Bertini theorems for hypersurface sections containing a subscheme over finite fields

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Introduction

Bertini theorems say that if a scheme $X \subseteq \mathbb{P}^n$ has a certain property, for example if it is smooth or geometrically irreducible, then there exists a hyperplane H such that the scheme-theoretic intersection $H \cap X$ has this property as well. For the projective space over an infinite field k, we have the following classical Bertini smoothness theorem:

Theorem 0.1 ([Jou83] Théorème 6.3). *Let k be an infinite field and $X \subseteq \mathbb{P}_k^n$ be a quasi-projective smooth scheme. Then there exists a hyperplane H such that the intersection $H \cap X$ is smooth.*

This can be shown in the following way. We have a parameterization of the hyperplanes in $\mathbb{P}_k^n$ by the dual projective space $(\mathbb{P}_k^n)^\vee$: a point $a = (a_0 : \ldots : a_n)$ corresponds to the hyperplane given by the equation $a_0x_0+\ldots+a_nx_n = 0$, where x_i denote the homogeneous coordinates of the projective space $\mathbb{P}_k^n$. Then for any field k, there is a dense Zariski open set $U_X \subseteq (\mathbb{P}_k^n)^\vee$ parameterizing the hyperplanes that intersect X smoothly. If k is infinite as in Theorem 0.1, the set $U_X(k)$ of k-rational points is non-empty, since $\mathbb{P}_k^n(k)$ is a Zariski dense set in $\mathbb{P}_k^n$. Hence we get the hyperplane we wanted.

Of course, one would like to have an analogue of Theorem 0.1 over finite fields as well. Unfortunately, if k is a finite field, it may happen that U_X does not have any k-rational points, and therefore none of the finitely many hyperplanes over k intersect X smoothly. But B. Poonen showed in [Poo04], that in this case there always exists a smooth hypersurface section of X.

Theorem 0.2 ([Poo04] Theorem 1.1). *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over a finite field k. Then there always exists a hypersurface H such that $H \cap X$ is smooth of dimension $m-1$.*

Independently, O. Gabber proved in Corollary 1.6 in [Gab01] the existence of good hypersurfaces of any sufficiently large degree that is divisible by the characteristic of the field k.

Poonen also proved in [Poo08] that the hypersurface H can be chosen such that it contains a given closed subscheme Z, if $Z \cap X$ is smooth and $\dim X > 2\dim(Z \cap X)$. It is already mentioned there, that it should be possible to prove a version for $Z \cap X$ non-smooth as well. The goal of this project was to show that there exists such an analogue.

In the first section of this thesis, we will present some basic results about intersections of schemes. Furthermore, the embedding dimension will be introduced and calculated in situations that are relevant for us. In this context, we will also look at the schemes $X_e = X(\Omega^1_{X|\mathbb{F}_q}, e)$ of the flattening stratification of a scheme X for the rank of the differential sheaf $\Omega^1_{X|\mathbb{F}_q}$, i.e. the locus in X where $\Omega^1_{X|\mathbb{F}_q}$ has rank e.

The second section contains the main result of this thesis, the requested analogue of Theorem 0.2:

Theorem 0.3. *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over a finite field $\mathbb{F}_q$. Let Z be a closed subscheme of $\mathbb{P}^n$, and let $V = Z \cap X$. Assume $\max_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, where V_e are the subschemes of the flattening stratification of V for the rank of the differential sheaf $\Omega^1_{V|\mathbb{F}_q}$. Then, for $d \gg 1$, there exists a hypersurface H of degree d containing Z such that $H \cap X$ is smooth of dimension $m - 1$.*

There is a similar result for infinite fields by Altman and Kleiman in [AK79]; the theorem there states the following:

Theorem 0.4. *([AK79] Theorem 7) Let k be an infinite field, let X be a smooth quasi-projective k-scheme, let Z be a subscheme of X and U a subscheme of Z. Assume the estimate*

$$\max_e \left\{\dim(U(\Omega^1_Z|U, e)) + e\right\} < \min_{x \in U}(\dim_x(X)).$$

Then there is a hypersurface section of X containing Z which is smooth along U and off the closure of Z in X.

If we take $Z = U$ closed in X, this gives the analogue of Theorem 0.3 for infinite fields, since the conditions $\max_{0 \le e \le m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$ of Theorem 0.3 coincide with the condition here. But in [AK79] the scheme Z must be contained in X, whereas in our case Z need not be contained in X; it does not even have to intersect X smoothly.

To prove Theorem 0.3, we will define the density $\mu_Z(\mathcal{P})$ of a subset $\mathcal{P}$ of all homogeneous polynomials in $\mathbb{F}_q[X_0, \ldots, X_n]$. Then we look at the set $\mathcal{P}$ of all homogeneous polynomials $f \in \mathbb{F}_q[X_0, \ldots, X_n]$ such that the hypersurface H_f given by f contains Z and intersects X smoothly. If the density $\mu_Z(\mathcal{P})$ is positive, the set is nonempty. Hence if we show that the density of $\mathcal{P}$ is larger than zero, we get the hypersurface section we want; more precisely, we will show that the density is equal to

$$\mu_Z(\mathcal{P}) = \frac{1}{\zeta_{X-V}(m+1) \prod_{e=0}^{m-1} \zeta_{V_e}(m-e)}.$$

For this calculation we apply the so called closed point sieve, which has been used by Poonen in [Poo04] for the proof of Theorem 0.2. The idea is to start with all homogeneous polynomials f of degree d that vanish at Z and, for each closed point $P \in X$, sieve out those polynomials f for which the intersection $H_f \cap X$ is singular at P. This works since smoothness can be tested locally, and since a scheme of finite type over a finite field is smooth if and only if it is regular at all closed points.

In a first step we consider only points of degree bounded by some $r > 0$ and calculate the density of the set $\mathcal{P}_r$ of the remaining polynomials, i.e. those polynomials that give a hypersurface containing Z and intersecting X smoothly at all closed points of degree bounded by r. Unfortunately, this does not generalize to all closed points: the fact that we only look at a finite set of points is crucial for the proof. But the points of degree $\ge r$ do not give a finite set. The main difficulty of the proof lies in its second step, in which we show that the set of polynomials that are sieved out at the infinitely many points of degree $\ge r$ is of density zero. Then the limit of $\mu_Z(\mathcal{P}_r)$ for $r \to \infty$ gives the correct density.

Finally, in the third section we prove a refined version of Theorem 0.3, in which we prescribe the first terms of the Taylor expansion of the polynomial f that give the hypersurface at finitely many points that are not in Z. Using this theorem, we show for a scheme X that is smooth in all but finitely many closed points, that there exists a hypersurface H that contains Z but none of those finitely many points, and intersects X smoothly.

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1 Scheme-theoretic intersections and embedding dimension

Let X and Z be two subschemes of a scheme Y with morphisms $i : X \to Y$ and $j : Z \to Y$. Then

$$X \cap Z := X \times_Y Z = i^{-1}(Z) = j^{-1}(X)$$

is the (scheme-theoretic) intersection of X and Z.

Remark 1.1. Let X and Z be closed subschemes of Y with ideal sheaves $\mathcal{I}_X$ and $\mathcal{I}_Z$, respectively. The intersection of X and Z is again a closed subscheme of Y and the ideal sheaf corresponding to it is given by $\mathcal{I}_{X\cap Z} = \mathcal{I}_X + \mathcal{I}_Z \subseteq \mathcal{O}_Y$, where the sum of $\mathcal{I}_X$ and $\mathcal{I}_Z$ is the sheaf associated to the presheaf $U \mapsto \mathcal{I}_X(U) + \mathcal{I}_Z(U) \subseteq \mathcal{O}_Y(U)$.

This can be proven in the following way: We can cover Y by affine open subsets and assume $Y = \operatorname{Spec} A$ to be affine. Let $X = \operatorname{Spec}(A/\mathfrak{a})$ and $Z = \operatorname{Spec}(A/\mathfrak{b})$. In this situation, the intersection of X and Z is given by

$$\begin{aligned} X \cap Z = X \times_Y Z &= \operatorname{Spec}(A/\mathfrak{a}) \times_{\operatorname{Spec} A} \operatorname{Spec}(A/\mathfrak{b}) \\ &= \operatorname{Spec}(A/\mathfrak{a} \otimes_A A/\mathfrak{b}) = \operatorname{Spec}(A/(\mathfrak{a}+\mathfrak{b})). \end{aligned}$$

Thus, in the local ring $\mathcal{O}_{Y,P}$ of a point $P \in Y$ given by the prime ideal $\mathfrak{p}$ we have

$$\mathcal{I}_{X\cap Z,P} = (\mathfrak{a}+\mathfrak{b})_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}} + \mathfrak{b}_{\mathfrak{p}} = (\mathcal{I}_X)_P + (\mathcal{I}_Z)_P = (\mathcal{I}_X + \mathcal{I}_Z)_P.$$

In particular, if X is quasi-projective and Z a closed subscheme of $\mathbb{P}^n$, as will be the case in the second section, then the local ring of the intersection $V = X \cap Z$ at a closed point P is given by $\mathcal{O}_{X,P} / \mathcal{I}_{Z,P}$, where $\mathcal{I}_Z$ is the sheaf of ideals of Z: To see this, we put $S = \mathbb{F}_q[x_0, \dots, x_n]$. In some affine open neighbourhood of P let X be given by $\operatorname{Spec} A$ and let Z, as a closed subscheme of $\mathbb{P}^n$, be given by $\operatorname{Spec}(S/\mathfrak{b})$. Then by definition, $X \cap Z = \operatorname{Spec} A \times_S \operatorname{Spec}(S/\mathfrak{b}) = \operatorname{Spec}(A \otimes_S S/\mathfrak{b}) = \operatorname{Spec}(A/\mathfrak{b})$. Hence the local ring of V at P is equal to $\mathcal{O}_{V,P} = \mathcal{O}_{X,P} / \mathcal{I}_{Z,P}$ with maximal ideal $\mathfrak{m}_{V,P} = \mathfrak{m}_{X,P} / \mathcal{I}_{Z,P}$, and we have an equality $\kappa_V(P) = \kappa(P)$.

Definition 1.2. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. We call the function $rk(\mathcal{F}) : X \to \mathbb{N}_0$ defined by*

$$rk(\mathcal{F})(x) = rk_x(\mathcal{F}) = \dim_{\kappa(x)} \mathcal{F}(x) = \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

the rank of $\mathcal{F}$.

Theorem 1.3. *([GW10] Theorem 11.17) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type and let $r \geq 0$ be an integer. Then there exists a unique subscheme $X(\mathcal{F}, r)$ of X such that a morphism of schemes $f : T \to X$ factors through $X(\mathcal{F}, r)$ if and only if $f^*(\mathcal{F})$ is locally free of rank r.*

By this theorem, a point $x \in X$ lies in $X(\mathcal{F}, r)$ if and only if $i_x^*\mathcal{F}$ is locally free of rank r, where $i_x : \mathrm{Spec}(\kappa(x)) \to X$ is the canonical morphism. Hence the underlying set of $X(\mathcal{F}, r)$ is $\{x \in X : \mathrm{rk}_x(\mathcal{F}) = r\}$. Set-theoretically, X is therefore the union of the locally closed subsets $X(\mathcal{F}, r)$. The family $X(\mathcal{F}, r)$ for $r \geq 0$ is called flattening stratification.

If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, $\mathrm{rk}(\mathcal{F})$ is a locally constant function. Conversely, we have the following corollary of Theorem 1.3 above:

Corollary 1.4. *([GW10] Corollary 11.18) Let X be a reduced scheme and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Then $\mathcal{F}$ is locally free if and only if $rk(\mathcal{F})$ is a locally constant function.*

Let k be a field, let X be a scheme locally of finite type over k and let $x \in X$ be a point. We define the embedding dimension $e(x)$ of X at x to be the integer

$$e(x) = \dim_{\kappa(x)}(\Omega^1_{X|k}(x)),$$

i.e. the rank of $\Omega^1_{X|k}$ at x. Then we have a flattening stratification of X given by the locally closed subschemes

$$X_e = X(\Omega^1_{X|k}, e).$$

By definition those are the subschemes such that for all points $x \in X_e$ the embedding dimension $e(x)$ of X at x is equal to e.

The situation in the next sections will be the following: let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over $\mathbb{F}_q$ and let Z be a closed subscheme of $\mathbb{P}^n$. Let $V = Z \cap X$ be the scheme-theoretical intersection of Z and X. In order to calculate the fraction of homogeneous polynomials $f \in \mathbb{F}_q[X_0, \ldots, X_n]$ of degree d that give us a hypersurface containing Z and intersecting X smoothly, we will need to know the embedding dimension $e_V(P)$ of V at a point $P \in V$. We will see that $e_V(P)$ equals $\dim_{\kappa(P)} \left(\mathfrak{m}_{X,P} / (\mathfrak{m}_{X,P}^2, \mathcal{I}_{Z,P}) \right)$. This dimension will arise naturally from the calculation of the fraction of the polynomials named above. For the calculation we need some properties of the sheaf of differentials.

Lemma 1.5. *([Har93] Proposition II 8.4A and Proposition II 8.7) Let A be a ring, let B be an A-algebra, and I be an ideal of B. Define $C = B/I$. Then there exists a canonical exact sequence of C-modules*

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B|A} \otimes_B C \to \Omega^1_{C|A} \to 0,$$

where for any $b \in I$ with image $\bar{b}$ in I/I^2 we have $\delta(\bar{b}) = db \otimes 1$.

If B is a local ring which contains a field k isomorphic to its residue field $B/\mathfrak{m}$, then the map $\delta : \mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{B|k} \otimes_B k$ is an isomorphism.

Lemma 1.6. *([Har93] Proposition II.8.12) Let $f : X \to Y$ be a morphism of schemes and let Z be a closed subscheme of X with ideal sheaf $\mathcal{I}$. Then there exists an exact sequence of sheaves on Z*

$$\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X|Y} \otimes \mathcal{O}_Z \to \Omega^1_{Z|Y} \to 0.$$

Lemma 1.7. *([Har93] Theorem II 8.25A) Let A be a complete local ring containing a field k. Assume that the residue field $\kappa(A) = A/\mathfrak{m}$ is a separably generated extension of k. Then there exists a subfield $K \subseteq A$, containing k, such that $K \to A/\mathfrak{m}$ is an isomorphism.*

Lemma 1.8. *(cf. [Har93] Exercise II 8.1) Let B be a local ring containing a field k such that the residue field $\kappa(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then there exists an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{B|k} \otimes_B \kappa(B)$.*

Proof. Since $B/\mathfrak{m}^2$ is a complete local ring, by Lemma 1.7 there exists a subfield $K \subseteq B/\mathfrak{m}^2$ and an isomorphism $K \cong \kappa(B)$. Now Lemma 1.5 yields an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{(B/\mathfrak{m}^2)|k} \otimes_{(B/\mathfrak{m}^2)} \kappa(B)$. By ([Mat70], p. 187, Theorem 58 (ii)) we have an isomorphism $\Omega^1_{B|k} \otimes_B \kappa(B) \cong \Omega^1_{(B/\mathfrak{m}^2)|k} \otimes_{(B/\mathfrak{m}^2)} \kappa(B)$; this shows the Proposition. □

Proposition 1.9. *Let X be a scheme of finite type over a perfect field k and let Z be a closed subscheme of $\mathbb{P}^n$. Let $V = Z \cap X$ be the intersection of Z and X. Then for a closed point $P \in V$,*

$$\Omega^1_{V|k}(P) \cong \mathfrak{m}_{V,P}/\mathfrak{m}^2_{V,P} \cong \mathfrak{m}_{X,P}/(\mathcal{I}_{Z,P}, \mathfrak{m}^2_{X,P}).$$

Proof. Since V is of finite type over k, the local ring $\mathcal{O}_{V,P}$ contains k and the residue field $\kappa(P)$ of X at P is a finite separable field extension of k. By Lemma 1.8, there are isomorphisms

$$\mathfrak{m}_{V,P}/\mathfrak{m}^2_{V,P} \cong \Omega^1_{\mathcal{O}_{V,P}|k} \otimes_{\mathcal{O}_{V,P}} \kappa_V(P) \cong \Omega^1_{V|k}(P),$$

where $\kappa_V(P)$ is the residue field of V at P. We have seen in Remark 1.1 that the local ring of V at P is equal to $\mathcal{O}_{V,P} = \mathcal{O}_{X,P}/\mathcal{I}_{Z,P}$ and we have an equality $\kappa_V(P) = \kappa(P)$. Now the Proposition follows from $(\mathfrak{m}_{X,P}/\mathcal{I}_{Z,P})/(\mathfrak{m}_{X,P}/\mathcal{I}_{Z,P})^2 \cong \mathfrak{m}_{X,P}/(\mathcal{I}_{Z,P}, \mathfrak{m}^2_{X,P})$. □

Remark 1.10. Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over $\mathbb{F}_q$, let Z be a closed subscheme of $\mathbb{P}^n$ and $V = Z \cap X$ be the intersection of Z and X. By Proposition 1.9, we can calculate the embedding dimension of V at P as

$$e_V(P) = \dim_{\kappa(P)} \Omega^1_{V|\mathbb{F}_q}(P) = \dim_{\kappa(P)} \mathfrak{m}_{X,P}/(\mathcal{I}_{Z,P}, \mathfrak{m}^2_{X,P}).$$

The points in the subschemes V_e of the flattening stratification of V are exactly the points $P \in V$ such that $\dim_{\kappa(P)} \mathfrak{m}_{X,P}/(\mathcal{I}_{Z,P}, \mathfrak{m}^2_{X,P}) = e$. In particular, since X is smooth, it is also regular and we get

$$\dim X \geq \dim \mathcal{O}_{X,P} = \dim_{\kappa(P)} \mathfrak{m}_{X,P} \,/\, \mathfrak{m}_{X,P}^2 \geq \dim_{\kappa(P)} \mathfrak{m}_{X,P} \,/ (\mathcal{I}_{Z,P}, \mathfrak{m}_{X,P}^2) = e_V(P),$$

i.e. the dimension of X is a uniform bound for the embedding dimension $e_V(P)$ for all closed points $P \in V$.

The relation between smoothness of a scheme over a field k at a point x and the stalk of the sheaf of differentials $\Omega^1_{X|k}$ at x that we will need is the following:

Lemma 1.11. *([Liu06] Proposition 6.2.2) Let X be a scheme of finite type over a field k and $x \in X$. Then the following properties are equivalent:*

(i) X is smooth in a neighbourhood of x.

(ii) X is smooth at x.

(iii) $\Omega^1_{X|k,x}$ is free of rank $\dim_x X := \inf\{\dim U |\ U$ is an open neighbourhood of $x\}$.

Note that for a closed point $x \in X$, we have $\dim_x X = \dim \mathcal{O}_{X,x}$.

Theorem 1.12. *([AK70] Theorem VII 5.7) Let S be a locally Noetherian scheme, X an S-scheme locally of finite type, Y a closed S-subscheme, and J its sheaf of ideals. Let x be a point of Y and $g_1, \dots, g_n$ local sections of $\mathcal{O}_X$. Suppose X is smooth over S at x. Then the following conditions are equivalent:*

(i) There exists an open neighbourhood X_1 of X such that $g_1, \dots, g_n$ define an étale morphism $g : X_1 \to \mathbb{A}^n_S$ and $g_1, \dots, g_p$ generate J on X_1.

(ii) (a) Y is smooth over S at x.

(b) $g_{1,x}, \dots, g_{p,x} \in J_x$.

(c) $dg_1(x), \dots, dg_n(x)$ form a basis of $\Omega^1_{X|S}(x)$.

(d) $dg_{p+1}(x), \dots, dg_n(x)$ form a basis of $\Omega^1_{Y|S}(x)$.

(iii) $g_{1,x}, \ldots, g_{p,x}$ *generate* J_x *and* $dg_1(x), \ldots, dg_n(x)$ *form a basis of* $\Omega^1_{X|S}(x)$.

(iv) Y *is smooth over* S *at* x, $g_{1,x}, \ldots, g_{p,x}$ *form a minimal set of generators of* J_x *and* $dg_{p+1}(x), \ldots, dg_n(x)$ *form a basis of* $\Omega^1_{Y|S}(x)$.

Furthermore, if these conditions hold, then, at x*, the sequence*

$$0 \to J/J^2 \to \Omega^1_{X|S} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \Omega^1_{Y|S} \to 0$$

is exact and composed of free $\mathcal{O}_Y$*-modules with bases that are induced by* $\{g_1, \ldots, g_p\}$, $\{dg_1, \ldots, dg_n\}$ *and* $\{dg_{p+1}, \ldots, dg_n\}$.

We also need the following property of coherent sheaves:

Lemma 1.13. *([Har93] Exercise II 5.7) Let* X *be a Noetherian scheme and* $\mathcal{F}$ *a coherent sheaf on* X*. If the stalk* $\mathcal{F}_x$ *at a point* $x \in X$ *is a free* $\mathcal{O}_{X,x}$*-module, then there exists a neighbourhood* U *of* x *such that* $\mathcal{F}\big|_U$ *is free.*

2 Smooth hypersurface sections containing a closed subscheme over a finite field

In this section we want to prove the analogue of Theorem 1.1 of [Poo08] in the case where the intersection of X and Z is not smooth. Let $\mathbb{F}_q$ be a finite field of $q = p^a$ elements. Let $S = \mathbb{F}_q[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of the projective space $\mathbb{P}^n$ over $\mathbb{F}_q$ and $S_d \subseteq S$ the $\mathbb{F}_q$-subspace of homogeneous polynomials of degree d. Let S'_d be the set of all polynomials in $\mathbb{F}_q[x_0, \ldots x_n]$ of degree $\leq d$ and $S_{\text{homog}} = \bigcup_{d \geq 0} S_d$.

Let X be a scheme of finite type over $\mathbb{F}_q$. The degree of a point $P \in X$ is defined as $\deg P = [\kappa(P) : \mathbb{F}_q]$. By [GW10] Proposition 3.33, a point P of a scheme locally of finite type over a field is closed if and only if the degree of P is finite. Furthermore, the schemes that we look at are always of finite type over a finite field $\mathbb{F}_q$, and therefore they are smooth over $\mathbb{F}_q$ if and only if they are regular at all closed points.

For a scheme X of finite type over $\mathbb{F}_q$ we define the zeta function

$$\zeta_X(s) := \prod_{P \in X \text{ closed}} (1 - q^{-s \deg P})^{-1}.$$

This product converges for $\mathrm{Re}(s) > \dim X$ by [Ser65] Chapter 1.6.

Let Z be a fixed closed subscheme of $\mathbb{P}^n$. For $d \in \mathbb{Z}_{\geq 0}$ let I_d be the $\mathbb{F}_q$-subspace of polynomials $f \in S_d$ vanishing on Z, and $I_{\text{homog}} = \bigcup_{d \geq 0} I_d$. We can identify S_d with S'_d by the dehomogenization $x_0 = 1$. We define the density of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ by

$$\mu_Z(\mathcal{P}) := \lim_{d \to \infty} \frac{\#(\mathcal{P} \cap I_d)}{\# I_d},$$

if the limit exists (cf. [Poo08]). We cannot measure the density using the definition of [Poo04], since if the dimension of Z is positive, the density of I_{homog} would always be zero (cf. Lemma 3.1 [CP13]), and hence we have to use this density relative to I_{homog}. We further define the upper and lower density $\overline{\mu}_Z(\mathcal{P})$ and $\underline{\mu}_Z(\mathcal{P})$ of a subset $\mathcal{P} \subseteq I_{\text{homog}}$ by

$$\overline{\mu}_Z(\mathcal{P}) := \limsup_{d \to \infty} \frac{\#(\mathcal{P} \cap I_d)}{\# I_d},$$

and using lim inf in place of lim sup. A set of density zero does not need to be nonempty; but if the density of a set is positive, then the set contains infinitely many polynomials. For a polynomial $f \in I_d$ let $H_f = \operatorname{Proj}(S/(f))$ be the hypersurface defined by f.

Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over $\mathbb{F}_q$. We will show that the density of the set of polynomials $f \in I_{\text{homog}}$, such that the hypersurface H_f contains Z and intersects X smoothly, is positive and therefore such a hypersurface always exists.

Theorem 2.1. *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over $\mathbb{F}_q$. Let Z be a closed subscheme of $\mathbb{P}^n$ and let $V := Z \cap X$ be the intersection. We define*

$$\mathcal{P} = \{f \in I_{\text{homog}} :\ H_f \cap X \textit{ is smooth of dimension } m-1\}.$$

(i) If $\max\limits_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, then

$$\mu_Z(\mathcal{P}) = \frac{\zeta_V(m+1)}{\zeta_X(m+1)\prod\limits_{e=0}^{m-1}\zeta_{V_e}(m-e)} = \frac{1}{\zeta_{X-V}(m+1)\prod\limits_{e=0}^{m-1}\zeta_{V_e}(m-e)}.$$

In particular, there exists a hypersurface H of degree $d \gg 1$ containing Z such that $H \cap X$ is smooth of dimension $m-1$.

(ii) If $\max\limits_{0 \leq e \leq m-1} \{e + \dim V_e\} \geq m$ or $V_m \neq \emptyset$, then $\mu_Z(\mathcal{P}) = 0$.

At the end of this section, we will give an example involving simple normal crossings, in which the conditions of Theorem 2.1 (i) are fulfilled.

Before we start the proof, we want to make a few remarks regarding the density. The first two remarks show that Theorem 2.1 implies both Theorem 1.1 of [Poo04] and Theorem 1.1 of [Poo08].

Remark 2.2. If we choose Z to be empty, then the density of a set $\mathcal{P} \subseteq I_{\text{homog}} = S_{\text{homog}}$ is just

$$\mu_\emptyset(\mathcal{P}) = \lim_{d\to\infty} \frac{\#(\mathcal{P} \cap S_d)}{\#S_d}.$$

This is the same density as used in [Poo04]. Furthermore, the conditions of Theorem 2.1(i) are fulfilled, since V is also empty, and the density

$$\mu_\emptyset(\mathcal{P}) = \frac{1}{\zeta_{X-V}(m+1)\prod_{e=0}^{m-1}\zeta_{V_e}(m-e)} = \zeta_X(m+1)^{-1}$$

given by Theorem 2.1(i) is the same density as in Theorem 1.1 of [Poo04].

Remark 2.3. If the intersection $V = Z \cap X$ is smooth of dimension $l \geq 0$ as required in [Poo08], then for a closed point $P \in V$,

$$l = \dim \mathcal{O}_{V,P} = \dim \mathcal{O}_{X,P} / \mathcal{I}_{Z,P} = \dim_{\kappa(P)} \mathfrak{m}_{X,P} / (\mathfrak{m}^2_{X,P}, \mathcal{I}_{Z,P}) = e_V(P).$$

Hence in this case, the embedding dimension of V at all points is equal to the dimension l of the intersection V and $V_e = \emptyset$ for all $e \neq l$. It follows that $\dim V_l = \dim V$ and the requirement $\max_{0\leq e\leq m-1} \{e + \dim V_e\} < m$ of Theorem 2.1 implies $l + \dim V = 2l < m$. Therefore, if this condition is fulfilled, Theorem 2.1 (i) yields the statement of Theorem 1.1 of [Poo08]

$$\mu_Z(\mathcal{P}) = \frac{\zeta_V(m+1)}{\zeta_X(m+1)\zeta_V(m-l)}.$$

Thus, Theorem 1.1 of [Poo08] is implied by Theorem 2.1.

Remark 2.4. The density in Theorem 2.1 is independent of the embedding $X \hookrightarrow \mathbb{P}^n$.

Remark 2.5. If X' is a subscheme of X, then obviously $\mu_Z(X') \leq \mu_Z(X)$.

We can say even more about the density of X if X is the union of two disjoint open subschemes X_1 and X_2 of X. Since the embedding dimension is calculated locally and X_1 is open in X, we have the equality $e_X(P) = e_{X_1}(P)$ for any point $P \in X_1$, and similarly for X_2. Therefore, the set of points in $(Z \cap X)_e$ is the union of the points in $(Z \cap X_1)_e$ and $(Z \cap X_2)_e$, and for $\mathrm{Re}(s) > \dim(Z \cap X)_e$ we have

$$\begin{aligned}\zeta_{(Z\cap X)_e}(s) &= \prod_{P\in(Z\cap X)_e \text{ closed}} (1-q^{-s\deg P})^{-1} \\ &= \prod_{P\in(Z\cap X_1)_e \text{ closed}} (1-q^{-s\deg P})^{-1} \cdot \prod_{P\in(Z\cap X_2)_e \text{ closed}} (1-q^{-s\deg P})^{-1} \\ &= \zeta_{(Z\cap X_1)_e}(s)\cdot\zeta_{(Z\cap X_2)_e}(s).\end{aligned}$$

The zeta function for X and for $Z \cap X$ is also multiplicative in the same way; hence if the requirements of Theorem 2.1 (i) are fulfilled,

$$\begin{aligned}\mu_Z(\mathcal{P}_X) &= \frac{\zeta_{Z\cap X}(m+1)}{\zeta_X(m+1)\prod_{e=0}^{m-1}\zeta_{(Z\cap X)_e}(m-e)} \\ &= \frac{\zeta_{Z\cap X_1}(m+1)\cdot\zeta_{Z\cap X_2}(m+1)}{\zeta_{X_1}(m+1)\cdot\zeta_{X_2}(m+1)\cdot\prod_{e=0}^{m-1}\zeta_{(Z\cap X_1)_e}(m-e)\cdot\prod_{e=0}^{m-1}\zeta_{(Z\cap X_2)_e}(m-e)} \\ &= \mu_Z(\mathcal{P}_{X_1})\cdot\mu_Z(\mathcal{P}_{X_2}),\end{aligned}$$

where $\mathcal{P}_X = \{f \in I_{\text{homog}} : H_f \cap X \text{ is smooth of dimension } m-1\}$ and $\mathcal{P}_{X_1}$ and $\mathcal{P}_{X_2}$ are defined similarly.

Remark 2.6. It is important that we fix the scheme Z at the beginning. Densities calculated for two different closed subschemes cannot be compared easily, because in the definition of the density we use the ideal sheaf of the closed subscheme; the density μ_Z is relative to I_{homog}. So in general, we cannot combine the result of Theorem 2.1 for two arbitrary but different closed subschemes Z_1 and Z_2 to get a result for example for the union of those subschemes. But if Z_1 and Z_2 are disjoint closed subschemes such that the requirements of Theorem 2.1 are fulfilled for Z_1, Z_2 and the union $Z_1 \cup Z_2$, then $\mu_{Z_1\cup Z_2}(\mathcal{P}) = \mu_{Z_1}(\mathcal{P}) \cdot \mu_{Z_2}(\mathcal{P})$. The reason for this is again the multiplicativity of the zeta function as in the remark above.

If Z_1 and Z_2 are two distinct closed subschemes of $\mathbb{P}^n$ with $Z_1 \cap X = Z = Z_2 \cap X$, such that the requirements of Theorem 2.1 (i) are fulfilled, then the density is in both cases given by

$$\mu_{Z_1}(\mathcal{P}) = \mu_{Z_2}(\mathcal{P}) = \frac{\zeta_V(m+1)}{\zeta_X(m+1)\prod_{e=0}^{m-1}\zeta_{V_e}(m-e)},$$

where again $V = Z \cap X$. Note that the density in Theorem 2.1 does not depend on the points of Z outside of X: if we consider two closed subschemes Z and $Z' := Z \cap X$ of $\mathbb{P}^n$, then the density $\mu_Z(\mathcal{P})$ must be equal to $\mu_{Z'}(\mathcal{P})$, since the right hand side of the equality in Theorem 2.1 (i) does not depend on the points in $Z - Z'$. This may seem suprising, since in general, for a fixed degree, there will be more hypersurfaces that contain Z' than hypersurfaces that contain Z; so one would expect the density calculated for Z' to be larger than that for Z. But as stated above, the two densities cannot be compared.

As mentioned in the introduction, the proof of Theorem 2.1 will use the closed point sieve introduced in [Poo04]. It will be parallel to the one in [Poo08]; but there the intersection of X and the closed subscheme Z is assumed to be smooth, which does not have to be the case here. Therefore we will have to make significant changes in almost every line of the proof.

For this closed point sieve we will consider closed points of X of low, medium and high degree in the next three sections. At first, we will calculate in Lemma 2.12 the density of the set $\mathcal{P}_r$ of polynomials $f \in I_{\text{homog}}$ that give a good hypersurface section in the points of low degree bounded by r. In the subsequent sections we will show in 2.15, 2.16 and 2.19, that this density does not change if we also consider points of medium and high degree. More precisely, we will see that $\mu_Z(\mathcal{P})$ differs from the density $\mu_Z(\mathcal{P}_r)$ of the polynomials that give a good hypersurface section at points of degree bounded by r at most by the upper density of the polynomials that do not give a good hypersurface section at points of medium and high degree. Hence we need to prove for $r \to \infty$, that this upper density is zero, and that the limit of $\mu_Z(\mathcal{P}_r)$ is the value that we claimed for

$\mu_Z(\mathcal{P})$. The requirements $\max\limits_{0\le e\le m-1}\{e+\dim V_e\}<m$ and $V_m=\emptyset$ of Theorem 2.1 (i) will be used in each lemma mentioned above.

2.1 Singular points of low degree

Let the notation be as in Theorem 2.1.

The goal in this section is to calculate the density of the set of polynomials that give a smooth hypersurface section in the points of low degree. For this, we need to study the zeroth Zariski-cohomology group of a finite subscheme of $\mathbb{P}^n$.

Lemma 2.7. *Let Y be a finite subscheme of $\mathbb{P}^n$ over the finite field $\mathbb{F}_q$. Then*

$$H^0(Y,\mathcal{O}_Y(d))\cong H^0(Y,\mathcal{O}_Y),$$

i.e. we may ignore the twist on finite schemes.

Proof. First we can assume $Y\subseteq\mathbb{A}^n=\{x_0\neq 0\}$: if this is not true, we can enlarge $\mathbb{F}_q$ if necessary and perform a linear change of variable to achieve that the finitely many points of Y are contained in $D_+(x_0)$. Hence the canonical morphism $\phi_d: H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d))\to H^0(Y,\mathcal{O}_Y(d))$ factors through $H^0(D_+(x_0),\mathcal{O}_{\mathbb{P}^n}\big|_{D_+(x_0)}(d))$. For the standard-open set $D_+(x_0)$ we have $(S(d))\tilde{}|_{D_+(x_0)}\cong(S(d)_{(x_0)})\tilde{}$ and $S_{(x_0)}\cong S(d)_{(x_0)}$ for all $d\in\mathbb{Z}$. Thus,

$$\mathcal{O}_{\mathbb{P}^n}(d)|_{D_+(x_0)}=(S(d))\tilde{}|_{D_+(x_0)}=(S(d)_{(x_0)})\tilde{}=(S_{(x_0)})\tilde{}=\tilde{S}|_{D_+(x_0)}=\mathcal{O}_{\mathbb{P}^n}|_{D_+(x_0)}.$$

This shows $H^0(D_+(x_0),\mathcal{O}_{\mathbb{P}^n}\big|_{D_+(x_0)}(d))\cong H^0(D_+(x_0),\mathcal{O}_{\mathbb{P}^n}\big|_{D_+(x_0)})$ and, since ϕ_d factors through $H^0(D_+(x_0),\mathcal{O}_{\mathbb{P}^n}\big|_{D_+(x_0)})$, we get $H^0(Y,\mathcal{O}_Y(d))\cong H^0(Y,\mathcal{O}_Y)$. □

Let $\mathcal{I}_Z\subseteq\mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of Z. We want to show $I_d=H^0(\mathbb{P}^n,\mathcal{I}_Z(d))$ (cf. [GW10] Remark 13.26). First of all, note that S is saturated as a graded S-module, i.e.

$$\alpha: S\to\Gamma_*(\tilde{S})=\bigoplus_{n\in\mathbb{Z}}\Gamma(\mathbb{P}^n,\tilde{S}(n))$$

is an isomorphism of graded S-modules. This is true because we have isomorphisms $S_d\cong\Gamma(\mathbb{P}_n,\mathcal{O}_{\mathbb{P}^n}(d))$ in every grade d. Therefore, by [GW10] Proposition 13.24, there exists a unique saturated homogeneous ideal $J \subseteq S$ such that $Z = \mathrm{Proj}(S/J)$; in particular, $\tilde{J} = \mathcal{I}_Z$. As J is saturated, we have an isomorphism $\alpha_J : J \to \Gamma_*(\tilde{J})$ of graded S-modules and hence an isomorphism

$$J_d \cong \Gamma(\mathbb{P}^n, \tilde{J}(d)) \cong \Gamma(\mathbb{P}^n, \mathcal{I}_Z(d))$$

for any d. By writing $Z = \mathrm{Proj}(S/J)$, we can interpret Z to be the intersection of hypersurfaces given by the polynomials that generate the ideal $J \subseteq S$. In particular, J_d is the set of homogeneous polynomials of degree d that vanish on Z, and thus J_d equals I_d.

Next we consider the surjection

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)$$
$$(f_0, \ldots, f_n) \mapsto x_0 f_0 + \ldots + x_n f_n.$$

Tensoring it with $\mathcal{I}_Z$ gives a surjection $\varphi : \mathcal{I}_Z^{\oplus(n+1)} \to \mathcal{I}_Z(1)$. By a vanishing theorem of Serre ([Har93] III.5.2), if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, there exists an integer d_0, depending on $\mathcal{F}$, such that $H^i(X, \mathcal{F}(d)) = 0$ for each $i > 0$ and each $d \geq d_0$. The ideal sheaf $\mathcal{I}_Z$ and therefore also the finite direct sum $\mathcal{I}_Z^{\oplus(n+1)}$ is coherent, since $\mathbb{P}^n$ is Noetherian and hence the category of coherent $\mathcal{O}_{\mathbb{P}^n}$-modules is an abelian category ([Har93] Proposition 5.9). Thus we can apply the above theorem to get $H^1(\mathbb{P}^n, \mathcal{I}_Z^{\oplus(n+1)}(d)) = 0$ for each $d \geq d_0$. This yields a short exact sequence

$$0 \to H^0(\mathbb{P}^n, \ker \phi(d)) \to H^0(\mathbb{P}^n, \mathcal{I}_Z^{\oplus(n+1)}(d)) \to H^0(\mathbb{P}^n, \mathcal{I}_Z(d+1)) \to 0,$$

and therefore a surjection for $d \gg 1$

$$I_d^{\oplus(n+1)} = H^0(\mathbb{P}^n, \mathcal{I}_Z^{\oplus(n+1)}(d)) \to H^0(\mathbb{P}^n, \mathcal{I}_Z(d+1)) = I_{d+1}.$$

Since $x_0 f_0 + \ldots + x_n f_n \in S_1 I_d$ for $f_i \in I_d$, we get $S_1 I_d = I_{d+1}$ for $d \gg 1$. We fix an integer c such that $S_1 I_d = I_{d+1}$ for all $d \geq c$.

Lemma 2.8. *([Poo08], Lemma 2.1.) Let Y be a finite subscheme of $\mathbb{P}^n$ over $\mathbb{F}_q$. Let*

$$\phi_d : I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \to H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$$

be the map induced by the map of sheaves $\mathcal{I}_Z \to \mathcal{I}_Z \cdot \mathcal{O}_Y$. Then ϕ_d is surjective for $d \geq c + \dim H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$.

Proof. For reasons of completeness, we add the proof following the one of [Poo08]. The map of sheaves $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y$ is surjective, so the induced map $\mathcal{I}_Z \to \mathcal{I}_Z \cdot \mathcal{O}_Y$ is surjective as well. Taking cohomology and using the vanishing theorem of Serre ([Har93] III.5.2) as in the remark previous to this lemma, we can show that ϕ_d is surjective for $d \gg 1$.

As seen in the proof of Lemma 2.7, we can assume $Y \subseteq \mathbb{A}^n = \{x_0 \neq 0\}$. Dehomogenization by setting $x_0 = 1$ identifies S_d with the space S'_d of polynomials in $\mathbb{F}_q[x_1, \dots, x_n]$ of degree $\leq d$ and I_d with the image I'_d of I_d under this dehomogenization. This identifies ϕ_d with a map

$$I'_d \to B = H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y).$$

The dimension b of B is finite as Y is a finite scheme and therefore the local ring at each of its finitely many points is a local finite $\mathbb{F}_q$-algebra.

For $d \geq c$, let B_d be the image of I'_d in B. By definition of c, we have $S'_1 I'_d = I'_{d+1}$ and hence $S'_1 B_d = B_{d+1}$ for $d \geq c$. Since $1 \in S'_1$, we get

$$B_c \subseteq B_{c+1} \subseteq \dots \subseteq B.$$

There exists a $j \in [c, c+b]$ such that $B_j = B_{j+1}$: suppose this is not true. Then $\dim B_{j+1} \geq \dim B_j + 1$ for all $j \in [c, c+b]$ and therefore $\dim B_{c+b+1} \geq \dim B_c + b + 1 \geq b + 1$, which contradicts $B_{c+b+1} \subseteq B$ since b is the dimension of B. Using $S'_1 B_d = B_{d+1}$ for $d \geq c$, we get

$$B_{j+2} = S'_1 B_{j+1} = S'_1 B_j = B_{j+1}.$$

Similarly $B_{j+2} = B_{j+3} = \dots$, and thus $B_j = B_{j+k}$ for all $k \in \mathbb{N}$. Now by the first paragraph of this proof, ϕ_d is surjective for some $d \gg 1$. Thus there exists an $l \in \mathbb{N}$ such that $B_{j+l} = B$ and hence $B_j = B$. This shows that ϕ_d is surjective for $d \geq j$, and in particular for $d \geq c + b = c + \dim H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$. □

Lemma 2.9. *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over $\mathbb{F}_q$. Let P be a closed point of X and let $f \in I_{\text{homog}}$. Then $H_f \cap X$ is smooth of dimension $m-1$ at P if and only if $f \notin \mathfrak{m}_{X,P}^2$.*

Further let $\mathfrak{m} \subseteq \mathcal{O}_X$ be the ideal sheaf of P and let $Y \subseteq X$ be the closed subscheme of $\mathbb{P}^n$ corresponding to the ideal sheaf $\mathfrak{m}^2 \subseteq \mathcal{O}_X$. Then $H_f \cap X$ is smooth of dimension $m-1$ at P if and only if the restriction of f to a section of $\mathcal{I}_Z \cdot \mathcal{O}_Y(d)$ is not equal to zero.

Proof. Let $P \in H_f \cap X$ and thus $f \in \mathfrak{m}_{X,P}$. Since $\mathbb{F}_q$ is perfect, $H_f \cap X$ is smooth of dimension $m-1$ at P if and only if $H_f \cap X$ is regular of dimension $m-1$ at P, i.e. $\mathcal{O}_{X,P}/f$ is regular, where f also denotes the image of f under the map $S \to \mathcal{O}_{X,P}$. By Krull's principal ideal theorem, $\dim(\mathcal{O}_{X,P}/f) = m-1$. Since $f \in \mathfrak{m}_{X,P} - \{0\}$, Corollary 2.12 in [Liu06] yields $\mathcal{O}_{X,P}/f$ is regular if and only if $f \notin \mathfrak{m}_{X,P}^2$. This shows the first claim.

For the second claim, we observe that Y is the support of the quotient sheaf given by $\mathcal{O}_X / \mathfrak{m}^2$. Hence $Y = \operatorname{Spec}(\mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2)$. Because both $\mathcal{O}_{X,P} / \mathfrak{m}_{X,P}$ and $\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$ are finitely generated $\mathbb{F}_q$-modules, $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}^2$ is also a finitely generated $\mathbb{F}_q$-module and Y is a finite scheme. Thus Lemma 2.7 yields $H^0(Y, \mathcal{O}_Y(d)) = H^0(Y, \mathcal{O}_Y) = \mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2$. By what we have shown above, $H_f \cap X$ is smooth at P if and only if $f \in I_d$ is not an element of $\mathfrak{m}_{X,P}^2$, i.e. f is not zero in $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$. □

Let P be a closed point of X. If we define the scheme Y as above, we have seen in the proof of Lemma 2.9 that $Y = \operatorname{Spec}(\mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2)$ is a finite scheme. Hence we can apply Lemma 2.8 to Y to get a surjective homomorphism $\phi_d : I_d \to H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$ and in particular an isomorphism $I_d/\ker \phi_d \cong H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$. Then Lemma 2.9 shows that the polynomials $f \in I_d$, which are not zero in $I_d/\ker \phi_d$ and thus not in the kernel of ϕ_d, are exactly the polynomials that give us a hypersurface H containing Z such that

$H \cap X$ is smooth of dimension $m-1$ at the point P. Therefore, if we want to calculate the fraction of those polynomials, we need to know the size of $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) - \{0\}$.

Lemma 2.10. *Let $\mathfrak{m} \subseteq \mathcal{O}_X$ be the ideal sheaf of a closed point $P \in X$. Let $Y \subseteq X$ be the closed subscheme of $\mathbb{P}^n$, which corresponds to the ideal sheaf $\mathfrak{m}^2 \subseteq \mathcal{O}_X$. Then for all $d \in \mathbb{Z}_{\geq 0}$*

$$\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) = \begin{cases} q^{(m+1)\deg P}, & \text{if } P \notin V, \\ q^{(m-e_V(P))\deg P}, & \text{if } P \in V. \end{cases}$$

Proof. As seen in the proof of Lemma 2.9, $Y = \operatorname{Spec}(\mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2)$ is a finite scheme. So we have $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) = H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$.

We have an exact sequence of sheaves

$$0 \to \mathcal{I}_Z \cdot \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_{Z \cap Y} \to 0.$$

By a vanishing theorem of Grothendieck ([Har93] Theorem III 2.7), $H^i(Y, \mathcal{F}) = 0$ for all $i > \dim Y = 0$ and all sheaves of abelian groups $\mathcal{F}$ on Y. Thus, taking cohomology of this sequence on Y yields an exact sequence

$$0 \to H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_{Z \cap Y}) \to 0.$$

Now we calculate $\#H^0(Y, \mathcal{O}_Y)$ and $\#H^0(Y, \mathcal{O}_{Z \cap Y})$ to get $\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$.

There is a filtration of $H^0(Y, \mathcal{O}_Y) = \mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2$ given by

$$0 \to \mathfrak{m}_{X,P} / \mathfrak{m}_{X,P}^2 \to \mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2 \to \mathcal{O}_{X,P} / \mathfrak{m}_{X,P} \to 0,$$

whose quotients are vector spaces of dimensions m and 1 respectively over the residue field $\kappa(P)$ of P since X is smooth and hence regular at the point P. So by additivity of length of modules, $\#H^0(Y, \mathcal{O}_Y) = \#\kappa(P)^{m+1} = q^{(m+1)\deg P}$.

Next we determine $\#H^0(Y, \mathcal{O}_{Z \cap Y})$. Since $Y = \operatorname{Spec}(\mathcal{O}_{X,P} / \mathfrak{m}_{X,P}^2)$, Remark 1.1 shows $H^0(Y, \mathcal{O}_{Z \cap Y}) = \mathcal{O}_{X,P} / (\mathcal{I}_{Z,P}, \mathfrak{m}_{X,P}^2)$. If $P \in X - V$, then $\mathcal{I}_{Z,P}$ is not contained in $\mathfrak{m}_{X,P}$ and $H^0(Y, \mathcal{O}_{Z \cap Y}) = 0$. If $P \in V$, then $H^0(Y, \mathcal{O}_{Z \cap Y})$ has a filtration given by

$$0 \to \mathfrak{m}_{X,P} / (\mathcal{I}_{Z,P}, \mathfrak{m}_{X,P}^2) \to H^0(Y, \mathcal{O}_{Z \cap Y}) \to \mathcal{O}_{X,P} / \mathfrak{m}_{X,P} \to 0.$$

We have seen in Remark 1.10, that $e_V(P) = \dim_{\kappa(P)} \mathfrak{m}_{X,P} / (\mathcal{I}_{Z,P}, \mathfrak{m}_{X,P}^2)$. Hence,

$$\dim_{\kappa(P)} H^0(Y, \mathcal{O}_{Z\cap Y}) = 1 + \dim_{\kappa(P)} \mathfrak{m}_{X,P} / (\mathcal{I}_{Z,P}, \mathfrak{m}_{X,P}^2) = 1 + e_V(P).$$

Thus,

$$\begin{aligned} \# H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) &= \frac{\# H^0(Y, \mathcal{O}_Y)}{\# H^0(Y, \mathcal{O}_{Z\cap Y})} \\ &= \begin{cases} q^{(m+1)\deg P}, & \text{if } P \notin V, \\ q^{(m+1)\deg P} / q^{(e_V(P)+1)\deg P}, & \text{if } P \in V, \end{cases} \end{aligned}$$

which is what we wanted to show. □

For a scheme X of finite type over $\mathbb{F}_q$ we define $X_{<r}$ to be the set of closed points of X of degree $< r$. Let $X_{>r}$ be defined similarly.

Remark 2.11. $X_{<r}$ is a finite set: since X is of finite type over $\mathbb{F}_q$, there exists a finite covering of X by affine open subschemes X_i, where $X_i = \operatorname{Spec}(\mathbb{F}_q[x_1, \ldots, x_n] / \mathfrak{a}_i)$ for an ideal $\mathfrak{a}_i \subseteq \mathbb{F}_q[x_1, \ldots, x_n]$ and $n \in \mathbb{N}$. Then $X_i(\mathbb{F}_{q^r}) = \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{F}_q[x_1, \ldots, x_n] / \mathfrak{a}_i, \mathbb{F}_{q^r}) \subseteq \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{F}_q[x_1, \ldots, x_n], \mathbb{F}_{q^r}) = \mathbb{F}_{q^r}^n$. The number of closed points of X_i with degree r is less than or equal to the number of elements in $X_i(\mathbb{F}_{q^r})$, because for every such point P there exists an $\mathbb{F}_q$-homomorphism $\operatorname{Spec} \mathbb{F}_{q^r} \to X$ mapping the unique point of $\operatorname{Spec} \mathbb{F}_{q^r}$ to P. Since $X_i(\mathbb{F}_{q^r})$ is finite, it follows that $X_{<r}$ is a finite set.

Lemma 2.12 (Singularities of low degree). *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m \geq 0$ over $\mathbb{F}_q$ and let Z be a closed subscheme of $\mathbb{P}^n$. Let $V := Z \cap X$ be the intersection. Define*

$$\mathcal{P}_r := \{ f \in I_{\text{homog}} : \ H_f \cap X \text{ is smooth of dimension } m-1 \text{ at all points } P \in X_{<r} \}.$$

Then

$$\mu_Z(\mathcal{P}_r) = \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}) \cdot \prod_{e=0}^{m} \prod_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P}).$$

Proof. Let $X_{<r} = \{P_1, \dots, P_s\}$. Let $\mathfrak{m}_i$ be the ideal sheaf of P_i on X and let Y_i be the closed subscheme of X whose ideal sheaf is $\mathfrak{m}_i^2 \subseteq \mathcal{O}_X$. Let $Y = \bigcup_{i=1}^{s} Y_i$. By Lemma 2.9, the intersection $H_f \cap X$ is not smooth of dimension $m-1$ at P_i if and only if the restriction of f to a section of $\mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)$ is zero. Hence $\mathcal{P}_r \cap I_d$ is the inverse image of $\prod_{i=1}^{s}(H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)) - \{0\})$ under the $\mathbb{F}_q$-linear map

$$\phi_d : I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \to H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) = \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)).$$

We can ignore the twist by Lemma 2.7, and we may further assume that the condition $d \geq c + \dim H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$ of Lemma 2.8 is fulfilled, since in the density that we want to calculate we only look at the limit $d \to \infty$. Hence Lemma 2.8 implies that ϕ_d is surjective and the inverse image of $\prod_{i=1}^{s}(H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)) - \{0\})$ is the disjoint union of $\# \prod_{i=1}^{s}(H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)) - \{0\})$ cosets of the kernel of ϕ_d. Thus

$$\#(\mathcal{P}_r \cap I_d) = \# \prod_{i=1}^{s} \left(H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)) - \{0\}\right) \cdot \# \ker \phi_d.$$

Again the surjectivity of ϕ_d yields

$$\# I_d = \# \ker \phi_d \cdot \# \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}(d)).$$

Inserting this into the definition of density and applying Lemma 2.10, we get

$$\begin{aligned} \mu_Z(\mathcal{P}_r) &= \prod_{i=1}^{s} \frac{\# H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) - 1}{\# H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})} \\ &= \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}) \cdot \prod_{P \in V_{<r}} (1 - q^{-(m - e_V(P))\deg(P)}) \\ &= \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}) \cdot \prod_{e=0}^{m} \prod_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P}) \end{aligned}$$ □

Note, that this proof only works since there are only finitely many points in $X_{<r}$ and hence Y is a finite subscheme of $\mathbb{P}^n$. If we wanted to use the same argument for the set of polynomials $\mathcal{P}$ defined as in Theorem 2.1, and therefore considered points of X of

arbitrary degree, we would have to let r tend to infinity before we calculate the density $\mu_Z(\mathcal{P})$, i.e. before we let d tend to infinity. But the proof of Lemma 2.12 does not work there anymore, as then we would have infinitely many points to deal with. So as mentioned in the introduction, we see now, that first we need to look only at points of some bounded degree r as above. Then we show that when $d \gg r \gg 1$, the number of polynomials $f \in I_d$ of degree d that do not give a smooth intersection at the infinitely many points of degree at least r is insignificant, i.e. the upper density of this set of polynomials is zero.

Corollary 2.13. *Let* $\max\limits_{0 \le e \le m-1} \{e + \dim V_e\} < m$ *and* $V_m = \emptyset$, *then*

$$\lim_{r \to \infty} \mu_Z(\mathcal{P}_r) = \frac{\zeta_V(m+1)}{\zeta_X(m+1) \prod\limits_{e=0}^{m-1} \zeta_{V_e}(m-e)}.$$

Proof. The first product in Lemma 2.12 converges anyway, since $m+1 > \dim(X - V)$.

The factor for $e = m$ in the second product in this lemma does not appear since V_m is empty. For all $0 \le e \le m-1$, the product $\prod\limits_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P})$ is just the partial product used in the definition of the zeta function of V_e. This converges for $m - e > \dim V_e$, i.e. for $\dim V_e + e < m$. Since we want every product in Lemma 2.12 to converge, we need $\dim V_e + e < m$ for all $e \ge 0$. □

Proof of Theorem 2.1 (ii). If $\max\limits_{0 \le e \le m-1} \{e + \dim V_e\} \ge m$, then there exists a $0 \le e_0 < m$ such that $e_0 + \dim V_{e_0} \ge m$, i.e. $m - e_0 \le \dim V_{e_0}$. Applying Lemma 2.12 gives

$$\mu_Z(\mathcal{P}_r) \le \prod_{P \in (V_{e_0})_{<r}} (1 - q^{-(m-e_0)\deg P}) \le \prod_{P \in (V_{e_0})_{<r}} (1 - q^{-\dim V_{e_0} \deg P}).$$

This is the inverse of the partial product used in the definition of the zeta function of V_{e_0}. This zeta function has a pole at $\dim V_{e_0}$ (cf. [Tat65] §4), thus the product tends to zero for $r \to \infty$.

As a locally closed subscheme of the Noetherian scheme X, the scheme V_m is again Noetherian. Therefore, if it is nonempty, it contains a closed point P and the factor

$(1-q^{-(m-m)\deg P})$ in the density of $\mathcal{P}_r$ in Lemma 2.12 is equal to zero; hence the density $\mu(\mathcal{P}_r)$ is zero for $V_m \neq \emptyset$.

The inclusion $\mathcal{P} \subseteq \mathcal{P}_r$ implies

$$\mu_Z(\mathcal{P}) \leq \mu_Z(\mathcal{P}_r).$$

We have seen above that the density of $\mathcal{P}_r$ tends to zero for $r \to \infty$ if $\max_{e\geq 0}\{e + \dim V_e\} \geq m$ or $V_m \neq \emptyset$. Hence the result follows. $\square$

From now on, we assume $\max_{0\leq e\leq m-1}\{e + \dim V_e\} < m$ and $V_m = \emptyset$.

2.2 Singular points of medium degree

Lemma 2.14. *Let $P \in X$ be a closed point of degree $\leq \frac{d-c}{m+1}$. Then the fraction of polynomials $f \in I_d$ such that $H_f \cap X$ is not smooth of dimension $m-1$ at P is equal to*

$$\begin{cases} q^{-(m+1)\deg P}, & \textit{if } P \notin V, \\ q^{-(m-e_V(P))\deg P}, & \textit{if } P \in V. \end{cases}$$

Proof. Let Y be defined as in Lemma 2.9. Then $H_f \cap X$ is not smooth of dimension $m-1$ at P if and only if the restriction of f to a section of $\mathcal{I}_Z \cdot \mathcal{O}_Y(d)$ is equal to zero. Applying Lemma 2.10 and using $\deg P \leq \frac{d-c}{m+1}$ we obtain

$$\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) \leq \begin{cases} q^{(d-c)}, & \text{if } P \notin V, \\ q^{(m-e_V(P))(d-c)/(m+1)}, & \text{if } P \in V. \end{cases}$$

Now $\frac{m-e_V(P)}{m+1} \leq 1$ and hence $\dim H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) \leq d-c$. Therefore we can apply Lemma 2.8 and get an isomorphism $H^0(\mathbb{P}^n, \mathcal{I}_Z(d))/\ker\phi_d \cong H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$, where ϕ_d is defined as in Lemma 2.8. As the polynomials we consider are exactly those with image zero in $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))$, the fraction we want to calculate equals

$$\frac{\#\ker\phi_d}{\#H^0(\mathbb{P}^n, \mathcal{I}_Z(d))} = \frac{1}{\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d))}.$$

But by Lemma 2.10 we get

$$\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(d)) = \begin{cases} q^{(m+1)\deg P}, & \text{if } P \notin V, \\ q^{(m-e_V(P))\deg P}, & \text{if} P \in V. \end{cases}$$

This shows the lemma. □

Lemma 2.15 (Singularities of medium degree). *Let*

$$\mathcal{Q}_r^{\text{medium}} := \bigcup_{d \geq 0} \{f \in I_d : \textit{there exists a point } P \in X \textit{ with } r \leq \deg P \leq \frac{d-c}{m+1} \textit{ such}$$
$$\textit{that } H_f \cap X \textit{ is not smooth of dimension } m-1 \textit{ at } P\}.$$

Then $\lim_{r \to \infty} \overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) = 0$.

Proof. If we take the union over all sets of polynomials that we considered in Lemma 2.14 for all points $P \in X$ with degree between r and $\frac{d-c}{m+1}$, we get $\mathcal{Q}_r^{\text{medium}}$. Hence, $\overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}})$ is at most equal to the sum over all those points of all fractions that we calculated in Lemma 2.14. Moreover, just like in this lemma, we can split this sum to get a sum for points in V and one for points in $X - V$. For a subscheme U of $\mathbb{P}^n_{\mathbb{F}_q}$ the number of points P of degree g in U is less than or equal to $\#U(\mathbb{F}_{q^g}) = \#\operatorname{Hom}(\operatorname{Spec}\mathbb{F}_{q^g}, U)$, thus

$$\begin{aligned}
\frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\#I_d} &\leq \sum_{e=1}^{m} \sum_{g=r}^{\lfloor (d-c)/(m+1) \rfloor} (\text{number of points of degree } g \text{ in } V_e)\ q^{-(m-e)g} \\
&\quad + \sum_{g=r}^{\lfloor (d-c)/(m+1) \rfloor} (\text{number of points of degree } g \text{ in } X - V)\ q^{-(m+1)g} \\
&\leq \sum_{e=1}^{m} \sum_{g=r}^{\lfloor (d-c)/(m+1) \rfloor} \#V_e(\mathbb{F}_{q^g})\ q^{-(m-e)g} \\
&\quad + \sum_{g=r}^{\lfloor (d-c)/(m+1) \rfloor} \#(X - V)(\mathbb{F}_{q^g})\ q^{-(m+1)g}.
\end{aligned}$$

By [LW54] Lemma 1, there exist constants C_e and C for V_e and $X - V$ that depend only on V_e and $X - V$, respectively, such that $\#V_e(\mathbb{F}_{q^g}) \leq C_e q^{g \dim V_e}$ and $\#(X - V)(\mathbb{F}_{q^g}) \leq$

$Cq^{g\dim(X-V)}$. Then, using the assumption $V_m = \emptyset$, we obtain

$$\begin{aligned}\frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\#I_d} &\leq \sum_{e=1}^{m-1}\sum_{g=r}^{\infty} C_e q^{g\dim V_e} q^{-(m-e)g} + \sum_{g=r}^{\infty} C q^{gm} q^{-(m+1)g} \\ &\leq \sum_{e=1}^{m-1} C_e \sum_{g=r}^{\infty} q^{g(\dim V_e + e - m)} + C\sum_{g=r}^{\infty} q^{-g}.\end{aligned}$$

The other assumption $\max\limits_{0\leq e\leq m-1}\{e+\dim V_e\} < m$ yields $\dim V_e + e - m \leq -1$ for all e, hence

$$\begin{aligned}\frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\#I_d} &\leq \sum_{e=1}^{m-1} C_e \sum_{g=r}^{\infty} q^{-g} + \frac{Cq^{-r}}{1-q^{-1}} \\ &\leq \sum_{e=1}^{m-1} \frac{C_e q^{-r}}{1-q^{-1}} + \frac{Cq^{-r}}{1-q^{-1}} = q^{-r}(\sum_{e=1}^{m-1}\frac{C_e}{1-q^{-1}} + \frac{C}{1-q^{-1}}).\end{aligned}$$

Since this tends to zero for $r \to \infty$, we get

$$\lim_{r\to\infty} \overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) = \lim_{r\to\infty}\limsup_{d\to\infty}\frac{\#(\mathcal{Q}_r^{\text{medium}} \cap I_d)}{\#I_d} = 0,$$

which is what we claimed. □

2.3 Singular points of high degree

In this section we will show that the upper density for polynomials $f \in I_{\text{homog}}$ that do not give a smooth intersection at a point $P \in X$ of high degree is equal to zero. We will split this up in two problems: First we prove that this upper density is zero if we only consider points in $X - V$. This is just a result of [Poo08]. Then we show the same for points in V.

Lemma 2.16 (Singularities of high degree off V). *Define*

$$\mathcal{Q}_{X-V}^{\text{high}} := \bigcup_{d\geq 0}\{\, f \in I_d : \textit{there exists a point } P \in (X-V)_{>\frac{d-c}{m+1}} \textit{ such that } H_f \cap X \textit{ is not smooth of dimension } m-1 \textit{ at } P\}.$$

Then $\overline{\mu}_Z(\mathcal{Q}_{X-V}^{\text{high}}) = 0$.

Proof. This is the statement of Lemma 4.2. in [Poo08] for the case in which the intersection V of X and Z is smooth, whereas in our case, V does not have to be smooth. But the proof does not use the fact that V is smooth, since in this lemma only the points that are not in V are considered; hence it also shows Lemma 2.16. □

The proof for the analogue of this lemma for points on V will use the following version of Bézout's theorem, and a lemma that counts the polynomials $f \in S'_d$ that vanish at some closed point $P \in \mathbb{A}^n$.

Lemma 2.17. *([Ful98] Example 12.3.1) Let $V_1, \ldots V_r$ be equidimensional closed subschemes of $\mathbb{P}^n$ over $\mathbb{F}_q$. Let $W_1, \ldots, W_s$ be the irreducible components of $\bigcup_{j=1}^{r} V_j$. Then*

$$r \leq \sum_{i=1}^{s} \deg W_i \leq \prod_{j=1}^{r} \deg V_j.$$

Lemma 2.18. *([Poo04] Lemma 2.5) Let P be a closed point in $\mathbb{A}^n$ over $\mathbb{F}_q$. Then the fraction of $f \in S'_d$ that vanish at P is at most $q^{-\min(d+1, \deg P)}$.*

The last lemma we need for the proof of Theorem 2.1 is the following:

Lemma 2.19 (Singularities of high degree on V). *Define*

$$\mathcal{Q}_V^{\text{high}} := \bigcup_{d \geq 0} \{\, f \in I_d : \textit{there exists a point } P \in V_{> \frac{d-c}{m+1}} \textit{ such that } H_f \cap X \textit{ is not smooth of dimension } m-1 \textit{ at } P \}.$$

Then $\overline{\mu}_Z(\mathcal{Q}_V^{\text{high}}) = 0$.

For this lemma we cannot use a result of [Poo08] as we did above: an analogue of this lemma does exist there as well, namely Lemma 4.3. But the fact that V is assumed to be smooth there is crucial for the proof. Therefore we need to prove this lemma in a different way, but we will use the same technique as in [Poo08], i.e. the induction argument introduced by Poonen in [Poo04], Lemma 2.6.

Proof. If the lemma is proven for all subsets X_i of a finite affine open cover of X, then it holds for X as well, because the sum of the corresponding upper densities for the X_i is an upper bound for $\overline{\mu}_Z(\mathcal{Q}_V^{\text{high}})$. Hence we can assume without loss of generality, that $X \subseteq \mathbb{A}^n_{\mathbb{F}_q} = \{x_0 \neq 0\} \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ is affine. Again we identify S_d, i.e. the homogeneous polynomials in $\mathbb{F}_q[x_0, \ldots, x_n]$ of degree d, with the space of polynomials $S'_d \subseteq \mathbb{F}_q[x_1, \ldots, x_n] = A$ of degree $\leq d$ by setting $x_0 = 1$. This dehomogenization also identifies I_d with a subspace $I'_d \subseteq S'_d$.

Let P be a closed point of X. Since X is smooth, we can choose a system of local parameters $t_1, \ldots, t_n \in A$ on $\mathbb{A}^n$ such that $t_{m+1} = \ldots = t_n = 0$ defines X locally at P. Then $dt_1, \ldots, dt_n$ are a basis for the stalk of $\Omega^1_{\mathbb{A}^n|\mathbb{F}_q}$ at P and by Theorem 1.12, $dt_1, \ldots, dt_m$ are a basis for the stalk of $\Omega^1_{X|\mathbb{F}_q}$ at P. By using those local parameters we now want to find suitable derivations $D_1, \ldots, D_m : A \to A$ such that for $f \in I'_d$, the intersection $H_f \cap X$ is not smooth at a point $P \in V$ if and only if $(D_1 f)(P) = \ldots = (D_m f)(P) = 0$. We will then show that the probability that $H_f \cap X$ is not smooth at a point in V_e tends to zero for $d \to \infty$ and any e. Since we only have finitely many subschemes V_e for $0 \leq e \leq m$ in the flattening stratification for $\Omega^1_{X|\mathbb{F}_q}$, the upper density $\overline{\mu}_Z(\mathcal{Q}^{\text{high}})$, that we actually want to calculate, is a finite sum of those probabilities, and hence zero.

As we have seen in the first section, V is the disjoint union of the locally closed subsets $V_e = V(\Omega^1_{V|\mathbb{F}_q}, e)$ for $0 \leq e \leq m$. By Proposition 3.52 of [GW10], we can give V_e the structure of a reduced subscheme of V. For all $P \in V_e$, by definition of the embedding dimension and the V_e, we have

$$\dim_{\kappa(P)} \Omega^1_{V|\mathbb{F}_q} \otimes_{\mathcal{O}_V} \mathcal{O}_{V_e}(P) = \dim_{\kappa(P)} \Omega^1_{V|\mathbb{F}_q} \otimes_{\mathcal{O}_V} \kappa(P) = e_V(P) = e.$$

Thus the rank of $\Omega^1_{V|\mathbb{F}_q} \otimes \mathcal{O}_{V_e}$ on V_e is a constant function. Because V_e is reduced and $\Omega^1_{V|\mathbb{F}_q} \otimes \mathcal{O}_{V_e}$ is a quasi-coherent $\mathcal{O}_{V_e}$-module of finite type, Corollary 1.4 shows that $\Omega^1_{V|\mathbb{F}_q} \otimes \mathcal{O}_{V_e}$ is a locally free $\mathcal{O}_{V_e}$-module.

Next we consider the map $\Omega^1_{X|\mathbb{F}_q} \otimes \mathcal{O}_V \to \Omega^1_{V|\mathbb{F}_q}$, which is surjective by Lemma 1.6.

Tensoring it with $\mathcal{O}_{V_e}$ gives a surjective map $\phi : \Omega^1_{X|\mathbb{F}_q} \otimes \mathcal{O}_{V_e} \to \Omega^1_{V|\mathbb{F}_q} \otimes \mathcal{O}_{V_e}$ where $\Omega^1_{X|\mathbb{F}_q} \otimes \mathcal{O}_{V_e}$ is a locally free sheaf of rank m and $\Omega^1_{V|\mathbb{F}_q} \otimes \mathcal{O}_{V_e}$ is a locally free sheaf of rank e. At P, the sequence

$$0 \to \ker\phi \to \Omega^1_{X|\mathbb{F}_q} \otimes \mathcal{O}_{V_e} \to \Omega^1_{V|\mathbb{F}_q} \otimes \mathcal{O}_{V_e} \to 0$$

is an exact sequence of free modules and splits, since $\Omega^1_{V|\mathbb{F}_q,P} \otimes \mathcal{O}_{V_e,P}$ is free and therefore projective. Hence, $dt_1, \ldots, dt_{m-e}$ form a basis of the kernel of ϕ at P and $dt_{m-e+1}, \ldots, dt_m$ a basis of $\Omega^1_{V|\mathbb{F}_q,P} \otimes \mathcal{O}_{V_e,P}$. In particular, $t_1, \ldots, t_{m-e}$ all vanish on V, since by Proposition 1.9 $\Omega^1_{V|\mathbb{F}_q} \otimes \kappa(P) \cong \mathfrak{m}_{V,P} \,/\, \mathfrak{m}^2_{V,P}$.

We want to show, that $t_1, \ldots, t_{m-e}$ can be assumed to vanish even on Z. For this, let the closed scheme Z be given by $\operatorname{Spec} A/I_Z$. Because the quasi-projective scheme X can be assumed to be projective in a neighbourhood of P and the above calculation was done locally, we can assume without loss of generality $X = \operatorname{Spec} A/J$, such that P corresponds to a maximal ideal of A/J. Since $t_1, \ldots, t_m$ are local parameters of X at P, it follows that $t_{m+1}, \ldots, t_n$ generate J_P, and $t_1, \ldots, t_m$ generate the maximal ideal $\mathfrak{m}_{X,P}$. In particular, $t_1, \ldots, t_m$ are not in $\mathfrak{m}^2_{X,P}$, because X is regular at P and therefore the images of $t_1, \ldots, t_m$ in $\mathfrak{m}_{X,P} \,/\, \mathfrak{m}^2_{X,P}$ are a basis of $\mathfrak{m}_{X,P} \,/\, \mathfrak{m}^2_{X,P}$ as a $\kappa(P)$-vector space. Thus, $t_1, \ldots, t_m$ are not in J_P. The intersection V of X and Z is given by $\operatorname{Spec} A/(J+I_Z)$. By what we have shown above, $t_1, \ldots, t_{m-e}$ vanish on V and thus are elements of the ideal $J + I_Z$ localized at P. Since $t_1, \ldots, t_{m-e}$ are not in J_P, there exist $a_i \in J_P$ and $b_i \in I_{Z,P} \setminus \{0\}$ such that $t_i = a_i + b_i$ for all $1 \leq i \leq m - e$. Then $t_i \equiv b_i \mod J_P$ and therefore $b_1, \ldots, b_{m-e}, t_{m-e+1}, \ldots, t_m$ are again local parameters of X at P. Furthermore, $db_1, \ldots, db_{m-e}$ are still a basis of the kernel of ϕ at P and $dt_{m-e+1}, \ldots, dt_m$ are a basis of $\Omega^1_{V|\mathbb{F}_q,P} \otimes \mathcal{O}_{V_e,P}$. Since $t_i \in A$, we can choose $b_1, \ldots, b_{m-e} \in I_Z$ and therefore assume that they vanish at Z. Hence we may also assume that $t_1, \ldots, t_{m-e}$ already vanish at Z.

Let $\partial_1, \ldots, \partial_n \in \mathcal{T}_{\mathbb{A}^n|\mathbb{F}_q,P}$ be the basis of the stalk of the tangent sheaf, dual to $dt_1, \ldots, dt_n$. We can find an $s \in A$ with $s(P) \neq 0$ such that $D_i = s\partial_i$ gives a global derivation $A \to A$ for $i = 1, \ldots, n$. Since $\Omega^1_{\mathbb{A}^n|\mathbb{F}_q}$ is a locally free and coherent $\mathcal{O}_{\mathbb{A}^n}$-module, by Lemma 1.13 there exists a neighbourhood N_P of P in $\mathbb{A}^n$ such that $N_P \cap X = N_P \cap \{t_{m+1} = \ldots = t_n = 0\}$ and $\Omega^1_{\mathbb{A}^n\,|\,\mathbb{F}_q}\big|_{N_P} = \bigoplus_{i=1}^{n} \mathcal{O}_{N_P}\, dt_i$. Furthermore, we can choose $s \in A$ such that $s \in \mathcal{O}(N_P)^*$ holds, due to the fact that $s(P)$ is not equal to zero. Since X is quasi-compact, we can cover X with finitely many N_P and assume $X \subseteq N_P$. Hence in particular, $\Omega^1_{X|\mathbb{F}_q} = \bigoplus_{i=1}^{m} \mathcal{O}_X\, dt_i$.

Let $P \in V_e$ be a closed point. Lemma 2.9 shows that for a polynomial $f \in I_d$, the hypersurface section $H_f \cap X$ is not smooth at P if and only if $f \in \mathfrak{m}^2_{X,P}$. By definition of the derivations D_i, this is equivalent to $(D_1 f)(P) = \ldots = (D_m f)(P) = 0$. Note that we do not have to demand $f(P)$ to be zero, since Z is contained in the hypersurface H_f for $f \in I'_d$, and thus f vanishes at all points in $V_e \subseteq Z$ anyway.

Now we want to bound the $f \in I'_d$ for which there exists such a point by using the induction argument in Lemma 2.6 of [Poo04].

Let $\tau = \max_{1 \le i \le l_e+1} (\deg t_i)$ and $\gamma = \lfloor (d-\tau)/p \rfloor$ where $l_e = \dim V_e$. We select $f_0 \in I'_d$ and $g_1 \in S'_\gamma, \ldots, g_{l_e+1} \in S'_\gamma$ uniformly and independently at random. Then the distribution of

$$f = f_0 + g_1^p t_1 + \ldots + g_{l_e+1}^p t_{l_e+1}$$

is uniform over I'_d: first of all, we have to show that the sum on the right hand side is again a polynomial in I'_d. By our assumption we have $e + l_e < m$ for all $0 \le e \le m$ and therefore $l_e + 1 \le m - e$. But $t_1, \ldots, t_{m-e}$ and consequently $t_1, \ldots, t_{l_e+1}$ all vanish on Z; hence the sum vanishes as well and defines an element in I'_d since the degree of f is $\le d$.

To prove that the distribution is uniform, note that every set mentioned above is finite, and thus we only need to show that all $f \in I'_d$ have the same number of representations of this kind. First, every polynomial $f \in I'_d$ can be constructed in this way because we can choose $f_0 = f$ and $g_1 = \ldots = g_e = 0$. Now let f and F be any two polynomials in I'_d and let $f = f_0 + g_1^p t_1 + \ldots + g_{l_e+1}^p t_{l_e+1}$. Then, $F = (f_0 - f_1) + g_1^p t_1 + \ldots + g_{l_e+1}^p t_{l_e+1}$ where $f_1 = f - F \in I'_d$. That way, we get for any two different representations of f also two different representations of F and similarly vice versa. Hence any two polynomials have the same number of representations of this kind.

Since the distribution of the polynomials f in this representation is uniform over I'_d, it is enough to bound the probability for an f constructed in this way to have a point $P \in V_{e,>\frac{d-c}{m+1}}$ such that $(D_1 f)(P) = \ldots = (D_m f)(P) = 0$. We will see, that we can even consider only the first $l_e + 1$ derivations D_i and still show the claim. Here we are using the construction above because by definition of $D_i = s\partial_i$ with $s \in \mathcal{O}(N_P)^*$ we have $D_i f = D_i f_0 + g_i^p s$. Therefore $D_i f$ does only depend on f_0 and g_i. We will select the polynomials $f_0, g_1, \ldots, g_{l_e+1}$ one at a time.

For $0 \leq i \leq l_e + 1$, define

$$W_i = V_e \cap \{D_1 f = \ldots = D_i f = 0\}.$$

Then $W_{l_e+1} \cap V_{e,>\frac{d-c}{m+1}}$ is the set of points $P \in V_e$ of degree $> \frac{d-c}{m+1}$ where $H_f \cap X$ may be singular. Because we want to show that for $d \to \infty$ the upper density of the set of polynomials f of degree d that have such a point is equal to zero, we will show by an induction argument that $W_{l_e+1} \cap V_{e,>\frac{d-c}{m+1}}$ is empty with probability $1 - o(1)$. Again note that we do not have to intersect this with the hypersurface H_f or demand $f(P)$ to be zero since $V_{e,>\frac{d-c}{m+1}}$, is already contained in H_f. At first, we will use the polynomials $f_0, g_1, \ldots g_{l_e}$ to show that the dimension of W_{l_e} is bounded. In a second claim we will show by using the polynomial g_{l_e+1} in the construction of f above that, if W_{l_e} is finite, the next derivation D_{l_e+1} does not vanish for f at P with probability $1 - o(1)$ as $d \to \infty$.

Claim 2.19.1. *If the polynomials $f_0, g_1, \ldots, g_i$ for $0 \leq i \leq l_e$ have been chosen such that $\dim(W_i) \leq l_e - i$ holds, then the probability for $\dim(W_{i+1}) \leq l_e - i - 1$ is equal to $1 - o(1)$ as $d \to \infty$. The function of d represented by $o(1)$ depends on V_e and the derivations D_i.*

Proof of Claim 2.19.1. Let $Y_1, \ldots, Y_s$ be the $(l_e - i)$ - dimensional $\mathbb{F}_q$ - irreducible components of $(W_i)_{\text{red}}$. The degree of a hypersurface generated by a polynomial of degree d is equal to d; hence Bézout's theorem Lemma 2.17 yields

$$s \leq (\deg \overline{V_e})(\deg D_1 f) \ldots (\deg D_i f) = O(d^i)$$

as $d \to \infty$, where $\overline{V_e}$ is the projective closure of V_e. Since $W_{i+1} = W_i \cap \{D_{i+1}f = 0\}$, we need to bound the set

$$G_k^{\mathrm{bad}} = \left\{ g_{i+1} \in S'_\gamma :\ D_{i+1}f = D_{i+1}f_0 + g_{i+1}^p s \equiv 0 \text{ on } Y_k \right\}.$$

For $g, g' \in G_k^{\mathrm{bad}}$ we have the equality $g^p s = -D_{i+1}f_0 = g'^p s$ on Y_k, i.e. $g^p s - g'^p s = 0$. Since $s \in \mathcal{O}(N_P)^*$, we can multiply by the inverse of s and get $g - g' = 0$ on Y_k because the characteristic of $\mathbb{F}_q$ is p. Thus, if G_k^{bad} is not empty, it is a coset of the subspace of functions in S'_γ that vanish on Y_k. The codimension of this subspace is the dimension of the image of S'_γ in the regular functions on Y_k. To calculate this dimension, let x_j be a coordinate depending on k such that the projection $x_j(Y_k)$ has dimension 1; such a coordinate exists since $\dim Y_k \geq 1$. A nonzero polynomial in x_j alone does not vanish on Y_k because $x_j \neq 0$ and $Y_k \subseteq (W_i)_{\mathrm{red}}$. Therefore the dimension of the image of S'_γ in the regular functions on Y_k must be at least $\gamma + 1$.

Hence the probability for $D_{i+1}f \equiv 0$ on some Y_k is at most the number of irreducible components Y_k multiplied by the probability that $D_{i+1}f$ vanishes on one Y_k, thus $sq^{-\gamma-1} = O(d^i q^{-(d-\tau)/p}) = o(1)$ as $d \to \infty$. This shows Claim 2.19.1. □

For $i = 0$ the requirements of Claim 2.19.1 are fulfilled since $\dim W_0 = \dim V_e = l_e$. Therefore with probability $\prod_{j=0}^{i}(1 - o(1)) = 1 - o(1)$ the dimension of W_i is at most $l_e - i$ for $0 \leq i \leq l_e$ and an arbitrary choice of $f_0, g_1, \ldots, g_{l_e}$. In particular, $\dim W_{l_e} = 0$ with probability $1 - o(1)$.

Now we show the second claim:

Claim 2.19.2. *Conditioned on a choice of $f_0, g_1, \ldots, g_{l_e}$ for which W_{l_e} is finite, the probability for $W_{l_e+1} \cap V_{e,>\frac{d-c}{m+1}}$ to be empty is equal to $1 - o(1)$ as $d \to \infty$.*

Proof of Claim 2.19.2. Since W_{l_e} is finite, the irreducible components of W_{l_e} are the closed points of W_{l_e}. Again by Lemma 2.17, the number s of irreducible components is

$$\#W_{l_e} = s \leq (\deg \overline{V_e})(\deg D_1 f) \ldots (\deg D_{l_e} f) = O(d^{l_e}).$$

Choose $f_0, g_1, \ldots, g_{l_e}$ such that W_{l_e} is finite. We define H^{bad} to be the set of polynomials $g_{l_e+1} \in S'_\gamma$ such that $(D_{l_e+1}f)(P) = 0$. As in Claim 2.19.1 we need to bound the set H^{bad} since $W_{l_e+1} = W_{l_e} \cap \{D_{l_e+1}f = 0\}$, and again we can show that H^{bad} is either empty or a coset of $\ker(\text{ev}_P)$, where $\text{ev}_P : S'_\gamma \to \kappa(P)$. If $\deg P > \frac{d-c}{m+1}$, then by Lemma 2.18 $\#H^{\text{bad}}/\#S'_\gamma \leq q^{-\nu}$ where $\nu = \min(\gamma, \frac{d-c}{m+1})$. Hence the probability for $W_{l_e} \cap V_{e,>\frac{d-c}{m+1}}$ to be nonempty is smaller than or equal to the number of points in W_{l_e} multiplied by the probability that $(D_{l_e+1}f)(P) = 0$ for any point $P \in W_{l_e}$; thus we get $\#W_{l_e} q^{-\nu} = O(d^{l_e} q^{-\nu}) = o(1)$ as $d \to \infty$ since ν grows linearly in d. This shows the second claim. □

Now we choose $f \in I'_d$ uniformly at random. The two claims show, that with probability $\prod_{i=0}^{l_e}(1-o(1))(1-o(1)) = 1-o(1)$ for $d \to \infty$, the dimension of W_i is equal to $l_e - i$ for $0 \leq i \leq l_e + 1$ and $W_{l_e+1} \cap V_{e,>\frac{d-c}{m+1}}$ is empty. As we have seen before, $W_{l_e} \cap V_{e,>\frac{d-c}{m+1}}$ is the set of points $P \in V_e$ of degree larger than $\frac{d-c}{m+1}$ where $H_f \cap X$ may not be smooth. We have just shown that this is of density zero, as we claimed. □

Proof of Theorem 2.1 (i). By definition,

$$\mathcal{P} \subseteq \mathcal{P}_r \subseteq \mathcal{P} \cup \mathcal{Q}_r^{\text{medium}} \cup \mathcal{Q}_{X-V}^{\text{high}} \cup \mathcal{Q}_V^{\text{high}}.$$

The first inclusion is clear; the second holds since for any polynomial $f \in \mathcal{P}_r$ the intersection $H_f \cap X$ is either smooth and therefore f must be in $\mathcal{P}$, or the intersection is at least at one point $P \in X - V$ or $P \in V$ not smooth of dimension $m-1$. As $f \in \mathcal{P}_r$, the degree of this point must be at least r and the second inclusion follows.

Therefore $\overline{\mu}(\mathcal{P})$ and $\underline{\mu}(\mathcal{P})$ differ from $\mu(\mathcal{P}_r)$ at most by $\overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) + \overline{\mu}_Z(\mathcal{Q}_{X-V}^{\text{high}}) + \overline{\mu}_Z(\mathcal{Q}_V^{\text{high}})$. The Lemmas 2.15, 2.16 and 2.19 yield $\lim_{r\to\infty} \overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}}) + \overline{\mu}_Z(\mathcal{Q}_{X-V}^{\text{high}}) + \overline{\mu}_Z(\mathcal{Q}_V^{\text{high}}) = 0$. Thus Corollary 2.13 shows

$$\mu_Z(\mathcal{P}) = \lim_{r\to\infty} \mu_Z(\mathcal{P}_r) = \frac{\zeta_V(m+1)}{\zeta_X(m+1) \prod_{e=0}^{m} \zeta_{V_e}(m-e)},$$

which is what we claimed. □

Definition 2.20. *Let V be a subscheme of $\mathbb{P}^n$. Let $W_1, \ldots, W_s$ be the irreducible components of V. We say that V has simple normal crossings if W_i is smooth for any i, $\bigcap_{i \in I} W_i$ is smooth and codim$_V \bigcap_{i \in I} W_i = \#I - 1$ for any subset $I \subseteq \{1, \ldots, s\}$.*

Corollary 2.21. *Let the notation be as in Theorem 2.1. Suppose V is equidimensional of dimension l and has simple normal crossings. If furthermore $2l < m$ holds, then there exists a hypersurface H containing Z such that $H \cap X$ is smooth of dimension $m - 1$.*

Proof. First, note that we have a lower bound for the embedding dimensions of closed points of V given by the dimension l of V: as V is equidimensional, $\dim \mathcal{O}_{V,P} = \dim V$ for all closed points $P \in V$, and $\dim \mathcal{O}_{V,P} \leq e_V(P)$. To prove this corollary, we just need to show that the conditions of Theorem 2.1 (i) are fulfilled. Therefore, we need to determine the schemes $V_e = V_{l+k}$ of the flattening stratification of V, where $0 \leq k \leq m - l$. We will prove by induction that V_{l+k} is contained in the union of all intersections of $k + 1$ irreducible components of V; more precisely, we show that if a point P is in the intersection of exactly k irreducible components of V, then $e_V(P) \leq l + k - 1$.

Let $W_1, \ldots, W_s$ be the irreducible components of V and let P be a closed point of V. Then by Remark 1.10, $e_V(P) = \dim_{\kappa(P)} \mathfrak{m}_{V,P} / \mathfrak{m}_{V,P}^2$. If $P \in W_i \backslash \bigcup_{j \neq i} W_j$, i.e. P lies in exactly one irreducible component of V, then $\mathcal{O}_{V,P}$ is regular of dimension l, and therefore $e_V(P) = l$. Hence, $\bigcup_{i=1}^{s} \big(W_i \backslash \bigcup_{j \neq i} W_j\big) \subseteq V_l$. The scheme V is not regular at all other closed points; thus, the embedding dimension cannot be equal to l and we get $\bigcup_{i=1}^{s} \big(W_i \backslash \bigcup_{j \neq i} W_j\big) = V_l$.

Next we show that if a closed point P is in exactly two irreducible components of V, then $P \in V_{l+1}$. For this, we need to calculate the $\kappa(P)$-dimension of $\mathfrak{m}_{V,P} / \mathfrak{m}_{V,P}^2$ for a closed point $P \in W_i \cap W_j \backslash \bigcup_{s \neq i,j} W_k$. We can assume V to be the union of W_i and W_j, since P lies in no other irreducible component. Because W_i and W_j intersect tranversally, we get

$$\mathfrak{m}_{V,P} / \mathfrak{m}_{V,P}^2 = \mathfrak{m}_{W_i,P} / \mathfrak{m}_{W_i,P}^2 + \mathfrak{m}_{W_j,P} / \mathfrak{m}_{W_j,P}^2$$

as $\kappa(P)$-vector spaces, i.e. the cotangent space of V at P is the sum of the l-dimensional cotangent spaces of W_i and W_j at P. But as $W_i \cap W_j$ is smooth of dimension $l-1$, $\mathfrak{m}_{W_i \cap W_j,P} \,/\, \mathfrak{m}^2_{W_i \cap W_j,P}$ is a $l-1$-dimensional subspace of $\mathfrak{m}_{W_i,P} \,/\, \mathfrak{m}^2_{W_i,P}$ and we can extend a $\kappa(P)$-basis $t_1, \ldots, t_{l-1}$ of $\mathfrak{m}_{W_i \cap W_j,P} \,/\, \mathfrak{m}^2_{W_i \cap W_j,P}$ to get a basis $t_1, \ldots, t_l$ of $\mathfrak{m}_{W_i,P} \,/\, \mathfrak{m}^2_{W_i,P}$. In the same way we get a basis $t_1, \ldots, t_{l-1}, t'_l$ of $\mathfrak{m}_{W_j,P} \,/\, \mathfrak{m}^2_{W_j,P}$. Then t_l and t'_l must be linearly independent, because otherwise, V would be smooth at P. Therefore $t_1, \ldots, t_l, t'_l$ is a basis of $\mathfrak{m}_{V,P} \,/\, \mathfrak{m}^2_{V,P}$. Thus, the embedding dimension of V at P is equal to $l+1$.

Now assume that if a point P is in the intersection of exactly k irreducible components of V, then $e_V(P) \leq l + k - 1$. For the induction step, let P be in the intersection of exactly $k+1$ irreducible components, say $P \in W_1 \cap \ldots \cap W_{k+1}$. Let $U_{k+1} = \bigcup_{i=1}^{k+1} W_i$. Then

$$e_V(P) = \dim \mathfrak{m}_{U_{k+1},P} \,/\, \mathfrak{m}^2_{U_{k+1},P},$$

since P lies in no other irreducible component. Let $U_k := \bigcup_{i=1}^{k} W_i$. As in the calculation above, $\mathfrak{m}_{U_{k+1},P} \,/\, \mathfrak{m}^2_{U_{k+1},P} = \mathfrak{m}_{U_k,P} \,/\, \mathfrak{m}^2_{U_k,P} + \mathfrak{m}_{W_{k+1},P} \,/\, \mathfrak{m}^2_{W_{k+1},P}$. By induction for the scheme U_k, we get $\dim \mathfrak{m}_{U_k,P} \,/\, \mathfrak{m}^2_{U_k,P} \leq l - k + 1$. Since W_{k+1} is smooth of dimension l, $\dim \mathfrak{m}_{W_{k+1},P} \,/\, \mathfrak{m}^2_{W_{k+1},P} = l$. Furthermore, $\mathfrak{m}_{W_1 \cap W_{k+1},P} \,/\, \mathfrak{m}^2_{W_1 \cap W_{k+1},P}$ can be interpreted as subspace of all the vector spaces above, and is of dimension $l-1$ since $W_1 \cap W_{k+1}$ is smooth of dimension $l-1$. Thus,

$$\begin{aligned} \dim \mathfrak{m}_{U_{k+1},P} \,/\, \mathfrak{m}^2_{U_{k+1},P} &= \dim \mathfrak{m}_{U_k,P} \,/\, \mathfrak{m}^2_{U_k,P} + \dim \mathfrak{m}_{W_{k+1},P} \,/\, \mathfrak{m}^2_{W_{k+1},P} \\ &\leq l - k + 1 + l - l + 1 = l + k, \end{aligned}$$

and therefore $e_V(P) \leq l + k$. Hence, we have shown that if $P \in V_{l+k}$, then P lies in the intersection of at least $k+1$ irreducible components of V. Using this, we can give an estimate for the dimension of V_{l+k}.

Since the codimension of the intersection of $k+1$ irreducible components of V is k, the union of those intersections has dimension $l-k$. As we have shown above, V_{l+k} is contained in this union and therefore $\dim V_{l+k} \leq l - k$.

Combining the above results, we get $\dim V_{l+k} + l + k \leq 2l$ for $0 \leq l \leq m$. Hence, if $2l < m$ holds, then the conditions of Theorem 2.1 are satisfied and the corollary follows. Note, that V_m is empty, because by what we have shown above, V_{m-1} is contained in the union of the intersections of $m - l$ irreducible components, and this union is already of dimension zero. As V has simple normal crossings, the intersection of $m - l + 1$ components, which contains V_m, must be empty. □

3 Bertini with Taylor conditions

In the last section we want to prove that we can find a polynomial f that gives a smooth hypersurface section containing the given closed subscheme Z even if we prescribe the first few terms of the Taylor expansion of the dehomogenization of f at finitely many closed points that are not in Z. We can use this to show that the hypersurface can be assumed to avoid finitely many points in which X is not smooth. We follow [Poo04] Theorem 1.2 and first define the restriction of a polynomial $f \in I_d$ to those finitely many points. Let Y be a finite subscheme of $\mathbb{P}^n$. For a polynomial $f \in I_d$ we define $f\big|_Y \in H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y)$ as follows: on each connected component Y_i of Y let $f\big|_Y$ be equal to the restriction of $x_j^{-d} f$ to Y_i, where $j = j(i)$ is the smallest $j \in \{0, 1, \ldots, n\}$ such that the coordinate x_j is invertible on Y_i.

Theorem 3.1. *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ of dimension $m \geq 0$ over $\mathbb{F}_q$ and let Z be a closed subscheme of $\mathbb{P}^n$. Let C be a finite subscheme of $\mathbb{P}^n$, such that $U := X - (X \cap C)$ is smooth of dimension $m \geq 0$ and $C \cap Z = \emptyset$. Let $V = Z \cap U$ be the intersection and let T be a subset of $H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C)$. Define*

$$\mathcal{P} = \left\{ f \in I_{\text{homog}} : \ H_f \cap U \text{ is smooth of dimension } m-1 \text{ and } f\big|_C \in T \right\}.$$

(i) If $\max\limits_{0 \leq e \leq m-1} \{e + \dim V_e\} < m$ and $V_m = \emptyset$, then

$$\mu_Z(\mathcal{P}) = \frac{\#T}{\#H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C)} \frac{\zeta_V(m+1)}{\zeta_U(m+1) \prod\limits_{e=0}^{m} \zeta_{V_e}(m-e)}.$$

(ii) If $\max\limits_{0 \leq e \leq m-1} \{e + \dim V_e\} \geq m$ or $V_m = \emptyset$, then $\mu_Z(\mathcal{P}) = 0$.

This is a version of Theorem 1.2 of [Poo04] where the hypersurface has to contain Z; as in Remark 2.2 we can show that Poonen's theorem follows by Theorem 3.1. The proof of Theorem 3.1 is parallel to the one of Theorem 2.1, only Lemma 2.12 for singularities of low degree needs to be changed:

Lemma 3.2 (Singularities of low degree). *Let the notation and hypotheses be as in Theorem 3.1. Define*

$$\mathcal{P}_r := \{f \in I_{\text{homog}} : H_f \cap U \text{ is smooth of dimension } m-1 \\ \text{at all points } P \in U_{<r} \text{ and } f\big|_C \in T\}.$$

Then

$$\mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C)} \prod_{e=0}^{m} \prod_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P}) \cdot \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}).$$

Proof. Let $U_{<r} = \{P_1, \ldots, P_s\}$. Let $\mathfrak{m}_i$ be the ideal sheaf of P_i on U and let Y_i be the closed subscheme of U corresponding to the ideal sheaf $\mathfrak{m}_i^2 \subseteq \mathcal{O}_X$. Then $H_f \cap U$ is not smooth of dimension $m-1$ at P_i if and only if the restriction of f to a section of $\mathcal{O}_{Y_i}(d)$ is equal to zero, by Lemma 2.9.

Since we also want $f\big|_C$ to be in T, the set $\mathcal{P}_r \cap I_d$ is the inverse image of

$$T \times \prod_{i=1}^{s} (H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) \setminus \{0\})$$

under the $\mathbb{F}_q$-linear composition

$$\begin{aligned} \phi_d : I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) &\to H^0(C \cup Y, \mathcal{I}_Z \cdot \mathcal{O}_{C \cup Y}(d)) \\ &\cong H^0(C \cup Y, \mathcal{I}_Z \cdot \mathcal{O}_{C \cup Y}) \cong H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C) \times \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}), \end{aligned}$$

where $Y := \bigcup_{i=1}^{s} Y_i$. The first isomorphism is the untwisting by multiplication by x_j^{-d} component by component as in the definition of $f\big|_Z$, and the second follows from

$$H^0(C \cup Y, \mathcal{I}_Z \cdot \mathcal{O}_{C \cup Y}) = \prod_{C_j \in C} (\mathcal{I}_Z \cdot \mathcal{O}_C)(C_j) \prod_{Y_i \in Y} (\mathcal{I}_Z \cdot \mathcal{O}_C)(Y_i),$$

where C_j are the connected components, i.e. the points of C. Note that at this point, we need the restriction $C \cap Z = \emptyset$. The map ϕ_d is surjective for $d \geq c + \dim H^0(C \cup Y, \mathcal{O}_{C \cup Y})$ by Lemma 2.8; as we want to calculate the density $\mu(\mathcal{P}_r)$ and therefore only consider the limit $d \to \infty$, we can assume ϕ_d to be surjective. Hence $H^0(\mathbb{P}^n, \mathcal{I}_Z(d))/\ker \phi_d \cong H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C) \times \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})$ and

$$\#I_d = \#(\ker \phi_d) \cdot \#\big(H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C) \times \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})\big).$$

Since $\mathcal{P}_r \cap I_d$ is the inverse image of $T \times \prod_{i=1}^{s} (H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) \setminus \{0\})$ under ϕ_d, it is the disjoint union of $\#\big(T \times \prod_{i=1}^{s} (H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) \setminus \{0\})\big)$ cosets of the kernel of ϕ_d. It follows that $\#(\mathcal{P}_r \cap I_d) = \#(\ker \phi_d) \cdot \#\big(T \times \prod_{i=1}^{s} (H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) \setminus \{0\})\big)$, and this shows

$$\begin{aligned}
\mu_Z(\mathcal{P}_r) &= \lim_{d \to \infty} \frac{\#(\mathcal{P}_r \cap I_d)}{\#I_d} \\
&= \frac{\#(\ker \Phi_d) \cdot \#\big(T \times \prod_{i=1}^{s} (H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) \setminus \{0\})\big)}{\#(\ker \Phi_d) \cdot \#\big(H^0(C, \mathcal{I}_Z \cdot \mathcal{O}_C) \times \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})\big)} \\
&= \frac{\#T}{\#H^0(C, \mathcal{O}_C)} \frac{\# \prod_{i=1}^{s} (H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i}) \setminus \{0\})}{\# \prod_{i=1}^{s} H^0(Y_i, \mathcal{I}_Z \cdot \mathcal{O}_{Y_i})}.
\end{aligned}$$

Applying Lemma 2.10 yields

$$\mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(C, \mathcal{O}_C)} \prod_{e=0}^{m} \prod_{P \in (V_e)_{<r}} (1 - q^{-(m-e)\deg P}) \cdot \prod_{P \in (X-V)_{<r}} (1 - q^{-(m+1)\deg P}),$$

and the result follows. □

Proof of Theorem 3.1. The proof is equal to the one of Theorem 2.1. As in Corollary 2.13 we can show

$$\lim_{r \to \infty} \mu_Z(\mathcal{P}_r) = \frac{\#T}{\#H^0(C, \mathcal{O}_C)} \frac{\zeta_V(m+1)}{\zeta_X(m+1) \prod_{e=0}^{m} \zeta_{V_e}(m-e)}.$$

Define $\mathcal{P}_r$, $\mathcal{Q}_r^{\text{medium}}$, $\mathcal{Q}_{X-V}^{\text{high}}$ and $\mathcal{Q}_V^{\text{high}}$ as in Section 2. Then again

$$\mathcal{P} \subseteq \mathcal{P}_r \subseteq \mathcal{P} \cup \mathcal{Q}_r^{\text{medium}} \cup \mathcal{Q}_{X-V}^{\text{high}} \cup \mathcal{Q}_V^{\text{high}}.$$

Thus $\overline{\mu}(\mathcal{P})$ and $\underline{\mu}(\mathcal{P})$ differ from $\mu(\mathcal{P}_r)$ at most by $\overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}})+\overline{\mu}_Z(\mathcal{Q}_{X-V}^{\text{high}})+\overline{\mu}_Z(\mathcal{Q}_V^{\text{high}})$. The Lemmas 2.15, 2.16 and 2.19 for singularities of medium and high degrees show also in this case that $\lim\limits_{r\to\infty}\overline{\mu}_Z(\mathcal{Q}_r^{\text{medium}})+\overline{\mu}_Z(\mathcal{Q}_{X-V}^{\text{high}})+\overline{\mu}_Z(\mathcal{Q}_V^{\text{high}})=0$. Hence

$$\mu_Z(\mathcal{P})=\lim_{r\to\infty}\mu_Z(\mathcal{P}_r)=\frac{\#T}{\#H^0(C,\mathcal{O}_C)}\frac{\zeta_V(m+1)}{\zeta_X(m+1)\prod\limits_{e=0}^{m}\zeta_{V_e}(m-e)},$$

what we wanted to show for (i).

For the second part where $\max\limits_{0\le e\le m-1}\{e+\dim V_e\}\ge m$ or $V_m=\emptyset$, we can show just as in the proof of Theorem 2.1 (ii) that the density of $\mathcal{P}_r$ tends to zero for $r\to\infty$. Since $\mathcal{P}\subseteq\mathcal{P}_r$, it follows that $\mu_Z(\mathcal{P})=0$. □

Actually, Theorem 2.1 is just the special case where C is the empty set and $T=\{0\}$. At last, we want to show that we can find a hypersurface that intersects X smoothly, contains a given closed subscheme and avoids the singular locus of X, if it is finite.

Corollary 3.3. *Let X be a quasi-projective subscheme of $\mathbb{P}^n$ that is smooth of dimension $m\ge 0$ over $\mathbb{F}_q$ at all but finitely many closed points $P_1,\dots,P_r$. Let Z be a closed subscheme of $\mathbb{P}^n$ that does not contain any of those points and let $V=Z\cap X$ be the intersection. Suppose $\max\limits_{0\le e\le m-1}\{e+\dim V_e\}<m$ and $V_m=\emptyset$. Then for $d\gg 1$, there exists a hypersurface H of degree d that contains Z but none of the points $P_1,\dots,P_r$, such that $H\cap X$ is smooth of dimension $m-1$.*

Proof. Let $C_i=\operatorname{Spec}\kappa(P_i)$ and $C=\bigcup\limits_{i=1}^{r}C_i$. Then C is a finite scheme since $\kappa(P_i)=\mathcal{O}_{X,P_i}/\mathfrak{m}_{X,P_i}$ are local finite $\mathbb{F}_q$-algebras, and $U=X-(X\cap C)$ is smooth of dimension $m\ge 0$. Furthermore, Z does not contain any point P_i and therefore $C\cap Z=\emptyset$. By definition, $H^0(C,\mathcal{I}_Z\cdot\mathcal{O}_C)=\prod\limits_{i=1}^{r}\mathcal{I}_{Z,P_i}\cdot\kappa(P_i)$. Now we define $T\subseteq H^0(C,\mathcal{I}_Z\cdot\mathcal{O}_C)$ to be the nonempty set of elements that are nonzero in every component of the above product. If f is a polynomial in I_d such that $f\big|_C\in T$, then the restriction of $f\cdot x_j^{-d}$ to C_i is not equal to zero in every field $\kappa(P_i)$, where $j=j(i)$ is the smallest $j\in\{0,1,\dots,n\}$ such that the coordinate x_j is invertible on C_i, as in the definition of $f\big|_C$. Hence f is not zero in $\kappa(P_i)$ and therefore $f \notin \mathfrak{m}_{X,P_i}$ for all $1 \leq i \leq r$. This shows that if $f\big|_C \in T$, then $P_i \notin H_f$ for all $1 \leq i \leq r$, and thus $H_f \cap C = \emptyset$.

Applying Theorem 3.1 to this situation shows the existence of hypersurface H of degree $d \gg 1$ that does not intersect C and therefore contains none of the points $P_1, \ldots, P_r$; further it intersects U and thus also X smoothly, as stated. □

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