A refinement of Hironaka's additive group schemes for an extended invariant

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Introduction

Hironaka introduced certain additive group schemes in [H5] to obtain information about the locus of near points under a permissible blow up in resolution of singularities in positive characteristic. The aim of this thesis is to introduce new additive group schemes, adapted to the locus of very near points, and to show that they have properties comparable to Hironaka's group schemes.

Let X be a scheme, say reduced and excellent. In resolution of singularities one considers the question if it is possible to find a proper and birational morphism $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is regular. In his famous paper [H1] Hironaka proved the existence of resolution of singularities for algebraic varieties of arbitrary dimension over a field of characteristic zero. He was honored with the Fields medal for this work in 1970. Originally the proof was not constructive and very technical. Building on Hironaka's ideas several results were accomplished during the last decades, leading to constructive and accessible proofs. This movement began with Villamayor [Vi] and Bierstone and Milman [BM1], [BM2] and was continued by Encinas and Hauser [EH], Hauser [Ha], Cutkosky [Cu1], and Włodarczyk [Wl] to name but a few.

In positive characteristic the first proof of resolution of singularities of surfaces goes back to Abhyankar [Ab1]. He showed resolution of singularities of threefolds in positive characteristic over an algebraically closed field of characteristic $p \neq 2, 3, 5$ in 1966. The proof in [Ab2], [Ab3], [Ab4], [Ab5] and [Ab6] is extremely long and difficult. Abhyankar's results have been simplified by Cutkosky [Cu2], [Cu3]. Lipman proved resolution of two-dimensional excellent schemes ([Li]). He used not only blow ups but also normalizations, so this does not give embedded resolution. Cossart, Jannsen and Saito proved canonical resolution of singularities for excellent schemes of dimension two based on an idea of Hironaka only with blow ups ([CJS] 2009), and hence embedded resolution. Cossart and Piltant showed the existence of a birational and global resolution in dimension three under the condition that the base field is differentially finite over a perfect field ([CP1] 2008, [CP2] 2009). They announced a similar result for the arithmetic case, see [CP3]. Hitherto no approach succeeded in higher dimensions. A weaker kind of resolution in positive characteristic was obtained by de Jong using alterations ([dJ]).

Invariants

Let X be a locally noetherian scheme. A standard approach to resolve its singularities is a blow up $\pi : X' \to X$ of X in a center $D \subseteq X$. Let $C_{X,x}$ be the tangent cone of X at x, which contains the tangent space $T_{D,x}$ of D at x. Then the blow up leaves $X \setminus D$ unchanged and a point x on D is replaced with the projective space $\mathbb{P}(C_{X,x}/T_{D,x})$ associated to $C_{X,x}/T_{D,x}$, at least if D is regular at x and X is normally flat along D at x. D is then called permissible at x. One uses invariants to measure singularities and to see if the situation at a point $x' \in \pi^{-1}(x)$ has improved.

As a first invariant we use the Hilbert series $H_{X,x} \in \mathbb{N}[[T]]$ of the tangent cone $C_{X,x}$. With the notations from above one wants the estimate

$$H^{(d)}_{X',x'} \leq H_{X,x} \tag{I}$$

in order to assure that the singularity at x' does not become worse. Here $H^{(d)}_{X',x'}$ is the Hilbert series of $C_{X',x'} \times \mathbb{A}^d$, where $C_{X',x'}$ is the tangent cone of X' at x' and $d = \operatorname{tr.deg}(\kappa(x')/\kappa(x))$. Property (I) was proved by Bennett ([Be]) and by Hironaka ([H4, Th. I]) in the slightly weaker form $H^{(d+1)}_{X',x'} \leq H^{(1)}_{X,x}$. Singh could show (I) in its full strength ([Si1]). In [H1] Hironaka uses another invariant, the ν-invariant $\nu^*_x(X) \in \mathbb{N}^{\mathbb{N}}$ that behaves differently from the Hilbert series. He proves $\nu^*_{x'}(X') \leq \nu^*_x(X)$ and $H^{(d)}_{X',x'} = H_{X,x}$ if and only if $\nu^*_{x'}(X') = \nu^*_x(X)$ ([H4, Th. II, III]).

(I) can be an equality. In this case x' is called near to x. The Hilbert series has to be extended to a subtler invariant to see also smaller improvements in the singularity under blow ups. As a second invariant we use the dimension of the ridge of $C_{X,x}$. The ridge $\operatorname{Rid}(C)$ of a cone C is the largest homogeneous additive group that leaves the cone invariant under translation inside some surrounding vector space. In characteristic zero, ore more generally over perfect fields, it always coincides with the directrix, at least up to reducedness. The directrix of the cone C is the largest vector space $\operatorname{Dir}(C)$ that translates the cone C onto itself. The inclusion $\operatorname{Dir}(C) \subseteq \operatorname{Rid}(C)$ can be strict in positive characteristic. The invariant $\dim \operatorname{Rid}(C_{X,x})$ was employed by Hironaka in [H1] and he proved ([H2, Th. (1,A)]) that for near points

$$\dim \operatorname{Rid}(C_{X',x'}) + d \leq \dim \operatorname{Rid}(C_{X,x}). \tag{II}$$

The point x' is called very near to x if (II) is an equality as well. Resolution is achieved if one can show that there is no infinite sequence of singular very near points under continued blow ups. For this purpose Hironaka associated a polyhedron to the local ring $\mathcal{O}_{X,x}$ ([H3]). In this work we only deal with the first two invariants and our objective is to gain as much information about the singularities from them as possible.

Hironaka schemes

Hironaka made an attempt to gain more information about the locus of near points inside $\pi^{-1}(x)$ in positive characteristic by introducing certain group schemes in [H5]. They are called Hironaka schemes now. To a point y of an affine space $V = \operatorname{Spec}(S), S = k[X_0, ..., X_n]$ one associates a subgroup B_y of V: The ring of invariants $\mathcal{U}_y$ of B_y in S is generated by those homogeneous polynomials $f \in S$ with $H^{(d)}_{Y,y} = H_{Y,0}$, where $Y = \operatorname{Spec}(S/\langle f \rangle)$ and $d = \operatorname{tr.deg}(\kappa(y)/k)$. With the notations from above let V be some vector space containing $C_{X,x}/T_{D,x}$. Then $x' \in \mathbb{P}V$ and we write $B_{x'}$ for the Hironaka scheme B_y, where $y \in V$ and $x' \in \mathbb{P}V$ are defined by the same prime ideal in S. Hironaka proved that $B_{x'}$ is contained in the ridge of $C_{X,x}/T_{D,x}$ if (I) is an equality ([H4, Th. IV]).

Therefore $x' \in \mathbb{P}(\operatorname{Rid}(C_{X,x})/T_{D,x})$ if x' is near to x. This can be proved without the use of Hironaka schemes, but Hironaka schemes are very special and rare group schemes and one can say more: Hironaka proved that all Hironaka schemes of dimension $\leq p$ are vector spaces in characteristic p and that there is precisely one type of non-vector space Hironaka scheme of dimension 3, namely

$$\mathrm{Spec}(k[X_0, ..., X_3]/\langle X_0^2 + a_2X_1^2 + a_1X_2^2 + a_1a_2X_3^2\rangle)$$

for $\mathrm{char}(k) = 2$ and $[k^2(a_1, a_2) : k^2] = 4$ ([H5], see Type 3 in 10.8 of this thesis). Oda was able to characterize Hironaka schemes via their Dieudonné modules using differential operators and classified Hironaka schemes up to dimension 5 ([Od], see also 10.8). Mizutani sharpened the original bound of Hironaka. He showed that in characteristic p all Hironaka schemes of dimension $\leq 2p - 2$ are vector spaces and that there is precisely one type of non-vector space Hironaka schemes of dimension $2p - 1$ ([Mi], see 10.9).

The benefit of these observations lies in the following: If $B_{x'}$ is a vector space, then a near point x' must lie in the subspace $\mathbb{P}(\mathrm{Dir}(C_{X,x})/T_{D,x}) \subseteq \mathbb{P}(\mathrm{Rid}(C_{X,x})/T_{D,x})$. Therefore $\dim X \leq 2\,\mathrm{char}(\kappa(x)) - 2$ or $\mathrm{char}(\kappa(x)) = 0$ imply that all points near to x must lie in $\mathbb{P}(\mathrm{Dir}(C_{X,x})/T_{D,x})$ (cf. [CJS, 2.14]). Thus the locus of near points is narrowed down. Near points can lie outside of $\mathbb{P}(\mathrm{Dir}(C_{X,x})/T_{D,x})$ if $\dim X \geq 2\,\mathrm{char}(\kappa(x)) - 1$. In [CJS] this is considered as one of the main obstructions to a generalization of their proof to higher dimensions. The more severe obstruction is the missing of a tertiary invariant in dimension ≥ 3.

Mizutani conjectured that Hironaka schemes of exponent e must have dimension at least $2p^e - 1$, where p is the characteristic of the ground field ([Mi]). The exponent e measures how far away a Hironaka scheme is from being a vector space (see 10.6). The author was able to show that a Hironaka scheme of exponent e must have dimension at least $e(p-1)+p$ ([Di], Th. E). Remark: In [Ru, 5.2] Russell claims the existence of Hironaka schemes of dimension $4p - 2$ with any exponent $e \geq 2$. This cannot be true in view of the dimensional bound $e(p-1)+p$ which depends on e.

Refined Hironaka schemes

Hironaka schemes are constructed with respect to the Hilbert series. In this work we introduce refined Hironaka schemes with respect to the extended invariant consisting of the Hilbert series and the dimension of the ridge. To a point y of an affine space $V = \mathrm{Spec}(S), S = k[X_0, ..., X_n]$ we associate a subgroup $F_y \subseteq V$: Its ring of invariants $\mathcal{V}_y$ is generated by those homogeneous additive polynomials $f \in \mathcal{U}_y$ for which also the initial form of f at y is additive. We will show that at least in low dimensions $\mathcal{V}_y$ is generated by those homogeneous polynomials $f \in S$ with the following property: For the hypersurface $Y := \mathrm{Spec}(S/\langle f\rangle)$ one has the equalities $H^{(d)}_{Y,y} = H_{Y,0}$ and $\dim \mathrm{Rid}(C_{Y,y}) + d = \dim \mathrm{Rid}(C_{Y,0})$, where $d = \mathrm{tr.deg}(\kappa(y)/k)$. The natural inclusion $B_y \subseteq F_y$ is an equality if B_y already is a vector space. This always holds in characteristic zero, so we will not discuss this case. The inclusion $B_y \subseteq F_y$ is strict if B_y is not a vector space in all examples known to the author.

Let still be X a locally noetherian scheme and $\pi : X' \to X$ a blow up with center $D \subseteq X$, permissible at $x \in D$ and $x' \in \pi^{-1}(x)$. If $C_{X,x}/T_{D,x} \subseteq V$, then $x' \in \mathbb{P}V$ and we write $F_{x'}$ for the refined Hironaka scheme F_y, where x' and y are defined by the same prime ideal. We are able to transfer [H4, Th. IV], [Mi, Theorem 2.8] and [CJS, 2.14] to some extent to this new situation and get the following results:

Main Theorem A. *If (I) and (II) are equalities and*

$$\text{(III)} \qquad \dim X \leq 5 \quad \textit{or} \quad \dim X \leq 2\operatorname{char}(\kappa(x)) - 1,$$

then $C_{X,x}/T_{D,x}$ *is invariant under the action of the refined Hironaka scheme* $F_{x'}$.

Main Theorem B. *Let* F *be a refined Hironaka scheme over a field* k *of positive characteristic. If* $\dim F \leq 5$ *or* $\dim F \leq 2\operatorname{char}(k) - 1$, *then* F *is is a vector space.*

Main Theorem C. *If (I) and (II) are equalities and (III) holds, then*

$$x' \in \mathbb{P}(\operatorname{Dir}(C_{X,x})/T_{D,x}).$$

C diminishes one of the obstructions to a generalization of the proof of [CJS] to higher dimensions.

For the proof of these theorems we will introduce a certain new kind of good coordinates (dissecting variables) at points of an affine space. To prove A in the presence of such variables, we will roughly show the following: If (I) is an equality, then certain equations giving rise to the ridge of $C_{X,x}/T_{D,x}$ at the origin, give rise to the ridge of this cone at the point y under taking their initial forms. With 'giving rise' we mean that equations of the ridge can be computed from these equations via differential operators. The necessity to ensure the existence of dissecting variables as well as the proof of B will then be reduced to a few situations: In the end we succeed in proving A and B in low dimensions using the classification of Oda in a case by case analysis. C is a direct consequence of A and B.

The obstruction to generalize the main theorems to higher dimensions lies in the fact that Hironaka schemes become increasingly intransparent in higher dimensions and our proof depends on analyzing all types of them. Theoretically, the main theorems (at least A) could be proved in any dimension if one could investigate the behavior of all Hironaka schemes up to that dimension. But instead of this seemingly inaccessible approach one rather should try to give a better description of the refined Hironaka schemes, maybe in form of a criterion describing their rings of invariants with differential operators.

In this work we find exactly one example of a non-vector space refined Hironaka scheme, namely the hypersurface of dimension 7 defined in $\operatorname{Spec}(k[X_0, ..., X_7])$ by the equation

$$X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_1a_2X_4^2 + a_1a_3X_5^2 + a_2a_3X_6^2 + a_1a_2a_3X_7^2,$$

where $\operatorname{char}(k) = 2$ and $[k^2(a_1, a_2, a_3) : k^2] = 8$ (see 10.8, Type 4-4). In view of the similarity to the minimal non-vector space Hironaka scheme in dimension 3 from above, it seems likely that this is the smallest non-vector space refined Hironaka scheme.

Content

In this work we try to give a comprehensive account of the overall situation and proceed as self-contained as possible. In chapters 1 to 6 the technical framework will be settled.

The most important tool we will use are the differential operators discussed in chapter 2. We determine them in positive characteristic in 2.2. Furthermore we are able to prove a very general version of a Jacobian criterion using these operators (see (2.3.5)): For a prime $\mathfrak{p}$ of a formally smooth A-algebra B such that also $\mathrm{Quot}(B/\mathfrak{p})$ is formally smooth over A one has $b \in \mathfrak{p}^{(n)}$ if and only if $\mathrm{Diff}_A^{\leq n-1}(B)(b) \subseteq \mathfrak{p}$. This criterion seems to be new and can be used to compute the locus of higher orders.

Hironaka schemes as well as our refined Hironaka schemes are algebraic groups of a certain specific type. In chapter 3 we characterize these homogeneous additive groups ((3.3.8)). Along the way we show how the mentioned differential operators can be computed on such groups in (3.2.3). In particular we introduce a basis of these differential operators with respect to additive polynomials in (3.3.2). This basis seems to be new and plays an important role in the proof of the main theorems.

After recalling filtrations, Hilbert series and bifiltrations in chapters 4 and 5, we deal with the ridge and the directrix of a cone in chapter 6. We generalize Giraud bases to σ-Giraud bases and show that also the latter ones can be used to compute the ridge. After recalling permissible blow ups in chapter 7, we will give a modified proof of (I) and (II) in chapter 8. This proof emphasizes the properties of cones and only consideres blow ups in the last step.

In chapter 9 we present Hironaka schemes and particularly investigate [H4, Th. IV]. We develop refined Hironaka schemes and give a proof of the main theorems in chapter 10.

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Conventions and notation

Unless mentioned otherwise a ring will always refer to a commutative ring with unit. The natural numbers include zero: $\mathbb{N} = \{0, 1, 2, ...\}$. If $B = \bigoplus_{n \geq 0} B_n$ is a graded object we denote $B_+ = \bigoplus_{n>0} B_n \subseteq B$. All schemes considered are locally noetherian. For a point x of a scheme X we denote with $(\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ the local ring of X at x and with $\kappa(x)$ its residue field. If $D \subseteq X$ is a closed subscheme, we write $\mathcal{I}_{X,D}$ for the sheaf of ideals defining D and $\mathcal{I}_{X,D,x}$ for the stalk of this sheaf at $x \in X$. If X is an S-scheme and $S \to T$ is a morphism of schemes, we write X_T for $T \times_S X$. If $X = \mathrm{Spec}(A)$ is an affine scheme and A a graded algebra, we write $\mathbb{P}X$ for the scheme $\mathrm{Proj}(A)$. $\mathbb{A}^n_k$ is the n-dimensional affine space over k.

1 Polynomials

After clarifying our use of multiindex notations in 1.1 for the following chapters, we present a short approach to Gröbner bases in 1.2 and 1.3. Although we are not going to make computational use of them, they are a good tool for theoretical work with ridges of cones in 6.1. Finally we shift our view to additive polynomials in 1.4. These are a technical key to homogeneous additive groups such as the mentioned ridges or the Hironaka group schemes. In 3.4 we will continue their treatment.

1.1 Multiindices

We introduce our multiindex notation, which will be applied at various points in the following chapters. Afterwards we focus on some orders that will lead to monomial orders in 1.3. Throughout this section let $\Lambda := \mathbb{N}^{(I)}$. Here I is an arbitrary index set at first. Λ is a monoid with respect to addition. The elements of Λ are called **multiindices**.

Definition (1.1.1). *For $M \in \Lambda$ we will denote with $M_i \in \mathbb{N}$ the entry of M at $i \in I$. The* ***degree*** *of $M \in \Lambda$ is the integer $|M| := \sum_{i\in I} M_i \geq 0$. For a system of elements of a ring (not nec. independent nor nec. pairwise different) $(x_i)_{i\in I}$ we will use the notation $x^M := \prod_{i\in I} x_i^{M_i}$ to denote* ***monomials****. For two multiindices M, N we define a* ***multiindex binomial coefficient***

$$\binom{N}{M} := \prod_{i\in I} \binom{N_i}{M_i}.$$

(This makes sense since almost all binomial coefficients involved in the product are 1). We define $\binom{n}{m}$ to be zero if $m < 0$ or $m > n$. For two multiindices q, M we define their ***product***

$$qM = \sum_{i\in I} q_i \cdot M_i \in \mathbb{N}.$$

For $p \in \mathbb{N}$ and a multiindex M we define the multiindex pM by $(pM)_i = p \cdot M_i$.

(1.1.2) We are going to introduce several orders $\leq$ on Λ. All of them have the following property: For $L, M, N \in \Lambda$ one has

$$\text{(1.1.2.A)} \qquad L \leq M \quad \Rightarrow \quad L \leq L + N \leq M + N.$$

This implies in general for $N \neq 0$ also that

$$L < M \quad \Rightarrow \quad L < L + N < M + N.$$

(1.1.2.1) The **componentwise (partial) order** $\leq_c$ where $M \leq_c N$ if and only if $M_i \leq N_i$ for all $i \in I$.

For the following orders we always assume that $I = \{1, ..., n\}$.

(1.1.2.2) The **lexicographic (total) order** $\leq_{lex}$ where $M \leq_{lex} N$ if and only if $M_i < N_i$ for the lowest integer i with $M_i \neq N_i$.

(1.1.2.3) The **homogeneous lexicographic (total) order** $\leq_{hlex}$ where $M \leq_{hlex} N$ iff $|M| < |N|$ or $|M| = |N|$ and $M \leq_{lex} N$. $\leq_{hlex}$ refines the order by degree.

(1.1.2.4) The **weighted homogeneous lexicographic (total) order** $\leq_{whlex}$ with respect to some multiindex $q \in \Lambda$ where $M \leq_{whlex} N$ iff $qM < qN$ or $qM = qN$ and $M \leq_{lex} N$. $\leq_{whlex}$ is a generalization of $\leq_{hlex}$ (take $q = (1, ..., 1)$) and at the same time also a generalization of $\leq_{lex}$ (take $q = 0$).

Definition (1.1.3). *Let $S = k[X_1, ..., X_n]$ be a polynomial ring over a field k. For a multiindex $q \in \Lambda = \mathbb{N}^n$ we can equip S with the structure of a graded k-algebra via*

$$S_d = \bigoplus_{qM=d} kX^M.$$

This includes the standard graduation for $q = (1, ..., 1)$.

1.2 Wellordered vector spaces

Although we will be dealing with the more explicit monomial orders on polynomial rings in the next section, we have to linger for a moment in a more general setting. We introduce a concept of wellordered vector spaces, which is necessary for a certain step in the proof of the main theorems. We also introduce the concepts of exponents and initial ideals, which play an important role in 1.3.

Definition (1.2.1). *A **wellorder** on a set is a total order on a set (which is then called **wellordered**) such that every non-empty subset has a least element, or equivalently every descending chain of elements becomes stationary.*

Lemma (1.2.2). *Let $(I_1, \leq_1), ..., (I_n, \leq_n)$ be wellordered sets. Then $I := I_1 \times \cdots \times I_n$ together with the lexicographic order $\leq$ with respect to the $\leq_i$ is wellordered.*

Proof. It is clear that the lexicographic order is total. A descending chain in I must stabilize at some point in the first component and after this in the second one and so on. Finally it becomes stationary in all components. □

Example (1.2.3). The lexicographic, homogeneous lexicographic and weighted homogeneous lexicographic order from (1.1.2) are wellorders.

Definition (1.2.4). *A **wellordered k-vector space** is a k-vector space V together with a fixed k-basis $(v_i)_{i \in I}$ indexed by a set I on which a wellorder $\leq$ is given. For an element $0 \neq v = \sum_{i \in I} \lambda_i v_i$ with unique $\lambda_i \in k$ we define its **exponent***

*$\exp(v)$ to be the largest element $i \in I$ with $\lambda_i \neq 0$ and its **initial term** $\mathrm{in}(v)$ to be $\lambda_{\exp(v)} v_{\exp(v)}$. We set $\mathrm{in}(0) = 0$. For a k-subspace $W \subseteq V$ we define the set*

$$\exp(W) := \{\exp(w) | 0 \neq w \in W\} \subseteq I$$

and the k-subspace

$$\mathrm{in}(W) := \langle \mathrm{in}(w) \rangle_{w \in W} = \langle v_i \rangle_{i \in \exp(W)} \subseteq V.$$

Lemma (1.2.5). *Let V be a wellordered k-vector space with basis $(v_i)_{i \in I}$ and $W \subseteq V$ a k-subspace. Then $(\overline{v_i})_{i \in I \setminus \exp(W)}$ is a k-basis of V/W.*

Proof. Assume we have $\sum_{i \in I \setminus \exp(W)} \lambda_i \overline{v_i} = 0$ in V/W with $\lambda_i \in k$, not all zero. Then $0 \neq v := \sum_{i \in I \setminus \exp(W)} \lambda_i v_i \in W$ and therefore $\exp(v) \in (I \setminus \exp(W)) \cap \exp(W)$ which is impossible. Thus the $\overline{v_i}$ are k-linearly independent and it remains to show that they generate V/W. Assume that $V' := W + \langle (v_i)_{i \in I \setminus \exp(W)} \rangle_k \subsetneq V$ and let $J := \{\exp(v) | v \in V \setminus V'\} \neq \emptyset$. Since $\leq$ is a wellorder, there is a least element in J and we pick $v \in V \setminus V'$ with this exponent. But we also find $v' \in V'$ with $\exp(v') = \exp(v)$. There is some $\lambda \in k$ such that either $v - \lambda v' = 0$ or $\exp(v - \lambda v') < \exp(v)$. The first one implies $v \in V'$ where for the second we get $\exp(v - \lambda v') \notin J$, which means $v - \lambda v' \in V'$. This gives the contradiction $v \in V'$. □

Definition (1.2.6). *A **wellordered graded k-vector space** is a graded k-vector space $V = \bigoplus_{d \geq 0} V_d$ such that V is a wellordered k-vector space with basis $(v_i)_{i \in I}$ where all v_i are homogeneous with respect to the graduation on V. We further require $\dim_k(V_d) < \infty$ for all $d \in \mathbb{N}$. The **Hilbert series** $H(V)$ of a graded k-vector space $V = \bigoplus_{d \geq 0} V_d$ is the series*

$$H(V) = \sum_{d \geq 0} \dim_k(V_d) T^d \in \mathbb{Z}[[T]].$$

Lemma (1.2.7). *Let V be a wellordered graded k-vector space with basis $(v_i)_{i \in I}$ and $W \subseteq V$ a homogeneous subspace. Then $\mathrm{in}(W) \subseteq V$ is a homogeneous subspace and*

$$H(V) = H(W) + H(V/W), \tag{1.2.7.A}$$

$$H(W) = H(\mathrm{in}(W)). \tag{1.2.7.B}$$

Proof. Since all v_i are homogeneous it is clear that $\mathrm{in}(W)$ is homogeneous. (1.2.7.A) is clear since $\dim_k$ is additive on finite dimensional k-vector spaces. If $v_i \in V_d$ we set $\deg(v_i) := d$. By (1.2.5) we have

$$H(V) = \sum_{i \in I} T^{\deg(v_i)} = \sum_{i \in \exp(W)} T^{\deg(v_i)} + \sum_{i \in I \setminus \exp(W)} T^{\deg(v_i)} =$$

$$= H(\mathrm{in}(W)) + H(V/W) \overset{(1.2.7.A)}{=} H(\mathrm{in}(W)) + H(V) - H(W)$$

which shows (1.2.7.B). □

1.3 Monomial orders

We introduce Gröbner bases and especially reduced Gröbner bases. As we will see, the last ones are Giraud bases and can be used to compute the ridge of a cone in 6.1. We are mainly interested in the existence of reduced Gröbner bases and will not deal with computational algorithms for them. In our approach we follow [Ei, ch. 15]. Throughout this section let $S = k[X_1, ..., X_n]$ be a polynomial ring over a field k and $\Lambda := \mathbb{N}^n$ as in 1.1.

Definition (1.3.1). *For $M \in \Lambda$ we define the k-linear map $\lambda_M : S \to k$ by $\lambda_M(X^N) := \delta_{M,N}$ (Kronecker delta). The* ***monomials*** *of S are the elements X^M of S for $M \in \Lambda$. The* ***terms*** *of S are the elements aX^M of S for $M \in \Lambda$ and $a \in k$. We say that a monomial X^M resp. a term $0 \neq aX^M$ is* ***involved*** *in $f \in S$ if $\lambda_M(f) \neq 0$.*

Definition (1.3.2). *A* ***monomial order*** *on S is a total order $\leq$ on Λ such that for monomials $L, M, N \in \Lambda$ we have*

(1.3.2.A) $$L \leq M \quad \Rightarrow \quad L \leq L + N \leq M + N.$$

This implies that for $L, M \in \Lambda$ we have

(1.3.2.B) $$L \leq_c M \quad \Rightarrow \quad L \leq M.$$

The lexicographic, homogeneous lexicographic and weighted homogeneous lexicographic orders on Λ are monomial (see (1.1.2)). We identify Λ with the set of monomials of S and write $X^L \leq X^M$ if $L \leq M$. For terms we write $a_L X^L \leq a_M X^M$ if $L \leq M$ or $a_L = 0$. We adopt the notions of (1.2.4): The exponent $\exp(f)$ of $0 \neq f \in S$ is the highest exponent M with respect to $\leq$ such that X^M is involved in f. The initial term of $f \in S$ with respect to $\leq$ is $\mathrm{in}(f) := \lambda_{\exp(f)}(f) X^{\exp(f)}$. In particular $\mathrm{in}(0) := 0$. For an ideal $I \subseteq S$ we define

$$\exp(I) := \{\exp(f) | 0 \neq f \in I\}, \qquad \mathrm{in}(I) := \langle \mathrm{in}(f) | f \in I \rangle_S.$$

Remark (1.3.3). *For $f, g \in S$ and a monomial order $\leq$ on S we have:*

(i) $\mathrm{in}(f + g) \leq \max_{\leq}\{\mathrm{in}(f), \mathrm{in}(g)\}$ *and this is an equality if and only if* $\mathrm{in}(f) \neq -\mathrm{in}(g)$ *or* $f = 0$ *or* $g = 0$.

(ii) $\mathrm{in}(f \cdot g) = \mathrm{in}(f) \cdot \mathrm{in}(g)$.

Lemma (1.3.4) (cf. [Ei, Lemma 15.2])**.** *A monomial order $\leq$ on S is a wellorder.*

Proof. Let $\Gamma \subseteq \Lambda$ be a subset. The ideal $I := \langle X^L | L \in \Gamma \rangle_S$ is finitely generated since S is noetherian. Therefore $I = \langle X^L | L \in \Gamma' \rangle_S$ for a finite subset $\Gamma' \subseteq \Gamma$. Let $M \in \Gamma'$ be the least element with respect to $\leq$. For every multiindex $L \in \Gamma$ there exists $M' \in \Gamma'$ with $X^L \in S \cdot X^{M'}$ and therefore $L \geq_c M' \geq M$. By (1.3.2.B) we get $L \geq M$ and M in fact is the least element of Γ. □

Definition (1.3.5). *A **Gröbner basis** of an ideal $I \subseteq S$ with respect to a monomial order $\leq$ on S is a system of elements $g_1, ..., g_t \in I$ such that*

$$\mathrm{in}(I) = \langle \mathrm{in}(g_1), ..., \mathrm{in}(g_t) \rangle.$$

*The basis is called **minimal** if $\exp(g_i) \leq_c \exp(g_j)$ implies $i = j$. A minimal Gröbner basis is obtained from a Gröbner basis by simply omitting some elements. The basis is called **reduced** if $\mathrm{in}(g_i)$ does not divide any term of g_j for $i \neq j$. Clearly a reduced basis is minimal. The basis is called **monic** if $\mathrm{in}(g_1), ..., \mathrm{in}(g_t)$ have coefficient 1.*

Lemma (1.3.6) (cf. [Ei, Lemma 15.5])**.** *If $I \subseteq J \subseteq S$ are ideals and $\leq$ is a monomial order on S with $\mathrm{in}(I) = \mathrm{in}(J)$, then $I = J$. In particular: If $g_1, ..., g_t$ is a Gröbner basis of I, then $I = \langle g_1, ..., g_t \rangle$.*

Proof. If $I \subsetneq J$, there would be an element $f \in J \setminus I$ with minimal $\exp(f)$ by (1.3.4). But there exists $g \in I$ with $\mathrm{in}(g) = \mathrm{in}(f)$ and therefore $\mathrm{in}(f - g) < \mathrm{in}(f)$. Now $f - g \in J$ together with the choice of f shows $f - g \in I$ and we get $f \in I$. If $g_1, ..., g_t$ is a Gröbner basis of I, then let $I' := \langle g_1, ..., g_t \rangle_S$. From $I' \subseteq I$ and $\mathrm{in}(I') = \mathrm{in}(I)$ we get $I' = I$. □

Proposition (1.3.7) (cf. [Ei, Proposition 15.6])**.** *Let $\leq$ be a monomial order on S and let $f, g_1, ..., g_t \in S$. Then there exist $f_1, ..., f_t, f' \in S$ with the following properties:*

(i) $f = \sum_{i=1}^{t} f_i g_i + f'$.

(ii) *None of the monomials involved in f' lies in $\langle \mathrm{in}(g_1), ..., \mathrm{in}(g_t) \rangle_S$.*

(iii) $\mathrm{in}(f) \geq \mathrm{in}(f_i g_i)$ *for all* $i \in \{1, ..., t\}$ *and* $\mathrm{in}(f) \geq \mathrm{in}(f')$.

If $f \in S_d$ is homogeneous for some graduation on S as in (1.1.3) and $g_1 \in S_{d_1}, ..., g_t \in S_{d_t}$ are homogeneous, then $f_1, ..., f_t$ and f' can be chosen homogeneous with $f_i \in S_{d-d_i}$ for all $i \in \{1, ..., t\}$ and $f' \in S_d$.

Proof. We prove the existence by an algorithm. At the beginning let $f_1 = \cdots = f_t = 0$ and $f' = f$. (i) and (iii) are fulfilled and will be true after each step of the algorithm (and the same holds for the additional condition that $f_1, ..., f_t, f'$ are homogeneous). Step: Assume that (ii) does not hold. Then we find a term m and an integer i such that $\mathrm{in}(m g_i)$ is the highest term of f' with respect to $\leq$ lying in $\langle \mathrm{in}(g_1), ..., \mathrm{in}(g_t) \rangle$. We replace f' with $f' - m g_i$ and f_i with $f_i + m$ and (i) and (iii) still hold (and still all polynomials are homogeneous). Since the highest term of f' lying in $\langle \mathrm{in}(g_1), ..., \mathrm{in}(g_t) \rangle$ does decrease strictly in each step, the process must end by (1.3.4) and then (ii) also holds. □

Lemma (1.3.8) (cf. [Ei, Theorem 15.3])**.** *Let I be an ideal of S and $\leq$ a monomial order on S. Then the set of monomials whose exponent does not lie in $\exp(I)$ forms a basis of S/I.*

Proof. Immediate from (1.2.5). □

Lemma (1.3.9). *Let $\leq$ be a monomial order on S and $g_1, ..., g_t$ and $h_1, ..., h_r$ be two monic minimal Gröbner bases of an ideal $I \subseteq S$. Then $t = r$ and after reindexing we have $\text{in}(g_i) = \text{in}(h_i)$ for $i = 1, ..., t$.*

Proof. Since $\text{in}(h_i) \in \text{in}(I) = \langle \text{in}(g_1), ..., \text{in}(g_t) \rangle$ we have $\text{in}(h_i) \geq_c \text{in}(g_j)$ for some j. This argument of course also works the other way round. Since $\text{in}(h_i) \geq_c \text{in}(h_l)$ implies $i = l$, we can derive the claimed equalities. □

Lemma (1.3.10). *Let I be an ideal of S and $\leq$ a monomial order on S. There exists a unique monic reduced Gröbner basis of I with respect to $\leq$.*

Proof. First we prove existence. Suppose that $g_1, ..., g_t$ is a monic minimal Gröbner basis of I such that $\text{in}(g_1) < \cdots < \text{in}(g_t)$. We present an algorithm computing polynomials $h_1, ..., h_t \in I$ such that for all $1 \leq i \leq t$ we will have $\text{in}(h_i) = \text{in}(g_i)$. Thus $h_1, ..., h_t$ will still be a monic Gröbner basis of I. Assume that $h_1, ..., h_{r-1}$ are already computed. We now apply (1.3.7) to obtain $g_r = \sum_{i=1}^{r-1} f_i h_i + h_r$ with $\text{in}(g_r) \geq \text{in}(f_i h_i)$ for all i and such that none of the monomials of h_r is divisible by $\text{in}(h_1), ..., \text{in}(h_{r-1})$. If we had for some i that $\text{in}(g_r)$ involves the same monomial as $\text{in}(f_i h_i) = \text{in}(f_i g_i)$, we would get a contradiction since we know that $\text{in}(g_r)$ is not divisible by $\text{in}(g_i)$. Therefore $\text{in}(g_r) > \text{in}(f_i h_i)$ for all i, proving that $\text{in}(h_r) = \text{in}(g_r)$ and h_r is also monic. Assume that one of the monomials m of h_i would be divisible by $\text{in}(h_r)$. Then we would have $\text{in}(h_r) \leq m \leq \text{in}(h_i)$ which is absurd since we have $\text{in}(h_r) = \text{in}(g_r) > \text{in}(h_i)$. After finishing this process, $h_1, ..., h_t$ is a monic reduced Gröbner basis of I. Now assume that $g_1, ..., g_t$ is another monic reduced Gröbner basis of I (with the same number of elements by (1.3.9)) and also $\text{in}(g_1) < \cdots < \text{in}(g_t)$ and $\text{in}(g_i) = \text{in}(h_i)$. Then we can prove inductively that $h_i = g_i$. Assume we have $h_i = g_i$ for all $i \leq r - 1$ and $h_r \neq g_r$. Then $\text{in}(g_r) = \text{in}(h_r) > \text{in}(h_r - g_r) \in \text{in}(I)$ and therefore $0 \neq \text{in}(h_r - g_r)$ is divisible by some $\text{in}(g_i)$ and some $\text{in}(h_j)$. So $\text{in}(h_r - g_r) \geq \text{in}(g_i), \text{in}(h_j)$ and therefore $i, j < r$. The coefficient of the monomial corresponding to $\text{in}(h_r - g_r)$ must have been non zero in at least one of h_r or g_r, but is divisible by $\text{in}(g_i)$ resp. $\text{in}(h_j)$, which is not possible since both bases are reduced. Therefore we must have $h_r = g_r$. □

Lemma (1.3.11). *Let $\leq$ be the weighted homogeneous lexicographic order on S with respect to some multiindex q as in (1.1.2.4) and I a homogeneous ideal of S with respect to the graduation as in (1.1.3) associated to q. Then the elements of a reduced Gröbner basis of I all are homogeneous.*

Proof. Let $g_1, ..., g_t$ be a reduced Gröbner basis of I. Assume that g_1 is not homogeneous. Then we can write $g_1 = h_1 + \cdots + h_s$ where h_i is of degree i. Let $\text{in}(h_i) = \text{in}(g_1)$ and $h_j \neq 0$ with $j \neq i$. Since $h_j \in I$ we have $\text{in}(h_j) \in \langle \text{in}(g_1), ..., \text{in}(g_t) \rangle$. Therefore $\text{in}(h_j) \geq_c \text{in}(g_l)$ for some l. For $l > 1$ this contradicts the reducednes of the basis. For $l = 1$ we get $\text{in}(h_j) \geq \text{in}(g_1) = \text{in}(h_i)$ which also is a contradiction. □

1.4 Additive polynomials

The last part of this chapter deals with additive polynomials, an absolutely central theme of this work. A polynomial $f(X)$ is called additive if $f(X + Y) = f(X) +$

$f(Y)$ for second indeterminates Y. We will study this property in detail in chapter 3 and will see that it coincides with definition (1.4.1) which we are going to use here. Throughout this section k is a field of positive characteristic $p > 0$ and $S = k[X_1, ..., X_n]$. S is graded in the standard way.

Definition (1.4.1). *A* ***homogeneous additive polynomial*** *(also called* ***totally inseparable form****) in S is a polynomial*

$$\sigma = a_1 X_1^q + \cdots + a_n X_n^q$$

where q is a power of p. We will denote the k-vector space spanned by all totally inseparable forms of S with $L = L(S)$. The Frobenius $F : S \to S, f \mapsto f^p$ restricts to L. Therefore L is a (left-)$k[F]$-module. $k[F]$ is not commutative: for $a \in k$ we have $Fa = a^p F$. We can regard $k[F]$ as a graded k-vector space with $(k[F])_d = kF^d$. Then L becomes a graded $k[F]$-module since $FL_d \subseteq L_{d+1}$, where L_d is the k-vector space of all homogeneous additive polynomials of degree p^d. $L = \bigoplus_{d \geq 0} L_d$ and in particular $L_0 = S_1$, where we always regard S with the standard graduation in this context. A system of elements $\sigma = (\sigma_1, ..., \sigma_m)$ of L is called ***arranged*** *if*

$$\sigma_i = X_i^{q_i} + \sum_{j=i+1}^{n} a_{ij} X_j^{q_i}$$

such that $q_1 \leq \cdots \leq q_m$. It is called ***well arranged*** *if additionally $a_{ij} = 0$ whenever $q_i = q_j$. For an arranged system we define $\Lambda' := \mathbb{N}^m$ and*

$$\Lambda'' := \{M \in \mathbb{N}^n | M_1 < q_1, ..., M_m < q_m\}.$$

Remark (1.4.2). *We could make our definitions more intrinsic in the following way: For a field k of positive characteristic p and a finite dimensional k-vector space $\mathfrak{V}$ let $S := \mathrm{Sym}_k(\mathfrak{V})$. The absolute Frobenius F acts on S and we define $L_d := kF^d(S_1)$ and $L = \bigoplus_{d \geq 0} L_d$. What we call arranged system was already used by Hironaka [H5, (1.2)] and Giraud [Gi, I 5.4].*

Remark (1.4.3). *We will frequently study graded $k[F]$-submodules Q of L (cf. [Od]). Such a module is always a free $k[F]$-module. In fact k-linearly independent elements $\tau_1, ..., \tau_m \in L_d$ also are $k[F]$-independent. Thus the following algorithm yields a homogeneous $k[F]$-basis of Q: Choose a k-basis $\tau_1, .., \tau_{e_0}$ of Q_0. Complete $F(\tau_1), ..., F(\tau_{e_0})$ with $\tau_{e_0+1}, ..., \tau_{e_0+e_1} \in Q_1$ to a k-basis of Q_1. Go on like this. The process finally stops since $\dim_k Q_d$ is bounded by n. The resulting τ_i then are are $k[F]$-basis of Q. By renumbering the variables and taking suitable linear combinations one can transform such a basis into a well arranged system.*

Let us study the behaviour of $k[F]$-bases first in the easy case of an arranged system.

Lemma (1.4.4). *If σ is an arranged system in L as in (1.4.1), then the monomials $(\sigma^N X^M)_{N \in \Lambda', M \in \Lambda''}$ form a k-basis of S and explicitely for the lexicographic order on Λ*

(1.4.4.A) $$\exp(\sigma^N X^M) = M + (q_1 N_1, ..., q_m N_m, 0, ..., 0).$$

The polynomials of σ are algebraically independent over k and σ is also a $k[F]$-basis of the graded $k[F]$-module $\langle\sigma\rangle_L$. σ is a minimal Gröbner basis for the lexicographic order with $X_1 > \cdots > X_n$ of the ideal $\langle\sigma\rangle_S$ and if σ is well arranged it is the unique monic reduced Gröbner basis of $\langle\sigma\rangle_S$. The monomials $(\overline{X^M})_{M\in\Lambda''}$ form a k-basis of $S/\langle\sigma\rangle_S$.

Proof. (1.4.4.A) is clear from (1.3.3) and the exponents on the right side of (1.4.4.A) precisely span Λ without any recurrences, proving that $(\sigma^N X^M)$ is a k-basis of S. Therefore also $\sigma_1, ..., \sigma_m$ are algebraically independent. Assume there would be $f \in \langle\sigma\rangle_S$ with $\text{in}(f) \notin \langle \text{in}(\sigma_1), ..., \text{in}(\sigma_m)\rangle_S$. Then by (1.4.4.A) we must have $\text{in}(f) = X^M$ for some $M \in \Lambda''$. But if we develop f in the k-basis above we must get $f = \sum_{0<_c N\in\Lambda', M\in\Lambda''} a_{N,M}\sigma^N X^M$ and this is a contradiction. Thus σ is a Gröbner basis of $\langle\sigma\rangle_S$. It is minimal again by (1.4.4.A). If σ is well arranged, then by definition σ is reduced. The $(\overline{X^M})_{M\in\Lambda''}$ are a k-basis as claimed by (1.3.8). □

In almost all situations we could assume without loss of generality that, after reindexing the variables, we find for a graded $k[F]$-module $Q \subseteq L$ a basis in form of an arranged (or well arranged) system. However we also state the following more general result which can be used in the case that certain $k[F]$-independent elements have to be fixed. This will be useful in the proof of the main theorems.

Lemma (1.4.5). *Let $\sigma = (\sigma_1, ..., \sigma_m)$ be a system of homogeneous additive $k[F]$-independent polynomials in $S = k[X_1, ..., X_n]$ of degrees $q = (q_1, ..., q_m)$ with $q_1 \le \cdots \le q_m$. Then, after renumbering the X_i, the following hold with Λ', Λ'' as in (1.4.1):*

(i) *The $(\sigma^N X^M)_{N\in\Lambda', M\in\Lambda''}$ form a k-basis of S making S into a wellordered k-vector space. Here $(N, M) \le (N', M')$ if $qN + |M| < qN' + |M'|$ or $qN + |M| = qN' + |M'|$ and $(N, M) \le_{\text{lex}} (N', M')$ (componentwise lexicographic order and lexicographic order on the product).*

(ii) *The images of the monomials $(\overline{X^M})_{M\in\Lambda''}$ form a k-basis of $S/\langle\sigma\rangle_S$.*

In particular $\langle\sigma\rangle_S \cap \bigoplus_{M\in\Lambda''} kX^M = 0$ and σ is algebraically independent over k. If $\sigma_i = \sigma_i' + \sigma_i''$ are decompositions with $\sigma_i' \in k[X_1, ..., X_{n'}], \sigma_i'' \in k[X_{n'+1}, ..., X_n]$ homogeneous and additive of the same degree such that $\sigma' = (\sigma_1', ..., \sigma_m')$ is $k[F]$-independent, then the renumbering can be achieved in such a way that $X_{n'+1}, ..., X_n$ are unchanged.

Proof. Consider the $k[F]$-module $Q := \langle\sigma\rangle_{k[F]}$. Since $Q_0 + kX_1 + \cdots + kX_n = L_0$, after renumbering the variables, we find $1 \le i_0 \le n+1$ with $Q_0 \oplus kX_{i_0} \oplus \cdots \oplus kX_n = L_0$. We proceed inductively. Assume we already have $Q_j \oplus kX_{i_j}^{p^j} \oplus \cdots \oplus kX_n^{p^j} = L_j$ for all $j = 0, ..., k-1$ with $1 \le i_0 \le \cdots \le i_{k-1} \le n+1$. By applying kF we have $Q_k + kX_{i_{k-1}}^{p^k} + \cdots + kX_n^{p^k} = L_k$ and therefore find after renumbering $X_{i_{k-1}}, ..., X_n$ an i_k with $i_{k-1} \le i_k \le n+1$ and $Q_k \oplus kX_{i_k}^{p^k} \oplus \cdots \oplus kX_n^{p^k} = L_k$. The sequence $i_0 \le i_1 \le \cdots$ then terminates. In the case of the decompositions $\sigma_i = \sigma_i' + \sigma_i''$ we use this algorithm for σ' in $k[X_1, ..., X_{n'}]$ and renumber only the variables $X_1, ..., X_{n'}$. Adding the other variables again we get

$$L_k = (\langle\sigma'\rangle_{k[F]})_k \oplus kX_{i_k}^{p^k} \oplus \cdots \oplus kX_n^{p^k} = (\langle\sigma\rangle_{k[F]})_k + kX_{i_k}^{p^k} + \cdots + kX_n^{p^k}$$

and hence also the latter sum is direct (by the number of generators). We come to the proof of (i) and (ii). Note that $q_i \le p^j$ is equivalent to $k\sigma_1^{p^j/q_1} \oplus \cdots \oplus k\sigma_i^{p^j/q_i} \subseteq Q_j$ which is equivalent to $i < i_j$ and therefore $q_i = \min\{p^j | i < i_j, j \ge 0\}$ for $i = 1, ..., m$. For $q_i = p^j$ we have $i < i_j$ and $X_i^{q_i} \in Q_j \oplus kX_{i_j}^{p^j} \oplus \cdots \oplus kX_n^{p^j}$. Thus we have a system of generators in (i). Λ' and Λ'' only depend on Q. Therefore

$$N_d := \#\{(N, M) \in \Lambda' \times \Lambda'' | qN + |M| = d\},$$

i.e. the number of terms $(\sigma^N X^M)_{N\in\Lambda', M\in\Lambda''}$ with a given degree d, only depends on Q. By choosing an arranged system as a $k[F]$-basis of Q we find with (1.4.4) that $\dim_k S_d = N_d$ and therefore we have linear independency in (i). With respect to the order in (i) we have $\exp(\langle\sigma\rangle_S) = (\Lambda' \setminus \{0\}) \times \Lambda''$ and (ii) follows from (1.2.5). □

The following two lemmas present a toolbox for relating ideals of S, $k[F]$-modules and certain subrings of S, that will be completed by (3.4.4).

Lemma (1.4.6). *If $U \subseteq S$ is a graded subalgebra generated by additive polynomials, then we have inclusion preserving inverse bijections*

$$\boxed{\begin{array}{l}\textit{homogeneous ideals } I \textit{ of}\\ S \textit{ with } S\cdot(U\cap I) = I\end{array}} \;\underset{S\cdot J \;\longleftarrow\!\shortmid\; J}{\overset{I \;\mapsto\; U\cap I}{\rightleftarrows}}\; \boxed{\begin{array}{l}\textit{homogeneous ideals of}\\ \textit{the graded ring } U\end{array}}$$

Proof. It is clear that both maps are well-defined and inclusion preserving. It remains to show that for a homogeneous ideal $J \subseteq U$ we have $U \cap (S \cdot J) = J$, where the inclusion $\supseteq$ is clear. $Q := U \cap L \subseteq L$ is a graded $k[F]$-submodule and we can find, after renumbering the variables, a $k[F]$-basis σ of Q in form of an arranged system. For $f \in S \cdot J$ we can write $f = \sum_{M\in\Lambda''} X^M g_M$ for certain $g_M \in J \subseteq U$. From the structure of the basis of S in (1.4.4) we see that $f \in U$ implies $g_M = 0$ whenever $M \ne 0$ and then $f = g_0 \in J$. □

Lemma (1.4.7). *Let Q be a graded $k[F]$-submodule of L. We have $SQ \cap L = Q$ (cf. [Od, 2.3 (b)]) and the maps $\mathfrak{a} \mapsto \mathfrak{a} \cap L$ and $Q \mapsto SQ$ are inverse inclusion preserving bijections between the set of ideals of S generated by homogeneous additive polynomials and the set of graded $k[F]$-submodules of L.*

Proof. Let $\sigma = (\sigma_1, ..., \sigma_m)$ be an arranged system generating a $k[F]$-module $Q \subseteq L$ (after renumbering the variables). Then L has a k-basis consisting of all p-powers of the σ_i and the monomials $X_i^{p^l}$ for $i \le m$ and $p^l < \deg(\sigma_i)$ or $i > m$. On the other hand SQ has the k-basis (see (1.4.4)) $\sigma^N X^M$ where $N >_c 0$ and M is arbitrary. Both bases are subbases of the whole basis $(\sigma^N X^M)_{N\in\Lambda', M\in\Lambda''}$. Therefore $SQ \cap L = Q$. If $\mathfrak{a}$ is generated by homogeneous additive polynomials, then $S(\mathfrak{a} \cap L) = \mathfrak{a}$. □

2 Differential Operators

We are introducing the concept of derivations and differential operators as in [EGA, 0_{IV} 20, IV 16] (see also [Gi, III §1], [H6, II]). We are mainly interested in absolute differential operators of polynomial rings over fields of positive characteristic. Their analysis splits into two parts: Finding the differential operators with respect to the variables will we dealt with in 3.3 using the group structure of an affine space. The absolute differential operators of a field will be treated in 2.2. In 2.3 we present a very general version of a Jacobian criterion using absolute differential operators. This criterion can be used not only to determine the singular locus of a variety but also the locus of higher orders.

2.1 Derivations and differential operators

Throughout this section let A be a ring, B an A-algebra and M a B-module. We are introducing the concepts of derivations and differential operators. For the latter we will give three equivalent definitions (see [EGA, IV (16.8.8)]), all of them useful in certain situations.

Definition (2.1.1). *An A-linear* ***derivation*** *from B to M is an A-linear map*

$$D : B \to M$$

which satisfies the Leibniz rule, i.e. for all $b, b' \in B$:

$$D(bb') = b'D(b) + bD(b'). \tag{2.1.1.A}$$

The A-derivations from B to M form a B-module, the ***module of relative derivations*** *of B over A with values in M, which we will denote $\mathrm{Der}_A(B, M)$. The multiplication of D with a scalar $b \in B$ is the obvious one: $(bD)(b') = bD(b')$. In the case $M = B$ we also write $\mathrm{Der}_A(B)$. Each ring B is in a unique way a $\mathbb{Z}$-algebra und we call $\mathrm{Der}_{\mathbb{Z}}(B, M)$ the* ***absolute derivations*** *of B with values in M.*

This concept of derivations will now be generalized to derivations 'of higher oder', the so-called differential operators, through an analogue of formula (2.1.1.A) for several factors.

Definition (2.1.2) (cf. [EGA, IV (16.8.8) c)]). *An A-linear* ***differential operator*** *from B to M of order $\leq n$ is an A-linear map*

$$D : B \to M$$

which satisfies the generalized Leibniz rule, i.e. for elements $b = (b_0, b_1, ..., b_n)$ of B

and the multiindex $\mathbf{1} = (1, ..., 1) \in \Lambda := \mathbb{N}^{n+1}$*:*

(2.1.2.A)
$$\sum_{L,M \in \Lambda, L+M=\mathbf{1}} (-1)^{|L|} b^L D(b^M) = 0.$$

These differential operators form a B*-module* $\mathrm{Diff}_A^{\leq n}(B, M)$*, the* ***module of relative differential operators*** *from* B *over* A *of order* $\leq n$ *with values in* M*. In the case* $M = B$ *we also write* $\mathrm{Diff}_A^{\leq n}(B)$*. Again we have the module of* ***absolute differential operators*** $\mathrm{Diff}_{\mathbb{Z}}^{\leq n}(B, M)$*.*

Lemma (2.1.3). *We have*

$$\mathrm{Diff}_A^{\leq 0}(B, M) = \mathrm{Hom}_B(B, M) \cong M$$

and there is a canonical isomorphism of B*-modules*

$$\mathrm{Diff}_A^{\leq 1}(B, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M) \oplus M$$

$$D \longmapsto ((b \mapsto D(b) - bD(1)), D(1))$$

$$(b \mapsto D(b) + bm) \longleftarrow (D, m).$$

In particular $\mathrm{Der}_A(B, M) \subseteq \mathrm{Diff}_A^{\leq 1}(B, M)$*.*

Proof. This is a straightforward computation (see [Di, (1.1.5), (1.1.6)]). □

(2.1.4) The humongous formula (2.1.2.A) is not very useful to derive properties of differential operators. Therefore we introduce universal properties characterizing derivations and differential operators (cf. [EGA, IV (16.8.1), (16.3.7)]). They will be used in 2.3.

(2.1.4.1) Let $I_{B/A}$ be the kernel of the multiplication map

$$m : B \otimes_A B \to B, \quad b \otimes b' \mapsto b \cdot b'.$$

In the following we view $B \otimes_A B$ (and derived objects) always as a B-module via the left factor of the tensor product. The B-module $I_{B/A}$ is generated by the elements $d(b) := 1 \otimes b - b \otimes 1$ for $b \in B$: If $\sum_j x_j \otimes y_j \in I_{B/A}$, i.e. $\sum_j x_j y_j = 0$, with $x_j, y_j \in B$, then

$$\sum_j x_j \otimes y_j = \sum_j x_j(1 \otimes y_j - y_j \otimes 1) + \sum_j x_j y_j \otimes 1 = \sum_j x_j d(y_j).$$

(2.1.4.2) Now we have the B-module of (relative) (Kähler-)differentials of B over A

$$\Omega^1_{B/A} := I_{B/A}/I^2_{B/A}$$

together with the derivation

$$d_{B/A} : B \to \Omega^1_{B/A}, \quad d_{B/A}(b) = d(b) \bmod I^2_{B/A} = 1 \otimes b - b \otimes 1 \bmod I^2_{B/A}.$$

$d_{B/A} \in \mathrm{Der}_A(B, \Omega^1_{B/A})$ since the image of $d_{B/A}$ is contained in $I_{B/A}$ and for $b, b' \in B$

we see that

$$d(bb') - bd(b') - b'd(b) = 1 \otimes bb' - bb' \otimes 1 - b \otimes b' + bb' \otimes 1 - b' \otimes b + bb' \otimes 1 =$$

$$= (1 \otimes b - b \otimes 1) \cdot (1 \otimes b' - b' \otimes 1) = d(b)d(b') \in I^2_{B/A}.$$

(2.1.4.3) Further we have for all $n \in \mathbb{N}$ the B-module

$$\mathcal{P}^n_{B/A} := B \otimes_A B / I^{n+1}_{B/A}$$

together with the differential operator

$$d^n_{B/A} : B \to \mathcal{P}^n_{B/A}, \quad d^n_{B/A}(b) = 1 \otimes b \bmod I^{n+1}_{B/A}.$$

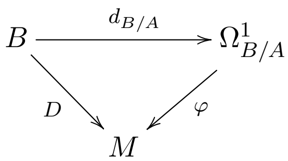
$d^n_{B/A} \in \mathrm{Diff}^{\leq n}_A(B, \mathcal{P}^n_{B/A})$ since for elements $b = (b_0, ..., b_n)$ of B (2.1.2.A) is fulfilled:

$$I^{n+1}_{B/A} \ni d(b_0) \cdots d(b_n) = (1 \otimes b_0 - b_0 \otimes 1) \cdots (1 \otimes b_n - b_n \otimes 1) =$$

$$= \sum_{L+M=\mathbf{1}} (-1)^{|L|} b^L \otimes b^M = \sum_{L+M=\mathbf{1}} (-1)^{|L|} b^L \cdot d^n_{B/A}(b^M).$$

Proposition (2.1.5) (Universal properties of $(\Omega^1_{B/A}, d_{B/A})$ and $(\mathcal{P}^n_{B/A}, d^n_{B/A})$).

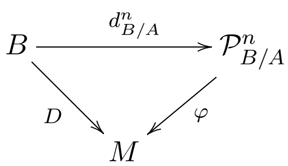
(i) $(\Omega^1_{B/A}, d_{B/A})$ *has the following universal property:*
For every B-module M and every A-derivation $D : B \to M$ there exists exactly one B-module homomorphism $\varphi : \Omega^1_{B/A} \to M$ with $\varphi \circ d_{B/A} = D$:



This yields an isomorphism of B-modules

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M), \qquad \varphi \mapsto \varphi \circ d_{B/A}.$$

(ii) $(\mathcal{P}^n_{B/A}, d^n_{B/A})$ *has the following universal property:*
For every B-module M and every A-differential operator $D : B \to M$ of order $\leq n$ there exists exactly one B-module homomorphism $\varphi : \mathcal{P}^n_{B/A} \to M$ with $\varphi \circ d^n_{B/A} = D$:



This yields an isomorphism of B-modules

$$\mathrm{Hom}_B(\mathcal{P}^n_{B/A}, M) \xrightarrow{\sim} \mathrm{Diff}^{\leq n}_A(B, M), \qquad \varphi \mapsto \varphi \circ d^n_{B/A}.$$

Proof. See [EGA, IV (16.8.8)]. ☐

We come to a third characterization of differential operators that is particularly useful in proofs using induction.

Definition (2.1.6). *For a map $D : B \to M$ and an element $b \in B$ we denote by $[D, b] : B \to M$ the* ***commutator*** *map given by $[D, b](b') = D(bb') - bD(b')$. If D is A-linear, then so is $[D, b]$. In the case $M = B$ we will use commutators $[D, E]$ also for maps $D, E : B \to B$. In the following we make the convention $\mathrm{Diff}_A^{\leq -1}(B, M) = \{0\}$ for consistency.*

Lemma (2.1.7) (cf. [EGA, IV (16.8.8) b)])**.** *Let $D : B \to M$ be an A-linear map. For $n \in \mathbb{N}$ the following are equivalent:*

(i) *D is a differential operator of order $\leq n$.*

(ii) *For all $b \in B$ the commutator $[D, b]$ is a differential operator of order $\leq n - 1$.*

Proof. (i) ⇒ (ii): For elements $c = (c_0, ..., c_{n-1})$ of B and $b \in B$ we set $c' = (c_0, ..., c_{n-1}, b)$ and find

$$\sum_{L,M \in \mathbb{N}^n, L+M=\mathbf{1}} (-1)^{|L|} c^L [D, b](c^M) = \sum_{L,M \in \mathbb{N}^n, L+M=\mathbf{1}} (-1)^{|L|} \left[c^L D(bc^M) - c^L b D(c^M) \right]$$

$$= \sum_{L,M \in \mathbb{N}^{n+1}, L+M=\mathbf{1}} (-1)^{|L|} {c'}^L D({c'}^M) = 0.$$

For (ii) ⇒ (i) just read the calculation backwards. ☐

Corollary (2.1.8). *We have the chain of inclusions*

$$\mathrm{Diff}_A^{\leq 0}(B, M) \subseteq \mathrm{Diff}_A^{\leq 1}(B, M) \subseteq \mathrm{Diff}_A^{\leq 2}(B, M) \subseteq ... \ .$$

Proof. Let $D \in \mathrm{Diff}_A^{\leq n}(B, M)$. It is obvious from the Leibniz rule that for any element $b \in B$ also the maps $b' \mapsto D(b \cdot b')$ and $b' \mapsto bD(b')$ are differential operators of order $\leq n$. Therefore for any $b \in B$ we know that $[D, b]$ is a differential operator of order $\leq n$. By (2.1.7) D is a differential operator of order $\leq n + 1$. ☐

Proposition (2.1.9) (cf. [EGA, IV (16.8.9)])**.** *If $D \in \mathrm{Diff}_A^{\leq m}(B)$ and $D' \in \mathrm{Diff}_A^{\leq n}(B)$, then $D' \circ D \in \mathrm{Diff}_A^{\leq m+n}(B)$.*

Proof. For three A-linear maps $D, E, F : B \to B$ we have the identity

$$[DE, F] = DEF - FDE = D([E, F] + FE) + ([D, F] - DF)E = D[E, F] + [D, F]E.$$

The proposition will be proved by induction on $d := m+n$. For the induction step we use criterion (2.1.7). We have to show that for all $b \in B$ the commutator $[D'D, b]$ is a differential operator of order $\leq d-1$. This is clear from $[D'D, b] = D'[D, b] + [D', b]D$ and the induction hypothesis: the differential operators $[D, b]$ and $[D', b]$ have order $\leq m - 1$ resp. $\leq n - 1$. (Note that the only differential operator of order ≤ -1 is the zero map, this also starts the induction with $d = 0$.) ☐

(2.1.8) and (2.1.9) mean that the module of all A-linear differential operators on B

$$\mathrm{Diff}_A(B) := \bigcup_{n\geq 0} \mathrm{Diff}_A^{\leq n}(B)$$

is a (noncommutative) filtered B-algebra.

Lemma (2.1.10). *For $D \in \mathrm{Diff}_A^{\leq n}(B, M)$ and a subring $C \subseteq B$ the following are equivalent:*

(i) *D is C-linear, i.e. $D \in \mathrm{Diff}_C^{\leq n}(B, M)$.*

(ii) *$D(c) = c \cdot D(1)$ for all $c \in C$ and $[D, b]$ is C-linear for all $b \in B$.*

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i): For elements $b \in B$ and $c \in C$ we find $D(cb) - cbD(1) = D(cb) - bD(c) = [D, b](c) = c(D(b) - bD(1)) = cD(b) - cbD(1)$, thus $D(cb) = cD(b)$. $\square$

2.2 Differential operators in positive characteristic

On a polynomial ring over a field of characteristic zero there exist only differential operators with respect to the variables. But in positive characteristic there exist, at least over a non-perfect field, also differential operators with respect to elements of the field itself.

Lemma (2.2.1) (cf. [Gi, III 1.2.2]). *If B is an A-algebra of prime characteristic $p > 0$ and M is a B-module, then for $n < p^e$*

$$\mathrm{Diff}_A^{\leq n}(B, M) = \mathrm{Diff}_{A[B^{p^e}]}^{\leq n}(B, M).$$

Proof. We fix e and proceed by induction on $n = 0, ..., p^e - 1$, where the claim is certainly true for $n = 0$. For $b \in B$ we use the Leibniz rule (2.1.2.A) with $b_0 = \cdots = b_{p^e-1} = b$ for $D \in \mathrm{Diff}_A^{\leq p^e-1}(B)$ and get

$$\sum_{i=0}^{p^e} \binom{p^e}{i} (-1)^i b^i D(b^{p^e-i}) = 0.$$

Thus $D(b^{p^e}) = b^{p^e} D(1)$. The induction step is therefore done with (2.1.10). $\square$

Definition (2.2.2) (cf. [EGA, 0_{IV} (21.1.9)]). *Let B be a ring of prime characteristic p. A family of elements $(x_i)_{i\in I}$ of B is called p-**independent** (resp. system of p-**generators**, resp. p-**basis**) of B if the family of monomials x^N in x with $N \in \Lambda := \mathbb{N}^{(I)}, 0 \leq N_i < p$ for all $i \in I$ is a free family (resp. system of generators, resp. basis) of the B^p-module B.*

Remark (2.2.3). *Let B be a ring of prime characteristic p, assume that the Frobenius $F : B \to B, b \mapsto b^p$ is injective and let $(x_i)_{i\in I}$ be a p-basis of B. Then the monomials x^N with $N \in (p^e\mathbb{N})^{(I)}, 0 \leq N_i < p^{e+1}$ for all $i \in I$ form a p-basis of B^{p^e}*

since $F^e : B \to B^{p^e}$ is an isomorphism. By induction therefore the monomials x^N with $N \in \Lambda, 0 \leq N_i < p^e$ for all $i \in I$ form a B^{p^e}-basis of B.

Remark (2.2.4). *Every field of positive characteristic has a p-basis and one also has the usual basis extension properties. For details see [EGA, 0_{IV} 21.4].*

Proposition (2.2.5). *Let $(b_i)_{i\in I}$ be a p-basis of B and suppose that $F : B \to B$ is injective. For every multiindex $M \in \Lambda = \mathbb{N}^{(I)}$ there exists $D_M \in \mathrm{Diff}_{\mathbb{Z}}^{\leq |M|}(B)$ with*

$$D_M(b^N) = \binom{N}{M} b^{N-M} \tag{2.2.5.A}$$

for all $N \in \Lambda$. Every $D \in \mathrm{Diff}_{\mathbb{Z}}^{\leq n}(B)$ is a (possibly infinite) sum

$$D = \sum_{|M|\leq n} c_M D_M \tag{2.2.5.B}$$

with unique coefficients $c_M \in B$.

Proof. Let $M \in \Lambda$ be given with $|M| < p^e$ for some integer e. With $C = B^{p^e}[X_i]_{i\in I}$ we have by (2.2.3) an isomorphism $C/\mathfrak{c} \cong B$ identifying X_i with b_i, where we take $\mathfrak{c} = \langle X_i^{p^e} - b_i^{p^e} \rangle_C$. From (3.3.1) we get a differential operator $D'_M \in \mathrm{Diff}_{B^{p^e}}^{\leq |M|}(C)$ with $D'_M(X^N) = \binom{N}{M} X^{N-M}$ for all $N \in \Lambda$. By (2.2.1) this operator is $B^{p^e}[C^{p^e}] = B^{p^e}[X_i^{p^e}]_{i\in I}$-linear and therefore $D'_M(\mathfrak{c}) \subseteq \mathfrak{c}$. Thus D'_M factors to the claimed differential operator on $B \cong C/\mathfrak{c}$. The coefficients in (2.2.5.B) are unique: Assume we had $D = 0$ and $c_L \neq 0$ for some $L \in \Lambda$. Then we could assume that $c_M = 0$ whenever $M <_c L$, but this would lead to the contradiction $0 = D(b^L) = c_L$. Some $D \in \mathrm{Diff}_{\mathbb{Z}}^{\leq n}(B)$ is by (2.1.2.A) and (2.2.1) determined from the values $D(b^M)$ for $|M| \leq n$. Therefore it suffices to show that for any family $(d_M)_{|M|\leq n}$ in B there exist $(c_M)_{|M|\leq n}$ in B with $D(b^M) = d_M$ for all $|M| \leq n$ and $D = \sum_M c_M D_M$. We do this by induction on n, where the case $n = 0$ is obvious. Let now $(d_M)_{|M|\leq n+1}$ be given in B. By the induction hypothesis we find a linear combination D' of the $D_M, |M| \leq n$ with $D'(b^M) = d_M$ for $|M| \leq n$. Then $D := D' + \sum_{|M|=n+1}(d_M - D'(b^M))D_M$ satisfies $D(b^M) = d_M$ for all $|M| \leq n + 1$. □

Remark (2.2.6). *The differential operators D_M from (2.2.5) all commute with each other, but not with elements of B. This follows from (3.2.6) and the proof of (2.2.5). They furthermore satisfy the following relation with the Frobenius F: $F^j \circ D_M = D_{p^j M} \circ F^j$. This follows from a simple calculation in the p-basis and the identiy of binomial coefficients $\binom{pa}{pb} \equiv \binom{a}{b}$ modulo p.*

Remark (2.2.7). *If $(a_i)_{i\in I}$ is a p-basis of a field k of positive characteristic, then it can be extended with the variables $X_1, ..., X_n$ to a p-basis of the polynomial ring $S = k[X_1, ..., X_n]$. Due to the fact that the operators D_M with respect to different basis elements commute, we eventually conclude that differential operators on S can be decomposed into such operators that are k-linear and differentiate the variables and such operators that are linear in the variables and only differentiate the elements of the p-basis of the field.*

Lemma (2.2.8). *Let $x = (x_i)_{i\in I}$ be a p-basis of a field k of positive characteristic p. Let $y = (y_j)_{j\in J}$ be a transcendence basis of a separable extension K/k. Then x together with y is a p-basis of K.*

Proof. First assume that K/k is algebraic. The set $\mathbb{E}$ of subfields $K/E/k$ such that x is a p-basis of E is inductively ordered and nonempty ($k \in \mathbb{E}$). In fact, a chain $(E_h)_{h\in H}$ in $\mathbb{E}$ has the upper bound $E := \bigcup_{h\in H} E_h$: If there is an E-linear combination of the x^M with $0 \leq M_i < p$ for all $i \in I$ to zero, all the coefficients can be found in one of the E_h, therefore are zero. Every $e \in E$ lies in one of the E_h and therefore is generated by the x^M over $E_h^p \subseteq E^p$. Thus we find a maximal element E in $\mathbb{E}$. To show that $E = K$ we assume the contrary and find a non-trivial primitive separable extension F/E inside K. Then also F^p/E^p is separable and E and F^p are linearly disjoint over E^p. Therefore the E^p-basis x^M of E is also an F^p-basis of F and E cannot have been maximal. In the general case we now can assume that K/k is purely transcendental. An element $\frac{f(y)}{g(y)}$ of K with $f(y), g(y) \in k[y_j | j \in J]$ can be written as $\frac{f(y)g(y)^{p-1}}{g(y)^p}$. Therefore x, y is a system of p-generators of K. But the $x^M y^N$ with $0 \leq M_i, N_j < p$ for all $i \in I, j \in J$ are also K^p-independent as is easily seen. □

Remark (2.2.9). *With (2.2.8) and (2.2.5) we see that every differential operator $D \in \mathrm{Diff}_{\mathbb{Z}}^{\leq n}(k)$ for a field of positive characteristic k extends to a differential operator $D' \in \mathrm{Diff}_{\mathbb{Z}}^{\leq n}(K)$ for a separable extension K/k. This extension is unique if K/k is algebraic.*

2.3 Jacobian criteria with differential operators

In this most technical section of the chapter we establish a connection between regularity and differential operators in form of the Jacobian criteria (2.3.4) and (2.3.5).

Lemma (2.3.1) (cf. [Gi, III 1.2.3]). *Let B be an A-algebra and $D \in \mathrm{Diff}_A^{\leq n}(B)$. For an ideal $I \subseteq B$ and $m \geq n$ we have*

$$D(I^m) \subseteq I^{m-n}.$$

Proof. This is obvious for $m = n$ and for $n = 0$. We proceed by a double induction on m and n for which we assume the claim to be true for differential operators of order $\leq n-1$ with arbitrary ideal powers, and differential operators of order n with ideal powers $\leq m$. We then show the claim for the ideal power $m+1$: Let therefore $b \in I, c \in I^m$ so that $bc \in I^{m+1}$ and let D be a differential operator of order n. We have to show that $D(bc) \in I^{m+1-n}$. But we know that $[D, b]$ is a differential operator of order $\leq n-1$ and therefore the induction hypothesis yields $D(bc) = [D,b](c) + bD(c) \in [D,b](I^m) + I \cdot D(I^m) \subseteq I^{m-(n-1)} + I \cdot I^{m-n} = I^{m+1-n}$. □

In the case of a prime ideal $\mathfrak{p} \subseteq B$ this result extends to its symbolic powers $\mathfrak{p}^{(n)}$, i.e. to the preimages of the powers of the maximal ideal of $B_{\mathfrak{p}}$ in B.

Lemma (2.3.2). *Let B be an A-algebra, $\mathfrak{p}$ a prime ideal of B and $b \in B$. Then*

$$b \in \mathfrak{p}^{(n)} \quad \Longrightarrow \quad \mathrm{Diff}_A^{\leq n-1}(B)(b) \subseteq \mathfrak{p}.$$

In other words

$$\mathrm{Diff}_A^{\leq n-1}(B)(\mathfrak{p}^{(n)}) \subseteq \mathfrak{p}.$$

Proof. If $b \in \mathfrak{p}^{(n)}$, then $sb \in \mathfrak{p}^n$ for some $s \notin \mathfrak{p}$. By (2.3.1) $\mathrm{Diff}_A^{\leq n-1}(B)(sb) \subseteq \mathfrak{p}$. For $D \in \mathrm{Diff}_A^{\leq m}(B)$ we have $[D,s] \in \mathrm{Diff}_A^{\leq m-1}(B)$ and $[D,s](b) = D(sb) - sD(b)$ shows that $[D,s](b) \in \mathfrak{p}$ if and only if $D(b) \in \mathfrak{p}$ as long as $m \leq n-1$ since $s \notin \mathfrak{p}$. This concludes the proof inductively because $\mathrm{Diff}_A^{\leq -1}(B) = 0$. □

To understand for what situations the converse of this lemma also holds, we are going to deduce two Jacobian criteria relying on formal smoothness. Let us first recall this property.

Definition (2.3.3). *Let B be an A-algebra. We say that B is* ***formally smooth*** *over A (for the discrete topologies) if the following holds: For every A-algebra C and every nilpotent ideal I of C all A-algebra homomorphisms $\mathrm{Hom}_A(B, C/I)$ can be lifted to A-algebra homomorphisms $\mathrm{Hom}_A(B,C)$. More precisely we say that the structure morphism $A \to B$ is formally smooth. We will only have to consider the discrete topology for all our applications. So, if no topology is mentioned, we are refering to the discrete topology.*

Theorem (2.3.4) (Jacobian criterion)**.** *Let B be an A-algebra and $\mathfrak{b} \subseteq B$ an ideal. Suppose that B and $B/\mathfrak{b}$ are formally smooth A-algebras. Then for every $b \in B$ and $n \geq 1$ we have*

$$b \in \mathfrak{b}^n \quad \Longleftrightarrow \quad \mathrm{Diff}_A^{\leq n-1}(B)(b) \subseteq \mathfrak{b}.$$

Theorem (2.3.5) (Local Jacobian criterion)**.** *Let B be an A-algebra and $\mathfrak{p} \subseteq B$ a prime ideal. Suppose that B and $\mathrm{Quot}(B/\mathfrak{p})$ are formally smooth A-algebras. Then for every $b \in B$ and $n \geq 1$ we have*

$$b \in \mathfrak{p}^{(n)} \quad \Longleftrightarrow \quad \mathrm{Diff}_A^{\leq n-1}(B)(b) \subseteq \mathfrak{p}.$$

In the case that B is a polynomial ring over a field, (2.3.5) can be found in [Gi, III 1.2.7] with a different kind of proof using completions or in [Od, 2.2] with a proof similar to ours. Before we come to our proof of the theorems, let us deduce some corollaries to see where they might be applied. For this we mention first a few facts about formal smoothness:

Theorem (2.3.6) (Cohen, [EGA, 0_{IV} (19.6.1)])**.** *Let K/k be a field extension. Then K is a formally smooth k-algebra if and only if K/k is a (not necessarily algebraic) separable extension.*

Lemma (2.3.7) ([EGA, 0_{IV} (19.3.5) (ii), (iv) and (19.3.3)]:)**.** *Let B be an A-algebra, C a B-algebra and $S \subseteq A, T \subseteq B$ compatible multiplicatively closed subsets.*

(i) *If $A \to B$ and $B \to C$ are formally smooth, then so is the composition $A \to C$.*

(ii) *If $A \to B$ is formally smooth, then so is $S^{-1}A \to T^{-1}B$.*

(iii) *$A \to A[X_i]_{i \in I}$ is formally smooth for arbitrary I.*

Corollary (2.3.8). *Let B be an A-algebra and $\mathfrak{p}$ a prime ideal of B. Suppose that B and $B/\mathfrak{p}$ are formally smooth A-algebras. Then for all $n \geq 1$ we have $\mathfrak{p}^{(n)} = \mathfrak{p}^n$.*

Proof. By (2.3.7) (ii) $\mathrm{Quot}(B/\mathfrak{p})$ is formally smooth over A. Now use (2.3.4) and (2.3.5) together. □

Corollary (2.3.9). *Consider an ideal I in a polynomial ring $S = K[X_i]_{i\in I}$ over a field K. Let k be a perfect subfield (e.g. the prime field) of K. Then for all prime ideals $\mathfrak{p} \subseteq S$ and all $n \geq 1$ the following are equivalent:*

(i) $I \subseteq \mathfrak{p}^{(n)}$,

(ii) $I_\mathfrak{p} \subseteq \mathfrak{p}^n_\mathfrak{p}$,

(iii) $\mathrm{Diff}_k^{\leq n-1}(S)(I) \subseteq \mathfrak{p}$,

(iv) $\mathrm{Diff}_\mathbb{Z}^{\leq n-1}(S)(I) \subseteq \mathfrak{p}$.

Proof. The equivalence of (i) and (ii) is clear. Since k is perfect, K/k is separable and therefore formally smooth by (2.3.6), but then so are S/k by (2.3.7) and also $\mathrm{Quot}(S/\mathfrak{p})/k$. Therefore we get (i) $\Leftrightarrow$ (iii) from (2.3.5). Choosing k to be the prime field gives the equivalence of (i) and (iv). This is immediate in positive characteristic and for characteristic zero we remark that $\mathbb{Q}/\mathbb{Z}$ is formally smooth. □

Definition (2.3.10). *Let $(R, \mathfrak{m})$ be a noetherian local ring. The order of an ideal $0 \neq \mathfrak{r} \subseteq R$ is defined as*

$$\nu_R(\mathfrak{r}) := \max\{n \in \mathbb{N}_0 | \mathfrak{r} \subseteq \mathfrak{m}^n\}.$$

For a point x of a scheme X we write ν_x instead of $\nu_{\mathcal{O}_{X,x}}$.

From (2.3.9) we get the following application:

Corollary (2.3.11). *Consider an ideal I in a polynomial ring $S = K[X_1, ..., X_n]$, $X = \mathrm{Spec}(S)$, over a field K. Denote by k either a perfect subfield (e.g. the prime field) of K or $k = \mathbb{Z}$. Then for any $n \geq 1$ we have*

$$\{x \in X | \nu_x(I_x) \geq n\} = V(\langle \mathrm{Diff}_k^{\leq n-1}(S)(I)\rangle).$$

For example, if $I = (f)$ is a principal ideal, we get

$$\mathrm{Sing}(\mathrm{Spec}(S/(f))) = V(\langle f, \mathrm{Der}_k(S)(f)\rangle).$$

The rest of this section is devoted to the proof of the Jacobian criteria. We are going to recall some facts from [EGA] and remind the reader of the notations from (2.1.4).

Lemma (2.3.12). *Let B be an A-algebra and $\mathfrak{b}$ an ideal of B. Suppose that B and $B/\mathfrak{b}$ are formally smooth A-algebras. Then $\mathfrak{b}/\mathfrak{b}^2$ is a projective $B/\mathfrak{b}$-module and the canonical homomorphism*

$$\mathrm{Sym}_{B/\mathfrak{b}}(\mathfrak{b}/\mathfrak{b}^2) \to \mathrm{gr}_\mathfrak{b}(B)$$

is an isomorphism.

Proof. B is a formally smooth A-algebra also for the $\mathfrak{b}$-preadic topology on B (e.g. see [EGA, 0_{IV} (19.3.8)]). Then the lemma follows immediately from [EGA, 0_{IV} (19.5.4)]. □

Lemma (2.3.13). *Let B be a formally smooth A-algebra. Then $\Omega^1_{B/A}$ and all $\mathcal{P}^n_{B/A}$ are projective B-modules and the canonical map*

$$\mathrm{Sym}_B(\Omega^1_{B/A}) \longrightarrow \bigoplus_{i\geq 0} I^i_{B/A}/I^{i+1}_{B/A}$$

is an isomorphism of graded B-algebras.

Proof. In fact the isomophism and the projectivity of $\Omega^1_{B/A}$ follow immediately from (the proof of) [EGA, IV (16.10.2)] . Then $\mathrm{Sym}_B(\Omega^1_{B/A})$ also is projective (e.g. see [BA, chap. III, §6, no. 6, Corollary after Theorem 1]) and so all $I^i_{B/A}/I^{i+1}_{B/A}$ are projective. From the canonical exact sequences

$$0 \to I^i_{B/A}/I^{i+1}_{B/A} \to \mathcal{P}^i_{B/A} \to \mathcal{P}^{i-1}_{B/A} \to 0 \qquad \text{(2.3.13.A)}$$

follows immediately the projectivity of the $\mathcal{P}^i_{B/A}$ by induction on i. □

Lemma (2.3.14) (Jacobian criterion of formal smoothness)**.** *Let B be an A-algebra and $\mathfrak{b}$ an ideal of B. Suppose that B and $B/\mathfrak{b}$ are formally smooth A-algebras. Then the canonical morphism*

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B B/\mathfrak{b}, \qquad \bar{b} \mapsto d_{B/A}(b) \otimes 1$$

is injective. It induces an injective morphism

$$\mathrm{Sym}_{B/\mathfrak{b}}(\mathfrak{b}/\mathfrak{b}^2) \to \mathrm{Sym}_B(\Omega^1_{B/A}) \otimes_B B/\mathfrak{b}.$$

Proof. The injectivity of the first morphism is part of theorem [EGA, 0_{IV} (22.6.1)]. In fact this morphism is left invertible since we are dealing with the discrete topologies (see [EGA, 0_{IV} (19.1.5)]). Therefore $\mathfrak{b}/\mathfrak{b}^2$ is a direct factor in $\Omega^1_{B/A} \otimes_B B/\mathfrak{b}$ and so the injectivity is inherited to the symmetric algebras (e.g. see [BA, chap. III, §6, no. 2, comment after Proposition 4]). Finally $\mathrm{Sym}_{B/\mathfrak{b}}(\Omega^1_{B/A} \otimes_B B/\mathfrak{b}) \cong \mathrm{Sym}_B(\Omega^1_{B/A}) \otimes_B B/\mathfrak{b}$ (e.g. see [BA, chap. III, §6, no. 4, Proposition 7]). □

Now we bring this together with the modules $\mathcal{P}^n_{B/A}$.

(2.3.15) Let B be an A-algebra and $\mathfrak{b}$ an ideal of B. Then for every $n \geq 0$ there is a unique A-linear map j_n making the following diagram commutative:

$$\begin{array}{ccc} B & \xrightarrow{d^n_{B/A}} & \mathcal{P}^n_{B/A} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes 1} \\ B/\mathfrak{b}^{n+1} & \xrightarrow{j_n} & \mathcal{P}^n_{B/A} \otimes_B B/\mathfrak{b} \end{array}$$

In fact $(\mathrm{Id} \otimes 1)d^n_{B/A}(\mathfrak{b}^{n+1}) = 0$. To see this take $b_1, ..., b_{n+1} \in \mathfrak{b}$. For $d(b_i) = 1 \otimes b_i - b_i \otimes 1$ we have $d(b_1) \cdots d(b_{n+1}) \in I^{n+1}_{B/A}$ and therefore this term is zero in $\mathcal{P}^n_{B/A} \otimes_B B/\mathfrak{b}$. On the other hand expanding the term results in $(1 \otimes b_1 \cdots b_{n+1}) \otimes 1 = (\mathrm{Id} \otimes 1)(d^n_{B/A})(b_1 \cdots b_{n+1})$.

Proposition (2.3.16). *Let the situation be as in (2.3.15) and suppose that B and $B/\mathfrak{b}$ are formally smooth over A. Then the map j_n in the above diagram is injective.*

Proof. We are going to explain the morphisms and their properties in the following commutative diagram which can be considered for everey $0 \leq l \leq n$:

$$\begin{array}{ccccccc}
\mathfrak{b}^l/\mathfrak{b}^{n+1} & \hookrightarrow & B/\mathfrak{b}^{n+1} & \xrightarrow{j_n} & \mathcal{P}^n_{B/A} \otimes_B B/\mathfrak{b} & \longrightarrow & \mathcal{P}^l_{B/A} \otimes_B B/\mathfrak{b} \\
\downarrow & & & & & & \uparrow \delta \\
\mathfrak{b}^l/\mathfrak{b}^{l+1} & \xrightarrow[\sim]{\alpha} & \mathrm{Sym}^l_B(\mathfrak{b}/\mathfrak{b}^2) & \xhookrightarrow{\beta} & \mathrm{Sym}^l_B(\Omega^1_{B/A}) \otimes_B B/\mathfrak{b} & \xrightarrow[\sim]{\gamma} & (I^l_{B/A}/I^{l+1}_{B/A}) \otimes_B B/\mathfrak{b}
\end{array}$$

The unnamed arrows are the obvious inclusions resp. projections. Once this diagram is established the claim is obvious. α comes from (2.3.12) and γ from (2.3.13). The injective morphism β was derived in the Jacobian criterion (2.3.14). The exact sequences (2.3.13.A) split and therefore remain exact after tensoring with $B/\mathfrak{b}$ wich yields δ. The commutativity is now easily seen from the explicit descriptions of the morphisms. □

Putting everything together we get

Lemma (2.3.17). *Let B be an A-algebra and $\mathfrak{b}$ an ideal of B such that B and $B/\mathfrak{b}$ are formally smooth A-algebras. Then the following are equivalent for an element $b \in B$ and some $n \geq 1$:*

(i) $\mathrm{Diff}^{n-1}_A(B)(b) \subseteq \mathfrak{b}$,

(ii) $\mathrm{Hom}_B\left(\mathcal{P}^{n-1}_{B/A}, B\right)\left(d^{n-1}_{B/A}(b)\right) \subseteq \mathfrak{b}$,

(iii) $j_{n-1}(b \bmod \mathfrak{b}^n) = 0$,

(iv) $b \in \mathfrak{b}^n$.

Proof. (i) implies (ii) by the universal property of differential operators (2.1.5). Assume that (ii) holds and that $j_{n-1}(b \bmod \mathfrak{b}^n) \neq 0$. We showed in (2.3.13) that $\mathcal{P}^{n-1}_{B/A}$ is a projective B-module. So $\mathcal{P}^{n-1}_{B/A} \otimes_B B/\mathfrak{b}$ is a projective $B/\mathfrak{b}$-module and there exists a $\varphi \in \mathrm{Hom}_{B/\mathfrak{b}}(\mathcal{P}^{n-1}_{B/A} \otimes_B B/\mathfrak{b}, B/\mathfrak{b})$ which sends $j_{n-1}(b \bmod \mathfrak{b}^n)$ not to zero. Therefore we get a $\varphi' \in \mathrm{Hom}_B\left(\mathcal{P}^{n-1}_{B/A}, B/\mathfrak{b}\right)$ which sends $d^{n-1}_{B/A}(b)$ not to zero. By the projectivity of $\mathcal{P}^{n-1}_{B/A}$ we also get a $\varphi'' \in \mathrm{Hom}_B\left(\mathcal{P}^{n-1}_{B/A}, B\right)$ which sends $d^{n-1}_{B/A}(b)$ not to $\mathfrak{b}$ and this contradicts (ii). (iii) implies (iv) by proposition (2.3.16). Finally (iv) implies (i) as was pointed out in (2.3.1). □

Proof of the Jacobian criteria (2.3.4) and (2.3.5). Lemma (2.3.17) now immediately implies (2.3.4). We suppose that $\mathfrak{p}$ is a prime ideal of B. Then $b \in \mathfrak{p}^{(n)}$ implies $\mathrm{Diff}_A^{n-1}(B)(b) \subseteq \mathfrak{p}$ as was shown in (2.3.2). By the universal property of differential operators (2.1.5) this in turn yields

$$\mathrm{Hom}_B\left(\mathcal{P}_{B/A}^{n-1}, B\right)\left(d_{B/A}^{n-1}(b)\right) \subseteq \mathfrak{p}.$$

Note that $\mathcal{P}_{B_\mathfrak{p}/A}^{n-1} \cong (\mathcal{P}_{B/A}^{n-1})_\mathfrak{p}$ (see [EGA, IV (16.4.14)]) and $\mathcal{P}_{B/A}^{n-1}$ is projective by lemma (2.3.13). Therefore the following lemma (2.3.18) implies

$$\mathrm{Hom}_{B_\mathfrak{p}}\left(\mathcal{P}_{B_\mathfrak{p}/A}^{n-1}, B_\mathfrak{p}\right)\left(d_{B_\mathfrak{p}/A}^{n-1}(b/1)\right) \subseteq \mathfrak{p}_\mathfrak{p}.$$

Now we use (ii) $\Rightarrow$ (iv) of (2.3.17) for the local ring $B_\mathfrak{p}$ and get $b/1 \in \mathfrak{p}_\mathfrak{p}^n$ and therefore $b \in \mathfrak{p}^{(n)}$ again. This is possible since $B_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} = \mathrm{Quot}(B/\mathfrak{p})$ and $B_\mathfrak{p}$ are formally smooth over A; the first by assumption and the latter by (2.3.7) since B is formally smooth over A. □

Lemma (2.3.18). *Let C be a ring and $T \subseteq C$ a muliplicatively closed subset. Let $M \subseteq N$ and $P \subseteq Q$ be C-modules. If N is a projective C-module we have*

$$\mathrm{Hom}_C(N,Q)M \subseteq P \quad \Longrightarrow \quad \mathrm{Hom}_{T^{-1}C}(T^{-1}N, T^{-1}Q)T^{-1}M \subseteq T^{-1}P.$$

Proof. Since N is projective we have

$$\mathrm{Hom}_C(N,Q)M \subseteq P \quad \Longrightarrow \quad \mathrm{Hom}_C(N,Q/P)M = 0$$

and we can take $P = 0 = T^{-1}P$ from the beginning. Further we can suppose that M is finitely generated and even that $M =< m >_C$ is generated by one element. First suppose that N is a free C-module. Then we can assume that N is finitely generated and get $T^{-1}\,\mathrm{Hom}_C(N,Q) \cong \mathrm{Hom}_{T^{-1}C}(T^{-1}N, T^{-1}Q)$ (see [BAC, chap. II, §2, no. 7, Proposition 19 (i)]). This concludes the proof in the case that N is a free C-module. If N is projective there exists a C-module F such that $N \oplus F$ is free. Then $T^{-1}N \oplus T^{-1}F$ is a free $T^{-1}C$-module. Regard M and $T^{-1}M$ as submodules of $N \oplus F$ resp. $T^{-1}N \oplus T^{-1}F$. We have $\mathrm{Hom}_C(N \oplus F, Q)M = 0$ since $\mathrm{Hom}_C(N \oplus F, Q) = \mathrm{Hom}_C(N,Q) \oplus \mathrm{Hom}_C(F,Q)$. So by the above argument we have $\mathrm{Hom}_{T^{-1}C}(T^{-1}N \oplus T^{-1}F, T^{-1}Q)T^{-1}M = 0$ and therefore $\mathrm{Hom}_{T^{-1}C}(T^{-1}N, T^{-1}Q)T^{-1}M = 0$. □

3 Groups

One aim of this chapter is to give easy access to the computation of the differential operators on a polynomial ring in (3.3.1). These differential operators are well known (e.g. see [EGA, IV 16.8, 16.11], [Gi, III §1], [H5, 2.]). Our access via groups may be less well known. We develop a new basis of these differential operators with respect to additive polynomials in (3.3.2). We also will characterize the homogeneous additive subgroups of affine space, which we just will call groups for simplicity ((3.3.8) f.). The ridges (ch. 6) as well as the Hironaka schemes (ch. 9) and our refinement for them (ch. 10) all are such groups. We complete this chapter with some technical preparations concerning the rings of invariants of groups in 3.4.

3.1 Cogroups

Let us first recall cogroups. We are not interested in a broad class of them. Therefore we use the notion of a cogroup for a specific type of cogroup. While mainly interested in the case of algebras of finite type over a field, we will begin in a more general setting.

(3.1.1) Throughout this section let A be a ring and $B = \bigoplus_{n\geq 0} B_n$ a graded A-algebra with $B_0 = A$ which is generated as an A-algebra by elements of degree 1. In this situation the projection $\eta : B \to B_0 = A$ is a homomorphism of A-algebras. Another homomorphism of A-algebras $\iota : B \to B$ which respects the graduation is given by

$$\iota((b_n)_{n\in\mathbb{N}}) = ((-1)^n b_n)_{n\in\mathbb{N}}.$$

Evidently ι is A-linear and further it respects multiplication:

$$\iota\left((b_n)_{n\in\mathbb{N}}\right) \cdot \iota((b'_n)_{n\in\mathbb{N}}) = \left(\sum_{i+j=n} (-1)^i b_i (-1)^j b'_j\right)_{n\in\mathbb{N}} = \iota((b_n)_{n\in\mathbb{N}} \cdot (b'_n)_{n\in\mathbb{N}}).$$

Definition (3.1.2). *A* ***coproduct*** *on B as above is a morphism of graded A-algebras*

$$\Delta : B \to B \otimes_A B$$

such that for every $b \in B_1$

$$\Delta(b) = b \otimes 1 + 1 \otimes b. \tag{3.1.2.A}$$

*B admits at most one coproduct as it is generated by B_1 as an A-algebra. If B admits a coproduct, we call it an A-**cogroup**. We are considering a special class of Hopf algebras here. An element $b \in B$ with the property (3.1.2.A) is called **primitive**. In particular all elements of B_1 are primitive. A morphism of A-cogroups just is a morphism of graded A-algebras.*

Example (3.1.3). Let M be an A-module. Then $\mathrm{Sym}_A(M)$ is an A-cogroup. In particular polynomial rings over A with the standard graduation are A-cogroups.

Lemma (3.1.4). *Let B be an A-cogroup and $\mathfrak{b} \subseteq B_+$ a homogeneous ideal. Then the following are equivalent:*

(i) *$B/\mathfrak{b}$ is an A-cogroup.*

(ii) *For all homogeneous elements $b \in \mathfrak{b}$ we have*

$$\Delta(b) \in \mathfrak{b} \otimes_A B + B \otimes_A \mathfrak{b},$$

where the tensor products stand for their natural images in $B \otimes_A B$.

(iii) *$\mathfrak{b}$ is generated by homogeneous elements b with $\Delta(b) \in \mathfrak{b} \otimes_A B + B \otimes_A \mathfrak{b}$.*

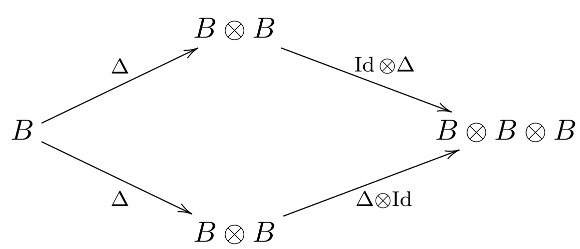
Proof. $B/\mathfrak{b}$ is an A-cogroup if and only if $\Delta : B \to B \otimes B$ factors to $B/\mathfrak{b} \to B/\mathfrak{b} \otimes B/\mathfrak{b}$. This means $\Delta(\mathfrak{b}) \subseteq \ker(\beta \otimes \beta)$, where $\beta : B \to B/\mathfrak{b}$ is the projection, which is equivalent to (ii). The equivalence of (ii) and (iii) is clear since Δ is a homomorphism of rings. □

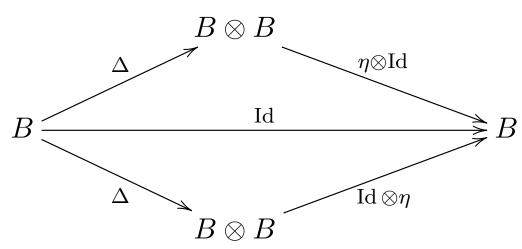
Corollary (3.1.5). *Let $\mathfrak{b}$ be an ideal of an A-cogroup B. If $\mathfrak{b}$ is generated by homogeneous primitive elements, then $B/\mathfrak{b}$ is an A-cogroup.*

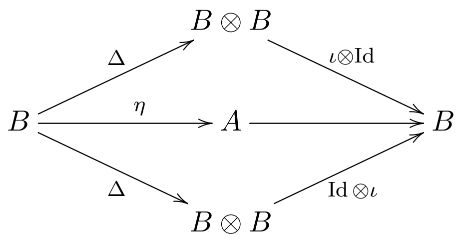
Proof. By definition of primitivity criterion (iii) of (3.1.4) is fulfilled. □

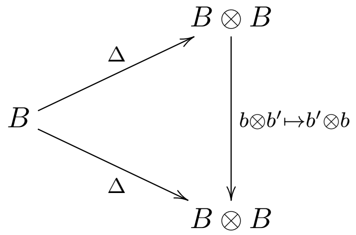
We are going to study under which circumstances the condition in (3.1.5) is necessary if $B/\mathfrak{b}$ is a cogroup in (3.3.5) ff.. Now we focus on the algebraic structure of a cogroup (cf. [Sc, 11], [KS, 8.1]).

Lemma (3.1.6). *If B is an A-cogroup, then the following diagrams commute*









where all tensor products are taken over A.

Proof. Since all arrows are A-algebra homomorphisms and all diagrams have B as source, it is enough to check the commutativities on B_1, which is easy. □

We now come to the group valued functors associated to cogroups.

Corollary (3.1.7). *Assume that B is an A-cogroup. For all A-algebras C the set of A-algebra homomorphisms $\mathrm{Hom}_A(B,C)$ becomes an abelian group with multiplication*

$$\mathrm{Hom}_A(B,C) \times \mathrm{Hom}_A(B,C) \to \mathrm{Hom}_A(B,C), \qquad (\varphi,\psi) \mapsto (B \xrightarrow{\Delta} B \otimes B \xrightarrow{\varphi\otimes\psi} C),$$

inversion

$$\mathrm{Hom}_A(B,C) \to \mathrm{Hom}_A(B,C), \qquad \varphi \mapsto (B \xrightarrow{\iota} B \xrightarrow{\varphi} C)$$

and neutral element

$$B \xrightarrow{\eta} A \to C.$$

Proof. All axioms of an abelian group follow from (3.1.6). □

Example (3.1.8). Let B be an A-cogroup. In the group $\mathrm{Hom}_A(B,B)$ we have the relation

$$\mathrm{Id} \circ \iota = \eta = \iota \circ \mathrm{Id},$$

i.e. the elements Id and ι are inverse to each other. This follows from the third diagram of (3.1.6).

Lemma (3.1.9). *Let B be an A-cogroup, C a B-algebra and M a C-module. Then M is by scalar restriction also an A-module. The group $\mathrm{Hom}_A(B,C)$ operates on the set $\mathrm{Hom}_A(B,M)$ via*

$$\mathrm{Hom}_A(B,C) \times \mathrm{Hom}_A(B,M) \to \mathrm{Hom}_A(B,M),$$

$$(\varphi,\zeta) \mapsto (B \xrightarrow{\Delta} B \otimes B \xrightarrow{\varphi\otimes\zeta} C \otimes M \xrightarrow[c\otimes m\mapsto cm]{\mu} M).$$

Proof. We verify the axioms of an operation again with (3.1.6). □

3.2 Differential operators on cogroups

We introduced differential operators in chapter 2. In (3.2.3) we will show how they can be computed on cogroups. Analogous proceedings are well known at least for derivations. Throughout this section B is an A-cogroup and M is a B-module.

Lemma (3.2.1). *The two maps*

$$\omega, \omega^* \in \mathrm{End}_B(\mathrm{Hom}_A(B, M))$$

defined for $\lambda \in \mathrm{Hom}_A(B, M)$ *by*

$$\omega(\lambda) := \ (B \xrightarrow{\Delta} B \otimes B \xrightarrow{\mathrm{Id}\otimes\lambda} B \otimes M \to M),$$

$$\omega^*(\lambda) := \ (B \xrightarrow{\Delta} B \otimes B \xrightarrow{\iota\otimes\lambda} B \otimes M \to M)$$

are inverse to each other. For elements $b = (b_1, ..., b_r)$ *of* B_1 *and* $\boldsymbol{1} = (1, ..., 1) \in \Lambda := \mathbb{N}^r$ *we have the explicit formulas*

$$\omega(\lambda)(b^{\boldsymbol{1}}) = \sum_{L,M\in\Lambda, L+M=\boldsymbol{1}} b^L \lambda(b^M),$$

$$\omega^*(\lambda)(b^{\boldsymbol{1}}) = \sum_{L,M\in\Lambda, L+M=\boldsymbol{1}} (-1)^{|L|} b^L \lambda(b^M).$$

Proof. The maps ω and ω^* are the actions of $\mathrm{Id}, \iota \in \mathrm{Hom}_A(B, B)$ on $\mathrm{Hom}_A(B, M)$ as in (3.1.9) and Id and ι are inverse to each other by (3.1.8). The formulas immediately come by definition and (3.1.2.A). □

Definition (3.2.2). *For* $n \in \mathbb{N}$ *we have an inclusion*

$$\mathrm{Hom}^n_A(B, M) := \mathrm{Hom}_A(B_n, M) \subseteq \mathrm{Hom}_A(B, M)$$

by extending $\lambda : B_n \to M$ *with* $\lambda(B_m) = 0$ *for* $m \neq n$. *We also will use*

$$\mathrm{Hom}^{\leq n}_A(B, M) := \bigoplus_{d=0}^{n} \mathrm{Hom}^d_A(B, M) \subseteq \mathrm{Hom}_A(B, M)$$

and

$$\mathrm{Hom}^b_A(B, M) := \bigcup_{n\in\mathbb{N}} \mathrm{Hom}^{\leq n}_A(B, M) = \bigoplus_{n\geq 0} \mathrm{Hom}^n_A(B, M) \subseteq \mathrm{Hom}_A(B, M),$$

where b *stands for bounded. The last inclusion will usually not be an equality.*

Theorem (3.2.3). ω *from (3.2.1) yields an isomorphism*

$$\omega : \mathrm{Hom}^b_A(B, M) \to \mathrm{Diff}_A(B, M)$$

of B-modules. In particular for every $n \in \mathbb{N}$ we have the isomorphism

$$\omega : \mathrm{Hom}_A^{\leq n}(B, M) \to \mathrm{Diff}_A^{\leq n}(B, M)$$

of B-modules.

Proof. It is enough to prove the lower isomorphisms for all $n \in \mathbb{N}$. By (3.2.1) it remains to show the first equality in

$$\mathrm{Diff}_A^{\leq n}(B, M) = \left\{ D \in \mathrm{Hom}_A(B, M) \,\middle|\, \omega^*(D) \left(\bigoplus_{d>n} B_d \right) = 0 \right\} =$$

$$= \{D \in \mathrm{Hom}_A(B, M) | \omega^*(D) \in \mathrm{Hom}_A^{\leq n}(B, M)\} = \omega(\mathrm{Hom}_A^{\leq n}(B, M)).$$

If $D \in \mathrm{Diff}_A^{\leq n}(B, M)$, then it satisfies the Leibniz rule (2.1.2.A) of order d for all $d \geq n$ (see (2.1.8)), i.e. for elements $b = (b_0, b_1, ..., b_d)$ of B and $\mathbf{1} = (1, ..., 1) \in \Lambda := \mathbb{N}^{d+1}$

(3.2.3.A)
$$\sum_{L,M \in \Lambda, L+M=\mathbf{1}} (-1)^{|L|} b^L D(b^M) = 0.$$

By linearity of $\omega^*(D)$ and the last formula of (3.2.1) we must have $\omega^*(D)(B_d) = 0$ for all $d > n$ since B is generated as an A-algebra by B_1. Let on the other hand $D \in \mathrm{Hom}_A(B, M)$ satisfy $\omega^*(D)(B_d) = 0$ for all $d > n$. Then (3.2.3.A) holds whenever $b_0, ..., b_d \in B_1$ and $d \geq n$ by (3.2.1) again. We show that this implies $D \in \mathrm{Diff}_A^{\leq n}(B, M)$ by induction on n, where the case $n = -1$ is trivial. For $b \in B_1$ we see that $[D, b]$ satisfies the statement for $n - 1$ and is therefore by the induction hypothesis a differential operator of order $\leq n - 1$. By criterion (2.1.7) it remains to show that

$$C := \{c \in B | [D, c] \in \mathrm{Diff}_A^{\leq n-1}(B, M)\}$$

is a ring since B is generated by B_1 and $A \subseteq C$ because D is A-linear. C is obviously stable under addition. For $c, c' \in C$ and $b \in B$ the calculation

$$[D, cc'](b) = D(cc'b) - cD(c'b) + cD(c'b) - cc'D(b) = [D, c](c'b) + c[D, c'](b)$$

shows that $[D, cc'] \in \mathrm{Diff}_A^{\leq n-1}(B, M)$. □

There are two decompositions of differential operators we want to mention.

(3.2.4) (Order decomposition) Using the isomorphism from (3.2.3) we define the B-submodules

$$\mathrm{Diff}_A^n(B, M) := \omega(\mathrm{Hom}_A^n(B, M)) \subseteq \mathrm{Diff}_A^{\leq n}(B, M)$$

and call them differential operators of order n. So every $D \in \mathrm{Diff}_A(B, M)$ with order $\leq n$ can be written uniquely as a sum

$$D = D_0 + \cdots + D_n, \qquad D_i \in \mathrm{Diff}_A^n(B, M).$$

In other words

$$\mathrm{Diff}_A^{\leq n}(B,M) = \bigoplus_{d\leq n} \mathrm{Diff}_A^d(B,M), \qquad \mathrm{Diff}_A(B,M) = \bigoplus_{d\in\mathbb{N}} \mathrm{Diff}_A^d(B,M).$$

There is also a degree decomposition in the case $M = B$, which is related to $\omega(\mathrm{Hom}_A(B_n, B_m))$ (cf. [H5, 2.]). We do not go into all details here but only present one certain subclass of operators that we will call of degree 0. Exactly these operators will be used to compute the ridge of a cone (see 6.1).

(3.2.5) (Operators of degree 0) We regard $\bigoplus_{d\geq 0} \mathrm{Hom}_A(B_d, A) \subseteq \mathrm{Hom}_A(B,B)$ and define the differential operators of degree 0 on $B \to B$ to be $\mathrm{Diff}_{A,0}(B) := \omega(\bigoplus_{d\geq 0} \mathrm{Hom}_A(B_d, A))$. This is compatible with the order decomposition and we can define $\mathrm{Diff}^n_{A,0}(B)$ and $\mathrm{Diff}^{\leq n}_{A,0}(B)$ analogously and $\mathrm{Diff}_{A,0}(B)$ is a graded A-module.

Lemma (3.2.6). $\mathrm{Diff}_{A,0}(B)$ *is a commutative graded A-algebra with unit.*

Proof. For $\lambda, \mu \in \bigoplus_{d\geq 0} \mathrm{Hom}_A(B_d, A)$ we find

$$\omega(\lambda)\circ\omega(\mu) = B \xrightarrow{\Delta} B\otimes B \xrightarrow{\mathrm{Id}\otimes\mu} B \xrightarrow{\Delta} B\otimes B \xrightarrow{\mathrm{Id}\otimes\lambda} B =$$

$$= B \xrightarrow{\Delta} B\otimes B \xrightarrow{\Delta\otimes\mathrm{Id}} B\otimes B\otimes B \xrightarrow{\mathrm{Id}\otimes\lambda\otimes\mu} B =$$

$$\overset{(3.1.6)}{=} B \xrightarrow{\Delta} B\otimes B \xrightarrow{\mathrm{Id}\otimes((\lambda\otimes\mu)\circ\Delta)} B = \omega((\lambda\otimes\mu)\circ\Delta) \overset{(3.1.6)}{=} \omega((\mu\otimes\lambda)\circ\Delta).$$

This multiplication respects the graduation: If $\lambda \in \mathrm{Hom}_A(B_k, A), \mu \in \mathrm{Hom}_A(B_l, A)$, then one easily sees that $(\lambda\otimes\mu)\circ\Delta \in \mathrm{Hom}_A(B_{k+l}, A)$. It is clear that Id is the unit of this ring. □

3.3 Examples

(3.2.3) allows us to compute differential operators without much effort. Let us begin with the standard differential operators on a polynomial ring.

Example (3.3.1). $B = A[X_i]_{i\in I}$ is an A-cogroup for an arbitrary ring A. Consider the A-linear maps $\lambda_M : B \to B$ defined by $\lambda_M(X^N) = \delta_{M,N}$ (Kronecker delta) for $M, N \in \Lambda := \mathbb{N}^{(I)}$. Then every $\lambda \in \mathrm{Hom}_A(B,B)$ can be written as $\lambda = \sum_{L\in\Lambda} b_L\lambda_L$ for unique $\lambda(X^L) = b_L \in B$ (note that this sum may be infinite). Here λ lies in $\mathrm{Hom}_A^{\leq n}(B,B)$ iff $\lambda = \sum_{|L|\leq n} b_L\lambda_L$. With (3.2.1) we get

$$\omega(\lambda)(X^L) = \sum_{M\leq_c L} \binom{L}{M} X^{L-M}\lambda(X^M) = \sum_{M\leq_c L} \binom{L}{M} b_M X^{L-M}$$

and (3.2.3) implicates that every $D \in \mathrm{Diff}_A^{\leq n}(B)$ has an expression in terms of the **standard differential operators** $D_M = \omega(\lambda_M), M \in \Lambda$ as a (infinite) sum

$$D = \sum_{|M|\leq n} b_M D_M,$$

where the standard differential operators satisfy

$$(3.3.1.A) \qquad D_M(X^L) = \binom{L}{M} X^{M-L}.$$

D is of order n iff $D = \sum_{|M|=n} b_M D_M$ and D is of degree 0 iff $b_M \in A$ for all M. For any $b \in B$ we have

$$(3.3.1.B) \qquad \Delta(b) = \sum_{M\in\Lambda} D_M(b) \otimes X^M = \sum_{M\in\Lambda} X^M \otimes D_M(b).$$

To see this we can assume, since all terms are A-linear in b, that $b = X^L$; then (3.3.1.B) is apparent.

The same differential operators can be expressed in a different basis, namely with respect to some additive polynomials, as follows:

Example (3.3.2). Let $S = k[X_1, ..., X_n]$, k a field of positive characteristic and σ a homogeneous $k[F]$-independent system of additive polynomials in S (e.g. an arranged system). Then, after renumbering the variables, we have with the notations of (1.4.5) differential operators $D_{N,M} \in \mathrm{Diff}_k^{\leq qN+|M|}(S)$ for $N \in \Lambda', M \in \Lambda''$ with the property

$$(3.3.2.A) \qquad D_{N,M}(\sigma^{N'} X^{M'}) = \binom{N'}{N}\binom{M'}{M} \sigma^{N'-N} X^{M'-M}$$

for $N' \in \Lambda', M' \in \Lambda''$. Every $D \in \mathrm{Diff}_k(S)$ is a unique S-linear combination of these $D_{N,M}$. The $D_{N,M}$ also are a k-basis of $\mathrm{Diff}_{k,0}(S)$. We further have for all $f \in S$

$$(3.3.2.B) \qquad \Delta(f) = \sum_{N\in\Lambda', M\in\Lambda''} D_{N,M}(f) \otimes \sigma^N X^M.$$

Proof. By (1.4.5) the $(\sigma^N X^M)_{N\in\Lambda', M\in\Lambda''}$ are a k-basis of S. We take $\lambda_{N,M}$ to be the elements of the corresponding dual basis and set $D_{N,M} := \omega(\lambda_{N,M})$. For $N' \in \Lambda'$ and $M' \in \Lambda''$ we use that all σ_i are primitive (cf. (3.3.5)) and conclude

$$D_{N,M}(\sigma^{N'} X^{M'}) = (\mathrm{Id} \otimes \lambda_{N,M}) \left(\Delta(\sigma)^{N'} \Delta(X)^{M'}\right) =$$

$$= \sum_{N'',M''} \binom{N'}{N''}\binom{M'}{M''} \lambda_{N,M}(\sigma^{N''} X^{M''}) \sigma^{N'-N''} X^{M'-M''} =$$

$$= \binom{N'}{N}\binom{M'}{M} \sigma^{N'-N} X^{M'-M}.$$

For the last formula we can assume by linearity that $f = \sigma^{N'} X^{M'}$ which makes it apparent. For the other claims use (3.2.3) and (3.2.5). □

We are also able to compute differential operators on quotients of polynomial rings as in the following example:

Example (3.3.3). In the situation of (3.3.2) $B := S/\langle\sigma\rangle_S$ is a k-cogroup by (3.3.5) and (3.1.5). As seen in (1.4.5) the $\overline{X^M}, M \in \Lambda''$ form a k-basis of B, denote the dual basis by $\overline{\lambda_M}$. Analogously to (3.3.2) we get a B-basis of $\mathrm{Diff}_k(B)$ (resp. a k-basis of $\mathrm{Diff}_{k,0}(B)$) by differential operators $\overline{D_M} = \omega(\overline{\lambda_M}) \in \mathrm{Diff}_k^{\leq |M|}(B)$ for $M \in \Lambda''$. For $\overline{X^{M'}}, M' \in \Lambda''$ they have the property

$$\overline{D_M}(\overline{X^{M'}}) = \binom{M'}{M}\overline{X^{M'-M}}.$$

Lemma (3.3.4). *Let $B = A[X_i]_{i\in I}$ as in (3.3.1). Then for all $L \in \Lambda$ and $b, b' \in B$ we have*

(3.3.4.A)
$$D_M(bb') = \sum_{L\leq_c M} D_L(b)D_{M-L}(b').$$

Proof. By (3.3.1.B) we find

$$D_M(bb') = (\mathrm{Id}\otimes\lambda_M)(\Delta(b)\Delta(b')) = \sum_{N,N'\in\Lambda} D_N(b)D_{N'}(b')\lambda_M(X^{N+N'})$$

and this gives (3.3.4.A). □

Now we are going to finish our discussion about primitive elements and the characterization of cogroups from (3.1.4) f..

Proposition (3.3.5). *Let $B = A[X_i]_{i\in I}$ and A a domain. For $0 \neq b \in B_n, n \neq 0$ the following are equivalent:*

(i) *b is primitive, i.e. $\Delta(b) = b\otimes 1 + 1 \otimes b$.*

(ii) *$D_M(b) = 0$ for all $M \in \Lambda$ with $0 < |M| < n$.*

(iii) *$b \in B_1$ or $n = p^l, l \geq 1$, $\mathrm{char}(A) = p$ is prime and $b = \sum_{i\in I} a_i X_i^{p^l}$ for $a_i \in A$.*

Proof. (i) ⇒ (ii): $D_M(b) = \omega(\lambda_M)(b) = (\mathrm{Id}\otimes\lambda_M)(b\otimes 1 + 1\otimes b) = b\cdot\lambda_M(1) + \lambda_M(b) = 0$. (ii) ⇒ (iii): Suppose that $n > 1$ and $b = \sum_{|N|=n} a_N X^N$ for $a_N \in A$. For a multiindex $N, |N| = n$ with two non zero entries we can write $N = N' + N''$ with $N', N'' \neq 0$ such that at every entry at most one of N' or N'' is not zero and $0 < |N'| < n$. By (ii) $0 = D_{N'}(b) = \sum_{|N|=n} a_N \binom{N}{N'} X^{N-N'}$. Since $\binom{N}{N'} = 1$, this implies $a_N = 0$. Thus $b = \sum_{i\in I} a_i X_i^n$. Again by applying differential operators as in (ii) we see that a_i is annihilated by the greatest common divisor of the binomial coefficients $\binom{n}{1}, ..., \binom{n}{n-1}$. This is well known to be 1 in the case that n is not a prime power. But then we would have $b = 0$. In the case $n = p^l, l \geq 1$ for a prime p this greatest common divisor is p and we have $pb = 0$. Since A is a domain and $b \neq 0$ we must have $p = \mathrm{char}(A)$. (iii) ⇒ (i) is obvious. □

Lemma (3.3.6). *Let $\beta : B \to B'$ be a surjective morphism of A-cogroups with kernel $\mathfrak{b} \subseteq B$. Then ω and ω^* induce inverse automorphisms of the B-module*

$$\{\lambda \in \mathrm{Hom}_A(B, B) | \lambda(\mathfrak{b}) \subseteq \mathfrak{b}\}.$$

If $\mathfrak{b}_0 = \cdots = \mathfrak{b}_{n-1} = 0$, *then* $\mathrm{Diff}_A^{\leq n-1}(B)(\mathfrak{b}) \subseteq \mathfrak{b}$.

Proof. $\lambda \in \mathrm{Hom}_A(B,B)$ with $\lambda(\mathfrak{b}) \subseteq \mathfrak{b}$ yields a $\lambda' \in \mathrm{Hom}_A(B',B')$ such that

$$\begin{array}{ccccc} B & \xrightarrow{\Delta} & B \otimes B & \xrightarrow{\mathrm{Id}\otimes\lambda} & B \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta\otimes\beta} & & \downarrow{\scriptstyle\beta} \\ B' & \xrightarrow[\Delta']{} & B' \otimes B' & \xrightarrow[\mathrm{Id}\otimes\lambda']{} & B' \end{array}$$

commutes. Thus $\omega(\lambda)(\mathfrak{b}) \subseteq \mathfrak{b}$ and by replacing Id with ι also $\omega^*(\lambda)(\mathfrak{b}) \subseteq \mathfrak{b}$. Let now $\mathfrak{b}_0 = \cdots = \mathfrak{b}_{n-1} = 0$ and $\lambda \in \mathrm{Hom}_A^{\leq n-1}(B,B)$. Then $\lambda(\mathfrak{b}) = 0$ and therefore $\omega(\lambda)(\mathfrak{b}) \subseteq \mathfrak{b}$. □

Lemma (3.3.7). *Let B be an A-cogroup and $\mathfrak{b} \subseteq B_+$ a homogeneous ideal such that $B/\mathfrak{b}$ also is an A-cogroup. For $b \in \mathfrak{b}_n, n > 1$ the following hold:*

(i) *If n is not a prime power, then $b \in B \cdot \mathfrak{b}_{\leq n-1}$.*

(ii) *If $n = p^l$ for some prime number p and $1 \leq l$, then $pb \in B \cdot \mathfrak{b}_{\leq n-1}$.*

Proof. By (3.1.4) (ii) we must have $\Delta(b) \in \mathfrak{b} \otimes B + B \otimes \mathfrak{b}$ and clearly $\Delta(b) \in \bigoplus_{i+j=n} B_i \otimes B_j$. With the projections $\pi_i : \bigoplus_{i+j=n} B_i \otimes B_j \to B_i \otimes B_{n-i}$ and the multiplication map $\mu : B \otimes_A B \to B$ we see that the composition

$$B_n \xrightarrow{\Delta} B \otimes_A B \xrightarrow{\pi_i} B_i \otimes_A B_{n-i} \xrightarrow{\mu} B_n$$

is the multiplication by $\binom{n}{i}$. In fact for elements $b = (b_1, ..., b_n)$ of B_1 and $\mathbf{1} = (1, ..., 1) \in \Lambda := \mathbb{N}^n$ we find

$$\mu(\pi_i(\Delta(b^{\mathbf{1}})) = \mu\left(\pi_i\left(\sum_{L+M=\mathbf{1}} b^L \otimes b^M\right)\right) = \sum_{L+M=\mathbf{1},|L|=i} b^{\mathbf{1}} = \binom{n}{i} b^{\mathbf{1}}.$$

For $0 < i < n$ we get $\binom{n}{i} b = \mu(\pi_i(\Delta(b))) \in B \cdot \mathfrak{b}_{n-i} + B \cdot \mathfrak{b}_i \subseteq B \cdot \mathfrak{b}_{\leq n-1}$. Now we use the fact that the greatest common divisor of $\binom{n}{1}, ..., \binom{n}{n-1}$ is 1 if n is not a prime power and is p if n is a power of the prime p. □

Proposition (3.3.8). *Let B be an A-cogroup.*

(i) *If $\mathbb{Q} \subseteq A$, then $B \cong \mathrm{Sym}_A(B_1)$.*

(ii) *If A is a field of characteristic $p > 0$ and B is of finite type over A, then $B \cong S/\mathfrak{s}$, where S is a polynomial ring of finite type over A and $\mathfrak{s}$ is an ideal generated by homogeneous additive polynomials.*

(iii) *If A is a field, B is of finite type over A and $\mathfrak{b}$ is a homogeneous ideal of B, then $B/\mathfrak{b}$ is an A-cogroup if and only if $\mathfrak{b}$ is generated by primitive elements.*

Proof. Let $\mathfrak{s}$ be the kernel of the morphism of A-cogroups $S := \mathrm{Sym}_A(B_1) \to B$. Clearly $\mathfrak{s}_0 = \mathfrak{s}_1 = 0$. In (i) $\mathfrak{s} = 0$ follows immediately from (3.3.7). In (ii) we let $\mathfrak{s}'$ be the ideal in S generated by all homogeneous primitive elements of $\mathfrak{s}$. By (3.3.5) this ideal is generated by additive polynomials, so we have to show $\mathfrak{s} = \mathfrak{s}'$. We may assume that $\mathfrak{s}' = \langle\sigma\rangle$ for a well arranged system σ of additive polynomials. By (3.1.5) also $S/\mathfrak{s}'$ is an A-cogroup. Assume that $\mathfrak{s}' \neq \mathfrak{s}$ and choose a homogeneous $f \in \mathfrak{s} \setminus \mathfrak{s}'$ with minimal degree $n > 1$. In $S/\mathfrak{s}'$ we have a basis representation $\overline{f} = \sum_{M\in\Lambda'',|M|=n} f_M \overline{X^M}, f_M \in A$ (see (1.4.4)). By (3.3.7) n is a power of p. Thus we find $f_M \neq 0$ for some M with two non zero entries and can choose $M = M' + M''$ such that $0 < |M'| < n$ and $\overline{D_{M'}}(\overline{f}) \neq 0$ (see (3.3.3)). But by (3.3.6) we come to the contradiction $\overline{D_{M'}}(\overline{f}) \in (\mathfrak{s}/\mathfrak{s}')_{n-|M'|} = 0$. (iii): It remains to show that for a cogroup $B/\mathfrak{b}$ the ideal $\mathfrak{b}$ is generated by primitive elements. This is clear in the case $\mathrm{char}(A) = 0$. Let $\mathrm{char}(A) > 0$ and $S = \mathrm{Sym}_A(B_1)$. Then we already proved that $\ker(S \to B \to B/\mathfrak{b})$ is generated by primitive elements of S and therefore $\mathfrak{b}$ is generated by primitive elements of B. □

(3.3.9) Let k be a field. Whenever B is a k-cogroup and B is of finite type over k we call $G = \mathrm{Spec}(B)$ a **group** over k. More precisely we should say homogeneous additive group scheme; note that by (3.1.7) the functor $\mathrm{Hom}_k(-, G)$ takes values in abelian groups and the graded structure of B defines an action of the multiplicative group over k on G. In fact we could have started our discussion taking this as a definition (cf. [Sc, 11.4]). G will be called a **vector space** if it is isomorphic to an affine space over k. Note that the group structure of G is independent of any embedding of G into a vector space. In (3.3.8) we showed that all groups are vector spaces in characteristic zero and we characterized the subgroups of a vector space in positive characteristic (cf. [Gi, I 5.4]).

3.4 Rings of invariants

Througout this section let k be a field of characteristic $p > 0$, $S = k[X_1, ..., X_n]$ and $\Lambda := \mathbb{N}^n$. Let L be the additive polynomials of S as in (1.4.1). We use differential operators as in (3.3.1).

Definition (3.4.1). *For a homogeneous ideal* $\mathfrak{a}$ *of* S *we define*

$$\mathcal{U}(\mathfrak{a}) := \{f \in S | \Delta(f) - 1 \otimes f \in \mathfrak{a} \otimes_k S\}.$$

If $S/\mathfrak{a}$ is a k-cogroup, then $\mathcal{U}(\mathfrak{a})$ is called the **ring of invariants** of the group $G = \mathrm{Spec}(S/\mathfrak{a})$ in the vector space $V = \mathrm{Spec}(S)$ (cf. [Gi, I 5.4.2 (4) ff.]). We need some technical characterizations as in [Gi, I 5.4.2 (6), 5.4.3] of such rings for later.

Lemma (3.4.2). *In (3.4.1)* $\mathcal{U}(\mathfrak{a})$ *is a graded* k*-subalgebra of* S *and* $L \cap \mathfrak{a} = L \cap \mathcal{U}(\mathfrak{a})$ *and* $\mathcal{U}_+(\mathfrak{a}) \subseteq \mathfrak{a}$. *For all* $m \geq 0$ *we have*

$$\mathcal{U}_m(\mathfrak{a}) = \{f \in S_m | \mathrm{Diff}_k^{\leq m-1}(S)(f) \subseteq \mathfrak{a}\}.$$

Proof. It is clear that $\mathcal{U}(\mathfrak{a})$ is a graded k-submodule of S. For $f, g \in \mathcal{U}(\mathfrak{a})$ set $f' := \Delta(f) - 1 \otimes f, g' := \Delta(g) - 1 \otimes g \in \mathfrak{a} \otimes_k S$ and then $f \cdot g \in \mathcal{U}(\mathfrak{a})$ is clear since

$$\Delta(fg) - 1 \otimes fg = (f' + 1 \otimes f) \cdot (g' + 1 \otimes g) - 1 \otimes fg \in \mathfrak{a} \otimes_k S.$$

For $f \in L$ we have $\Delta(f) - 1 \otimes f = f \otimes 1$, therefore $L \cap \mathfrak{a} = L \cap \mathcal{U}(\mathfrak{a})$. As in (3.3.1) we have for $f \in S_m$ that $\Delta(f) - 1 \otimes f = \sum_{|M|<m} D_M(f) \otimes X^M \in \mathcal{U}(\mathfrak{a})$ iff $D_M(f) \in \mathfrak{a}$ for all $|M| < m$ and these D_M are an S-basis of $\mathrm{Diff}_k^{\leq m-1}(S)$. For $f \in \mathcal{U}_+(\mathfrak{a})$ we have $f = D_0(f) \in \mathfrak{a}$. □

Proposition (3.4.3). *Let H be a graded k-subalgebra of S. The following are equivalent:*

(i) *There exists a subset T of S such that for all $m \geq 0$*

$$H_m = \{f \in S_m | \, \mathrm{Diff}_k^{\leq m-1}(S)(f) \subseteq T\}.$$

(ii) *$D_M H_m \subseteq H$ for all $M \in \Lambda$ with $|M| \leq m-1$ and all $m \in \mathbb{N}$.*

(iii) *$D_M H \subseteq H$ for all $M \in \Lambda$.*

(iv) *H is generated by additive polynomials.*

(v) *$H = k[\sigma]$ for a well arranged system of additive polynomials σ after possibly renumbering the variables.*

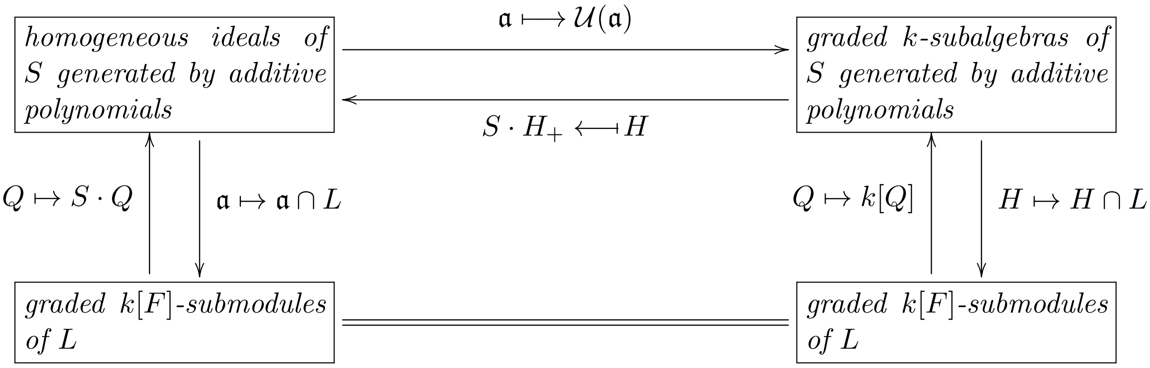
(vi) *$\mathcal{U}(SH_+) = H$.*

(vii) *There are homogeneous polynomials $f_1, ..., f_m \in S$ such that H is generated as a k-algebra by all $D_M(f_i)$ for $i = 1, ..., m$ and $M \in \Lambda$.*

Proof. (i) ⇒ (ii): Let $f \in H_m$ and $|M| \leq m-1$. Then $D_M(f)$ is homogeneous of degree $m - |M| \geq 1$ and by (2.1.9) $\mathrm{Diff}_k^{\leq m-|M|-1}(S) D_M(f) \subseteq \mathrm{Diff}_k^{\leq m-1}(S)(f) \subseteq T$ and therefore $D_M(f) \in H_{m-|M|}$. (ii) ⇒ (iii): Let $f \in H_m$. If $|M| \leq m-1$, then $D_M(f) \in H$ by (ii). If $|M| \geq m$, then $D_M(f) \in k = H_0$. (iii) ⇒ (iv): Let $K := k[H \cap L] \subseteq H$. Assume $K \subsetneq H$ and let m be minimal with $K_m \neq H_m$. Then $m \geq 2$ and we find $f = \sum_{|A|=m} f_A X^A \in H_m \setminus K_m$ with $f_A \in k$ and $f_A = 0$ whenever $A = \exp(g)$ for some $g \in K_m$; here and in the following all exponents are taken with respect to the lexicographical order, see 1.3. Let A be the largest exponent such that $f_A \neq 0$ and X^A is not additive. Let $A = (0, ..., 0, m_j, ..., m_n)$ with $m_j > 0$ and $m_j + \cdots + m_n = m$. If $m_j = m$, then there exists $0 < r < m$ with $0 \neq \binom{m}{r} = \binom{m}{m-r}$ since X^A is not additive. Then define $B := (0, ..., 0, r, 0, ..., 0)$ with r at position j. And if $m_j < m$ define $B := (0, ..., m_j, 0, ..., 0)$ with m_j at position j. Let $B' := A - B$. Then $0 < |B|, |B'| < |A| = m$ and in every case $D_B(X^A) \neq 0 \neq D_{B'}(X^A)$. But if X^C is additive with $C > A$, then $D_B(X^C) = D_{B'}(X^C) = 0$. Therefore $\exp(D_B(f)) = \exp(D_B(X^A)) = A - B$ and $\exp(D_{B'}(f)) = A - B'$. By (iii) we have $D_B(f), D_{B'}(f) \in H_{\leq m-1} = K_{\leq m-1}$. But then $D_B(f) \cdot D_{B'}(f) \in K$ is homogeneous of degree $m - |B| + m - |B'| = 2m - |A| = m$. Then $\exp(D_B(f) \cdot D_{B'}(f)) = \exp(D_B(f)) + \exp(D_{B'}(f)) = A - B + A - B' = A$ contradicts $f_A \neq 0$ by our special choice of f. Therefore $K = H$. (iv) ⇔ (v) is clear since $L \cap H$ is a $k[F]$-module (see (1.4.3) f.). (v) ⇒ (vi): $Q := H \cap L$ is a graded $k[F]$-module with $SH_+ = SQ$ and $L \cap SQ = Q$ by (1.4.7). By (3.4.2) and what is proved so far we know that $\mathcal{U}(SH_+)$ is generated by additive polynomials and by (3.4.2) we get $\mathcal{U}(SH_+) = k[L \cap \mathcal{U}(SH_+)] = k[L \cap SH_+] = k[L \cap SQ] = k[Q] = H$. (vi) ⇒ (i) follows from (3.4.2) for $\mathfrak{a} = SH_+$. (v) $\Rightarrow$ (vii): Take $(f_1, ..., f_m) = \sigma$. (vii) $\Rightarrow$ (iii) is clear from (3.3.4). $\square$

Note that with (3.4.2) and (3.4.3) we have proved that rings of invariants are generated by additive polynomials. The following corollary completes the train of thought we began in (1.4.6) f. At the same time it yields the connection between groups $G = \mathrm{Spec}(S/\mathfrak{a})$, their rings of invariants $\mathcal{U}(\mathfrak{a})$ and their Dieudonné modules $L/(\mathfrak{a} \cap L)$ (cf. [Od, 1.]).

Corollary (3.4.4). *In the following diagram all arrows are inclusion preserving bijections, the inverse is always the corresponding arrow in the opposite direction. The diagram commutes.*



If σ is a system of homogeneous additive polynomials (arranged or not), the objects $\langle\sigma\rangle_S$, $k[\sigma]$ and $\langle\sigma\rangle_{k[F]}$ correspond to each other under the bijections of the diagram.

Proof. For the bijections on the left see (1.4.7). For the upper arrows see (3.4.3) (vi) and note that $S \cdot \mathcal{U}_+(\mathfrak{a}) = \mathfrak{a}$ is clear since $\mathcal{U}_+(\mathfrak{a}) \subseteq \mathfrak{a}$ and $L \cap \mathfrak{a} \subseteq \mathcal{U}(\mathfrak{a})$ (see (3.4.2)). On the right side it is clear that $k[H \cap L] = H$ and $Q \subseteq k[Q] \cap L \subseteq SQ \cap L = Q$. It is obvious that all maps are inclusion preserving. For the commutativity we just have to check one compatibility and it is clear that $S \cdot k[Q]_+ = S \cdot Q$. For the last statement we now just have to remark that $\langle\sigma\rangle_S = S \cdot \langle\sigma\rangle_{k[F]}$ and $k[\langle\sigma\rangle_{k[F]}] = k[\sigma]$. $\square$

4 Filtrations and Graduations

Goal of this chapter is to introduce Hilbert series of filtered modules in order to define an invariant for resolution of singularities in chapter 8. These series are a powerful tool for many technical problems. We will first recall the notion of filtered rings and filtered modules in great detail and in a very broad perspective, followed by graded rings and modules. Most of the ideas presented here come from [Gi, I], where mainly the local case is treated, and [AM, 10] and are well known. We try to formulate everything as general as possible. The most important result is proposition (4.6.5).

4.1 Filtered rings and modules

Let us recall filtered rings and modules.

Definition (4.1.1). *A* ***filtration on a ring*** *A is a sequence of ideals*

$$\mathcal{A}: \qquad A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

satisfying $A_n \cdot A_m \subseteq A_{n+m}$ for all $n, m \geq 0$. The pair $(A, \mathcal{A})$ is called a ***filtered ring****. Sometimes it is convenient to extend the sequence of ideals from $\mathbb{N}$ to $\mathbb{Z}$ by defining $A_n = A$ for $n \leq 0$. The relation $A_n \cdot A_m \subseteq A_{n+m}$ then is true for all $n, m \in \mathbb{Z}$. For an ideal I of a ring A there is the* ***I-adic filtration*** *$A_n := I^n$ (where we also use $I^n = A$ for $n \leq 0$). With letters $\mathcal{A}, \mathcal{B}, ...$ we will refer to arbitrary filtrations. In the case of an I-adic filtration we will write (A, I) instead of $(A, \mathcal{A})$. A* ***morphism*** *of filtered rings $\varphi : (A, \mathcal{A}) \to (B, \mathcal{B})$ is a morphism of rings $\varphi : A \to B$ satisfying $\varphi(A_n) \subseteq B_n$ for all n.*

Definition (4.1.2). *Let $(A, \mathcal{A})$ be a filtered ring and M an A-module. A* ***filtration on the module*** *M (more precisely an $(A, \mathcal{A})$-filtration) is a sequence*

$$\mathcal{M}: \qquad \cdots \supseteq M_{-2} \supseteq M_{-1} \supseteq M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

of A-submodules of M such that $A_m M_n \subseteq M_{m+n}$ for all $m, n \in \mathbb{Z}$. The pair $(M, \mathcal{M})$ is called a ***filtered*** *$(A, \mathcal{A})$****-module****. $\mathcal{M}$ is called* ***bounded*** *if there exists $N \in \mathbb{Z}$ such that $M_n = M$ for all $n \leq N$. It is called* ***positive*** *if this holds for $N = 0$. The filtration is called* ***exhaustive*** *if $\bigcup_{n \in \mathbb{Z}} M_n = M$. The filtration is called* ***separated*** *if $\bigcap_{n \in \mathbb{Z}} M_n = \{0\}$. A* ***morphism of filtered*** *$(A, \mathcal{A})$****-modules*** *$\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ is an A-module homomorphism $\varphi : M \to N$ such that $\varphi(M_d) \subseteq N_d$ for all $d \in \mathbb{Z}$. This implies $\varphi(M_d) \subseteq \varphi(M) \cap N_d$. φ is called* ***strict*** *if $\varphi(M_d) = \varphi(M) \cap N_d$ for all $d \in \mathbb{Z}$. A sequence of filtered $(A, \mathcal{A})$-modules is called exact if the sequence of underlying A-modules is exact. It is called strict if all morphisms are strict. If M is an A-module, then $(M, \mathcal{M})$ with $M_n := A_n M$ is a filtered $(A, \mathcal{A})$-module. A sequence $\cdots \supseteq M_{-2} \supseteq M_{-1} \supseteq M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of A-submodules of M yields an*

(A, I)-filtration on M iff $IM_i \subseteq M_{i+1}$ for all $i \geq 0$. Such a filtration is called an ***I-filtration*** *on M. If furthermore $IM_i = M_{i+1}$ for all sufficiently large integers i and the filtration $\mathcal{M}$ is bounded, we call the filtration* ***I-stable****. In particular we get the* ***I-adic filtration*** *on M by putting $M_i := I^iM$. This is a positive I-stable filtration and we will write (M, I) if we refer to M together with this filtration. A* ***free filtered $(A, \mathcal{A})$-module*** *is a filtered $(A, \mathcal{A})$-module which is isomorphic (as a filtered $(A, \mathcal{A})$-module) to a direct sum of $(A, \mathcal{A}(n))$ for various $n \in \mathbb{Z}$. (see (4.1.5) and (4.1.6)). It is called additionally* ***of finite type*** *if the direct sum is finite. A* ***subfiltration*** *of an $(A, \mathcal{A})$-filtration $\mathcal{M}$ on M is an $(A, \mathcal{A})$-filtration $\mathcal{M}'$ on M such that $M'_d \subseteq M_d$ for all $d \in \mathbb{Z}$. We simply write $\mathcal{M}' \subseteq \mathcal{M}$ in this case.*

We now discuss several situations in which a filtration induces a filtration on a related object. We omit the easy proofs.

(4.1.3) (Pullback) Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$-module and $\varphi : L \to M$ a homomorphism of A-modules. For all $d \in \mathbb{Z}$ put $L_d := \varphi^{-1}(M_d)$. This makes $(L, \mathcal{L})$ into a filtered $(A, \mathcal{A})$-module and $\varphi : (L, \mathcal{L}) \to (M, \mathcal{M})$ is a strict morphism. We call $\mathcal{L}$ the **pullback filtration** on L and denote it with $\varphi^*(\mathcal{M})$. If $\mathcal{M}$ is bounded, then so is $\mathcal{L}$. If $\mathcal{M}$ is exhaustive, then so is $\mathcal{L}$. If $\mathcal{M}$ is separated and φ is injective, then $\mathcal{L}$ is separated.

(4.1.4) (Pushforward) Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$-module and $\varphi : M \to N$ a homomorphism of A-modules. For all $d \in \mathbb{Z}$ put $N_d := \varphi(M_d)$. This makes $(N, \mathcal{N})$ into a filtered $(A, \mathcal{A})$-module and $\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ is a strict morphism. We call $\mathcal{N}$ the **pushforward filtration** on N and denote it with $\varphi_*(\mathcal{M})$. If $\mathcal{M}$ is bounded and φ is surjective, then $\mathcal{N}$ is bounded. If $\mathcal{M}$ is exhaustive and φ is surjective, then $\mathcal{N}$ is exhaustive.

(4.1.5) (Shift) Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$-module and $n \in \mathbb{Z}$. Define a new filtration $\mathcal{M}(n)$ on M by $M(n)_i := M_{n+i}$. Then $(M, \mathcal{M}(n))$ is a filtered $(A, \mathcal{A})$-module and $\mathcal{M}(n)$ is called a **shifted filtration** of $\mathcal{M}$. $\mathcal{M}$ is bounded (resp. exhaustive, resp. separated) iff $\mathcal{M}(n)$ is bounded (resp. exhaustive, resp. separated). For $n \geq 0$ we have $\mathcal{M}(n) \subseteq \mathcal{M}$.

(4.1.6) (Direct sum) Let $(M^i, \mathcal{M}^i)_{i \in I}$ be a family of filtered $(A, \mathcal{A})$-modules. We define a filtration $\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}^i$ on the A-module $M := \bigoplus_{i \in I} M^i$ by setting $M_n := \bigoplus_{i \in I} M^i_n \subseteq M$ for $n \in \mathbb{Z}$. Then $(M, \mathcal{M})$ is a filtered $(A, \mathcal{A})$-module and $\mathcal{M}$ is called the **direct sum filtration**. All inclusions $\varphi_i : M^i \to M$ and all projections $\pi^i : M \to M^i$ are strict morphisms of filtered $(A, \mathcal{A})$-modules and $\varphi_i^*(\mathcal{M}) = \mathcal{M}^i$ and $\pi^i_*(\mathcal{M}) = \mathcal{M}^i$. If all $\mathcal{M}^i$ are bounded and I is finite, then $\mathcal{M}$ is bounded. If all $\mathcal{M}^i$ are exhaustive, then so is $\mathcal{M}$. If all $\mathcal{M}^i$ are separated, then so is $\mathcal{M}$.

(4.1.7) (Localization) Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$-module and $S \subseteq A$ multiplicatively closed. Then $(A_S, \mathcal{A}_S)$ is a filtered ring with $(\mathcal{A}_S)_n := (\mathcal{A}_n)_S$ and $(M_S, \mathcal{M}_S)$ is a filtered $(A, \mathcal{A}_S)$-module with $(\mathcal{M}_S)_n := (\mathcal{M}_n)_S$. The filtrations $\mathcal{A}_S$ and $\mathcal{M}_S$ are called the **localized filtrations**. In the case that $S = A \setminus \mathfrak{p}$ for a prime $\mathfrak{p}$ of A we also use the notations $\mathcal{A}_\mathfrak{p}$ and $\mathcal{M}_\mathfrak{p}$. If $\mathcal{M}$ is bounded, then so is $\mathcal{M}_S$. If $\mathcal{M}$ is exhaustive, then so is $\mathcal{M}_S$.

Lemma (4.1.8) (cf. [AM, 10.6]). *Let $\mathcal{M}'$ be an I-stable filtration and $\mathcal{M}''$ a bounded I-filtration of the A-module M. Then there exists an integer $n_0 \in \mathbb{Z}$ with $\mathcal{M}'(n_0) \subseteq \mathcal{M}''_n$. (One can choose $n_0 \geq 0$.)*

Proof. Choose $n_1, n_2 \leq 0$ such that $\mathcal{M}'(n_1)$ and $\mathcal{M}''(n_2)$ are positive (remaining I-stable, resp. I-filtered). Choose $n_3 \geq 0$ such that for all $n \geq n_3$ we have $\mathcal{M}'(n_1)_{n+1} = I\mathcal{M}'(n_1)_n$. Then for $n \geq 0$ we get $\mathcal{M}'(n_1 + n_3)_n = \mathcal{M}'(n_1)_{n_3+n} = I^n\mathcal{M}'(n_1)_{n_3} \subseteq I^n M \subseteq \mathcal{M}''(n_2)_n$. For $n < 0$ we also have $\mathcal{M}'(n_1+n_3)_n \subseteq M = \mathcal{M}''(n_2)_n$. Therefore $\mathcal{M}'(n_1 + n_3) \subseteq \mathcal{M}''(n_2)$ and $\mathcal{M}'(n_1 - n_2 + n_3) \subseteq \mathcal{M}''$, define $n_0 := n_1 - n_2 + n_3$. If $n_0 \leq 0$ we get $\mathcal{M}' \subseteq \mathcal{M}'(n_0) \subseteq \mathcal{M}''$ and therefore can also choose $n_0 = 0$. □

We end this section with two useful remarks.

Remark (4.1.9). *Let $\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ be a morphism of filtered $(A, \mathcal{A})$-modules. Then φ is strict if and only if the pushforward filtration on $\varphi(M)$ via $M \to \varphi(M)$ is the same as the pullback filtration on $\varphi(M)$ via $\varphi(M) \to N$. In particular: If φ is injective, then φ is strict iff $\mathcal{M} = \varphi^*\mathcal{N}$. If φ is surjective, then φ is strict iff $\varphi_*\mathcal{M} = \mathcal{N}$.*

Proof. This is clear since the pushforward filtration on $\varphi(M)$ is given by $\varphi(M_d)$ and the pullback filtration by $\varphi(M) \cap N_d$. □

Remark (4.1.10). *Let $(A, \mathcal{A})$ be a filtered ring and $(L, \mathcal{L}) \xrightarrow{\varphi} (M, \mathcal{M}) \xrightarrow{\psi} (N, \mathcal{N})$ an exact sequence of filtered A-modules, where φ is strict. Then the induced sequence $L_d \xrightarrow{\varphi_d} M_d \xrightarrow{\psi_d} N_d$ is exact for all $d \in \mathbb{Z}$.*

Proof. $\ker(\psi_d) = \ker(\psi) \cap M_d = \varphi(L) \cap M_d = \varphi(L_d)$. □

4.2 Good filtrations and their inheritance

We discuss important classes of filtered rings and modules with noetherian properties.

Definition (4.2.1). *To a filtered ring $(A, \mathcal{A})$ we associate two graded A-algebras: The **blow up algebra** $\mathrm{Bl}(\mathcal{A}) := \bigoplus_{d\in\mathbb{N}} A_d$ and the **total blow up algebra** $\mathrm{Bl}^\dagger(\mathcal{A}) := \bigoplus_{d\in\mathbb{Z}} A_d$. They can be regarded as subalgebras of $A[T]$, resp. $A[T, T^{-1}]$. $\mathrm{Bl}^\dagger(\mathcal{A})$ is an exception to our convention that all graded rings are positively graded.*

Lemma (4.2.2). *For a filtered ring $(A, \mathcal{A})$ the following are equivalent:*

(i) $\mathrm{Bl}^\dagger(\mathcal{A})$ *is a noetherian ring.*

(ii) $A = A_0$ *is a noetherian ring and* $\mathrm{Bl}^\dagger(\mathcal{A})$ *is an* A_0*-algebra of finite type.*

(iii) $\mathrm{Bl}(\mathcal{A})$ *is a noetherian ring.*

(iv) $A = A_0$ *is a noetherian ring and* $\mathrm{Bl}(\mathcal{A})$ *is an* A_0*-algebra of finite type.*

Proof. For (ii) ⇒ (i) and (iv) ⇒ (iii) just use Hilbert's basis theorem. The ring homomorphism $\mathrm{Bl}(\mathcal{A}) \to \mathrm{Bl}^\dagger(\mathcal{A})$ extends to a surjective morphism $\mathrm{Bl}(\mathcal{A})[T] \to \mathrm{Bl}^\dagger(\mathcal{A})$ sending T to $z := 1 \in A_{-1}$, proving (iii) ⇒ (i) and (iv) ⇒ (ii). It remains to prove

(i) $\Rightarrow$ (iv): Let $\mathfrak{a}_0$ be an ideal in A_0 and $\mathfrak{a} := \mathrm{Bl}^\dagger(\mathcal{A}) \cdot \mathfrak{a}_0$. Then $\mathfrak{a} = \bigoplus_{d\in\mathbb{Z}} A_d\mathfrak{a}_0$ and since $A_0\mathfrak{a}_0 = \mathfrak{a}_0$, the ideal $\mathfrak{a}_0$ can be regained from $\mathfrak{a}$. This proves that every ascending sequence of ideals in A_0 must stabilize, so A_0 is noetherian. Consider the ideal $\mathfrak{b} \subseteq \mathrm{Bl}^\dagger(\mathcal{A})$ generated by $\bigoplus_{d\geq 1} A_d$. Then $\mathfrak{b}$ is a homogeneous ideal with $\mathfrak{b}_n = A_1$ for $n \leq 1$ and $\mathfrak{b}_n = A_n$ for $n \geq 1$. Take homogeneous generators $x_1, ..., x_s$ of this ideal with degrees $k_1, ..., k_s$. Take homogeneous generators $y_1, ..., y_r$ of A_1 as an ideal in A_0 and regard them as elements of $\mathrm{Bl}^\dagger(\mathcal{A})$ in $A_1 \subseteq \mathfrak{b}$. Now $x_1, ..., x_s, y_1, ..., y_r$ generate the ideal $\mathfrak{b}$ and using $z \in \mathrm{Bl}^\dagger(\mathcal{A})$ as above we can suppose that $k_1, ..., k_s > 0$. Let B be the A_0-subalgebra of $\mathrm{Bl}(\mathcal{A})$ generated over A_0 by $y_1, ..., y_r$ and for $1 \leq i \leq s$ the elements $x_i, zx_i, ..., z^{k_i-1}x_i$, which is contained in $\mathrm{Bl}(\mathcal{A})$. We show that $B = \mathrm{Bl}(\mathcal{A})$ which finishes the proof. We do this by proving inductively $B_n = \mathrm{Bl}(\mathcal{A})_n$ and start with $n = 0$. Suppose $n > 0$. Let $x \in \mathrm{Bl}(\mathcal{A})_n \subseteq \mathfrak{b}$. Then $x = f_1x_1 + \cdots + f_sx_s + g_1y_1 + \cdots + g_ry_r$ with $f_i \in A_{n-k_i}$ and $g_j \in A_{n-1}$. Therefore $g_j \in B$ and $g_jy_j \in B$. If $n - k_i \geq 0$, then $f_i \in B$ and $f_ix_i \in B$. If $n - k_i < 0$, then we write f_i' for the element f_i in the homogeneous part $A_0 (= A_{n-k_i})$. But now $f_ix_i = f_i'x_iz^{k_i-n}$ with $x_iz^{k_i-n} \in B$ by definition. This shows $x \in B$. □

Definition (4.2.3). *Let $(A, \mathcal{A})$ be a filtered ring. The filtration $\mathcal{A}$ is called **n-good** and $(A, \mathcal{A})$ is called an **n-good filtered ring** if $\mathrm{Bl}^\dagger(\mathcal{A})$ is a noetherian ring, i.e. $(A, \mathcal{A})$ satisfies the equivalent conditions of (4.2.2). The filtration $\mathcal{A}$ is called **good** and $(A, \mathcal{A})$ is called a **good filtered ring** (cf. [Gi, I 1.]) if $\mathcal{A}$ is n-good and there exists an $n > 0$ such that A_n is contained in the radical of A.*

Example (4.2.4). If A is a noetherian ring and $I \subseteq A$ an ideal, then (A, I) is an n-good filtered ring. If furthermore I is contained in the radical of A (e.g. A a local ring and $I \neq A$), then the filtration is good.

Definition (4.2.5). *Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$-module. Its associated **total blow up module** is the graded $\mathrm{Bl}^\dagger(\mathcal{A})$-module $\mathrm{Bl}^\dagger(\mathcal{M}) := \bigoplus_{d\in\mathbb{Z}} M_d$.*

Lemma (4.2.6) (cf. [Gi, I 1.3, 1.3.3])**.** *For a filtered module $(M, \mathcal{M})$ over an n-good filtered ring $(A, \mathcal{A})$ the following are equivalent:*

(i) $\mathrm{Bl}^\dagger(\mathcal{M})$ *is a finitely generated* $\mathrm{Bl}^\dagger(\mathcal{A})$*-module.*

(ii) *There exist $x_1 \in M_{n_1}, ..., x_r \in M_{n_r}$ such that $M_n = \sum_{i=1}^r A_{n-n_i}x_i$ for all $n \in \mathbb{Z}$.*

(iii) *All $(M_n)_{n\in\mathbb{Z}}$ are finitely generated A-modules, there exists an $s \in \mathbb{Z}$ such that $M_n = \sum_{i\leq s} A_{n-i}M_i$ for all $n \in \mathbb{Z}$ and there exists $t \in \mathbb{Z}$ such that $\cdots = M_{t-2} = M_{t-1} = M_t$.*

(iv) *There exists a free filtered $(A, \mathcal{A})$-module $(L, \mathcal{L})$ of finite type and a surjective strict morphism of filtered $(A, \mathcal{A})$-modules $(L, \mathcal{L}) \to (M, \mathcal{M})$.*

Proof. Start with (i) and let $x_1 \in M_{n_1}, ..., x_r \in M_{n_r}$ be homogeneous generators of the $\mathrm{Bl}^\dagger(\mathcal{A})$-module $\mathrm{Bl}^\dagger(\mathcal{M})$. Then for all $n \in \mathbb{Z}$ clearly (ii) holds, showing that all M_i are finitely generated (all A_i are finitely generated $A = A_0$-modules by (4.2.2)), and with $s := \max\{n_1, ..., n_r\}$ we find

$$M_n = \sum_{i=1}^{r} A_{n-n_i} x_i \subseteq \sum_{i=1}^{r} A_{n-n_i} M_{n_i} \subseteq \sum_{i \leq s} A_{n-i} M_i \subseteq M_n$$

and (iii) follows since for any $n \leq t := \min\{n_1, ..., n_r\}$ we have $M_n = \sum_{i=1}^{r} A_{n-n_i} x_i = \sum_{i=1}^{r} A x_i$. (iii) $\Rightarrow$ (i): Let $x_1, ..., x_r \in \mathrm{Bl}^\dagger(\mathcal{M})$ be generators of the A-modules $M_t, ..., M_s$ regarded in the respective degrees. Let N be the $\mathrm{Bl}^\dagger(\mathcal{A})$-submodule of $\mathrm{Bl}^\dagger(\mathcal{M})$ generated by these elements. Since $A = (\mathrm{Bl}^\dagger(\mathcal{A}))_0$, at least $M_t, ..., M_s$ are contained in N. $A = (\mathrm{Bl}^\dagger(\mathcal{A}))_{-1}$ implies then $M_n \subseteq N$ for all $n \leq t$. But now the formula in (iii) assures that $N = \mathrm{Bl}^\dagger(\mathcal{M})$. It is easy to see that (iv) $\Leftrightarrow$ (ii). □

Definition (4.2.7). *Let $(M, \mathcal{M})$ be a filtered module over an n-good filtered ring $(A, \mathcal{A})$. The filtration $\mathcal{M}$ is called a **good filtration** and $(M, \mathcal{M})$ is called a **good filtered** $(A, \mathcal{A})$**-module** if $\mathrm{Bl}^\dagger(\mathcal{M})$ is a finitely generated $\mathrm{Bl}^\dagger(\mathcal{A})$-module and $\mathcal{M}$ is exhaustive. (4.2.6) implies that a good filtration is bounded.*

Example (4.2.8). Let $(A, \mathcal{A})$ be an n-good filtered ring and M a finitely generated A-module. Then $M_n := A_n M$ yields a good filtration on M (use criterion (iv) of (4.2.6) for a surjection $A^n \to M$). In particular (M, I) is a good filtered (A, I)-module.

Lemma (4.2.9) (cf. [AM, 10.8]). *For a filtered (A, I)-module $(M, \mathcal{M})$ over a noetherian ring A the following are equivalent:*

(i) *$\mathcal{M}$ is good.*

(ii) *$\mathcal{M}$ is I-stable and M is a finitely generated A-module.*

Proof. (i) $\Rightarrow$ (ii): By (4.2.6) (iii) all M_n are finitely generated and there exists an integer $s \in \mathbb{Z}$ such that for all $n \geq s$

$$M_{n+1} = \sum_{i \leq s} A_{n+1-i} M_i = \sum_{i \leq s} I I^{n-i} M_i = I \sum_{i \leq s} A_{n-i} M_i = I M_n$$

which means that $\mathcal{M}$ is I-stable. (ii) $\Rightarrow$ (i): Use the same criterion and note that I-stable filtrations are bounded by definition. There exists $s \in \mathbb{Z}$ such that for all $n \geq s$ one has $M_{n+1} = I M_n$. Then for $n \leq s$ we have $M_n = A_0 M_n \subseteq \sum_{i \leq s} A_{n-i} M_i \subseteq M_n$ and for $n \geq s$ we find $M_n = I^{n-s} M_s = I^{n-s} \sum_{i \leq s} I^{s-i} M_i = \sum_{i \leq s} A_{n-i} M_i$. □

The noetherian properties discussed so far are passed to related objects under certain circumstances.

Lemma (4.2.10). *Let $(A, \mathcal{A})$ be an n-good filtered ring and $(M, \mathcal{M})$ a good filtered $(A, \mathcal{A})$-module.*

(i) *(Artin-Rees)(cf. [AM, 10.9]) For a monomorphism $\iota : N \to M$ also $(N, \iota^* \mathcal{M})$ is a good filtered $(A, \mathcal{A})$-module.*

(ii) *For an epimorphism $\pi : M \to N$ also $(N, \pi_* \mathcal{M})$ is a good filtered $(A, \mathcal{A})$-module.*

(iii) *For any $n \in \mathbb{Z}$ also $(M, \mathcal{M}(n))$ is a good filtered $(A, \mathcal{A})$-module.*

(iv) *For a finite family $(M^i, \mathcal{M}^i)_{i \in I}$ of good filtered $(A, \mathcal{A})$-modules, also $(\bigoplus_{i \in I} M^i, \bigoplus_{i \in I} \mathcal{M}^i)$ is a good filtered $(A, \mathcal{A})$-module.*

(v) *For a multiplicatively closed subset $S \subseteq A$ also $(A_S, \mathcal{A}_S)$ is an n-good filtered ring and $(M_S, \mathcal{M}_S)$ is a good filtered $(A_S, \mathcal{A}_S)$-module.*

Proof. (i) The module $\mathrm{Bl}^\dagger(\iota^*\mathcal{M})$ clearly is a naturally graded $\mathrm{Bl}^\dagger(\mathcal{A})$-submodule of $\mathrm{Bl}^\dagger(\mathcal{M})$. Since $\mathcal{A}$ is n-good, the ring $\mathrm{Bl}^\dagger(\mathcal{A})$ is noetherian. Since $\mathrm{Bl}^\dagger(\mathcal{M})$ is a finitely generated $\mathrm{Bl}^\dagger(\mathcal{A})$-module, the same holds for $\mathrm{Bl}^\dagger(\iota^*\mathcal{M})$. Since $\mathcal{M}$ is exhaustive, the same holds for the pullback filtration $\iota^*\mathcal{M}$.

(ii) Clearly $\pi_*\mathcal{M}$ is exhaustive. The induced $\mathrm{Bl}^\dagger(\mathcal{A})$-linear map from $\mathrm{Bl}^\dagger(\mathcal{M})$ to $\mathrm{Bl}^\dagger(\pi_*\mathcal{M})$ is surjective. Therefore also $\mathrm{Bl}^\dagger(\pi_*\mathcal{M})$ is finitely generated.

(iii) This is obvious for example by criterion (iii) of of (4.2.6).

(iv) $\bigoplus_{i \in I} \mathcal{M}^i$ is exhaustive since all $\mathcal{M}^i$ are exhaustive and $\mathrm{Bl}^\dagger(\bigoplus_{i \in I} \mathcal{M}^i)$ is isomorphic to $\bigoplus_{i \in I} \mathrm{Bl}^\dagger(\mathcal{M}^i)$ and therefore a finitely generated $\mathrm{Bl}^\dagger(\mathcal{A})$-module.

(v) Since $(\mathrm{Bl}^\dagger(\mathcal{A}))_S \cong \mathrm{Bl}^\dagger(\mathcal{A}_S)$, also $(A_S, \mathcal{A}_S)$ is n-good. $\mathrm{Bl}^\dagger(\mathcal{M})$ is a finitely generated $\mathrm{Bl}^\dagger(\mathcal{A})$-module. This property also is preserved under localization and $(\mathrm{Bl}^\dagger(\mathcal{M}))_S \cong \mathrm{Bl}^\dagger(\mathcal{M}_S)$ shows that $(M_S, \mathcal{M}_S)$ is good. □

Let us finish this section with an application.

Lemma (4.2.11). *Let $(A, \mathcal{A})$ be an n-good filtered ring and $(M, \mathcal{M})$ a good filtered $(A, \mathcal{A})$-module. Let $L := \bigcap_{m \in \mathbb{Z}} M_m$. Then for all $n \in \mathbb{Z}$ we have $A_n L = L$.*

Proof. Let $\mathcal{L}$ be the pullback filtration on L via the inclusion $L \to M$. Then by (4.2.10) also $(L, \mathcal{L})$ is a good filtered $(A, \mathcal{A})$-module and $L_m = L$ for all $m \in \mathbb{Z}$. With (4.2.6) (iii) we get $L = L_{n+s} = \sum_{i \leq s} A_{n+s-i} L_i = \sum_{j \geq 0} A_{n+j} L = A_n L$. □

Corollary (4.2.12). *Let $(A, \mathcal{A})$ be a good filtered ring and $(M, \mathcal{M})$ a good filtered $(A, \mathcal{A})$-module. Then $\mathcal{M}$ is separated.*

Proof. Since A is noetherian and M is finitely generated, also $L := \bigcap_{m \in \mathbb{Z}} M_m$ is finitely generated. By (4.2.11) there exists an ideal A_n which is contained in the radical of A such that $A_n L = L$ and the Nakayama lemma yields $L = 0$. □

Corollary (4.2.13) (Krull's intersection theorem, cf. [AM, 10.19])**.** *Let I be an ideal in a noetherian ring A contained in its radical and M a finitely generated A-module. Then $\bigcap_{n=0}^{\infty} I^n M = 0$.*

Proof. Use (4.2.12) for the good filtered module (M, I) over the good filtered ring (A, I) ((4.2.4) and (4.2.8)). □

Corollary (4.2.14). *Let I be an ideal in a noetherian ring A contained in its radical and M a finitely generated A-module with submodules $K, L \subseteq M$. Then $\bigcap_{n=0}^{\infty}(K + I^n L) = K$.*

Proof. With $L_n := (K + I^n L)/K \subseteq M/K$ we have $I L_n = L_{n+1}$ and (4.2.13) yields $\bigcap_{n \geq 0} L_n = \bigcap_{n \geq 0} I^n L_0 = 0$. Note that $K = \ker(\bigcap_{n \geq 0}(K + I^n L) \to \bigcap_{n \geq 0} L_n)$. □

4.3 Graded rings and modules

We are going to use the associated graded rings and modules of their filtered counterparts. Let us recall some definitions and facts about graded rings and modules in general first.

Definition (4.3.1). *A* ***graded ring*** *is a ring A together with a subgroup decomposition $A = \bigoplus_{n\geq 0} A_n$ such that $A_n A_m \subseteq A_{n+m}$ for all $m, n \geq 0$. Thus A_0 is a subring and each A_n is an A_0-module. We denote with A_+ the ideal $\bigoplus_{n\geq 1} A_n$ of A. A morphism of graded rings is a morphism of rings which respects the graduation.*

Lemma (4.3.2) (cf. [AM, 10.7])**.** *Let A be a graded ring. Then A is noetherian if and only if A_0 is noetherian and A is of finite type over A_0.*

Proof. If A is noetherian, then so is $A_0 \cong A/A_+$. If $A_+ = \langle x_1, ..., x_s\rangle_A$ with homogeneous x_i, then one easily sees by induction that $A = A_0[x_1, ..., x_s]$. Hilbert's basis theorem yields the other implication. □

Definition (4.3.3). *A* ***graded module*** *is a module M over a graded ring A together with a subgroup decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ such that $A_m M_n \subseteq M_{m+n}$ for all $m \geq 0, n \in \mathbb{Z}$. All M_n are A_0-modules. The graduation is called* ***bounded*** *if there exists $N \in \mathbb{Z}$ such that $M_n = 0$ for $n < N$ and is called* ***positive*** *if $M_n = 0$ for $n < 0$. Morphisms of graded modules respect the graduation.*

Lemma (4.3.4). *Let M be a finitely generated graded module over the noetherian graded ring A. Then all M_n are finitely generated A_0-modules and the graduation on M is bounded.*

Proof. M is generated by homogeneous elements $m_1, ..., m_s$ of degrees $k_1 \leq \cdots \leq k_s$. Then $M_n = 0$ for $n < k_1$. By (4.3.2) all A_i are finitely generated A_0-modules. Since $M_n = A_{n-k_1} m_1 + \cdots + A_{n-k_s} m_s$, also M_n is a finitely generated A_0-module. □

Remark (4.3.5). *A sequence of graded modules $L \to M \to N$ is* ***exact*** *if and only if all induced sequences of A_0-modules $L_n \to M_n \to N_n$ for $n \in \mathbb{Z}$ are exact. One can give a graded A-module M a* ***shifted graduation*** *$M(n)$ for $n \in \mathbb{Z}$ by $M(n)_m := M_{m+n}$. $M(n)$ again is a graded A-module.*

Lemma (4.3.6). *Let M be a bounded graded module over a graded ring A. Then $A_+ M = M$ implies $M = 0$. If $N \subseteq M$ is a graded submodule with $M = A_+ M + N$, then $M = N$. If $\varphi : L \to M$ is a morphism of graded A-modules such that the induced morphism $L/A_+ L \to M/A_+ M$ is surjective, then already $L \to M$ was surjective.*

Proof. Let $M_n = 0$ for $n < N$. Then $M = A_+^n M \subseteq M_{\geq N+n}$ and therefore $M \subseteq \bigcap_n M_{\geq N+n} = 0$. Assume that $M = A_+ M + N$. Then M/N is bounded and $A_+(M/N) = A_+ M + N/N = M/N$ and therefore $M/N = 0$. If $L/A_+ L \to M/A_+ M$ is surjective, we have $M = A_+ M + \varphi(L)$ and therefore get $M = \varphi(L)$. □

Let us also mention the idea of a standard basis, widely used in resolution of singularities (cf. [CJS, 1.3]).

Definition (4.3.7). *A system of elements $(m_1, ..., m_n)$ of a graded A-module M is called a* ***standard basis*** *of M if m_i is homogeneous of degree s_i for $1 \leq i \leq n$, $s_1 \leq s_2 \leq ... \leq s_n$, $m_i \notin \langle m_1, ..., m_{i-1}\rangle_A$ for $i = 1, ..., n$ and $\langle m_1, ..., m_n\rangle_A = M$. A standard basis can be obtained from a homogeneous system of generators by reordering and thrashing out superfluous elements.*

4.4 Graduations associated to filtrations

Definition (4.4.1). *The* ***associated graded ring*** *of a filtered ring $(A, \mathcal{A})$*

$$\mathrm{gr}(\mathcal{A}) := \bigoplus_{d\geq 0} A_d/A_{d+1} = A/A_1 \oplus A_1/A_2 \oplus \cdots$$

will be denoted by $\mathrm{gr}_I(A)$ for the I-adic filtration. gr *is functorial. If $(M, \mathcal{M})$ is a filtered $(A, \mathcal{A})$-module, its* ***associated graded module*** *is the graded $\mathrm{gr}(\mathcal{A})$-module*

$$\mathrm{gr}(\mathcal{M}) := \bigoplus_{d\in\mathbb{Z}} M_d/M_{d+1}$$

and in the case of the I-adic filtration we write $\mathrm{gr}_I(M)$. gr *is functorial in filtered $(A, \mathcal{A})$-modules. One has canonical isomorphisms of graded $\mathrm{gr}(\mathcal{A})$-modules $\mathrm{gr}(\mathcal{M})(n) \cong \mathrm{gr}(\mathcal{M}(n))$ for $n \in \mathbb{Z}$.* gr *also commutes with direct sums.*

Lemma (4.4.2) (cf. [Gi, I 2.1])**.** *Let $(A, \mathcal{A})$ be a filtered ring and $(L, \mathcal{L}) \xrightarrow{\varphi} (M, \mathcal{M}) \xrightarrow{\psi} (N, \mathcal{N})$ a strict exact sequence of filtered $(A, \mathcal{A})$-modules. Then the induced sequence $\mathrm{gr}(\mathcal{L}) \to \mathrm{gr}(\mathcal{M}) \to \mathrm{gr}(\mathcal{N})$ is an exact sequence of $\mathrm{gr}(\mathcal{A})$-modules.*

Proof. We have to show that $L_d/L_{d+1} \to M_d/M_{d+1} \to N_d/N_{d+1}$ is exact for all $d \in \mathbb{Z}$. From (4.1.10) we know that $L_d \to M_d \to N_d$ is exact for all d. Let $m \in M_d$ with $\psi(m) \in N_{d+1}$. Then $\psi(m) = \psi(m')$ for some $m' \in M_{d+1}$. Since now $\psi(m - m') = 0$, we know that $m - m' = \varphi(l)$ for some $l \in L_d$. But then $m \equiv \varphi(l) \bmod M_{d+1}$. □

Remark (4.4.3). *Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$ module. If $\mathcal{M}$ is exhaustive and separated and $\mathrm{gr}(\mathcal{M}) = 0$, then $M = 0$.*

Proof. $\mathrm{gr}(\mathcal{M}) = 0$ implies $M_n = M_{n+1}$ and $M = \bigcup_{n\in\mathbb{Z}} M_n = \bigcap_{n\in\mathbb{Z}} M_n = \{0\}$. □

Proposition (4.4.4). *Let $\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ be a morphism of filtered $(A, \mathcal{A})$-modules.*

(i) *If $\mathrm{gr}(\varphi)$ is injective and $\mathcal{M}$ is exhaustive, then φ is strict. If $\mathcal{M}$ is additionally separated, then φ is injective.*

(ii) *If $\mathrm{gr}(\varphi)$ is surjective and the pushforward filtration on $N/\varphi(M)$ via $N \to N/\varphi(M)$ is separated and $\mathcal{N}$ is exhaustive, then φ is surjective.*

Proof. (i): $M_d/M_{d+1} \to N_d/N_{d+1}$ is injective and we have to show that $\varphi(M_d) = N_d \cap \varphi(M)$. Since $\mathcal{M}$ is exhaustive, $x \in N_d \cap \varphi(M)$ lies in $\varphi(M_e)$ for some e. If $e \geq d$, we are done. If $e < d$, then x maps to zero in N_e/N_{e+1} and therefore

$x \in N_d \cap \varphi(M_{e+1})$. Inductively we get $x \in \varphi(M_d)$. Make $0 \to K \xrightarrow{\psi} M \xrightarrow{\varphi} N$ into a strict exact sequence of filtered $(A, \mathcal{A})$-modules using the pullback filtration $\mathcal{K}$ on K, which is exhaustive and separated (cf. (4.1.3)). Then $\mathrm{gr}(\varphi) \circ \mathrm{gr}(\psi) = 0$ and the injectivity of $\mathrm{gr}(\varphi)$ and $\mathrm{gr}(\psi)$ (see (4.4.2)) imply $\mathrm{gr}(\mathcal{K}) = 0$ and with (4.4.3) we conclude that $K = 0$. (ii): Let $M \xrightarrow{\varphi} N \xrightarrow{\psi} P \to 0$ be an exact sequence of A-modules and equip P with the exhaustive (cf. (4.1.4)) and separated pushforward filtration via ψ. Then $\mathrm{gr}(\psi) \circ \mathrm{gr}(\varphi) = 0$ and since $\mathrm{gr}(\varphi)$ is surjective we must have $\mathrm{gr}(\psi) = 0$. ψ is strict and surjective, so (4.4.2) implies $\mathrm{gr}(\mathcal{P}) = 0$ and (4.4.3) yields $P = 0$. □

Lemma (4.4.5). *Let $(M, \mathcal{M})$ be a good filtered module over an n-good filtered ring $(A, \mathcal{A})$. $\mathrm{gr}^0(\mathcal{A})$ and $\mathrm{gr}(\mathcal{A})$ are noetherian rings and $\mathrm{gr}(\mathcal{A})$ is a $\mathrm{gr}^0(\mathcal{A})$-algebra of finite type. $\mathrm{gr}(\mathcal{M})$ is a finitely generated $\mathrm{gr}(\mathcal{A})$-module and $\mathrm{gr}^n(\mathcal{M})$ is a finitely generated $\mathrm{gr}^0(\mathcal{A})$-module for all $n \in \mathbb{Z}$.*

Proof. $\mathrm{gr}(\mathcal{A})$ is a quotient of $\mathrm{Bl}(\mathcal{A})$ and $\mathrm{gr}^0(\mathcal{A})$ is a quotient of A, now use (4.2.2). $\mathrm{gr}(\mathcal{M}) \cong \mathrm{Bl}^\dagger(\mathcal{M})/\mathrm{Bl}^\dagger(\mathcal{M}(1))$ and $\mathrm{Bl}^\dagger(\mathcal{M})$ is a finitely generated $\mathrm{Bl}^\dagger(\mathcal{A})$-module. Also all M_n are finitely generated A-modules, see (4.2.6). □

Corollary (4.4.6) (cf. [Gi, I 2.3]). *Let $(A, \mathcal{A})$ be a good filtered ring and $\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ a morphism of good filtered $(A, \mathcal{A})$-modules. If $\mathrm{gr}(\varphi)$ is injective (resp. surjective, resp. bijective), then so is φ and φ is strict.*

Proof. $\mathcal{M}$ and $\mathcal{N}$ are bounded and separated (see (4.2.12)). The injective case follows from (4.4.4) (i). Let $\mathrm{gr}(\varphi)$ be surjective. Pushforwards of good filtrations are good again (see (4.2.10)) and φ is surjective by (4.4.4) (ii). All induced morphisms $M_n \to N_n$ are also surjective by carrying over the argument to the induced filtrations on them which are good again by (4.2.10). This proves the strictness. □

Corollary (4.4.7) ([Gi, I 2.4]). *Let $(A, \mathcal{A})$ be a good filtered ring and $(M, \mathcal{M})$ a good filtered $(A, \mathcal{A})$-module. If $\mathrm{gr}(\mathcal{M}) = 0$, then $M = 0$.*

Proof. Apply (4.4.6) to $0 \to M$. □

Definition (4.4.8). *Let $(M, \mathcal{M})$ be a filtered $(A, \mathcal{A})$-module, let $\mathcal{M}$ be exhaustive. The* ***order*** *of an element $m \in M$ is defined as*

$$\nu(m) := \nu_{\mathcal{M}}(m) := \sup\{n \in \mathbb{Z} | m \in M_n\} \in \mathbb{Z} \cup \{\infty\}.$$

In the case $\nu(m) = \infty$ we have $m \in \bigcap_{n \in \mathbb{Z}} M_n$, i.e. $m = 0$ if $\mathcal{M}$ is separated. If $\nu(m) = \infty$ we define $\mathrm{in}(m) := 0$, otherwise we declare the ***initial form*** *of m to be*

$$\mathrm{in}(m) := m \bmod M_{\nu(m)+1} \in \mathrm{gr}^{\nu(m)}(\mathcal{M}).$$

The elements $\mathrm{in}(m)$ for $m \in M$ are precisley the homogeneous elements of $\mathrm{gr}(\mathcal{M})$.

Remark (4.4.9). *In the situation of (4.4.8) we have for elements $m, m' \in M$ $\nu(m + m') \geq \min\{\nu(m), \nu(m')\}$ and equality holds if $\nu(m) \neq \nu(m')$. For $m \in M$ and $a \in A$ one has $\nu(am) \geq \nu(a) + \nu(m)$. If $\mathrm{gr}(\mathcal{A})$ is a domain, then for all $a, a' \in A$ we get $\nu(a \cdot a') = \nu(a) + \nu(a')$ and $\mathrm{in}(a \cdot a') = \mathrm{in}(a) \cdot \mathrm{in}(a')$.*

The notion of a standard basis is carried over to filtered modules (cf. [CJS, 1.17]).

Definition (4.4.10). *Let $(M, \mathcal{M})$ be an exhaustive filtered $(A, \mathcal{A})$-module. A **standard basis** of $(M, \mathcal{M})$ is a system of elements $(m_1, ..., m_n)$ of M such that $(\mathrm{in}(m_1), ..., \mathrm{in}(m_n))$ is a standard basis of the $\mathrm{gr}(\mathcal{A})$-module $\mathrm{gr}(\mathcal{M})$ (see (4.3.7)). If $\mathrm{gr}(\mathcal{M})$ is finitely generated over $\mathrm{gr}(\mathcal{A})$, then there exists such a standard basis.*

Lemma (4.4.11). *Let $\varphi : (M, \mathcal{M}) \to (N, \mathcal{N})$ be an injective strict morphism of exhaustive filtered $(A, \mathcal{A})$-modules. Then for any $m \in M$ we have $\nu(\varphi(m)) = \nu(m)$ and $gr(\varphi)(\mathrm{in}(m)) = \mathrm{in}(\varphi(m))$.*

Proof. This is clear if $\nu(m) = \infty$. If $\nu(m) = n$, then $\varphi(m) \in N_n$. If $\varphi(m) \in N_{n+1}$, then $\varphi(m) \in N_{n+1} \cap \varphi(M) = \varphi(M_{n+1})$ and since φ is injective we would have $m \in M_{n+1}$ which contradicts $\nu(m) = n$. Therefore also $\nu(\varphi(m)) = n$. □

Corollary (4.4.12). *Let $(M, \mathcal{M})$ be a good filtered module over a good filtered ring $(A, \mathcal{A})$. Let $(m_1, ..., m_n)$ be a standard basis of M. Then $M = \langle m_1, ..., m_n \rangle_A$.*

Proof. Let $N := \langle m_1, ..., m_n \rangle_A \subseteq M$ and equip N with the pullback filtration $\mathcal{N}$, so $\varphi : (N, \mathcal{N}) \to (M, \mathcal{M})$ is injective and strict. By (4.4.11) we have $\mathrm{in}(m_1), ..., \mathrm{in}(m_n) \in \mathrm{im}\, \mathrm{gr}(\varphi)$ and hence $\mathrm{gr}(\varphi)$ is surjective. By (4.4.6) also φ is surjective. □

4.5 Hilbert series of graded modules and cones

We are going to define Hilbert series for graded modules and filtered modules as elements of the ring of formal Laurent series $\mathbb{Z}((T))$. Hilbert series will be used to measure singularities later. Therefore we introduce an order on $\mathbb{Z}((T))$ and focus on estimations between series. Since we have to compare singularities in different dimensions, we work with series $H^{(n)}$, where one has n dimensions added in comparison to H (cf. (4.5.9)).

Definition (4.5.1). *For $H(T) \in \mathbb{Z}((T))$ and $n \in \mathbb{Z}$ we define series $H^{(n)}(T) := (1-T)^{-n} H(T) \in \mathbb{Z}((T))$. In particular $H^{(1)}(T) = (1 + T + T^2 + \cdots) \cdot H(T)$. We introduce a **partial order** on $\mathbb{Z}((T))$ by $\sum_{n \in \mathbb{Z}} a_n T^n \leq \sum_{n \in \mathbb{Z}} b_n T^n$ if $a_n \leq b_n$ for all $n \in \mathbb{Z}$. In particular $\sum_{n \in \mathbb{Z}} a_n T^n \geq 0$ means that $a_n \geq 0$ for all $n \in \mathbb{Z}$. We say that $\sum_{n \in \mathbb{Z}} a_n T^n$ is **increasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{Z}$. For two series $H(T), K(T) \in \mathbb{Z}((T))$ and $n \in \mathbb{Z}$ we write $H(T) \equiv K(T) \bmod T^n$ if $H(T) - K(T) \in T^n \mathbb{Z}[[T]]$.*

Remark (4.5.2). *Let $H(T), K(T) \in \mathbb{Z}((T))$. One easily sees the following:*

(i) *If $H(T) = \sum_{n \in \mathbb{Z}} a_n T^n$, then $H^{(1)}(T) = \sum_{n \in \mathbb{Z}} \left(\sum_{m \leq n} a_m \right) T^n$.*

(ii) *If $H(T) \geq 0$, then $H^{(1)}(T)$ is increasing.*

(iii) *If $H(T)$ is increasing, then $H(T) \leq T^{-e} H(T)$ and $(1 - T^e) H(T) \geq 0$ for $e \geq 0$.*

(iv) *If $H(T) \leq K(T)$, then $H^{(n)}(T) \leq K^{(n)}(T)$ for all $n \geq 0$. This does not hold in general for $n < 0$.*

For the following we fix a noetherian graded ring $A = \bigoplus_{n\geq 0} A_n$ (then also A_0 is noetherian, see (4.3.2)) and an additive function λ on the category of finitely generated A_0-modules that takes values in $\mathbb{N}$. We have in mind the length function for λ. For any finitely generated graded A-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ all M_n are finitely generated A_0-modules (see (4.3.4)). The graduation on M is bounded and $\lambda(M_n) = 0$ for $n \ll 0$. Therefore the next definition makes sense:

Definition (4.5.3). *The* ***Hilbert series*** *of a finitely generated graded A-module M is the power series*

$$H(M) := H(M,T) := \sum_{n\in\mathbb{Z}} \lambda(M_n)T^n \in \mathbb{Z}((T)).$$

The same information can be stored in the **Hilbert function** $H : \mathbb{Z} \to \mathbb{N}, H(n) = \lambda(M_n)$, but we take the point of view of power series to use the structure of $\mathbb{Z}((T))$.

Remark (4.5.4). *For a finitely generated graded A-module M and $m \in \mathbb{Z}$ we have the identity $H(M(m),T) = T^{-m}H(M,T)$. For an exact sequence of finitely generated graded A-modules $0 \to M^0 \to M^1 \to M^2 \to \cdots \to M^r \to 0$ one gets from the additivity of λ that $\sum_{i=0}^{r}(-1)^i H(M^i,T) = 0$. For a short exact sequence $0 \to L \to M \to N \to 0$ we therefore have $H(L,T), H(N,T) \leq H(M,T)$.*

Remark (4.5.5). *If A is generated by homogeneous elements $x_1, ..., x_s$ of degrees $k_1, ..., k_s > 0$, then*

$$H(M,T) = (1 - T^{k_1})^{-1} \cdots (1 - T^{k_s})^{-1} \cdot f(T)$$

for some $f(T) \in \mathbb{Z}[T,T^{-1}]$ (see [AM, 11.1], [Gi, I 3.3]). We do not discuss this any further since we are not interested in Hilbert polynomials.

Lemma (4.5.6). *Let M be a finitely generated graded A-module and $x \in A_k$. Let K be the kernel of $M \xrightarrow{\cdot x} M$. We have*

$$H(M/xM) \geq (1 - T^k)H(M)$$

and this is an equality if $K = 0$. If $\lambda = \dim_k$, k a field, then equality holds iff $K = 0$.

Proof. The graduation on M/xM is $(M/xM)_n = M_n/xM_{n-k}$ and the exact sequence of finitely generated graded A-modules

$$0 \to K \to M \xrightarrow{\cdot x} M(k) \to M/xM(k) \to 0$$

yields by (4.5.4)

$$T^{-k}H(M/xM) = T^{-k}H(M) - H(M) + H(K)$$

and therefore $H(M/xM) = (1 - T^k)H(M) + T^k H(K)$. If the above equality holds, then $H(K) = 0$ and this implies $K = 0$ at least if $\lambda = \dim_k$. □

Lemma (4.5.7). *Let M be a finitely generated graded A-module. Suppose that*

$A_0 = k$ is a field and $\lambda = \dim_k$. Then

$$H(M) \leq H(A) \cdot H(M/A_+M)$$

and this is an equality if and only if M is free.

Proof. We find a morphism $L \to M$ of finitely generated graded A-modules, where L is free, such that $L/A_+L \to M/A_+M$ is an isomorphism. By (4.3.6) then already $L \to M$ was surjective and we consider the exact sequence of graded A-modules $0 \to K \to L \to M \to 0$. $H(M) = H(L) - H(K) \leq H(L)$ and $H(L) = H(A) \cdot H(L/A_+L) = H(A) \cdot H(M/A_+M)$. Equality holds iff $H(K) = 0$, i.e. $K = 0$. □

We carry over Hilbert series to cones and make a few easy observations. A cone stands for the spectrum of a standard graded k-algebra for a field k (see def. (6.1.1)).

Definition (4.5.8). *If $C = \mathrm{Spec}(A)$ is a cone over the field k with graded k-algebra A we define $H(C) := H(A)$ for $\lambda = \dim_k$. We also use $H^{(n)}(C) := H^{(n)}(A)$.*

Lemma (4.5.9). *We consider cones over a field k.*

(i) *$H(C_1 \times_k C_2) = H(C_1) \cdot H(C_2)$ for two cones C_1, C_2.*

(ii) *$H(\mathbb{A}^n_k) = (1-T)^{-n}$ and for a cone C we have $H(C \times_k \mathbb{A}^n_k) = H^{(n)}(C)$.*

Proof. (i): Let $C_i = \mathrm{Spec}(A_i)$ for graded k-algebras A_i for $i = 1, 2$. Then

$$H(A_1 \otimes_k A_2) = \sum_{d \geq 0} \dim_k \left(\bigoplus_{i+j=d} (A_i \otimes_k A_j) \right) T^d =$$

$$= \sum_{d \geq 0} \sum_{i+j=d} (\dim_k(A_i) T^i) \cdot (\dim_k(A_j) T^j) = H(A_1) \cdot H(A_2).$$

(ii): $H(\mathbb{A}_k) = H(k[X]) = (1-T)^{-1}$ and $\mathbb{A}^{n+1}_k \cong \mathbb{A}^n_k \times_k \mathbb{A}_k$; now use (i). □

Example (4.5.10). Let k be a field of positive characteristic, $S = k[X_1, ..., X_n]$ with the standard graduation and $\lambda = \dim_k$. Let $\sigma = (\sigma_1, ..., \sigma_m)$ be a $k[F]$-independent system of homogeneous additive polynomials of degrees $q_1, ..., q_m$. Then

$$H(\mathrm{Spec}(S/\langle\sigma\rangle)) = H(S/\langle\sigma\rangle) = \frac{(1-T^{q_1}) \cdots (1-T^{q_m})}{(1-T)^n}.$$

In particular we can compute the Hilbert series of any subgroup of $V = \mathrm{Spec}(S)$.

Proof. Since $H(S) = (1-T)^{-n}$, this follows from (4.5.6) as soon as σ_i is not a zero divisor in $S/\langle\sigma_1, ..., \sigma_{i-1}\rangle$. Let $f \in S$ with $\sigma_i f \in \langle\sigma_1, ..., \sigma_{i-1}\rangle$. We can develop f in the basis from (1.4.5) as $f = \sum_{N \in \Lambda', M \in \Lambda''} a_{N,M} \sigma^N X^M$ with $a_{N,M} \in k$. Automatically $\sigma_i f$ is developed in this basis too. Whenever $a_{N,M} \neq 0$ we must have $\sigma^N \in \langle\sigma_1, ..., \sigma_{i-1}\rangle$ because $\sigma_i f \in \langle\sigma_1, ..., \sigma_{i-1}\rangle$. Hence $f \in \langle\sigma_1, ..., \sigma_{i-1}\rangle$. □

4.6 Hilbert series of good filtered modules

We carry over Hilbert series to filtered modules.

Definition (4.6.1). *Suppose that $(A, \mathcal{A})$ is an n-good filtered ring, $\operatorname{gr}^0(\mathcal{A})$ is artinian and $(M, \mathcal{M})$ is a good filtered $(A, \mathcal{A})$-module. By (4.4.5) $\operatorname{gr}(\mathcal{A})$ is noetherian and the length function ℓ is an additive function on the category of finitely generated $\operatorname{gr}^0(\mathcal{A})$-modules. By (4.4.5) $\operatorname{gr}(\mathcal{M})$ is a finitely generated graded $\operatorname{gr}(\mathcal{A})$-module. Therefore we can apply the theory of Hilbert series from 4.5 here and define*

$$H(\mathcal{M}) := H(\mathcal{M}, T) := H(\operatorname{gr}(\mathcal{M}), T) = \sum_{n \in \mathbb{Z}} \ell(M_n/M_{n+1})T^n.$$

We will also use the notation $H(M, T)$ or $H(M)$ for it and write $H(M, I)$ if we refer to the I-adic filtration on M.

Lemma (4.6.2). *In the situation of (4.6.1) the following are true:*

(i) *$H^{(m)}(\mathcal{M}(n)) = T^{-n}H^{(m)}(\mathcal{M})$ for all $m, n \in \mathbb{Z}$.*

(ii) *For a strict exact sequence of filtered $(A, \mathcal{A})$-modules $0 \to (L, \mathcal{L}) \to (M, \mathcal{M}) \to (N, \mathcal{N}) \to 0$ we have $H(\mathcal{M}) = H(\mathcal{L}) + H(\mathcal{N})$.*

(iii) *$H^{(1)}(\mathcal{M}) = \sum_{n \in \mathbb{Z}} \ell(M/M_{n+1})T^n$ and $H^{(1)}(\mathcal{M})$ is increasing. $H^{(1)}$ is sometimes called the* ***Hilbert-Samuel function****.*

(iv) *If $\mathcal{M}'$ is another good $(A, \mathcal{A})$-filtration on M such that $\mathcal{M} \subseteq \mathcal{M}'$, then we have $H^{(1)}(\mathcal{M}', T) \leq H^{(1)}(\mathcal{M}, T)$ and equality holds iff $\mathcal{M} = \mathcal{M}'$.*

Proof. (i) and (ii) are immediate from (4.5.4) by observing (4.4.1) and (4.4.2) (note that $\mathcal{L}, \mathcal{N}$ are good by (4.2.10)). For (iii) assume that $M_e = M$. The coefficient of T^n in $H^{(1)}(\mathcal{M}) = (1 + T + T^2 + \cdots)H(\mathcal{M})$ equals $\ell(M_n/M_{n+1}) + \ell(M_{n-1}/M_n) + \cdots + \ell(M_e/M_{e+1}) = \ell(M/M_{n+1})$ and clearly $\ell(M/M_{n+2}) \geq \ell(M/M_{n+1})$. (iii) yields

$$H^{(1)}(\mathcal{M}') = \sum_{n \in \mathbb{Z}} \ell(M/M'_{n+1})T^n \leq \sum_{n \in \mathbb{Z}} \ell(M/M_{n+1})T^n = H^{(1)}(\mathcal{M})$$

since there is an exact sequence $0 \to M'_{n+1}/M_{n+1} \to M/M_{n+1} \to M/M'_{n+1} \to 0$. This also shows that $\ell(M/M'_{n+1}) = \ell(M/M_{n+1})$ implies $\ell(M'_{n+1}/M_{n+1}) = 0$ which means $M'_{n+1} = M_{n+1}$. □

Remark (4.6.3). *Suppose that $I \subseteq A$ is an ideal in a noetherian ring A, A/I is artinian and $(M, \mathcal{M})$ is a good filtered (A, I)-module (equivalently M is a finitely generated A-module and $\mathcal{M}$ is an I-stable filtration on M (see (4.2.9))). Then all conditions of (4.6.1) hold and we can work with $H(\mathcal{M})$.*

Lemma (4.6.4). *Let M be finitely generated over a noetherian local ring $(A, \mathfrak{m})$ with residue field k. Then*

$$H(M, \mathfrak{m}) \leq H(A, \mathfrak{m}) \cdot \dim_k(M/\mathfrak{m}M)$$

and this is an equality if and only if M is a free A-module.

Proof. We find a free A-module L of finite rank and an A-module homomorphism $\varphi : L \to M$ which becomes an isomorphism after tensoring with $A/\mathfrak{m}$, i.e. $L/\mathfrak{m}L \cong M/\mathfrak{m}M$. By the Nakayama lemma φ is already surjective. Since $\varphi(\mathfrak{m}^n L) = \mathfrak{m}^n \varphi(L) = \mathfrak{m}^n M$ we see that φ is a strict morphism of good filtered $(A, \mathfrak{m})$-modules if we equip all modules with the $\mathfrak{m}$-adic filtrations. Then we get a strict exact sequence $0 \to K \to L \to M \to 0$ of good filtered $(A, \mathfrak{m})$-modules. Therefore $H(M, \mathfrak{m}) = H(L, \mathfrak{m}) - H(\mathcal{K})$. $H(L, \mathfrak{m}) = \mathrm{rank}_A(L) \cdot H(A, \mathfrak{m})$ and $\mathrm{rank}_A(L) = \dim_k(L/\mathfrak{m}L) = \dim_k(M/\mathfrak{m}M)$ show that $H(M, \mathfrak{m}) \leq H(A, \mathfrak{m}) \cdot \dim_k(M/\mathfrak{m}M)$ and if equality holds, then $H(\mathcal{K}) = 0, \mathrm{gr}(\mathcal{K}) = 0$ and $K = 0$. □

The following result is crucial for the proof that our invariant for singularities will not increase under a permissible blow up. It goes back to Bennett.

Proposition (4.6.5) (cf. [Gi, I 3.9], [H4, Proposition 6]). *In the situation of (4.6.3) suppose additionally that I is contained in the radical of A. Let $x \in I^k, k > 0$ and define $X := x \bmod I^{k+1} \in \mathrm{gr}_I^k(A)$. The morphism of filtered (A, I)-modules $\varphi : M(-k) \xrightarrow{\cdot x} M$ induces a morphism of graded $\mathrm{gr}_I(A)$-modules $\mathrm{gr}(\varphi) : \mathrm{gr}(\mathcal{M})(-k) \xrightarrow{\cdot X} \mathrm{gr}(\mathcal{M})$. The following hold:*

(i) $H^{(1)}(M/xM) \geq (1 - T^k) H^{(1)}(M) \geq H(M)$.

(ii) $H^{(1)}(M/xM) = (1 - T^k) H^{(1)}(M)$ *if and only if* $\mathrm{gr}(\varphi)$ *is injective, in which case also φ is injective and strict and we have an isomorphism*

$$\mathrm{gr}(M)/X\,\mathrm{gr}(M) \cong \mathrm{gr}(M/xM).$$

Proof. We get strict exact sequences of good filtered (A, I)-modules

$$0 \to (K, \mathcal{K}) \to (M, \mathcal{M}(-k)) \to (N, \mathcal{N}') \to 0$$

$$0 \to (N, \mathcal{N}'') \to (M, \mathcal{M}) \to (M/xM, \mathcal{M}') \to 0$$

with $N = \mathrm{im}(\varphi)$ and $\mathcal{N}' \subseteq \mathcal{N}''$. Using (4.6.2) we find

$$H^{(1)}(\mathcal{M}') = (1 - T^k) H^{(1)}(\mathcal{M}) + [H^{(1)}(\mathcal{N}') - H^{(1)}(\mathcal{N}'')] + H^{(1)}(\mathcal{K})$$

and $[H^{(1)}(\mathcal{N}') - H^{(1)}(\mathcal{N}'')] \geq 0$. Therefore $H^{(1)}(\mathcal{M}') \geq (1 - T^k) H^{(1)}(\mathcal{M}) \geq (1 - T) H^{(1)}(\mathcal{M}) = H(\mathcal{M})$ and (i) is proved. For (ii) assume first that $\mathrm{gr}(\varphi)$ is injective. Then by (4.4.6) also φ is injective and strict which means $\mathcal{N}' = \mathcal{N}''$ and $K = 0$ and therefore $H^{(1)}(\mathcal{M}') = (1 - T^k) H^{(1)}(\mathcal{M})$. Assume on the other hand, that this is an equality. Then we must have $H^{(1)}(\mathcal{K}) = 0$ and $H^{(1)}(\mathcal{N}') = H^{(1)}(\mathcal{N}'')$. This implies $K = 0$ by (4.4.7) and $\mathcal{N}' = \mathcal{N}''$ by (4.6.2) (iv), so φ is strict and injective and therefore $\mathrm{gr}(\varphi)$ is injective. In this case we have a strict exact sequence $M(-k) \xrightarrow{\varphi} M \to M/xM \to 0$ and hence $\mathrm{gr}(\mathcal{M})(-k) \xrightarrow{\mathrm{gr}(\varphi)} \mathrm{gr}(\mathcal{M}) \to \mathrm{gr}(\mathcal{M}') \to 0$ is exact, giving the last isomorphism by (4.4.2). □

5 Bifiltrations

We recall the notion of bifiltrations from [Gi, II 2.3 ff.]. Bifiltrations are useful for the comparison of two different filtrations on a module. Later we use them to compare data between two points of a scheme. We begin in a broader perspective than [Gi], where the local case is treated from the beginning, but later also bifiltrations on polynomial rings are used ([Gi, III 2.5.2 f.], cf. 5.4). The central result is (5.3.3) for which we develop an analog over polynomial rings in (5.4.3).

5.1 Bifiltered modules and harmonious modules

Throughout this section A is a noetherian ring and $\mathfrak{p} \subseteq \mathfrak{q} \subseteq A$ are ideals. Then $(A, \mathfrak{p})$ and $(A, \mathfrak{q})$ are n-good filtered rings.

Definition (5.1.1). *A* ***bifiltered*** $(A, \mathfrak{p}, \mathfrak{q})$***-module*** $(E, \mathcal{E}', \mathcal{E}'')$ *is an A-module E equipped with the following data and conditions:*

(i) *a good* $\mathfrak{p}$*-filtration* $\mathcal{E}'$ *on* E.

(ii) *a good* $\mathfrak{q}$*-filtration* $\mathcal{E}''$ *on* E.

(iii) $\mathcal{E}' \subseteq \mathcal{E}''$, *i.e.* $\mathcal{E}'_n \subseteq \mathcal{E}''_n$ *for all* $n \in \mathbb{Z}$.

E is called ***harmonious*** *if for all n one has* $E''_n = \sum_i \mathfrak{q}^{n-i} E'_i$. *$E$ is called* ***free*** *if there exist a basis $e_1, ..., e_p$ of E and integers $n_1, ..., n_p$ such that* $E'_n = \sum \mathfrak{p}^{n-n_i} e_i$ *and* $E''_n = \sum \mathfrak{q}^{n-n_i} e_i$ *for all n, i.e. E is a direct sum of modules* $(A, \mathfrak{p}(-n_i), \mathfrak{q}(-n_i))$. *A* ***morphism*** *of bifiltered modules is a morphism of modules that respects both filtrations, it is called* ***bistrict*** *if it is strict for both filtrations.*

Remark (5.1.2). *Let* $(E, \mathcal{E}', \mathcal{E}'')$ *be a bifiltered* $(A, \mathfrak{p}, \mathfrak{q})$*-module.*

(i) *The natural morphism* $\mathrm{gr}(\mathcal{E}') \to \mathrm{gr}(\mathcal{E}'')$ *is compatible with* $\mathrm{gr}_{\mathfrak{p}}(A) \to \mathrm{gr}_{\mathfrak{q}}(A)$ *and we get a canonical morphism*

(5.1.2.A) $$\mathrm{gr}_{\mathfrak{q}}(A) \otimes_{\mathrm{gr}_{\mathfrak{p}}(A)} \mathrm{gr}(\mathcal{E}') \to \mathrm{gr}(\mathcal{E}'')$$

of $\mathrm{gr}_{\mathfrak{q}}(A)$*-modules.*

(ii) *If E is free, then E is harmonious (e.g. use (5.1.3) (iv)) and (5.1.2.A) is an isomorphism.*

(iii) *E is free if* $\mathcal{E}'$ *and* $\mathcal{E}''$ *are the* $\mathfrak{p}$*- and* $\mathfrak{q}$*-adic filtrations on a free A-module.*

(iv) *For fixed* $\mathcal{E}'$ *there exists a unique* $\mathfrak{q}$*-good filtration* $\mathcal{E}''$ *of E such that* $(\mathcal{E}', \mathcal{E}'')$ *is harmonious.*

We enlarge the list of characterizations of harmonious bifiltered modules from [Gi, II 2.3.1]:

Lemma (5.1.3). *For a bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-module $(E, \mathcal{E}', \mathcal{E}'')$ the following are equivalent:*

(i) *E is harmonious.*

(ii) *$E''_n = \sum_{i \le n} \mathfrak{q}^{n-i} E'_i$ for all $n \in \mathbb{Z}$.*

(iii) *$E''_n = E'_n + \mathfrak{q} E''_{n-1}$ for all $n \in \mathbb{Z}$.*

(iv) *There is a free bifiltered module $(L, \mathcal{L}', \mathcal{L}'')$ and a bistrict surjective morphism $p : L \to E$.*

If these conditions hold, also the following are true:

(v) *(5.1.2.A) is surjective.*

(vi) *There exists a standard basis $(e_1, ..., e_n)$ of $(E, \mathcal{E}'')$ with $\nu_{\mathcal{E}'}(e_i) = \nu_{\mathcal{E}''}(e_i)$ for $i = 1, ..., n$.*

(vii) *$E''_n = \sum_i \mathfrak{q}^{n-i} E'_i + E''_{n+1}$ for all $n \in \mathbb{Z}$.*

If $\mathfrak{q}$ is contained in the radical of A or A is a graded ring with $\mathfrak{q} \subseteq A_+$ and E is a bounded graded A-module, then the last conditions are equivalent to all the others.

Proof. (i) $\Rightarrow$ (ii): For $i \ge n$ we have $\mathfrak{q}^{n-i} E'_i = E'_i$. (ii) $\Rightarrow$ (iii): $E''_n = E'_n + \sum_{i \le n-1} \mathfrak{q}^{n-i} E'_i = E'_n + \mathfrak{q} \sum_{i \le n-1} \mathfrak{q}^{n-1-i} E_i = E'_n + \mathfrak{q} E''_{n-1}$. (iii) $\Rightarrow$ (i): $\mathcal{E}'$ and $\mathcal{E}''$ are good, so there is some e such that $E'_n = E''_n = E$ for all $n \le e$. Hence in proving $E''_n = \sum_i \mathfrak{q}^{n-i} E'_i$ we can proceed by induction on n and get $E''_n = E'_n + \mathfrak{q} \sum_i \mathfrak{q}^{n-1-i} E'_i = \sum_i \mathfrak{q}^{n-i} E'_i$. (i) $\Rightarrow$ (iv): Since $\mathcal{E}'$ is $\mathfrak{p}$-good, there exists a free filtered $(A, \mathfrak{p})$-module $(L, \mathcal{L}')$ (see (4.2.6)) together with a strict surjection $p : L \to E$ for the $\mathfrak{p}$-filtrations. Since L is harmonious, $p(L''_n) = p(\sum_i \mathfrak{q}^{n-i} L'_i) = \sum_i \mathfrak{q}^{n-i} p(L'_i) = \sum_i \mathfrak{q}^{n-i} E'_i = E''_i$ shows that p is also strict for the $\mathfrak{q}$-filtrations. (iv) $\Rightarrow$ (i): $E''_n = p(L''_n) = p(\sum_i \mathfrak{q}^{n-i} L'_i) = \sum_i \mathfrak{q}^{n-i} p(L'_i) = \sum_i \mathfrak{q}^{n-i} E'_i$. (i) $\Rightarrow$ (v) $\Rightarrow$ (vii) are clear. (v) $\Leftrightarrow$ (vi): (v) is equivalent to: there exist elements $f_1, ..., f_m$ of $\mathrm{gr}(\mathcal{E}')$ that generate $\mathrm{gr}(\mathcal{E}'')$ as a $\mathrm{gr}_{\mathfrak{q}}(A)$-module. Of course these can be chosen homogeneous and with increasing order such that none can be omitted, i.e. as a standard basis. (vii) $\Rightarrow$ (i): Let $F_n := \sum_i \mathfrak{q}^{n-i} E'_i$. Clearly $F_{n+1} \subseteq F_n$. Since $(E, \mathcal{E}'')$ is good, $\mathcal{E}''$ is $\mathfrak{q}$-stable (see (4.2.9)), i.e. there exists $N > n$ such that $E''_{m+1} = \mathfrak{q} E''_m$ for all $m \ge N$. Now $E''_n = F_n + E''_{n+1} = F_n + F_{n+1} + E''_{n+2} = \cdots = F_n + E''_m$ and with (4.2.14) in the case that $\mathfrak{q}$ is contained in the radical of A, resp. by using the graduation, we see that

$$E''_n = \bigcap_{m \ge N} F_n + E''_m = \bigcap_{m \ge 0} F_n + \mathfrak{q}^m E''_N = F_n,$$

which ends the proof. □

(5.1.4) Let $(E, \mathcal{E}', \mathcal{E}'')$ be a bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-module. We are going do derive some other filtrations on related modules. They are essential to study the relation between $\mathcal{E}'$ and $\mathcal{E}''$. Suppose additionally that $A/\mathfrak{q}$ has finite length. We also write $\mathcal{E}'_m$ for E'_m and $\mathcal{E}''_m$ for E''_m.

(5.1.4.1) Fix $n \in \mathbb{Z}$. E'_n becomes a good filtered $(A, \mathfrak{q})$-module when we equip it with the filtration $\mathcal{E}^n$ induced by $\mathcal{E}''$ (see (4.2.10) (i)), explicitely $\mathcal{E}^n_m = \mathcal{E}'_n \cap \mathcal{E}''_m$. As in 4.6 we can form the Hilbert series $H(\mathcal{E}^n)$. Fix an integer n_0 such that $\mathcal{E}'_{n_0} = E$. Then for $n \leq n_0$ we have $\mathcal{E}^n = \mathcal{E}''$ since $\mathcal{E}^n_m = \mathcal{E}'_n \cap \mathcal{E}''_m = \mathcal{E}''_m$. Also note that for $m \leq n$ in general $\mathcal{E}'_n \subseteq \mathcal{E}''_n \subseteq \mathcal{E}''_m$ and therefore $\mathcal{E}^n_m = \mathcal{E}'_n$, so $H(\mathcal{E}^n) \in T^n\mathbb{Z}[[T]]$.

(5.1.4.2) Fix $n \in \mathbb{Z}$. $\underline{E}^n := \mathrm{gr}^n(\mathcal{E}') = E'_n/E'_{n+1}$ becomes a good filtered $(A, \mathfrak{q})$-module (see (4.2.10)) if we equip it with the filtration $\underline{\mathcal{E}}^n$ defined by $\underline{\mathcal{E}}^n_m$ being the image of $\mathcal{E}'_n \cap \mathcal{E}''_{m+n}$ in $\underline{E}^n$, explicitely $\underline{\mathcal{E}}^n_m = (\mathcal{E}'_n \cap \mathcal{E}''_{m+n}) + \mathcal{E}'_{n+1}/\mathcal{E}'_{n+1}$.

(5.1.4.A) $$0 \to (E'_{n+1}, \mathcal{E}^{n+1}) \to (E'_n, \mathcal{E}^n) \to (\underline{E}^n, \underline{\mathcal{E}}^n(-n)) \to 0$$

is a strict exact sequence of good filtered $(A, \mathfrak{q})$-modules and yields

(5.1.4.B) $$T^n H(\underline{\mathcal{E}}^n) = H(\mathcal{E}^n) - H(\mathcal{E}^{n+1}).$$

Note that $\underline{E}^n = 0$ for $n \leq n_0 - 1$. The $\mathfrak{q}$-adic filtration of $\underline{E}^n$ is contained in $\underline{\mathcal{E}}^n$:

$$\mathfrak{q}^m \cdot (\mathcal{E}'_n/\mathcal{E}'_{n+1}) = (\mathfrak{q}^m \mathcal{E}'_n + \mathcal{E}'_{n+1})/\mathcal{E}'_{n+1} \subseteq (\mathcal{E}'_n \cap \mathcal{E}''_{m+n}) + \mathcal{E}'_{n+1}/\mathcal{E}'_{n+1}.$$

(5.1.4.3) A morphism of bifiltered modules $(E, \mathcal{E}', \mathcal{E}'') \to (F, \mathcal{F}', \mathcal{F}'')$ induces morphisms of filtered modules $(E'_n, \mathcal{E}^n) \to (F'_n, \mathcal{F}^n)$ and $(\underline{E}^n, \underline{\mathcal{E}}^n) \to (\underline{F}^n, \underline{\mathcal{F}}^n)$.

(5.1.4.4) Let $\mathfrak{p}$ be a prime ideal. Then $E_\mathfrak{p}$ is finitely generated over the noetherian local ring $(A_\mathfrak{p}, \mathfrak{n} := \mathfrak{p}_\mathfrak{p})$ and we equip $E_\mathfrak{p}$ with a good $\mathfrak{n}$-filtration $\mathcal{E}'_\mathfrak{p}$: For $n \in \mathbb{Z}$ let $E'_{\mathfrak{p},n} := A_\mathfrak{p} \cdot E'_n = (E'_n)_\mathfrak{p} \subseteq E_\mathfrak{p}$. This filtration is good (see (4.2.10)). Localiziation commutes with quotients and therefore $\mathrm{gr}(\mathcal{E}'_\mathfrak{p}) \cong \mathrm{gr}(\mathcal{E}')_\mathfrak{p}$. We also can form the Hilbert series $H(\mathcal{E}'_\mathfrak{p})$.

Lemma (5.1.5) (cf. [Gi, II 2.3.4, 2.3.5]). *In the situation of (5.1.4) the following hold:*

(i) $H(\mathcal{E}'') = \sum_{n\in\mathbb{Z}} T^n H(\underline{\mathcal{E}}^n)$.

(ii) $H^{(1)}(\underline{\mathcal{E}}^n) \leq H^{(1)}(\mathrm{gr}^n(\mathcal{E}'), \mathfrak{q})$ *and this is an equality if and only if* $\underline{\mathcal{E}}^n$ *is the* $\mathfrak{q}$*-adic filtration on* $\mathrm{gr}^n(\mathcal{E}')$.

(iii) *If* $\mathrm{gr}^n(\mathcal{E}')$ *is a free* $A/\mathfrak{p}$*-module, then* $H(\mathrm{gr}^n(\mathcal{E}'), \mathfrak{q}) = \mathrm{rank}_{A/\mathfrak{p}}(\mathrm{gr}^n(\mathcal{E}'))H(A/\mathfrak{p}, \mathfrak{q})$.

(iv) *If all* $\mathrm{gr}^n(\mathcal{E}')$ *are free* $A/\mathfrak{p}$*-modules, then*

$$H^{(1)}(\mathcal{E}'') \leq \sum_n T^n \,\mathrm{rank}_{A/\mathfrak{p}}(\mathrm{gr}^n(\mathcal{E}'))H^{(1)}(A/\mathfrak{p}, \mathfrak{q})$$

and this is an equality if and only if $\underline{\mathcal{E}}^n$ *is the* $\mathfrak{q}$*-adic filtration on* $\mathrm{gr}^n(\mathcal{E}')$ *for all* n.

Proof. (i): We show by induction on $N \in \mathbb{Z}$ that

$$H(\mathcal{E}'') = H(\mathcal{E}^N) + \sum_{n<N} T^n H(\underline{\mathcal{E}}^n).$$

This will prove (i) since $H(\underline{\mathcal{E}}^n) \in \mathbb{Z}[[T]]$ and $H(\mathcal{E}^N) \in T^N\mathbb{Z}[[T]]$. The equation is clear for $N \leq n_0$ since then $\underline{E}^n = 0$ for all $n < N$ and $\mathcal{E}'' = \mathcal{E}^N$. The induction step is done by (5.1.4.B): $H(\mathcal{E}^N) = H(\mathcal{E}^{N+1}) + T^N H(\underline{\mathcal{E}}^N)$. (ii): As remarked in (5.1.4.2), the $\mathfrak{q}$-adic filtration of $\mathrm{gr}^n(\mathcal{E}')$ is contained in the filtration $\underline{\mathcal{E}}^n$. Therefore $H^{(1)}(\underline{\mathcal{E}}^n) \leq H^{(1)}(\mathrm{gr}^n(\mathcal{E}'), \mathfrak{q})$ (see (4.6.2)). (iii) is clear. (iv) follows from (i) - (iii). □

5.2 Exact sequences

Remark (5.2.1) (cf. [Gi, II 2.3.2]). *A bistrict short exact sequence of bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-modules $0 \to K \to L \to E \to 0$ is an exact sequence of A-modules which are bifiltered that is strict for both filtrations. Note that if $(L, \mathcal{L}', \mathcal{L}'')$ is a free bifiltered module, then $(E, \mathcal{E}', \mathcal{E}'')$ is harmonious by (5.1.3) (iv). We get exact sequences of the graded modules*

$$0 \to \mathrm{gr}(\mathcal{K}') \to \mathrm{gr}(\mathcal{L}') \to \mathrm{gr}(\mathcal{E}') \to 0,$$

$$0 \to \mathrm{gr}(\mathcal{K}'') \to \mathrm{gr}(\mathcal{L}'') \to \mathrm{gr}(\mathcal{E}'') \to 0$$

and morphisms of filtered modules as in (5.1.4.3). Suppose that $(E, \mathcal{E}', \mathcal{E}'')$ is some harmonious bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-module. Then by (5.1.3) (iv) there exists a strict surjection from a free bifiltered module $(L, \mathcal{L}', \mathcal{L}'') \to (E, \mathcal{E}', \mathcal{E}'')$ and its kernel K can be equipped with the two pullback filtrations $\mathcal{K}'$ and $\mathcal{K}''$ from L, which are good again (see (4.2.10)), and $(K, \mathcal{K}', \mathcal{K}'')$ is a bifiltered module. Thus we obtain a bistrict short exact sequence as described above.

Lemma (5.2.2) (cf. [Gi, II 2.4.5]). *Let $0 \to K \to L \to E \to 0$ be a bistrict short exact sequence of bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-modules. If L is a free bifiltered module, then the following are equivalent:*

(i) $\mathrm{gr}_{\mathfrak{q}}(A) \otimes_{\mathrm{gr}_{\mathfrak{p}}(A)} \mathrm{gr}(\mathcal{K}') \to \mathrm{gr}(\mathcal{K}'')$ *is surjective.*

(ii) $\mathrm{gr}_{\mathfrak{q}}(A) \otimes_{\mathrm{gr}_{\mathfrak{p}}(A)} \mathrm{gr}(\mathcal{E}') \to \mathrm{gr}(\mathcal{E}'')$ *is an isomorphism.*

Proof. Tensoring the first exact sequence of graded modules from (5.2.1) over $\mathrm{gr}_{\mathfrak{p}}(A)$ with $\mathrm{gr}_{\mathfrak{q}}(A)$ we get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathrm{gr}_{\mathfrak{q}}(A) \otimes \mathrm{gr}(\mathcal{K}') & \longrightarrow & \mathrm{gr}_{\mathfrak{q}}(A) \otimes \mathrm{gr}(\mathcal{L}') & \longrightarrow & \mathrm{gr}_{\mathfrak{q}}(A) \otimes \mathrm{gr}(\mathcal{E}') & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & \mathrm{gr}(\mathcal{K}'') & \longrightarrow & \mathrm{gr}(\mathcal{L}'') & \longrightarrow & \mathrm{gr}(\mathcal{E}'') & \longrightarrow & 0
\end{array}$$

γ is surjective since E is harmonious ((5.1.3) (v)) and β is an isomorphism since L is free. By the snake lemma therefore α is surjective (i.e. (i) holds) if and only if γ is injective (i.e. (ii) holds). □

Lemma (5.2.3) (cf. [Gi, II 3.2.2]). *Let $0 \to K \to E \to L \to 0$ be a bistrict short exact sequence of bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-modules and $n \in \mathbb{Z}$. Consider the exact sequence of $A/\mathfrak{p}$-modules:*

(5.2.3.A) $$0 \to \mathrm{gr}^n(\mathcal{K}') \to \mathrm{gr}^n(\mathcal{L}') \to \mathrm{gr}^n(\mathcal{E}') \to 0.$$

Suppose that $\underline{\mathcal{L}}^n$ is the $\mathfrak{q}$-adic filtration on $\operatorname{gr}^n(\mathcal{L}')$ and that $\operatorname{gr}^n(\mathcal{E}')$ is a projective $A/\mathfrak{p}$-module. Then $\underline{\mathcal{K}}^n$ is the $\mathfrak{q}$-adic filtration on $\operatorname{gr}^n(\mathcal{K}')$ and $\operatorname{gr}^n(\mathcal{K}') \to \operatorname{gr}^n(\mathcal{L}')$ is strict, i.e. $\underline{\mathcal{K}}^n_m = \underline{\mathcal{L}}^n_m \cap \operatorname{gr}^n(\mathcal{K}')$ for all m.

Proof. We explain the following chain of inclusions that immediately gives the claims:

$$\mathfrak{q}^m \operatorname{gr}^n(\mathcal{K}') \overset{(a)}{\subseteq} \underline{\mathcal{K}}^n_m \overset{(b)}{\subseteq} \underline{\mathcal{L}}^n_m \cap \operatorname{gr}^n(\mathcal{K}') \overset{(c)}{=} \mathfrak{q}^m \operatorname{gr}^n(\mathcal{L}') \cap \operatorname{gr}^n(\mathcal{K}') \overset{(d)}{=} \mathfrak{q}^m \operatorname{gr}^n(\mathcal{K}').$$

(a): See (5.1.4.2). (b): Clear. (c): $\underline{\mathcal{L}}^n$ is the $\mathfrak{q}$-adic filtration. (d): Since $\operatorname{gr}^n(\mathcal{E}')$ is projective, the sequence (5.2.3.A) splits, hence $\operatorname{gr}^n(\mathcal{K}')$ is a direct summand of $\operatorname{gr}^n(\mathcal{L}')$ and therefore inherits the $\mathfrak{q}$-adic filtration. □

Lemma (5.2.4) (cf. [Gi, II 3.2.3]). *Let $0 \to K \to E \to L \to 0$ be a bistrict short exact sequence of bifiltered $(A, \mathfrak{p}, \mathfrak{q})$-modules and $n \in \mathbb{Z}$. Consider for $p \in \mathbb{Z}$ the sequence of filtered $(A/\mathfrak{p}, \mathfrak{q})$-modules*

(5.2.4.A) $$0 \to (\operatorname{gr}^p(\mathcal{K}'), \underline{\mathcal{K}}^p) \to (\operatorname{gr}^p(\mathcal{L}'), \underline{\mathcal{L}}^p) \to (\operatorname{gr}^p(\mathcal{E}'), \underline{\mathcal{E}}^p) \to 0.$$

Suppose that for all $p \leq n-1$ we know that $\underline{\mathcal{L}}^p$ is the $\mathfrak{q}$-adic filtration on $\operatorname{gr}^p(\mathcal{L}')$ and $\operatorname{gr}^p(\mathcal{E}')$ is a projective $A/\mathfrak{p}$-module. Then the following hold:

(i) *For all $p \leq n$ and $q \geq 0$ the image of $\underline{\mathcal{L}}^p_q$ in $\operatorname{gr}^n(\mathcal{E}')$ is $\underline{\mathcal{E}}^p_q$.*

(ii) *In particular for all $p \leq n-1$ all filtrations in (5.2.4.A) are the $\mathfrak{q}$-adic filtrations and the sequence is strict exact.*

Proof. (ii) follows immediately from (i) by (5.2.3). For (i) let $\underline{a} \in \underline{\mathcal{E}}^p_q$. Then there exists an $a \in E'_p \cap E''_{p+q}$ such that $\underline{a}$ is the class of a. There exists $b \in L''_{p+q}$ such that a is its image. We have to show that we can choose b in $L'_p \cap L''_{p+q}$. If this would not be possible, we still could choose $b \in L'_r \cap L''_{p+q}$ with r maximal and $r < p$. Let $\underline{b}$ be the class of b in $\underline{L}^r$ (which is not zero since r is maximal). We have $\underline{b} \in \underline{L}^r_{q+p-r}$. Since $r < p$, we know that $\underline{b}$ is mapped to zero in $\underline{E}^r$ because $\underline{a} \in \underline{E}^p$. Therefore we find $\underline{b} \in \underline{K}^r$. Since $r < p$, $\operatorname{gr}^r(\mathcal{E}')$ is projective and we can apply (5.2.3) which yields $\underline{b} \in \underline{K}^r \cap \underline{L}^r_{q+p-r} = \underline{K}^r_{q+p-r}$. Therefore we find $c \in K \cap L'_r \cap L''_{q+p}$ such that the class of c in $\underline{L}^r$ equals $\underline{b}$. Then $b - c \in L'_{r+1}$ and also $b - c \in L''_{q+p}$. Since $c \in K$, the image of $b - c$ in E is that of b. Therefore r cannot have been maximal and we can suppose that $b \in L'_p \cap L''_{p+q}$ in which case the class of b in $\underline{L}^p$ appears in $\underline{\mathcal{L}}^p_q$ and its image is $\underline{a}$, which finishes this proof. □

5.3 Bifiltered modules over local rings

We now come to the important result (5.3.3) about local rings. Throughout this section $(R, \mathfrak{m})$ is a noetherian local ring and $\mathfrak{p} \subseteq \mathfrak{m}$ is some prime ideal.

Lemma (5.3.1) (weak semi-continuity, cf. [Gi, II 2.3.6]). *If $(E, \mathcal{E}', \mathcal{E}'')$ is a harmonious bifiltered $(R, \mathfrak{p}, \mathfrak{m})$-module and $R/\mathfrak{p}$ is regular of dimension d, then*

$$H^{(1)}(\mathcal{E}'') \geq H^{(d+1)}(\mathcal{E}'_{\mathfrak{p}}).$$

Proof. We argue by induction on d. For $d = 0$ we have $\mathfrak{p} = \mathfrak{m}$ and $\mathcal{E}'' = \mathcal{E}'$ since E was harmonious and also $\mathcal{E}' \cong \mathcal{E}'_{\mathfrak{p}}$. For the induction step we introduce another prime ideal $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ such that $R/\mathfrak{q}$ is regular of dimension 1. Let $\mathcal{E}'''$ be the unique $\mathfrak{q}$-good filtration on E such that $(E, \mathcal{E}', \mathcal{E}''')$ is a harmonious bifiltered $(R, \mathfrak{p}, \mathfrak{q})$-module. By localization we get the harmonious bifiltered $(R_{\mathfrak{q}}, \mathfrak{p}_{\mathfrak{q}}, \mathfrak{q}_{\mathfrak{q}})$-module $(E_{\mathfrak{q}}, \mathcal{E}'_{\mathfrak{q}}, \mathcal{E}'''_{\mathfrak{q}})$. Since $R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}} \cong (R/\mathfrak{p})_{\mathfrak{q}}$ is a regular local ring of smaller dimension than $R/\mathfrak{p}$, we can apply the induction hypothesis and get $H^{(1)}(\mathcal{E}'''_{\mathfrak{q}}) \geq H^{(d)}(\mathcal{E}'_{\mathfrak{p}})$ and it remains to prove $H^{(1)}(\mathcal{E}'') \geq H^{(2)}(\mathcal{E}'''_{\mathfrak{q}})$. Therefore it suffices to treat the case $d = 1$ from the beginning. Let us take a bistrict short exact sequence of bifiltered modules $0 \to K \to L \to E \to 0$ with free L as in (5.2.1). Since $(L, \mathcal{L}', \mathcal{L}'')$ is free and $\mathrm{gr}_{\mathfrak{p}}(R)$ is free over $R/\mathfrak{p}$, also $\mathrm{gr}(\mathcal{L}')$ is free over $R/\mathfrak{p}$. $R/\mathfrak{p}$ is regular of dimension 1, hence a principal ideal domain and therefore also $\mathrm{gr}(\mathcal{K}') \subseteq \mathrm{gr}(\mathcal{L}')$ is free over $R/\mathfrak{p}$. (5.1.5) (iv) yields $H^{(1)}(\mathcal{K}'') \leq \sum_n T^n \operatorname{rank}_{R/\mathfrak{p}}(\mathrm{gr}^n(\mathcal{K}'))H^{(1)}(R/\mathfrak{p}, \mathfrak{m})$. Since $R/\mathfrak{p}$ is a discrete valuation ring, we have $H(R/\mathfrak{p}, \mathfrak{q}) = (1-T)^{-1}$. With $\operatorname{rank}_{R/\mathfrak{p}}(\mathrm{gr}^n(\mathcal{K}')) = \dim_{\mathrm{Quot}(R/\mathfrak{p})}(\mathrm{gr}^n(\mathcal{K}')_{\mathfrak{p}})$ we get (see (5.1.4.4)) $H^{(1)}(\mathcal{K}'') \leq H^{(2)}(\mathcal{K}'_{\mathfrak{p}})$. Since L is free we get in the same way $H^{(1)}(\mathcal{L}'') = H^{(2)}(\mathcal{L}'_{\mathfrak{p}})$. The exact sequences from (5.2.1) yield

$$H^{(1)}(\mathcal{E}'') = H^{(1)}(\mathcal{L}'') - H^{(1)}(\mathcal{K}'') \geq H^{(2)}(\mathcal{L}'_{\mathfrak{p}}) - H^{(2)}(\mathcal{K}'_{\mathfrak{p}}) = H^{(2)}(\mathcal{E}'_{\mathfrak{p}})$$

which finishes the proof. □

Lemma (5.3.2). *Let $(E, \mathcal{E}', \mathcal{E}'')$ be a harmonious bifiltered $(R, \mathfrak{p}, \mathfrak{m})$-module. Suppose that $R/\mathfrak{p}$ is regular of dimension d and $\mathrm{gr}(\mathcal{E}')$ is flat over $R/\mathfrak{p}$. Then the following hold:*

(i) $H(\mathcal{E}'') = H^{(d)}(\mathcal{E}'_{\mathfrak{p}})$.

(ii) *The filtration $\underline{\mathcal{E}}^n$ on $\mathrm{gr}^n(\mathcal{E}')$ is the $\mathfrak{m}$-adic filtration for all n.*

Proof. By (5.3.1) we have $H^{(1)}(\mathcal{E}'') \geq H^{(d+1)}(\mathcal{E}'_{\mathfrak{p}})$ and since all $\mathrm{gr}^n(\mathcal{E}')$ are free over $R/\mathfrak{p}$, (5.1.5) (iv) yields

$$H^{(1)}(\mathcal{E}'') \leq \sum_n T^n \operatorname{rank}_{R/\mathfrak{p}}(\mathrm{gr}^n(\mathcal{E}'))H^{(1)}(R/\mathfrak{p}, \mathfrak{m}) =$$

$$= \sum_n T^n \dim_{\mathrm{Quot}(R/\mathfrak{p})}(\mathrm{gr}^n(\mathcal{E}')_{\mathfrak{p}})(1-T)^{-d-1} = (1-T)^{-d-1} \cdot H(\mathcal{E}'_{\mathfrak{p}}) = H^{(d+1)}(\mathcal{E}'_{\mathfrak{p}}).$$

Together we get (i). The inequality at the begin of our calculation must be an equality and therefore we get (ii) from (5.1.5) (iv). □

Theorem (5.3.3) (cf. [Gi, II 2.4, 3.2])**.** *Let $(E, \mathcal{E}', \mathcal{E}'')$ be a harmonious bifiltered $(R, \mathfrak{p}, \mathfrak{m})$-module, where R is regular and $R/\mathfrak{p}$ is regular of dimension d. The following are equivalent:*

(i) $\mathrm{gr}(\mathcal{E}')$ *is flat over* $R/\mathfrak{p}$.

(ii) *The natural morphism* $\mathrm{gr}_{\mathfrak{m}}(R) \otimes_{\mathrm{gr}_{\mathfrak{p}}(R)} \mathrm{gr}(\mathcal{E}') \to \mathrm{gr}(\mathcal{E}'')$ *is an isomorphism.*

(iii) $H(\mathcal{E}'') = H^{(d)}(\mathcal{E}'_{\mathfrak{p}})$.

Proof. (i) $\Rightarrow$ (iii) was proved in (5.3.2).

(5.3.3.1) Since E is harmonious, the morphism in (ii) is surjective (see (5.1.3) (v)). To show that it is bijective, it is sufficent to show that both sides have the same Hilbert series. Choose regular parameters $z_1, ..., z_r, x_1, ..., x_d$ of R such that $\mathfrak{p} = \langle z_1, ..., z_r\rangle$. With

$$R/\mathfrak{p}[Z_1, ..., Z_r] \cong \mathrm{gr}_{\mathfrak{p}}(R) \to \mathrm{gr}_{\mathfrak{m}}(R) \cong R/\mathfrak{m}[Z_1, ..., Z_r, X_1, ..., X_d], \qquad Z_i \mapsto Z_i$$

we get

$$\mathrm{gr}_{\mathfrak{m}}(R) \otimes_{\mathrm{gr}_{\mathfrak{p}}(R)} \mathrm{gr}(\mathcal{E}') \cong R/\mathfrak{m}[X_1, ..., X_d] \otimes_{R/\mathfrak{m}} R/\mathfrak{m}[Z_1, ..., Z_r] \otimes_{R/\mathfrak{p}[Z_1,...,Z_r]} \mathrm{gr}(\mathcal{E}')$$

and therefore

$$H(\mathrm{gr}_{\mathfrak{m}}(R) \otimes_{\mathrm{gr}_{\mathfrak{p}}(R)} \mathrm{gr}(\mathcal{E}')) = H^{(d)}(\mathrm{gr}(\mathcal{E}')/\mathfrak{m}\,\mathrm{gr}(\mathcal{E}')). \tag{5.3.3.A}$$

(5.3.3.2) (i) $\Rightarrow$ (ii): $\mathrm{gr}(\mathcal{E}')$ is flat, hence free over $R/\mathfrak{p}$. For a free $R/\mathfrak{p}$-module M we have $\dim_{R/\mathfrak{m}}(M/(\mathfrak{m}/\mathfrak{p})M) = \mathrm{rank}_{R/\mathfrak{p}}(M) = \dim_{\mathrm{Quot}(R/\mathfrak{p})}(M_{\mathfrak{p}})$. This shows $H(\mathrm{gr}(\mathcal{E}')/\mathfrak{m}\,\mathrm{gr}(\mathcal{E}')) = H(\mathcal{E}'_{\mathfrak{p}})$ and this part of the proof is finished since we can use the equality in (iii).

(5.3.3.3) (ii) $\Rightarrow$ (i): By (5.3.3.A) we have

$$H(\mathcal{E}'') = (1-T)^{-d} \sum_{n\in\mathbb{Z}} T^n \dim_{R/\mathfrak{m}}(\mathrm{gr}^n(\mathcal{E}')/\mathfrak{m}\,\mathrm{gr}^n(\mathcal{E}'))$$

and for every n (4.6.4) implies

$$\begin{aligned} H(\mathrm{gr}^n(\mathcal{E}'), \mathfrak{m}) &\leq H(R/\mathfrak{p}, \mathfrak{m}) \cdot \dim_{R/\mathfrak{m}}(\mathrm{gr}^n(\mathcal{E}')/\mathfrak{m}\,\mathrm{gr}^n(\mathcal{E}')) = \\ &= (1-T)^{-d} \dim_{R/\mathfrak{m}}(\mathrm{gr}^n(\mathcal{E}')/\mathfrak{m}\,\mathrm{gr}^n(\mathcal{E}')). \end{aligned} \tag{5.3.3.B}$$

Together we find

$$H(\mathcal{E}'') \geq \sum_{n\in\mathbb{Z}} T^n H(\mathrm{gr}^n(\mathcal{E}'), \mathfrak{m}). \tag{5.3.3.C}$$

In (5.1.5) we saw that $H^{(1)}(\mathcal{E}'') \leq \sum_{n\in\mathbb{Z}} T^n H^{(1)}(\mathrm{gr}^n(\mathcal{E}'), \mathfrak{m})$. Therefore we have equality in (5.3.3.C) and then also in (5.3.3.B). From (4.6.4) we see that all $\mathrm{gr}^n(\mathcal{E}')$ are free.

(5.3.3.4) (iii) $\Rightarrow$ (i): Since $(E, \mathcal{E}', \mathcal{E}'')$ is harmonious, we find a bistrict short exact sequence of bifiltered modules $0 \to K \to L \to E \to 0$ as in (5.2.1) where L is free. We prove by induction on n, that $\mathrm{gr}^n(\mathcal{E}')$ is flat over $R/\mathfrak{p}$. Suppose that $\mathrm{gr}^p(\mathcal{E}')$ is flat for $p \leq n-1$. Since L is free, hence a direct sum of copies of $(R, \mathfrak{p}, \mathfrak{q})$, we know that $\mathrm{gr}(\mathcal{L}')$ is free over $R/\mathfrak{p}$ since R and $R/\mathfrak{p}$ are regular. Therefore by (5.3.2) the filtration $\underline{\mathcal{L}}^p$ on $\mathrm{gr}^p(\mathcal{L}')$ is the $\mathfrak{m}$-adic filtration for all p. Hence by (5.2.4) the filtration $\underline{\mathcal{E}}^p$ on $\mathrm{gr}^p(\mathcal{E}')$ is the $\mathfrak{m}$-adic filtration for all $p \leq n$. With (5.1.5) we therefore see that on the one hand

$$H^{(1)}(\mathcal{E}'') \equiv \sum_{p\leq n} T^p H^{(1)}(\underline{\mathcal{E}}^p) \equiv$$

$$\equiv \sum_{p<n} T^p(1-T)^{-d-1} \operatorname{rank}_{R/\mathfrak{p}}(\operatorname{gr}^p(\mathcal{E}')) + T^n \dim_{R/\mathfrak{m}}(\operatorname{gr}^n(\mathcal{E}')/\mathfrak{m} \operatorname{gr}^n(\mathcal{E}')) \bmod T^{n+1}$$

and on the other hand

$$H^{(1)}(\mathcal{E}'') \equiv H^{(d+1)}(\mathcal{E}'_{\mathfrak{p}}) \equiv \sum_{p \le n} T^p(1-T)^{-d-1} \dim_{\operatorname{Quot}(R/\mathfrak{p})}(\operatorname{gr}^p(\mathcal{E}')_{\mathfrak{p}}) \equiv$$

$$\equiv \sum_{p<n} T^p(1-T)^{-d-1} \operatorname{rank}_{R/\mathfrak{p}}(\operatorname{gr}^p(\mathcal{E}')) + T^n \dim_{\operatorname{Quot}(R/\mathfrak{p})}(\operatorname{gr}^n(\mathcal{E}')_{\mathfrak{p}}) \bmod T^{n+1}.$$

By comparing the two results we get $\dim_{R/\mathfrak{m}}(\operatorname{gr}^n(\mathcal{E}')/\mathfrak{m} \operatorname{gr}^n(\mathcal{E}')) = \dim_{\kappa(\mathfrak{p})}(\operatorname{gr}^n(\mathcal{E}')_{\mathfrak{p}})$ and therefore $\operatorname{gr}^n(\mathcal{E}')$ is free. (Remember that for a finitely generated R-module M $\dim(M \otimes \operatorname{Quot}(R)) \le \dim(M \otimes R/\mathfrak{m})$ is an equality if and only if M is free.) □

5.4 Bifiltered modules over polynomial rings

We carry over some of the techniques to a special setting on polynomial rings that will be valuable in (9.2.4). This setting is also used in [Gi, III 2.5]. We carve out the connection to ridges in greater clarity, see the remark after (5.4.3). Throughout this section $S = k[X_1, ..., X_n]$ is a polynomial ring over a field k of positive characteristic and $U = k[\sigma]$ is an additive algebra in S with a homogeneous $k[F]$-independent system of additive polynomials $\sigma = (\sigma_1, ..., \sigma_e)$ of degrees $q_1 \le \cdots \le q_e$. Let $J := SU_+$.

Lemma (5.4.1). *We fix the following filtrations on S:*

$$\mathcal{S}'_n := SU_{\ge n}, \qquad \mathcal{S}''_n := S_{\ge n} = S_+^n.$$

Then

(i) $(S, \mathcal{S}', \mathcal{S}'')$ *is a harmonious bifiltered (S, J, S_+)-module.*

(ii) $\operatorname{gr}^n(\mathcal{S}')$ *is a free S/J-module with* $\operatorname{rank}_{S/J}(\operatorname{gr}^n(\mathcal{S}')) = \dim_k U_n$ *for all n.*

(iii) *The filtrations $\underline{\mathcal{S}}^n$ are the S_+-filtrations for all n.*

Proof. (i): Since $J \subseteq S_+$, we have $J\mathcal{S}'_n \subseteq \mathcal{S}'_{n+1}$, so $\mathcal{S}'$ is a J-filtration. The harmonious property is clear from $S_+^n \subseteq \sum_i S_+^{n-i}(SU_{\ge i}) \subseteq S_+^n$. It remains to show that $(S, \mathcal{S}')$ is a good filtered (S, J)-module. Clearly $\mathcal{S}'$ is bounded and we have to show (see (4.2.7)) that $\operatorname{Bl}^\dagger(\mathcal{S}')$ is a finitely generated $\operatorname{Bl}^\dagger(S, J)$-module. We claim that it is generated by the elements $1 \in \mathcal{S}'_{-1}$ and $\sigma_i \in \mathcal{S}'_j$ for $i = 1, ..., e$ and $j = 1, ..., q_e$. It is sufficient to show that $U_{\ge n} \subseteq \mathcal{S}'_n$ is generated by these elements since $\operatorname{Bl}^\dagger(S, J)_0 = S$. We do this by induction on n, the case $n = 0$ is trivial. Suppose that $n > 0$ and take $\sigma^A \in U_m, m \ge n$. Then $\sigma^A = \sigma_i \sigma^B$ for some i ($B = 0$ is possible). Since $\deg(\sigma^B) = m - q_i$ we find $\sigma^B \in U_{\ge n-q_i} \subseteq \mathcal{S}'_{n-q_i}$. By the induction hypothesis σ^B is in the span of the claimed generators, therefore so is σ^A. (ii), (iii): We introduce the filtration $\mathcal{U}_n = U_{\ge n}$ on the ring U. Since $U \to S$ is flat (see (1.4.5)), we have $S \otimes_U U_{\ge n} \cong SU_{\ge n}$ and

$$S \otimes_U \operatorname{gr}^n(\mathcal{U}) \cong S \otimes_U (U_{\ge n}/U_{\ge n+1}) \cong SU_{\ge n}/SU_{\ge n+1} = \operatorname{gr}^n(\mathcal{S}').$$

But $U_+ \operatorname{gr}^n(\mathcal{U}) = 0$ yields $S \otimes_U \operatorname{gr}^n(\mathcal{U}) \cong S/SU_+ \otimes_k \operatorname{gr}^n(\mathcal{U})$. Since $\dim_k(\operatorname{gr}^n(\mathcal{U})) = \dim_k U_n$ we see that $\operatorname{gr}^n(\mathcal{S}')$ is a free S/J-module of rank $\dim_k U_n$. By (5.1.5) we get therefore that

$$(1-T)^{-n-1} = H^{(1)}(\mathcal{S}'') \le \sum_n T^n \dim_k(U_n) H^{(1)}(S/J, S_+) = H(U)H^{(1)}(S/J, S_+).$$

Since U is a polynomial algebra with generators of degrees $q_1,..,q_e$, we have $H(U) = \prod_{i=1}^e (1-T^{q_i})^{-1}$ and we also know from (4.5.10) that $H(S/J, S_+) = (1-T)^{-n} \prod_{i=1}^e (1-T^{q_i})$. Therefore the above inequality is an equality, which means by (5.1.5) that the filtrations $\underline{\mathcal{S}}^n$ are the S_+-filtrations. □

If we use Hilbert series in the following, we take them with respect to graded k-vector spaces.

Lemma (5.4.2). *Let E be a finitely generated graded S-module and let $\mathcal{E}'$ be an $\mathcal{S}'$-filtration by graded S-submodules on E such that $\mathcal{E}'_n \subseteq E_{\ge n}$.*

(i) *For the graded S/J-module $\operatorname{gr}^n(\mathcal{E}')$ we have*

$$H(\operatorname{gr}^n(\mathcal{E}')) \le H(S/J) \cdot H(\operatorname{gr}^n(\mathcal{E}')/S_+ \operatorname{gr}^n(\mathcal{E}'))$$

and this is an equality if and only if $\operatorname{gr}^n(\mathcal{E}')$ is a free S/J-module.

(ii) *The graded morphism $\operatorname{gr}(\mathcal{E}') \to E, \bar{e} \in \operatorname{gr}^n(\mathcal{E}') \mapsto e \bmod E_{\ge n+1}$ induces a natural morphism*

$$S \otimes_{\operatorname{gr}(\mathcal{S}')} \operatorname{gr}(\mathcal{E}') \to E.$$

It is surjective if $(E, \mathcal{E}')$ is a harmonious $(S, \mathcal{S}')$-module, i.e. $E_{\ge n} = \sum_i S_+^{n-i} \mathcal{E}'_i$ for all n.

(iii) *In general*

$$H(S \otimes_{\operatorname{gr}(\mathcal{S}')} \operatorname{gr}(\mathcal{E}')) = H(S/J) \cdot H(\operatorname{gr}(\mathcal{E}')/S_+ \operatorname{gr}(\mathcal{E}')).$$

Proof. For (i) see (4.5.7). (ii) follows as in the case of bifiltered modules (cf. (5.1.3) (v)). (iii): We can identify (see (5.4.1)) $\operatorname{gr}(\mathcal{S}')$ with $S/J \otimes_k U$, where the graduation comes from U. The canonical map $\operatorname{gr}(\mathcal{S}') \to \operatorname{gr}(\mathcal{S}'') \cong S$ identifies then with

$$S/J \otimes_k U \to S, \qquad (f \bmod J) \otimes \sigma \mapsto (f \bmod S_+) \cdot \sigma.$$

The first factor S/J of the tensor product therefore factors over k. We find

$$S \otimes_{\operatorname{gr}(\mathcal{S}')} \operatorname{gr}(\mathcal{E}') \cong S \otimes_{S/J \otimes_k U} \operatorname{gr}(\mathcal{E}') \cong S \otimes_U (S/S_+ \otimes_{S/J} \operatorname{gr}(\mathcal{E}')) \cong$$

$$\cong S \otimes_U (\operatorname{gr}(\mathcal{E}')/S_+ \operatorname{gr}(\mathcal{E}')).$$

Now S is a free U-module with basis $X^M, M \in \Lambda''$ as in (1.4.5). On the other hand the same monomials module J form a k-basis of S/J. Hence

$$H(S \otimes_{\operatorname{gr}(\mathcal{S}')} \operatorname{gr}(\mathcal{E}')) = H(\bigoplus_{M \in \Lambda''} U \cdot X^M \otimes_U (\operatorname{gr}(\mathcal{E}')/S_+ \operatorname{gr}(\mathcal{E}'))) =$$

$$= \sum_{M \in \Lambda''} T^{|M|} H(\operatorname{gr}(\mathcal{E}')/S_+ \operatorname{gr}(\mathcal{E}')) = H(S/J) \cdot H(\operatorname{gr}(\mathcal{E}')/S_+ \operatorname{gr}(\mathcal{E}'))$$

as claimed. □

Let $I \subseteq S$ be a homogeneous ideal and $A := S/I$. $0 \to I \to S \to A \to 0$ becomes a strict exact sequence of filtered $(S, \mathcal{S}')$-modules by equipping I and A with the induced filtrations $\mathcal{I}'$ and $\mathcal{A}'$. These filtrations are compatible with the graduations inherited from S. In analogy to (5.3.3) we find:

Proposition (5.4.3). *The following are equivalent:*

(i) *All* $\operatorname{gr}^n(\mathcal{A}')$ *are free* S/J*-modules.*

(ii) $H(A) = H(S/J) \cdot H(\operatorname{gr}(\mathcal{A}')/S_+ \operatorname{gr}(\mathcal{A}'))$.

(iii) $S \otimes_{\operatorname{gr}(\mathcal{S}')} \operatorname{gr}(\mathcal{A}') \to A$ *is an isomorphism.*

(iv) $S(U \cap I) = I$.

Proof. Since $A = \mathcal{A}'_0 \supseteq \mathcal{A}'_1 \supseteq \mathcal{A}'_2 \cdots$ and $H(\mathcal{A}'_n) \in T^n\mathbb{N}[[T]]$, we have $H(\operatorname{gr}^n(\mathcal{A}')) = H(\mathcal{A}'_n) - H(\mathcal{A}'_{n+1})$ and get $H(A) = \sum_n H(\operatorname{gr}^n(\mathcal{A}'))$. (i) and (ii) are therefore equivalent by (5.4.2) (i). For (iii) it is sufficient by (5.4.2) (ii) to show that both sides have the same Hilbert series since $(A, \mathcal{A}')$ is harmonious as a quotient of the harmonious $(S, \mathcal{S}')$. This is equivalent to (ii) by (5.4.2) (iii). (iii) is equivalent by an argument similar as in (5.2.2) to

$$S \otimes_{\operatorname{gr}(\mathcal{S}')} \operatorname{gr}(\mathcal{I}') \to I$$

being surjective. The image of this map is the S-submodule of I generated by $\operatorname{gr}(\mathcal{I}')$. Observe that $\mathcal{I}'_n = I \cap \mathcal{S}'_n = I \cap SU_{\geq n}$. Therefore the image of this morphism is generated by the elements $f \bmod I_{\geq n+1}$ for $f \in I \cap SU_{\geq n}$. We see that $f \bmod I_{\geq n+1}$ is zero if f is a homogeneous polynomial of degree $\geq n+1$. Thus the image of the morphism is generated by the elements in $U \cap I$. Therefore (iii) is equivalent to (iv). □

With (5.4.3) we have a tool that can be used to relate a criterion (i), comparable to normal flatness (cf. chapter 7), via Hilbert series to (iv), which means $\operatorname{Spec}(S/J) \subseteq \operatorname{Rid}(\operatorname{Spec}(S/I))$ (cf. chapter 6) (see also (9.2.4)).

6 Ridge and Directrix

One of the invariants we want to analyze measures the dimension of the ridge of the tangent cone at a point of a scheme. In close connection to the ridge stands the directrix. Both were introduced by Hironaka. We explain what ridge and directrix of a cone are and show how they can be computed. For the proof of our main theorems we generalize the notion of Giraud bases to σ-Giraud bases, see (6.1.4). Among other properties we will especially mention what happens with the ridge when the cone is intersected with a vector space. This is important for the behavior of our invariant.

6.1 Ridge

Definition (6.1.1). *With the notion of a* ***cone*** *we usually will refer to the spectrum of a standard graded algebra of finite type over a field. Hence a cone C is given as* $\mathrm{Spec}(A)$ *for a graded k-algebra $A = \bigoplus_{d\geq 0} A_d$ with a field k, $A_0 = k$ and A is generated as a k-algbebra by A_1. A* ***morphism of cones*** *is a morphism respecting the graduation on the respective graded algebras. With a* ***vector space*** *we mean an affine space over a field (see also (3.3.9)).*

Throughout this section let k be a field, $S = k[X_1, ..., X_n]$ and $V = \mathrm{Spec}(S)$. As mentioned in (3.3.9) the vector space V carries the structure of an algebraic group. For a point v of V and a closed subcone $C \subseteq V$ one can translate C by v. The ridge of the cone C (inside V) consists of all points v in V that translate C onto itself. Since C is a cone, with v and w also $v+w$ and $-v$ will belong to the ridge of C. The ridge of C should therefore be a subgroup of V contained in C, namely the largest subgroup that translates C onto itself. This is of course far from being the largest group just contained in C. We are going to make this heuristic remark more precise in the following. We will not make much use of the functorial language of algebraic groups but rather will stick to the point of view of commutative algebra for we are mostly interested in explicit ways of calculation. We remind the reader of 3.2.

Definition (6.1.2) (cf. [BHM, 2.1], [Gi, I 5.2]). *Let $i : C \to V$ be a closed subcone of V.*

(i) *For any k-scheme X we define the X-points of the* ***ridge*** $\mathrm{Rid}_{V,C}$ *of C in V to be the subset* $\mathrm{Rid}_{V,C}(X)$ *of the X-valued points $V(X) := \mathrm{Hom}_k(X, V)$ of V consisting of those $v \in V(X)$ for which the translation with v*

$$X \times_k C \to V$$

maps C to itself.

(ii) *If C is defined in $V = \mathrm{Spec}(S)$ by the ideal I, then we define* $\mathrm{IRid}(I)$ *to be the ideal in S generated by the elements $\omega(\lambda)(f)$ (see (3.2.1)) where f varies over*

I and λ varies over

$$H := \{\lambda \in \bigoplus_{d\geq 0} \mathrm{Hom}_k(S_d, k) | \lambda(I) = 0\} \subseteq \mathrm{Hom}_k(S, S).$$

Theorem (6.1.3) (cf. [BHM, 2.1], [Gi, I 5.3])**.** *Let $C = \mathrm{Spec}(S/I)$ be a cone of V. Then $\mathrm{Rid}_{V,C}$ is a functor, represented by the closed subscheme $\mathrm{Rid}(C) := \mathrm{Spec}(S/\,\mathrm{IRid}(I))$ of C. $\mathrm{Rid}(C)$ is a group.*

Proof. It suffices to verify the definition of $\mathrm{Rid}_{V,C}(X)$ for $X = \mathrm{Spec}(B)$, B a k-algebra. Let $v \in V(X)$ be represented by a morphism of k-algebras $v^{\#} : S \to B$. Then v belongs to $\mathrm{Rid}_{V,C}(X)$ if and only if

$$S \xrightarrow{\Delta} S \otimes_k S \xrightarrow{v^{\#}\otimes\pi} B \otimes_k S/I$$

factors over S/I, i.e. maps I to zero (π is the canonical projection). An element of $B \otimes_k S/I$ is zero if and only if $(\mathrm{Id} \otimes \lambda)$ maps it to zero for all $\lambda \in \mathrm{Hom}_k(S/I, k)$. Therefore $v \in \mathrm{Rid}_{V,C}(X)$ iff for all $f \in I$ and all $\lambda : S/I \to k$ we have

$$0 = (\mathrm{Id} \otimes \lambda)(v^{\#} \otimes \pi)\Delta(f) = v^{\#}(\omega(\lambda)(f)),$$

i.e. $v^{\#}(\mathrm{IRid}(I)) = 0$. Thus $\mathrm{Rid}_{V,C}(X) = \mathrm{Rid}(C)(X)$ and now it is clear that $\mathrm{Rid}_{V,C}$ is a functor. By taking λ as the projection $S \to S_0 = k$, it follows that $I \subseteq \mathrm{IRid}(I)$ and therefore $\mathrm{Rid}(C) \subseteq C$. Since C is a cone and $-C = C$, one sees that $\mathrm{Rid}_{V,C}$ takes values in additive groups, hence $\mathrm{Rid}(C)$ is a group (see [Sc, 11.4]). □

So far we verified the existence of the group $\mathrm{Rid}(C)$. Now we want to see how it can be computed. For this we use Giraud bases as in [BHM, 2.4]. We generalize them to σ-Giraud bases to obtain a similar tool with respect to some additive polynomials.

Definition (6.1.4). *Let $I \subseteq S$ be a homogeneous ideal. A system of homogeneous polynomials $\gamma = (\gamma_1, ..., \gamma_l)$ in S with $I = \langle\gamma\rangle_S$ is called*

(i) *a **Giraud basis** of I if $D_M(\gamma_i) = 0$ for all $M \in \exp(I)$ with $|M| < \deg(\gamma_i)$ for $i = 1, ..., l$, where $\exp(I)$ is taken with respect to an arbitrary monomial order on S (see 1.3) and D_M are the standard differential operators with respect to the variables of S as in (3.3.1).*

(ii) *a σ**-Giraud basis** of I for a homogeneous $k[F]$-independent system of additive polynomials $\sigma = (\sigma_1, ..., \sigma_m)$ of degrees $q = (q_1, ..., q_m)$ ($q_1 \leq \cdots \leq q_m$) if $D_{N,M}(\gamma_i) = 0$ for all $(N, M) \in \exp(I)$ with $qN + |M| < \deg(\gamma_i)$ for $i = 1, ..., l$, where $\exp(I)$ is taken with respect to the order of (1.4.5) and $D_{N,M}$ are the differential operators of S like in (3.3.2).*

Lemma (6.1.5). *If $I = \langle\gamma\rangle, \gamma = (\gamma_1, ..., \gamma_l)$ is a homogeneous ideal of S, then $\mathrm{IRid}(I)$ is generated by $\omega(\lambda)(\gamma_i)$ for $i = 1, ..., l$ and $\lambda \in H$ (as in (6.1.2)). In particular every $\omega(\lambda)(\gamma_i)$ is of the form $\sum_{M\in\Lambda} a_M D_M(\gamma_i)$ for certain coefficients $a_M \in k$.*

(i) *(cf. [BHM, 2.3]) If γ is a Giraud basis of I, then*

$$\mathrm{IRid}(I) = \langle D_M(\gamma_i) | |M| < \deg(\gamma_i), i = 1, ..., l\rangle_S.$$

(ii) *If γ is a σ-Giraud basis of I, then*

$$\mathrm{IRid}(I) = \langle D_{N,M}(\gamma_i) | qN + |M| < \deg(\gamma_i), i = 1, ..., l\rangle_S.$$

Proof. For $x \in S_1$ and $f \in S$ we have

$$\omega(\lambda)(xf) = (\mathrm{Id} \otimes \lambda)((x \otimes 1 + 1 \otimes x) \cdot \Delta(f)) = x\omega(\lambda)(f) + \omega(\lambda \circ x)(f),$$

where $\circ x$ stands for the multiplication map with x. Note that with λ also $\lambda \circ x$ lies in H. Since all $\omega(\lambda)$ are k-linear and S is generated as an algebra over k by S_1, this proves the first claim. Since γ is a Giraud basis in (i), we have

$$\omega(\lambda)(\gamma_i) \overset{(3.3.1.B)}{=} \sum_{M \in \Lambda} \lambda(X^M) D_M(\gamma_i) = \lambda(\gamma_i) + \sum_{|M| < \deg \gamma_i, M \notin \exp(I)} \lambda(X^M) D_M(\gamma_i).$$

Because $\lambda(\gamma_i) = 0$ and the monomials $\overline{X^M}$ for $M \notin \exp(I)$ form a k-basis of S/I (see (1.3.8)), we get the claimed equality from the definition of H. (ii) is proved analogously with

$$\omega(\lambda)(\gamma_i) \overset{(3.3.2.B)}{=} \sum_{(N,M) \in \Lambda' \times \Lambda''} \lambda(\sigma^N X^M) D_{N,M}(\gamma_i) =$$

$$= \lambda(\gamma_i) + \sum_{(N,M) \notin \exp(I), qN + |M| < \deg(\gamma_i)} \lambda(\sigma^N X^M) D_{N,M}(\gamma_i)$$

and the k-basis $\overline{\sigma^N X^M}$ for $(N, M) \notin \exp(I)$ of S/I (see (1.2.5)). □

Lemma (6.1.6) (cf. [BHM, 3.2]). *A reduced Gröbner basis (say homogeneous as in (1.3.11)) of the homogeneous ideal $I \subseteq S$ also is a Giraud basis.*

Proof. Assume that γ is a reduced Gröbner basis of I and let $M \in \exp(I)$ with $|M| < \deg(\gamma_i)$ and $D_M(\gamma_i) \neq 0$. Then X^M divides a monomial of γ_i. Since $M \in \exp(I)$, there exists γ_j such that $\mathrm{in}(\gamma_j)$ divides X^M and therefore also a monomial of γ_i. Since γ is a reduced Gröbner basis, this implies $i = j$ which is absurd since $|M| < \deg(\gamma_i)$. □

There are more Giraud bases than reduced Gröbner bases (cf. [BHM]).

Example (6.1.7). In (6.1.2) we originally defined the ridge of a cone C as a functor on schemes. Note that the ridge is not at all functorial in the cone C itself. Take for example $S = \mathbb{F}_2[X, Y, Z]$. The line C defined by $I = \langle X, Y\rangle$ is a subgroup of $V = \mathrm{Spec}(S)$ and therefore $\mathrm{Rid}(C) = C$. C is contained in C' defined by $I' = \langle X^2 - YZ\rangle$. $\mathrm{Rid}(C')$ is the origin (not reduced): $X^2 - YZ$ is a Giraud basis of I' and therefore we can apply the derivations with respect to X, Y, Z to $X^2 - YZ$ and eventually find $\mathrm{IRid}(I') = \langle X^2, Y, Z\rangle$. Thus $\mathrm{Rid}(C') \subsetneq \mathrm{Rid}(C) = C \subsetneq C'$.

We also can compute the ring of invariants (see 3.4) of the group $\mathrm{Rid}(C)$:

Lemma (6.1.8). *Let $I \subseteq S$ be a homogeneous ideal with $I = \langle \gamma \rangle, \gamma = (\gamma_1, ..., \gamma_l)$.*

(i) *(cf. [BHM, 2.10]) If γ is a Giraud basis of I, then*

$$\mathcal{U}(\mathrm{IRid}(I)) = k[D_M(\gamma_i) |\, |M| < \deg(\gamma_i), i = 1, ..., l].$$

(ii) *If γ is a σ-Giraud basis of I, then*

$$\mathcal{U}(\mathrm{IRid}(I)) = k[D_{N,M}(\gamma_i) |\, qN + |M| < \deg(\gamma_i), i = 1, ..., l].$$

Proof. Since $\mathrm{Rid}(C)$ is a group, $\mathrm{IRid}(I)$ is generated by additive polynomials (see (3.3.8)). (i): By (3.4.3) (vii) $U := k[D_M(\gamma_i) |\, |M| < \deg(\gamma_i), i = 1, ..., l]$ is generated by additive polynomials ($D_M(\gamma_i) \in k$ if not $|M| < \deg(\gamma_i)$). From (6.1.5) (i) we have $S \cdot U_+ = \mathrm{IRid}(I)$ and by (3.4.4) $U = \mathcal{U}(S \cdot U_+) = \mathcal{U}(\mathrm{IRid}(I))$. (ii) is proved in absolute analogy; note that the D_M from (i) as well as the $D_{N,M}$ from (ii) are k-bases of $\mathrm{Diff}_{k,0}(S)$ and therefore it makes no difference which type of differential operators are applied. □

There is a different possibility to define the ridge of a cone $C = \mathrm{Spec}(S/I)$: The ring of invariants U of $\mathrm{Rid}(C)$ in S is the smallest graded k-subalgebra of S generated by additive polynomials with the property $S \cdot (U \cap I) = I$. The inclusion $S \cdot (U \cap I) \subseteq I$ always holds. From this point of view, computing the ridge means to look for the smallest additively generated subalgebra of S containing generators of I.

Lemma (6.1.9) (cf. [BHM, 2.12]). *Let I be a homogeneous ideal of S. If H is a graded k-subalgebra of S generated by additive polynomials, then*

(6.1.9.A) $$S \cdot (H \cap I) = I$$

if and only if $\mathcal{U}(\mathrm{IRid}(I)) \subseteq H$.

Proof. (6.1.9.A) holds for $H = \mathcal{U}(\mathrm{IRid}(I))$ as one can see for example from (6.1.5) and (6.1.8), this proves the if part. Assume (6.1.9.A) holds for H, i.e. there are $f_1, ..., f_l \in H$ that generate I. By (6.1.5) $\mathrm{IRid}(I)$ is generated by elements of the form $\sum_{M \in \Lambda} a_M D_M(f_i)$ with $a_A \in k$, but these all lie in H by (3.4.3). Therefore $\mathrm{IRid}(I) \subseteq S \cdot H_+$ and by (3.4.4) and (3.4.3) we get $\mathcal{U}(\mathrm{IRid}(I)) \subseteq \mathcal{U}(S \cdot H_+) = H$. □

The ridge of a cone is well known to be stable under any field extension:

Lemma (6.1.10). *Let C be a cone in the vector space V over the field k. For any field extension k'/k we have*

$$\mathrm{Rid}(C_{k'}) \cong \mathrm{Rid}(C)_{k'}.$$

Proof. One can see this on the one hand from the functorial definition of the ridge in (6.1.2) (i). On the other hand there is an easy computational way to prove this: If $C = \mathrm{Spec}(S/I)$ and γ is a Giraud basis of I, then it also is a Giraud basis of

$k' \otimes_k I \subseteq k' \otimes_k S$ and the $D_M(\gamma_i)$ generating $\mathrm{IRid}(I)$ and $\mathrm{IRid}(k' \otimes_k I)$ are unchanged by any field extension. □

Another point of view on the ridge is via the Dieudonné module of the group $\mathrm{Rid}(C)$:

Definition (6.1.11). *For a homogeneous ideal $I \subseteq S$ we define the graded $k[F]$-module $\mathfrak{Q}(I) := \mathcal{U}(\mathrm{IRid}(I)) \cap L$ and in particular for $f \in S$ we write $\mathfrak{Q}(f) := \mathfrak{Q}(\langle f \rangle)$. Note that $\mathcal{U}(\mathrm{IRid}(I)) = k[\mathfrak{Q}(I)]$ (see (3.4.4)).*

Lemma (6.1.12). *For a homogeneous ideal $I \subseteq S$ and a graded $k[F]$-submodule $Q \subseteq L$ we have*

$$S \cdot (k[Q] \cap I) = I$$

if and only if $\mathfrak{Q}(I) \subseteq Q$.

Proof. If $S(k[Q] \cap I) = I$, then $\mathcal{U}(\mathrm{IRid}(I)) \subseteq k[Q]$ by (6.1.9) and therefore $\mathfrak{Q}(I) \subseteq Q$ by (3.4.4). If on the other hand $\mathfrak{Q}(I) \subseteq Q$, then $\mathcal{U}(\mathrm{IRid}(I)) = k[\mathfrak{Q}(I)] \subseteq k[Q]$; now use (6.1.9). □

6.2 Directrix

The directrix of a cone C is the largest vector space that translates C onto itself. The directrix is not functorial as we will see in (6.3.5) f.. Therefore we cannot take an approach as in (6.1.2) (i).

Definition (6.2.1) (cf. [CJS, 1.7])**.** *Let k be a field, $\mathfrak{V}$ a finite dimensional vector space over k and I an ideal of $S := \mathrm{Sym}_k(\mathfrak{V})$. We say that I is defined over a subspace $\mathfrak{W} \subseteq \mathfrak{V}$ if*

$$S \cdot (\mathrm{Sym}_k(\mathfrak{W}) \cap I) = I,$$

where we understand $\mathrm{Sym}_k(\mathfrak{W})$ as a subring of S. This means that I is generated by elements of $\mathrm{Sym}_k(\mathfrak{W})$. We will also write $k[\mathfrak{W}]$ for $\mathrm{Sym}_k(\mathfrak{W})$.

Lemma (6.2.2). *In the situation of (6.2.1) there is a smallest k-subspace $\mathfrak{T}(I) \subseteq \mathfrak{V}$ such that I is defined over $\mathfrak{T}(I)$, i.e. $S \cdot (k[\mathfrak{W}] \cap I) = I$ if and only if $\mathfrak{T}(I) \subseteq \mathfrak{W}$.*

Proof. Let $\mathfrak{T}$ be a k-subspace of $\mathfrak{V}$ with I being defined over $\mathfrak{T}$ and with smallest possible dimension. Let I be also defined over the subspace $\mathfrak{T}' \subseteq \mathfrak{V}$. We have to show that $\mathfrak{T} \subseteq \mathfrak{T}'$. Assume that $\mathfrak{T} \not\subseteq \mathfrak{T}'$. Let $x = (x_1, ..., x_k)$ be a basis of $\mathfrak{T} \cap \mathfrak{T}'$, choose $y = (y_1, ..., y_l)$ such that x, y is a basis of $\mathfrak{T}$, choose $z = (z_1, ..., z_m)$ such that x, z is a basis of $\mathfrak{T}'$, choose $t = (t_1, ..., t_n)$ such that x, y, z, t is a basis of $\mathfrak{V}$. Then we have $I = \langle f_1, ..., f_p \rangle$ with $f_i \in k[x, y]$ and $I = \langle g_1, ..., g_q \rangle$ with $g_i \in k[x, z]$. We want to show that I is defined over $\mathfrak{T} \cap \mathfrak{T}'$. We can assume that $l = 1, y = y_1$ and have to show that I is defined over $\langle x_1, ..., x_k \rangle$. Write

$$f_i(x, y) = \sum_r f_i^r(x) y^r$$

and

$$f_i(x, y) = \sum_{s=1}^{q} g_s(x, z) \cdot \sum_r h_{s,i}^r(x, z, t) y^r$$

and we get

$$f_i^r(x) = \sum_{s=1}^{q} g_s(x,z) \cdot h_{s,i}^r(x,z,t) \in I.$$

Therefore $I = \langle f_i \rangle = \langle f_i^r \rangle$ and this concludes the proof. □

Definition (6.2.3) (cf. [CJS, 1.8]). *In (6.2.2) $\mathfrak{T}(I)$ is uniquely determined by I. For the closed subscheme $X := \mathrm{Spec}(S/I)$ of the vector space $V = \mathrm{Spec}(S)$ we call*

$$\mathrm{Dir}(X) := \mathrm{Spec}(S/\langle \mathfrak{T}(I) \rangle_S) \subseteq X$$

*the **directrix** of X. For a polynomial $f \in S$ we will use the notation $\mathfrak{T}(f) := \mathfrak{T}(\langle f \rangle)$.*

6.3 Comparing ridge and directrix

Let us make a few easy observations first.

Lemma (6.3.1). *For a cone C the following hold:*

(i) $\mathrm{Dir}(C) \subseteq \mathrm{Rid}(C) \subseteq C$ *is a sequence of closed immersions.*

(ii) $\mathrm{Dir}(C)$ *is the largest vector space contained in* $\mathrm{Rid}(C)$.

(iii) *Neither* $\mathrm{Rid}(C)$ *nor* $\mathrm{Dir}(C)$ *depend on the embedding of C into a vector space.*

(iv) $\mathrm{Dir}(\mathrm{Rid}(C)) = \mathrm{Dir}(C)$.

(v) *If C is defined over a field of characteristic zero, then* $\mathrm{Dir}(C) = \mathrm{Rid}(C)$.

Proof. Let $C = \mathrm{Spec}(S/I)$ be embedded into the vector space $V = \mathrm{Spec}(S)$. A vector space $W = \mathrm{Spec}(S/S \cdot \mathfrak{W})$ for a k-space $\mathfrak{W} \subseteq S_1$ is contained in $\mathrm{Rid}(C)$ iff $\mathrm{IRid}(I) \subseteq S \cdot \mathfrak{W}$ which is equivalent (see (3.4.4)) to $k[\mathfrak{Q}(I)] = \mathcal{U}(\mathrm{IRid}(I)) \subseteq k[\mathfrak{W}]$. By (6.1.9) this means $S(k[\mathfrak{W}] \cap I) = I$. This holds by definition for $\mathfrak{W} = \mathfrak{T}(I)$, proving (i) and (ii). (iii) comes from the fact that the group structure on a surrounding vector space is unique. (iv) follows from (ii) because $\mathrm{Rid}(\mathrm{Rid}(C)) = \mathrm{Rid}(C)$. (v) is clear since in characteristic zero every group is a vector space (see (3.3.8)). □

The directrix of a cone can be computed from its ridge by (6.3.1) (iv). To see how this can be done we introduce two operators on $k[F]$-modules. Assume in the following that k is a field of characteristic $p > 0$.

Definition (6.3.2). *For a graded $k[F]$-module $Q \subseteq L$ we define*

(i) $\mathcal{R}(Q) := \{f \in L | F^j(f) \in Q$ *for some* $j \in \mathbb{N}\}$, *the radical of Q.*

(ii) $\mathcal{E}(Q) := \bigoplus_{d \geq 0} \mathrm{Hom}_{k^{p^d}}(k,k)(Q_d)$.

The homomorphisms in (ii) are meant to be acting on the coefficients of the polynomials. $\mathcal{E}(Q)$ does not depend on the choice of the variables in a graded polynomial ring.

Lemma (6.3.3). *$\mathcal{R}$ and $\mathcal{E}$ are inclusion preserving operators on graded $k[F]$-submodules of L. For such a module Q the following hold:*

(i) $\mathcal{ERE}(Q) = \mathcal{RE}(Q)$.

(ii) *If $Q \subseteq k[\mathfrak{W}]$ for a subspace $\mathfrak{W} \subseteq S_1$, then $\mathcal{R}(Q), \mathcal{E}(Q) \subseteq k[\mathfrak{W}]$.*

(iii) *There is a k-subspace $\mathfrak{T} \subseteq S_1$ with $\mathcal{RE}(Q) = \langle \mathfrak{T} \rangle_{k[F]}$.*

Proof. It is clear that $\mathcal{R}(Q)$ is graded and for $f, f' \in \mathcal{R}(Q)$ and $\lambda \in k$ there are $i, j \in \mathbb{N}$ with $F^i(f), F^j(f') \in Q$. Then also $F^{i+j}(f + f') \in Q$ and $F^i(\lambda f) = \lambda^{p^i} F^i(f) \in Q$ and therefore $f + f', \lambda f \in \mathcal{R}(Q)$. $\mathcal{E}(Q)$ is clearly a graded k-module and we have to show that $F(\mathcal{E}(Q)_d) \subseteq \mathcal{E}(Q)_{d+1}$. Let $f \in Q_d$ and $\lambda \in \mathrm{Hom}_{k^{p^d}}(k, k)$. We extend λ arbitrarily to some $\lambda' \in \mathrm{Hom}_{k^{p^d}}(k^{1/p}, k^{1/p})$. Then $\mu := F \circ \lambda' \circ F^{-1} \in \mathrm{Hom}_{k^{p^{d+1}}}(k, k)$ and $F(\lambda(f)) = \mu(F(f)) \in \mathcal{E}(Q)_{d+1}$. $\mathcal{RE}(Q) \subseteq \mathcal{ERE}(Q)$ is clear in (i). Let on the other hand $f \in \mathcal{RE}(Q)_d$ and $\lambda \in \mathrm{Hom}_{k^{p^d}}(k, k)$. Then $F^i(f) \in \mathcal{E}(Q)$ for some $i \in \mathbb{N}$. Extend λ to some $\lambda' \in \mathrm{Hom}_{k^{p^d}}(k^{1/p^i}, k^{1/p^i})$ and set $\mu := F^i \circ \lambda' \circ F^{-i} \in \mathrm{Hom}_{k^{p^{d+i}}}(k, k)$. We get $F^i(\lambda(f)) = \mu(F^i(f)) \in \mathcal{EE}(Q) = \mathcal{E}(Q)$. This shows $\lambda(f) \in \mathcal{RE}(Q)$ and proves (i). Let $Q \subseteq k[\mathfrak{W}]$ for (ii). If $f \in L, F^i(f) \in Q \subseteq k[\mathfrak{W}]$, then already $f \in k[\mathfrak{W}]$. $\mathcal{E}(Q) \subseteq \mathcal{E}(L \cap k[\mathfrak{W}]) = L \cap k[\mathfrak{W}]$ since the homomorphisms in $\mathcal{E}$ only act on the coefficients of a polynomial. In (iii) set $\mathfrak{T} := \mathcal{RE}(Q) \cap L_0 \subseteq S_1$. By (i) we can apply the operators $\mathcal{R}$ and $\mathcal{E}$ arbitrarily to $\mathcal{RE}(Q)$ without altering the result. By applying these operators to some homogeneous $f \in L$ one gets linear polynomials whose $k[F]$-span contains f. This proves (iii). □

Lemma (6.3.4). *For a homogeneous ideal $I \subseteq S$ one has $\mathfrak{T}(I) = S_1 \cap \mathcal{RE}\mathfrak{Q}(I)$.*

Proof. Since $S(k[\mathfrak{T}(I)] \cap I) = I$, we have $\mathfrak{Q}(I) \subseteq k[\mathfrak{Q}(I)] \subseteq k[\mathfrak{T}(I)]$ (see (6.1.9)) and get the inclusion $\mathcal{RE}\mathfrak{Q}(I) \subseteq k[\mathfrak{T}(I)]$ from (6.3.3) (ii). On the other hand we know from (6.3.3) (iii) that $\mathfrak{Q}(I) \subseteq \mathcal{RE}\mathfrak{Q}(I) = \langle \mathfrak{T} \rangle_{k[F]}$ for a subspace $\mathfrak{T} \subseteq S_1$ and from $S(k[\mathfrak{Q}(I)] \cap I) = I$ we get $S(k[\mathfrak{T}] \cap I) = I$, hence $\mathfrak{T}(I) \subseteq \mathfrak{T} = S_1 \cap \mathcal{RE}\mathfrak{Q}(I)$. □

The directrix can therefore be computed from $\mathfrak{Q}(I)$ by linear algebra. In fact, it is not even necessary to compute all of $\mathcal{RE}\mathfrak{Q}(I)$: $\dim_k(\mathfrak{Q}(I)_d)$ is constant for large d and one gets $\mathfrak{T}(I)$ from $\mathrm{Hom}_{k^{p^d}}(k, k)(\mathfrak{Q}(I)_d)$ by taking the radical. Since the ridge can be computed using differential operators, i.e. special linear maps, one can ask if the directrix also can be computed with differential operators. This is true, but is not as easy as only to apply for example differential operators on k of order say $p^d - 1$ to $\mathfrak{Q}(I)_d$ (cf. $\mathcal{D}$ in 10.6). The coefficients of a polynomial $f \in \mathfrak{Q}(I)_d$ can span a k^{p^d}-space of any dimension, only bounded by the number of variables and $\dim_{k^{p^d}}(k)$, and $\mathrm{Diff}_{\mathbb{Z}}^{\leq p^d - 1}(k)$ is smaller than $\mathrm{Hom}_{k^{p^d}}(k, k)$ in general (e.g. for finite $\dim_{k^{p^d}}(k)$ since the dimensions do not coincide) (see 2.2). On the other hand it is absurd to apply differential operators with higher orders than indicated. For example even such a harmless element as $X_1 + aX_2 \in \mathfrak{Q}(I)_0$ could be split up by a derivation with respect to a. The right thing to do would be to apply to $\mathfrak{Q}(I)_d$ the subring of $\mathrm{Diff}_{\mathbb{Z}}(k)$ generated by $\mathrm{Diff}_{\mathbb{Z}}^{\leq p^d - 1}(k)$. This formulation is quite complicated, so we prefer to use the operator $\mathcal{E}$.

The behavior of the ridge under field extensions (see (6.1.10)) is much better than that of the directrix:

Lemma (6.3.5). *Let C be a cone over the field k and K/k a field extension.*

(i) $\mathrm{Dir}(C)_K \subseteq \mathrm{Dir}(C_K)$ *and this is an equality if K/k is separable (see [CJS, 1.10] for a different proof).*

(ii) *If K is perfect, then* $\mathrm{Dir}(C_K) = \mathrm{Rid}(C_K)_{\mathrm{red}}$ *and* $\dim \mathrm{Rid}(C) = \dim \mathrm{Dir}(C_K)$.

Proof. Let $C = \mathrm{Spec}(S/I)$, $V = \mathrm{Spec}(S)$ a vector space and $S' = K \otimes_k S, I' = K \otimes_k I \subseteq S'$. (i): From (6.1.10) we have $\mathfrak{Q}(I') = K\mathfrak{Q}(I)$. Hence $\mathcal{R}_K\mathcal{E}_K\mathfrak{Q}(I') \subseteq K\mathcal{R}_k\mathcal{E}_k\mathfrak{Q}(I)$, where we denote the field over which the operator is used in the subscript. This is clear since every $\mu \in \mathrm{Hom}_{K^{p^d}}(K,K)$ restricts to $\mathrm{Hom}_{k^{p^d}}(k,K)$. By (6.3.4) we get $\mathfrak{T}(I') \subseteq K\mathfrak{T}(I)$, hence $\mathrm{Dir}(C)_K \subseteq \mathrm{Dir}(C_K)$. If K/k is separable we get from (2.2.8) and (2.2.3) a k^{p^d}-basis of k, that also is a K^{p^d}-basis of K. Hence every $\lambda \in \mathrm{Hom}_{k^{p^d}}(k,k)$ extends to some $\mu \in \mathrm{Hom}_{K^{p^d}}(K,K)$. Therefore $\mathcal{R}_K\mathcal{E}_K\mathfrak{Q}(I') = K\mathcal{R}_k\mathcal{E}_k\mathfrak{Q}(I)$ and (6.3.4) shows $\mathfrak{T}(I') = K\mathfrak{T}(I)$. (ii): Over a perfect field every additive polynomial is a power of a variable, hence $\mathrm{Dir}(C_K) = \mathrm{Rid}(C_K)_{\mathrm{red}}$. Note that $\dim(\mathrm{Rid}(C)) = \dim(\mathrm{Rid}(C_K))$ by (6.1.10). □

Example (6.3.6). Take for example $C = \mathrm{Spec}(k[X,Y]/\langle X^p + aY^p\rangle)$ for an element $a \in k \setminus k^p$. Then $\mathrm{Dir}(C) = \mathrm{Spec}(k)$. But for $K = k(\sqrt[p]{a})$ we have $\mathrm{Dir}(C_K) = \mathrm{Spec}(k[X,Y]/\langle X + \sqrt[p]{a}Y\rangle)$. The dimension of the directrix went up from zero to one and reached the dimension of the ridge $\mathrm{Rid}(C) = C$.

In the following we describe how directrix and ridge of $C_{f\cdot g} := \mathrm{Spec}(S/\langle f \cdot g\rangle)$ are related to those of $C_f := \mathrm{Spec}(S/\langle f\rangle)$ and $C_g := \mathrm{Spec}(S/\langle g\rangle)$ for $f, g \in S$.

Lemma (6.3.7). *Let $\mathfrak{W} \subseteq \mathfrak{V}$ be subspace and $f, g \in S = k[\mathfrak{V}]$. If $f \cdot g \in k[\mathfrak{W}]$ and $f, g \neq 0$, then $f, g \in k[\mathfrak{W}]$.*

Proof. Choose a basis $X_{m+1}, ..., X_n$ of $\mathfrak{W}$ such that $X_1, ..., X_n$ is a basis of $\mathfrak{V}$. We use the lexicographic order on S like in (1.1.2.2) with $X_1 > \cdots > X_n$. Then $\mathrm{in}(f)\cdot\mathrm{in}(g) = \mathrm{in}(f \cdot g) \in k[X_{m+1}, ..., X_n]$, hence $\mathrm{in}(f), \mathrm{in}(g) \in k[X_{m+1}, ..., X_n]$ since $f, g \neq 0$. This implies $f, g \in k[X_{m+1}, ..., X_n] = k[\mathfrak{W}]$. □

Corollary (6.3.8). *For $0 \neq f, g \in S$ we have*

$$\mathfrak{T}(f \cdot g) = \mathfrak{T}(f) + \mathfrak{T}(g),$$

$$\mathrm{Dir}(C_{f\cdot g}) = \mathrm{Dir}(C_f) \cap \mathrm{Dir}(C_g).$$

Proof. The second equality is a direct consequence of the first one. Since $f \in k[\mathfrak{T}(f)]$ and $g \in k[\mathfrak{T}(g)]$, we have $f \cdot g \in k[\mathfrak{T}(f) + \mathfrak{T}(g)]$, hence $\mathfrak{T}(f \cdot g) \subseteq \mathfrak{T}(f) + \mathfrak{T}(g)$. On the other hand $f \cdot g \in k[\mathfrak{T}(f \cdot g)]$ and (6.3.7) yields $f, g \in k[\mathfrak{T}(f \cdot g)]$. Therefore $\mathfrak{T}(f), \mathfrak{T}(g) \subseteq \mathfrak{T}(f \cdot g)$ and $\mathfrak{T}(f) + \mathfrak{T}(g) \subseteq \mathfrak{T}(f \cdot g)$. □

It is not true that $f \cdot g$ being additive with f, g homogeneous implies f and g being additive as the following calculation shows:

$$(X + Y) \cdot (X^{2^n-1} + X^{2^n-2}Y + \cdots + Y^{2^n-1}) = X^{2^n} + Y^{2^n} \in \mathbb{F}_2[X,Y].$$

Hence one has to do a bit more work for an analog of (6.3.8) for the ridge. But we see that every factor itself in our example is a power of an additive polynomial. The problem is solved in the end by taking a radical.

Lemma (6.3.9). *For $f \in S$ and $j \in \mathbb{N}$ we have*

$$\mathfrak{Q}(F^j(f)) = kF^j(\mathfrak{Q}(f)).$$

Proof. It suffices to prove this for $j = 1$. We have $k[\mathfrak{Q}(f)] = k[D_M(f)|M \in \Lambda]$. If $M \notin p\Lambda$, then $D_M(F(f)) = 0$. But $D_{pM}(F(f)) = F(D_M(f))$ (see (2.2.6)). So

$$k[\mathfrak{Q}(F(f))] = k[D_M(F(f))|M \in \Lambda] = k[F(D_M(f))|M \in \Lambda] = k[F(\mathfrak{Q}(f))]$$

and thus the claim follows from (3.4.4). □

Corollary (6.3.10). *Let $Q \subseteq L$ be a graded $k[F]$-module. For $f \in S$ one has $F^j(f) \in k[Q]$ for some $j \geq 0$ if and only if $f \in k[\mathcal{R}(Q)]$.*

Proof. $F^j(f) \in k[Q]$ implies $\mathfrak{Q}(F^j(f)) \subseteq Q$ and (6.3.9) gives $kF^j(\mathfrak{Q}(f)) \subseteq Q$. Therefore $f \in k[\mathfrak{Q}(f)] \subseteq k[\mathcal{R}(Q)]$. The other implication is clear. □

Lemma (6.3.11). *Let $Q \subseteq L$ be a graded $k[F]$-module. If $f \cdot g \in k[Q]$ and $f, g \neq 0$, then $f, g \in k[\mathcal{R}(Q)]$.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Then $f \cdot g \in \overline{k}[Q] \subseteq \overline{k}[\mathcal{R}_{\overline{k}}(Q)]$ which is a standard graded $\overline{k}$-subalgebra of $\overline{k} \otimes_k S$. By (6.3.7) we have $f, g \in \overline{k}[\mathcal{R}_{\overline{k}}(Q)]$. Then we find $j \geq 0$ with $F^j(f) \in \overline{k}[Q] \cap S = k[Q]$ and (6.3.10) gives $f \in k[\mathcal{R}(Q)]$. Similarly for g. □

Corollary (6.3.12). *For homogeneous $0 \neq f, g \in S$ we have*

$$\mathfrak{Q}(f \cdot g) \subseteq \mathfrak{Q}(f) + \mathfrak{Q}(g) \subseteq \mathcal{R}\mathfrak{Q}(f \cdot g).$$

$$\mathrm{Rid}(C_{f \cdot g})_{\mathrm{red}} = (\mathrm{Rid}(C_f) \cap \mathrm{Rid}(C_g))_{\mathrm{red}}.$$

Proof. The second equality comes from the first one. Since $f \in k[\mathfrak{Q}(f)]$ and $g \in k[\mathfrak{Q}(g)]$ we have $f \cdot g \in k[\mathfrak{Q}(f) + \mathfrak{Q}(g)]$, hence $\mathfrak{Q}(f \cdot g) \subseteq \mathfrak{Q}(f) + \mathfrak{Q}(g)$. On the other hand $f \cdot g \in k[\mathfrak{Q}(f \cdot g)]$, and (6.3.11) yields $f, g \in k[\mathcal{R}\mathfrak{Q}(f \cdot g)]$. Therefore $\mathfrak{Q}(f), \mathfrak{Q}(g) \subseteq \mathcal{R}\mathfrak{Q}(f \cdot g)$. □

6.4 Intersections of ridge and directrix

When we are going to study the ridge as an invariant in chapter 8, we will have to compare the ridge of the tangent cone after a blow up with the ridge of the tangent cone in the exceptional divisor. The latter cone will be the intersection of the first cone with a hyperplane. In this section we investigate similar situations for cones in general.

For two closed subschemes, not necessarily with the reduced structure, X, Y of Z we mean with their intersection $X \cap Y$ the fiber product $X \times_Z Y$. It is a closed subscheme of Z again. Let us recall some facts. In the affine case $Z = \operatorname{Spec}(S), X = \operatorname{Spec}(S/I)$ and $Y = \operatorname{Spec}(S/J)$ we simply have $X \cap Y \cong \operatorname{Spec}(S/I+J)$. If Z is an S-scheme and T is another S-scheme, then $T \times_S X$ and $T \times_S Y$ are closed subschemes of $T \times_S Z$ and $(T \times_S X) \cap (T \times_S Y) \cong T \times_S (X \cap Y)$.

In some cases of intersections one does not lose much information about the cone, even if its dimension decreases. Let us introduce some notions to enable us to express this behavior. We also introduce quotients of cones.

Lemma (6.4.1). *Let C be a cone over the field k. Then there is a morphism of cones $\pi : C \to C/\operatorname{Dir}(C)$ with the following universal property: For any morphism of cones $c : C \to C'$ over k such that $\operatorname{Dir}(C) \to C \to C'$ is the zero morphism there exists a unique morphism $\overline{c}$ making the following diagram commutative:*

$$\begin{array}{ccc} C & \xrightarrow{\;c\;} & C' \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{\overline{c}} & \\ C/\operatorname{Dir}(C) & & \end{array}$$

There is a non-canonical isomorphism $C \cong C/\operatorname{Dir}(C) \times_k \operatorname{Dir}(C)$ and

$$H(C) = H^{(\dim \operatorname{Dir}(C))}(C/\operatorname{Dir}(C)).$$

Proof. Let $C = \operatorname{Spec}(S/I)$ for a polynomial ring S over k. Then

$$C/\operatorname{Dir}(C) = \operatorname{Spec}(k[\mathfrak{T}(I)]/k[\mathfrak{T}(I)] \cap I)$$

and π is defined via the canonical injection $k[\mathfrak{T}(I)] \to S$. Now let $c : \operatorname{Spec}(S/I) \to \operatorname{Spec}(R)$ be given as a morphism $c^{\#} : R \to S/I$ of standard graded k-algebras. $\operatorname{Dir}(C) \to C \to C'$ being zero means that $\varphi(R_1) \subseteq \mathfrak{T}(I)$ and therefore we find a unique $\overline{c}^{\#}$ making the following diagram commutative:

$$\begin{array}{ccc} S/I & \xleftarrow{\;c^{\#}\;} & R \\ {\scriptstyle \pi^{\#}}\big\uparrow & \swarrow_{\overline{c}^{\#}} & \\ k[\mathfrak{T}(I)]/k[\mathfrak{T}(I)] \cap I & & \end{array}$$

This is the required universal property. If $\mathfrak{T}(I) \oplus \mathfrak{T} = S_1$ (there is no canonical choice for $\mathfrak{T}$), then $S/I \cong (k[\mathfrak{T}(I)]/k[\mathfrak{T}(I)] \cap I) \otimes k[\mathfrak{T}]$ proves the last claims (see (4.5.9)). $\square$

Analogously we can form quotients C/W for any vector space $W \subseteq \operatorname{Dir}(C)$.

Lemma (6.4.2). *Let $c : C \to C'$ be a morphism of cones over k with $c(\operatorname{Dir}(C)) \subseteq \operatorname{Dir}(C')$. Then there is a unique morphism $\overline{c}$ of cones making the following diagram commutative:*

$$\begin{array}{ccc} C & \xrightarrow{\;c\;} & C' \\ {\scriptstyle \pi_C}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{C'}} \\ C/\operatorname{Dir}(C) & \xrightarrow{\;\overline{c}\;} & C'/\operatorname{Dir}(C') \end{array}$$

Proof. Let $C = \mathrm{Spec}(S/I), C' = \mathrm{Spec}(S'/I')$ and c be given by $c^{\#} : S'/I' \to S/I$. Then $c(\mathrm{Dir}(C)) \subseteq \mathrm{Dir}(C')$ means that $S'/I' \to S/I \to S/S\mathfrak{T}(I)$ factors over $S'/S'\mathfrak{T}(I')$, i.e. $c^{\#}(\mathfrak{T}(I')) \subseteq \mathfrak{T}(I)$. But then the composition $\mathrm{Dir}(C) \to C \to C' \to C'/\mathrm{Dir}(C')$ is the zero morphism and we get a unique morphism $\overline{c}$ from the universal property in (6.4.1). □

Definition (6.4.3). *Let $c : C \to C'$ be a morphism of cones over k.*

(i) *c is called a **core-morphism** if $c(\mathrm{Dir}(C)) \subseteq \mathrm{Dir}(C')$, i.e. c induces a unique morphism*

$$\overline{c} : C/\mathrm{Dir}(C) \to C'/\mathrm{Dir}(C') \tag{6.4.3.A}$$

as in (6.4.2).

(ii) *c is called a **core-isomorphism** if it is a core-morphism such that the induced morphism (6.4.3.A) is an isomorphism.*

(iii) *c is called a **k'-core-isomorphism** if k'/k is a field extension such that the induced morphism $C_{k'} \to C'_{k'}$ is a core-isomorphism.*

Note that the compositions of core morphisms (resp. core-isomorphisms, resp. k'-core-isomorphisms) are again such morphisms.

Lemma (6.4.4). *Let $c : C \to C'$ be a morphism of cones over k and let k'/k be some field extension. If c is a core-isomorphism, then so is the induced morphism $C_{k'} \to C'_{k'}$.*

Proof. The induced morphism $C_{k'} \to C'_{k'}$ still factors to a morphism $C_{k'}/(\mathrm{Dir}(C))_{k'} \to C'_{k'}/(\mathrm{Dir}(C'))_{k'}$ which identifies naturally with $(C/\mathrm{Dir}(C))_{k'} \to (C'/\mathrm{Dir}(C'))_{k'}$, so the last two morphisms are isomorphisms. Therefore $\mathrm{Dir}((C/\mathrm{Dir}(C))_{k'})$ identifies with $\mathrm{Dir}((C'/\mathrm{Dir}(C'))_{k'})$ and we get an isomorphism

$$(C/\mathrm{Dir}(C))_{k'}/\mathrm{Dir}((C/\mathrm{Dir}(C))_{k'}) \to (C'/\mathrm{Dir}(C'))_{k'}/\mathrm{Dir}((C'/\mathrm{Dir}(C'))_{k'})$$

that identifies with $C_{k'}/\mathrm{Dir}(C_{k'}) \to C'_{k'}/\mathrm{Dir}(C'_{k'})$. □

Now we present a list of intersection properties, first for the directrix, then for the ridge. The most important part is (iii) in both cases. Similar results for the directrix can be found in [Gi, I 6.9.2].

Lemma (6.4.5). *Let C be a cone in the vector space V over the field k and W a vector subspace of V.*

(i) *There are closed immersions $C \cap W \subseteq C$ and*

$$\mathrm{Dir}(C) \cap W \subseteq \mathrm{Dir}(C \cap W). \tag{6.4.5.A}$$

(ii) *The following inequality holds:*

$$\mathrm{codim}_{\mathrm{Dir}(C)}(\mathrm{Dir}(C) \cap W) \leq \mathrm{codim}_V(W). \tag{6.4.5.B}$$

(iii) *The inequality*

$$\dim \operatorname{Dir}(C) \leq \dim \operatorname{Dir}(C \cap W) + \operatorname{codim}_V(W) \tag{6.4.5.C}$$

is an equality if and only if (6.4.5.A) and (6.4.5.B) are equalities and in this case there is a natural isomorphism

$$(C \cap W)/\operatorname{Dir}(C \cap W) \to C/\operatorname{Dir}(C) \tag{6.4.5.D}$$

such that $C \cap W \to C$ is a core-ismorphism yielding a non-canonical isomorphism

$$C \cong (C \cap W) \times_k \mathbb{A}_k^{\operatorname{codim}_V(W)}. \tag{6.4.5.E}$$

(iv) *If $\operatorname{Dir}(C) \subseteq W$, then*

$$\dim \operatorname{Dir}(C) \leq \dim \operatorname{Dir}(C \cap W). \tag{6.4.5.F}$$

Proof. Let $V = \operatorname{Spec}(S), S = \operatorname{Sym}_k(\mathfrak{V})$, $C = \operatorname{Spec}(S/I)$ and $W = \operatorname{Spec}(S/S\mathfrak{W})$ for a vector subspace $\mathfrak{W} \subseteq \mathfrak{V}$. Then $C \cap W = \operatorname{Spec}(S/J)$ for $J = I + S\mathfrak{W}$ and $\operatorname{Dir}(C) \cap W = \operatorname{Spec}(S/S(\mathfrak{T}(I) + \mathfrak{W}))$.

(i) Clearly $C \cap W \subseteq C$ is a closed immersion. We have $S(k[\mathfrak{T}(I)] \cap I) = I$ and therefore also $S(k[\mathfrak{T}(I) + \mathfrak{W}] \cap J) = J$, hence $\mathfrak{T}(J) \subseteq \mathfrak{T}(I) + \mathfrak{W}$, yielding the closed immersion (6.4.5.A).

(ii) $\operatorname{codim}_{\operatorname{Dir}(C)}(\operatorname{Dir}(C) \cap W) = \dim \operatorname{Dir}(C) - \dim(\operatorname{Dir}(C) \cap W) = \dim_k(\mathfrak{T}(I) + \mathfrak{W}) - \dim_k(\mathfrak{T}(I)) \leq \dim_k(\mathfrak{W}) = \dim V - \dim W = \operatorname{codim}_V(W)$.

(iii) By (6.4.5.A) and (6.4.5.B) we have

$$\dim \operatorname{Dir}(C) = \dim(\operatorname{Dir}(C) \cap W) + \operatorname{codim}_{\operatorname{Dir}(C)}(\operatorname{Dir}(C) \cap W) \leq$$

$$\leq \dim \operatorname{Dir}(C \cap W) + \operatorname{codim}_V(W)$$

and it is clear that this is an equality if and only if (6.4.5.A) and (6.4.5.B) both are equalities. Assume from now on that these equalities hold. S/J arises from S/I by dividing out $\operatorname{codim}_V(W)$ elements of S_1 (generators of $\mathfrak{W}$). Using the inequality from (4.5.6) several times and multiplying with $(1-T)^{-\operatorname{codim}_V(W)}$ gives

$$H^{(\operatorname{codim}_V(W))}(C \cap W) \geq H(C)$$

and therefore

$$H^{(\dim \operatorname{Dir}(C \cap W) + \operatorname{codim}_V(W))}((C \cap W)/\operatorname{Dir}(C \cap W)) = H^{(\operatorname{codim}_V(W))}(C \cap W) \geq$$

$$\geq H(C) = H^{(\dim \operatorname{Dir}(C))}(C/\operatorname{Dir}(C)).$$

We have

$$C/\operatorname{Dir}(C) \cong \operatorname{Spec}(k[\mathfrak{T}(I)]/k[\mathfrak{T}(I)] \cap I),$$

$$(C \cap W)/\operatorname{Dir}(C \cap W) \cong \operatorname{Spec}(k[\mathfrak{T}(J)]/k[\mathfrak{T}(J)] \cap J).$$

Since (6.4.5.A) is an equality, we must have (see the proof of (i)) $\mathfrak{T}(J) = \mathfrak{T}(I) + \mathfrak{W}$. Hence $\mathfrak{T}(I) \subseteq \mathfrak{T}(J)$ induces (6.4.5.D) $C \cap W/\operatorname{Dir}(C \cap W) \to C/\operatorname{Dir}(C)$. Since this is a closed immersion, it must be an isomorphism by the above inequality of Hilbert series. In particular the closed immersion $C \cap W \to C$ is a core-isomorphism. We finally get (6.4.5.E):

$$C \cong C/\operatorname{Dir}(C) \times_k \mathbb{A}_k^{\dim \operatorname{Dir}(C)} \cong C \cap W/\operatorname{Dir}(C \cap W) \times_k \mathbb{A}_k^{\dim \operatorname{Dir}(C \cap W) + \operatorname{codim}_V(W)} \cong$$

$$\cong (C \cap W) \times_k \mathbb{A}_k^{\operatorname{codim}_V(W)}.$$

(iv) For $\operatorname{Dir}(C) \subseteq W$ we have $\operatorname{codim}_{\operatorname{Dir}(C)}(\operatorname{Dir}(C) \cap W) = 0$ and get (6.4.5.F) by examining the proof of (iii). □

Lemma (6.4.6). *Let C be a cone in the vector space V over the field k and W a vector subspace of V.*

(i) *There are closed immersions $C \cap W \subseteq C$ and*

(6.4.6.A) $$\operatorname{Rid}(C) \cap W \subseteq \operatorname{Rid}(C \cap W).$$

(ii) *The following inequality holds:*

(6.4.6.B) $$\operatorname{codim}_{\operatorname{Rid}(C)}(\operatorname{Rid}(C) \cap W) \leq \operatorname{codim}_V(W).$$

(iii) *The inequality*

(6.4.6.C) $$\dim \operatorname{Rid}(C) \leq \dim \operatorname{Rid}(C \cap W) + \operatorname{codim}_V(W)$$

is an equality if and only if (6.4.6.A) and (6.4.6.B) are equalities and in this case there is a natural isomorphism

(6.4.6.D) $$(C \cap W)_K/\operatorname{Dir}((C \cap W)_K) \to C_K/\operatorname{Dir}(C_K)$$

such that $C \cap W \to C$ is a K-core-isomorphism, inducing a non-natural isomorphism

(6.4.6.E) $$C_K \cong (C \cap W) \times_k \mathbb{A}_K^{\operatorname{codim}_V(W)},$$

for any perfect field K/k.

(iv) *If $\operatorname{Rid}(C) \subseteq W$, then*

(6.4.6.F) $$\dim \operatorname{Rid}(C) \leq \dim \operatorname{Rid}(C \cap W).$$

Proof. Let $V = \operatorname{Spec}(S), S = \operatorname{Sym}_k(\mathfrak{V})$, $C = \operatorname{Spec}(S/I)$ and $W = \operatorname{Spec}(S/S\mathfrak{W})$ for a vector subspace $\mathfrak{W} \subseteq \mathfrak{V}$. Then $C \cap W = \operatorname{Spec}(S/J)$ for $J = I + S\mathfrak{W}$ and $\operatorname{Rid}(C) \cap W = \operatorname{Spec}(S/S(\mathfrak{Q}(I) + \mathfrak{W}))$.

(i) We have $S(k[\mathfrak{Q}(I)] \cap I) = I$ and therefore also $S(k[\mathfrak{Q}(I) + \mathfrak{W}] \cap J) = J$, hence $\mathfrak{Q}(J) \subseteq \mathfrak{Q}(I) + \langle \mathfrak{W} \rangle_{k[F]}$, yielding (6.4.6.A).

(ii) $\operatorname{codim}_{\operatorname{Rid}(C)}(\operatorname{Rid}(C)\cap W) = \dim\operatorname{Rid}(C) - \dim(\operatorname{Rid}(C)\cap W) = \operatorname{rank}_{k[F]}(\mathfrak{Q}(I) + \langle\mathfrak{W}\rangle_{k[F]}) - \operatorname{rank}_{k[F]}(\mathfrak{Q}(I)) \leq \operatorname{rank}_{k[F]}(\langle\mathfrak{W}\rangle_{k[F]}) = \dim V - \dim W = \operatorname{codim}_V(W)$.

(iii) By (6.4.6.A) and (6.4.6.B) we see that $\dim\operatorname{Rid}(C) = \dim(\operatorname{Rid}(C)\cap W) + \operatorname{codim}_{\operatorname{Rid}(C)}(\operatorname{Rid}(C)\cap W) \leq \dim\operatorname{Rid}(C\cap W) + \operatorname{codim}_V(W)$ and it is clear that this is an equality if and only if (6.4.6.A) and (6.4.6.B) both are equalities. Assume the equality and let K/k be a perfect field. Then

$$\dim\operatorname{Dir}(C_K) = \dim\operatorname{Rid}(C_K) = \dim\operatorname{Rid}(C) = \dim\operatorname{Rid}(C\cap W) + \operatorname{codim}_V(W) =$$

$$= \dim\operatorname{Rid}((C\cap W)_K) + \operatorname{codim}_V(W) = \dim\operatorname{Dir}(C_K\cap W_K) + \operatorname{codim}_{V_K}(W_K).$$

By (6.4.6) (iii) we get (6.4.6.D) and the non-canonical isomorphism (6.4.6.E)

$$C_K \cong (C_K\cap W_K)\times_K \mathbb{A}_K^{\operatorname{codim}_V(W)} \cong (C\cap W)\times_k \mathbb{A}_K^{\operatorname{codim}_V(W)}.$$

(iv) For $\operatorname{Rid}(C)\subseteq W$ we have $\operatorname{codim}_{\operatorname{Rid}(C)}(\operatorname{Rid}(C)\cap W) = 0$ and get (6.4.6.F) by examining the proof of (iii). □

7 Permissible Blow Ups

We discuss permissible blow ups, i.e. blow ups in normally flat regular centers. In the next chapter we will analyze the behavior of our invariants under such transformations. We carry over the defintion of the ridge of a cone to tangent cones of schemes, recall regularity, normal cones and blow ups, say some words on the reduction to the embedded case and eventually come to the different characterizations of normal flatness.

7.1 Tangent cones and normal cones

We recollect well-known properties of tangent cones and normal cones. All schemes are supposed to be locally noetherian.

Definition (7.1.1). *For a point x of a scheme X the* ***tangent cone*** *of X at x is the cone*

$$C_{X,x} := \mathrm{Spec}(\mathrm{gr}_{X,x})$$

over x (resp. $\kappa(x)$) for the graded $\kappa(x)$-algebra

$$\mathrm{gr}_{X,x} := \mathrm{gr}_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) = \bigoplus_{d\geq 0} \mathfrak{m}^d_{X,x}/\mathfrak{m}^{d+1}_{X,x}.$$

The ***tangent space*** *of X at x is the vector space*

$$T_{X,x} := \mathrm{Spec}(\mathrm{Sym}_{X,x})$$

over x for the graded $\kappa(x)$-algebra

$$\mathrm{Sym}_{X,x} := \mathrm{Sym}_{\kappa(x)}(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}).$$

$C_{X,x}$ is a closed subscheme of $T_{X,x}$. As described in chapter 6 one can associate to the cone $C_{X,x}$ its ***ridge*** *and* ***directrix***

$$\mathrm{Rid}_{X,x} := \mathrm{Rid}(C_{X,x}),$$

$$\mathrm{Dir}_{X,x} := \mathrm{Dir}(C_{X,x})$$

and we have the sequence of closed subcones of $T_{X,x}$

(7.1.1.A) $$\mathrm{Dir}_{X,x} \subseteq \mathrm{Rid}_{X,x} \subseteq C_{X,x} \subseteq T_{X,x}.$$

Remember that X is called ***regular*** *at x if $\mathcal{O}_{X,x}$ is a regular local ring. This is the case if and only if $\mathrm{gr}_{X,x}$ is a polynomial ring over $\kappa(x)$. X is called regular if it is regular at all points. We also could say that X is regular at x if $C_{X,x}$ is a vector* *space. In this case* $C_{X,x}$ *itself is a group and therefore* $\mathrm{Rid}_{X,x} = C_{X,x}$*. All objects in the sequence (7.1.1.A) then are the same.*

Lemma (7.1.2). *Let* $f : X \to Y$ *be a morphism of schemes and* $x \in X$ *with* $y := f(x)$*. Then there is a functorial canonical morphism*

$$C_{f,x} : C_{X,x} \to C_{Y,y}$$

inducing a morphism of cones over $\kappa(x)$

$$C^{\kappa}_{f,x} : C_{X,x} \to \kappa(x) \times_{\kappa(y)} C_{Y,y}.$$

If f *is a closed immersion, then so is* $C_{f,x}$*.*

Proof. f induces a filtered morphism of local rings $(\mathcal{O}_{Y,y}, \mathfrak{m}_{Y,y}) \to (\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ and therefore a morphism of graded rings $\mathrm{gr}_{Y,y} \to \mathrm{gr}_{X,x}$ yielding $C_{X,x} \to C_{Y,y}$. From the commutative diagram

$$\begin{array}{ccc} C_{X,x} & \longrightarrow & C_{Y,y} \\ \downarrow & & \downarrow \\ \kappa(x) & \longrightarrow & \kappa(y) \end{array}$$

and the universal property of the fiber product we get a morphism $C_{X,x} \to \kappa(x) \times_{\kappa(y)} C_{Y,y}$. If f is a closed immersion, then $(\mathcal{O}_{Y,y}, \mathfrak{m}_{Y,y}) \to (\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ and therefore also $\mathrm{gr}_{Y,y} \to \mathrm{gr}_{X,x}$ are epimorphisms. □

Definition (7.1.3). *For a closed subscheme* X *of a scheme* Y *there is a natural structure of graded* $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}_{Y,X}$*-algebra on*

$$\mathrm{gr}_{Y,X} := \mathrm{gr}_{\mathcal{I}_{Y,X}}(\mathcal{O}_Y) = \bigoplus_{d \geq 0} \mathcal{I}^d_{Y,X}/\mathcal{I}^{d+1}_{Y,X}.$$

The ***normal cone*** *of* Y *along* X

$$N_{Y,X} := \mathrm{Spec}(\mathrm{gr}_{Y,X})$$

therefore comes with a projection $N_{Y,X} \to X$*. For a point* $x \in X$ *we denote the fiber of the morphism* $N_{Y,X} \to X$ *above* x *as* $N_{Y,X,x}$*. Explicitely* $N_{Y,X,x} = \mathrm{Spec}(\mathrm{gr}_{Y,X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x))$*, where* $\mathrm{gr}_{Y,X,x} = \bigoplus_{d \geq 0} \mathcal{I}^d_{Y,X,x}/\mathcal{I}^{d+1}_{Y,X,x}$ *is the stalk of* $\mathrm{gr}_{Y,X}$ *at* x*.*

Lemma (7.1.4). *Let* $X \subseteq Y \subseteq Z$ *be a chain of closed subschemes and* x *a point of* X*. There is a canonical cartesian diagram in which the horizontal arrows are closed immersions:*

$$\begin{array}{ccc} N_{Y,X,x} & \longrightarrow & N_{Z,X,x} \\ \downarrow & & \downarrow \\ N_{Y,X} & \longrightarrow & N_{Z,X} \end{array}$$

Proof. We have $\mathcal{I}_{Z,Y} \subseteq \mathcal{I}_{Z,X}$ and can identify $\mathcal{I}_{Y,X} \cong \mathcal{I}_{Z,X}/\mathcal{I}_{Z,Y}$. From the epimorphism $\mathcal{I}_{Z,X} \to \mathcal{I}_{Y,X}$ we get an epimorphism

$$\mathrm{gr}_{Z,X} = \bigoplus_{d\geq 0} \mathcal{I}^d_{Z,X}/\mathcal{I}^{d+1}_{Z,X} \to \bigoplus_{d\geq 0} \mathcal{I}^d_{Y,X}/\mathcal{I}^{d+1}_{Y,X} = \mathrm{gr}_{Y,X}$$

which gives the lower closed immersion in the diagram.

$$N_{Y,X} \times_{N_{Z,X}} N_{Z,X,x} = N_{Y,X} \times_{N_{Z,X}} (N_{Z,X} \times_X \{x\}) = N_{Y,X} \times_X \{x\} = N_{Y,X,x}$$

shows that the diagram is cartesian. Remark that the composition $N_{Y,X} \to N_{Z,X} \to X$ gives the canonical $N_{Y,X} \to X$. $\square$

Lemma (7.1.5). *Let $X \subseteq Y$ be a closed subscheme and x a point of X. There is a canonical morphism of cones*

$$C_{Y,x} \to N_{Y,X,x}.$$

If $X = \overline{\{x\}}$, this is an isomorphism.

Proof. We have $\mathcal{I}_{Y,X,x} \subseteq \mathfrak{m}_{Y,x}$ and $C_{Y,x} \to N_{Y,X,x}$ is given by

$$\mathrm{gr}_{Y,X,x} = \bigoplus_{d\geq 0} \mathcal{I}^d_{Y,X,x}/\mathcal{I}^{d+1}_{Y,X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \cong$$

$$\cong \bigoplus_{d\geq 0} \mathcal{I}^d_{Y,X,x}/\mathfrak{m}_{Y,y}\mathcal{I}^d_{Y,X,x} \to \bigoplus_{d\geq 0} \mathfrak{m}^d_{Y,x}/\mathfrak{m}^{d+1}_{Y,x} = \mathrm{gr}_{Y,y}\,.$$

In the case $X = \overline{\{x\}}$ we have $\mathcal{I}_{Y,X,x} = \mathfrak{m}_{Y,x}$. $\square$

Lemma (7.1.6). *If $D \subseteq Z$ is a closed subscheme and both D and Z are regular, then $N_{Z,D}$ is a locally trivial vector bundle over D. In particular for a point x of D we have an exact sequence of vector spaces (cf. [H4, p. 153])*

$$0 \longrightarrow T_{D,x} \xrightarrow{\alpha} T_{Z,x} \xrightarrow{\beta} N_{Z,D,x} \longrightarrow 0.$$

This sequence is split exact, i.e. we have morphisms $N_{Z,D,x} \xrightarrow{\beta'} T_{Z,x} \xrightarrow{\alpha'} T_{D,x}$ such that $\beta\beta' = \mathrm{id}_{N_{Z,D,x}}$ and $\alpha'\alpha = \mathrm{id}_{T_{D,x}}$ and

$$\begin{array}{ccc} T_{Z,x} & \xrightarrow{\beta} & N_{Z,D,x} \\ \downarrow{\scriptstyle\alpha'} & & \downarrow \\ T_{D,x} & \longrightarrow & \kappa(x) \end{array}$$

is cartesian.

Proof. We have to verify the exact sequence and that $N_{Z,D,x}$ is a vector space. Suppose that $Z = \mathrm{Spec}(A)$ with a regular local ring $(A, \mathfrak{m})$, $D = \mathrm{Spec}(A/\mathfrak{p})$ with a regular prime $\mathfrak{p}$ and x is the closed point of Z. We can choose regular parameters

$x_1, ..., x_n$ of $\mathfrak{m}$ and can assume that $\mathfrak{p} = \langle x_1, ..., x_m \rangle$. Then

$$T_{Z,x} = \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}}(A)) \cong \mathrm{Spec}(\kappa(x)[X_1, ..., X_n]), \qquad x_i \bmod \mathfrak{m}^2 \mapsto X_i$$

induces

$$T_{D,x} = \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}/\mathfrak{p}}(A/\mathfrak{p})) \cong \mathrm{Spec}(\kappa(x)[X_{m+1}, ..., X_n])$$

and per definition $N_{Z,D,x} \cong \mathrm{Spec}(\mathrm{gr}_{\mathfrak{p}}(A) \otimes_{A/\mathfrak{p}} A/\mathfrak{m})$, where

$$\mathrm{gr}_{\mathfrak{p}}(A) \cong A/\mathfrak{p}[X_1, ..., X_m], \qquad x_i \bmod \mathfrak{p}^2 \mapsto X_i$$

and therefore $\mathrm{gr}_{\mathfrak{p}}(A) \otimes_{A/\mathfrak{p}} A/\mathfrak{m} \cong \kappa(x)[X_1, ..., X_m]$. We can then choose α' corresponding to the inclusion $\kappa(x)[X_{m+1}, ..., X_n] \to \kappa(x)[X_1, ..., X_n]$ and β' corresponding to the projection $\kappa(x)[X_1, ..., X_n] \to \kappa(x)[X_1, ..., X_m]$ with $X_i \mapsto 0$ for $i > m$. $\square$

7.2 Blow ups

Definition (7.2.1). *Let $D \subseteq X$ be a closed subscheme defined by the sheaf of ideals $\mathcal{I}_{X,D}$. The **blow up** of X in D, resp. with **center** D, is the scheme*

$$\mathrm{Bl}_D(X) := \mathrm{Proj}\left(\mathrm{Bl}_{\mathcal{I}_{X,D}}(\mathcal{O}_X)\right), \qquad \mathrm{Bl}_{\mathcal{I}_{X,D}}(\mathcal{O}_X) := \bigoplus_{d \geq 0} \mathcal{I}_{X,D}^d$$

together with the canonical morphism

$$\pi : \mathrm{Bl}_D(X) \to X$$

*defined via $\mathcal{O}_X = \mathcal{I}_{X,D}^0$. The closed subscheme $\pi^{-1}(D) = \mathrm{Bl}_D(X) \times_X D$ of $\mathrm{Bl}_D(X)$ is called the **exceptional divisor** of the blow up. Explicitely we have*

$$\pi^{-1}(D) \cong \mathrm{Proj}\left(\bigoplus_{d \geq 0} \mathcal{I}_{X,D}^d / \mathcal{I}_{X,D}^{d+1}\right) = \mathrm{Proj}(\mathrm{gr}_{X,D}) = \mathbb{P}(N_{X,D})$$

and for a point $x \in D$

$$\pi^{-1}(x) = \mathrm{Proj}(\mathrm{gr}_{X,D,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)) = \mathbb{P}(N_{X,D,x}).$$

This is clear since $\mathrm{gr}_{X,D} \cong \mathrm{Bl}_{\mathcal{I}_{X,D}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}_{X,D}$.

Lemma (7.2.2). *Let $f : Y \to X$ be a morphism of schemes, $D \subseteq X$ a closed subscheme and $E := Y \times_X D \to Y$ the corresponding closed subscheme of Y. There exists a canonical morphism $f' : \mathrm{Bl}_E(Y) \to \mathrm{Bl}_D(X)$ making the following diagram commutative:*

(7.2.2.A)
$$\begin{array}{ccc} \mathrm{Bl}_E(Y) & \xrightarrow{f'} & \mathrm{Bl}_D(X) \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\ Y & \xrightarrow{f} & X \end{array}$$

If f is flat, then the diagram is cartesian.

Proof. $\mathcal{I}_{Y,E} = \mathcal{I}_{X,D} \cdot \mathcal{O}_Y$ yields a morphism of graded algebras $\mathrm{Bl}_{\mathcal{I}_{X,D}}(\mathcal{O}_X) \to \mathrm{Bl}_{\mathcal{I}_{Y,E}}(\mathcal{O}_Y)$. Let f be flat and $X = \mathrm{Spec}(A), D = \mathrm{Spec}(A/I), Y = \mathrm{Spec}(B), E = \mathrm{Spec}(B/J)$, $J = BI$. Then $B \otimes_A I^d \cong J^d$ for all $d \geq 0$. □

Remark (7.2.3). *Let $D = \mathrm{Spec}(A/I)$ be a closed subscheme of $X = \mathrm{Spec}(A)$. For $x \in I = \mathrm{Bl}_I^1(A)$ the homogeneous localization*

$$\mathrm{Bl}_I(A)_{(x)} = \left\{ \frac{y}{x^d} \middle| \, y \in \mathrm{Bl}_I^d(A) = I^d, d \in \mathbb{N} \right\}$$

yields a chart $\mathrm{Spec}(\mathrm{Bl}_I(A)_{(x)})$ of the blow up $\mathrm{Bl}_D(X)$. The exceptional divisor is given in this chart as

$$\mathrm{Spec}(\mathrm{Bl}_I(A)_{(x)} \otimes_A A/I)) \cong \mathrm{Spec}(\mathrm{Bl}_I(A)_{(x)}/I\,\mathrm{Bl}_I(A)_{(x)})$$

and

$$I\,\mathrm{Bl}_I(A)_{(x)} = \left\{ \frac{y}{x^d} \middle| \, y \in I^{d+1}, d \in \mathbb{N} \right\} = x \cdot \mathrm{Bl}_I(A)_{(x)}$$

because $\frac{y}{x^d} = x \cdot \frac{y}{x^{d+1}}$. Therefore the exceptional divisor is defined by x in this chart and is in fact a Cartier divisor (x is not a zero divisor since $\mathrm{Bl}_I(A)_{(x)} \subseteq \mathrm{Bl}_I(A)_x$).

Definition (7.2.4). *Let $D \subseteq X$ be a closed subscheme. X is called* ***normally flat*** *along D at a point $x \in D$ if the normal cone $N_{X,D}$ is flat over D at x, i.e. $\mathrm{gr}_{\mathcal{I}_{X,D,x}}(\mathcal{O}_{X,x})$ is a flat $\mathcal{O}_{D,x}$-module. In fact this is equivalent to say that $\mathrm{gr}_{X,D,x} = \mathrm{gr}_{\mathcal{I}_{X,D,x}}(\mathcal{O}_{X,x})$ is a free $\mathcal{O}_{D,x}$-module. X is normally flat along D if it is normally flat at every point of D. D is called* ***permissible*** *at a point $x \in D$ if X is normally flat along D at x and D is regular at x, i.e. $\mathcal{O}_{D,x}$ is regular. D is called permissible if it is permissible at all points.*

The property of normal flatness was used by Hironaka in his proof of resolution of singularities in characteristic zero [H1]. The notion of permissible blow up or Hironaka permissible blow up is widely used in resolution of singularities.

7.3 Permanence properties

Many properties can be expressed easier if a scheme is embedded into a regular scheme. For our purposes we always can assume to be in this situation: This follows from the discussion in this section together with the Cohen structure theorem (e.g. [EGA, 0_{IV} 19.8]). Throughout this section consider a morphism $f : Y \to X$ of locally noetherian schemes and fix a closed subscheme $D \subseteq X$, a point $x \in D$ and a point $y \in Y$ with $f(y) = x$. With $E = Y \times_X D$ the point y lies on E and we have a morphism $f' : Y' \to X'$ as in (7.2.2) with $Y' = \mathrm{Bl}_E(Y)$ and $X' = \mathrm{Bl}_D(X)$. See [CJS, 1.27 (i)] for a similar discussion.

Definition (7.3.1). *f is said to be quasi-étale (in the sense of Bennett) at y if f is flat at y and $\mathfrak{m}_{X,x}\mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$.*

Lemma (7.3.2). *If f is quasi-étale at y, then the following hold:*

(i) $C^{\kappa}_{f,y} : C_{Y,y} \cong \kappa(y) \times_{\kappa(x)} C_{X,x}$ *is an isomorphism.*

(ii) X *is regular at* x *iff* Y *is regular at* y.

(iii) $E \to D$ *is quasi-étale at* y.

(iv) D *is regular at* x *iff* E *is regular at* y.

(v) *There is a canonical isomorphism* $N_{Y,E,y} \cong \kappa(y) \times_{\kappa(x)} N_{X,D,x}$ *inducing a canonical isomorphism* $\pi_Y^{-1}(y) \cong \kappa(y) \times_{\kappa(x)} \pi_X^{-1}(x)$.

(vi) X *is normally flat along* D *at* x *iff* Y *is normally flat along* E *at* y.

(vii) $D \subseteq X$ *is permissible at* x *iff* $E \subseteq Y$ *is permissible at* y.

Proof. We get (i) since f is quasi-étale at y:

$$\operatorname{gr}_{X,x} \otimes_{\kappa(x)} \kappa(y) \cong \left(\bigoplus_{d \geq 0} \mathfrak{m}^d_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \right) \otimes_{\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} \cong$$

$$\cong \bigoplus_{d \geq 0} \mathfrak{m}^d_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} \cong \bigoplus_{d \geq 0} \left(\mathfrak{m}^d_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y} \right) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} \cong$$

$$\cong \bigoplus_{d \geq 0} \mathfrak{m}_{X,x}\mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} \cong \operatorname{gr}_{Y,y} .$$

(i) implies (ii) because $x \in X$ is regular iff $C_{X,x}$ is a vector space and analogously for $y \in Y$. We can assume that $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B), D = \operatorname{Spec}(A/I)$, x corresponds to a prime $\mathfrak{p}$ of A and y to a prime $\mathfrak{P}$ of B. Then $E = \operatorname{Spec}(B \otimes_A A/I) = \operatorname{Spec}(B/IB)$ and since $x \in D$ we have $I \subseteq \mathfrak{p}$ and get $IB \subseteq \mathfrak{p}B \subseteq \mathfrak{P}$, i.e. $y \in E$. Passing to the local rings yields the diagram

$$\begin{array}{ccc} B_{\mathfrak{P}}/(IB)_{\mathfrak{P}} & \longleftarrow & A_{\mathfrak{p}}/I_{\mathfrak{p}} \\ \uparrow & & \uparrow \\ B_{\mathfrak{P}} & \longleftarrow & A_{\mathfrak{p}} \end{array}$$

which is cocartesian since $I_{\mathfrak{p}} B_{\mathfrak{P}} = (IB)_{\mathfrak{P}}$. The lower morphism in this diagram is flat and therefore also the upper one, this proves (iii). Applying (ii) to $X = D, Y = E$ yields (iv). Tensoring

(7.3.2.A) $$\operatorname{gr}_{(IB)_{\mathfrak{P}}}(B_{\mathfrak{P}}) = \operatorname{gr}_{I_{\mathfrak{p}} B_{\mathfrak{P}}}(B_{\mathfrak{P}}) \cong \operatorname{gr}_{I_{\mathfrak{p}}}(A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}/I_{\mathfrak{p}}} B_{\mathfrak{P}}/I_{\mathfrak{p}} B_{\mathfrak{P}}$$

with $B/\mathfrak{P}B$ yields on the left $\operatorname{gr}_{IB}(B) \otimes_B \kappa(y)$ which is the affine ring of $N_{Y,E,y}$. Since $IB \subseteq \mathfrak{P}$ we have $B/IB \otimes_B B/\mathfrak{P} \cong B/\mathfrak{P}$ and get on the right $\operatorname{gr}_I(A) \otimes_A \kappa(x) \otimes_{\kappa(x)} \kappa(y)$. This yields (v) since $\pi_Y^{-1}(y) = \mathbb{P}(N_{Y,E,y})$ and $\pi_X^{-1}(x) = \mathbb{P}(N_{X,D,x})$. The morphism of local rings $\mathcal{O}_{D,x} \to \mathcal{O}_{E,y}$ is flat and therefore also faithfully flat, so by (7.3.2.A) $\operatorname{gr}_{\mathcal{I}_{Y,E,y}}(\mathcal{O}_{Y,y})$ is free iff $\operatorname{gr}_{\mathcal{I}_{X,D,x}}(\mathcal{O}_{X,x})$ is free, proving (vi). (vii) is clear from (iv) and (vi). □

Consider a point $x' \in \pi_X^{-1}(x)$ and a point $y' \in Y'$ mapping to y resp. x'. Such a point exists if f is flat and hence the diagram (7.2.2.A) is cartesian.

Corollary (7.3.3). *In the case $Y = \operatorname{Spec}(\mathcal{O}_{X,x})$ with closed point $y \in Y$ the canonical morphism $f : Y \to X$ is flat, hence also f' is flat. We have $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$ and $\mathcal{O}_{Y',y'} \cong \mathcal{O}_{X',x'}$. So f is quasi-étale at y and f' is quasi-étale at y' and both residue field extensions are trivial. The statements of (7.3.2) are applicable.*

Corollary (7.3.4). *In the case $X = \operatorname{Spec}(A)$ with a noetherian local ring $(A, \mathfrak{m})$ and closed point x and $Y = \operatorname{Spec}(\widehat{A})$ (completion) with closed point y the morphism f is quasi-étale at y and f' is quasi-étale at y' and both residue field extensions are trivial. All statements of (7.3.2) are applicable.*

Proof. It is well known that $A \to \widehat{A}$ is flat, $\operatorname{gr}_{\mathfrak{m}}(A) \cong \operatorname{gr}_{\widehat{\mathfrak{m}}}(\widehat{A})$ and $\mathfrak{m}\widehat{A} = \widehat{\mathfrak{m}}$. Therefore f is quasi-étale at y. Since x' lies over x, i.e. $\mathfrak{m}$ is mapped to zero under $A \to \kappa(x')$ we find

$$Y' \times_{X'} x' \cong Y \times_X X' \times_{X'} x' \cong Y \times_X x' \cong \operatorname{Spec}(\widehat{A} \otimes_A \kappa(x')) \cong$$

$$\cong \operatorname{Spec}(\widehat{A}/\mathfrak{m}\widehat{A} \otimes_{\kappa(x)} \kappa(x')) \cong x'.$$

The canonical map $\operatorname{Spec}(\mathcal{O}_{Y',y'}) \times_{\operatorname{Spec}(\mathcal{O}_{X',x'})} x' \to Y' \times_{X'} x' \cong x'$ is given as a localization $\kappa(x') \to \mathcal{O}_{Y',y'}/\mathfrak{m}_{X',x'}\mathcal{O}_{Y',y'}$. Since $\kappa(x')$ is a field, this map must be an isomorphism. This proves not only $\kappa(x') = \kappa(y')$ but also $\mathfrak{m}_{X',x'}\mathcal{O}_{Y',y'} = \mathfrak{m}_{Y',y'}$ and since f' is flat, this shows that f' is quasi-étale at y' with trivial residue extension. □

If we use (7.3.2) with $\kappa(x) = \kappa(y)$ like in the two preceding corollaries, the isomorphism $C_{Y,y} \to C_{X,x}$ of course also induces isomorphisms $\operatorname{Dir}_{Y,y} \to \operatorname{Dir}_{X,x}$ and $\operatorname{Rid}_{Y,y} \to \operatorname{Rid}_{X,x}$. If we had a non-trivial residue extension we still would have an isomorphism for the ridge (see (6.1.10)), but the directrix could change in the case of an inseparable extension (cf. (6.3.6)).

7.4 Characterizations of normal flatness

We give four equivalent characterizations of normal flatness under the assumption that the closed subscheme at hand is regular as was required for a permissible blow up. We reap the fruits of our discussion of bifiltrations and follow [Gi, II 2.] in our proof.

Theorem (7.4.1) (cf. [Gi, II 2.2], [CJS, 2.2 (2)]). *Let $x \in D \subseteq X$ be a point of a closed subscheme. Suppose that D is regular at the point x. Then the following conditions are equivalent:*

(i) *X is normally flat along D at x.*

(ii) *$T_{D,x} \subseteq \operatorname{Dir}_{X,x}$ and the natural morphism $C_{X,x} \to N_{X,D,x}$ from (7.1.5) induces an isomorphism*

$$C_{X,x}/T_{D,x} \to N_{X,D,x},$$

and in particular $C_{X,x} \to N_{X,D,x}$ is a core-isomorphism.

If X is a closed subscheme of a regular scheme Z, then the above are equivalent to the following further equivalent conditions:

(iii) *The canonical diagram*

$$\begin{array}{ccc} C_{X,x} & \longrightarrow & N_{X,D} \\ \downarrow & & \downarrow \\ C_{Z,x} & \longrightarrow & N_{Z,D} \end{array}$$

is cartesian.

(iv) *Let $\mathcal{O}_{Z,x} = \mathrm{Spec}(R)$ for a regular local ring R with maximal ideal $\mathfrak{m}$, $\mathcal{O}_{D,x} = \mathrm{Spec}(R/\mathfrak{p})$ and $\mathcal{O}_{X,x} = \mathrm{Spec}(R/I)$ for ideals $I \subseteq \mathfrak{p} \subseteq \mathfrak{m}$. Let $\mathcal{I}'$ and $\mathcal{I}''$ be the pullbacks of the $\mathfrak{p}$-adic and $\mathfrak{m}$-adic filtrations on R via $I \to R$. Then there exists a standard basis $(f_1, ..., f_m)$ of $(I, \mathcal{I}'')$ such that $\nu_{\mathcal{I}'}(f_i) = \nu_{\mathcal{I}''}(f_i)$ for $1 \leq i \leq m$.*

Proof. Conditions (i) and (ii) are stable by passing from X to $\mathrm{Spec}(\mathcal{O}_{X,x})$ and also by further passing to $\mathrm{Spec}(\widehat{\mathcal{O}_{X,x}})$ as seen in 7.3. Therefore, using the Cohen structure theorem, we can assume from the beginning that X is a closed subscheme of a regular scheme Z and we show the equivalence of (i) to (iv) in this situation.

(7.4.1.1) In the commutative diagram

$$\begin{array}{ccccccc} T_{D,x} & \longrightarrow & C_{X,x} & \longrightarrow & N_{X,D,x} & \longrightarrow & N_{X,D} \\ \| & (I) & \downarrow & (II) & \downarrow & (III) & \downarrow \\ T_{D,x} & \longrightarrow & C_{Z,x} & \longrightarrow & N_{Z,D,x} & \longrightarrow & N_{Z,D} \\ & & \downarrow & (IV) & \downarrow & & \\ & & T_{D,x} & \longrightarrow & \kappa(x) & & \end{array}$$

all morphisms in (I), (II) and (III) are the obvious natural morphisms. As remarked in (7.1.4) (III) is cartesian. The diagram from (iii) is the composition of (II) and (III). Therefore (iii) is equivalent to (II) being cartesian. (IV) is the cartesian diagram from (7.1.6). Therefore (II) is cartesian iff the composition of (II) and (IV)

$$\begin{array}{ccc} C_{X,x} & \longrightarrow & N_{X,D,x} \\ \downarrow & (V) & \downarrow \\ T_{D,x} & \longrightarrow & \kappa(x) \end{array}$$

is cartesian. So (iii) is equivalent to (V) being cartesian.

(7.4.1.2) Let A, B be $\kappa(x)$-algebras such that $C_{X,x} \cong \mathrm{Spec}(A)$ and $N_{X,D,x} \cong \mathrm{Spec}(B)$. (V) being cartesian is equivalent to $A \cong \kappa(x)[X_{m+1}, ..., X_n] \otimes_{\kappa(x)} B$ (notation from (7.1.6)). This is equivalent to $T_{D,x} \subseteq \mathrm{Dir}_{X,x}$ and $C_{X,x}/T_{D,x} \cong N_{X,D,x}$.

(7.4.1.3) For what remains we can of course again assume that $Z = \mathrm{Spec}(R)$ for a regular local ring R, x is the closed point of Z corresponding to the maximal ideal $\mathfrak{m}$ of R, $D = \mathrm{Spec}(R/\mathfrak{p})$ such that $R/\mathfrak{p}$ is regular and $X = \mathrm{Spec}(R/I)$ for an ideal

$I \subseteq \mathfrak{p} \subseteq \mathfrak{m}$. The diagram from (iii) then corresponds to the commutative diagram of graded $\kappa(x)$-algebras

$$\begin{array}{ccc} \mathrm{gr}_{\mathfrak{m}/I}(R/I) & \longleftarrow & \mathrm{gr}_{\mathfrak{p}/I}(R/I) \\ \uparrow & (VI) & \uparrow \\ \mathrm{gr}_{\mathfrak{m}}(R) & \longleftarrow & \mathrm{gr}_{\mathfrak{p}}(R) \end{array}$$

(i) means that $\mathrm{gr}_{\mathfrak{p}/I}(R/I)$ is a flat $R/\mathfrak{p}$-algebra. On the R-module $E := R/I$ we have the induced $\mathfrak{p}$-adic filtration $\mathcal{E}'$ and the induced $\mathfrak{m}$-adic filtration $\mathcal{E}''$. By definition (5.1.1) we see that $(E, \mathcal{E}', \mathcal{E}'')$ is a harmonious bifiltered $(R, \mathfrak{p}, \mathfrak{m})$-module. Now (5.3.3) says that $\mathrm{gr}(\mathcal{E}')$ is flat over $R/\mathfrak{p}$ if and only if the natural morphism $\mathrm{gr}_{\mathfrak{m}}(R) \otimes_{\mathrm{gr}_{\mathfrak{p}}(R)} \mathrm{gr}(\mathcal{E}') \to \mathrm{gr}(\mathcal{E}'')$ is an isomorphism. This shows the equivalence of (i) and (iii). The same is also equivalent by (5.2.2) and (5.1.3) to (iv). □

Lemma (7.4.2). *Let $f : Y \to X$ be a morphism of schemes, $D \subseteq X$ a closed subscheme and $E := Y \times_X D$. Let $x \in D, y \in E$ with $f(y) = x$. If X is normally flat along D at x and f is flat at y, then Y is normally flat along E at y and*

$$\mathrm{gr}_{Y,E,y} \cong \mathrm{gr}_{X,D,x} \otimes_{\mathcal{O}_{D,x}} \mathcal{O}_{E,y}.$$

Proof. Let D be defined in $\mathcal{O}_{X,x}$ by the ideal I, then E is defined in $\mathcal{O}_{Y,y}$ by $J := I \cdot \mathcal{O}_{Y,y}$. Since X is normally flat along D at x, we know that $\mathrm{gr}_I(\mathcal{O}_{X,x})$ is a free $\mathcal{O}_{X,x}/I$-module. $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat and therefore the same holds for $\mathcal{O}_{X,x}/I \to \mathcal{O}_{Y,y}/J$. Thus $\mathrm{gr}_I(\mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}/I} \mathcal{O}_{Y,y}/J \cong \mathrm{gr}_J(\mathcal{O}_{Y,y})$ is a free $\mathcal{O}_{Y,y}/J$-module, i.e. Y is normally flat along E at y. □

Corollary (7.4.3) (cf. [Gi, II 2.6]). *Let V be a vector space over the field k, C a closed subcone of V and W a vector space contained in C. Then the following conditions are equivalent:*

(i) *C is normally flat along W at the origin 0 of V.*

(ii) *$W \subseteq \mathrm{Dir}(C)$.*

(iii) *C is normally flat along W at every point of W.*

Proof. If (i) holds, then by (7.4.1) we have $W = T_{W,0} \subseteq \mathrm{Dir}_{C,0} = \mathrm{Dir}(C)$ and this is (ii). $W \subseteq \mathrm{Dir}(C)$ implies $C \cong W \times_k C'$ compatible with the embedding into V. The projection $C \to C/W$ is flat and the fiber above $0 \in C/W$ equals $W \subseteq C$. Clearly C/W is normally flat along 0, hence by (7.4.2) C is normally flat along W. (iii) $\Rightarrow$ (i) is obvious. □

8 The Invariants

We define our invariants: The Hilbert series and the dimension of the ridge of the tangent cone at a point of a scheme. We describe their meaning and prove that these invariants do not increase under permissible blow ups. Our strategy to achieve this result focuses on cones.

8.1 Overview

Our invariants only depend on the tangent cone at a point of a scheme. We use them also for cones in general.

Definition (8.1.1). *Let C be a cone over some field k. Let x be a point of a locally noetherian scheme X.*

(i) *Our first invariant is the Hilbert series. For the cone $C = \mathrm{Spec}(A), A = \bigoplus_{n\geq 0} A_n$, over the field k it is the series*

$$H(C) = H(A) = \sum_{n\geq 0} \dim_k(A_n)T^n \in \mathbb{N}[[T]]$$

(see 4.5). The Hilbert series of X at x is defined as

$$H_{X,x} := H(C_{X,x}) = H(\mathrm{gr}_{X,x}) = H(\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$$

(see 4.6 for the last equality). We use the notations $H^{(d)}(C) = (1-T)^{-d}H(C)$ and $H^{(d)}_{X,x} = H^{(d)}(C_{X,x})$.

(ii) *The second invariant we use is the dimension of the ridge of a cone*

$$R(C) := \dim(\mathrm{Rid}(C)) \in \mathbb{N},$$

where $\mathrm{Rid}(C)$ *is the largest subgroup that translates C onto itself (see 6.1). We use this invariant also for schemes with the notation*

$$R_{X,x} = \dim(\mathrm{Rid}(C_{X,x}))$$

and extend the notations to $R^{(d)}(C) = R(C) + d$ and $R^{(d)}_{X,x} := R^{(d)}(C_{X,x})$.

(iii) *We sometimes combine these two invariants to*

$$HR(C) := (H(C), R(C)) \in \mathbb{N}[[T]] \times \mathbb{N},$$

$$HR_{X,x} := (H_{X,x}, R_{X,x}).$$

Again we extend our notations to $HR^{(d)}(C) = (H^{(d)}(C), R^{(d)}(C))$ *and* $HR^{(d)}_{X,x} = HR^{(d)}(C_{X,x})$.

Instead of the Hilbert series also the Hilbert function $H(C) : \mathbb{N} \to \mathbb{N}, H(C)(n) = \dim_k(A_n)$ often is used. It contains the same information. The aim of this chapter is to give a comprehensive proof of the following well-known result:

Theorem (8.1.2). *If* $\pi : X' \to X$ *is a permissible blow up in a center* $D \subseteq X$ *and* $x' \in \pi^{-1}(x)$ *for a point* $x \in D$*, then*

$$HR^{(d+1)}_{X',x'} \leq HR^{(1)}_{X,x},$$

where $d = \operatorname{tr.deg}(\kappa(x')/\kappa(x))$.

Let us explain the order used in the theorem to compare the two invariants. For two series $K = \sum_n a_n T^n, K' = \sum_n a'_n T^n \in \mathbb{Z}[[T]]$ the relation $K \leq K'$ means $a_n \leq a'_n$ for all n (we used this already in (4.5.1) f.). This only is a partial order and is not wellfounded on $\mathbb{N}[[T]]$. It is wellfounded when restricted to the subset of $\mathbb{N}[[T]]$ which comes under our consideration, namely the set of all Hilbert series of standard graded k-algebras (see [CJS, 1.15 f.]). On $\mathbb{N}$ we use the usual order and equip $\mathbb{N}[[T]] \times \mathbb{N}$ with the lexicographic order with respect to the orders just described. We obtain a wellfounded order. Therefore the invariant HR cannot drop infinitely many times under continued permissible blow ups. Resolution is achieved if one can show that it has to drop at a singular point after some blow ups.

Theorem (8.1.2) consists of two statements:

(H) $H^{(d+1)}_{X',x'} \leq H^{(1)}_{X,x}$.

(R) If (H) is an equality, then $R^{(d)}_{X',x'} \leq R_{X,x}$.

In (R) it makes no difference if we say $R^{(d+1)}_{X',x'} \leq R^{(1)}_{X,x}$. For series $K, K' \in \mathbb{Z}[[T]]$ clearly $K \leq K'$ implies $K^{(1)} \leq K'^{(1)}$, but the converse does not hold in general. In our special situation also $H^{(d)}_{X',x'} \leq H_{X,x}$ would be true. This was proved by Singh [Si1, Th. 1 f.]. The proof is a little harder and we see no advantage in the stronger inequality for our purposes since we are mainly interested in the case of equality. So we are satisfied with (H). Together with (H) and (R) we also will prove:

(N) If (H) is an equality, then $x' \in \mathbb{P}\operatorname{Rid}(C_{X,x}/T_{D,x}) \subseteq \pi^{-1}(x)$.

(I) If (H) and (R) are equalities, then there is an isomorphism

$$(C_{X,x})_K / \operatorname{Dir}((C_{X,x})_K) \to (C_{X',x'})_K / \operatorname{Dir}((C_{X',x'})_K)$$

for some perfect field K.

(N) gives information about the locus of near points, i.e. points x' where (H) is an equality. (I) means that the tangent cones $C_{X,x}$ and $C_{X',x'}$ are very similar: There is not much information left for additional invariants depending only on the tangent cone. The invariant HR will not be sufficient for resolution of singularities. It only depends on the tangent cone $C_{X,x}$, which is defined by the initial forms of equations defining X inside some regular scheme Z. Therefore HR does not see any terms of higher order. The invariant HR has to be extended by further invariants constructed directly from the local setting, such as polyhedra.

(8.1.3) Let us make a few simple observations concerning our invariants.

(i) If C is a cone, then there is a canonical isomorphism $C_{C,0} \cong C$ and therefore

$$H_{C,0} = H(C), \qquad R_{C,0} = R(C).$$

(ii) Our invariants, seen as invariants of a cone C over a field k, do not change under field extensions k'/k. For the Hilbert series this is clear since it measures vector space dimensions. For the ridge we have seen in (6.1.10) that $\mathrm{Rid}(C_{k'}) \cong \mathrm{Rid}(C)_{k'}$.

(iii) The shifted invariants $H^{(d)}$ and $R^{(d)}$ represent additional d dimensions:

$$H(C \times \mathbb{A}^d) = H^{(d)}(C), \qquad R(C \times \mathbb{A}^d) = R^{(d)}(C).$$

(8.1.4) We take a look at the meaning of our invariants.
Let k be a field, $S = k[X_1, ..., X_n]$, $I \subseteq S$ a homogeneous ideal and $C = \mathrm{Spec}(S/I)$ a cone in $V = \mathrm{Spec}(S)$. We use (4.5.6) to look at some examples. If $I = \langle f \rangle_S$ for a single homogeneous polynomial f of degree m, then $H(C) = (1 - T^m)(1 - T)^{-n}$. So in the case of a hypersurface the Hilbert series contains precisely the information about the degree of f, i.e. the multiplicity of the hypersurface at the origin. The same is true from the local point of view: If $(R, \mathfrak{m})$ is a regular local ring of dimension n and $t \in R$, then $H(R/\langle t \rangle, \mathfrak{m}) = (1 - T^m)(1 - T)^{-n}$ with the order $m = \nu_{\mathfrak{m}}(t)$ (see (4.6.5) and (4.4.8)). If we only were interested in hypersurfaces, we could use the easier invariant of multiplicity. But if I is not a principal ideal, things become more complicated. With (4.5.6) we see that one can simply calculate the Hilbert series of C if I is generated by a regular sequence $f_1, ..., f_r$ of homogeneous elements. This means that f_i is not a zero divisor in $S/\langle f_1, ..., f_{i-1} \rangle$ for $i = 2, ..., r$. In this case we still have $H(C) = (1 - T^{\deg(f_1)}) \cdots (1 - T^{\deg(f_r)})(1 - T)^{-n}$. But we also see that this holds only if $f_1, ..., f_r$ is a regular sequence. Not every ideal can be generated by a regular sequence, for example take $n = 2, I = \langle X_1^2, X_1X_2 \rangle$. Then $H(C) = T + (1 - T)^{-1}$. I cannot be generated by a single element and if I would be generated by a regular sequence with at least two elements, we would have $H(C) \in \mathbb{N}[T]$ which is not the case. A good generalization of the multiplicity to non-hypersurfaces is the Hilbert series. Another possible generalization would be to measure the degrees of a minimal system of generators of I: If $f_1, ..., f_m$ is a standard basis of the ideal I (see (4.3.7)), one looks at the array of degrees $\nu^*(I) = (\deg(f_1), ..., \deg(f_m), \infty, \cdots)$. This invariant is widely used also (cf. [H4, p. 155], [CJS, 1.1 ff.]). It behaves rather differently than the Hilbert series in general. In fact it is not possible to compute the one invariant from the other, the connection is more subtle: The Hilbert series can be computed from the ν^*-invariants of each module of a graded minimal free resolution of I ([Si2]). But ν^* also does not increase under a permissible blow up and in the critical cases the

invariants behave alike: (H) is an equality if and only if the corresponding equality for ν^* holds. This was proved by Hironaka [H4, Th. II, Th. III].
We now come to our other invariant R. Recall that $\mathcal{U}(\mathrm{IRid}(I))$ is the ring of invariants of $\mathrm{Rid}(C)$ in S. Its dimension is the minimal number of additive polynomials that are necessary to denote equations generating I (see (6.1.9)). Since $\mathrm{IRid}(I) = S \cdot \mathcal{U}(\mathrm{IRid}(I))$ we see that $\mathrm{codim}_V(\mathrm{Rid}(C))$ is again the same number of additive polynomials and $R(C)$ is the maximal number of independent additive polynomials that do not appear in equations generating I. So we can say: The lower $R(C)$ becomes, the more complex the ideal I looks like. It may seem counterproductive to wish that $R(C)$ drops, i.e. to make I more complex. But as a matter of fact, this will make the further process of resolution easier! Just take a look at property (N) above. The smaller the ridge becomes, the smaller the place gets where near points might appear. One gets another hint in the same direction, when one looks at the strategy to obtain resolution of threefolds in positive characteristic in [CP1],[CP2]: In [CP1] the general case is narrowed down to the special cases of purely inseparable and Artin-Schreier equations. The longest part of the work ([CP2]) has to deal with these two troublesome cases by defining further invariants. A good deal of consideration is given to equations that begin with a p-th power in resolution of singularities. These inseparable equations are a main obstruction to resolution of singularities. Inseparable equations correspond to a ridge that is rather large. Therefore one hopes that the dimension of the ridge will drop eventually in a resolution process.

(8.1.5) Let us compare our way to prove (H), (R), (N) and (I) to some standard references.

[Be] Bennett proves (H) in Theorem (2). His prove was simplified by Hironaka:

[H4] Hironaka proves (H) in Th. I loc. cit. by the sequence of inequalities (4.1) loc. cit.

$$H^{(1+d)}_{X',x'} \leq H^{(2+d+s)}_{\pi^{-1}(x),x'} \leq H^{(1+s)}_{N_{X,D},x} = H^{(1)}_{X,x}$$

with $s = \dim T_{D,x}$. We prove the first inequality in the same way as Hironaka (cf. (5.2) loc. cit.). The second inequality is proved in [H4] in several steps: The residually rational case $\kappa(x') = \kappa(x)$ (Lem. 8 loc. cit.), the purely transcendental, separable algebraic and purely inseparable cases (Case 1-3 p. 164 loc. cit.). Hironaka proves these cases using blow ups and has to go to a new local ring after every step. We will argue in comparable steps, but will do this without leaving our original object that far: It is clear that one has to compare a point of a cone to the origin of this cone at least implicitely somewhere in the proof. We make exactly this to our objective. We first state our main result for this comparison (8.2.6) without speaking about blow ups by considering a sequence of points under consecutive base changes of a cone with simple field extensions. Only in the last step we will look at a blow up and extend our result. This way seems easier to the author.

[H2] Hironaka proves an inequality implying (R) in Th. (1,A) loc. cit. under the assumption that the ν^*-invariant has not changed. This is equivalent to an unchanged Hilbert series by [H4, Th. III].

[Si1] As already mentioned, Singh proved an amelioration of (H). His prove follows roughly the strategy used by Hironaka.

[Gi] Our proof of (H) and (N) is inspired in many points from this work of Giraud. However, he denotes a similar proof only in the case that $x' \in \pi^{-1}(x)$ is a closed point (II 3.8 loc. cit) and does not focus that clearly on the tangent cone. Under the same assumption he also proves (R) and (I) (II 4.1 ff. loc. cit.).

Our proof combines different properties in one line of thought with a view focused on cones rather than blow ups. We do not use the ν^*-invariant.

8.2 Behavior of the invariants under permissible blow ups

We are going to prove the statements (H), (R), (N) and (I) in (8.2.7). For the beginning we exploit the characterizations of normal flatness from (7.4.1).

Lemma (8.2.1) (cf. [Gi, II 3.1]). *Let X be a locally noetherian scheme and x, y points of X such that $x \in Y := \overline{\{y\}}$. Suppose that $\mathcal{O}_{Y,x}$ is regular of dimension d. Then*

$$H^{(d+1)}_{X,y} \leq H^{(1)}_{X,x}.$$

Equality holds if and only if X is normally flat along Y at x.

Proof. Using the permanence properties of 7.3 we can assume that X is the spectrum of a complete local ring. The Cohen structure theorem allows us then to take $X = \mathrm{Spec}(R/I)$ for a regular local ring R with maximal ideal $\mathfrak{m}$ and some ideal $I \subseteq \mathfrak{m}$. Since $\mathcal{O}_{Y,y}$ is regular, we have $Y = \mathrm{Spec}(R/\mathfrak{p})$ such that $R/\mathfrak{p}$ is regular of dimension d. We equip $E := R/I$ with the $\mathfrak{p}$-adic and $\mathfrak{m}$-adic filtrations $\mathcal{E}'$ and $\mathcal{E}''$. $(E, \mathcal{E}', \mathcal{E}'')$ is then a harmonious bifiltered $(R, \mathfrak{p}, \mathfrak{m})$-module. The weak semi-continuity (5.3.1) states

$$H^{(1)}_{X,x} = H^{(1)}(\mathrm{gr}_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x})) = H^{(1)}(\mathrm{gr}_{\mathfrak{m}/I}(R/I)) = H^{(1)}(\mathcal{E}'') \geq H^{(d+1)}(\mathcal{E}'_{\mathfrak{p}}) =$$

$$= H^{(d+1)}(\mathrm{gr}_{\mathfrak{p}/I}(R/I)_{\mathfrak{p}}) = H^{(d+1)}(\mathrm{gr}_{\mathfrak{m}_{X,y}}(\mathcal{O}_{X,y})) = H^{(d+1)}_{X,y}.$$

By theorem (5.3.3) equality holds if and only if $\mathrm{gr}(\mathcal{E}') = \mathrm{gr}_{X,Y,x}$ is flat over $R/\mathfrak{p}$. This means by definition precisely the normal flatness of X along Y at x. □

Proposition (8.2.2) (cf. [Gi, II 3.4]). *Let $f : X \to Y$ be a morphism of locally noetherian schemes and $x \in X$ with image $y = f(x)$. Assume that $\mathcal{O}_{f^{-1}(y),x}$ is regular of dimension d and f is flat at x. Then $C^{\kappa}_{f,x} : C_{X,x} \to C_{Y,y} \times_{\kappa(y)} \kappa(x)$ is a core-isomorphism (see definition (6.4.3)) and induces an isomorphism*

(8.2.2.A) $$C_{X,x}/T_{f^{-1}(y),x} \cong C_{Y,y} \times_{\kappa(y)} \kappa(x).$$

We have non-canonical isomorphisms

$$C_{X,x} \cong C_{Y,y} \times_{\kappa(y)} T_{f^{-1}(y),x} \cong C_{Y,y} \times_{\kappa(y)} \mathbb{A}^d_{\kappa(x)}$$

and the equality

$$H_{X,x} = H^{(d)}_{Y,y}.$$

Proof. We can replace X and Y by the spectra of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$. This does not alter the cones involved nor the Hilbert series (see 7.3). Also $\mathcal{O}_{f^{-1}(y),x}$ is unchanged. To see this we can assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine. Let x be given by the prime ideal $\mathfrak{P} \subseteq B$ and y be the prime $\mathfrak{p} \subseteq A$. Then $f^{-1}(y) = \mathrm{Spec}(B \otimes_A A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p})$ and $\mathcal{O}_{f^{-1}(y),x} = B_\mathfrak{P} \otimes_A A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. On the other hand, if we take the induced morphism $A_\mathfrak{p} \to B_\mathfrak{P}$, the fiber over the closed point y which is $B_\mathfrak{P} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ already is the local ring we are looking for and is in fact the same as in the first case. From now on we assume to be in the local situation. Then $f^{-1}(y)$ is a closed subscheme of X, namely $f^{-1}(y) = \mathrm{Spec}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y)) = \mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x})$ which is the spectrum of a regular local ring of dimension d. Since f is flat, we have

$$C_{Y,y} \times_{\kappa(y)} f^{-1}(y) \cong N_{X,f^{-1}(y)} \tag{8.2.2.B}$$

as is seen from the following calculation:

$$\mathfrak{m}^n_{Y,y}/\mathfrak{m}^{n+1}_{Y,y} \otimes_{\kappa(y)} \mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} \cong \mathfrak{m}^n_{Y,y}/\mathfrak{m}^{n+1}_{Y,y} \otimes_{\kappa(y)} \mathcal{O}_{X,x} \cong \mathfrak{m}^n_{Y,y}\mathcal{O}_{X,x}/\mathfrak{m}^{n+1}_{Y,y}\mathcal{O}_{X,x}.$$

Since

$$N_{X,f^{-1}(y)} \cong \mathrm{Spec}(\bigoplus_{n\geq 0} \mathfrak{m}^n_{Y,y}/\mathfrak{m}^{n+1}_{Y,y} \otimes_{\kappa(y)} \mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x})$$

already comes from the stalk of $\mathrm{gr}_{X,f^{-1}(y)}$ at x, we see that X is normally flat along $f^{-1}(y)$ at x. The $\kappa(y)$-algebra $\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$ is tensored over the field $\kappa(y)$ with the $\kappa(y)$-modules $\mathfrak{m}^n_{Y,y}/\mathfrak{m}^{n+1}_{Y,y}$. Of course this yields free $\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x}$-modules. By (7.4.1) (ii) $N_{X,f^{-1}(y),x} \cong C_{X,x}/T_{f^{-1}(y),x}$ (here we used the regularity of $\mathcal{O}_{f^{-1}(y),x}$) and we get the claimed natural isomorphism (8.2.2.A) since $N_{X,f^{-1}(y),x} \cong C_{Y,y} \times_{\kappa(y)} f^{-1}(y) \times_X \kappa(x)$ and $f^{-1}(y) \times_X \kappa(x) \cong \mathrm{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)) \cong \kappa(x)$. We therefore also have $C_{X,x} \cong N_{X,f^{-1}(y),x} \times_{\kappa(x)} T_{f^{-1}(y),x} \cong C_{Y,y} \times_{\kappa(y)} \kappa(x) \times_{\kappa(x)} \mathbb{A}^d_{\kappa(x)}$. The composition

$$C_{X,x} \to N_{X,f^{-1}(y),x} \to C_{Y,y}$$

is the canonical morphism $C_{f,x}$ from (7.1.2). Hence $C^\kappa_{f,x}$ is a core-isomorphism. □

We study the effect of a base change with a field extension for our invariants. First we do this in the algebraic case, then in the transcendental case. We begin to use the full strength of the essential technical tool (4.6.5). Our motivation for the next result is [Gi, II 3.5, 4.2.3].

Proposition (8.2.3). *Let k'/k be a finite field extension and X a locally noetherian k-scheme. Consider a point x' of $X' := X_{k'} = X \times_k k'$ that maps to $x \in X$ under the natural morphism $f : X' \to X$.*

(i) *One has the inequality*

$$H^{(1)}_{X,x} \leq H^{(1)}_{X',x'} \tag{8.2.3.A}$$

and equality holds if k'/k is separable.

(ii) *If (8.2.3.A) is an equality, then*

$$R_{X,x} \leq R_{X',x'}. \tag{8.2.3.B}$$

(iii) *If (8.2.3.A) and (8.2.3.B) are equalities, then we have a natural isomorphism*

$$(C_{X',x'})_K / \operatorname{Dir}((C_{X',x'})_K) \to (C_{X,x})_K / \operatorname{Dir}((C_{X,x})_K) \tag{8.2.3.C}$$

such that in particular $C^{\kappa}_{f,x'}$ is a K-core-isomorphism and there is an isomorphism

$$C_{X,x} \times_{\kappa(x)} K \cong C_{X',x'} \times_{\kappa(x')} K, \tag{8.2.3.D}$$

where $K/\kappa(x')$ is any perfect field.

Proof. We can assume that $k' = k(a)$ is a primitive algebraic field extension. The general case follows then by applying this special case a finite number of times (k'/k is finite). All claims we have to prove are compatible with these intermediate steps. Let p be the minimal polynomial of a in $k[Y]$. With $k \to k[Y] \to k' \cong k[Y]/\langle p \rangle$ we get the sequence of locally noetherian schemes

$$X' \xrightarrow{h} X'' \xrightarrow{g} X,$$

where $X'' := X \times_k \operatorname{Spec}(k[Y])$. Let $x'' := h(x')$. We have

$$g^{-1}(x) \cong \kappa(x) \times_X X'' \cong \kappa(x) \times_X X \times_k \operatorname{Spec}(k[Y]) \cong \kappa(x) \times_k \operatorname{Spec}(k[Y]) \cong \operatorname{Spec}(\kappa(x)[Y])$$

and for $f = gh$

$$f^{-1}(x) \cong \kappa(x) \times_X X' \cong \kappa(x) \times_X X \times_k k' \cong \operatorname{Spec}(\kappa(x)[Y]/\langle p \rangle).$$

Residue fields are unchanged under passing to a fiber; $\kappa(x')/\kappa(x)$ and also $\kappa(x'')/\kappa(x)$ are finite, hence x'' must be closed in $g^{-1}(x)$. Clearly g is flat and since $g^{-1}(x)$ is regular we see that $\mathcal{O}_{g^{-1}(x),x''}$ is regular of dimension 1. By (8.2.2) therefore we have a natural isomorphism

$$C_{X'',x''} / T_{g^{-1}(x),x''} \cong C_{X,x} \times_{\kappa(x)} \kappa(x''), \tag{8.2.3.E}$$

such that in particular $C^{\kappa}_{g,x''}$ is a core-isomorphism and have an isomorphism

$$C_{X'',x''} \cong C_{X,x} \times_{\kappa(x)} \mathbb{A}^1_{\kappa(x'')} \tag{8.2.3.F}$$

and in particular $H_{X'',x''} = H^{(1)}_{X,x}$. We now prove (i) - (iii).

(i) We look at the closed immersion h and find $\mathcal{O}_{X',x'} = \mathcal{O}_{X'',x''}/\langle p \rangle$. Then we apply (4.6.5) and get

$$H^{(1)}_{X',x'} \geq (1 - T^r) H^{(1)}_{X'',x''} \geq (1 - T) H^{(1)}_{X'',x''} = H_{X'',x''} = H^{(1)}_{X,x},$$

where $r = \nu_{\mathfrak{m}_{X'',x''}}(p)$. This shows (8.2.3.A). If k'/k is separable algebraic, the base change $X' \to X$ also is étale, a fortiori quasi-étale and we can use the properties of permanence in 7.3 to get the desired equality even without the work done so far in this proof.

(ii) If $H^{(1)}_{X',x'} = H^{(1)}_{X,x}$, then also $H^{(1)}_{X',x'} = H_{X'',x''}$ and we must have $r = 1$ in the proof of (i). (4.6.5) moreover yields the isomorphism $\mathrm{gr}_{X',x'} \cong \mathrm{gr}_{X'',x''}/\langle \mathrm{in}_{x''}(p)\rangle$. The one graded ring therefore arises from the other by dividing out an element of degree one. Geometrically this means that $C_{X',x'}$ equals the intersection of $C_{X'',x''}$ with a hyperplane W in some surrounding vector space V (take $T_{X'',x''}$). We now are in the situation of (6.4.6) with $\mathrm{codim}_V(W) = 1$ and therefore get (8.2.3.B)

$$\dim \mathrm{Rid}_{X,x} \overset{(8.2.3.F)}{=} \dim \mathrm{Rid}_{X'',x''} - 1 \leq \dim \mathrm{Rid}_{X',x'} .$$

(iii) If additionally $\dim \mathrm{Rid}_{X,x} = \dim \mathrm{Rid}_{X',x'}$, then $\dim \mathrm{Rid}_{X'',x''} = \dim \mathrm{Rid}_{X',x'} + 1$ and by (6.4.6) the closed immersion $C_{h,x'} = C^{\kappa}_{h,x'} : C_{X',x'} \to C_{X'',x''}$ is a K-core-isomorphism for any perfect field $K/\kappa(x')$. We already know that $C^{\kappa}_{g,x''}$ is a core-isomorphism. Hence the composition $C^{\kappa}_{f,x'} = C^{\kappa}_{g,x''} \circ C^{\kappa}_{h,x'}$ is a K-core-isomorphism. But this just means (8.2.3.C). Finally we get (8.2.3.D) with (8.2.3.F). □

Lemma (8.2.4). *Let x be a point of a locally noetherian scheme X over the field k and $\lambda \in \kappa(x)$ an element that is transcendental over k. Consider the projection $p : \widetilde{X} := k(\Lambda) \times_k X \to X$ for a purely transcendental extension $k(\Lambda)/k$. Then there exists a point $\widetilde{x} \in p^{-1}(x)$ with $\kappa(\widetilde{x}) = \kappa(x)$ and a natural isomorphism*

$$C_{\widetilde{X},\widetilde{x}}/T_{p^{-1}(x),\widetilde{x}} \to C_{X,x} \tag{8.2.4.A}$$

such that $C_{p,\widetilde{x}}$ is a core isomorphism. In particular we have an isomorphism

$$C_{\widetilde{X},\widetilde{x}} \cong C_{X,x} \times_{\kappa(x)} \mathbb{A}_{\kappa(x)}. \tag{8.2.4.B}$$

Proof. We are going to show that $\widetilde{x}$ can be chosen such that $\mathcal{O}_{p^{-1}(x),\widetilde{x}}$ is regular of dimension one. Then we get (8.2.4.A) and (8.2.4.B) with $\kappa(\widetilde{x}) = \kappa(x)$ from (8.2.2) since p is flat. For this purpose we pick $\widetilde{x}$ as the point of

$$p^{-1}(x) \cong \widetilde{X} \times_X \kappa(x) \cong k(\Lambda) \times_k X \times_X \kappa(x) \cong k(\Lambda) \times_k \kappa(x)$$

corresponding to the surjection (defining a prime ideal $\mathfrak{p} = \ker(\varphi)$ representing $\widetilde{x}$)

$$\varphi : k(\Lambda) \otimes_k \kappa(x) \to \kappa(x), \qquad \Lambda \mapsto \lambda.$$

Thus clearly $\kappa(\widetilde{x}) = \kappa(x)$. With the multiplicatively closed set $S := k[\Lambda] \setminus \{0\}$ we find

$$k(\Lambda) \otimes_k \kappa(x) \cong k[\Lambda]_S \otimes_k \kappa(x) \cong \kappa(x)[\Lambda]_S$$

with $\varphi : \kappa(x)[\Lambda]_S \to \kappa(x), \Lambda \mapsto \lambda$. Observe that the kernel $\mathfrak{p}'$ of $\varphi|_{\kappa(x)[\Lambda]}$ is the principal ideal generated by the element $\Lambda - \lambda \in \kappa(x)[\Lambda]$. One easily sees that $S \cap \mathfrak{p}' = \emptyset$. We therefore get isomorphisms

$$\mathcal{O}_{p^{-1}(x),\widetilde{x}} \cong (\kappa(x)[\Lambda]_S)_{\mathfrak{p}} \cong \kappa(x)[\Lambda]_{\langle \Lambda - \lambda \rangle}$$

and apparently $\mathcal{O}_{p^{-1}(x),\widetilde{x}}$ is in fact a regular local ring of dimension one. □

The fiber over a point in the center of a permissible blow up is the projective space associated to a quotient of the tangent cone at that point. We will have to switch between the projective space associated to a cone and the cone itself at several points, so we have to talk about a comparison morphism between these two objects (cf. [Gi, II 3.6.1]).

Proposition (8.2.5). *Let C be a cone over the field k. There is a canonical smooth morphism of relative dimension one*

$$\gamma : C^* \to \mathbb{P}C,$$

where $C^* := C \setminus \{0\}$ is the punctured cone. It has the following properties:

(i) *For $x \in \mathbb{P}C$ we have $\gamma^{-1}(x) \cong (\mathbb{A}^1_{\kappa(x)})^*$ (the affine line with the origin removed). If x is defined by some homogeneous prime ideal in the graded ring $R, C = \operatorname{Spec}(R)$, then the same prime ideal defines a point of C^* and this is the generic point of $\gamma^{-1}(x)$. We will call the points of C^* arising in this way* ***homogeneous points*** *of C^*.*

(ii) *If $y \in C^*$ is a homogeneous point, there is a natural isomorphism*

$$C_{C,y} \cong C_{\mathbb{P}C,\gamma(y)} \times_{\kappa(\gamma(y))} \kappa(y)$$

and $\kappa(y)/\kappa(\gamma(y))$ is purely transcendental with transcendence degree one.

(iii) *If $z \in C^*$ is not homogeneous, there is a natural isomorphism*

$$C_{C,z}/T_{\gamma^{-1}(\gamma(z)),z} \cong C_{\mathbb{P}C,\gamma(z)} \times_{\kappa(\gamma(z))} \kappa(z)$$

inducing an isomorphism

$$C_{C,z} \cong C_{\mathbb{P}C,\gamma(z)} \times_{\kappa(\gamma(z))} \mathbb{A}^1_{\kappa(z)}$$

and $\kappa(z)/\kappa(\gamma(z))$ is algebraic.

For any point $w \in C^*$ the morphism $C^{\kappa}_{\gamma,w}$ is a core-isomorphism.

Proof. Let $C = \operatorname{Spec}(R)$ for a standard graded k-algebra R. Locally γ corresponds to the morphism of rings $R_{(f)} \to R_f$ for $f \in R_1$ (R is generated by R_1), where

$$R_{(f)} = \left\{ \frac{g}{f^d} \in R_f \,\middle|\, g \in R_d, d \in \mathbb{N} \right\}.$$

This morphism factors to

$$R_{(f)} \xrightarrow{\varphi^{\#}} R_{(f)}[X] \xrightarrow{\rho^{\#}} R_{(f)}[X, X^{-1}] \xrightarrow{\sim} R_f.$$

$\varphi^{\#}$ is the inclusion and $\rho^{\#}$ the localization at the element X. The isomorphism at the end stands for $X \mapsto f^{-1}$ and is easily checked. On the level of schemes this corresponds to

$$V \xrightarrow{\rho} \mathbb{A}^1_U \xrightarrow{\varphi} U,$$

where $V \subseteq C^*$ and $U \subseteq \mathbb{P}C$ are open affines and ρ is an open immersion, hence étale. Therefore γ is smooth of relative dimension one. In particular

$$\gamma^{-1}(x) \cong \operatorname{Spec}(R_{(f)}[X, X^{-1}]) \times_{\operatorname{Spec}(R_{(f)})} \kappa(x) \cong \operatorname{Spec}(\kappa(x)[X, X^{-1}]) \cong (\mathbb{A}^1_{\kappa(x)})^*.$$

This shows (i). Explicitely one sees easily that the map γ is given by $\operatorname{Spec}(R) \to \operatorname{Proj}(R), \mathfrak{p} \mapsto \mathfrak{p}^h$, where $\mathfrak{p}^h$ is the homogeneous prime ideal of R generated by the homogeneous elements in $\mathfrak{p}$. Therefore the generic point of the fiber over x (defined by a homogeneous prime ideal $\mathfrak{p} \subseteq R$) is the point of C^* corresponding to the same prime ideal $\mathfrak{p} = \mathfrak{p}^h$. A homogeneous point $y \in C^*$ is the generic point of the fiber $\gamma^{-1}(\gamma(y))$ and a non-homogeneous point $z \in C^*$ is closed in the fiber $\gamma^{-1}(\gamma(z))$. Therefore (ii) and (iii) follow directly from (8.2.2). □

We assemble the tools prepared so far and compare the origin of a cone with some other point on it.

Theorem (8.2.6). *Let C be a cone over the field k, x a point of C^* and $d = \operatorname{tr.deg}(\kappa(x)/k)$.*

(i) *One has the inequality*

(8.2.6.A) $$H^{(1)}(C) \geq H^{(d+1)}_{C,x}.$$

(ii) *If (8.2.6.A) is an equality, then*

(8.2.6.B) $$R(C) \geq R^{(d)}_{C,x}$$

and $x \in \operatorname{Rid}(C)$.

(iii) *If (8.2.6.A) and (8.2.6.B) are equalities, then there is an isomorphism*

(8.2.6.C) $$C_K / \operatorname{Dir}(C_K) \to (C_{C,x})_K / \operatorname{Dir}((C_{C,x})_K)$$

for any perfect field $K/\kappa(x)$.

Proof. We proceed in several steps.

(8.2.6.1) The field extension $\kappa(x)/k$ is finitely generated since C is of finite type over k. Let k' be an intermediate field of this extension such that $\kappa(x)/k'$ is algebraic and k'/k is purely transcendental with $\operatorname{tr.deg}(k'/k) = d$. We apply (8.2.4) d times, each time making a base change with a purely transcendental extension of transcendence degree 1, and gain a point $x' \in C' := C_{k'}$ lying over x with $\kappa(x') = \kappa(x)$ such that $C_{C',x'} \to C_{C,x}$ is a core isomorphism, i.e.

$$C_{C',x'} / \operatorname{Dir}(C_{C',x'}) \to C_{C,x} / \operatorname{Dir}(C_{C,x})$$

is an isomorphism. In particular we get an isomorphism

(8.2.6.D) $$C_{C',x'} \cong C_{C,x} \times_{\kappa(x)} \mathbb{A}^d_{\kappa(x)}.$$

(8.2.6.2) Now C' is a k'-scheme and $\kappa(x')/k'$ is a finitely generated algebraic, hence a finite field extension. Let $p : C'' := C_{\kappa(x)} \to C' = C_{k'}$ be the natural projection. Then we find in the fiber $p^{-1}(x') \cong \kappa(x) \times_{k'} \kappa(x')$ a point x'' corresponding to the kernel of the multiplication map $\kappa(x) \otimes_{k'} \kappa(x') \to \kappa(x)$ with residue field $\kappa(x'') = \kappa(x)$. We apply (8.2.3) and gain the inequality

(8.2.6.E) $$H^{(d+1)}_{C,x} \overset{(8.2.6.D)}{=} H^{(1)}_{C',x'} \leq H^{(1)}_{C'',x''}.$$

If this is an equality, still by (8.2.3) we have

(8.2.6.F) $$R^{(d)}_{C,x} \overset{(8.2.6.D)}{=} R_{C',x'} \leq R_{C'',x''}.$$

If this also is an equality we know furthermore that $C_{C'',x''} \to C_{C',x'}$ is a K-core isomorphism for any perfect field $K/\kappa(x'')$ and we therefore have natural isomorphisms
(8.2.6.G)
$$(C_{C'',x''})_K/\operatorname{Dir}((C_{C'',x''})_K) \to (C_{C',x'})_K/\operatorname{Dir}((C_{C',x'})_K) \to (C_{C,x})_K/\operatorname{Dir}((C_{C,x})_K).$$

Also the composition $C_{C'',x''} \to C_{C,x}$ will then be a K-core isomorphism.

(8.2.6.3) Now x'' is a rational point on C''. We can embed C'' into an affine space, say $V = \operatorname{Spec}(\kappa(x)[X_0, ..., X_n])$, and can suppose that x'' has coordinates $(1, 0, ..., 0)$, i.e. is defined by the maximal ideal $\langle X_0 - 1, X_1, ..., X_n\rangle$ (we excluded the case that x'' is the origin). We apply (8.2.5) and get a morphism $\gamma : C''^* \to \mathbb{P}C''$ as constructed there. $\gamma(x'')$ is the homogeneous prime $\langle X_1, ..., X_n\rangle$ defining a homogeneous point y of C''. From (8.2.5) (ii) and (iii) we get canonical isomorphisms

(8.2.6.H) $$C_{C'',y} \cong C_{\mathbb{P}C'',\gamma(y)} \times_{\kappa(\gamma(y))} \kappa(y),$$

(8.2.6.I) $$C_{C'',x''}/T_{\gamma^{-1}(\gamma(x'')),x''} \cong C_{\mathbb{P}C'',\gamma(x'')} \times_{\kappa(\gamma(x''))} \kappa(x'')$$

and an isomorphism

(8.2.6.J) $$C_{C'',x''} \cong C_{\mathbb{P}C'',\gamma(x'')} \times_{\kappa(\gamma(x''))} \mathbb{A}^1_{\kappa(x'')}.$$

Note that $\gamma(y) = \gamma(x'')$.

(8.2.6.4) Consider in the regular local ring $R := \mathcal{O}_{V,0}$ with the maximal ideal $\mathfrak{m}_R$ the regular prime ideal $\mathfrak{p}_R := \langle X_1, ..., X_n\rangle_R$. Let the closed subscheme C'' of V be defined by the ideal J in R. We consider the exact sequence $0 \to J \to R \to E \to 0$, i.e. $E = R/J$. It becomes a bistrict short exact sequence of bifiltered $(R, \mathfrak{p}_R, \mathfrak{m}_R)$-modules when we equip J and E with the induced filtrations from R (see (5.2.1)). $(E, \mathcal{E}', \mathcal{E}'')$ (standard notation as in chapter 5) is harmonious.
By (5.3.1) we have

(8.2.6.K) $$H^{(1)}(C) = H^{(1)}(C'') = H^{(1)}(\mathcal{O}_{C'',0}, \mathfrak{m}_{C'',0}) = H^{(1)}(\mathcal{E}'') \geq H^{(2)}(\mathcal{E}'_{\mathfrak{p}_R}) =$$
$$= H^{(2)}(\mathcal{O}_{C'',y}, \mathfrak{m}_{C'',y}) = H^{(2)}_{C'',y}.$$

Altogether we can prove (8.2.6.A):

$$H^{(1)}(C) \overset{(8.2.6.K)}{\geq} H^{(2)}_{C'',y} \overset{(8.2.6.H)}{=} H_{\mathbb{P}C'',\gamma(y)} \overset{(8.2.6.J)}{=} H^{(1)}_{C'',x''} \overset{(8.2.6.E)}{\geq} H^{(d+1)}_{C,x}.$$

(8.2.6.5) From now on we suppose that (8.2.6.A) is an equality. Then (8.2.6.K) and (8.2.6.E) also must be equalities. Since $H^{(1)}(\mathcal{E}'') = H^{(2)}(\mathcal{E}'_{\mathfrak{p}_R})$ we get from theorem (5.3.3) and (5.2.2) that the canonical morphism $\mathrm{gr}_{\mathfrak{m}_R}(R) \otimes_{\mathrm{gr}_{\mathfrak{p}_R}(R)} \mathrm{gr}(\mathcal{J}') \to \mathrm{gr}(\mathcal{J}'')$ is surjective ($\mathcal{J}', \mathcal{J}''$ are the filtrations on J induced from R). Since R is the local ring of V at the origin, we can identify $\mathrm{gr}_{\mathfrak{m}_R}(R) \cong \kappa(x)[X_0, ..., X_n]$ and $\mathrm{gr}(\mathcal{J}'') \cong I$, the ideal defining C'' in V. Since $\mathrm{gr}(\mathcal{I}') \subseteq \mathrm{gr}_{\mathfrak{p}_R}(R)$ we see that I is generated by some elements in the image of

$$\kappa(x)[x_0][X_1, ..., X_n] \cong \mathrm{gr}_{\mathfrak{p}_R}(R) \to \mathrm{gr}_{\mathfrak{m}_R}(R) \cong \kappa(x)[X_0, ..., X_n]$$

$$x_0 \mapsto 0, X_1 \mapsto X_1, \cdots, X_n \mapsto X_n.$$

Let L be the line joining 0 and x'' in V defined by the ideal $\langle X_1, ..., X_n\rangle$ in the ring $\kappa(x)[X_0, ..., X_n]$. Since $\kappa(x)[X_0, ..., X_n](\kappa(x)[X_1, ..., X_n] \cap I) = I$ we see that by definition (see 6.2) $L \subseteq \mathrm{Dir}(C'')$ and therefore also $L \subseteq \mathrm{Rid}(C'')$. In particular $x'' \in \mathrm{Rid}(C'')$. The projection $p : C'' \to C''/L$ is smooth of relative dimension 1 (it is given by

$$\kappa(x)[X_1, ..., X_n]/\kappa(x)[X_1, ..., X_n] \cap I \to$$

$$\to (\kappa(x)[X_1, ..., X_n]/\kappa(x)[X_1, ..., X_n] \cap I)[X_0] \cong \kappa(x)[X_0, ..., X_n]/I.)$$

We have $p(x'') = p(0)$ and both 0 and x'' are closed in $p^{-1}(0) = L$. Hence by (8.2.2) we have a natural isomorphism

$$C_{C'',0}/T_{L,0} \cong C_{C'',x''}/T_{L,x''}. \tag{8.2.6.L}$$

We can therefore show (8.2.6.B) (note that the ridge is stable under field extensions as seen in (6.1.10)):

$$R(C) = R_{C,0} = R_{C'',0} \overset{(8.2.6.L)}{=} R_{C'',x''} \overset{(8.2.6.F)}{\geq} R^{(d)}_{C,x}.$$

Since $C'' = C_{\kappa(x)}$ we have $x'' \in \mathrm{Rid}(C'') = \mathrm{Rid}(C)_{\kappa(x)}$ and therefore $x \in \mathrm{Rid}(C)$.

(8.2.6.6) Finally suppose that also (8.2.6.B) is an equality. Then (8.2.6.F) is an equality. (8.2.6.L) induces first an isomorphism

$$C_{\kappa(x)}/\,\mathrm{Dir}(C_{\kappa(x)}) \to C_{C'',0}/\,\mathrm{Dir}(C_{C'',0}) \to C_{C'',x''}/\,\mathrm{Dir}(C'', x'')$$

and then for any perfect field $K/\kappa(x)$ an isomorphism

$$C_K/\,\mathrm{Dir}(C_K) \to (C_{C'',x''})_K/\,\mathrm{Dir}((C_{C'',x''})_K)$$

that yields (8.2.6.C) when combined with (8.2.6.G). □

Theorem (8.2.7). *Let D be a closed subscheme of a locally noetherian scheme X such that D is permissible in X at the point x. Let $x' \in \pi^{-1}(x)$ for the blow up $\pi : X' := \mathrm{Bl}_D(X) \to X$ and $d := \mathrm{tr.deg}(\kappa(x')/\kappa(x))$.*

(i) *One has the inequality*

$$H^{(1)}_{X,x} \geq H^{(d+1)}_{X',x'}. \tag{8.2.7.A}$$

(ii) *If (8.2.7.A) is an equality, then*

$$R_{X,x} \geq R^{(d)}_{X',x'} \tag{8.2.7.B}$$

and $x' \in \mathbb{P}\,\mathrm{Rid}(C_{X,x}/T_{D,x}) \subseteq \pi^{-1}(x)$.

(iii) *If (8.2.7.A) and (8.2.7.B) are equalities, then there is an isomorphism*

$$(C_{X,x})_K/\,\mathrm{Dir}((C_{X,x})_K) \to (C_{X',x'})_K/\,\mathrm{Dir}((C_{X',x'})_K) \tag{8.2.7.C}$$

for some perfect field $K/\kappa(x')$.

Proof. With the techniques of 7.3 and the Cohen structure theorem we may again suppose that X is embedded into a regular scheme Z for all questions at hand. We also may restrict to the case that x is a closed point.

(8.2.7.1) Consider the diagram with cartesian squares

$$\begin{array}{ccccc}
\mathrm{Spec}(\mathcal{O}_{\pi^{-1}(x),x'}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{\pi^{-1}(D),x'}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X',x'}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi^{-1}(x) & \longrightarrow & \pi^{-1}(D) & \longrightarrow & X' \\
\downarrow & & \downarrow & & \downarrow \pi \\
\{x\} & \longrightarrow & D & \longrightarrow & X
\end{array}$$

in which all horizontal arrows are closed immersions. The closed subscheme $\pi^{-1}(D) \subseteq X'$ is the exceptional divisor we already discussed in (7.2.3). Hence we can find a local equation $v \in \mathcal{O}_{X',x'}$ of this divisor with $\mathcal{I}_{X',\pi^{-1}(D),x'} = v\mathcal{O}_{X',x'}$ so that $\mathcal{O}_{\pi^{-1}(D),x'} \cong \mathcal{O}_{X',x'}/\langle v\rangle$. Furthermore the maximal ideal of $\mathcal{O}_{D,x}$ is generated by let us say $u_1, ..., u_q$ and therefore

$$\mathcal{O}_{\pi^{-1}(x),x'} \cong \mathcal{O}_{\pi^{-1}(D),x'}/\langle u_1, ..., u_q\rangle \cong \mathcal{O}_{X',x'}/\langle v, u_1, ..., u_q\rangle.$$

Let $r_0 = \nu_{\mathfrak{m}_{X',x'}}(v)$ and $r_i = \nu_{\mathfrak{m}_{X',x'}}(u_i)$ for $i = 1, ..., q$. Applying (4.6.5) successively for $v, u_1, ..., u_q$ we find

$$H^{(q+2)}_{\pi^{-1}(x),x'} \geq (1 - T^{r_0}) \cdots (1 - T^{r_q}) \cdot H^{(q+1)}_{X',x'} \geq H^{(1)}_{X',x'}. \tag{8.2.7.D}$$

If this is an equality we must have $r_0 = \cdots = r_q = 1$ and further know from (4.6.5) that

$$\mathrm{gr}_{\pi^{-1}(x),x'} = \mathrm{gr}_{X',x'}/\langle \mathrm{in}_{\mathfrak{m}_{X',x'}}(v), \mathrm{in}_{\mathfrak{m}_{X',x'}}(u_1), ..., \mathrm{in}_{\mathfrak{m}_{X',x'}}(u_q)\rangle.$$

Therefore $C_{\pi^{-1}(x),x'}$ arises from $C_{X',x'}$ (inside some vector space like $T_{X',x'}$) as an intersection with a $(q+1)$-codimensional subspace. Thus we can use (6.4.6) and get

$$R^{(q+1)}_{\pi^{-1}(x),x'} \geq R_{X',x'}. \tag{8.2.7.E}$$

If here also equality holds, we have a canonical isomorphism

$$(C_{\pi^{-1}(x),x'})_K / \operatorname{Dir}((C_{\pi^{-1}(x),x'})_K) \to (C_{X',x'})_K / \operatorname{Dir}((C_{X',x'})_K) \tag{8.2.7.F}$$

for any perfect field $K/\kappa(x')$.

(8.2.7.2) Since D is permissible at x we get from the characterizations of normal flatness (7.4.1) that $T_{D,x} \subseteq \operatorname{Dir}(C_{X,x})$ and an isomorphism $C := C_{X,x}/T_{D,x} \cong N_{X,D,x}$ of cones over $\kappa(x)$. D is regular at x and therefore $\dim T_{D,x} = q$, q as above. So we have an isomorphism

$$C_{X,x} \cong C \times_{\kappa(x)} T_{D,x}. \tag{8.2.7.G}$$

We saw in (7.2.1) that $\pi^{-1}(x) \cong \mathbb{P}N_{X,D,x} \cong \mathbb{P}C$. Consider the morphism $\gamma : C^* \to \mathbb{P}C$ from (8.2.5). We denote with $y \in C^*$ the generic point of the fiber $\pi^{-1}(x')$, where we regard x' as a point of $\mathbb{P}C$. Then y is a homogeneous point of C^*, i.e. corresponds to a homogeneous prime ideal, and we have a natural isomorphism (see (8.2.5))

$$C_{C,y} \cong C_{\mathbb{P}C,x'} \times_{\kappa(x')} \kappa(y), \tag{8.2.7.H}$$

where $\operatorname{tr.deg}(\kappa(y)/\kappa(x')) = 1$ and therefore $\operatorname{tr.deg}(\kappa(y)/\kappa(x)) = d+1$.

(8.2.7.3) Now we can apply theorem (8.2.6) to compare the tangent cone at $y \in C$ with the cone C itself and deduce (8.2.7.A) in the following way:

$$H^{(1)}_{X,x} \overset{(8.2.7.G)}{=} H^{(q+1)}(C) \overset{(8.2.6.A)}{\geq} H^{(q+d+2)}_{C,y} \overset{(8.2.7.H)}{=} H^{(q+d+2)}_{\pi^{-1}(x),x'} \overset{(8.2.7.D)}{\geq} H^{(d+1)}_{X',x'}.$$

If this is an equality, then (8.2.6.A) and (8.2.7.D) both must be equalities and we prove (8.2.7.B) again by using theorem (8.2.6):

$$R_{X,x} = R(C_{X,x}) \overset{(8.2.7.G)}{=} R^{(q)}(C) \overset{(8.2.6.B)}{\geq} R^{(q+d+1)}_{C,y} \overset{(8.2.7.H)}{=} R^{(q+d+1)}_{\pi^{-1}(x),x'} \overset{(8.2.7.E)}{\geq} R^{(d)}_{X',x'}.$$

From (8.2.6) (ii) we also get $y \in \operatorname{Rid}(C)$. Thus $x' \in \mathbb{P}\operatorname{Rid}(C) \cong \mathbb{P}\operatorname{Rid}(C_{X,x}/T_{D,x})$. If the last estimation is also an equality, we find the isomorphism (8.2.7.C) for a perfect field $K/\kappa(y)$ with the use of (8.2.6) again (the first isomorphism comes from the core isomorphism $C_{X,x} \to C$):

$$(C_{X,x})_K / \operatorname{Dir}((C_{X,x})_K) \to C_K / \operatorname{Dir}(C_K) \overset{(8.2.6.C)}{\to} (C_{C,y})_K / \operatorname{Dir}((C_{C,y})_K)$$

$$\overset{(8.2.7.H)}{\to} (C_{\pi^{-1}(x),x'})_K / \operatorname{Dir}((C_{\pi^{-1}(x),x'})_K) \overset{(8.2.7.F)}{\to} (C_{X',x'})_K / \operatorname{Dir}((C_{X',x'})_K).$$

This finishes the proof. □

9 Hironaka Schemes

Hironaka schemes are group schemes associated to points of the projective or affine space over a field. They were introduced by Hironaka ([H5]) in order to obtain information about the locus of near points of a blow up with [H4, Th. IV]. We recall their definition, say some words on their characterization via differential operators and will take a look at the mentioned theorem IV of Hironaka: In the author's point of view it seems desirable to find a proof of this theorem that works with Hilbert series instead of the ν^*-invariant Hironaka uses in his complex proof. We do not succeed in this, but will present our results and describe what is missing to complete our line of thought. At least we give a complete proof in characteristic zero.

9.1 Definition and characterization via differential operators

We give the definition of Hironaka schemes via their rings of invariants and present a very useful description for these rings via differential operators. Throughout this section let V be a vector space over a field k, i.e. $V = \mathrm{Spec}(S)$ for a polynomial ring $S = k[X_1, ..., X_n]$. For any point $x \in V$ given by a prime ideal $\mathfrak{p} \subseteq S$ we consider the $\mathfrak{m}_{V,x}$-adic filtration on $\mathcal{O}_{V,x}$ and the order function ν_x associated to it as in (4.4.8).

Remark (9.1.1). *Via the morphism $\gamma : V^* \to \mathbb{P}V$ we get a point $\xi = \gamma(x)$ as long as $x \neq 0$ (see (8.2.5)). We also can consider the order function ν_ξ of the $\mathfrak{m}_{\mathbb{P}V,\xi}$-adic filtration on $\mathcal{O}_{\mathbb{P}V,\xi}$. An element $f \in S_d$ defines a hypersurface* $\mathrm{Proj}(S/\langle f\rangle) \subseteq \mathbb{P}V$*. Its multiplicity at the point ξ is given by $\nu_\xi(f/T^d)$ for some $T \in S_1 \setminus \mathfrak{p}$ where it does not matter which T is chosen. In particular $\nu_\xi(f/T^d) = \nu_x(f)$.*

Proof. First note that ξ corresponds to the homogeneous prime ideal $\mathfrak{p}^h$ that is generated by all homogeneous elements of $\mathfrak{p}$. Therefore $S_1 \setminus \mathfrak{p} = S_1 \setminus \mathfrak{p}^h$. For some other $T' \in S_1 \setminus \mathfrak{p}$ we have $T'/T \in S^\times_{(\mathfrak{p}^h)}$ and thus $\nu_\xi(f/T^d) = \nu_\xi(f/T'^d)$. It is easy to see that $\nu_\xi(f/T^d) = \nu_x(f)$. □

Corollary (9.1.2) (cf. [Od, 2.2], [Gi, III 2.2.2 f.]). *For $f \in S$ we have*

$$\nu_x(f) \geq m \qquad \Longleftrightarrow \qquad \mathrm{Diff}^{\leq m-1}_{\mathbb{Z}}(S)(f) \subseteq \mathfrak{p}$$

and

$$\nu_x(f) \leq \deg(f).$$

Proof. The first description is immediate from the local Jacobian criterion (2.3.5). Assume $\nu_x(f) \geq \deg(f) + 1$. Then we would have $\mathrm{Diff}^{\leq \deg(f)}_{\mathbb{Z}}(S)(f) \subseteq \mathfrak{p}$. For a monomial X^M of degree $\deg(f)$ appearing in f with coefficient $a \neq 0$ we therefore have with the standard differential operator D_M corresponding to the variables X (cf. (3.3.1)) the impossibility $D_M(f) = a \in k^\times \cap \mathfrak{p}$. □

Definition (9.1.3). *The* ***Hironaka ring of invariants*** *associated to x in V is the graded algebra (see (9.1.4)) $\mathcal{U}_{V,x} \subseteq S$ (if no confusion is possible also denoted $\mathcal{U}_x$) with*

(9.1.3.A) $$(\mathcal{U}_x)_d = \{f \in S_d | \nu_x(f) = d\}.$$

We will see immediately that $\mathcal{U}_x$ is generated by additive polynomials, hence we also will study the graded $k[F]$-module

$$\mathcal{Q}_x := \mathcal{Q}_{V,x} := \mathcal{U}_{V,x} \cap L.$$

Remember that L stands for the $k[F]$-module of additive polynomials in S, see 1.4.

Note that the construction of $\mathcal{Q}_x$ only makes sense if $\mathrm{char}(k) > 0$, the case we are mainly interested in. In characteristic zero $\mathcal{U}_x$ will be generated by polynomials of degree one. We will use $\mathcal{Q}_x$ without further comments; in the case of characteristic zero there always will be an obvious analogous statement at hand.

Lemma (9.1.4). *$\mathcal{U}_x$ is in fact a graded k-subalgebra of S with*

(9.1.4.A) $$(\mathcal{U}_x)_d = \{f \in S_d | \operatorname{Diff}_{\mathbb{Z}}^{\leq d-1}(S)(f) \subseteq \mathfrak{p}\}$$

and $\mathcal{U}_x$ is generated by homogeneous additive polynomials (cf. [Gi, III 2.2.4]), i.e.

$$\mathcal{U}_x = k[\mathcal{Q}_x].$$

We have $\mathcal{R}\mathcal{Q}_x = \mathcal{Q}_x$ and for $0 \neq f, g \in S$

(9.1.4.B) $$f \cdot g \in \mathcal{U}_x \qquad \Longrightarrow \qquad f, g \in \mathcal{U}_x.$$

Proof. It is clear from (9.1.2) that (9.1.4.A) holds and $\mathcal{U}_x$ is a graded k-algebra (for $f \in (\mathcal{U}_x)_d$ and $g \in (\mathcal{U}_x)_e$ one has $d+e = \nu_x(f)+\nu_x(g) = \nu_x(f \cdot g) \leq \deg(f \cdot g) = d+e$). To show that $\mathcal{U}_x$ is generated by additive polynomials we use criterion (ii) of (3.4.3): Let D_M be a standard differential operator with respect to the variables X and $f \in (\mathcal{U}_x)_d$ such that $|M| \leq d-1$, i.e. $D_M \in \operatorname{Diff}_k^{\leq d-1}(S)$. Then $D_M(f) \in S_{d-|M|}$ and

$$\operatorname{Diff}_{\mathbb{Z}}^{\leq d-|M|-1}(S)(D_M(f)) \overset{(2.1.9)}{\subseteq} \operatorname{Diff}_{\mathbb{Z}}^{\leq d-1}(S)(f) \subseteq \mathfrak{p}$$

shows that $D_M(f) \in (\mathcal{U}_x)_{d-|M|}$. Let $f \in L$ be homogeneous with $F(f) \in \mathcal{Q}_x$. Then $\nu_x(f) = \nu_x(F(f))/p = \deg(F(f))/p = \deg(f)$ and therefore $f \in \mathcal{Q}_x$. This proves $\mathcal{R}\mathcal{Q}_x = \mathcal{Q}_x$ which implies the last property by (6.3.11). In characteristic zero the last property holds since $\mathcal{U}_x$ is generated by polynomials of degree one. □

(9.1.4.B) is a simple observation if f and g are homogeneous.

Definition (9.1.5) (cf. [Gi, III])**.** *The* ***Hironaka scheme*** *associated to x in V is the subgroup of V*

$$B_x := B_{V,x} := \operatorname{Spec}(S/S(\mathcal{U}_{V,x})_+).$$

Remark (9.1.6). *The Hironaka scheme $B_{V,x}$ only depends on the point $\xi = \gamma(x)$ of the projective space $\mathbb{P}V$. We therefore also write $B_{V,\xi}$ for $B_{V,x}$. Originally Hironaka associated his groups to points of the projective space in [H5]. This approach leads to the same result since $\nu_x(f) = \nu_\xi(f/T^d)$ as we saw in (9.1.1).*

(i) *$B_{V,x}$ contains the point x.*

(ii) *For points $x, y \in V$ with $y \in \overline{\{x\}}$ we have $\mathcal{U}_x \subseteq \mathcal{U}_y$ and therefore $B_y \subseteq B_x$. This is clear from (9.1.4.A) since the prime ideal defining x is contained in the prime ideal defining y.*

Let us examine a simple example.

Example (9.1.7). If $\mathfrak{p} = \langle X_m, ..., X_n\rangle$, then $X_m, ..., X_n$ are regular parameters of $S_\mathfrak{p}$ and $X_1, ..., X_{m-1}$ become units in this ring. Therefore

$$\mathcal{U}_{V,x} = k[X_m, ..., X_n].$$

Lemma (9.1.8) (cf. [Gi, III 2.2.5])**.** *Hironaka schemes are independent of the embedding into a vector space, i.e. if $x \in W$ and $W \subseteq V$ is a subspace, then $B_{W,x} = B_{V,x}$.*

Proof. Let $V = \mathrm{Spec}(S), S = k[X_1, ..., X_n]$ and $W = \mathrm{Spec}(T), T = k[X_{n'+1}, ..., X_n]$. We can suppose that the inclusion $\iota : W \to V$ is defined by $\iota^\# : S \to T, X_i \mapsto 0$ for $i = 1, ..., n'$ and $X_i \mapsto X_i$ for $i = n' + 1, ..., n$. We also have a projection $\pi : V \to W$ defined by $\pi^\# : T \to S, X_i \mapsto X_i$ for $i = n' + 1, ..., n$. Since $\iota^\# \circ \pi^\# = \mathrm{id}_T$ we have $\pi \circ \iota = \mathrm{id}_W$. For $f \in (\mathcal{U}_{V,x})_d$ we have $f \in \mathfrak{m}_{V,x}^d$ and since $(\mathcal{O}_{V,x}, \mathfrak{m}_{V,x}) \to (\mathcal{O}_{W,x}, \mathfrak{m}_{W,x})$ is a morphism of local rings, we also have $\iota^\#(f) \in \mathfrak{m}_{W,x}^d$. Therefore $\iota^\#(f) \in \mathcal{U}_{W,x}$ and we get $\iota^\#(\mathcal{U}_{V,x}) \subseteq \mathcal{U}_{W,x}$. With a similar argument we also find $\pi^\#(\mathcal{U}_{W,x}) \subseteq \mathcal{U}_{V,x}$. Since $\pi^\#$ is just an inclusion we will drop it in our notations. For $i = 1, ..., n'$ we have $X_i \in \mathfrak{m}_{V,x}$ and therefore $\nu_x(X_i) \geq \deg(X_i)$ which proves $X_i \in \mathcal{U}_{V,x}$. Thus $\mathcal{U}_{W,x}[X_1, ..., X_{n'}] \subseteq \mathcal{U}_{V,x}$. We show that this is in fact an equality, from this we get $B_{W,x} = B_{V,x}$. We can assume $n' = 1$ by inserting an adequate sequence of vector spaces between W and V. Let $f \in \mathcal{U}_{V,x}$ and write $f = g + X_1 h$ with $g \in T$ and $h \in S$. Since $g = \pi^\#(\iota^\#(f)) \in \mathcal{U}_{W,x} \subseteq \mathcal{U}_{V,x}$, also $X_1 h \in \mathcal{U}_{V,x}$ and by (9.1.4) also $h \in \mathcal{U}_{V,x}$. Inductively we can show that $f \in \mathcal{U}_{W,x}[X_1]$. $\square$

9.2 Hironaka's theorem IV

The reason why Hironaka schemes are studied is the following theorem of Hironaka:

Theorem (9.2.1) ([H4, Th. IV])**.** *Let $x \in D \subseteq X \subseteq Z$, where Z is a regular scheme, X is a closed subscheme of Z and D a closed regular subscheme of X such that X is normally flat along D at the point x. Consider the blow up $\pi : X' \to X$ with the center D and a point $x' \in X'$ over x. If $H^{(d+1)}_{X',x'} = H^{(1)}_{X,x}$ with $d = \mathrm{tr.deg}(\kappa(x')/\kappa(x))$, then the tangent cone $C_{X,x}$ is invariant under the subgroup $B_{g,x'}$ of $T_{Z,x}$.*

In our terminology $B_{g,x'}$ is the subgroup of $T_{Z,x}$ whose directrix contains $T_{D,x}$ such that $B_{g,x'}/T_{D,x} = B_{V,x'}$ with $V = T_{Z,x}/T_{D,x}$ (cf. [H4, p. 154]). Note that $x' \in \pi^{-1}(x) \cong \mathbb{P}(C_{X,x}/T_{D,x}) \subseteq \mathbb{P}V$. We also can restate the conclusion of the theorem as

$$B_{V,x'} \subseteq \mathrm{Rid}(C_{X,x}/T_{D,x}) = \mathrm{Rid}_{X,x}/T_{D,x}.$$

Note that with the techniques of 7.3 one can drop the assumption that there is a surrounding regular scheme Z and state the theorem intrinsically by replacing $T_{Z,x}$ with $T_{X,x}$; the Hironaka scheme does not depend on the embedding in a vector space, see (9.1.8):

Theorem (9.2.2) (cf. [Gi, III 2.4]). *Let D be a closed subscheme of a locally noetherian scheme X such that D is permissible in X at the point x. Let $x' \in \pi^{-1}(x)$ for the blow up $\pi : X' := \mathrm{Bl}_D(X) \to X$ and $d := \mathrm{tr.deg}(\kappa(x')/\kappa(x))$. If $H^{(d)}_{X',x'} = H_{X,x}$, then $B_{T_{X,x}/T_{D,x},x'} \subseteq \mathrm{Rid}_{X,x}/T_{D,x}$.*

Since in particular $x' \in B_{V,x'}, V := T_{X,x}/T_{D,x}$, the theorem implies

$$x' \in \mathbb{P}(\mathrm{Rid}_{X,x}/T_{D,x}).$$

This was already proved in (8.2.7) (ii). Hironaka proves (9.2.1) using the ν^*-invariant and standard bases, the proof is rather complex and hard to understand. Only with his theorem [H4, Th. III] that relates the ν^*-invariant to the Hilbert series for near points, it becomes possible to relate Hironaka schemes to Hilbert series and obtain the version of the theorem in which we stated it. From the author's point of view it seems desirable to have a proof of (9.2.2) only using Hilbert series as invariants. In fact, Giraud already worked in this direction. Let us recollect his results. First of all, Hironaka's theorem can be restated equivalently for cones. This very well fits into our perspective in chapter 8 to see everything from the point of view of a cone.

Theorem (9.2.3) (cf. [Gi, III 2.3]). *Let C be a cone in a vector space V over the field k and $0 \neq x \in C$ with $d = \mathrm{tr.deg}(\kappa(x)/k)$. If $H^{(d)}_{C,x} = H(C)$, then $B_{V,x} \subseteq \mathrm{Rid}(C)$.*

Proof of equivalence. We show that (9.2.3) is equivalent to (9.2.2) (cf. [Gi, III 2.4.1]). Assume that (9.2.3) holds and let $\pi : X' \to X$ be a permissible blow up with center D and $x \in D, x' \in \pi^{-1}(x)$. From (8.2.7.3) with the notations from there we see with $C = C_{X,x}/T_{D,x}$ that $H^{(d+1)}_{C,y} = H(C)$. But $\mathrm{tr.deg}(\kappa(y)/\kappa(x)) = d+1$ and therefore $B_{T_{X,x}/T_{D,x},x'} \subseteq \mathrm{Rid}(C_{X,x}/T_{D,x})$. Assume on the other hand that (9.2.2) holds and let $0 \neq x \in C$ with $H^{(d)}_{C,x} = H(C)$ for $d = \mathrm{tr.deg}(\kappa(x)/k)$. Let $\pi : C' \to C$ be the blow up in the origin (which is permissible). Let $x' \in \pi^{-1}(0) \cong \mathbb{P}C$ be the image of x under $C^* \to \mathbb{P}C$. In any case we have $H^{(d'+1)}_{\mathbb{P}C,x'} = H(C)$ for $d' = \mathrm{tr.deg}(\kappa(x')/k)$, see (8.2.5). If one looks at the t-chart of the blow up $C' \to C$ for some $t \in S_1$, $V = \mathrm{Spec}(S)$, then t is an equation of the exceptional divisor in this chart and does not appear in any transformed equation of C since C is defined by homogeneous equations. Therefore we have (cf. (8.2.7.1)) $H^{(1)}_{\mathbb{P}C,x'} = H_{C',x'}$. Together this yields $H^{(d')}_{C',x'} = H(C) = H_{C,0}$ and (9.2.2) gives $B_{V,x} = B_{T_{C,0},x'} \subseteq \mathrm{Rid}_{C,0} = \mathrm{Rid}(C)$. □

Hironaka's theorem is an easy observation in the case of a hypersurface C. Then $H^{(d)}_{C,x} = H(C)$ means that the multiplicity of the hypersurface $C \subseteq V$ is the same at 0 and at the point x, i.e. $\nu_x(f) = \deg(f)$ for a homogeneous equation f of the hypersurface. Then by definition $f \in \mathcal{U}_{V,x}$ and the ring of invariants of $\operatorname{Rid}(C)$ is contained in $\mathcal{U}_{V,x}$. It seems that the difficulty in Hironaka's theorem lies in the passage to non-hypersurfaces. Therefore Hilbert series might be a good tool to handle the problem.
The other result of Giraud proves that $H^{(d)}_{C,x} = H(C)$ is in fact equivalent to $B_{V,x} \subseteq \operatorname{Rid}(C)$:

Proposition (9.2.4) (cf. [Gi, III 2.5])**.** *Let C be a cone in a vector space V over the field k and $0 \neq x \in C$ with $d = \operatorname{tr.deg}(\kappa(x)/k)$. Let $G := \operatorname{Rid}(C)$. Then the following are equivalent:*

(i) $H^{(d)}_{C,x} = H(C)$.

(ii) $H^{(d)}_{G,x} = H(G)$.

(iii) $B_{V,x} \subseteq G$.

Proof. Let $n := \dim(V), V = \operatorname{Spec}(S)$. Denote with U the ring of invariants of G in V. Let $\sigma = (\sigma_1, ..., \sigma_e)$ be a $k[F]$-basis of $U \cap L$. Let $q_i := \deg(\sigma_i)$. Let I resp. J be the ideals of C resp. G in the polynomial ring S. Thus $I \subseteq J = SU_+ = S\sigma$. Let $\mathfrak{p} \subseteq S$ be the prime ideal defining x and $R := \mathcal{O}_{V,x} = S_\mathfrak{p}$ with maximal ideal $\mathfrak{m}$.

(9.2.4.1) (i) $\Rightarrow$ (iii) follows from (9.2.3) and so does (ii) $\Rightarrow$ (iii). We show (iii) $\Rightarrow$ (ii). From (4.5.10) we know

$$H_{G,0} = (1 - T^{q_1}) \cdots (1 - T^{q_e})(1 - T)^{-n}.$$

(iii) means that $U \subseteq \mathcal{U}_{V,x}$ and therefore $\nu_x(\sigma_i) = q_i$ and we get from (4.6.5) that

$$H^{(d+1)}_{G,x} \geq (1-T^{q_1}) \cdots (1-T^{q_e}) H^{(d+1)}_{V,x} = (1-T^{q_1}) \cdots (1-T^{q_e})(1-T)^{-(n+1)} = H^{(1)}_{G,0}.$$

By (8.2.6) (i) we also have $H^{(1)}_{G,0} \geq H^{(d+1)}_{G,x}$, hence we get the claimed equality in (ii).

(9.2.4.2) It remains to show (ii) + (iii) $\Rightarrow$ (i). Assume that (ii) and (iii) hold. We have the S_+-adic filtration $\mathcal{S}''$ on S given by $\mathcal{S}''_n = S_{\geq n}$. Further we have the filtration $\mathcal{S}'$ on S defined by $\mathcal{S}'_n = S \cdot U_{\geq n}$. As seen in (5.4.1) $(S, \mathcal{S}', \mathcal{S}'')$ is a bifiltered (S, J, S_+)-module, $\operatorname{gr}^n(\mathcal{S}')$ is free over S/J with rank $\dim_k U_n$ and the filtrations $\underline{\mathcal{S}}^n$ are the S_+-filtrations for all n. On R we have the $\mathfrak{m}$-adic filtration $\mathcal{R}''$ and define the filtration $\mathcal{R}'$ by $\mathcal{R}'_n = (\mathcal{S}'_n)_\mathfrak{p}$. Then $(R, \mathcal{R}', \mathcal{R}'')$ is a bifiltered $(R, J_\mathfrak{p}, \mathfrak{m})$-module, since by (iii) we have $\nu_x(f) = \deg(f)$ for all $s \in U$ and hence $\mathcal{R}' \subseteq \mathcal{R}''$ ($\mathcal{R}'$ is good by (4.2.10)). $\operatorname{gr}^n(\mathcal{R}') \cong \operatorname{gr}^n(\mathcal{S}')_\mathfrak{p}$ is free of rank $\dim_k U_n$ over $(S/J)_\mathfrak{p}$.

(9.2.4.3) By (5.1.5) we get

$$H^{(1)}_{V,0} = \sum_n T^n \operatorname{rank}_{S/J}(\operatorname{gr}^n(\mathcal{S}')) H^{(1)}(S/J, S_+) = H(U) H^{(1)}_{G,0},$$

$$H^{(1)}_{V,x} \le \sum_n T^n \operatorname{rank}_{(S/J)_{\mathfrak{p}}}(\operatorname{gr}^n(\mathcal{R}'))H^{(1)}((S/J)_{\mathfrak{p}}, \mathfrak{m}) = H(U)H^{(1)}_{G,x} =$$

$$= H^{(1)}_{V,0}H^{(1)}_{G,x}/H^{(1)}_{G,0} \overset{(ii)}{=} H^{(1)}_{V,0}H^{(1)}_{G,x}/H^{(d+1)}_{G,x} = H^{(1)}_{V,x}$$

and hence by (5.1.5) the filtration $\underline{\mathcal{R}}^n$ on $\operatorname{gr}^n(\mathcal{R}')$ is the $\mathfrak{m}$-adic filtration for all n.

(9.2.4.4) Let $A = S/I$. Equip I and A with the induced filtrations from $(S, \mathcal{S}', \mathcal{S}'')$ such that $0 \to I \to S \to A \to 0$ becomes a bistrict short exact sequence of bifiltered (S, J, S_+)-modules. $S(U \cap I) = I$ implies by (5.4.3) that all $\operatorname{gr}^n(\mathcal{A}')$ are free S/J-modules. By (5.2.4) therefore all $\underline{\mathcal{A}}^n$ are the S_+-adic filtrations and by (5.1.5) (iv) we get

$$H_{C,0} = H(\mathcal{A}'') = H_{G,0} \sum_n T^n \operatorname{rank}_{S/J}(\operatorname{gr}^n(\mathcal{A}')).$$

(9.2.4.5) Let $K = I_{\mathfrak{p}}$ and $B = A_{\mathfrak{p}}$. Then we get a bistrict short exact sequence $0 \to K \to R \to B \to 0$ of bifiltered $(R, J_{\mathfrak{p}}, \mathfrak{m})$-modules. $\mathcal{B}'_n = (\mathcal{S}'_n)_{\mathfrak{p}} + I_{\mathfrak{p}}/I_{\mathfrak{p}}$ and thus $\operatorname{gr}^n(\mathcal{B}') \cong \operatorname{gr}^n(\mathcal{A}')_{\mathfrak{p}}$. Therefore $\operatorname{gr}^n(\mathcal{B}')$ is a free $(S/J)_{\mathfrak{p}}$-module of rank $\operatorname{rank}_{S/J}(\operatorname{gr}^n(\mathcal{A}'))$ and $\underline{\mathcal{B}}^n$ is the $\mathfrak{m}$-adic filtration for all n again by (5.2.4). Using (5.1.5) (iv) a last time we get

$$H_{C,x} = H(\mathcal{B}'') = H_{G,x} \sum_n T^n \operatorname{rank}_{S/J}(\operatorname{gr}^n(\mathcal{A}')) = H_{C,0}H_{G,x}/H_{G,0} \overset{(ii)}{=} H^{(-d)}_{C,0}.$$

This shows (i). □

With (5.4.3) we gave a tool whose strength was only partially used in the last proof. Maybe it could also be applied in a proof for (i) ⇒ (iii) of the proposition and therefore in a proof of (9.2.2).

Let us now come to a different approach via differential operators. We are able to narrow down a proof of (9.2.2) to a very special situation, which is - not to our surprise - a problem of inseparability. We study the behavior of Hironaka schemes under field extensions. Let us begin with an easy observation.

Lemma (9.2.5). *Let C be a cone in a vector space V over a field k. Let k'/k be a field extension and $x' \in C' := C_{k'} \subseteq V' := V_{k'}$ a point lying over $x \in C$. The following hold:*

(i) $k' \otimes_k \mathcal{U}_{V,x} \subseteq \mathcal{U}_{V',x'}$.

(ii) $B_{V',x'} \subseteq (B_{V,x})_{k'}$.

(iii) *If $B_{V,x} \subseteq \operatorname{Rid}(C)$, then $B_{V',x'} \subseteq \operatorname{Rid}(C')$.*

Proof. Let $V = \operatorname{Spec}(S)$ for a polynomial ring S over k. For $f \in (\mathcal{U}_{V,x})_d$ we have $f \in \mathfrak{m}^d_{V,x}$ and the morphism of local rings $(\mathcal{O}_{V,x}, \mathfrak{m}_{V,x}) \to (\mathcal{O}_{V',x'}, \mathfrak{m}_{V',x'})$ shows that $f \in \mathfrak{m}^d_{V',x'}$. Therefore $f \in \mathcal{U}_{V',x'}$ and (i) is proved. This implies (ii) and we get (iii) since $\operatorname{Rid}(C)_{k'} = \operatorname{Rid}(C_{k'})$ (see (6.1.10)). □

Proposition (9.2.6). *Let $V = \operatorname{Spec}(S)$ be a vector space over the field k and k'/k a separable (not necessarily algebraic) field extension, $V' := V_{k'}$. Let $C \subseteq V$ be* *a cone and* $C' := C_{k'} \subseteq V'$. *Let* $x' \in C'$ *be a point mapping to* $x \in C$. *The following hold:*

(i) $\mathcal{U}_{V',x'} \cap S = \mathcal{U}_{V,x}$.

(ii) $B_{V,x} \subseteq \mathrm{Rid}(C)$ *if and only if* $B_{V',x'} \subseteq \mathrm{Rid}(C')$.

Proof. By (2.2.7) and (2.2.8) we know that a p-basis $(x_i)_{i\in I}$ of S extends to a p-basis $(x_j)_{j\in J}$ of $S' := k' \otimes_k S$ with $I \subseteq J$. Denote the differential operators on S (resp. S') as described in (2.2.5) with respect to these p-bases by $D_M, M \in \mathbb{N}^{(I)}$ (resp. $M \in \mathbb{N}^{(J)}$). They are clearly compatible with the restriction from S' to S, so there is no problem in using the same notation on S and S'. Denote with $\mathfrak{p} \subseteq S$ resp. $\mathfrak{p}' \subseteq S'$ the prime ideals corresponding to x resp. x'. Let $f \in S_d$. Then $f \in \mathcal{U}_{V',x'}$ if and only if $\mathrm{Diff}_{\mathbb{Z}}^{\leq d-1}(S')(f) \subseteq \mathfrak{p}'$ which is equivalent to $D_M(f) \in \mathfrak{p}'$ for every $M \in \mathbb{N}^{(J)}$ with $|M| < d$. Since $f \in S$ we have $D_M(f) = 0$ whenever $M \notin \mathbb{N}^{(I)}$. Hence this is equivalent to $D_M(f) \in \mathfrak{p}' \cap S = \mathfrak{p}$ for every $M \in \mathbb{N}^{(I)}$ with $|M| < d$ which in turn means $\mathrm{Diff}_{\mathbb{Z}}^{\leq d-1}(S)(f) \subseteq \mathfrak{p}$, i.e. $f \in \mathcal{U}_{V,x}$. This proves (i). (ii): Let U be the ring of invariants of $\mathrm{Rid}(C)$ in S. Then $k' \otimes_k U$ is the ring of invariants of $\mathrm{Rid}(C')$. Assume that $B_{V',x'} \subseteq \mathrm{Rid}(C')$, i.e. $k' \otimes_k U \subseteq \mathcal{U}_{V',x'}$. With (i) we find $U = (k' \otimes_k U) \cap S \subseteq \mathcal{U}_{V',x'} \cap S = \mathcal{U}_{V,x}$ and therefore $B_{V,x} \subseteq \mathrm{Rid}(C)$. □

We show that theorem (9.2.2) is equivalent to:

Lemma (9.2.7). *Let* x *be a closed point of a cone* C *contained in a vector space* V *over* k *such that* $\kappa(x)/k$ *is purely inseparable. Let* k'/k *be a field extension generated by a single element* $a \in \kappa(x) \setminus k$. *Let* $x' \in C' := C_{k'}$ *be the closed point lying over* x *and* $V' := V_{k'}$. *If* $B_{V',x'} \subseteq \mathrm{Rid}(C')$ *and* $H_{C',x'} = H_{C,x} = H(C)$, *then* $B_{V,x} \subseteq \mathrm{Rid}(C)$.

Proof of equivalence. If $H_{C,x} = H(C)$, then (9.2.3) immediately yields $B_{V,x} \subseteq \mathrm{Rid}(C)$. For the other direction first observe that the lemma also holds for $k' = \kappa(x)$ since $H_{C_K,x''} = H(C_K)$ automatically will hold for a closed point x'' over x on C_K for any subfield $\kappa(x)/K/k$ by (8.2.3.A) and (8.2.6.A); now use induction. We show (9.2.3). Let C be a cone over k and $0 \neq y \in C$ with $H^{(d)}_{C,y} = H(C)$ for $d = \mathrm{tr.deg}(\kappa(y)/k)$. Let $k \subseteq k' \subseteq k'' \subseteq \kappa(y)$ such that $\kappa(y)/k''$ is finite and inseparable, k''/k' is finite and separable and k'/k is purely transcendental of transcendence degree d. As in (8.2.4) we find $y' \in C_{k'}$ with $\kappa(y') = \kappa(y)$ and $H_{C_{k'},y'} = H^{(d)}_{C,y} = H(C) = H(C_{k'})$. We find $y'' \in C_{k''}$ over y' with $\kappa(y'') = \kappa(y')$ (cf. (8.2.6.2)) and know from (8.2.3.A) and (8.2.6.A) that $H_{C_{k''},y''} = H(C_{k''})$. Since now we can use the extended version of the lemma and (9.2.6) it remains to prove (9.2.3) in the case that $y \in C$ is rational. As in (8.2.6.3) ff. we can assume that y corresponds to the prime ideal $\mathfrak{p} = \langle X_0 - 1, X_1, ..., X_n\rangle$ in $S = k[X_0, ..., X_n], V = \mathrm{Spec}(S)$ and that C is defined by an ideal $I \subseteq S$ with $S \cdot (k[X_1, ..., X_n] \cap I) = I$. Let U be the ring of invariants of $\mathrm{Rid}(C)$ in S. Then $U \subseteq k[X_1, ..., X_n]$. Since $X_0 - 1, X_1, ..., X_n$ are regular parameters at y we have $k[X_1, ..., X_n] \subseteq \mathcal{U}_{V,y}$. Now $U \subseteq \mathcal{U}_{V,y}$ proves $B_{V,y} \subseteq \mathrm{Rid}(C)$. □

Unfortunately at this point we are stuck in our line of thougt and do not know how to prove this lemma. One could try to examine the proof of (8.2.3) (i), (ii). As one sees we prove (9.2.2) at least in the residually separable case and therefore get a full proof for schemes over fields of characteristic zero.

Let us point out one of the problems arising, namely what keeps us from using differential operators as in (9.2.6) for inseparable extensions: If k is a field of characteristic p and $a \in k \setminus k^p$, one has a derivation ∂_a on k with the property $\partial_a(a^n) = na^{n-1}$ for all n. ∂_a extends to a differential operator on $k' := k(\sqrt[p]{a})$. In fact $\sqrt[p]{a} \in k'$ is p-independent and we get a differential operator D on k' of order p with $D(\sqrt[p]{a}^n) = \binom{n}{p}\sqrt[p]{a}^{n-p}$. This coincides with ∂_a on k since $D(a^n) = D(\sqrt[p]{a}^{pn}) = \binom{pn}{p}\sqrt[p]{a}^{pn-p} = na^{n-1}$. The issue is the following: the order of the differential operator ∂_a is 1 on k but rises to p on k'. Hence we get a problem in controlling the order of some polynomial in view of (9.1.2).

10 Refined Hironaka Schemes

Before we come to the actual definition of our refined Hironaka schemes, we introduce our concept of the initial map. After presenting dissecting variables, we come to the general part of the proof of the main theorems. This proof is finished by taking a look at all types of Hironaka schemes in low dimensions.

10.1 The initial map

Let x be a point in a vector space $V = \operatorname{Spec}(S)$ over k. Taking initial forms $S \to \operatorname{gr}_{V,x}, f \mapsto \operatorname{in}_x(f)$ is not a homomorphism of rings in general. But it is rather obvious that we get a nice morphism of graded rings when we restrict to the Hironaka ring of invariants $\mathcal{U}_{V,x}$:

Lemma (10.1.1). *The initial map*

$$\operatorname{In}_x : \mathcal{U}_{V,x} \to \operatorname{gr}_{V,x} = \operatorname{gr}_{\mathfrak{m}_{V,x}}(\mathcal{O}_{V,x}), \qquad f = \sum_i f_i \mapsto \sum_i \operatorname{in}_x(f_i)$$

is an injective k-linear morphism of graded rings. f_i is the i-th homogeneous component of f and $\operatorname{in}_x(f_i) = f_i \bmod \mathfrak{m}_{V,x}^{\nu_x(f_i)+1} = f_i \bmod \mathfrak{m}_{V,x}^{i+1}$.

Proof. Since $f_i \in \mathcal{U}_{V,x}$, we have $\nu_x(f_i) = \deg(f_i) = i$ if $f_i \neq 0$ and therefore $\operatorname{in}_x(f_i) \in \operatorname{gr}^i_{V,x}$. $\operatorname{In}_x |_{(\mathcal{U}_{V,x})_i}$ is injective since for $f_i \neq 0$ we have $f_i \in \mathfrak{m}^i_{V,x} \setminus \mathfrak{m}^{i+1}_{V,x}$. The k-linearity of this map is clear. For $0 \neq f_i \in (\mathcal{U}_{V,x})_i$ and $0 \neq f_j \in (\mathcal{U}_{V,x})_j$ we have $0 \neq f_i f_j \in (\mathcal{U}_{V,x})_{i+j}$ and hence $\operatorname{in}_x(f_i f_j) = \operatorname{in}_x(f_i)\operatorname{in}_x(f_j) \in \operatorname{gr}^{i+j}_{V,x}$. Therefore In_x is a morphism of graded rings. □

Lemma (10.1.2). *Let $V = \operatorname{Spec}(S)$ be a vector space over the field k and k'/k some field extension. Let $x' \in V' := V_{k'} = \operatorname{Spec}(S')$ with $S' = k' \otimes_k S$ and $x = \pi(x')$ with the projection $\pi : V' \to V$. Then the canonical morphism of graded rings $S \to S'$ induces a morphism of graded rings*

(10.1.2.A) $$\mathcal{U}_{\pi,x'} : \quad \mathcal{U}_{V,x} \to \mathcal{U}_{V',x'}$$

such that the diagramm of graded rings

(10.1.2.B) $$\begin{array}{ccc} \operatorname{gr}_{V',x'} & \xleftarrow{\operatorname{In}_{x'}} & \mathcal{U}_{V',x'} \\ {\scriptstyle \operatorname{gr}_{\pi,x'}}\uparrow & & \uparrow{\scriptstyle \mathcal{U}_{\pi,x'}} \\ \operatorname{gr}_{V,x} & \xleftarrow{\operatorname{In}_{x}} & \mathcal{U}_{V,x} \end{array}$$

is commutative, where $\operatorname{gr}_{\pi,x'}$ is the canonical morphism.

Proof. The canonical diagram

$$\begin{array}{ccc} \mathcal{O}_{V',x'} & \longleftarrow & S' \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,x} & \longleftarrow & S \end{array}$$

of course is commutative. $(\mathcal{O}_{V,x}, \mathfrak{m}_{V,x}) \to (\mathcal{O}_{V',x'}, \mathfrak{m}_{V',x'})$ is a morphism of filtered rings and therefore for $f \in \mathcal{U}_{V,x}$

$$\deg(f) \overset{(9.1.2)}{\geq} \nu_{x'}(f) \geq \nu_x(f) = \deg(f),$$

i.e. $\nu_{x'}(f) = \deg(f)$ and hence $f \in \mathcal{U}_{V',x'}$. This shows that $\mathcal{U}_{\pi,x'}$ is well-defined and for $f \in \mathcal{U}_{V,x}$ homogeneous of degree d

$$\mathrm{gr}_{\pi,x'} \circ \mathrm{In}_x(f) = \mathrm{gr}_{\pi,x'}(f \bmod \mathfrak{m}_{V,x}^{d+1}) = f \bmod \mathfrak{m}_{V',x'}^{d+1} = \mathrm{In}_{x'} \circ \mathcal{U}_{\pi,x'}(f)$$

shows that the diagramm is commutative. □

The observations of the following corollary will lead us to the definition of our dissecting variables in (10.3.6). Remember that $\mathcal{Q}_{V,x} = \mathcal{U}_{V,x} \cap L$.

Corollary (10.1.3). *Let x be a point of a vector space V over a field k of positive characteristic. We can choose variables $(Y, Z) = (Y_1, ..., Y_r, Z_1, ..., Z_s)$ of $\mathrm{gr}_{V,x}$ (i.e. $\mathrm{gr}_{V,x} = \kappa(x)[Y, Z]$ and for $f \in \mathcal{U}_{V,x}$ we have $\mathrm{In}_x(f)(Y, Z) \in \mathrm{gr}_{V,x}$) such that the following hold:*

(i) *For $\rho \in \mathcal{Q}_{V,x}$ the polynomial $\mathrm{In}_x(\rho)(Y, 0) \in \kappa(x)[Y]$ is additive.*

(ii) *If $\sigma = (\sigma_1, ..., \sigma_m)$ is a $k[F]$-independent system in $\mathcal{Q}_{V,x}$, then*

$$(\mathrm{In}_x(\sigma_1)(Y, 0), ..., \mathrm{In}_x(\sigma_m)(Y, 0))$$

is $\kappa(x)[F]$-independent in $\kappa(x)[Y]$.

Proof. We use (10.1.2.A) with $k' = \kappa(x)$. The kernel $\mathfrak{p}'$ of the obvious morphism $S' = \kappa(x) \otimes_k S \to \kappa(x) \otimes_k \kappa(x) \to \kappa(x)$ defines a point x' on V' lying above x with $\kappa(x') = \kappa(x) = k'$. Therefore x' is a rational point on V'. We can suppose that $\mathfrak{p}' = \langle X_0 - 1, X_1, ..., X_n \rangle \subseteq S' = k'[X_0, ..., X_n]$. The ring of invariants $\mathcal{U}_{V',x'}$ only depends on the image of x' under $(V')^* \to \mathbb{P}V'$ which is given by the prime ideal $\langle X_1, ..., X_n \rangle$. As in (9.1.7) we see that $\mathcal{U}_{V',x'} = k'[X_1, ..., X_n]$. $X_0 - 1, X_1, ..., X_n$ are regular parameters at x' and therefore $\mathrm{gr}_{V',x'} = k'[\overline{X_0 - 1}, \mathrm{In}_{x'}(X_1), ..., \mathrm{In}_{x'}(X_n)]$. Thus $\mathrm{In}_{x'}$ is an injection of polynomial rings over k'. $\mathrm{gr}_{\pi,x'} : \mathrm{gr}_{V,x} \to \mathrm{gr}_{V',x'}$ is a graded morphism of polynomial rings over the same field k'. Hence we can choose variables (Y, Z) of $\mathrm{gr}_{V,x}$ such that $\ker(\mathrm{gr}_{\pi,x'}) = \langle Z \rangle$. Then $\mathrm{gr}_{\pi,x'}$ induces an injection of polynomial rings $k'[Y] \subseteq \mathrm{gr}_{V,x} \to \mathrm{gr}_{V',x'}$ over k'. To prove (i) let $\rho \in \mathcal{Q}_{V,x}$. Clearly $\mathcal{U}_{\pi,x'}(\rho) \in \mathcal{U}_{V',x'}$ still is additive and by what we just showed also $\mathrm{In}_{x'}(\mathcal{U}_{\pi,x'}(\rho))$ is additive. With (10.1.2.B) $\mathrm{gr}_{\pi,x'}(\mathrm{In}_x(\rho))$ is additive and this polynomial can be viewed as $\mathrm{In}_x(\rho)(Y, 0)$. For (ii) let σ be a system of additive polynomials in $\mathcal{Q}_{V,x}$. Assume that $(\mathrm{In}_x(\sigma_1)(Y,0),...,\mathrm{In}_x(\sigma_m)(Y,0))$ are $k'[F]$-dependent. By (10.1.2.B) then also $(\mathrm{In}_{x'}(\mathcal{U}_{\pi,x'})(\sigma_1),...,\mathrm{In}_{x'}(\mathcal{U}_{\pi,x'})(\sigma_m))$ are $k'[F]$-dependent. By our analysis of $\mathrm{In}_{x'}$ then already $(\mathcal{U}_{\pi,x'}(\sigma_1),...,\mathcal{U}_{\pi,x'}(\sigma_m))$ were $k'[F]$-dependent. Since $\sigma_1,...,\sigma_m \in S$, this in turn implies that $(\sigma_1,...,\sigma_m)$ are $k[F]$-dependent. This shows (ii). □

10.2 Refined Hironaka schemes

The idea behind our refined Hironaka schemes is the following: Replace the invariant H with the extended invariant HR and copy the theory of Hironaka schemes. Especially desirable are an equivalent of Hironaka's theorem (9.2.1) (resp. (9.2.2)) which we give with main theorem A and a result about the linearity of refined Hironaka schemes which is main theorem B.

We will restrict our discussion to fields of positive characteristic. Of course one can do everything we have in mind similarly in characteristic zero. On the one hand this would be tedious since we permanently would have to distinguish both cases in our notations. On the other hand one would not gain anything since in characteristic zero our refined Hironaka schemes would be just the old ones (cf. (10.2.1)). From now on we always assume to be in positive characteristic.

A delicate point is the question how refined Hironaka schemes should be defined. For a vector space $V = \mathrm{Spec}(S)$ over the field k and a point $x \in V$ we should define a graded subring of S generated by additive polynomials $\mathcal{V}_{V,x}$ (or $\mathcal{V}_x$ if no confusion is possible), the **ring of invariants of the refined Hironaka scheme**. The **refined Hironaka scheme** can then be defined as the homogeneous additive group

$$F_x := F_{V,x} := \mathrm{Spec}(S/S(\mathcal{V}_{V,x})_+).$$

Of course we also can look at the graded $k[F]$-submodule

$$\mathcal{P}_x := \mathcal{P}_{V,x} := \mathcal{V}_{V,x} \cap L$$

of the additive polynomials L in S generated by the additive polynomials in $\mathcal{V}_{V,x}$. The first idea is to imitate directly the original definition of $\mathcal{U}_{V,x}$ as in [H5]. We can do this as well in the affine setting, where Hironaka uses the projective point of view, see (9.1.1), (9.1.6). What was formulated in the original definition with multiplicities of hypersurfaces can of course be expressed also in terms of Hilbert series, cf. our discussion in (8.1.4). Therefore we could make the following definition:

(D1) $\mathcal{V}_{V,x}$ is the subring of S generated by those homogeneous polynomials f with the following property: For the hypersurface $X := \mathrm{Spec}(S/\langle f \rangle)$ in V one has the equality $HR^{(d)}_{X,x} = HR(X) = HR_{X,0}$ with $d = \mathrm{tr.deg}(\kappa(x)/k)$.

Two questions naturally arise after this definition:

(Q1) Is $\mathcal{V}_{V,x}$ generated by additive polynomials? Only in this case F_x will be a group and the statement of main theorem A will make sense.

(Q2) Do all homogeneous polynomials of $\mathcal{V}_{V,x}$ have the property $HR^{(d)}_{X,x} = HR(X)$ from (D1)? For Hironaka schemes the analog statement holds.

In order to have an easy definition at hand and to assure that F_x is a group, we therefore take as our definition the following one:

(D2) $\mathcal{P}_{V,x}$ is the $k[F]$-submodule of $\mathcal{Q}_{V,x}$ consisting of those additive polynomials $\rho \in \mathcal{Q}_{V,x}$ for which also $\mathrm{In}_x(\rho)$ is additive.

It is clear that this definition really yields a graded $k[F]$-module and we define $\mathcal{V}_{V,x} := k[\mathcal{P}_{V,x}]$ and F_x as above. If $X = \mathrm{Spec}(S/\langle\rho\rangle)$, then $H^{(d)}_{X,x} = H_{X,0}$ since ρ lies in $\mathcal{U}_{V,x}$. Since ρ is additive, we have $\mathrm{Rid}(X) = X$ and since $\mathrm{In}_x(\rho)$ also is additive, we get in the same way $\mathrm{Rid}(C_{X,x}) = C_{X,x}$. Therefore

$$R^{(d)}_{X,x} = \dim T_{V,x} - 1 + d = \dim V - 1 = R_{X,0}.$$

This shows that $\mathcal{V}_{V,x}$ of (D2) lies in $\mathcal{V}_{V,x}$ of (D1). We always will work with (D2). We show with (10.4.10) that at least in low dimensions, or in general if there exist good coordinates at x (dissecting variables, see 10.3), both definitions in fact coincide and (Q1) and (Q2) can be answered positively.

Obviously $\mathcal{V}_{V,x} \subseteq \mathcal{U}_{V,x}$ and therefore $B_{V,x} \subseteq F_{V,x}$. So we enlarged the original Hironaka schemes. That is precisely what we want. There could be far less cones $C \subseteq V$ with $F_{V,x} \subseteq \mathrm{Rid}(C)$ than there are cones with $B_{V,x} \subseteq \mathrm{Rid}(C)$. This represents the improvement of the invariant from H to HR. As for Hironaka schemes we have $x \in F_{V,x}$. $F_{V,x}$ also does not depend on the embedding into a vector space (see (10.3.9)). For a point $x' \in \mathbb{P}V$ we define $F_{V,x'} := F_{V,x}$ for the homogeneous point $x \in V$ corresponding to x'.

Remark (10.2.1). *If B_x is a vector space, then $B_x = F_x$ and F_x also is a vector space.*

Proof. If B_x is a vector space, then $\mathcal{Q}_x$ is generated by linear forms $l \in S_1$. Of course $\mathrm{In}_x(l) \in \mathrm{gr}^1_{V,x}$ is again a linear form. This shows $\mathcal{P}_x = \mathcal{Q}_x$. □

Lemma (10.2.2). *We have $\mathcal{RP}_x = \mathcal{P}_x$, where $\mathcal{RP}_x$ is the radical of the $k[F]$-module $\mathcal{P}_x$, and for $0 \neq f, g \in S$*

$$f \cdot g \in \mathcal{V}_x \qquad \Longrightarrow \qquad f, g \in \mathcal{V}_x.$$

Proof. In (9.1.4) we saw that $\mathcal{RQ}_x = \mathcal{Q}_x$. Let $\sigma \in L$ be homogeneous with $F(\sigma) \in \mathcal{P}_x$. Then $\sigma \in \mathcal{Q}_x$ and $F(\mathrm{In}_x(\sigma)) = \mathrm{In}_x(F(\sigma))$ is additive, hence already $\mathrm{In}_x(\sigma)$ is additive, i.e. $\sigma \in \mathcal{P}_x$. The last implication follows from (6.3.11). □

Lemma (10.2.3). *If $\dim B_x = \dim F_x$, then $B_x = F_x$.*

Proof. Since $\mathcal{P}_x \subseteq \mathcal{Q}_x$ and $\dim_{k[F]} \mathcal{P}_x = n - \dim F_x = n - \dim B_x = \dim_{k[F]} \mathcal{Q}_x$, there exists $e \in \mathbb{N}$ with $(\mathcal{P}_x)_g = (\mathcal{Q}_x)_g$ for all $g \geq e$. For large enough i and g we have $\mathcal{P}_x = \{f \in L | F^i(f) \in \mathcal{P}_x\} = \{f \in L | F^i(f) \in \langle(\mathcal{P}_x)_g\rangle_{k[F]}\} = \mathcal{Q}_x$. □

10.3 Dissecting variables

In (10.1.3) we saw that the initial forms of elements of $\mathcal{U}_{V,x}$ will have a special structure. To be able to prove our main theorems we will need additional information.

This is precisely the reason why our proof of main theorem A does not generalize to all dimensions in the end. We introduce the necessary additional information with the concept of dissecting variables. Let V be a vector space over k and $x \in V$.

Definition (10.3.1). *For chosen variables $(Y,Z) = (Y_1,...,Y_r,Z_1,...,Z_s)$ at x, i.e. $\mathrm{gr}_{V,x} = \kappa(x)[Y,Z]$, we will use the morphism of graded rings*

$$\mathrm{In}_{x,Y} : \mathcal{U}_x \xrightarrow{\mathrm{In}_x} \mathrm{gr}_{V,x} = \kappa(x)[Y,Z] \xrightarrow{Y\mapsto Y, Z\mapsto 0} \kappa(x)[Y]$$

and also use the map

(10.3.1.A) $$\mathrm{In}_{x,Z} := \mathrm{In}_x - \mathrm{In}_{x,Y} \quad : \mathcal{U}_x \to \mathrm{gr}_{V,x}$$

which will not be a morphism of rings but k-linear. Here we use $\kappa(x)[Y]$ as a subring of $\mathrm{gr}_{V,x}$.

Lemma (10.3.2). *For variables (Y,Z) at x and $j \geq 0$ we have*

(10.3.2.A) $$\mathrm{In}_{x,Y} \circ F^j = F^j \circ \mathrm{In}_{x,Y},$$

(10.3.2.B) $$\mathrm{In}_{x,Z} \circ F^j = F^j \circ \mathrm{In}_{x,Z}.$$

Proof. (10.3.2.A) is clear since $\mathrm{In}_{x,Y}$ is a morphism of rings. (10.3.2.B) follows from the definition (10.3.1.A) since the Frobenius F commutes with differences. □

With L we denote the additive polynomials in the polynomial ring S, $V = \mathrm{Spec}(S)$. With L_x we will denote the additive polynomials in $\mathrm{gr}_{V,x}$.

Lemma (10.3.3). *For variables (Y,Z) at x the following are equivalent:*

(i) *There exists a system of $k[F]$-generators σ of $\mathcal{Q}_x$ with* $\mathrm{In}_{x,Y}(\sigma) \subseteq L_x$.

(ii) $\mathrm{In}_{x,Y}(\mathcal{Q}_x) \subseteq L_x$.

Proof. If (i) holds, then $\mathcal{Q}_x = \langle\sigma\rangle_{k[F]}$ and (ii) follows from (10.3.2.A). (ii) ⇒ (i) is clear. □

Lemma (10.3.4). *For variables (Y,Z) at x the following are equivalent:*

(i) *There exists a system of $k[F]$-generators σ of $\mathcal{Q}_x$ with* $\mathrm{In}_{x,Z}(\sigma) \subseteq \kappa(x)[Z]$.

(ii) $\mathrm{In}_{x,Z}(\mathcal{Q}_x) \subseteq \kappa(x)[Z]$.

Proof. If (i) holds, then $\mathcal{Q}_x = \langle\sigma\rangle_{k[F]}$ and (ii) follows from (10.3.2.B). (ii) ⇒ (i) is clear. □

Lemma (10.3.5). *Let (Y,Z) be variables at x and assume the conditions of (10.3.3) hold. Then the following are equivalent:*

(i) *There exists a $k[F]$-basis σ of $\mathcal{Q}_x$ such that the* $\mathrm{In}_{x,Y}(\sigma)$ *are $\kappa(x)[F]$-independent.*

(ii) *For every $k[F]$-basis σ of $\mathcal{Q}_x$ the* $\mathrm{In}_{x,Y}(\sigma)$ *are $\kappa(x)[F]$-independent.*

(iii) *The morphism of graded $\kappa(x)[F]$-modules*

$$\kappa(x) \otimes_k \mathcal{Q}_x \to \langle \mathrm{In}_{x,Y}(\mathcal{Q}_x) \rangle_{\kappa(x)[F]}$$

is an isomorphism.

Proof. (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is easy to see. □

Definition (10.3.6). ***Dissecting variables*** *at x are variables (Y, Z) at x such that the conditions of (10.3.3), (10.3.4) and (10.3.5) hold, i.e.*

(i) $\mathrm{In}_{x,Y}(\mathcal{Q}_x) \subseteq L_x$.

(ii) $\mathrm{In}_{x,Z}(\mathcal{Q}_x) \subseteq \kappa(x)[Z]$.

(iii) $\kappa(x) \otimes_k \mathcal{Q}_x \to \langle \mathrm{In}_{x,Y}(\mathcal{Q}_x) \rangle_{\kappa(x)[F]}$ *is an isomorphism.*

For a $k[F]$-basis $\sigma = (\sigma_1, ..., \sigma_m)$ of $\mathcal{Q}_x$ we denote with $\Sigma = (\Sigma_1, ..., \Sigma_m)$ the polynomials $\mathrm{In}_{x,Y}(\sigma_1), ..., \mathrm{In}_{x,Y}(\sigma_m)$. If (Y, Z) are dissecting variables, then Σ are $\kappa(x)[F]$-independent additive polynomials.

Note that (10.1.3) only assures the existence of variables (Y, Z) with properties (i) and (iii) of the definition.

Lemma (10.3.7). *Assume that (Y, Z) are dissecting variables at x and that (Y', Z') are other variables at x such that $\langle Z \rangle_{\kappa(x)} = \langle Z' \rangle_{\kappa(x)}$. Then also (Y', Z') are dissecting variables.*

Proof. Let $\rho \in \mathcal{Q}_x$. Then $\mathrm{In}_x(\rho) = \mathrm{In}_{x,Y}(\rho) + \mathrm{In}_{x,Z}(\rho)$ with $\mathrm{In}_{x,Y}(\rho) \in L_x$ and $\mathrm{In}_{x,Z}(\rho) \in \kappa(x)[Z]$. If we change the variables to (Y', Z') we get from $\mathrm{In}_{x,Z}(\rho)$ only terms in $\kappa(x)[Z']$ since $\langle Z \rangle_{\kappa(x)} = \langle Z' \rangle_{\kappa(x)}$. From $\mathrm{In}_{x,Y}(\rho)$ there also can arise terms in Z', but since this polynomial is additive, it will be the sum of an additive polynomial in $\kappa(x)[Y']$ (which is $\mathrm{In}_{x,Y'}(\rho)$) and an additive polynomial in $\kappa(x)[Z']$ (which yields $\mathrm{In}_{x,Z'}(\rho)$ together with the transformed $\mathrm{In}_{x,Z}(\rho)$). Therefore conditions (i) and (ii) of definition (10.3.6) hold for (Y', Z'). One easily sees that the assignment $\mathrm{In}_{x,Y}(\rho) \mapsto \mathrm{In}_{x,Y'}(\rho)$ is $\kappa(x)$-linear. Since this also works the other way round, condition (iii) holds for (Y', Z'). □

Definition (10.3.8). *A point $x \in V$ is called* ***dissected*** *if there exist dissecting variables (Y, Z) at x.*

It is in our interest to show that all homogeneous points of $\mathbb{A}_k^n$ are dissected. We prove this in the end for $n \leq 5$. To enable us to accomplish this, we have to reduce the problem to a few cases. The next lemmas will help us in this.

Lemma (10.3.9). *Let $x \in W$ be a point of a vector space over k and $W \subseteq V$ contained in a larger vector space. If $x \in W$ is dissected, then $x \in V$ is dissected. Refined Hironaka schemes are independent of the embedding into a vector space, i.e. $F_{W,x} = F_{V,x}$.*

Proof. Let us recall the situation of the proof of (9.1.8): $V = \mathrm{Spec}(S)$ for $S = k[X_1, ..., X_n]$ and $W = \mathrm{Spec}(T), T = k[X_{n'+1}, ..., X_n]$, $\iota : W \to V$ is defined by $\iota^\# : S \to T$ which sends $X_1, ..., X_{n'}$ to zero. The projection $\pi : V \to W$ is defined by the inclusion $\pi^\# : T \to S$ and $\iota^\# \circ \pi^\# = \mathrm{id}_T, \pi \circ \iota = \mathrm{id}_W$. It was proved that $\mathcal{U}_{V,x} = \mathcal{U}_{W,x}[X_1, ..., X_{n'}]$ and therefore $\mathcal{Q}_{V,x} = \mathcal{Q}_{W,x} \oplus \langle X_1, ..., X_{n'}\rangle_{k[F]}$. We have morphisms of graded rings

$$(10.3.9.A) \qquad \mathrm{gr}_{W,x} \xrightarrow{\mathrm{gr}_{\pi,x}} \mathrm{gr}_{V,x} \xrightarrow{\mathrm{gr}_{\iota,x}} \mathrm{gr}_{W,x}$$

such that the composition is the identity on $\mathrm{gr}_{W,x}$. These morphisms are compatible with In_x on $\mathcal{U}_{W,x} \subseteq \mathcal{U}_{V,x}$. Since $\mathrm{gr}_{\pi,x}, \mathrm{gr}_{\iota,x}$ preserve additive polynomials, we have by definition $\mathrm{gr}_{\pi,x}(\mathcal{P}_{W,x}) \subseteq \mathcal{P}_{V,x}$ and $\mathrm{gr}_{\iota,x}(\mathcal{P}_{V,x}) \subseteq \mathcal{P}_{W,x}$. Since $X_1, ..., X_{n'}$ are of degree one, also $\mathrm{In}_x(X_1), ..., \mathrm{In}_x(X_{n'})$ are additive, so $X_1, ..., X_{n'} \in \mathcal{P}_{V,x}$. This proves $\mathcal{P}_{V,x} = \mathcal{P}_{W,x} \oplus \langle X_1, ..., X_{n'}\rangle_{k[F]}$. Thus $F_{W,x} = F_{V,x}$. Let now (Y, Z) be dissecting variables at $x \in W$. The variables (Y, Z) are independent in $\mathrm{gr}_{V,x}$ by (10.3.9.A). They have to be completed by $\dim \mathrm{gr}_{V,x} - \dim \mathrm{gr}_{W,x} = \dim V - \mathrm{tr.deg}(\kappa(x)/k) - (\dim W - \mathrm{tr.deg}(\kappa(x)/k)) = \dim V - \dim W = n'$ independent variables in $\ker(\mathrm{gr}_{\iota,x})$. This is done by $Y^* = (\mathrm{In}_x(X_1), ..., \mathrm{In}_x(X_{n'}))$. (Y', Z') with $Y' = (Y, Y^*)$ and $Z' = Z$ then are variables at $x \in V$. $\rho \in \mathcal{Q}_{V,x}$ is of the form $\rho = \rho' + \rho''$ with $\rho' \in \mathcal{Q}_{W,x}$ and $\rho'' \in \langle X_1, ..., X_{n'}\rangle_{k[F]}$. We have $\mathrm{In}_{x,Y'}(\rho) = \mathrm{In}_{x,Y}(\rho') + \mathrm{In}_x(\rho'')$ which is additive. Also $\mathrm{In}_{x,Z'}(\rho) = \mathrm{In}_x(\rho) - \mathrm{In}_{x,Y'}(\rho) = \mathrm{In}_{x,Z}(\rho')$ lies in $\kappa(x)[Z']$. Let σ be a $k[F]$-basis of $\mathcal{Q}_{W,x}$. Then (σ, Y^*) is a $k[F]$-basis of $\mathcal{Q}_{V,x}$ and $\mathrm{In}_{x,Y'}(\sigma) = \mathrm{In}_{x,Y}(\sigma)$ is $\kappa(x)[F]$-independent. Therefore also $\mathrm{In}_{x,Y'}(\sigma), \mathrm{In}_{x,Y'}(Y^*)$ are $\kappa(x)[F]$-independent. □

Lemma (10.3.10). *Let $S = k[X_1, ..., X_n], T = k[X_{n'+1}, ..., X_n] \subseteq S$ and $\pi : V = \mathrm{Spec}(S) \to W = \mathrm{Spec}(T)$ the canonical projection. Let x be a point of V with image $y = \pi(x)$.*

(i) *Then $\mathcal{U}_{W,y} = \mathcal{U}_{V,x} \cap T$.*

(ii) *If $y \in W$ is dissected and $\mathcal{U}_{V,x} \subseteq T$, then $\mathcal{U}_{V,x} = \mathcal{U}_{W,y}$, $\mathcal{V}_{V,x} = \mathcal{V}_{W,y}$ and $x \in V$ is dissected.*

Proof. Let $\mathfrak{p}$ be the prime ideal corresponding to x; then $\mathfrak{p}' := \mathfrak{p} \cap T$ is the prime ideal corresponding to y. By (9.1.4.A) we have for all $d \in \mathbb{N}$

$$(\mathcal{U}_{V,x})_d = \{f \in S_d | \mathrm{Diff}_{\mathbb{Z}}^{\le d-1}(S)(f) \subseteq \mathfrak{p}\},$$

$$(\mathcal{U}_{W,y})_d = \{f \in T_d | \mathrm{Diff}_{\mathbb{Z}}^{\le d-1}(T)(f) \subseteq \mathfrak{p}'\}.$$

For $f \in (\mathcal{U}_{W,y})_d \subseteq T$ we have with our structure results on differential operators (2.2.5) (see also (2.2.7))

$$\mathrm{Diff}_{\mathbb{Z}}^{\le d-1}(S)(f) = S \cdot \mathrm{Diff}_{\mathbb{Z}}^{\le d-1}(T)(f) \subseteq S \cdot \mathfrak{p}' \subseteq \mathfrak{p}$$

and therefore $f \in \mathcal{U}_{V,x}$. This proves $\mathcal{U}_{W,y} \subseteq \mathcal{U}_{V,x}$. On the other hand let $f \in (\mathcal{U}_{V,x})_d \cap T$. By using (2.2.5) again we find

$$\mathrm{Diff}_{\mathbb{Z}}^{\le d-1}(T)(f) \subseteq \mathrm{Diff}_{\mathbb{Z}}^{\le d-1}(S)(f) \cap T \subseteq \mathfrak{p} \cap T = \mathfrak{p}'$$

which proves $f \in \mathcal{U}_{W,y}$ and we showed (i). In (ii) we get $\mathcal{U}_{V,x} = \mathcal{U}_{W,x}$ from the condition $\mathcal{U}_{V,x} \subseteq T$ by (i). Then also $\mathcal{Q}_{V,x} = \mathcal{Q}_{W,y}$. All fibers of π are regular and with (8.2.2) we see that the canonical morphism

$$g : \kappa(x) \otimes_{\kappa(y)} \mathrm{gr}_{W,y} \to \mathrm{gr}_{V,x}$$

is injective and we regard $\mathrm{gr}_{W,y} \subseteq \mathrm{gr}_{V,x}$ as a subring. Now let (Y, Z) be dissecting variables at $y \in W$. By the above injectivity we see that (Y, Z) are also $\kappa(x)$-independent and therefore can be completed by variables Z^* to variables of $\mathrm{gr}_{V,x}$. We claim that (Y', Z') with $Y' = Y, Z' = (Z, Z^*)$ are dissecting variables at $x \in V$. From $g \circ \mathrm{In}_y = \mathrm{In}_x : \mathcal{U}_{W,y} \to \mathrm{gr}_{V,x}$ we see that also $g \circ \mathrm{In}_{y,Y} = \mathrm{In}_{x,Y'}$ and therefore $g \circ \mathrm{In}_{y,Z} = \mathrm{In}_{x,Z'}$. Then $\mathrm{In}_{x,Y'}(\mathcal{Q}_{V,x}) = \mathrm{In}_{y,Y}(\mathcal{Q}_{W,y})$ are additive polynomials and $\mathrm{In}_{x,Z'}(\mathcal{Q}_{V,x}) = \mathrm{In}_{y,Z}(\mathcal{Q}_{W,y}) \subseteq \kappa(y)[Z] \subseteq \kappa(x)[Z']$. If σ is a $k[F]$-basis of $\mathcal{Q}_{V,x}$, then $\mathrm{In}_{x,Y'}(\sigma) = \mathrm{In}_{y,Y}(\sigma)$ are $\kappa(y)[F]$-independent, therefore also $\kappa(x)[F]$-independent. Thus (Y', Z') are dissecting variables at $x \in V$. □

Lemma (10.3.11). *Let $x \in V$ be a point of a vector space and consider the projection $\pi : V \to W := V/\operatorname{Dir}(\overline{\{x\}})$. Then*

$$\mathcal{U}_{V,x} \cong \mathcal{U}_{W,\pi(x)}.$$

Proof. Using a chain of vector spaces $0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_j = \operatorname{Dir}(\overline{\{x\}})$ such that $\operatorname{codim}_{D_{i+1}} D_i = 1$ and the projections $V = V/D_0 \to V/D_1 \to \cdots \to V/D_j$, it suffices to prove the claim in the following situation: Let $S = k[X_1, ..., X_n], T = k[X_2, ..., X_n]$, $\mathfrak{p} \subseteq T$ a prime ideal and $\mathfrak{P} \subseteq S$ the prime ideal generated by $\mathfrak{p} \subseteq T \subseteq S$. The point $x \in V := \operatorname{Spec}(S)$ corresponding to $\mathfrak{P}$ is then mapped to the point corresponding to $\mathfrak{p}$ in $W := \operatorname{Spec}(T)$ under the canonical projection $\pi : V \to W$. By (10.3.10) (i) we have $\mathcal{U}_{W,\pi(x)} = \mathcal{U}_{V,x} \cap T$ and it remains to show that $\mathcal{U}_{V,x} \subseteq T$. Assume this is not an equality. Then we find a homogeneous $f \in \mathcal{U}_{V,x} \setminus T$ of degree d such that $(\mathcal{U}_{V,x})_{\leq d-1} = (\mathcal{U}_{W,\pi(x)})_{\leq d-1}$. Develop $f = \sum_{i=e}^{d} f_i X_1^i$ with $f_i \in T_{d-i}$ and $f_e \neq 0$. In the case $e = d$ we would have $X_1^d \in \mathcal{U}_{V,x}$ and therefore $X_1 \in \mathfrak{P}$ which is impossible. Thus $e < d$. There exists a differential operator $D = \lambda D_M \in \operatorname{Diff}_k^{\leq d-e}(T)$ (see (3.3.1)) with respect to the variables of T with $\lambda \in k^\times$ such that $D(f_e) = 1$. It extends X_1-linearly to a differential operator of the same degree on S and $D(f) = X_1^e + \sum_{i=e+1}^{d} D(f_i) X_1^i$ which lies in $(\mathcal{U}_{V,x})_e = (\mathcal{U}_{W,\pi(x)})_e \subseteq T$ since $\mathcal{U}_{V,x}$ is generated by additive polynomials by (3.4.3). But this is impossible. □

10.4 Main theorem A for cones

Goal of this section is to establish results that allow a proof of the main theorems in the presence of dissecting variables. We prove the equivalent of main theorem A in the case of a cone. Throughout this section we fix a point x in a vector space $V = \operatorname{Spec}(S), S = k[X_1, ..., X_n]$ over the field k, dissecting variables $(Y, Z) = (Y_1, ..., Y_r, Z_1, ..., Z_s)$ at x and a homogeneous $\kappa(x)[F]$-basis $\sigma = (\sigma_1, ..., \sigma_m)$ of $\mathcal{Q}_x$ with $q_i = \deg(\sigma_i)$ and $q_1 \leq \cdots \leq q_m$. Set $\Sigma := \mathrm{In}_{x,Y}(\sigma)$.

(10.4.1) We have to introduce some notations.

(i) By (1.4.5) we find (after possibly renumbering the X_i) a homogeneous k-basis $\sigma^N X^M$ of S, where N ranges over $\Lambda' = \mathbb{N}^m$ and M ranges over

$$\Lambda'' = \{M \in \mathbb{N}^n | M_1 < q_1, ..., M_m < q_m\}.$$

We use the order on $\Lambda := \Lambda' \times \Lambda''$ defined by $(N, M) \leq (N', M')$ if $qN + |M| < qN' + |M'|$ or $qN + |M| = qN' + |M'|$ and $(N, M) \leq_{\text{lex}} (N', M')$ (componentwise lexicographic order and lexicographic order on the product). This is a wellorder. By restricting this order onto $\Lambda' = \Lambda' \times 0 \subseteq \Lambda' \times \Lambda''$ we get a wellorder on Λ' which is a weighted homogeneous lexicographical order (see (1.1.2.4)) and therefore a monomial order on $\mathcal{U}_x = k[\sigma]$, where we regard the σ_i as the variables for the monomial order.

(ii) Σ is $\kappa(x)[F]$-independent since (Y, Z) are dissecting variables. By (1.4.5) again we therefore find (after possibly renumbering the Y_j) a homogeneous $\kappa(x)$-basis $\Sigma^N Y^M$ of $\kappa(x)[Y_1, ..., Y_r]$, where N ranges over $\Gamma' = \mathbb{N}^m$ and M ranges over

$$\Gamma'' = \{M \in \mathbb{N}^r | M_1 < q_1, ..., M_m < q_m\}.$$

We take the order on $\Gamma := \Gamma' \times \Gamma''$ defined by $(N, M) \leq (N', M')$ if $qN + |M| < qN' + |M'|$ or $qN + |M| = qN' + |M'|$ and $(N, M) \leq_{\text{lex}} (N', M')$. As before this is a wellorder and by restricting it onto $\Gamma' \subseteq \Gamma$ we get the same order on $\Lambda' = \Gamma'$ as in (i).

(iii) Finally we get a homogeneous $\kappa(x)$-basis $\Sigma^N Y^M Z^L$ of $\kappa(x)[Y, Z]$, where N ranges over Γ', M ranges over Γ'' and L ranges over $\Gamma''' := \mathbb{N}_0^s$. We take the order on $\Gamma^* := \Gamma' \times \Gamma'' \times \Gamma'''$ defined by $(N, M, L) \leq (N', M', L')$ if $qN + |M| + |L| < qN' + |M'| + |L'|$ or $qN + |M| + |L| = qN' + |M'| + |L'|$ and $(N, M, L) \leq_{\text{lex}} (N', M', L')$. This is a wellorder and by restricting it onto $\Gamma = \Gamma \times 0 \subseteq \Gamma^*$ we obtain the order of (ii).

When we take exponents of polynomials with respect to these ordered bases as in (1.2.4) we denote the exponents with $\exp_\Lambda, \exp_{\Lambda'}, \exp_\Gamma$ and so on. We make the same convention and use the same notation for the vector spaces of initial terms $\text{in}_\Lambda, ...$.

It would be nice if the orders just introduced would be monomial orders. But this is not the case as the following example shows: Take $\sigma_1 = X_1^2, \sigma_2 = X_2^2$ in characteristic two. Then $X_1 \geq X_2$ but $X_1 X_2 \leq X_2^2 = \sigma_2$.

For the following lemma we need the property $\text{In}_{x,Z}(\mathcal{Q}_x) \subseteq \kappa(x)[Z]$ of our dissecting variables.

Lemma (10.4.2). *For $0 \neq f \in \mathcal{U}_x$ we have*

$$\exp_{\Lambda'}(f) = \exp_{\Gamma^*}(\text{In}_x(f)) \in \Lambda' = \Gamma'.$$

Proof. Since In_x is injective, we have $0 \neq \text{In}_x(f)$ and therefore can form the exponent of $\text{In}_x(f)$. The orders on Λ and Γ^* use the degree first and In_x is graded, so we can assume that f is homogeneous of degree d. Write $f = \sum_{N \in \Lambda', qN = d} a_N \sigma^N$ with

$a_N \in k$. Then

$$\mathrm{In}_x(f) = \sum_{N\in\Lambda', qN=d} a_N(\Sigma + \mathrm{In}_{x,Z}(\sigma))^N =$$

$$= \sum_{N\in\Lambda'} a_N \left(\Sigma^N + \sum_{N'\in\Lambda', N'<_c N} \binom{N}{N'} \Sigma^{N'} \mathrm{In}_{x,Z}(\sigma)^{N-N'} \right).$$

For fixed N we have $N' <_{\mathrm{lex}} N$ for N' as in the last sum and since $\mathrm{In}_{x,Z}(\sigma)^{N-N'}$ lies in $\kappa(x)[Z]$ we have

$$\exp_{\Gamma^*}(\Sigma + \mathrm{In}_{x,Z}(\sigma))^N = (N, 0, 0).$$

The claim follows since $\Lambda' = \Gamma'$ both are equipped with the lexicographic order. $\square$

For the rest of this section let us fix a homogeneous ideal $I \subseteq S$ defining a cone $C := \mathrm{Spec}(S/I) \subseteq V$. We also use the ideal $I' := \mathcal{U}_x \cap I$ and the homogeneous ideal J of $\mathrm{gr}_{V,x}$ defining the cone $C_{C,x}$ in $T_{V,x} = \mathrm{Spec}(\mathrm{gr}_{V,x})$.

Proposition (10.4.3). *If I is defined by homogeneous polynomials in $\mathcal{U}_x$, i.e.*

(10.4.3.A) $$S \cdot (\mathcal{U}_x \cap I) = I,$$

then

(10.4.3.B) $$\exp_\Lambda(I) = \exp_{\Lambda'}(I') \times \Lambda'',$$

(10.4.3.C) $$\exp_{\Gamma^*}(J) = \exp_{\Lambda'}(I') \times \Gamma'' \times \Gamma'''.$$

Whenever $\alpha = (\alpha_1, ..., \alpha_l) \subseteq \mathcal{U}_x$ is a homogeneous system of generators of I, the transformed ideal J can be computed by

(10.4.3.D) $$J = \langle \mathrm{In}_x(\alpha) \rangle_{\mathrm{gr}_{V,x}}.$$

Proof. Note that $x \in C$ since $(\mathcal{U}_x)_+$ lies in the prime ideal $\mathfrak{p} \subseteq S$ corresponding to x and therefore $I = S(\mathcal{U}_x \cap I) \subseteq \mathfrak{p}$. The right side of (10.4.3.B) is contained in the left side since for $M \in \Lambda''$ and $0 \neq f' \in I'$ we have $\exp_\Lambda(f'X^M) = (\exp_{\Lambda'}(f'), M)$. Let $0 \neq f \in I$. Then we can expand $f = \sum_{M\in\Lambda''} X^M g_M$ with $g_M \in I'$. Then we find $M \in \Lambda''$ with $\exp_\Lambda(f) = (\exp_{\Lambda'}(g_M), M)$. This shows (10.4.3.B). $SI' = S\langle\alpha\rangle_{\mathcal{U}_x}$ and therefore by (1.4.6) already $I' = \langle\alpha\rangle_{\mathcal{U}_x}$. There exists a homogeneous system of generators $\beta = (\beta_N)_{N\in\exp(I')} \subseteq \mathcal{U}_x$ of I' with $\exp(\beta_N) = N \in \Lambda'$ for all $N \in \exp(I')$. Then also $\langle\beta\rangle_S = I$ and since $\mathrm{In}_x : \mathcal{U}_x \to \mathrm{gr}_{V,x}$ is a ring homomorphism, we have $\langle \mathrm{In}_x(\alpha)\rangle_{\mathrm{gr}_{V,x}} = \langle \mathrm{In}_x(\beta)\rangle_{\mathrm{gr}_{V,x}}$. Therefore it suffices to prove (10.4.3.D) for β. It is clear that $\mathrm{In}_x(\beta) \subseteq J$ and by (10.4.2) we get $\exp_{\Lambda'}(I') \times 0 \times 0 \subseteq \exp_{\Gamma^*}(J)$. Since $\mathrm{In}_x(\beta_N) \in \kappa(x)[\Sigma, Z]$ we have $\exp_{\Gamma^*}(Y^M Z^L \mathrm{In}_x(\beta_N)) = (N, M, L)$ for $M \in \Gamma'', L \in \Gamma'''$. Let $J' := \langle \mathrm{In}_x(\beta)\rangle_{\mathrm{gr}_{V,x}}$. We have $J' \subseteq J$ and

$$\exp_{\Lambda'}(I') \times \Gamma'' \times \Gamma''' \subseteq \exp_{\Gamma^*}(J') \subseteq \exp_{\Gamma^*}(J).$$

With the notion of Hilbert series from 1.2 we have

$$H(S/I) \overset{(1.2.7)}{=} H(S) - H(I) = H(S) - H(\mathrm{in}_\Lambda(I)) \overset{(10.4.3.B)}{=}$$

$$= (1-T)^{-n} - \sum_{(N,M)\in \exp_{\Lambda'}(I')\times\Lambda''} T^{qN+|M|} \overset{(4.5.10)}{=}$$

$$= (1-T)^{-n} - \left(\sum_{N\in\exp(I')} T^{qN}\right)\cdot(1-T)^{-n}\prod_{i=1}^{m}(1-T^{q_i}).$$

(10.4.3.A) means that $B_x \subseteq \mathrm{Rid}(C)$. (9.2.4) therefore yields with $\mathrm{tr.deg}(\kappa(x)/k) = \dim(V) - \dim(T_{V,x}) = n - r - s$

$$H(\mathrm{gr}_{V,x}/J) = (1-T)^{\mathrm{tr.deg}(\kappa(x)/k)} H(S/I) =$$

$$= (1-T)^{-r-s} - \left(\sum_{N\in\exp(I')} T^{qN}\right)\cdot(1-T)^{-r-s}\prod_{i=1}^{m}(1-T^{q_i}).$$

We also have

$$H(\mathrm{gr}_{V,x}/J) \le H(\mathrm{gr}_{V,x}/J') = H(\mathrm{gr}_{V,x}) - H(J') = H(\mathrm{gr}_{V,x}) - H(\mathrm{in}_{\Gamma^*}(J')) \le$$

$$\le H(\mathrm{gr}_{V,x}) - H(\langle \Sigma^N X^M Z^L | (N,M,L)\in \exp_{\Lambda'}(I')\times\Gamma''\times\Gamma'''\rangle_{\kappa(x)}) =$$

$$= (1-T)^{-r-s} - \left(\sum_{N\in\exp(I')} T^{qN}\right)\cdot(1-T)^{-r}\prod_{i=1}^{m}(1-T^{q_i})\cdot(1-T)^{-s} =$$

$$= H(\mathrm{gr}_{V,x}/J).$$

Therefore $H(\mathrm{gr}_{V,x}/J) = H(\mathrm{gr}_{V,x}/J')$ and we get $J = J'$ and proved (10.4.3.D). Then

$$\mathrm{in}_{\Gamma^*}(J) = \mathrm{in}_{\Gamma^*}(J') = \langle \Sigma^N X^M Z^L | (N,M,L)\in \exp_{\Lambda'}(I')\times\Gamma''\times\Gamma'''\rangle_{\kappa(x)}$$

proves (10.4.3.C). □

We introduce differential operators that allow us to compute the ridges of the cones C and $C_{C,x}$.

(10.4.4) (i) As in (3.3.2) we have the k-basis $(D_{N,M})_{(N,M)\in\Lambda}$ of differential operators of degree 0 on S with the property

$$D_{N,M}(\sigma^{N'}X^{M'}) = \binom{N'}{N}\binom{M'}{M}\sigma^{N'-N}X^{M'-M}$$

for $(N',M')\in\Lambda$. $D_{N,M}$ has order $qN+|M|$ and

$$D_{N,M} = D_{N,0}D_{0,M} = D_{0,M}D_{N,0}.$$

(ii) Simlilary we have the $\kappa(x)$-basis $(E_{N,M,L})_{(N,M,L)\in\Gamma^*}$ of differential operators of degree 0 on $\mathrm{gr}_{V,x}$ with the property

$$E_{N,M,L}(\Sigma^{N'}Y^{M'}Z^{L'}) = \binom{N'}{N}\binom{M'}{M}\binom{L'}{L}\Sigma^{N'-N}Y^{M'-M}Z^{L'-L}$$

for $(N',M',L') \in \Gamma^*$. $E_{N,M,L}$ has order $qN + |M| + |L|$ and

$$E_{N,M,L} = E_{N,0,0}E_{0,M,0}E_{0,0,L},$$

where the last three maps commute with each other.

All the information is carried over from S to $\mathrm{gr}_{V,x}$ via In_x on $\mathcal{U}_x$. To get a connection between the ridges in both cases we need a relation between the differential operators used to compute the ridge.

Lemma (10.4.5). *For all $N \in \Lambda'$ we have the relation on $\mathcal{U}_x$*

(10.4.5.A) $$\mathrm{In}_x \circ D_{N,0} = E_{N,0,0} \circ \mathrm{In}_x \,.$$

Proof. By k-linearity it suffices to check this for $\sigma^{N'}$ with $N' \in \Lambda'$. On the one side we find

$$\mathrm{In}_x(D_{N,0}(\sigma^{N'})) = \mathrm{In}_x\left(\binom{N'}{N}\sigma^{N'-N}\right) = \binom{N'}{N}\mathrm{In}_x(\sigma)^{N'-N} =$$

$$= \sum_{N''\in\Lambda'}\binom{N'}{N}\binom{N'-N}{N''}\Sigma^{N'-N-N''}\,\mathrm{In}_{x,Z}(\sigma)^{N''}$$

and on the other side we get

$$E_{N,0,0}(\mathrm{In}_x(\sigma^{N'})) = E_{N,0,0}\left(\sum_{N''\in\Lambda'}\binom{N'}{N''}\Sigma^{N'-N''}\,\mathrm{In}_{x,Z}(\sigma)^{N''}\right) =$$

$$= \sum_{N''\in\Lambda'}\binom{N'}{N''}\binom{N'-N''}{N}\Sigma^{N'-N''-N}\,\mathrm{In}_{x,Z}(\sigma)^{N''}$$

and the relation for integers $n, n', n'' \in \mathbb{N}$

$$\binom{n'}{n}\binom{n'-n}{n''} = \frac{n'!}{n!n''!(n'-n-n'')!} = \binom{n'}{n''}\binom{n'-n''}{n}$$

that immediately generalizes to multiindex binomial coefficients finishes the proof. □

From now on denote with $\mathcal{U}_I$ the ring of invariants of $\mathrm{Rid}(C)$ in S and with $\mathcal{U}_J$ the ring of invariants of $\mathrm{Rid}(C_{C,x})$ in $\mathrm{gr}_{V,x}$. Λ' defines a monomial order on $\mathcal{U}_x$, where σ are the variables of $\mathcal{U}_x$. As seen in (1.3.10) and (1.3.11) we find a homogeneous reduced Gröbner basis $\gamma = \langle\gamma_1, ..., \gamma_l\rangle$ of $I' \subseteq \mathcal{U}_x$ with respect to this order. We fix this basis.

Lemma (10.4.6). *Assume that $S \cdot (\mathcal{U}_x \cap I) = I$. Then $I' = \langle \gamma \rangle_{\mathcal{U}_x}$, $I = \langle \gamma \rangle_S$ and*

$$\mathcal{U}_I = k[D_{N,0}(\gamma_i) | N \in \Lambda', 1 \le i \le l]. \tag{10.4.6.A}$$

Proof. Let $1 \le i \le l$ and $(N, M) \in \exp_\Lambda(I)$ with $qN + |M| < \deg(\gamma_i)$. By (10.4.3.B) we have $N \in \exp(I')$ and since γ is a Gröbner basis of I' we find $1 \le j \le l$ with $N \ge_c \exp(\gamma_j)$. Assume that $D_{N,M}(\gamma_i) \ne 0$. Then $M = 0$ since $\gamma_i \in \mathcal{U}_x$ and in γ_i there appears a monomial $\sigma^{N'}$ with $N' \ge_c N$. But now $N' \ge_c \exp(\gamma_j)$ and the Gröbner basis γ is reduced, hence we must have $i = j$. This is absurd since $\deg(\gamma_i) > qN + |M| = qN \ge \deg(\gamma_j)$. Therefore γ is a σ-Giraud basis (see definition (6.1.4) (ii)) and we get (10.4.6.A) from (6.1.8) (ii). □

The basis γ can not only be used to compute the ridge of C, but also to compute the ridge of $C_{C,x}$. This is a central step on the way to prove the main theorems.

Lemma (10.4.7). *If $S \cdot (\mathcal{U}_x \cap I) = I$, then*

$$\mathcal{U}_J = \kappa(x)[E_{N,0,L}(\mathrm{In}_x(\gamma_i)) | N \in \Gamma', L \in \Gamma''', 1 \le i \le l]. \tag{10.4.7.A}$$

Proof. Since $I = \langle \gamma \rangle_S$, we get from (10.4.3.D) that $J = \langle \mathrm{In}_x(\gamma) \rangle_{\mathrm{gr}_{V,x}}$. For a tuple $(N, M, L) \in \Gamma^*$ we have $E_{N,M,L}(\mathrm{In}_x(\gamma_i)) = 0$ if $M \ne 0$ since $\mathrm{In}_x(\gamma_i) \in \kappa(x)[\Sigma, Z]$. If $(N, 0, L) \in \exp_{\Gamma^*}(J)$ and $qN + |L| < \deg(\mathrm{In}_x(\gamma_i)) = \deg(\gamma_i)$, then we have by (10.4.3.C) that $N \in \exp(I')$ and $qN < \deg(\gamma_i)$. Therefore

$$E_{N,0,L}(\mathrm{In}_x(\gamma_i)) = E_{0,0,L}(E_{N,0,0} \circ \mathrm{In}_x)(\gamma_i) \overset{(10.4.5)}{=} E_{0,0,L}(\mathrm{In}_x(D_{N,0}(\gamma_i))) = 0$$

as in the proof of (10.4.6). This shows that $\mathrm{In}_x(\gamma)$ is a Σ-Giraud basis and (10.4.7.A) follows from (6.1.8) (ii). □

Let us further fix a homogeneous $k[F]$-basis $\rho = (\rho_1, ..., \rho_t)$ of $\mathcal{U}_I \cap L$. Then $\mathcal{U}_I = k[\rho]$.

Lemma (10.4.8). *If $S \cdot (\mathcal{U}_x \cap I) = I$, then*

$$\mathcal{U}_J = \kappa(x)[E_{0,0,L}(\mathrm{In}_x(\rho_j)) | L \in \Gamma''', 1 \le j \le t]. \tag{10.4.8.A}$$

Proof. By (10.4.6.A) we have $\rho_j \in \mathcal{U}_I = k[D_{N,0}(\gamma_i) | N \in \Lambda', 1 \le i \le l]$ and therefore can find polynomials $p_1, ..., p_t \in k[T_N^i]_{N \in \Lambda', 1 \le i \le l}$ with $\rho_j = p_j(D_{N,0}(\gamma_i))$ where we plug in $D_{N,0}(\gamma_i)$ for T_N^i. $S \cdot (\mathcal{U}_x \cap I) = I$ implies $\mathcal{U}_I \subseteq \mathcal{U}_x$. Since In_x is a ring homomorphism on $\mathcal{U}_x$ we find with (10.4.5.A) and (10.4.7.A)

$$\mathrm{In}_x(\rho_j) = \mathrm{In}_x(p_j(D_{N,0}(\gamma_i))) = p_j(\mathrm{In}_x(D_{N,0}(\gamma_i))) = p_j(E_{N,0,0}(\mathrm{In}_x(\gamma_i))) \in \mathcal{U}_J.$$

Since $\mathcal{U}_J$ is generated by additive polynomials and $E_{0,0,L}$ is of degree zero, we have by (3.4.3) also $E_{0,0,L}(\mathrm{In}_x(\rho_j)) \in \mathcal{U}_J$; this shows $\mathcal{U}_J \supseteq \kappa(x)[E_{0,0,L}(\mathrm{In}_x(\rho_j)) | L \in \Gamma''', 1 \le j \le t] =: \mathcal{U}'_J$. On the other hand $\gamma_i \in k[\rho]$ shows that $\mathrm{In}_x(\gamma_i) \in \kappa(x)[\mathrm{In}_x(\rho_j) | j = 1, ..., t] \subseteq \mathcal{U}'_J$. Because $J = \langle \mathrm{In}_x(\gamma) \rangle_{\mathrm{gr}_{V,x}}$ ((10.4.3.D)), we have

$$\mathrm{gr}_{V,x} \cdot (\mathcal{U}'_J \cap J) = J. \tag{10.4.8.B}$$

$\mathrm{In}_x(\rho_j) \in \kappa(x)[\Sigma, Z]$ shows that $E_{N,M,L}(\mathrm{In}_x(\rho_j)) = 0$ whenever $M \ne 0$. Further

$E_{N,0,L}(\mathrm{In}_x(\rho_j)) = E_{0,0,L}E_{N,0,0}(\mathrm{In}_x(\rho_j)) = E_{0,0,L}\,\mathrm{In}_x(D_{N,0}(\rho_j))$ is zero if $N \neq 0$ and $qN \neq \deg(\rho_j)$ ($D_{N,0}$ is a differential operator of order qN and ρ_j is an additive polynomial) and lies in $\kappa(x)$ if $qN = \deg(\rho_j)$. Hence $\mathcal{U}'_J$ is generated by the elements one gets if all differential operators of degree zero on $\mathrm{gr}_{V,x}$ are applied to the polynomials $\mathrm{In}_x(\rho)$. With criterion (3.4.3) (vii) we see that $\mathcal{U}'_J$ is generated by homogeneous additive polynomials. (10.4.8.B) therefore implies $\mathcal{U}_J \subseteq \mathcal{U}'_J$ which finishes the proof. □

We now take into consideration the invariant R, the dimension of the ridge, and describe what it means if this invariant does not change. Let $d = \mathrm{tr.deg}(\kappa(x)/k)$.

Proposition (10.4.9). *If $S \cdot (\mathcal{U}_x \cap I) = I$, then the following are equivalent:*

(i) $R^{(d)}_{C,x} = R(C)$.

(ii) $\dim(\mathcal{U}_J) = \dim(\mathcal{U}_I)$.

(iii) $\mathrm{In}_x(\rho_j)$ *is additive for* $j = 1, ..., t$.

Proof. The equivalence of (i) and (ii) follows from

$$\dim \mathrm{Rid}(\mathrm{Spec}(\mathrm{gr}_{V,x}/J)) + \mathrm{tr.deg}(\kappa(x)/k) - \dim \mathrm{Rid}(\mathrm{Spec}(S/I)) =$$

$$= \dim(\mathrm{gr}_{V,x}) - \dim(\mathcal{U}_J) + \mathrm{tr.deg}(\kappa(x)/k) - (\dim(S) - \dim(\mathcal{U}_I)) = \dim(\mathcal{U}_I) - \dim(\mathcal{U}_J).$$

We have $\mathcal{U}_I = k[\rho_1, ..., \rho_t]$ and $\dim(\mathcal{U}_I) = t$. Assume that (iii) holds. Since applying differential operators to an additive polynomial only yields multiples of this polynomial or elements of the field, we get from (10.4.8.A) that

$$\mathcal{U}_J = \kappa(x)[\mathrm{In}_x(\rho_1), ..., \mathrm{In}_x(\rho_t)].$$

We make use of the dissecting variables (Y, Z) at x and have $\mathrm{In}_x(\rho_j) = \mathrm{In}_{x,Y}(\rho_j) + \mathrm{In}_{x,Z}(\rho_j)$. $\mathrm{In}_{x,Y}(\rho_j)$ is additive by the defining properties of dissecting variables, hence also $\mathrm{In}_{x,Z}(\rho_j)$ is additive. Since $\rho_1, ..., \rho_t$ are $k[F]$-independent, we see by (10.3.5) (iii) that $\mathrm{In}_{x,Y}(\rho_1), ..., \mathrm{In}_{x,Y}(\rho_t)$ are $\kappa(x)[F]$-independent. Since the $\mathrm{In}_{x,Z}(\rho_j)$ lie in $\kappa(x)[Z]$, also $\mathrm{In}_x(\rho_1), ..., \mathrm{In}_x(\rho_t)$ are $\kappa(x)[F]$-independent. This shows $\dim(\mathcal{U}_J) = t$. Assume that (iii) does not hold. For $j \in \{1, ..., t\}$ let $\mathcal{U}_j$ be the ring of invariants of $\mathrm{Spec}(\mathrm{gr}_{V,x}/\langle \mathrm{In}_x(\rho_j)\rangle)$ in $\mathrm{gr}_{V,x}$. In the proof of (10.4.8) we saw that $\mathcal{U}_J$ contains all applications of a differential operator of degree zero on $\mathrm{In}_x(\rho_j)$. Therefore $\mathcal{U}_j \subseteq \mathcal{U}_J$ ($\mathrm{In}_x(\rho_j)$ is necessarily a Giraud basis of the ideal it generates). $\mathrm{In}_x(\rho_j)$ lies in the algebra $\kappa(x)[\mathrm{In}_{x,Y}(\rho_j), Z]$ which is generated by homogeneous additive polynomials, hence $\mathcal{U}_j \subseteq \kappa(x)[\mathrm{In}_{x,Y}(\rho_j), Z]$. Since the polynomials $\mathrm{In}_{x,Y}(\rho)$ are $\kappa(x)[F]$-independent, we must have $\mathrm{In}_{x,Y}(\rho_j) \neq 0$. Therefore $\mathrm{In}_x(\rho_j)$ does not lie in $\kappa(x)[Z]$ and $\mathcal{U}_j$ must contain a homogeneous additive polynomial in which $\mathrm{In}_{x,Y}(\rho_j)$ appears. If we choose such a polynomial of minimal degree, it must have the same degree as $\mathrm{In}_x(\rho_j)$. Therefore $\mathcal{U}_j$ contains a homogeneous additive polynomial $\tau_j = \mathrm{In}_{x,Y}(\rho_j) + \tau'_j$ with $\tau'_j \in \kappa(x)[Z]$. We must have $\dim \mathcal{U}_j \geq 1$. Assume that $\dim \mathcal{U}_j = 1$. Then $\mathcal{U}_j$ is generated by a homogeneous additive polynomial that is up to a scalar multiple a root of τ_j. But since $\deg(\mathrm{In}_x(\rho_j))$ is a p-power, this polynomial already must be τ_j. In particular $\mathrm{In}_x(\rho_j) = \tau_j$ must be additive. This cannot be true for all

j and we find some $h \in \{1, ..., t\}$ with $\dim \mathcal{U}_h \geq 2$. So $\mathcal{U}_h$ contains a $\kappa(x)[F]$-independent polynomial to τ_h. Since $\mathcal{U}_h \subseteq \kappa(x)[\mathrm{In}_{x,Y}(\rho_j), Z]$, we can find a polynomial $0 \neq \tau \in \mathcal{U}_h \cap \kappa(x)[Z]$. Altogether we have $\tau_1, ..., \tau_t, \tau \in \mathcal{U}_J$. If we had an equation $c\tau + \sum_j c_j \tau_j = 0$ with $c, c_j \in \kappa(x)[F]$, we find by setting $Z = 0$ an equation $\sum_j c_j \mathrm{In}_{x,Y}(\rho_j) = 0$. This implies $c_1 = \cdots = c_t = 0$ and since $\tau \neq 0$ also c must be 0. Therefore $\mathcal{U}_J$ contains $t+1$ $\kappa(x)[F]$-independent homogeneous additive polynomials and we have $\dim \mathcal{U}_J \geq t+1 > t = \dim \mathcal{U}_I$. □

Theorem (10.4.10). *The central results for cones are the following:*

(i) *For a homogeneous polynomial $f \in S$ let $X := \mathrm{Spec}(S/\langle f \rangle) \subseteq V$. Then we have*

$$f \in \mathcal{V}_x \qquad \Longleftrightarrow \qquad HR^{(d)}_{X,x} = HR(X).$$

(ii) *We have the following equivalence:*

$$F_x \subseteq \mathrm{Rid}(C) \qquad \Longleftrightarrow \qquad HR^{(d)}_{C,x} = HR(C).$$

Proof. We show (ii). Assume that $HR^{(d)}_{C,x} = HR(C)$. Then $H^{(d)}_{C,x} = H(C)$ implies by (9.2.4) that $B_x \subseteq \mathrm{Rid}(C)$, i.e. $S \cdot (\mathcal{U}_x \cap I) = I$. Therefore we can apply (10.4.9) and get from $R^{(d)}_{C,x} = R(C)$ that $\mathrm{In}_x(\rho_1), ..., \mathrm{In}_x(\rho_t)$ are additive. By definition this means $\rho_1, ..., \rho_t \in \mathcal{P}_x$. Thus $\mathcal{U}_I = k[\rho] \subseteq k[\mathcal{P}_x] = \mathcal{V}_x$ and we get $F_x \subseteq \mathrm{Rid}(C)$. Assume on the other hand that $F_x \subseteq \mathrm{Rid}(C)$. Then $B_x \subseteq F_x \subseteq \mathrm{Rid}(C)$. By (9.2.4) again we get $H^{(d)}_{C,x} = H(C)$. Since $\mathcal{U}_I \subseteq \mathcal{V}_x$ we have $\rho_1, ..., \rho_t \in \mathcal{P}_x$ and by (10.4.9) we get $R^{(d)}_{C,x} = R(C)$. To derive (i) from this we take $I = \langle f \rangle$. Note that $f \in \mathcal{V}_x$ is equivalent to $\mathcal{U}_I \subseteq \mathcal{V}_x$ which means $F_x \subseteq \mathrm{Rid}(C)$. □

Note that (i) of the theorem answeres the questions (Q1) and (Q2) in 10.2 positively.

Corollary (10.4.11). *If $HR^{(d)}_{C,x} = HR(C)$ and F_x is a vector space, then $x \in \mathrm{Dir}(C)$.*

Proof. From (10.4.10) we get $F_x \subseteq \mathrm{Rid}(C)$. Since F_x is a vector space, we must have $F_x \subseteq \mathrm{Dir}(C)$. Note that $x \in F_x$. □

10.5 Proof of the main theorems I

We begin with the proof of the main theorems and will be left in the end with two open questions: Are all refined Hironaka schemes up to dimension 5 resp. $2p-1$ linear and do we find dissecting variables always when we need them? We will give the answer to this in 10.8 f..

Let us recall the main theorems: For a locally noetherian scheme X and a closed subscheme $D \subseteq X$ which is permissible at a point $x \in D$ let $\pi : X' \to X$ be the blow up in the center D. Take a point x' in the fiber $\pi^{-1}(x)$, set $d := \mathrm{tr.deg}(\kappa(x')/\kappa(x))$ and $C := C_{X,x}/T_{D,x}$. Assume that $\kappa(x)$ has positive characteristic.

Main Theorem A. *If* $HR^{(d)}_{X',x'} = HR_{X,x}$ *and also* $\dim X \leq 5$ *or* $\dim X \leq 2\operatorname{char}(\kappa(x))-1$, *then* $C_{X,x}/T_{D,x}$ *is invariant under the action of the refined Hironaka scheme* $F_{T_{X,x}/T_{D,x},x'}$.

Main Theorem B. *Let* F *be a refined Hironaka scheme over a field* k *of positive characteristic. If* $\dim F \leq 5$ *or* $\dim F \leq 2\operatorname{char}(k) - 1$, *then* F *is linear.*

Main Theorem C. *If* $HR^{(d)}_{X',x'} = HR_{X,x}$ *and also* $\dim X \leq 5$ *or* $\dim X \leq 2\operatorname{char}(\kappa(x)) - 1$, *then* $x' \in \mathbb{P}(\operatorname{Dir}(C_{X,x}/T_{D,x}))$.

To see what one could do to prove the main theorems also in higher dimensions N we reformulate them to the following statements depending also on a prime p:

$(\mathrm{A})^p_{\mathrm{N}}$ If $HR^{(d)}_{X',x'} = HR_{X,x}$, $\dim C \leq N$ and $\operatorname{char}(\kappa(x)) = p$, then $F_{T_{X,x}/T_{D,x},x'} \subseteq \operatorname{Rid}(C)$.

$(\mathrm{B})^p_{\mathrm{N}}$ A refined Hironaka scheme of dimension $\leq N$ over a field of characteristic p is linear.

$(\mathrm{C})^p_{\mathrm{N}}$ If $HR^{(d)}_{X',x'} = HR_{X,x}$, $\dim C \leq N$ and $\operatorname{char}(\kappa(x)) = p$, then $x' \in \mathbb{P}(\operatorname{Dir}(C))$.

The main theorems will be proved when we showed $(\mathrm{A})^p_5$, $(\mathrm{B})^p_5$, $(\mathrm{C})^p_5$, $(\mathrm{A})^p_{2\mathrm{p}-1}$, $(\mathrm{B})^p_{2\mathrm{p}-1}$ and $(\mathrm{C})^p_{2\mathrm{p}-1}$ for all p since $\dim C \leq \dim X$. Let us consider also the following statement:

$(\mathrm{D})^p_{\mathrm{N}}$ If x is a homogeneous point in a vector space V over a field k with $\dim B_x \leq N$ and $\operatorname{char}(k) = p$, then x is dissected.

After we proved the following implications, it will remain to verify $(\mathrm{B})^p_5$, $(\mathrm{D})^p_5$, $(\mathrm{B})^p_{2\mathrm{p}-1}$ and $(\mathrm{D})^p_{2\mathrm{p}-1}$ to obtain the main theorems.

(i) $(\mathrm{D})^p_{\mathrm{N}}$ implies $(\mathrm{A})^p_{\mathrm{N}}$.

(ii) $(\mathrm{A})^p_{\mathrm{N}}$ and $(\mathrm{B})^p_{\mathrm{N}}$ imply $(\mathrm{C})^p_{\mathrm{N}}$.

Note that $\dim V \leq N$ implies $\dim B_x \leq N$ and we will have proved that all homogeneous points of $\mathbb{A}^n_k$ are dissected for $n \leq 5$ or $n \leq 2\operatorname{char}(k) - 1$.

Proof of the implications. (i): Let $HR^{(d)}_{X',x'} = HR_{X,x}$ and also $\dim C \leq N$ and $\operatorname{char}(\kappa(x)) = p$. By examining (8.2.7.3) we see that also $HR^{(e)}_{C,x'} = HR(C)$, where we regard x' as a homogeneous point of C and $e = \operatorname{tr.deg}(\kappa(x')/\kappa(x)) = d + 1$ in this sense. x' is a homogeneous point of the vector space $V = T_{X,x}/T_{D,x}$. By (9.2.4) we have $B_{V,x'} \subseteq \operatorname{Rid}(C)$ and therefore $\dim B_{V,x'} \leq \dim C \leq N$. By $(\mathrm{D})^p_{\mathrm{N}}$ the point $x' \in V$ is dissected and we can use theorem (10.4.10) which finishes the proof. (ii): $F_{V,x'} \subseteq \operatorname{Rid}(C)$ by $(\mathrm{A})^p_{\mathrm{N}}$ and since $\dim F_{V,x'} \leq \dim C \leq N$, the refined Hironaka scheme is linear by $(\mathrm{B})^p_{\mathrm{N}}$. Hence $F_{V,x'} \subseteq \operatorname{Dir}(C)$ and we have $x' \in \mathbb{P}(F_{V,x'}) \subseteq \mathbb{P}(\operatorname{Dir}(C))$. □

The statements $(\mathrm{B})^p_{\mathrm{N}}$ and $(\mathrm{D})^p_{\mathrm{N}}$ together can also be expressed as:

$(\mathrm{BD})^p_{\mathrm{N}}$ Let x be a homogeneous point in a vector space V over a field k of characteristic p. If $\dim B_x \leq N$, then x is dissected and if $\dim F_x \leq N$, then F_x is linear.

To show $(\mathrm{BD})^p_{\mathrm{N}}$ one can assume the following additional hypotheses:

(E) $(\mathcal{Q}_x)_0 = 0$, i.e. $\mathcal{Q}_x$ (resp. $\mathcal{U}_x$) does not contain a non zero linear polynomial.

(F) $\mathrm{Dir}(B_x) = 0$.

With the additional conditions (E) and (F) we will have to check $(\mathrm{BD})^p_5$ and $(\mathrm{BD})^p_{2\mathrm{p}-1}$ only for a handful of examples of Hironaka schemes. We prove $(\mathrm{BD})^p_5$ in 10.8 and $(\mathrm{BD})^p_{2\mathrm{p}-1}$ in 10.9. This will finish the proof of the main theorems.

Proof. First we reduce to (E). Let $V = \mathrm{Spec}(S), S = \mathrm{Sym}_k(\mathfrak{V})$ and $\mathfrak{W} := (\mathcal{Q}_x)_0 \subseteq \mathfrak{V}_1$. Then $W := \mathrm{Spec}(S/\langle \mathfrak{W} \rangle_S)$ is a vector space contained in V. The prime ideal of S defining $x \in V$ contains $\mathfrak{W}$ and therefore $x \in W$. By (9.1.8) we have $B_{V,x} = B_{W,x}$ and in (10.3.9) we saw that also $F_{V,x} = F_{W,x}$. Because of $\mathcal{Q}_{W,x} \cong \mathcal{Q}_{V,x}/\langle \mathfrak{W} \rangle_{k[F]}$ we have $(\mathcal{Q}_{W,x})_0 = 0$ and condition (E) holds for $x \in W$. Assume that $\dim F_{V,x} \leq N$. Then also $\dim F_{W,x} \leq N$. If $(\mathrm{B})^p_{\mathrm{N}}$ holds under the condition (E), then $F_{V,x} = F_{W,x}$ is linear. Assume on the other hand that $\dim B_{V,x} \leq N$. Thus $\dim B_{W,x} \leq N$. If $(\mathrm{D})^p_{\mathrm{N}}$ holds under the condition (E), then $x \in W$ is dissected. With (10.3.9) also $x \in V$ is dissected.
Now we reduce further to (F) and have to keep the condition (E) unchanged. Let us start over with a point x of a vector space V satisfying (E). Define $W := V/\mathrm{Dir}(B_x)$ and consider the canonical projection $\pi : V \to W$, let $y = \pi(x)$. Explicitely $W = \mathrm{Spec}(k[\mathfrak{T}(I)])$ for $I = S \cdot (\mathcal{U}_x)_+$. Since $S(k[\mathfrak{T}(I)] \cap I) = I$, we have by (6.1.9) $\mathcal{U}_{V,x} \subseteq k[\mathfrak{T}(I)]$. From (10.3.10) (i) we get $\mathcal{U}_{V,x} = \mathcal{U}_{W,y}$. Hence conditions (E) and (F) hold for $y \in W$. Let $\dim B_{V,x} \leq N$. Then we have $\dim B_{W,y} = \dim W - \dim \mathcal{U}_{W,y} \leq \dim V - \dim \mathcal{U}_{V,x} = \dim B_{V,x} \leq N$. If $(\mathrm{D})^p_{\mathrm{N}}$ holds under the conditions (E) and (F), then $y \in W$ is dissected. (10.3.10) (ii) shows that $x \in V$ is dissected. Let on the other hand $\dim F_{V,x} \leq N$. Then $\dim B_{V,x} \leq \dim F_{V,x} \leq N$ and $y \in W$ is dissected as just mentioned. By (10.3.10) also $\mathcal{V}_{W,y} = \mathcal{V}_{V,x}$ and $\dim F_{W,x} = \dim W - \dim \mathcal{V}_{W,y} \leq \dim V - \dim \mathcal{V}_{V,x} = \dim F_{V,x} \leq N$. If $(\mathrm{B})^p_{\mathrm{N}}$ holds under the conditions (E) and (F), then $F_{W,y}$ is linear. Since $\mathcal{V}_{W,y} = \mathcal{V}_{V,x}$, also $F_{V,x}$ is a vector space. □

10.6 The setting for the examples

For an easy handling of examples we establish some technical tools. In the next section we will look at a special class of examples that are particularly easy to compute. All the examples necessary to finish our proof belong to this class, except for one.

Let us first mention Oda's characterization of Hironaka schemes. For this we have to explain some terminology for groups: The Dieudonné module of an additive group G is the $\mathrm{Hom}_{\mathrm{Gr}}(\mathbb{A}^1, \mathbb{A}^1) \cong k[F]$-module $\mathrm{Hom}_{\mathrm{Gr}}(G, \mathbb{A}^1)$ (morphisms of groups). The group $G = \mathrm{Spec}(S/\langle Q \rangle)$ for a graded $k[F]$-module $Q \subseteq L$ has the Dieudonné module L/Q, where L are the additive polynomials in a polynomial ring S. An invariant of a graded $k[F]$-module N is its exponent

$$\exp(N) := \min\{i \in \mathbb{N} | \dim_k N_i = \dim_k N_j \text{ for all } j \geq i\}.$$

For a minimal system of homogeneous generators of the module $N \subseteq L$ the number $\exp(N)$ is the maximum of the degrees of these generators. This also defines an invariant for a group $G = \operatorname{Spec}(S/\langle Q\rangle)$ by $\exp(G) = \exp(Q) = \exp(L/Q)$.

Theorem (10.6.1) ([Od, 2.5])**.** *Let N be a graded $k[F]$-submodule of L. L/N is the Dieudonné module of a Hironaka scheme of exponent e if and only if the following hold:*

(i) $N_e \subsetneq L_e$,

(ii) $k(FL_{e-1} \cap N_e) \subsetneq N_e$ *if* $e > 0$,

(iii) $\mathcal{N}_e\mathcal{D}_e(N_e) = N_e$,

(iv) $N = \mathcal{R}(\langle N_e\rangle_{k[F]})$ *($\mathcal{R}$ is the radical as in (6.3.2)).*

Moreover $\operatorname{rad}(\langle \mathcal{D}_e(N_e)\rangle_S)$ *is the generic point of the locus of such points $\mathfrak{p}$ that for each i the hypersurface defined by every element in N_i has multiplicity $\geq p^i$ at $\mathfrak{p}$. The Hironaka scheme associated to* $\operatorname{rad}(\langle \mathcal{D}_e(N_e)\rangle_S)$ *has the Dieudonné module L/N.*

The operators $\mathcal{D}_e, \mathcal{N}_e$ are defined in the following way for a k-subspace $Q_e \subseteq L_e$:

$$\mathcal{D}_e(Q_e) = \operatorname{Diff}_{\mathbb{Z}}^{\leq p^e-1}(k)(Q_e),$$

$$\mathcal{N}_e(Q_e) = \{f \in L_e | \mathcal{D}_e(\langle f\rangle) \subseteq Q_e\},$$

where the differential operators of the field k act only on the coefficients of the polynomials in L_e. One of the assets of (10.6.1) is the fact that in order to classify Hironaka schemes one can forget about the point $x \in V$ at which a Hironaka scheme arises. If one looks for such a point, it can be obtained as the radical of the ideal $\langle \mathcal{D}_e(N_e)\rangle_S$.

(10.6.2) We fix the notations introduced in the following for all examples. A Hironaka scheme B in a vector space $V = \operatorname{Spec}(S), S = k[X], \operatorname{char}(k) = p$ will be defined in our discussion always via the equations σ defining its Dieudonné module L/N with $N = \langle \sigma\rangle_{k[F]}$. As mentioned, there always is a 'generic' point $x \in V$ such that $B_x = B$ and $y \in \overline{\{x\}}$ for all points $y \in V$ with $B = B_y$. For the proof of our main theorems we would have to deal with all the points y with $B = B_y$. We can eliminate this difficulty with the condition (F). We will also be able to avoid some unessential computations as we can assume that none of the σ is linear by condition (E). Our proceeding is the following: The generic point x, or rather its corresponding prime ideal $\mathfrak{p} \subseteq S$, is computed via $P_e = \mathcal{D}_e N_e$ and $\mathfrak{p} = \operatorname{rad}(\langle P_e\rangle_S)$, where e is the exponent of N and $\mathcal{D}_e = \operatorname{Diff}_{\mathbb{Z}}^{\leq p^e-1}(k)$. Then we will have to look for regular parameters at x, they will be denoted by ψ and ζ. With these we can find the initial forms of the σ, where we write Y resp. Z for the initial forms of ψ resp. ζ.

Whenever $P \subseteq L$ is a graded $k[F]$-module, the radical $\operatorname{rad}(\langle P\rangle_S)$ is a prime ideal in S. We get this as a byproduct of the next lemma which is useful to compute this prime.

Lemma (10.6.3). *Let $P \subseteq L$ be a graded $k[F]$-submodule. Then $\mathfrak{p} = \operatorname{rad}(\langle P \rangle_S) \subseteq S$ is a homogeneous prime ideal. For a well arranged $k[F]$-basis $\tau = (\tau_1, ..., \tau_m)$ of P, i.e.*

$$\tau_i = X_i^{p^{e_i}} + \sum_{j=m+1}^{n} a_{ij} X_j^{p^{e_i}}$$

with $e_1 \leq \cdots \leq e_m$ and $a_{ij} \in k$, the ideal $\mathfrak{p}$ is the kernel of the evaluation homomorphism

$$S \to \overline{k}[X_{m+1}, ..., X_n], \qquad f \mapsto f(a).$$

Here $\overline{k}$ is an algebraic closure of k and a has the following coordinates in the ring $\overline{k}[X_{m+1}, ..., X_n]$:

$$a = (X_1 - F^{-e_1}(\tau_1), ..., X_m - F^{-e_m}(\tau_m), X_{m+1}, ..., X_n).$$

Proof. For $f \in S$ we have $f \in \operatorname{rad}(\langle P \rangle_S)$ if and only if $F^e(f) \in \langle P \rangle_S$ for some $e \in \mathbb{N}$, where we can assume $e \geq e_m$. If $f = \sum_{M \in \mathbb{N}^n} c_M X^M$ is an expansion with $c_M \in k$, we get the following equality modulo $\langle P \rangle$:

$$F^e(f) = \sum_{M \in \mathbb{N}^n} F^e(c_M) \cdot \left(\prod_{i=1}^{m} F^{e-e_i}(X_i^{p^{e_i}} - \tau_i)^{M_i} \right) \cdot F^e(X_{m+1}^{M_{m+1}} \cdots X_n^{M_n}).$$

Here we understand $X_i^{p^{e_i}} - \tau_i$ as a polynomial in the variables $X_{m+1}, ..., X_n$. The right hand side lies in $\langle P \rangle_S$ resp. is zero in $S/\langle P \rangle_S$ if and only if it is zero expressed in the monomials in the variables $X_{m+1}, ..., X_n$ (see (1.4.4)). To check this we can apply F^{-e} and get $f(a)$ on the right side. Therefore f lies in $\operatorname{rad}(\langle P \rangle_S)$ iff $f(a) = 0$. $\square$

(10.6.4) On the k-vector space L_e we will use the nondegenerate symmetric bilinear form (cf. [Od]) $\langle ., . \rangle$ defined by

$$\langle X_i^{p^e}, X_j^{p^e} \rangle = \delta_{ij}.$$

For a subspace $N_e \subseteq L_e$ we can take its orthogonal complement $N_e^{\perp}$ with respect to this bilinear form and have

$$\dim N_e + \dim N_e^{\perp} = \dim L_e, \qquad (N_e^{\perp})^{\perp} = N_e.$$

Lemma (10.6.5). *If $\dim P_e^{\perp} = 1$, then the generic point x of B is the only homogeneous point on V with $B = B_x$.*

Proof. We use (10.6.3) for $P = \langle P_e \rangle_{k[F]}$, where we can find a basis of P in form of a well arranged system $\tau_1, ..., \tau_{n-1}$. The quotient field of the image of the evaluation map $S \to \overline{k}[X_n]$ has transcendence degree one over k. So $\operatorname{tr.deg}(\kappa(x)/k) = 1$ for the generic point x of B. Therefore x is a closed point of $\mathbb{P}V$. Since all other homogeneous points y with $B = B_y$ lie in $\overline{\{x\}}$, x itself is the only point with this property. $\square$

10.7 Strategy for a certain class of examples

To outline a strategy based on 10.6 in full detail, we restrict to examples satisfying additional conditions (G) and (H) below. These examples are easier to handle and we will apply the technique of this section to finish the proof of the main theorems. Let us fix the following notations: $\sigma = (\sigma_1, ..., \sigma_m)$ is a homogeneous $k[F]$-basis of N such that L/N is the Dieudonné module of a Hironaka scheme $B \subseteq V, V = \mathrm{Spec}(S), S = k[X_1, ..., X_n]$. Set $P_e = \mathcal{D}_e Q_e$ for the exponent e of N, $\mathfrak{p} = \mathrm{rad}(\langle P_e \rangle_S)$ and let $x \in V$ be the point corresponding to $\mathfrak{p}$. Throughout this section we assume that the following additional conditions hold:

(G) All σ_i are of degree $p = \mathrm{char}(k)$.

(H) $P_e^\perp = \langle \tau' \rangle$ with a single polynomial $\tau' = \sum_i t_i X_i^p$ such that the t_i are k^p-linearly independent.

(10.7.1) To show that $P_e^\perp = \langle \tau' \rangle$ for a given τ' one has to do the following:

(i) Show that $\mathcal{D}_e N_e$ contains at least $n-1$ k-linearly independent elements. This is obvious if we find differential operators $D_1, ..., D_{n-1}$ on k of degree $\leq p-1$ such that $D_1(\sigma_{i_1}), ..., D_{n-1}(\sigma_{i_{n-1}})$ is an arranged system of homogeneous additive polynomials.

(ii) Check that $D_j(\sigma_{i_j}) \perp \tau'$ for $j = 1, ..., n-1$.

By (i) we have $\dim P_e \geq n-1$, thus $\dim P_e^\perp \leq 1$. Then $\dim P_e^\perp = 1$ must hold since $P_e = L_e$ would imply $N_e = L_e$ which contradicts (10.6.1). (ii) therefore shows $P_e^\perp = \langle \tau' \rangle$.

(10.7.2) To get a clear view we choose a new basis of P_e as simple as possible:

(i) Choose one of the variables, say X_n.

(ii) For $i = 1, ..., n-1$ take the polynomial

$$\tau_i := X_i^p - \frac{t_i}{t_n} X_n^p.$$

Clearly $\tau_i \perp \tau'$ for $i = 1, ..., n-1$ and the system of polynomials $\tau = (\tau_1, ..., \tau_{n-1})$ is obviously linearly independent. So τ is a basis of P_e since

$$\langle \tau \rangle = \langle \tau' \rangle^\perp = P_e.$$

(10.7.3) To compute $\mathfrak{p}$ we use (10.6.3) and get $\mathfrak{p}$ as the kernel of the map

$$\varphi : S \to \overline{k}[X_n], \qquad X_1 \mapsto \sqrt[p]{t_1/t_n} X_n, \cdots, X_{n-1} \mapsto \sqrt[p]{t_{n-1}/t_n} X_n, \ X_n \mapsto X_n.$$

Fortunately it is not necessary to compute generators of $\mathfrak{p}$ itself.

(10.7.4) We have to compute parameters of the local ring $R = S_\mathfrak{p}$. Since $\dim S_\mathfrak{p} = n-1$, we know that a set of $n-1$ generators of the maximal ideal $\mathfrak{m} = \mathfrak{p} S_\mathfrak{p}$

of the regular local ring $S_{\mathfrak{p}}$ will be a system of parameters. We use that $t_1, ..., t_n$ are k^p-independent and therefore $\mathfrak{p}_1 = 0$ follows from the definition of φ. Thus $X_1, ..., X_n \in R^*$. Our parameters will come in two types:

(Y) Parameters $\psi_i = X_n^{n_i} X_i + \psi_i'$ with $\psi_i' \in k[X_{i+1}, ..., X_n]$ for $i = 1, ..., r$.

(Z) Parameters $\zeta_j = X_j^p - t_j/t_n X_n^p$ for $j = r+1, ..., n-1$.

Clearly $\varphi(\zeta_j) = 0$ and it will be easy to verify that $\varphi(\psi_i) = 0$. Then $\psi_i, \zeta_j \in \mathfrak{p}$. To show that $\mathfrak{m}$ is generated by the ψ_i, ζ_j let us calculate in $\mathfrak{m}$ modulo $\langle \psi_i, \zeta_j \rangle$. Take some $f/s \in \mathfrak{m}$ with $f \in \mathfrak{p}, s \in S \backslash \mathfrak{p}$. By multiplying f with a high enough power of X_n we can assume that $X_n^N f \in \mathfrak{p} \cap k[X_{r+1}, ..., X_n]$ by using the relations $\psi_1, ..., \psi_r$. Using the relations $\zeta_{r+1}, ..., \zeta_{n-1}$ we can suppose that $X_{r+1}, ..., X_{n-1}$ appear in f only up to the $(p-1)-$st power. If now $t_{r+1}, ..., t_{n-1}$ are p-independent, we see by using the map φ that we must have $f = 0$. Note that $\kappa(x) = k[\sqrt[p]{t_{r+1}/t_n}, ..., \sqrt[p]{t_{n-1}/t_n}](X_n)$. We will denote the initial form of ψ_i with Y_i and the initial form of ζ_j with Z_j. Of course we will adapt the notations (order and names of the variables) always to our example at hand.

(10.7.5) It then remains to develop $\sigma_1, ..., \sigma_m$ in these parameters and to denote the initial forms. For this we actually can use the technique described in (10.7.4). To see that the variables (Y, Z) are dissecting, the following will have be to confirmed:

(i) $\mathrm{In}_{x,Y}(\sigma)$ are additive and $\kappa(x)[F]$-independent.

(ii) $\mathrm{In}_{x,Z}(\sigma) \in \kappa(x)[Z]$.

(10.7.6) Finally we compute $\mathcal{P}_x$. If it is generated by linear equations, then F_x is a vector space. We will also say a word on the dimension of the ridge $R_{B,x}$ of $C_{B,x}$ at the end of each example to see how far it has dropped compared to $R_{B,0} = \dim B$. We denote the ideal of $\mathrm{Rid}_{B,x}$ by J.

10.8 Examples up to dimension 5 — Proof of the main theorems II

We prove $(\mathrm{BD})_5^p$ and have to show the following: Let y be a point in a vector space V over k, $\mathrm{char}(k) = p$ with $B = B_{V,y}$. If $\dim B_{V,y} \leq 5$, then y is dissected and if $\dim F_{V,y} \leq 5$, then $F_{V,y}$ is a vector space. We can assume by (E) that $N_0 = 0$ and by (F) that $\mathrm{Dir}(B) = 0$.

Assume that B is a vector space. Then N is generated by linear equations and $N_0 = 0$ implies $B = V$. On the other hand $B = \mathrm{Dir}(B) = 0$. There is nothing to prove since $V = 0$.

Let us now assume that B is not a vector space. By the classification of the Dieudonné modules L/N (resp. N) of Hironaka schemes by Oda [Od, 3.14] we are up to isomorphism in one of the following cases since $\dim B \leq 5$; note that $\dim B_{V,y} \leq \dim F_{V,y}$. We follow the order in which Oda presents his types of Hironaka schemes, but adapt the notations a little. We will arrange the order and names of the variables to our convenience. We indicate $p = \mathrm{char}(k)$, $d = \dim B$ and $e = \exp(B)$ in every example.

Type 3. $p = 2,\ d = 3,\ e = 1.\quad S = k[X_0, ..., X_n],\ N = \langle \sigma, X_4, ..., X_n\rangle,$

$$\sigma = X_0^2 + a_2 X_1^2 + a_1 X_2^2 + a_1 a_2 X_3^2,$$

where a_1, a_2 are 2-independent elements of k. We present this example very detailed. The argumentation for the other examples will be similar. A priori we have to assume that the dimension of V could be very large with variables up to X_n in S: Oda says that the Dieudonné module L/N is isomorphic to $\langle X_0, ..., X_3\rangle / \langle \sigma \rangle$. Therefore we can assume that any variable beyond X_3 appears in N. But (E) immediately gives $n = 3$. We will not mention this argument in the other examples again. Therefore $N_1 = k \cdot \sigma$. By (2.2.5.A) there are derivations $\partial_{a_1}, \partial_{a_2}$ on k, where we extend (a_1, a_2) to some p-basis of k whose other elements we do not need to denote. Hence the following elements lie in $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 1}(N_1)$:

$$\sigma, \quad \partial_{a_1}\sigma = X_2^2 + a_2 X_3^2, \quad \partial_{a_2}\sigma = X_1^2 + a_1 X_3^2.$$

With

$$\tau' := a_1 a_2 X_0^2 + a_1 X_1^2 + a_2 X_2^2 + X_3^2$$

we immediately see that $\tau' \perp \langle \sigma, \partial_{a_1}\sigma, \partial_{a_2}\sigma \rangle$. Properties (G) and (H) are verified and this example falls under the class we talked about in 10.7. By (10.6.5) y must be the generic point x of B whose prime ideal $\mathfrak{p}$ is the kernel of

$$S \to \overline{k}[X_3], \qquad X_0 \mapsto \sqrt{a_1 a_2} X_3, \quad X_1 \mapsto \sqrt{a_1} X_3, \quad X_2 \mapsto \sqrt{a_2} X_3, \quad X_3 \mapsto X_3.$$

We easily check that we have the following regular parameters at $S_{\mathfrak{p}}$:

$$\psi = X_3 X_0 + X_1 X_2, \qquad \zeta_1 = X_1^2 + a_1 X_3^2, \quad \zeta_2 = X_2^2 + a_2 X_3^2.$$

One sees that

$$X_3^2 \sigma = \psi^2 + \zeta_1 \zeta_2.$$

Therefore

$$\mathrm{In}_x(\sigma) = X_3^{-2}(Y^2 + Z_1 Z_2)$$

and (Y, Z) are dissecting variables. $\mathrm{In}_x(\sigma)$ is not additive and then clearly $\mathrm{In}_x(\rho)$ is never additive for $\rho \in N$. Thus $\mathcal{P}_{V,x} = 0$ and $F_{V,x} = V$ is a vector space. $R_{B,x} = 0$ since $J = \langle Y^2, Z_1, Z_2 \rangle$.

This type 3 is the most famous example of a Hironaka scheme as it already appeared in Hironaka's original article [H5]. It had also to be considered in the proof of resolution of threefolds of Cossart and Piltant in [CP2, II.5.3].

Type 4-1. $p = 2,\ d = 4,\ e = 1.\quad S = k[X_0, ..., X_4],\ N = \langle \sigma \rangle,$

$$\sigma = X_0^2 + a_2 X_1^2 + a_1 X_2^2 + a_1 a_2 X_3^2$$

as above. This Hironaka scheme is the product of Type 3 with $\mathbb{A}_k^1$. Its directrix has dimension one since X_4 does not appear in N. Hence condition (F) does not hold and we can skip this type. In other words: We did all the necessary calculations already in Type 3.

Type 4-2. $p = 2,\ d = 4,\ e = 1.$ $S = k[X_{01}, X_{02}, X_1, ..., X_4],\ N = \langle\sigma_1, \sigma_2\rangle,$

$$\sigma_1 = X_{01}^2 + a_3X_2^2 + a_2X_3^2 + a_2a_3X_4^2,$$

$$\sigma_2 = X_{02}^2 + a_3X_1^2 + a_1X_3^2 + a_1a_3X_4^2,$$

where a_1, a_2, a_3 are p-independent elements of k. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 1}(N_1)$ we find

$$\sigma_1, \quad \sigma_2, \quad \partial_{a_1}\sigma_2 = \partial_{a_2}\sigma_1 = X_3^2 + a_3X_4^2, \quad \partial_{a_3}\sigma_1 = X_2^2 + a_2X_4^2, \quad \partial_{a_3}\sigma_2 = X_1^2 + a_1X_4^2.$$

With

$$\tau' = a_2a_3X_{01}^2 + a_1a_3X_{02}^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + X_4^2$$

we are again in the class of 10.7 and the generic point $x = y$ is the kernel of

$$S \to \bar{k}[X_4], \qquad X_{01} \mapsto \sqrt{a_2a_3}X_4, \quad X_{02} \mapsto \sqrt{a_1a_3}X_4,$$

$$X_1 \mapsto \sqrt{a_1}X_4, \quad X_2 \mapsto \sqrt{a_2}X_4, \quad X_3 \mapsto \sqrt{a_3}X_4, \quad X_4 \mapsto X_4.$$

Parameters of $S_{\mathfrak{p}}$ are

$$\psi_1 = X_4X_{01} + X_2X_3, \quad \psi_2 = X_4X_{02} + X_1X_3,$$

$$\zeta_1 = X_1^2 + a_1X_4^2, \quad \zeta_2 = X_2^2 + a_2X_4^2, \quad \zeta_3 = X_3^2 + a_3X_4^2.$$

$$X_4^2\sigma_1 = \psi_1^2 + \zeta_2\zeta_3, \qquad X_4^2\sigma_2 = \psi_2^2 + \zeta_1\zeta_3.$$

$$\mathrm{In}_x(\sigma_1) = X_4^{-2}(Y_1^2 + Z_2Z_3), \qquad \mathrm{In}_x(\sigma_2) = X_4^{-2}(Y_2^2 + Z_1Z_3).$$

(Y, Z) are dissecting variables. The non additive terms $\mathrm{In}_{x,Z}(\sigma_1) = X_4^{-2}Z_2Z_3$ and $\mathrm{In}_{x,Z}(\sigma_2) = X_4^{-2}Z_1Z_3$ cannot be cancelled out by any $\kappa(x)[F]$-linear combination. Therefore $\mathcal{P}_{V,x} = 0$ and $F_{V,x} = V$ is a vector space. $R_{B,x} = 0$ since we have $J = \langle Y_1^2, Y_2^2, Z_1, Z_2, Z_3\rangle$.

Type 4-3. $p = 2,\ d = 4,\ e = 1.$ $S = k[X_{01}, X_{02}, X_{03}, X_1, ..., X_4],\ N = \langle\sigma_1, \sigma_2, \sigma_3\rangle,$

$$\sigma_1 = X_{01}^2 + a_3X_2^2 + a_2X_3^2 + a_2a_3X_4^2,$$

$$\sigma_2 = X_{02}^2 + a_3X_1^2 + a_1X_3^2 + a_1a_3X_4^2,$$

$$\sigma_3 = X_{03}^2 + a_2X_1^2 + a_1X_2^2 + a_1a_2X_4^2,$$

where a_1, a_2, a_3 are p-independent elements of k. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 1}(N_1)$ we find

$$\sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \partial_{a_1}\sigma_2 = \partial_{a_2}\sigma_1 = X_3^2 + a_3X_4^2,$$

$$\partial_{a_3}\sigma_1 = \partial_{a_1}\sigma_3 = X_2^2 + a_2X_4^2, \quad \partial_{a_3}\sigma_2 = \partial_{a_2}\sigma_3 = X_1^2 + a_1X_4^2.$$

With

$$\tau' = a_2a_3X_{01}^2 + a_1a_3X_{02}^2 + a_1a_2X_{03}^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + X_4^2$$

we are in the class of 10.7 and the generic point $x = y$ is the kernel of

$$S \to \bar{k}[X_4], \qquad X_{01} \mapsto \sqrt{a_2a_3}X_4, \quad X_{02} \mapsto \sqrt{a_1a_3}X_4, \quad X_{03} \mapsto \sqrt{a_1a_2}X_4,$$

$$X_1 \mapsto \sqrt{a_1}X_4, \quad X_2 \mapsto \sqrt{a_2}X_4, \quad X_3 \mapsto \sqrt{a_3}X_4, \quad X_4 \mapsto X_4.$$

Parameters of $S_\mathfrak{p}$ are

$$\psi_1 = X_4X_{01} + X_2X_3, \quad \psi_2 = X_4X_{02} + X_1X_3, \quad \psi_3 = X_4X_{03} + X_1X_2,$$

$$\zeta_1 = X_1^2 + a_1X_4^2, \quad \zeta_2 = X_2^2 + a_2X_4^2, \quad \zeta_3 = X_3^2 + a_3X_4^2.$$

$$X_4^2\sigma_1 = \psi_1^2 + \zeta_2\zeta_3, \quad X_4^2\sigma_2 = \psi_2^2 + \zeta_1\zeta_3, \quad X_4^2\sigma_3 = \psi_3^2 + \zeta_1\zeta_2.$$

$$\mathrm{In}_x(\sigma_1) = X_4^{-2}(Y_1^2 + Z_2Z_3), \quad \mathrm{In}_x(\sigma_2) = X_4^{-2}(Y_2^2 + Z_1Z_3),$$

$$\mathrm{In}_x(\sigma_3) = X_4^{-2}(Y_3^2 + Z_1Z_2).$$

(Y, Z) are dissecting variables. The non additive terms $\mathrm{In}_{x,Z}(\sigma_1) = X_4^{-2}Z_2Z_3$, $\mathrm{In}_{x,Z}(\sigma_2) = X_4^{-2}Z_1Z_3$ and $\mathrm{In}_{x,Z}(\sigma_3) = X_4^{-2}Z_1Z_2$ again cannot be cancelled out by any $\kappa(x)[F]$-linear combination. Therefore $\mathcal{P}_{V,x} = 0$ and $F_{V,x} = V$ is a vector space. $R_{B,x} = 0$ since $J = \langle Y_1^2, Y_2^2, Y_3^2, Z_1, Z_2, Z_3\rangle$.

Type 4-4. $p = 2,\ d = 4,\ e = 1.$ $S = k[X_{01}, ..., X_{04}, X_1, ..., X_4]$, $N = \langle\sigma_1, \sigma_2, \sigma_3, \sigma_4\rangle$,

$$\sigma_1 = X_{01}^2 + a_3X_2^2 + a_2X_3^2 + a_2a_3X_4^2,$$

$$\sigma_2 = X_{02}^2 + a_3X_1^2 + a_1X_3^2 + a_1a_3X_4^2,$$

$$\sigma_3 = X_{03}^2 + a_2X_1^2 + a_1X_2^2 + a_1a_2X_4^2,$$

$$\sigma_4 = X_{04}^2 + a_2a_3X_1^2 + a_1a_3X_2^2 + a_1a_2X_3^2,$$

where a_1, a_2, a_3 are p-independent elements of k. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 1}(N_1)$ we find

$$\sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \sigma_4, \quad \partial_{a_1}\sigma_2 = \partial_{a_2}\sigma_1 = X_3^2 + a_3X_4^2,$$

$$\partial_{a_3}\sigma_1 = \partial_{a_1}\sigma_3 = X_2^2 + a_2X_4^2, \quad \partial_{a_3}\sigma_2 = \partial_{a_2}\sigma_3 = X_1^2 + a_1X_4^2.$$

With

$$\tau' = a_2a_3X_{01}^2 + a_1a_3X_{02}^2 + a_1a_2X_{03}^2 + a_1a_2a_3X_{04}^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + X_4^2$$

we are in the class of 10.7 and the generic point $x = y$ is the kernel of

$$S \to \overline{k}[X_4], \qquad X_{01} \mapsto \sqrt{a_2a_3}X_4, \quad X_{02} \mapsto \sqrt{a_1a_3}X_4, \quad X_{03} \mapsto \sqrt{a_1a_2}X_4,$$

$$X_{04} \mapsto \sqrt{a_1a_2a_3}X_4, \quad X_1 \mapsto \sqrt{a_1}X_4, \quad X_2 \mapsto \sqrt{a_2}X_4, \quad X_3 \mapsto \sqrt{a_3}X_4, \quad X_4 \mapsto X_4.$$

Parameters of $S_\mathfrak{p}$ are

$$\psi_1 = X_4X_{01} + X_2X_3, \quad \psi_2 = X_4X_{02} + X_1X_3,$$

$$\psi_3 = X_4X_{03} + X_1X_2, \quad \psi_4 = X_4^2X_{04} + X_1X_2X_3,$$

$$\zeta_1 = X_1^2 + a_1X_4^2, \quad \zeta_2 = X_2^2 + a_2X_4^2, \quad \zeta_3 = X_3^2 + a_3X_4^2.$$

$$X_4^2\sigma_1 = \psi_1^2 + \zeta_2\zeta_3, \quad X_4^2\sigma_2 = \psi_2^2 + \zeta_1\zeta_3, \quad X_4^2\sigma_3 = \psi_3^2 + \zeta_1\zeta_2,$$

$$X_4^4\sigma_4 = \psi_4^2 + \zeta_1\zeta_2\zeta_3 + a_3X_4^2\zeta_1\zeta_2 + a_1X_4^2\zeta_2\zeta_3 + a_2X_4^2\zeta_1\zeta_3.$$

$$\mathrm{In}_x(\sigma_1) = X_4^{-2}(Y_1^2 + Z_2Z_3), \quad \mathrm{In}_x(\sigma_2) = X_4^{-2}(Y_2^2 + Z_1Z_3),$$
$$\mathrm{In}_x(\sigma_3) = X_4^{-2}(Y_3^2 + Z_1Z_2),$$
$$\mathrm{In}_x(\sigma_4) = X_4^{-4}Y_4^2 + a_3X_4^{-2}Z_1Z_2 + a_1X_4^{-2}Z_2Z_3 + a_2X_4^{-2}Z_1Z_3.$$

(Y, Z) are dissecting variables. The non additive terms

$$\mathrm{In}_{x,Z}(\sigma_1) = X_4^{-2}Z_2Z_3, \quad \mathrm{In}_{x,Z}(\sigma_2) = X_4^{-2}Z_1Z_3, \quad \mathrm{In}_{x,Z}(\sigma_3) = X_4^{-2}Z_1Z_2$$
$$\mathrm{In}_{x,Z}(\sigma_4) = a_3X_4^{-2}Z_1Z_2 + a_1X_4^{-2}Z_2Z_3 + a_2X_4^{-2}Z_1Z_3$$

can be cancelled out by a $\kappa(x)[F]$-linear combination, namely if we take

$$\xi := a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + \sigma_4 =$$
$$= X_{04}^2 + a_1X_{01}^2 + a_2X_{02}^2 + a_3X_{03}^2 + a_2a_3X_1^2 + a_1a_3X_2^2 + a_1a_2X_3^2 + a_1a_2a_3X_4^2.$$

We see that $\mathcal{P}_{V,x} = \langle\xi\rangle$ and $F_{V,x} = \mathrm{Spec}(S/\langle\xi\rangle)$ is not a vector space, but $\dim F_{V,x} = 7$, so there is no problem in view of the proof of the main theorems. $R_{B,x} = 0$ since $J = \langle Y_1^2, Y_2^2, Y_3^2, Y_4^2, Z_1, Z_2, Z_3\rangle$. Under a somewhat different objective the scheme $\mathrm{Spec}(S/\langle\xi\rangle)$ was already considered in [Gi, III 2.12]. While Types 4-2 and 4-3 were very similar to Type 3, Type 4-4 behaves differently. The reason why we find a non linear refined Hironaka scheme is the following: The term $\zeta_1\zeta_2\zeta_3$ in the transform of σ_4 is of higher order and does not appear in the initial form of σ_4. Our invariants do not see any improvement in this case. Nevertheless the situation at x has improved:

$$\mathrm{In}_x(\xi) = a_1\,\mathrm{In}_x(\sigma_1) + a_2\,\mathrm{In}_x(\sigma_2) + a_3\,\mathrm{In}_x(\sigma_3) + \mathrm{In}_x(\sigma_4) =$$
$$= X_4^{-2}(a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 + X_4^{-2}Y_4^2) = Y^2,$$

where we took $Y := X_4^{-1}(\sqrt{a_1}Y_1 + \sqrt{a_2}Y_2 + \sqrt{a_3}Y_3 + X_4^{-1}Y_4) \in \mathrm{gr}_{V,x}$. Therefore the situation at the origin

$$\mathrm{Dir}(B) = 0, \qquad \mathrm{Rid}(B) = B$$

has improved to

$$\mathrm{Dir}(C_{B,x}) = \mathrm{Rid}(C_{B,x})_{\mathrm{red}}.$$

Also in

$$X_4^2\xi = a_1\psi_1^2 + a_2\psi_2^2 + a_3\psi_3^2 + X_4^{-2}\psi_4^2 + X_4^{-2}\zeta_1\zeta_2\zeta_3$$

the monomial $\zeta_1\zeta_2\zeta_3$ looks quite manageable in view of the techniques in [CP2].

Type 5. $p = 3,\ d = 5,\ e = 1.\quad S = k[X_0, ..., X_5],\ N = \langle\sigma\rangle,$

$$\sigma = X_0^2 + a_1X_1^3 + a_2X_2^3 + a_1^2X_3^3 + a_1a_2X_4^3 + a_1^2a_2X_5^3,$$

where a_1, a_2 are p-independent elements of k. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 2}(N_1)$ we find

$$\sigma, \quad \partial_{a_1}\sigma = X_1^3 - a_1X_3^3 + a_2X_4^3 - a_1a_2X_5^3, \quad \partial_{a_2}\sigma = X_2^3 + a_1X_4^3 + a_1^2X_5^3,$$
$$\partial_{a_1^2}\sigma = X_3^3 + a_2X_5^3, \quad \partial_{a_1}\partial_{a_2}\sigma = X_4^3 - a_1X_5^3.$$

With
$$\tau' = a_1^2 a_2 X_0^2 + a_1 a_2 X_1^3 - a_1^2 X_2^3 + a_2 X_3^3 - a_1 X_4^3 - X_5^3$$
we are again in the class of 10.7 and the generic point $x = y$ is the kernel of
$$S \to \overline{k}[X_4], \qquad X_0 \mapsto -\sqrt[3]{a_1^2 a_2} X_5, \quad X_1 \mapsto -\sqrt[3]{a_1 a_2} X_5,$$
$$X_2 \mapsto \sqrt[3]{a_1^2} X_5, \quad X_3 \mapsto -\sqrt[3]{a_2} X_5, \quad X_4 \mapsto \sqrt[3]{a_1} X_5, \quad X_5 \mapsto X_5.$$
Parameters of $S_{\mathfrak{p}}$ are
$$\psi_1 = X_5 X_0 - X_3 X_2, \quad \psi_2 = X_5 X_1 - X_3 X_4, \quad \psi_3 = X_2 X_4 - a_1 X_5^2,$$
$$\zeta_1 = X_2^3 - a_1^2 X_5^3, \quad \zeta_2 = X_3^3 + a_2 X_5^3.$$
$$X_5^3 \sigma = \psi_1^3 + a_1 \psi_2^3 + \zeta_2 X_2^{-3}(\zeta_1^2 + a_1 \psi_3^3).$$
$$\mathrm{In}_x(\sigma) = X_5^{-3}(Y_1^3 + a_1 Y_2^3 + a_1^{-2} Z_1^2 Z_2).$$
(Y, Z) are dissecting variables. The non additive term $Z_1^2 Z_2$ cannot be cancelled out. Therefore $\mathcal{P}_{V,x} = 0$ and $F_{V,x} = V$ is a vector space.

Type 5-1. B is isomorphic to the product of $\mathbb{A}_k^2$ and Type 3. $\mathrm{Dir}(B) \cong \mathbb{A}_k^2$ and this case is excluded by condition (F).

Type 5-2. B is isomorphic to the product of $\mathbb{A}_k$ and Type 4-2. $\mathrm{Dir}(B) \cong \mathbb{A}_k$ and this case is excluded by condition (F).

Type 5-3. B is isomorphic to the product of $\mathbb{A}_k$ and Type 4-3. $\mathrm{Dir}(B) \cong \mathbb{A}_k$ and this case is excluded by condition (F).

Type 5-4. B is isomorphic to the product of $\mathbb{A}_k$ and Type 4-4. $\mathrm{Dir}(B) \cong \mathbb{A}_k$ and this case is excluded by condition (F).

The next type does not fall under the class of examples described in 10.7. With little adaptions we still can follow a similar line of thought.
Type 5-5. $p = 2,\ d = 5,\ e = 1.$ $\quad S = k[X_0, ..., X_5],\ N = \langle \sigma \rangle,$
$$\sigma = X_0^2 + a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_2 a_3 X_4^2 + a_1 a_3 X_5^2$$
where a_1, a_2, a_3 are p-independent elements of k. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 1}(k)(N_1)$ we find
$$\partial_{a_1}\sigma = X_1^2 + a_3 X_5^2, \quad \partial_{a_2}\sigma = X_2^2 + a_3 X_4^2, \quad \partial_{a_3}\sigma = X_3^2 + a_2 X_4^2 + a_1 X_5^2,$$
$$\sigma - a_1 \partial_{a_1}\sigma - a_2 \partial_{a_2}\sigma - a_3 \partial_{a_3}\sigma = X_0^2 + a_2 a_3 X_4^2 + a_1 a_3 X_5^2.$$
With (10.6.3) we see that the prime ideal of the generic point x of B is the kernel of
$$S \to \overline{k}[X_4, X_5], \qquad X_0 \mapsto \sqrt{a_2 a_3} X_4 + \sqrt{a_1 a_3} X_5, \quad X_1 \mapsto \sqrt{a_3} X_5,$$
$$X_2 \mapsto \sqrt{a_3} X_4, \quad X_3 \mapsto \sqrt{a_2} X_4 + \sqrt{a_1} X_5.$$
In this case y need not necessarily be the generic point x. However we still may

assume that $\mathfrak{p}_1 = 0$ by condition (E). We will show that

$$\psi_1 = X_5X_0 + X_1X_3, \qquad \psi_2 = X_5X_2 + X_1X_4,$$

$$\zeta_1 = X_1^2 + a_3X_5^2, \qquad \zeta_2 = X_3^2 + a_2X_4^2 + a_1X_5^2$$

are part of a regular sequence of parameters in the regular local ring ($R := S_{\mathfrak{p}}, \mathfrak{m} := \mathfrak{p}S_{\mathfrak{p}}$) at y. We have to prove that the classes of these elements in $\mathfrak{m}/\mathfrak{m}^2$ are $R/\mathfrak{m}$-linearly independent. It suffices to show: For $t_1, ..., t_4 \in S$ with

$$T := t_1\psi_1 + t_2\psi_2 + t_3\zeta_1 + t_4\zeta_2 \in \mathfrak{p}^{(2)}$$

we must have $t_1, ..., t_4 \in \mathfrak{p}$. By multiplying with a high enough power of $X_5 \notin \mathfrak{p}$ we can assume that $t_i \in K[X_1, X_3, X_4, X_5]$ by using the relations ψ_1, ψ_2. By (2.3.2) we get

$$\mathfrak{p} \ni \partial_{X_0}(T) = t_1X_5 \qquad \Rightarrow t_1 \in \mathfrak{p},$$

$$\mathfrak{p} \ni \partial_{X_2}(T) = t_2X_5 \qquad \Rightarrow t_2 \in \mathfrak{p},$$

$$\partial_{a_3}(T) \in \mathfrak{p} \quad \Rightarrow t_3\partial_{a_3}(\zeta_1) + t_4\partial_{a_3}(\zeta_2) \in \mathfrak{p} \quad \Rightarrow t_3X_5^2 \in \mathfrak{p} \quad \Rightarrow t_3 \in \mathfrak{p},$$

$$\partial_{a_1}(T) \in \mathfrak{p} \quad \Rightarrow t_4X_5^2 = t_4\partial_{a_1}(\zeta_2) \in \mathfrak{p} \quad \Rightarrow t_4 \in \mathfrak{p}.$$

Independently of the exact point y we can find

$$X_5^2\sigma = \psi_1^2 + a_2\psi_2^2 + \zeta_1\zeta_2.$$

$$\mathrm{In}_y(\sigma) = X_5^{-2}(Y_1^2 + a_2Y_2^2 + Z_1Z_2).$$

We find dissecting variables by potentially adding a further variable arbitrarily to Y or Z. The non additive term Z_1Z_2 cannot be cancelled out. Therefore $\mathcal{P}_{V,y} = 0$ and $F_{V,y} = V$ is a vector space.

The last kind of examples are not described as explicitely as all types hitherto.

Type 5-$\star$. $p = 2,\ d = 5,\ e = 1.$ $\quad S = k[X_{0,1}, ..., X_{0,v}, X_1, ..., X_5],\ N = \langle \sigma_1, ..., \sigma_v \rangle,$

$$\sigma_j = X_{0,j}^2 + \sum_{i=1}^{4}(\partial_i g_j)X_i^2 + \left(g_j + \sum_{i=1}^{4} a_i\partial_i g_j\right)X_5^2$$

for $j = 1, ..., v$, where $a_1, ..., a_4$ are p-independent elements of k (∂_i is the derivative with respect to a_i for a 2-basis containing $a_1, ..., a_4$) and $g_1, ..., g_v \in k^2(a_1, ..., a_4)$ such that the k^2-subspace spanned by $1, a_1, ..., a_4, g_1, ..., g_v$ in k has dimension $5 + v$ (therefore $v \leq 11$) and that the matrix

$$B = (\partial_i g_j)_{i=1,...,4; j=1,...,v}$$

has the property that the rows of dB as elements of $\Omega^1(k)^v$ are k-linearly independent. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\leq 1}(k)(N_1)$ we find

$$\sigma_j, \qquad \partial_h(\sigma_j) = \sum_{i=1}^{4}(\partial_h\partial_i g_j)X_i^2 + \left(\partial_h g_j + \sum_{i=1}^{4}(\delta_{ih}\partial_i g_j + a_i(\partial_h\partial_i g_j))\right)X_5^2 =$$

$$= \sum_{i=1}^{4}(\partial_h\partial_i g_j)X_i^2 + \left(\sum_{i=1}^{4} a_i(\partial_h\partial_i g_j)\right) X_5^2$$

for all $j = 1, ..., v$ and $h = 1, ..., 4$. Let $\tau' = \sum_{j=1}^{v} t_{0,j}X_{0,j}^2 + \sum_{i=1}^{4} t_i X_i + t_5 X_5^2 \in P_1^{\perp}$ and define $s_i := t_i + a_i t_5$ for $i = 1, ..., 4$. Then

$$0 = \langle \partial_h \sigma_j, \tau' \rangle = \sum_{i=1}^{4}(\partial_h\partial_i g_j)t_i + \left(\sum_{i=1}^{4} a_i(\partial_h\partial_i g_j)\right) t_5 = \sum_{i=1}^{4}(\partial_h\partial_i g_j)s_i$$

for all $h = 1, ..., 4$ and $j = 1, ..., v$. Therefore

$$0 = \sum_{i=1}^{4} s_i \cdot \left(\sum_{h=1}^{4}(\partial_h\partial_i g_j)d(a_h)\right)_{j=1,...,v} = \sum_{i=1}^{4} s_i \cdot ((dB)_{i,1}, ..., (dB)_{i,v}) \,.$$

In view of the above condition on B this implies $s_1 = \cdots = s_4 = 0$. Thus $t_i = a_i t_5$ for $i = 1, ..., 4$ and from

$$0 = \langle \sigma_j, \tau' \rangle = t_{0,j} + \sum_{i=1}^{4}(\partial_i g_j)a_i t_5 + \left(g_j + \sum_{i=1}^{4} a_i \partial_i g_j\right) t_5$$

we get $t_{0,j} = g_j t_5$. Therefore $P_1^{\perp} = \langle \tau' \rangle$ with

$$\tau' = \sum_{j=1}^{v} g_j X_{0,j}^2 + \sum_{i=1}^{4} a_i X_i^2 + X_5^2.$$

Since $1, a_1, ..., a_4, g_1, ..., g_v$ are k^2-indpendent, we are again in the case of 10.7. The generic point $x = y$ is the kernel of

$$S \to \overline{k}[X_5], \qquad X_{0,j} \mapsto \sqrt{g_j}X_5 \;\text{ for } j = 1, ..., v,$$

$$X_i \mapsto \sqrt{a_i}X_5 \;\text{ for } i = 1, ..., 4, \qquad X_5 \mapsto X_5.$$

We can expand $g_j \in k^2(a_1, ..., a_4)$ as $g_j = \sum_{M \le_c \mathbf{1}} \alpha_{M,j}^2 a^M$ for $\mathbf{1} = (1,1,1,1)$ with $\alpha_{M,j} \in k$. Define $G_j := \sum_{M+L=\mathbf{1}} \alpha_{M,j} X^M X_5^{|L|} \in k[X_1, ..., X_5]$ $(X = (X_1, ..., X_4))$. Then $X_5^3 X_{0,j} + G_j$ is mapped to zero under the above morphism. We find the following regular parameters at $S_{\mathfrak{p}}$:

$$\psi_j := X_5^3 X_{0,j} + G_j \quad \text{for} \quad j = 1, ..., v,$$

$$\zeta_i := X_i^2 + a_i X_5^2 \quad \text{for} \quad i = 1, ..., 4.$$

$$X_5^6 \sigma_j = (X_5^3 X_{0,j})^2 + X_5^6 \sum_{i=1}^{4}(\partial_i g_j)(\zeta_i + a_i X_5^2) + \left(g_j + \sum_{i=1}^{4} a_i(\partial_i g_j)\right) X_5^8 =$$

$$= \psi_j^2 + G_j^2 + X_5^6 \sum_{i=1}^{4}(\partial_i g_j)\zeta_i + g_j X_5^8.$$

For standard differential operators D_M with respect to $X_1, ..., X_4$ we have by (3.3.1.B)

$$G_j^2 = (G_j(X))^2 = (G_j(\sqrt{\zeta} + \sqrt{a}X_5))^2 =$$

$$= \left(\sum_M \sqrt{\zeta}^M D_M(G_j)(\sqrt{a}X_5) \right)^2 = \sum_M \zeta^M \left(D_M(G_j)(\sqrt{a}X_5) \right)^2 .$$

Denote with E_M the standard differential operators with respect to a. Then

$$(D_M(G_j)(\sqrt{a}X_5))^2 = \left(\left(\sum_{N \le_c \mathbf{1}} \binom{N}{M} \alpha_{N,j} X^{N-M} X_5^{4-|N|} \right) (\sqrt{a}X_5) \right)^2 =$$

$$= \sum_{N \le_c \mathbf{1}} \binom{N}{M} \alpha_{N,j}^2 a^{N-M} X_5^{8-2|M|} = E_M(g_j) X_5^{8-2|M|}.$$

Together we find

$$X_5^6 \sigma_j = \psi_j^2 + \sum_{|M| \ge 2} \zeta^M E_M(g_j) X_5^{8-2|M|}$$

and therefore

$$\mathrm{In}_x(\sigma_j) = X_5^{-6} Y_j^2 + X_5^{-2} \sum_{|M|=2} Z^M E_M(g_j) = X_5^{-6} Y_j^2 + X_5^{-2} \sum_{1 \le i < h \le 4} (\partial_h \partial_i g_j) Z_h Z_i .$$

Clearly (Y, Z) are dissecting variables. Assume that $\dim F_{V,x} \le 5$. Then $\dim F_{V,x} = 5 = \dim B_{V,x}$ and with (10.2.3) we must have $\mathcal{Q}_x = \mathcal{P}_x$. This implies $\mathrm{In}_x(\sigma_j)$ being additive for all j. For fixed j this means $\partial_h \partial_i g_j = 0$ which in turn implies $g_j \in \langle 1, a_1, ..., a_4 \rangle_{k^2}$. This is not possible. Therefore $\dim F_{V,x} \ge 6$. It may very well be that the non linear terms can be cancelled out by certain $\kappa(x)[F]$-combinations of the σ_j and that nonlinear refined Hironaka schemes appear in this class of types.

10.9 An example in dimension $2p-1$ — Proof of the main theorems III

We prove $(\mathrm{BD})^p_{2\mathrm{p}-1}$. Let k be a field of characteristic p. With the notation as in 10.8 we similarly have only to consider non linear Hironaka schemes B over k up to dimension $2p-1$. Mizutani proved in [Mi] that there is only one such type, namely the following in dimension $2p-1$. It generalizes Types 3 and 5.

The minimal nonlinear type in arbitrary characteristic. p arbitrary, $d = 2p-1,\ e = 1.\quad S = k[X_{0,0}, ..., X_{p-1,0}, X_{0,1}, ..., X_{p-1,1}],\ N = \langle \sigma \rangle$,

$$\sigma = \sum_{i=0}^{p-1} a_1^i X_{i,0}^p + a_1^i a_2 X_{i,1}^p ,$$

where a_1, a_2 are p-independent elements of k. In $P_1 = \mathrm{Diff}_{\mathbb{Z}}^{\le p-1}(N_1)$ we find using the differential operators D_M with $M = (j, l), j + l \le p-1$ with respect to the

p-independent system (a_1, a_2)

$$D_{j,0}(\sigma) = \sum_{i=0}^{p-1} \binom{i}{j} \left(a_1^{i-j} X_{i,0}^p + a_1^{i-j} a_2 X_{i,1}^p \right) = X_{j,0}^p + \cdots , \qquad j = 0, ..., p-1,$$

$$D_{j,1}(\sigma) = \sum_{i=0}^{p-1} \binom{i}{j} a_1^{i-j} X_{i,1}^p = X_{j,1}^p + \cdots , \qquad j = 0, ..., p-2.$$

With

$$\tau' = \sum_{i=0}^{p-1} a_1^{p-1-i} a_2 X_{i,0}^p - a_1^{p-1-i} X_{i,1}^p$$

we are again in the class of 10.7. In fact

$$\langle \tau', D_{j,0}(\sigma) \rangle = \sum_{i=0}^{p-1} \binom{i}{j} (a_1^{p-1-j} a_2 - a_1^{p-1-j} a_2) = 0,$$

$$\langle \tau', D_{j,1}(\sigma) \rangle = -\sum_{i=0}^{p-1} \binom{i}{j} a_1^{p-1-j} = a_1^{p-1-j} \sum_{i=0}^{p-1} \binom{i}{j+1} - \binom{i+1}{j+1} =$$

$$= a_1^{p-1-j} \left(\binom{p}{j+1} - \binom{j}{j+1} \right) = 0.$$

The generic point $x = y$ is the kernel of

$$S \to \overline{k}[X_{p-1,1}], \qquad X_{i,0} \mapsto -a_1^{p-1-i/p} a_2^{1/p} X_{p-1,1}, \quad X_{i,1} \mapsto a_1^{p-1-i/p} X_{p-1,1}.$$

Parameters of $S_{\mathfrak{p}}$ are

$$\psi_{i,0} = X_{p-1,1} X_{i,0} - X_{p-1,0} X_{i,1}, \qquad i = 0, ..., p-2.$$

$$\psi_{i,1} = X_{0,1}^{p-1-i} X_{i,1} - a_1^{p-1-i} X_{p-1,1}^{p-i}, \qquad i = 1, ..., p-2,$$

$$\zeta_1 = X_{0,1}^p - a_1^{p-1} X_{p-1,1}^p, \qquad \zeta_2 = X_{p-1,0}^p + a_2 X_{p-1,1}^p.$$

With

$$X_{p-1,1}^p \sigma = \sum_{i=0}^{p-1} a_1^i (X_{p-1,1} X_{i,0})^p + a_1^i a_2 X_{i,1}^p X_{p-1,1}^p =$$

$$= \sum_{i=0}^{p-2} a_1^i \psi_{i,0}^p + (X_{p-1,0}^p + a_2 X_{p-1,1}^p) \sum_{i=0}^{p-1} a_1^i X_{i,1}^p$$

and (using $(x-y)^{p-1} = \sum_{i=1}^p x^{i-1} y^{p-i}$)

$$X_{0,1}^{p(p-2)} \sum_{i=0}^{p-1} a_1^i X_{i,1}^p = X_{0,1}^{p(p-1)} + \sum_{i=1}^{p-2} a_1^i X_{0,1}^{p(i-1)} (X_{0,1}^{p-1-i} X_{i,1})^p + a_1^{p-1} X_{0,1}^{p(p-2)} X_{p-1,1}^p =$$

$$= \sum_{i=1}^{p-2} a_1^i \psi_{i,1}^p + \sum_{i=1}^{p} X_{0,1}^{p(i-1)} (a_1^{p-1} X_{p-1,1}^p)^{p-i} = \zeta_1^{p-1} + \sum_{i=1}^{p-2} a_1^i \psi_{i,1}^p$$

we get

$$X_{p-1,1}^p \sigma = \sum_{i=0}^{p-2} a_1^i \psi_{i,0}^p + \zeta_2 X_{0,1}^{-p(p-2)} \left(\zeta_1^{p-1} + \sum_{i=1}^{p-2} a_1^i \psi_{i,1}^p \right),$$

$$\mathrm{In}_x(\sigma) = X_{p-1,1}^{-p} \sum_{i=0}^{p-2} a_1^i Y_{i,0}^p + a_1^{(1-p)(p-2)} X_{p-1,1}^{p(1-p)} Z_1^{p-1} Z_2.$$

(Y, Z) are dissecting variables. The non additive term $Z_1^{p-1} Z_2$ in $\mathrm{In}_{x,Z}$ shows that $\mathcal{P}_{V,x} = 0$ and $F_{V,x} = V$ is a vector space.

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