progress!). *Let $X = T_1 \times \cdots \times T_n$ be the cartesian product of uniformly locally finite non-amenable trees. Let $A \in \{\mathbb{R}, \mathbb{Z}\}$. Then*

$$H_k^{\mathrm{uf}}(X; A) = \begin{cases} \textit{infinite dimensional} & \textit{if } k = n \\ 0 & \textit{otherwise.} \end{cases}$$

Notice that for $k > n$, the conjecture holds. Indeed, by choosing a suitable triangulation for X, we obtain an n-dimensional uniformly contractible simplicial complex X^t of bounded geometry. By Proposition 1.3.3, for $k > n$ we have that $H_k^{\mathrm{uf}}(X; A) = 0$. The case $k = n$ can be proved in a similar way as the case $k = 3$ of Theorem 3.3.3. Indeed, it is possible to construct infinitely many linearly independent non-trivial classes in $H_n^{\mathrm{uf}}(X; A)$ by taking products of bi-infinite paths constructed in every T_i. The vanishing of uniformly finite homology for $k < n$ could be proved with similar techniques as the one seen in the proof of Leamma 3.3.9 and Lemma 3.3.11. More precisely, for any $k < n$ one could develop a higher dimensional Eilenberg-swindle construction by assembling beams of $k+1$-dimensional simplices attached to k-simplices. The difficulty is that, in higher dimension, a detailed construction of such beams become rather cumbersome. In particular, one needs to find a convenient notation in order to make the strategy manageable. One possible idea would be to use cubical uniformly finite homology. As we pointed out in Remark 3.3.8, the space X has the structure of a cube complex and one could define cubical uniformly finite chains by taking sums of cubes, instead of sums of simplices. Notice that for any $k < n$ a k-cube in the product X of trees is given by direct product of k edges and $n-k$ vertices in the different components of X. Following Lemma 3.3.9, the idea would be to consider "special" cycles in $C_k^{\mathrm{uf}}(X;A)$ and to prove that they bound. These special cycles are the ones supported only on k-cubes whose first k components are edges in T_i for $i \in \{1,\ldots,k\}$. Then, following Lemma 3.3.10 and Lemma 3.3.11, with a proper induction argument, one could reduce every cycle in $C_k^{\mathrm{uf}}(X;A)$ to a special cycle which represents the same class in $H_k^{\mathrm{uf}}(X;A)$.

3.4 The Künneth-vanishing-Conjecture

The aim of this chapter is to understand the behavior of uniformly finite homology of products of spaces or groups. In Section 3.2 and in Section 3.3 we have proved vanishing results for the uniformly finite homology of products of some non-amenable spaces using the vanishing of the uniformly finite homology of the factors in degree zero. This motivates us to conjecture a more general vanishing result for uniformly finite homology of products:

Conjecture 3.4.1 (The Künneth-vanishing-Conjecture)**.** *Let X, Y be metric spaces of coarse bounded geometry and let $A \in \{\mathbb{R}, \mathbb{Z}\}$. Let $n_1, n_2 \in \mathbb{N}$. Assume that $H_k^{\mathrm{uf}}(X;A) = 0$ for all $k \leq n_1$ and $H_k^{\mathrm{uf}}(X;A) = 0$ for all $k \leq n_1$. Then $H_k^{\mathrm{uf}}(X \times Y;A) = 0$ for all $k \leq n_1 + n_2 + 1$.*

In Section 3.2 and in Section 3.3, we have chosen a geometric approach to deduce results for the uniformly finite homology of products. On the other hand, one could try to prove the conjecture using an algebraic approach; in particular, in the case of finitely generated groups, one could look at the homology with ℓ^∞-coefficients. However, as we have already mentioned in the introduction, there are some significant difficulties to relate the ℓ^∞-space of the product with the ℓ^∞-space of its factors. So, the Künneth-vanishing-Conjecture is not obvious even in the case of finitely generated groups.

Chapter 4

Semi-norm on uniformly finite homology

One way to study algebraic structures using metric information is to consider functional analytic versions of homology in which the chain complexes are enriched with a norm. In uniformly finite homology, one can take the natural supremum norm on the chains obtaining a corresponding semi-norm on homology classes. One of the main motivations to consider norms on uniformly finite chains is to be able to measure the size of certain classes: we know that in many cases (for example for amenable groups) uniformly finite homology tends to be rather big but we do not have more precise information. In many situations the supremum semi-norm tends to vanish, but if one restricts to some quasi-lattice in a metric space of coarse bounded geometry, then it is possible to find classes with non trivial semi-norm (Section 4.2). We have seen that quasi-isometries between metric spaces induces isomorphisms in uniformly finite homology. However, as we will see in this chapter, these isomorphisms do not always the supremum semi-norm.

We start by defining the supremum norm on the various uniformly finite chain complexes introduced in Chapter 1. In Section 4.2, we study the behavior of semi-norms with respect to the comparison maps between the Block-Weinberger and the Rips uniformly finite homology. In Section 4.3 and Section 4.4, we compute the semi-norm of uniformly finite homology classes in some cases. Based on Theorem 1.6.3, we use semi-norms to distinguish between quasi-isometries and bilipschitz equivalences in the case of finitely generated groups first (Section 4.5) and then, more generally, in the case of UDBG-spaces (Section 4.6).

4.1 Norm on chain complexes

Let A be a unital ring with norm as in (1.1) on page 3.

Definition 4.1.1. Let M be an A-module. A *norm on M* is a function

$$\| - \|: M \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following conditions:

(i) For any $m \in M$ we have $\|m\| \geq 0$ and $\|m\| = 0$ if and only if $m = 0$.

(ii) For any $m, m' \in M$ we have $\|m + m'\| \leq \|m\| + \|m'\|$.

(iii) For any $a \in A$ and $m \in M$ we have $\|a \cdot m\| = |a| \|m\|$.

A *semi-norm on M* is a function as above satisfying only condition (ii) and

(iii)′ For any $a \in A$ and $m \in M$ we have $\|a \cdot m\| \leq |a| \|m\|$.

We now define a norm on the A-module of uniformly finite chains. This induces a semi-norm on uniformly finite homology.

Definition 4.1.2 (Norm on the Block-Weinberger chain complex)**.** Let X be a metric space and let $n \in \mathbb{N}$. The *supremum norm on $C_n^{\text{uf}}(X; A)$* is the norm $\| - \|_\infty$ defined for any $c = \sum_{\overline{x} \in X^{n+1}} c(\overline{x}) \cdot \overline{x}$ as

$$\|c\|_\infty := \sup_{\overline{x} \in X^{n+1}} |c(\overline{x})|.$$

The supremum norm induces a semi-norm on $H_*^{\text{uf}}(X; A)$ defined for any class $\alpha \in H_n^{\text{uf}}(X; A)$ as

$$\|\alpha\|_\infty := \inf\{\|c\|_\infty \mid c \in C_n^{\text{uf}}(X; A), \partial(c) = 0, \alpha = [c]\}.$$

Similarly, we can define a supremum norm on the simplicial uniformly finite chains given in Definition 1.2.5.

Definition 4.1.3 (Norm on the simplicial uniformly finite chain complex)**.** Let X be a simplicial complex of bounded geometry and let $n \in \mathbb{N}$. The *supremum norm on $C_n^{\text{suf}}(X; A)$* is the norm $\| - \|_\infty$ defined for any chain $c = \sum_{\sigma \in \Delta_n(X)} c(\sigma) \cdot \sigma$ as

$$\|c\|_\infty := \sup_{\sigma \in \Delta_n(X)} |c(\sigma)|.$$

As above, the supremum norm induces a semi-norm on $H_*^{\text{suf}}(X; A)$ defined for any $\alpha \in H_n^{\text{suf}}(X; A)$ as

$$\|\alpha\|_\infty := \inf\{\|c\|_\infty \mid c \in C_n^{\text{suf}}(X; A), \partial(c) = 0, \alpha = [c]\}.$$

For a quasi-lattice $\Gamma \subseteq X$ in a metric space X of coarse bounded geometry we define a supremum norm on $C_*^{\text{R-uf}}(\Gamma; A)$.

Definition 4.1.4. Let X be a metric space of coarse bounded geometry and let $n \in \mathbb{N}$. Let $\Gamma \in \mathrm{QL}(X)$. The *supremum norm on* $C_n^{\text{R-uf}}(\Gamma; A)$ is the norm $\| - \|_\infty$ defined for any $c \in C_n^{\text{R-uf}}(\Gamma; A)$ as

$$\|c\|_\infty := \inf\{\|c_r\|_\infty \mid c_r \in C_n^{\text{suf}}(R_r(\Gamma); A) \text{ for some } r \in \mathbb{R}_{>0}, [c_r]_{\text{Rips}} = c\}.$$

As in the previous definitions one can define a semi-norm on $H_*^{\text{R-uf}}(\Gamma; A)$.

Definition 4.1.5 (Norm on the Rips uniformly finite chain complex)**.** Let X be a metric space of coarse bounded geometry and let $n \in \mathbb{N}$. The *supremum norm on* $C_n^{\text{R-uf}}(X; A)$ is the norm $\| - \|_\infty$ defined for $c \in C_n^{\text{R-uf}}(X; A)$ as

$$\|c\|_\infty := \inf\{\|c_\Gamma\|_\infty \mid c_\Gamma \in C_n^{\text{R-uf}}(\Gamma; A) \text{ for some } \Gamma \in \mathrm{QL}(X), [c_\Gamma]_{\mathrm{QL}} = c\}.$$

As in the previous definitions one can define a semi-norm on $H_*^{\text{R-uf}}(X; A)$.

Remark 4.1.6. *For any semi-norm defined above, one obtains the corresponding reduced uniformly finite homology by taking the quotient of the kernel of the boundary operator by the closure of the image.*

4.2 Semi-norms in the case of metric spaces

We want to investigate the behaviour of these semi-norms with respect to the comparison maps between the different uniformly finite chain complexes seen in Chapter 1. The following proposition shows that the isomorphism between the Block-Weinberger and the Rips uniformly finite homology established in Proposition 1.8.1 is indeed an isometry with respect to the semi-norms defined above. We consider, again, A to be any unital ring with norm.

Proposition 4.2.1. *Let X be a metric space with coarse bounded geometry and let $\Gamma \in \mathrm{QL}(X)$. The isomorphism $\phi_*\colon H_*^{\text{uf}}(\Gamma; A) \longrightarrow H_*^{\text{R-uf}}(\Gamma; A)$ established in Proposition 1.8.1 is an isometry.*

Proof. Consider the chain maps ϕ_* and ψ_* defined in the proof of Proposition 1.8.1. Let $a \in C_*^{\text{uf}}(\Gamma; \mathbb{R})$. The map ϕ_* sends a to its equivalence class in $C_*^{\text{R-uf}}(\Gamma; A)$, so by Definition 4.1.4, it follows that $\|\phi_*(a)\|_\infty \leq \|a\|_\infty$. On the other hand, it is easy to see that for any $[a]_{\text{Rips}} \in C_*^{\text{R-uf}}(\Gamma; A)$ we have $\|\psi_*([a]_{\text{Rips}})\|_\infty \leq \|[a]_{\text{Rips}}\|_\infty$. It follows that the induced isomorphism ϕ_* has norm at most 1 and has an inverse of norm at most 1. So $\phi_*\colon H_*^{\text{uf}}(\Gamma; A) \longrightarrow H_*^{\text{R-uf}}(\Gamma; A)$ is an isometry. $\square$

We want to compute the semi-norms of certain classes in uniformly finite homology. We always consider non-empty metric spaces. Recall that the *fundamental class* of a quasi-lattice $\Gamma \subseteq X$ in a metric space of coarse bounded geometry X is the class $[\Gamma] \in H_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$ (or in $H_0^{\mathrm{R\text{-}uf}}(\Gamma;\mathbb{R})$), represented by the element $\sum_{\gamma\in\Gamma}\gamma \in C_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$. For any quasi-lattice in a non-empty metric space of coarse bounded geometry, we have:

Lemma 4.2.2. *Let X be an amenable metric space with coarse bounded geometry and let $\Gamma \in \mathrm{QL}(X)$. Consider Γ as a metric space equipped with the induced metric. Then the fundamental class $[\Gamma]$ has semi-norm 1 in $H_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$ and in $H_0^{\mathrm{R\text{-}uf}}(\Gamma;\mathbb{R})$.*

Proof. The element $\sum_{\gamma\in\Gamma}\gamma \in C_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$ is a cycle representing the class $[\Gamma]$ whose coefficients are equal to 1 for any $\gamma \in \Gamma$ and whose supremum norm is obviously 1. Let $a = \sum_{\gamma\in\Gamma} a(\gamma)\cdot\gamma \in C_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$ be any other cycle representing the class $[\Gamma]$. Then $\left[\sum_{\gamma\in\Gamma}(1-a(\gamma))\cdot\gamma\right] = 0$ in $H_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$. In particular, since the inclusion $\Gamma \hookrightarrow X$ is a quasi-isometry, this class is zero in $H_0^{\mathrm{uf}}(X;\mathbb{R})$. Since X is amenable and Γ is a quasi-lattice in X, by Proposition 1.4.6, we have

$$\forall_{\epsilon>0} \quad \exists_{\gamma\in\Gamma} \quad (1-a(\gamma)) < \epsilon.$$

So, for any arbitrarily small $\epsilon > 0$ there exists $\gamma \in \Gamma$ such that $a(\gamma) > 1-\epsilon$. In particular

$$\|a\|_\infty = \sup_{\gamma\in\Gamma} |a(\gamma)| > 1-\epsilon.$$

It follows that $\|\,[\Gamma]\,\|_\infty = 1$ in $H_0^{\mathrm{uf}}(\Gamma;\mathbb{R})$. The same argument proves that $\|[\Gamma]\|_\infty = 1$ in $H_0^{\mathrm{R\text{-}uf}}(\Gamma;\mathbb{R})$. □

On the other hand, if we consider the uniformly finite homology of non-discrete metric spaces, then the semi-norm is almost always zero. More precisely, we have the following:

Proposition 4.2.3. *For any metric space X having no isolated points, we have that $\|-\|_\infty = 0$ in $H_*^{\mathrm{uf}}(X;\mathbb{R})$.*

Proof. Let $n \in \mathbb{N}$ and let $c = \sum_{\overline{x}\in X^{n+1}} c(\overline{x})\cdot\overline{x} \in C_n^{\mathrm{uf}}(X;\mathbb{R})$ be a cycle. Let $\mathrm{supp}(c)$ be the set of tuples $\overline{x} \in X^{n+1}$ such that $c(\overline{x}) \neq 0$. For any $x \in X$ we say that x *is a vertex of* $\mathrm{supp}(c)$ (and we write $x \in V_{\mathrm{supp}(c)}$) if x is the vertex of a simplex $\overline{x} \in \mathrm{supp}(c)$. Then, for any $r \in \mathbb{R}_{>0}$ there exists a constant $N > 0$ (depending on r and on c) such that

$$\forall_{x\in V_{\mathrm{supp}(c)}} \quad |B_r(x) \cap V_{\mathrm{supp}(c)}| < N.$$

Indeed, by contradiction, suppose this does not hold. Then there exists $\tilde{r} \in \mathbb{R}_{>0}$ such that for any $K > 0$ we can find a point $x \in V_{\operatorname{supp}(c)}$ with

$$|B_{\tilde{r}}(x) \cap V_{\operatorname{supp}(c)}| > K. \tag{4.1}$$

By condition (i)-(c) of Definition 1.1.4, all the tuples in $\operatorname{supp}(c)$ have uniformly bounded diameter by some uniform constant R. Now for any K consider a point $x \in V_{\operatorname{supp}(c)}$ for which (4.1) holds. Let $\overline{x} \in \operatorname{supp}(c)$ be a simplex having x as a vertex. We have

$$|\{\overline{y} \in B_{\tilde{r}+R}(\overline{x}) \mid \overline{y} \in \operatorname{supp}(c)\}| > K$$

were $B_{\tilde{r}+R}(\overline{x})$ is the ball in X^{n+1} considered with the maximum metric. But this contradicts condition (i)-(b) of Definition 1.1.4.

Now we fix $r \in \mathbb{R}_{>0}$ and we consider balls of radius r in X. We want to "separate" the points in $V_{\operatorname{supp}(c)}$ which are contained in these balls, by using balls of smaller radii. More precisely, since there exists a N (depending on r and on c) such that for any $x \in V_{\operatorname{supp}(c)}$ we have $|B_r(x) \cap V_{\operatorname{supp}(c)}| < N$, it follows that for any $x \in V_{\operatorname{supp}(c)}$ we can find $0 < r_x < r$ such that the following holds:

$$\forall_{x' \in V_{\operatorname{supp}(c)}, x' \neq x} \quad B_{r_x}(x) \cap B_{r_{x'}}(x') = \emptyset.$$

Let $k \in \mathbb{N}$. For any $x \in V_{\operatorname{supp}(c)}$ we choose k distinct points $x_1^x, \dots, x_k^x$ in $B_{r_x}(x)$ such that for all $i \in \{1, \dots, k\}$ we have $x_i^x \neq x$. Moreover, for any $x_i^x \in B_{r_x}(x)$ we consider the map

$$\begin{aligned} p_i : X &\longrightarrow X \\ x &\longmapsto \begin{cases} x_i^x & \text{if } x \in V_{\operatorname{supp}(c)} \\ x & \text{otherwise.} \end{cases} \end{aligned}$$

This map is well-defined. Indeed, for any $x \in X$ and $i \in \{1, \dots, k\}$ the point x_i^x is uniquely determined by x. Clearly for any $i \in \{1, \dots, n\}$ the map p_i is at bounded distance from the identity $\mathrm{id} : X \longrightarrow X$. Thus p_i is a quasi-isometry and it induces a map $p_i \colon C_n^{\mathrm{uf}}(X; \mathbb{R}) \longrightarrow C_n^{\mathrm{uf}}(X; \mathbb{R})$. Moreover, since p_i is at bounded distance from the identity, by Proposition 1.1.5, for any $i \in \{1, \dots, k\}$ we have that $[c] = [p_i(c)] \in H_n^{\mathrm{uf}}(X; \mathbb{R})$. Thus we can write the class $[c]$ as:

$$[c] = \sum_{i=1}^{k} \frac{1}{k} [p_i(c)].$$

It is easy to see that for any $i \in \{1, \dots, k\}$, the map p_i restricted to $V_{\operatorname{supp}(c)}$ is a bijection between $V_{\operatorname{supp}(c)}$ and $\bigcup_{x \in V_{\operatorname{supp}(c)}} x_i^x$. Thus $\|c\|_\infty = \|p_i(c)\|_\infty$. Moreover, for any $x \in V_{\operatorname{supp}(c)}$ and for any $i, j \in \{1, \dots, k\}$ $i \neq j$ we have $x_i^x \neq x_j^x$, so it follows that $\mathrm{supp}(p_i(c)) \cap \mathrm{supp}(p_j(c)) = \emptyset$. Thus, we have

$$\|[c]\|_\infty \leq \left\| \sum_{i=1}^{k} \frac{1}{k} p_i(c) \right\|_\infty = \frac{1}{k} \|c\|_\infty.$$

Since k can be arbitrary large, for $k \to \infty$, we have $\|[c]\|_\infty \to 0$. The same argument can be used for any $n \in \mathbb{N}$ and any cycle $c \in C_n^{\mathrm{uf}}(X;\mathbb{R})$. Thus $\| - \|_\infty = 0$ in $H_*^{\mathrm{uf}}(X;\mathbb{R})$. □

Remark 4.2.4. *One can prove the same in the case of the Rips uniformly finite homology. In other words, we have $\| - \|_\infty = 0$ in $H_*^{\text{R-uf}}(X;\mathbb{R})$ for any X metric space with no isolated point. Indeed, any class in $H_*^{\text{R-uf}}(X;\mathbb{R})$ is represented by a cycle supported on a certain quasi-lattice in X. As in the proof of Lemma 4.2.3, we can "move" the support of each cycle to another quasi-lattice by choosing points in balls of suitable radius. In this way, we obtain arbitrary many cycles representing the same class and having equal semi-norm.*

From Lemma 4.2.2 and Lemma 4.2.3 it follows that certain isomorphisms do not preserve the semi-norms in uniformly finite homology.

Proposition 4.2.5. *Let X be an amenable metric space with no isolated points having coarse bounded geometry and let $\Gamma \in \mathrm{QL}(X)$. We have*

(i) The isomorphism $H_0^{\mathrm{uf}}(\Gamma;\mathbb{R}) \cong H_0^{\mathrm{uf}}(X;\mathbb{R})$ induced by the canonical inclusion $i\colon \Gamma \hookrightarrow X$ is not an isometry.

(ii) The isomorphism $H_0^{\text{R-uf}}(\Gamma;\mathbb{R}) \cong H_0^{\text{R-uf}}(X;\mathbb{R})$ induced by the canonical map defined in Proposition 1.7.6 is not an isometry.

Proof. Claim (i) follows from Lemma 4.2.2 and Lemma 4.2.3. Claim (ii) follows from Lemma 4.2.2 and Remark 4.2.4. □

By Proposition 1.3.3, for any uniformly contractible simplicial complex of bounded geometry X, there is an isomorphism $H_*^{\mathrm{uf}}(X;\mathbb{R}) \cong H_*^{\mathrm{suf}}(X;\mathbb{R})$. This isomorphism is, in general, far from being an isometry. Indeed, in Lemma 4.2.3, we have seen that the semi-norm is trivial in the Block-Weinberger uniformly finite homology in many situations. On the other hand, for any n-dimensional simplicial complex of bounded geometry X, any non-zero cycle in $C_n^{\mathrm{suf}}(X;\mathbb{R})$ gives a class with non-trivial semi-norm. Indeed, we have $C_k^{\mathrm{suf}}(X;\mathbb{R}) = 0$ for any $k > n$, thus, there cannot be another representative with smaller norm for this class.

4.3 Semi-norms in the case of UDBG-spaces

As we have seen in the previous section, uniformly finite homology classes have zero semi-norm in most non-discrete metric spaces. We have seen examples of classes having non-zero semi-norm in the homology of a quasi-lattice in a metric space of coarse bounded geometry. In this section, we consider non-empty uniformly discrete metric spaces of bounded geometry (Definition 1.6.1): in particular, we use the rigidity result given by Whyte (Theorem 1.6.3) to study the behaviour of semi-norms of classes in the uniformly finite homology of UDBG-spaces.

4.3.1 Isometries for UDBG-spaces

In the case of UDBG-spaces we can easily find a lower and an upper bound for the norm of the isomorphisms induced by quasi-isometries.

Proposition 4.3.1. *Let X, Y be* UDBG*-spaces and let $f\colon X \longrightarrow Y$ be a quasi-isometry. Then for any $n \in \mathbb{N}$ there exists a $K_n \in \mathbb{R}_{>0}$ such that for any class $\alpha \in H_n^{\mathrm{uf}}(X; A)$ the following inequality holds*

$$\frac{1}{K_n} \cdot \|\alpha\|_\infty \leq \|f_n(\alpha)\|_\infty \leq K_n \cdot \|\alpha\|_\infty.$$

Proof. Let $f\colon X \longrightarrow Y$ be a (C, D)-quasi-isometry for some $C, D \in \mathbb{R}_{>0}$ and let $n \in \mathbb{N}$. We denote by f the map induced on the cartesian product

$$\begin{aligned} f\colon X^{n+1} &\longrightarrow Y^{n+1} \\ \overline{x} &\longmapsto f(\overline{x}) = (f(x_0), \dots, f(x_n)). \end{aligned}$$

The map $f\colon X \longrightarrow Y$ induces a map

$$f_n\colon C_n^{\mathrm{uf}}(X; A) \longrightarrow C_n^{\mathrm{uf}}(Y; A).$$

which sends any chain $c = \sum_{\overline{x} \in X^{n+1}} c(\overline{x}) \cdot \overline{x}$ to

$$f_n(c) = \sum_{\overline{y} \in f(X^{n+1})} \left(\sum_{\overline{x} \in f^{-1}(\overline{y})} c(\overline{x}) \right) \overline{y}.$$

By the bounded geometry condition on X, we know that balls in X have uniformly bounded cardinality. In particular for the balls of radius $C \cdot D$, there exists a $K_{C \cdot D} > 0$ such that for any $x \in X$ we have $|B_{C \cdot D}(x)| \leq K_{C \cdot D}$. By the same argument given in Proposition 1.7.3, for any $\overline{y} \in f(X^{n+1})$ we have

$$\left| f^{-1}(\overline{y}) \right| = \prod_{0 \leq i \leq n} \left| \{ x \in X \mid f(x) = y_i \} \right| \leq \prod_{0 \leq i \leq n} |B_{C \cdot D}(x_i)| \leq (K_{C \cdot D})^{n+1}.$$

Then, for any $c = \sum_{\overline{x} \in X^{n+1}} c(\overline{x}) \cdot \overline{x} \in C_n^{\mathrm{uf}}(X; A)$ we have

$$\begin{aligned}
\|f_n(c)\|_\infty &= \sup_{\overline{y} \in f(X^{n+1})} \left| \sum_{\overline{x} \in f^{-1}(\overline{y})} c(\overline{x}) \right| \\
&\leq \sup_{\overline{y} \in f(X^{n+1})} \left\{ \max_{\overline{x} \in f^{-1}(\overline{y})} |c(\overline{x})| \cdot \left| f^{-1}(\overline{y}) \right| \right\} \\
&\leq \sup_{\overline{x} \in X^{n+1}} |c(\overline{x})| \cdot (K_{C \cdot D})^{n+1} = \|c\|_\infty \cdot (K_{C \cdot D})^{n+1}.
\end{aligned}$$

Suppose there is a (C', D') quasi-isometry $g\colon Y \longrightarrow X$, quasi-inverse to f for some $C', D' \in \mathbb{R}_{>0}$. By the same argument, for any $b \in C_n^{\mathrm{uf}}(Y; A)$ we have

$$\|g_n(b)\|_\infty \leq \|b\|_\infty \cdot (N_{C' \cdot D'})^{n+1}$$

for some $N_{C' \cdot D'} > 0$. Now let $K_n := \max\{K_{C \cdot D}^{n+1}, N_{C' \cdot D'}^{n+1}\}$. Since g_n induces the inverse isomorphism of f_n in homology, for any class $\alpha \in H_n^{\mathrm{uf}}(X; A)$ we have:

$$\|f_n(\alpha)\|_\infty \leq K_n \cdot \|\alpha\|_\infty$$

and

$$\|\alpha\|_\infty = \|g_n(f_n(\alpha))\|_\infty \leq K_n \cdot \|f_n(\alpha)\|.$$

So the claim follows. □

We have the following immediate results on norms of quasi-isometries which are at bounded distance from bilipschitz equivalences.

Proposition 4.3.2. *Let $f\colon X \longrightarrow Y$ be a quasi-isometry between* UDBG*-spaces. If there is a bilipschitz equivalence at bounded distance from f, then the induced map $f_*\colon H_*^{\mathrm{uf}}(X; A) \longrightarrow H_*^{\mathrm{uf}}(Y; A)$ is an isometric isomorphism.*

Proof. Two quasi-isometric embeddings that are at bounded distance induce chain maps that are chain homotopic [7, Proposition 2.1]. In particular, if $f\colon X \longrightarrow Y$ is a quasi-isometry at bounded distance from a bilipschitz equivalence $g\colon X \longrightarrow Y$, then $f_* = g_*$. Since bilipschitz equivalence are bijections, it is easy to see that the induced map in homology preserves the semi-norm. □

We can make some observations regarding norms of maps in more general cases:

Remark 4.3.3. *(i) Proposition 4.3.2 clearly holds also in the case of bilipschitz equivalences between general metric spaces.*

(ii) *Proposition 4.3.1 can be also proved for quasi-isometries between quasi-lattices in metric spaces of coarse bounded geometry. The constant K_n, in this case, depends on the quasi-isometry considered and on the density constant of the quasi-lattice (Definition 1.4.2-(i)).*

(iii) *On the other hand, in the proof of Proposition 4.3.1 we have used the fact that balls in a* UDBG*-space have uniformly bounded cardinality. In a more general metric space of coarse bounded geometry, the uniform bound will depend on the quasi-lattice chosen, so it is not clear if the proof of Proposition 4.3.1 could work for more general metric spaces.*

4.3.2 Semi-norms for non-amenable UDBG-spaces

We consider now non-amenable UDBG-spaces. By Theorem 1.4.1, we know that in this case the zero degree uniformly finite homology is zero, so any result on semi-norms in degree zero is trivial. The following proposition shows that even in higher degree the semi-norm of uniformly finite homology classes is always zero in the non-amenable case.

Proposition 4.3.4. *Let X be a non-amenable* UDBG*-space. Then $\| - \|_\infty = 0$ on $H_*^{\mathrm{uf}}(X;\mathbb{R})$.*

Proof. Let d be the metric on X and consider the space $Y := X \times \{0,1\}$ endowed with the metric defined as follows:

$$\forall_{(x,k),(x',k')\in Y} \quad d_Y((x,k),(x',k')) = d_X(x,x') + |k-k'|.$$

Clearly, Y is a UDBG-space and the inclusion $i\colon X \hookrightarrow Y$, $x \longmapsto (x,0)$ has the canonical projection $p : Y \longrightarrow X$ as quasi-inverse. Since X is non-amenable, by Remark 1.6.4-(i), we know that i and p are both at bounded distance from bilipschitz equivalences. From Proposition 4.3.2, it follows that $i_*\colon H_*^{\mathrm{uf}}(X;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(Y;\mathbb{R})$ and $p_*\colon H_*^{\mathrm{uf}}(Y;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(X;\mathbb{R})$ are isometric isomorphisms. Let $n \in \mathbb{N}$ and let $\alpha \in H_n^{\mathrm{uf}}(X;\mathbb{R})$. Consider the class $\beta = \frac{1}{2}\cdot\alpha\times 0 + \frac{1}{2}\cdot\alpha\times 1 \in H_n^{\mathrm{uf}}(Y;\mathbb{R})$. Clearly, $p_n(\beta) = \alpha$ and since p_n is an isometry for any $n \in \mathbb{N}$, we have $\|\alpha\|_\infty = \|\beta\|_\infty$. Moreover, we have

$$\|\alpha\|_\infty = \|\beta\|_\infty = \left\| \frac{1}{2}\cdot\alpha\times 0 + \frac{1}{2}\cdot\alpha\times 1 \right\|_\infty = \frac{1}{2}\|\alpha\|_\infty.$$

So $\|\alpha\|_\infty = 0$ and the claim follows. □

4.4 Semi-norms for amenable groups

From Proposition 4.3.4 it follows that $\| - \|_\infty = 0$ on the uniformly finite homology of any finitely generated non-amenable group. We consider now the semi-norm in the amenable case. In Chapter 2, we have seen that for finitely generated amenable groups, every mean $m \in M(G)$, induces a map $m_*\colon H_*^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_*(G;\mathbb{R})$. In degree zero, we can consider the mean-invisible part $\widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R}) \subset H_0^{\mathrm{uf}}(G;\mathbb{R})$ (Definition 2.3.6). The following proposition is a consequence of an observation of Marcinkowski [30]:

Proposition 4.4.1. *Let G be a finitely generated amenable group. Then for any $\alpha \in H_0^{\mathrm{uf}}(G;\mathbb{R})$ we have that $\|\alpha\|_\infty = 0$ if and only if $\alpha \in \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})$.*

Notice that one direction of the proposition follows immediately from the fact that for any $\varphi \in \ell^\infty(G)$ and any invariant mean $m \in M(G)$, we have $m(\varphi) \leq \|\varphi\|_\infty$ [14, Proposition 4.1.6]. Indeed, if $\alpha \in H_0^{\mathrm{uf}}(G;\mathbb{R})$ has trivial semi-norm, then for any $m \in M(G)$, we have $m(\alpha) \leq \|\alpha\|_\infty = 0$, so $\alpha \in \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})$. In the case of higher degree uniformly finite homology of an amenable group, we have the following:

Proposition 4.4.2. *Let G be a finitely generated amenable group. For $n \in \mathbb{N}_{>0}$ we have $\|-\|_\infty = 0$ in $H_n^{\mathrm{uf}}(G;\mathbb{R})$.*

Proof. For any $n \in \mathbb{N}$ we define the supremum norm and the ℓ^1-norm of any $a = \sum_{\overline{g}=(e,g_1,\dots,g_n)\in G^{n+1}} \overline{g} \otimes \varphi_{\overline{g}} \in C_n(G;\ell^\infty(G))$ as follows:

$$\|a\|_\infty := \sup_{\overline{g}\in G^{n+1}} \|\varphi_{\overline{g}}\|_\infty$$

$$\|a\|_1 := \sum_{\overline{g}\in G^{n+1}} \|\varphi_{\overline{g}}\|_\infty$$

where for any $\overline{g} \in G^{n+1}$, $\|\varphi_{\overline{g}}\|_\infty$ is the standard supremum norm of $\varphi_{\overline{g}}$ in $\ell^\infty(G)$. We define the corresponding semi-norms on $H_*(G;\ell^\infty(G))$. Clearly, for any $\alpha \in H_n(G;\ell^\infty(G))$ we have

$$\|\alpha\|_\infty \leq \|\alpha\|_1. \tag{4.2}$$

Now let $c \in C_n^{\mathrm{uf}}(G;\mathbb{R})$ and let $\mathrm{supp}(c)$ be the set of tuples $\overline{g} \in G^{n+1}$ for which $c(\overline{g}) \neq 0$. From Lemma A.1.3, it follows that any $\overline{g} \in \mathrm{supp}(c)$ belongs to the G-orbit of some simplex of the form $(e, t_1, \dots, t_n)$. Notice that, since the simplices in $\mathrm{supp}(c)$ have all bounded diameter (Definition 1.1.4-(i)-(c)), there are only finitely many tuples of type $(e, t_1, \dots, t_n) \in G^{n+1}$ such that $g \cdot (e, t_1, \dots, t_n) \in \mathrm{supp}(c)$ for some $g \in G$ (Lemma A.1.3). It follows that

$$\|c\|_\infty = \sup_{g\in G} \sup_{(e,t_1,\dots,t_n)\in G^{n+1}} |c(g \cdot (e, t_1, \dots, t_n))|$$

From the chain isomorphism $\rho_n\colon C_n^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow C_n(G;\ell^\infty(G))$ given in Appendix A, for any $c \in C_n^{\mathrm{uf}}(G;\mathbb{R})$ it follows that:

$$\|c\|_\infty = \sup_{g\in G} \sup_{(e,t_1,\dots,t_n)\in G^{n+1}} |c(g\cdot(e,t_1,\dots,t_n))| \leq \sup_{(e,t_1,\dots,t_n)\in G^{n+1}} \|\varphi_{(e,t_1,\dots,t_n)}\|_\infty$$

where for any $(e,t_1,\dots,t_n) \in G^{n+1}$ we have

$$\begin{aligned}\varphi_{(e,t_1,\dots,t_n)}\colon G &\longrightarrow \mathbb{R}\\ g &\mapsto c(g^{-1}\cdot(e,t_1,\dots,t_n)).\end{aligned}$$

It follows that $\|c\|_\infty \leq \|\rho_n(c)\|_\infty$. In particular, for any $n \in \mathbb{N}$ and for any $\alpha \in H_n^{\mathrm{uf}}(G;\mathbb{R})$, we have

$$\|\alpha\|_\infty \leq \|\rho_n(\alpha)\|_\infty. \tag{4.3}$$

The proof is, then, an immediate consequence of the vanishing of l^1-homology in the case of amenable groups [29]. Indeed, the comparison map $H_*(G;\ell^\infty(G)) \longrightarrow H_*^{\ell^1}(G;\ell^\infty(G))$ induced by the inclusion is an isometry ([29, Proposition 2.4]). Moreover, for any $n > 0$ we have that $H_n^{\ell^1}(G;\ell^\infty(G)) = 0$ ([29, Corollary 5.5]). So, from (4.2) and from (4.3), for any $n > 0$ and any class $\alpha \in H_n^{\mathrm{uf}}(G;\mathbb{R})$ we have

$$\|\alpha\|_\infty \leq \|\rho_n(\alpha)\|_\infty \leq \|\rho_n(\alpha)\|_1 = 0.$$

So for any $n > 0$ we have $\|-\|_\infty = 0$ on $H_n^{\mathrm{uf}}(G;\mathbb{R})$. □

4.5 Rigidity for finitely generated groups via semi-norms

In Section 1.6, we have seen how one can distinguish between quasi-isometries and bilipschitz equivalences in the case of UDBG-spaces. In particular, Theorem 1.6.3 applies also to finitely generated groups. In this section, we investigate the difference between quasi-isometries and bilipschitz equivalences using semi-norms of classes in uniformly finite homology of groups. In particular we prove the following:

Theorem 4.5.1. *Let $f\colon G \longrightarrow H$ be a quasi-isometry between finitely generated groups. Then f is at bounded distance from a bilipschitz equivalence if and only if the induced map $f_0\colon H_0^{\mathrm{uf}}(G;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(H;\mathbb{Z})$ is an isometric isomorphism.*

We consider finitely generated discrete groups as UDBG-spaces and we take the uniformly finite homology with coefficients in $\mathbb{Z}$. We endow $\mathbb{Z}$ with the standard norm induced by the norm on $\mathbb{R}$ and we consider the supremum norm on the uniformly finite chains and the corresponding semi-norm on homology classes as in Definition 4.1.2. Notice that in the case of integer coefficients Proposition 4.3.1, 4.3.2 remain valid. Notice also that the semi-norm on $H_*^{\mathrm{uf}}(X;\mathbb{Z})$ is actually a norm: indeed if a class has norm zero, then there is a representative for this class which is zero everywhere, so the class must be trivial. However, the norm on $H_*^{\mathrm{uf}}(X;\mathbb{Z})$ for some UDBG-space X is, in general, not homogeneous. For example, for $X = \mathbb{Z}$ with the metric induced by the metric on $\mathbb{R}$, the class $[\sum_{z\in 2\mathbb{Z}} z] = [2\mathbb{Z}] \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ has clearly semi-norm 1. On the other hand, the class $2\cdot[2\mathbb{Z}]$ has also semi-norm 1. Indeed, the inclusion $i\colon 2\mathbb{Z} \hookrightarrow \mathbb{Z}$ is a quasi-isometry with inverse

$$\begin{aligned} p\colon \mathbb{Z} &\longrightarrow 2\mathbb{Z} \\ z &\mapsto \begin{cases} z & \text{if } z\in 2\mathbb{Z} \\ z+1 & \text{if } z\notin 2\mathbb{Z} \end{cases}. \end{aligned}$$

Thus, at the level of uniformly finite homology we have:

$$\begin{gathered} H_0^{\mathrm{uf}}(X;\mathbb{Z}) \xrightarrow{p_0} H_0^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \xrightarrow{i_0} H_0^{\mathrm{uf}}(X;\mathbb{Z}) \\ [\mathbb{Z}] \mapsto 2\cdot[2\mathbb{Z}] \mapsto 2\cdot[2\mathbb{Z}]. \end{gathered}$$

Since p_0 is an isomorphism having inverse i_0, it follows that $2\cdot[2\mathbb{Z}] = [\mathbb{Z}]$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$ and thus, the class $2\cdot[2\mathbb{Z}]$ has semi-norm 1.

We need some steps before proving Theorem 4.5.1. First of all, we give a criterion due to Whyte to detect trivial classes in the zero degree uniformly finite homology of any UDBG-space.

Theorem 4.5.2 ([43, Theorem 7.6])**.** *Let X be a* UDBG*-space. Let c be a cycle in $C_0^{\mathrm{uf}}(X;\mathbb{Z})$. Then $[c] = 0 \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ if and only if*

$$\exists_{C,r\in\mathbb{N}_{>0}} \quad \forall_{F\subseteq X \text{ finite}} \quad \left|\sum_{x\in F} c(x)\right| \leq C\cdot|\partial_r F|.$$

In a UDBG-space we distinguish the subsets which are coarsely dense from the ones which are sparse in a certain sense. Following Definition 1.4.2-(i), for $a\in\mathbb{R}_{>0}$ we say that a subset $S\subseteq X$ in a UDBG-space is *a-coarsely dense* in X if

$$\forall_{x\in X} \quad d(x,S)\leq a.$$

When we are not concerned with the value of a, we simply say that S is *coarsely dense* in X. We then have the following obvious definition for subsets which are not coarsely dense:

Definition 4.5.3. We say that a subset $S\subset X$ is *sparse* if it is not coarsely dense, more precisely if

$$\forall_{a\in\mathbb{R}_{>0}} \quad \exists_{x\in X} \quad \text{s.t. } d(x,S) > a.$$

The following lemma is an easy consequence of Theorem 1.6.3 and Theorem 4.5.2.

Lemma 4.5.4. *Let X be a* UDBG*-space. Let $D' \subseteq X$ be any coarsely dense subset with the induced metric. Supppose the inclusion $i\colon D' \hookrightarrow X$ is at bounded distance from a bilipschitz equivalence. Then for any other coarsely dense subset D for which $D' \subseteq D \subseteq X$, the inclusion $i\colon D \hookrightarrow X$ is also at bounded distance from a bilipschitz equivalence.*

Proof. Suppose that $i\colon D' \hookrightarrow X$ is at bounded distance from a bilipschitz equivalence, then by Theorem 1.6.3, we have

$$\begin{aligned} i_0\colon H_0^{\mathrm{uf}}(D';\mathbb{Z}) &\longrightarrow H_0^{\mathrm{uf}}(X;\mathbb{Z}) \\ [D'] &\longmapsto [X] \end{aligned}$$

So $[D'] = [X]$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$. In particular $0 = [X \setminus D'] = \left[\sum_{x \in X\setminus D'} x\right]$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$. By the characterization of Whyte (Theorem 4.5.2), we have

$$\exists_{C,r\in\mathbb{N}_{>0}} \quad \forall_{F\subseteq X \text{ finite}} \quad |(X\setminus D')\cap F| = \left|\sum_{x\in(X\setminus D')\cap F} 1\right| \leq C\cdot|\partial_r F|.$$

Now suppose $D' \subseteq D$ for some coarsely dense subset D of X. Then we have:

$$\exists_{C,r\in\mathbb{N}_{>0}} \quad \forall_{F\subseteq X \text{ finite}} \quad |(X\setminus D)\cap F| \leq |(X\setminus D')\cap F| \leq C\cdot|\partial_r F|.$$

Thus, by Theorem 4.5.2, the class $[X\setminus D] = \left[\sum_{x\in X\setminus D} x\right]$ must be zero in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$. It follows that the inclusion $i\colon D \hookrightarrow X$ induces a map $i_0\colon H_0^{\mathrm{uf}}(D;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(X;\mathbb{Z})$ such that $i_0([D]) = [D] = [X]$ in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$. Thus, by Theorem 1.6.3, there is a bilipschitz equivalence at bounded distance from $i\colon D \hookrightarrow X$. □

Following Whyte, we consider "positive" classes in $H_0^{\mathrm{uf}}(X;\mathbb{Z})$, for any UDBG-space X. These are classes supported on coarsely dense subsets of X and having positive integer coefficients.

Definition 4.5.5 ([43, Definition 3.3])**.** Let X be a UDBG-space.

(i) A chain in $c \in C_0^{\mathrm{uf}}(X;\mathbb{Z})$ is *positive* if the following two conditions hold:

- The support of c is a coarsely dense subset of X.
- For any $x \in \mathrm{supp}(c)$, we have $c(x) > 0$.

The space of positive chains is denoted by $C_0^{\mathrm{uf},+}(X;\mathbb{Z})$.

(ii) A class $\alpha \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ is called *positive* if it is represented by a positive cycle. The space of positive classes is denoted by $H_0^{\mathrm{uf},+}(X;\mathbb{Z})$ and it is called the *positive part* of $H_0^{\mathrm{uf}}(X;\mathbb{Z})$.

Remark 4.5.6. *From Proposition 1.4.6 it follows that a* UDBG*-space is amenable if and only if* $0 \notin H_0^{\mathrm{uf},+}(X,\mathbb{Z})$.

We want to consider sparse and coarsely dense subsets in finitely generated groups. The following is an easy consequence of the left invariance of the word metric on a finitely generated group (Definition 2.1.1)

Lemma 4.5.7. *Let G be a finitely generated amenable group. Let $F \subset G$ be a finite subset of G and let $C \subset G$ be sparse subset of G. Then there exists $g \in G$ such that $g \cdot F \cap C = \emptyset$.*

Proof. If $F \cap C = \emptyset$, then we can take $g = e$ and the claim follows. Otherwise, consider $K = \sup_{f \in F} d(f^{-1}, e)$. Since F is a finite set, K is finite. Suppose that for any $g \in G$ the set $g \cdot F \cap C$ is not empty. Then for any $g \in G$ there exists $f \in F$ and $c \in C$ such that $g = c \cdot f^{-1}$. Since the word metric on G with respect to any finite generating set is left invariant, for any $g = c \cdot f^{-1} \in G$ we have

$$d(g, C) = d(c \cdot f^{-1}, C) \leq d(f^{-1}, e) \leq K$$

This implies that C is K-coarsely dense, thus we have a contradiction. □

In the next lemma we see that in a finitely generated group positive classes cannot be represented by chains supported on sparse subsets.

Lemma 4.5.8. *Let G be a finitely generated amenable group and let α be a class in $H_0^{\mathrm{uf},+}(G;\mathbb{Z})$. Then there is no cycle $b = \sum_{g \in C \subset G} b(g) \cdot g \in C_0^{\mathrm{uf}}(G;\mathbb{Z})$ supported on a sparse set $C \subset G$ for which $\alpha = [b]$ in $H_0^{\mathrm{uf}}(G;\mathbb{Z})$.*

Proof. Let c be a positive cycle representing α. Suppose $\mathrm{supp}(c) \subset G$ is a-coarsely dense, for some $a > 0$. Suppose, for a contradiction, that there is a cycle b supported on a sparse subset $C \subset G$ such that $[b] = [c]$ in $H_0^{\mathrm{uf}}(G;\mathbb{Z})$. Since G is amenable, it admits a Følner sequence $\{F_j\}_{j \in \mathbb{N}}$ such that for any $r \in \mathbb{R}_{>0}$ the following holds:

$$\forall_{R \in \mathbb{R}_{>0}} \ \exists_{j_R \in \mathbb{N}} \ \text{ s.t. } \forall_{j \geq j_R} \quad |F_j| \geq R \cdot |\partial_r(F_j)|. \tag{4.4}$$

For any $r, R \in \mathbb{R}_{>0}$, fix a $\bar{j} \geq j_R \in \mathbb{N}$ for which $|F_{\bar{j}}| \geq R \cdot |\partial_r(F_{\bar{j}})|$. By Lemma 4.5.7, there exists $g_{\bar{j}} \in G$ such that $g_{\bar{j}} \cdot F_{\bar{j}} \cap C = \emptyset$. Clearly, we have $|g_{\bar{j}} \cdot F_{\bar{j}}| \geq R \cdot |\partial_r(g_{\bar{j}} \cdot F_{\bar{j}})|$. For any $r, R \in \mathbb{R}_{>0}$ consider $g_{\bar{j}} \cdot F_{\bar{j}} \subset G$ as above. Since $g_{\bar{j}} \cdot F_{\bar{j}} \cap C = \emptyset$ and since $\operatorname{supp}(c)$ is a-coarsely dense in G we have

$$\left| \sum_{g \in g_{\bar{j}} \cdot F_{\bar{j}}} c(g) - b(g) \right| = \left| \sum_{g \in g_{\bar{j}} \cdot F_{\bar{j}}} c(g) \right| \geq \frac{\left| g_{\bar{j}} \cdot F_{\bar{j}} \right|}{|B_a(e)|} \geq \frac{R}{|B_a(e)|} \cdot |\partial_r (g_{\bar{j}} \cdot F_{\bar{j}})|.$$

By Whyte's criterion (Theorem 4.5.2), it follows that the class $[c - b]$ is non-trivial. Thus we have a contradiction. $\square$

To understand the relation between quasi-isometries and bilipschitz equivalences better, Whyte introduced the so called *bilipschitz structures* on UDBG-spaces [43, Definition 3.1].

Definition 4.5.9. Let X be a UDBG-space. The *bilipschitz structure* on X is the "set"

$$S_{bilip}(X) := \big\{ (Y, f) \bigm| Y \text{ UDBG-space}, f \colon Y \longrightarrow X \text{ quasi-isometry} \big\} / \sim$$

where $(Y_1, f_1) \sim (Y_2, f_2)$ if and only if there exists a bilipschitz equivalence $h \colon Y_1 \longrightarrow Y_2$ for which the diagram

$$\begin{tikzcd} Y_1 \arrow[rr, "h"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

commutes up to bounded distance, i.e. there exists a K such that

$$\forall_{y \in Y_1} \quad d(f_2 \circ h(y), f_1(y)) \leq K.$$

We denote the element in $S_{bilip}(X)$ represented by (Y, f) as $\overline{(Y, f)}$.

The following theorem establishes a relation between the bilipschitz structures on a UDBG-space X and the positive classes in $H_0^{\mathrm{uf}}(X; \mathbb{Z})$.

Theorem 4.5.10. *[43, Theorem 3.5] Let (X, d) be a* UDBG*-space. There is a bijection*

$$\begin{aligned} \chi \colon H_0^{\mathrm{uf},+}(X; \mathbb{Z}) &\longrightarrow S_{bilip}(X) \\ \left[\sum_{x \in X} c(x) \cdot x \right] &\longmapsto \overline{(X_c, \pi)} \end{aligned}$$

where

$$X_c := \{ (x, n) \in \operatorname{supp}(c) \times \mathbb{N} \bigm| 1 \leq n \leq c(x) \}$$

is the UDBG*-space with the metric*

$$d_{X_c}((x,n),(y,m)) := d(x,y) + |n-m|$$

and $\pi\colon X_c \longrightarrow X$ *is the standard projection.*

Moreover, the map

$$\begin{aligned} S_{bilip}(X) &\longrightarrow H_0^{\mathrm{uf},+}(X;\mathbb{Z}) \\ \overline{(Y,f)} &\longmapsto f_*([Y]) \end{aligned}$$

is an inverse for χ.

Now we are ready to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. By Proposition 4.3.2, any quasi-isometry which is at bounded distance from a bilipschitz equivalence induces an isometric isomorphism in homology in degree zero, so one direction of the statement follows. If G and H are non-amenable then by Theorem 1.4.1, we have $H_0^{\mathrm{uf}}(G;\mathbb{Z}) = H_0^{\mathrm{uf}}(H;\mathbb{Z}) = 0$ and by Remark 1.6.4-(i), any quasi-isometry between G and H is at bounded distance from a bilipschitz equivalence, so the claim follows. So, we consider the amenable case. We consider both groups G and H with finite generating sets S and T respectively and with the corresponding word metrics.

Let $f\colon G \longrightarrow H$ be a quasi-isometry having quasi-inverse $l\colon H \longrightarrow G$. Suppose that f (and consequently l) is not at bounded distance from any bilipschitz equivalence. Then by Theorem 1.6.3, we have

$$[H] \neq f_0([G]) = \left[\sum_{h\in f(G)} |f^{-1}(h)|\cdot h\right], \tag{4.5}$$

$$[G] \neq l_0([H]) = \left[\sum_{g\in l(H)} |l^{-1}(g)|\cdot g\right]. \tag{4.6}$$

Notice that $f_0([G]) \in H_0^{\mathrm{uf},+}(H;\mathbb{Z})$ and $l_0([H]) \in H_0^{\mathrm{uf},+}(G;\mathbb{Z})$. We want to prove that the induced map $f_0\colon H_0^{\mathrm{uf}}(G;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(H;\mathbb{Z})$ does not preserve semi-norms. Assume, for a contradiction, that f_0 is an isometry. Then l_0 is also an isometry. It is clear that $[G] \in H_0^{\mathrm{uf}}(G;\mathbb{Z})$,$[H] \in H_0^{\mathrm{uf}}(H;\mathbb{Z})$ have semi-norm 1, since there cannot be other representatives of these classes having integer coefficients and smaller norms. Thus f_0 (respectively l_0) maps $[G]$ (respectively $[H]$) to a class of semi-norm 1 in $H_0^{\mathrm{uf}}(H;\mathbb{Z})$ (respectively $H_0^{\mathrm{uf}}(G;\mathbb{Z})$). The classes in $H_0^{\mathrm{uf}}(H;\mathbb{Z})$ having semi-norm 1 are of one of the following three types:

(i) A class represented by $\sum_{h \in V} h$, for some $V \subseteq H$, not empty;

(ii) A class represented by $\sum_{h \in V} -h$, for some $V \subseteq H$, not empty;

(iii) A class represented by $\sum_{h \in V_0} h - \sum_{h \in V_1} h$, for some $V_0, V_1 \subset H$ not empty such that $V_0 \cap V_1 = \emptyset$.

The same classification can be made for classes in $H_0^{\mathrm{uf}}(G;\mathbb{Z})$ having seminorm 1. We suppose that $f_0([G])$ and $l_0([H])$ are of one these three types and we will get a contradiction in all the possible cases. First, we can exclude immediately type (ii). Indeed, suppose $f_0([G]) = [\sum_{h \in V} -h]$, for some $V \subseteq H$. Then we have $f_0([G]) + [V] = 0$ in $H_0^{\mathrm{uf}}(H;\mathbb{Z})$. By Remark 4.5.6, this cannot happen because the class $f_0([G]) + [V]$ is positive and the group H is amenable. We are left with four possible situations:

Case 1. The classes $f_0([G])$ and $l_0([H])$ are be both of type (i).

Suppose $f_0([G]) = [V]$ for some $V \subseteq H$ and $l_0([H]) = [U]$ for some $U \subseteq G$. Since $f_0([G])$ and $l_0([H])$ are both positive classes, by (4.5) and (4.6) and by Lemma 4.5.8, the subsets $V \subset H$, $U \subset G$ must be proper and coarsely dense in H and in G respectively. By the bijection χ defined in Theorem 4.5.10, we can represent the class $f_0([G])$ as a bilipschitz structure in the following way:

$$f_0([G]) = \left[\sum_{h \in f(G)} |f^{-1}(h)| \cdot h \right] \overset{\chi}{\mapsto} \overline{(X_{f_0([G])}, \pi)}$$

where

$$X_{f_0([G])} = \left\{ (h, n) \in f(G) \times \mathbb{N} \;\middle|\; 1 \leq n \leq |f^{-1}(h)| \right\}$$

and $\pi \colon X_{f_0([G])} \longrightarrow H$ is the standard projection. On the other hand, applying χ to the class $[V] = [\sum_{h \in V \subset H} h]$ we have

$$[V] \overset{\chi}{\mapsto} \overline{(V \times \{1\}, \pi)}$$

where $\pi \colon V \times \{1\} \longrightarrow H$ is the standard projection. Notice that $\overline{(V \times \{1\}, \pi)} = \overline{(V, i)}$ where $i \colon V \hookrightarrow H$ is the standard inclusion. Since $f_0([G]) = [V]$, by Theorem 4.5.10 there is a bilipschitz equivalence $\mu_1 \colon X_{f_0([G])} \longrightarrow V$ such that the diagram

$$\begin{tikzcd} X_{f_0([G])} \arrow[rr, "\mu_1"] \arrow[dr, "\pi"'] & & V \arrow[dl, "i"] \\ & H & \end{tikzcd}$$

commutes up to bounded distance. By applying χ to $f_0(l_0([H]))$ in $H_0^{\mathrm{uf},+}(H;\mathbb{Z})$ we have

$$f_0(l_0([H])) = f_0([U]) = \left[\sum_{h\in f(U)} |f^{-1}(h)|\cdot h\right] \overset{\chi}{\mapsto} \overline{(X_{f_0 l_0([H])},\pi)}$$

where

$$X_{f_0 l_0([H])} = \left\{(h,n)\in f(U)\times\mathbb{N} \;\middle|\; 1\le n\le |f^{-1}(h)|\right\}$$

and $\pi\colon X_{f_0 l_0([H])} \longrightarrow H$ is the standard projection. Moreover, applying χ to the fundamental class $[H]\in H_0^{\mathrm{uf},+}(H;\mathbb{Z})$ we have

$$[H] \overset{\chi}{\mapsto} \overline{(H\times\{1\},\pi)}$$

where $\pi\colon H\times\{1\}\longrightarrow H$ is the standard projection. One can easily see that $\overline{(H\times\{1\},\pi)} = \overline{(H,\mathrm{id})}$ where $\mathrm{id}\colon H\longrightarrow H$ is the identity map. Moreover, since $f_0(l_0([H])) = [H]$, there is a bilipschitz equivalence $\mu_2\colon X_{f_0 l_0([H])}\longrightarrow H$ such that the diagram

$$\begin{tikzcd} X_{f_0 l_0([H])} \arrow[rr, "\mu_2"] \arrow[dr, "\pi"'] & & H \arrow[dl, "\mathrm{id}"] \\ & H & \end{tikzcd} \tag{4.7}$$

commutes up to bounded distance. Notice that $X_{f_0 l_0([H])}\subset X_{f_0([G])}$. So, restricting μ_1 to $X_{f_0 l_0([H])}$ we obtain a bilipschitz equivalence between $X_{f_0 l_0([H])}$ and a subspace $V'\subset V$. Thus we have

$$\begin{tikzcd} V' \arrow[r, "(\mu_1^{-1})_{|V'}"] \arrow[dr, "i"'] & X_{f_0 l_0([H])} \arrow[r, "\mu_2"] \arrow[d, "\pi"] & H \arrow[dl, "\mathrm{id}"] \\ & H & \end{tikzcd}$$

So $\mu := \mu_2\circ(\mu_1^{-1})_{|V'}$ is a bilipschitz equivalence from V' to H which is at bounded distance from the inclusion $i\colon V'\hookrightarrow H$. We have $V'\subset V\subset H$ thus, by Lemma 4.5.4, there must be a bilipschitz equivalence at bounded distance from the inclusion $i\colon V\longrightarrow H$. This implies that $\overline{(V,i)} = \overline{(H,\mathrm{id})}$ as elements in $S_{bilip}(H)$. It follows that $f_0([G]) = [V] = [H]$ which contradicts (4.5). So, Case 1 cannot occur.

Case 2. The class $f_0([G])$ is of type (iii) and the class $l_0([H])$ is of type (i).

Suppose $f_0([G]) = \left[\sum_{h\in V_0} h - \sum_{h\in V_1} h\right]$ for some $V_0, V_1 \subset H$ such that $V_0 \cap V_1 = \emptyset$ and suppose $l_0([H]) = [U]$ for some $U \subset G$. Since $f_0([G]) + [V_1]$ is a class in $H_0^{\mathrm{uf},+}(H;\mathbb{Z})$, by Lemma 4.5.8, it follows that $V_0 \subset H$ must be a coarsely dense subset of H. The same holds for $U \subset G$. Applying χ to $f_0([G]) + [V_1]$ we have

$$f_0([G]) + [V_1] = \left[\sum_{h\in f(G)} |f^{-1}(h)| \cdot h + \sum_{h\in V_1} h\right] \overset{\chi}{\longmapsto} \overline{(X_{f_0([G])+[V_1]}, \pi)}$$

where

$$X_{f_0([G])+[V_1]} = \left\{(h,n) \in f(G) \cup V_1 \times \mathbb{N} \;\middle|\; 1 \le n \le c(h)\right\}$$

and

$$c(h) := \begin{cases} |f^{-1}(h)| & \text{if } h \in f(G)\backslash V_1 \\ 1 & \text{if } h \in V_1 \backslash f(G) \\ |f^{-1}(h)| + 1 & \text{if } h \in f(G) \cap V_1. \end{cases}$$

The map $\pi\colon X_{f_0([G])+[V_1]} \longrightarrow H$ is the projection. Moreover, since $f_0([G]) + [V_1] = [V_0]$ in $H_0^{\mathrm{uf}}(H;\mathbb{Z})$, by the correspondence between positive classes and bilipschitz structures there is a bilipschitz equivalence $\mu_1\colon X_{f_0([G])+[V_1]} \longrightarrow V_0$ such that the diagram

$$\begin{tikzcd} X_{f_0([G])+[V_1]} \arrow[rr, "\mu_1"] \arrow[dr, "\pi"'] & & V_0 \arrow[dl, "i"] \\ & H & \end{tikzcd} \tag{4.8}$$

commutes up to bounded distance. As in Case 1, there is a bilipschitz equivalence $\mu_2\colon X_{f_0 l_0([H])} \longrightarrow H$ making the diagram (4.7) commuting up to bounded distance. Notice that $X_{f_0 l_0([H])} \subset X_{f_0([G])+[V_1]}$. In particular we can restrict μ_1 to $X_{f_0 l_0([H])}$ obtaining a bilipschitz equivalence between $X_{f_0 l_0([H])}$ and a subset $V_0' \subset V_0$. Similarly to Case 1, we have

$$\begin{tikzcd} V_0' \arrow[r, "(\mu_1^{-1})_{|V_0'}"] \arrow[dr, "i"'] & X_{f_0 l_0([H])} \arrow[r, "\mu_2"] \arrow[d, "\pi"] & H \arrow[dl, "\mathrm{id}"] \\ & H & \end{tikzcd}$$

So $\mu := \mu_2 \circ (\mu_1^{-1})_{|_{V_0'}}$ is a bilipschitz equivalence from V_0' to H which is at bounded distance from the inclusion $i\colon V_0' \hookrightarrow H$. Since we have $V_0' \subset V_0 \subset H$, by Lemma 4.5.4, , there must be a bilipschitz equivalence at bounded distance from the inclusion $i\colon V_0 \hookrightarrow H$. It follows that $[V_0] = [H]$ in $H_0^{\mathrm{uf}}(H;\mathbb{Z})$ which contradicts our assumption $V_0 \cap V_1 = \emptyset$. So we can exclude Case 2.

Case 3. By the same argument used in Case 2., we can exclude the case in which $f_0([G])$ is of type (i) and $l_0([H])$ is of type (iii). We are left with one last case:

Case 4. The classes $f_0([G])$ and $l_0([H])$ are be both of type (iii).

Suppose $f_0([G]) = \left[\sum_{h\in V_0} h - \sum_{h\in V_1} h\right]$ for some $V_0, V_1 \subset H$ such that $V_0 \cap V_1 = \emptyset$ and $l_0([H]) = \left[\sum_{g\in U_0\subset G} g - \sum_{g\in U_1\subset G} g\right]$ for some $U_0, U_1 \subset G$ such that $U_0 \cap U_1 = \emptyset$. By the same argument used in Case 1., the subsets $U_0 \subset G$, $V_0 \subset H$ must be coarsely dense subsets of G and H respectively. Proceeding as in Case 2., we obtain a bilipschitz equivalence $\mu_1\colon X_{f_0([G])+[V_1]} \longrightarrow V_0$ making diagram (4.8) commutative. Clearly, $[H] = f_0 l_0([H]) = f_0([U_0]) - f_0([U_1])$. Applying χ to $[H] + f_0([U_1]) \in H_0^{\mathrm{uf},+}(H,\mathbb{Z})$ we have

$$[H] + f_0([U_1]) = \left[\sum_{h\in H} h + \sum_{h\in f(U_1)} |f^{-1}(h)| \cdot h\right] \overset{\chi}{\mapsto} \overline{(X_{[H]+f_0([U_1])}, \pi)}$$

where $X_{[H]+f_0([U_1])} = \left\{(h,n) \in H \times \mathbb{N} \;\middle|\; 1 \le n \le c(h)\right\}$ and

$$c(h) := \begin{cases} |f^{-1}(h)| + 1 & \text{if } h \in f(U_1) \\ 1 & \text{otherwise.} \end{cases}$$

The map $\pi\colon X_{[H]+f_0([U_1])} \longrightarrow H$ is, again, the standard projection. On the other hand, applying χ to the class $f_0([U_0]) \in H_0^{\mathrm{uf},+}(H;\mathbb{Z})$, we obtain

$$f_0([U_0]) = \left[\sum_{h\in f(U_0)} |f^{-1}(h)| \cdot h\right] \overset{\chi}{\mapsto} \overline{(X_{f_0([U_0])}, \pi)}$$

where

$$X_{f_0([U_0])} = \left\{(h,n) \in f(U_0) \times \mathbb{N} \;\middle|\; 1 \le n \le |f^{-1}(h)|\right\}$$

and $\pi\colon X_{f_0([U_0])} \longrightarrow H$ is the standard projection. By Theorem 4.5.10, since $f_0([U_0]) = [H] + f_0([U_1])$, we can find a bilipschitz equivalence

$\mu_2 \colon X_{f_0([U_0])} \longrightarrow X_{[H]+f_0([U_1])}$ such that the diagram

$$\begin{tikzcd} X_{f_0([U_0])} \arrow[rr, "\mu_2"] \arrow[dr, "\pi"'] & & X_{[H]+f_0([U_1])} \arrow[dl, "\pi"] \\ & H & \end{tikzcd}$$

commutes up to bounded distance. Clearly $X_{f_0([U_0])} \subset X_{f_0([G])+[V_1]}$. In particular, we can restrict μ_1 to $X_{f_0([U_0])}$ obtaining a bilipschitz equivalence between $X_{f_0([U_0])}$ and a subset $V_0' \subset V_0$. Similarly to Case 1. and 2., we have

$$\begin{tikzcd} V_0' \arrow[r, "(\mu_1^{-1})_{|V_0'}"] \arrow[dr, "i"'] & X_{f_0([U_0])} \arrow[r, "\mu_2"] \arrow[d, "\pi"] & X_{[H]+f_0([U_1])} \arrow[dl, "\pi"] \\ & H & \end{tikzcd}$$

So $\mu := \mu_2 \circ (\mu_1^{-1})_{|V_0'}$ is a bilipschitz equivalence from $V_0' \subset V_0$ to $X_{[H]+f_0([U_1])}$ such that $d(\pi \circ \mu, i) < \infty$. Notice that $H \subset X_{[H]+f_0([U_1])}$, so restricting the image of μ to H we obtain a bilipschitz equivalence $\mu' \colon V_0'' \longrightarrow H$ for some $V_0'' \subset V_0' \subset H$. Since the diagram above commutes, it follows that μ' is at bounded distance from the inclusion $i \colon V_0'' \hookrightarrow H$. Again, using Lemma 4.5.4, we deduce that there must be a bilipschitz equivalence at bounded distance from the inclusion $i \colon V_0 \hookrightarrow H$. It follows that $[V_0] = [H]$ in $H_0^{\mathrm{uf}}(H;\mathbb{Z})$ which contradicts our assumption $V_0 \cap V_1 = \emptyset$. So we can exclude Case 4.

Since we have excluded all the possible cases, it follows that $f_0([G])$ and $l_0([H])$ do not have norm 1. So f_0 (resp. l_0) is not an isometry and the claim follows. $\square$

The following example shows that Theorem 4.5.1 does not hold for higher degree uniformly finite homology.

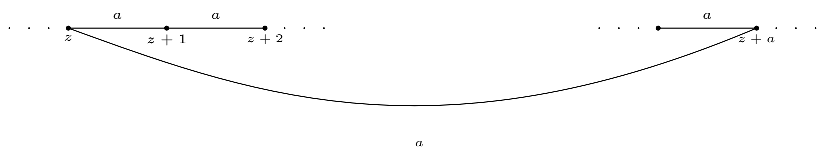
Example 4.5.11. The inclusion $i \colon 2\mathbb{Z} \hookrightarrow \mathbb{Z}$ is a quasi-isometry which is not at bounded distance from any bilipschitz equivalence. On the other hand, the induced map in homology $i_1 \colon H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is an isometric isomorphism.

Proof. By Remark 1.6.4-(ii), the inclusion of a subgroup of index $n > 1$ in a finitely generated amenable group induces a quasi-isometry which is not at bounded distance from a bilipschitz equivalence. To show that the map $i_1 : H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z}) \longrightarrow H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ is an isometry, we prove that any non-trivial class in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ and in $H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z})$ has semi-norm 1, so i_1 necessarily preserves the semi-norms. Notice that any non-trivial class in homology with $\mathbb{Z}$ coefficients has always at least semi-norm 1. We consider, first, classes represented by cycles of the form

$$\sum_{z\in\mathbb{Z}} a \cdot (z, z+1) \in C_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z}) \tag{4.9}$$

for some $a \in \mathbb{Z}\backslash\{0\}$. We have $[\sum_{z\in\mathbb{Z}} a \cdot (z, z+1)] = [\sum_{z\in a\mathbb{Z}} a \cdot (z, z+a)]$ in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$; indeed, one can easily see that for any $z \in a\mathbb{Z}$, the cycle $\left(\sum_{j=1}^{a}(z, z+j)\right) - (z, z+a)$ is bounded by sums of 2-simplices of the form

$$(z, z+1, z+a) + (z+1, z+2, z+a) + \cdots + (z+a-1, z+a, z+a).$$



More generally, for any $k = \{0, \ldots, a-1\}$, we have

$$\left[\sum_{z\in\mathbb{Z}} a \cdot (z, z+1)\right] = \left[\sum_{z\in a\cdot\mathbb{Z}+k} a \cdot (z, z+a+k)\right].$$

Thus, we have

$$\left[\sum_{z\in\mathbb{Z}} a \cdot (z, z+1)\right] = \sum_{k=0}^{a-1}\left[\sum_{z\in a\mathbb{Z}+k} (z, z+a+k)\right].$$

It follows that for any $a \in \mathbb{Z}\backslash\{0\}$, we have

$$\left\|\left[\sum_{z\in\mathbb{Z}} a \cdot (z, z+1)\right]\right\|_\infty = 1.$$

Now we consider any class in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ and we reduce it to a class represented by a cycle of type (4.9). Since $\mathbb{Z}$ and $\mathbb{R}$ are quasi-isometric, we have $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z}) \cong H_1^{\mathrm{uf}}(\mathbb{R};\mathbb{Z})$. Moreover, in Section 1.5.1 we have seen that $H_1^{\mathrm{uf}}(\mathbb{R};\mathbb{Z}) \cong H_1^{\mathrm{suf}}(\mathbb{R};\mathbb{Z}) \cong \mathbb{Z}$, where $\mathbb{R}$ is viewed as uniformly contractible simplicial complex with the standard triangulation. Following the proof of Proposition 1.3.3 given in Appendix B, we can find a chain map $j_*\colon C_*^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z}) \longrightarrow C_*^{\mathrm{suf}}(\mathbb{R};\mathbb{Z})$ which is chain homotopy inverse to the inclusion $i_*\colon C_*^{\mathrm{suf}}(\mathbb{R};\mathbb{Z}) \longrightarrow C_*^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$. In degree 1, the map j_1 is defined for any $(z,z') \in \mathbb{Z}^2$ as

$$j_1(z,z') = \begin{cases} [z,z+1]+[z+1,z+2]+\cdots+[z'-1,z'] & \text{if } z<z' \\ -[z',z'+1]-[z'+1,z'+2]-\cdots-[z-1,z] & \text{if } z>z'. \end{cases}$$

It follows that, for any cycle $c \in C_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ the cycle $i_1 j_1(c)$ is of the form (4.9). Moreover, since i_1 and j_1 are chain homotopy inverse, we have that $[c]=[i_1 j_1(c)]$ in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$. So the class of $[c]$ is equivalent to a class which has semi-norm 1. It follows that any non-trivial class in $H_1^{\mathrm{uf}}(\mathbb{Z};\mathbb{Z})$ has semi-norm 1. A similar argument shows that any non-trivial class in $H_1^{\mathrm{uf}}(2\mathbb{Z};\mathbb{Z})$ has semi-norm 1, so i_1 must be an isometric isomorphism in homology. □

It is not clear if Theorem 4.5.1 also holds in the case of uniformly finite homology with coefficients in $\mathbb{R}$ (or in other unitary normed rings). Indeed, in the case of real coefficients the classes of semi-norm 1 are much more complicated and cannot be classified in three types as in the case of integer coefficients. The following theorem shows that if one reduces to group homomorphisms with finite kernel and with finite index image, then one can say something about the behavior of the induced map with respect to the semi-norm.

Theorem 4.5.12. *Let $f\colon G \longrightarrow H$ be a group homomorphism between finitely generated amenable groups. Suppose $[H : f(G)] = n$ and $|f^{-1}(e)| = k$ for some $n,k<\infty$. The following are equivalent:*

(i) $k=n$.

(ii) The map f is at bounded distance from a bilipschitz equivalence.

(iii) The induced map $f_0\colon H_0^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_0^{\mathrm{uf}}(H;\mathbb{R})$ is an isometric isomorphism.

Notice that a group homomorphism $f\colon G \longrightarrow H$ is a quasi-isometry if and only if $[H : f(G)] < \infty$ and $|f^{-1}(e)| < \infty$.

The equivalence between (i) and (ii) was proved by Dymarz [17] and it is a consequence of Theorem 1.6.3 and of the fact that the inclusion of a finite index subgroup is not at bounded distance from a bilipschitz equivalence (Remark 1.6.4-(ii)). The implication $(ii) \Rightarrow (iii)$ is immediate. We prove the equivalences between the statements in one step.

Proof. The implication $(ii) \Rightarrow (iii)$ follows from Proposition 4.3.2. We have

$$f_0([G]) = \left[\sum_{h \in f(G)} |f^{-1}(h)| \cdot h \right] = k \cdot [f(G)] \,.$$

Moreover, we can write

$$[H] = \left[\sum_{h \cdot f(G) \in H \cdot f(G)} h \cdot f(G) \right] = n \cdot [f(G)] \,.$$

It follows that $f_0([G]) = \frac{k}{n}[H]$. Since the semi-norm of the fundamental class is 1 and the semi-norm on the uniformly finite homology with coefficients in $\mathbb{R}$ is homogeneous, we have $\|f_0([G])\|_\infty = \frac{k}{n}$. So, if f_0 is an isometry, then $k = n$. Thus $(iii) \Rightarrow (i)$ follows. Moreover, by Theorem 1.6.3, we have $k = n$ if and only if f_0 is at bounded distance from a bilipschitz equivalence, so $(ii) \Leftrightarrow (i)$. □

4.6 Rigidity for UDBG-spaces via semi-norms

In this section we prove a more general version of Theorem 4.5.1, that detects the difference between quasi-isometries and bilipschitz equivalences between UDBG-spaces using semi-norms in uniformly finite homology in degree zero. The proof is based on the fact that the fundamental class $[X] \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ for some a UDBG-space X is the "largest" class with semi-norm at most 1 (Lemma 4.6.2). The proof of Theorem 4.6.3 was suggested by Clara Löh and it makes uses of Whyte's rigidity result (Theorem 1.6.3).

For any UDBG-space X, we define a subspace of $H_0^{\mathrm{uf}}(X;\mathbb{Z})$ consisting of classes which are represented by cycles of non-negative coefficients.

Definition 4.6.1. For any UDBG-space X, define:

$$C_0^{\mathrm{uf}(+)}(X;\mathbb{Z}) := \left\{ \sum_{x \in X} c(x) \cdot x \;\middle|\; \forall_{x \in X} \quad c(x) \geq 0 \right\}.$$

We say that a class $\alpha \in H_0^{\mathrm{uf}(+)}(X;\mathbb{Z}) \subset H_0^{\mathrm{uf}}(X;\mathbb{Z})$ if α is represented by some cycle in $C_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$.

Clearly, for any UDBG-space X the fundamental class $[X]$ belongs to $H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$ and has semi-norm ≤ 1.

Lemma 4.6.2. *Let X be a* UDBG*-space. We have:*

(i) Let $\alpha \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$, $\alpha \neq 0$. Suppose $\|\alpha\|_\infty \leq 1$ and $\alpha \neq [X]$. Then, there exists $\beta \in H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$, $\beta \neq 0$ such that $\|\alpha + \beta\|_\infty \leq 1$.

(ii) For any $\beta \in H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$, $\beta \neq 0$ we have $\|[X] + \beta\|_\infty > 1$.

Proof. We first prove (i). Let $\alpha \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ with $\|\alpha\|_\infty \leq 1$ and $\alpha \neq [X]$. Then α is represented by some cycle that have only 1 or -1 as coefficients. Let $\beta := [X] - \alpha$. Clearly $\beta \in H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$ and $\|\alpha + \beta\|_\infty \leq 1$ and the claim follows. To prove (ii) we assume, for a contradiction, that there exists $\beta \in H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})\setminus\{0\}$ such that $\|[X] + \beta\|_\infty \leq 1$. In particular, suppose that $\beta = [\sum_{x\in X} c(x)\cdot x]$ with $c(x) \geq 0$ for any $x \in X$ and that $[X] + \beta = [\sum_{x\in X} \widetilde{c}(x)\cdot x]$ with $|\widetilde{c}(x)| \leq 1$ for any $x \in X$. We have

$$0 = [X] + \beta - ([X] + \beta) = \left[\sum_{x\in X} (1 + c(x) - \widetilde{c}(x))\cdot x\right].$$

By Theorem 4.5.2, it follows that

$$\exists_{C,r\in\mathbb{N}_{>0}} \quad \forall_{F\subseteq X \text{ finite}} \quad \left|\sum_{x\in F} (1 + c(x) - \widetilde{c}(x))\right| \leq C\cdot|\partial_r F|. \tag{4.10}$$

On the other hand, for any $x \in X$ we have $|\widetilde{c}(x)| \leq 1$ and $c(x) \geq 0$. Thus for any $x \in X$ we have $c(x) \leq 1 + c(x) - \widetilde{c}(x)$. From (4.10) it follows that

$$\exists_{C,r\in\mathbb{N}_{>0}} \quad \forall_{F\subseteq X \text{ finite}} \quad \left|\sum_{x\in F} c(x)\right| \leq \left|\sum_{x\in F} (1 + c(x) - \widetilde{c}(x))\right| \leq C\cdot|\partial_r F|.$$

Thus, by Theorem 4.5.2, we have $\beta = [\sum_{x\in X} c(x)\cdot x] = 0$, but this contradicts our assumption on β, so the claim follows. $\square$

We can, then, use semi-norms to deduce rigidity results for any UDBG-space.

Theorem 4.6.3. *Let $f\colon X \longrightarrow Y$ be a quasi-isometry between* UDBG-*spaces. Then f is at bounded distance from a bilipschitz equivalence if and only if the induced map $f_0\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ is an isometric isomorphism.*

Proof. By Proposition 4.3.2, any quasi-isometry that is at bounded distance from a bilipschitz equivalence induces an isometric isomorphism in uniformly finite homology in degree zero, so one direction of the statement follows. Suppose $f\colon X \longrightarrow Y$ is a quasi-isometry that is not at bounded distance from any bilipschitz equivalence and suppose, for a contradiction, that the induced map $f_0\colon H_0^{\mathrm{uf}}(X;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(Y;\mathbb{Z})$ is an isometry. From Theorem 1.6.3, it follows that $f_0([X]) \neq [Y] \in H_0^{\mathrm{uf}}(Y;\mathbb{Z})$. Moreover, since f_0 is an isometry and since $\|[X]\|_\infty \leq 1$, we have $\|f_0([X])\|_\infty \leq 1$. Thus, by Lemma 4.6.2-(i), there exists $\beta \in H_0^{\mathrm{uf}(+)}(Y;\mathbb{Z})$, $\beta \neq 0$ such that $\|f_0([X]) + \beta\| \leq 1$. Since f_0 is an isomorphism, there exists $\alpha \in H_0^{\mathrm{uf}}(X;\mathbb{Z})$ such that $f_0(\alpha) = \beta$. Notice that $f_0|_{H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})}$ gives a bijection between $H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$ and $H_0^{\mathrm{uf}(+)}(Y;\mathbb{Z})$, so $\alpha \in H_0^{\mathrm{uf}(+)}(X;\mathbb{Z})$. We have

$$\|f_0(\alpha + [X])\|_\infty = \|\beta + f_0([X])\|_\infty \leq 1$$

and since f_0 is an isometry, it follows that $\|\alpha + [X]\| \leq 1$. However, this contradicts Lemma 4.6.2-(ii), thus the claim follows. $\square$

Appendices

Appendix A

Uniformly finite homology and homology with ℓ^∞-coefficients

In this chapter, we give a detailed proof of Proposition 2.2.4 which follows from an observation of Brodski, Niblo and Wrigth [11]. We consider finitely generated infinite groups and we provide an explicit chain isomorphism between uniformly finite chains with values in a normed unital ring A and the chain complex given in Definition 2.2.1 where we consider $\ell^\infty(G, A)$ as coefficient module.

A.1 Uniformly finite homology defined using small simplices

Let G be a finitely generated infinite group and let S be a finite generating set. Consider the word metric $d = d_S$ with respect to S as given in Definition 2.1.1. We want to give a description of the uniformly finite chains on G in terms of bounded functions on sets of "small simplices" of G.

Definition A.1.1. Let $r \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$. A tuple $\overline{g} = (g_0, \dots, g_n) \in G^{n+1}$ is an *r-small n-simplex of G* if

$$\forall_{0 \leq i,j \leq n} \quad d(g_i, g_j) \leq r.$$

For all $r \in \mathbb{R}_{>0}$ and all $n \in \mathbb{N}$ we denote by $S_n^r(G)$ the set of r-small n-simplices in G. Now le A be a unital ring with norm as defined in (1.1) on page 3. Consider the uniformly finite chain complex $(C_*^{\mathrm{uf}}(G; A), \partial_*)$ as given in Definition 1.1.4. It is easy to see that for any $n \in \mathbb{N}$ the module $C_n^{\mathrm{uf}}(G; A)$ can be written as

$$C_n^{\mathrm{uf}}(G; A) = \bigcup_{r \in \mathbb{R}_{>0}} \ell^\infty(S_n^r(G), A).$$

Definition A.1.2. For any $n \in \mathbb{N}, r \in \mathbb{R}_{>0}, g \in G$ define

$$S_n^r(G)_g := \{(g_0, \dots, g_n) \in S_n^r(G) \mid g_0 = g\}.$$

Notice that, since the group G is finitely generated, the set $S_n^r(G)_g$ is finite for any $g \in G$, any $r \in \mathbb{R}_{>0}$ and any $n \in \mathbb{N}$.

Lemma A.1.3. *Let $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}$. Then*

$$S_n^r(G) = G \cdot S_n^r(G)_e$$

where the action of G acts diagonally on $S_n^r(G)_e$.

Proof. The proof is an easy consequence of the left invariance of the word metric (Remark 2.1.2). □

It follows that for all $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}$ any element $\overline{g} \in S_n^r(G)$ can be written as $\overline{g} = g\overline{t}$ for some $g \in G$ and some $\overline{t} \in S_n^r(G)_e$. By Definition 1.1.4-(i)-(c), for any $n \in \mathbb{N}$ every chain $c \in C_n^{\mathrm{uf}}(G; A)$ is supported on elements in G^{n+1} of diameter bounded by some constant $R_c > 0$ depending on c. Thus, for any $n \in \mathbb{N}$ we can write any $c = \sum_{\overline{g} \in G^{n+1}} c(\overline{g}) \cdot \overline{g} \in C_n^{\mathrm{uf}}(G; A)$ as

$$c = \sum_{\overline{t} \in S_n^{R_c}(G)_e} \sum_{g \in G} c(g\overline{t}) \cdot g\overline{t}.$$

Notice that the first sum is finite since $\left|S_n^{R_c}(G)_e\right| < \infty$.

A.2 Chain isomorphism

Let G be a finitely generated group and let A be a unital ring with norm.

Proof of Proposition 2.2.4. We start by constructing a chain map

$$\rho_* \colon C_*^{\mathrm{uf}}(G; A) \longrightarrow C_*(G; \ell^\infty(G, A)).$$

Let $n \in \mathbb{N}$ and let $c = \sum_{\overline{t} \in S_n^r(G)_e} \sum_{g \in G} c(g\overline{t}) \cdot g\overline{t} \in C_n^{\mathrm{uf}}(G; A)$ for some $r > 0$. For any $\overline{t} \in S_n^r(G)_e$, define

$$\rho_n \left(\sum_{g \in G} c(g\overline{t}) \cdot g\overline{t} \right) = \overline{t} \otimes \varphi_{\overline{t}}$$

where

$$\begin{aligned} \varphi_{\overline{t}} \colon G &\longrightarrow A \\ g &\longmapsto c(g^{-1}\overline{t}). \end{aligned}$$

Extending ρ_n linearly we obtain

$$\rho_n(c) = \sum_{\bar{t} \in S_n^r(G)_e} \bar{t} \otimes \varphi_{\bar{t}}.$$

By Definition 1.1.4, the chain $c \in C_n^{\mathrm{uf}}(G; A)$ is bounded. In particular, there exists $K_c > 0$ such that

$$\sup_{\bar{t} \in S_n^r(G)_e} \sup_{g \in G} |c(g\bar{t})| < K_c.$$

Thus for any $\bar{t} \in S_n^r(G)_e$ we have $\varphi_{\bar{t}} \in \ell^\infty(G, A)$. In particular, $\rho_n(c)$ is a well-defined element of $C_n(G; \ell^\infty(G, A))$.

On the other hand, for any $n \in \mathbb{N}$ we define the following map:

$$\begin{aligned} \mu_n \colon C_n(G; \ell^\infty(G, A)) &\longrightarrow C_n^{\mathrm{uf}}(G; A) \\ (e, t_1, \ldots, t_n) \otimes \varphi &\longmapsto \sum_{g \in G} \varphi(g^{-1}) \cdot g(e, t_1, \ldots, t_n). \end{aligned}$$

Since $\varphi \in \ell^\infty(G, A)$ the coefficients of $\mu_n((e, t_1, \ldots, t_n) \otimes \varphi)$ are uniformly bounded. Now for any tuple $\overline{g} \in G^{n+1}$ consider $r_{\overline{g}} := \max_{0 \leq i,j \leq n} d(g_i, g_j)$. By the left invariance of the word metric, for any $\bar{t} \otimes \varphi \in C_n(G; \ell^\infty(G, A))$ we have $\mu_n(\bar{t} \otimes \varphi) \in \ell^\infty\big(S_n^{r_{\bar{t}}}(G), A\big)$, in particular $\mu_n(\bar{t} \otimes \varphi)$ is a well-defined element of $C_n^{\mathrm{uf}}(G; A)$.

It is easy to see that μ_n is inverse of ρ_n for any $n \in \mathbb{N}$. So it remains to prove that $\rho_* \colon C_*^{\mathrm{uf}}(G; A) \longrightarrow C_*(G; \ell^\infty(G, A))$ is a chain map, i.e. that for any $n \in \mathbb{N}$ the following holds

$$\partial_n \circ \rho_n = \rho_{n-1} \circ \partial_n. \tag{A.1}$$

Let $c \in C_n^{\mathrm{uf}}(G; A)$. Suppose that $c \in \ell^\infty(S_n^r(G), A)$ for some $r \in \mathbb{R}_{>0}$ and for simplicity assume c is of the form

$$c = \sum_{g \in G} c(g\bar{t}) \cdot g\bar{t}$$

for some $\bar{t} \in S_n^r(G)$. Then we have

$$\begin{aligned} \partial_n \circ \rho_n(c) = \partial_n \left(\bar{t} \otimes \varphi_{\bar{t}}\right) &= \sum_{j=1}^{n} (-1)^j (e, t_1, \ldots, \widehat{t_j}, \ldots, t_n) \otimes \varphi_{\bar{t}} \\ &\quad + (t_1, \ldots, t_n) \otimes \varphi_{\bar{t}}. \end{aligned}$$

On the other hand

$$\begin{aligned} \rho_{n-1} \circ \partial_n(c) =& \rho_{n-1} \left(\sum_{g \in G} \sum_{j=1}^{n} (-1)^j c(g\bar{t}) \cdot g(e, t_1, \ldots, \widehat{t_j}, \ldots, t_n) \right) \\ &+ \rho_{n-1} \left(\sum_{g \in G} c(g\bar{t}) \cdot g(t_1, \ldots, t_n) \right). \end{aligned}$$

The equality

$$\rho_{n-1}\left(\sum_{g\in G}\sum_{j=1}^{n}(-1)^j c(g\bar{t})\cdot g(e,t_1,\dots,\widehat{t_j},\dots,t_n)\right)$$
$$=\sum_{j=1}^{n}(-1)^j(e,t_1,\dots,\widehat{t_j},\dots,t_n)\otimes\varphi_{\bar{t}}$$

is easily verified. So to prove that ρ_* is a chain map it remains to prove that for any $n\in\mathbb{N}$ the following equality holds:

$$\rho_{n-1}\left(\sum_{g\in G}c(g\bar{t})\cdot g(t_1,\dots,t_n)\right)=(t_1,\dots,t_n)\otimes\varphi_{\bar{t}}. \tag{A.2}$$

Notice that the left hand side of (A.2) can be written as:

$$\rho_{n-1}\left(\sum_{g\in G}c(g\bar{t})\cdot g(t_1,\dots,t_n)\right)=\rho_{n-1}\left(\sum_{g\in G}c(g\bar{t})\cdot gt_1(e,t_1^{-1}t_2\dots,t_1^{-1}t_n)\right)$$
$$=(e,t_1^{-1}t_2\dots,t_1^{-1}t_n)\otimes\psi_{(e,t_1^{-1}t_2\dots,t_1^{-1}t_n)}$$

where $\psi_{(e,t_1^{-1}t_2,\dots,t_1^{-1}t_n)}\in\ell^\infty(G,A)$ assigns to each $g\in G$ the coefficient associated to the simplex $g^{-1}(e,t_1^{-1}t_2,\dots,t_1^{-1}t_n)$. More precisely:

$$\begin{aligned}\psi_{(e,t_1^{-1}t_2,\dots,t_1^{-1}t_n)}\colon G&\longrightarrow R\\ g&\longmapsto c((t_1g)^{-1}(e,t_1,\dots,t_n)).\end{aligned}$$

On the other hand, the right hand side of (A.2) can be written as:

$$(t_1,\dots,t_n)\otimes\varphi_{\bar{t}}=t_1(e,t_1^{-1}t_2,\dots,t_1^{-1}t_n)\otimes\varphi_{\bar{t}}=(e,t_1^{-1}t_2,\dots,t_1^{-1}t_n)\otimes t_1^{-1}\varphi_{\bar{t}}.$$

and for all $g\in G$ we have

$$t_1^{-1}\varphi_{\bar{t}}(g)=\varphi_{\bar{t}}(t_1g)=c((t_1g)^{-1}(e,t_1,\dots,t_n))=\psi_{(e,t_1^{-1}t_2,\dots,t_1^{-1}t_n)}(g).$$

Thus (A.2) follows and ρ_* is a chain map. □

Remark A.2.1. *Notice that $C_0^{\mathrm{uf}}(G;A)=\ell^\infty(G,A)=C_0(G;\ell^\infty(G,A))$ so the map ρ_0 is just the canonical involution*

$$\begin{aligned}\ell^\infty(G,A)&\longrightarrow\ell^\infty(G,A)\\ \varphi&\longmapsto(g\longmapsto\varphi(g^{-1})).\end{aligned}$$

Appendix B

Uniformly finite homology and simplicial uniformly finite homology

Following Mosher [33, Section 2.1], we give a detailed proof of Proposition 1.3.3.

We consider an ordered simplicial complex X as a metric space by endowing it with the ℓ^1-path metric rescaled so that every simplex has length 1. Following Definition 1.1.4, we can define its Block-Weinberger uniformly finite homology. On the other hand, for any simplicial complex there is a notion of simplicial uniformly finite homology where the simplices in every chain are the ones given by the simplicial structure on X (Definition 1.2.5). We want to establish the isomorphism between the two uniformly finite homology groups in the case of uniformly contractible simplicial complexes.

Let $A \in \{\mathbb{R}, \mathbb{Z}\}$ and let $(X, \leq)$ be an ordered simplicial complex of bounded geometry. Recall that, for any $n \in \mathbb{N}$, the n-simplices X are of the form $[x_0, \ldots, x_n]$, where $x_0 \leq \cdots \leq x_n$ (Definition 1.2.3). We want to allow degenerate simplices in the simplicial uniformly finite chain complex; we consider a chain complex in which chains do not necessarily vanish on degenerate simplices:

Definition B.0.2. Let X be an ordered simplicial complex of bounded geometry. For each $n \in \mathbb{N}$ let $C_n^{\text{suf,deg}}(X; A) = \ell^\infty(\Delta_n(X), A)$ be the A-module of bounded functions $c \colon \Delta_n(X) \longrightarrow A$. We will write such functions as $c = \sum_{\sigma \in \Delta_n(X)} c(\sigma) \cdot \sigma$. For any $n \in \mathbb{N}$, denote by ∂_n the boundary operator as given in Definition 1.2.5-(ii). In this way we have a chain complex $(C_*^{\text{suf,deg}}(X; A), \partial_*)$. We denote by $H_*^{\text{suf,deg}}(X; A)$ the corresponding homology.

For any $n \in \mathbb{N}$ the vector space $C_n^{\mathrm{suf}}(X;A)$ is given by the quotient of $C_n^{\mathrm{suf,deg}}(X;A)$ by the degenerate simplices. Using standard arguments in homology, we can show that the map $p_*\colon C_*^{\mathrm{suf,deg}}(X;A) \longrightarrow C_*^{\mathrm{suf}}(X;A)$ which sends degenerate simplices to zero is a chain homotopy equivalence.

Lemma B.0.3. *The canonical projection*

$$p_*\colon C_*^{\mathrm{suf,deg}}(X;A) \longrightarrow C_*^{\mathrm{suf}}(X;A)$$

is chain homotopic inverse to the inclusion

$$i_*\colon C_*^{\mathrm{suf}}(X;A) \longrightarrow C_*^{\mathrm{suf,deg}}(X;A).$$

In particular $H_*^{\mathrm{suf}}(X;A) \cong H_*^{\mathrm{suf,deg}}(X;A)$.

Proof. It is clear that p_* can be extended to infinite sums of simplices with uniformly bounded coefficients. The same holds for i_*. Clearly, we have $p_* \circ i_* = \mathrm{id}_{C_*^{\mathrm{suf}}(X;A)}$. On the other hand we construct a chain homotopy between $i_* \circ p_*$ and the identity map on $C_*^{\mathrm{suf,deg}}(X;A)$, i.e. a map $H_*\colon C_*^{\mathrm{suf,deg}}(X;A) \longrightarrow C_{*+1}^{\mathrm{suf,deg}}(X;A)$ such that for any $n \in \mathbb{N}$ the following holds:

$$H_{n-1} \circ \partial_n + \partial_{n+1} \circ H_n = i_n \circ p_n - \mathrm{id}\,. \tag{B.1}$$

Let $n \in \mathbb{N}$. For any $k \in \{0, \dots, n\}$ and for any simplex $[x_0, \dots, x_n] \in \Delta_n(X)$ define

$$h_k([x_0, \dots, x_n]) := \begin{cases} [x_0, \dots, x_k, x_k, \dots, x_n] & \text{if } x_k < x_{k+1} < \cdots < x_n \text{ in } V_X \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, for any simplex $[x_0, \dots, x_n] \in \Delta_n(X)$ we can define:

$$H_n([x_0, \dots, x_n]) = \sum_{k=0}^{n} (-1)^k h_k([x_0, \dots, x_n]).$$

Extending H_n to any chain in $C_n^{\mathrm{suf,deg}}(X;A)$, it is easy to see that for any $n \in \mathbb{N}$ we get a well-defined map $H_n\colon C_n^{\mathrm{suf,deg}}(X;A) \longrightarrow C_{n+1}^{\mathrm{suf,deg}}(X;A)$ satisfying (B.1). Thus $i_* \circ p_*$ is chain homotopy equivalent to the identity on $C_*^{\mathrm{suf,deg}}(X;A)$ and the claim follows. $\square$

It is easy to see that the vertex set V_X of any simplicial complex of bounded geometry X is a quasi-lattice in X (Definition 1.4.2). In particular, V_X is quasi-isometric to X. So by the quasi-isometry invariance of uniformly finite homology established in Proposition 1.1.5, we can consider the uniformly finite chain complex of V_X to compute the uniformly finite homology of X. With an abuse of notation we denote the uniformly finite chain complex of V_X by $C_*^{\mathrm{uf}}(X, A)$ and its corresponding homology by $H_*^{\mathrm{uf}}(X; A)$.

Proof of Proposition 1.3.3. We prove the claim by constructing chain maps $i_*\colon C_*^{\mathrm{suf,deg}}(X; A) \longrightarrow C_*^{\mathrm{uf}}(X; A)$, $j_*\colon C_*^{\mathrm{uf}}(X; A) \longrightarrow C_*^{\mathrm{suf,deg}}(X; A)$ and by showing that they are chain homotopy inverse to each other.

Every simplex in X can be viewed as a (ordered) tuple of vertices in X, so we can define $i_*\colon C_*^{\mathrm{suf,deg}}(X; A) \longrightarrow C_*^{\mathrm{uf}}(X; A)$ to be just the inclusion map. Since the boundary operators in the uniformly finite and in the simplicial uniformly finite chain complexes are the same, this map is a well-defined chain map.

For $k = 0$, the map $j_0\colon C_0^{\mathrm{uf}}(X; A) \longrightarrow C_0^{\mathrm{suf,deg}}(X; A)$ is just the map induced by the identity. Recall that, by an abuse of notation, we are taking $H_*^{\mathrm{uf}}(X; A)$ to be the uniformly finite homology of the vertex set of X which is a quasi-lattice in X. For $k = 1$, we define $j_1\colon C_1^{\mathrm{uf}}(X; A) \longrightarrow C_1^{\mathrm{suf,deg}}(X; A)$ as follows (Figure B.1):

Let $(x, y) \in V_X^2$ with $d(x, y) = l$ for some $l \in \mathbb{N}$. Since X is uniformly contractible and hence connected, there are at most l simplices in $\Delta_1(X)$ "connecting" x with y. In other words, we can choose $x_0 = x, \dots, x_m = y$ in V_X for some $m \leq l$ such that for any $i \in \{0, \dots, m-1\}$ either $x_i \leq x_{i+1}$ or $x_{i+1} \leq x_i$. For any $i \in \{0, \dots, m-1\}$ we consider

$$\sigma_i = \begin{cases} [x_i, x_{i+1}] & \text{if } x_i \leq x_{i+1} \\ [x_{i+1}, x_i] & \text{if } x_i \geq x_{i+1} \end{cases}$$

and

$$\epsilon_{\sigma_i}^{(x,y)} = \begin{cases} 0 & \text{if } \sigma_i = [x_i, x_{i+1}] \\ 1 & \text{if } \sigma_i = [x_{i+1}, x_i]. \end{cases}$$

Then we define

$$j_1(x, y) = \sum_{i=0}^{m-1} (-1)^{\epsilon_{\sigma_i}^{(x,y)}} \sigma_i.$$

For any $(x, y) \in V_X^2$ and any $\sigma \in \Delta_1(X)$, we write $\sigma \subset j_1(x, y)$ if σ appears in the sum $j_1(x, y)$ defined above. If $\sigma \subset j_1(x, y)$, we define $\epsilon_\sigma^{(x,y)}$ as above (according to the orientation of σ with respect to (x, y)). Now we can extend the map j_1 to any $c = \sum_{(x,y)\in V_X^2} c(x, y) \cdot (x, y) \in C_1^{\mathrm{uf}}(X; A)$ as follows:

$$j_1(c) = \sum_{\sigma \in \Delta_1(X)} \left(\sum_{(x,y)\in V_X^2, \sigma \subset j_1(x,y)} (-1)^{\epsilon_\sigma^{(x,y)}} c(x, y) \right) \cdot \sigma.$$

By condition (i)-(c) of Definition 1.1.4, there exists $R_c \in \mathbb{R}_{>0}$ such that for any (x, y) in the support of c, we have $d(x, y) < R_c$. Since X has bounded geometry and it is uniformly contractible, for any $\sigma \in \Delta_1(X)$ there is a uniformly bounded number of elements $(x, y) \in \mathrm{supp}(c)$ such that $\sigma \subset j_1(x, y)$. It follows that $j_1(c)$ has uniformly bounded coefficients. Thus, $j_1(c) \in C_1^{\mathrm{suf}}(X; A)$.

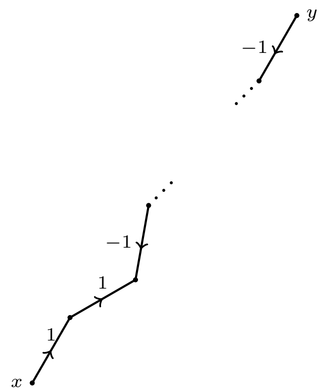


Figure B.1: For any $(x, y) \in V_X^2$, construct $j_1(x, y)$ as a sum of ordered 1-simplices.

We proceed by induction. So suppose j_k has been constructed for all $k \leq n - 1$. We define $j_n \colon C_n^{\mathrm{uf}}(X; A) \longrightarrow C_n^{\mathrm{suf,deg}}(X; A)$ as follows:

Let $\overline{x} = (x_0, \ldots, x_n) \in V_X^{n+1}$ and suppose that $\max_{0 \leq i,j \leq n} d(x_i, x_j) = l$ for some $l \in \mathbb{N}$. Consider now

$$a = \sum_{i=0}^{n} j_{n-1}(x_0, \ldots, \widehat{x}_i, \ldots, x_n).$$

By the induction hypothesis this is an element in $C_{n-1}^{\mathrm{suf}}(X; A)$ and its support is a subset of X having diameter $\leq \binom{n+1}{2} l$. Since X is uniformly contractible, there exists $S_{\binom{n+1}{2} l} > 0$ such that $\mathrm{supp}(a)$ can be contracted to a point inside $N_{S_{\binom{n+1}{2} l}}(\mathrm{supp}(a))$. In particular there exists a singular n-simplex $f \colon \Delta^n \longrightarrow |X|$ such that $\partial f = a$. By the simplicial approximation theorem [31, Theorem 6, Chapter 1.5], after subdividing Δ^n, one can find a simplicial map $g \colon \Delta^n \longrightarrow X$ which approximates f and such that

$$\forall_{i \in \{0, \ldots, n\}} \quad g([t_0, \ldots, \widehat{t}_i, \ldots, t_n]) = j_{n-1}(x_0, \ldots, \widehat{x}_i, \ldots, x_n).$$

Then $g(\Delta^n)$ is a sum of (possibly degenerate) n-simplices in X (Figure B.2). So we define

$$j_n(\overline{x}) = g(\Delta^n).$$

Any $\sigma \subset j_n(\overline{x})$ will appear in the sum with a sign $(-1)^{\epsilon_\sigma^{\overline{x}}}$ compatible with the sign of its faces on the faces of $\overline{x} = (x_0, \ldots, x_n)$. Since $\mathrm{supp}(a)$ is contractible to a point in $N_{S_{\binom{n+1}{2}l}}(\mathrm{supp}(a))$, we have that $g(\Delta^n)$ is a sum of uniformly finitely many simplices. Notice that if $n > \dim(X)$, the map j_n will assign to an $n+1$-tuple of points in V_X a (uniformly bounded) sum of degenerate simplices in X. Now we extend j_n to any chain $c = \sum_{\overline{x} \in V_X^{n+1}} c(\overline{x}) \cdot \overline{x} \in C_n^{\mathrm{uf}}(X;A)$ as follows:

$$j_n(c) = \sum_{\sigma \in \Delta_n(X)} \left(\sum_{\overline{x} \in V_X^{n+1}, \sigma \subset j_n(\overline{x})} (-1)^{\epsilon_\sigma^{\overline{x}}} c(\overline{x}) \right) \cdot \sigma.$$

As before, by condition (i)-(c) of Definition 1.1.4, there exists $R_c \in \mathbb{R}_{>0}$ such that $\max_{0 \le i,j \le n} d(x_i, x_j) \le R_c$ for any $\overline{x} = (x_0, \ldots, x_n) \in \mathrm{supp}(c)$ appearing in the sum. So since X is uniformly contractible and it has bounded geometry, the coefficients of $j_n(c)$ are uniformly bounded. Since the boundary operators are the same for $C_*^{\mathrm{uf}}(X;A)$ and for $C_*^{\mathrm{suf,deg}}(X;A)$ it follows that j_* is a well-defined chain map.

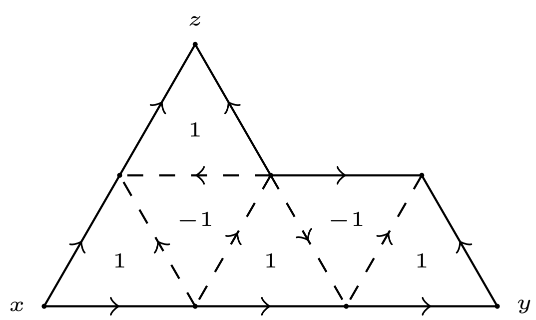


Figure B.2: For any $(x,y,z) \in V_X^3$, construct $j_2(x,y,z)$ as a sum of ordered 2-simplices.

Now it remains to show that the maps $j_* \circ i_*$ and $i_* \circ j_*$ are chain homopotopic to the identity on $C_*^{\mathrm{suf,deg}}(X;A)$ and on $C_*^{\mathrm{uf}}(X;A)$ respectively.

Clearly, $j_0 \circ i_0 = \mathrm{id}$. Moreover the map j_1 which is given by "connecting" points is the identity on every 1-simplex of X. By induction, we can suppose $j_k \circ i_k = \mathrm{id}$ for all $k \le n-1$. For $k = n$, for any $\sigma \in \Delta_n(X)$, we can take a simplicial map $g_\sigma \colon \Delta^n \longrightarrow X$ such that $g_\sigma(\Delta^n) = \sigma$. It follows that $j_* \circ i_* = \mathrm{id}$.

On the other hand, using classical arguments in algebraic topology, we construct a chain homotopy between $i_* \circ j_*$ and the identity, i.e. a map $h_*\colon C_*^{\mathrm{uf}}(X;A) \longrightarrow C_{*+1}^{\mathrm{uf}}(X;A)$ such that for every $n \in \mathbb{N}$

$$h_{n-1} \circ \partial_n + \partial_{n+1} \circ h_n = i_n \circ j_n - \mathrm{id}\,. \tag{B.2}$$

Let $n \in \mathbb{N}$ and $\overline{x} = (x_0, \ldots, x_n) \in V_X^{n+1}$. For any $k \in \{0, \ldots, n\}$ define $\overline{x}_k = (x_k, \ldots, x_n)$. Suppose that for any $k \in \{0, \ldots, n\}$ we have

$$i_{n-k} \circ j_{n-k}(\overline{x}_k) = (-1)^{\epsilon_{\sigma_1}^{\overline{x}_k}}(y_0^1, \ldots, y_{n-k}^1) + \cdots + (-1)^{\epsilon_{\sigma_{i_k}}^{\overline{x}_k}}(y_0^{i_k}, \ldots, y_{n-k}^{i_k})$$

for some $\sigma_1 = [y_0^1, \ldots, y_{n-k}^1], \ldots, \sigma_{i_k} = [y_0^{i_k}, \ldots, y_{n-k}^{i_k}] \in \Delta_{n-k}(X)$. Then define

$$h_n(\overline{x}) = \sum_{k=0}^{n} \sum_{i=1}^{i_k} (-1)^{k+\epsilon_{\sigma_i}^{\overline{x}_k}}(x_0, \ldots, x_k, y_0^i, \ldots, y_{n-k}^i).$$

Extending this to any chain in $C_n^{\mathrm{uf}}(X;A)$, for any $n \in \mathbb{N}$ we have a well-defined map $h_n\colon C_n^{\mathrm{uf}}(X;A) \longrightarrow C_{n+1}^{\mathrm{uf}}(X;A)$ satisfying (B.2). So $i_* \circ j_*$ is chain homotopic to $\mathrm{id}_{C_*^{\mathrm{uf}}(X;A)}$. It follows that $C_*^{\mathrm{uf}}(X;A)$ and $C_*^{\mathrm{suf,deg}}(X;A)$ are chain homotopy equivalent. Then, by Lemma B.0.3, the claim follows. $\square$

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