Fourier-Mukai Transforms from T-Duality

Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat.) der Fakultät für Mathematik der Universität Regensburg

vorgelegt von

Martin Ruderer

aus

Landau a.d. Isar

im

Oktober 2014

Promotionsgesuch wurde eingereicht am: 28.10.2014

Die Arbeit wurde angeleitet von: Prof. Dr. Ulrich Bunke

Prüfungsausschuss:

Vorsitzende:	Prof. Dr. Clara Löh
Erst-Gutachter:	Prof. Dr. Ulrich Bunke
Zweit-Gutachter:	Juniorprof. Dr. Markus Spitzweck
weiterer Prüfer:	Prof. Dr. Klaus Künnemann

Contents

1 Introduction

1.1 Fourier-Mukai Transforms

Fourier-Mukai transformations, although not called that at the time, were first introduced in S. Mukai's seminal paper *"Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves"* [19]. They consider the following situation. Let A be an abelian variety, $\hat{A}$ its dual and $\mathcal{P}$ the Poincaré bundle. Further, let

$$\begin{tikzcd} & A \times \hat{A} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A & & \hat{A} \end{tikzcd}$$

be the projections. He proved that the *Fourier-Mukai transformation*

$$FM\colon \mathcal{D}^b(A) \to \mathcal{D}^b(\hat{A}), \ \ \mathcal{F}^\bullet \mapsto Rq_* \left(\mathcal{P} \otimes^L Lq^* \mathcal{F}^\bullet\right)$$

is an equivalence between the bounded derived category of quasi-coherent sheaves on A and $\hat{A}$ respectively.

Later, U. Bruzzo, G. Marelli and F. Pioli [3] transported the theory to the real case, where $A = T^n$ is the real torus and $\hat{A} = \widehat{T}^n$ is the dual torus. They proved a somewhat weaker statement.

Theorem. *The Fourier-Mukai transformation induces an equivalence*

$$Loc(T^n) \to Sky(\widehat{T}^n),$$

where $Loc(T^n)$ is the category of (unitary) local systems on T^n and $Sky(\widehat{T}^n)$ is the category of skyscraper sheaves on $\widehat{T}^n$. The transformation fulfills

$$\mathcal{L}_x \mapsto \mathbb{C}(-x),$$

where $\mathcal{L}_x$ is the line bundle with holonomy $x \in \widehat{T}^n$ and $\mathbb{C}(-x)$ is the skyscraper sheaf of length 1 supported on $-x$.

The motivation of this thesis is to extend that result to a relative setting, namely to T^n-principal bundles.

An obvious idea is: Given two principal T^n-bundles $\pi\colon E \to B$ and

$\hat{\pi}\colon \hat{E} \to B$ over a common base manifold B, form the pullback

$$\begin{tikzcd}
& E \underset{B}{\times} \hat{E} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \\
& B &
\end{tikzcd}$$

and construct a *relative* Poincaré bundle $\mathcal{P}$ over the pullback $E \underset{B}{\times} \hat{E}$. The relative Poincaré bundle should have the property, that restricted to the fiber $(E \underset{B}{\times} \hat{E})|_b = T^n \times \widehat{T}^n$ over any point $b \in B$ it is a Poincaré bundle. The *relative Fourier-Mukai transform* then has the form

$$\mathcal{F}^\bullet \mapsto Rq_* \left(\mathcal{P} \otimes^L Lp^* \mathcal{F}^\bullet \right).$$

However, it is a little less clear what the source and target categories are supposed to be, such that the transformation becomes an equivalence.

An approach like that, but with fiber-bundles and taking geometry into account, can be found in [11]. Furthermore, [4] treats the case of a symplectic family of Lagrangian tori and sheaves supported on Lagrangian submanifolds.

One problem with that approach is that a relative Poincaré bundle might not exist. Even though the local construction is clear, it might not glue to give a global bundle. A solution to that problem is to not consider it as a bundle over the product $E \underset{B}{\times} \hat{E}$ but to allow for it to be twisted, which leads to *topological T-duality*.

1.2 Topological T-Duality

Definition. A *T-Duality triple*, as defined in [9], is a triple

$$\big((\mathcal{H}, E \to B), (\hat{\mathcal{H}}, \hat{E} \to B), u\big)$$

consisting of T^n-bundles E, $\hat{E}$ over a common base B, S^1-banded gerbes $\mathcal{H}$, $\hat{\mathcal{H}}$ over E and $\hat{E}$ respectively, and an isomorphism u as in diagram (1).

$$\begin{tikzcd}
 & p^*\mathcal{H} \arrow[rr, "u"] \arrow[dl, "q"'] \arrow[dr] & & \hat{p}^*\hat{\mathcal{H}} \arrow[dl] \arrow[dr, "\hat{q}"] & \\
\mathcal{H} \arrow[dr] & & E \underset{B}{\times} \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & & \hat{\mathcal{H}} \arrow[dl] \\
 & E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] & \\
 & & B & &
\end{tikzcd} \tag{1}$$

We require the following two conditions:

- $\mathcal{H}$ and $\hat{\mathcal{H}}$ are trivializable on the fibers E_b and $\hat{E}_b$ over every point $b \in B$,
- the isomorphism u is to satisfy the *Poincaré-condition*: for all $b \in B$, the restriction $u|_{E_b \times \hat{E}_b}$ is an isomorphism of trivializable gerbes on $E_b \times \hat{E}_b$. There have to exist trivializations of those gerbes such that the induced automorphism (of trivial gerbes on $E_b \times \hat{E}_b$) is represented by the *Poincaré bundle*.

One says that $(\mathcal{H}, E \to B)$ and $(\hat{\mathcal{H}}, \hat{E} \to B)$ are *T-dual pairs* if there is T-duality triple extending them.

Given any Lie group G, the sheaf of G-valued smooth functions $\mathcal{G}$ on the site of smooth submersions over E (see Section 2.3) has a subsheaf $\mathcal{G}_{E/B}$, whose sections are those functions that are fiberwise constant. In [6] it was established that the reduction of $\mathcal{H}$ to the sheaf $\mathcal{S}^1_{E/B}$ is exactly the data needed to construct a T-dual pair.

Theorem ([6])**.** *Given a $\mathcal{S}^1_{E/B}$-banded gerbe $\mathcal{H}$ on a principal torus bundle $\pi\colon E \to B$, the push-forward $\pi_*\mathcal{H}$ has as moduli-space a principal $\hat{T}$-bundle $\hat{\pi}\colon \hat{E} \to B$ over B and is a $\mathcal{S}^1_{\hat{E}/B}$-banded gerbe over $\hat{E}$. Furthermore, $H \to E$ and $\hat{H} \to \hat{E}$ are T-dual to each other and the isomorphism u is given by the universal section.*

1.3 Description of the Thesis

The thesis aims to derive a *twisted Fourier-Mukai transform* from T-duality that is an equivalence.

Let $\mathcal{H} \to E \to B$ be a $\mathcal{S}^1_{E/B}$-banded gerbe over a principal T^n-bundle as in the previous theorem. Among the $C^\infty_{E/B}$-module sheaves, where $C^\infty_{E/B} = \mathbb{C}_{E/B}$, there is a special class of sheaves, called the *twisted sheaves* (Definition 2.20). Note that on a $\mathcal{S}^1_{E/B}$-banded gerbe $\mathcal{H}$, a relative automorphism of an element $t\colon T \to \mathcal{H}$ is a map $\sigma\colon T \to S^1$ such that σ is constant in fiber direction, i.e. $\sigma \in \mathcal{S}^1_{E/B}(T)$. Twisted sheaves have the property that relative automorphisms act on them through multiplication of the corresponding S^1-valued function.

The category $\mathrm{Mod}_1(\mathcal{H})$ of twisted sheaves is a Grothendieck abelian category (Lemma 2.30) hence the derived category $\mathcal{D}_1(\mathcal{H}) := \mathcal{D}(\mathrm{Mod}_1(\mathcal{H}))$ of unbounded complexes is well behaved (Section 2.4). More generally, we set $\mathcal{D}_k(\mathcal{H}) := \mathcal{D}(\mathrm{Mod}_k(\mathcal{H}))$, the derived category of k-twisted sheaves for any natural number $k \in \mathbb{N}$.

Because $\pi\colon E \to B$ is representable, we can form the push-forward of $\mathcal{H}$ by setting

$$\pi_*\mathcal{H}(X \to B) = \mathcal{H}(E \underset{B}{\times} X \to E).$$

Instead of $\pi_*\mathcal{H}$ we often write $\widehat{\mathcal{H}}$. That stack is called the *stack of fiberwise trivializations* of $\mathcal{H}$. Using the notation of the previous theorem, we can form the following pull-back diagram, where the arrow

$$u\colon E \underset{B}{\times} \widehat{\mathcal{H}} = \pi^*\pi_*\mathcal{H} \to \mathcal{H} \underset{E}{\times} (\pi^*\pi_*\mathcal{H}) = \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}$$

is induced by the counit $\pi^*\pi_*\mathcal{H} \to \mathcal{H}$ of the adjunction $\pi^* \dashv \pi_*$ and the identity $\mathrm{id}\colon \pi^*\pi_*\mathcal{H} \to \pi^*\pi_*\mathcal{H}$.

$$
\begin{tikzcd}
 & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl] \arrow[dr, "q_2"] \arrow[ddll, bend right, "p"'] & & \\
 & \mathcal{H} \underset{B}{\times} \hat{E} \arrow[dl] \arrow[dr] & & E \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl] \arrow[dr, "q_1"] \arrow[ul, dashed, bend right, "u"'] & \\
\mathcal{H} \arrow[dr] & & E \underset{B}{\times} \hat{E} \arrow[dl] \arrow[dr] & & \widehat{\mathcal{H}} \arrow[dl] \\
 & E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] & \\
 & & B, & &
\end{tikzcd}
\tag{2}
$$

The stack $\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} = \pi^*\pi_*\mathcal{H} \underset{E}{\times} \mathcal{H}$ has a canonical map to $\mathcal{H} \underset{E}{\times} \mathcal{H}$, given as the product of the counit $\pi^*\pi_*\mathcal{H} \to \mathcal{H}$ and the identity $\mathrm{id}\colon \mathcal{H} \to \mathcal{H}$.

$\mathcal{H} \underset{E}{\times} \mathcal{H}$ is trivial and carries a 1-twisted $\mathcal{S}^1_{E/B}$-bundle. The associated $p^* C^\infty_{E/B}$-bundle to the pulled-back bundle to $\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}$ yields the kernel $\mathcal{L}$. It is 1-twisted with respect to $\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \to \hat{E} \underset{B}{\times} \mathcal{H}$ and (-1)-twisted with respect to $\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \to E \underset{B}{\times} \widehat{\mathcal{H}}$. We say that $\mathcal{L}$ is $(1,-1)$-twisted.

The *twisted Fourier-Mukai transform*

$$FM\colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

is defined to be the composition

$$\mathcal{D}_1(\mathcal{H}) \xrightarrow{p^*} \mathcal{D}_{(0,1)}(\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}) \xrightarrow{\otimes\mathcal{L}} \mathcal{D}_{(1,0)}(\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}) \xrightarrow{q_{2*}} \mathcal{D}_1(E \underset{B}{\times} \widehat{\mathcal{H}}) \xrightarrow{q_{1*}} \mathcal{D}_1(\widehat{\mathcal{H}}),$$

where push-forward along q_2 is the 0-twisted version. The kernel $\mathcal{L}$ also induces a functor in the opposite direction

$$\widehat{FM}\colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\mathcal{H}).$$

By definition, for every point $b \in B$ there exists an open neighborhood U of b such that every bundle and every gerbe is trivializable. The next diagram, the pullback of diagram (2) to a neighborhood U like that, illustrates that situation.

$$
\begin{tikzcd}
 & & \mathcal{H}_U \underset{U}{\times} \widehat{\mathcal{H}}_U \arrow[rrrrr, "t"] \arrow[dl, "q_2"'] \arrow[ddr, dashed, "\hat{q}_2"] & & & & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl, "p_2"'] \arrow[ddr, "\hat{p}_2"] & & \\
 & \mathcal{H}_U \underset{U}{\times} \hat{E}_U \arrow[ddl, "q_1"'] & & & & & \mathcal{H} \underset{B}{\times} \hat{E} \arrow[ddl, "p_1"'] & & & \\
 & & & E_U \underset{U}{\times} \widehat{\mathcal{H}}_U \arrow[ddr, dashed, "\hat{q}_1"] & & & & & E \underset{B}{\times} \widehat{\mathcal{H}} \arrow[ddr, "\hat{p}_1"] & \\
\mathcal{H}_U \arrow[rrrrr, "r"] \arrow[ddr] & & & & & \mathcal{H} \arrow[ddr] & & & & \\
 & & & & \widehat{\mathcal{H}}_U \arrow[rrrrr, dashed, "\hat{t}"] \arrow[ddl, dashed] & & & & & \widehat{\mathcal{H}} \arrow[ddl] \\
 & E_U \arrow[ddr] & & & & & E \arrow[ddr] & & & \\
 & & & \hat{E}_U \arrow[dl, dashed] & & & & & \hat{E} \arrow[dl] & \\
 & & U \arrow[rrrrr, "u"] & & & & & B & &
\end{tikzcd}
$$

It is the content of Section 4.2.2 to show that the local transformation is an equivalence. Given sections $s\colon E_U \to \mathcal{H}_U$ and $\hat{s}\colon \hat{E}_U \to \widehat{\mathcal{H}}_U$, we have the following

Proposition (Proposition 4.26). *On trivial gerbes, going back and forth is invertible, i.e. the composition*

$$\widehat{FM} \circ FM\colon \mathcal{D}_1(\mathcal{H}_U) \to \mathcal{D}_1(\mathcal{H}_U)$$

is an equivalence.

This is practically a formal consequence of the calculation of the push-forward of the kernel carried out in Section 4.1.2.

The last part of the thesis derives a global statement from the local equivalence.

Denote by $\mathcal{D}_1(\mathcal{H}) := \mathcal{D}(\mathrm{Ch}(\mathrm{Mod}_1(p^*\mathcal{S}^1_{E/B})))$ the derived category of the category $\mathrm{Mod}_1(p^*\mathcal{S}^1_{E/B})$ of 1-twisted sheaves on $\mathcal{H}$.

Let $u\colon U \to B$ be a cover such that $E_U := u^*E$ and $\mathcal{H}_U := \mathcal{H}|_{u^*E}$ are trivializable. Denote by $E_U^\bullet$ the Čech nerve whose n-th space is the n-fold pull-back $E_U \underset{E}{\times} \dots \underset{E}{\times} E_U$.

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\
\cdots \arrow[r] & E_U \underset{E}{\times} E_U \underset{E}{\times} E_U \arrow[r] & E_U \underset{E}{\times} E_U \arrow[r] & E_U \arrow[r] & E
\end{tikzcd}
$$

Pulling $\mathcal{H}$ along gives a simplicial stack $\mathcal{H}_U^\bullet$ and denote by $\mathcal{D}_1(\mathcal{H}_U^\bullet)$ its derived 1-twisted cosimplicial ∞-category.
For example, $\mathcal{D}_1(\mathcal{H}_U^0) = \mathcal{D}_1(\mathcal{H}_U)$. The cosimplicial ∞-category $\mathcal{D}_1(\mathcal{H}_U^\bullet)$ is an *augmented* cosimplicial ∞-category. It is augmented via the map $r^*\colon \mathcal{D}_1(\mathcal{H}_U^{-1}) = \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}_U)$.

We then use the theory of *descent* developed in [17, Section 6.2.4] to prove

Theorem (Theorem 5.2)**.** *The derived category $\mathcal{D}_1(\mathcal{H})$ fulfills descent with regard to trivializing covers, i.e. the map*

$$\mathcal{D}_1(\mathcal{H}) \xrightarrow{r^*} \lim_n \mathcal{D}_1(\mathcal{H}_U^n)$$

is an equivalence.

Pulling back $\mathcal{L}$ induces Fourier-Mukai transformations

$$FM^\bullet\colon \mathcal{D}_1(\mathcal{H}_U^\bullet) \to \mathcal{D}_1(\widehat{\mathcal{H}}_U^\bullet)$$

and

$$\widehat{FM}^\bullet\colon \mathcal{D}_1(\widehat{\mathcal{H}}_U^\bullet) \to \mathcal{D}_1(\mathcal{H}_U^\bullet)$$

that define maps of augmented cosimplical ∞-categories (see Lemma 5.7).

Lemma (Lemmas 5.7 and 5.8)**.** *The transforms FM^n assemble to give a map of cosimplicial ∞-categories. I.e., for any $\alpha\colon [n] \to [m]$ the following diagram of ∞-categories commutes.*

$$\begin{tikzcd}
\mathcal{D}_1(\mathcal{H}_U^m) \arrow[r, "FM^m"] \arrow[d, "\alpha_{\mathcal{H}}^*"'] & \mathcal{D}_1(\widehat{\mathcal{H}}_U^m) \arrow[d, "\alpha_{\widehat{\mathcal{H}}}^*"] \\
\mathcal{D}_1(\mathcal{H}_U^n) \arrow[r, "FM^n"] & \mathcal{D}_1(\widehat{\mathcal{H}}_U^n).
\end{tikzcd}$$

Even more is true, $FM^\bullet$ is a map of augmented *cosimplical ∞-categories. I.e. the diagram*

$$\begin{tikzcd}
\mathcal{D}_1(\mathcal{H}) \arrow[r, "FM"] \arrow[d, dashed, "G"'] & \mathcal{D}_1(\widehat{\mathcal{H}}) \arrow[d, dashed, "G"] \\
\mathcal{D}_1(\mathcal{H}_U^0) \arrow[r, "FM^0"] & \mathcal{D}_1(\widehat{\mathcal{H}}_U^0)
\end{tikzcd}$$

of ∞-categories commutes.

We proceed by showing that the compositions

$$\widehat{FM}^{\bullet} \circ FM^{\bullet} \colon \mathcal{D}_1(\mathcal{H}_U^{\bullet}) \to \mathcal{D}_1(\mathcal{H}_U^{\bullet})$$

and

$$FM^{\bullet} \circ \widehat{FM}^{\bullet} \colon \mathcal{D}_1(\widehat{\mathcal{H}}_U^{\bullet}) \to \mathcal{D}_1(\widehat{\mathcal{H}}_U^{\bullet})$$

are (levelwise) Fourier-Mukai transformations using Lemma 4.3.

Using descent and the fact that the composition is a local equivalence (Proposition 4.26) then yield the Main Theorem

Theorem (Theorem 5.9)**.** *The composition*

$$\widehat{FM} \circ FM \colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H})$$

is an equivalence.

Because the situation is entirely symmetrical, we may deduce

Theorem (Theorem 5.10)**.** *The twisted Fourier-Mukai transformations*

$$FM \colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

and

$$\widehat{FM} \colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\mathcal{H})$$

are equivalences.

Note that this theorem recovers the theorem of U. Bruzzo, G. Marelli and F. Pioli mentioned earlier in the introduction by restricting $\mathcal{H}$ to the trivial gerbe on the torus, where $B = \star$ is the one-point space.

Acknowledgements

First and foremost I thank my advisor Ulrich Bunke. He not only accepted me as a PhD student and suggested the topic, but also helped me writing this text in many many ways. I am very grateful to Markus Spitzweck, who showed me what twisted sheaves ought to be, at a time when they appeared to be somewhat magical. Overstating the impact Thomas Nikolaus had would be a very hard problem indeed, for he is the reason ∞-categories make an appearance.

If I had to, I would single out Tobias Sitte and Michael Völkl, for tirelessly listening to and correcting my ramblings – both mathematically and not. There is a whole parliament of people who had an impact on my thesis and on my life besides. It includes, but is not limited to, in no particular order, the Matthiases Nagel and Blank, Peter Arndt, Francesca Diana and David Gepner.

Then there is my family, my wife Leonie, my parents Monika and Otto and my sister Carola. Without them, nothing would have been possible.

2 Gerbes

This section introduces gerbes from the viewpoint of differentiable stacks. We assume that the reader knows what a topological/differentiable stack is. An introduction to those can be found in [12], [20] and [1]. We are mostly following section 2 of [16] in our description of $\mathcal{G}$-gerbes and twisted sheaves.

Definition 2.1. A map of stacks $f\colon \mathcal{M} \to \mathcal{N}$ is an epimorphism, if for any $T \to \mathcal{N}$ there is an open cover $U \to T$ and a section $s\colon U \to \mathcal{M}$:

$$\begin{tikzcd} & & \mathcal{M} \arrow[d, "f"] \\ U \arrow[r] \arrow[urr, dashed, "s"] & T \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \mathcal{N} \end{tikzcd}$$

Remark 2.2. That means that for every $t \in \mathcal{N}(T)$ there is an open cover $\{U_\alpha\}$ of T such that there are lifts $s_\alpha \in \mathcal{M}(U_\alpha)$ of the restrictions $t|_{U_\alpha}$. Here a lift is to be understood as s_α and an isomorphism $\varphi_\alpha\colon f(s_\alpha) \to t|_{U_\alpha}$ in $\mathcal{N}(U_\alpha)$.

Definition 2.3. A map of stacks $f\colon \mathcal{M} \to \mathcal{N}$ is called a gerbe if f and the diagonal $\mathcal{M} \to \mathcal{M} \underset{\mathcal{N}}{\times} \mathcal{M}$ are epimorphisms.

Remark 2.4. The first condition, that f is an epimorphism, means that there are local sections, whereas the second condition means that isomorphisms have local lifts. More concretely, given elements $s, s' \in \mathcal{M}(T)$ and an isomorphism $\phi\colon f(s) \to f(s')$, there is an open cover $\{U_\alpha\}$ of T and isomorphisms $\varphi_\alpha\colon s|_{U_\alpha} \to s'_{U_\alpha}$ lifting ϕ.

2.1 Relative Automorphisms

Before defining what a $\mathcal{G}$-gerbe is, we need to talk a little bit about automorphisms in a stack and about different ways to describe them. More details can be found in [20, section 3.4].

In this section, fix a morphism of stacks $f\colon \mathcal{M} \to \mathcal{N}$.

Definition 2.5. We define the *relative automorphism stack* $\mathrm{Aut}(-/f)$ to be the stack that to any space T assigns the category that has as

- objects: $\{(s, \varphi) \mid s \in \mathcal{M}(T),\ \varphi \in \mathrm{Aut}(s)\ f(\varphi) = \mathrm{id}\}$

- morphisms: a morphism from (s, φ) to (s', φ') is a morphism ψ such that

$$\begin{tikzcd} s \arrow[r, "\varphi"] \arrow[d, "\psi"'] & s \arrow[d, "\psi"] \\ s' \arrow[r, "\varphi'"] & s' \end{tikzcd}$$

commutes.

Definition 2.6. Define the relative inertia $\mathcal{I}_{\mathcal{M}/\mathcal{N}}$ by the following pullback:

$$\begin{tikzcd} \mathcal{I}_{\mathcal{M}/\mathcal{N}} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta"] & \\ \mathcal{M} \arrow[r, "\Delta"] & \mathcal{M} \times_{\mathcal{N}} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "f"] \\ & \mathcal{M} \arrow[r, "f"] & \mathcal{N} \end{tikzcd}$$

Lemma 2.7. $\mathrm{Aut}(-/f)$ *is equivalent to* $\mathcal{I}_{\mathcal{M}/\mathcal{N}}$.

Proof: Sketch: An object in $\mathcal{I}_{\mathcal{M}/\mathcal{N}}(T)$ is

- two objects $a, b \in \mathcal{M}(T)$ and
- a morphism $\Phi_{ab} \colon \Delta(a) \to \Delta(b)$ in $(\mathcal{M} \underset{\mathcal{N}}{\times} \mathcal{M})(T)$.

The morphism $\Phi_{ab} \colon \Delta(a) \to \Delta(b) \in (\mathcal{M} \underset{\mathcal{N}}{\times} \mathcal{M})(T)$ is given as a pair of morphisms $(\varphi_{ab} \colon a \to b, \varphi'_{ab} \colon a \to b)$ such that the following diagram commutes in the category $\mathcal{N}(T)$

$$\begin{tikzcd} f(a) \arrow[r, "f(\varphi_{ab})"] \arrow[d, "\mathrm{id}"'] & f(b) \arrow[d, "\mathrm{id}"] \\ f(a) \arrow[r, "f(\varphi'_{ab})"] & f(b) \end{tikzcd}$$

(Note that $\Delta(a) = (a, a, \mathrm{id}_{\mathcal{N}(T)} \colon f(a) \to f(a))$, that is where the identities in the diagram come from.) Hence

$$f(\phi_{ab}^{-1} \phi'_{ab}) = \mathrm{id}$$

and

$$\phi_{ab}^{-1} \phi'_{ab} \colon a \to a$$

is a relative automorphism.

A morphism $(a, b, \phi_{ab} \colon \Delta(a) \to \Delta(b)) \to (a', b', \phi_{a'b'} \colon \Delta(a') \to \Delta(b'))$ is a pair of morphisms $(\eta_1 \colon a \to a', \eta_2 \colon b \to b')$ such that the diagram

$$\begin{tikzcd} \Delta(a) \arrow[r, "\Delta(\eta_1)"] \arrow[d, "\phi_{ab}"'] & \Delta(a') \arrow[d, "\phi_{a'b'}"] \\ \Delta(b) \arrow[r, "\Delta(\eta_2)"] & \Delta(b') \end{tikzcd}$$

commutes. □

Lemma 2.8. *Let $f \colon \mathcal{M} \to \mathcal{N}$ be a map of stacks.* $\mathrm{Aut}(-/f)$ *is isomorphic to the pull-back P in*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{N} \arrow[d] \\ \mathrm{Aut}(\mathcal{M}/*) \arrow[r] & \mathrm{Aut}(\mathcal{N}/*) \end{tikzcd}$$

Definition 2.9. Let $s \colon X \to \mathcal{M}$ be a section, we define the sheaf $\mathrm{Aut}(s/f)$ on X as the assignment

$$(T \xrightarrow{t} X) \longmapsto \{\varphi \colon t^*s \to t^*s \mid f(\phi) = \mathrm{id}\}$$

Lemma 2.10. *The diagram*

$$\begin{tikzcd} \mathrm{Aut}(s/f) \arrow[r] \arrow[d] & \mathrm{Aut}(-/f) \arrow[d] \\ X \arrow[r, "s"] & \mathcal{M} \end{tikzcd}$$

is a pull-back.

Proof: Sketch: This is easy to verify as the pull-back

$$\begin{tikzcd} \mathrm{Aut}(s/f) \arrow[r] \arrow[d] & \mathrm{Aut}(-/f) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta"] \\ X \arrow[r, "s"] & \mathcal{M} \arrow[r, "\Delta"] & \mathcal{M} \times_{\mathcal{N}} \mathcal{M} \end{tikzcd}$$

because here the T-valued objects are:

- $t \colon T \to X$,
- $m \colon T \to \mathcal{M}$ and
- $\varphi = (\varphi_1, \varphi_2) \colon (t^*s, t^*s, \mathrm{id}_{\mathcal{N}}) \to (m, m, \mathrm{id}_{\mathcal{N}})$

such that

$$\begin{tikzcd}
f(t^*s) \arrow[r, "f(\varphi_1)"] \arrow[d, "\mathrm{id}"'] & f(m) \arrow[d, "\mathrm{id}"] \\
f(t^*s) \arrow[r, "f(\varphi_2)"] & f(m)
\end{tikzcd}$$

commutes. □

Lemma 2.11. *Let*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathcal{Y} \arrow[r] & \mathcal{N}
\end{tikzcd}$$

be a pull-back diagram in stacks, then

$$\mathrm{Aut}(-/\mathcal{X} \to \mathcal{Y}) \simeq F^*\mathrm{Aut}(-/\mathcal{M} \to \mathcal{N}).$$

Proof: In the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "\Delta"'] & \mathcal{M} \arrow[d, "\Delta"] \\
\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{M} \underset{\mathcal{N}}{\times} \mathcal{M} \arrow[d] \\
\mathcal{Y} \arrow[r] & \mathcal{N}
\end{tikzcd}$$

the large and the bottom diagrams are pull-backs, hence so is the top. Set $A = \mathrm{Aut}(-/\mathcal{M} \to \mathcal{N})$, then the question is equivalent is asking if the back-square in

$$\begin{tikzcd}
& F^*A \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{X} \arrow[dl, "F"'] \arrow[dd] \\
A \arrow[rr] \arrow[dd] & & \mathcal{M} \arrow[dd] & \\
& \mathcal{X} \arrow[rr] \arrow[dl] & & \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X} \arrow[dl] \\
\mathcal{M} \arrow[rr, dash] & & \mathcal{M} \underset{\mathcal{N}}{\times} \mathcal{M} &
\end{tikzcd}$$

is a pull-back square. Which it is, because all the other faces are. □

2.2 $\mathcal{G}$-Gerbes

Let $\mathcal{G}$ be a sheaf of abelian groups over a stack $\mathcal{N}$. The condition for $\mathcal{G}$ to be abelian is important, because for a general $\mathcal{G}$, the automorphisms of a $\mathcal{G}$-torsor are not $\mathcal{G}$ itself.

Definition 2.12. A $\mathcal{G}$-gerbe is a gerbe $\mathcal{M} \to \mathcal{N}$ together with an isomorphism $F\colon \mathcal{I}_{\mathcal{M}/\mathcal{N}} \simeq \mathcal{M} \times \mathcal{G}$ as stacks over $\mathcal{N}$.

Remark 2.13. The definition of a $\mathcal{G}$-gerbe implies that the relative automorphisms $\mathrm{Aut}(s/p)$, for a section $s\colon X \to \mathcal{M}$, are (canonically) isomorphic to $\mathcal{G}$. By lemma (2.10) we have the following pull-back:

$$\begin{tikzcd} X \times \mathcal{G} \simeq Aut(s/p) \arrow[r] \arrow[d] & \mathrm{Aut}(-/p) \simeq \mathcal{M} \times \mathcal{G} \arrow[d] \\ X \arrow[r] & \mathcal{M} \end{tikzcd}$$

Example 2.14. Let G be an abelian group and $\mathcal{G}$ its associated sheaf of G-valued functions, then

$$B\mathcal{G} \to *,$$

the stack of $\mathcal{G}$-torsors, is a $\mathcal{G}$-gerbe, as the following is a pull-back:

$$\begin{tikzcd} B\mathcal{G} \times \mathcal{G} \arrow[r] \arrow[d] & B\mathcal{G} \arrow[d, "\Delta"] \\ B\mathcal{G} \arrow[r, "\Delta"] & B\mathcal{G} \times_* B\mathcal{G} \end{tikzcd}$$

Remark 2.15. In fact, this is the prototypical example and any gerbe locally is of that form. One could have defined a $\mathcal{G}$-gerbe, for abelian $\mathcal{G}$, to be a stack over a space X, such that it locally is of the form $U \times B\mathcal{G}$ for small enough open sets $U \subset X$.

Example 2.16. Let $p\colon \mathcal{M} \to \mathcal{N}$ be a $\mathcal{G}$-gerbe and $f\colon \mathcal{H} \to \mathcal{N}$ a map of stacks. Form the pull-back

$$\begin{tikzcd} f^*\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "p"] \\ \mathcal{H} \arrow[r, "f"] & \mathcal{N}. \end{tikzcd}$$

Lemma (2.11) shows that $f^*\mathcal{M} \to \mathcal{H}$ is a $f^*\mathcal{G}$-gerbe.

Remark 2.17. Let $\mathcal{M} \to \mathcal{N}$ be a $\mathcal{G}$-gerbe via $\mathcal{G} \times \mathcal{M} \xrightarrow{F} \mathrm{Aut}(-/\mathcal{M} \to \mathcal{N})$ and let $a, b \in \mathcal{M}(T)$.

$$\mathrm{Hom}_{\mathcal{M}(T)}(a, b)$$

is both an $\mathrm{Aut}(a/\mathcal{M} \to \mathcal{N})$ and an $\mathrm{Aut}(b/\mathcal{M} \to \mathcal{N})$ torsor. Hence it is a $\mathcal{G}$-torsor in two ways: Let $\varphi \in \mathrm{Hom}_{\mathcal{M}(T)}(a, b)$, $g \in \mathcal{G}(T)$

- $g \odot_a \varphi = \varphi \circ F(g, a)$
- $g \odot_b \varphi = F(g, b) \circ \varphi$.

Note that this actually defines actions because $F(-, a)$ is an isomorphism between abelian groups.

Because F is a map of stacks, the diagram

$$\begin{tikzcd} a \arrow[r, "F(g,a)"] \arrow[d, "\varphi"'] & a \arrow[d, "\varphi"] \\ b \arrow[r, "F(g,b)"'] & b. \end{tikzcd}$$

commutes and the two $\mathcal{G}$-torsor structures agree:

$$g \odot_a \varphi = \varphi \circ F(g, a) = F(g, b) \circ \varphi = g \odot_b \varphi$$

Remark 2.18. In the situation above, we can further define a sheaf of $\mathcal{G}$-torsors $\mathrm{Hom}(t, t')$ by the assignment

$$(f: U \to T) \mapsto \mathrm{Hom}_{\mathcal{M}(U)}(f^*t, f^*t'),$$

where $t, t': T \to \mathcal{M}$.

2.3 Twisted Sheaves

For a general discussion of sheaves in this setting, see the very readable [22]. Somewhat closer to the context in question are [7] and [8].

For a differentiable stack $\mathcal{M}$ denote by $\underline{\mathcal{M}}$ the category whose objects are submersions $X \to \mathcal{M}$ where X is a smooth manifold and whose morphisms are triangles

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl] \\ & \mathcal{M}, & \end{tikzcd}$$

where $f: Y \to X$ is a smooth submersion, that commute up to a 2-cell. A cover $\{U_\alpha \to \mathcal{M}\}$ of $X \to \mathcal{M}$ is a jointly surjective collection of smooth submersions $U_\alpha \to \mathcal{M}$ such that the following diagram is a morphism in $\underline{\mathcal{M}}$:

$$\begin{tikzcd} U_\alpha \arrow[rr, "u"] \arrow[dr] & & X \arrow[dl] \\ & \mathcal{M}. & \end{tikzcd}$$

A sheaf on a differentiable stack $\mathcal{M}$ is a sheaf with respect to the site $\underline{\mathcal{M}}$.

There are at least two ways to define what a twisted sheaf is. The first, as can be seen in [10], defines it roughly as follows. Given a Čech 2-cocycle $g = \{g_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to \mathbb{C}^\times\}$ on X, a *g-twisted sheaf* $\mathcal{F}$ is a collection $(\{\mathcal{F}_\alpha\}, \varphi_{\alpha\beta})$, where $\mathcal{F}_\alpha$ is a sheaf on U_α and $\varphi_{\alpha\beta} \colon \mathcal{F}_\alpha|_{U_{\alpha\beta}} \to \mathcal{F}_\beta|_{U_{\alpha\beta}}$ are isomorphisms that do not quite glue, in the sense that

$$\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = g_{\alpha\beta\gamma}.$$

We prefer to follow [16] and define them as a full subcategory of sheaves on the gerbe representing g. The relation between these two definitions is explored in [16, Section 2.1.3].

Let $p \colon \mathcal{M} \to \mathcal{N}$ be a map of stacks.

Remark 2.19. The assignment

$$(f \colon T \to \mathcal{M}) \mapsto \mathrm{Aut}(f/p)$$

defines a sheaf on $\mathcal{M}$. It acts on any other sheaf $\mathcal{F}$ on $\mathcal{M}$ via the *inertia-action*:

$$\begin{array}{ccc} (\mathrm{Aut}(-/p) \times \mathcal{F})(T) & \to & \mathcal{F}(T) \\ \left(\begin{tikzcd} T \arrow[r, bend left, "s"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{M} \end{tikzcd}, f \right) & \mapsto & \mathcal{F}(\varphi)(f) \end{array}$$

Now let $p \colon \mathcal{M} \to \mathcal{N}$ be a $\mathcal{G}$-gerbe with a sheaf of rings $\mathcal{O}_\mathcal{M}$. Further, let $\chi \colon p^*\mathcal{G} \to \mathcal{O}_\mathcal{M}$ be a character and $\mathcal{F}$ a sheaf of $\mathcal{O}_\mathcal{M}$-modules.

Definition 2.20. A *k-fold χ-twisted sheaf* $\mathcal{F}$ on $\mathcal{M}$ is a sheaf of $\mathcal{O}_\mathcal{M}$-modules on $\mathcal{M}$, such that the following diagram commutes for some $k \in \mathbb{N}$:

$$\begin{tikzcd} \mathrm{Aut}(-/p) \times \mathcal{F} \arrow[r] \arrow[d, "\chi^k"'] & \mathcal{F} \arrow[d, "\mathrm{id}"] \\ \mathcal{O}_\mathcal{M} \times \mathcal{F} \arrow[r, "m"] & \mathcal{F} \end{tikzcd}$$

The character χ, although important, will generally be absent from the notation when there is no chance of confusion. A 1-twisted sheaf is simply called twisted. $\mathrm{Mod}_k(\mathcal{M}) = \mathrm{Mod}_k(\mathcal{O}_\mathcal{M})$ is the *category of k-twisted $\mathcal{O}_\mathcal{M}$-module sheaves on $\mathcal{M}$.*

Remark 2.21. We have a functor

$$(-)_k\colon \mathrm{Mod}(\mathcal{M}) \to \mathrm{Mod}_k(\mathcal{M}), \;\; \mathcal{F} \mapsto \mathcal{F}_k$$

defined by

$$\mathcal{F}_k(X \to \mathcal{M}) = \bigcap_{\varphi \in \mathrm{Aut}(X \to \mathcal{M})} \lim \left(\mathcal{F}(X \to \mathcal{M}) \underset{\chi(\varphi)^k}{\overset{\mathcal{F}(\varphi)}{\rightrightarrows}} \mathcal{F}(X \to \mathcal{M}) \right)$$

Lemma 2.22. *$(-)_k$ is right adjoint to the inclusion $i\colon \mathrm{Mod}_k(\mathcal{M}) \to \mathrm{Mod}(\mathcal{M})$.*

Proof: The triangle identities are easily checked for $\mathcal{F} \xrightarrow{\mathrm{id}} (i\mathcal{F})_k$ for a k-twisted sheaf $\mathcal{F}$ and $i(\mathcal{G}_k) \hookrightarrow \mathcal{G}$ the canonical inclusion for any sheaf $\mathcal{G}$. $\square$

Lemma 2.23. *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a smooth representable map of stacks and $\mathcal{H} \to \mathcal{N}$ be a gerbe. Further, let $F\colon \mathcal{G} \to \mathcal{H}$ be defined via the pull-back*

$$\begin{tikzcd} \mathcal{G} \arrow[r, "F"] \arrow[d] & \mathcal{H} \arrow[d] \\ \mathcal{M} \arrow[r, "f"] & \mathcal{N}, \end{tikzcd}$$

then push-forward and pull-back along F preserve twists and are an adjoint pair:

$$F^*\colon \mathrm{Mod}_k(\mathcal{H}) \rightleftarrows \mathrm{Mod}_k(\mathcal{G}) : F_*$$

Proof: For the case of the pull-back, let $\mathcal{F} \in \mathrm{Mod}_k(\mathcal{H})$ be a k-twisted sheaf on $\mathcal{H}$ and $\varphi \in \mathrm{Aut}(s/\mathcal{G} \to \mathcal{M})$ a relative automorphism of $s\colon T \to \mathcal{G}$:

$$\begin{tikzcd} T \arrow[r, bend left, "s"{name=U}] \arrow[r, bend right, "s"'{name=D}] & \mathcal{G} \arrow[from=U, to=D, phantom, "\Downarrow\varphi"] \end{tikzcd}$$

Then

$$\begin{tikzcd} F^*\mathcal{F}(T \xrightarrow{s} \mathcal{G}) \arrow[r, "F^*\mathcal{F}(\varphi)"] \arrow[d, "\mathrm{id}"] & F^*\mathcal{F}(T \xrightarrow{s} \mathcal{G}) \arrow[d, "\mathrm{id}"] \\ \mathcal{F}(T \xrightarrow{s} \mathcal{G} \to \mathcal{H}) \arrow[r, "\mathcal{F}(F(\varphi))"] \arrow[d, "\mathrm{id}"] & \mathcal{F}(T \xrightarrow{s} \mathcal{G} \to \mathcal{H}) \arrow[d, "\mathrm{id}"] \\ \mathcal{F}(T \xrightarrow{s} \mathcal{G} \to \mathcal{H}) \arrow[r, "\cdot\chi(\mathcal{F}(\varphi))^k"] & \mathcal{F}(T \xrightarrow{s} \mathcal{G} \to \mathcal{H}) \end{tikzcd}$$

commutes. The result follows with the identification

$$\mathrm{Aut}(-/\mathcal{G} \to \mathcal{M}) \simeq F^*\mathrm{Aut}(-/\mathcal{H} \to \mathcal{N})$$

of relative automorphisms of $\mathcal{G}$ with the pull-back of relative automorphisms of $\mathcal{H}$ form Lemma 2.11.

The proof in the case of the push-forward is similar. $\square$

Lemma 2.24. *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a gerbe. The functor*

$$f_*\colon \mathrm{Mod}_0(\mathcal{M}) \to \mathrm{Mod}(\mathcal{N})$$

is an equivalence of categories. This functor sometimes is also called $\mathrm{desc}(f)$.

Proof: This can also be seen in ([16, Lemma 2.1.1.17]). We prove that $f^*f_* \to \mathrm{id}$ is an equivalence.

We may do so locally and can assume that f has a section $s\colon \mathcal{N} \to \mathcal{M}$. Note that $f^*\colon \mathrm{Mod}_0(\mathcal{M}) \leftrightarrows \mathrm{Mod}(\mathcal{N}) : s^*$ is an equivalence of categories, hence $s^* = f_*$ and the claim follows.

The last step may need more justification. The equality $s^*f^* = \mathrm{id}$ is clear because s is a section. To give a map $f^*s^* \to \mathrm{id}$, we have to give a map

$$(f^*s^*)\mathcal{F}(X \to \mathcal{M}) = \mathcal{F}(X \to \mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{s} \mathcal{M}) \to \mathcal{F}(X \to \mathcal{M})$$

for any $\mathcal{F} \in \mathrm{Mod}(\mathcal{M})$ and any $X \to \mathcal{M}$. Since $f\colon \mathcal{M} \to \mathcal{N}$ is a gerbe and objects are locally isomorphic, we find a cover $U \to X$ such that the diagram

$$\begin{tikzcd}
U \arrow[rrr] \arrow[d] & & & X \arrow[d] \\
X \arrow[r] & \mathcal{M} \arrow[r, "f"] & \mathcal{N} \arrow[r, "s"] & \mathcal{M}
\end{tikzcd}$$

commutes up to a 2-cell σ. Since $\mathcal{F}$ is a sheaf and $U \to X$ a cover, the vertical sequences in the following diagram are exact.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{F}(X \to \mathcal{M}) \arrow[r, dashed] \arrow[d] & \mathcal{F}(X \to \mathcal{M} \to \mathcal{N} \to \mathcal{M}) \arrow[d] \\
\mathcal{F}(U \to X \to \mathcal{M}) \arrow[r, "\mathcal{F}(\sigma)"] \arrow[d] & \mathcal{F}(U \to X \to \mathcal{M} \to \mathcal{N} \to \mathcal{M}) \arrow[d] \\
\mathcal{F}(U \times_{\mathcal{M}} U \to \mathcal{M}) \arrow[r, "\mathcal{F}(\sigma')"] & \mathcal{F}(U \times_{\mathcal{M}} U \to \mathcal{M})
\end{tikzcd}$$

Since $\mathcal{F}$ is 0-twisted, $\mathcal{F}(\sigma)$ and $\mathcal{F}(\sigma')$ are identities (they are induced by 2-cells). The right square commutes, hence the left arrow also is id.

□

Lemma 2.25. *The category of twisted sheaves* $\mathrm{Mod}_k(\mathcal{O}_{\mathcal{M}})$ *is complete and co-complete.*

Proof: Note that colimits (resp. limits) are computed object-wise, followed by sheafification. Sheafification preserves k-twistedness and coproducts (resp. products) and coequalizers (resp. equalizers) of twisted presheaves are twisted. □

Lemma 2.26. *Let* $f\colon \mathcal{M} \to \mathcal{N}$ *be a representable map of smooth stacks and* $\mathcal{O}_{\mathcal{N}}$ *a sheaf of rings on* $\mathcal{N}$. *Set* $\mathcal{O}_{\mathcal{M}} = f^*\mathcal{O}_{\mathcal{N}}$. *The functor*

$$f^*\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{N}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{M}})$$

has a left adjoint

$$f_!\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{M}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{N}}).$$

Proof: This is [21, Tag 0797], where the map of sites is precomposition with f. □

Lemma 2.27. *Let* $\mathcal{H} \xrightarrow{p} E$ *be a* $\mathcal{G}$*-gerbe and* $U \xrightarrow{f} E$ *the inclusion of an open subset. Consider the pull-back*

$$\begin{tikzcd} f^*\mathcal{H} \arrow[r, "F"] \arrow[d] & \mathcal{H} \arrow[d, "p"] \\ U \arrow[r, "f"] & E \end{tikzcd}$$

Then $F_!\colon \mathrm{Mod}(\mathcal{H}) \to \mathrm{Mod}(f^*\mathcal{H})$ *preserves* 1*-twisted sheaves, i.e.*

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{H}) \arrow[d, "i"] & \mathrm{Mod}(f^*\mathcal{H}) \arrow[l, "F_!"'] \arrow[d, "i"] \\ \mathrm{Mod}_1(\mathcal{H}) & \mathrm{Mod}_1(f^*\mathcal{H}) \arrow[l, "F_!^1"] \end{tikzcd}$$

commutes, where $F_1^* = F^*|_{\mathrm{Mod}_1(\mathcal{H})}$.

Proof: See [10, Theorem 1.2.10] □

Lemma 2.28. *The category* $\mathrm{Mod}(\mathcal{O}_{\mathcal{N}})$ *of* $\mathcal{O}_{\mathcal{N}}$*-module sheaves on a stack* $\mathcal{N}$ *is a Grothendieck abelian category.*

Proof: See [15, Theorem 18.1.6.v]. □

Lemma 2.29. *Let $\mathcal{H} \xrightarrow{p} E$ be a $\mathcal{G}$-gerbe with section $E \xrightarrow{s} \mathcal{H}$. The category of 1-twisted sheaves is equivalent to the category of sheaves on E:*

$$s^*\colon \mathrm{Mod}_1(p^*\mathcal{O}_E) \xrightarrow{\simeq} \mathrm{Mod}(\mathcal{O}_E).$$

Proof: This can also be seen in ([16, Lemma 2.1.3.10]). We include a proof for the sake of convenience. Let $\mathcal{H}$ be the $p^*\mathcal{O}_E$-module associated to the $\mathcal{G}$-torsor $\mathrm{Hom}(-, s \circ p \circ -)$, which assigns to any $f\colon X \to \mathcal{H}$ the $\mathcal{G}$-torsor $\mathrm{Hom}_{\mathcal{H}(X)}(f, s \circ p \circ f)$. It is 1-twisted and its pull-back along s is trivial. Hence, we can define

$$u\colon \mathrm{Mod}(E) \to \mathrm{Mod}_1(\mathcal{H}), \;\; \mathcal{F} \mapsto p^*\mathcal{F} \otimes_{p^*\mathcal{O}_E} \mathcal{H}.$$

It is the inverse of the pull-back s^* along s. □

Lemma 2.30. *The category of twisted sheaves* $\mathrm{Mod}_1(\mathcal{O}_{\mathcal{M}})$ *on a gerbe* $\mathcal{M} \to M$ *is a Grothendieck abelian category.*

Proof: The category $\mathrm{Mod}(\mathcal{O}_{\mathcal{M}})$ of all $\mathcal{O}_{\mathcal{M}}$-modules is a Grothendieck abelian category, hence it remains to find a generator. Let $f\colon U \to M$ be a cover such that $f^*\mathcal{M} \to U$ admits a section $s\colon U \to f^*\mathcal{M}$. By 2.29, pulling back via s identifies $\mathrm{Mod}_1(f^*\mathcal{M})$ with $\mathrm{Mod}(U)$ which is a Grothendieck abelian category and has as generating family a family $\{G_{X\to U}\}_{X\to U \in \underline{U}}$ such that

$$\mathrm{Hom}(G_{X\to U}, \mathcal{F}) = \mathcal{F}(X \to U).$$

Let $F\colon f^*\mathcal{M} \to \mathcal{M}$ be the induced map. It follows that

$$G^{\mathcal{M}}_{X\to U} := F_!(s^*)^{-1} G_{X\to U}$$

defines a system of generators of $\mathrm{Mod}_1(\mathcal{O}_{\mathcal{M}})$, because the equality

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}(\mathcal{M})}(G^{\mathcal{M}}_{X\to U}, \mathcal{F}) &= \mathrm{Hom}_{\mathrm{Mod}(\mathcal{M})}((s^*)^{-1}G_{X\to U}, F^*\mathcal{F}) \\
&= \mathrm{Hom}_{\mathrm{Mod}(U)}(G_{X\to U}, s^*F^*\mathcal{F}) \\
&= s^*F^*\mathcal{F}(X \to U) \\
&= \mathcal{F}(X \to U \to f^*\mathcal{M} \to \mathcal{M}),
\end{aligned}$$

implies that the functor

$$\prod_{X\to U} \mathrm{Hom}(G^{\mathcal{M}}_{X\to U}, -)$$

is conservative. □

Example 2.31. Consider the universal bundle $u\colon pt \to BS^1$. Its associated C^∞-module is 1-twisted. When this is the case, one says that u has *weight* 1. For more on this, see [12, Lemma 5.6].

Lemma 2.32. *The tensor product of two twisted sheaves is twisted with the sum of the twists.*

Example 2.33. Let $\mathcal{H} \to E$ be a $\mathcal{G}$-banded gerbe. Consider the following pull-back:

$$\begin{tikzcd} \mathcal{H}\underset{E}{\times}\mathcal{H} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathcal{H} \arrow[d] \\ \mathcal{H} \arrow[u, bend left, "s"] \arrow[ur, "id"] \arrow[r] & E \end{tikzcd}$$

There is a canonical section $s\colon \mathcal{H} \to \mathcal{H}\underset{E}{\times}\mathcal{H}$ given by the identity $\mathcal{H} \to \mathcal{H}$. This section induces a $\mathcal{G}$-torsor on $\mathcal{H}\underset{E}{\times}\mathcal{H}$ given by a map $\mathcal{H}\underset{E}{\times}\mathcal{H} \to \mathcal{H} \times B\mathcal{G}$ (this map is actually an isomorphism). An object of $\mathcal{H}\underset{E}{\times}\mathcal{H}(T)$ is a pair (t_1, t_2) with $t_1, t_2 \in \mathcal{H}(T)$ such that t_1 and t_2 are equal in $E(T)$. That pair is mapped to $(t_1, \mathrm{Hom}(t_1, t_2)) \in H \times B\mathcal{G}(T)$. A map of pairs $\varphi\colon (t_1, t_2) \to (t_1', t_2')$ is a pair $(\varphi_1\colon t_1 \to t_1', \varphi_2\colon t_2 \to t_2')$ such that the induced maps in $E(T)$ are equal. The map $(t_1, \mathrm{Hom}(t_1, t_2)) \to (t_1', \mathrm{Hom}(t_1', t_2'))$ is explained as the pair of maps $\varphi_1\colon t_1 \to t_1'$ and

$$\mathrm{Hom}(t_1, t_2) \xrightarrow{\varphi_{2,*}} \mathrm{Hom}(t_1, t_2') \xrightarrow{(\varphi_1^{-1})^*} \mathrm{Hom}(t_1', t_2'). \tag{3}$$

Given $\psi \in \mathrm{Hom}(t_1, t_2)$ this map is the following composition:

$$\begin{tikzcd}[column sep=huge] T \arrow[r, bend left=70, "t_1'"{name=A}] \arrow[r, bend left=20, "t_1"{name=B}] \arrow[r, bend right=20, "t_2"'{name=C}] \arrow[r, bend right=70, "t_2'"'{name=D}] & \mathcal{H} \arrow[from=A, to=B, phantom, "\Downarrow\varphi_1^{-1}"] \arrow[from=B, to=C, phantom, "\Downarrow\psi"] \arrow[from=C, to=D, phantom, "\Downarrow\varphi_2"] \end{tikzcd}$$

In our situation, we are considering $\varphi = (\varphi_1, \varphi_2) \in \mathrm{Aut}_{\mathcal{H}\underset{E}{\times}\mathcal{H}}(t_1, t_2)$. Under the composition $c\colon \mathcal{H}\underset{E}{\times}\mathcal{H} \to \mathcal{H} \times B\mathcal{G} \to B\mathcal{G}$, φ is mapped to a morphism $\mathrm{Hom}(t_1, t_2) \xrightarrow{c(\varphi)} \mathrm{Hom}(t_1, t_2)$.

Notice that homomorphisms $\mathrm{Hom}(t_1, t_2) \to \mathrm{Hom}(t_1', t_2')$ are pairs of morphisms $(t_1' \to t_1, t_2 \to t_2')$.

We now get a $\mathcal{G}$-torsor L of weight 1 (that is, the associated module is 1-twisted) on $\mathcal{H} \underset{E}{\times} \mathcal{H}$ as the following pull-back:

$$\begin{tikzcd}
pt \arrow[d] & \mathcal{H} \times pt \arrow[l] \arrow[d] & L \arrow[l] \arrow[d] \\
B\mathcal{G} \arrow[d] & \mathcal{H} \times B\mathcal{G} \arrow[l] \arrow[d] & \mathcal{H} \underset{E}{\times} \mathcal{H} \arrow[l] \arrow[d, "\pi_1"] \\
pt & \mathcal{H} \arrow[l] & \mathcal{H} \arrow[l]
\end{tikzcd}$$

Because the map $\mathcal{H} \underset{E}{\times} \mathcal{H} \to \mathcal{H} \times B\mathcal{G}$ is an isomorphism, we see that L is isomorphic to $\mathcal{H}$. It is easy to see that it is a 1-twisted sheaf on $\mathcal{H} \underset{E}{\times} \mathcal{H}$ with respect to $\mathcal{H} \underset{E}{\times} \mathcal{H} \xrightarrow{\pi_1} \mathcal{H}$.

Example 2.34. To extend the previous example, consider $\mathcal{H} \underset{E}{\times} \mathcal{H}$ as a gerbe not over π_1, but π_2. The relative automorphism groups have the form

$$\mathrm{Aut}(-/\pi_1) = \{(id, \varphi_2) \mid (id, \varphi_2) \in \mathrm{Aut}(-)\}$$
$$\mathrm{Aut}(-/\pi_2) = \{(\varphi_1, id) \mid (\varphi_1, id) \in \mathrm{Aut}(-)\}.$$

Denote the composition $\mathcal{H} \underset{E}{\times} \mathcal{H} \to \mathcal{H} \times B\mathcal{G} \to B\mathcal{G}$ as c. Using formula (3) we see that $c(\varphi_1, id)$ acts as $\psi \circ \varphi_1^{-1}$ for $\psi \in \mathrm{Hom}(t_1, t_2)$.

Expressing that as the $\mathcal{G}$-action on the $\mathcal{G}$-torsor, by remark (2.17), this is $g^{-1} \cdot \psi$ where $g \in \mathcal{G}(T)$ is the image of $\psi \in \mathrm{Hom}(t_1, t_2)$ under $\mathrm{Aut}(-/\pi_2)\colon \mathcal{H} \underset{E}{\times} \mathcal{H} \to \mathcal{H} \underset{E}{\times} \mathcal{H} \times \mathcal{G}$.

Hence,

$$\begin{tikzcd}
c^* pt(T \to \mathcal{H} \underset{E}{\times} \mathcal{H}) \arrow[r, "c^* pt(\varphi, id)"] \arrow[d, equal] & c^* pt(T \to \mathcal{H} \underset{E}{\times} \mathcal{H}) \arrow[d, equal] \\
pt(T \to B\mathcal{G}) \arrow[r, "c(\varphi, id)", "=(-\cdot g^{-1})"'] & pt(T \to B\mathcal{G})
\end{tikzcd}$$

commutes and L is (-1)-twisted relative to π_2.

2.4 The Derived Category

In this section we record the necessary ingredients of [17, Section 1.3] which are needed for our main result.

Given a Grothendieck abelian category $\mathcal{A}$, by [17, Proposition 1.3.5.4] there is a left proper combinatorial model structure on $\mathrm{Ch}(\mathcal{A})$ such that

- cofibrations are level-wise monomorphisms and
- weak equivalences are quasi-isomorphisms.

The category of chain complexes with values in $\mathcal{A}$ can be viewed as a differential graded category ([17, Definition 1.3.2.1]). The *Derived Category* of $\mathcal{A}$ then is defined to be the differential graded nerve $\mathcal{D}(\mathcal{A}) := N_{dg}(\mathrm{Ch}(\mathcal{A})^\circ)$ of $\mathrm{Ch}(\mathcal{A})^\circ$, the category generated by the fibrant objects of $\mathrm{Ch}(\mathcal{A})$. By [17, Proposition 1.3.5.9] it is a stable ∞-category.

Classically, the derived category is the category of chain-complexes localized at the weak equivalences. [17, Proposition 1.3.5.15] tells us, that this is indeed the same as $\mathcal{D}(\mathcal{A})$. Precisely, the composition

$$N(\mathrm{Ch}(\mathcal{A})) \to N_{dg}(\mathrm{Ch}(\mathcal{A})) \to \mathcal{D}(\mathcal{A})$$

exhibits $\mathcal{D}(\mathcal{A})$ as the underlying ∞-category of the model category $\mathrm{Ch}(\mathcal{A})$, i.e. $N(\mathrm{Ch}(\mathcal{A})^c)[W^{-1}] \simeq \mathcal{D}(\mathcal{A})$. By [17, Proposition 1.3.5.21] this category is presentable. In particular it is cocomplete and has all geometric realizations, i.e. colimits of simplicial objects in $\mathcal{D}(\mathcal{A})$.

Remark 2.35. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a smooth map between smooth stacks such that $f^*\mathcal{O}_\mathcal{N} \simeq \mathcal{O}_\mathcal{M}$. Because $f^*\colon \mathrm{Ch}(\mathrm{Mod}(\mathcal{N})) \to \mathrm{Ch}(\mathrm{Mod}(\mathcal{M}))$ is exact, it preserves cofibrations and trivial cofibrations, so it is a *left Quillen functor*, i.e.

$$f^*\colon \mathrm{Ch}(\mathrm{Mod}(\mathcal{N})) \leftrightarrows \mathrm{Ch}(\mathrm{Mod}(\mathcal{M})) : f_*$$

is a *Quillen adjunction*. In particular, this implies, by Quillen's total derived functor theorem, that

$$f^*\colon \mathcal{D}(\mathcal{N}) \leftrightarrows \mathcal{D}(\mathcal{M}) : f_*$$

are adjoint. In particular, f_*, being a right Quillen functor, preserves fibrant objects.

Because $\mathcal{L}$ is locally free, forming the tensor-product $- \otimes_{C^\infty_{E/B}} \mathcal{L}$ is exact, hence its derived functor exists and is adjoint to $\mathrm{Hom}(\mathcal{L}, -)$.

Definition 2.36. Let $\mathcal{M}$ be a gerbe with structure sheaf $\mathcal{O}_\mathcal{M}$ and $n \in \mathbb{Z}$ a natural number. The derived category of $\mathrm{Mod}_n(\mathcal{O}_\mathcal{M})$ is denoted $\mathcal{D}_n(\mathcal{M})$ or $\mathcal{D}_n(\mathcal{O}_\mathcal{M})$.

Definition 2.37. Let $\mathcal{A}$ be an abelian category and $F\colon \mathcal{A} \to \mathcal{A}$ a left exact additive functor such that its derived functor exists. It has *finite cohomological dimension* if there is an $n \in \mathbb{N}$ such that $R^iF\colon \mathcal{A} \to \mathcal{A}$ vanishes for all $i > n$.

Proposition 2.38 (Projection Formula)**.** *Let $f\colon \mathcal{M} \to \mathcal{N}$ be a smooth representable map of smooth gerbes, $\mathcal{P} \in \mathcal{D}_k(\mathcal{N})$ perfect and $\mathcal{F}^\bullet \in \mathcal{D}_k(\mathcal{M})$ arbitrary, then*

$$f_*\mathcal{F} \otimes \mathcal{P} \simeq Rf_* \left(\mathcal{F} \otimes f^*\mathcal{P}\right)$$

Proof: Again, there is a map

$$f_*\mathcal{F} \otimes \mathcal{P} \to Rf_* \left(\mathcal{F} \otimes f^*\mathcal{P}\right)$$

given as the adjoint of

$$f^* \left(Rf_*\mathcal{F} \otimes \mathcal{P}\right) \simeq f^*Rf_*\mathcal{F} \otimes f^*\mathcal{P} \to \mathcal{F} \otimes f^*\mathcal{P}$$

on the level of ∞-categories. The proof of [21, Tag 0943] shows that this is a weak equivalence when $\mathcal{P}$ is perfect. $\square$

Remark 2.39. Let $f\colon U \to \mathcal{M}$ be an object in the site of a stack $\mathcal{M}$. Let (U) be the small site on U, i.e. open sets as objects, inclusions as morphism and open covers as coverings. Denote by $\mathrm{Sh}^s(U)$ and $\mathcal{D}^s(U)$ the categories with regard to the small site. By [18, II.3.18] and [8, Lemma 6.1.11], associated to f is an adjoint pair

$$\mathrm{ext}\colon \mathrm{Sh}^s(U) \leftrightarrows \mathrm{Sh}(\mathcal{M}) : \mathrm{res}_U,$$

with ext being exact. Note that $\mathrm{res}_U(\mathcal{F})$ has as shorthand notation $\mathcal{F}_U$. These functors induce a Quillen adjunction

$$\mathrm{ext}\colon \mathcal{D}^s(U) \leftrightarrows \mathcal{D}(\mathcal{M}) : \mathrm{res}_U.$$

Here is a list of some of its properties. See [8, 6.1.14] for more.

- For $\mathcal{F},\ \mathcal{G} \in \mathcal{D}(\mathcal{M})$, a map $\mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $res_U(\mathcal{F} \to \mathcal{G})$ is one for all U.
- For a representable map $f\colon \mathcal{N} \to \mathcal{M}$ and $U \to \mathcal{M}$ in the site of $\mathcal{M}$, denote by $f'\colon V := \mathcal{N} \underset{\mathcal{M}}{\times} U \to U$ the projection, then, for $\mathcal{F} \in \mathcal{D}(\mathcal{N})$, we have
$$(f_*\mathcal{F})_U = f'_*(\mathcal{F}_V)$$
This is easy to see on the level of sheaves. The derived version follows from the fact that res_U preserves fibrant objects.

Definition 2.40. Let $F\colon \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$ be a left exact additive functor such that its derived functor exists. We say it has *locally finite cohomological dimension* if for any $U \to Y$ in the site of Y, the functor $\mathrm{res}_{U\to Y} \circ F$ has finite cohomological dimension.

Lemma 2.41. *Let $f\colon M \to N$ be a smooth representable map of smooth manifolds and $\mathcal{H} \to N$ a stack over N. Consider the pullback-square*

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f'"] \arrow[d] & \mathcal{H} \arrow[d] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

The functor $f'_*\colon \mathrm{Mod}(\mathcal{G}) \to \mathrm{Mod}(\mathcal{H})$ has locally finite cohomological dimension if f is a topological submersion in the sense of [14, Definition 3.3.1].

Proof: For any $V \to \mathcal{H}$, the map g in the pullback diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d] & V \arrow[d] \\ \mathcal{G} \arrow[r, "f'"] \arrow[d] & \mathcal{H} \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

is a topological submersion, and has finite cohomological dimension. This implies

$$R^i f'_*(\mathcal{F})_V = R^i g_*(\mathcal{F}_P) = 0$$

for $i > n$ for some $n \in \mathbb{N}$. Hence f'_* has locally finite cohomological dimension. □

Definition 2.42. A stack $\mathcal{M}$ is called *locally compact* if it admits an atlas $A \to \mathcal{M}$ such that A and $A \underset{\mathcal{M}}{\times} A$ are locally compact as topological spaces (i.e. Hausdorff and every point admits a compact neighborhood).

Proposition 2.43 (Base Change). *Consider the cartesian diagram of locally compact stacks over a space B,*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "v"] \arrow[d, "g"'] & \mathcal{G} \arrow[d, "f"] \\ \mathcal{N} \arrow[r, "u"'] & \mathcal{H} \end{tikzcd}$$

where all maps are smooth and u and v are representable. Further assume that the structure sheaves of $\mathcal{M}$, $\mathcal{N}$, $\mathcal{G}$ and $\mathcal{H}$ are pulled back from B. Then the statement is: If $g_*\colon \mathrm{Mod}(\mathcal{M}) \to \mathrm{Mod}(\mathcal{N})$ has locally finite cohomological dimension and is representable, then the base change-morphism

$$u^* Rf_* \to Rg_* v^*$$

is an equivalence.

Proof: The proof is heavily inspired by [8, Lemma 6.5.7].

Choosing fibrant replacement functors

$$I_{\mathcal{M}} \colon \mathrm{Ch}(\mathrm{Mod}(\mathcal{M})) \to \mathrm{Ch}(\mathrm{Mod}(\mathcal{M})), \quad I_{\mathcal{G}} \colon \mathrm{Ch}(\mathrm{Mod}(\mathcal{G})) \to \mathrm{Ch}(\mathrm{Mod}(\mathcal{G}))$$

we can write the base-change morphism on the level of chain complexes as

$$u^* f_* I_{\mathcal{G}} \simeq g_* v^* I_{\mathcal{G}} \to g_* I_{\mathcal{M}} v^* I_{\mathcal{G}} \leftarrow g_* I_{\mathcal{M}} v^* \ .$$

For the first isomorphism we need the assumption on the structure sheaves. It ensures that pull-backs can be formed as if we dealt with sheaves of abelian groups. There is no need to tensorize with the structure sheaf. The proof then is the same as in [7, Lemma 2.16].

The second map is a quasi-isomorphism because the functors v^* and $g_* I_{\mathcal{M}}$ preserve quasi-isomorphisms and $\mathrm{id} \to \mathcal{I}_{\mathcal{G}}$ is one.

To show that the first map is a quasi-isomorphism we need the finiteness condition. By [17, Proposition 1.3.5.6] the complex $I_{\mathcal{G}}\mathcal{F}$ consists of injective, hence flabby, sheaves. Because v is representable the push-forward v^* preserves flabby sheaves (see [8, Lemma 3.1.5]) and $v^* I_{\mathcal{G}}\mathcal{F}$ is a complex of flabby sheaves.

Hence, we have to show that for any complex $A \in \mathrm{Ch}(\mathrm{Mod}(\mathcal{G}))$ of flabby sheaves the map $g_* A \to g_* I_{\mathcal{M}} A$ is a quasi-isomorphism. This can be checked locally. Let $U \to \mathcal{M}$ be in the site of $\mathcal{M}$ and form the pull-back

$$\begin{tikzcd} V \arrow[r] \arrow[d, "g'"'] & \mathcal{M} \arrow[d, "g"] \\ U \arrow[r] & \mathcal{N}. \end{tikzcd}$$

Recall that $(-)_U$ is a right Quillen functor, hence we need to show that

$$g'_*(A_U) = (g_* A)_U \to (g_* I_{\mathcal{G}} A)_U = g'_*(I_U A_U)$$

is a quasi isomorphism. Note that $(-)_U$ preserves flabby sheaves by [7, Lemma 2.4.9]. Hence the mapping cone C of $A_U \to I_U A_U$ is an exact complex of flabby sheaves. Setting $Z^n := \ker(C^n \to C^{n+1})$ we can decompose C into short exact sequences

$$0 \to Z^n \to C^n \to Z^{n+1} \to 0.$$

Applying g'_* yields the exact sequence

$$0 \to g'_*(Z^n) \to g'_*(C^n) \to g'_*(Z^{n+1}) \to R^1 g'_*(Z^n) \to 0$$

and isomorphisms

$$R^k g'_*(Z^{n+1}) \simeq R^{k+1} g'_*(Z^n)$$

for all $k \geq 1$. This is because flabby sheaves are g_*-acyclic by [8, Lemma 3.1.4]. By induction one sees that

$$R^k g'_*(Z^n) \simeq R^{k+1} g'_*(Z^{n-l})$$

for all $k \geq 1$ and all $l \in \mathbb{N}$. But g'_* has finite cohomological dimension and we may conclude that $R^k g'_*(Z^n) = 0$ for all $n \in \mathbb{Z}$ and $k \geq 1$. Hence the sequences

$$0 \to g'_*(Z^n) \to g'_*(C^n) \to g'_*(Z^{n+1}) \to 0$$

are exact for all $n \in \mathbb{Z}$ and thus C is exact. □

3 T-Duality

3.1 A Rather General Setting

From now on, the site will be smooth submersions with coverings the jointly surjective smooth submersions. Let $\pi\colon E \to B$ be a representable smooth map of smooth manifolds. Let $\mathcal{R}$ be a sheaf of rings on B and $\mathcal{G}$ a sheaf of groups on B together with a character $\chi\colon \mathcal{G} \to \mathcal{R}$. Further, let $\mathcal{H}$ be a $\pi^*\mathcal{G}$-gerbe on E.

Since π is representable, we can define the push-forward $\pi_*\mathcal{H}$ of $\mathcal{H}$. It is a stack on B and defined by the following formula

$$\pi_*\mathcal{H}(M \to B) = \mathcal{H}(M \underset{B}{\times} E \to E).$$

Definition 3.1. Like in [6], the stack $\widehat{\mathcal{H}}$ is called the *stack of local sections of* $\mathcal{H}$.

Given an element $t \in \pi_*\mathcal{H}(X \xrightarrow{f} B)$, its automorphisms are

$$\mathrm{Aut}(t) = \pi^*\mathcal{G}(E \underset{B}{\times} X \to E) = \pi_*\pi^*\mathcal{G}(X \to B).$$

Let $\hat{E}$ be the sheaf associated to the presheaf $\pi_0(\pi_*\mathcal{H})$ of isomorphism classes of $\pi_*\mathcal{H}$. Further let $\hat{\pi}\colon \hat{E} \to B$ be the canonical map. Viewed as a gerbe over $\hat{E}$, it is a $\hat{\pi}^*\pi_*\pi^*\mathcal{G}$-gerbe. Assume, as is the case with $\mathcal{G} = \mathcal{S}^1_{B/B}$ (compare with the next section), that $\pi_*\pi^*\mathcal{G} = \mathcal{G}$. Forming pull-backs, we have the following diagram

$$\begin{tikzcd}
& & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl] \arrow[dr] & & \\
& \mathcal{H} \underset{B}{\times} \hat{E} \arrow[dl] \arrow[dr] & & E \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl] \arrow[dr] \arrow[ul, dashed, bend right, "u"'] & \\
\mathcal{H} \arrow[dr] & & E \underset{B}{\times} \hat{E} \arrow[dl] \arrow[dr] & & \widehat{\mathcal{H}} \arrow[dl] \\
& E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] & \\
& & B. & &
\end{tikzcd}$$

The map $\mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \to E \underset{B}{\times} \widehat{\mathcal{H}}$ has a *universal section* $u \colon E \underset{B}{\times} \widehat{\mathcal{H}} \to \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}$. It is induced by the counit $\widehat{\mathcal{H}} \underset{B}{\times} E = \pi^*\pi_*\mathcal{H} \to \mathcal{H}$, together with the projection $\widehat{\mathcal{H}} \underset{B}{\times} E \to \widehat{\mathcal{H}}$. That map will be discussed in more detail in the next section.

3.2 Specializing to T-Duality

We put ourselves in the situation of [6], which we quickly recall here. Let $\pi \colon E \to B$ be a principal T^n-bundle and $f \colon M \to E$ a smooth submersion, then we can define a subbundle of TM whose fiber over $m \in M$ is given by

$$\mathcal{F}_{E/B,m} = \{\phi \in T_m M \mid d\pi(df(\phi)) = 0\}\,.$$

Given a Lie group G consider the associated sheaf of functions $\mathcal{G}$ over E and define the sheaf $\mathcal{G}_{E/B}$ as the assignment

$$(M \to E) \mapsto \{\varphi \colon M \to G \mid d\varphi X = 0 \text{ for all } X \in \mathcal{F}_{E/B}\}\,,$$

the subsheaf of fiberwise constant G-valued functions.

Let $p \colon \mathcal{H} \to E$ be a $\mathcal{S}^1_{E/B}$-banded gerbe on E.

Lemma 3.2. *Provided that $\mathcal{H}$ is fiberwise trivial, as defined in [6, (Definition 5.5)], the objects $\hat{E}$ and $\widehat{\mathcal{H}}$ constructed in the previous section lie in the correct categories, i.e., $\hat{p} \colon \hat{E} \to B$ is a T^n-bundle and $\hat{q} \colon \widehat{\mathcal{H}} \to \hat{E}$ is a $\mathcal{S}^1_{\hat{E}/B}$-banded gerbe.*

Proof: See [6, Theorem 6.2] and note that for $t\colon M \to \widehat{\mathcal{H}}$ we have

$$\operatorname{Aut}(t) = \mathcal{S}^1_{E/B}(E \underset{B}{\times} M \to E) \simeq \mathcal{S}^1_{B/B}(M \to \hat{E} \to B) = \mathcal{S}^1_{\hat{E}/B}(M \to \hat{E}).$$

$\square$

Remark 3.3. We are going to consider twisted sheaves on $\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}$. Recall that $\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}$ is a gerbe in two ways, namely $\hat{p}''\colon \widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H} \to \mathcal{H} \underset{B}{\times} \hat{E}$ and $p''\colon \widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H} \to \widehat{\mathcal{H}} \underset{B}{\times} E$. The full subcategory of $\operatorname{Mod}(\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H})$ consisting of sheaves that are a-twisted with respect to $\hat{p}''$ and b-twisted with respect to p'' will be denoted $\operatorname{Mod}_{(a,b)}(\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H})$. Sometimes we say that a sheaf $\mathcal{F}$ on $\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}$ is (a, b)-twisted if it is an element of $\operatorname{Mod}_{(a,b)}(\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H})$.

Remark 3.4. On $\mathcal{H} \underset{E}{\times} \mathcal{H}$ there is a natural $\mathcal{G}$-torsor: $\mathcal{H} \xrightarrow{(\mathrm{id},\mathrm{id})} \mathcal{H} \underset{E}{\times} \mathcal{H}$. The composition

$$E \underset{B}{\times} \widehat{\mathcal{H}} \xrightarrow{u} \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \to \mathcal{H}$$

of the universal section u and the projection is a morphism of gerbes. Hence, using the character χ, pulling back determines a locally free $(1, -1)$-twisted $\mathcal{R}$-module on $\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}$, henceforth denoted $\mathcal{L}$.

Denote by $\mathcal{S}^1_\delta$ the sheaf of locally constant S^1-valued functions on M; temporarily let ${}^s\mathcal{S}^1_\delta$ be the small sheaf of locally constant S^1-valued functions on a manifold M.

Lemma 3.5. *The category* $\operatorname{Tors}({}^s\mathcal{S}^1_\delta)$ *of (small)* ${}^s\mathcal{S}^1_\delta$*-torsors on M is equivalent to the category* $\operatorname{Tors}(\mathcal{S}^1_\delta)$ *of (large)* $\mathcal{S}^1_\delta$ *torsors on M.*

Proof: First note that the functors ext and res_M preserve torsors, hence we have an adjoint pair

$$\operatorname{ext}\colon \operatorname{Tors}({}^s\mathcal{S}^1_\delta) \leftrightarrows \operatorname{Tors}(\operatorname{ext}({}^s\mathcal{S}^1_\delta)) : \operatorname{res}_M,$$

such that $\operatorname{res}_M \circ \operatorname{ext} = \operatorname{id}$. Further, the counit $(\operatorname{ext} \circ \operatorname{res})(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for any torsor $\mathcal{F}$, so that the adjoint pair is an equivalence of categories. The Lemma then follows from the fact that ${}^s\mathcal{S}^1_\delta = \operatorname{res}(\mathcal{S}^1_\delta)$ and $\operatorname{ext}(\operatorname{res}(\mathcal{S}^1_\delta)) = \mathcal{S}^1_\delta$. $\square$

Remark 3.6. Note that for $M = T^n$ a torus the category $\operatorname{Tors}({}^s\mathcal{S}^1_\delta)$ of small ${}^s\mathcal{S}^1_\delta$- torsors on M is parametrized by the dual torus $\widehat{T}^n$. Hence, using the previous Lemma, the dual torus $\widehat{T}^n$ also parametrizes (large) $\mathcal{S}^1_\delta$ torsors.

Definition 3.7. By that remark, a $\mathcal{S}^1_\delta$-Torsor $\mathcal{F}$ on a torus T^n is given by an element $\varphi \in \widehat{T}^n$ – that element is called the *holonomy* of $\mathcal{F}$.

The following Lemma shows that locally everything behaves as expected, and, in particular, locally, for $\mathcal{H} = B\mathcal{S}^1_{T\times B/B} \times T \times B$ and the push forward $\widehat{\mathcal{H}} = B\mathcal{S}^1_{\widehat{T}\times B/B} \times \widehat{T} \times B$, the diagram

$$\begin{tikzcd}
 & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{H} \underset{B}{\times} (\widehat{T} \times B) \arrow[dl] \arrow[dr] & & (T \times B) \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl] \arrow[dr] \arrow[ul, bend right, "u"'] & \\
\mathcal{H} \arrow[dr] & & T \times \widehat{T} \times B \arrow[dl] \arrow[dr] & & \widehat{\mathcal{H}} \arrow[dl] \\
 & T \times B \arrow[dr, "\pi"] & & \widehat{T} \times B \arrow[dl] & \\
 & & B & &
\end{tikzcd}$$

is a T-duality diagram . This identification is going to be important later.

Lemma 3.8. *For $E = T^n \times B$ and $\pi\colon T^n \times B \to B$ the projection, we have*

$$\widehat{B\mathcal{S}^1}_{T^n\times B/B} \simeq B\mathcal{S}^1_{B/B} \times \widehat{T}^n \times B.$$

Proof: Restricting an element

$$\begin{aligned} P \in & (\widehat{B\mathcal{S}^1}_{B\times T^n/B})(M \xrightarrow{f} B) \\ & = (B\mathcal{S}^1_{B\times T^n/B})(M \times T^n \xrightarrow{f\times \mathrm{id}} B \times T^n) \end{aligned}$$

to $m \in M$ and reading off holonomy gives a map $\varphi\colon M \to \widehat{T}$. Let $s\colon M \to T^n \times M$ be the section $m \mapsto (m, e)$. The pullback s^*P is a $\mathcal{S}^1_{B/B}$-torsor on M. Hence we get a map

$$\widehat{B\mathcal{S}^1}_{T^n\times B/B} \to B\mathcal{S}^1_{B/B} \times \widehat{T}^n \times B$$

Going in the other direction, given a map $\varphi\colon B \to \widehat{T}$, we can construct a small $\pi^{-1}\mathcal{S}^1_B$-torsor ${}^s\mathcal{T}_\varphi$ on $B \times T$ as the sheaf of fiberwise locally constant sections of the following S^1-bundle.

Let $\widetilde{T}$ be the universal cover of T. The fundamental group $\pi_1(T)$ acts on $\widetilde{T} \times B \times S^1$ through φ, giving the bundle $(\widetilde{T} \times B \times S^1)/\sim_\varphi \to T \times B$.

Extending ${}^{s}\mathcal{T}_{\varphi}$ to the large site yields a large $\mathrm{ext}(\pi^{-1}\mathcal{S}^1_B)$-torsor on M. Using the inclusion $\mathrm{ext}(\pi^{-1}\mathcal{S}^1_B) \hookrightarrow \mathcal{S}^1_{T^n \times B/B}$ we get a $\mathcal{S}^1_{T^n\times B/B}$-torsor, henceforth called $\mathcal{T}_{\varphi}$ on $T^n \times B$. Now, given any $\mathcal{S}^1_{B/B}$-torsor $\mathcal{F}$, taking the concentrated product $\pi^*\mathcal{F} \wedge \mathcal{T}_{\varphi}$ defines the inverse

$$B\mathcal{S}^1_{B/B} \times \widehat{T}^n \times B \to \widehat{B\mathcal{S}^1}_{T^n\times B/B}.$$

See [5, Definition and Proposition 5.1.5 and the following paragraphs] for an introduction to torsors and operations on them. $\square$

Definition 3.9. Let $\mathcal{F}$ be a $\mathcal{S}^1_{T^n\times B/B}$ torsor on the product $T^n \times B$ of a torus T^n and a smooth manifold B. By the previous Lemma it is given by a map $\varphi\colon B \to \widehat{T}$ and a $\mathcal{S}^1$-torsor $\mathcal{T}$ on B. The map $\varphi\colon B \to \widehat{T}$ is called the *relative holonomoy* and $\mathcal{T}$ the *carrier* of $\mathcal{F}$.

Next, we describe the Poincaré-sheaf.

Definition 3.10 (Poincaré-bundle/sheaf)**.** The module $\mathcal{L}$ described in Remark 3.4 is called the *Poincaré-bundle*. It is a locally free, rank 1, $(1,-1)$-twisted $p^*C^\infty_{E/B}$-module.

3.2.1 The Case of a Point

Let $\mathcal{H} = B\mathcal{S}^1_{T/\star} \times T = B\mathcal{S}^1_{\delta} \times T$, then, by (3.8), $\pi_*\mathcal{H} = B\mathcal{S}^1_{\widehat{T}/\star} \times \widehat{T} = B\mathcal{S}^1_{\delta} \times \widehat{T}$, where $\widehat{T}^n$ is the moduli space of $\widehat{\mathcal{H}}$.

Given the diagram

$$\begin{tikzcd}
 & & & \mathcal{H}\times\widehat{\mathcal{H}} \arrow[dl] \arrow[dr] & & \\
 & & \mathcal{H}\times\widehat{T} \arrow[dl] \arrow[dr] & & T\times\widehat{\mathcal{H}} \arrow[dl] \arrow[dr] \arrow[ul, bend right, "u"'] & \\
 & \mathcal{H} \arrow[dr] & & T\times\widehat{T} \arrow[dl] \arrow[dr] & & \widehat{\mathcal{H}} \arrow[dl] \\
 & & T \arrow[ul, dashed, bend left, "s"] \arrow[ur, dashed, bend right, "r"'] \arrow[dr, "\pi"'] & & \widehat{T} \arrow[ul, dashed, bend left, "\hat{r}"] \arrow[ur, dashed, bend right, "\hat{s}"'] \arrow[dl] & \\
 & & & \star & &
\end{tikzcd}$$

with $\hat{s}\colon \widehat{T} \to \widehat{\mathcal{H}} = B\mathcal{S}^1_{\widehat{T}/\star} \times \widehat{T}$ given as the identity on $\widehat{T}$ and the trivial torsor on $\widehat{T}$, we would like to see that the induced L has the holonomy we would expect a Poincaré bundle to have.

Remark 3.11. Given two sections $s\colon T \to \mathcal{H}$ and $\hat{s}\colon \widehat{T} \to \widehat{\mathcal{H}}$ as in the diagram above, there are induced sections $s'\colon T \times \widehat{T} \to \mathcal{H} \times \widehat{T}$ and $\hat{s}'\colon T \times \widehat{T} \to \widehat{\mathcal{H}} \times T$. We would like to understand the bundle defined as in the pullback:

$$\begin{tikzcd} \mathcal{H} \arrow[d] & L \arrow[l] \arrow[d] & L(s,\hat{s}) \arrow[l] \arrow[d] \\ \mathcal{H} \underset{T}{\times} \mathcal{H} & \mathcal{H} \times \widehat{\mathcal{H}} \arrow[l] & T \times \widehat{T}. \arrow[l] \end{tikzcd}$$

In order to do so, we need to understand the map

$$T \times \widehat{T} \to T \times \widehat{\mathcal{H}} \to \mathcal{H} = B\mathcal{S}^1_{T/*} \times T,$$

where the last map is the counit. Note that the map $\hat{s}\colon \widehat{T} \to \pi_*\mathcal{H}$ defines an $\mathcal{S}^1_{T/\star}$-torsor on $T \times \widehat{T}$. That torsor is exactly the map $T \times \widehat{T} \to \mathcal{H}$ from above. Concretely, it is the torsor $\mathcal{T}_{\mathrm{id}\colon \widehat{T} \to \widehat{T}}$ of Lemma 3.8, i.e., a Poincaré bundle.

Remark 3.12. Given sections $s\colon T \to \mathcal{H}$ and $\hat{s}\colon \widehat{T} \to \widehat{\mathcal{H}}$, there are two maps $f_s\colon T \times \widehat{T} \to \mathcal{H}$ and $f_{\hat{s}}\colon T \times \widehat{T} \to \mathcal{H}$. Let $L(s,\hat{s})$ be defined by the pull-back

$$\begin{tikzcd} \mathcal{H} \arrow[d] & L \arrow[l] \arrow[d] & L(s,\hat{s}) \arrow[l] \arrow[d] \\ \mathcal{H} \underset{T}{\times} \mathcal{H} & \mathcal{H} \times \widehat{\mathcal{H}} \arrow[l] & T \times \widehat{T}. \arrow[l] \end{tikzcd}$$

Then

$$L(s,s') = \mathrm{Hom}_{\mathcal{H}(T \times \widehat{T})}(f_s, f_{\hat{s}})$$

4 Relative Fourier-Mukai Transformation

Now suppose we have a T-duality diagram as depicted below and a $(1,-1)$-twisted $C^\infty_{E/B}$-module $\mathcal{L}$ on $\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}$.

$$
\begin{tikzcd}
 & & \widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H} \arrow[dl, "p_1"] \arrow[dr, "q_1"] & & \\
 & \mathcal{H} \underset{B}{\times} \hat{E} \arrow[dl, "p_0"] \arrow[dr, "q_1'"] & & E \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl, "p_1'"] \arrow[dr, "q_0"] & \\
\mathcal{H} \arrow[dr, "q_1''"] & & E \underset{B}{\times} \hat{E} \arrow[dl, "p_0'"] \arrow[dr, "q_0'"] & & \hat{\mathcal{H}} \arrow[dl, "p_1''"] \\
 & E \arrow[dr, "q_0''"] & & \hat{E} \arrow[dl, "p_0''"] & \\
 & & B & &
\end{tikzcd}
$$

Definition 4.1. Given that data, we can define two *relative Fourier-Mukai-Transformation* functors with *kernel* $\mathcal{L}$.

$$
\begin{aligned}
FM^{\mathcal{L}}_{\mathcal{H}\to\widehat{\mathcal{H}}}(-) &:= q_{0*}q_{1*}(\mathcal{L} \otimes_{(p_0p_1)^*C^\infty_{E/B}} (p_0p_1)^*-) \colon \mathcal{D}_1(\mathcal{H}) \to D_1(\widehat{\mathcal{H}}) \\
\widehat{FM}^{\mathcal{L}}_{\widehat{\mathcal{H}}\to\mathcal{H}}(-) &:= p_{0*}p_{1*}(\mathcal{L} \otimes_{(q_0q_1)^*C^\infty_{\hat{E}/B}} (q_0q_1)^*-) \colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to D_1(\mathcal{H}),
\end{aligned}
$$

where all these operations are the derived twisted versions. The kernel $\mathcal{L}$ usually is dropped from the notation.

Remark 4.2. FM and $\widehat{FM}$ implicitly use the twisted operations. FM, for example, is the composition:

$$
\mathcal{D}_1(\mathcal{H}) \xrightarrow{(p_0p_1)^*} \mathcal{D}_{(1,0)}(\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}) \xrightarrow{\otimes\mathcal{L}} \mathcal{D}_{(0,1)}(\widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H}) \xrightarrow{q_{1*}} \mathcal{D}_1(\widehat{\mathcal{H}} \underset{B}{\times} E) \xrightarrow{q_{0*}} \mathcal{D}_1(\widehat{\mathcal{H}})
$$

Next we study the composition of Fourier-Mukai-Transformations as given by the following diagram

$$
\begin{tikzcd}[column sep=tiny]
 & & & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H} \arrow[dl, "p_3"] \arrow[dr, "q_3"] & & & & \\
 & & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \underset{B}{\times} E \arrow[dl, "p_2"] \arrow[dr, "q_3'"] & & E \underset{B}{\times} \widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H} \arrow[dl, "p_3'"] \arrow[dr, "q_2"] & & & \\
 & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl, "p_1"] \arrow[dr, "q_3''"] & & E \underset{B}{\times} \widehat{\mathcal{H}} \underset{B}{\times} E \arrow[dl, "p_2'"] \arrow[dr, "q_2'"] & & \widehat{\mathcal{H}} \underset{B}{\times} \mathcal{H} \arrow[dl, "p_3''"] \arrow[dr, "q_1"] & & \\
 & \mathcal{H} \underset{B}{\times} \hat{E} \arrow[dl, "p_0"] & & \widehat{\mathcal{H}} \underset{B}{\times} E \arrow[dr, "q_2''"] & & \widehat{\mathcal{H}} \underset{B}{\times} E \arrow[dl, "p_2''"] & & \hat{E} \underset{B}{\times} \mathcal{H} \arrow[dr, "q_0"] & \\
\mathcal{H} & & & & \widehat{\mathcal{H}} & & & & \mathcal{H}.
\end{tikzcd}
$$

Given the projections

$$\mathcal{H}\underset{B}{\times}\widehat{\mathcal{H}}\underset{B}{\times}\mathcal{H}\xrightarrow{r_3}\mathcal{H}\underset{B}{\times}\hat{E}\underset{B}{\times}\mathcal{H}\xrightarrow{r_2}\mathcal{H}\underset{B}{\times}\mathcal{H},$$

define

$$\mathcal{R} := r_{2*}r_{3*}\left(p_3^*p_2^*\mathcal{L}\otimes q_3^*q_2^*\mathcal{L}'\right)$$

(Note that the r_{3*} here is the 0-twisted version). That sheaf turns out to be the kernel of the composition.

Lemma 4.3 (FMs compose)**.** *Assume that $\mathcal{L}$ is locally trivial (being a perfect complex would be enough), then the composition*

$$\mathcal{D}_1(\mathcal{H})\xrightarrow{FM^{\mathcal{L}}_{\mathcal{H}\to\widehat{\mathcal{H}}}}D_1(\widehat{\mathcal{H}})\xrightarrow{F\hat{M}^{\mathcal{L}}_{\widehat{\mathcal{H}}\to\mathcal{H}}}\mathcal{D}_1(\mathcal{H})$$

is isomorphic to

$$\mathcal{D}_1(\mathcal{H})\xrightarrow{FM^{\mathcal{R}}_{\mathcal{H}\to\mathcal{H}}}\mathcal{D}_1(\mathcal{H}).$$

Proof: This is formal – base-change and projection-formula are applicable because we have finite cohomological dimension of the push-forward along a principal bundle, exactness of the push-forward of 0-twisted sheaves and a locally free/perfect kernel. See [13, Proposition 5.10]. □

Remark 4.4. The locally-freeness (resp. perfectness) condition can be dropped when one works in the derived category of complexes that are bounded below. This is due to the limitation of our projection formula (Proposition 2.38). To the best of the author's knowledge, no more general projection formula is known.

Now suppose both gerbes have sections. In what follows we show that the transformation can be calculated locally.

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{H} \times_B \widehat{\mathcal{H}} \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & \\
 & \mathcal{H} \times_B \hat{E} \arrow[ur, bend right, "\hat{s}''"'] \arrow[dl, "p_0"] \arrow[dr, "q_1'"] & & E \times_B \widehat{\mathcal{H}} \arrow[ul, bend left, "s''"] \arrow[dl, "p_1'"'] \arrow[dr, "q_0"] & \\
\mathcal{H} \arrow[dr, "q_1''"] & & E \times_B \hat{E} \arrow[ul, bend left, "s'"] \arrow[ur, bend right, "\hat{s}'"'] \arrow[dl, "p_0'"'] \arrow[dr, "q_0'"] & & \hat{\mathcal{H}} \arrow[dl, "p_1''"'] \\
 & E \arrow[ul, bend left, "s"] \arrow[dr, "q_0''"'] & & \hat{E} \arrow[ur, bend right, "\hat{s}"'] \arrow[dl, "p_0''"] & \\
 & & B & &
\end{tikzcd}$$

Lemma 4.5 (Locality). *Using the notation from the diagram above, we have*

$$\hat{s}^* FM^{\mathcal{L}}_{\mathcal{H} \to \widehat{\mathcal{H}}}(-) \simeq q'_{0*}(p'^*_0 s^* - \otimes (s, \hat{s})^* \mathcal{L})$$

as functors $\mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$.

Proof: Note that by 2.24 we have $s'^* \mathcal{F} = q'_{1*} \mathcal{F}$ and $s''^* \mathcal{F} = q_{1*} \mathcal{F}$ for 0-twisted sheaves $\mathcal{F}$. Using that and base change, we have the following chain of isomorphisms. Note that the base change is along q_0, which is a bundle. Push-forward along a T^n-principal bundle has finite cohomological dimension. Hence we can apply the base change formula from Proposition 2.43.

$$\begin{aligned}
\hat{s}^* FM^{\mathcal{L}}_{\mathcal{H} \to \widehat{\mathcal{H}}}(\mathcal{F}) &= \hat{s}^* q_{0*} q_{1*}(\mathcal{L} \otimes (p_0 p_1)^* \mathcal{F}) \\
&\simeq q'_{0*} \hat{s}'^* q_{1*}(\mathcal{L} \otimes (p_0 p_1)^* \mathcal{F}) \\
&\simeq q'_{0*} \hat{s}'^* s''^*(\mathcal{L} \otimes (p_0 p_1)^* \mathcal{F}) \\
&\simeq q'_{0*} s'^* \hat{s}''^*(\mathcal{L} \otimes (p_0 p_1)^* \mathcal{F}) \\
&\simeq q'_{0*}(s'^* \hat{s}''^* \mathcal{L} \otimes s'^* \hat{s}''^* (p_0 p_1)^* \mathcal{F}) \\
&\simeq q'_{0*}(s'^* \hat{s}''^* \mathcal{L} \otimes s'^* p_0^* \mathcal{F}) \\
&\simeq q'_{0*}(s'^* \hat{s}''^* \mathcal{L} \otimes p'^*_0 s^* \mathcal{F})
\end{aligned}$$

For the first isomorphism, note that the following diagram is a pull-back:

$$\begin{tikzcd}
& E \underset{B}{\times} \hat{E} \arrow[r] \arrow[d, "\hat{s}'"] & \hat{E} \arrow[d, "\hat{s}"] & \\
& \widehat{\mathcal{H}} \underset{B}{\times} E \arrow[r] \arrow[dl] \arrow[dd] & \widehat{\mathcal{H}} \arrow[dr] \arrow[dd] & \\
\hat{E} \underset{B}{\times} E \arrow[dr] & & & \hat{E} \arrow[dl] \\
& E \arrow[r] & B &
\end{tikzcd}$$

□

4.1 The Kernel $\mathcal{L}$

4.1.1 Resolution

The following definitions are taken from [6, Section 2].

Given a smooth submersion $\pi\colon E \to B$ and a smooth map $f\colon X \to E$, recall the definition of the subbundle $\mathcal{F}_{E/B}X$ of TX, which is given at a point $x \in X$ by

$$\mathcal{F}_{E/B,x}X = \{\phi \in T_xX \mid d\pi(df(\phi)) = 0\}.$$

This allows us to define the complex of complex valued forms:

$$\Omega^{\bullet}_{E/B}(X \to E) = C^{\infty}(X, \Lambda^{\bullet}\mathcal{F}^{*}_{E/B}X).$$

Analogous to [6, Lemma 2.5],

$$0 \to C^{\infty}_{E/B} \to \Omega^{*}_{E/B}$$

is an acyclic resolution and $\Omega^{n}_{E/B}$ is flabby, because it's a C^{∞}-module. Hence, by [7, Lemma 2.30], the resolution is good enough to calculate the push-forward.

Let $\mathcal{L}$ be the $p^{*}C^{\infty}_{E/B}$-module from Definition 3.10. Because p^{*} is exact and $\mathcal{L}$ is a flat $p^{*}C^{\infty}_{E/B}$-module (because it is locally free), the exact sequence

$$0 \to \mathcal{L} \to \mathcal{L} \otimes_{p^{*}C^{\infty}_{E/B}} p^{*}\Omega^{*}_{E/B}$$

is an acyclic, flabby (still C^∞-modules) resolution of $\mathcal{L}$. Specializing this to the 1-dimensional case, we get a resolution

$$0 \to \mathcal{L} \to \mathcal{L} \otimes p^*\Omega^0_{E/B} \to \mathcal{L} \otimes p^*\Omega^1_{E/B} \to 0$$

of $\mathcal{L}$ in the category of $p^*C^\infty_{E/B}$-modules.

4.1.2 Push-Forward

To separate the formal from the non-formal parts, we first consider the case of a 1-dimensional torus.

4.1.2.1 The 1-Dimensional Case

Many of the following arguments are adaptions from [3]. Consider the projection $p\colon \hat{T} \times T \times U \to \hat{T} \times U$. We would like to calculate the push-forward $p_*\mathcal{L}$ of $\mathcal{L}$ along p.

Lemma 4.6. *Let $f\colon W \to T \times U$ be arbitrary and $\pi\colon W \times \mathbb{R}^n \to W$ be the projection. Then*

$$\mathcal{L}(W \to T \times \hat{T} \times U) \xrightarrow{\pi^*} \mathcal{L}(W \times \mathbb{R}^n \to T \times \hat{T} \times U)$$

is an isomorphism with inverse $s\colon W \to W \times \mathbb{R}^n, \quad (w) \mapsto (w, 0)$.

Proof: Note that $\mathcal{S}^1_{E/B}$, $C^\infty_{E/B}$ and every sheaf in the image of ext share that property. $\mathcal{L}$ is built as the associated sheaf of an extended sheaf. Hence the result follows. □

Lemma 4.7. *Let $\pi\colon W \times \mathbb{R}^n \to W$ be the projection and $f\colon W \to T \times \hat{T} \times U$ arbitrary. π induces a chain homotopy equivalence*

$$\left(\mathcal{L} \otimes \Omega^*_{E/B}\right)(W \to U) \to \left(\mathcal{L} \otimes \Omega^*_{E/B}\right)(W \times \mathbb{R}^n \to W \to U),$$

with inverse $s\colon W \to W \times \mathbb{R}^n, \quad w \mapsto (w, 0)$.

Proof: This is pretty much a word by word translation of [2, Proposition 4.1]. □

Lemma 4.8. *The push-forward vanishes,*

$$R^0 p_*\mathcal{L} \simeq 0.$$

Proof: This is a local question, so we can check it on submersions of the form $\mathbb{R}^n \times V \to V \hookrightarrow \widehat{T} \times U$, where the first map is the projection to V and $i\colon V \hookrightarrow \widehat{T} \times U$ is the inclusion of an open subset. By the previous Lemma, we can ignore the factor $\mathbb{R}^n$. An element

$$\varphi \in R^0 p_* \mathcal{L}(V \to \widehat{T} \times U) = \mathcal{L}(T \times V \to T \times \widehat{T} \times U)$$

induces global sections $\varphi|_{T\times\{v\}}$ of $\mathcal{L}|_{T\times\{v\}}$ for all $v \in V$. Hence $\varphi|_{T\times\{v\}}$ has to vanish whenever

$$(i \circ pr_{\widehat{T}})(v) \neq e \in \widehat{T},$$

because a $\mathcal{S}^1_\delta$-torsor with non-trivial holonomy has no global sections.

Since φ is smooth, it has to vanish everywhere. □

Lemma 4.9. *The higher direct images vanish for degrees higher than* 1*, i.e.,*

$$R^n p_* \mathcal{L} \simeq 0 \quad \text{for} \quad n > 1.$$

Proof: Let $f\colon X \to \hat{T} \times U$ be a submersion. It is enough to check that locally. Since f is a submersion, locally it is of the form

$$\begin{tikzcd} X \arrow[r, "f"] & \hat{T} \times U \\ \mathbb{R}^m \times W \arrow[u, hook, "i"] \arrow[r] & W \arrow[u, hook] \end{tikzcd}$$

where W is an open subset of $\hat{T} \times U$ and $\mathbb{R}^m \times W \to W$ is the projection.

$$\begin{aligned} R^n p_* \mathcal{L} &= H^n\left((\mathcal{L} \otimes \Omega^*_{E/B})(\mathbb{R}^m \times T \times W)\right) \\ &= H^n\left((\mathcal{L} \otimes \Omega^*_{E/B})(T \times W)\right) \end{aligned}$$

Hence the latter complex vanishes for all degrees greater than 1,

$$\left(\mathcal{L} \otimes \Omega^i_{E/B}\right)(T \times W) = 0 \quad \text{for} \quad i > 1.$$

□

Definition 4.10. Let $s\colon U \to \hat{T} \times U$ be a section. The sheaf

$$k(s) := s_* C^\infty_{U/U} \in \mathrm{Mod}(\hat{T} \times U)$$

is called a *fiberwise skyscraper sheaf*.

Remark 4.11. Note the analogy to the non-relative case here: A skyscraper sheaf on a manifold M is the push forward $i_*\mathcal{O}_\star$ of the structure sheaf along the inclusion of a point $i\colon \star \hookrightarrow M$.

Theorem 4.12. *Let $\mathcal{L}$ be the Poincaré bundle on $T \times \widehat{T} \times U$ and $q\colon T \times \widehat{T} \times U \to \widehat{T} \times U$ the projection. Then there exists an isomorphism*

$$q_*\mathcal{L} \simeq k(s)[1],$$

where $s\colon U \to \widehat{T} \times U$, $u \mapsto (e, u)$ is the constant section picking out the identity.

Proof: The map $F\colon q_*\mathcal{L} \to s_*C^\infty_{U/U}[1]$ is given as the composition

$$\begin{aligned} q_*(\mathcal{L} \otimes p^*\Omega^*_{E/B}) &\to s_*s^*q_*(\mathcal{L} \otimes p^*\Omega^*_{E/B}) \\ &\simeq s_*\pi_*\widetilde{s}^*(\mathcal{L} \otimes p^*\Omega^*_{E/B}) \\ &\simeq s_*\pi_*(\widetilde{s}^*\mathcal{L} \otimes \widetilde{s}^*p^*\Omega^*_{E/B}) \\ &\overset{\text{①}}{\simeq} s_*\pi_*\Omega^*_{E/B} \\ &\xrightarrow{\int_T} s_*\Omega^*_{U/U}[1] \\ &\simeq s_*C^\infty_{U/U}[1] \end{aligned}$$

Note that the isomorphism ① comes about, because the restriction of $\mathcal{L}$ to $T \times \{e\} \times U$ is trivializable.

Given $\varphi \in q_*(\mathcal{L} \otimes p^*\Omega^1_{E/B})(X \to \widehat{T} \times U)$ with $F(\varphi) = 0$, we need to find $\psi \in q_*(\mathcal{L} \otimes p^*\Omega^0_{E/B})(X \to \widehat{T} \times U)$ such that $\varphi = d\psi$. Choose an open cover $\coprod_\alpha W_\alpha \xrightarrow{\{f_\alpha\}} U$ such that

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R} \times W_\alpha \arrow[r] \arrow[d, "i_\alpha"] & \mathbb{R} \times W_\alpha \arrow[r] \arrow[d] & W_\alpha \arrow[d] \\ X \arrow[r, "f"] & \widehat{T} \times U \arrow[r] & U. \end{tikzcd}$$

commutes and the vertical map in the middle is a product of maps $\mathbb{R} \to \widehat{T}$ and $W_\alpha \to U$. This induces covers of $\widehat{T} \times U$ and X. We will find $\psi_\alpha \in q_*(\mathcal{L} \otimes p^*\Omega^0_{E/B})(\mathbb{R}^n \times \mathbb{R} \times W_\alpha \to \widehat{T} \times U)$ such that $i^*_\alpha\varphi = d\psi_\alpha$. Gluing these forms using a partition of unity subordinate to that open cover yields the form ψ, that we set out to find. Hence we are interested in the module

$$q_*(\mathcal{L} \otimes p^*\Omega^0_{E/B})(\mathbb{R}^n \times \mathbb{R} \times W_\alpha) = (\mathcal{L} \otimes p^*\Omega^0_{E/B})(T \times \mathbb{R}^n \times \mathbb{R} \times W_\alpha).$$

An element in the latter may be written as $g_\alpha(t, r_n, r, v)dt$, where the map $g_\alpha\colon \widetilde{T} \times \mathbb{R}^n \times \mathbb{R} \times V_\alpha \to \mathbb{C}$, is a complex valued function such that for any $\gamma \in \pi_1(T)$, we have

$$g_\alpha(t+\gamma, r_n, r, v) = g_\alpha(t, r_n, r, v) \cdot [r](\gamma).$$

Here we identify $\pi_1(T)$ with $\mathbb{Z}$, the universal cover $\widetilde{T}$ with $\mathbb{R}$ and $[r] \in \widehat{T}$ is the image of $r \in \mathbb{R}$ in $\widehat{T}$.

Suppose g_α is in the image of d, that is, there exists some h_α with

$$h_\alpha(u, r_n, r, v) = \int_0^u g_\alpha(t, r_n, r, v)dt + c(r_n, r, v).$$

Let $\gamma \in \pi_1(T)$ be a generator, then

$$\begin{aligned}
h_\alpha(u+\gamma, r_n, r, v) &= \int_0^{u+\gamma} g_\alpha(t, r_n, r, v)dt + c(r_n, r, v)\\
&= \int_0^{\gamma} g_\alpha(t, r_n, r, v)dt + \int_\gamma^{u+\gamma} g_\alpha(t, r_n, r, w)dt + c(r_n, r, v)\\
&= \int_0^{\gamma} g_\alpha(t, r_n, r, v)dt + \int_0^{u} g_\alpha(t+\gamma, r_n, r, v)dt + c(r_n, r, v)\\
&= \int_0^{\gamma} g_\alpha(t, r_n, r, v)dt + [r](\gamma)\int_0^{u} g_\alpha(t, r_n, r, v)dt + c(r_n, r, v)\\
&= \int_0^{\gamma} g_\alpha(t, r_n, r, v)dt + [r](\gamma)\left(h_\alpha(u, w) - c(r_n, r, v)\right) + c(r_n, r, v)\\
&= \int_0^{\gamma} g_\alpha(t, r_n, r, v)dt + h_\alpha(u+\gamma, r_n, r, v) - [r](\gamma)c(r_n, r, v) + c(r_n, r, v)
\end{aligned}$$

It follows that, with

$$c(r_n, r, v) = -\int_0^{\gamma} g_\alpha(t, r_n, r, v)dt/(1-[r](\gamma)),$$

this equation is solvable.

$c(r, v)$ is smooth if $[r](\gamma) \neq 1$, that is, if $[r] \neq e \in \widehat{T}$. Since $F(\psi)$ vanishes, the integral is zero for $[r] = e$. Because $1 - [r](\gamma)$ vanishes with order 1, this $c(r_n, r, v)$ is smooth everywhere. □

Remark 4.13. Recall the situation at hand

$$\begin{tikzcd}
 & T \times \widehat{T} \times U \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\
T \times U \arrow[dr, "\pi"'] & & \widehat{T} \times U \arrow[dl, "\hat{\pi}"] \\
 & U. &
\end{tikzcd}$$

Hence, for the structure sheaves we have the equalities

$$p^*\mathcal{S}^1_{T\times U/U} = p^*\pi^*\mathcal{S}^1_{U/U} = \hat{p}^*\hat{\pi}^*\mathcal{S}^1_{U/U} = \hat{p}^*\mathcal{S}^1_{\hat{T}\times U/U}.$$

The Poincaré bundle $\mathcal{L}$ can thus also be seen as a Poincaré bundle for the dual torus $\widehat{T}$. Hence the situation has become entirely symmetrical and the same calculation as above yields that

$$R\hat{p}_*\mathcal{L} \simeq k(s)[1]$$

with $s\colon U \to T \times U,\ \ u \mapsto (e, u)$.

4.1.2.2 Higher Tori

The Poincaré bundle for tori of higher dimensions is just the external product of the Poincaré bundles of 1-dimensional tori. The following results show that calculating the push-forward in the higher dimensional case is a formal consequence of the 1-dimensional case.

Before we can do anything, we need the following result.

Definition 4.14. Let X, Y and U be manifolds and consider the pull-back square

$$\begin{tikzcd}
 & X \times Y \times U \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \times U \arrow[dr] & & X \times U \arrow[dl] \\
 & U. &
\end{tikzcd}$$

For $\mathcal{F} \in \mathrm{Mod}(C^\infty_{X\times U/U})$ and $\mathcal{G} \in \mathrm{Mod}(C^\infty_{Y\times U/U})$, define

$$\mathcal{F} \boxtimes_U \mathcal{G} = p^*\mathcal{F} \otimes q^*\mathcal{G} \in \mathrm{Mod}(X \times Y \times U)$$

Lemma 4.15. *Let X and Y be compact manifolds and suppose we are given smooth representable maps $f\colon X \to X'$ and $g\colon Y \to Y'$. The maps f and g induce maps $\widetilde{f}\colon X \times U \to X' \times U$ and $\widetilde{g}\colon Y \times U \to Y' \times U$. For flat $\mathcal{F} \in \mathrm{Mod}(C^\infty_{X\times U/U})$ and flat $\mathcal{G} \in \mathrm{Mod}(C^\infty_{Y\times U/U})$, we get*

$$(\widetilde{f} \times_U \widetilde{g})_*(\mathcal{F} \boxtimes_U \mathcal{G}) \simeq (\widetilde{f}_*\mathcal{F} \boxtimes_U \widetilde{g}_*\mathcal{G}) \in \mathcal{D}(X' \times Y' \times U).$$

Proof: First note that we can decompose $\widetilde{f} \times_U \widetilde{g}$,

$$\begin{tikzcd} X \times Y \times U \arrow[r, "\widetilde{f}\times_U \mathrm{id}"'] \arrow[rr, bend left, "\widetilde{f}\times_U\widetilde{g}"] & X' \times Y \times U \arrow[r, "\mathrm{id}\times_U\widetilde{g}"'] & X' \times Y' \times U. \end{tikzcd}$$

So we can reduce the question to $\widetilde{f} \times_U \mathrm{id}$. Consider the two pullbacks:

$$\begin{tikzcd} Y \times U \arrow[r, "\mathrm{id}"] & Y \times U \\ X \times Y \times U \arrow[u, "p_{Y\times U}"] \arrow[r, "\widetilde{f}\times_U \mathrm{id}"] \arrow[d, "p_{X\times U}"'] & X' \times Y \times U \arrow[u, "p'_{Y\times U}"'] \arrow[d, "p'_{X'\times U}"] \\ X \times U \arrow[r, "\widetilde{f}"] & X' \times U \end{tikzcd}$$

It follows that

$$\begin{aligned} (\widetilde{f} \times_U \mathrm{id})_*(\mathcal{F} \boxtimes_U \mathcal{G}) &= (\widetilde{f} \times_U \mathrm{id})_*(p^*_{X\times U}\mathcal{F} \otimes p^*_{Y\times U}\mathcal{G}) \\ &= (\widetilde{f} \times_U \mathrm{id})_* \left(p^*_{X\times U}\mathcal{F} \otimes (p'_{Y\times U} \circ (\widetilde{f} \times_U \mathrm{id}))^*\mathcal{G}\right) \\ &\overset{\mathrm{I}}{\simeq} (\widetilde{f} \times_U \mathrm{id})_* p^*_{X\times U}\mathcal{F} \otimes p'^*_{Y\times U}\mathcal{G} \\ &\overset{\mathrm{II}}{\simeq} p'^*_{X'\times U}\widetilde{f}_*\mathcal{F} \otimes p'^*_{Y\times U}\mathcal{G} \\ &= (\widetilde{f}_*\mathcal{F}) \boxtimes_U (id_*\mathcal{G}). \end{aligned}$$

Step I is the projection formula, which is applicable because both $\mathcal{F}$ and $\mathcal{G}$ are flat. Step II is an application of basechange (Lemma 2.43). Note that the condition on the structure sheaf is fulfilled by assumption. □

Theorem 4.16. *Let $\mathcal{L}$ be the Poincaré bundle on $T^n \times \widehat{T}^n \times U$ and $q\colon T^n \times \widehat{T}^n \times U \to \widehat{T}^n \times U$ the projection, then*

$$p_*\mathcal{L} \simeq k(s)[n],$$

where $s\colon U \to \widehat{T}^n \times U,\ \ u \mapsto (e, u)$ is the section picking out the identity.

Proof: Note that $\mathcal{L} = \mathcal{L}_1 \boxtimes \mathcal{L}_2 \boxtimes \cdots \boxtimes \mathcal{L}_n$. By the above, we see that

$$\begin{aligned} q_*\mathcal{L} &= (q_1 \times_U q_2 \times_U \cdots \times_U q_n)_* (\mathcal{L}_1 \boxtimes \mathcal{L}_2 \boxtimes \cdots \boxtimes \mathcal{L}_n) \\ &= (q_{1,*}\mathcal{L}_1) \boxtimes (q_{2,*}\mathcal{L}_2) \boxtimes \cdots \boxtimes (q_{n,*}\mathcal{L}_n) \\ &= (k(s_1)[1]) \boxtimes (k(s_2)[1]) \boxtimes \cdots \boxtimes (k(s_n)[1]) \\ &= k(s)[n] \end{aligned}$$

□

4.2 The Trivial Case

In this section we are going to study the Fourier-Mukai Transform in the case where E is a trivial bundle $E = T \times B$ and $\mathcal{H}$ is the trivial gerbe on E, $\mathcal{H} = B\mathcal{S}^1_{E/B} \times E$.

4.2.1 Skyscraper to Local Systems

Let $s\colon U \to \widehat{T} \times U$ be a section and $s'\colon T \times U \to \hat{T} \times T \times U$ be the induced section as defined in the pull-back

$$\begin{tikzcd}
& T \times \hat{T} \times U \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\hat{T} \times U \arrow[dr, "\hat{\pi}"] & & T \times U \arrow[dl, "\pi"] \arrow[ul, dashed, bend left, "s'"] \\
& U. \arrow[ul, dashed, bend left, "s"] &
\end{tikzcd}$$

Definition 4.17. The sheaf $\mathcal{L}_s := s'^*\mathcal{L}$ is called a *fiberwise local system*.

Lemma 4.18. *Let $s\colon U \to \widehat{T} \times U$ and $s'\colon T \times U \to T \times \widehat{T} \times U$ be as above. Fiberwise skyscraper sheaves are mapped to fiberwise local systems, that is,*

$$\widehat{FM}(k(s)) \simeq \mathcal{L}_s.$$

Proof: This follows by the following chain of isomorphisms.

$$\begin{aligned}
FM(k(s)) &\simeq p_*(q^*k(s) \otimes_{p^*C^\infty_{E/B}} \mathcal{L}) \\
&\simeq p_*(q^*s_*C^\infty_{U/U} \otimes_{p^*C^\infty_{E/B}} \mathcal{L}) \\
&\simeq p_*(s'_*\pi^*C^\infty_{U/U} \otimes_{p^*C^\infty_{E/B}} \mathcal{L}) \\
&\simeq p_*s'_*(\pi^*C^\infty_{U/U} \otimes_{s'^*p^*C^\infty_{E/B}} s'^*\mathcal{L}) \\
&\simeq p_*s'_*(C^\infty_{E/B} \otimes_{C^\infty_{E/B}} s'^*\mathcal{L}) \\
&\simeq s'^*\mathcal{L}
\end{aligned}$$

□

4.2.2 Local Systems to Local Systems – The Composite

The aim of this section is to show that the composition

$$FM \circ \widehat{FM}\colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

is invertible in the case of trivial bundles and gerbes. Most of the arguments are heavily inspired by [13, Proposition 9.19], which proves an equivalence of $\mathcal{D}^b(A) \simeq \mathcal{D}^b(\widehat{A})$ for A an abelian variety and $D^b(-)$ the derived category of quasi coherent sheaves.

In this section, we consider the diagram of the composition.

$$
\begin{tikzcd}
 & & & \hat{T} \times \hat{T} \times U \arrow[dddlll, bend right, "p_c"'] \arrow[dddrrr, bend left, "q_c"] & & & \\
 & & & \hat{T} \times T \times \hat{T} \times U \arrow[u, "pr_{13}"] \arrow[dll, "pr_{12}"] \arrow[drr, "pr_{23}"'] & & & \\
 & \hat{T} \times T \times U \arrow[dl, "p"'] \arrow[drr, "q"] & & & & T \times \hat{T} \times U \arrow[dll, "p"'] \arrow[dr, "q"] & \\
\hat{T} \times U \arrow[dr] & & & T \times U \arrow[ull, bend left, "s'"] \arrow[dll] \arrow[drr] & & & \hat{T} \times U \arrow[ul, bend left, "s'"] \arrow[dl] \\
 & U \arrow[ul, bend left, "s"] & & & & U. \arrow[ull, bend left, "s"] &
\end{tikzcd}
$$

By 4.3, the Fourier-Mukai transformation of the above diagram has the sheaf

$$pr_{13,*}(pr_{12}^{*}\mathcal{P} \otimes pr_{23}^{*}\mathcal{P})$$

as kernel.

Remark 4.19. Note that given a bundle P on X with holonomy

$$h \colon \pi_1(X) \to S^1$$

and a map $f \colon Y \to X$, the pull-back f^*P has holonomy

$$h \circ \pi_1(f) \colon \pi_1(Y) \to \pi_1(X) \to S^1.$$

Also, given bundles P and P' with holonomies $h, h' \colon \pi_1(X) \to S^1$, the tensor product $P \otimes P'$ has holonomy $h \cdot h' \colon \pi_1(X) \to S^1$.

Corollary 4.20. *Given the maps*

$$
\begin{tikzcd}
\hat{T} \times T \times \hat{T} \times U \arrow[r, bend left, "pr_{23}"] \arrow[r, "m_{13}"] \arrow[r, bend right, "pr_{12}"'] & \hat{T} \times T \times U,
\end{tikzcd}
$$

where pr_{12} and pr_{23} are the evident projections and m_{13} is induced by the multiplication $\widehat{T} \times \widehat{T} \to \widehat{T}$, we have

$$(pr_{12}^*\mathcal{L} \otimes pr_{23}^*\mathcal{L}) \simeq m_{13}^*\mathcal{L}.$$

Proof: This is a straight forward calculation using the previous remark. □

Remark 4.21. The multiplication map fits into the following pull-back diagram

$$\begin{tikzcd} \hat{T} \times T \times \hat{T} \times U \arrow[r, "m_{13}"] \arrow[d, "pr_{13}"] & \hat{T} \times T \times U \arrow[d, "pr_1"] \\ \hat{T} \times \hat{T} \times U \arrow[r, "m"] & \hat{T} \times U. \end{tikzcd}$$

Corollary 4.22. *It follows that*

$$pr_{13,*}m_{13}^*\mathcal{L} \simeq m^* pr_{1,*}\mathcal{L}.$$

Corollary 4.23. *With 4.12, we can conclude that*

$$m^* pr_{1,*}\mathcal{L} \simeq m^* k(s)[n].$$

Corollary 4.24. *Looking at the pull-back*

$$\begin{tikzcd} \widehat{T}^n \times U \arrow[r, "\Gamma_\iota"] \arrow[d, "pr_U"] & \widehat{T}^n \times \widehat{T}^n \times U \arrow[d, "m"] \\ U \arrow[r, "s"] & \widehat{T}^n \times U, \end{tikzcd}$$

where Γ_ι is $(\hat{t}, u) \mapsto (\hat{t}, -\hat{t}, u)$, we see that

$$m^*k(s) = m^*s_*C^\infty_{U/U} = \Gamma_{\iota*}pr_U^*C^\infty_{U/U} = \Gamma_{\iota*}C^\infty_{\hat{E}/B}$$

Note that $\Gamma_{\iota*}$ has locally finite cohomological dimension, because it is a topological submersion.

Theorem 4.25. *Denote by*

$$FM_{triv}\colon \mathcal{D}(E) \to \mathcal{D}(\hat{E}) \ \text{ and } \ \widehat{FM}_{triv}\colon \mathcal{D}(\hat{E}) \to \mathcal{D}(E)$$

the Fourier-Mukai transformation with kernel the Poincaré bundle. The composition $\widehat{FM}_{triv} \circ FM_{triv}$ is the same as pushing forward along ι, i.e.

$$\widehat{FM}_{triv} \circ FM_{triv} \simeq \iota_*[n].$$

Proof: Using the results from this section, we get the following chain of isomorphisms.

$$\begin{aligned}(\widehat{FM}_{\mathrm{triv}} \circ FM_{\mathrm{triv}})(\mathcal{F}) &\simeq q_{c,*}\left(p_c^*(\mathcal{F}) \otimes pr_{13,*}(pr_{12}^*\mathcal{L} \otimes pr_{23}^*\mathcal{L})\right) \\ &\simeq q_{c,*}\left(p_c^*(\mathcal{F}) \otimes pr_{13,*}m_{13}^*\mathcal{L})\right) \\ &\simeq q_{c,*}\left(p_c^*(\mathcal{F}) \otimes m^* pr_{1,*}\mathcal{L}\right) \\ &\simeq q_{c,*}\left(p_c^*(\mathcal{F}) \otimes m^* k(s)[n]\right) \\ &\simeq q_{c,*}\left(p_c^*(\mathcal{F}) \otimes \Gamma_{\iota,*} C^\infty_{\hat{E}/B}[n]\right) \\ &\simeq q_{c,*}\Gamma_{\iota,*}\left(\Gamma_\iota^* p_c^*(\mathcal{F}) \otimes C^\infty_{\hat{E}/B}[n]\right) \\ &\simeq \iota_*\mathcal{F}[n]\end{aligned}$$

The last use of the projection formula requires some justification. Consider the underived setting. The formula reduces to the equivalence

$$(p_c^*\mathcal{F}) \otimes \Gamma_{\iota*} C^\infty_{\hat{E}/B} = \Gamma_{\iota*}\Gamma_\iota^* p_c^*\mathcal{F} = \Gamma_{\iota*}\mathcal{F}$$

Since $\Gamma_{\iota*}$ is the inclusion of a closed subset this is clearly true. For the derived statement, replace $\mathcal{F}$ by a K-flat resolution. There exists a K-flat resolution that is preserved by the pullback $\mathcal{F}^*$ ([21, Tag 06YW]). Hence the statement follows from the underived version. $\square$

Proposition 4.26. *On trivial gerbes, going back and forth is invertible, i.e. the composition*

$$\widehat{FM} \circ FM \colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H})$$

is an equivalence.

Proof: By Lemma 4.5, the diagram

$$\begin{tikzcd} \mathcal{D}_1(\mathcal{H}) \arrow[r, "\widehat{FM}\circ FM"] & \mathcal{D}_1(\mathcal{H}) \\ \mathcal{D}(E)'' \arrow[u, "s^*"', "\simeq"] \arrow[r, "\widehat{FM}_{\mathrm{triv}}\circ FM_{\mathrm{triv}}", "\simeq"'] & \mathcal{D}(E) \arrow[u, "s^*"', "\simeq"] \end{tikzcd}$$

commutes. By Lemma 2.24, the vertical maps are equivalences and so is the lower horizontal, by the previous Theorem. $\square$

Corollary 4.27. *Again, the situation is entirely symmetrical. Using the same proofs as above and remark 4.13 yields that the composition*

$$FM \circ \widehat{FM} \colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

is an equivalence.

Using the last two results, we get the following

Theorem 4.28. *The Fourier-Mukai transformation with kernel $\mathcal{L}$ is an equivalence. That is, the functors*

$$FM\colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

and

$$\widehat{FM}\colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\mathcal{H})$$

are equivalences.

5 Local to Global, Or: The Equivalence

In this section, we derive a global statement from the local statements gathered so far. The heavy lifting is done by the ∞-categorical machinery presented in [17]. In particular, the derived categories appearing are ∞-categories. That was not needed in the previous section, because locally we only need an equivalence of ∞-categories, which can be checked in the homotopy category, i.e. in the usual derived category.

5.1 Setup

Denote by $\mathcal{D}_1(\mathcal{H}) := \mathcal{D}(\mathrm{Ch}(\mathrm{Mod}_1(p^*\mathcal{S}^1_{E/B})))$ the derived category of the Grothendieck abelian category $\mathrm{Mod}_1(p^*\mathcal{S}^1_{E/B})$ of 1-twisted sheaves on $\mathcal{H}$.

Let $u\colon U \to B$ be a cover such that $E_U := u^*E$ and $\mathcal{H}_U := \mathcal{H}|_{u^*E}$ are trivializable. Denote by $E_U^\bullet$ the Čech nerve whose n-th space is the n-fold pull-back $E_U \underset{E}{\times} \dots \underset{E}{\times} E_U$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\
\cdots \arrow[r] & E_U \underset{E}{\times} E_U \underset{E}{\times} E_U \arrow[r] & E_U \underset{E}{\times} E_U \arrow[r] & E_U \arrow[r] & E
\end{tikzcd}$$

Pulling $\mathcal{H}$ along gives a simplicial stack $\mathcal{H}_U^\bullet$ and denote by $\mathcal{D}_1(\mathcal{H}_U^\bullet)$ its derived 1-twisted cosimplicial ∞-category. For example, $\mathcal{D}_1(\mathcal{H}_U^0) = \mathcal{D}_1(\mathcal{H}_U)$. The cosimplicial ∞-category $\mathcal{D}_1(\mathcal{H}_U^\bullet)$ is an *augmented* cosimplicial ∞-category. It is augmented with $G\colon \mathcal{D}_1(\mathcal{H}_U^{-1}) = \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}_U)$.

5.2 Descent

In this section we only prove the following theorem.

Theorem. *The derived category $\mathcal{D}_1(\mathcal{H})$ fulfills descent with regard to trivializing covers, i.e. the map*

$$\mathcal{D}_1(\mathcal{H}) \xrightarrow{r^*} \lim_n \mathcal{D}_1(\mathcal{H}_U^n)$$

is an equivalence.

The proof is an application of a higher-categorical version of *Descent* and the *Beck-Chevalley Condition*. See [17, Section 6.2] for all the details.

The primary tool in its most convenient form is

Theorem 5.1 (Corollary 6.2.4.3 [17])**.** *Let $\mathcal{C}^\bullet \colon N(\Delta_*) \to Cat_\infty$ be an augmented cosimplicial ∞-category and set $\mathcal{C} = \mathcal{C}^{-1}$. Let $G \colon \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that*

- *The ∞-category $\mathcal{C}^{-1}$ admits geometric realizations of G-split simplicial objects, and those are preserved by G.*
- *For every morphism $\alpha \colon [m] \to [n]$ in Δ_+, the diagram*

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1} \end{array}$$

  is left adjointable.

Then the canonical map $\Theta \colon \mathcal{C} \to \lim_{n \in \Delta} \mathcal{C}^n$ admits a fully faithful left adjoint. If G is conservative, then Θ is an equivalence.

Remark 5.2. The conditions in the previous Theorem require a bit of explanation.

In our application, G preserves all colimits, in particular geometric realizations of G-split simplicial objects. The latter are just colimits of certain simplicial objects in $\mathcal{C}$.

Left adjointability of a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{C}' \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V'} \\ \mathcal{D} & \xrightarrow{G'} & \mathcal{D}' \end{array}$$

of ∞-categories means that the functors G and G' admit left adjoints F and F' respectively and the transformation

$$F' \circ V' \to F' \circ V' \circ G \circ F \simeq F' \circ G' \circ V \circ F \to V \circ F$$

is an equivalence.

We are going to apply that theorem with $\mathcal{C}^\bullet = \mathcal{D}_1(\mathcal{H}_U^\bullet)$. The rest of this section is checking the assumptions.

Lemma 5.3. *Using the notation above, the diagram*

$$\begin{tikzcd} \mathcal{D}_1(\mathcal{H}_U^m) \arrow[r, "d^{0,*}"] \arrow[d, "\alpha^*"'] & \mathcal{D}_1(\mathcal{H}_U^{m+1}) \arrow[d, "(1+\alpha)^*"] \\ \mathcal{D}_1(\mathcal{H}_U^n) \arrow[r, "d^{0,*}"] & \mathcal{D}_1(\mathcal{H}_U^{n+1}) \end{tikzcd}$$

is left-adjointable.

Proof: First note that by [17, Remark 6.2.3.14] left-adjointability of that diagram is equivalent to right-adjointability of

$$\begin{tikzcd} \mathcal{D}_1(\mathcal{H}_U^m) \arrow[r, "\alpha^*"'] \arrow[d, "d^{0,*}"'] & \mathcal{D}_1(\mathcal{H}_U^n) \arrow[d, "d^{0,*}"] \arrow[l, bend right, dotted, "\alpha_*"'] \\ \mathcal{D}_1(\mathcal{H}_U^{m+1}) \arrow[r, "(1+\alpha)^*"] & \mathcal{D}_1(\mathcal{H}_U^{n+1}) \arrow[l, bend left, dotted, "(1+\alpha)_*"] \end{tikzcd}$$

It is left to check that

$$d^{0,*}\alpha_* \simeq (1+\alpha)_* d^{0,*},$$

but this is true by basechange and the next lemma, because the underlying square of spaces is cartesian:

$$\begin{tikzcd}
\mathcal{H}_U^{n+1} \arrow[r] \arrow[d] & \mathcal{H}_U^{n} \arrow[d] \\
\mathcal{H}_U^{m+1} \arrow[r] \arrow[d] & \mathcal{H}_U^{m} \arrow[d] \\
\mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\
E_U \arrow[r] & E
\end{tikzcd}$$

It is clear that the lower and outer squares are pull-back squares, then so is the upper. (For this, see that $E_U \leftarrow {E_U}^{m+1} \xrightarrow{d^0} {E_U}^m$ is the projection and the left vertical composition is just the projection). Also, the horizontal maps are representable and everything is smooth. $\square$

Lemma 5.4. $(1+\alpha)_*$ *has locally finite cohomological dimension.*

Proof: $1+\alpha$ is a composition of face and degeneracy maps. The former are induced by diagonal inclusions, the latter by projections. All these maps are induced by pullback along topological submersions, for example, the first degeneracy map is induced from the pullback

$$\begin{tikzcd}
\mathcal{H}_U \underset{H}{\times} \mathcal{H}_U \arrow[r, "\simeq"] & \mathcal{H}_U \underset{E_U}{\times} E_U \underset{E}{\times} E_U \arrow[d] & \mathcal{H}_U \underset{E_U}{\times} E_U \underset{E}{\times} E_U \underset{E_U \underset{E}{\times} E_U}{\times} E_U \arrow[l] \arrow[d] & \mathcal{H}_U \arrow[l, "\simeq"'] \\
& E_U \underset{E}{\times} E_U & E_U, \arrow[l] &
\end{tikzcd}$$

while the first face map is induced from the pullback

$$\begin{tikzcd}
\mathcal{H}_U \underset{\mathcal{H}}{\times} \mathcal{H}_U \arrow[r] \arrow[d] & \mathcal{H}_U \arrow[d] \\
E_U \underset{E}{\times} E_U \arrow[r] & E_U.
\end{tikzcd}$$

Hence, by Lemma 2.41, the respective push-forwards have locally finite cohomological dimension. Note that compositions of functors of locally finite cohomological dimensions are of locally finite cohomological dimension. $\square$

Lemma 5.5. *The category $\mathcal{C}^{-1} = \mathcal{D}_1(\mathcal{H})$ admits geometric realizations and those are preserved by $G\colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}_U)$.*

Proof: Note that the category of 1-twisted $\mathcal{S}^1_{E/B}$-modules $\mathrm{Mod}_1(\mathcal{S}^1_{E/B})$ on $\mathcal{H}$ is Grothendieck abelian, hence by [17, Proposition 1.3.5.21] the derived category $\mathcal{D}_1(\mathcal{H})$ is presentable (cocomplete). Thus geometric realizations exist and G preserves those because it is a left adjoint. □

Lemma 5.6. *The functor $G\colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}_U)$ is conservative.*

Proof: Recall that a functor is called conservative if it reflects weak equivalences. Note that u^* is exact and conservative on the level of module-sheaves. Assume that $u^*\mathcal{F}^\bullet \xrightarrow{u^*f} u^*\mathcal{G}^\bullet$ is an equivalence. That means that $H^*u^*\mathcal{F}^\bullet \xrightarrow{Hu^*f} H^*u^*\mathcal{F}^\bullet$ is an isomorphism of module-sheaves. The following diagram commutes because u^* is exact.

$$\begin{array}{ccc} H^*u^*\mathcal{F}^\bullet & \xrightarrow{Hu^*f} & H^*u^*\mathcal{F}^\bullet \\ \| & & \| \\ u^*H^*\mathcal{F}^\bullet & \xrightarrow{u^*Hf} & u^*H^*\mathcal{G}^\bullet \end{array}$$

The lower map is an isomorphism of sheaves, u^* reflects those, hence $\mathcal{F}^\bullet \xrightarrow{f} \mathcal{G}^\bullet$ is an equivalence. □

5.3 Application

In this section we study the composition

$$\widehat{FM} \circ FM\colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}).$$

The strategy is to use the descent results from the previous section and realize the relative Fourier-Mukai transform as a map of cosimplicial ∞-categories. To show that this map is an equivalence it is enough to show that it is so at each level. Picking the simplicial resolution such that the T-duality diagram trivializes allows us to use the results of Section 4.2 to show that it is a levelwise equivalence.

Let $u\colon U \to B$ be a trivializing cover. Similarly to the simplicial stack $\mathcal{H}^\bullet_U$, we can form the simplicial stack $\widehat{\mathcal{H}}^\bullet_U$ and the simplicial bundles $E^\bullet_U$, $\hat{E}^\bullet_U$. Taking levelwise pullbacks, we get an induced diagrams, in which every square happens to be cartesian,

$$
\begin{tikzcd}[column sep=tiny]
 & & \mathcal{H}^n_U \underset{U^n}{\times} \widehat{\mathcal{H}}^n_U \arrow[dl, "q_2^n"'] \arrow[dr, "\hat{q}_2^n"] & & \\
 & \mathcal{H}^n_U \underset{U^n}{\times} \hat{E}^n_U \arrow[dl, "q_1^n"'] \arrow[dr] & & E^n_U \underset{U^n}{\times} \widehat{\mathcal{H}}^n_U \arrow[dl] \arrow[dr, "\hat{q}_1^n"] & \\
\mathcal{H}^n_U \arrow[dr] & & E^n_U \underset{U^n}{\times} \hat{E}^n_U \arrow[dl] \arrow[dr] & & \widehat{\mathcal{H}}^n_U \arrow[dl] \\
 & E^n_U \arrow[dr] & & \hat{E}^n_U \arrow[dl] & \\
 & & U^n & &
\end{tikzcd}
$$

for each $n \in \mathbb{N}$. The stack $\mathcal{H}^n_U \underset{U^n}{\times} \widehat{\mathcal{H}}^n_U$ has a canonical map

$$t^n \colon \mathcal{H}^n_U \underset{U^n}{\times} \widehat{\mathcal{H}}^n_U \to \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}$$

via which it receives an induced kernel $\mathcal{L}^n = t^{n*}\mathcal{L}$ through pull-back. Hence we get Fourier-Mukai transformations

$$
\begin{aligned}
FM^n &\colon \mathcal{D}_1(\mathcal{H}^n_U) \to \mathcal{D}_1(\widehat{\mathcal{H}}^n_U),\\
\widehat{FM}^n &\colon \mathcal{D}_1(\widehat{\mathcal{H}}^n_U) \to \mathcal{D}_1(\mathcal{H}^n_U)
\end{aligned}
$$

using that kernel.

Lemma 5.7. *The transforms FM^n assemble to give a map of cosimplicial ∞-categories. I.e., for any $\alpha \colon [n] \to [m]$ the following diagram of ∞-categories commutes.*

$$
\begin{tikzcd}
\mathcal{D}_1(\mathcal{H}^m_U) \arrow[r, "FM^m"] \arrow[d, "\alpha^*_{\mathcal{H}}"'] & \mathcal{D}_1(\widehat{\mathcal{H}}^m_U) \arrow[d, "\alpha^*_{\widehat{\mathcal{H}}}"] \\
\mathcal{D}_1(\mathcal{H}^n_U) \arrow[r, "FM^n"] & \mathcal{D}_1(\widehat{\mathcal{H}}^n_U)
\end{tikzcd}
$$

Proof: Note that the diagrams

$$\begin{tikzcd}
\mathcal{H}^n \arrow[rr, "\alpha_{\mathcal{H}}"] \arrow[dr] & & \mathcal{H}^m \arrow[dl] \\
& \mathcal{H} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\widehat{\mathcal{H}}^n \arrow[rr, "\alpha_{\widehat{\mathcal{H}}}"] \arrow[dr] & & \widehat{\mathcal{H}}^m \arrow[dl] \\
& \widehat{\mathcal{H}} &
\end{tikzcd}$$

commute. Hence so does

$$\begin{tikzcd}
\mathcal{H}^n \underset{U^n}{\times} \widehat{\mathcal{H}}^n \arrow[rr, "\alpha_{\mathcal{H}} \times \alpha_{\widehat{\mathcal{H}}}"] \arrow[dr] & & \mathcal{H}^m \underset{U^m}{\times} \widehat{\mathcal{H}}^m \arrow[dl] \\
& \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}}, &
\end{tikzcd}$$

which implies that the induced kernels are compatible:

$$(\alpha_{\mathcal{H}} \times \alpha_{\widehat{\mathcal{H}}})^* \mathcal{L}^m \simeq \mathcal{L}^n.$$

The same argument shows that the squares involving an α in the following diagram are pull-back squares.

$$\begin{tikzcd}
& & \mathcal{H}^n_U \underset{U^n}{\times} \widehat{\mathcal{H}}^n_U \arrow[dl, "q_2^n"'] \arrow[ddr, dashed, "\hat{q}_2^n"] \arrow[rrrr, "\alpha_{\widehat{\mathcal{H}}} \times \alpha_{\mathcal{H}}"] & & & & \mathcal{H}^m_U \underset{U^m}{\times} \widehat{\mathcal{H}}^m_U \arrow[dl, "q_2^m"'] \arrow[ddr, "\hat{q}_2^m"] & \\
& \mathcal{H}^n_U \underset{U^n}{\times} \hat{E}_{U^n} \arrow[ddl, "q_1^n"'] \arrow[rrrr, "\alpha'_{\mathcal{H}}"] & & & & \mathcal{H}^m_U \underset{U^m}{\times} \hat{E}_{U^m} \arrow[ddl, "q_1^m"'] & & \\
& & & E_{U^n} \underset{U^n}{\times} \widehat{\mathcal{H}}^n_U \arrow[dd, dashed, "\hat{q}_1^n"] \arrow[rrrr, dashed, "\alpha'_{\widehat{\mathcal{H}}}"] & & & & E_{U^m} \underset{U^m}{\times} \widehat{\mathcal{H}}^m_U \arrow[dd, "\hat{q}_1^m"] \\
\mathcal{H}^n_U \arrow[ddr] \arrow[rrrr, "\alpha_{\mathcal{H}}"] & & & & \mathcal{H}^m_U \arrow[ddr] & & & \\
& & & \widehat{\mathcal{H}}^n_U \arrow[ddl, dashed] \arrow[rrrr, dashed, "\alpha_{\widehat{\mathcal{H}}}"] & & & & \widehat{\mathcal{H}}^m_U \arrow[ddl] \\
& E_{U^n} \arrow[dd] & & & & E_{U^m} \arrow[dd] & & \\
& & \hat{E}_{U^n} \arrow[dl, dashed] & & & & \hat{E}_{U^m} \arrow[dl] & \\
& U^n \arrow[rrrr, "u^n"] & & & & U^m & &
\end{tikzcd}$$

The statement then is the following chain of isomorphisms.

$$
\begin{aligned}
\alpha^*_{\widehat{\mathcal{H}}} FM^m(\mathcal{F}) &= \alpha^*_{\widehat{\mathcal{H}}} \circ \hat{q}^m_{1*} \circ \hat{q}^m_{2*}\big((q_1^{m*} q_2^{m*} \mathcal{F}) \otimes \mathcal{L}^m\big) \\
&\simeq \hat{q}^n_{1*} \circ \alpha'^*_{\widehat{\mathcal{H}}} \circ \hat{q}^m_{2*}\big((q_1^{m*} q_2^{m*} \mathcal{F}) \otimes \mathcal{L}^m\big) \\
&\simeq \hat{q}^n_{1*} \circ \hat{q}^n_{2*} \circ (\alpha_{\widehat{\mathcal{H}}} \times \alpha_{\mathcal{H}})^*\big((q_1^{m*} q_2^{m*} \mathcal{F}) \otimes \mathcal{L}^m\big) \\
&\simeq \hat{q}^n_{1*} \circ \hat{q}^n_{2*} \circ \big((\alpha_{\widehat{\mathcal{H}}} \times \alpha_{\mathcal{H}})^*(q_1^{m*} q_2^{m*} \mathcal{F}) \otimes (\alpha_{\widehat{\mathcal{H}}} \times \alpha_{\mathcal{H}})^*\mathcal{L}^m\big) \\
&\simeq \hat{q}^n_{1*} \circ \hat{q}^n_{2*} \circ \big((\alpha_{\widehat{\mathcal{H}}} \times \alpha_{\mathcal{H}})^*(q_1^{m*} q_2^{m*} \mathcal{F}) \otimes \mathcal{L}^n\big) \\
&\simeq \hat{q}^n_{1*} \circ \hat{q}^n_{2*} \circ \big((q_2^{n*} q_1^{n*} \alpha^*_{\mathcal{H}} \mathcal{F}) \otimes \mathcal{L}^n\big) \\
&= FM^n(\alpha^*_{\mathcal{H}} \mathcal{F})
\end{aligned}
$$

□

Lemma 5.8. *$FM^\bullet$ is a map of* augmented *cosimplical ∞-categories. I.e. the diagram*

$$
\begin{tikzcd}
\mathcal{D}_1(\mathcal{H}) \arrow[r, "FM"] \arrow[d, dashed, "G"'] & \mathcal{D}_1(\widehat{\mathcal{H}}) \arrow[d, dashed, "G"] \\
\mathcal{D}_1(\mathcal{H}^0_U) \arrow[r, "FM^0"] & \mathcal{D}_1(\widehat{\mathcal{H}}^0_U)
\end{tikzcd}
$$

of ∞-categories commutes.

Proof: The proof is very similar to the previous one. We include it for completeness. Consider the diagram

$$
\begin{tikzcd}
 & & \mathcal{H}_U \underset{U}{\times} \widehat{\mathcal{H}}_U \arrow[rrrrr, "t"] \arrow[dl, "q_2"'] \arrow[ddr, dashed, "\hat{q}_2"] & & & & & \mathcal{H} \underset{B}{\times} \widehat{\mathcal{H}} \arrow[dl, "p_2"'] \arrow[ddr, "\hat{p}_2"] & & \\
 & \mathcal{H}_U \underset{U}{\times} \hat{E}_U \arrow[rrrrr, "s"] \arrow[ddl, "q_1"'] & & & & & \mathcal{H} \underset{B}{\times} \hat{E} \arrow[ddl, "p_1"'] & & & \\
 & & & E_U \underset{U}{\times} \widehat{\mathcal{H}}_U \arrow[rrrrr, dashed, "\hat{s}"] \arrow[ddr, dashed, "\hat{q}_1"'] & & & & & E \underset{B}{\times} \widehat{\mathcal{H}} \arrow[ddr, "\hat{p}_1"] & \\
\mathcal{H}_U \arrow[rrrrr, "r"] \arrow[ddr] & & & & & \mathcal{H} \arrow[ddr] & & & & \\
 & & & & \widehat{\mathcal{H}}_U \arrow[rrrrr, dashed, "\hat{t}"] \arrow[ddl, dashed] & & & & & \widehat{\mathcal{H}} \arrow[ddl] \\
 & E_U \arrow[rrrrr] \arrow[ddr] & & & & & E \arrow[ddr] & & & \\
 & & & \hat{E}_U \arrow[rrrrr, dashed] \arrow[dl, dashed] & & & & & \hat{E} \arrow[dl] & \\
 & & U \arrow[rrrrr, "u"] & & & & & B & &
\end{tikzcd}
$$

This follows by the following chain of isomorphisms.

$$\begin{aligned}
\hat{r}^* FM(\mathcal{F}) &= r^* p_{1*} \hat{p}_{2*} \left(p_2^* p_1^* \mathcal{F} \otimes \mathcal{L}\right) \\
&= \hat{q}_{1*} \hat{s}^* \hat{p}_{2*} \left(p_2^* p_1^* \mathcal{F} \otimes \mathcal{L}\right) \\
&= \hat{q}_{1*} \hat{q}_{2*} t^* \left(p_2^* p_1^* \mathcal{F} \otimes \mathcal{L}\right) \\
&= \hat{q}_{1*} \hat{q}_{2*} \left(t^* p_2^* p_1^* \mathcal{F} \otimes t^* \mathcal{L}\right) \\
&= \hat{q}_{1*} \hat{q}_{2*} \left(q_2^* q_1^* r^* \mathcal{F} \otimes t^* \mathcal{L}\right) \\
&= FM_U(r^* \mathcal{F})
\end{aligned}$$

The second isomorphism is Proposition 2.43, because $\hat{q}_2$ is a trivial T^n-bundle. The third is base-change on the level of chain-complexes of 0-twisted sheaves, because $\hat{p}_{2*}$ and $\hat{s}^*$ are exact in that category. $\square$

Summarizing the discussion, we get maps of augmented cosimplicial ∞-categories

$$\begin{aligned}
FM^\bullet &\colon \mathcal{D}_1(\mathcal{H}_U^\bullet) \to \mathcal{D}_1(\widehat{\mathcal{H}}_U^\bullet), \\
\widehat{FM}^\bullet &\colon \mathcal{D}_1(\widehat{\mathcal{H}}_U^\bullet) \to \mathcal{D}_1(\mathcal{H}_U^\bullet)
\end{aligned}$$

and we are finally ready to prove our main results.

Theorem 5.9. *The composition*

$$\widehat{FM} \circ FM \colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H})$$

is an equivalence.

Proof: In the previous section we have shown that the derived category satisfies descent, i.e.

$$\mathcal{D}_1(\mathcal{H}) \simeq \lim_n \mathcal{D}_1(\mathcal{H}_U^n).$$

Hence, to prove that the map $\widehat{FM} \circ FM \colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H})$ is an equivalence it is enough to show that

$$\widehat{FM}_U^\bullet \circ FM_U^\bullet \colon \mathcal{D}_1(\mathcal{H}_U^\bullet) \to \mathcal{D}_1(\mathcal{H}_U^\bullet)$$

is a levelwise equivalence, i.e., for each $n \in \mathbb{N}$ the composition

$$\widehat{FM}_U^n \circ FM_U^n \colon \mathcal{D}_1(\mathcal{H}_U^n) \to \mathcal{D}_1(\mathcal{H}_U^n)$$

is an equivalence. It suffices to know that it is an isomorphism in the homotopy category, so the results of Section 4.2 are applicable.

On each level the gerbes and bundles in question are trivializable, and we can use Lemma 4.3 to see that at each level, the Fourier-Mukai transformations compose and we have a canonical kernel. Theorem 4.26 shows that in this case, with the canonical kernel, the Fourier-Mukai transformation is an equivalence. □

With the same reasoning and Corollary 4.27, we can show that the composition

$$FM \circ \widehat{FM} \colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

is an equivalence, hence we have the following theorem.

Theorem 5.10. *The twisted Fourier-Mukai transformations*

$$FM \colon \mathcal{D}_1(\mathcal{H}) \to \mathcal{D}_1(\widehat{\mathcal{H}})$$

and

$$\widehat{FM} \colon \mathcal{D}_1(\widehat{\mathcal{H}}) \to \mathcal{D}_1(\mathcal{H})$$

are equivalences.

References

[1] Kai Behrend and Ping Xu. Differentiable stacks and gerbes. *J. Symplectic Geom.*, 9(3):285–341, 2011.

[2] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[3] U. Bruzzo, G. Marelli, and F. Pioli. A Fourier transform for sheaves on real tori. I. The equivalence $\mathrm{Sky}(T) \simeq \mathrm{Loc}(\hat{T})$. *J. Geom. Phys.*, 39(2):174–182, 2001.

[4] U. Bruzzo, G. Marelli, and F. Pioli. A Fourier transform for sheaves on real tori. II. Relative theory. *J. Geom. Phys.*, 41(4):312–329, 2002.

[5] Jean-Luc Brylinski. *Loop spaces, characteristic classes and geometric quantization*, volume 107 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[6] Ulrich Bunke and Thomas Nikolaus. T-duality via gerby geometry and reductions. *ArXiv e-prints*, May 2013.

[7] Ulrich Bunke, Thomas Schick, and Markus Spitzweck. Sheaf theory for stacks in manifolds and twisted cohomology for S^1-gerbes. *Algebr. Geom. Topol.*, 7:1007–1062, 2007.

[8] Ulrich Bunke, Thomas Schick, and Markus Spitzweck. Periodic twisted cohomology and T-duality. *Astérisque*, (337):vi+134, 2011.

[9] Ulrich Bunke, Thomas Schick, Markus Spitzweck, and Andreas Thom. Duality for topological abelian group stacks and T-duality. In *K-theory and noncommutative geometry*, EMS Ser. Congr. Rep., pages 227–347. Eur. Math. Soc., Zürich, 2008.

[10] Andrei Horia Caldararu. *Derived categories of twisted sheaves on Calabi-Yau manifolds*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Cornell University.

[11] James F. Glazebrook, Marcos Jardim, and Franz W. Kamber. A Fourier-Mukai transform for real torus bundles. *J. Geom. Phys.*, 50(1-4):360–392, 2004.

[12] J. Heinloth. Notes on differentiable stacks. In *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Winter Term 2004/2005*, pages 1–32. Universitätsdrucke Göttingen, Göttingen, 2005.

[13] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[14] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel.

[15] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[16] Max Lieblich. Moduli of twisted sheaves. *Duke Math. J.*, 138(1):23–118, 2007.

[17] Jacob Lurie. *Higher Algebra*. 2012.

[18] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[19] Shigeru Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[20] Behrang Noohi. Foundations of topological stacks i. *ArXiv e-prints*, May 2013.

[21] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2014.

[22] Günter Tamme. *Introduction to étale cohomology*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.