Characterization of Non-Smooth Pseudodifferential Operators

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Abstract

In the last decades the theory of pseudodifferential operators was established as an auxiliary tool for solving problems in the field of partial differential equations. However, proving that the inverse of a partial differential operator is a pseudodifferential operator is often difficult. In order to simplify this task, R. Beals and J. Ueberberg developed a characterization of pseudodifferential operators with smooth symbols. In applications also non-smooth pseudodifferential operators occur. Hence such a characterization would be useful for non-smooth pseudodifferential operators, too. Therefore, we show that every linear operator P, which satisfies some specific continuity assumptions, is a non-smooth pseudodifferential operator whose coefficients are in a Hölder space.

With the characterization of non-smooth pseudodifferential operators at hand, we are in the position to verify under suitable conditions for a sufficiently large $p_0 > 1$ the following result: the L^p-spectrum of a non-smooth pseudodifferential operator P, whose coefficient is in a Hölder space, is independent of the choice of $p \geq p_0$. With P being continuously invertible as a linear operator on certain Bessel potential spaces, the inverse of P is a non-smooth pseudodifferential operator of the same symbol-class under suitable conditions.

In order to reach these goals we make use of the central ideas of the analogous results from R. Beals and J. Ueberberg in the smooth case. The main new difficulties are the limited mapping properties of pseudodifferential operators with non-smooth symbols.

Zusammenfassung

In den letzten Jahrzehnten etablierte sich die Theorie der Pseudodifferentialoperatoren als Hilfsmittel zum Lösen von Problemen aus dem Gebiet der partiellen Differentialgleichungen. Hierfür muss überprüft werden, dass der Inverse Operator eines zugehörigen partiellen Differentialoperators ein Pseudodifferentialoperator ist. Zur Vereinfachung dieser Aufgabe entwickelten R. Beals und J. Ueberberg eine Charakterisierung von Pseudodifferentialoperatoren mit glatten Symbolen. In den Anwendungen treten allerdings auch Pseudodifferentialoperatoren mit nicht glatten Symbolen auf. Ein Ziel dieser Arbeit ist es daher eine solche Charakterisierung für nicht glatte Pseudodifferentialoperatoren zu erarbeiten. Es wird gezeigt, dass jeder lineare Operator P, der bestimmte Stetigkeitsbedingungen erfüllt, ein nicht glatter Pseudodifferentialoperator ist, dessen Koeffizienten in einem Hölderraum liegen.

Mit Hilfe dieser Charakterisierung kann unter bestimmten Vorraussetzun-

gen für genügend große $p_0 > 1$ folgendes Ergebnis nachgewiesen werden: Das L^p-Spektrum eines nicht glatten Pseudodifferentialoperators P, dessen Koeffizienten in einem Hölderraum liegen, ist unabhängig von der Wahl von $p \geq p_0$. Falls P als Operator auf gewissen Besselpotential Räumen stetig invertierbar ist, ist auch P^{-1} ein nicht glatter Pseudodifferentialoperator.

Um diese Ziele zu erreichen, werden die Hauptideen der entsprechenden Ergebnisse von R. Beals und J. Ueberberg für den glatten Fall benutzt. Die neuen Schwierigkeiten, die sich ergeben, resultieren aus den begrenzten Abbildungseigenschaften von Pseudodifferentialoperatoren mit nicht glatten Symbolen.

Contents

Chapter 1

Introduction

In medicine great effort has been invested in improving and optimizing their methods of diagnostics and treatment by combining their knowledge with the results of mathematical modeling. An example of particular interest and hopes is the modeling of the tumor growth, especially during radiotherapy, which should help to substantially improve the success of treatment of irradiations. In particular radiation damages to healthy tissue should be minimized. A consistent mathematical model which contains both the tumor growth and the effects of radiotherapy could be the solution in order to make the treatment more effective. In [35] two essential processes of the tumor growth are modeled by means of a partial differential equation: The growth of tissue and the diffusion of cancer cells into the surrounding tissue. Whenever this happens healthy tissue is displaced or destroyed. To describe the tumor growth and other problems in medicine, biology, physics and other scientific fields is an important application of the theory of partial differential equations. On the mathematical side, this requires to determine the solution of partial differential equations and investigate their properties.

Since 1965 J.J. Kohn and L. Nirenberg, L. Hörmander and others developed a new tool, that helps to solve or simplify the treatment of certain problems belonging to the field of linear partial differential equations: the theory of pseudodifferential operators. Pseudodifferential operators have been constructed as a generalization of the linear partial differential operators

$$p(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha,$$

where $m \in \mathbb{N}_0$, $a_\alpha \in C^\infty(\mathbb{R}^n)$ and $D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha$ for all multi-indices α with $|\alpha| \leq m$. Elementary properties of the Fourier transformation $\mathscr{F}$ and its inverse enable us to write

$$p(x, D_x)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \mathscr{F}[u](\xi) d\xi \tag{1.1}$$

for certain functions u and all $x \in \mathbb{R}^n$. Here $p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ is called symbol of $p(x, D_x)$. Pseudodifferential operators are defined by the above-mentioned characterization (1.1) of the linear differential operators for a given symbol. Such pseudodifferential operators are also called pseudodifferential operators of the Kohn-Nirenberg form. In contrast to the linear differential operators the symbols of pseudodifferential operators do not have to be polynomials in ξ. There are a lot of different symbol-classes for pseudodifferential operators. At first, only smooth symbols fulfilling certain estimates with respect to their derivatives have been considered. The most common smooth symbol-classes are the Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. We will get to know them in this thesis, cf. Definition 3.1 below.

However, pseudodifferential operators do not only serve as an auxiliary tool in the field of linear partial differential equations. They are also used in the field of time frequency analysis. V. Turunen recently spent time on denoising the sound recorded inside an MRI machine in collaboration with engineers. The main idea for solving this problem was to use the Born-Jordan transformation $Q(f,g)$ instead of the short-time Fourier transformation for certain functions f and g. Here $Q(f,g) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is defined by

$$Q(f,g)(x,\eta) = \int\limits_{\mathbb{R}^n} e^{i2\pi y\cdot\eta} y^{-1} \int_{x-y/2}^{x+y/2} f(t+y/2)g(t-y/2)^* dtdy$$

for all $x, \eta \in \mathbb{R}^n$. The Born-Jordan transformation enables us to define a pseudodifferential operator A_σ of the Born-Jordan form for each symbol $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ via

$$\langle f, A_\sigma g\rangle_{L^2(\mathbb{R}^n)} := \langle Q(f,g), \sigma\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}.$$

V. Turunen verified the properties of the Born-Jordan transformation by means of the properties of the pseudodifferential operators A_σ as this turned out to be much easier. A paper about this topic will be published soon.

Showing that the inverse of a pseudodifferential operator P exists provided certain conditions are given and that the inverse is a pseudodifferential operator again, is one purpose of the theory of pseudodifferential operators. Based on the existence of such an inverse, we are in the position to solve the partial differential equation $Pu = f$, if the inverse of P: $u = P^{-1}f$ can be applied on f. Since pseudodifferential operators are linear and bounded as maps between certain function spaces, this theory even allows us to show some regularity results.

Proving the maximal L^p-regularity of parabolic mixed order systems was the task of R. Denk and J. Seiler in [27]. Additionally there are a lot of other regularity results and applications of the theory of smooth pseudodifferential operators, cf. e.g. [40] and [58].

Let us point out another big field of applications besides the determination of regularity. Making use of pseudodifferential methods allows us to more easily calculate the spectrum of some operators, cf. e.g. [22] and [57]. Moreover, pseudodifferential operators also serve as an auxiliary tool in the index theory. One of the most important statements proved there is certainly the Atiyah–Singer index theorem, in which the equality of the topological index and the analytic index is verified. For the proof and for applications of this statement we warmly recommend [21].

Beyond the theory of smooth pseudodifferential operators also a theory for non-smooth pseudodifferential operators has been developed over the last 40 years. There are several classes of non-smooth pseudodifferential operators, cf. e.g. [53], [54] and [67]. The most common ones are non-smooth pseudodifferential operators with coefficients in the Hölder spaces, cf. Definition 4.10 below. For the first time they were presented by H. Kumano-Go and M. Nagase in [43]. Non-smooth pseudodifferential operators with coefficients in the Hölder spaces, which are even non-smooth in ξ, were introduced by J.D. Alvarez-Alonso and A.P. Calderon in [10] and investigated by R.R. Coifman and Y. Meyer in [23] and J. Marschall in [53].

Making use of the technique of non-smooth pseudodifferential operators, many interesting results in the field of nonlinear partial differential equations have already been proved. Just to mention a few of them: H. Kumano-Go and M. Nagase constructed the fundamental solution for hyperbolic operators with non-smooth coefficients and proved the sharp Garding inequality for a differential operator with non-smooth coefficients in [43]. Moreover, H. Abels and M. Kassmann treated the Cauchy problem and the Martingale problem for integro-differential operators with non-smooth kernels in [7]. Other applications can be found in [67]. In Chapter 8 of this reference for instance, M.E. Taylor established estimates and the regularity for solutions to nonlinear elliptic boundary problems. Moreover, he also treated nonlinear hyperbolic systems in Chapter 5 and nonlinear parabolic systems in Chapter 7 of the same reference. For further applications to boundary value problems see [3], [6] and the references given therein.

We already mentioned that for the determination of the spectrum methods of the field of smooth pseudodifferential operators can be used. In the field of partial differential equations usually the L^p-spectrum for $p > 2$ is needed. Being equipped with a Hilbert space structure it is mostly easier to calculate the L^2-spectrum. Hence the spectral invariance of pseudodifferential operators is of particular interest. In the regularity theory the spectral invariance of pseudodifferential operators is an important tool, too. There have been several observations in the smooth case yet. Results of R. Beals [16] and J. Ueberberg

[74] allow us to directly obtain the following statement: The spectrum of each pseudodifferential operator whose symbol is in the symbol-class $S^{\mu}(\Phi, \varphi)$ or in the Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ acting on the Bessel potential space $H^s_q(\mathbb{R}^n)$ is independent of the choice of $s \in \mathbb{R}$ for $q = 2$. We refer to [16] for the definition of $S^{\mu}(\Phi, \varphi)$. For symbols of the Hörmander classes the choice of $q \in (1, \infty)$ is also possible. The spectrum of the associated pseudodifferential operator is even independent of $q \in (1, \infty)$, cf. [74]. However, R. Beals and J. Ueberberg even checked in [16] and [74] that the inverse of a pseudodifferential operator P with a symbol in the symbol-class $S^{\mu}(\Phi, \varphi)$ or in the Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ is again a pseudodifferential operator with its symbol being in the same symbol-class if P is invertible as an operator on $L^2(\mathbb{R}^n)$. In the literature this property is often called spectral invariance. E. Schrohe investigated in [64] that the spectrum of pseudodifferential operators in the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ considered as linear maps on anisotropically weighted L^p-Sobolev spaces

$$H^{s,t}_p = \{\gamma^{-1} u : u \in H^s_p(\mathbb{R}^n)\}$$

does not depend on the choice of $s, t \in \mathbb{R}$, $1 < p < \infty$ and on the weight function $\gamma \in C^{\infty}(\mathbb{R}^n)$, which has to be bounded away from zero with all derivatives bounded. J. Alvarez and J. Hounie extended this observation in [9]. For a specific subclass of these pseudodifferential operators E. Schrohe established a stronger result in [65]: The inverse of a pseudodifferential operator P with symbol in a Grushin class $\tilde{S}^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \leq \delta < \rho \leq 1$, of slowly varying symbols is a pseudodifferential operator with its symbol in the same symbol-class again, whenever P is continuously invertible as an operator on a weighted Sobolev space, cf. [65] for more details. Due to H.-G. Leopold and E. Schrohe [46] a similar result holds for pseudodifferential operators whose symbols are in the symbol-class $S^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ on Besov spaces of variable order of differentiation $B^{s,a}_{p,q}(\mathbb{R}^n)$: Being continuously invertible as a linear operator on $B^{s,a}_{p,q}(\mathbb{R}^n)$, the inverse of P is a pseudodifferential operator of the same symbol-class. This implies that the spectrum of P is independent of the chosen space $B^{s,a}_{p,q}(\mathbb{R}^n)$. Moreover, it coincides with the L^2-spectrum, cf. [46]. For $\rho = 1$ and $\delta < 1$ H.-G. Leopold and E. Schrohe verified in [47] that the same statement remains to be true while exchanging the spaces $B^{s,a}_{p,q}(\mathbb{R}^n)$ with Besov spaces or Triebel-Lizorkin spaces. For pseudodifferential operators whose symbols are in the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ acting on Hölder-Zygmund spaces we get the spectral invariance, too. This was confirmed by V.D. Kryakvin in [41]. Let us consider a pseudodifferential operator P whose symbols are in the symbol-class $S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $\delta < 1$ as a linear and bounded operator on weighted Besov spaces or on weighted Triebel-Lizorkin spaces. In [49] H.-G. Leopold and H. Triebel stated that the spectrum of P on these function spaces is the same as the spectrum of P as a linear and bounded operator in $L^2(\mathbb{R}^n)$.

Furthermore, the invariance of the L^p-spectrum for suitable hypoelliptic pseudodifferential operators has been revised by H.-G. Leopold and E. Schrohe in [48]. J.-M. Bony extended the spectral invariance results of R. Beals [16] in [20] for pseudodifferential operators with symbols in the so-called Weyl-Hörmander classes. For the spectral invariance of boundary-value problems we refer to the paper [33] of G. Grubb. The spectral invariance is not only treated for pseudodifferential operators on $\mathbb{R}^n$, but also for pseudodifferential operators on e.g. the unit circle $\mathbb{S}^1$. This was done in [56] written by S. Molahajloo and M.W. Wong.

For an abstract approach to the spectral invariance we refer to the paper [32] of B. Gramsch, J. Ueberberg and K. Wagner.

Thus the spectral invariance is a significant property which many classes of pseudodifferential operators have. However, it already fails in closely related situations, cf. e.g. [26], [30], [31] and [76].

The spectral invariance is mostly verified by means of characterizations of pseudodifferential operators. We distinguish between two different types of characterizations: The C^∞-elements approach of H.O. Cordes in [24] and the characterization via the iterated commutators $\operatorname{ad}(-ix_j)$ and $\operatorname{ad}(D_{x_j})$ developed by R. Beals in [16], [17]. The iterated commutators $\operatorname{ad}(-ix_j)$ and $\operatorname{ad}(D_{x_j})$ are defined by

$$\operatorname{ad}(-ix_j)P := -ix_jP + P(ix_j) \qquad \text{and} \qquad \operatorname{ad}(D_{x_j})P := D_{x_j}P - PD_{x_j}$$

for all linear operators P and all $j \in \{1, \dots, n\}$. M.E. Taylor combined both types to establish a characterization of pseudodifferential operators on the unit sphere S^n in the $(n+1)$-dimensional space in [68]. This turned out to be a simplification of the characterization of pseudodifferential operators on a compact C^∞-manifold M established by H.O. Cordes in [24] in the case $M = S^n$.

However, let us look more closely at the characterization of R. Beals in [16] and [17]: The set of all pseudodifferential operators, whose symbols are in the symbol-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ, is equivalent to the set of all linear operators which satisfy some specific continuity assumptions of their iterated commutators with respect to some weighted Sobolev spaces $H^m(\rho)$. For the definition of $H^m(\rho)$ we refer to [16]. Eleven years later J. Ueberberg extended this characterization for pseudodifferential operators with symbols in the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ in [74]. Here the iterated commutators have to satisfy some specific continuity assumptions with respect to some Bessel potential spaces. His work is based on the characterization of R. Beals in [16], [17] and some methods developed by R.R. Coifman and Y. Meyer in [23] and by H. O. Cordes in [24] and [25]. The results of R. Beals and J. Ueberberg enabled E. Schrohe to prove another similar characterization of pseudodifferential operators whose symbols are in the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ, cf. [64], Theorem 3.3: Whenever the above-mentioned specific continuity conditions for the iterated commutators of a linear operator P holds with respect to the anisotropically weighted Sobolev spaces $H_p^{s,t}(\mathbb{R}^n)$ instead of the Bessel potential space $H_p^s(\mathbb{R}^n)$, then P is a pseudodifferential operator in the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable ρ and δ. Together with H.G. Leopold, E. Schrohe verified in [47] that the same statement is still true if Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ or Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are used instead of anisotropically weighted Sobolev spaces. Here $1 < p < \infty$ and $0 < q \leq \infty$ are fixed but arbitrary. This provides another characterization of the pseudodifferential operators with symbols in the Hörmander class. The choice of $p = q = \infty$ is also possible in the case $\rho = 1$ for Besov spaces. This was confirmed by V.D. Kryakvin in [41]. Thus there is a characterization of pseudodifferential operators whose symbols are in the Hörmander class $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $\delta < 1$ via iterated commutators which have to fulfill certain continuity properties concerning the Hölder-Zygmund spaces $C^s_*(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n)$. Moreover, J.-M. Bony extended the Beals-type characterization for pseudodifferential operators with symbols in the so-called Weyl-Hörmander class in [20]. At last let us mention that a commutator characterization of periodic pseudodifferential operators has already been proved in [73] by V. Turunen.

We have seen, that there are already many characterizations and spectral invariance results in the smooth case. Since the tool of non-smooth pseudodifferential operators becomes more and more a standard technique in the field of nonlinear partial differential equations, such a characterization is also useful in the non-smooth case. One purpose of this dissertation is to establish a characterization of non-smooth pseudodifferential operators via iterated commutators. This is done in Lemma 5.46 and Theorem 5.47. With this characterization at hand we are in the position to show the second goal of this thesis, cf. Theorem 6.12: We verify that the L^q-spectrum of a non-smooth pseudodifferential operator P, whose coefficient is in a Hölder space, is independent of the choice of q under suitable conditions. With P being continuously invertible as a linear operator on certain Bessel potential spaces, the inverse of P is a non-smooth pseudodifferential operator whose symbol is in the same symbol-class under suitable conditions. In order to reach these goals we make use of the central ideas of the analogous results of J. Ueberberg [74] and R. Beals [16] in the smooth case. The main new difficulties one is confronted with are the limited mapping properties of pseudodifferential operators with non-smooth symbols.

Let us give an outline of this dissertation:

We start with presenting the mathematical basics needed in this thesis in Chapter 2. We establish some general conventions like the notation of some frequently used sets first. Section 2.1 is dedicated to fix the notation concerning some familiar function spaces and often used functions. In Section 2.2 we give a short introduction to a standard tool in the theory of function spaces:

the dyadic partition of unity. In the next three sections we focus on the investigation of the Schwartz space, the space of tempered distributions, the Hölder spaces, the Hölder-Zygmund spaces and the Bessel potential spaces. In Chapter 5 we characterize non-smooth pseudodifferential operators with certain mapping properties of their iterated commutators. Therefore it is the task of Section 2.6 to introduce the so-called iterated commutators of a linear operator. The just-mentioned characterization is proved by means of a kernel theorem, which we present in Section 2.7.

Having treated the mathematical background we turn towards the theory of smooth pseudodifferential operators in Chapter 3. In Section 3.1 we define the most common symbol-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, which is called Hörmander class, and its associated pseudodifferential operators. Moreover, we show some first results on the way to the field of pseudodifferential operators. We often restrict ourselves to the case $\rho = 1$ and $\delta = 0$ since it is the important one for us. An interesting question is whether the composition of two pseudodifferential operators is a pseudodifferential operator again. This is the topic of Section 3.2. After that we present a kernel representation of smooth pseudodifferential operators in Section 3.3. Finally, we focus on some boundedness statements of pseudodifferential operators in Section 3.4. Such boundedness results are essential for applications. For instance we prove a characterization of the Bessel potential spaces.

Chapter 4 is devoted to the investigation of observations concerning non-smooth pseudodifferential operators which serve as ingredients for the characterization of non-smooth pseudodifferential operators and spectral invariance in Chapter 5 and Chapter 6. As in the smooth case we begin with the introduction of the non-smooth symbol-classes needed later on and their associated non-smooth pseudodifferential operators in Section 4.1. These are the most common non-smooth symbol-class with coefficients in the Hölder spaces (Subsection 4.1.1), the non-smooth symbol-class with coefficients in the uniformly local Sobolev space (Subsection 4.1.3) and the non-smooth symbol-class with coefficients in Bessel potential spaces (Subsection 4.1.4). Having not treated the uniformly local Sobolev spaces yet, we present the basic properties of these spaces in Subsection 4.1.2. In applications in the field of partial differential equations many pseudodifferential operators are classical ones. Hence the restriction to the so-called classical pseudodifferential operators is not a big disadvantage. Working with classical pseudodifferential operators is mostly much easier. Consequently classical symbol-classes are introduced in Subsection 4.1.5. The main goals of this chapter is to prove a kernel representation (Section 4.5) and the most important mapping properties of non-smooth pseudodifferential operators (Section 4.4). Moreover, we spend time on the composition of two non-smooth pseudodifferential operators in Section 4.3. In contrast to the smooth case, the composition of non-smooth pseudodifferential operators is in general not a pseudodifferential operator with the same regularity with respect to its coefficient. However, there is an asymptotic expansion for the composition of non-smooth pseudodifferential operators. In the smooth case the oscillatory integral

$$\mathrm{Os}\text{-}\iint e^{iy\cdot\eta} f(y,\eta)dyd\eta := \lim_{\varepsilon\to 0}\iint \chi(\varepsilon y;\varepsilon\eta)e^{iy\cdot\eta} f(y,\eta)dyd\eta$$

served for many purposes as auxiliary tool, where χ has to be a rapidly decreasing smooth function. The oscillatory integral is well-defined for all functions in the space of amplitudes, the set of all smooth functions of polynomial growth, to non-smooth functions such that the oscillatory integral is well-defined for all functions of this extension. Hence the topic of Section 4.2 is to extend the space of amplitudes. We also convince ourselves that the properties of the oscillatory integral even hold for these functions. While verifying the characterization of smooth pseudodifferential operators one is confronted with the task to reduce a double symbol to a single symbol. In order to obtain a characterization of non-smooth pseudodifferential operators in an analogous way as in the smooth case, we introduce non-smooth double symbols in Section 4.6.

The main purpose of Chapter 5 is to verify a characterization via iterated commutators for non-smooth pseudodifferential operators with symbols in the class $C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n;M)$, $\rho=1$. This is done in Section 5.5. We will see that the symbol has to be sufficiently smooth in the second variable. In analogy to the proof of J. Ueberberg in the smooth case we reduce this statement to the characterization of non-smooth pseudodifferential operators whose symbols are in the class $C^{\tilde{m},s}S^m_{0,0}(\mathbb{R}^n\times\mathbb{R}^n;M)$. Details for deriving this result are explained in Section 5.4. The first three sections of Chapter 5 serve to develop some auxiliary tools needed for the proof of this statement. In Section 5.1 we start with showing that a bounded sequence in $C^{\tilde{m},s}S^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n;M)$ has a convergent subsequence in the symbol-class $C^{\tilde{m},s}S^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n;M-1)$. Section 5.2 is devoted to the symbol reduction of non-smooth double symbols to non-smooth single symbols. Details for the third tool are proved in Section 5.3: There a family of operators $(T_\varepsilon)_{\varepsilon\in(0,1]}$ fulfilling the following three properties is constructed: $T_\varepsilon:\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is continuous for all $\varepsilon\in(0,1]$ and converges pointwise if $\varepsilon\to 0$. Moreover, all iterated commutators of T_ε are uniformly bounded with respect to ε as maps from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. With this auxiliary tool at hand it is possible to show the characterization in the non-smooth case. We are even able to improve this characterization in Section 5.6: Linear operators which satisfy some specific continuity assumptions of their iterated commutators are not only non-smooth pseudodifferential operators whose symbols are in the symbol-class $C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n;M)$, $\rho\in\{0,1\}$, but even non-smooth pseudodifferential operators with coefficients in an uniformly local Sobolev space. Section 5.7 is devoted to an illustration of the usefulness of such a characterization: We show that the composition PQ of two non-smooth pseudodifferential operators P and Q is a non-smooth pseudodifferential operator again, if Q is smooth enough.

This is done by means of the characterization of non-smooth pseudodifferential operators.

Chapter 6 is devoted to the study of the inverse of a non-smooth pseudodifferential operator P whose symbol is in the symbol-class $C^\tau S^0_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$, where $\rho \in \{1, 0\}$. In analogy to the proof of J. Ueberberg in the smooth case, we use the characterization of pseudodifferential operators via iterated commutators in order to show that the inverse of P is also a pseudodifferential operator provided suitable conditions are given. In Section 6.1 we derive this result for a non-smooth pseudodifferential operator P with symbol in the symbol-class $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$. We show that P^{-1} is also a non-smooth pseudodifferential operator whose symbol is in the symbol-class $C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$, where $s < \tau$. It turns out that some smoothness of the coefficients is lost in contrast to the smooth case. Our next goal is to verify the spectral invariance of non-smooth pseudodifferential operators whose symbol is in the symbol-class $C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ for sufficiently large N. To be more precise, we arrive at the following statement, cf. Theorem 6.12: The inverse of a non-smooth pseudodifferential operator of the order zero with coefficients in the Hölder space $C^{\tilde{m},\tau}(\mathbb{R}^n)$ is also a non-smooth pseudodifferential operator if its inverse is an element of $\mathscr{L}(H^r_q(\mathbb{R}^n))$ for one $|r| < \tilde{m} + \tau$. Hence the L^q-spectrum of P is independent of $q \in (\tilde{q}, \infty)$ for sufficiently large $\tilde{q}$. This is the topic of Section 6.3. Beyond the characterization of non-smooth pseudodifferential operators we also use the technique of difference quotients for the proof of the above mentioned statement. We introduce this technique in Section 6.2. We improve the results of Section 6.3 in Section 6.4 for non-smooth pseudodifferential operators of the order zero with coefficients in the uniformly local Sobolev spaces. Here we consider merely symbols which are smooth in ξ, as in regularity applications the symbols are usually polynomials in ξ. It turns out that we even get a better result for some subsets of the set of all non-smooth pseudodifferential operators with coefficients in the uniformly local Sobolev spaces.

Appendix A serves to prove an easy consequence of the basic results in the topics of measure theory needed in this thesis. Additionally we introduce the Banach space valued Sobolev and Hölder spaces and present those properties of these spaces needed in this work in Appendix B. Finally, Appendix C is devoted to the proof of an interpolation result for Hölder spaces needed in this thesis.

Chapter 2

Preliminaries

The present chapter serves to introduce the notation and the mathematical basics for this thesis. We establish some general conventions like the notation of some frequently used sets first. Section 2.1 is dedicated to fix the notation concerning some familiar function spaces and often used functions. In Section 2.2 we give a short introduction to a standard tool in the theory of function spaces: the dyadic partition of unity. In the next three sections we focus on the investigation of the Schwartz space, the space of tempered distributions, the Hölder spaces, the Hölder-Zygmund spaces and the Bessel potential spaces. We will characterize non-smooth pseudodifferential operators with certain mapping properties of their iterated commutators in Chapter 5. Therefore it is the task of Section 2.6 to introduce the so-called iterated commutators of a linear operator. The just-mentioned characterization will be proved by means of a kernel theorem, which we present in Section 2.7. The present chapter is mainly based on [5] and [75].

Regarding constants appearing in estimates we adopt the following convention: All constants are denoted by C. Indices on constants, for instance C_α, indicate that they depend on other variables. This convention is chosen for all inequalities, that means, even if constants change from line to line, their notation is kept fix. Furthermore, $\mathbb{N}$ is the set of all natural numbers without zero, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The set of all integers is denoted by $\mathbb{Z}$, the set of all real numbers by $\mathbb{R}$ and the set of all complex numbers by $\mathbb{C}$. All positive real numbers belong to the set $\mathbb{R}^+$. Additionally we denote $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. For $r > 0$ and $x_0 \in \mathbb{R}^n$ we set

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

as the open ball of radius r around x_0. Here $|.|$ denotes the Euclidean length. During the whole thesis, we consider $n \in \mathbb{N}$ except when stated otherwise. Considering $x \in \mathbb{R}$ we define

$$x^+ := \max\{0; x\} \qquad \text{and} \qquad \lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}.$$

Partial derivatives with respect to a variable $x \in \mathbb{R}$ are denoted by ∂_x. We use the shorter convention ∂_x^m, $m \in \mathbb{N}_0$, if we apply the partial derivative with respect to x for m times. The divergence with respect to $x \in \mathbb{R}^n$ is denoted by ∇_x. Additionally we scale partial derivatives with respect to a variable $x \in \mathbb{R}^n$ with the factor $-i$ and denote it by

$$D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and is called *multi-index.* The *length* $|\alpha|$ of the multi-index α is defined by $|\alpha| := \alpha_1 + \dots + \alpha_n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ we define $x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Then x^α is a polynomial of degree $|\alpha|$. We write $\alpha \leq \beta$ for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ if $\alpha_i \leq \beta_i$ for every $i \in \{1, \dots, n\}$. The summation with respect to all muli-indices α with $|\alpha| \leq m$, $m \in \mathbb{N}_0$, is denoted by $\sum_{|\alpha|\leq m}$. For arbitrary $j \in \{1, \dots, n\}$, $e_j \in \mathbb{N}_0^n$ is defined as the vector which has only one non-zero component:

$$e_j^t = (0, \dots, 0, \underset{\substack{\uparrow \\ \text{j-th component}}}{1}, 0, \dots, 0).$$

If the integration area is the whole Euclidean space $\mathbb{R}^n$, we often skip the integration area and write $\int$ instead of $\int\limits_{\mathbb{R}^n}$.

For two Banach spaces X, Y the set $\mathscr{L}(X, Y)$ consists of all linear and bounded operators $A : X \to Y$. We also write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$. Furthermore, $\mathrm{GL}(n)$ is the set of all invertible $n \times n$-matrices.

We finally note that the dual space of a topological vector space V is denoted by V'. If V is even a Banach space the duality product V is denoted by $\langle ., . \rangle_{V;V'}$.

2.1 Functions on $\mathbb{R}^n$

In this section we fix conventions for well-known function spaces and frequently used functions. In particular we present some properties of the Fourier transformation.

During the whole work we adopt the following notations for all $k \in \mathbb{N}_0$, $1 \leq q < \infty$ and each open set $\Omega \subseteq \mathbb{R}^n$:

- $C^k(\overline{\Omega}) := \{f : \Omega \to \mathbb{C} : f$ is k-times continuously differentiable and for all $\alpha \in \mathbb{N}_0^n, |\alpha| \leq k$ there is a continuous extension of $\partial_x^\alpha f$ on $\overline{\Omega}\}$,
- $C^\infty(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} : f$ is smooth$\}$,

- $C_b^k(\mathbb{R}^n) := \{f \in C^k(\mathbb{R}^n) : \partial_x^\alpha f$ is bounded for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k\}$,
- $C_b^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \partial_x^\alpha f$ is bounded for all$\alpha \in \mathbb{N}_0^n\}$,
- $C_c^k(\mathbb{R}^n) := \{f \in C^k(\mathbb{R}^n) :$ supp f is a compact subset of $\mathbb{R}^n\}$,
- $C_c^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) :$ supp f is a compact subset of $\mathbb{R}^n\}$,
- $L^q(\Omega) := \Big\{f$ is measurable $: \|f\|_{L^q(\Omega)} := \big[\int_\Omega |f(x)|^q dx\big]^{1/q} < \infty\Big\}$,
- $W_q^k(\Omega) := \{f \in L^q(\Omega) : \partial_x^\alpha f \in L^q(\Omega)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k\}$.

Here the support of f is defined by supp $f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$. For all $1 \leq q < \infty$ the spaces $L^q(\Omega)$ are Banach-spaces. In particular $L^2(\Omega)$ is even a Hilbert space. We denote the scalar product of this space by

$$(u, v)_{L^2(\Omega)} := \int_\Omega u(x)\overline{v(x)}dx \qquad \text{for all } u, v \in L^2(\Omega).$$

If $\Omega = \mathbb{R}^n$, we also write L^q instead of $L^q(\mathbb{R}^n)$. .

For every $k \in \mathbb{N}_0$, $1 \leq q < \infty$, all open sets $\Omega \subseteq \mathbb{R}^n$ and all open and bounded sets $\Omega_1 \subseteq \mathbb{R}^n$ the spaces $C_b^k(\mathbb{R}^n)$, $C^k(\overline{\Omega}_1)$ and the Sobolev spaces $W_q^k(\Omega)$ are Banach spaces which can be normed by

$$\begin{aligned}
\|f\|_{C_b^k} &:= \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha f(x)| && \text{for all } f \in C_b^k(\mathbb{R}^n),\\
\|f\|_{C^k(\overline{\Omega}_1)} &:= \max_{|\alpha|\leq k} \sup_{x\in\overline{\Omega}_1} |\partial_x^\alpha f(x)| && \text{for all } f \in C^k(\overline{\Omega}_1),\\
\|f\|_{W_q^k(\Omega)} &:= \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_{L^q(\Omega)} && \text{for all } f \in W_q^k(\Omega).
\end{aligned}$$

Additionally $C_c^k(\mathbb{R}^n)$ can be normed with the norm

$$\|f\|_{C_c^k} := \|f\|_{C_b^k} \qquad \text{for all } f \in C_c^k(\mathbb{R}^n).$$

Since this space is not complete, $C_c^k(\mathbb{R}^n)$ is not a Banach space. Moreover, $C_b^\infty(\mathbb{R}^n)$ can be considered as a Fréchet space with respect to the semi-norms

$$|f|_{k,C_b^\infty} := \|f\|_{C_b^k} \qquad \text{for all } f \in C_b^\infty(\mathbb{R}^n) \text{ and } k \in \mathbb{N}_0.$$

For a short introduction into the theory of Fréchet spaces we recommend [5], Section A.5. For more details see e.g. [70] or [37].

We choose open and bounded sets $\Omega_i \subseteq \mathbb{R}^n$, $i \in \mathbb{N}_0$ such that $\Omega_i \subseteq \Omega_{i+1}$ for all $i \in \mathbb{N}_0$ and $\bigcup_{i=0}^{\infty} \Omega_i = \mathbb{R}^n$. Then the spaces $C^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$ are Fréchet spaces with the semi-norms

$$|f|_{i,C^k} := \|f\|_{C^k(\overline{\Omega}_i)} \qquad \text{for all } i \in \mathbb{N}_0 \text{ and } f \in C^k(\mathbb{R}^n).$$

Similarly $C^\infty(\mathbb{R}^n)$ can be considered as a Fréchet space with respect to the semi-norms $(|f|_{i,C^k})_{i,k\in\mathbb{N}_0}$. We equip the space $C_c^\infty(\mathbb{R}^n)$ with the topology which is induced by the semi-norms

$$|f|_{k,C_c^\infty} := \|f\|_{C_c^k} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n) \text{ and } k \in \mathbb{N}_0.$$

Note that it is not a Fréchet space since this space is not complete. We define the convergence in $C_c^\infty(\mathbb{R}^n)$ in the following way: The sequence $(f_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ converges to $f \in C_c^\infty(\mathbb{R}^n)$ if and only if there is a compact set $K \subseteq \mathbb{R}^n$ such that

$$\operatorname{supp} f_k, \operatorname{supp} f \subseteq K \text{ for all } k \in \mathbb{N} \qquad \text{and} \qquad f_k \xrightarrow{k\to\infty} f \text{ in } C^\infty(\mathbb{R}^n).$$

For elements $f : \mathbb{R}^n \to \mathbb{C}$ of a certain function space, e.g. $f \in L^1(\mathbb{R}^n)$, we also write $f(x) \in L^1(\mathbb{R}^n_x)$. Additionally the convolution between two functions $f, g : \mathbb{R}^n \to \mathbb{C}$ is denoted by

$$f * g(x) := \int f(y)g(x-y)dy \qquad \text{for all } x \in \mathbb{R}^n.$$

Since we often use the translation function of functions in $L^1(\mathbb{R}^n)$, we introduce the following notation:

Notation 2.1. For $g \in L^1(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ we define the translation function $\tau_y(g) : \mathbb{R}^n \to \mathbb{C}$ as

$$\tau_y(g)(x) := g(x-y) \qquad \text{for all } x \in \mathbb{R}^n.$$

A key role in the theory of pseudodifferential operators has the Fourier transformation $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$. These functions are defined by

$$\hat{f}(\xi) := \mathscr{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx \qquad \text{for all } f \in L^1(\mathbb{R}^n) \text{ and all } \xi \in \mathbb{R}^n,$$

$$\mathscr{F}^{-1}[f](x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)đ\xi \qquad \text{for all } f \in L^1(\mathbb{R}^n) \text{ and all } x \in \mathbb{R}^n,$$

where $đ\xi := (2\pi)^{-n}d\xi$. During the whole thesis all integrals with respect to a *phase variable* $\xi, \eta, \ldots$ are taken with respect to the scaled Lebesgue measure $đ\xi, đ\eta, \ldots$, while the usual Lebesgue measure is used for the integration with respect to a *space variable* $x, y, z \ldots$.

Some important properties of the Fourier transformation are:

Theorem 2.2. *Let $j \in \{1, \dots, n\}$. Then*

i) $\|\hat{f}\|_{C_b^0} \le \|f\|_{L^1}$ *for all* $f \in L^1(\mathbb{R}^n)$.

ii) For every continuously differentiable function $f : \mathbb{R}^n \to \mathbb{C}$ *with the property* $f, \partial_{x_j} f \in L^1(\mathbb{R}^n)$ *we obtain*

$$\mathscr{F}[\partial_{x_j} f](\xi) = i\xi_j \hat{f}(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

iii) For $f \in L^1(\mathbb{R}^n)$ *with* $x_j f(x) \in L^1(\mathbb{R}^n_x)$ *the function* $\hat{f}$ *is the continuous partial differentiable with respect to the* j*-th component and*

$$\partial_{\xi_j} \hat{f} = \mathscr{F}[-ix_j f(x)].$$

iv) Let $f \in L^1(\mathbb{R}^n)$ *and* $y \in \mathbb{R}^n$. *Then we have for each* $\xi \in \mathbb{R}^n$:

$$\mathscr{F}[\tau_y(f)](\xi) = e^{-iy\cdot\xi} \hat{f}(\xi).$$

v) Let $f \in L^1(\mathbb{R}^n)$ *and let* $(\rho_\varepsilon f)(x) := f(\varepsilon x)$, $\varepsilon > 0$, *denote the dilation of* f *by* ε. *Then we get for each* $\xi \in \mathbb{R}^n$:

$$\mathscr{F}[\rho_\varepsilon f](\xi) = \varepsilon^{-n} \left(\rho_{\varepsilon^{-1}} \hat{f}\right)(\xi).$$

vi) If $f, g \in L^1(\mathbb{R}^n)$, *then we obtain for every* $\xi \in \mathbb{R}^n$:

$$\hat{f}(\xi)\hat{g}(\xi) = \mathscr{F}[f * g](\xi).$$

We refer to [5], Theorem 2.1 for the proof.

Since $\mathscr{F}^{-1}[f](x) = (2\pi)^n \mathscr{F}[f](-x)$ for all $f \in L^1(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$, the statements of the previous theorem also hold for $\mathscr{F}^{-1}$ instead of $\mathscr{F}$ with minor modifications.

We warmly recommend [1], [5], Chapter 2 and the references given there for a good introduction in the theory of the Fourier transformation.

The term $\sqrt{1+|\xi|^2}$ is often needed while working with pseudodifferential operators. Hence for short we write:

Notation 2.3. For $\xi \in \mathbb{R}^n$ we set

$$\langle \xi \rangle := \sqrt{1+|\xi|^2}.$$

Let us verify some useful estimates of this term. To begin with, we present Peetre's inequality now:

Lemma 2.4. *Let $m \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$. Then the following inequalities hold:*

i) $\langle \xi + \eta \rangle^m \leq 2^{|m|} \langle \xi \rangle^m \langle \eta \rangle^{|m|}$ ***(Peetre's inequality)****,*

ii) $\langle \xi \rangle \leq (1 + |\xi|) \leq \sqrt{2} \langle \xi \rangle$.

For the proof see e.g. [5], Lemma 3.7.

As a consequence of Peetre's inequality, we obtain:

Corollary 2.5. *For all $m \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$ there exists a constant $C_{\eta,m}$, which is independent of $\xi \in \mathbb{R}^n$, such that*

$$\langle \xi + \eta \rangle^m \leq C_{\eta,m} \langle \xi \rangle^m.$$

Proof: Defining $C_{\eta,m} := 2^{|m|} \langle \eta \rangle^{|m|}$ the corollary is a direct consequence of Peetre's inequality, cf. Lemma 2.4. □

Moreover, we are able to calculate the next estimate:

Remark 2.6. Considering $\xi, \eta \in \mathbb{R}^n$, we get

$$\langle \xi + \eta \rangle \leq \langle \xi \rangle + |\eta|.$$

Proof: Let $\xi, \eta \in \mathbb{R}^n$ be arbitrary. Then we have

$$\begin{aligned} \langle \xi + \eta \rangle^2 &= 1 + |\xi + \eta|^2 \leq 1 + |\xi|^2 + 2|\xi||\eta| + |\eta|^2 \leq \langle \xi \rangle^2 + 2\langle \xi \rangle |\eta| + |\eta|^2 \\ &\leq \left(\langle \xi \rangle + |\eta| \right)^2. \end{aligned}$$

□

Again Peetre's inequality provides:

Lemma 2.7. *For $\xi \in \mathbb{R}^n, \eta \in \mathbb{R}^k$ and $m \geq 0$ there is a constant C_m, independent of ξ, η, such that*

$$\langle \xi \rangle^{-m} \langle \eta \rangle^{-m} \leq C_m \langle (\xi, \eta) \rangle^{-m}.$$

Proof: Using Peetre's inequality for $m \geq 0$ and $(\xi, \eta) = (\xi, 0) + (0, \eta)$, we get the claim at once. □

A useful estimate for all derivatives of $\langle x \rangle^s$ with respect to x is presented in the following remark. We refer to e.g. [5], Exercise 2.51 for the proof.

Remark 2.8. Let $\alpha \in \mathbb{N}_0^n$, $s \in \mathbb{R}$. Then

$$|D_x^\alpha \langle x \rangle^s| \leq C_{s,\alpha} \langle x \rangle^{s-|\alpha|} \qquad \text{for all } x \in \mathbb{R}^n$$

for some constants $C_{s,\alpha}$.

Next we expand Notation 2.3 to derivatives:

Notation 2.9. For $f \in C^{2l}(\mathbb{R}^n)$ with $l \in \mathbb{N}_0$ we set

$$\langle D_x \rangle^{2l} f := \left(1 + \sum_{j=1}^{n} D_{x_j}^2\right)^l f.$$

We now state first properties of $\langle D_x \rangle^{2l}$ with $l \in \mathbb{N}_0$:

Remark 2.10. Let $l \in \mathbb{N}_0$. Then we have for some constants $a_{\alpha,l} \in \mathbb{N}_0$:

i) $\langle D_x \rangle^{2l} = \sum_{|\alpha| \le l} a_{\alpha,l} D_x^{2\alpha}$,

ii) $e^{-ix\cdot\xi} = \langle \xi \rangle^{-2l} \langle D_x \rangle^{2l} e^{-ix\cdot\xi}$ for all $x, \xi \in \mathbb{R}^n$.

Proof: The claim follows directly from the definition of $\langle D_x \rangle^2$. □

A frequently used ingredient for verifying boundedness results of pseudo-differential operators is described in the next theorem. We refer to e.g. [5], Lemma A.9 or [59], Theorem 1.3 for the proof.

Theorem 2.11. *Let $s > n$. Then $\langle x \rangle^{-s} \in L^1(\mathbb{R}^n_x)$ and $(1 + |x|)^{-s} \in L^1(\mathbb{R}^n_x)$.*

We also need the following estimate later on:

Lemma 2.12. *Assuming $f \in C^1(\mathbb{R}^n)$, the following estimate holds:*

$$|f(x) - f(y)| \le \sup_{0 \le t \le 1} |Df((1-t)x + ty)||x - y| \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Proof: Using the fundamental theorem of calculus, one immediately gets:

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_0^1 \frac{d}{dt} \{f((1-t)x + ty)\}\, dt \right| \le \int_0^1 |(Df)((1-t)x + ty)|\, |x - y|\, dt \\ &\le \sup_{0 \le t \le 1} |Df((1-t)x + ty)||x - y|. \end{aligned}$$

□

2.2 Partitions of Unity

One of the well-known tools in the theory of function spaces is the dyadic partition of unity. This tool especially turned out to be useful when studying the mapping properties of certain operators. There a partition of unity was often able to provide the link between a local result and a more generalized, global result. We refer to [38], Chapter II.10 for a short introduction to this theory.

We start this section with the definition of a partition of unity on $\mathbb{R}^n$:

Definition 2.13. A *partition of unity* on $\mathbb{R}^n$ is a family of continuous functions $\varphi_j : \mathbb{R}^n \to [0,1]$, $j \in \mathbb{N}_0$ with the properties:

- For each $x \in \mathbb{R}^n$ there is a neighbourhood U_x of x such that $\varphi_j|_{U_x} \equiv 0$ for all except a finite number of $j \in \mathbb{N}_0$.
- $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

A dyadic partition of unity is a partition of unity with an additional property:

Definition 2.14. A *dyadic partition of unity* is a partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ on $\mathbb{R}^n$ with the properties

$$\text{supp } \varphi_0 \subseteq \overline{B_2(0)} \quad \text{and} \quad \text{supp } \varphi_j \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\} \tag{2.1}$$

for all $j \in \mathbb{N}$.

A dyadic partition of unity can be constructed in the following way: We take $\varphi_0 \in C^\infty(\mathbb{R}^n)$ with $\varphi_0(\xi) = 1$ for all $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ for $|\xi| \geq 2$. Then we set $\varphi_j(\xi) := \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$ for all $\xi \in \mathbb{R}^n$ and $j \in \mathbb{N}$. Consequently condition (2.1) holds and for a fixed but arbitrary $\xi \in \mathbb{R}^n$ we obtain

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = \sum_{j=0}^{k} \varphi_j(\xi) = \varphi_0(2^{-k}\xi) = 1,$$

where $k = \max\{l \in \mathbb{N}_0 : \xi \in \text{supp } \varphi_l\}$.

Later we will need a partition of unity $(\psi_j)_{j\in\mathbb{Z}^n}$ where each ψ_j is a translation of the function ψ_0 and the support of ψ_0 is contained in a cube. Such a partition of unity exists as the next lemma shows:

Lemma 2.15. *There is a partition of unity $(\psi_j)_{j\in\mathbb{Z}^n}$ such that for an $a \in \mathbb{R}^+$ and every $j \in \mathbb{Z}^n$ we have*

- $\psi_j \in C_c^\infty(\mathbb{R}^n)$,
- $\psi_j(x) = \psi_0(x - j)$ *for all* $x \in \mathbb{R}^n$,
- *supp* $\psi_0 \subseteq [-a, a]^n$.

Proof: Let $0 < \varepsilon < 1$. The characteristic function $\chi_{[-1,1]} : \mathbb{R} \to \mathbb{R}$ of the interval $[-1,1]$ is defined by $\chi_{[-1,1]}(x) := 1$ for all $x \in [-1,1]$ and $\chi_{[-1,1]}(x) := 0$ else. Taking $\varphi \in C_c^\infty(\mathbb{R})$ with supp $\varphi = [-\varepsilon, \varepsilon]$ and $\varphi(x) > 0$ for all $x \in (-\varepsilon, \varepsilon)$, we define the function $\eta_0 : \mathbb{R} \to \mathbb{R}$ by

$$\eta_0(x) := \left(\varphi * \chi_{[-1,1]}\right)(x) := \int \varphi(y)\chi_{[-1,1]}(x-y)dy \qquad \text{for all } x \in \mathbb{R}.$$

The properties of the convolution and the non-negativity of φ provides that $\text{supp}\ \eta_0 \subseteq \{x+y : x \in \text{supp}\ \varphi, y \in \text{supp}\ \chi_{[-1,1]}\} \subseteq [-1-\varepsilon, 1+\varepsilon]$ and that η_0 is a non-negative function, cf. e.g. [38], Theorem 107. Since φ is a non-negative function, which is even positive on $(-\varepsilon, \varepsilon)$, we obtain for all $x \in [-1,1]$:

$$\eta_0(x) = \int_{\text{supp}\ \varphi} \varphi(y)\chi_{[-1,1]}(x-y)dy = \int_{\text{supp}\ \varphi \cap [-1+x, 1+x]} \varphi(y)dy > 0.$$

Now we set for each $j \in \mathbb{Z}$ the functions $\eta_j, \Phi : \mathbb{R} \to \mathbb{R}_0^+$ by

- $\eta_j(x) := \eta_0(x-j) \qquad \text{for all } x \in \mathbb{R},$
- $\Phi(x) := \sum_{j\in\mathbb{Z}} \eta_j(x) \qquad \text{for all } x \in \mathbb{R}.$

By means of the definition of the functions η_j, $j \in \mathbb{Z}$, the previous sum is finite for every $x \in \mathbb{R}$. Additionally the definition yields for all $j \in \mathbb{Z}$ that the non-negative function $\eta_j(x)$ is positive for each $x \in [-1+j, 1+j]$. Hence for every $x \in \mathbb{R}$ one can find a $j \in \mathbb{Z}$ such that $\eta_j(x) > 0$. Therefore Φ is a positive function. We also get the following translation property for arbitrary $k \in \mathbb{Z}$:

$$\Phi(x+k) = \sum_{j\in\mathbb{Z}} \eta_0(x+k-j) = \sum_{j\in\mathbb{Z}} \eta_{j-k}(x) = \sum_{j\in\mathbb{Z}} \eta_j(x) = \Phi(x) \quad \text{for all } x \in \mathbb{R}^n.$$

The positivity of Φ enables us to define for all $j \in \mathbb{Z}$ the functions $\tilde{\psi}_j : \mathbb{R} \to \mathbb{R}_0^+$ by

$$\tilde{\psi}_j(x) := \frac{\eta_j(x)}{\Phi(x)} \qquad \text{for all } x \in \mathbb{R}.$$

Therefore the support of $\tilde{\psi}_j \in C_c^\infty(\mathbb{R})$ is a subset of $[-1-\varepsilon+j, 1+\varepsilon+j]$. Moreover, $(\tilde{\psi}_j)_{j\in\mathbb{Z}}$ is a partition of unity. Now we define for all $j \in \mathbb{Z}^n$ the functions $\psi_j : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi_j(x) := \prod_{i=1}^n \tilde{\psi}_{j_i}(x_i) \qquad \text{for all } x \in \mathbb{R}^n.$$

Then the properties of the functions $\tilde{\psi}_k$, $k \in \mathbb{N}$ provide that ψ_j is a non-negative function, $\psi_j \in C_c^\infty(\mathbb{R}^n)$, $\text{supp}\ \psi_0 \in [-1-\varepsilon, 1+\varepsilon]^n$ and $\psi_j(x) = \psi_0(x-j)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}^n$. Furthermore, we get for each $x \in \mathbb{R}^n$:

$$\sum_{j\in\mathbb{Z}^n} \psi_j(x) = \sum_{j\in\mathbb{Z}^n} \prod_{i=1}^n \tilde{\psi}_{j_i}(x_i) = \prod_{i=1}^n \sum_{j\in\mathbb{Z}^n} \tilde{\psi}_{j_i}(x_i) = 1.$$

Hence $(\psi_j)_{j\in\mathbb{Z}^n}$ is a partition of unity, which fulfills all required properties. □

Our next goal is to introduce some function spaces which play a central role during this thesis.

2.3 The Schwartz Space and its Dual Space

This section is devoted to the introduction of the Schwartz space, the space of all rapidly decreasing smooth functions, and its dual space. We discuss several conditions for a function being an element of the Schwartz space. In particular we look at the properties of the Fourier transformation on the Schwartz space and its dual space. After showing some convergence results for rapidly decreasing smooth functions, we show two continuous embedding theorems needed later on. We start with the definition of the Schwartz space:

Definition 2.16. The space of all *rapidly decreasing smooth functions* $\mathcal{S}(\mathbb{R}^n)$ is the set of all smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for every $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ there is a constant $C_{\alpha,N}$ such that

$$|\partial_x^\alpha f(x)| \leq C_{\alpha,N}(1+|x|)^{-N} \qquad \text{for all } x \in \mathbb{R}^n.$$

Another name for $\mathcal{S}(\mathbb{R}^n)$ is *Schwartz space.* A function $f \in \mathcal{S}(\mathbb{R}^n)$ is also called *Schwartz function.* For $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathbb{N}_0$, we define the semi-norm:

$$|f|_{m,\mathcal{S}} := \sup_{|\alpha|+|\beta|\leq m} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|.$$

An important property of the Schwartz space is that for arbitrary Schwartz functions f and g the function $f \cdot g$ is an element of the Schwartz space, too. We get this statement from the following lemma, which is proved e.g. in [5], Lemma 2.5:

Lemma 2.17. *Let $C^\infty_{poly}(\mathbb{R}^n)$ be the set of all smooth polynomially bounded functions, i.e. the set of all smooth $f : \mathbb{R}^n \to \mathbb{C}$ such that for all $\alpha \in \mathbb{N}_0^n$ there exist some $m_\alpha \in \mathbb{N}_0$ and $C_\alpha > 0$ with*

$$|\partial_x^\alpha f(x)| \leq C_\alpha(1+|x|)^{m_\alpha} \qquad \textit{for all } x \in \mathbb{R}^n.$$

Then for every $f \in C^\infty_{poly}(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$ we have $f \cdot g \in \mathcal{S}(\mathbb{R}^n)$.

Additionally we have $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n)$. Note that $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space with respect to the semi-norms $|.|_{m,\mathcal{S}}$, $m \in \mathbb{N}_0$, cf. e.g. [5], Lemma A.20. Another set of semi-norms $(|.|'_{m,\mathcal{S}})_{m\in\mathbb{N}_0}$ on the Schwartz space is denoted by

$$|f|'_{m,\mathcal{S}} := \sup_{k+|\beta|\leq m} \sup_{x\in\mathbb{R}^n} (1+|x|)^k |\partial_x^\beta f(x)| \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n) \text{ and } m \in \mathbb{N}_0.$$

This set of semi-norms is equivalent to $(|.|_{m,\mathcal{S}})_{m\in\mathbb{N}_0}$ in the following sence: For each $m \in \mathbb{N}_0$, there is a $k(m) \in \mathbb{N}_0$ and two constants $C_m, C'_m > 0$, such that

$$|f|'_{m,\mathcal{S}} \leq C_m |f|_{k(m),\mathcal{S}} \qquad \text{and} \qquad |f|_{m,\mathcal{S}} \leq C'_m |f|'_{k(m),\mathcal{S}} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

In particular the embedding $\mathrm{Id} : C_c^\infty(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous. One immediately gets this by using the fact that $C_c^\infty(\mathbb{R}^n)$ is equipped with the semi-norms $(|.|_m)_{m\in\mathbb{N}_0}$, where

$$|f|_m := \sup_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha f(x)| \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

Similarly we can show for every $k \in \mathbb{N}_0$ the continuity of the embedding

$$\mathrm{Id} : \mathcal{S}(\mathbb{R}^n) \hookrightarrow C_b^k(\mathbb{R}^n).$$

The Fourier transformation of a Schwartz function is a Schwartz function again as we see in the next lemma:

Lemma 2.18. *The Fourier transformation $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear mapping. Moreover, for all $m \in \mathbb{N}_0$ there is a $C_m > 0$ such that*

$$|\hat{f}|_{m,\mathcal{S}} \le C_m |f|_{m+n+1,\mathcal{S}} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

Hence $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is bounded. Moreover, $\mathscr{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism with inverse $\mathscr{F}^{-1}$.

For the proof see e.g. [5], Lemma 2.7. and Lemma 2.9.

We also mention Plancherel's Theorem since it is often an ingredient for proving statements.

Theorem 2.19 (Plancherel's Theorem)**.**
For every $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\int f(x)\overline{g(x)}dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}đ\xi.$$

In particular $\mathscr{F}$ extends to a linear isomorphismus $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

We refer to e.g [5], Theorem 2.11 for the proof.

Fixing one variable of a Schwartz function with two variables does not change being an element of the Schwartz space:

Remark 2.20. Let $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $y \in \mathbb{R}^n$. We define $\tilde{\chi} : \mathbb{R}^n \to \mathbb{C}$ and $\hat{\chi} : \mathbb{R}^n \to \mathbb{C}$ via

$$\begin{aligned} \tilde{\chi}(x) &:= \chi(x, y) \qquad \text{for all } x \in \mathbb{R}^n, \\ \hat{\chi}(x) &:= \chi(y, x) \qquad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

Then $\tilde{\chi}, \hat{\chi} \in \mathcal{S}(\mathbb{R}^n)$.

Proof: The claim follows directly from $\langle (x,y)\rangle^{-N} \leq \langle x\rangle^{-N}$ for every $N \in \mathbb{N}$. $\square$

All derivatives of Schwartz functions belong to the Schwartz space. This is a direct consequence of the definition of the Schwartz space.

Remark 2.21. Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ be arbitrary. We define the functions $\tilde{\chi} : \mathbb{R}^n \to \mathbb{C}$ and $\eta : \mathbb{R}^n \to \mathbb{C}$ as

$$\tilde{\chi}(x) := \partial_x^\alpha \chi(x), \qquad \eta(x) := D_x^\alpha \chi(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

Then $\tilde{\chi}, \eta \in \mathcal{S}(\mathbb{R}^n)$.

An often used ingredient for the verification whether a function is in the Schwartz space arises from the next proposition. It states that every linear transformation of a Schwartz function is a Schwartz function again:

Proposition 2.22. *Let $\chi \in \mathcal{S}(\mathbb{R}^n)$, $A \in \mathrm{GL}(n)$ and $b \in \mathbb{R}^n$. We define the function $\tilde{\chi} : \mathbb{R}^n \to \mathbb{C}$ via*

$$\tilde{\chi}(x) := \chi(Ax + b) \qquad \textit{for all } x \in \mathbb{R}^n.$$

Then $\tilde{\chi} \in \mathcal{S}(\mathbb{R}^n)$.

Proof: Let $m \leq 0$ be arbitrary. Since $A \in \mathrm{GL}(n)$, A and A^{-1} are continuous. Therefore there is a constant $\tilde{C} > 0$ such that

$$|x + A^{-1}b| = |A^{-1}(Ax + b)| \leq \tilde{C}|Ax + b| \qquad \text{for all } x \in \mathbb{R}^n.$$

Using this inequality we get

$$1 + |Ax + b| \geq 1 + \tilde{C}^{-1}|x + A^{-1}b| \geq C(1 + |x + A^{-1}b|),$$

where $C := \min\{1; \tilde{C}^{-1}\}$. Together with Lemma 2.4 and Corollary 2.5 this implies

$$\begin{aligned}(1 + |Ax + b|)^m &\leq C_m(1 + |x + A^{-1}b|)^m \leq C_m\langle x + A^{-1}b\rangle^m \leq C_m\langle x\rangle^m \\ &\leq C_m(1 + |x|)^m \qquad \text{for all } x \in \mathbb{R}^n, m \leq 0. \end{aligned} \tag{2.2}$$

Finally, we have to check for arbitrary $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ the existence of a constant $C_{\alpha,N} > 0$ such that

$$|\partial_x^\alpha \tilde{\chi}(x)| \leq C_{\alpha,N}(1 + |x|)^{-N} \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.3}$$

We obtain the previous inequality by mathematical induction with respect to $|\alpha|$ by using the properties of Schwartz functions together with the estimate (2.2) and the chain rule. $\square$

The following notation is used during the whole thesis:

Notation 2.23. Let $\xi \in \mathbb{R}^n$. Then we define the function $e_\xi : \mathbb{R}^n \to \mathbb{C}$ by

$$e_\xi(x) := e^{ix\cdot\xi} \qquad \text{for all } x \in \mathbb{R}^n.$$

Let us point out one consequence of the previous lemma we need later on:
Remark 2.24. Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $y, \xi \in \mathbb{R}^n$. Then $\tau_y(g), e_\xi \tau_y(g) \in \mathcal{S}(\mathbb{R}^n)$.

Proof: Using Lemma 2.22 for $A = I$ and $b = -y$ provides $\tau_y(g) \in \mathcal{S}(\mathbb{R}^n)$. The calculation $\partial_x^\alpha e_\xi = (i\xi)^\alpha e_\xi$ for every $\alpha \in \mathbb{N}_0^n$ immediately provides $e_\xi \in C^\infty_{poly}(\mathbb{R}^n)$. Thus $e_\xi \tau_y(g)$ is a Schwartz function on account of Lemma 2.17. □

For the product of two Schwartz functions we obtain the following result:

Lemma 2.25. *Let $m \in \mathbb{N}$, $\chi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $\chi_2 \in \mathcal{S}(\mathbb{R}^m)$. We define the function $\chi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ via*

$$\chi(x, y) := \chi_1(x)\chi_2(y) \qquad \text{for all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Then $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof: By means of the product rule and of $\langle x\rangle^{-N}\langle y\rangle^{-N} \leq \langle (x, y)\rangle^{-N}$ for every $N \in \mathbb{N}_0$, cf. Lemma 2.7, we get the lemma at once. □

Now we take a look at the convergence of rapidly decreasing smooth functions:

Lemma 2.26. *Let $0 < \varepsilon < 1$ and $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$. Then there is a constant C_α, independent of ε, such that*

i) $\chi(\varepsilon x) \xrightarrow{\varepsilon\to 0} 1$ *uniformly on any compact set in* $\mathbb{R}^n$,

ii) $\partial_x^\alpha \chi(\varepsilon x) \xrightarrow{\varepsilon\to 0} 0$ *uniformly for each* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \neq 0$,

iii) $|\partial_x^\alpha \chi(\varepsilon x)| \leq C_\alpha \varepsilon^\sigma \langle x\rangle^{-(|\alpha|-\sigma)}$ *for all* $x \in \mathbb{R}^n$ *and each* $0 \leq \sigma \leq |\alpha|$.

This lemma has been proved for example in [42], Lemma 6.3.

Next we treat the dual space of $\mathcal{S}(\mathbb{R}^n)$ and some of its properties.

Definition 2.27. The space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^n) := (\mathcal{S}(\mathbb{R}^n))'$ is the space of all linear and bounded functions $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. An element of $\mathcal{S}'(\mathbb{R}^n)$ is called *Schwartz distribution.* We equip $\mathcal{S}'(\mathbb{R}^n)$ with the strong dual topology, cf. e.g. [70], Chapter 19, Example IV). Consequently a sequence $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{S}'(\mathbb{R}^n)$ converges to $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if for all bounded sets $\mathscr{B} \subseteq \mathcal{S}(\mathbb{R}^n)$ we have

$$\lim_{k\to\infty} \sup_{\varphi\in\mathscr{B}} \langle f_k - f, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = 0,$$

where $\langle f, \varphi\rangle_{\mathcal{S}',\mathcal{S}} := f(\varphi)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ denotes the duality product.

In [70], p.376 the next property of the Schwartz space and its dual space is verified:

Remark 2.28. The spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are reflexive. Consequently we have the isomorphism $(\mathcal{S}'(\mathbb{R}^n))' \cong \mathcal{S}(\mathbb{R}^n)$.

For a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the Fourier transformation $\mathscr{F}[f]$ and its inverse $\mathscr{F}^{-1}[f]$ are defined by duality:

$$\begin{aligned} \langle \mathscr{F}[f], \varphi \rangle_{\mathcal{S}',\mathcal{S}} &:= \langle f, \mathscr{F}[\varphi] \rangle_{\mathcal{S}',\mathcal{S}} && \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n), \\ \langle \mathscr{F}^{-1}[f], \varphi \rangle_{\mathcal{S}',\mathcal{S}} &:= \langle f, \mathscr{F}^{-1}[\varphi] \rangle_{\mathcal{S}',\mathcal{S}} && \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Analogous to the result on the Schwartz space $\mathscr{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an isomorphism with inverse $\mathscr{F}^{-1}$, cf. [5], Proposition 2.27.

A convergent sequence of Schwartz distributions fulfills the following estimate:

Proposition 2.29. *Let $(u_l)_{l\in\mathbb{N}} \subseteq \mathcal{S}'(\mathbb{R}^n)$ be a sequence which converges to the tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$. Then there is a $\kappa \in \mathbb{N}_0$ and a constant $C > 0$, independent of $l \in \mathbb{N}$ and of $\varphi \in \mathcal{S}(\mathbb{R}^n)$, such that for each $l \in \mathbb{N}$ we have*

$$|\langle u_l, \varphi \rangle_{\mathcal{S}',\mathcal{S}}| \leq C|\varphi|_{\kappa,\mathcal{S}} \quad \text{and} \quad |\langle u, \varphi \rangle_{\mathcal{S}',\mathcal{S}}| \leq C|\varphi|_{\kappa,\mathcal{S}}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

For the proof of this proposition we need some further results. Before listing them, we recall the definition of an equicontinuous set:

Definition 2.30. We suppose that X and Y are topological vector spaces and that Γ is a collection of linear mappings from X to Y. Then Γ is *equicontinuous* if for every neighbourhood W of 0 in Y there exists a neighbourhood V of 0 in X such that $\Lambda(V) \subseteq W$ for all $\Lambda \in \Gamma$.

With the previous definition at hand, we get the next two theorems:

Theorem 2.31. *Let Γ be a collection of continuous linear mappings from a Fréchet space X into a topological vector space Y. If the sets*

$$\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$$

are bounded in Y for every $x \in X$, then Γ is equicontinuous.

Theorem 2.32. *We assume that $\mathscr{P}$ is a separating family of semi-norms on a topological vector space X. For each $p \in \mathscr{P}$ and each $n \in \mathbb{N}$ we set*

$$V(p, n) := \left\{ x \in X : p(x) < \frac{1}{n} \right\}.$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$. Then $\mathscr{B}$ is a (convex balanced) local base for a topology τ on X.

The proofs can be found for example in [61], Theorem 1.37 and Theorem 2.6.

On account of the previous two theorems, we are able to check Proposition 2.29.

Proof of Proposition 2.29. Since $u_l \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ if $l \to \infty$, we have the convergence of $(\langle u_l, \varphi\rangle_{\mathcal{S}',\mathcal{S}})_{l\in\mathbb{N}}$ to $\langle u, \varphi\rangle_{\mathcal{S}',\mathcal{S}}$ for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This implies the boundedness of the set $\{\langle u_l, \varphi\rangle_{\mathcal{S}',\mathcal{S}} : l \in \mathbb{N}\} \cup \{\langle u, \varphi\rangle_{\mathcal{S}',\mathcal{S}}\}$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Defining $\Gamma := \{u_l : l \in \mathbb{N}\} \cup \{u\}$, Theorem 2.31 yields that Γ is equicontinuous. Thus for all neighbourhoods $\Omega \subseteq \mathbb{C}$ of 0 there is a neighbourhood $V \subseteq \mathcal{S}(\mathbb{R}^n)$ of 0, such that

$$u(V) \subseteq \Omega \qquad \text{and} \qquad u_l(V) \subseteq \Omega \quad \text{for all } l \in \mathbb{N}. \tag{2.4}$$

In particular we can choose $\Omega = B_1(0)$. Let $V \subseteq \mathcal{S}(\mathbb{R}^n)$ be a neighbourhood of 0, which fulfills property (2.4). By means of Theorem 2.32, we obtain that

$$\mathscr{B}^0 := \left\{B_n^k(0) : n \in \mathbb{N}; k \in \mathbb{N}_0\right\}$$

is a local base of 0, where $B_n^k(0) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : |\varphi|_{k,\mathcal{S}} < \frac{1}{n}\}$ for all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Therefore there is a $\kappa \in \mathbb{N}_0$ and an $n \in \mathbb{N}$ such that $B_n^\kappa(0) \subseteq V$ because of the definition of a local base. This implies together with (2.4):

$$|\langle u, \varphi\rangle_{\mathcal{S}',\mathcal{S}}| \leq 1 \quad \text{and} \quad |\langle u_l, \varphi\rangle_{\mathcal{S}',\mathcal{S}}| \leq 1 \tag{2.5}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $|\varphi|_{\kappa,\mathcal{S}} < \frac{1}{n}$ and each $l \in \mathbb{N}$. Choosing an arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define $\psi := \frac{\varphi}{(n+1)|\varphi|_{\kappa,\mathcal{S}}}$. Then we have $|\psi|_{\kappa,\mathcal{S}} < \frac{1}{n}$. Using inequality (2.5), we get $|\langle u, \psi\rangle_{\mathcal{S}',\mathcal{S}}| \leq 1$. This provides $|\langle u, \varphi\rangle_{\mathcal{S}',\mathcal{S}}| \leq (n+1)|\varphi|_{\kappa,\mathcal{S}}$. In the same way we can show the inequality $|\langle u_l, \varphi\rangle_{\mathcal{S}',\mathcal{S}}| \leq (n+1)|\varphi|_{\kappa,\mathcal{S}}$ for all $l \in \mathbb{N}$ and every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. $\square$

At last, we list two continuous embedding theorems for later purposes:

Corollary 2.33. *For every $1 \leq r \leq \infty$ the embedding $i : \mathcal{S}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ is continuous.*

Proof: For $1 \leq r < \infty$ take a $k \in \mathbb{N}$ such that $-rk < -n$. We obtain for arbitrary $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned}\|f\|^r_{L^r(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |f(x)|^r dx = \int_{\mathbb{R}^n} (1+|x|)^{-rk}(1+|x|)^{rk}|f(x)|^r dx \\ &\leq \int_{\mathbb{R}^n} (1+|x|)^{-rk} \left(|f|'_{k,\mathcal{S}}\right)^r dx \leq C_k \left(|f|'_{k,\mathcal{S}}\right)^r.\end{aligned}$$

Here the last inequality holds because of Theorem 2.11. Therefore the map $i : \mathcal{S}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ defined by $i(f) := f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ is linear and bounded. This implies the continuity of i.
If $r = \infty$, we get $\|f\|_{L^\infty(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |f(x)| \le |f|_{0,\mathcal{S}}$ for each $f \in \mathcal{S}(\mathbb{R}^n)$ directly. □

Corollary 2.34. *For all $1 \le q \le \infty$ the embedding $i : L^q(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous.*

Proof: We set $q' := 1$ if $q = \infty$ and $q' := \left(1 - \frac{1}{q}\right)^{-1}$ else. Then $q' \in [1,\infty)\cup\{\infty\}$, too. We choose an arbitrary $f \in L^q(\mathbb{R}^n)$ and a sequence $(f_k)_{k\in\mathbb{N}} \subseteq L^q(\mathbb{R}^n)$, which converges to f. An application of the Hölder inequality and of Corollary 2.33 provides the existence of a $\kappa \in \mathbb{N}$ and of a constant C, independent of $f, f_k \in L^q(\mathbb{R}^n)$ and $k \in \mathbb{N}$, such that we have for all bounded sets $\mathscr{B} \subseteq \mathcal{S}(\mathbb{R}^n)$:

$$
\begin{aligned}
\sup_{\varphi\in\mathscr{B}} |\langle f_k - f, \varphi\rangle_{\mathcal{S}',\mathcal{S}}| &= \sup_{\varphi\in\mathscr{B}} \left| \int (f_k - f)(x)\varphi(x)dx \right| \\
&\le \sup_{\varphi\in\mathscr{B}} \|\varphi\|_{L^{q'}(\mathbb{R}^n)} \|f_k - f\|_{L^q(\mathbb{R}^n)} \\
&\le C \sup_{\varphi\in\mathscr{B}} \|f_k - f\|_{L^q(\mathbb{R}^n)} |\varphi|_{\kappa,\mathcal{S}} \le C\|f_k - f\|_{L^q(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.
\end{aligned}
$$

Thus we proved the convergence of $(f_k)_{k\in\mathbb{N}}$ to f in $\mathcal{S}'(\mathbb{R}^n)$. □

2.4 Hölder Space and Hölder-Zygmund Space

Another important class of spaces are the Hölder spaces. Analysing the relationship of Hölder spaces to the space of all bounded k-times differentiable functions, we see that the Hölder space $C^{m,s}(\mathbb{R}^n)$ always lies between the spaces $C_b^m(\mathbb{R}^n)$ and $C_b^{m+1}(\mathbb{R}^n)$. In some cases the Hölder space is equivalent to an Hölder-Zygmund space. We also give a short introduction to the Hölder-Zygmund spaces in the present section.

To begin with, we define the Hölder spaces:

Definition 2.35. Let $0 < s \le 1$ and $m \in \mathbb{N}_0$. Additionally let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the *Hölder space* $C^{m,s}(\overline{\Omega})$ is defined as

$$
\begin{aligned}
C^{0,s}(\overline{\Omega}) &:= \left\{ f \in C_b^0(\overline{\Omega}) : \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^s} < \infty \right\}, \\
C^{m,s}(\overline{\Omega}) &:= \left\{ f \in C_b^m(\overline{\Omega}) : \partial_x^\alpha f \in C^{0,s}(\overline{\Omega}) \text{ for all } \alpha \in \mathbb{N}_0^n, |\alpha| \le m \right\}.
\end{aligned}
$$

The norm $\|.\|_{C^{m,s}(\overline{\Omega})}$ of the Hölder spaces is defined by

$$\|f\|_{C^{m,s}(\overline{\Omega})} := \max_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{L^\infty} + \max_{|\alpha|\leq m} \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)|}{|x-y|^s}$$

for all $f \in C^{m,s}(\overline{\Omega})$. In the case $\Omega = \mathbb{R}^n$ we often write $C^{m,s}$ and $\|.\|_{C^{m,s}}$ instead of $C^{m,s}(\mathbb{R}^n)$ and $\|.\|_{C^{m,s}(\mathbb{R}^n)}$.

In literature usually the equivalent norm $\|.\|'_{C^{m,s}(\overline{\Omega})}$ defined via

$$\|f\|'_{C^{m,s}(\overline{\Omega})} := \|f\|_{C_b^m(\overline{\Omega})} + \sum_{|\alpha|=m} \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)|}{|x-y|^s} \qquad \text{for all } f \in C^{m,s}(\overline{\Omega})$$

is used instead of $\|.\|_{C^{m,s}(\overline{\Omega})}$. The norm equivalence can be verified by means of the mean value theorem.

Let us remark that Hölder spaces are Banach spaces, cf. e.g. [8], p.44. The following notation for these spaces is often used:

Notation 2.36. If $s > 0$ and $s \notin \mathbb{N}_0$, we set

$$C^s(\mathbb{R}^n) := C^{m,\tau}(\mathbb{R}^n),$$

where $m = \max\{r \in \mathbb{N}_0 : r < s\}$ and $\tau = s - m$.

As a direct consequence of the definition of the Hölder spaces we get:

Remark 2.37. Let $m \in \mathbb{N}$, $0 < s \leq 1$ and $j \in \{1, \dots, n\}$. Then we obtain

$$\|D_{x_j} f\|_{C^{m-1,s}} \leq \|f\|_{C^{m,s}} \qquad \text{for all } f \in C^{m,s}(\mathbb{R}^n).$$

Proof: The claim follows directly from the definition of the Hölder spaces. □

The Hölder space $C^{m,s}(\mathbb{R}^n)$ is a subset of $C_b^m(\mathbb{R}^n)$ for all $m \in \mathbb{N}_0$ and $0 < s \leq 1$ due to the definition of these spaces. However, we even are able to show the following statement:

Lemma 2.38. *Let $k \in \mathbb{N}_0$ and $\mathscr{B} \subseteq C_b^{k+1}(\mathbb{R}^n)$ be a bounded subset. Then $\mathscr{B} \subseteq C^{k,1}(\mathbb{R}^n)$ is bounded, too.*

Proof: First of all we show the existence of a constant C, independent of $p \in \mathscr{B}$, such that

$$\max_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\leq k}} \sup_{x\neq y} \frac{|\partial_x^\alpha p(x) - \partial_x^\alpha p(y)|}{|x-y|} \leq C \qquad \text{for all } p \in \mathscr{B}. \tag{2.6}$$

For $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ the boundedness of the subset $\mathscr{B} \subseteq C_b^{k+1}(\mathbb{R}^n)$ and Lemma 2.12 yield the existence of a constant C_α, independent of $p \in \mathscr{B}$ and of $x, y \in \mathbb{R}^n$ with $x \neq y$, such that

$$\begin{aligned}\frac{|\partial_x^\alpha p(x) - \partial_x^\alpha p(y)|}{|x-y|} &\leq \sup_{0\leq t\leq 1} |D(\partial_x^\alpha p)(x + t(y-x))| \\ &\leq \sup_{0\leq t\leq 1} \sum_{j=1}^n |\partial_{x_j}\partial_x^\alpha p(x + t(y-x))| \leq \sum_{j=1}^n \sup_{x\in\mathbb{R}^n} |\partial_{x_j}\partial_x^\alpha p(x)| \\ &\leq C\|p\|_{C_b^{k+1}(\mathbb{R}^n)} \leq C_\alpha\end{aligned}$$

for all $x, y \in \mathbb{R}^n$ with $x \neq y$ and all $p \in \mathscr{B}$. Consequently there is a constant C_α, independent of $p \in \mathscr{B}$, such that

$$\sup_{x\neq y} \frac{|\partial_x^\alpha p(x) - \partial_x^\alpha p(y)|}{|x-y|} \leq C_\alpha \qquad \text{for all } p \in \mathscr{B}.$$

Since $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ was chosen arbitrary, we get (2.6). Therefore it remains to use the boundedness of $\mathscr{B} \subseteq C_b^{k+1}(\mathbb{R}^n) \subseteq C_b^k(\mathbb{R}^n)$ and the estimate (2.6) to conclude

$$\|p\|_{C^{k,1}(\mathbb{R}^n)} = \|p\|_{C_b^k(\mathbb{R}^n)} + \max_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\leq k}} \sup_{x\neq y} \frac{|\partial_x^\alpha p(x) - \partial_x^\alpha p(y)|}{|x-y|} \leq C \qquad \text{for all } p \in \mathscr{B},$$

where C is independent of $p \in \mathscr{B}$. □

Now we turn to another function space needed later on:

Definition 2.39. Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a dyadic partition of unity on $\mathbb{R}^n$ and $s > 0$. Then the *Hölder-Zygmund space* $C_*^s(\mathbb{R}^n)$ is defined by

$$C_*^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{C_*^s} < \infty\},$$

where $\|f\|_{C_*^s} := \sup_{j\in\mathbb{N}_0} 2^{js}\|\mathscr{F}^{-1}[\varphi_j \hat{f}]\|_{L^\infty}$.

Note, that the previous definition is independent of the choice of the dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$.

Properties of the Hölder-Zygmund spaces are discussed e.g. in [67], Section A.1. We mention a few of them:

- Hölder-Zygmund spaces are Banach spaces.
- $C^s(\mathbb{R}^n) \subseteq C_*^s(\mathbb{R}^n)$ for all $s > 0$.
- $C_*^s(\mathbb{R}^n) = C^s(\mathbb{R}^n)$ for all $s > 0$ with $s \notin \mathbb{N}$.

- $C_b^k(\mathbb{R}^n) \subseteq C_*^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$.

The last property follows form [51], Example 1.22 and [50], Theorem 1.2.17.

Additionally the next multiplication property holds for elements of Hölder-Zygmund spaces, cf. [72], Remark 2.8.2.1:

Lemma 2.40. *Let $s > 0$. Then there is a constant $C > 0$ such that*

$$\|fg\|_{C_*^s} \leq C\|f\|_{C_*^s}\|g\|_{C_*^s} \qquad \text{for all } f, g \in C_*^s(\mathbb{R}^n).$$

For later purposes we also refer to the next interpolation result for Hölder-Zygmund spaces:

Lemma 2.41. *Let $k, m \in \mathbb{N}$ with $k \leq m$, $0 < \tau < 1$ and $0 < \theta < 1$. Setting $\theta_1 := \frac{k}{m+\tau}$ we obtain*

i) $\|f\|_{C_*^{\theta m}(\mathbb{R}^n)} \leq C_\theta \|f\|_{C_b^0(\mathbb{R}^n)}^{1-\theta} \|f\|_{C_b^m(\mathbb{R}^n)}^{\theta}$ *for all* $f \in C_b^m(\mathbb{R}^n)$.

ii) $\|f\|_{C_b^k(\mathbb{R}^n)} \leq C \|f\|_{C_b^0(\mathbb{R}^n)}^{1-\theta_1} \|f\|_{C^{m,\tau}(\mathbb{R}^n)}^{\theta_1}$ *for all* $f \in C^{m,\tau}(\mathbb{R}^n)$.

Proof: Claim *i)* is a direct consequence of [50], Theorem 1.2.17 and [71], Theorem 1.3.3. We will verify the second claim in Appendix C, Lemma C.1. □

2.5 Bessel Potential Space

The present section serves as a brief summary of all properties of the Bessel potential spaces that are important for our purpose. If the order of a Bessel potential space is a non-negative integer number, this space turns out to be equal to a Sobolev space. Additionally, we investigate the relationship of these spaces to the Schwartz space and its dual space. Moreover, some important embedding results and characterizations for functions of a Bessel potential space are mentioned. At last we also would like to point out an important interpolation theorem for Bessel potential spaces.

In order to define the Bessel potential spaces we set for every $s \in \mathbb{R}$:

$$\langle D_x \rangle^s f := \mathscr{F}^{-1}[\langle \xi \rangle^s \hat{f}] \qquad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n).$$

For even $s \in \mathbb{N}_0$ and $f \in C^s(\mathbb{R}^n)$, the linearity of $\mathscr{F}^{-1}$ and the properties of the Fourier transformation imply that the previous definition of $\langle D_x \rangle^s f$ is consistent with that one of Notation 2.9.

Definition 2.42. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then the *(L^p-)Bessel potential space* $H^s_p(\mathbb{R}^n)$ of order s is defined by

$$H^s_p(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x\rangle^s f \in L^p(\mathbb{R}^n)\}.$$

The norm of $H^s_p(\mathbb{R}^n)$ is defined via

$$\|f\|_{H^s_p(\mathbb{R}^n)} := \|\langle D_x\rangle^s f\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } f \in H^s_p(\mathbb{R}^n).$$

For short we also write $\|.\|_{H^s_p}$ instead of $\|.\|_{H^s_p(\mathbb{R}^n)}$.

Some important properties of the Bessel potential spaces are listed in the next lemma:

Lemma 2.43. *Let $s \in \mathbb{R}$ and $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Then*

i) $H^s_p(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$ *with equivalent norms if* $s \in \mathbb{N}_0$,

ii) $H^s_p(\mathbb{R}^n)$ *is a reflexive Banach space,*

iii) $\mathcal{S}(\mathbb{R}^n) \subseteq H^s_p(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$,

iv) $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $H^s_p(\mathbb{R}^n)$,

v) the dual space of $H^s_p(\mathbb{R}^n)$ *is isomorphic to* $H^{-s}_{p'}(\mathbb{R}^n)$ *with canonical isomorphism.*

Proof: For the proof of *i)* we refer to e.g. [5], Theorem 6.8. Since $H^s_p(\mathbb{R}^n)$ is isomorph to $L^p((0,1))$ for each $1 < p < \infty$ and $s \in \mathbb{R}$, $H^s_p(\mathbb{R}^n)$ is a reflexive Banach space in this case. For the proof see [71], Theorem 2.11.2.a). Moreover, claim *iii)* and *iv)* hold because of Theorem 2.3.3 (ii) of [72]. Finally, we refer to [19], Corollary 6.2.8 for the proof of *v)*. □

Some well-known results for Sobolev spaces are the continuous embeddings in the Hölder spaces and in the Sobolev spaces. These results can be generalized for Bessel potential spaces:

Lemma 2.44. *Let $1 < p < \infty$, $s \in \mathbb{R}$ with $s > n/p$ and $0 < \tau \leq s - n/p$ with $\tau \notin \mathbb{N}$. Then we obtain the continuous embedding*

$$H^s_p(\mathbb{R}^n) \hookrightarrow C^\tau(\mathbb{R}^n).$$

Proof: The claim is a consequence of Corollary 6.13 in [5], Lemma 6.5 in [5], Theorem 6.15 in [5] and Remark 6.4 in [5]. □

Lemma 2.45. *For every $s, r \in \mathbb{R}$ with $s < r$ and $1 < p < \infty$ we have the continuous embedding*

$$H^r_p(\mathbb{R}^n) \hookrightarrow H^s_p(\mathbb{R}^n).$$

This lemma was proved e.g. in [5], Corollary 6.14.

In the next proposition we give a first characterization for Bessel potential spaces. For the proof we refer to [59], Proposition 1.14.

Proposition 2.46. *For all $s \in \mathbb{R}$ we have*

$$u \in H_2^{s+1}(\mathbb{R}^n) \Leftrightarrow u, D_{x_1}u, \ldots, D_{x_n}u \in H_2^s(\mathbb{R}^n)$$

and the equality $\|u\|^2_{H_2^{s+1}(\mathbb{R}^n)} = \|u\|^2_{H_2^s(\mathbb{R}^n)} + \sum_{j=1}^n \|D_{x_j}u\|^2_{H_2^s(\mathbb{R}^n)}$. Additionally for any $k \in \mathbb{N} \cup \{\infty\}$ we obtain:

i) $u \in H_2^k(\mathbb{R}^n) \Leftrightarrow D_x^\alpha u \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

ii) If $s > k + n/2$ and $u \in H_2^s(\mathbb{R}^n)$, then for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ the function $D_x^\alpha u$ is bounded and continuous with $\|D_x^\alpha u\|_\infty \leq C_{s,k}\|u\|_{H_2^s(\mathbb{R}^n)}$.

The next proposition is devoted to the question which conditions Schwartz distributions have to satisfy in order to be a Bessel potential function.

Proposition 2.47. *If $u \in H_2^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$|\langle u, \varphi \rangle_{\mathcal{S}',\mathcal{S}}| \leq \|u\|_{H_2^s(\mathbb{R}^n)} \|\varphi\|_{H_2^{-s}(\mathbb{R}^n)}.$$

Conversely, if $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $|\langle u, \varphi \rangle_{\mathcal{S}',\mathcal{S}}| \leq C\|\varphi\|_{H_2^{-s}(\mathbb{R}^n)}$ for some constants C and for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $u \in H_2^s(\mathbb{R}^n)$ with $\|u\|_{H_2^s(\mathbb{R}^n)} \leq C$.

This statement was shown e.g. in [59], Proposition 1.15. Making use of the previous two propositions, enables us to prove the next technical proposition which can be verified in the same way as Proposition 1.16 in [59].

Proposition 2.48. *Let $\mathscr{B} \subseteq \mathcal{S}'(\mathbb{R}^n)$ be a subset with the following property: There is a $k \in \mathbb{N}$ and a $C_k > 0$, independent of $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, with $|\langle u, \varphi \rangle_{\mathcal{S}';\mathcal{S}}| \leq C_k |\varphi|_{k,\mathcal{S}}$ for all $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then there is an $N \in \mathbb{N}$ such that*

$$(1 + |x|^2)^{-N} u(x) \in H_2^{-N}(\mathbb{R}_x^n) \qquad \text{for each } u \in \mathscr{B}.$$

In particular there is a constant C_k, independent of $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, such that

$$|\langle (1 + |x|^2)^{-N} u, \varphi \rangle_{\mathcal{S}';\mathcal{S}}| \leq C_k \|\varphi\|_{H_2^N(\mathbb{R}^n)} \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ and } u \in \mathscr{B}.$$

Proof: According to the assumptions there exists a constant $C_k > 0$, independent of $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and a $k \in \mathbb{N}$ such that $|\langle u, \varphi \rangle_{\mathcal{S}';\mathcal{S}}| \leq C_k |\varphi|_{k,\mathcal{S}}$ for every $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. For all $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$ we arise from Remark 2.8 the existence of a constant C_α such that

$$|\partial_x^\alpha\{(1+|x|^2)^{-N}\}| \leq C_\alpha(1+|x|^2)^{-N-|\alpha|/2} \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.7}$$

We choose a $k' \in \mathbb{N}$ with $|f|_{k,\mathcal{S}} \leq C|f|'_{k',\mathcal{S}}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Such a $k' \in \mathbb{N}$ exists due to the equivalence of $(|.|_{k,\mathcal{S}})_{k\in\mathbb{N}}$ and $(|.|'_{k,\mathcal{S}})_{k\in\mathbb{N}}$. Taking $N \in \mathbb{N}$ with $N \geq \max\{\frac{k'}{2}; n+k\}$, estimate (2.7) and Peetre's inequality, cf. Lemma 2.4, yield the existence of a constant $C_{k'}$, such that

$$\begin{aligned}
|(1+|x|^2)^{-N}|_{k,\mathcal{S}} &\leq C|(1+|x|^2)^{-N}|'_{k',\mathcal{S}} \\
&= C \sup_{m+|\alpha|\leq k'} \sup_{x\in\mathbb{R}^n} (1+|x|)^m \left|\partial_x^\alpha \left\{(1+|x|^2)^{-N}\right\}\right| \\
&\leq C \sup_{m+|\alpha|\leq k'} \sup_{x\in\mathbb{R}^n} (1+|x|)^m (1+|x|^2)^{-N-|\alpha|/2} \\
&\leq \sup_{x\in\mathbb{R}^n} C_{k'}(1+|x|^2)^{\frac{k'}{2}-N} \leq C_{k'}.
\end{aligned}$$

Hence there is a constant C_k, independent of $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and each $u \in \mathscr{B}$ we have

$$\begin{aligned}
|\langle(1+|x|^2)^{-N}u, \varphi\rangle_{\mathcal{S}';\mathcal{S}}| &= |\langle u, (1+|x|^2)^{-N}\varphi\rangle_{\mathcal{S}';\mathcal{S}}| \leq C_k|(1+|x|^2)^{-N}\varphi|_{k,\mathcal{S}} \\
&\leq C_k|(1+|x|^2)^{-N}|_{k,\mathcal{S}} \max_{|\beta|\leq k} |\partial_x^\beta\varphi|_{0,\mathcal{S}} \leq C_k \max_{|\beta|\leq k} |\partial_x^\beta\varphi|_{0,\mathcal{S}}.
\end{aligned}$$

The penultimate inequality can be calculated by means of the Leibnitz-formula and the estimate $|\partial_x^\beta\varphi(x)| \leq \max_{|\beta|\leq k} |\partial_x^\beta\varphi|_{0,\mathcal{S}}$ for each $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k$.

On the other hand, we know due to Lemma 2.43 that $\varphi \in \mathcal{S}(\mathbb{R}^n) \subseteq H_2^N(\mathbb{R}^n)$. Therefore we obtain by Proposition 2.46:

$$|\partial_x^\beta\varphi|_{0,\mathcal{S}} = \|\partial_x^\beta\varphi\|_\infty \leq C_k\|\varphi\|_{H_2^N(\mathbb{R}^n)} \qquad \text{for all } |\beta| \leq k.$$

Collecting all estimates, we conclude for every $u \in \mathscr{B}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$|\langle(1+|x|^2)^{-N}u, \varphi\rangle_{\mathcal{S}';\mathcal{S}}| \leq C_k \max_{|\beta|\leq k} |\partial_x^\beta\varphi|_{0,\mathcal{S}} \leq C_k\|\varphi\|_{H_2^N(\mathbb{R}^n)}.$$

Hence an application of Proposition 2.47 finally provides

$$(1+|x|^2)^{-N}u(x) \in H_2^{-N}(\mathbb{R}_x^n) \qquad \text{for all } u \in \mathscr{B}.$$

□

Let us mention another characterization of functions in a Bessel potential space needed later on:

Lemma 2.49. *Let* $1 < p < \infty$, $s < 0$ *and* $m := -\lfloor s \rfloor$. *Then for each* $f \in H_p^s(\mathbb{R}^n)$ *there are functions* $g_\alpha \in H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n)$, *where* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \leq m$, *such that*

- $f = \sum\limits_{|\alpha|\leq m} \partial_x^\alpha g_\alpha$,
- $\sum\limits_{|\alpha|\leq m} \|g_\alpha\|_{H_p^{s-\lfloor s\rfloor}} \leq C\|f\|_{H_p^s}$.

The proof of the above-mentioned lemma is based on the next statement:

Proposition 2.50. *Let $1 < q < \infty$, $m \in \mathbb{N}_0$ and $s < 0$. We define N by $N := \sharp\{\alpha \in \mathbb{N}_0^n : |\alpha| \leq m\}$ and the operator $T : H_q^{-s}(\mathbb{R}^n) \to \left(H_q^{\lfloor s\rfloor - s}(\mathbb{R}^n)\right)^N$ via:*

$$T(\varphi) = (\partial_x^\alpha \varphi)_{|\alpha|\leq m}.$$

Then $T : H_q^{-s}(\mathbb{R}^n) \to T(H_q^{-s}(\mathbb{R}^n)) =: Y$ is invertible and $T^{-1} \in \mathscr{L}(Y, H_q^{-s}(\mathbb{R}^n))$.

Proof: Due to the definition of T, $Tg = 0$ implies $g = 0$. Hence the operator $T : H_q^{-s}(\mathbb{R}^n) \to Y$ is invertible. An application of the bounded inverse theorem, cf. e.g. [61], Corollary 2.12, yields $T^{-1} \in \mathscr{L}(Y, H_q^{-s}(\mathbb{R}^n))$ if Y is a Banach space. Thus it remains to show that Y is a closed subspace of $X_q := \left(H_q^{\lfloor s\rfloor - s}(\mathbb{R}^n)\right)^N$. In view of verifying this claim let $(f_l)_{l\in\mathbb{N}} \subseteq Y$ be a sequence, which converges to f in X_q. Therefore there is a sequence $(\varphi_l)_{l\in\mathbb{N}} \subseteq H_q^{-s}(\mathbb{R}^n)$ with $f_l = T\varphi_l$ for all $l \in \mathbb{N}$. Since $(f_l)_{l\in\mathbb{N}}$ is bounded in Y, there is a constant C, independent of $l \in \mathbb{N}$ such that

$$\begin{aligned}\|\varphi_l\|_{H_q^{-s}} = \|\langle D_x\rangle^{\lfloor s\rfloor - s}\varphi_l\|_{H_q^{-\lfloor s\rfloor}} &\leq C \sum_{|\alpha|\leq m} \left\|\partial_x^\alpha\{\langle D_x\rangle^{\lfloor s\rfloor - s}\varphi_l\}\right\|_{L^q} \\ &= C \sum_{|\alpha|\leq m} \|\partial_x^\alpha \varphi_l\|_{H_q^{\lfloor s\rfloor - s}} = C\|f_l\|_{X_q} \leq C.\end{aligned}$$

Here the inequality holds due to the norm equivalence of $\|.\|_{H_q^m}$ and $\|.\|_{W_q^m}$, cf. Lemma 2.43. Thus we have proved the boundedness of $(\varphi_l)_{l\in\mathbb{N}} \subseteq H_q^{-s}(\mathbb{R}^n)$. According to Lemma 2.43, the Banach space $H_q^{-s}(\mathbb{R}^n)$ is reflexive. Hence there is a subsequence $(\varphi_{l_k})_{k\in\mathbb{N}} \subseteq (\varphi_l)_{l\in\mathbb{N}}$ which converges weakly to an element of $H_q^{-s}(\mathbb{R}^n)$, denoted by φ. This implies the weak convergence of $(\partial_x^\alpha \varphi_{l_k})_{k\in\mathbb{N}}$ to $\partial_x^\alpha\varphi$ in $H_q^{\lfloor s\rfloor - s}(\mathbb{R}^n)$. Additionally the choice of $(f_l)_{l\in\mathbb{N}}$ yields for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$:

$$\lim_{k\to\infty} (\partial_x^\alpha \varphi_{l_k})_{|\alpha|\leq m} = \lim_{k\to\infty} (T\varphi_{l_k}) = \lim_{k\to\infty} f_{l_k} = \lim_{n\to\infty} f_l = f \qquad \text{in } X_q.$$

Denoting the α-th component of f by f^α, the function $\partial_x^\alpha \varphi_{l_k}$ converges to f^α in $H_q^{\lfloor s\rfloor - s}(\mathbb{R}^n)$ if $k \to \infty$ due to the previous convergence. Consequently we have the weak convergence of $(\varphi_{l_k})_{k\in\mathbb{N}}$ to f^α in $H_q^{\lfloor s\rfloor - s}(\mathbb{R}^n)$. Due to the uniqueness of the weak limit we get $f^\alpha = \partial_x^\alpha \varphi$ for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. This implies $f \in Y$:

$$f = (f^\alpha)_{|\alpha|\leq m} = (\partial_x^\alpha \varphi)_{|\alpha|\leq m} = T\varphi.$$

Hence Y is a closed subspace of X_q. □

Now we are able to prove Lemma 2.49:

Proof of Lemma 2.49: We define the operator $T : H_q^{-s}(\mathbb{R}^n) \to \left(H_q^{\lfloor s\rfloor -s}(\mathbb{R}^n)\right)^N$ with $1/p+1/q=1$ and $N := \sharp\{\alpha \in \mathbb{N}_0^n : |\alpha| \le m\}$ in the following way:

$$T(\varphi) = (\partial_x^\alpha \varphi)_{|\alpha|\le m}.$$

The norm $\|.\|_{X_q}$ of $X_q := \left(H_q^{\lfloor s\rfloor -s}(\mathbb{R}^n)\right)^N$ is defined by

$$\|f\|_{X_q} := \sum_{i=1}^N \|f_i\|_{H_q^{\lfloor s\rfloor -s}} \qquad \text{for all } f \in \left(H_q^{\lfloor s\rfloor -s}(\mathbb{R}^n)\right)^N.$$

Because of Proposition 2.50 the operator $T : H_q^{-s}(\mathbb{R}^n) \to T(H_q^{-s}(\mathbb{R}^n)) =: Y$ is invertible. Therefore we are able to define the linear functional $\tilde{f} \in Y'$ by $\tilde{f} := f \circ T^{-1}$ for each $f \in H_p^s(\mathbb{R}^n)$. In order to use the Theorem of Hahn Banach we have to show that $|\tilde{f}(g)| \le C\|g\|_{X_q}$ for all $g \in Y$. Thus we choose an arbitrary $g \in Y$. Denoting g^α as the α-th component of g we get in view of the norm equivalence of $\|.\|_{H_q^m}$ and $\|.\|_{W_q^m}$, cf. Lemma 2.43 and by means of $T^{-1} \in \mathscr{L}(Y, H_q^{-s}(\mathbb{R}^n))$, cf. Proposition 2.50:

$$|\tilde{f}(g)| = |f \circ T^{-1}(g)| = |\langle f, T^{-1}g\rangle_{H_p^s;H_q^{-s}}| \le \|f\|_{H_p^s}\|T^{-1}g\|_{H_q^{-s}} \le C\|f\|_{H_p^s}\|g\|_{X_q}.$$

Here the constant C is independent of $f \in H_p^s(\mathbb{R}^n)$ and $g \in Y$. An application of the Theorem of Hahn Banach provides the existence of a linear functional $F \in (X_q)' = \left(H_p^{s-\lfloor s\rfloor}(\mathbb{R}^n)\right)^N$ such that

i) $F|_Y = \tilde{f}$,

ii) $|F(g)| \le C\|f\|_{H_p^s}\|g\|_{X_q}$ for all $g \in X_q$.

We denote the α-th component of the linear functional F by F^α. On account of the definition of the distributional derivative and of property *i)* we get for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^n) \subseteq H_q^{-s}(\mathbb{R}^n)$, cf. Lemma 2.43:

$$\begin{aligned}\Big\langle \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial_x^\alpha F^\alpha, \varphi\Big\rangle_{H_p^s;H_q^{-s}} &= \sum_{|\alpha|\le m} (-1)^{|\alpha|}\langle \partial_x^\alpha F^\alpha, \varphi\rangle_{H_p^s;H_q^{-s}} \\ &= \sum_{|\alpha|\le m} \langle F^\alpha, \partial_x^\alpha\varphi\rangle_{H_p^s;H_q^{-s}} = \langle F, (\partial_x^\alpha\varphi)_{|\alpha|\le m}\rangle_{(X_q)';X_q} = \langle F, T\varphi\rangle_{(X_q)';X_q} \\ &= \langle \tilde{f}, T\varphi\rangle_{(X_q)';X_q} = f\circ T^{-1}(T\varphi) = \langle f, \varphi\rangle_{H_p^s;H_q^{-s}}.\end{aligned}$$

In view of $T\varphi \in Y$ we can apply property $i)$ in the previous equality. Due to the density of $\mathcal{S}(\mathbb{R}^n)$ in $H_q^{-s}(\mathbb{R}^n)$, cf. Lemma 2.43 we obtain

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial_x^\alpha F^\alpha = f \qquad \text{in } H_p^s(\mathbb{R}^n).$$

Additionally we get the claim due to $ii)$:

$$\sum_{|\alpha|\leq m} \|(-1)^{|\alpha|}F^\alpha\|_{H_p^{s-\lfloor s\rfloor}} = \|F\|_{(X_q)'} = \sup_{\|g\|_{X_q}\leq 1} |\langle F, g\rangle_{(X_q)';X_q}| \leq C\|f\|_{H_p^s}.$$

□

Being a linear and bounded operator on a Bessel potential space implies the linearity and boundedness of its adjoint operator:

Lemma 2.51. *Let $s \in \mathbb{R}$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$. For a linear operator $P \in \mathscr{L}(H_p^s(\mathbb{R}^n))$ we have $P^* \in \mathscr{L}(H_q^{-s}(\mathbb{R}^n))$.*

We refer to e.g. [8], Section 10.1 for the proof.

The next two interpolation results for Bessel potential spaces are listed for later purposes:

Theorem 2.52. *Let $s_1, s_2 \in \mathbb{R}$, $1 < p_1, p_2 < \infty$ and $0 < \theta < 1$. We define $s := (1-\theta)s_1 + \theta s_2$ and p via $1/p = (1-\theta)/p_1 + \theta/p_2$. Then we obtain*

$$\|f\|_{H_p^s} \leq C\|f\|_{H_{p_1}^{s_1}}^{1-\theta}\|f\|_{H_{p_2}^{s_2}}^{\theta} \qquad \text{for all } f \in H_{p_1}^{s_1}(\mathbb{R}^n) \cap H_{p_2}^{s_2}(\mathbb{R}^n).$$

Proof: We get the claim due to [72], Theorem 2.4.7 and [71], Theorem 1.9.3. □

Theorem 2.53. *Let $1 < p < \infty$ and $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$. If T is an element of $\mathscr{L}(H_p^{s_0}(\mathbb{R}^n), H_p^{s_0}(\mathbb{R}^n))$ and $T \in \mathscr{L}(H_p^{s_1}(\mathbb{R}^n), H_p^{s_1}(\mathbb{R}^n))$ we have*

$$T \in \mathscr{L}(H_p^s(\mathbb{R}^n), H_p^s(\mathbb{R}^n)) \qquad \text{for all } s_0 \leq s \leq s_1.$$

Proof: The theorem is a consequence of e.g. [71], Theorem 1.9.3 and Remark 2.4.2.2 d) or of [2], Theorem 2.10 and Theorem 4.23. □

For further studies of the Bessel potential spaces we refer to the books of H. Triebel about the theory of function spaces, cf. [72] and [71].

2.6 The Operators $\operatorname{ad}(-ix_j)$ and $\operatorname{ad}(D_{x_j})$

One of the main goals of this thesis is the characterization of non-smooth pseudo-differential operators. For this characterization we will need the operators $\operatorname{ad}(-ix_j)$ and $\operatorname{ad}(D_{x_j})$. Therefore, we define them now:

Definition 2.54. Let X be either $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$ and $T : X \to \mathcal{S}'(\mathbb{R}^n)$ be a linear operator. We define the linear operators $\operatorname{ad}(-ix_j)T : X \to \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{ad}(D_{x_j})T : X \to \mathcal{S}'(\mathbb{R}^n)$ for all $j \in \{1, \ldots, n\}$ and $u \in X$ by

$$\operatorname{ad}(-ix_j)Tu := -ix_jTu + T(ix_ju) \quad \text{and} \quad \operatorname{ad}(D_{x_j})Tu := D_{x_j}(Tu) - T\left(D_{x_j}u\right).$$

Additionally we define for all $j \in \{1, \ldots, n\}$ and $m \in \mathbb{N}_0$:

$$\operatorname{ad}(-ix_j)^m T := [\operatorname{ad}(-ix_j)]^m T \quad \text{and} \quad \operatorname{ad}(D_{x_j})^m T := \left[\operatorname{ad}(D_{x_j})\right]^m T.$$

For arbitrary multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ we denote the *iterated commutator* of T as

$$\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta T := \operatorname{ad}(-ix_1)^{\alpha_1} \ldots \operatorname{ad}(-ix_n)^{\alpha_n} \operatorname{ad}(D_{x_1})^{\beta_1} \ldots \operatorname{ad}(D_{x_n})^{\beta_n} T.$$

We assume an operator $T : X \to \mathcal{S}(\mathbb{R}^n)$, where X is either $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$. Since $u \in \mathcal{S}(\mathbb{R}^n)$ provides for all $\alpha, \beta \in \mathbb{N}_0^n$ that $(-ix)^\alpha u(x), D_x^\beta u(x) \in \mathcal{S}(\mathbb{R}^n_x)$, we get

$$\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta T : X \to \mathcal{S}(\mathbb{R}^n) \qquad \text{for each } \alpha, \beta \in \mathbb{N}_0^n.$$

For every iterated commutator of the composition of two operators we can calculate the following formula:

Proposition 2.55. *Let $X, Y, Z \in \{\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)\}$ and $\alpha, \beta \in \mathbb{N}_0^n$. For two linear operators $P : X \to Y$ and $Q : Y \to Z$ we have*

$$\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta (QP) = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} \left[\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} Q\right] \left[\operatorname{ad}(-ix)^{\alpha_2} \operatorname{ad}(D_x)^{\beta_2} P\right].$$

Proof: Let $\beta_1, \beta_2, \gamma \in \mathbb{N}_0^n$ with $|\gamma| = 1$ be arbitrary. Denote $\tilde{P} := \operatorname{ad}(D_x)^{\beta_2} P$ and $\tilde{Q} := \operatorname{ad}(D_x)^{\beta_1} Q$. Applying the definition of iterated commutators twice, we obtain

$$\begin{aligned}
\operatorname{ad}(D_x)^\gamma (\tilde{Q}\tilde{P})u &= D_x^\gamma\{\tilde{Q}\tilde{P}u\} - \tilde{Q}\tilde{P}(D_x^\gamma u) \\
&= D_x^\gamma\{\tilde{Q}\tilde{P}u\} - \tilde{Q}\{D_x^\gamma[\tilde{P}u]\} + \tilde{Q}\{D_x^\gamma[\tilde{P}u]\} - \tilde{Q}\tilde{P}(D_x^\gamma u) \\
&= (\operatorname{ad}(D_x)^\gamma \tilde{Q})\tilde{P}u + \tilde{Q}(\operatorname{ad}(D_x)^\gamma \tilde{P})u.
\end{aligned}$$

By mathematical induction with respect to $|\beta|$ we can easily prove that

$$\operatorname{ad}(D_x)^\beta (QP) = \sum_{\beta_1+\beta_2=\beta} \left[\operatorname{ad}(D_x)^{\beta_1} Q\right] \left[\operatorname{ad}(D_x)^{\beta_2} P\right] \qquad \text{for all } \beta \in \mathbb{N}_0^n. \tag{2.8}$$

Finally we can show the claim by mathematical induction with respect to $|\alpha|$ by means of equality (2.8). □

A well-known result in functional analysis provides the following:

Remark 2.56. Let $\alpha, \beta \in \mathbb{N}_0^n$. Furthermore, let $X, Y \subseteq \mathcal{S}'(\mathbb{R}^n)$ be two Banach spaces, where $\mathcal{S}(\mathbb{R}^n)$ is dense in X. We assume that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear operator with the property:

$$\| \operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta T u\|_Y \leq C\|u\|_X \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

Then $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta T$ can be extended to a linear and bounded operator $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta T : X \to Y$.

Next we show a technical statement needed later on:

Proposition 2.57. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $P : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear operator. Moreover, we denote for every $y \in \mathbb{R}^n$ the translation function $\tau_y(g)$ by g_y. Then we obtain for each $x, y, \xi \in \mathbb{R}^n$ and $\beta, \gamma \in \mathbb{N}_0^n$ with $|\beta|, |\gamma| \leq 1$:*

$$D_x^\beta \left\{ e^{-ix\cdot\xi} P\left[e_\xi D_x^\gamma g_y\right](x) \right\} = \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1} e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{\beta_1} P\left[e_\xi D_x^{\beta_2+\gamma} g_y\right](x).$$

Proof: If $|\beta| = 0$, the claim follows immediately. Thus we assume $\beta = e_j$ for $j \in \{1, \dots, n\}$. First we note that

$$\begin{aligned} &D_{x_j} \left\{ e^{-ix\cdot\xi} P\left[e_\xi D_x^\gamma g_y\right](x) \right\} \\ &\quad = -\xi_j e^{-ix\cdot\xi} P\left[e_\xi D_x^\gamma g_y\right](x) + e^{-ix\cdot\xi} D_{x_j} \left\{ P\left[e_\xi D_x^\gamma g_y\right](x) \right\} \\ &\quad = e^{-ix\cdot\xi} (D_x^\beta - \xi_j) \left\{ P\left[e_\xi D_x^\gamma g_y\right](x) \right\} \qquad \text{for all } x \in \mathbb{R}^n. \end{aligned} \tag{2.9}$$

The definition of the iterated commutator yields for each $x \in \mathbb{R}^n$:

$$\begin{aligned} &e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{e_j} P\left[e_\xi D_x^\gamma g_y\right](x) = e^{-ix\cdot\xi} \operatorname{ad}(D_{x_j}) P\left[e_\xi D_x^\gamma g_y\right](x) \\ &\quad = e^{-ix\cdot\xi} D_{x_j} \left\{ P\left[e_\xi D_x^\gamma g_y\right](x) \right\} - e^{-ix\cdot\xi} P D_{x_j} \left\{ \left[e_\xi D_x^\gamma g_y\right](x) \right\} \\ &\quad = e^{-ix\cdot\xi} D_{x_j} \left\{ P\left[e_\xi D_x^\gamma g_y\right](x) \right\} - e^{-ix\cdot\xi} P \left\{ \left[\xi_j e_\xi D_x^\gamma g_y\right](x) \right\} \\ &\qquad - e^{-ix\cdot\xi} P \left\{ \left[e_\xi D_x^{\gamma+e_j} g_y\right](x) \right\} \\ &\quad = e^{-ix\cdot\xi} (D_{x_j} - \xi_j) \left\{ P\left[e_\xi D_x^\gamma g_y\right](x) \right\} - e^{-ix\cdot\xi} P\left[e_\xi D_x^{\gamma+e_j} g_y\right](x). \end{aligned}$$

Consequently we have for every $x \in \mathbb{R}^n$:

$$\begin{aligned} &e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{e_j} P\left[e_\xi D_x^\gamma g_y\right](x) + e^{-ix\cdot\xi} P\left[e_\xi D_x^{\gamma+e_j} g_y\right](x) \\ &\quad = e^{-ix\cdot\xi} (D_{x_j} - \xi_j) \left\{ P\left[e_\xi D_x^\gamma g_y\right](x) \right\}. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) we conclude

$$\begin{aligned} &D_x^\beta \left\{ e^{-ix\cdot\xi} P\left[e_\xi D_x^\gamma g_y\right](x) \right\} = D_{x_j} \left\{ e^{-ix\cdot\xi} P\left[e_\xi D_x^\gamma g_y\right](x) \right\} \\ &\quad = e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{e_j} P\left[e_\xi D_x^\gamma g_y\right](x) + e^{-ix\cdot\xi} P\left[e_\xi D_x^{\gamma+e_j} g_y\right](x) \\ &\quad = \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1} e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{\beta_1} P\left[e_\xi D_x^{\beta_2+\gamma} g_y\right](x). \end{aligned}$$

□

2.7 Kernel Theorem

The purpose of this section is to get a kernel representation of a linear and bounded operator which maps $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. First of all, we look at the definitions and statements needed for the proof of this kernel representation:

Definition 2.58. Let E be a locally convex Hausdorff topological vector space. Then E is *nuclear* if to every continuous semi-norm p on E there is another continuous semi-norm $q \geq p$ on E with the following property:

Let X and Y be the completions of the normed spaces $E/\ker q$ respectively $E/\ker p$. Additionally let $i : X \to Y$ be the canonical mapping. Then there is a sequence $(f_j')_{j\in\mathbb{N}} \subseteq X'$, where $\{f_j' : j \in \mathbb{N}\}$ is equicontinuous, a bounded sequence $(b_j)_{j\in\mathbb{N}} \subseteq Y$ and a sequence $(\lambda_j)_{j\in\mathbb{N}} \subseteq \mathbb{C}$ with $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, such that

$$i(f) = \sum_{j=1}^{\infty} \lambda_j \langle f_j', f\rangle_{X';X} b_j \qquad \text{for all } f \in X.$$

Moreover, E is called *conuclear* if its dual is nuclear.

One example for a nuclear and conuclear space can be found in [70], p. 530:

Lemma 2.59. *The spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are nuclear. Hence $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are also conuclear due to Remark 2.28.*

We also need the projective topology on the tensor product of two locally convex topological vector spaces:

Definition 2.60. Let E and F be two locally convex topological vector spaces. The *projective topology* on $E \otimes F$ is the strongest locally convex topology on this vector space for which the canonical bilinear mapping $(x, y) \to x \otimes y$ of $E \times F$ into $E \otimes F$ is continuous.

The completion of the tensor product $E \otimes F$ with respect to the projective topology is denoted by $E\hat{\otimes}F$.

For more details we refer to [70].

As an ingredient for the kernel representation we use the next already known isomorphism, which has been proved e.g. in [70], Theorem 51.6:

Theorem 2.61. *The following isomorphism holds:*

$$\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \cong \mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^n).$$

In particular we can extend the mapping $\varphi : \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ *to an isomorphism from* $\mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^n)$ *to* $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$*. Here* φ *is defined for each* $u \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n)$ *by*

$$\varphi(u)(x,y) := \sum_{j=1}^{N} u_j(x)v_j(y) \qquad \text{for all } x, y \in \mathbb{R}^n,$$

where $u_j, v_j \in \mathcal{S}(\mathbb{R}^n)$ *are defined by the representation* $u = \sum_{j=1}^{N} u_j \otimes v_j$ *for every* $j \in \{1, \dots, N\}$*.*

Another isomorphism has been discussed in [11], Theorem 1.4.8:

Theorem 2.62. *Let* F *and* G *be complete locally convex spaces such that* F *is reflexive, nuclear and conuclear. Additionally we define an injective linear map* $\tau : F \otimes G \to \mathscr{L}(F', G)$ *by*

$$\tau(f \otimes g) := \langle ., f \rangle_{F';F}\, g \qquad \text{for all } f \otimes g \in F \otimes G.$$

Then τ *extends to a linear isomorphism which maps* $F\hat{\otimes}G$ *to* $\mathscr{L}(F', G)$*.*

Making use of these statements enables us to show the next kernel theorem:

Theorem 2.63. *Every continuous linear operator* $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *has a Schwartz kernel* $t(x,y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$*. Thus for every* $u \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$Tu(x) = \int_{\mathbb{R}^n} t(x,y)u(y)dy \qquad \text{for all } x \in \mathbb{R}^n.$$

Proof: First of all we define the space X, needed later on, in the following way:

$$X := \left\{ u \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \;\middle|\; \begin{array}{c} \exists N \in \mathbb{N} \text{ and } \exists u_j, v_j \in \mathcal{S}(\mathbb{R}^n), j \in \{1, \dots N\} \\ \text{with } u(x,y) := \sum_{j=1}^{N} u_j(x)v_j(y) \ \forall x, y \in \mathbb{R}^n \end{array} \right\}.$$

The space $\mathcal{S}(\mathbb{R}^n)$ is a nuclear and conuclear, reflexive locally convex vector space because of Lemma 2.59 and Remark 2.28. Therefore we can apply Theorem 2.62 and get that the injective map $\tau : \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n) \to \mathscr{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, which is denoted by

$$\tau(f \otimes g) := \langle ., f \rangle_{\mathcal{S}';\mathcal{S}}\, g \qquad \text{for all } f \otimes g \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n),$$

extends to a linear isomorphism $\tau : \mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^n) \to \mathscr{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$. Now we define the mapping $\varphi : \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ for each $u \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n)$ by

$$\varphi(u)(x,y) := \sum_{j=1}^{N} u_j(x)v_j(y) \qquad \text{for all } x, y \in \mathbb{R}^n,$$

where $u_j, v_j \in \mathcal{S}(\mathbb{R}^n)$ are defined by the representation $u = \sum_{j=1}^N u_j \otimes v_j$ for every $j \in \{1, \ldots, N\}$. Theorem 2.61 provides the extension of φ to an isomorphism from $\mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. In particular φ is injective. Hence the isomorphism $\Phi : \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathscr{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, defined by $\Phi := \tau \circ \varphi^{-1}$, is the extension of the mapping $\Phi : X \to \mathscr{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ defined as

$$\begin{aligned}(\Phi(u)(\psi))(y) &:= \left(\tau(\varphi^{-1}(u))(\psi)\right)(y) = \sum_{j=1}^N \tau\left(u_j \otimes v_j\right)(\psi)(y) \\ &= \sum_{j=1}^N \langle \psi, u_j \rangle_{\mathcal{S}',\mathcal{S}} v_j(y) \qquad \text{for all } \psi \in \mathcal{S}'(\mathbb{R}^n),\ y \in \mathbb{R}^n,\ u \in X.\end{aligned}$$

Here the representation of u is $u(x,y) := \sum_{j=1}^N u_j(x)v_j(y)\ \forall x, y \in \mathbb{R}^n$ where $u_j, v_j \in \mathcal{S}(\mathbb{R}^n)$ and $N \in \mathbb{N}$.

In the case that ψ is a regular distribution, we have $\langle \psi, \eta \rangle_{\mathcal{S}',\mathcal{S}} = \int \psi(x)\eta(x)dx$ for all $\eta \in \mathcal{S}(\mathbb{R}^n)$. Therefore we obtain for all regular distributions ψ, for every $y \in \mathbb{R}^n$ and $u \in X$ with the representation $u(x_1, x_2) = \sum_{j=1}^N u_j(x_1)v_j(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$, where $N \in \mathbb{N}$ and $u_j, v_j \in \mathcal{S}(\mathbb{R}^n)$, $j \in \{1, \ldots N\}$:

$$\begin{aligned}(\Phi(u)(\psi))(y) &= \sum_{j=1}^N \langle \psi, u_j \rangle_{\mathcal{S}',\mathcal{S}}\, v_j(y) = \sum_{j=1}^N \int_{\mathbb{R}^n} \psi(x)u_j(x)dx\, v_j(y) \\ &= \int_{\mathbb{R}^n} \psi(x) \sum_{j=1}^N u_j(x)v_j(y)dx = \int_{\mathbb{R}^n} \psi(x)u(x,y)dx \qquad (2.11)\end{aligned}$$

for all regular distributions ψ. Since Φ is an isomorphism, there is a unique $t \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\Phi(t) = T$ for each $T \in \mathscr{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$. Then equality (2.11) implies for all regular distributions ψ:

$$T\psi = \Phi(t)(\psi) = \int_{\mathbb{R}^n} \psi(x)t(x,.)dx.$$

Thus we have checked that T has the Schwartz kernel $t \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. □

We can extend the previous kernel theorem for the iterated commutators of the linear and bounded operator $P : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$:

Corollary 2.64. *Let $\alpha, \beta \in \mathbb{N}_0^n$ and $P : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear operator. Then the operator $\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta P : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ has a Schwartz kernel $f^{\alpha,\beta} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, i.e. for all $u \in \mathcal{S}(\mathbb{R}^n)$*

$$\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta P u(x) = \int_{\mathbb{R}^n} f^{\alpha,\beta}(x,y)u(y)dy \qquad \textit{for all } x \in \mathbb{R}^n. \qquad (2.12)$$

Proof: Since P maps $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, all iterated commutators of P have the same property as we have seen in Subsection 2.6. Therefore an application of Theorem 2.63 provides the corollary at once. $\square$

Next we mention an application of the kernel theorem needed for the proof of the characterization in Chapter 5:

Lemma 2.65. *Let $g \in \mathcal{S}(\mathbb{R}^n)$. For all $y \in \mathbb{R}^n$ we denote $g_y := \tau_y(g)$. Moreover, let $P : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear and continuous operator with the following property: The function $p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which is defined by*

$$p(x,\xi,y) := e^{-ix\cdot\xi} P\left(e_\xi g_y\right)(x) \qquad \text{for all } x,\xi,y \in \mathbb{R}^n$$

is an element of $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Then differentiation in the sense of tempered distributions yields for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ with $|\alpha|, |\beta|, |\gamma| \leq 1$:

$$\begin{aligned} &\partial_\xi^\alpha D_x^\beta D_y^\gamma p(x,\xi,y) \\ &\quad = (-1)^\gamma \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1} e^{-ix\cdot\xi} \left(\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^{\beta_1} P\right) \left(e_\xi D_x^{\beta_2+\gamma} g_y\right)(x). \end{aligned}$$

Before proving this lemma, we recall the conditions for interchanging a derivative and an integral:

Lemma 2.66. *Let $N \in \mathbb{N}$. Additionally let $k : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{C}$ be a smooth function and $\gamma \in \mathbb{N}_0^N$ be such that*

$$|\partial_x^{\tilde{\gamma}} k(x,z)| \leq g_{\tilde{\gamma}}(z) \in L^1(\mathbb{R}^n_z)$$

for all $x \in \mathbb{R}^N$ and $\tilde{\gamma} \in \mathbb{N}_0^N$ with $\tilde{\gamma} \leq \gamma$. Then

$$\partial_x^\gamma \int\limits_{\mathbb{R}^n} k(x,z)dz = \int\limits_{\mathbb{R}^n} \partial_x^\gamma k(x,z)dz \qquad \text{for all } x \in \mathbb{R}^N.$$

Proof: The claim follows immediately by mathematical induction with respect to $|\gamma|$ if we apply the theorem about interchanging derivatives and integrals, cf. e.g. [44], p.283. $\square$

The previous lemma enables us to verify the following proposition:

Proposition 2.67. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. For all $y \in \mathbb{R}^n$ we denote $g_y := \tau_y(g)$. Then we get for every $x, y, \xi \in \mathbb{R}^n$ and $\beta, \gamma \in \mathbb{N}_0^n$:*

$$\partial_\xi^\gamma D_y^\beta \int\limits_{\mathbb{R}^n} f(x,z) e^{iz\cdot\xi} g_y(z) dz = \int\limits_{\mathbb{R}^n} f(x,z)(iz)^\gamma e^{iz\cdot\xi} D_y^\beta g_y(z) dz.$$

Proof: Let $x \in \mathbb{R}^n$ and $\beta, \gamma \in \mathbb{N}_0^n$ be fixed. Note that $f(x,z)e^{iz\cdot\xi}g_y(z)$ is a function of $C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi \times \mathbb{R}^n_z)$. If we use the definition of Schwartz functions and Corollary 2.5, we obtain for $N \in \mathbb{N}_0$ with $-N < -n$:

$$|f(x,z)| \leq C_N \langle (x,z)\rangle^{-N-|\gamma|} \leq C_{N,x}\langle z\rangle^{-N-|\gamma|} \qquad \text{for all } z \in \mathbb{R}^n, \tag{2.13}$$

where $C_{N,x}$ is independent of $z \in \mathbb{R}^n$. Moreover, $g \in \mathcal{S}(\mathbb{R}^n)$ implies the existence of a constant C_β, independent of $y, z \in \mathbb{R}^n$, such that

$$|\partial_y^\beta g_y(z)| = |(\partial_z^\beta g)(z-y)| \leq C_\beta \qquad \text{for all } y, z \in \mathbb{R}^n. \tag{2.14}$$

Combining (2.13) and (2.14) and using Lemma 2.11 yields

$$\begin{aligned}\left|\partial_\xi^\gamma \partial_y^\beta \left[f(x,z)e^{iz\cdot\xi}g_y(z)\right]\right| &= |f(x,z)| \left|\partial_\xi^\gamma e^{iz\cdot\xi}\right| \left|\partial_y^\beta g_y(z)\right| \\ &\leq C_{N,x,\beta}\langle z\rangle^{-N-|\gamma|}|z|^{|\gamma|} \leq C_{N,x,\beta}\langle z\rangle^{-N} \in L^1(\mathbb{R}^n_z)\end{aligned}$$

where $C_{N,x,\beta}$ is independent of $y, z, \xi \in \mathbb{R}^n$. Finally, an application of Lemma 2.66 finishes the proof. □

With all the work done in the present section we are now in the position to prove Lemma 2.65:

Proof of Lemma 2.65. First of all note that we have shown $e_\xi g_y \in \mathcal{S}(\mathbb{R}^n)$ in Remark 2.24. Due to the continuity of $P : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ the assumptions of Theorem 2.63 hold. Hence P has the Schwartz kernel $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Consequently we have for every $x \in \mathbb{R}^n$:

$$P\left(e_\xi g_y\right)(x) = \int\limits_{\mathbb{R}^n} f(x,z)e^{iz\cdot\xi}g_y(z)dz. \tag{2.15}$$

Additionally one has the following connection between the differentiation with respect to different variables of the function g_y:

$$\begin{aligned}D_y^\gamma g_y(z) &= D_y^\gamma g(z-y) = (-1)^{|\gamma|}\left(D_x^\gamma g\right)(z-y) = (-1)^{|\gamma|}D_z^\gamma g(z-y) \\ &= (-1)^{|\gamma|}D_z^\gamma g_y(z) \qquad \text{for all } z \in \mathbb{R}^n.\end{aligned} \tag{2.16}$$

Applying Proposition 2.67, (2.15) and (2.16), we obtain for all $x \in \mathbb{R}^n$:

$$\begin{aligned}D_y^\gamma\left\{e^{-ix\cdot\xi}P(e_\xi g_y)(x)\right\} &= e^{-ix\cdot\xi}D_y^\gamma \int f(x,z)e^{iz\cdot\xi}g_y(z)dz \\ &= e^{-ix\cdot\xi}\int f(x,z)e^{iz\cdot\xi}D_y^\gamma g_y(z)dz = (-1)^{|\gamma|}e^{-ix\cdot\xi}\int f(x,z)e^{iz\cdot\xi}D_z^\gamma g_y(z)dz \\ &= (-1)^{|\gamma|}e^{-ix\cdot\xi}P(e_\xi D_x^\gamma g_y)(x).\end{aligned}$$

Together with Proposition 2.57 we get for each $x \in \mathbb{R}^n$:

$$\begin{aligned} D_x^\beta D_y^\gamma \left\{e^{-ix\cdot\xi} P(e_\xi g_y)(x)\right\} &= (-1)^{|\gamma|} D_x^\beta \left\{e^{-ix\cdot\xi} P(e_\xi D_x^\gamma g_y)(x)\right\} \\ &= (-1)^{|\gamma|} \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1} e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{\beta_1} P\left(e_\xi D_x^{\beta_2+\gamma} g_y\right)(x). \end{aligned} \tag{2.17}$$

On account of Corollary 2.64 the iterated commutator $\operatorname{ad}(D_x)^{\beta_1}P$ has a Schwarz kernel $f^{\beta_1} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Using Lemma 2.21 first and Remark 2.24 afterwards one can show $e_\xi D_x^{\beta_2+\gamma} g_y \in \mathcal{S}(\mathbb{R}^n)$. Since $(ix)^{\alpha_2} \in C^\infty_{poly}(\mathbb{R}^n_x)$ for each $\alpha_2 \in \mathbb{N}_0^n$, we have $(ix)^{\alpha_2} e_\xi D_x^{\beta_2+\gamma} g_y(x) \in \mathcal{S}(\mathbb{R}^n_x)$ due to Lemma 2.17. Therefore we conclude for every $x \in \mathbb{R}^n$:

$$\begin{aligned} \operatorname{ad}(D_x)^{\beta_1} P\left(e_\xi D_x^{\beta_2+\gamma} g_y\right)(x) &= \int_{\mathbb{R}^n} f^{\beta_1}(x,z) e^{iz\cdot\xi} D_z^{\beta_2+\gamma} g_y(z)\,dz, \\ \operatorname{ad}(D_x)^{\beta_1} P\left((i.)^{\alpha_2} e_\xi D_x^{\beta_2+\gamma} g_y\right)(x) &= \int_{\mathbb{R}^n} f^{\beta_1}(x,z)(iz)^{\alpha_2} e^{iz\cdot\xi} D_z^{\beta_2+\gamma} g_y(z)\,dz. \end{aligned}$$

An application of the Leibniz rule and of Proposition 2.67 yields for all $x \in \mathbb{R}^n$:

$$\begin{aligned} &\partial_\xi^\alpha \left\{e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{\beta_1} P\left(e_\xi D_x^{\beta_2+\gamma} g_y\right)(x)\right\} \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} \left\{\partial_\xi^{\alpha_1} e^{-ix\cdot\xi}\right\} \partial_\xi^{\alpha_2} \int f^{\beta_1}(x,z) e^{iz\cdot\xi} D_z^{\beta_2+\gamma} g_y(z)\,dz \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} (-ix)^{\alpha_1} e^{-ix\cdot\xi} \int f^{\beta_1}(x,z)(iz)^{\alpha_2} e^{iz\cdot\xi} D_z^{\beta_2+\gamma} g_y(z)\,dz \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} (-ix)^{\alpha_1} e^{-ix\cdot\xi} (\operatorname{ad}(D_x)^{\beta_1} P)\left((i.)^{\alpha_2} e_\xi D_x^{\beta_2+\gamma} g_y\right)(x) \\ &= e^{-ix\cdot\xi} (\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^{\beta_1} P)\left(e_\xi D_x^{\beta_2+\gamma} g_y\right)(x). \end{aligned} \tag{2.18}$$

Finally, the combination of (2.17) and (2.18) finishes the proof:

$$\begin{aligned} \partial_\xi^\alpha D_x^\beta D_y^\gamma p(x,\xi,y) &= \partial_\xi^\alpha D_x^\beta D_y^\gamma \left\{e^{-ix\cdot\xi} P\left(e_\xi g_y\right)(x)\right\} \\ &= (-1)^{|\gamma|} \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1} \partial_\xi^\alpha \left\{e^{-ix\cdot\xi} \operatorname{ad}(D_x)^{\beta_1} P\left[e_\xi D_x^{\beta_2+\gamma} g_y\right](x)\right\} \\ &= (-1)^{|\gamma|} \sum_{\beta_1+\beta_2=\beta} \binom{\beta}{\beta_1} e^{-ix\cdot\xi} (\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^{\beta_1} P)\left(e_\xi D_x^{\beta_2+\gamma} g_y\right)(x) \end{aligned}$$

for all $x, \xi, y \in \mathbb{R}^n$. □

Altogether, having presented the basic framework for this thesis, we are now in the position to introduce smooth pseudodifferential operators in the next chapter.

Chapter 3

Smooth Pseudodifferential Operators

Smooth pseudodifferential operators have been introduced by J.J. Kohn and L. Nirenberg, L. Hörmander and others in a natural way by rewriting the symbol of a differential operator in terms of its asymptotic expansion. For a self-contained introduction to the theory of smooth pseudodifferential operators of the class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we refer to the book of H. Abels [5]. An elementary introduction to the theory of smooth pseudodifferential operators is given of X.S. Raymond in [59]. A standard reference about this topic was written by L. Hörmander in [36]. Also e.g. R. Beals [15] and E.M. Stein [66] studied the symbol-class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Additionally E.M. Stein treated some exotic symbol-classes in [66]. As a reference for dealing with the more general Hörmander class we mention e.g. the book of H. Kumano-Go [42].

The present chapter is mainly based on the books of H. Abels [5], H. Kumano-Go [42] and X.S. Raymond [59] and a paper of J. Ueberberg [74]. We introduce smooth pseudodifferential operators with symbols of the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ in this chapter. Mostly we restrict ourselves to the case $\rho = 1$ and $\delta = 0$. In Section 3.1 we define pseudodifferential operators with smooth symbols and their associated symbol-classes. Moreover, we show some first properties of pseudodifferential operators there. Compositions of two pseudodifferential operators are treated in Section 3.2. After that, we present a kernel representation of smooth pseudodifferential operators in Section 3.3. Finally, we focus on some boundedness results of pseudodifferential operators in Section 3.4.

3.1 Smooth Symbol-Classes

This section is devoted to the definition of smooth symbols and their associated pseudodifferential operators. Additionally we prove some first statements for smooth pseudodifferential operators. We begin with the definition of the smooth symbol-class:

Definition 3.1. Let $m \in \mathbb{R}$, $N \in \mathbb{N}$ and $0 \leq \rho, \delta \leq 1$. Then the *Hörmander class* $S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n)$ is the vector space of all smooth functions $p : \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \qquad \text{for all } x \in \mathbb{R}^N, \xi \in \mathbb{R}^n \tag{3.1}$$

holds for all $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^N$. A function $p \in S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n)$ is called *(pseudodifferential) symbol* and m is called *order* of p. Moreover, we define

$$\begin{aligned} S^\infty_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n) &:= \bigcup_{m \in \mathbb{R}} S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n) \text{ and} \\ S^{-\infty}_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n) &:= \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n). \end{aligned}$$

For short we also write $S^m_{\rho,\delta}$ instead of $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

In literature the inequality (3.1) is often exchanged by

$$|\partial_\xi^\alpha D_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \qquad \text{for all } x \in \mathbb{R}^N, \xi \in \mathbb{R}^n.$$

Since just absolute values are verified, the derivatives D_x^β can be exchanged by∂_x^β, so these inequalities are equivalent.

Using the definition of Schwartz functions and of the Hörmander class, it is not difficult to comprehend the following example, cf. e.g. [5], Example 3.3 for *i)* and *ii)*:

Example 3.2.

i) Let $m \in \mathbb{R}$. Then $q(\xi) := \langle \xi \rangle^m \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.

ii) Let $m \in \mathbb{N}_0$ and $c_\alpha \in C_b^\infty(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. Then

$$q(x,\xi) := \sum_{|\alpha| \leq m} c_\alpha(x) \xi^\alpha \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

iii) $\mathcal{S}(\mathbb{R}^n_\xi) \subseteq S^{-\infty}_{1,0}(\mathbb{R}^N_x \times \mathbb{R}^n_\xi)$ for every $N \in \mathbb{N}$.

In the next step we define a family of semi-norms $(|.|_k^{(m)})_{k \in \mathbb{N}_0}$ on the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n)$ in a natural way:

$$|p|_k^{(m)} := \max_{|\alpha|,|\beta| \leq k} \sup_{x \in \mathbb{R}^N, \xi \in \mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle \xi \rangle^{-(m-\rho|\alpha|+\delta|\beta|)}, \tag{3.2}$$

where $p \in S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n)$ and $k \in \mathbb{N}_0$ are arbitrary. It can be shown that these semi-norms turn $S^m_{\rho,\delta}(\mathbb{R}^N \times \mathbb{R}^n)$ into a Fréchet space, cf. e.g. [5], Chapter 3.

Remark 3.3. In the literature the function $\langle\xi\rangle$ in the estimates (3.1) and (3.2) is replaced by $(1+|\xi|)$ several times. This can be done without changing the symbol-classes because of Peetre's inequality, cf. Lemma 2.4. Hence the semi-norm $(|.|_k^{(m)})_{k\in\mathbb{N}_0}$ is equivalent to the semi-norm $(|.|_k^{\prime(m)})_{k\in\mathbb{N}_0}$ which is defined via

$$|p|_k^{\prime(m)} := \max_{|\alpha|,|\beta|\le k} \sup_{x\in\mathbb{R}^N,\xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|(1+|\xi|)^{-(m-\rho|\alpha|+\delta|\beta|)}.$$

J. Ueberberg proved in [74], Lemma 2.1 the following technical lemma we will need later for the characterization of pseudodifferential operators:

Lemma 3.4. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function with $\varphi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\varphi(x) = 0$ for $|x| \ge 1$. Additionally let $0 < \varepsilon \le 1$. We define the functions $p_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ and $q_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$p_\varepsilon(x,\xi) := \varphi(\varepsilon x) \qquad \text{and} \qquad q_\varepsilon(x,\xi) := \varphi(\varepsilon\xi)$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then $\{p_\varepsilon | 0 < \varepsilon \le 1\}$ and $\{q_\varepsilon | 0 < \varepsilon \le 1\}$ are bounded subsets of $S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof: Let $k \in \mathbb{N}_0$ be arbitrary. Since $\varphi \in C_c^\infty(\mathbb{R}^n)$, there is a constant $C_k > 0$ such that the following inequality holds:

$$\max_{|\beta|\le k} \sup_{x\in\mathbb{R}^n} |\partial_x^\beta \varphi(x)| \le C_k. \tag{3.3}$$

Additionally because of the independence of p_ε of ξ we know that $\partial_\xi^\alpha \partial_x^\beta p_\varepsilon \equiv 0$ for all $\alpha, \beta \in \mathbb{N}_0$ with $|\alpha| \ne 0$. On account of estimate (3.3) and the definition of the semi-norms we obtain:

$$\begin{aligned} |p_\varepsilon|_k^{(0)} &= \max_{|\alpha|,|\beta|\le k} \sup_{x,\xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta p_\varepsilon(x,\xi)|\langle\xi\rangle^{|\alpha|} \le \max_{|\beta|\le k} \sup_{x,\xi\in\mathbb{R}^n} |\partial_x^\beta p_\varepsilon(x,\xi)| \\ &= \max_{|\beta|\le k} \sup_{x\in\mathbb{R}^n} |\partial_x^\beta \{\varphi(\varepsilon x)\}| = \max_{|\beta|\le k} \sup_{x\in\mathbb{R}^n} |(\partial_x^\beta \varphi)(\varepsilon x) \cdot \varepsilon^{|\beta|}| \\ &\le \max_{|\beta|\le k} \sup_{x\in\mathbb{R}^n} |(\partial_x^\beta \varphi)(\varepsilon x)| \le C_k, \end{aligned}$$

where C_k is independent of $0 < \varepsilon \le 1$. Hence

$$\{p_\varepsilon | 0 < \varepsilon \le 1\} \subseteq S_{1,0}^0 = \{p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : |p|_k^{(0)} < \infty \text{ for all } k \in \mathbb{N}_0\}$$

is bounded.
The independence of q_ε of $x \in \mathbb{R}^n$ provides for $\alpha, \beta \in \mathbb{N}_0^n$, $|\beta| \ne 0$ and $\xi \in \mathbb{R}^n$: $\partial_\xi^\alpha \partial_x^\beta q_\varepsilon(x,\xi) = 0$. Using the definition of the semi-norms and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we get for arbitrary $0 < \varepsilon \le 1$:

$$|q_\varepsilon|_k^{(0)} = \max_{|\alpha|,|\beta|\le k} \sup_{x,\xi\in\mathbb{R}^n} \{|\partial_\xi^\alpha \partial_x^\beta q_\varepsilon(x,\xi)|\langle\xi\rangle^{|\alpha|}\}$$

$$= \max_{|\alpha|\le k} \sup_{x,\xi\in\mathbb{R}^n} \{|\partial_\xi^\alpha q_\varepsilon(x,\xi)|\langle\xi\rangle^{|\alpha|}\} \le \max_{|\alpha|\le k} \sup_{\xi\in\mathbb{R}^n} |\partial_\xi^\alpha\{\varphi(\varepsilon\xi)\}|\langle\xi\rangle^{|\alpha|}$$
$$= \max_{|\alpha|\le k} \sup_{\xi\in\mathbb{R}^n} |(\partial_\xi^\alpha\varphi)(\varepsilon\xi)\cdot\varepsilon^{|\alpha|}|\langle\xi\rangle^{|\alpha|} \le \max_{|\alpha|\le k} \sup_{\xi\in\mathbb{R}^n} |(\partial_\xi^\alpha\varphi)(\varepsilon\xi)|\langle\varepsilon\xi\rangle^{|\alpha|} \le C_k.$$

Here C_k is not dependent on $0 < \varepsilon \le 1$. The last inequality implies the boundedness of $\{q_\varepsilon | 0 < \varepsilon \le 1\}$ as a subset of $S^0_{1,0}$. □

If $u \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$, too. Hence $p(x,\xi)\hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n_\xi)$ for every fixed $x \in \mathbb{R}^n$ and each symbol $p \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$ due to Lemma 2.17. Therefore $f(x) := \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) đ\xi$ exists for all $x \in \mathbb{R}^n$. One even can check that $f \in \mathcal{S}(\mathbb{R}^n)$, see for example [5], Section 3.1. This enables us to define a linear operator in the following way:

Definition 3.5. For every symbol $p \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \le \rho, \delta \le 1$ and $m \in \mathbb{R}$, we define the *pseudodifferential operator* $p(x, D_x) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ with the symbol p by

$$p(x,D_x)u(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)đ\xi \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n.$$

We also write $\mathrm{OP}(p)$ instead of $p(x, D_x)$. The set of all pseudodifferential operators with symbols in $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is denoted by $\mathrm{OP}S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

For elements of $\mathrm{OP}S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ we also say pseudodifferential operators of the symbol-class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

We mention one well-known special case pseudodifferential operators:

Example 3.6. Let $m \in \mathbb{N}_0$ and $c_\alpha \in C_b^\infty(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$. The associated pseudodifferential operator to the symbol q of Example 3.2, *ii*) is

$$q(x,D_x) := \sum_{|\alpha|\le m} c_\alpha(x) D_x^\alpha.$$

This pseudodifferential operator is a linear differential operator. Consequently every linear differential operator, whose coefficient is in $C_b^\infty(\mathbb{R}^n)$, is a pseudodifferential operator.

Every iterated commutator of a pseudodifferential operator is again a pseudodifferential operator, only the symbol-class can change, as we see in the following remark, cf. e.g. [74], p.461:

Remark 3.7. Let $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$. We assume $p \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. Then for each $\alpha, \beta \in \mathbb{N}_0^n$ the operator $\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta p(x, D_x)$ is a pseudodifferential operator of the class $S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with the symbol $\partial_\xi^\alpha D_x^\beta p(x,\xi)$.

Proof: This claim can be proved by mathematical induction with respect to $|\alpha| + |\beta|$. The case $|\alpha| + |\beta| = 0$ is trivial. We assume that the claim holds for $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq l$, $l \in \mathbb{N}_0$. Let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| = l$ and $j \in \{1, \dots, n\}$ be arbitrary. Then we have to show the claim for $\alpha' := \alpha$ and $\beta' := \beta + e_j$ respectively $\alpha' := \alpha + e_j$ and $\beta' := \beta$. Due to the induction hypothesis, $\tilde{P} := \mathrm{ad}(-ix)^\alpha \, \mathrm{ad}(D_x)^\beta p(x, D_x)$ is an operator of the class $\mathrm{OP}S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}$ with symbol $\tilde{p}(x,\xi) := \partial_\xi^\alpha D_x^\beta p(x,\xi)$. Using integration by parts and some properties of the Fourier transformation, one can calculate for each $u \in \mathcal{S}(\mathbb{R}^n)$ at once that

$$\begin{aligned} \mathrm{ad}(-ix_j)\tilde{P}u(x) &= -ix_j\tilde{P}u(x) + \tilde{P}[ix_j u(x)] = (\partial_{\xi_j}\tilde{p})(x, D_x)u(x), \\ \mathrm{ad}(D_{x_j})\tilde{P}u(x) &= D_{x_j}\{\tilde{P}u(x)\} - \tilde{P}[D_{x_j}u(x)] = (D_{x_j}\tilde{p})(x, D_x)u(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$. Applying $\tilde{p}(x,\xi) \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, we obtain

$$\begin{aligned} (\partial_{\xi_j}\tilde{p})(x,\xi) &\in S^{m-\rho(|\alpha|+1)+\delta|\beta|}_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \qquad \text{and} \\ (D_{x_j}\tilde{p})(x,\xi) &\in S^{m-\rho|\alpha|+\delta(|\beta|+1)}_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi). \end{aligned}$$

□

As a direct consequence of the previous remark, we obtain

Remark 3.8. Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. Moreover, let $p(x), q(\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ be two symbols. Denoting the pseudodifferential operator with symbol p by P and the pseudodifferential operator with symbol q by Q, we have

$$\begin{aligned} \mathrm{ad}(-ix)^\alpha \, \mathrm{ad}(D_x)^\beta P = 0 \qquad &\text{for all } \alpha, \beta \in \mathbb{N}_0^n \text{ with } |\alpha| \neq 0, \\ \mathrm{ad}(-ix)^\alpha \, \mathrm{ad}(D_x)^\beta Q = 0 \qquad &\text{for all } \alpha, \beta \in \mathbb{N}_0^n \text{ with } |\beta| \neq 0. \end{aligned}$$

3.2 Composition of Pseudodifferential Operators

An important question is whether the composition of two pseudodifferential operators is a pseudodifferential operator again. To answer this question, we need the so called oscillatory integral. This special kind of integral is defined on functions of the so-called space of amplitudes which we define now:

Definition 3.9. Let $m, \tau \in \mathbb{R}$. The *space of amplitudes* $\mathscr{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ is the set of all smooth functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial_\eta^\alpha \partial_y^\beta a(y,\eta)| \leq C_{\alpha,\beta}(1+|\eta|)^m(1+|y|)^\tau$$

uniformly in $y, \eta \in \mathbb{R}^n$ for all $\alpha, \beta \in \mathbb{N}_0^n$.

The associated sequence of monotone increasing semi-norms

$$|a|_{\mathscr{A}^m_\tau, k} := \max_{|\alpha|+|\beta| \leq k} \sup_{y,\eta \in \mathbb{R}^n} (1+|\eta|)^{-m}(1+|y|)^{-\tau}|\partial_\eta^\alpha \partial_y^\beta a(y,\eta)|, \quad k \in \mathbb{N}_0$$

turn $\mathscr{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ into a Fréchet space.

We easily can convince ourself that the space of amplitudes is not empty: Considering arbitrary $m, \tau \in \mathbb{R}$ the function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is defined by $a(y,\eta) := \langle y\rangle^\tau \langle \eta\rangle^m$ is an element of $\mathscr{A}_\tau^m(\mathbb{R}^n \times \mathbb{R}^n)$ due to Remark 2.8 and Lemma 2.4, *ii*). In the previous section we also treated some elements of the space of amplitudes:

Example 3.10. For $m \in \mathbb{R}$ and $l \in \mathbb{N}_0$ we have due to the definitions of the symbol-class, of the Schwartz space and of the space of amplitudes:

i) $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq \mathscr{A}_0^m(\mathbb{R}^n \times \mathbb{R}^n)$,

ii) $\mathcal{S}(\mathbb{R}^n_y) \subseteq \mathscr{A}^0_{-l}(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$

with continuous embedding.

Moreover, Definition 3.9 provides for $a \in \mathscr{A}_\tau^m(\mathbb{R}^n \times \mathbb{R}^n)$ with $m, \tau \in \mathbb{R}$ and for all $\alpha \in \mathbb{N}_0^n$:

- $D^\alpha_\eta a(y,\eta), D^\alpha_y a(y,\eta) \in \mathscr{A}_\tau^m(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$,
- $y^\alpha a(y,\eta) \in \mathscr{A}^m_{\tau+|\alpha|}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$,
- $\eta^\alpha a(y,\eta) \in \mathscr{A}_\tau^{m+|\alpha|}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$.

Now we are able to define the oscillatory integral for elements of the space of amplitudes:

Theorem 3.11. *Let $m, \tau \in \mathbb{R}$ and $a \in \mathscr{A}_\tau^m(\mathbb{R}^n \times \mathbb{R}^n)$. Additionally we choose an arbitrary $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. Then the* ***oscillatory integral*** *of a defined by*

$$Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta := \lim_{\varepsilon\to 0} \iint \chi(\varepsilon y, \varepsilon\eta) e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta$$

exists. In particular the definition does not depend on the choice of χ. For all $l, l' \in \mathbb{N}_0$ with $2l > n+m$ and $2l' > n+\tau$ the oscillatory integral has the following property:

$$Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta = \iint e^{-iy\cdot\eta} \langle y\rangle^{-2l'} \langle D_\eta\rangle^{2l'} [\langle\eta\rangle^{-2l} \langle D_y\rangle^{2l} a(y,\eta)]\,dy\,đ\eta,$$

where integrand is in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. In particular the next estimate holds:

$$\left| Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta \right| \le C_{m,\tau} |a|_{\mathscr{A}_\tau^m, 2(l+l')}.$$

Here $C_{m,\tau} > 0$ is independent of $a \in \mathscr{A}_\tau^m(\mathbb{R}^n \times \mathbb{R}^n)$.

The proof of this theorem can be found for example in [5], Theorem 3.9. Note that we also can define the oscillatory integral of functions $a \in L^1(\mathbb{R}^n; L^1(\mathbb{R}^n))$ as in the previous theorem. Applying Lebesgue's theorem, we can verify that

$$\text{Os-}\iint e^{-iy\cdot\eta} a(y,\eta)dy đ\eta = \iint e^{-iy\cdot\eta} a(y,\eta)dy đ\eta. \tag{3.4}$$

As an ingredient for the characterization of non-smooth pseudodifferential operators in Section 5.4 the next technical remark is needed:

Remark 3.12 (Inversion formula). Let $u \in C_b^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then the function $(y,\eta) \mapsto a(y,\eta) = e^{ix\cdot\eta}u(y)$ is an element of $\mathcal{A}_0^0(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)$ and the inversion formula holds:

$$\text{Os-}\iint e^{i(x-y)\cdot\eta} u(y)dy đ\eta = u(x).$$

For the proof we refer to [5], Example 3.11.

Next we turn to the main topic of the present section: Is the composition of two pseudodifferential operators again a pseudodifferential operator? An answer has been given for example in [5], Theorem 3.16:

Theorem 3.13. *Let $p_j \in S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m_j \in \mathbb{R}$ for $j \in \{1,2\}$. Then there is a symbol $p_1 \# p_2 \in S^{m_1+m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$p_1(x,D_x)p_2(x,D_x) = (p_1\#p_2)(x,D_x).$$

Moreover, if $p_1(x,D_x)$ is a differential operator of the order $m_1 \in \mathbb{N}_0$ with coefficients in $C_b^\infty(\mathbb{R}^n)$, then

$$(p_1\#p_2)(x,\xi) = \sum_{|\alpha|\le m_1} \frac{1}{\alpha!}\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) \qquad \text{for all } x,\xi \in \mathbb{R}^n.$$

3.3 Kernel Representation of a Pseudodifferential Operator

The present section is devoted to the kernel representation of smooth pseudodifferential operators, cf. e.g. [5], Chapter 5.4. This turned out to be an important tool for dealing with smooth pseudodifferential operators. Making use of this kernel representation allows us to prove the boundedness of a smooth pseudodifferential operator between certain Bessel potential spaces. Details can be found in Section 3.4.

Theorem 3.14. *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \in \mathbb{R}$. Then there is a smooth function $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$p(x, D_x)u(x) = \int k(x, x-y)u(y)dy \qquad \text{for all } x \notin \text{supp } u.$$

If the symbol p just depends on the second variable ξ, then the kernel k is independent of the first variable. Moreover, for every $\alpha, \beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ the kernel k satisfies

$$|\partial_x^\beta \partial_z^\alpha k(x,z)| \leq \begin{cases} C_{\alpha,\beta,N}|z|^{-n-m-|\alpha|}\langle z\rangle^{-N} & \text{if } n+m+|\alpha| > 0, \\ C_{\alpha,\beta,N}(1+|\log|z||)\langle z\rangle^{-N} & \text{if } n+m+|\alpha| = 0, \\ C_{\alpha,\beta,N}\langle z\rangle^{-N} & \text{if } n+m+|\alpha| < 0 \end{cases}$$

uniformly in $x, z \in \mathbb{R}^n$, $z \neq 0$.

We refer e.g. to [5], Theorem 5.12 for the proof.

To conclude, this theorem gives a good estimate of the kernel of a smooth pseudodifferential operator off the diagonal.

3.4 Boundedness on Different Function Spaces

Being linear and bounded as a map between different function spaces is an important property of pseudodifferential operators. Here we present boundedness results on Hölder spaces, on Bessel potential spaces, on the Schwartz space and on its dual space. Making use of these results, we are able to prove a characterization of the Bessel potential spaces.

To begin with, we treat the boundedness of pseudodifferential operators on Schwartz spaces. For the proof see e.g. [5], Theorem 3.6.

Theorem 3.15. *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \in \mathbb{R}$, be a pseudodifferential symbol. Then*

$$p(x, D_x) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a bounded mapping. More precisely, for every $k \in \mathbb{N}_0$ there is a constant $C_k > 0$ such that

$$|p(x,D_x)f|_{k,\mathcal{S}} \leq C_k |p|_k^{(m)} |f|_{\tilde{m},\mathcal{S}} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n),$$

where $\tilde{m} := \max\{0, m+2(n+1)+k\}$ if $m \in \mathbb{Z}$ and $\tilde{m} := \max\{0, \lfloor m \rfloor + 2n+3+k\}$ else.

Up to now, pseudodifferential operators are just defined as linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. Our next goal is to extend pseudodifferential operators to linear maps from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. This purpose can be reached using a duality argument. For this we need to know that the adjoint operator of a pseudodifferential operator is a pseudodifferential operator again as stated in the following lemma.

Lemma 3.16. *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \in \mathbb{R}$. Then the adjoint operator $p(x, D_x)^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ defined by*

$$(p(x, D_x)u, v)_{L^2(\mathbb{R}^n)} = (u, p(x, D_x)^* v)_{L^2(\mathbb{R}^n)} \qquad \text{for all } u, v \in \mathcal{S}(\mathbb{R}^n)$$

is a pseudodifferential operator with symbol

$$p^*(x,\xi) = Os\text{-}\iint e^{-iy\cdot\eta}\overline{p(x+y,\xi+\eta)}dy\,đ\eta \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

We refer to [5], Corollary 3.34 for the proof.

The previous lemma enables us to extend the definition of pseudodifferential operators to linear maps from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$:

Definition 3.17. Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \in \mathbb{R}$. Then we define the pseudodifferential operator $p(x, D_x) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle p(x, D_x)u, \overline{v}\rangle_{\mathcal{S}';\mathcal{S}} := \langle u, \overline{p(x, D_x)^* v}\rangle_{\mathcal{S}';\mathcal{S}}$$

for all $u \in \mathcal{S}'(\mathbb{R}^n), v \in \mathcal{S}(\mathbb{R}^n)$.

For functions $u \in \mathcal{S}(\mathbb{R}^n)$ the previous definition of $p(x, D_x)u$ coincides with our first definition of this term, cf. Definition 3.5. This is shown for example in [5], Section 3.8.

Due to the previous definition we know that an application of pseudodifferential operators on functions belonging to the Bessel potential space is possible. The boundedness of pseudodifferential operators as maps between two Bessel potential spaces is treated in the next theorem:

Theorem 3.18. *Let $m \in \mathbb{R}$, $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $1 < q < \infty$. Then $p(x, D_x)$ extends to a bounded linear operator*

$$p(x, D_x) : H^{s+m}_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n) \qquad \text{for all } s \in \mathbb{R}.$$

More precisely, there are some $C_{s,m,q} > 0$ and $k \in \mathbb{N}_0$, independent of p, such that

$$\|p(x, D_x)\|_{\mathscr{L}(H^{s+m}_q, H^s_q)} \le C_{s,m,q}|p|^{(m)}_k \qquad \text{for all } s \in \mathbb{R}.$$

For the proof we refer to e.g. [5], Theorem 5.20 and Remark 5.21.

The previous theorem enables us to prove a characterization of the Bessel potential spaces. The last missing piece towards this result is the following proposition:

Proposition 3.19. *Let $1 < p < \infty$ and $s < 0$. We assume a partition of unity $(\psi_j)_{j\in\mathbb{Z}^n} \subseteq C_c^\infty(\mathbb{R}^n)$ with the following property:*

$$\psi_j(x) = \psi_0(x-j) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{Z}^n.$$

Such a partition of unity exists due to Lemma 2.15. Then for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq -\lfloor s \rfloor$ we have

$$\|[\partial_x^\alpha, \psi_j]f\|_{H_p^s} \leq C\|f\|_{H_p^{s-\lfloor s \rfloor}} \qquad \text{for all } f \in H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n), \tag{3.5}$$

where C is independent of $j \in \mathbb{Z}^n$ and of $f \in H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n)$.

Proof: Let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$ be arbitrary. Then we get for all $j \in \mathbb{Z}^n$ and all $x, \xi \in \mathbb{R}^n$:

- $|\partial_\xi^\alpha \partial_x^\beta \psi_j(x)| = 0 \leq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}$,
- $|\partial_x^\beta \psi_j(x)| = |(\partial_x^\beta \psi_0)(x-j)| \leq C_\beta$.

Since $C_{\alpha,\beta}$ and C_β are independent of $j \in \mathbb{Z}^n$ and $x, \xi \in \mathbb{R}^n$, the previous inequalities imply the boundedness of $\{\psi_j\}_{j\in\mathbb{Z}^n} \subseteq S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Now let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq -\lfloor s \rfloor$ be arbitrary. Then ∂_x^α is a pseudodifferential operator with symbol $(i\xi)^\alpha \in S^{|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. An application of Theorem 3.18 provides for all $f \in H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n)$:

$$\begin{aligned}\|[\partial_x^\alpha, \psi_j]f\|_{H_p^s} &\leq \|\partial_x^\alpha(\psi_j f)\|_{H_p^s} + \|\psi_j(\partial_x^\alpha f)\|_{H_p^s} \\ &\leq C\|\psi_j f\|_{H_p^{s+|\alpha|}} + C\|\partial_x^\alpha f\|_{H_p^s} \leq C\|f\|_{H_p^{s+|\alpha|}} \leq C\|f\|_{H_p^{s-\lfloor s \rfloor}}.\end{aligned}$$

Here the constant C is independent of $j \in \mathbb{Z}^n$ and $f \in H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n)$. □

With the previous proposition at hand, we are able to check the next characterization of Bessel potential spaces. This characterization already exists for Bessel potential spaces of the order $s \geq 0$. For the proof see for example [54], Theorem 1.3.

Proposition 3.20. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. Moreover, let $(\psi_j)_{j\in\mathbb{Z}^n} \subseteq C_c^\infty(\mathbb{R}^n)$ be a partition of unity with the properties:*

- $\psi_0(x) = 1$ *for all* $x \in [0,1]^n$,

- *$\psi_j(x) = \psi_0(x-j)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}^n$.*

Such a partition of unity exists due to Lemma 2.15. Then we obtain the following norm equivalence:

$$\|f\|_{H_p^s(\mathbb{R}^n)} \simeq \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s(\mathbb{R}^n)}^p \right)^{1/p}$$

Proof: The case $s \geq 0$ follows directly from [54], Theorem 1.3. Therefore let $s < 0$ be arbitrary. First of all we show

$$\|f\|_{H_p^s(\mathbb{R}^n)} \leq C \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s(\mathbb{R}^n)}^p \right)^{1/p} \tag{3.6}$$

by duality. Let $f \in H_p^s(\mathbb{R}^n)$ and $g \in H_q^{-s}(\mathbb{R}^n)$ with $1/p + 1/q = 1$ be arbitrary. Moreover, we define $\eta_j : \mathbb{R}^n \to \mathbb{C}$ for all $j \in \mathbb{Z}^n$ by

$$\eta_0 = \sum_{k\in Z} \psi_k, \qquad \text{where } Z := \{k \in \mathbb{Z}^n : \operatorname{supp} \psi_0 \cap \operatorname{supp} \psi_k \neq \varnothing\}$$

and $\eta_j(x) := \eta_0(x-j)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}^n$. An application of the partition of unity and Hölder's inequality for sequence spaces first and the case $-s > 0$ afterwards provides

$$\begin{aligned}
\left|\langle f, g\rangle_{H_p^s;H_q^{-s}}\right| &\leq \sum_{j\in\mathbb{Z}^n} \left|\langle \psi_j f, g\rangle_{H_p^s;H_q^{-s}}\right| = \sum_{j\in\mathbb{Z}^n} \left|\langle \eta_j \psi_j f, g\rangle_{H_p^s;H_q^{-s}}\right| \\
&= \sum_{j\in\mathbb{Z}^n} \left|\langle \psi_j f, \eta_j g\rangle_{H_p^s;H_q^{-s}}\right| \leq \sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s} \|\eta_j g\|_{H_q^{-s}} \\
&\leq \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s}^p \right)^{1/p} \left(\sum_{j\in\mathbb{Z}^n} \|\eta_j g\|_{H_q^{-s}}^q \right)^{1/q} \\
&\leq C_{q,Z} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s}^p \right)^{1/p} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j g\|_{H_q^{-s}}^q \right)^{1/q} \\
&\leq C_{q,Z} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s}^p \right)^{1/p} \|g\|_{H_q^{-s}}.
\end{aligned}$$

Consequently we get (3.6) by duality:

$$\|f\|_{H_p^s} = \sup_{\|g\|_{H_q^{-s}} \leq 1} \left|\langle f, g\rangle_{H_p^s;H_q^{-s}}\right| \leq C_{q,Z} \left(\sum_{j\in\mathbb{Z}^n} \|\psi_j f\|_{H_p^s}^p \right)^{1/p}$$

for all $f \in H_p^s(\mathbb{R}^n)$. Thus it remains to check for each $f \in H_p^s(\mathbb{R}^n)$:

$$\left(\sum_{j \in \mathbb{Z}^n} \|\psi_j f\|_{H_p^s}^p \right)^{1/p} \leq C\|f\|_{H_p^s}. \tag{3.7}$$

We define the functions η_j for every $j \in \mathbb{Z}^n$ as before and $m := -\lfloor s \rfloor$. Additionally we choose an arbitrary $f \in H_p^s(\mathbb{R}^n)$. The existence of some functions $g_\alpha \in H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, fulfilling the two properties $f = \sum_{|\alpha| \leq m} \partial_x^\alpha g_\alpha$ and $\sum_{|\alpha| \leq m} \|g_\alpha\|_{H_p^{s-\lfloor s \rfloor}} \leq C\|f\|_{H_p^s}$ was verified in Lemma 2.49. Together with Proposition 3.19 and $\eta_j \equiv 1$ on supp ψ_j we obtain

$$\begin{aligned}
\|\psi_j f\|_{H_p^s}^p &\leq \sum_{|\alpha| \leq m} \|\psi_j \partial_x^\alpha g_\alpha\|_{H_p^s}^p = \sum_{|\alpha| \leq m} \|\psi_j \partial_x^\alpha \{g_\alpha \eta_j\}\|_{H_p^s}^p \\
&\leq \sum_{|\alpha| \leq m} \left\{ \|\partial_x^\alpha \{\psi_j g_\alpha \eta_j\}\|_{H_p^s} + \|[\partial_x^\alpha, \psi_j](g_\alpha \eta_j)\|_{H_p^s} \right\}^p \\
&\leq \sum_{|\alpha| \leq m} \left\{ \|\partial_x^\alpha \{\psi_j g_\alpha\}\|_{H_p^s} + C\|g_\alpha \eta_j\|_{H_p^{s-\lfloor s \rfloor}} \right\}^p \\
&\leq \sum_{k \in Z+j} \sum_{|\alpha| \leq m} \left\{ \|\partial_x^\alpha \{\psi_k g_\alpha\}\|_{H_p^s} + C\|g_\alpha \psi_k\|_{H_p^{s-\lfloor s \rfloor}} \right\}^p .
\end{aligned}$$

On account of $\partial_x^\alpha \in \mathrm{OP}S_{1,0}^{|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n)$, Theorem 3.18 yields the boundedness of $\partial_x^\alpha : H_p^{s-\lfloor s \rfloor}(\mathbb{R}^n) \to H_p^{-|\alpha|+s-\lfloor s \rfloor}(\mathbb{R}^n) \subseteq H_p^s(\mathbb{R}^n)$. Hence the previous inequality can be estimated by

$$\begin{aligned}
\|\psi_j f\|_{H_p^s}^p &\leq \sum_{k \in Z+j} \sum_{|\alpha| \leq m} \left\{ \|\partial_x^\alpha \{\psi_k g_\alpha\}\|_{H_p^s} + C\|g_\alpha \psi_k\|_{H_p^{s-\lfloor s \rfloor}} \right\}^p \\
&\leq C \sum_{k \in Z+j} \sum_{|\alpha| \leq m} \|\psi_k g_\alpha\|_{H_p^{s-\lfloor s \rfloor}}^p .
\end{aligned}$$

Using the case $s \geq 0$ again provides (3.7):

$$\begin{aligned}
\sum_{j \in \mathbb{Z}^n} \|\psi_j f\|_{H_p^s} &\leq C \sum_{j \in \mathbb{Z}^n} \sum_{k \in Z+j} \sum_{|\alpha| \leq m} \|\psi_k g_\alpha\|_{H_p^{s-\lfloor s \rfloor}}^p \leq C \sum_{|\alpha| \leq m} \sum_{j \in \mathbb{Z}^n} \|\psi_j g_\alpha\|_{H_p^{s-\lfloor s \rfloor}}^p \\
&\leq C \sum_{|\alpha| \leq m} \|g_\alpha\|_{H_p^{s-\lfloor s \rfloor}}^p \leq C\|f\|_{H_p^s}.
\end{aligned}$$

□

Additionally pseudodifferential operators are linear and bounded as maps between two Hölder-Zygmund spaces. This has been proved for example in [5], Theorem 6.19.

Theorem 3.21. *Let $m \in \mathbb{R}$ and $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then $p(x, D_x)$ extends to a bounded linear operator*

$$p(x, D_x) : C^{s+m}_*(\mathbb{R}^n) \to C^s_*(\mathbb{R}^n) \qquad \text{for all } s > 0 \text{ with } s + m > 0.$$

If the symbol p is an element of $S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we even get the next result:

Corollary 3.22. *For $p \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we have*

$$p(x, D_x) : \mathcal{S}'(\mathbb{R}^n) \to C^\infty_{poly}(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n).$$

X.S. Raymond proved this corollary in [59], Corollary 3.8. We even are able to show the continuity of $p(x, D_x) : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ for symbols p being in the class $S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. This is the topic of the next lemma:

Lemma 3.23. *We consider $C^\infty(\mathbb{R}^n)$ as a Fréchet space with the semi-norms $(|.|_{m,K})_{m \in \mathbb{N}_0, K \subseteq \mathbb{R}^n \text{ compact}}$, where*

$$|f|_{m,K} := \sup_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha_x f(x)| \qquad \text{for all } f \in C^\infty(\mathbb{R}^n).$$

Assuming $p \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ provides the continuity of the map

$$p(x, D_x) : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n).$$

The proof of this lemma is based on the main idea of the proof of [59], Corollary 3.8.

Additionally we need the next remark:

Remark 3.24. For any polynomial p and each symbol $a \in S^\infty_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, one has for all $x \in \mathbb{R}^n$, $u \in \mathcal{S}'(\mathbb{R}^n)$:

$$a(x, D_x)[pu](x) = \sum_{\beta \in \mathbb{N}^n_0} \frac{1}{\beta!} \left(D^\beta_x p(x)\right) \left(\partial^\beta_\xi a\right)(x, D_x) u(x).$$

This remark has been proved for example in [59], Example 3.5.

With this remark at hand, we are able to show Lemma 3.23:

Proof of Lemma 3.23: On account of Corollary 3.22, $p(x, D_x)$ maps $\mathcal{S}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. Hence it remains to show the continuity of this operator. Due to Remark 2.28, Corollary 1 of page 154 in [62] and Lemma 8.3 in [62], Chapter II we just have to show the convergence of $(p(x, D_x)u_l)_{l \in \mathbb{N}}$ in $C^\infty(\mathbb{R}^n)$ for each convergent sequence $(u_l)_{l \in \mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$. For this purpose we choose an arbitrary sequence $(u_l)_{l\in\mathbb{N}} \subseteq \mathcal{S}'(\mathbb{R}^n)$, which converges to $u \in \mathcal{S}'(\mathbb{R}^n)$. Applying Proposition 2.29, there is a $\kappa \in \mathbb{N}$ and a constant $C > 0$ such that

$$|\langle u, \varphi\rangle_{\mathcal{S}';\mathcal{S}}| \leq C|\varphi|_{\kappa,\mathcal{S}} \quad \text{and} \quad |\langle u_l, \varphi\rangle_{\mathcal{S}';\mathcal{S}}| \leq C|\varphi|_{\kappa,\mathcal{S}} \qquad \text{for all } l \in \mathbb{N}, \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Due to Proposition 2.48, there is an $N \in \mathbb{N}$ with the property

$$\begin{aligned} v(x) &:= (1+|x|^2)^{-N} u(x) \in H_2^{-N}(\mathbb{R}^n_x) \qquad \text{and} \\ v_l(x) &:= (1+|x|^2)^{-N} u_l(x) \in H_2^{-N}(\mathbb{R}^n_x) \end{aligned}$$

for every $l \in \mathbb{N}$. We define the symbol $b_\alpha := \xi^\alpha \# p$ for every $\alpha \in \mathbb{N}_0^n$ as in Theorem 3.13. Then we can write for all $l \in \mathbb{N}$:

$$D_x^\alpha\left[p(x, D_x)u_l\right] = b_\alpha(x, D_x)u_l \qquad \text{and} \qquad D_x^\alpha\left[p(x, D_x)u\right] = b_\alpha(x, D_x)u.$$

Here $\xi^\alpha \in S^{|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ because of Example 3.2, *ii*). Since $p \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we arise from Theorem 3.13 that $b_\alpha \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Consequently $\partial_\xi^\beta b_\alpha$ is an element of $S^{-\infty}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ for each $\beta \in \mathbb{N}_0^n$. Therefore the definition of v and Remark 3.24 provides for all $x \in \mathbb{R}^n$:

$$\begin{aligned} b_\alpha(x, D_x)u(x) &= b_\alpha(x, D_x)\left[(1+|x|^2)^N v\right](x) \\ &= \sum_{|\beta|\leq 2N} \frac{1}{\beta!} D_x^\beta (1+|x|^2)^N (\partial_\xi^\beta b_\alpha)(x, D_x)v(x). \end{aligned}$$

In the same way one gets for every $l \in \mathbb{N}$ and for all $x \in \mathbb{R}^n$:

$$b_\alpha(x, D_x)u_l(x) = \sum_{|\beta|\leq 2N} \frac{1}{\beta!} D_x^\beta (1+|x|^2)^N (\partial_\xi^\beta b_\alpha)(x, D_x)v_l(x).$$

Now we choose $m \in \mathbb{Z}$ such that $-N - m - \frac{n}{2} > 0$. Since $\partial_\xi^\beta b_\alpha \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we in particular have $\partial_\xi^\beta b_\alpha \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. On account of Theorem 3.18 we obtain that $(\partial_\xi^\beta b_\alpha)(x, D_x)$ is an element of $\mathscr{L}(H_2^{-N}, H_2^{-N-m})$.

Next we show the weak convergence of the sequence $(v_l)_{l\in\mathbb{N}}$ to v in $H_2^{-N}(\mathbb{R}^n)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary. Then the definition of v_l provides:

$$\begin{aligned} \langle v_l, \psi\rangle_{H_2^{-N};H_2^N} &= \langle v_l, \psi\rangle_{\mathcal{S}';\mathcal{S}} = \langle (1+|x|^2)^{-N} u_l, \psi\rangle_{\mathcal{S}';\mathcal{S}} = \langle u_l, (1+|x|^2)^{-N}\psi\rangle_{\mathcal{S}';\mathcal{S}} \\ &\xrightarrow{l\to\infty} \langle u, (1+|x|^2)^{-N}\psi\rangle_{\mathcal{S}';\mathcal{S}} = \langle v, \psi\rangle_{\mathcal{S}';\mathcal{S}} = \langle v, \psi\rangle_{H_2^{-N};H_2^N}. \qquad (3.8) \end{aligned}$$

In view of Proposition 2.29, we can apply Proposition 2.48 and get for all $l \in \mathbb{N}$

$$|\langle v_l, \psi\rangle_{H_2^{-N};H_2^N}| = |\langle v_l, \psi\rangle_{\mathcal{S}';\mathcal{S}}| \leq C\|\psi\|_{H_2^N(\mathbb{R}^n)} \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^n), \qquad (3.9)$$

where C is not dependent on $l \in \mathbb{N}$. Now we choose an arbitrary $\varphi \in H_2^N(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_2^N(\mathbb{R}^n)$, as we have seen in Lemma 2.43, there is a sequence $(\varphi_k)_{k\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^n)$, which converges to φ. The inequality (3.9) yields $|\langle v_l, \varphi_k\rangle_{H_2^{-N};H_2^N}| \leq C\|\varphi_k\|_{H_2^N(\mathbb{R}^n)}$ for all $l,k \in \mathbb{N}$. If k converges to ∞, the last inequality implies

$$|\langle v_l, \varphi\rangle_{H_2^{-N};H_2^N}| \leq C\|\varphi\|_{H_2^N(\mathbb{R}^n)} \qquad \text{for all } l \in \mathbb{N}. \tag{3.10}$$

Combining (3.8) and (3.10), we have for all $k \in \mathbb{N}$:

$$\begin{aligned}|\langle v_l - v, \varphi\rangle_{H_2^{-N};H_2^N}| &\leq |\langle v_l - v, \varphi_k\rangle_{H_2^{-N};H_2^N}| + |\langle v_l - v, \varphi - \varphi_k\rangle_{H_2^{-N};H_2^N}| \\ &\leq |\langle v_l - v, \varphi_k\rangle_{H_2^{-N};H_2^N}| + C\|\varphi - \varphi_k\|_{H_2^N} \xrightarrow{l\to\infty} C\|\varphi - \varphi_k\|_{H_2^N}.\end{aligned}$$

Consequently we get, if k converges to ∞:

$$\langle v_l, \varphi\rangle_{H_2^{-N};H_2^N} \xrightarrow{l\to\infty} \langle v, \varphi\rangle_{H_2^{-N};H_2^N}.$$

Hence the sequence $(v_l)_{l\in\mathbb{N}}$ converges weakly to v. Since $(\partial_\xi^\beta b_\alpha)(x,D_x)$ is an element of $\mathscr{L}(H_2^{-N}, H_2^{-N-m})$, we obtain for every $\alpha, \beta \in \mathbb{N}_0^n$:

$$(\partial_\xi^\beta b_\alpha)(x,D_x)v_l \rightharpoonup (\partial_\xi^\beta b_\alpha)(x,D_x)v \qquad \text{in } H_2^{-N-m}(\mathbb{R}^n) \text{ if } l \to \infty. \tag{3.11}$$

Now let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open and bounded set. Then the embedding theorem from Sobolev spaces in Hölder spaces, cf. e.g. [8], Theorem 8.13, implies the continuous embedding $H_2^{-N-m}(\Omega) \subseteq C^\tau(\overline{\Omega})$ if $\tau \notin \mathbb{Z}$ with the property $0 < \tau \leq -N - m - \frac{n}{2}$. Additionally the embedding theorem in Hölder spaces, which could be found e.g. in [8], Theorem 8.6, provides the compactness of the embedding $C^\tau(\overline{\Omega}) \subseteq C^0(\overline{\Omega})$. The composition of a compact and a continuous embedding is compact again. Hence we also get the compactness of the embedding $H_2^{-N-m}(\Omega) \subseteq C^0(\overline{\Omega})$. Let us recall that $C^0(\mathbb{R}^n)$ is a Fréchet space with the semi-norms $(\|.\|_{C^0(\overline{\Omega}_i)})_{i\in\mathbb{N}}$, where $\Omega_i \subseteq \mathbb{R}^n$ is an open and bounded set with $\Omega_i \subseteq \Omega_{i+1}$ for all $i \in \mathbb{N}$ and $\bigcup_{i=1}^\infty \Omega_i = \mathbb{R}^n$. Hence we obtain the compactness of the embedding $H_2^{-N-m}(\mathbb{R}^n) \subseteq C^0(\mathbb{R}^n)$. Compact mappings map weak convergent sequences to strong convergent sequences. Therefore (3.11) implies for each $\alpha, \beta \in \mathbb{N}_0^n$:

$$(\partial_\xi^\beta b_\alpha)(x,D_x)v_l \xrightarrow{l\to\infty} (\partial_\xi^\beta b_\alpha)(x,D_x)v \qquad \text{in } C^0(\mathbb{R}^n).$$

An application of this convergence gives us for every $\alpha \in \mathbb{N}_0^n$ and all $x \in \mathbb{R}^n$:

$$\begin{aligned}b_\alpha(x,D_x)u_l(x) &= \sum_{|\beta|\leq 2N} \frac{1}{\beta!} D_x^\beta (1+|x|^2)^N (\partial_\xi^\beta b_\alpha)(x,D_x)v_l(x) \\ &\xrightarrow{l\to\infty} \sum_{|\beta|\leq 2N} \frac{1}{\beta!} D_x^\beta (1+|x|^2)^N (\partial_\xi^\beta b_\alpha)(x,D_x)v(x) = b_\alpha(x,D_x)u(x)\end{aligned}$$

in $C^0(\mathbb{R}^n_x)$. The chosen topology of $C^\infty(\mathbb{R}^n)$ implies the statement of the lemma:

$$p(x, D_x)u_l \xrightarrow{l\to\infty} p(x, D_x)u \qquad \text{in } C^\infty(\mathbb{R}^n).$$

□

With this lemma at hand, we are able to show the continuity of a special kind of pseudodifferential operator, which is called smoothing operator, as a map from $\mathcal{S}'(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, cf. [74], Lemma 2.1:

Lemma 3.25. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function which satisfies $\varphi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ for $|x| \geq 1$. Additionally let $0 < \varepsilon \leq 1$. We define $q_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ via*

$$q_\varepsilon(x, \xi) := \varphi(\varepsilon\xi)$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then $q_\varepsilon(D_x) : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is continuous.

Proof: On account of Example 3.2, *ii*), we have each fixed $\varepsilon \in (0, 1]$:

$$\varphi(\varepsilon\xi) \in C_c^\infty(\mathbb{R}^n_\xi) \subseteq \mathcal{S}(\mathbb{R}^n_\xi) \subseteq S^{-\infty}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

Then we get the lemma by applying Lemma 3.23. □

We will use this result in order to get a characterization of non-smooth pseudodifferential operators with coefficients in the Hölder spaces.

This closes this section and also the chapter. Beyond the composition of two smooth pseudodifferential operators and the kernel representation we presented some boundedness results of smooth pseudodifferential operators. All these statements will serve as ingredients for later derivations.

Chapter 4

Non-Smooth Pseudodifferential Operators

Beyond the theory of smooth pseudodifferential operators also a theory for non-smooth pseudodifferential operators has been developed during the last 40 years. Making use of the technique of non-smooth pseudodifferential operators, many interesting results in the field of nonlinear partial differential equations have already been investigated, cf. e.g. [3], [7], [67] and the references given there.

The purpose of this chapter is to introduce and prove results concerning non-smooth pseudodifferential operators which will serve as ingredients for the characterization of non-smooth pseudodifferential operators and the spectral invariance in Chapter 5 and Chapter 6.

As in the smooth case, we start with the introduction of the non-smooth symbol-classes needed later on and their associated non-smooth pseudodifferential operators in Section 4.1. The main goals in this context are to prove a kernel representation, cf. Section 4.5, and to investigate the most important mapping properties of non-smooth pseudodifferential operators, cf. Section 4.4. Moreover, we treat the composition of two non-smooth pseudodifferential operators in Section 4.3. In contrast to the smooth case the composition of non-smooth pseudodifferential operators is in general not a pseudodifferential operator with the same regularity with respect to its coefficients. However, we are able to show an asymptotic expansion for the composition of non-smooth pseudodifferential operators. In the smooth case the oscillatory integral serves as an auxiliary tool for many purposes. It is well-defined for all functions in the space of amplitudes. Hence the topic of Section 4.2 is to extend the space of amplitudes to non-smooth functions such that the oscillatory integral is well-defined for all functions of this extension. We make sure of the fact that the properties of the oscillatory integral even hold for these functions. While verifying the characterization of smooth pseudodifferential operators, one is confronted with the task to reduce a double symbol to a single symbol. Since we want to reuse the main idea of the smooth case in order to obtain a characterization of non-smooth pseudodifferential operators, it remains to introduce non-smooth double symbols in Section 4.6.

4.1 Non-Smooth Symbol-Classes

The present section serves to give an overview of the non-smooth symbol-classes needed in this work. We start with the most common non-smooth symbol-class with coefficients in the Hölder spaces in Subsection 4.1.1. Our next goal is to define a specific subclass of this non-smooth symbol-class. Hence we introduce the uniformly local Sobolev spaces in Subsection 4.1.2. We also want to present the basic properties of these spaces in this subsection. After that we are in the position to define non-smooth symbols with coefficients in an uniformly local Sobolev space in Subsection 4.1.3. In Chapter 6 we also prove the spectral invariance of pseudodifferential operators with coefficients in Sobolev spaces. Therefore Subsection 4.1.4 is devoted to the non-smooth symbol-class with coefficients in Bessel potential spaces.

However, beyond the non-smooth symbol-classes we have already mentioned there are a lot of other non-smooth symbol-classes. For every Banach space X, fulfilling $C_c^\infty(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$, we can define the non-smooth symbol-class $XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. This is done in e.g. [67], Section 1.1 for the case $\rho = 1$ and $\delta = 0$. But, since we already list up all non-smooth symbol-classes needed in this work, we refer to the above-mentioned book for further studies.

In applications to partial differential equations many pseudodifferential operators are classical ones. Hence the restriction to the so-called classical pseudodifferential operators is not a big disadvantage. However, working with classical pseudodifferential operators is mostly much easier. Consequently classical symbol-classes are treated in Subsection 4.1.5.

4.1.1 Non-Smooth Symbol-Classes with Coefficients in $C^{\tilde{m},\tau}$

For the first time, non-smooth pseudodifferential operators with coefficients in the Hölder spaces were treated by H. Kumano-Go and M. Nagase [43] in 1978. We warmly recommend e.g. [69] and [67] for a good summary of results concerning non-smooth pseudodifferential operators of the class $C^{m,s}S^{\tilde{m}}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Non-smooth symbol-classes with coefficients in the Hölder spaces, which are even non-smooth in ξ, were investigated by J.D. Alvarez-Alonso and A.P. Calderon in [10], by R.R. Coifman and Y. Meyer in [23] and by J. Marschall in [53]. The present subsection is mainly based on [53] and [67].

After the introduction of the non-smooth symbol-class with coefficients in Hölder spaces or in Hölder-Zygmund spaces we establish a link to the smooth symbol-classes. Moreover, we verify some technical statements before defining the associated pseudodifferential operator to a non-smooth symbol.

To begin with, we define the non-smooth symbol-class with coefficients in Hölder spaces or in Hölder-Zygmund spaces:

Definition 4.1. Let $0 < s \leq 1$, $m \in \mathbb{N}_0$ and $\tilde{m} \in \mathbb{R}$. Additionally let $X^{m,s}$ be either $C^{m,s}$ or C_*^{m+s}. Furthermore, let $0 \leq \rho, \delta \leq 1$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $N \in \mathbb{N}$. Then the *symbol-class* $X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$ is the set of all functions $p : \mathbb{R}^n_x \times \mathbb{R}^N_\xi \to \mathbb{C}$ such that

i) $\partial_x^\beta p(x, .) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,

ii) $\partial_x^\beta \partial_\xi^\alpha p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^N_\xi)$,

iii) $|\partial_\xi^\alpha p(x, \xi)| \leq C_\alpha \langle \xi \rangle^{\tilde{m}-\rho|\alpha|}$ for all $\xi \in \mathbb{R}^N$ and $x \in \mathbb{R}^n$,

iv) $\|\partial_\xi^\alpha p(., \xi)\|_{X^{m,s}(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{\tilde{m}-\rho|\alpha|+\delta(m+s)}$ for all $\xi \in \mathbb{R}^N$

holds for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq M$ and for each $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$, where C_α is independent of $\xi \in \mathbb{R}^N$ and $x \in \mathbb{R}^n$. The function p is called *(non-smooth) pseudodifferential symbol* or just *(non-smooth) symbol* and $\tilde{m}$ is called *order* of p. If $M = \infty$, the symbols are smooth in ξ. In this case we write $X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$ instead of $X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; \infty)$.

Now we define for all $k \in \mathbb{N}$ with $k \leq M$ the semi-norms

$$|a|_k^{(\tilde{m})} := \sup_{\xi \in \mathbb{R}^N} \max_{|\alpha| \leq k} \|\partial_\xi^\alpha p(., \xi)\|_{X^{m,s}(\mathbb{R}^n)} \langle \xi \rangle^{-\tilde{m}+\rho|\alpha|-\delta(m+s)}$$

for all $a \in X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$. Equipped with the family of semi-norms $(|.|_k^{(\tilde{m})})_{k \in \{0,\ldots,M\}}$ the symbol-class $X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$ is a topological vector space.

Note that condition iii) is a consequence of condition iv) in the case $\delta = 0$. Due to $C^{m,s} = C_*^{m+s}$ for all $s \in (0, 1)$, we obtain

$$C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M) = C_*^{m+s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$$

if $s \neq 1$. These symbol-classes are not empty as we see in the next remark:

Remark 4.2. Let $0 < s \leq 1$, $m \in \mathbb{N}_0$ and $\tilde{m} \in \mathbb{R}$. Moreover, let $N \in \mathbb{N}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho, \delta \leq 1$. Then we have

$$S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N) \subseteq C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M).$$

Proof: The definition of the smooth symbol-class implies the conditions i)-iii) of the previous definition. Thus it remains to verify condition iv). Let $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq M$ and $\xi \in \mathbb{R}^N$ be arbitrary. Defining $\theta := \frac{m+s}{m+1}$, an application of

Lemma 2.41 in the case $s < 1$ and Lemma 2.38 else provides the existence of a constant C_α such that

$$\begin{aligned}\|\partial_\xi^\alpha a(.,\xi)\|_{C^{m,s}} &\leq \|\partial_\xi^\alpha a(.,\xi)\|_{C_b^0}^{1-\theta} \|\partial_\xi^\alpha a(.,\xi)\|_{C_b^{m+1}}^{\theta} \\ &\leq C_\alpha \langle\xi\rangle^{(\tilde{m}-\rho|\alpha|)(1-\theta)} \max_{|\beta|\leq m+1} \|\partial_x^\beta \partial_\xi^\alpha a(.,\xi)\|_{C_b^0}^{\theta} \leq C_\alpha \langle\xi\rangle^{\tilde{m}-\rho|\alpha|+\delta(m+s)}\end{aligned}$$

for all $\xi \in \mathbb{R}^N$. □

There is also a certain ordering with respect to the order of the coefficients of the symbols:

Remark 4.3. Let $\tau_1 > \tau_2 > 0$ and $\tilde{m} \in \mathbb{R}$. Moreover, let $N \in \mathbb{N}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho, \delta \leq 1$. Then

$$C^{\tau_1} S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M) \subseteq C^{\tau_2} S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$$

Proof: Similarly to the proof of Remark 4.2 we get the claim by using the definition of the symbol-classes and Lemma 2.41. □

In the literature p is also called *non-regular* symbol instead of non-smooth symbol. During this work, we always work with the usual case $n = N$. In order to illustrate that the non-smooth symbol-class contains also non-smooth symbols which are not smooth, we give an example for such a non-smooth symbol:

Example 4.4. Let $0 < s \leq 1$, $m \in \mathbb{N}_0$ and $\tilde{m} \in \mathbb{R}$. Additionally let $0 \leq \rho, \delta \leq 1$. If $a \in C^{m,s}(\mathbb{R}^n)$ and $p(\xi) \in S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, we obtain

$$a(x)p(\xi) \in C^{m,s} S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

Proof: Let $\alpha \in \mathbb{N}_0^n$ be arbitrary. Since the conditions i) and ii) of the definition hold obviously, we just have to verify the conditions iii) and iv). On account of p being an element of $S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ there is a constant C_α, independent of $x, \xi \in \mathbb{R}^n$, such that we have for all $\xi \in \mathbb{R}^n$:

$$\|\partial_\xi^\alpha [a(x)p(\xi)]\|_{C^{m,s}(\mathbb{R}^n_x)} = \|a\|_{C^{m,s}} |\partial_\xi^\alpha p(\xi)| \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|} \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|+\delta(m+s)}.$$

Hence condition iv) is true. In the same way we obtain condition iii) due to the continuous embedding of $C_b^0(\mathbb{R}^n) \subseteq C^{m,s}(\mathbb{R}^n)$:

$$|\partial_\xi^\alpha [a(x)p(\xi)]| = |a(x)||\partial_\xi^\alpha p(\xi)| \leq C_\alpha \|a\|_{C^{m,s}} \langle\xi\rangle^{m-\rho|\alpha|} \leq C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}$$

for all $x, \xi \in \mathbb{R}^n$. Here C_α is independent of $x, \xi \in \mathbb{R}^n$. □

Now we take a look at Definition 4.1 again: The partial derivatives $\partial_x^\beta \partial_\xi^\alpha p$ of the non-smooth symbol p are well-defined since the theorem of Schwarz enables us to change the derivatives:

Lemma 4.5. *Let $N \in \mathbb{N}$, $m \in \mathbb{N}_0$, $M \in \mathbb{N}_0 \cup \{\infty\}$, $0 < s \leq 1$, $\tilde{m} \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. Furthermore, let $p \in C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$. Then for all $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^N$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_1| + |\beta_1| = 1, \ldots, |\alpha_l| + |\beta_l| = 1$, $|\alpha_1 + \ldots + \alpha_l| \leq M$ and $|\beta_1 + \ldots + \beta_l| \leq m$ we get*

$$\partial_\xi^{\alpha_1}\partial_x^{\beta_1} \ldots \partial_\xi^{\alpha_l}\partial_x^{\beta_l} p = \partial_\xi^{\alpha_{\pi(1)}}\partial_x^{\beta_{\pi(1)}} \ldots \partial_\xi^{\alpha_{\pi(l)}}\partial_x^{\beta_{\pi(l)}} p$$

for each permutation $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$.

Proof: The claim is a consequence of Schwarz's theorem and condition ii) of the definition of the symbols. □

Our next goal is to establish a link between smooth and non-smooth symbol-classes. We already know that smooth symbols are elements of some non-smooth symbol-classes. However, each function that is an element of every non-smooth symbol-class, whose coefficients are in a Hölder space, is a smooth symbol. This is the topic of the next remark.

Remark 4.6. Let $\tilde{m} \in \mathbb{R}$, $N \in \mathbb{N}$ and $0 \leq \rho, \delta \leq 1$. Then

$$S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N) = \bigcap_{\substack{m \in \mathbb{N}_0 \\ 0<s\leq 1}} C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N).$$

Proof: Due to Remark 4.2 it remains to show

$$A := \bigcap_{\substack{m \in \mathbb{N}_0 \\ 0<s\leq 1}} C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N) \subseteq S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N).$$

Thus let $a \in A$ be arbitrary. The definition of the non-smooth symbol-classes and Lemma 4.5 yield $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$. We take arbitrary $\alpha \in \mathbb{N}_0^N$, $\beta \in \mathbb{N}_0^n$ with $|\beta| \neq 0$ and $0 < s < 1$ and set $\theta := \frac{|\beta|}{|\beta|+s}$. Since $a \in C^{|\beta|,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$, Lemma 2.41 provides the existence of a constant $C_{\alpha,\beta}$ such that

$$\begin{aligned}|\partial_\xi^\alpha D_x^\beta a(x,\xi)| &\leq \|\partial_\xi^\alpha a(.,\xi)\|_{C_b^{|\beta|}(\mathbb{R}^n)} \leq C_{\alpha,\beta}\|\partial_\xi^\alpha a(.,\xi)\|^{1-\theta}_{C_b^0(\mathbb{R}^n)}\|\partial_\xi^\alpha a(.,\xi)\|^\theta_{C^{|\beta|,s}(\mathbb{R}^n)} \\ &\leq C_{\alpha,\beta}\langle\xi\rangle^{\tilde{m}-\rho|\alpha|+\delta|\beta|} \qquad \text{for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^N.\end{aligned}$$

Additionally we obtain from $a \in C^{|\beta|,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$:

$$|\partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{\tilde{m}-\rho|\alpha|} \qquad \text{for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^N.$$

Hence $a \in S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$. □

We mention another property of non-smooth symbols needed in Chapter 6:

Lemma 4.7. *Let $0 \leq \rho_i, \delta_i \leq 1$, $\tau_i > 0$ with $\tau_i \notin \mathbb{N}$, $M_i \in \mathbb{N}_0 \cup \{\infty\}$ and $m_i \in \mathbb{R}$ for $i \in \{1,2\}$. We define $\tau := \min\{\tau_1, \tau_2\}$, $m := \max\{m_1, m_2\}$, $\rho := \min\{\rho_1, \rho_2\}$, $M := \min\{M_1, M_2\}$ and $\delta := \max\{\delta_1, \delta_2\}$. Assuming $p_i \in C^{\tau_i} S^{m_i}_{\rho_i,\delta_i}(\mathbb{R}^n \times \mathbb{R}^n; M_i)$ for $i \in \{1,2\}$, we obtain*

$$p_1 + p_2 \in C^{\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M).$$

Proof: The claim follows directly from the definition of the symbols and from Remark 4.3. □

Next, we would like to show that under certain conditions the partial derivatives of a symbol in the class $C^{m,s} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is an element of the Hölder space $C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$. As an ingredient for the proof we use the following proposition:

Proposition 4.8. *Let $m \in \mathbb{N}_0$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 < s \leq 1$. Moreover, let $\mathscr{B} \subseteq C^{m,s} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be a bounded subset. Then for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ and $|\beta| \leq M - 1$, there exists a constant $C_{\alpha,\beta}$, independent of $x, \xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that*

$$|(\nabla_\xi [\partial_x^\alpha \partial_\xi^\beta a])(x,\xi)| \leq C_{\alpha,\beta} \qquad \text{for all } x, \xi \in \mathbb{R}^n \text{ and } a \in \mathscr{B}.$$

Proof: Applying the definition of ∇_ξ and Lemma 4.5 immediately provides the claim. □

Making use of the previous proposition yields:

Lemma 4.9. *Let $m \in \mathbb{N}_0$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 < s \leq 1$. Additionally let $\mathscr{B} \subseteq C^{m,s} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be a bounded subset. Considering $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq M - 1$ and $|\alpha| \leq m$, the set $\{\partial_x^\alpha \partial_\xi^\gamma a : a \in \mathscr{B}\} \subseteq C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ is bounded.*

Proof: First of all we choose arbitrary $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq M-1$ and $|\alpha| \leq m$. Since $\mathscr{B} \subseteq C^{m,s} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a bounded subset, we have

$$\|\partial_x^\alpha \partial_\xi^\gamma a(.,\xi)\|_{L^\infty(\mathbb{R}^n)} \leq \|\partial_\xi^\gamma a(.,\xi)\|_{C^{m,s}(\mathbb{R}^n)} < C_\gamma \quad \text{for all } a \in \mathscr{B}, \xi \in \mathbb{R}^n. \tag{4.1}$$

The definition of the symbol-class yields $\partial_x^\alpha \partial_\xi^\gamma a \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$. Hence the previous inequality implies the boundedness of $\{\partial_x^\alpha \partial_\xi^\gamma a : a \in \mathscr{B}\}$ in $C^0_b(\mathbb{R}^n \times \mathbb{R}^n)$. On account of

$$|\partial_x^\alpha \partial_\xi^\gamma a(x,\eta) - \partial_x^\alpha \partial_\xi^\gamma a(y,\eta)| = 0 \qquad \text{if } x = y$$

and the boundedness of $\mathscr{B} \subseteq C^{m,s} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, we get for all $a \in \mathscr{B}$:

$$\sup_{(x,\xi) \neq (y,\eta)} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\eta) - \partial_x^\alpha \partial_\xi^\gamma a(y,\eta)|}{|(x,\xi) - (y,\eta)|^s} \leq \sup_{x \neq y} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\eta) - \partial_x^\alpha \partial_\xi^\gamma a(y,\eta)|}{|x-y|^s}$$

$$\leq \|\partial_\xi^\gamma a(.,\eta)\|_{C^{m,s}(\mathbb{R}^n)} < C_\gamma. \tag{4.2}$$

Here C_γ is independent of $a \in \mathscr{B}$. Applying the fundamental theorem of calculus and Proposition 4.8 provides together with the Cauchy-Schwarz inequality on $\mathbb{R}^n$ that for all $x, \xi, \eta \in \mathbb{R}^n$ and all $a \in \mathscr{B}$

$$\begin{aligned}
|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)| &= \left| \int_0^1 \frac{d}{dt} \left[\partial_x^\alpha \partial_\xi^\gamma a(x, t\xi + (1-t)\eta)\right] dt \right| \\
&= \left| \int_0^1 \nabla_\xi [\partial_x^\alpha \partial_\xi^\gamma a](x, t\xi + (1-t)\eta) \cdot (\xi - \eta) dt \right| \\
&\leq \int_0^1 \left| \nabla_\xi [\partial_x^\alpha \partial_\xi^\gamma a](x, t\xi + (1-t)\eta) \right| dt \, |\xi - \eta| \leq C_\gamma |\xi - \eta|,
\end{aligned}$$

where the constant C_γ is independent of $x, y, \xi, \eta \in \mathbb{R}^n$ and of $a \in \mathscr{B}$. Therefore we obtain for $x \in \mathbb{R}^n$ and $|\xi - \eta| < 1$ with $\xi \neq \eta$:

$$\begin{aligned}
\frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|(x,\xi) - (y,\eta)|^s} &\leq \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|\xi - \eta|^s} \\
&\leq C_\gamma |\xi - \eta|^{1-s} \leq C_\gamma.
\end{aligned} \tag{4.3}$$

On the other hand we get for $x, y \in \mathbb{R}^n$ and $|\xi - \eta| \geq 1$ the existence of a constant C_γ, which is again independent of $x, y, \xi, \eta \in \mathbb{R}^n$ and of $a \in \mathscr{B}$, such that:

$$\begin{aligned}
\frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|(x,\xi) - (y,\eta)|^s} &\leq \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|\xi - \eta|^s} \\
&\leq |\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)| \leq 2\|\partial_x^\alpha \partial_\xi^\gamma a\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\
&\leq C_\gamma.
\end{aligned} \tag{4.4}$$

Now we use $|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)| = 0$ for $\xi = \eta$. On account of (4.3) and (4.4) we conclude the following inequality:

$$\begin{aligned}
\sup_{(x,\xi) \neq (y,\eta)} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|(x,\xi) - (y,\eta)|^s} &\leq \sup_{\substack{x,y \in \mathbb{R}^n \\ |\xi - \eta| < 1,\, \xi \neq \eta}} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|(x,\xi) - (y,\eta)|^s} \\
&+ \sup_{\substack{(x,\xi) \neq (y,\eta) \\ |\xi - \eta| \geq 1}} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|(x,\xi) - (y,\eta)|^s} \leq C_\gamma \qquad \text{for all } a \in \mathscr{B}.
\end{aligned} \tag{4.5}$$

Here C_γ is independent of $a \in \mathscr{B}$. Collecting the estimates (4.1),(4.2) and (4.5) we finally obtain by means of the triangle inequality:

$$\begin{aligned}
&\|\partial_x^\alpha \partial_\xi^\gamma a\|_{C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)} \\
&\quad = \|\partial_x^\alpha \partial_\xi^\gamma a\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} + \sup_{(x,\xi) \neq (y,\eta)} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(y,\eta)|}{|(x,\xi) - (y,\eta)|^s}
\end{aligned}$$

$$\leq C_\gamma + \sup_{(x,\xi)\neq(y,\eta)} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi) - \partial_x^\alpha \partial_\xi^\gamma a(x,\eta)|}{|(x,\xi)-(y,\eta)|^s} + \sup_{(x,\xi)\neq(y,\eta)} \frac{|\partial_x^\alpha \partial_\xi^\gamma a(x,\eta) - \partial_x^\alpha \partial_\xi^\gamma a(y,\eta)|}{|(x,\xi)-(y,\eta)|^s} \leq C_\gamma \qquad \text{for all } a \in \mathscr{B}.$$

The constant C_γ is independent of $a \in \mathscr{B}$. This shows the boundedness of $\{\partial_x^\alpha \partial_\xi^\gamma a : a \in \mathscr{B}\} \subseteq C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$. □

In analogy to the smooth case we want to define an associated operator for every non-smooth symbol: We choose $m, s, \tilde{m}, M, \rho$ and δ as in Definition 4.1. Additionally let $X^{m,s}$ be either $C^{m,s}$ or C_*^{m+s}. For $u \in \mathcal{S}(\mathbb{R}^n)$, a non-smooth symbol $p \in X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ and $x \in \mathbb{R}^n$ we get due to the definition of non-smooth symbols, $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$ and Theorem 2.11:

$$|p(x,\xi)\hat{u}(\xi)| \leq C\langle\xi\rangle^{-n-1} \in L^1(\mathbb{R}^n_\xi).$$

Therefore $f(x) := \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) đ\xi$ exists for all $x \in \mathbb{R}^n$. Unfortunately f is in general not a Schwartz function as in the smooth case. However, we will see in Lemma 4.48 below that $f \in X^{m,s}(\mathbb{R}^n)$. Consequently we are able to define the associated operator in the following way:

Definition 4.10. Let $0 < s \leq 1$, $m \in \mathbb{N}_0$, $\tilde{m} \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. Moreover, let $M \in \mathbb{N}_0 \cup \{\infty\}$ and $X^{m,s}$ be either $C^{m,s}$ or C_*^{m+s}. For every symbol $p \in X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, we define the associated *pseudodifferential operator* $p(x, D_x)$ by

$$p(x,D_x)u(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) đ\xi \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n.$$

We also write $\mathrm{OP}(p)$ instead of $p(x, D_x)$. The set of all non-smooth pseudodifferential operators with symbols in the set $X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is denoted by $\mathrm{OP}X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. If $M = \infty$, we write $\mathrm{OP}X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $\mathrm{OP}X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

For elements of $\mathrm{OP}X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$ we also say pseudodifferential operators of the symbol-class $X^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N; M)$.

Having treated the most common non-smooth symbol-class, our next goal is to define a subclass of this non-smooth symbol-class.

4.1.2 Uniformly Local Sobolev Spaces

In Section 5.6 non-smooth pseudodifferential operators with coefficients in a uniformly local Sobolev space become a key part of the improvement of the characterization. This section serves to introduce the uniformly local Sobolev spaces and to give an overview about the basic properties of these spaces needed later on. We use Banach space valued Sobolev and Hölder spaces in this subsection. The definition and properties of these function spaces needed here are listed in the Appendix B. For more details we refer to e.g. [11], [12] and [63].

We begin with the definition of the uniformly local Sobolev spaces:

Definition 4.11. Let $1 \leq q \leq \infty$, $m \in \mathbb{N}_0$, $U \subseteq \mathbb{R}^n$ be an open subset and X be a Banach space. Then the space of all functions, which belong *uniformly local* to $L^q(U;X)$ or even $W^m_q(U;X)$ is denoted by

$$L^q_{uloc}(U;X) := \{f \in L^q_{loc}(U;X) : \|f\|_{L^q_{uloc}(U;X)} < \infty\},$$
$$W^{m,q}_{uloc}(U;X) := \{f \in L^q_{uloc}(U;X) : \partial^\alpha_x f \in L^q_{uloc}(U;X) \text{ for all } |\alpha| \leq m\},$$

where

$$\|f\|_{L^q_{uloc}(U;X)} := \sup_{x\in U} \|f\|_{L^q(B_1(x)\cap U;X)} \qquad \text{for } f \in L^q_{uloc}(U;X),$$
$$\|f\|_{W^{m,q}_{uloc}(U;X)} := \sum_{|\alpha|\leq m} \|\partial^\alpha_x f\|_{L^q_{uloc}(U;X)} \qquad \text{for } f \in W^{m,q}_{uloc}(U;X).$$

If $X = \mathbb{C}$, we write $L^q_{uloc}(U)$ instead of $L^q_{uloc}(U;\mathbb{C})$ and $W^{m,q}_{uloc}(U)$ instead of $W^{m,q}_{uloc}(U;\mathbb{C})$. Moreover, we also write $\|.\|_{L^q_{uloc}}$ and $\|.\|_{W^{m,q}_{uloc}}$ instead of $\|.\|_{L^q_{uloc}(\mathbb{R}^n;\mathbb{C})}$ and $\|.\|_{W^{m,q}_{uloc}(\mathbb{R}^n;\mathbb{C})}$.

The spaces $L^q_{uloc}(U;X)$ and $W^{m,q}_{uloc}(U;X)$ are Banach spaces. This can be verified by using the fact that $L^q(U;X)$ is a Banach space due to Definition B.8. The norm $\|.\|_{W^{m,q}_{uloc}(U;X)}$ is equivalent to the norm $\|.\|'_{W^{m,q}_{uloc}(U;X)}$ defined by

$$\|f\|'_{W^{m,q}_{uloc}(U;X)} := \max_{|\alpha|\leq m} \|\partial^\alpha_x f\|_{L^q_{uloc}(U;X)} \qquad \text{for } f \in W^{m,q}_{uloc}(U;X).$$

This can be easily proved by the norm equivalence in $\mathbb{R}^n$.

The locality of this new function space is illustrated in the next remark:

Remark 4.12. Let $1 \leq q \leq \infty$, $U \subseteq \mathbb{R}^n$ be an open subset and $V \subseteq U$ be a compact set. Moreover, let X be a Banach space. Then

$$\|f\|_{L^q(V;X)} \leq C\|f\|_{L^q_{uloc}(U;X)} \qquad \text{for all } f \in L^q_{uloc}(U;X).$$

Proof: Let $(B_1(x_i))_{i=1}^N$ be a finite cover of V with open balls of radius 1, where $N \in \mathbb{N}$ and $x_i \in U$ for all $i \in \{1,\dots,N\}$. Then we get

$$\|f\|^q_{L^q(V;X)} = \int_V \|f(x)\|^q_X \, dx \leq \sum_{i=1}^N \int_{B_1(x_i)\cap U} \|f(x)\|^q_X \, dx \leq N\|f\|^q_{L^q_{uloc}(U;X)}.$$

□

The definition of the uniformly local Sobolev spaces implies the embedding $W_q^m(U;X) \subseteq W_{uloc}^{m,q}(U;X)$. This enables us to extend some well-known results for Sobolev spaces to the uniformly local Sobolev spaces. We can estimate the derivative of a function in $W_{uloc}^{m,q}(U;X)$ by its norm:

Remark 4.13. Let $1 < q \leq \infty$, $m \in \mathbb{N}_0$ and $j \in \{1, \ldots, n\}$. Additionally let $U \subseteq \mathbb{R}^n$ be an open subset and X be a Banach space. Then

$$\|D_{x_j} f\|_{W_{uloc}^{m,q}(U;X)} \leq \|f\|_{W_{uloc}^{m+1,q}(U;X)} \qquad \text{for all } f \in W_{uloc}^{m+1,q}(U;X).$$

Proof: The claim follows at once by using the definition of the uniformly local Sobolev spaces. □

In some way there is an order of the uniformly local Sobolev spaces $L_{uloc}^q(U;X)$ with respect to q:

Lemma 4.14. *Let $1 \leq q \leq r < \infty$. Additionally let $U \subseteq \mathbb{R}^n$ be an open subset and X be a Banach space. Then we have the embedding*

$$L_{uloc}^r(U;X) \subseteq L_{uloc}^q(U;X).$$

Proof: We define p by $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$ if $q < r$ and $p = \infty$ else. The Hölder inequality for bounded sets yields:

$$\begin{aligned} \|f\|_{L_{uloc}^q(U;X)} &= \sup_{x \in U} \|f\|_{L^q(B_1(x)\cap U;X)} \leq \sup_{x \in U} \left\{ \|f\|_{L^r(B_1(x)\cap U;X)} \|1\|_{L^p(B_1(x)\cap U)} \right\} \\ &\leq C_{q,r,n} \sup_{x \in U} \|f\|_{L^r(B_1(x)\cap U;X)} = C_{q,r,n} \|f\|_{L_{uloc}^r(U;X)}. \end{aligned}$$

□

Additionally under certain conditions there exists a continuous embedding of uniformly local Sobolev spaces in Hölder spaces:

Lemma 4.15. *Let $1 < q \leq \infty$, $m \in \mathbb{N}_0$ and X be a Banach space. Considering $0 < \tau \leq m - n/q$ with $\tau \notin \mathbb{N}$, we get the continuous embedding*

$$W_{uloc}^{m,q}(\mathbb{R}^n;X) \hookrightarrow C^\tau(\mathbb{R}^n;X).$$

Proof: Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \lfloor \tau \rfloor$ be arbitrary. Then we obtain

$$\sup_{x \neq y} \frac{\|\partial_x^\alpha f(x) - \partial_y^\alpha f(y)\|_X}{|x-y|^{\tau - \lfloor \tau \rfloor}} \leq \sup_{x \in \mathbb{R}^n} \|\partial_x^\alpha f\|_{C^{\tau - \lfloor \tau \rfloor}(\overline{B_1(x)};X)} + 2\|\partial_x^\alpha f\|_{C_b^0(\mathbb{R}^n;X)} \tag{4.6}$$

if we split the supremum of the left side in the supremum over $|x-y| > 1$ and the rest. Using inequality (4.6) and $\|f\|_{C_b^{\lfloor \tau \rfloor}(\mathbb{R}^n;X)} \leq \sup_{x \in \mathbb{R}^n} \|f\|_{C^\tau(\overline{B_1(x)};X)}$ for all functions $f \in C^\tau(\mathbb{R}^n;X)$, we get the existence of a constant C, independent of $f \in C^\tau(\mathbb{R}^n;X)$, such that

$$\|f\|_{C^\tau(\mathbb{R}^n;X)} \leq C \sup_{x \in \mathbb{R}^n} \|f\|_{C^\tau(\overline{B_1(x)};X)} \qquad \text{for all } f \in C^\tau(\mathbb{R}^n;X).$$

Making use of Theorem B.11 and the previous estimate yields:

$$\begin{aligned}\|f\|_{C^\tau(\mathbb{R}^n;X)} &\leq \sup_{x\in\mathbb{R}^n} \|f\|_{C^\tau(\overline{B_1(x)};X)} = \sup_{x\in\mathbb{R}^n} \|f(x+.)\|_{C^\tau(\overline{B_1(0)};X)} \\ &\leq C \sup_{x\in\mathbb{R}^n} \|f(x+.)\|_{W^{m,q}(B_1(0);X)} = C\|f\|_{W^{m,q}_{uloc}(\mathbb{R}^n;X)}.\end{aligned}$$

□

Now we want to discuss a special case of the Banach space X. Choosing $X = L^q_{uloc}(\mathbb{R}^m;\mathbb{C})$, it turns out that all functions of the set $L^q_{uloc}(\mathbb{R}^n;X)$ are measurable with respect to the product measure of $\mathbb{R}^m \times \mathbb{R}^n$.

Proposition 4.16. *Let $1 \leq q < \infty$ and $m \in \mathbb{N}$. For $a \in L^q_{uloc}(\mathbb{R}^n; L^q_{uloc}(\mathbb{R}^m;\mathbb{C}))$ we define $\tilde{a} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}$ via*

$$\tilde{a}(x,y) := a(y)(x) \qquad \text{for all } y \in \mathbb{R}^n \text{ and } x \in \mathbb{R}^m.$$

Then $\tilde{a}$ is measurable with respect to the product measure of $\mathbb{R}^m \times \mathbb{R}^n$.

Proof: The assumptions imply the strong measurability of $a : \mathbb{R}^n \to L^q_{uloc}(\mathbb{R}^m;\mathbb{C})$. Hence Corollary 4.14 in Chapter 3 of [28] provides the existence of a sequence of step functions $(s_l)_{l\in\mathbb{N}}$ with the property $s_l : \mathbb{R}^n \to L^q_{uloc}(\mathbb{R}^m;\mathbb{C})$ for all $l \in \mathbb{N}$ and

$$s_l \xrightarrow{l\to\infty} a \qquad \text{pointwise.} \tag{4.7}$$

Since s_l is a step function for every $l \in \mathbb{N}$, there are some $m_l \in \mathbb{N}$, disjoint Borel sets $X^l_i \in \mathscr{B}(\mathbb{R}^n)$ and $\alpha^l_i \in L^q_{uloc}(\mathbb{R}^m;\mathbb{C})$ for all $i \in \{1,\dots,m_l\}$ such that

$$s_l = \sum_{i=1}^{m_l} \alpha^l_i \, \chi_{X^l_i}.$$

Now we choose an arbitrary $l \in \mathbb{N}$. On account of Proposition A.1 we get for all $i \in \{1,\dots,m_l\}$ the measurability with respect to the product measure of $\mathbb{R}^m \times \mathbb{R}^n$ of the function $\varphi_i : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}$ defined by $\varphi_i(x,y) := \alpha^l_i(x)\chi_{X^l_i}(y)$. Together with Corollary 4.8 in Chapter 3 of [28] we obtain the measurability of s_l with respect to the product measure of $\mathbb{R}^m \times \mathbb{R}^n$. A combination with (4.7) and Theorem 4.3 in Chapter 3 of [28] yields the claim of the proposition. □

With this statement at hand, we obtain the next continuous embedding:

Remark 4.17. Let $1 \leq q \leq r < \infty$ and $m \in \mathbb{N}$. Then the following continuous embedding holds:

$$L^q_{uloc}(\mathbb{R}^n; L^q_{uloc}(\mathbb{R}^m)) \subseteq L^q_{uloc}(\mathbb{R}^n \times \mathbb{R}^m).$$

Proof: Since $L^q_{uloc}(\mathbb{R}^m)$ is a Banach space, we immediately obtain the claim by Proposition 4.16, the definition of these spaces and by using the fact that $B_1(x,y) \subseteq B_1(x) \times B_1(y)$ for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. □

An interesting question is in which space the derivative of an element of $W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n))$ is included. This is the topic of the following statement:

Remark 4.18. Let $m, \tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \tilde{m}$. Assuming $a \in W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n))$, we obtain

$$\partial_x^\alpha a \in W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m}-|\alpha|,q}_{uloc}(\mathbb{R}^n_x)).$$

This remark is a direct consequence of the next proposition:

Proposition 4.19. *Let $1 < q < \infty$, $U \subseteq \mathbb{R}^n$ be an open subset and X, Y be two Banach spaces. Moreover, let $T : X \to Y$ be a linear bounded operator and $f \in W^{m,q}_{uloc}(U, X)$. If we define $\tilde{T}f : U \to Y$ by*

$$(\tilde{T}f)(x) := T(f(x)) \qquad \text{for all } x \in U$$

we obtain $\tilde{T}f \in W^{m,q}_{uloc}(U, Y)$.

As an ingredient for the proof, we use the following theorem.

Theorem 4.20. *Let X, Y be Banach spaces, $U \subseteq \mathbb{R}^n$ be an open subset and $T \in \mathscr{L}(X, Y)$. For $f \in L^1(U, X)$, we get $Tf \in L^1(U, Y)$ and*

$$T \int_U f(x)dx = \int_U Tf(x)dx.$$

For the proof of this theorem we refer e.g. to [13], Theorem 2.11 *iii*).

Making use of the previous theorem, we are able to show Proposition 4.19:

Proof of Proposition 4.19. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$ and $\varphi \in C_c^\infty(U)$ be arbitrary. First of all we show that $(\partial_x^\alpha f)\, \varphi$ is an element of $L^1(U; X)$. In order to check this claim we choose a finite cover $\bigcup_{i=1}^N B_1(x_i)$ of $U \cap \operatorname{supp} \varphi$ of open balls with radius 1. Then an application of Lemma 4.14 yields

$$\begin{aligned}
\int_U \| (\partial_x^\alpha f(x))\, \varphi(x)\|_X dx &\le \sup_{x\in U} |\varphi(x)| \int_{U\cap \operatorname{supp}\, \varphi} \|\partial_x^\alpha f(x)\|_X dx \\
&\le C \sum_{i=1}^N \int_{B_1(x_i)\cap U} \|\partial_x^\alpha f(x)\|_X dx \le C\|\partial_x^\alpha f\|_{L^1_{uloc}(U;X)} \le C\|\partial_x^\alpha f\|_{L^q_{uloc}(U;X)} \\
&\le C_q \|f\|_{W^{m,q}_{uloc}(U;X)} \le C_q.
\end{aligned}$$

Therefore $(\partial_x^\alpha f)\, \varphi \in L^1(U; X)$. In the same way we can prove that $f\, (\partial_x^\alpha \varphi)$ is an element of $L^1(U; X)$, too. Additionally $f \in W^{m,q}_{uloc}(U; X)$ implies

$$\int_U (\partial_x^\alpha f(x))\, \varphi(x)dx = (-1)^{|\alpha|} \int_U f(x)\, (\partial_x^\alpha \varphi(x))\, dx. \tag{4.8}$$

On account of $(\partial_x^\alpha f)\,\varphi, f\,(\partial_x^\alpha \varphi) \in L^1(U;X)$ we can apply Theorem 4.20. Together with (4.8) and the linearity of T we obtain

$$\begin{aligned}\int_U \tilde{T}\,(\partial_x^\alpha f)\,(x)\varphi(x)dx &= \int_U T\,[\partial_x^\alpha f(x)\varphi(x)]\,dx = T\int_U (\partial_x^\alpha f(x))\,\varphi(x)dx \\ &= (-1)^{|\alpha|}T\int_U f(x)\partial_x^\alpha\varphi(x)dx = (-1)^{|\alpha|}\int_U T\,[f(x)\partial_x^\alpha\varphi(x)]\,dx \\ &= (-1)^{|\alpha|}\int_U (\tilde{T}f)(x)\partial_x^\alpha\varphi(x)dx.\end{aligned}$$

Since $\varphi \in C_c^\infty(U)$ was arbitrary, the α-th weak derivative of $\tilde{T}f$ exists and is given by $\partial_x^\alpha(\tilde{T}f) = \tilde{T}(\partial_x^\alpha f)$. Hence we get $\tilde{T}f \in W^{m,q}_{uloc}(U;Y)$ because of the next inequality:

$$\begin{aligned}\|\tilde{T}f\|_{W^{m,q}_{uloc}(U;Y)} &= \max_{|\alpha|\le m}\|\partial_x^\alpha(\tilde{T}f)\|_{L^q_{uloc}(U;Y)} = \max_{|\alpha|\le m}\|\tilde{T}(\partial_x^\alpha f)\|_{L^q_{uloc}(U;Y)} \\ &= \max_{|\alpha|\le m}\sup_{x\in U}\int_{B_1(x)\cap U}\|T\,(\partial_x^\alpha f(x))\,\|_Y dx \\ &\le C\max_{|\alpha|\le m}\sup_{x\in U}\int_{B_1(x)\cap U}\|\partial_x^\alpha f(x)\|_X dx = C\|f\|_{W^{m,q}_{uloc}(U;X)} \le C.\end{aligned}$$

□

Another important property of the spaces $W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n))$ is that we are allowed to interchange the order of the derivatives. The last missing piece towards this result is to show that we can interchange the order of the integration:

Remark 4.21. Let $m, \tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ and $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. Moreover, let $f \in W^{m,q}_{uloc}(\mathbb{R}^n_y; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x))$. Then

$$\iint f(x,y)\varphi(x)\psi(y)dxdy = \iint f(x,y)\varphi(x)\psi(y)dydx.$$

Proof: Since $f \in W^{m,q}_{uloc}(\mathbb{R}^n_y; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x))$, we have

$$f \in L^q_{uloc}(\mathbb{R}^n_y; L^q_{uloc}(\mathbb{R}^n_x)) \subseteq L^1_{uloc}(\mathbb{R}^n_y; L^1_{uloc}(\mathbb{R}^n_x)) \subseteq L^1_{uloc}(\mathbb{R}^n_y \times \mathbb{R}^n_x) \tag{4.9}$$

due to Lemma 4.14 and Remark 4.17. Hence f is locally integrable. Now we define the compact sets $A := \text{supp}\ \varphi$ and $B := \text{supp}\ \psi$. Using the boundedness of φ and ψ, Remark 4.12 and the embedding (4.9), we get the integrability of the function $(x,y) \mapsto f(x,y)\varphi(x)\psi(y)$:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x,y)\varphi(x)\psi(y)|\,d(x,y) \le C\int_{A\times B}|f(x,y)|\,d(x,y)$$

$$\leq C\|f\|_{L^1_{uloc}(\mathbb{R}^n\times\mathbb{R}^n)} \leq C. \tag{4.10}$$

In the same way we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x,y)\varphi(x)\psi(y)|\,dxdy \leq C\int_B\int_A |f(x,y)|\,dxdy$$
$$\leq C\int_B \|f(x,y)\|_{L^1_{uloc}(\mathbb{R}^n_x)}\,dy \leq C\,\|f(x,y)\|_{L^1_{uloc}(\mathbb{R}^n_y;L^1_{uloc}(\mathbb{R}^n_x))} \leq C. \tag{4.11}$$

Because of (4.10) and (4.11) the assumptions of Fubini's theorem, cf. [44], Section 8.5, are fulfilled. An application of this theorem provides the claim:

$$\iint f(x,y)\varphi(x)\psi(y)dxdy = \iint f(x,y)\varphi(x)\psi(y)dydx.$$

□

Now that we have verified the previous remark, we are in the position to prove the next statement:

Remark 4.22. Let $m, \tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m}$. Additionally let $a \in W^{m,q}_{uloc}(\mathbb{R}^n; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n))$. Now we choose $\alpha_1,\ldots,\alpha_l \in \mathbb{N}_0^n$ and $\beta_1,\ldots,\beta_l \in \mathbb{N}_0^n$ with $|\alpha_1|+|\beta_1| = \ldots = |\alpha_l|+|\beta_l| = 1$, $|\alpha| \leq \tilde{m}$ and $|\beta| \leq m$. Here $\alpha := \alpha_1+\ldots+\alpha_l$ and $\beta := \beta_1+\ldots+\beta_l$. Then we can change the order of the derivatives:

$$\partial_x^{\alpha_1}\partial_y^{\beta_1}\ldots\partial_x^{\alpha_l}\partial_y^{\beta_l}a(x,y) = \partial_x^{\alpha}\partial_y^{\beta}a(x,y) \qquad \text{for almost all } x,y \in \mathbb{R}^n.$$

Proof: We prove the remark by mathematical induction with respect to l. For $l = 1$ there is nothing to show. In order to check the induction step, we choose an arbitrary $l \in \mathbb{N}$. Assuming the induction hypothesis, it remains to prove for all α_1, β_1, $\tilde{\alpha}$, $\tilde{\beta} \in \mathbb{N}_0^n$ with $|\alpha_1|+|\beta_1| = 1$, $|\tilde{\alpha}|+|\tilde{\beta}| = l$, $|\tilde{\alpha}|+|\alpha_1| \leq \tilde{m}$ and $|\tilde{\beta}|+|\beta_1| \leq m$:

$$\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_x^{\tilde{\alpha}}\partial_y^{\tilde{\beta}}a(x,y) = \partial_x^{\tilde{\alpha}+\alpha_1}\partial_y^{\tilde{\beta}+\beta_1}a(x,y) \qquad \text{for almost all } x,y \in \mathbb{R}^n. \tag{4.12}$$

Here we distinguish two different cases: First let $\beta_1 = 0$. Then the claim follows by using the definition of the weak derivative and the Lemma of Schwarz. In the second case we have to look at $\alpha_1 = 0$. Then Remark 4.17 provides:

$$b := \partial_y^{\tilde{\beta}}a \in W^{m-|\tilde{\beta}|,q}_{uloc}(\mathbb{R}^n_y; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)) \subseteq L^q_{uloc}(\mathbb{R}^n_y; L^q_{uloc}(\mathbb{R}^n_x)) \subseteq L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y).$$

In the same way we get $\partial_x^{\tilde{\alpha}}b \in W^{m-|\tilde{\beta}|,q}_{uloc}(\mathbb{R}^n_y; W^{\tilde{m}-|\tilde{\alpha}|,q}_{uloc}(\mathbb{R}^n_x)) \subseteq L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)$ due to Remark 4.18. In particular this embedding implies the existence of the derivatives $\partial_y^{\beta_1}b$ and $\partial_y^{\beta_1}\left(\partial_x^{\tilde{\alpha}}b\right)$ in the sense of distributions. Hence we get for each $\chi \in C_c^\infty(\mathbb{R}^n)$ and almost all $x \in \mathbb{R}^n$:

$$\int b(x,y)\partial_y^{\beta_1}\chi(y)dy = -\int \left(\partial_y^{\beta_1}b\right)(x,y)\chi(y)dy, \tag{4.13}$$

$$\int \left(\partial_x^{\tilde{\alpha}} b\right)(x,y)\partial_y^{\beta_1}\chi(y)dy = -\int \left(\partial_y^{\beta_1}\partial_x^{\tilde{\alpha}} b\right)(x,y)\chi(y)dy. \tag{4.14}$$

Since $\partial_y^{\beta_1} b \in W_{uloc}^{m-|\tilde{\beta}|-1,q}(\mathbb{R}_y^n; W_{uloc}^{\tilde{m}}(\mathbb{R}_x^n))$, we obtain for almost all $y \in \mathbb{R}^n$ that $\partial_y^{\beta_1} b(.,y) \in W_{uloc}^{\tilde{m}}(\mathbb{R}_x^n)$. Therefore the next equality holds for every $\chi \in C_c^\infty(\mathbb{R}^n)$ and almost all $y \in \mathbb{R}^n$:

$$\int \left(\partial_y^{\beta_1} b\right)(x,y)\partial_x^{\tilde{\alpha}}\chi(x)dx = (-1)^{|\tilde{\alpha}|}\int \left(\partial_x^{\tilde{\alpha}}\partial_y^{\beta_1} b\right)(x,y)\chi(x)dx. \tag{4.15}$$

In the same way we get for almost all $y \in \mathbb{R}^n$:

$$\int b(x,y)\partial_x^{\tilde{\alpha}}\chi(x)dx = (-1)^{|\tilde{\alpha}|}\int \left(\partial_x^{\tilde{\alpha}} b\right)(x,y)\chi(x)dx. \tag{4.16}$$

Now let $\psi, \varphi \in C_c^\infty(\mathbb{R}^n)$ be arbitrary. If we use (4.13) - (4.16) and Remark 4.21, we obtain the following equality:

$$\iint \left(\partial_x^{\tilde{\alpha}}\partial_y^{\beta_1} b\right)(x,y)\varphi(x)\psi(y)dxdy = \iint \left(\partial_y^{\beta_1}\partial_x^{\tilde{\alpha}} b\right)(x,y)\varphi(x)\psi(y)dxdy.$$

Since $\psi, \varphi \in C_c^\infty(\mathbb{R}^n)$ were arbitrary, we derive from two applications of the fundamental lemma of the calculus of variations, cf. e.g. [5], Theorem A.7, for almost all $x, y \in \mathbb{R}^n$:

$$\partial_x^{\tilde{\alpha}}\partial_y^{\beta_1+\tilde{\beta}} a(x,y) = \partial_x^{\tilde{\alpha}}\partial_y^{\beta_1} b(x,y) = \partial_y^{\beta_1}\partial_x^{\tilde{\alpha}} b(x,y) = \partial_y^{\beta_1}\partial_x^{\tilde{\alpha}}\partial_y^{\tilde{\beta}} a(x,y).$$

Thus (4.12) holds which finishes the proof. □

We are also able to list sufficient conditions for functions to belong to the space $W_{uloc}^{N,q}(\mathbb{R}_y^n; W_{uloc}^{m,q}(\mathbb{R}_x^n))$:

Lemma 4.23. *Let $m \in \mathbb{N}_0$, $1 < q < \infty$ and $N \in \mathbb{N}_0$. We consider a measurable function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the following property: for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ we have*

- *$\partial_y^\alpha a(.,y) \in W_{uloc}^{m,q}(\mathbb{R}^n)$ for all $y \in \mathbb{R}^n$,*
- *$a(x,.) \in W_{uloc}^{N,q}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,*
- *$\sup_{y \in \mathbb{R}^n} \|\partial_y^\alpha a(.,y)\|_{W_{uloc}^{m,q}(\mathbb{R}^n)} < C_\alpha$ for a constant $C_\alpha > 0$.*

Then $a \in W_{uloc}^{N,q}(\mathbb{R}_y^n; W_{uloc}^{m,q}(\mathbb{R}_x^n))$.

Proof: First of all we note that the weak α-th derivative of a in the sense of $\mathscr{D}'(\mathbb{R}^n_y; W^{m,q}_{uloc}(\mathbb{R}^n_x))$ exists for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq N$. The reason of this is that for every $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq N$ we have

$$\int a(x,y)\partial^\alpha_y \varphi(y)dy = (-1)^{|\alpha|} \int \partial^\alpha_y a(x,y)\varphi(y)dy \qquad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n),$$

because of $a(x,.) \in W^{N,q}_{uloc}(\mathbb{R}^n)$. This implies for all $\varphi \in C^\infty_c(\mathbb{R}^n)$

$$\int a(.,y)\partial^\alpha_y \varphi(y)dy = (-1)^{|\alpha|} \int \partial^\alpha_y a(.,y)\varphi(y)dy \qquad \text{in } W^{m,q}_{uloc}(\mathbb{R}^n),$$

which shows the existence of the weak α-th derivative of a in the sence of $\mathscr{D}'(\mathbb{R}^n_y; W^{m,q}_{uloc}(\mathbb{R}^n_x))$ for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq N$. By means of the next norm estimate the claim holds:

$$\begin{aligned}\|a\|_{W^{N,q}_{uloc}(\mathbb{R}^n_y;W^{m,q}_{uloc}(\mathbb{R}^n_x))} &\leq \sum_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} \left\{ \int_{B_1(y)} \|\partial^\alpha_y a(.,z)\|^q_{W^{m,q}_{uloc}(\mathbb{R}^n)} dz \right\}^{1/q} \\ &\leq \sum_{|\alpha|\leq N} C_{\alpha,q} \sup_{y\in\mathbb{R}^n} |B_1(y)|^{1/q} \leq C_{N,q,n}.\end{aligned}$$

□

With all these results at hand, we are able to show the following technical result:

Lemma 4.24. *Let $1 < q < \infty$ and $\tilde{m}, N \in \mathbb{N}_0$. Furthermore, let $\mathscr{B}$ be a set of all measurable functions $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the following property:*

- $\partial^\alpha_y a(.,\xi,y) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ *for all* $\xi, y \in \mathbb{R}^n$ *and each* $\alpha \in \mathbb{N}^n_0$ *with* $|\alpha| \leq N$,
- $a(x,\xi,.) \in W^{N,q}_{uloc}(\mathbb{R}^n)$ *for all* $x, \xi \in \mathbb{R}^n$.

Additionally we assume that there is an $m \in \mathbb{N}_0$ fulfilling the following property: for every $\xi \in \mathbb{R}^n$, $a \in \mathscr{B}$ and $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq N$ we have

$$\sup_{y\in\mathbb{R}^n} \|\partial^\alpha_y a(.,\xi,y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_{\alpha,q}\langle\xi\rangle^m.$$

Here $C_{\alpha,q}$ is independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$. If we define $b : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ for a fixed but arbitrary $a \in \mathscr{B}$ by

$$b(x,\xi,y) := a(x,\xi,x+y) \qquad \text{for all } x,\xi,y \in \mathbb{R}^n,$$

we get for each $\alpha, \beta \in \mathbb{N}^n_0$ with $|\beta| \leq \tilde{m}$ and $|\alpha| + |\beta| \leq N$ and for all $\xi \in \mathbb{R}^n$:

$$\|\partial^\alpha_y \partial^\beta_x b(x,\xi,y)\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)} < C_{\alpha,q}\langle\xi\rangle^m,$$

where $C_{\alpha,q}$ is independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$.

Proof: Lemma 4.23 and Remark 4.18 imply

$$\partial_y^\alpha \partial_x^\beta a(x,\xi,y) \in W^{N-|\alpha|,q}_{uloc}\left(\mathbb{R}^n_y; W^{\tilde{m}-|\beta|,q}_{uloc}(\mathbb{R}^n_x)\right) \subseteq L^q_{uloc}(\mathbb{R}^n_y; L^q_{uloc}(\mathbb{R}^n_x))$$

for all $\xi \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $|\beta| \leq \tilde{m}$. On account of Proposition 4.16 we derive the measurability of $\partial_y^\alpha \partial_x^\beta a(x,\xi,y)$ with respect to the product measure of $\mathbb{R}^n_x \times \mathbb{R}^n_y$ for a fixed $\xi \in \mathbb{R}^n$. Hence the assumptions of Tonelli's theorem, cf. e.g. [75], Theorem 6.10 and the related Remark, are fulfilled. Using Tonelli's theorem twice and substituting $\hat{y} := z + \tilde{y}$, we obtain for each $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $|\beta| \leq \tilde{m}$:

$$\begin{aligned}
&\int_{B_1(x,y)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,z+\tilde{y})\right|^q d(z,\tilde{y}) \leq \int_{B_1(x)\times B_1(y)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,z+\tilde{y})\right|^q d(z,\tilde{y}) \\
&\quad = \int_{B_1(x)} \int_{B_1(y)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,z+\tilde{y})\right|^q d\tilde{y}\, dz \\
&\quad = \int_{B_1(x)} \int_{B_1(y+z)} \left|\left(\partial_y^\alpha \partial_x^\beta a\right)(z,\xi,\hat{y})\right|^q d\hat{y}\, dz \leq \int_{B_1(x)} \int_{B_2(y+x)} \left|\partial_{\hat{y}}^\alpha \partial_z^\beta a(z,\xi,\hat{y})\right|^q d\hat{y}\, dz \\
&\quad = \int_{B_2(y+x)} \int_{B_1(x)} \left|\partial_{\hat{y}}^\alpha \partial_z^\beta a(z,\xi,\hat{y})\right|^q dz\, d\hat{y} \leq \int_{B_2(y+x)} \left\|\partial_{\hat{y}}^\alpha \partial_x^\beta a(x,\xi,\hat{y})\right\|^q_{L^q_{uloc}(\mathbb{R}^n_x)} d\hat{y} \\
&\quad \leq \int_{B_2(y+x)} \sup_{\hat{y}\in\mathbb{R}^n} \left\|\partial_{\hat{y}}^\alpha a(x,\xi,\hat{y})\right\|^q_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} d\hat{y} \leq C_{\alpha,q}\langle\xi\rangle^m |B_2(y+x)| \leq C_{\alpha,q,n}\langle\xi\rangle^m,
\end{aligned}$$

where $C_{\alpha,q,n}$ is independent of $x, \xi, y \in \mathbb{R}^n$ and $a \in \mathscr{B}$. Here we have used that for arbitrary $x, y, z \in \mathbb{R}^n$ with $z \in B_1(x)$ the inclusions $B_1(x,y) \subseteq B_1(x) \times B_1(y)$ and $B_1(y+z) \subseteq B_2(y+x)$ hold. Finally, let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$ and $|\alpha|+|\beta| \leq N$ be arbitrary. We get the existence of a constant $C_{\alpha,\beta,q,n}$, independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$, by an application of Remark 4.22, the Leibniz rule and of the previous inequality:

$$\begin{aligned}
\|\partial_y^\alpha \partial_x^\beta b(x,\xi,y)\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)} &\leq \sum_{\beta_1+\beta_2=\beta} \sup_{x,y\in\mathbb{R}^n} \left\{ \int_{B_1(x,y)} \left|\left(\partial_y^{\alpha+\beta_1}\partial_x^{\beta_2} a\right)(z,\xi,z+\tilde{y})\right|^q \right\}^{1/q} \\
&\leq C_{\alpha,\beta,q,n}\langle\xi\rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n.
\end{aligned}$$

□

The previous lemma enables us to proof the next technical statement:

Lemma 4.25. *Let $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $\mathscr{B}$ be a set of all measurable functions $a : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the following property:*

- $\partial_y^\alpha a(.,\xi,y) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ *for all $\xi, y \in \mathbb{R}^n$ and each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2\tilde{m}$,*

- *$a(x,\xi,.) \in W^{2\tilde{m},q}_{uloc}(\mathbb{R}^n)$ for all $x, \xi \in \mathbb{R}^n$.*

Additionally we assume that there is an $m \in \mathbb{N}_0$, such that for every $\xi \in \mathbb{R}^n$, $a \in \mathscr{B}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2\tilde{m}$ we have

$$\sup_{y\in\mathbb{R}^n} \|\partial_y^\alpha a(.,\xi,y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_{\alpha,q}\langle\xi\rangle^m,$$

where $C_{\alpha,q}$ is independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$. Then there exists a constant $C_{m,q}$, independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$, such that for all $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$ the following inequality holds:

$$\sup_{y\in\mathbb{R}^n} \|a(x,\xi,x+y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C_{m,q}\langle\xi\rangle^m.$$

Proof: First of all we show the existence of a constant $C_q > 0$, independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$, such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq 2\tilde{m}$ and $|\alpha| \leq \tilde{m}$, $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$ the following inequality holds:

$$\sup_{x,y\in\mathbb{R}^n} \left[\int_{B_2(x,y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha a(z,\xi,z+\tilde{y})\right|^q d(\tilde{y},z)\right]^{1/q} \leq C_q \|\partial_y^\beta \partial_x^\alpha a(x,\xi,x+y)\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)}. \tag{4.17}$$

In order to check this inequality we choose an arbitrary finite cover $(U_i)_{i=1}^N$ of $B_2(0,0)$ with open balls of radius 1. Denoting for all $i \in \{1,\ldots,n\}$ and each $x, y \in \mathbb{R}^n$ the set $U_i(x,y)$ as the shifting of U_i around (x,y), $(U_i(x,y))_{i=1}^N$ is a finite cover of $B_2(x,y)$ with open balls of radius 1. For all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq 2\tilde{m}$ and $|\alpha| \leq \tilde{m}$, $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$ we obtain:

$$\begin{aligned}
&\sup_{x,y\in\mathbb{R}^n} \left[\int_{B_2(x,y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha a(z,\xi,z+\tilde{y})\right|^q d(\tilde{y},z)\right]^{1/q} \\
&\qquad \leq C_q \sum_{i=1}^N \sup_{x,y\in\mathbb{R}^n} \left[\int_{U_i(x,y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha a(z,\xi,z+\tilde{y})\right|^q d(\tilde{y},z)\right]^{1/q} \\
&\qquad \leq C_q \sum_{i=1}^N \|\partial_y^\beta \partial_x^\alpha a(x,\xi,x+y)\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)} \\
&\qquad \leq C_q \|\partial_y^\beta \partial_x^\alpha a(x,\xi,x+y)\|_{L^q_{uloc}(\mathbb{R}^n_x\times\mathbb{R}^n_y)}.
\end{aligned}$$

Since C_q is independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$ we get the inequality (4.17). Additionally Lemma 4.23 provides that $a \in W^{2\tilde{m},q}_{uloc}(\mathbb{R}^n_y; W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x))$. Due to Remark 4.18 we know that

$$\partial_y^\alpha \partial_x^\beta a(x,\xi,y) \in W^{2\tilde{m}-|\alpha|,q}_{uloc}\left(\mathbb{R}^n_y; W^{\tilde{m}-|\beta|,q}_{uloc}(\mathbb{R}^n_x)\right) \subseteq L^q_{uloc}(\mathbb{R}^n_y; L^q_{uloc}(\mathbb{R}^n_x))$$

for all $\xi \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq 2\tilde{m}$ and $|\beta| \leq \tilde{m}$. This implies the measurability of $\partial_y^\alpha \partial_x^\beta a(x,\xi,y)$ with respect to the product measure of $\mathbb{R}_x^n \times \mathbb{R}_y^n$ for a fixed $\xi \in \mathbb{R}^n$ as stated in Proposition 4.16. Hence the assumptions of Tonelli's theorem, cf. e.g. [75], Theorem 6.10 and the related remark, are fulfilled. Now we define $b : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ via $b(x,\xi,y) := a(x,\xi,x+y)$ for all $x,\xi,y \in \mathbb{R}^n$. Using Sobolev embedding theorem first and Tonelli's theorem afterwards, we obtain for each $a \in \mathscr{B}$, $x,y,\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m}$:

$$
\begin{aligned}
\int\limits_{B_1(x)} \sup_{\tilde{y} \in B_1(y)} |\partial_z^\alpha b(z,\xi,\tilde{y})|^q \, dz &= \int\limits_{B_1(x)} \|\partial_z^\alpha b(z,\xi,.)\|^q_{C^0(\overline{B_1(y)})} \, dz \\
&\leq C_q \int\limits_{B_1(x)} \|\partial_z^\alpha b(z,\xi,.)\|^q_{W_q^{\tilde{m}}(B_1(y))} \, dz \\
&\leq C_q \sum_{|\beta| \leq \tilde{m}} \int\limits_{B_1(x)} \int\limits_{B_1(y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y})\right|^q d\tilde{y} \, dz \\
&= C_q \sum_{|\beta| \leq \tilde{m}} \int\limits_{B_1(x) \times B_1(y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y})\right|^q d(\tilde{y},z) \\
&\leq C_q \sum_{|\beta| \leq \tilde{m}} \int\limits_{B_2(x,y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y})\right|^q d(\tilde{y},z). \qquad (4.18)
\end{aligned}
$$

Here C_q is independent of $a \in \mathscr{B}$ and $x,y,\xi \in \mathbb{R}^n$. The last inequality holds because of $B_1(x) \times B_1(y) \subseteq B_2(x,y)$ for every $x,y \in \mathbb{R}^n$. Therefore the inequalities (4.18), (4.17) and Lemma 4.24 yield for all $\xi \in \mathbb{R}^n$ and all $a \in \mathscr{B}$

$$
\begin{aligned}
&\sup_{y \in \mathbb{R}^n} \|a(x,\xi,x+y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq \sum_{|\alpha| \leq \tilde{m}} \sup_{y \in \mathbb{R}^n} \|\partial_x^\alpha b(x,\xi,y)\|_{L^q_{uloc}(\mathbb{R}^n_x)} \\
&\quad = \sum_{|\alpha| \leq \tilde{m}} \sup_{y \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} \left\{ \int_{B_1(x)} |\partial_z^\alpha b(z,\xi,y)|^q \, dz \right\}^{1/q} \\
&\quad \leq \sum_{|\alpha| \leq \tilde{m}} \sup_{x,y \in \mathbb{R}^n} \left\{ \int_{B_1(x)} \sup_{\tilde{y} \in B_1(y)} |\partial_z^\alpha b(z,\xi,\tilde{y})|^q \, dz \right\}^{1/q} \\
&\quad \leq C_q \sum_{|\alpha| \leq \tilde{m}} \sum_{|\beta| \leq \tilde{m}} \sup_{x,y \in \mathbb{R}^n} \left\{ \int_{B_2(x,y)} \left|\partial_{\tilde{y}}^\beta \partial_z^\alpha b(z,\xi,\tilde{y})\right|^q d(\tilde{y},z) \right\}^{1/q} \\
&\quad \leq C_q \sum_{|\alpha| \leq \tilde{m}} \sum_{|\beta| \leq \tilde{m}} \left\|\partial_y^\beta \partial_x^\alpha b(x,\xi,y)\right\|_{L^q_{uloc}(\mathbb{R}^n_x \times \mathbb{R}^n_y)} \leq C_{m,q,n} \langle \xi \rangle^m,
\end{aligned}
$$

where constant $C_{m,q,n}$, independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$. □

4.1.3 Non-Smooth Symbol-Classes with Coefficients in $W^{\tilde{m},q}_{uloc}$

Now that we have defined the uniformly local Sobolev spaces in the previous subsection, we are in the position to define the non-smooth symbol-class with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$. J. Marschall already introduced the non-smooth symbol-class with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ in [54]. We will see that these symbol-classes are subsets of those ones we have defined in Subsection 4.1.1.

Analogous to the definition of $C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we define

Definition 4.26. Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Then the *symbol-class* $W^{\tilde{m},q}_{uloc}S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is the set of all functions $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ such that

i) $\partial_x^\beta p(x,.) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,

ii) $\partial_x^\beta \partial_\xi^\alpha p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$,

iii) $|\partial_\xi^\alpha p(x,\xi)| \leq C_\alpha \langle \xi \rangle^{m-\rho|\alpha|}$ for all $x, \xi \in \mathbb{R}^n$,

iv) $\|\partial_\xi^\alpha p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}} \leq C_\alpha \langle \xi \rangle^{m-\rho|\alpha|}$ for all $\xi \in \mathbb{R}^n$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\beta| < \tilde{m} - n/q$. Here the constant C_α is independent of $x, \xi \in \mathbb{R}^n$. The function p is called *(non-smooth) pseudodifferential symbol* or just *(non-smooth) symbol* and m is called *order* of p. If $M = \infty$, the symbols are smooth in ξ. In this case we write $W^{\tilde{m},q}_{uloc}S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $W^{\tilde{m},q}_{uloc}S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

Now we define for all $k \in \mathbb{N}$ with $k \leq M$ the semi-norms

$$|a|_k^{(\tilde{m})} := \sup_{\xi \in \mathbb{R}^n} \max_{|\alpha| \leq k} \|\partial_\xi^\alpha p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \langle \xi \rangle^{-\tilde{m}+\rho|\alpha|}$$

for all $a \in W^{\tilde{m},q}_{uloc}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Equipped with the family of semi-norms $(|.|_k^{(\tilde{m})})_{k \in \{0,\ldots,M\}}$ the symbol-class $W^{\tilde{m},q}_{uloc}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a topological vector space.

Note, that condition iii) of the previous definition is a direct consequence of condition iv) due to Lemma 4.15. Hence we can skip condition iii) of the previous definition.

The symbol-class $W^{\tilde{m},q}_{uloc}S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a subset of a non-smooth symbol-class with coefficients in the Hölder spaces. This is the topic of the next remark:

Remark 4.27. Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Assuming $0 < \tau \leq \tilde{m} - n/q$, $\tau \notin \mathbb{N}$ we have

$$W^{\tilde{m},q}_{uloc}S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \subseteq C^\tau S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M).$$

Proof: The claim is a consequence of the Lemma 4.15 and the definition of these symbol-classes. □

Due to the last remark the associated pseudodifferential operator $p(x, D_x)$ to a non-smooth symbol $p \in W^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ was already defined in Definition 4.10. The set of all non-smooth pseudodifferential operators with symbols in $W^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is denoted by $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. If $M = \infty$ we write $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

The iterated commutators of a non-smooth pseudodifferential operator with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ are also non-smooth pseudodifferential operators. We see this in the following remark:

Remark 4.28. Let $1 < q < \infty$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $m \in \mathbb{R}$, $0 \leq \rho \leq 1$ and $p \in W^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Moreover, let $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, n\}$, $|\alpha| \leq M$ and $|\beta| < \tilde{m} - n/q$, where α and β are defined by $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Then the operator

$$\mathrm{ad}(-ix)^{\alpha_1}\,\mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l}\,\mathrm{ad}(D_x)^{\beta_l} p(x, D_x)$$

is a pseudodifferential operator with the symbol

$$\partial_\xi^\alpha D_x^\beta p(x,\xi) \in W^{\tilde{m}-|\beta|,q}_{uloc} S^{m-\rho|\alpha|}_{\rho,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; M - |\alpha|).$$

Proof: The proof this statement is essentially the same as that one of Remark 3.7. We just have to take care of the limited smoothness in x and ξ. Moreover, we have to use Remark 4.13, while verifying

$$D_{x_j}\tilde{p} \in W^{\tilde{m}-(|\beta|+1),q}_{uloc} S^{m-\rho|\alpha|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M - |\alpha|)$$

for all $j \in \{1, \ldots, n\}$. Here $\tilde{p}$ is defined as in the proof of Remark 3.7. □

4.1.4 Non-Smooth Symbol-Classes with Coefficients in $H^{\tilde{m}}_q$

Non-smooth pseudodifferential operators with coefficients in the Bessel potential spaces were investigated in [14] and [54]. Here we want to give a short introduction to the symbol-class $H^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ of these pseudodifferential operators.

To begin with, we define non-smooth symbols with coefficients in the Bessel potential spaces:

Definition 4.29. Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{R}$ with $\tilde{m} > n/q$. Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Then the *symbol-class* $H^{\tilde{m}}_q S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is the set of all functions $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ such that

i) $\partial_x^\beta p(x,.) \in C^M(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,

ii) $\partial_x^\beta \partial_\xi^\alpha p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$,

iii) $|\partial_\xi^\alpha p(x,\xi)| \leq C_\alpha \langle \xi \rangle^{m-\rho|\alpha|}$ for all $x, \xi \in \mathbb{R}^n$,

iv) $\|\partial_\xi^\alpha p(.,\xi)\|_{H_q^{\tilde{m}}} \leq C_\alpha \langle \xi \rangle^{m-\rho|\alpha|}$ for all $\xi \in \mathbb{R}^n$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq M$ and $|\beta| < \tilde{m} - n/q$. Here the constant C_α is independent of $x, \xi \in \mathbb{R}^n$. The function p is called *(non-smooth) pseudodifferential symbol* or just *(non-smooth) symbol* and m is called *order* of p. If $M = \infty$, the symbol-class is smooth in ξ. In this case we write $H_q^{\tilde{m}} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $H_q^{\tilde{m}} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

In the previous definition condition iii) is a direct consequence of condition iv) due to Lemma 2.44. Hence we are allowed to skip condition iii) of the previous definition.

Now we define for all $k \in \mathbb{N}$ with $k \leq M$ the semi-norms

$$|a|_k^{(\tilde{m})} := \sup_{\xi \in \mathbb{R}^n} \max_{|\alpha| \leq k} \|\partial_\xi^\alpha p(.,\xi)\|_{H_q^{\tilde{m}}(\mathbb{R}^n)} \langle \xi \rangle^{-\tilde{m}+\rho|\alpha|}$$

for all $a \in H_q^{\tilde{m}} S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Equipped with the family of semi-norms $(|.|_k^{(\tilde{m})})_{k \in \{0,\dots,M\}}$ the symbol-class $H_q^{\tilde{m}} S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a topological vector space.

The symbol-class $H_q^{\tilde{m}} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a subset of a non-smooth symbol-class with coefficients in the Hölder spaces. This is shown next remark:

Remark 4.30. Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{R}$ with $\tilde{m} > n/q$. Moreover, let $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Assuming $0 < \tau \leq \tilde{m} - n/q$, $\tau \notin \mathbb{N}$ we have

$$H_q^{\tilde{m}} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \subseteq C^\tau S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M).$$

Proof: The claim is a consequence of the Lemma 2.44 and the definition of these symbol-classes. □

Since $H_q^{\tilde{m}} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a subset of $C^\tau S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for certain $\tau > 0$, the associated pseudodifferential operator $p(x, D_x)$ to $p \in H_q^{\tilde{m}} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ was already defined in Definition 4.10. The set of all non-smooth pseudodifferential operators with symbols in the symbol-class $H_q^{\tilde{m}} S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is denoted by $\mathrm{OP} H_q^{\tilde{m}} S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. If $M = \infty$, we write $\mathrm{OP} H_q^{\tilde{m}} S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of $\mathrm{OP} H_q^{\tilde{m}} S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; \infty)$.

The iterated commutators of a non-smooth pseudodifferential operator with coefficients in $H_q^{\tilde{m}}(\mathbb{R}^n)$ are non-smooth pseudodifferential operators again:

Remark 4.31. Let $1 < q < \infty$, $0 \leq \rho \leq 1$, $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $\tilde{m} \in \mathbb{R}$ with $\tilde{m} > n/q$. We assume that $p \in H_q^{\tilde{m}} S_{\rho,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M)$. Additionally let $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, n\}$, $|\alpha| \leq M$ and $|\beta| < \tilde{m} - n/q$. Here α and β are denoted by $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Then the operator

$$\mathrm{ad}(-ix)^{\alpha_1} \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \mathrm{ad}(D_x)^{\beta_l} p(x, D_x)$$

is a pseudodifferential operator with the symbol

$$\partial_\xi^\alpha D_x^\beta p(x,\xi) \in H_q^{\tilde{m}-|\beta|} S_{\rho,0}^{m-\rho|\alpha|}(\mathbb{R}^n \times \mathbb{R}^n; M - |\alpha|).$$

Proof: On account of Example 3.2 and Example 3.6 we know that $D_{x_j} = \mathrm{OP}(\xi_j)$ is an element of $\mathrm{OP}S_{1,0}^1(\mathbb{R}^n \times \mathbb{R}^n)$ for all $j \in \{1, \ldots, n\}$. An application of Theorem 3.18 provides for each $s \in \mathbb{R}$ and every $j \in \{1, \ldots, n\}$:

$$\|D_{x_j} f\|_{H_q^s} \leq C \|f\|_{H_q^{s+1}} \qquad \text{for all } f \in H_q^{s+1}(\mathbb{R}^n). \tag{4.19}$$

We get the claim in the same way as the statement of Remark 4.28. We just have to use inequality (4.19) instead of Remark 4.13. □

4.1.5 Classical Non-Smooth Symbol-Classes

In applications to partial differential equations, many pseudodifferential operators are classical ones. Hence the restriction to so-called classical pseudodifferential operators is not a big disadvantage. This subsection is devoted to the introduction of classical non-smooth symbol-classes. We refer the reader to [67].

In order to define these symbol-classes we need the notion of a function being "homogeneous of degree d":

Definition 4.32. Considering $d \in \mathbb{R}$, a function $f : \mathbb{R}^n \to \mathbb{C}$ is *homogeneous of degree d (for $|x| \geq 1$)* if

$$f(rx) = r^d f(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \geq 1 \text{ and } r \geq 1.$$

Functions which are homogeneous of degree d have the following important property:

Remark 4.33. Let $d \in \mathbb{R}$ and $k \in \mathbb{N}$. Moreover, let $f \in C^k(\mathbb{R}^n)$ be homogeneous of degree d (for $|x| \geq 1$). Then $\partial_x^\alpha f$ is homogeneous of degree $d - |\alpha|$ (for $|x| \geq 1$) for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

Proof: The definition of derivatives and the previous definition provide the claim by mathematical induction with respect to $|\alpha|$. □

As mentioned before, we are able to define for every Banach space X with $C_c^\infty(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$ the non-smooth symbol-class $XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, cf. e.g. [67], Chapter 1. For all these non-smooth symbol-classes the previous definition enables us to define classical non-smooth symbols:

Definition 4.34. Let $m \in \mathbb{R}$ and $C_c^\infty(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$ be any Banach space. Then $p \in XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a *classical symbol of the order m* if p has a classical expansion

$$p(x,\xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x,\xi),$$

where p_j are homogeneous of degree $m-j$ in ξ (for $|\xi| \geq 1$) for all $j \in \mathbb{N}_0$ in the sense, that for all $N \in \mathbb{N}$ we have

$$p(x,\xi) - \sum_{j<N} p_j(x,\xi) \in XS^{m-N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

The set of all classical symbols of the order m is denoted by $XS^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$.

As an immediate consequence of the previous definition we obtain for all Banach spaces X with $C_c^\infty(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$ and all $m \in \mathbb{R}$:

$$XS^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

For our purposes we just need the case $X = W^{\tilde{m},q}_{uloc}$. Thus we restrict ourselves to this case now. In Remark 4.28 we have verified that the iterated commutators of a non-smooth pseudodifferential operator with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ are such operators, too. For iterated commutators of non-smooth pseudodifferential operators with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ which have classical symbols the following remark holds:

Remark 4.35. Let $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $m \in \mathbb{R}$. We assume that $p \in W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, let $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, n\}$ and $|\beta| < \tilde{m} - n/q$. Here α and β are defined by $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Then the operator

$$\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} \ldots \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} p(x, D_x)$$

is a pseudodifferential operator with the symbol

$$\partial_\xi^\alpha D_x^\beta p(x,\xi) \in W^{\tilde{m}-|\beta|,q}_{uloc} S^{m-|\alpha|}_{cl}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

Proof: Let $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ be as in the assumptions. Since $p \in W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, Remark 4.28 provides that

$\mathrm{ad}(-ix)^{\alpha_1}\,\mathrm{ad}(D_x)^{\beta_1}\dots\mathrm{ad}(-ix)^{\alpha_l}\,\mathrm{ad}(D_x)^{\beta_l}p(x,D_x)$ is a pseudodifferential operator with the symbol

$$\partial_\xi^\alpha D_x^\beta p(x,\xi)\in W^{\tilde{m}-|\beta|,q}_{uloc}S^{m-|\alpha|}_{1,0}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi).$$

Therefore it remains to show that the symbol $\partial_\xi^\alpha D_x^\beta p(x,\xi)$ is even an element of $W^{\tilde{m}-|\beta|,q}_{uloc}S^{m-|\alpha|}_{cl}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$. Because of $p\in W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n\times\mathbb{R}^n)$, there is a classical expansion, where p_k are homogeneous of degree $m-k$ in ξ (for $|\xi|\geq 1$) such that for all $N\in\mathbb{N}$ we have:

$$p(x,\xi)-\sum_{k<N}p_k(x,\xi)\in W^{\tilde{m},q}_{uloc}S^{m-N}_{1,0}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi).$$

Let $N\in\mathbb{N}$ be arbitrary. On account of Remark 4.28 we have

$$\partial_\xi^\alpha D_x^\beta p(x,\xi)-\sum_{k<N}\partial_\xi^\alpha D_x^\beta p_k(x,\xi)\in W^{\tilde{m}-|\beta|,q}_{uloc}S^{m-|\alpha|-N}_{1,0}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi).$$

Iteratively we are able to show that $D_x^\beta p(x,\xi)$ is homogeneous of degree $m-k$ with respect to ξ (for $|\xi|\geq 1$) by using the definition of homogeneous functions. Consequently $\partial_\xi^\alpha D_x^\beta p_k(x,\xi)$ is homogeneous of degree $m-|\alpha|-k$ in ξ (for $|\xi|\geq 1$) for every $k\in\mathbb{N}_0$ due to Remark 4.33. Hence $\partial_\xi^\alpha D_x^\beta p(x,\xi)$ is an element of $W^{\tilde{m}-|\beta|,q}_{uloc}S^{m-|\alpha|}_{cl}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$. □

4.2 Extension of the Space of Amplitudes

The space of amplitudes was introduced in Section 3.2. We have seen that the oscillatory integral is well-defined for all functions of this set. At first we just were able to define the pseudodifferential operator $p(x,D_x)$ on the Schwartz space for a given smooth symbol p. The oscillatory integral allows us to extend the definition of $p(x,D_x)u$ on functions $u\in C_b^\infty(\mathbb{R}^n)$, cf. e.g. [5], Chapter 3.7. But this is not the only application of oscillatory integrals. By means of the properties of these integrals, we can calculate the symbol of the composition of two smooth pseudodifferential operators explicitly, cf. e.g [5], Chapter 3.5. As these oscillatory integrals are an important technique for working with smooth pseudodifferential operators, we would like to use the oscillatory integral in the non-smooth case in order to obtain similar results.

We consider an arbitrary non-smooth symbol $p\in C^{\tilde{m},\tau}S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n;M)$ and a Schwartz function u. If we choose an arbitrary $l\in\mathbb{N}_0$ with $-2l+m<-n$, we obtain for all $x\in\mathbb{R}^n$ by using Remark 2.10 and integration by parts with respect to y:

$$p(x,D_x)u(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)đ\xi=\iint e^{i(x-y)\cdot\xi}p(x,\xi)u(y)dyđ\xi$$

$$= \iint e^{i(x-y)\cdot\xi} \langle \xi \rangle^{-2l} p(x,\xi) \langle D_y \rangle^{2l} u(y) dy đ\xi$$
$$= \mathrm{Os}\text{-}\iint e^{i(x-y)\cdot\xi} \langle \xi \rangle^{-2l} p(x,\xi) \langle D_y \rangle^{2l} u(y) dy đ\xi.$$

Here we have used the identity, according to which we are able to write iterated integrals as oscillatory integrals due to (3.4). Hence for each $x \in \mathbb{R}^n$ the oscillatory integral of f, defined by $f(y,\xi) := p(x,\xi)u(y)$ for all $y, \xi \in \mathbb{R}^n$, is well-defined. But in the case $N \neq \infty$ this function is not an element of the space of amplitudes since f is not smooth with respect to ξ. Therefore the possibility arises to extend the space of amplitudes to non-smooth functions in the following way: the oscillatory integral is well-defined for all functions of this extension. This is the topic of the next definition:

Definition 4.36. Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. We define $\mathscr{A}_\tau^{m,N}(\mathbb{R}^n \times \mathbb{R}^n)$ as the set of all functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the following properties: For all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ we have

i) $\partial_\eta^\alpha \partial_y^\beta a(y,\eta) \in C^0(\mathbb{R}_y^n \times \mathbb{R}_\eta^n)$,

ii) $\left|\partial_\eta^\alpha \partial_y^\beta a(y,\eta)\right| \leq C_{\alpha,\beta}(1+|\eta|)^m(1+|y|)^\tau$ for all $y, \eta \in \mathbb{R}^n$,

where $C_{\alpha,\beta}$ is independent of $y, \eta \in \mathbb{R}^n$.

Note that $\mathscr{A}_\tau^{m,\infty}(\mathbb{R}^n \times \mathbb{R}^n) = \mathscr{A}_\tau^{m}(\mathbb{R}^n \times \mathbb{R}^n)$.

We mention one important example for the extension of the space of amplitudes:

Example 4.37. Let $m \in \mathbb{R}$, $\tilde{m} \in \mathbb{N}_0$, $0 < s \leq 1$ and $0 \leq \rho, \delta \leq 1$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$. Assuming $a \in C^{\tilde{m},s} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; N)$ and $u \in \mathcal{S}(\mathbb{R}^n)$, we obtain for all $x \in \mathbb{R}^n$ and $l \in \mathbb{N}_0$

$$a(x,\xi)u(y) \in \mathscr{A}_{-l}^{m,N}(\mathbb{R}_y^n \times \mathbb{R}_\xi^n).$$

Proof: Let $x \in \mathbb{R}^n$, $l \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ be arbitrary. Then the following inequality holds due to $u \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned} \left|\partial_\xi^\alpha \partial_y^\beta [a(x,\xi)u(y)]\right| &= \left|\partial_\xi^\alpha a(x,\xi)\right| \left|\partial_y^\beta u(y)\right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|} \langle y \rangle^{-l} \\ &\leq C_{\alpha,\beta,l} \langle \xi \rangle^m \langle y \rangle^{-l} \qquad \text{for all } \xi, y \in \mathbb{R}^n. \end{aligned}$$

Since $\partial_\xi^\alpha \partial_y^\beta a(x,\xi)u(y)$ is also an element of $C^0(\mathbb{R}_y^n \times \mathbb{R}_\xi^n)$ for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \leq N$, we conclude the claim. □

The previous definition enables us to extend the definition of the oscillatory integral for functions in the set $\mathscr{A}_\tau^{m,N}(\mathbb{R}^n \times \mathbb{R}^n)$:

Theorem 4.38. *Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $\tilde{l} \in \mathbb{N}_0$ with the property $N \geq 2\tilde{l} > n+\tau$. Moreover, let $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$ be arbitrary. Then the* ***oscillatory integral***

$$Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta) dy đ\eta := \lim_{\varepsilon \to 0} \iint \chi(\varepsilon y, \varepsilon \eta) e^{-iy\cdot\eta} a(y,\eta) dy đ\eta$$

exists for each $a \in \mathscr{A}^{m,N}_{\tau}(\mathbb{R}^n \times \mathbb{R}^n)$. Additionally for all $l, l' \in \mathbb{N}_0$ with $2l > n+m$ and $N \geq 2l' > n+\tau$ we have the following equality:

$$Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta) dy đ\eta = \iint e^{-iy\cdot\eta} \langle y \rangle^{-2l'} \langle D_\eta \rangle^{2l'} [\langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} a(y,\eta)] dy đ\eta.$$

Therefore the definition does not depend on the choice of χ.

This theorem is an extension of Theorem 3.11 and can be proved in the same way.

Next, we want to convince ourselves that the properties of the oscillatory integral even hold for all functions of the set $\mathscr{A}^{m,N}_{\tau}(\mathbb{R}^n \times \mathbb{R}^n)$. Analogous to the proof of [5], Theorem 3.13 we obtain the ability to interchange derivatives with the oscillatory integral. To be more precise:

Theorem 4.39. *Let $m, \tau \in \mathbb{R}$, $N \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \mathbb{N}$ be such that there is an $\tilde{l} \in \mathbb{N}_0$ with $N \geq 2\tilde{l} > k+\tau$. We set $\tilde{\tau} := \tau$ if $\tau \geq -k$, $\tilde{\tau} := -k-0.5$ if $\tau \in \mathbb{Z}$ and $\tau < -k$ and $\tilde{\tau} := -k - (|\tau| - \lfloor -\tau \rfloor)/2$ else. Moreover, we define $\hat{\tau} := \tau_+$ if $\tau \geq -k$ and $\hat{\tau} := \tau - \tilde{\tau}$ else. Assuming an $a \in \mathscr{A}^{m,N}_{\tau}(\mathbb{R}^{n+k} \times \mathbb{R}^{n+k})$, we define the function $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by integration with respect to $\mathbb{R}^k \times \mathbb{R}^k$:*

$$b(y,\eta) := Os\text{-}\iint e^{-iy'\cdot\eta'} a(y,y',\eta,\eta') dy' đ\eta' \qquad \text{for all } y, \eta \in \mathbb{R}^n.$$

Let $M := \max\{m \in \mathbb{N}_0 : N - m \geq 2l > k + \tilde{\tau}$ for one $l \in \mathbb{N}_0\}$. Then b is an element of $\mathscr{A}^{m_+,M}_{\hat{\tau}}(\mathbb{R}^n \times \mathbb{R}^n)$ and for each $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq M$ we have:

$$\partial_y^\alpha \partial_\eta^\beta b(y,\eta) = Os\text{-}\iint e^{-iy'\cdot\eta'} \partial_y^\alpha \partial_\eta^\beta a(y,y',\eta,\eta') dy' đ\eta' \qquad \text{for all } y, \eta \in \mathbb{R}^n.$$

Proof: On account of $\langle (\eta,\eta') \rangle^2 \geq \langle \eta \rangle^2$ for all $\eta, \eta' \in \mathbb{R}^n$ and of Peetre's inequality, cf. Lemma 2.4, we get

$$\langle (\eta,\eta') \rangle^m \langle (y,y') \rangle^\tau \leq C \langle \eta \rangle^{m_+} \langle \eta' \rangle^m \langle y \rangle^{\hat{\tau}} \langle y' \rangle^{\tilde{\tau}} \qquad \text{for all } \eta, \eta', y, y' \in \mathbb{R}^n.$$

Hence we obtain for fixed $y, \eta \in \mathbb{R}^n$ and for all $\alpha, \beta \in \mathbb{N}_0^n$, $\tilde{\alpha}, \tilde{\beta} \in N_0^k$ with $|\beta| \leq M$ and $|\tilde{\beta}| \leq N - |\beta|$:

$$|\partial_y^{\tilde{\alpha}} \partial_\eta^{\tilde{\beta}} \partial_y^\alpha \partial_\eta^\beta a(y,y',\eta,\eta')| \leq C_{y,\eta} \langle \eta' \rangle^m \langle y' \rangle^{\tilde{\tau}} \qquad \text{for all } \eta', y' \in \mathbb{R}^n.$$

Using the previous inequality the theorem can be verified in the same way as [5], Theorem 3.13. □

We also get Fubini's theorem for oscillatory integrals. It can be shown in the same manner as Theorem 3.13 in [5].

Theorem 4.40. *Let $m, \tau \in \mathbb{R}$ and $k \in \mathbb{N}$. We define $\tilde{\tau} := \tau$ if $\tau \geq -k$, $\tilde{\tau} := -k - 0.5$ if $\tau \in \mathbb{Z}$ and $\tau < -k$ and $\tilde{\tau} := -k - (|\tau| - \lfloor -\tau \rfloor)/2$ else. Moreover, we define $\hat{\tau} := \tau_+$ if $\tau \geq -k$ and $\hat{\tau} := \tau - \tilde{\tau}$ else. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $\tilde{l} \in \mathbb{N}_0$ with $M \geq 2\tilde{l} > n + \hat{\tau}$, where $M := \max\{m \in \mathbb{N}_0 : N - m \geq 2l > k + \tilde{\tau}$ for one $l \in \mathbb{N}_0\}$. Assuming an $a \in \mathscr{A}^{m,N}_{\tau}(\mathbb{R}^{n+k} \times \mathbb{R}^{n+k})$ we obtain*

$$
\begin{aligned}
Os\text{-}\iiiint & e^{-iy\cdot\eta - iy'\cdot\eta'} a(y, y', \eta, \eta') dy dy' đ\eta đ\eta' \\
&= Os\text{-}\iint e^{-iy\cdot\eta} \left[Os\text{-}\iint e^{-iy'\cdot\eta'} a(y, y', \eta, \eta') dy' đ\eta' \right] dy đ\eta.
\end{aligned}
$$

For functions in the set $\mathscr{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ integration by parts is possible. We refer to e.g. [42], Theorem 6.8 for a proof. In the same way we can verify that the statement is also true for elements in the set $\mathscr{A}^{m,N}_\tau(\mathbb{R}^n \times \mathbb{R}^n)$:

Theorem 4.41. *Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l' \in \mathbb{N}_0$ with $N \geq 2l' > n + \tau$. Moreover, let $l_0, \tilde{l}_0 \in \mathbb{N}_0$ with the property $2\tilde{l}_0 \leq N$. Then*

$$
Os\text{-}\iint e^{-iy\cdot\eta} a(y, \eta) dy đ\eta = Os\text{-}\iint e^{-iy\cdot\eta} \langle y \rangle^{-2\tilde{l}_0} \langle D_\eta \rangle^{2\tilde{l}_0} [\langle \eta \rangle^{-2l_0} \langle D_y \rangle^{2l_0} a(y, \eta)] dy đ\eta
$$

for every $a \in \mathscr{A}^{m,N}_\tau(\mathbb{R}^n \times \mathbb{R}^n)$.

Another important property is the ability to interchange limit and oscillatory integral. This is the topic of the next corollary. It can be checked in the same manner as the analogous result in the smooth case, cf. e.g. [5], Corollary 3.10.

Corollary 4.42. *Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l' \in \mathbb{N}_0$ with $N \geq 2l' > n + \tau$. Additionally let $(a_j)_{j \in \mathbb{N}} \subseteq \mathscr{A}^{m,N}_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ be a bounded sequence, i.e for all $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \leq N$ and all $j \in \mathbb{N}$:*

$$
\left| \partial^\beta_y \partial^\alpha_\eta a_j(y, \eta) \right| \leq C_{\alpha,\beta} \langle \eta \rangle^m \langle y \rangle^\tau \qquad \text{for all } y, \eta \in \mathbb{R}^n.
$$

Here the constant $C_{\alpha,\beta}$ is independent of $j \in \mathbb{N}$ and $y, \eta \in \mathbb{R}^n$. Moreover, there is an $a \in \mathscr{A}^{m,N}_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$
\lim_{j \to \infty} \partial^\alpha_\eta \partial^\beta_y a_j(y, \eta) = \partial^\alpha_\eta \partial^\beta_y a(y, \eta) \qquad \text{for all } y, \eta \in \mathbb{R}^n \tag{4.20}
$$

for each $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \leq N$. Then

$$
\lim_{j \to \infty} Os\text{-}\iint e^{-iy\cdot\eta} a_j(y, \eta) dy đ\eta = Os\text{-}\iint e^{-iy\cdot\eta} a(y, \eta) dy đ\eta.
$$

Additionally we mention that the oscillatory integral of functions being of the set $\mathscr{A}_\tau^m(\mathbb{R}^n \times \mathbb{R}^n)$ is translation invariant. We refer to e.g. [42], Theorem 6.8 for a proof. Similary we are able to show the analogous result for the set $\mathscr{A}_\tau^{m,N}(\mathbb{R}^n \times \mathbb{R}^n)$:

Theorem 4.43. *Let $m, \tau \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}_0$ with $N \geq 2l > n + \tau$. For $a \in \mathscr{A}_\tau^{m,N}(\mathbb{R}^n \times \mathbb{R}^n)$ we have the following invariance of the oscillatory integral with respect to translations:*

$$Os\text{-}\iint e^{-i(y+y_0)\cdot(\eta+\eta_0)} a(y+y_0, \eta+\eta_0) dy đ\eta = Os\text{-}\iint e^{-iy\cdot\eta} a(y,\eta) dy đ\eta.$$

4.3 Symbol Composition

A calculus for non-smooth pseudodifferential operators in the non-smooth symbol-class $C^\tau S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ was first developed by H. Kumano-Go and M. Nagase in [43]. Y. Meyer and J. Marschall improved this calculus in [55] and [52], Chapter 6. Later J. Marschall adapted the arguments given there to obtain a calculus for the general case $C^\tau S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; N)$ in [53]. Most recent, H. Abels treated a calculus for operator-valued pseudodifferential operators with non-smooth symbols of class $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \mathscr{L}(X_1, X_2))$ in [3].

We recall that the composition of two smooth pseudodifferential operators is also a smooth pseudodifferential operator, cf. Theorem 3.13 in Section 3.2. For the composition of a non-smooth pseudodifferential operator with a special smooth pseudodifferential operator we obtain a similar result:

Remark 4.44. Let $m_1, m_2 \in \mathbb{R}$, $\tilde{m} \in \mathbb{N}_0$, $0 < s \leq 1$ and $0 \leq \rho \leq 1$. Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$. We choose an $a \in C^{\tilde{m},s} S^{m_1}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ and a symbol $p \in S^{m_2}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$, which is independent of the first variable x. Then the pseudodifferential operator

$$a(x, D_x)p(x, D_x) \in \mathrm{OP}C^{\tilde{m},s} S^{m_1+m_2}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$$

has the symbol $a(x,\xi)p(\xi)$.

Proof: Since p is independent of x, we can write $p(D_x)$ instead of $p(x, D_x)$. One immediately gets $a(x,\xi)p(\xi) \in C^{\tilde{m},s} S^{m_1+m_2}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ due to the choice of the symbols a and p. Using $\mathscr{F}[p(D_x)u](\xi) = p(\xi)\hat{u}(\xi)$ for all $\xi \in \mathbb{R}^n$, $u \in \mathcal{S}(\mathbb{R}^n)$ provides the remark. □

We are also able to show that the iterated commutators of a non-smooth pseudodifferential operator are pseudodifferential operators provided that suitable conditions are fulfilled.

Remark 4.45. Let $\tilde{m} \in \mathbb{N}_0$, $M \in \mathbb{N}_0 \cup \{\infty\}$, $0 < \tau \leq 1$, $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. We assume that $p \in C^{\tilde{m},\tau} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Moreover, let $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, n\}$, $|\alpha| \leq M$ and $|\beta| \leq \tilde{m}$. Here α and β are defined by $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Then the operator

$$\mathrm{ad}(-ix)^{\alpha_1} \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \mathrm{ad}(D_x)^{\beta_l} p(x, D_x)$$

is a pseudodifferential operator with the symbol

$$\partial_\xi^\alpha D_x^\beta p(x,\xi) \in C^{\tilde{m}-|\beta|,\tau} S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; M - |\alpha|).$$

Proof: We get the claim in the same way as the statement of Remark 4.28. We just have to use Remark 2.37 instead of Remark 4.13. □

But in contrast to the smooth case, the composition of two non-smooth pseudodifferential operators is in general not a pseudodifferential operator with the same regularity with respect to its coefficient, cf. [3], p.1465. To illustrate this, let $j \in \{1, \ldots, n\}$ and $p \in C^\tau S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ with $\tau \in (0,1)$, $m \in \mathbb{R}$ and $p(x,\xi) \notin C^1(\mathbb{R}^n_x)$ for all $\xi \in \mathbb{R}^n$. Remark 4.44 provides that $p(x, D_x)\mathrm{OP}(\xi_j)$ is an element of $\mathrm{OP}C^\tau S^{m+1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ due to $\xi_j \in S^1_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore the question arises whether $\mathrm{OP}(\xi_j)p(x, D_x)$ is also a pseudodifferential operator. If this would be the case, the iterated commutator

$$\mathrm{ad}(D_{x_j})p(x, D_x) = \mathrm{OP}(\xi_j)p(x, D_x) - p(x, D_x)\mathrm{OP}(\xi_j)$$

would be a pseudodifferential operator, too. In the same manner as in the previous remark we are able to calculate $\mathrm{ad}(D_{x_j})p(x, D_x)$ formally and we obtain

$$\mathrm{ad}(D_{x_j})p(x, D_x) = (\partial_{x_j} p)(x, D_x). \tag{4.21}$$

Because of the choice of τ, $\partial_{x_j} p$ does not exist. Hence $\mathrm{ad}(D_{x_j})p(x, D_x)$ is not a pseudodifferential operator just like $\mathrm{OP}(\xi_j)p(x, D_x)$.

However, there is a finite expansion for the composition of two non-smooth pseudodifferential operators: let $p_i \in C^{\tilde{m}_i,\tau_i} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $i \in \{1, 2\}$ be two symbols, where $\tilde{m}_i \in \mathbb{N}_0$, $0 < \tau_i < 1$ and $m_i \in \mathbb{R}$ for $i \in \{1, 2\}$. For every $k \in \mathbb{N}_0$ we define the symbol $p_1 \#_k p_2$ by

$$(p_1 \#_k p_2)(x, \xi) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial_\xi^\alpha p_1(x, \xi) D_x^\alpha p_2(x, \xi) \qquad \text{for all } x, \xi \in \mathbb{R}^n.$$

The next theorem shows for each $\theta \in (0, \tau_2)$ that the operator

$$R_\theta(p_1, p_2) := p_1(x, D_x)p_2(x, D_x) - (p_1 \#_{\lfloor \theta \rfloor} p_2)(x, D_x)$$

is of the order $m_1 + m_2 - \theta$ in the sense of mapping properties in Bessel potential spaces.

Theorem 4.46. *For $i \in \{1,2\}$ let $\mathscr{B}_i \subseteq C^{\tilde{m}_i,\tau_i} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be a bounded subset with $\tilde{m}_i \in \mathbb{N}_0$, $m_i \in \mathbb{R}$ and $0 < \tau_i < 1$. Moreover, let $1 < q < \infty$ and $\theta \in (0, \tau_2)$. Setting $\tau := \min\{\tau_1, \tau_2 - \lfloor \theta \rfloor\}$ we get for all $s \in (-\tau, \tau)$ with $s - \theta > -\tau_2$ and $-\tau_2 + \theta < s + m_1 < \tau_2$ the boundedness of*

$$\{R_\theta(p_1, p_2) : p_i \in \mathscr{B}_i\} \subseteq \mathscr{L}(H^{s+m_1+m_2-\theta}_q(\mathbb{R}^n), H^s_q(\mathbb{R}^n)).$$

Proof: On account of Theorem 3.6 in [3] we obtain the boundedness of $R_\theta(p_1, p_2)$ as a map from $H^{s+m_1+m_2-\theta}_q(\mathbb{R}^n)$ to $H^s_q(\mathbb{R}^n)$. Verifying the proof of this theorem, we obtain the independence of the constant C from $p_i \in \mathscr{B}_i$, $i \in \{1,2\}$. □

4.4 Boundedness of Non-Smooth Pseudodifferential Operators

This section is devoted to the boundedness of non-smooth pseudodifferential operators between different function spaces. Let us give a short outline of this section: At first we show the boundedness of non-smooth pseudodifferential operators with coefficients in a Banach space X. Here X has to be such that $C^\infty_c(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$. In Subsection 4.4.1 we treat the most common class of non-smooth pseudodifferential operators: non-smooth pseudodifferential operators with coefficients in the Hölder spaces. After that we focus on boundedness results of non-smooth pseudodifferential operators with coefficients in $H^{\tilde{m}}_q(\mathbb{R}^n)$ and $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ in the Subsections 4.4.2 and 4.4.3.

To begin with, we prove the boundedness of non-smooth pseudodifferential operators with coefficients in a Banach space X. During this thesis we just need the cases $X \in \{C^{\tilde{m},\tau}, C^{\tilde{m}+\tau}_*, H^{\tilde{m}}_q, W^{\tilde{m},q}_{uloc}\}$. Hence during this section we assume $X, Y \in \{C^{\tilde{m},\tau}, C^{\tilde{m}+\tau}_*, H^{\tilde{m}}_q, W^{\tilde{m},q}_{uloc}\}$ with $\tilde{m} \in \mathbb{N}_0$, $0 < \tau \leq 1$ and $1 < q < \infty$, unless otherwise noted. Additionally we assume $\delta = 0$ in the case $X \notin \{C^{\tilde{m},\tau}, C^{\tilde{m}+\tau}_*\}$ and $\tilde{m} > n/q$ if $X \in \{H^{\tilde{m}}_q, W^{\tilde{m},q}_{uloc}\}$.

For these spaces the following property holds:

Remark 4.47. For $0 \leq \rho, \delta \leq 1$ and $M \in \mathbb{N}_0 \cup \{\infty\}$ we choose an arbitrary $a \in XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Moreover let $N = \tilde{m} + 2$ in the case $X = C^{\tilde{m}+\tau}_*$, $N = \tilde{m} + 1$ in the case $X = C^{\tilde{m},\tau}$ and $N = \tilde{m}$ else. Then there is a constant $C_{\tilde{m},\tau} > 0$ such that

$$\|e_\xi \cdot a(.,\xi)\|_X \leq C_{\tilde{m},\tau} \langle \xi \rangle^N \|a(.,\xi)\|_X \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Proof: For $X \in \{H^{\tilde{m}}_q, W^{\tilde{m},q}_{uloc}\}$ the claim can be verified by using the definition of these spaces and the Leibniz rule.

Our next goal is to calculate the norm $\|e_\xi\|_{C_b^{\tilde{m}+\lfloor\tau\rfloor+1}}$. Since $\|\partial_x^\alpha e^{ix\cdot\xi}\|_{L^\infty} \leq \langle\xi\rangle^{|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m} + \lfloor\tau\rfloor + 1$, we have

$$\|e_\xi\|_{C_b^{\tilde{m}+\lfloor\tau\rfloor+1}} \leq C \sum_{|\alpha|\leq\tilde{m}+\lfloor\tau\rfloor+1} \|\partial_x^\alpha e^{ix\cdot\xi}\|_{L^\infty} \leq C\langle\xi\rangle^{\tilde{m}+\lfloor\tau\rfloor+1}. \tag{4.22}$$

With Lemma 2.40, (4.22) and the embedding $C_b^{\tilde{m}+\lfloor\tau\rfloor+1}(\mathbb{R}^n) \hookrightarrow C_*^{\tilde{m}+\tau}(\mathbb{R}^n)$ at hand, we are in the position to prove the remark for $X = C_*^{\tilde{m},\tau}$:

$$\begin{aligned}\|e_\xi \cdot a(.,\xi)\|_{C_*^{\tilde{m},\tau}} &\leq C_{\tilde{m},\tau}\|e_\xi\|_{C_*^{\tilde{m},\tau}}\|a(.,\xi)\|_{C_*^{\tilde{m},\tau}} \leq C_{\tilde{m},\tau}\|e_\xi\|_{C_b^{\tilde{m}+\lfloor\tau\rfloor+1}}\|a(.,\xi)\|_{C_*^{\tilde{m},\tau}}\\ &\leq C_{\tilde{m},\tau}\langle\xi\rangle^{\tilde{m}+2}\|a(.,\xi)\|_{C_*^{\tilde{m},\tau}} \qquad \text{for all } \xi \in \mathbb{R}^n.\end{aligned}$$

It remains to prove the case $X = C^{\tilde{m},\tau}$. Using the mean value theorem in the case $|x_1 - x_2| \leq 1$, $x_1 \neq x_2$ we obtain

$$\max_{x_1\neq x_2} \frac{|e^{ix_1\cdot\xi} - e^{ix_2\cdot\xi}|}{|x_1 - x_2|^\tau} \leq 2\langle\xi\rangle \qquad \text{for all } \xi \in \mathbb{R}^n.$$

If we add the term $e^{ix_1\cdot\xi}\partial_x^\alpha a(x_2,\xi) - e^{ix_1\cdot\xi}\partial_x^\alpha a(x_2,\xi)$ in the numerator, we are able to verify the next estimate by means of the previous inequality and the Leibniz rule:

$$\max_{|\alpha|\leq\tilde{m}} \sup_{x_1\neq x_2} \frac{|e^{ix_1\cdot\xi}\partial_x^\alpha a(x_1,\xi) - e^{ix_2\cdot\xi}\partial_x^\alpha a(x_2,\xi)|}{|x_1 - x_2|^\tau} \leq C_{\tilde{m},\tau}\langle\xi\rangle\|a(.,\xi)\|_{C_*^{\tilde{m},\tau}} \tag{4.23}$$

for all $\xi \in \mathbb{R}^n$. In the same way as in estimate (4.22), we are able to show

$$\|e_\xi \cdot a(.,\xi)\|_{C_b^{\tilde{m}}} \leq C_{\tilde{m}}\langle\xi\rangle^{\tilde{m}}\|a(.,\xi)\|_{C^{\tilde{m},\tau}} \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{4.24}$$

A combination of the inequalities (4.23) and (4.24) yields

$$\|e_\xi \cdot a(.,\xi)\|_{C^{\tilde{m},\tau}} \leq C_{\tilde{m},\tau}\langle\xi\rangle^{\tilde{m}+1}\|a(.,\xi)\|_{C^{\tilde{m},\tau}} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

□

The previous remark enables us to prove the next boundedness result:

Lemma 4.48. *Let $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho, \delta \leq 1$. Assuming $p \in XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, we obtain the continuity of $p(x, D_x) : \mathcal{S}(\mathbb{R}^n) \to X$.*

Proof: Let $u \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary. An application of $p \in XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, $u \in \mathcal{S}(\mathbb{R}^n)$, Remark 4.47 and Lemma 2.18 yields

$$\begin{aligned}\|p(x, D_x)u(x)\|_X &\leq \int \|e_\xi p(.,\xi)\|_X|\hat{u}(\xi)|đ\xi \leq C\int \langle\xi\rangle^{-(n+1)}đ\xi|\hat{u}|_{\hat{m}+(n+1),\mathcal{S}}\\ &\leq C|u|_{\hat{m}+2(n+1),\mathcal{S}} \qquad \text{for all } x \in \mathbb{R}^n\end{aligned}$$

for some $\hat{m} \in \mathbb{N}$. □

In the case $X = C^{\tilde{m},\tau}$ this statement was already checked in [45], Theorem 3.6. For a bounded subset of $\mathscr{B} \subseteq XS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$, where X, m, ρ, δ and M are defined as in the previous lemma, we even are able to improve the statement of Lemma 4.48: Verifying the proof of Lemma 4.48 yields the boundedness of

$$\{p(x, D_x) : p \in \mathscr{B}\} \subseteq \mathscr{L}(\mathcal{S}(\mathbb{R}^n); X). \tag{4.25}$$

In the literature such problems are mostly not investigated. Usually just boundedness results are shown in different cases. Verifying these proofs in order to get similar results as (4.25) is often very complex. With the next lemma at hand, such problems are much easier to prove.

Lemma 4.49. *Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. We consider that $\mathscr{B}$ is the topological vector space $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ or $YS^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; N)$. In the case $\mathscr{B} = S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ we set $N := \infty$. Moreover, let X_1, X_2 be two Banach spaces with the following properties:*

i) $\mathcal{S}(\mathbb{R}^n) \subseteq X_1, X_2 \subseteq \mathcal{S}'(\mathbb{R}^n)$,

ii) $\mathcal{S}(\mathbb{R}^n)$ *is dense in* X_1 *and in* X_2',

iii) $a(x, D_x) \in \mathscr{L}(X_1, X_2)$ *for all* $a \in \mathscr{B}$.

Then there is a $k \in \mathbb{N}$ with $k \leq N$ and a constant $C > 0$, independent of $a \in \mathscr{B}$, such that

$$\|a(x, D_x)f\|_{\mathscr{L}(X_1;X_2)} \leq C|a|^{(m)}_k \qquad \text{for all } a \in \mathscr{B}.$$

Proof: First of all we define for $f, g \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{X_1} \leq 1$ and $\|g\|_{X_2'} \leq 1$ the operator $\mathrm{OP}_{f,g} : \mathscr{B} \to \mathbb{C}$ by $\mathrm{OP}_{f,g}(a) := \langle a(x, D_x)f, g\rangle_{X_2,X_2'}$. Using *iii)* we get the existence of a constant C, independent of $f, g \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{X_1} \leq 1$ and $\|g\|_{X_2'} \leq 1$, such that

$$\begin{aligned} |\langle a(x, D_x)f, g\rangle_{X_2,X_2'}| &\leq \|a(x, D_x)f\|_{X_2} \|g\|_{X_2'} \leq C \|a(x, D_x)\|_{\mathscr{L}(X_1;X_2)} \|f\|_{X_1} \|g\|_{X_2'} \\ &\leq C \|a(x, D_x)\|_{\mathscr{L}(X_1;X_2)} . \end{aligned}$$

Consequently the set

$$\left\{\mathrm{OP}(a)_{f,g} : f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with } \|f\|_{X_1} \leq 1 \text{ and } \|g\|_{X_2'} \leq 1\right\} \subseteq \mathbb{C}$$

is bounded for each $a \in \mathscr{B}$. Hence all assumptions of the theorem of Banach-Steinhaus, cf. e.g. [61], Theorem 2.5 hold. An application of this theorem provides that

$$\left\{\mathrm{OP}_{f,g} : f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with } \|f\|_{X_1} \leq 1 \text{ and } \|g\|_{X_2'} \leq 1\right\}$$

is equicontinuous. With the equicontinuity of the previous set at hand, we get the existence of a $k \in \mathbb{N}$ with $k \leq N$ and a constant $C > 0$ such that

$$|\mathrm{OP}_{f,g}(a)| \leq C|a|_k^{(m)} \quad \text{for all } a \in \mathscr{B}, f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with } \|f\|_{X_1} \leq 1, \|g\|_{X_2'} \leq 1.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in X_1 and in X_2', the previous inequality even holds for all $f \in X_1$ and $g \in X_2'$ with $\|f\|_{X_1} \leq 1$ and $\|g\|_{X_2'} \leq 1$. This implies the claim:

$$\begin{aligned}\|a(x, D_x)\|_{\mathscr{L}(X_1;X_2)} &= \sup_{\|f\|_{X_1}\leq 1} \|a(x, D_x)f\|_{X_2} = \sup_{\|f\|_{X_1}\leq 1} \sup_{\|g\|_{X_2'}\leq 1} |\mathrm{OP}_{f,g}(a)| \\ &\leq C|a|_k^{(m)} \qquad \text{for all } a \in \mathscr{B}.\end{aligned}$$

□

We already mentioned that the symbol-classes $XS^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ can be defined for arbitrary Banach spaces X fulfilling $C_c^\infty(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$, $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$, too. Hence Lemma 4.48 and 4.49 also hold for arbitrary Banach spaces X, Y with the property $C_c^\infty(\mathbb{R}^n) \subseteq X, Y \subseteq C^0(\mathbb{R}^n)$.

4.4.1 Boundedness of Pseudodifferential Operators with Coefficients in the Hölder Space

After verifying an estimate for non-smooth pseudodifferential operators with coefficients in the Hölder spaces applied to a Schwartz function, we have a look at the boundedness of these operators with respect to the Bessel potential spaces.

Lemma 4.50. *Let $s \in \mathbb{R}^+$ with $s \notin \mathbb{N}$, $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$. Moreover, $\mathscr{B} \subseteq C^s S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ should be a bounded subset and $u \in \mathcal{S}(\mathbb{R}^n)$. For every $N \in \mathbb{N}_0$ with $2N \leq M$ there is a constant $C_{N,n}$, independent of $x \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that*

$$|a(x, D_x)u(x)| \leq C_{N,n}\langle x\rangle^{-2N} \qquad \text{for all } x \in \mathbb{R}^n \text{ and } a \in \mathscr{B}.$$

Note that $C_{N,n}$ is dependent on $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof: Let $N \in \mathbb{N}_0$ with $2N \leq M$ be arbitrary. Since $u \in \mathcal{S}(\mathbb{R}^n)$, we also have $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$. Due to Remark 2.21 we have $D_\xi^{\alpha_2}\hat{u} \in \mathcal{S}(\mathbb{R}^n)$, too. Choosing an arbitrary $M_{m,n} \in \mathbb{N}$ with $-M_{m,n} < -n - |m|$, we get for all $\alpha_2 \in \mathbb{N}_0^n$ with $|\alpha_2| \leq 2N$ and for all $\xi \in \mathbb{R}^n$:

$$|D_\xi^{\alpha_2}\hat{u}(\xi)| \leq C_{\alpha_2,n}\langle\xi\rangle^{-M_{m,n}},$$

where $C_{\alpha_2,n}$ is independent of $\xi \in \mathbb{R}^n$. Using the last inequality and the boundedness of $\mathscr{B} \subseteq C^s S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; M)$ first and Theorem 2.11 afterwards, we have for all $a \in \mathscr{B}$ and all $x \in \mathbb{R}^n$:

$$\left|\langle D_\xi\rangle^{2N}\left[a(x,\xi)\hat{u}(\xi)\right]\right| \leq \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\leq N}} \sum_{\alpha_1+\alpha_2=2\alpha} C_{\alpha,\alpha_1}\left|D_\xi^{\alpha_1}a(x,\xi)\right|\left|D_\xi^{\alpha_2}\hat{u}(\xi)\right|$$

$$\leq C_{N,n}\langle\xi\rangle^{m-\rho|\alpha_1|-M_{m,n}} \in L^1(\mathbb{R}^n_\xi). \tag{4.26}$$

Here $C_{N,n}$ is independent of $x, \xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$. Additionally integration by parts with respect to ξ yields for all $a \in \mathscr{B}$ and all $x \in \mathbb{R}^n$:

$$\int \left(\langle D_\xi\rangle^{2N} e^{ix\cdot\xi}\right) a(x,\xi)\hat{u}\, đ\xi = \int e^{ix\cdot\xi}\langle D_\xi\rangle^{2N}\left[a(x,\xi)\hat{u}\right] đ\xi. \tag{4.27}$$

On account of (4.27), (4.26) and $\langle x\rangle^{2N}e^{ix\cdot\xi} = \langle D_\xi\rangle^{2N}e^{ix\cdot\xi}$ we conclude the claim:

$$\begin{aligned}\left|\langle x\rangle^{2N}a(x,D_x)u(x)\right| &= \left|\int \langle x\rangle^{2N}e^{ix\cdot\xi}a(x,\xi)\hat{u}(\xi)\,đ\xi\right| \\ &= \left|\int e^{ix\cdot\xi}\langle D_\xi\rangle^{2N}\left[a(x,\xi)\hat{u}(\xi)\right] đ\xi\right| \leq C_{N,n}\int\langle\xi\rangle^{-n-1}đ\xi \leq C_{N,n}\end{aligned}$$

for all $a \in \mathscr{B}$ and $x \in \mathbb{R}^n$, where $C_{N,n}$ is independent of $x, \xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$. The last inequality holds due to Theorem 2.11. □

Now we discuss another boundedness result we often need later on:

Theorem 4.51. *Let $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ with $\rho > 0$ and $1 < p < \infty$. Additionally let $\tau > \frac{1-\rho}{1-\delta}\cdot\frac{n}{2}$ if $\rho < 1$ and $\tau > 0$ if $\rho = 1$ respectively. Moreover, let $N \in \mathbb{N}\cup\{\infty\}$ with $N > n/2$ for $2 \leq p < \infty$ and $N > n/p$ else. Denoting $k_p := (1-\rho)n\,|1/2 - 1/p|$, let $\mathscr{B} \subseteq C^\tau_* S^{m-k_p}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n; N)$ be a bounded subset. Then for each real number s with the property*

$$(1-\rho)\frac{n}{p} - (1-\delta)\tau < s < \tau$$

there is a constant $C_s > 0$, independent of $a \in \mathscr{B}$, such that

$$\|a(x,D_x)f\|_{H^s_p} \leq C_s\|f\|_{H^{s+m}_p} \qquad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \text{ and } a \in \mathscr{B}.$$

Proof: In the case $2 \leq p < \infty$ the theorem was shown in [53], Theorem 2.7 for $\sharp\mathscr{B} = 1$. The case $1 < p < 2$ has been proved in [53], Theorem 4.2 for $\sharp\mathscr{B} = 1$. Thus it remains to verify whether the constant C_s is independent of $a \in \mathscr{B}$. We define p' by $1/p + 1/p' = 1$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s+m}_p(\mathbb{R}^n)$ and $H^{-s}_{p'}(\mathbb{R}^n)$ due to Lemma 2.43, the theorem holds because of Lemma 4.49. □

In the case $\sharp\mathscr{B} = 1$, the previous theorem also holds for $p = 1$ or $p = \infty$, cf. [53], Theorem 2.7 and Theorem 4.2. In the case $\rho = 1$, $\delta = 0$ and $p = 2$, H. Abels extended this result for operator-valued non-smooth symbols in [34], Theorem 3.7.

For $\rho = \delta = 0$ there is a similar boundedness result as in the previous theorem:

Theorem 4.52. *Let $m \in \mathbb{R}$ and $\tau > \frac{n}{2}$. Moreover, let $N \in \mathbb{N} \cup \{\infty\}$ with $N > n/2$. Additionally let $a \in C^\tau_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$. Then for each real number s with the property*

$$\frac{n}{2} - \tau < s < \tau$$

there is a constant $C_s > 0$ such that

$$\|a(x, D_x)f\|_{H^s_2} \leq C_s \|f\|_{H^{s+m}_2} \qquad \text{for all } f \in H^{s+m}_2(\mathbb{R}^n).$$

Proof: The theorem was checked in [53], Theorem 2.1. □

Theorem 4.53. *Let $m \in \mathbb{R}$, $N > n/2$, $\tau > 0$ be arbitrary. Moreover let P be an element of $OPC^\tau_* S^{m-n/2}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$. Then the operator*

$$P : H^{s+m}_2(\mathbb{R}^n) \to H^s_2(\mathbb{R}^n) \qquad \text{is continuous for all } -\tau < s < \tau.$$

Proof: An application of [53], Lemma 2.9 provides the claim. □

4.4.2 Boundedness of Pseudodifferential Operators with Coefficients in $H^{\tilde{m}}_q$

In this subsection we focus on the boundedness of non-smooth pseudodifferential operators with coefficients in $H^{\tilde{m}}_q(\mathbb{R}^n)$ as maps between two Bessel potential spaces.

Theorem 4.54. *Let $1 < p, q < \infty$ and $m, \tilde{m} \in \mathbb{R}$ with $\tilde{m} > n/q$. Moreover, let $\mathscr{B} \subseteq H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be a bounded subset. Then for each real number s with the property*

$$n(1/p + 1/q - 1)^+ - \tilde{m} < s \leq \tilde{m} - n(1/q - 1/p)^+$$

there is a constant C_s, independent of $a \in \mathscr{B}$, such that

$$\|a(x, D_x)f\|_{H^s_p} \leq C_s \|f\|_{H^{s+m}_p} \qquad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \text{ and all } a \in \mathscr{B}.$$

Proof: Let s be as in the assumptions. J. Marschall proved in [54], Theorem 2.2 the boundedness of a pseudodifferential operator with symbol in $H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ as a map from $H^{s+m}_p(\mathbb{R}^n)$ to $H^s_p(\mathbb{R}^n)$. Thus it remains to verify whether the constant C_s is independent of $a \in \mathscr{B}$. We define p' by $1/p + 1/p' = 1$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s+m}_p(\mathbb{R}^n)$ and $H^{-s}_{p'}(\mathbb{R}^n)$ due to Lemma 2.43, the theorem holds because of Lemma 4.49. □

Note that the last theorem even holds for $0 < p \leq \infty$ and $q \in \{1, \infty\}$ if $\sharp\mathscr{B} = 1$ due to J. Marschall, cf. [54], Theorem 2.2. J. Marschall verified a similar result for pseudodifferential operators of the class $H^{\tilde{m}}_q S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \leq \delta \leq 1$, if $\sharp\mathscr{B} = 1$ in [54], Theorem 2.2.

As a consequence of the previous theorem we obtain an estimate for the product of two Bessel potential functions:

Lemma 4.55. *Let $1 < p, q < \infty$ and $m \in \mathbb{R}$ with $m > n/q$. Assuming $n\,(1/p + 1/q - 1)^+ - m < s \leq m - n\,(1/q - 1/p)^+$, we get the existence of a constant C_s, independent of $a \in H^m_q(\mathbb{R}^n)$ and $b \in H^s_p(\mathbb{R}^n)$, such that*

$$\|ab\|_{H^s_p} \leq C_s \|a\|_{H^m_q} \|b\|_{H^s_p} \qquad \text{for all } a \in H^m_q(\mathbb{R}^n), b \in H^s_p(\mathbb{R}^n). \tag{4.28}$$

Proof: Considering a symbol $a(x) \in H^m_q(\mathbb{R}^n_x)$, we know that $\|\partial^\alpha_\xi a\|_{H^m_q} \leq \|a\|_{H^m_q}$ for all $\alpha \in \mathbb{N}^n_0$ because a is independent of $\xi \in \mathbb{R}^n$. Therefore

$$\mathscr{B} := \left\{a \in H^m_q(\mathbb{R}^n) : \|a\|_{H^m_q} = 1\right\} \subseteq H^m_q S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

is a bounded subset. Let m and s be as in the assumptions. On account of Theorem 4.54 we obtain the existence of a constant C_s, independent of $a \in \mathscr{B}$, such that

$$\|ab\|_{H^s_p} = \|a(x, D_x)b\|_{H^s_p} \leq C_s \|b\|_{H^s_p} \qquad \text{for all } a \in H^m_q(\mathbb{R}^n), b \in H^s_p(\mathbb{R}^n).$$

Since $\tilde{a} := a \cdot (\|a\|_{H^m_q})^{-1} \in \mathscr{B}$ for each $a \in H^m_q(\mathbb{R}^n)$ with $\|a\|_{H^m_q} \neq 0$, the last inequality provides the claim:

$$\|ab\|_{H^s_p} = \|a\|_{H^m_q} \|\tilde{a}b\|_{H^s_p} \leq C_s \|a\|_{H^m_q} \|b\|_{H^s_p} \quad \text{for all } a \in H^m_q(\mathbb{R}^n), b \in H^s_p(\mathbb{R}^n).$$

□

4.4.3 Boundedness of Pseudodifferential Operators with Coefficients in $W^{\tilde{m},q}_{uloc}$

Here we investigate the boundedness of non-smooth pseudodifferential operators with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ as maps between two Bessel potential spaces. We even get a better result for classical symbols.

Theorem 4.56. *Let $m \in \mathbb{R}$, $1 < p, q < \infty$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$. We suppose that $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a bounded subset. Then for each real number s with the property*

$$-\tilde{m} + n/q < s \leq \tilde{m} - n(1/q - 1/p)^+ \tag{4.29}$$

there is a constant $C_s > 0$, independent of $a \in \mathscr{B}$, such that

$$\|a(x, D_x)f\|_{H^s_p} \leq C_s \|f\|_{H^{s+m}_p} \qquad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \text{ and } a \in \mathscr{B}.$$

Proof: J. Marschall proved in [54], Theorem 2.6 the boundedness of $a(x, D_x)$ as a map from $H_p^{s+m}(\mathbb{R}^n)$ to $H_p^s(\mathbb{R}^n)$ for every $a \in \mathscr{B}$. Therefore it remains to show that C_s is independent of $a \in \mathscr{B}$. We define p' by $1/p + 1/p' = 1$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{s+m}(\mathbb{R}^n)$ and $H_{p'}^{-s}(\mathbb{R}^n)$ due to Lemma 2.43, the theorem holds true because of Lemma 4.49. □

We remark that the previous theorem even holds for $0 < p < \infty$ and $q = 1$ if $\sharp\mathscr{B} = 1$ and if $s \in \mathbb{R}$ fulfills

$$n(\max\{1, 1/p\} - 1) - \tilde{m} + n/q < s \leq \tilde{m} - n(1/q - 1/p)^+$$

instead of the assumption (4.29) due to J. Marschall [54], Theorem 2.6.

Our next goal is to improve the previous statement for classical pseudodifferential operators of the symbol-class $W^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. For reaching this goal we have to develop some further tools. We begin with the definition of a family of Banach spaces which is called scale:

Definition 4.57. Let Σ be of the form $[\sigma, \infty)$ or (σ, ∞) with $\sigma \in \mathbb{R} \cup \{-\infty\}$. Then a family of Banach spaces $\{X^s : s \in \Sigma\}$ is called *scale*, provided the following properties hold:

i) $C_c^\infty(\mathbb{R}^n) \subseteq X_s \subseteq \mathcal{S}'(\mathbb{R}^n)$ for all $s \in \Sigma$,

ii) $X^s \subseteq X^t$ for all $s, t \in \Sigma$ with $t < s$,

iii) $s + m \in \Sigma$ for all $s \in \Sigma$ and $m \in \mathbb{N}$,

iv) $P : X^{s+m} \to X^s$ for all differential operators P of the order $m \in \mathbb{N}_0$ with smooth coefficients and all $s \in \Sigma$.

With the previous definition at hand, we are in the position to define the term microlocalizable set of Banach spaces:

Definition 4.58. A scale $\{X^s : s \in \Sigma\}$ is called *microlocalizable* if for every symbol $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ the operator

$$p(x, D_x) : X^{s+m} \to X^s$$

is bounded for all $s \in \Sigma$ with $s + m \in \Sigma$.

We already know a set of microlocalizable Banach spaces:

Example 4.59. Let $1 < p < \infty$. Then the set $\{H_p^s(\mathbb{R}^n) : s \in \mathbb{R}\}$ and the set $\{C_*^s(\mathbb{R}^n) : 0 < s < \infty\}$ are microlocalizable.

Proof: The claim follows directly from Theorem 3.18 and Theorem 3.21. □

Non-smooth pseudodifferential operators with classical symbols are bounded maps between some elements of a microlocalizable set of Banach spaces. This is the topic of the next proposition:

Proposition 4.60. *Let* $Y \in \{C^{\tilde{m},\tau}, C_*^{\tilde{m}+\tau}, H_q^{\tilde{m}}, W_{uloc}^{\tilde{m},q}\}$ *with* $\tilde{m} \in \mathbb{N}_0$, $0 < \tau \leq 1$ *and* $1 < q < \infty$. *If* $\{X^s : s \in \Sigma\}$ *is microlocalizable and* $p \in YS_{cl}^m$, *then*

$$p(x, D_x) : X^{s+m} \to X^s$$

is bounded for $s \in \Sigma$, *provided that* $s + m \in \Sigma$ *and that there is some* $C > 0$ *such that*

$$\|ab\|_{X^s} \leq C\|a\|_{X^s}\|b\|_Y \qquad \text{for all } a \in X^s \text{ and } b \in Y.$$

We refer to [67], Proposition 1.1B for the proof. In this reference the last proposition was verified for a more general setting: The previous statement holds for an arbitrary Banach space Y with $C_c^\infty(\mathbb{R}^n) \subseteq Y \subseteq C^0(\mathbb{R}^n)$.

The last missing piece towards the improvement of Theorem 4.56 for classical non-smooth pseudodifferential operators is treated in the next theorem.

Theorem 4.61. *Let* $m \in \mathbb{R}$, $1 < p, q < \infty$ *and* $\tilde{m} \in \mathbb{N}$ *with* $\tilde{m} > n/q$. *Assuming* $p \in W_{uloc}^{\tilde{m},q} S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$, *we get for each real number* s *with the property*

$$n\,(1/p + 1/q - 1)^+ - \tilde{m} < s \leq \tilde{m} - n\,(1/q - 1/p)^+ \tag{4.30}$$

the existence of a constant $C_s > 0$ *such that*

$$\|p(x, D_x)f\|_{H_p^s} \leq C_s\|f\|_{H_p^{s+m}} \qquad \text{for all } f \in H_p^{s+m}(\mathbb{R}^n).$$

Proof: On account of Example 4.59 we know that $\{H_p^s(\mathbb{R}^n) : s \in \mathbb{R}\}$ is microlocalizable for each $1 < p < \infty$. Hence the theorem follows directly from Proposition 4.60 if the following inequality holds for every $1 < p < \infty$ and each $s \in \mathbb{R}$ fulfilling (4.30):

$$\|fg\|_{H_p^s} \leq C_s\|f\|_{H_p^s}\|g\|_{W_{uloc}^{\tilde{m},q}} \qquad \text{for all } f \in H_p^s(\mathbb{R}^n), g \in W_{uloc}^{\tilde{m},q}(\mathbb{R}^n). \tag{4.31}$$

Thus it remains to prove inequality (4.31). To this let $1 < p < \infty$ and $s \in \mathbb{R}$ with $n\,(1/p + 1/q - 1)^+ - \tilde{m} < s \leq \tilde{m} - n\,(1/q - 1/p)^+$ be arbitrary. In view of Proposition 3.20 we may choose a partition of unity $(\psi_j)_{j\in\mathbb{Z}^n} \subseteq C_c^\infty(\mathbb{R}^n)$ with the properties $\operatorname{supp} \psi_0 \subseteq [-\varepsilon, \varepsilon]^n$ for one fixed $\varepsilon > 0$ and $\psi_j(x) = \psi_0(x - j)$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}^n$. With $Z_j := \{k \in \mathbb{Z}^n : \operatorname{supp} \psi_k \cap \operatorname{supp} \psi_j \neq \varnothing\}$, we set $\eta_j : \mathbb{R}^n \to \mathbb{R}$ as

$$\eta_j(x) := \sum_{k \in Z_j} \psi_k(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and every } j \in \mathbb{Z}^n.$$

In a similar way as in the proof of Remark 4.12 we first want to verify the existence of a constant C, independent of $j \in \mathbb{Z}^n$ and $g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$, such that

$$\|\eta_j g\|_{H^{\tilde{m}}_q(\mathbb{R}^n)} \le C\|g\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \qquad \text{for all } g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n) \text{ and all } j \in \mathbb{Z}^n. \tag{4.32}$$

Choosing a finite cover $(B_1(x_i))_{i=1}^N$, $N \in \mathbb{N}$, of the compact set supp η_0 with open balls of radius 1 provides a finite cover $(B_1(x_i + j))_{i=1}^N$ of supp η_j with open balls of radius 1. Hence N is independent of $j \in \mathbb{Z}^n$. We obtain for each $g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ and all $j \in \mathbb{Z}^n$ by means of the Leibniz rule:

$$\begin{aligned}\|\eta_j g\|^q_{H^{\tilde{m}}_q(\mathbb{R}^n)} &\le \sum_{|\alpha|\le\tilde{m}} \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} \int_{\text{supp } \eta_j} |\partial_x^{\alpha_1}\eta_j(x)|^q |\partial_x^{\alpha_2} g(x)|^q dx \\ &\le \sum_{|\alpha|\le\tilde{m}} \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} \sum_{i=1}^N \int_{B_1(x_i+j)} |\partial_x^{\alpha_2} g(x)|^q dx \\ &\le \sum_{|\alpha|\le\tilde{m}} C_\alpha \|\partial_x^\alpha g(x)\|^q_{L^q_{uloc}(\mathbb{R}^n_x)} \le C_{\tilde{m}} \|g\|^q_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)}.\end{aligned}$$

Since $C_{\tilde{m}}$ is independent of $j \in \mathbb{Z}^n$, estimate (4.32) holds. Together with Proposition 3.20 and Lemma 4.55 we get for all $f \in H^s_p(\mathbb{R}^n)$ and $g \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$:

$$\begin{aligned}\|fg\|^p_{H^s_p} &\le C_s \sum_{j\in\mathbb{Z}^n} \|\psi_j(fg)\|^p_{H^s_p} = C_s \sum_{j\in\mathbb{Z}^n} \|(\psi_j f)(\eta_j g)\|^p_{H^s_p} \\ &\le C_{s,\tilde{m}} \sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p} \|\eta_j g\|^p_{H^{\tilde{m}}_q} \le C_{s,\tilde{m}} \|g\|^p_{W^{\tilde{m},q}_{uloc}} \sum_{j\in\mathbb{Z}^n} \|\psi_j f\|^p_{H^s_p} \\ &\le C_{s,\tilde{m}} \|g\|^p_{W^{\tilde{m},q}_{uloc}} \|f\|^p_{H^s_p}.\end{aligned}$$

Hence inequality (4.31) is true. This completes the proof of the theorem. □

Making use of the previous theorem and of Lemma 4.49 enables us to improve Theorem 4.56 for classical non-smooth pseudodifferential operators:

Theorem 4.62. *Let $m \in \mathbb{R}$, $1 < p, q < \infty$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$. Assuming a bounded subset $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ we get for each real number s with the property*

$$n\,(1/p + 1/q - 1)^+ - \tilde{m} < s \le \tilde{m} - n\,(1/q - 1/p)^+$$

the existence of a constant $C_s > 0$, independent of $a \in \mathscr{B}$, such that

$$\|a(x, D_x)f\|_{H^s_p} \le C_s \|f\|_{H^{s+m}_p} \qquad \text{for all } f \in H^{s+m}_p(\mathbb{R}^n) \text{ and } a \in \mathscr{B}.$$

Proof: Let s be as in the assumptions. Theorem 4.61 yields the boundedness of a pseudodifferential operator in the non smooth symbol-class $W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ as a map from $H^{s+m}_p(\mathbb{R}^n)$ to $H^s_p(\mathbb{R}^n)$. Thus it remains to verify whether the constant C_s is independent of $a \in \mathscr{B}$. We define p' by $1/p + 1/p' = 1$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s+m}_p(\mathbb{R}^n)$ and in $H^{-s}_{p'}(\mathbb{R}^n)$ due to Lemma 2.43, the theorem holds because of Lemma 4.49. □

So far we have presented some boundedness results for non-smooth pseudodifferential operators in this section. They will play a key role for proving the characterization of non-smooth pseudodifferential operators and for verifying the spectral invariance of non-smooth pseudodifferential operators in Chapter 5 and Chapter 6.

4.5 Kernel Representation

In this section we focus on the kernel representation of a non-smooth pseudodifferential operator $p(x, D_x)$, whose symbol is in the class $XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for certain Banach spaces X. In this thesis we just treated the symbol-classes $X \in \{C^{\tilde{m},\tau}, C^{\tilde{m}+\tau}_*, H^{\tilde{m}}_q, W^{\tilde{m},q}_{uloc}\}$ with $\tilde{m} \in \mathbb{N}_0$, $0 < \tau \leq 1$ and $1 < q < \infty$. However, a kernel representation can be verified in a more general setting. We only have to assume, that $C^\infty_c(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$ holds for the Banach space X. In the same manner as in the smooth case we are able to estimate the kernel of $p(x, D_x)$ except for the diagonal $\{(x, x) : x \in \mathbb{R}^n\}$:

Theorem 4.63. *Let $p \in XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ where $C^\infty_c(\mathbb{R}^n) \subseteq X \subseteq C^0(\mathbb{R}^n)$ is a Banach space and $m \in \mathbb{R}$. Then there is a function $k : \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{C}$ such that $k(x, .) \in C^\infty(\mathbb{R}^n \backslash \{0\})$ for all $x \in \mathbb{R}^n$ and*

$$p(x, D_x)u(x) = \int k(x, x - y)u(y)dy \qquad \text{for all } x \notin \text{supp } u$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{N}^n_0$ and each $N \in \mathbb{N}_0$ the kernel k satisfies

$$\|\partial^\alpha_z k(., z)\|_X \leq \begin{cases} C_{\alpha,N}|z|^{-n-m-|\alpha|}\langle z\rangle^{-N} & \text{if } n + m + |\alpha| > 0, \\ C_{\alpha,N}(1 + |\log |z||)\langle z\rangle^{-N} & \text{if } n + m + |\alpha| = 0, \\ C_{\alpha,N}\langle z\rangle^{-N} & \text{if } n + m + |\alpha| < 0 \end{cases}$$

uniformly in $z \in \mathbb{R}^n \backslash \{0\}$.

Proof: This theorem can be checked in a similar way as the kernel representation in the smooth case, cf. Theorem 3.14. The main idea of the proof is to decompose the pseudodifferential operator

$$p(x, D_x)f = \sum_{j=0}^\infty p(x, D_x)\varphi_j(D_x)f \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^n)$$

where $(\varphi_j)_{j\in\mathbb{N}_0}$ is a dyadic partition of unity. The series converges in X, since $p(x,D_x) : \mathcal{S}(\mathbb{R}^n) \to X$ is continuous due to Lemma 4.48. First of all we want to construct a kernel k_j of $p_j(x,D_x) := p(x,D_x)\varphi_j(D_x)$ for each $j \in \mathbb{N}_0$. This can be made in the same way as in the smooth case. We just have to use $\|\partial_\xi^\alpha p(.,\xi)\|_X \leq C_\alpha \langle\xi\rangle^{m-|\alpha|}$ instead of $|\partial_\xi^\alpha p(.,\xi)| \leq C_\alpha\langle\xi\rangle^{m-|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$ and all $\xi \in \mathbb{R}^n$. Afterwards we use this kernel decompositions in order to construct the kernel of $p(x,D_x)$ as in the smooth case. By means of $X \subseteq C^0(\mathbb{R}^n)$ we get the absolute and uniform convergence of $k(x,z) = \sum_{j=0}^{\infty} k_j(x,z)$. □

This kernel representation of a non-smooth pseudodifferential operator will enable us to construct an example for P^{-1} being not a pseueoddifferential operator, whose symbol is smooth in the second variable, if this is the case for the pseudodifferential operator P in Chapter 6.

4.6 Non-Smooth Double Symbols

Smooth double symbols were introduced in order to prove the following statement: the product and the formal adjoint of a smooth pseudodifferential operator are also smooth pseudodifferential operators, cf. e.g. [5], Section 3.4 or [42], Section 2.2. For this purpose an important technique was established: The symbol reduction of a smooth double symbol to a smooth single symbol. Since the topic of Section 5.2 will be the development of such a symbol reduction for non-smooth pseudodifferential operators, we have to define double symbols also in the non-smooth case. This is done in the present section.

Definition 4.64. Let $0 < s \leq 1$, $m \in \mathbb{N}_0$ and $\tilde{m}, m' \in \mathbb{R}$. Furthermore, let $N \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Then the space of *non-smooth double (pseudodifferential) symbols* $C^{m,s}S^{\tilde{m},m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is the set of all functions $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'} \to \mathbb{C}$ such that

i) $\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p \in C^s(\mathbb{R}^n_x)$ and $\partial_x^\beta \partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'})$,

ii) $\|\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p(.,\xi,x',\xi')\|_{C^s(\mathbb{R}^n)} \leq C_{\alpha,\beta',\alpha'} \langle\xi\rangle^{\tilde{m}-\rho|\alpha|} \langle\xi'\rangle^{m'-\rho|\alpha'|}$

for all $\xi, x', \xi' \in \mathbb{R}^n$ and arbitrary $\beta, \alpha, \beta', \alpha' \in \mathbb{N}_0^n$ with $|\beta| \leq m$ and $|\alpha| \leq N$. Here the constant $C_{\alpha,\beta',\alpha'}$ is independent of $\xi, x', \xi' \in \mathbb{R}^n$. In the case $N = \infty$ we write $C^{m,s}S^{\tilde{m},m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ instead of $C^{m,s}S^{\tilde{m},m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \infty)$. Furthermore, we define the set of semi-norms $\{|.|_k^{\tilde{m},m'} : k \in \mathbb{N}_0\}$ by

$$|p|_k^{\tilde{m},m'} := \max_{\substack{|\alpha|+|\beta'|+|\alpha'| \leq k \\ |\alpha| \leq N}} \sup_{\xi,x',\xi' \in \mathbb{R}^n} \|\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p(.,\xi,x',\xi')\|_{C^{m,s}(\mathbb{R}^n)} \langle\xi\rangle^{-(\tilde{m}-\rho|\alpha|)} \langle\xi'\rangle^{-(m'-\rho|\alpha'|)}.$$

Due to the previous definition a non-smooth symbol $p \in C^{m,s}S^{\tilde{m}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^N)$ is often called a *non-smooth single symbol.*

Next we would like to define an associated operator for every non-smooth double symbol. Since we want to use the oscillatory integral for the definition, we first have to verify the existence of the oscillatory integral. This is the topic of the next remark:

Remark 4.65. Let $N \in \mathbb{N}_0 \cup \{\infty\}$, $0 \leq \delta \leq 1$, $0 < s \leq 1$, $\tilde{m}, m' \in \mathbb{R}$ and $m \in \mathbb{N}_0$. We define $\hat{m} := \tilde{m}_+ + m'_+$. Additionally let $\mathscr{B} \subseteq C^\infty_b(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ be a bounded set. Considering a double symbol $p \in C^{m,s}S^{\tilde{m},m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ and a function $u \in \mathcal{S}(\mathbb{R}^n)$ we have for every fixed $x \in \mathbb{R}^n$ and every $l \in \mathbb{N}_0$ the boundedness of the set

$$\left\{ \langle y' \rangle^{-2l} \langle D_{\xi'} \rangle^{2l} \left[\chi(y, y', \xi, \xi') p(x, \xi, x + y, \xi') u(x + y + y') \right] : \chi \in \mathscr{B} \right\}$$

in $\mathscr{A}^{\hat{m},N}_{-l}(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\xi')})$.

Proof: Let $x \in \mathbb{R}^n$ and $l \in \mathbb{N}_0$ be arbitrary but fixed. On account of Remark 2.8 we have for all $\alpha' \in \mathbb{N}^n_0$:

$$|\partial^{\alpha'}_{y'} \langle y' \rangle^{-2l}| \leq C \langle y' \rangle^{-2l-|\alpha'|} \leq C \langle y' \rangle^{-2l} \qquad \text{for all } y' \in \mathbb{R}^n. \tag{4.33}$$

Additionally Peetre's inequality, cf. Lemma 2.4, provides for all $y, y' \in \mathbb{R}^n$:

$$\langle y' \rangle^{-2l} \langle x + y + y' \rangle^{-l} \leq C_x \langle y' \rangle^{-l} \langle y \rangle^{-l} \leq C_x \langle (y, y') \rangle^{-l}. \tag{4.34}$$

Moreover, we obtain for all $\xi, \xi' \in \mathbb{R}^n$:

$$\langle \xi \rangle^{\tilde{m}} \langle \xi' \rangle^{m'} \leq \langle \xi \rangle^{\tilde{m}_+} \langle \xi' \rangle^{m'_+} \leq \langle (\xi, \xi') \rangle^{\tilde{m}_+ + m'_+} = \langle (\xi, \xi') \rangle^{\hat{m}}. \tag{4.35}$$

Let $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n_0$ with $|\alpha| \leq N$ be arbitrary. By means of the Leibniz rule, Remark 2.10, inequality (4.33) and the assumptions we can verify the following estimate:

$$\begin{aligned} &|\partial^\alpha_\xi \partial^{\alpha'}_{\xi'} \partial^\beta_y \partial^{\beta'}_{y'} \{ \langle y' \rangle^{-2l} \langle D_{\xi'} \rangle^{2l} \chi(y, y', \xi, \xi') p(x, \xi, x + y, \xi') u(x + y + y') \}| \\ &\quad \leq C_{\alpha,\alpha',\beta,\beta'} \langle y' \rangle^{-2l} \langle \xi \rangle^{\tilde{m}} \langle \xi' \rangle^{m'} \langle x + y + y' \rangle^{-l} \\ &\quad \leq C_{x,\alpha,\alpha',\beta,\beta'} \langle (\xi, \xi') \rangle^{\hat{m}} \langle (y, y') \rangle^{-l} \qquad \text{for all } \xi, y, \xi', y' \in \mathbb{R}^n, \chi \in \mathscr{B}. \end{aligned}$$

Here the last inequality holds due to the estimates (4.34) and (4.35).

Since p is an element of $C^{m,s}S^{\tilde{m},m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we obtain that $\partial^\alpha_\xi \partial^{\alpha'}_{\xi'} \partial^\beta_y \partial^{\beta'}_{y'} \{ p(x, \xi, x + y, \xi') u(x + y + y') \}$ is an element of $C^0(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\xi')})$ for all $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n_0$ with $|\alpha| \leq N$. Hence the claim holds due to Definition 4.36. □

With the previous remark at hand, we can show the next lemma:

Lemma 4.66. *Let $N \in \mathbb{N}_0 \cup \{\infty\}$, $0 \leq \delta \leq 1$. For every $u \in \mathcal{S}(\mathbb{R}^n)$ and every double symbol $p \in C^{m,s}S^{\tilde{m},m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ with $0 < s \leq 1$, $\tilde{m}, m' \in \mathbb{R}$ and $m \in \mathbb{N}_0$ the oscillatory integral*

$$Os\text{-}\iiiint e^{-i(y\cdot\xi+y'\cdot\xi')}p(x,\xi,x+y,\xi')u(x+y+y')dydy'đ\xi đ\xi'$$

exists for all $x \in \mathbb{R}^n$. Additionally we have for all $x \in \mathbb{R}^n$ and all $l \in \mathbb{N}_0$ with $l > 2n$:

$$\begin{aligned}Os\text{-}\iiiint & e^{-i(y\cdot\xi+y'\cdot\xi')}\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}[p(x,\xi,x+y,\xi')u(x+y+y')]dydy'đ\xi đ\xi' \\ &= Os\text{-}\iiiint e^{-i(y\cdot\xi+y'\cdot\xi')}p(x,\xi,x+y,\xi')u(x+y+y')dydy'đ\xi đ\xi'.\end{aligned}$$

Proof: Let $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0,0,0) = 1$ be arbitrary. For all $\varepsilon \in (0,1]$ we define $\chi_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$\chi_\varepsilon(y,y',\xi,\xi') := \chi(\varepsilon y, \varepsilon y', \varepsilon\xi, \varepsilon\xi') \qquad \text{for all } y, y'\xi, \xi' \in \mathbb{R}^n.$$

Choosing $l \in \mathbb{N}_0$ with $l > 2n$ we define the functions $p_\varepsilon, \hat{p} : \mathbb{R}^{5n} \to \mathbb{C}$ by

$$\begin{aligned}p_\varepsilon(x,y,y',\xi,\xi') &:= \langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}\{\chi(\varepsilon y, \varepsilon y', \varepsilon\xi, \varepsilon\xi')p(x,\xi,x+y,\xi')u(x+y+y')\}, \\ \hat{p}(x,y,y',\xi,\xi') &:= \langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}\{p(x,\xi,x+y,\xi')u(x+y+y')\}\end{aligned}$$

for all $x, y, y', \xi, \xi' \in \mathbb{R}^n$. Now let $x \in \mathbb{R}^n$ be arbitrary. Since χ_ε is an element of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for each fixed $\varepsilon \in (0,1]$, all assumptions of Fubini's theorem hold and integration by parts is possible. Consequently we get by means of Fubini's theorem and integration by parts for each fixed $\varepsilon \in (0,1]$:

$$\begin{aligned}\iiiint & e^{-i(y\cdot\xi+y'\cdot\xi')}\chi(\varepsilon y, \varepsilon y', \varepsilon\xi, \varepsilon\xi')p(x,\xi,x+y,\xi')u(x+y+y')dydy'đ\xi đ\xi' \\ &= \iiiint e^{-i(y\cdot\xi+y'\cdot\xi')}p_\varepsilon(x,y,y',\xi,\xi')dydy'đ\xi đ\xi'. \qquad (4.36)\end{aligned}$$

On account of Remark 4.65 and of the boundedness of $\{\chi_\varepsilon : \varepsilon \in (0,1]\}$ as a subset of $C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ we have the boundedness of the set

$$\{p_\varepsilon(x,y,y',\xi,\xi') : \varepsilon \in (0,1]\} \subseteq \mathscr{A}^{\hat{m},N}_{-l}(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\xi')}). \qquad (4.37)$$

Moreover, Remark 4.65 also provides that

$$\hat{p}(x,y,y',\xi,\xi') \in \mathscr{A}^{\hat{m},N}_{-l}(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\xi')}). \qquad (4.38)$$

Using the Leibniz rule and Lemma 2.26 we obtain for all $\alpha, \beta, \gamma, \delta \in \mathbb{N}_0^n$ with $|\gamma| + |\delta| \leq N$ the pointwise convergence of

$$\partial_y^\alpha \partial_{y'}^\beta \partial_\xi^\gamma \partial_{\xi'}^\delta p_\varepsilon(x, y, y', \xi, \xi') \xrightarrow{\varepsilon \to 0} \partial_y^\alpha \partial_{y'}^\beta \partial_\xi^\gamma \hat{p}(x, \xi, x + y, \xi') \tag{4.39}$$

for all $y, y', \xi, \xi' \in \mathbb{R}^n$. Due to (4.37) and the choice of l, the assumptions of Theorem 4.38 hold. Consequently the oscillatory integral of p_ε is well-defined for all $\varepsilon \in (0, 1]$. By means of (4.37)- (4.39) we are able to apply Corollary 4.42 and get:

$$\begin{aligned}\lim_{\varepsilon \to 0} \text{Os -} \iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} p_\varepsilon(x, y, y', \xi, \xi') dy dy' đ\xi đ\xi \\ = \text{Os -} \iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} \hat{p}(x, y, y', \xi, \xi') dy dy' đ\xi đ\xi.\end{aligned} \tag{4.40}$$

A combination of (4.36) and (4.40) yields the next convergence:

$$\begin{aligned}&\iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} \chi(\varepsilon y, \varepsilon y', \varepsilon \xi, \varepsilon \xi') p(x, \xi, x + y, \xi') u(x + y + y') dy dy' đ\xi đ\xi' \\ &= \text{Os -} \iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} p_\varepsilon(x, y, y', \xi, \xi') dy dy' đ\xi đ\xi' \\ &\xrightarrow{\varepsilon \to 0} \text{Os -} \iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} \hat{p}(x, y, y', \xi, \xi') dy dy' đ\xi đ\xi.\end{aligned}$$

Consequently the oscillatory integral of $p(x, \xi, x+y, \xi') u(x+y+y')$ with respect to (y, y') and (ξ, ξ') exists and

$$\begin{aligned}\text{Os -} \iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} p(x, \xi, x + y, \xi') u(x + y + y') dy dy' đ\xi đ\xi' \\ = \text{Os -} \iiiint e^{-i(y \cdot \xi + y' \cdot \xi')} \hat{p}(x, y, y', \xi, \xi') dy dy' đ\xi đ\xi'.\end{aligned}$$

□

The previous lemma enable us to define for every non-smooth double symbol an operator in the following way:

Definition 4.67. Let $0 < s \leq 1$, $m \in \mathbb{N}_0$, $0 \leq \rho \leq 1$ and $\tilde{m}, m' \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$. Assuming $p \in C^{m,s} S_{\rho,0}^{\tilde{m},m'}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we define the pseudodifferential operator $P = p(x, D_x, x', D_{x'})$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$Pu(x) := \text{Os -} \iiiint e^{-i(y \cdot \xi + y' \xi')} p(x, \xi, x + y, \xi') u(x + y + y') dy dy' đ\xi đ\xi'.$$

The set of all non-smooth pseudodifferential operators whose double symbols are in the symbol-class $C^{m,s} S_{\rho,0}^{\tilde{m},m'}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is denoted by

$$\text{OP}C^{m,s} S_{\rho,0}^{\tilde{m},m'}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N).$$

For later purposes we will need a special subset of the non-smooth double symbols $C^{m,s}S^{\tilde{m},0}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$: For $0 < s \leq 1$, $m \in \mathbb{N}_0$, $N \in \mathbb{N}_0 \cup \{\infty\}$ and $\tilde{m} \in \mathbb{R}$ we denote the space $C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ as the set of all non-smooth symbols $p \in C^{m,s}S^{\tilde{m},0}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ with the following property:

$$p(x,\xi,x',\xi') = p(x,\xi,x') \qquad \text{for all } x,\xi,x',\xi' \in \mathbb{R}^n.$$

Then we define the pseudodifferential operator $p(x,D_x,x')$ by

$$p(x,D_x,x') := p(x,D_x,x',D_{x'}).$$

The set of all non-smooth pseudodifferential operators whose double symbols are in $C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is denoted by $\mathrm{OP}C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. As usual write $C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ instead of the set $C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \infty)$ and $\mathrm{OP}C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ instead of the set $\mathrm{OP}C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \infty)$.

Pseudodifferential operators of the symbol-class $C^{m,s}S^{\tilde{m}}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ applied on a Schwartz function can be presented in the following way:

Lemma 4.68. *Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$, $0 \leq \rho \leq 1$, $m \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. Considering $a \in C^{\tilde{m},s}S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we obtain for all $u \in \mathcal{S}(\mathbb{R}^n)$:*

$$a(x,D_x,x')u(x) = Os\text{-}\iint e^{i(x-y)\cdot\xi} a(x,\xi,y)u(y)\,dy\,đ\xi \qquad \textit{for all } x \in \mathbb{R}^n.$$

Proof: Let $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be arbitrary. Due to Remark 4.65 we know that for each $l \in \mathbb{N}$

$$\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l} a(x,\xi,x+y)u(x+y+y') \in \mathscr{A}^{m_+,N}_{-l}(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\xi')}). \tag{4.41}$$

Now we set $l = 2n+2$. Then (4.41) provides

$$\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l} a(x,\xi,z)u(z+y') \in \mathscr{A}^{m_+,N}_{-2n-2}(\mathbb{R}^{2n}_{(z,y')} \times \mathbb{R}^{2n}_{(\xi,\xi')}). \tag{4.42}$$

With Lemma 4.66, Theorem 4.43 and Theorem 4.40 at hand, we get

$$\begin{aligned}
&a(x,D_x,x')u(x)\\
&\quad= \mathrm{Os}\text{-}\iiiint e^{-i(y\cdot\xi+y'\cdot\xi')}\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}[a(x,\xi,x+y)u(x+y+y')]\,dy\,dy'\,đ\xi\,đ\xi'\\
&\quad= \mathrm{Os}\text{-}\iiiint e^{-i(z-x)\cdot\xi}e^{-iy'\cdot\xi'} a(x,\xi,z)\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}u(z+y')\,dz\,dy'\,đ\xi\,đ\xi'\\
&\quad= \mathrm{Os}\text{-}\iint e^{i(x-z)\cdot\xi}a(x,\xi,z)\left[\mathrm{Os}\text{-}\iint e^{-iy'\cdot\xi'}\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}u(z+y')\,dy'\,đ\xi'\right]dz\,đ\xi
\end{aligned}$$

Here an application of Theorem 4.40 is possible since all assumptions are fulfilled due to (4.42). By means of Proposition 2.22, Remark 2.8 and Lemma 2.17 we can show that $\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}u(z+y') \in \mathcal{S}(\mathbb{R}^n_{y'})$ for each $z \in \mathbb{R}^n$. Let $k \in \mathbb{N}$ with $k > n$. Due to Example 3.10 we get

$$\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}u(z+y') \in \mathcal{S}(\mathbb{R}^n_{y'}) \subseteq \mathscr{A}^0_{-k}(\mathbb{R}^n_{y'} \times \mathbb{R}^n_{\xi'}).$$

This enables us to apply Theorem 4.41 and Theorem 4.43 which provides

$$\begin{aligned}\text{Os-}\iint e^{-iy'\cdot\xi'}\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}u(z+y')dy'đ\xi' &= \text{Os-}\iint e^{-iy'\cdot\xi'}u(z+y')dy'đ\xi'\\ &= \text{Os-}\iint e^{-i(\tilde{z}-z)\cdot\xi'}u(\tilde{z})d\tilde{z}đ\xi'.\end{aligned}$$

Combining all these results we conclude the proof by an application of Remark 3.12:

$$\begin{aligned}&a(x,D_x,x')u(x)\\ &= \text{Os-}\iint e^{i(x-z)\cdot\xi}a(x,\xi,z)\left[\text{Os-}\iint e^{-iy'\cdot\xi'}\langle y'\rangle^{-2l}\langle D_{\xi'}\rangle^{2l}u(z+y')dy'đ\xi'\right]dzđ\xi\\ &= \text{Os-}\iint e^{i(x-z)\cdot\xi}a(x,\xi,z)\left[\text{Os-}\iint e^{-i(\tilde{z}-z)\cdot\xi'}u(\tilde{z})d\tilde{z}đ\xi'\right]dzđ\xi\\ &= \text{Os-}\iint e^{i(x-z)\cdot\xi}a(x,\xi,z)u(z)dzđ\xi.\end{aligned}$$

□

As in the case of single symbols the derivatives of double symbols are under certain conditions double symbols again:

Remark 4.69. Let $0 < s < 1$, $m \in \mathbb{R}$, $\tilde{m} \in \mathbb{N}_0$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. The boundedness of the subset $\mathscr{B} \subseteq C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, $0 \leq \rho \leq 1$, implies the boundedness of

$$\left\{\partial_x^\delta\partial_\xi^\gamma a : a \in \mathscr{B}\right\} \subseteq C^{\tilde{m}-|\delta|,s}S^{m-\rho|\gamma|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N-|\gamma|)$$

for each $\gamma, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$ and $|\gamma| \leq N$.

Proof: The claim is a direct consequence of the definition of the double symbols.

□

In order to improve the characterization of non-smooth pseudodifferential operators in Section 5.6, we will need a special subclass of the non-smooth double symbols with coefficients in the Hölder spaces:

Definition 4.70. Let $\tilde{m} \in \mathbb{N}_0$, $1 < q < \infty$ with $\tilde{m} > n/q$ and $m, m' \in \mathbb{R}$. Furthermore, let $N \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Then the space of *non-smooth double (pseudodifferential) symbols* $W^{\tilde{m},q}_{uloc}S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is the set of all functions $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'} \to \mathbb{C}$ such that

i) $\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p(.,\xi) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ and $\partial_x^\beta \partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi'})$,

ii) $\|\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p(.,\xi,x',\xi')\|_{W^{\tilde{m},q}_{uloc}} \leq C_{\alpha,\beta',\alpha'} \langle \xi \rangle^{m-\rho|\alpha|} \langle \xi' \rangle^{m'-\rho|\alpha'|}$

for all $x, \xi, x', \xi' \in \mathbb{R}^n$ and $\beta, \alpha, \beta', \alpha' \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $|\beta| < \tilde{m} - n/q$. Here the constant $C_{\alpha,\beta',\alpha'}$ is independent of $\xi, x', \xi' \in \mathbb{R}^n$. In the case $N = \infty$ we write $W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ instead of $W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \infty)$. We define the set of semi-norms $\{|.|_k^{m,m'} : k \in \mathbb{N}_0\}$ by

$$|p|_k^{m,m'} := \max_{\substack{|\alpha|+|\beta'|+|\alpha'| \leq k \\ |\alpha| \leq N}} \sup_{\xi, x', \xi' \in \mathbb{R}^n} \|\partial_\xi^\alpha \partial_{x'}^{\beta'} \partial_{\xi'}^{\alpha'} p(.,\xi,x',\xi')\|_{W^{\tilde{m},q}_{uloc}} \langle \xi \rangle^{-(m-\rho|\alpha|)} \langle \xi' \rangle^{-(m'-\rho|\alpha'|)}.$$

The non-smooth double symbol-class $W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is indeed a subclass of $C^\tau S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ for certain $\tau > 0$ as we see in the following remark. Consequently the associated non-smooth pseudodifferential operators are already defined. The set of all these non-smooth pseudodifferential operators is denoted by $\mathrm{OP}W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ and in the case $N = \infty$ as $\mathrm{OP}W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

Remark 4.71. Let $1 < q < \infty$, $m, m' \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $N \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Assuming $0 < \tau \leq \tilde{m} - n/q$, $\tau \notin \mathbb{N}$, we have

$$W^{\tilde{m},q}_{uloc} S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N) \subseteq C^\tau S^{m,m'}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N).$$

Proof: The claim is a consequence of the Lemma 4.15 and the definition of these symbol-classes. □

Later we mainly will work on a special subset of the non-smooth double symbol-class $W^{\tilde{m},q}_{uloc} S^{m,0}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$: Let $1 < q < \infty$, $m \in \mathbb{R}$, $0 \leq \rho \leq 1$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$. Then we define the non-smooth double symbol-class $W^{\tilde{m},q}_{uloc} S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ as the set of all non-smooth symbols p with

$$p \in W^{\tilde{m},q}_{uloc} S^{m,0}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N) \cap C^\tau S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$$

for some $\tau > 0$. Again we are able to show that derivatives of double symbols of the class $W^{\tilde{m},q}_{uloc} S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ are elements of this class, too:

Remark 4.72. Let $m \in \mathbb{R}$, $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Moreover, let $0 \leq \rho \leq 1$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. If $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^{m}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is a bounded subset, we get the boundedness of

$$\mathscr{B}' := \left\{ \partial_y^\gamma \partial_\xi^\delta a : a \in \mathscr{B} \right\} \subseteq W^{\tilde{m},q}_{uloc} S^{m-\rho|\delta|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N - |\delta|)$$

for each $\gamma, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq N$.

Proof: This remark follows directly by means of Schwarz's theorem and the boundedness of $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. □

With all the knowledge of the last sections we are able to show the next estimate for non-smooth double symbols with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$:

Lemma 4.73. *Let $m \in \mathbb{R}$, $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with the property $\tilde{m} > n/q$. Moreover, let $0 \leq \rho \leq 1$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. Assuming a bounded subset $\mathscr{B}$ of $W^{\tilde{m},q}_{uloc} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we can show for each $\gamma, \delta \in \mathbb{N}^n_0$ with $|\delta| \leq N$ the following inequality:*

$$\sup_{y \in \mathbb{R}^n} \|\partial^\gamma_y \partial^\delta_\xi a(x, \xi, x + y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C_{\tilde{m},q,\gamma,\delta} \langle \xi \rangle^{m-\rho|\delta|} \qquad \text{for all } a \in \mathscr{B}, \xi \in \mathbb{R}^n.$$

Here $C_{\tilde{m},q,\gamma,\delta}$ is independent of $a \in \mathscr{B}$ and $\xi \in \mathbb{R}^n$.

Proof: Let $\gamma, \delta \in \mathbb{N}^n_0$ with $|\delta| \leq N$ be arbitrary. Since $\mathscr{B}$ is a bounded subset of $W^{\tilde{m},q}_{uloc} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we get the boundedness of

$$\mathscr{B}' := \left\{ \partial^\gamma_y \partial^\delta_\xi a : a \in \mathscr{B} \right\} \subseteq W^{\tilde{m},q}_{uloc} S^{m-\rho|\delta|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N - |\delta|)$$

due to Remark 4.72. Therefore an application of Lemma 4.25 provides the claim. □

Altogether, this chapter was devoted to the study of non-smooth pseudodifferential operators. They serve as ingredients for the characterization of non-smooth pseudodifferential operators in the next chapter.

Chapter 5

Characterization of Non-Smooth Pseudodifferential Operators with Coefficients in Hölder Spaces

In the smooth case some characterizations of pseudodifferential operators are already proved: In 1977 R. Beals [16] proved a characterization of smooth pseudodifferential operators, for example of the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Eleven years later J. Ueberberg [74] generalized this characterization for pseudodifferential operators of the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. In the literature there are some other characterizations in the smooth case, e.g. [41], [47] or [64]. But the most important one for this chapter is the one of J. Ueberberg, cf. [74]. It is based on the method for characterizing algebras of pseudodifferential operators developed by R. Beals [16], [18], R.R. Coifman, Y. Meyer [23] and H.O. Cordes [24], [25]. Since non-smooth pseudodifferential operators are used in order to obtain regularity results for partial differential equations, such a characterization is also useful in the non-smooth case. We use the main ideas of the characterization of J. Ueberberg in the smooth case, cf. [74], in order to derive a characterization for non-smooth pseudodifferential operators. Motivated by the characterization in the smooth case, we define the following set of operators:

Definition 5.1. Let $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho \leq 1$. Additionally let $\tilde{m} \in \mathbb{N}_0 \cup \{\infty\}$ and $1 < q < \infty$. Then we define $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$ as the set of all linear and bounded functions $P : H^m_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, such that for all $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_1| + |\beta_1| = \ldots = |\alpha_l| + |\beta_l| = 1$, $|\alpha| \leq M$ and $|\beta| \leq \tilde{m}$ the function

$$\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} \ldots \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} P : H^{m-\rho|\alpha|}_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

is continuous. Here $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. In the case $M = \infty$ we write $\mathcal{A}^m_{\rho,0}(\tilde{m}, q)$ instead of $\mathcal{A}^{m,\infty}_{\rho,0}(\tilde{m}, q)$.

Choosing $M = \tilde{m} = \infty$ the proof of the characterization in the smooth case of J. Ueberberg, cf. [74], Chapter 3, provides that each $T \in \mathcal{A}^{m,\infty}_{\rho,0}(\infty, q)$ is a smooth pseudodifferential operator of the class $S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$. But we even get more: Smooth pseudodifferential operators of the symbol-class $S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ are elements of $\mathcal{A}^{m,\infty}_{\rho,0}(\infty, q)$ due to Remark 3.7 and Theorem 3.18. Thus we have $\mathcal{A}^{m,\infty}_{\rho,0}(\infty, q) = \mathrm{OP}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In the case $\tilde{m} \neq \infty$ we obtain a similar result: Non-smooth pseudodifferential operators of the class $C^\tau S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\rho \in \{0, 1\}$ are elements of such sets:

Example 5.2. Let $\tau > 0$, $\tau \notin \mathbb{N}$, $m \in \mathbb{R}$ and $\rho \in \{0, 1\}$. Considering a non-smooth symbol $p \in C^\tau_* S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we get for $\tilde{m} := \max\{k \in \mathbb{N}_0 : \tau - k > n/2\}$ and $1 < q < \infty$:

i) $p(x, D_x) \in \mathcal{A}^{m+n/2}_{0,0}(\lfloor \tau \rfloor, 2)$ if $\rho = 0$,

ii) $p(x, D_x) \in \mathcal{A}^m_{0,0}(\tilde{m}, 2)$ if $\rho = 0$,

iii) $p(x, D_x) \in \mathcal{A}^m_{1,0}(\lfloor \tau \rfloor, q)$ if $\rho = 1$.

Proof: First of all let $\rho = 0$. Moreover, let $l \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\beta| \leq \lfloor \tau \rfloor$. Here $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Then Remark 4.45 implies

$$\mathrm{ad}(-ix)^{\alpha_1} \,\mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \,\mathrm{ad}(D_x)^{\beta_l} p(x, D_x) \in \mathrm{OP}C^{\tau - |\beta|}_* S^{m-\rho|\alpha|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Since $\tau - |\beta| \geq \tau - \lfloor \tau \rfloor > 0$, Theorem 4.53 provides the continuity of

$$\mathrm{ad}(-ix)^{\alpha_1} \,\mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \,\mathrm{ad}(D_x)^{\beta_l} p(x, D_x) : H^{m+n/2}_2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Hence $p(x, D_x) \in \mathcal{A}^{m+n/2}_{0,0}(\lfloor \tau \rfloor, 2)$. The claims *ii)* and *iii)* can be checked in a similar way. □

As already mentioned the characterization of non-smooth pseudodifferential operators of the symbol-class $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is reduced to the characterization of those ones of the symbol-class $C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. To this end the following property of the set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$ is needed:

Lemma 5.3. *Let $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \rho_1 < \rho_2 \leq 1$. Furthermore, let $\tilde{m} \in \mathbb{N}_0$ and $1 < q < \infty$. Then*

$$\mathcal{A}^{m,M}_{\rho_2,0}(\tilde{m}, q) \subseteq \mathcal{A}^{m,M}_{\rho_1,0}(\tilde{m}, q).$$

Proof: On account of $H^{m-\rho_1|\alpha|}_q(\mathbb{R}^n) \hookrightarrow H^{m-\rho_2|\alpha|}_q(\mathbb{R}^n)$, cf. Lemma 2.45, the claim holds. □

The main goal of this chapter is to show that each element of $\mathcal{A}^{m,M}_{1,0}(\tilde{m},q)$ is a non-smooth pseudodifferential operator with coefficients in a Hölder space. This is the topic of Section 5.5. We will see that M has to be sufficiently large. In analogy to the proof of J. Ueberberg in the smooth case one reduces this statement to the following: Each element of the set $\mathcal{A}^{m,M}_{0,0}(\tilde{m},q)$ is a non-smooth pseudodifferential operator with coefficients in a Hölder space. Details for deriving this result are explained in Section 5.4. Making use of order reducing pseudodifferential operators we obtain the characterization of non-smooth pseudodifferential operators of arbitrary order m from that.

The first three sections serve to develop some auxiliary tools needed for the proof of the case $m = 0$. In Section 5.1 we start by showing that a bounded sequence in $C^{\tilde{m},s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ has a subsequence which converges in the symbol-class $C^{\tilde{m},s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. Section 5.2 is devoted to the symbol reduction of non-smooth double symbols to non-smooth single symbols. Details for the third tool are proved in Section 5.3. There a family $(T_\varepsilon)_{\varepsilon\in(0,1]}$ fulfilling the following three properties is constructed: $T_\varepsilon : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous for all $\varepsilon \in (0,1]$ and converges pointwise if $\varepsilon \to 0$. Moreover, all iterated commutators of T_ε are uniformly bounded with respect to ε as maps from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

With these auxiliary tools at hand it is possible to show the characterization in the non-smooth case. In other words every operator of the set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m},q)$, $\rho \in \{0,1\}$ is a non-smooth pseudodifferential operator with coefficients in the Hölder space $C^\tau(\mathbb{R}^n)$ where $\tau \in (0, \tilde{m} - n/q]$ with $\tau \notin \mathbb{N}$. Unfortunately we loose some regularity with respect to the order of the Hölder space $C^\tau(\mathbb{R}^n)$ with τ being strictly smaller than $\tilde{m}$. Therefore the question arises whether we can improve our result. This is done in Section 5.6: Linear operators in the set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m},q)$, $\rho \in \{0,1\}$, are even pseudodifferential operators with coefficients in the uniformly local Sobolev space $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$. Section 5.7 is devoted to illustrate the usefulness of such a characterization: We show that the composition PQ of two non- smooth pseudodifferential operators P and Q is a non-smooth pseudodifferential operator again if Q is smooth enough. This is done by means of the characterization of non-smooth pseudodifferential operators.

In the whole chapter we use the following notation for an often used pseudodifferential operator:

Notation 5.4. For every $m \in \mathbb{R}$ we define the order reducing pseudodifferential operator $\Lambda^m := \lambda^m(D_x)$, where $\lambda^m(\xi) := \langle\xi\rangle^m$.

As proved in Example 3.2, we know that $\Lambda^m \in \mathrm{OP}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $m \in \mathbb{R}$. Moreover, $(\varphi_j)_{j\in\mathbb{N}_0}$ is an arbitrary dyadic partition of unity on $\mathbb{R}^n$, which is also fixed in the whole chapter.

5.1 Pointwise Convergence in $C^{m,s}S^0_{0,0}$

Assuming a bounded sequence $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{m,s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, we show the existence of a subsequence of $(p_\varepsilon)_{\varepsilon>0}$ which converges pointwise in the symbol-class $C^{m,s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. An important auxiliary tool for deriving this result is the compact embedding theorem in Hölder spaces. We refer to e.g. example [8], Theorem 8.6 for the proof.

Lemma 5.5 (Embedding theorem in Hölder spaces). *Let $0 < s_1, s_2 \leq 1$ and $m_1, m_2 \in \mathbb{N}_0$ with $m_1 + s_1 > m_2 + s_2$. Furthermore, let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. We assume that Ω has a Lipschitz boundary if $m_1 \geq 1$. Then the embedding*

$$Id : C^{m_1,s_1}(\bar{\Omega}) \to C^{m_2,s_2}(\bar{\Omega})$$

is compact.

As a first step to reach the goal of this section, we prove that each bounded sequence in $C^{m,s}(\mathbb{R}^n)$ has a subsequence which converges pointwise in $C^{m,s}(\mathbb{R}^n)$. In order to prove this statement we use the next remark:

Remark 5.6. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $(\Omega_j)_{j\in\mathbb{N}}$ be a countable open cover of bounded sets of $\mathbb{R}^n$. If $p : \mathbb{R}^n \to \mathbb{C}$ is an element of $C^k(\bar{\Omega}_j)$ for all $j \in \mathbb{N}$, then $p \in C^k(\mathbb{R}^n)$.

This remark is a direct consequence of the definition of the spaces $C^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0 \cup \{\infty\}$. With this remark at hand, we obtain

Proposition 5.7. *Let $0 < s \leq 1$, $m \in \mathbb{N}_0$ and $(\Omega_j)_{j\in\mathbb{N}}$ be a countable open cover of bounded sets of $\mathbb{R}^n$. Moreover, let $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{m,s}(\mathbb{R}^n)$ be a bounded sequence and $p : \mathbb{R}^n \to \mathbb{C}$ be a function such that*

$$p_\varepsilon \xrightarrow{\varepsilon\to 0} p \qquad \text{in } C^m(\bar{\Omega}_j) \text{ for all } j \in \mathbb{N}.$$

Then $p \in C^{m,s}(\mathbb{R}^n)$.

Proof: One assumption of the proposition is that $p_\varepsilon \to p$ in $C^m(\bar{\Omega}_j)$, if ε converges to 0 for every $j \in \mathbb{N}$. This implies the pointwise convergence of

$$\partial_x^\alpha p_\varepsilon \xrightarrow{\varepsilon\to 0} \partial_x^\alpha p \qquad \text{on } \mathbb{R}^n \tag{5.1}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. We already know that p is an element of $C^m(\mathbb{R}^n)$ due to Remark 5.6. Now let $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ be arbitrary. Because of the boundedness of $(p_\varepsilon)_{\varepsilon>0}$ in $C^{m,s}(\mathbb{R}^n)$, there is a constant $C > 0$, independent of $\varepsilon > 0$, $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ such that $|\partial_x^\alpha p_\varepsilon(x)| \leq \|p_\varepsilon\|_{C^{m,s}} \leq C$. Therefore $\varepsilon \to 0$ provides

$$|\partial_x^\alpha p(x)| = \lim_{\varepsilon\to 0} |\partial_x^\alpha p_\varepsilon(x)| \leq C.$$

The independence of C with respect to $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ yields $p \in C_b^m(\mathbb{R}^n)$:

$$\|p\|_{C_b^m(\mathbb{R}^n)} = \max_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\leq m}} \sup_{x\in\mathbb{R}^n} |\partial_x^\alpha p(x)| \leq C. \tag{5.2}$$

Thus it remains to show $p \in C^{m,s}(\mathbb{R}^n)$. From the boundedness of the sequence $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{m,s}(\mathbb{R}^n)$ we arise

$$\max_{|\alpha|\leq m} \sup_{x\neq y} \frac{|\partial_x^\alpha p_\varepsilon(x) - \partial_x^\alpha p_\varepsilon(y)|}{|x-y|^s} \leq \|p_\varepsilon\|_{C^{m,s}(\mathbb{R}^n)} \leq C,$$

where C is independent of $\varepsilon > 0$. On account of the previous inequality we obtain for arbitrary $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$:

$$|\partial_x^\alpha p(x) - \partial_x^\alpha p(y)| = \lim_{\varepsilon\to 0} |\partial_x^\alpha p_\varepsilon(x) - \partial_x^\alpha p_\varepsilon(y)| \leq C|x-y|^s.$$

Here C is independent of $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. An application of the last inequality and (5.2) yields the claim:

$$\|p\|_{C^{m,s}(\mathbb{R}^n)} = \|p\|_{C_b^m(\mathbb{R}^n)} + \max_{|\alpha|\leq m} \sup_{x\neq y} \frac{|\partial_x^\alpha p(x) - \partial_x^\alpha p(y)|}{|x-y|^s} \leq C.$$

Therefore $p \in C^{m,s}(\mathbb{R}^n)$. □

If we even choose a countable cover of bounded sets with Lipschitz boundary of $\mathbb{R}^n$, we are able to improve the last result:

Lemma 5.8. *Let* $m \in \mathbb{N}_0$, $0 < s \leq 1$ *and* $(\Omega_j)_{j\in\mathbb{N}}$ *be a countable open cover of bounded sets with Lipschitz boundary of* $\mathbb{R}^n$. *Additionally let* $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{m,s}(\mathbb{R}^n)$ *be a bounded sequence. Then there is a subsequence* $(p_{\varepsilon_k})_{k\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ *with* $\varepsilon_k \to 0$ *for* $k \to \infty$ *and a function* $p \in C^{m,s}(\mathbb{R}^n)$ *with the following property: For all* $\beta \in \mathbb{N}_0^n$ *with* $|\beta| \leq m$

$$\partial_x^\beta p_{\varepsilon_k} \xrightarrow{k\to\infty} \partial_x^\beta p$$

converges uniformly on each $\bar{\Omega}_j$, $j \in \mathbb{N}$.

Proof: Let $0 < s_1 < s$ and $j \in \mathbb{N}$. The boundedness of $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{m,s}(\mathbb{R}^n)$ yields the boundedness of $(p_\varepsilon|_{\bar{\Omega}_j})_{\varepsilon>0} \subseteq C^{m,s}(\bar{\Omega}_j)$. The embedding

$$\mathrm{Id} : C^{m,s}(\bar{\Omega}_j) \to C^{m,s_1}(\bar{\Omega}_j)$$

is compact due to Lemma 5.5. We know that the composition of a compact embedding and of a continuous embedding is compact again. Consequently the continuity of $\mathrm{Id} : C^{m,s_1}(\bar{\Omega}_j) \hookrightarrow C^m(\bar{\Omega}_j)$ gives us

$$\mathrm{Id} : C^{m,s}(\bar{\Omega}_j) \to C^m(\bar{\Omega}_j) \qquad \text{is compact for all } j \in \mathbb{N}. \tag{5.3}$$

Now we prove the lemma iteratively with respect to j. So first observe $j = 1$. Since $(p_\varepsilon|_{\bar{\Omega}_1})_{\varepsilon>0}$ is a bounded sequence in $C^{m,s}(\bar{\Omega}_1)$, we can apply (5.3). This gives us the existence of a subsequence $(p_{\varepsilon_k})_{k\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_k \to 0$ for $k \to \infty$ and of a unique function $p_{\Omega_1} \in C^m(\bar{\Omega}_1)$ such that

$$p_{\varepsilon_k} \xrightarrow{k\to\infty} p_{\Omega_1} \qquad \text{in } C^m(\bar{\Omega}_1).$$

We assume that the claim holds for an arbitrary $j \in \mathbb{N}$. Our goal is to verify that the claim also holds for $j+1$. As in the base case we can use (5.3) due to the boundedness of $(p_\varepsilon|_{\bar{\Omega}_{j+1}})_{\varepsilon>0} \subseteq C^{m,s}(\bar{\Omega}_{j+1})$. Consequently there is a subsequence of $(p_{\varepsilon_k})_{k\in\mathbb{N}}$, which we again denote by $(p_{\varepsilon_k})_{k\in\mathbb{N}}$, and a unique $p_{\Omega_{j+1}} \in C^m(\bar{\Omega}_{j+1})$ such that

$$p_{\varepsilon_k} \xrightarrow{k\to\infty} p_{\Omega_{j+1}} \qquad \text{in } C^m(\bar{\Omega}_{j+1}).$$

We define $p : \mathbb{R}^n \to \mathbb{C}$ via $p(x) := p_{\Omega_j}(x)$ for all $x \in \bar{\Omega}_j$ and each $j \in \mathbb{N}$. Note that p is well defined due to the uniqueness of the limit. Moreover, we have $p \in C^m(\bar{\Omega}_j)$ and

$$p_{\varepsilon_k} \xrightarrow{k\to\infty} p \qquad \text{in } C^m(\bar{\Omega}_j) \text{ for each } j \in \mathbb{N}.$$

This implies the uniform convergence of

$$\partial_x^\beta p_{\varepsilon_k} \xrightarrow{k\to\infty} \partial_x^\beta p \qquad \text{on } \bar{\Omega}_j$$

for all $j \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^n$ with $|\beta| \le m$. At last we arise from Proposition 5.7 that $p \in C^{m,s}(\mathbb{R}^n)$. □

As a direct consequence of the previous lemma we get

Corollary 5.9. *Let $0 < s \le 1$ and $M \in \mathbb{N} \cup \{\infty\}$. Additionally let $(\Omega_j)_{j\in\mathbb{N}}$ be a countable open cover of bounded sets with Lipschitz boundary of $\mathbb{R}^n$. Furthermore, let $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{k,s}(\mathbb{R}^n)$ be a bounded sequence for all $k \in \mathbb{N}_0$ with $k \le M$. Then there is a subsequence $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_l \to 0$ for $l \to \infty$ and a function $p \in C^M(\mathbb{R}^n)$ such, that*

$$\partial_x^\beta p_{\varepsilon_l} \xrightarrow{l\to\infty} \partial_x^\beta p$$

converges uniformly on $\bar{\Omega}_j$ for all $j \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^n$ with $|\beta| \le M$.

Proof: It is not difficult to check the corollary by a diagonal sequence argument if one uses Lemma 5.8. □

The previous results enable us to show the next claim:

Lemma 5.10. *Let $m \in \mathbb{N}_0$ and $0 < s \leq 1$. Additionally let $(\Omega_j \times A_i)_{i,j\in\mathbb{N}}$ be a countable open cover of bounded sets with Lipschitz boundary of $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore, let $(\partial_x^\beta p_\varepsilon)_{\varepsilon>0} \subseteq C^{0,s}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ be a bounded sequence for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$. Then there is a subsequence $(p_{\varepsilon_k})_{k\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_k \to 0$ for $k \to \infty$ and a function $p \in C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ with the following properties: For all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$ we have*

i) $\partial_x^\beta p \in C^{0,s}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$,

ii) $\partial_x^\beta p_{\varepsilon_k} \xrightarrow{k\to\infty} \partial_x^\beta p$ *converges uniformly on each* $\overline{\Omega_j \times A_i}$, $i, j \in \mathbb{N}$.

Proof: We verify the claim by mathematical induction with respect to $|\beta|$ with $|\beta| \leq m$. An application of Lemma 5.8 provides the case $|\beta| = 0$. In order to prove the induction step, we assume that the claim holds for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq l$, $l < m$. Considering an arbitrary $\beta \in \mathbb{N}_0^n$ with $|\beta| = l$ and $\tilde{i} \in \{1, \ldots, n\}$ we have to show *i*) and *ii*) for $\tilde{\beta} := \beta + e_{\tilde{i}}$. Since the subset $(\partial_x^{\tilde{\beta}} p_\varepsilon)_{\varepsilon>0}$ is bounded in $C^{0,s}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, we arise from Lemma 5.8 the existence of a subsequence of $(p_{\varepsilon_k})_{k\in\mathbb{N}}$, which we again denote with $(p_{\varepsilon_k})_{k\in\mathbb{N}}$, and a function $q_{\tilde{\beta}} \in C^{0,s}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ such that

$$\partial_x^{\tilde{\beta}} p_{\varepsilon_k} \xrightarrow{k\to\infty} q_{\tilde{\beta}} \qquad \text{uniformly in } \overline{\Omega_j \times A_i} \text{ for all } i, j \in \mathbb{N}. \tag{5.4}$$

Iteratively the same argument yields the existence of a subsequence of $(p_{\varepsilon_k})_{k\in\mathbb{N}}$, which we again denote with $(p_{\varepsilon_k})_{k\in\mathbb{N}}$, and functions $q_{\tilde{\beta}} \in C^{0,s}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ such that (5.4) holds for each $\tilde{\beta} \in \mathbb{N}_0^n$ with $|\tilde{\beta}| = l + 1$. Choosing an arbitrary but fixed $\xi \in \mathbb{R}^n$, the induction hypothesis and (5.4) implies the uniformly convergence of

$$\partial_x^{\tilde{\beta}} p_{\varepsilon_k}(.,\xi) \xrightarrow{k\to\infty} q_{\tilde{\beta}}(.,\xi) \qquad \text{and} \qquad \partial_x^\delta p_{\varepsilon_k}(.,\xi) \xrightarrow{k\to\infty} \partial_x^\delta p(.,\xi) \tag{5.5}$$

in $\overline{\Omega}_j$ for all $\tilde{\beta}, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq l$ and $|\tilde{\beta}| = l + 1$ and all $j \in \mathbb{N}$. Hence $(p_{\varepsilon_k}(.,\xi))_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^{l+1}(\overline{\Omega}_j)$. Due to the completeness of $C^{l+1}(\overline{\Omega}_j)$ we have the convergence of $(p_{\varepsilon_k}(.,\xi))_{k\in\mathbb{N}}$ to q_0 in $C^{l+1}(\overline{\Omega}_j)$. Consequently we obtain for all $\tilde{\beta}, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq l$ and $|\tilde{\beta}| = l + 1$ and each $j \in \mathbb{N}$:

$$\partial_x^{\tilde{\beta}} p_{\varepsilon_k}(.,\xi) \xrightarrow{k\to\infty} \partial_x^{\tilde{\beta}} q_0 \qquad \text{and} \qquad \partial_x^\delta p_{\varepsilon_k}(.,\xi) \xrightarrow{k\to\infty} \partial_x^\delta q_0 \tag{5.6}$$

in $C^0(\overline{\Omega}_j)$. Because of the uniqueness of the strong limit we get together with (5.5) that $q_0 = p(.,\xi)$ and $\partial_x^{\tilde{\beta}} q_0 = q_{\tilde{\beta}}(.,\xi)$ for each $\tilde{\beta} \in \mathbb{N}_0^n$ with $|\tilde{\beta}| = l + 1$. Therefore we have $q_{\tilde{\beta}} = \partial_x^{\tilde{\beta}} p(.,\xi)$ for each $\tilde{\beta} \in \mathbb{N}_0^n$ with $|\tilde{\beta}| = l + 1$. Since $\xi \in \mathbb{R}^n$ was arbitrary, the induction step holds. □

Finally we are able to show the main theorem of this subsection: Considering a bounded sequence in the symbol-class $C^{m,s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ there is a subsequence which converges pointwise to a symbol of the same symbol-class.

Theorem 5.11. *Let $m \in \mathbb{N}_0$, $M \in \mathbb{N} \cup \{\infty\}$ and $0 < s \leq 1$. Additionally let $(\Omega_j \times A_{\tilde{j}})_{\tilde{j},j\in\mathbb{N}}$ be a countable open cover of bounded sets of $\mathbb{R}^n \times \mathbb{R}^n$ with Lipschitz boundary. Furthermore, let $(p_\varepsilon)_{\varepsilon>0} \subseteq C^{m,s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ be a bounded sequence. Then there is a subsequence $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_l \to 0$ for $l \to \infty$ and a function $p : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ such that for all $\alpha, \beta \in \mathbb{N}^n_0$ with $|\beta| \leq m$ and $|\alpha| \leq M - 1$ we get*

i) $\partial^\beta_x \partial^\alpha_\xi p$ *exists and* $\partial^\beta_x \partial^\alpha_\xi p \in C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$,

ii) $\partial^\beta_x \partial^\alpha_\xi p_{\varepsilon_l} \xrightarrow{l\to\infty} \partial^\beta_x \partial^\alpha_\xi p$ *is uniformly convergent on each* $\overline{\Omega_j \times A_{\tilde{j}}}$, $\tilde{j}, j \in \mathbb{N}$.

In particular $p \in C^{m,s}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$.

Proof: We show $i)$ and $ii)$ by mathematical induction with respect to $|\alpha|$ with $|\alpha| \leq M-1$. Applying Lemma 4.9 we get the boundedness of the sequence $(\partial^\beta_x \partial^\gamma_\xi p_\varepsilon)_{\varepsilon>0} \subseteq C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\beta, \gamma \in \mathbb{N}^n_0$ with $|\beta| \leq m$ and $|\gamma| \leq M-1$. Thus by Lemma 5.10 we obtain the existence of a subsequence $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ and a function $p \in C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ with the following properties: For all $\tilde{j}, j \in \mathbb{N}$ and $\beta \in \mathbb{N}^n_0$ with $|\beta| \leq m$ we have $\partial^\beta_x p \in C^{0,s}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and

$$\partial^\beta_x p_{\varepsilon_l} \xrightarrow{l\to\infty} \partial^\beta_x p \tag{5.7}$$

converges uniformly on the set $\overline{\Omega_j \times A_{\tilde{j}}}$. So we have checked the base case $|\alpha| = 0$. Next we prove the induction step. We assume that the claim holds for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{k}$, $\tilde{k} < M-1$. We define $\tilde{\alpha} := \alpha + e_i$ for some $i \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{N}^n_0$ with $|\alpha| = \tilde{k}$. Using the boundedness of the subsequence $(\partial^\beta_x \partial^{\tilde{\alpha}}_\xi p_{\varepsilon_l})_{l\in\mathbb{N}}$ in $C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\beta \in \mathbb{N}^n_0$ with $|\beta| \leq m$, Lemma 5.10 provides the existence of a subsequence of $(p_{\varepsilon_l})_{l\in\mathbb{N}}$, which we again denote by $(p_{\varepsilon_l})_{l\in\mathbb{N}}$, and a function $q_{\tilde{\alpha}}$ with $\partial^\beta_x q_{\tilde{\alpha}} \in C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\beta \in \mathbb{N}^n_0$ with $|\beta| \leq m$ such that

$$\partial^\beta_x \partial^{\tilde{\alpha}}_\xi p_{\varepsilon_l} \xrightarrow{l\to\infty} \partial^\beta_x q_{\tilde{\alpha}} \tag{5.8}$$

converges uniformly on $\overline{\Omega_j \times A_{\tilde{j}}}$ for all $\tilde{j}, j \in \mathbb{N}$ and $\beta \in \mathbb{N}^n_0$ with $|\beta| \leq m$. Now we choose an arbitrary but fixed $k \in \mathbb{N}_0$ with $k \leq M-1$ and $x \in \mathbb{R}^n$. The boundedness of $(\partial^\gamma_\xi p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq C^{0,s}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\gamma \in \mathbb{N}^n_0$ with $|\gamma| \leq k$ leads to

$$\begin{aligned}\|p_{\varepsilon_l}(x,.)\|_{C^{k,s}(\mathbb{R}^n)} &= \max_{\substack{\gamma\in\mathbb{N}^n_0\\|\gamma|\leq k}} \sup_{\xi\in\mathbb{R}^n} \|\partial^\gamma_\xi p_{\varepsilon_l}(x,.)\|_{C^{0,s}(\mathbb{R}^n)} \leq \max_{\substack{\gamma\in\mathbb{N}^n_0\\|\gamma|\leq k}} \sup_{\xi\in\mathbb{R}^n} \|\partial^\gamma_\xi p_{\varepsilon_l}\|_{C^{0,s}(\mathbb{R}^n\times\mathbb{R}^n)}\\ &\leq C_k \qquad \text{for all } x \in \mathbb{R}^n,\end{aligned}$$

where C_k is independent of $l \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This implies the boundedness of $(p_{\varepsilon_l}(x,.))_{l\in\mathbb{N}} \subseteq C^{k,s}(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ with $k \leq M-1$. Applying Corollary 5.9 arises the existence of a subsequence of $(p_{\varepsilon_l})_{l\in\mathbb{N}}$ denoted by $(p_{\varepsilon_{l_r}})_{r\in\mathbb{N}}$ and of a function $\tilde{p} \in C^{M-1}(\mathbb{R}^n)$ with the property

$$\partial^\gamma_\xi p_{\varepsilon_{l_r}}(x,\xi) \xrightarrow{r\to\infty} \partial^\gamma_\xi \tilde{p}(\xi) \qquad \text{pointwise for all } \xi \in \mathbb{R}^n \tag{5.9}$$

and every $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq M-1$. The convergence (5.7) provides the pointwise convergence of $p_{\varepsilon_{l_r}}$ to p if $r \to \infty$. On account of the uniqueness of the limit we have $p(x,.) = \tilde{p}$. This implies $p(x,.) \in C^{M-1}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$. Therefore it remains to show $q_{\tilde{\alpha}} = \partial_\xi^{\tilde{\alpha}} p$. Let $x \in \mathbb{R}^n$ be arbitrary. Using $p(x,.) = \tilde{p}$, we get by means of (5.8) and (5.9):

$$\partial_\xi^{\tilde{\alpha}} p_{\varepsilon_{l_r}}(x,\xi) \xrightarrow{r\to\infty} q_{\tilde{\alpha}}(x,\xi) \quad \text{and} \quad \partial_\xi^{\tilde{\alpha}} p_{\varepsilon_{l_r}}(x,\xi) \xrightarrow{r\to\infty} \partial_\xi^{\tilde{\alpha}} p(x,\xi)$$

pointwise for all $\xi \in \mathbb{R}^n$. Since $x \in \mathbb{R}^n$ was chosen arbitrary, this gives $q_{\tilde{\alpha}} = \partial_\xi^{\tilde{\alpha}} p$. Hence the induction step holds. Note that (i) implies the existence of a constant C_γ, independent of $\xi \in \mathbb{R}^n$, such that for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq M-1$ we have

$$\|\partial_\xi^\gamma p(.,\xi)\|_{C^{m,s}(\mathbb{R}^n)} \leq \max_{|\beta|\leq m} \|\partial_x^\beta \partial_\xi^\gamma p\|_{C^{0,s}(\mathbb{R}^n\times\mathbb{R}^n)} \leq C_\gamma.$$

Together with $p(x,.) \in C^{M-1}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ the previous inequality yields $p \in C^{m,s} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. □

5.2 Reduction of Non-Smooth Pseudodifferential Operators with Double Symbol

In this section we derive a formula representing an operator with a non-smooth double symbol as an operator with a non-smooth single symbol. In the smooth case a symbol reduction of pseudodifferential operators with double symbols already is developed, cf. e.g. [42], Theorem 2.5. During the development of this work (however independent) D. Köppl generalized this result in his diploma thesis, cf. [45], Theorem 3.33, for non-smooth double symbols of the symbol-class $C^{\tilde{m},\tau} S^{m,m'}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ where $N = \infty$. However, as we will show, the assumption $N = \infty$ (as assumed in the diploma thesis) may be weakend. Then the smoothness in ξ is reduced by the order of n.

For the proof of the characterization of non-smooth pseudodifferential operators only the case $\rho = \delta = 0$ is required. Thus we limit the symbol reduction to this case. This significantly simplifies some proofs. The main idea of the symbol reduction is taken from the smooth case. Since the symbols are non-smooth in both variables, the proof has to be adopted to this modified condition. The first step to reach this goal is to construct an integral representation of a non-smooth pseudodifferential operator with a double symbol applied to a Schwartz function. In the second step a non-smooth single symbol is defined in terms of a given non-smooth double symbol and it is checked that above said function really is a single symbol. Finally, we have to verify the equality of the two resulting pseudodifferential operators.

We begin with the construction of an integral representation of a non-smooth pseudodifferential operator with double symbol applied on a function of $\mathcal{S}(\mathbb{R}^n)$. For this we need the next technical remark:

Remark 5.12. Let $C \in \mathbb{R}$ and $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. For each $0 < \varepsilon < 1$ we define the function $\chi_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$\chi_\varepsilon(\xi, \eta, x, y) := \chi(\varepsilon\xi, \varepsilon\eta, \varepsilon x, \varepsilon y) \qquad \text{for all } \xi, \eta, x, y \in \mathbb{R}^n.$$

Then we have for every $m, m' \in \mathbb{R}$ and each $\alpha, \beta, \gamma, \delta \in \mathbb{N}_0^n$:

$$\begin{aligned} C\langle\xi\rangle^m \langle\eta\rangle^{m'} (\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma \partial_y^\delta \chi_\varepsilon)(\xi, \eta, x' - x, x'' - x') &\in \mathcal{S}(\mathbb{R}^n_\xi \times \mathbb{R}^n_\eta \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{x''}) \\ &\subseteq L^1(\mathbb{R}^n_{x''} \times \mathbb{R}^n_\eta \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_\xi). \end{aligned}$$

Proof: An application of Remark 2.21 and Lemma 2.22 provides

$$(\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma \partial_y^\delta \chi_\varepsilon)(\xi, \eta, x' - x, x'' - x') \in \mathcal{S}(\mathbb{R}^n_\xi \times \mathbb{R}^n_\eta \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{x''}).$$

By means of Remark 2.8 we can show $C\langle\xi\rangle^m \langle\eta\rangle^{m'} \in C^\infty_{poly}(\mathbb{R}^n_\xi \times \mathbb{R}^n_\eta \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{x''})$. Therefore Lemma 2.17 yields that $C\langle\xi\rangle^m \langle\eta\rangle^{m'} (\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma \partial_y^\delta \chi_\varepsilon)(\xi, \eta, x' - x, x'' - x')$ is an element of $\mathcal{S}(\mathbb{R}^n_\xi \times \mathbb{R}^n_\eta \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{x''})$. In particular we have

$$C\langle\xi\rangle^m \langle\eta\rangle^{m'} (\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma \partial_y^\delta \chi_\varepsilon)(\xi, \eta, x' - x, x'' - x') \in L^1(\mathbb{R}^n_{x''} \times \mathbb{R}^n_\eta \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_\xi).$$

□

On account of Remark 5.12 we now are able to prove an integral representation of a non-smooth pseudodifferential operator with double symbol:

Lemma 5.13. *Let $s > 0$, $s \notin \mathbb{N}_0$, $m, m' \in \mathbb{R}$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. Additionally we choose $l, l_0, l_0' \in \mathbb{N}_0$ such that*

$$-2l + m < -n, \quad -2l_0 < -n \quad \textit{and} \quad -2l_0' + 2l + m' < -n.$$

Furthermore, let $P := p(x, D_x, x', D_{x'}) \in OPC^s_* S^{m,m'}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be a non-smooth pseudodifferential operator with double symbol. For $u \in \mathcal{S}(\mathbb{R}^n)$ we define the function $\tilde{p} : \mathbb{R}^{5n} \to \mathbb{C}$ by

$$\tilde{p}(x, \xi, x', \xi', x'') := \langle -\xi + \xi' \rangle^{-2l} \langle D_{x'} \rangle^{2l} \hat{p}(x, \xi, x', \xi', x'')$$

for all $x, \xi, x', \xi', x'' \in \mathbb{R}^n$, where

$$\hat{p}(x, \xi, x', \xi', x'') := \langle x' - x'' \rangle^{-2l_0} \langle D_{\xi'} \rangle^{2l_0} \left[\langle \xi' \rangle^{-2l_0'} \langle D_{x''} \rangle^{2l_0'} p(x, \xi, x', \xi') u(x'') \right]$$

for all $x, \xi, x', \xi', x'' \in \mathbb{R}^n$. Then we can write for all $x \in \mathbb{R}^n$:

$$Pu(x) = \iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} \tilde{p}(x, \xi, x', \xi', x'') dx'' đ\xi' dx' đ\xi.$$

Proof: Let $x \in \mathbb{R}^n$ be arbitrary but fixed. Additionally let χ_ε be as in Remark 5.12 and $u \in \mathcal{S}(\mathbb{R}^n)$. We define $p_{\varepsilon,u} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ for every $0 < \varepsilon < 1$ by

$$p_{\varepsilon,u}(\tilde{x}, \xi, x', \xi', x'') := \chi_\varepsilon(\xi, \xi', x' - x, x'' - x') p(\tilde{x}, \xi, x', \xi') u(x'').$$

for all $\tilde{x}, \xi, x', \xi', x'' \in \mathbb{R}^n$. Assuming arbitrary $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, there is a constant $C_{\alpha,\beta,\gamma}$, independent of $\xi, x', \xi', x'' \in \mathbb{R}^n$, such that

$$\begin{aligned} |\partial_{x''}^\alpha \partial_{\xi'}^\beta \partial_{x'}^\gamma \{p(x,\xi,x',\xi')u(x'')\}| &= |\partial_{x''}^\alpha \{u(x'')\} \partial_{\xi'}^\beta \partial_{x'}^\gamma \{p(x,\xi,x',\xi')\}| \\ &\le C_{\alpha,\beta,\gamma} \langle \xi \rangle^m \langle \xi' \rangle^{m'} \le C_{\alpha,\beta,\gamma} \langle \xi \rangle^{|m|} \langle \xi' \rangle^{|m'|} \le C_{\alpha,\beta,\gamma} \langle (x'', \xi', x', \xi) \rangle^{|m|+|m'|}. \end{aligned}$$

Here we have used $u \in \mathcal{S}(\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n)$ and $p \in C_*^s S_{0,0}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. Using Leibniz's rule and the previous inequality first and Remark 5.12 afterwards provides for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$:

$$\partial_{x''}^\alpha \partial_{\xi'}^\beta \partial_{x'}^\gamma p_{\varepsilon,u}(x, \xi, x', \xi', x'') \in L^1(\mathbb{R}_\xi^n \times \mathbb{R}_{x'}^n \times \mathbb{R}_{\xi'}^n \times \mathbb{R}_{x''}^n).$$

Due to the definition of the oscillatory integral, the change of variables $x' := x+y$ and $x'' := x' + y'$ gives us

$$\begin{aligned} Pu(x) &= \text{Os-}\iiiint e^{-i(y\cdot\xi + y'\cdot\xi')} p(x, \xi, x+y, \xi') u(x+y+y') dy dy' đ\xi đ\xi' \\ &= \lim_{\varepsilon\to 0} \iiiint e^{-i(y\cdot\xi + y'\cdot\xi')} \chi_\varepsilon(\xi, \xi', y, y') p(x, \xi, x+y, \xi') u(x+y+y') dy dy' đ\xi đ\xi' \\ &= \lim_{\varepsilon\to 0} \iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} p_{\varepsilon,u}(x, \xi, x', \xi', x'') dx' dx'' đ\xi đ\xi' \\ &= \lim_{\varepsilon\to 0} \iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} p_{\varepsilon,u}(x, \xi, x', \xi', x'') dx'' đ\xi' dx' đ\xi, \qquad (5.10) \end{aligned}$$

where we have applied Fubini's theorem in the last equality. Here the assumptions of Fubini's theorem are fulfilled, since $p_{\varepsilon,u} \in L^1(\mathbb{R}_\xi^n \times \mathbb{R}_{x'}^n \times \mathbb{R}_{\xi'}^n \times \mathbb{R}_{x''}^n)$. Now we choose $l, l_0, l_0' \in \mathbb{N}_0$ as in the assumptions. Then we define for every $0 < \varepsilon < 1$ the function $\tilde{p}_\varepsilon : \mathbb{R}^{5n} \to \mathbb{C}$ by

$$\tilde{p}_\varepsilon(\tilde{x}, \xi, x', \xi', x'') := \langle -\xi + \xi' \rangle^{-2l} \langle D_{x'} \rangle^{2l} \hat{p}_\varepsilon(\tilde{x}, \xi, x', \xi', x'')$$

for all $\tilde{x}, \xi, x', \xi', x'' \in \mathbb{R}^n$, where the function $\hat{p}_\varepsilon : \mathbb{R}^{5n} \to \mathbb{C}$ is defined by

$$\hat{p}_\varepsilon(\tilde{x}, \xi, x', \xi', x'') := \langle x' - x'' \rangle^{-2l_0} \langle D_{\xi'} \rangle^{2l_0} \left[\langle \xi' \rangle^{-2l_0'} \langle D_{x''} \rangle^{2l_0'} p_{\varepsilon,u}(\tilde{x}, \xi, x', \xi', x'') \right]$$

for each $\tilde{x}, \xi, x', \xi', x'' \in \mathbb{R}^n$. Additionally because of (5.10) and Remark 2.10 integration by parts yields

$$Pu(x) = \lim_{\varepsilon\to 0} \iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} p_{\varepsilon,u}(x, \xi, x', \xi', x'') dx'' đ\xi' dx' đ\xi$$

$$= \lim_{\varepsilon\to 0} \iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} \tilde{p}_\varepsilon(x,\xi,x',\xi',x'') dx'' đ\xi' dx' đ\xi. \tag{5.11}$$

Since we checked $\partial_{x''}^\alpha \partial_{\xi'}^\beta \partial_{x'}^\gamma p_{\varepsilon,u}(x,\xi,x',\xi',x'') \in L^1(\mathbb{R}^n_\xi \times \mathbb{R}^n_{\xi'} \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{x''})$ for all $\alpha,\beta,\gamma \in \mathbb{N}_0^n$ before, integration by parts is possible. Our goal is to simplify (5.11) by means of Lebesgue's theorem. In order to prove the assumptions of this theorem, we need the following claim: Let $M \in \mathbb{N}$ with $-M + 2l_0 < -n$. For all $\beta,\gamma,\delta \in \mathbb{N}_0^n$ there exists a constant $C_{\beta,\gamma,\delta}$, which is independent of $0 < \varepsilon < 1$ and $\xi, x', \xi', x'' \in \mathbb{R}^n$, such that

$$|D_{x'}^\beta D_{\xi'}^\gamma D_{x''}^\delta p_{\varepsilon,u}(x,\xi,x',\xi',x'')| \leq C_{\beta,\gamma,\delta} \langle\xi\rangle^m \langle\xi'\rangle^{m'} \langle x''\rangle^{-M} \tag{5.12}$$

for all $\xi, x', \xi', x'' \in \mathbb{R}^n$. An application of $p \in C_*^s S_{0,0}^{m,m'}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, $u \in \mathcal{S}(\mathbb{R}^n)$ and Lemma 2.26 *iii)* yields (5.12), due to the Leibniz rule. If we apply the Leibniz rule again, we obtain together with Remark 2.8 and (5.12) the following inequality:

$$\begin{aligned}
&\left|\langle -\xi+\xi'\rangle^{-2l} D_{x'}^\beta \left[\langle x'-x''\rangle^{-2l_0} D_{\xi'}^\gamma [\langle\xi'\rangle^{-2l_0'} D_{x''}^\delta p_{\varepsilon,u}(x,\xi,x',\xi',x'')]\right]\right| \\
&\quad \leq C \langle -\xi+\xi'\rangle^{-2l} \langle x'-x''\rangle^{-2l_0} \langle\xi'\rangle^{-2l_0'} \langle\xi\rangle^m \langle\xi'\rangle^{m'} \langle x''\rangle^{-M} \\
&\quad \leq C \langle\xi\rangle^{m-2l} \langle\xi'\rangle^{m'+2l-2l_0'} \langle x'\rangle^{-2l_0} \langle x''\rangle^{2l_0-M} \qquad \text{for all } \xi, x', \xi', x'' \in \mathbb{R}^n.
\end{aligned}$$

On account of Remark 2.10, Peetre's inequality and Theorem 2.11, the previous inequality provides the existence of a constant C, which is independent of $0 < \varepsilon < 1$ and $\xi, x', \xi', x'' \in \mathbb{R}^n$, such that

$$\begin{aligned}
&|e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} \tilde{p}_\varepsilon(x,\xi,x',\xi',x'')| \leq C \langle\xi\rangle^{m-2l} \langle\xi'\rangle^{-2l_0'+m'+2l} \langle x'\rangle^{-2l_0} \langle x''\rangle^{2l_0-M} \\
&\quad \in L^1(\mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'} \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_\xi).
\end{aligned} \tag{5.13}$$

Moreover, we can prove for arbitrary $\beta,\gamma,\delta \in \mathbb{N}_0^n$ the pointwise convergence

$$D_{x'}^\beta D_{\xi'}^\gamma D_{x''}^\delta p_{\varepsilon,u}(x,\xi,x',\xi',x'') \xrightarrow{\varepsilon\to 0} D_{x'}^\beta D_{\xi'}^\gamma D_{x''}^\delta \{p(x,\xi,x',\xi')u(x'')\}, \tag{5.14}$$

if we use the Leibniz rule and Lemma 2.26. Analogously we obtain the pointwise convergence of

$$\tilde{p}_\varepsilon(x,\xi,x',\xi',x'') \xrightarrow{\varepsilon\to 0} \tilde{p}(x,\xi,x',\xi',x'') \tag{5.15}$$

by using Remark 2.10, the Leibniz rule and (5.14). Because of (5.13) and (5.15) the assumptions of Lebesgue's theorem hold. While applying Lebesgue's theorem to (5.11), we conclude the proof:

$$\begin{aligned}
Pu(x) &= \lim_{\varepsilon\to 0} \iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} \tilde{p}_\varepsilon(x,\xi,x',\xi',x'') dx'' đ\xi' dx' đ\xi \\
&\iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'} \tilde{p}(x,\xi,x',\xi',x'') dx'' đ\xi' dx' đ\xi.
\end{aligned}$$

□

Making use of this integral representation we are able to estimate the absolute value of a non-smooth pseudodifferential operator with double symbol:

Lemma 5.14. *Let $s > 0$, $s \notin \mathbb{N}_0$ and $m, m' \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ and $l' \in \mathbb{N}_0$ with $2l' \leq N$. Furthermore, let $\mathscr{B} \subset C^s_* S^{m,m'}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be a bounded subset and $u \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function. Assuming a symbol $p \in C^s_* S^{m,m'}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we denote $P := p(x, D_x, x', D_{x'})$. Then we obtain the existence of a constant C, independent of $x \in \mathbb{R}^n$ and $p \in \mathscr{B}$, such that*

$$|Pu(x)| \leq C\langle x\rangle^{-2l'} \qquad \text{for all } x \in \mathbb{R}^n.$$

Proof: Let $u \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary. In order to check the claim we use the statement of Lemma 5.13 and estimate this integral representation of Pu. Thus we choose arbitrary $l, l_0, l'_0 \in \mathbb{N}_0$ such that

$$-2l + m < -n, \qquad 2l' - 2l_0 < -n, \qquad \text{and} \qquad -2l'_0 + 2l + m' < -n.$$

Additionally we define the functions $\tilde{p}, p' : \mathbb{R}^{5n} \to \mathbb{C}$ as

$$\tilde{p}(x,\xi,x',\xi',x'') := \langle -\xi + \xi'\rangle^{-2l}\langle D_{x'}\rangle^{2l}\hat{p}(x,\xi,x',\xi',x''),$$
$$p'(x,\xi,x',\xi',x'') := \langle x - x'\rangle^{-2l'}\langle D_\xi\rangle^{2l'}\left[\langle -\xi + \xi'\rangle^{-2l}\langle D_{x'}\rangle^{2l}\hat{p}(x,\xi,x',\xi',x'')\right]$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$, where we define

$$\hat{p}(x,\xi,x',\xi',x'') := \langle x' - x''\rangle^{-2l_0}\langle D_{\xi'}\rangle^{2l_0}\left[\langle \xi'\rangle^{-2l'_0}\langle D_{x''}\rangle^{2l'_0}p(x,\xi,x',\xi')u(x'')\right]$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. In the same way as in inequality (5.13) we obtain for all $M \in \mathbb{N}_0$:

$$\begin{aligned}|p'(x,\xi,x',\xi',x'')| &\leq C\langle -x' + x\rangle^{-2l'}\langle -\xi + \xi'\rangle^{-2l}\langle x' - x''\rangle^{-2l_0}\langle \xi'\rangle^{m'-2l'_0}\langle \xi\rangle^{m}\langle x''\rangle^{-M}\\ &\leq C\langle x\rangle^{-2l'}\langle x'\rangle^{2l'-2l_0}\langle \xi\rangle^{m-2l}\langle \xi'\rangle^{2l+m'-2l'_0}\langle x''\rangle^{2l_0-M}\end{aligned}$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. Here C is independent of $p \in \mathscr{B}$ and $x, \xi, x', \xi', x'' \in \mathbb{R}^n$. We choose M with the property $2l_0 - M < -n$. On account of Theorem 2.11 and the last inequality we have $p' \in L^1(\mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'} \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi})$. Since all assumptions of Lemma 5.13 hold, an application of Lemma 5.13, integration by parts and Theorem 2.11 provides

$$\begin{aligned}|Pu(x)| &\leq \left|\iiiint e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'}\tilde{p}(x,\xi,x',\xi',x'')dx''đ\xi'dx'đ\xi\right|\\ &\leq \iiiint |e^{-i(x'-x)\cdot\xi - i(x''-x')\cdot\xi'}p'(x,\xi,x',\xi',x'')|dx''đ\xi'dx'đ\xi\\ &\leq \iiiint C\langle x\rangle^{-2l'}\langle x'\rangle^{2l'-2l_0}\langle \xi\rangle^{m-2l}\langle \xi'\rangle^{2l+m'-2l'_0}\langle x''\rangle^{2l_0-M}dx''đ\xi'dx'đ\xi\\ &\leq C\langle x\rangle^{-2l'}\end{aligned}$$

for all $x \in \mathbb{R}^n$. Here the constant C is independent of $p \in \mathscr{B}$ and $x \in \mathbb{R}^n$. $\square$

The next step towards the symbol reduction is the construction of a non-smooth single symbol out of a non-smooth double symbol. In the smooth case the single symbol can be defined as $a_L(x,\xi) := \mathrm{Os}\text{-}\iint e^{-iy\cdot\eta} a(x,\eta+\xi,x+y)dyđ\eta$ for $a \in S^{m,m'}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ cf. e.g. [42]. We will use the same definition. In order to check whether a_L is still a single symbol in the non-smooth case we need the next results:

Proposition 5.15. *Let $m \in \mathbb{R}$ and X be a Banach space with $X \hookrightarrow L^\infty(\mathbb{R}^n)$. Considering an $l_0 \in \mathbb{N}_0$ with the property $-2l_0 < -n$, let $\mathscr{B}$ be a set of functions $r : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ which are smooth with respect to the fourth variable such that the next inequality holds for all $l \in \mathbb{N}_0$:*

$$\|\langle D_y\rangle^{2l} r(.,\xi,\eta,y)\|_X \leq C_l \langle y\rangle^{-2l_0}\langle \xi+\eta\rangle^m \qquad \text{for all } \xi,\eta,y \in \mathbb{R}^n.$$

Here the constant C_l is independent of $y,\xi,\eta \in \mathbb{R}^n$ and of $r \in \mathscr{B}$. Then $\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \in L^1(\mathbb{R}^n_\eta)$ for all $x,\xi \in \mathbb{R}^n$. If we define

$$I(x,\xi) := \int \left[\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy\right] đ\eta$$

for arbitrary $x,\xi \in \mathbb{R}^n$ and $r \in \mathscr{B}$, there is a constant C, independent of $\xi \in \mathbb{R}^n$ and $r \in \mathscr{B}$, such that

$$\|I(.,\xi)\|_X \leq C\langle \xi\rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Proof: Let $\xi \in \mathbb{R}^n$ and $r \in \mathscr{B}$ be arbitrary. The assumptions of the proposition imply the existence of a constant C_l, independent of $x,y,\xi,\eta \in \mathbb{R}^n$ and $r \in \mathscr{B}$, such that

$$\|\langle D_y\rangle^{2l} r(.,\xi,\eta,y)\|_X \leq C_l \langle y\rangle^{-2l_0}\langle \xi+\eta\rangle^m \leq C_l \langle y\rangle^{-2l_0}\langle \xi\rangle^m \langle \eta\rangle^{|m|} \tag{5.16}$$

for all $y,\eta \in \mathbb{R}^n, l \in \mathbb{N}_0$. The last inequality holds because of Peetre's inequality, cf. Lemma 2.4. Together with the equation $e^{-iy\cdot\eta} = \langle\eta\rangle^{-2l}\langle D_y\rangle^{2l} e^{-iy\cdot\eta}$ we obtain by means of integration by parts for all $l \in \mathbb{N}_0$ and $x,\eta \in \mathbb{R}^n$:

$$\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy = \langle\eta\rangle^{-2l}\int e^{-iy\cdot\eta}\langle D_y\rangle^{2l} r(x,\xi,\eta,y)dy. \tag{5.17}$$

Note that integration by parts is possible since $\langle D_y\rangle^{2\tilde{l}} r(x,\xi,\eta,y) \in L^1(\mathbb{R}^n_y)$ due to (5.16) and Theorem 2.11 for each $x,\xi,\eta \in \mathbb{R}^n$ and $\tilde{l} \in \mathbb{N}_0$. Now we choose an $l \in \mathbb{N}_0$ with $|m| - 2l < -n$. Then we conclude

$$\begin{aligned}\|I(.,\xi)\|_X &= \left\|\int \langle\eta\rangle^{-2l}\int e^{-iy\cdot\eta}\langle D_y\rangle^{2l} r(.,\xi,\eta,y)dyđ\eta\right\|_X \\ &\leq \int \langle\eta\rangle^{-2l}\int \left\|\langle D_y\rangle^{2l} r(.,\xi,\eta,y)\right\|_X dyđ\eta\end{aligned}$$

$$\leq C\langle\xi\rangle^m \int \langle\eta\rangle^{-2l+|m|} \int \langle y\rangle^{-2l_0} dy đ\eta \leq C\langle\xi\rangle^m \int \langle\eta\rangle^{-2l+|m|} đ\eta \leq C\langle\xi\rangle^m.$$

Here C is independent of $\xi \in \mathbb{R}^n$ and $r \in \mathscr{B}$. In particular this provides $\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \in L^1(\mathbb{R}^n_\eta)$ for all $x, \xi \in \mathbb{R}^n$. □

For a given bounded subset $\mathscr{B} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ the previous proposition provides:

Proposition 5.16. *Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $n < 2l \leq N$. Moreover, let $\mathscr{B} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded and $a \in \mathscr{B}$. Considering $l_0 \in \mathbb{N}_0$ with $n < 2l_0 \leq N$, we define $r : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$r(x,\xi,\eta,y) := \langle y\rangle^{-2l_0} \langle D_\eta\rangle^{2l_0} a(x, \xi+\eta, x+y) \qquad \text{for all } x,\xi,\eta,y \in \mathbb{R}^n.$$

Then $\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \in L^1(\mathbb{R}^n_\eta)$ for all $x, \xi \in \mathbb{R}^n$. If we define

$$I(x,\xi) := \int \left[\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy\right] đ\eta$$

for arbitrary $x, \xi \in \mathbb{R}^n$, there is a constant C, independent of $x, \xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that

$$|I(x,\xi)| \leq C\langle\xi\rangle^m \qquad \text{for all } x,\xi \in \mathbb{R}^n.$$

Proof: Let $x, \xi \in \mathbb{R}^n$ be arbitrary. For every $\tilde{\gamma} \in \mathbb{N}^n_0$ we get due to the boundedness of $\mathscr{B} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$:

$$\begin{aligned}\left|\partial^{\tilde{\gamma}}_y \left\{\langle D_\eta\rangle^{2l_0} a(x,\xi+\eta,x+y)\right\}\right| &\leq \sum_{|\alpha|\leq l_0} C_\alpha \left|\partial^{\tilde{\gamma}}_y D^{2\alpha}_\eta a(x,\xi+\eta,x+y)\right| \\ &\leq C_{\tilde{\gamma}} \langle\xi+\eta\rangle^m \qquad \text{for all } y,\eta \in \mathbb{R}^n,\end{aligned}$$

where $C_{\tilde{\gamma}}$ is independent of $x, y, \xi, \eta \in \mathbb{R}^n$ and $a \in \mathscr{B}$. Now the Leibniz rule immediately provides by using the last inequality and Remark 2.8 the existence of a constant C_l, independent of $x, y, \xi, \eta \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that

$$|\langle D_y\rangle^{2l} r(x,\xi,\eta,y)| \leq C_l \langle y\rangle^{-2l_0} \langle\xi+\eta\rangle^m$$

for all $\xi, \eta \in \mathbb{R}^n, l \in \mathbb{N}_0$. Hence all assumptions of Proposition 5.15 are fulfilled. By means of Proposition 5.15 we conclude the proof. □

Another statement for the function r of the previous proposition is

Proposition 5.17. *Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $n < 2l \leq N$. Moreover, let $a \in C^{\tilde{m},s}_{*} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. Considering an $l_0 \in \mathbb{N}_0$ with $n < 2l_0 \leq N$, we define $r : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ as in Proposition 5.16 by*

$$r(x,\xi,\eta,y) := \langle y\rangle^{-2l_0} \langle D_\eta\rangle^{2l_0} a(x,\xi+\eta,x+y) \qquad \text{for all } x,\xi,\eta,y \in \mathbb{R}^n.$$

Then $\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \in L^1(\mathbb{R}^n_\eta)$ for all $x,\xi \in \mathbb{R}^n$ and we obtain

$$Os\text{-}\iint e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy đ\eta = \int \left[\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy\right] đ\eta.$$

In order to prove the proposition, we need the following remark:

Remark 5.18. Let $\mathscr{B} \subseteq C^{\tilde{m},s}_{*} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be a bounded subset with $N \in \mathbb{N}_0 \cup \{\infty\}$, $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m \in \mathbb{R}$. If we choose arbitrary $x,\xi \in \mathbb{R}^n$, we obtain the boundedness of

$$\{f : \mathbb{R}^{2n} \to \mathbb{C} : f(y,\eta) := a(x,\eta+\xi,x+y) \text{ for all } y,\eta \in \mathbb{R}^n, \text{ and } a \in \mathscr{B}\}$$

in the space $\mathscr{A}^{m,N}_0(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$.

Proof: Let $x,\xi \in \mathbb{R}^n$ be arbitrary. The assumption $\partial^\gamma_\xi \partial^\beta_y a(x,\xi,y) \in C^0(\mathbb{R}^n_\xi \times \mathbb{R}^n_y)$ implies $\partial^\gamma_\eta \partial^\beta_y a(x,\eta+\xi,x+y) \in C^0(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$ for all $\beta,\gamma \in \mathbb{N}^n_0$ with $|\gamma| \leq N$. Thus it remains to calculate for every $\alpha,\beta \in \mathbb{N}^n_0$ with $|\alpha| \leq N$ the following inequality:

$$\begin{aligned} |\partial^\alpha_\eta \partial^\beta_y a(x,\eta+\xi,x+y)| &\leq \|\partial^\alpha_\eta \partial^\beta_y a(.,\eta+\xi,x+y)\|_{C^{\tilde{m},s}(\mathbb{R}^n)} \leq C_{\alpha,\beta}\langle \xi+\eta\rangle^m \\ &\leq C_{\alpha,\beta,\xi}\langle\eta\rangle^m \qquad \text{for all } y,\eta \in \mathbb{R}^n \text{ and } a \in \mathscr{B}. \end{aligned} \tag{5.18}$$

Here we have used the boundedness of $\mathscr{B}$ and Corollary 2.5. The independence of $C_{\alpha,\beta,\xi}$ of $y,\eta \in \mathbb{R}^n$ and $a \in \mathscr{B}$ provides the claim. □

Now that we have verified the previous remark, we are in the position to show Proposition 5.17:

Proof of Proposition 5.17: Note that the oscillatory integral exists due to Theorem 4.41 and Remark 5.18. Assuming an arbitrary $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$, we get for fixed $x,\eta,\xi \in \mathbb{R}^n$:

$$e^{-iy\cdot\eta}\chi(\varepsilon y) r(x,\xi,\eta,y) \xrightarrow{\varepsilon\to 0} e^{-iy\cdot\eta} r(x,\xi,\eta,y) \quad \text{pointwise for all } y \in \mathbb{R}^n. \tag{5.19}$$

Now let $0 < \varepsilon \leq 1$. In the same way as inequality (5.16), we can prove the next two estimates if we use $\chi \in \mathcal{S}(\mathbb{R}^n) \subseteq C^\infty_b(\mathbb{R}^n)$:

$$|\partial^\alpha_y r(x,\xi,\eta,y)| \leq C_{\alpha,m}\langle y\rangle^{-2l_0}\langle\xi\rangle^m\langle\eta\rangle^{|m|} \qquad \text{for all } \alpha \in \mathbb{N}^n_0, \tag{5.20}$$

$$|\langle D_y\rangle^{2l'}[\chi(\varepsilon y) r(x,\xi,\eta,y)]| \leq C_{l',m}\langle y\rangle^{-2l_0}\langle\xi\rangle^m\langle\eta\rangle^{|m|} \qquad \text{for all } l' \in \mathbb{N}_0, \tag{5.21}$$

uniformly in $x, \xi, \eta, y \in \mathbb{R}^n$ and in $0 < \varepsilon \leq 1$. The inequality (5.21) implies $\langle D_y \rangle^{2l'} \chi(\varepsilon y) r(x, \xi, \eta, y) \in L^1(\mathbb{R}^n_y)$ for every fixed $x, \xi, \eta \in \mathbb{R}^n$, $\varepsilon \in (0,1]$ and for all $l' \in \mathbb{N}_0$ due to Theorem 2.11. Therefore we can integrate by parts and get by means of Remark 2.10 for an arbitrary $l \in \mathbb{N}_0$ with $|m| - 2l < -n$:

$$\int e^{-iy\cdot\eta} \chi(\varepsilon y) r(x, \xi, \eta, y) dy = \int e^{-iy\cdot\eta} \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} [\chi(\varepsilon y) r(x, \xi, \eta, y)] dy \quad (5.22)$$

Using $\chi \in \mathcal{S}(\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n)$ and (5.22) first and (5.21) and Theorem 2.11 afterwards provides for fixed $x, \xi \in \mathbb{R}^n$:

$$\begin{aligned} \left| \chi(\varepsilon\eta) \int e^{-iy\cdot\eta} \chi(\varepsilon y) r(x, \xi, \eta, y) dy \right| &\leq C \int \left| e^{-iy\cdot\eta} \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} [\chi(\varepsilon y) r(x, \xi, \eta, y)] \right| dy \\ \leq C_{l,m,\xi} \langle \eta \rangle^{-2l+|m|} \int \langle y \rangle^{-2l_0} dy &\leq C_{l,m,\xi} \langle \eta \rangle^{-2l+|m|} \in L^1(\mathbb{R}^n_\eta). \end{aligned} \quad (5.23)$$

Here the constant $C_{l,m,\xi}$ is independent of $\varepsilon \in (0,1]$. Setting $l' = 0$, (5.21) and Theorem 2.11 provide for each fixed $x, \xi, \eta \in \mathbb{R}^n$, that

$$\{ y \mapsto \chi(\varepsilon y) r(x, \xi, \eta, y) : 0 < \varepsilon \leq 1 \}$$

has a $L^1(\mathbb{R}^n_y)$-majorant. Together with (5.19) we have verified all assumptions of Lebesgue's theorem. Applying this theorem we obtain the pointwise convergence of

$$\chi(\varepsilon\eta) \int e^{-iy\cdot\eta} \chi(\varepsilon y) r(x, \xi, \eta, y) dy \xrightarrow{\varepsilon\to 0} \int e^{-iy\cdot\eta} r(x, \xi, \eta, y) dy \quad (5.24)$$

for all $x, \xi, \eta \in \mathbb{R}^n$. Thus it only remains to apply Lebesgue's theorem again in order to get for all $x, \xi \in \mathbb{R}^n$:

$$\begin{aligned} \text{Os-}\iint e^{-iy\cdot\eta} r(x, \xi, \eta, y) dy đ\eta &= \lim_{\varepsilon\to 0} \int \chi(\varepsilon\eta) \int e^{-iy\cdot\eta} \chi(\varepsilon y) r(x, \xi, \eta, y) dy đ\eta \\ &= \int \left[\int e^{-iy\cdot\eta} r(x, \xi, \eta, y) dy \right] đ\eta. \end{aligned}$$

Note that all assumptions of Lebesgue's theorem are fulfilled because of (5.23) and (5.24). □

The previous results enable us to show that a_L fulfills one property of non-smooth symbols:

Lemma 5.19. *Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m \in \mathbb{R}$. Additionally let $\mathscr{B}$ be a bounded subset of $C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ and $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $n < 2l \leq N$. We define for each $a \in \mathscr{B}$ the function $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x, \xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \eta + \xi, x + y) dy đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n.$$

Then we can estimate the absolute value of some derivatives of this function by a constant C, independent of $x, \xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$:

$$|\partial_x^\delta a_L(x,\xi)| \leq C\langle\xi\rangle^m \qquad \textit{for each } \delta \in \mathbb{N}_0^n \textit{ with } |\delta| \leq \tilde{m}.$$

Before proving the lemma, we verify the existence of $a_L(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$. This follows directly from Theorem 4.38 since $a(x, \eta+\xi, x+y) \in \mathscr{A}_0^{m,N}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$ due to Remark 5.18. In order to prove Lemma 5.19 we also need the ability to change the oscillatory integral and the derivative with respect to x:

Lemma 5.20. *Let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with the property $n < 2l \leq N$. Assuming a symbol $a \in C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ with $\tilde{m} \in \mathbb{N}_0$, $m \in \mathbb{R}$, $0 \leq \rho \leq 1$ and $0 < s < 1$, we define $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \eta+\xi, x+y) dy đ\eta \qquad \textit{for all } x, \xi \in \mathbb{R}^n.$$

Then we get for each $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$:

$$\partial_x^\beta a_L(x,\xi) = Os\text{-}\iint e^{-iy\cdot\eta} \partial_x^\beta \{a(x, \eta+\xi, x+y)\} dy đ\eta \qquad \textit{for all } x, \xi \in \mathbb{R}^n.$$

Proof: In the case $\tilde{m} = 0$ there is nothing to show. So let $\tilde{m} > 0$. Moreover, let $j \in \{1, \ldots, n\}$ and $x, \xi \in \mathbb{R}^n$ be arbitrary. For $h \in \mathbb{R}$ with $0 < |h| \leq 1$ we define the functions $a_h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$a_h(y,\eta) := \frac{a(x+he_j, \xi+\eta, x+he_j+y) - a(x, \xi+\eta, x+y)}{h} \quad \text{for all } y, \eta \in \mathbb{R}^n.$$

First of all we show that

$$\{a_h : 0 < |h| \leq 1\} \subseteq \mathscr{A}_0^{m,N}(\mathbb{R}^n \times \mathbb{R}^n) \qquad \text{is bounded.} \tag{5.25}$$

Therefore let $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ be arbitrary. Since the symbol a is an element of $C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, Lemma 2.12 provides the existence of a constant $C_{\alpha,\gamma,m,\xi}$, such that

$$\begin{aligned}
|\partial_\eta^\alpha \partial_y^\gamma a_h(y,\eta)| &\leq \frac{1}{|h|} \sup_{0\leq t\leq 1} \sum_{j=1}^n \Big\{ \big|(\partial_{x_j}\partial_\eta^\alpha \partial_y^\gamma a)(x+the_j, \xi+\eta, x+the_j+y)\big| \\
&\qquad + \big|(\partial_\eta^\alpha \partial_y^{\gamma+e_j} a)(x+the_j, \xi+\eta, x+the_j+y)\big| \Big\} |h| \\
&\leq C_{\alpha,\gamma}\langle\xi+\eta\rangle^{m-\rho|\alpha|} \leq C_{\alpha,\gamma,m,\xi}\langle\eta\rangle^m
\end{aligned}$$

for all $y, \eta \in \mathbb{R}^n$. Here $C_{\alpha,\gamma,m,\xi}$ is independent of $y, \eta \in \mathbb{R}^n$ and $0 < |h| \leq 1$. The last inequality holds because of Corollary 2.5. This implies (5.25). Additionally we get for each $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and $y, \eta \in \mathbb{R}^n$:

$$|\partial_\eta^\alpha \partial_y^\gamma \partial_{x_j} \{a(x, \xi+\eta, x+y)\}|$$

$$\begin{aligned}
&\leq |\left(\partial_{x_j}\partial_\eta^\alpha\partial_y^\gamma a\right)(x,\xi+\eta,x+y)| + |\left(\partial_\eta^\alpha\partial_y^{\gamma+e_j} a\right)(x,\xi+\eta,x+y)| \\
&\leq C_{j,\alpha,\beta}\langle\xi+\eta\rangle^{m-\rho|\alpha|} \leq C_{j,\alpha,\beta,m,\xi}\langle\eta\rangle^m,
\end{aligned}$$

where $C_{j,\alpha,\beta,m,\xi}$ is independent of $y,\eta \in \mathbb{R}^n$. The last two inequalities hold because a is an element of $C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n;N)$ and because of Corollary 2.5. Hence we have checked $\partial_{x_j}\{a(x,\xi+\eta,x+y)\} \in \mathscr{A}_0^{m,N}(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)$ for all $x,\xi\in\mathbb{R}^n$. Together with (5.25) and $\partial_\eta^\alpha\partial_y^\gamma a_h(y,\eta) \to \partial_\eta^\alpha\partial_y^\gamma\partial_{x_j}\{a(x,\xi+\eta,x+y)\}$ for all $\alpha,\gamma\in\mathbb{N}_0^n$ with $|\alpha|\leq N$ if $h\to 0$, we verified the assumptions of Corollary 4.42, now. Finally, an application of Corollary 4.42 implies

$$\partial_{x_j}a_L(x,\xi) = \lim_{h\to 0}\text{Os-}\iint a_h(y,\eta)dy\,đ\eta = \text{Os-}\iint \partial_{x_j}\{a(x,\xi+\eta,x+y)\}dy\,đ\eta.$$

For a general $\beta\in\mathbb{N}_0^n$ with $|\beta|\leq\tilde{m}$ one gets the lemma by mathematical induction. □

Now we are able to prove Lemma 5.19:

Proof of Lemma 5.19. First of all we show the claim in the case $\delta = 0$: Using Theorem 4.41 we can write for each $x,\xi\in\mathbb{R}^n$ and $l_0\in\mathbb{N}_0$ with $n < 2l_0 \leq N$:

$$\begin{aligned}
a_L(x,\xi) &= \text{Os-}\iint e^{-iy\cdot\eta}\langle y\rangle^{-2l_0}\langle D_\eta\rangle^{2l_0}a(x,\xi+\eta,x+y)dy\,đ\eta \\
&= \iint e^{-iy\cdot\eta}\langle y\rangle^{-2l_0}\langle D_\eta\rangle^{2l_0}a(x,\xi+\eta,x+y)dy\,đ\eta.
\end{aligned}$$

The last equality holds because of Proposition 5.17. Hence we obtain by Proposition 5.16 the existence of a constant C, which is independent of $x,\xi\in\mathbb{R}^n$ and of $a\in\mathscr{B}$, such that

$$|a_L(x,\xi)| = \left|\iint e^{-iy\cdot\eta}\langle y\rangle^{-2l_0}\langle D_\eta\rangle^{2l_0}a(x,\xi+\eta,x+y)dy\,đ\eta\right| \leq C\langle\xi\rangle^m.$$

Therefore we have checked the theorem in the case $\delta = 0$. Now we assume an arbitrary $\delta\in\mathbb{N}_0^n$ with $|\delta|\leq\tilde{m}$. On account of Lemma 5.20 we have for $x,\xi\in\mathbb{R}^n$:

$$\partial_x^\delta a_L(x,\xi) = \text{Os-}\iint e^{-iy\cdot\eta}\partial_x^\delta a(x,\eta+\xi,x+y)dy\,đ\eta.$$

We know that $\mathscr{B}^\delta := \left\{\partial_x^\delta a : a\in\mathscr{B}\right\} \subseteq C_*^{\tilde{m}-|\delta|,s}S^m_{0,0}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n;N)$ is bounded due to Remark 4.69. Hence the first case, applied on the set $\mathscr{B}^\delta$, gives us the existence of a constant C, which is independent of $x,\xi\in\mathbb{R}^n$ and of $a\in\mathscr{B}$, such that

$$|\partial_x^\delta a_L(x,\xi)| \leq C\langle\xi\rangle^m \qquad \text{for all } x,\xi\in\mathbb{R}^n.$$

□

Our goal is to verify that a_L is a non-smooth symbol whose coefficient is in the Hölder space $C^{s,\tilde{m}}(\mathbb{R}^n)$. Therefore it is not sufficient to know for each $\delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$ that $|\partial_x^\delta a_L(x,\xi)| \leq C\langle\xi\rangle^m$, cf. Lemma 5.19. We even have to show whether the inequality $\|\partial_x^\delta a_L(x,\xi)\|_{C^{\tilde{m},s}(\mathbb{R}^n_x)} \leq C\langle\xi\rangle^m$ holds for each $\delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$. Having in mind the definition of the Hölder spaces, we need the next statement:

Proposition 5.21. *Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$, $N \in \mathbb{N}_0 \cup \{\infty\}$ and $m \in \mathbb{R}$. Moreover, let $\mathscr{B} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded. Then there is a constant C, independent of $x_1, x_2, y, \xi, \eta \in \mathbb{R}^n$ with $x_1 \neq x_2$ and $a \in \mathscr{B}$, such that for each $\gamma, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq N$ we have*

$$\max_{|\alpha|\leq\tilde{m}} \left\{ \frac{|\partial^\alpha_{x_1}\partial^\gamma_y\partial^\beta_\eta a(x_1,\xi+\eta,x_1+y) - \partial^\alpha_{x_2}\partial^\gamma_y\partial^\beta_\eta a(x_2,\xi+\eta,x_2+y)|}{|x_1-x_2|^s} \right\} \leq C\langle\xi+\eta\rangle^m$$

for all $x_1, x_2, y, \xi, \eta \in \mathbb{R}^n$ with $x_1 \neq x_2$.

Proof: First of all we choose arbitrary $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{m}$ and $|\beta| \leq N$. Additionally let $x_1, x_2 \in \mathbb{R}^n$ be arbitrary. The boundedness of $\mathscr{B}$ in the set $C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N) = C^{\tilde{m},s} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ implies

$$\begin{aligned}\sup_{\substack{x,\tilde{x}\in\mathbb{R}^n\\ x\neq\tilde{x}}} &\left\{ \frac{|\partial^\alpha_x\partial^\gamma_y\partial^\beta_\eta a(x,\xi+\eta,x_1+y) - \partial^\alpha_{\tilde{x}}\partial^\gamma_y\partial^\beta_\eta a(\tilde{x},\xi+\eta,x_1+y)|}{|x-\tilde{x}|^s} \right\} \\ &\leq \|\partial^\gamma_y\partial^\beta_\eta a(x,\xi+\eta,x_1+y)\|_{C^{\tilde{m},s}(\mathbb{R}^n_x)} \leq C_{\gamma,\beta}\langle\xi+\eta\rangle^m \qquad (5.26)\end{aligned}$$

for all $\xi, \eta, y \in \mathbb{R}^n$ and all $a \in \mathscr{B}$, where $C_{\gamma,\beta}$ is independent of $x, \xi, \eta, x_1, y \in \mathbb{R}^n$ and of $a \in \mathscr{B}$. Furthermore, an application of Lemma 2.12 yields, if we use the boundedness of $\mathscr{B}$ again,

$$\begin{aligned}&|\partial^\alpha_x\partial^\gamma_y\partial^\beta_\eta a(x,\xi+\eta,x_1+y) - \partial^\alpha_x\partial^\gamma_y\partial^\beta_\eta a(x,\xi+\eta,x_2+y)| \\ &\quad\leq \sup_{0\leq t\leq 1}\sum_{j=1}^n \left|(\partial_{y_j}\partial^\alpha_x\partial^\gamma_y\partial^\beta_\eta a)(x,\xi+\eta,tx_1+y+(1-t)x_2)\right| |x_1-x_2| \\ &\quad\leq C_{\beta,\gamma}\langle\xi+\eta\rangle^m\,|x_1-x_2|\end{aligned}$$

for all $a \in \mathscr{B}$ and $x, \xi, \eta, y, x_1, x_2 \in \mathbb{R}^n$. Here $C_{\beta,\gamma}$ is independent of $a \in \mathscr{B}$ and $x, \xi, \eta, y, x_1, x_2 \in \mathbb{R}^n$. Hence in the case $|x_1 - x_2| < 1$, $x_1 \neq x_2$, the inequality above provides for all $x, \xi, \eta, x_1, y \in \mathbb{R}^n$ and all $a \in \mathscr{B}$:

$$\begin{aligned}&\frac{|\partial^\alpha_x\partial^\gamma_y\partial^\beta_\eta a(x,\xi+\eta,x_1+y) - \partial^\alpha_x\partial^\gamma_y\partial^\beta_\eta a(x,\xi+\eta,x_2+y)|}{|x_1-x_2|^s} \\ &\quad\leq C_{\beta,\gamma}\langle\xi+\eta\rangle^m\,|x_1-x_2|^{1-s} \leq C_{\beta,\gamma}\langle\xi+\eta\rangle^m.\end{aligned}$$

Using the boundedness of the set $\mathscr{B} \subseteq C^{\tilde{m},s} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, one gets an analog estimate for $|x_1 - x_2| \geq 1$ with $x_1, x_2 \in \mathbb{R}^n$, $x, \xi, \eta, y \in \mathbb{R}^n$ and $a \in \mathscr{B}$:

$$\frac{|\partial_x^\alpha \partial_y^\gamma \partial_\eta^\beta a(x, \xi + \eta, x_1 + y) - \partial_x^\alpha \partial_y^\gamma \partial_\eta^\beta a(x, \xi + \eta, x_2 + y)|}{|x_1 - x_2|^s}$$
$$\leq \left|\partial_x^\alpha \partial_y^\gamma \partial_\eta^\beta a(x, \xi + \eta, x_1 + y)\right| + \left|\partial_x^\alpha \partial_y^\gamma \partial_\eta^\beta a(x, \xi + \eta, x_2 + y)\right| \leq C_{\beta,\gamma} \langle \xi + \eta \rangle^m,$$

where $C_{\beta,\gamma}$ is independent of $a \in \mathscr{B}$ and $x, \xi, \eta, x_1, x_2, y \in \mathbb{R}^n$ with $|x_1 - x_2| \geq 1$. Combining these two inequalities, we obtain the existence of a constant $C_{\beta,\gamma}$, independent of $a \in \mathscr{B}$ and of $x, \xi, \eta, x_1, x_2, y \in \mathbb{R}^n$, $x_1 \neq x_2$, such that

$$\frac{|\partial_x^\alpha \partial_y^\gamma \partial_\eta^\beta a(x, \xi + \eta, x_1 + y) - \partial_x^\alpha \partial_y^\gamma \partial_\eta^\beta a(x, \xi + \eta, x_2 + y)|}{|x_1 - x_2|^s} \leq C_{\beta,\gamma} \langle \xi + \eta \rangle^m \quad (5.27)$$

for all $a \in \mathscr{B}$ and $x, \xi, \eta, x_1, x_2, y \in \mathbb{R}^n$, $x_1 \neq x_2$. Finally, the proposition follows from (5.26) and (5.27) if we choose $x = x_1$, $\tilde{x} = x_2$ in (5.26) and $x = x_2$ in (5.27) by means of the triangle inequality. $\square$

For every non-smooth symbol certain continuity conditions have to hold. Therefore we prove them now:

Lemma 5.22. *Let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $N \geq 2l > n$. Moreover, we define $\tilde{N} := N - (n+2)$ if n is even and $\tilde{N} := N - (n+1)$ else. For a non-smooth symbol $a \in C^{\tilde{m},s} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ with $\tilde{m} \in \mathbb{N}_0$, $m \in \mathbb{R}$, $0 \leq \rho \leq 1$ and $0 < s < 1$, we define $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x, \xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \eta + \xi, x + y) dy đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n.$$

Then $\partial_x^\delta \partial_\xi^\gamma a_L \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$ for every $\gamma, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$ and $|\gamma| \leq \tilde{N}$.

In order to check this lemma we need the next remark:

Remark 5.23. Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ and $0 \leq \delta \leq 1$. For every $a \in C^{\tilde{m},s}_* S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ we have

$$a(x, \xi + \eta, x + y) \in \mathscr{A}_0^{m^+, N}(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\eta)}).$$

Proof: Due to $(|\xi| - |\eta|)^2 \geq 0$ we have $|\xi|^2 + |\eta|^2 \geq 2|\xi||\eta|$ for all $\xi, \eta \in \mathbb{R}^n$. Consequently we get for all $\xi, \eta \in \mathbb{R}^n$:

$$\langle \xi + \eta \rangle^2 = 1 + |\xi + \eta|^2 \leq 1 + |\xi|^2 + |\eta|^2 + 2|\xi||\eta| \leq 1 + 2(|\xi|^2 + |\eta|^2)$$
$$\leq 2 + 2|(\xi, \eta)|^2 \leq 2\langle (\xi, \eta) \rangle^2.$$

Choosing $x \in \mathbb{R}^n$ arbitrary, the last inequality and $a \in C^{s,\tilde{m}}_* S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ provides for all $\alpha, \alpha', \beta, \beta' \in \mathbb{N}_0^n$ with $|\alpha| + |\alpha'| \leq N$:

$$|\partial_\xi^\alpha \partial_\eta^{\alpha'} \partial_y^\beta \partial_{y'}^{\beta'} a(x, \xi + \eta, x + y)| \leq C_{\alpha,\alpha',\beta,\beta'} \langle \xi + \eta \rangle^{m - \rho(|\alpha| + |\alpha'|)} \leq C_{\alpha,\alpha',\beta,\beta'} \langle \xi + \eta \rangle^{m^+}$$

$$\leq C_{\alpha,\alpha',\beta,\beta'}\langle(\xi,\eta)\rangle^{m^+} \qquad \text{for all } \xi,\eta,y \in \mathbb{R}^n.$$

Since $a \in C_*^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, we obtain that $\partial_\xi^\alpha \partial_\eta^{\alpha'} \partial_y^\beta \partial_{y'}^{\beta'} a(x, \xi+\eta, x+y)$ is an element of $C^0(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\eta)})$ for all $\alpha, \alpha', \beta, \beta' \in \mathbb{N}_0^n$ with $|\alpha| + |\alpha'| \leq N$. Therefore the claim holds. □

Now we are able to verify the statement of Lemma 5.22:

Proof of Lemma 5.22: Let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$ and $|\alpha| \leq \tilde{N}$. In Remark 4.69 we have checked that $\partial_\xi^\alpha a \in C^{\tilde{m},s}S^{m-\rho|\alpha|}_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N - |\alpha|)$. Hence the definition of the double symbol implies that $\partial_x^\beta \partial_\xi^\alpha a \in C^0(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Additionally we get by Theorem 4.39 and Lemma 5.20 for all $x, \xi \in \mathbb{R}^n$:

$$\begin{aligned}
\partial_x^\beta \partial_\xi^\alpha a_L(x,\xi) &= \partial_x^\beta \partial_\xi^\alpha \text{Os -}\iint e^{-iy\cdot\eta} a(x, \eta+\xi, x+y) dy đ\eta \\
&= \partial_x^\beta \text{Os -}\iint e^{-iy\cdot\eta} \partial_\xi^\alpha a(x, \eta+\xi, x+y) dy đ\eta \\
&= \text{Os -}\iint e^{-iy\cdot\eta} \partial_x^\beta \{\partial_\xi^\alpha a(x, \eta+\xi, x+y)\} dy đ\eta. \qquad (5.28)
\end{aligned}$$

Here we are able to apply Theorem 4.39 since $N - |\alpha| \geq N - \tilde{N} \geq 2l > n$ and $a(x, \xi+\eta, x+y) \in \mathscr{A}_0^{m^+,N}(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\eta)})$ as we have seen in Remark 5.23. In order to show the continuity of $\partial_x^\beta \partial_\xi^\alpha a_L$, we want to apply Corollary 4.42. Hence we have to prove the assumptions of this corollary, now: let $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ be arbitrary. Additionally let $(x', \xi') \in \mathbb{R}^n \times \mathbb{R}^n$ with $|x - x'|, |\xi - \xi'| < 1$. Then we derive from Peetre's inequality, cf. Theorem 2.4:

$$\langle \eta + \xi' \rangle^m \leq \langle \eta \rangle^m \langle \xi' \rangle^{|m|} \leq \langle \eta \rangle^m \langle \xi' - \xi \rangle^{|m|} \langle \xi \rangle^{|m|} \leq C \langle \eta \rangle^m \langle \xi \rangle^{|m|}.$$

For every $\beta_1, \beta_2, \gamma, \delta \in \mathbb{N}_0^n$ with $\beta_1 + \beta_2 = \beta$ and $|\delta| \leq N - |\alpha|$ an application of $a \in C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ first and of the previous inequality afterwards provides

$$\begin{aligned}
|\partial_y^\gamma \partial_\eta^\delta (\partial_x^{\beta_1} \partial_\xi^\alpha \partial_y^{\beta_2} a)(x', \eta+\xi', x'+y)| &\leq C_{\alpha,\beta,\gamma,\delta} \langle \eta + \xi' \rangle^{m-\rho(|\alpha|+|\delta|)} \\
&\leq C_{\alpha,\beta,\gamma,\delta} \langle \eta \rangle^m \langle \xi \rangle^{|m|}.
\end{aligned}$$

Here $C_{\alpha,\beta,\gamma,\delta}$ is independent of $x', \eta, \xi', y \in \mathbb{R}^n$. This yields the boundedness of

$$\{(\partial_x^{\beta_1} \partial_\xi^\alpha \partial_y^{\beta_2} a)(x', \eta+\xi', x'+y) : x', \xi' \in \mathbb{R}^n \text{ with } |x - x'|, |\xi - \xi'| < 1\}$$

in $\mathscr{A}_0^{m, N-|\alpha|}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$. Moreover, we obtain for all $y, \eta \in \mathbb{R}^n$ and for each $\beta_1, \beta_2, \gamma, \delta \in \mathbb{N}_0^n$ with $\beta_1 + \beta_2 = \beta$ and $|\delta| \leq N - |\alpha|$ the following convergence:

$$\partial_y^\gamma \partial_\eta^\delta (\partial_x^{\beta_1} \partial_\xi^\alpha \partial_y^{\beta_2} a)(x', \eta+\xi', x'+y) \xrightarrow[x' \to x]{\xi' \to \xi} \partial_y^\gamma \partial_\eta^\delta (\partial_x^{\beta_1} \partial_\xi^\alpha \partial_y^{\beta_2} a)(x, \eta+\xi, x+y)$$

due to the definition of $a \in C^{\tilde{m},s}S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. Therefore all assumptions of Corollary 4.42 are fulfilled. Using Leibniz's rule first and Corollary 4.42 afterwards yields

$$\lim_{\substack{\xi' \to \xi \\ x' \to x}} \text{Os-}\iint e^{-iy\cdot\eta}\partial^\beta_{x'}\{\partial^\alpha_\xi a(x', \eta + \xi', x' + y)\} dy đ\eta$$
$$= \text{Os-}\iint e^{-iy\cdot\eta}\partial^\beta_x\{\partial^\alpha_\xi a(x, \eta + \xi, x + y)\} dy đ\eta.$$

Consequently we get together with (5.28) the continuity of $\partial^\beta_x \partial^\alpha_\xi a_L$. Hence the lemma is verified. □

Now we have proved all utilities to show that a_L is a non-smooth symbol whose coefficient is in the Hölder space $C^{s,\tilde{m}}(\mathbb{R}^n)$. Unfortunately we loose some smoothness with respect to ξ of the double symbol:

Theorem 5.24. *Let $0 < s < 1$, $\tilde{m} \in \mathbb{N}_0$ and $m \in \mathbb{R}$. Additionally we choose $N \in \mathbb{N}_0 \cup \{\infty\}$ such that there is an $l \in \mathbb{N}$ with $N \geq 2l > n$. Moreover, we define $\tilde{N} := N - (n+2)$ if n is even and $\tilde{N} := N - (n+1)$ else. Furthermore, let $\mathscr{B} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded. If we define for each $a \in \mathscr{B}$ the function $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x, \xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \eta + \xi, x + y) dy đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n,$$

we get $a_L \in C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$ for all $a \in \mathscr{B}$ and the existence of a constant C_β, independent of $a \in \mathscr{B}$, such that

$$\|\partial^\beta_\xi a_L(., \xi)\|_{C^{\tilde{m},s}_*(\mathbb{R}^n)} \leq C_\beta \langle \xi \rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and } \beta \in \mathbb{N}^n_0 \text{ with } |\beta| \leq \tilde{N}.$$

This implies the boundedness of $\{a_L : a \in \mathscr{B}\} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$.

Proof: First of all note that for every $\gamma, \delta \in \mathbb{N}^n_0$ with $|\gamma| \leq \tilde{N}$ and $|\delta| \leq \tilde{m}$ we have $\partial^\delta_x \partial^\gamma_\xi a_L \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$ due to Lemma 5.22. In Remark 5.23 we have verified that $a(x, \xi + \eta, x + y)$ is an element of $\mathscr{A}^{m^+,N}_0(\mathbb{R}^{2n}_{(y,y')} \times \mathbb{R}^{2n}_{(\xi,\eta)})$. Since $N - |\alpha| \geq N - \tilde{N} > n$, we derive from Theorem 4.39 for each $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{N}$:

$$\partial^\alpha_\xi a_L(x, \xi) = \text{Os-}\iint e^{-iy\cdot\eta} \partial^\alpha_\xi a(x, \eta + \xi, x + y) dy đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n.$$

Now let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{N}$ be arbitrary. On account of Remark 4.69 the boundedness of $\mathscr{B}$ implies the boundedness of

$$\tilde{\mathscr{B}} := \left\{\partial^\alpha_\xi a : a \in \mathscr{B}\right\} \subseteq C^{\tilde{m},s}_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, N - |\alpha|).$$

Hence it remains to show for all $\xi \in \mathbb{R}^n$ and $a \in \tilde{\mathscr{B}}$:

$$\|a_L(.,\xi)\|_{C_*^{\tilde{m},s}(\mathbb{R}^n)} \leq C\langle\xi\rangle^m, \tag{5.29}$$

where C is independent of $\xi \in \mathbb{R}^n$ and $a \in \tilde{\mathscr{B}}$. Inequality (5.29) implies $\|\partial_\xi^\alpha a_L(.,\xi)\|_{C_*^{\tilde{m},s}(\mathbb{R}^n)} \leq C_\alpha\langle\xi\rangle^m$, where C_α is independent of $\xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$. This yields the boundedness of $\{a_L : a \in \mathscr{B}\} \subseteq C_*^{\tilde{m},s}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$ since $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \tilde{N}$ was chosen arbitrary. In order to prove (5.29), note that there is a constant C, independent of $a \in \tilde{\mathscr{B}}$ and $x, \xi \in \mathbb{R}^n$, such that

$$|\partial_x^\delta a_L(x,\xi)| \leq C\langle\xi\rangle^m \qquad \text{for all } x, \xi \in \mathbb{R}^n \text{ and } a \in \tilde{\mathscr{B}} \tag{5.30}$$

for each $\delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$ because of Lemma 5.19. Now we choose an arbitrary $l \in \mathbb{N}_0$ with $-2l + |m| < -n$ and $2l_0 := N - \tilde{N}$. An application of Lemma 5.20 and Theorem 4.38 provides for every $\delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$:

$$\begin{aligned}\partial_x^\delta a_L(x,\xi) &= \text{Os-}\iint e^{-iy\cdot\eta}\partial_x^\delta \{a(x,\eta+\xi,x+y)\}\, dy đ\eta \\ &= \iint e^{-iy\cdot\eta}\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\left\{\langle y\rangle^{-2l_0}\langle D_\eta\rangle^{2l_0}\partial_x^\delta[a(x,\xi+\eta,x+y)]\right\} dy đ\eta. \end{aligned}\tag{5.31}$$

On account of Remark 2.8 and Proposition 5.21, we obtain for $x_1, x_2, y, \xi, \eta \in \mathbb{R}^n$ with $x_1 \neq x_2$ and $\tilde{\alpha}, \beta, \gamma, \delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$ and $|\beta| \leq N - \tilde{N}$:

$$\begin{aligned}&\langle\eta\rangle^{-2l}\left|\partial_y^{\tilde{\alpha}}\langle y\rangle^{-2l_0}\right| \frac{|\partial_{x_1}^\delta\partial_y^\gamma\partial_\eta^\beta a(x_1,\xi+\eta,x_1+y) - \partial_{x_2}^\delta\partial_y^\gamma\partial_\eta^\beta a(x_2,\xi+\eta,x_2+y)|}{|x_1-x_2|^s} \\ &\quad \leq C\langle\eta\rangle^{-2l}\langle y\rangle^{-2l_0-|\tilde{\alpha}|}\langle\xi+\eta\rangle^m \leq C\langle y\rangle^{-2l_0}\langle\xi\rangle^m\langle\eta\rangle^{-2l+|m|},\end{aligned}$$

where C is independent of $x_1, x_2, y, \xi, \eta \in \mathbb{R}^n$ with the property $x_1 \neq x_2$ and of $a \in \tilde{\mathscr{B}}$. The last inequality holds because of Peetre's inequality, cf. Lemma 2.4. Now we plug in $\langle D_y\rangle^{2l} = \sum_{|\alpha|\leq l} a_{\alpha,l}D_y^{2\alpha}$ and $\langle D_\eta\rangle^{2l_0} = \sum_{|\beta|\leq l_0} a_{\beta,l_0}D_\eta^{2\beta}$ in (5.31) first, and use the Leibniz rule and the previous inequality afterwards. Then we can estimate for each $x_1, x_2, \xi \in \mathbb{R}^n$ with $x_1 \neq x_2$ and $\delta \in \mathbb{N}_0^n$ with $|\delta| \leq \tilde{m}$:

$$\frac{|\partial_x^\delta a_L(x_1,\xi) - \partial_x^\delta a_L(x_2,\xi)|}{|x_1-x_2|^s} \leq \iint C\langle y\rangle^{-2l_0}\langle\xi\rangle^m\langle\eta\rangle^{-2l+|m|} dy đ\eta \leq C\langle\xi\rangle^m,$$

where C is independent of $x_1, x_2, \xi \in \mathbb{R}^n$ with $x_1 \neq x_2$ and $a \in \tilde{\mathscr{B}}$. Here we get the last inequality by means of Theorem 2.11. Finally, we only have to combine this result with (5.30) to get (5.29):

$$\begin{aligned}&\|a_L(.,\xi)\|_{C_*^{\tilde{m},s}(\mathbb{R}^n)} = \|a_L(.,\xi)\|_{C^{\tilde{m},s}(\mathbb{R}^n)} \\ &\quad = \max_{|\delta|\leq\tilde{m}}\|\partial_x^\delta a_L(.,\xi)\|_{L^\infty} + \max_{|\delta|\leq\tilde{m}}\max_{\substack{x_1,x_2\in\mathbb{R}^n\\x_1\neq x_2}}\left\{\frac{|\partial_x^\delta a_L(x_1,\xi) - \partial_x^\delta a_L(x_2,\xi)|}{|x_1-x_2|^s}\right\} \leq C\langle\xi\rangle^m.\end{aligned}$$

Here C is independent of $\xi \in \mathbb{R}^n$ and $a \in \tilde{\mathscr{B}}$. ☐

Now we have developed a single symbol out of a non-smooth double symbol. Thus it remains to check whether the pseudodifferential operators of these two symbols are the same. In order to prove this statement we need the next technical result:

Proposition 5.25. *Let $\tilde{m} \in \mathbb{N}_0$, $m \in \mathbb{R}$, $0 < s < 1$ and $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $N \geq 2l > n$. Moreover, we choose $l, l_0, l_0' \in \mathbb{N}_0$ with*

$$-2l + m < -n, \qquad -2l_0 < -n, \qquad -2l_0' + 2l < -n.$$

Assuming $0 < \varepsilon' < 1$, $a \in C^{\tilde{m},s} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ and $u \in \mathcal{S}(\mathbb{R}^n)$ we define for every $0 < \varepsilon < 1$ the functions $a_0, \hat{a}, a_\varepsilon, \tilde{a}_0 : \mathbb{R}^{5n} \to \mathbb{C}$ by

$$\begin{aligned} a_0(x,x',x'',\xi,\xi') &:= \chi(\varepsilon' x'', \varepsilon' \xi') a(x,\xi,x') u(x'') \\ \hat{a}(x,x',x'',\xi,\xi') &:= \langle x' - x'' \rangle^{-2l_0} \langle D_{\xi'} \rangle^{2l_0} \left[\langle \xi' \rangle^{-2l_0'} \langle D_{x''} \rangle^{2l_0'} a_0(x,x',x'',\xi,\xi') \right], \\ a_\varepsilon(x,x',x'',\xi,\xi') &:= \chi(\varepsilon x', \varepsilon \xi) \hat{a}(x,x',x'',\xi,\xi'), \\ \tilde{a}_0(x,x',x'',\xi,\xi') &:= \langle -\xi + \xi' \rangle^{-2l} \langle D_{x'} \rangle^{2l} \hat{a}(x,x',x'',\xi,\xi'). \end{aligned}$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. Then

$$\begin{aligned} &\lim_{\varepsilon \to 0} \iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} a_\varepsilon(x,x',x'',\xi,\xi') dx'' đ\xi' dx' đ\xi \\ &\quad = \iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} \tilde{a}_0(x,x',x'',\xi,\xi') dx'' đ\xi' dx' đ\xi. \end{aligned}$$

Proof: First of all we define for each $0 < \varepsilon < 1$ the function $\tilde{a}_\varepsilon : \mathbb{R}^{5n} \to \mathbb{C}$ by

$$\tilde{a}_\varepsilon(x,x',x'',\xi,\xi') := \langle -\xi + \xi' \rangle^{-2l} \langle D_{x'} \rangle^{2l} a_\varepsilon(x,x',x'',\xi,\xi')$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. In the same way as inequality (5.13), one can show for a fixed $x \in \mathbb{R}^n$ and for arbitrary $M \in \mathbb{N}_0$:

$$\begin{aligned} |a_\varepsilon(x,x',x'',\xi,\xi')| &\leq C|\chi(\varepsilon x', \varepsilon\xi)| \langle x' - x'' \rangle^{-2l_0} \langle \xi' \rangle^{-2l_0'} \langle \xi \rangle^m \langle x'' \rangle^{-M}, && (5.32) \\ |\tilde{a}_\varepsilon(x,x',x'',\xi,\xi')| &\leq C \langle -\xi + \xi' \rangle^{-2l} \langle x' - x'' \rangle^{-2l_0} \langle \xi' \rangle^{-2l_0'} \langle \xi \rangle^m \langle x'' \rangle^{-M} \\ &\leq C \langle x' \rangle^{-2l_0} \langle x'' \rangle^{2l_0 - M} \langle \xi' \rangle^{-2l_0' + 2l} \langle \xi \rangle^{-2l+m}, && (5.33) \\ |\tilde{a}_0(x,x',x'',\xi,\xi')| &\leq C \langle -\xi + \xi' \rangle^{-2l} \langle x' - x'' \rangle^{-2l_0} \langle \xi' \rangle^{-2l_0'} \langle \xi \rangle^m \langle x'' \rangle^{-M} \\ &\leq C \langle x' \rangle^{-2l_0} \langle x'' \rangle^{2l_0 - M} \langle \xi' \rangle^{-2l_0' + 2l} \langle \xi \rangle^{-2l+m}, && (5.34) \end{aligned}$$

where C is independent of $x, x', x'', \xi, \xi' \in \mathbb{R}^n$ and of $0 < \varepsilon < 1$. The last estimates of the inequalities (5.33) and (5.34) hold because of Corollary 2.5. Since $\chi(\varepsilon x', \varepsilon\xi) \in \mathcal{S}(\mathbb{R}^n_{x'} \times \mathbb{R}^n_\xi)$ for every fixed $0 < \varepsilon < 1$, the definition of the Schwartz space arises

$$|\chi(\varepsilon x', \varepsilon\xi)| \leq C_\varepsilon \langle (x', \xi) \rangle^{-M_1 - M_2} \leq C_\varepsilon \langle x' \rangle^{-M_1} \langle \xi \rangle^{-M_2}, \qquad (5.35)$$

where $M_1, M_2 \in \mathbb{N}$ are such that $-2l_0 - M_1 < -n$ and $m - M_2 < -n$. Here the constant C_ε is independent of $x', \xi \in \mathbb{R}^n$. If we insert (5.35) into (5.32), an application of Petree's inequality yields

$$|a_\varepsilon(x, x', x'', \xi, \xi')| \leq C_\varepsilon \langle x' \rangle^{-2l_0 - M_1} \langle \xi \rangle^{m - M_2} \langle x'' \rangle^{2l_0 - M} \langle \xi' \rangle^{-2l_0'}.$$

Choosing $M \in \mathbb{N}$ with the property $2l_0 - M < -n$, Theorem 2.11 provides $a_\varepsilon(x, x', x'', \xi, \xi') \in L^1(\mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'} \times \mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi})$. Hence we are able to use Fubini's theorem and get

$$\begin{aligned} &\iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} a_\varepsilon(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi \\ &\quad = \iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} a_\varepsilon(x, x', x'', \xi, \xi') dx' đ\xi dx'' đ\xi'. \end{aligned} \tag{5.36}$$

We remember that $a_\varepsilon(x, x', x'', \xi, \xi') \in L^1(\mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi})$ for each fixed $x, x'', \xi' \in \mathbb{R}^n$ and $0 < \varepsilon < 1$. In the same way we can prove $\tilde{a}_\varepsilon(x, x', x'', \xi, \xi') \in L^1(\mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi})$ for every fixed $x, x'', \xi' \in \mathbb{R}^n$ and $0 < \varepsilon < 1$. Hence we are able to integrate by parts with respect to x' and ξ. Using Remark 2.10 integration by parts in (5.36) provides

$$\begin{aligned} &\iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} a_\varepsilon(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi \\ &\quad = \iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} \tilde{a}_\varepsilon(x, x', x'', \xi, \xi') dx' đ\xi dx'' đ\xi'. \end{aligned} \tag{5.37}$$

The next step is to apply Fubini's theorem to the last equation. Therefore we have to check the assumptions of this theorem. For every $x \in \mathbb{R}^n$ we get by (5.33) and Theorem 2.11, that

$$\begin{aligned} &|e^{-ix''\cdot\xi' - ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} \tilde{a}_\varepsilon(x, x', x'', \xi, \xi')| \\ &\qquad \leq C_x \langle x' \rangle^{-2l_0} \langle x'' \rangle^{2l_0 - M} \langle \xi' \rangle^{-2l_0' + 2l} \langle \xi \rangle^{-2l + m} \\ &\qquad \in L^1(\mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi} \times \mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'}). \end{aligned} \tag{5.38}$$

Here C_x is independent of $x', x'', \xi, \xi' \in \mathbb{R}^n$ and of $0 < \varepsilon < 1$. Consequently we can use Fubini's theorem in (5.37) and obtain:

$$\begin{aligned} &\iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} a_\varepsilon(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi \\ &\quad = \iiiint e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} \tilde{a}_\varepsilon(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi. \end{aligned} \tag{5.39}$$

Thus it remains to calculate the limit of this function, if $\varepsilon \to 0$. Since (5.34) holds, Theorem 2.11 provides that $e^{-ix''\cdot\xi'} e^{-ix'\cdot\xi + ix'\cdot\xi' + ix\cdot\xi} \tilde{a}_0(x, x', x'', \xi, \xi')$ is an element of $L^1(\mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi} \times \mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'})$. Using the definition of $\langle D_{x'} \rangle^{2l}$ and the

Leibniz rule, one easily obtains the following pointwise convergence by Lemma 2.26:

$$e^{-ix''\cdot\xi'}e^{-ix'\cdot\xi+ix'\cdot\xi'+ix\cdot\xi}\tilde{a}_\varepsilon(x,x',x'',\xi,\xi') \to e^{-ix''\cdot\xi'}e^{-ix'\cdot\xi+ix'\cdot\xi'+ix\cdot\xi}\tilde{a}_0(x,x',x'',\xi,\xi')$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$ if $\varepsilon \to 0$. Together with (5.38), we have checked the assumptions of Lebesgue's theorem. Finally, we conclude the proposition by applying this theorem to (5.39). □

A combination of all results of this section enables us to show the symbol reduction of non-smooth double symbols:

Theorem 5.26. *Let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with the property $N \geq 2l > n$. Moreover, we define $\tilde{N} := N - (n+2)$ if n is even and $\tilde{N} := N-(n+1)$ else. Assuming an $a \in C^{\tilde{m},s}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ with $\tilde{m} \in \mathbb{N}_0$, $m \in \mathbb{R}$ and $0 < s < 1$, we define $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta}a(x,\eta+\xi,x+y)dy\,đ\eta \in C^{\tilde{m},s}S^m_{0,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; \tilde{N})$$

for all $x, \xi \in \mathbb{R}^n$. Then we have for every $u \in \mathcal{S}(\mathbb{R}^n)$

$$a(x, D_x, x')u = a_L(x, D_x)u.$$

Proof: First of all note, that we already have proved $a_L \in C^{\tilde{m},s}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$ in Theorem 5.24. Now we choose $u \in \mathcal{S}(\mathbb{R}^n)$ and $l, l_0, l_0' \in \mathbb{N}_0^n$ with the property

$$-2l + m < -n, \qquad -2l_0 < -n, \qquad -2l_0' + 2l < -n. \tag{5.40}$$

Example 4.37 provides $a_L(x,\xi')u(x'') \in \mathscr{A}^{m,\tilde{N}}_{-k}(\mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'})$ for all $k \in \mathbb{N}_0$. Hence we know due to Theorem 4.38 that $a_L(x, D_x)u$ exists. Because of Remark 5.18 the function $a(x,,\eta+\xi',x+y)$ is an element of $\mathscr{A}^{m,N}_0(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$ for every fixed $x, \xi' \in \mathbb{R}^n$. Therefore we can apply Theorem 4.43 and get

$$\begin{aligned} a_L(x,D_x)u(x) &= \mathrm{Os}\text{-}\iint e^{i(x-x'')\cdot\xi'}a_L(x,\xi')u(x'')dx''đ\xi' \\ &= \mathrm{Os}\text{-}\iint e^{i(x-x'')\cdot\xi'}\mathrm{Os}\text{-}\iint e^{-iy\cdot\eta}a(x,\eta+\xi',x+y)u(x'')dy\,đ\eta\, dx''đ\xi' \\ &= \mathrm{Os}\text{-}\iint e^{i(x-x'')\cdot\xi'}\mathrm{Os}\text{-}\iint e^{-i(x'-x)\cdot(\xi-\xi')}a(x,\xi,x')u(x'')dx'đ\xi dx''đ\xi'. \end{aligned} \tag{5.41}$$

Now we choose an arbitrary $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. On account of the definition of the oscillatory integral and equality (5.41) we have:

$$a_L(x,D_x)u(x) = \lim_{\varepsilon'\to 0}\iint e^{i(x-x'')\cdot\xi'}\chi(\varepsilon'x'',\varepsilon'\xi')$$

$$\lim_{\varepsilon\to 0}\iint e^{-i(x'-x)\cdot(\xi-\xi')}a(x,\xi,x')\chi(\varepsilon x',\varepsilon\xi)u(x'')dx'đ\xi dx''đ\xi'. \tag{5.42}$$

Since $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, we are able to integrate by parts. Thus we obtain for arbitrary $0<\varepsilon<1$ and $k,k'\in\mathbb{N}_0$ with $-N\leq -2k<-n$ and $-2k'+m<-n$:

$$\begin{aligned}\iint & e^{-i(x'-x)\cdot(\xi-\xi')}a(x,\xi,x')\chi(\varepsilon x',\varepsilon\xi)u(x'')dx'đ\xi\\ &=\iint e^{-i(x'-x)\cdot(\xi-\xi')}b_\varepsilon(x,\xi,x',x'')dx'đ\xi,\end{aligned} \tag{5.43}$$

where the function $b_\varepsilon:\mathbb{R}^{5n}\to\mathbb{C}$ is defined by

$$b_\varepsilon(x,\xi,x',\xi',x''):=\langle x'-x\rangle^{-2k}\langle D_\xi\rangle^{2k}\Big[\langle\xi-\xi'\rangle^{-2k'}\langle D_{x'}\rangle^{2k'}\chi(\varepsilon x',\varepsilon\xi)a(x,\xi,x')u(x'')\Big]$$

for all $x,\xi,x',\xi',x''\in\mathbb{R}^n$ and each $0\leq\varepsilon<1$. We choose $M_1,M_2\in\mathbb{N}$ with $-M_2<-2n$ and $-M_1+M_2<-n$. Using Leibniz's rule, Petree's inequality and Theorem 2.11 we are able to estimate the absolute value of b_ε for arbitrary but fixed $x,\xi',x''\in\mathbb{R}^n$ in the same way as in (5.13):

$$\begin{aligned}|b_\varepsilon(x,\xi,x',\xi',x'')| &\leq C\langle x'-x\rangle^{-2k}\langle\xi-\xi'\rangle^{-2k'}\langle\xi\rangle^m\langle x''\rangle^{-M_1}\\ &\leq C_x\langle x'\rangle^{-2k}\langle\xi'\rangle^{2k'}\langle\xi\rangle^{m-2k'}\langle x''\rangle^{-M_1}\in L^1(\mathbb{R}^n_{x'}\times\mathbb{R}^n_\xi).\end{aligned} \tag{5.44}$$

Here the constant C_x is independent of $0<\varepsilon<1$ and of $x',\xi,\xi',x''\in\mathbb{R}^n$. Since $b_\varepsilon(x,\xi,x',\xi',x'')$ converges to $b_0(x,\xi,x',\xi',x'')$ if $\varepsilon\to 0$ and (5.44) holds we are able to apply Lebesgue's theorem and get the pointwise convergence of

$$\begin{aligned}\iint & e^{-i(x'-x)\cdot(\xi-\xi')}b_\varepsilon(x,\xi,x',\xi',x'')dx'đ\xi\\ &\xrightarrow{\varepsilon\to 0}\iint e^{-i(x'-x)\cdot(\xi-\xi')}b_0(x,\xi,x',\xi',x'')dx'đ\xi.\end{aligned}$$

We obtain together with (5.43) the pointwise convergence of

$$\begin{aligned}e^{i(x-x'')\cdot\xi'}&\chi(\varepsilon'x'',\varepsilon'\xi')\iint e^{-i(x'-x)\cdot(\xi-\xi')}a(x,\xi,x')\chi(\varepsilon x',\varepsilon\xi)u(x'')dx'đ\xi\\ &\xrightarrow{\varepsilon\to 0} e^{i(x-x'')\cdot\xi'}\chi(\varepsilon'x'',\varepsilon'\xi')\iint e^{-i(x'-x)\cdot(\xi-\xi')}b_0(x,\xi,x',\xi',x'')dx'đ\xi\end{aligned} \tag{5.45}$$

for every $x,x'',\xi'\in\mathbb{R}^n$ and $0<\varepsilon'<1$. We need this convergence later on. Additionally using (5.43), (5.44), $\chi\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$ and Theorem 2.11 yields for fixed but arbitrary $0<\varepsilon'<1$:

$$\left|e^{i(x-x'')\cdot\xi'}\chi(\varepsilon'x'',\varepsilon'\xi')\iint e^{-i(x'-x)\cdot(\xi-\xi')}a(x,\xi,x')\chi(\varepsilon x',\varepsilon\xi)u(x'')dx'đ\xi\right|$$

$$\leq C_{\varepsilon'} \iint \left| \langle (x'', \xi') \rangle^{-M_2-2k'} e^{-i(x'-x)\cdot(\xi-\xi')} b_\varepsilon(x, \xi, x', \xi', x'') \right| dx' đ\xi$$
$$\leq C_{\varepsilon'} \iint \langle x' \rangle^{-2k} \langle \xi \rangle^{m-2k'} \langle (x'', \xi') \rangle^{-M_2} dx' đ\xi$$
$$\leq C_{\varepsilon'} \langle (x'', \xi') \rangle^{-M_2} \in L^1(\mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi'}). \tag{5.46}$$

Due to (5.45) and (5.46) we are able to apply Lebesgue's theorem again. This implies together with (5.42) and (5.43):

$$a_L(x, D_x)u(x) = \lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \iint e^{i(x-x'')\cdot\xi'} \chi(\varepsilon' x'', \varepsilon' \xi')$$
$$\iint e^{-i(x'-x)\cdot(\xi-\xi')} a(x, \xi, x') \chi(\varepsilon x', \varepsilon \xi) u(x'') dx' đ\xi dx'' đ\xi'$$
$$= \lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \iiiint e^{i(x-x'')\cdot\xi'} \chi(\varepsilon' x'', \varepsilon' \xi') e^{-i(x'-x)\cdot(\xi-\xi')} a(x, \xi, x')$$
$$\chi(\varepsilon x', \varepsilon \xi) u(x'') dx'' đ\xi' dx' đ\xi. \tag{5.47}$$

In the last equation an application of Fubini's theorem is possible because the function $\chi(\varepsilon' x'', \varepsilon' \xi') \chi(\varepsilon x', \varepsilon \xi) \in \mathscr{S}(\mathbb{R}^n_{x'} \times \mathbb{R}^n_{\xi} \times \mathbb{R}^n_{x''} \times \mathbb{R}^n_{\xi})$ for each fixed ε and ε'. Now we define the functions $\tilde{a}_{\varepsilon'}, a_{\varepsilon'}, \hat{a}_{\varepsilon'} : \mathbb{R}^{5n} \to \mathbb{C}$ for each $0 < \varepsilon < 1$ by

$$\tilde{a}_{\varepsilon'}(x, x', x'', \xi, \xi') := \chi(\varepsilon' x'', \varepsilon' \xi') a(x, \xi, x') u(x''),$$
$$a_{\varepsilon'}(x, x', x'', \xi, \xi') := \langle x' - x'' \rangle^{-2l_0} \langle D_{\xi'} \rangle^{2l_0} \left[\langle \xi' \rangle^{-2l_0'} \langle D_{x''} \rangle^{2l_0'} \tilde{a}_{\varepsilon'}(x, x', x'', \xi, \xi') \right],$$
$$\hat{a}_{\varepsilon'}(x, x', x'', \xi, \xi') := \langle -\xi + \xi' \rangle^{-2l} \langle D_{x'} \rangle^{2l} a_{\varepsilon'}(x, x', x'', \xi, \xi')$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. Since $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ we are able to integrate by parts in (5.47) and get:

$$a_L(x, D_x)u(x)$$
$$= \lim_{\varepsilon' \to 0} \lim_{\varepsilon \to 0} \iiiint e^{i(x-x'')\cdot\xi' - i(x'-x)\cdot(\xi-\xi')} \chi(\varepsilon x', \varepsilon \xi) a_{\varepsilon'}(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi$$
$$= \lim_{\varepsilon' \to 0} \iiiint e^{-ix''\cdot\xi' - i(x'-x)\cdot\xi + ix'\cdot\xi'} \hat{a}_{\varepsilon'}(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi. \tag{5.48}$$

Here the last equality holds due to Proposition 5.25. We define the function $\hat{a} : \mathbb{R}^{5n} \to \mathbb{C}$ by

$$\hat{a}(x, x', x'', \xi, \xi') := \langle -\xi + \xi' \rangle^{-2l} \langle D_{x'} \rangle^{2l} a_0(x, x', x'', \xi, \xi'),$$
$$a_0(x, x', x'', \xi, \xi') := \langle x' - x'' \rangle^{-2l_0} \langle D_{\xi'} \rangle^{2l_0} \left[\langle \xi' \rangle^{-2l_0'} \langle D_{x''} \rangle^{2l_0'} a(x, \xi, x') u(x'') \right]$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. Then we have the pointwise convergence of

$$\hat{a}_{\varepsilon'}(x, x', x'', \xi, \xi') \xrightarrow{\varepsilon' \to 0} \hat{a}(x, x', x'', \xi, \xi') \tag{5.49}$$

for all $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. Similarly to (5.44) we get due to Leibniz's rule and Petree's inequality:

$$|\hat{a}_{\varepsilon'}(x, x', x'', \xi, \xi')| \leq C\langle x'\rangle^{-2l_0}\langle x''\rangle^{2l_0-M}\langle\xi\rangle^{m-2l}\langle\xi'\rangle^{2l-2l'_0}.$$

Here the constant C is independent of $0 < \varepsilon' < 1$ and $x, x', x'', \xi, \xi' \in \mathbb{R}^n$. On account of Theorem 2.11 we obtain the existence of a L^1-majorant of the set $\{\hat{a}_{\varepsilon'} : \varepsilon' \in (0,1)\}$ with respect to (x', ξ, x'', ξ'). Since we have already checked (5.49) the assumptions of Lebesgue's theorem hold. An application of Lebesgue's theorem to (5.48) provides:

$$\begin{aligned} a_L(x, D_x)u(x) &= \lim_{\varepsilon' \to 0} \iiiint e^{-ix''\cdot\xi' - i(x'-x)\cdot\xi + ix'\cdot\xi'} \hat{a}_{\varepsilon'}(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi \\ &= \iiiint e^{-ix''\cdot\xi' - i(x'-x)\cdot\xi + ix'\cdot\xi'} \hat{a}(x, x', x'', \xi, \xi') dx'' đ\xi' dx' đ\xi. \end{aligned}$$

Hence we get the claim by using Lemma 5.13. □

With all the work done in this section we gained an important technique dealing with non-smooth pseudodifferential operators. Comparing our result with the smooth case, one looses some smoothness in ξ of the order n if $N \neq \infty$.

5.3 Properties of the Operator T_ε

Beyond the pointwise convergence in $C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ of a subsequence of a bounded sequence in $C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and the symbol reduction of non-smooth double symbols to non-smooth single symbols, a third tool is necessary in order to prove the characterization of non-smooth pseudodifferential operators. For a given operator $T \in \mathcal{A}^{0,M}_{0,0}(\tilde{m}, q)$ we want to construct a sequence of operators $(T_\varepsilon)_{\varepsilon \in (0,1]}$ such that the following properties hold for all $\varepsilon \in (0,1]$:

- $T_\varepsilon : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous,
- The iterated commutators of T_ε are uniformly bounded with respect to ε as maps from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$,
- T_ε converges pointwise to T if $\varepsilon \to 0$.

We start with some general assumptions for the whole section: Let $1 < q < \infty$ and $M \in \mathbb{N}_0 \cup \{\infty\}$ be arbitrary. Moreover, let $T \in \mathcal{A}^{0,M}_{0,0}(\tilde{m}, q)$ with $\tilde{m} \in \mathbb{N}_0$. We choose a function $\varphi \in C^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ for all $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ for all $|x| \geq 1$. Then we define for $0 < \varepsilon \leq 1$ the pseudodifferential operators

$$P_\varepsilon := \tilde{p}_\varepsilon(x, D_x) \qquad \text{and} \qquad Q_\varepsilon := q_\varepsilon(x, D_x),$$

where the symbols $\tilde{p}_\varepsilon$ and q_ε are defined as $\tilde{p}_\varepsilon(x,\xi) := \varphi(\varepsilon x)$ and $q_\varepsilon(x,\xi) := \varphi(\varepsilon\xi)$. Note that for arbitrary $u \in \mathcal{S}(\mathbb{R}^n)$ we have $P_\varepsilon u = \tilde{p}_\varepsilon u$ since

$$P_\varepsilon u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\tilde{p}_\varepsilon(x)\hat{u}(\xi)đ\xi = \tilde{p}_\varepsilon(x)\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)đ\xi = \tilde{p}_\varepsilon(x)u(x)$$

for each $x \in \mathbb{R}^n$. Additionally the continuity of multiplication operators with C_c^∞-functions imply the continuity of the operator $P_\varepsilon : C^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$. Moreover, we define the linear operator T_ε by

$$T_\varepsilon := P_\varepsilon Q_\varepsilon T P_\varepsilon Q_\varepsilon.$$

Now we cover some properties of this operator: To begin with, we verify the pointwise L^q-convergence of the operators T_ε if $\varepsilon \to 0$. We need this statement for the characterization of non-smooth pseudodifferential operators.

Lemma 5.27. *For all $u \in L^q(\mathbb{R}^n)$ we have the following convergence:*

$$L^q - \lim_{\varepsilon\to 0} T_\varepsilon u = Tu.$$

Proof: The first step in order to prove the claim is to show the following convergence for each $u \in L^q(\mathbb{R}^n)$:

$$Q_\varepsilon u \xrightarrow{\varepsilon\to 0} u \qquad \text{and} \qquad P_\varepsilon u \xrightarrow{\varepsilon\to 0} u \qquad \text{in } L^q(\mathbb{R}^n). \tag{5.50}$$

Due to the theorem of Banach-Steinhaus, cf. e.g. [39], Theorem 8.6, it remains to show that

i) $\{Q_\varepsilon : 0 < \varepsilon \leq 1\}$ and $\{P_\varepsilon : 0 < \varepsilon \leq 1\}$ are bounded subsets of $\mathscr{L}(L^q(\mathbb{R}^n))$,

ii) $\|Q_\varepsilon u - Qu\|_{L^q(\mathbb{R}^n)} \xrightarrow{\varepsilon\to 0} 0$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

iii) $\|P_\varepsilon u - Pu\|_{L^q(\mathbb{R}^n)} \xrightarrow{\varepsilon\to 0} 0$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

We start with verifying claim *i)*. Lemma 3.4 implies the boundedness of the subsets $\{p_\varepsilon | 0 < \varepsilon \leq 1\}$ and $\{q_\varepsilon | 0 < \varepsilon \leq 1\}$ of $S^0_{1,0}$. Therefore there are $k, k' \in \mathbb{N}_0$ and a constant C_q, independent of $\varepsilon \in (0,1]$, such that

$$\|Q_\varepsilon u\|_{L^q(\mathbb{R}^n)} \leq C_q |q_\varepsilon|_k^{(0)} \|u\|_{L^q(\mathbb{R}^n)} \leq C_q \|u\|_{L^q(\mathbb{R}^n)}, \tag{5.51}$$

$$\|P_\varepsilon u\|_{L^q(\mathbb{R}^n)} \leq C_q |\tilde{p}_\varepsilon|_{k'}^{(0)} \|u\|_{L^q(\mathbb{R}^n)} \leq C_q \|u\|_{L^q(\mathbb{R}^n)} \tag{5.52}$$

for all $u \in L^q(\mathbb{R}^n)$ if we use Theorem 3.18.

In order to prove claim *ii)*, we choose an fixed but arbitrary $u \in \mathcal{S}(\mathbb{R}^n)$. Since $\{q_\varepsilon | 0 < \varepsilon \leq 1\}$ is a bounded subset of $S^0_{1,0}$, we derive from Lemma 4.50 for each $N \in \mathbb{N}$ with $-2Nq < -n$ the existence of a constant $C_{N,n}$, independent of $x \in \mathbb{R}^n$ and $\varepsilon \in (0,1]$, such that

$$|q_\varepsilon(D_x)u(x)| \leq C_{N,n}\langle x\rangle^{-2N} \qquad \text{for all } x \in \mathbb{R}^n.$$

Using $u \in \mathcal{S}(\mathbb{R}^n)$ first and Theorem 2.11 afterwards, we get by means of the previous inequality:

$$|q_\varepsilon(D_x)u(x) - u(x)|^q \leq (|q_\varepsilon(D_x)u(x)| + |u(x)|)^q \leq C_{N,n}\langle x\rangle^{-2Nq} \in L^1(\mathbb{R}^n_x).$$

Here $C_{N,n}$ is independent of $\varepsilon \in (0,1]$. We define for each $\varepsilon \in (0,1]$ the function $\psi_\varepsilon : \mathbb{R}^n \to \mathbb{C}$ by $\psi_\varepsilon(x) := \varepsilon^{-n}\mathscr{F}^{-1}[\varphi](x/\varepsilon)$ for all $x \in \mathbb{R}^n$. Then the properties of the inverse Fourier transformation, cf. Theorem 2.2, provide $\mathscr{F}^{-1}[\varphi(\varepsilon\xi)] = \psi_\varepsilon$. In view of Theorem 2.2 we have

$$Q_\varepsilon u = \mathscr{F}^{-1}[\varphi(\varepsilon\xi)\hat{u}(\xi)] = \mathscr{F}^{-1}[\varphi(\varepsilon\xi)] * u = \psi_\varepsilon * u.$$

On account of Example B.6 and Lemma B.5 we obtain to pointwise convergence of $Q_\varepsilon u = q_\varepsilon(D_x)u \to u$ if $\varepsilon \to 0$. So we have checked all assumptions of Lebesgue's theorem, which provides claim *ii*):

$$\|Q_\varepsilon u - u\|_{L^q(\mathbb{R}^n)} = \left[\int |q_\varepsilon(D_x)u(x) - u(x)|^q dx\right]^{1/q} \xrightarrow{\varepsilon\to 0} 0.$$

Moreover, an application of Lebesgue's theorem yields *iii*):

$$\|P_\varepsilon u - u\|_{L^q(\mathbb{R}^n)} \xrightarrow{\varepsilon\to 0} 0,$$

if one uses $P_\varepsilon u = \tilde{p}_\varepsilon u$ and $\tilde{p}_\varepsilon u \to u$ if ε converges to 0. By means of (5.50) and (5.52) we get for all $u \in L^q(\mathbb{R}^n)$:

$$\begin{aligned}\|P_\varepsilon Q_\varepsilon u - u\|_{L^q(\mathbb{R}^n)} &\leq \|P_\varepsilon Q_\varepsilon u - P_\varepsilon u\|_{L^q(\mathbb{R}^n)} + \|P_\varepsilon u - u\|_{L^q(\mathbb{R}^n)}\\ &\leq C\|Q_\varepsilon u - u\|_{L^q(\mathbb{R}^n)} + \|P_\varepsilon u - u\|_{L^q(\mathbb{R}^n)} \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

Consequently we obtain together with (5.51) and (5.52) for all $u \in L^q(\mathbb{R}^n)$:

$$\begin{aligned}\|T_\varepsilon u - Tu\|_{L^q(\mathbb{R}^n)} &= \|P_\varepsilon Q_\varepsilon T P_\varepsilon Q_\varepsilon u - Tu\|_{L^q(\mathbb{R}^n)}\\ &\leq \|P_\varepsilon Q_\varepsilon T P_\varepsilon Q_\varepsilon u - P_\varepsilon Q_\varepsilon Tu\|_{L^q(\mathbb{R}^n)} + \|P_\varepsilon Q_\varepsilon Tu - Tu\|_{L^q(\mathbb{R}^n)}\\ &\leq C\|T P_\varepsilon Q_\varepsilon u - Tu\|_{L^q(\mathbb{R}^n)} + \|P_\varepsilon Q_\varepsilon Tu - Tu\|_{L^q(\mathbb{R}^n)}\\ &\leq C\|P_\varepsilon Q_\varepsilon u - u\|_{L^q(\mathbb{R}^n)} + \|P_\varepsilon Q_\varepsilon Tu - Tu\|_{L^q(\mathbb{R}^n)} \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

□

Another important property of the operator T_ε is its continuity as a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$:

Lemma 5.28. *For each* $0 < \varepsilon \leq 1$*, the operator* $T_\varepsilon : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is continuous.*

Proof: At the beginning of this section we verified the continuity of the operator $P_\varepsilon : C^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$. Due to Lemma 3.25, $T \in \mathcal{A}_{0,0}^{0,M}(\tilde{m}, q)$, Corollary 2.33 and Corollary 2.34 we obtain the continuity of $Q_\varepsilon : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, $T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $\mathrm{Id}_1 : C_c^\infty(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathrm{Id}_2 : \mathcal{S}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ and $\mathrm{Id}_3 : L^q(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Combining these results, we get the claim because of

$$T_\varepsilon = \mathrm{Id}_1 P_\varepsilon Q_\varepsilon \mathrm{Id}_3 T \mathrm{Id}_2 \mathrm{Id}_1 P_\varepsilon Q_\varepsilon.$$

□

Moreover, T_ε is also a linear bounded operator from $L^q(\mathbb{R}^n)$ to $C_b^{k+1}(\mathbb{R}^n)$:

Lemma 5.29. $T_\varepsilon \in \mathscr{L}(L^q(\mathbb{R}^n), C_b^{k+1}(\mathbb{R}^n))$ *for every* $0 < \varepsilon \leq 1$ *and* $k \in \mathbb{N}_0$.

Proof: Let $k \in \mathbb{N}_0$ and $0 < \varepsilon \leq 1$ be arbitrary. Due to Corollary 2.34 and Section 2.3 we have the continuity of the embedding $\mathrm{Id}_2 : L^q(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{Id}_1 : \mathcal{S}(\mathbb{R}^n) \to C_b^{k+1}(\mathbb{R}^n)$. Combining these results and Lemma 5.28 provides the continuity of the map

$$T_\varepsilon = \mathrm{Id}_1 T_\varepsilon \mathrm{Id}_2 : L^q(\mathbb{R}^n) \to C_b^{k+1}(\mathbb{R}^n).$$

□

Furthermore, one can prove the following uniform continuity in ε:

Lemma 5.30. *Let* $\alpha, \beta \in \mathbb{N}_0^n$ *with* $|\beta| \leq \tilde{m}$ *and* $|\alpha| \leq M$*. Then there is a constant* $C_{\alpha,\beta}$*, independent of* $0 < \varepsilon \leq 1$*, such that*

$$\| \mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta T_\varepsilon \|_{\mathscr{L}(L^q(\mathbb{R}^n))} \leq C_{\alpha,\beta}.$$

Proof: Let $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \leq \tilde{m}$ and $|\alpha| \leq M$ be arbitrary. $T \in \mathcal{A}_{0,0}^{0,M}(\tilde{m}, q)$ implies that the operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is linear. Since all pseudodifferential operators are linear as maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and as maps from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ by definition, the operator $T_\varepsilon : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is linear for all $\varepsilon \in (0, 1]$. Hence the assumptions of Proposition 2.55 hold. Now we define $R^{\beta_1,\beta_2,\beta_3}_{\alpha_1,\alpha_2,\alpha_3}$ by

$$R^{\beta_1,\beta_2,\beta_3}_{\alpha_1,\alpha_2,\alpha_3} := \left[\mathrm{ad}(D_x)^{\beta_1} P_\varepsilon\right] \left[\mathrm{ad}(-ix)^{\alpha_1} Q_\varepsilon\right] T^{\alpha_2,\beta_2} \left[\mathrm{ad}(D_x)^{\beta_3} P_\varepsilon\right] \left[\mathrm{ad}(-ix)^{\alpha_3} Q_\varepsilon\right],$$

where $T^{\alpha_2,\beta_2} := \mathrm{ad}(-ix)^{\alpha_2} \,\mathrm{ad}(D_x)^{\beta_2} T$. By means of Proposition 2.55 and of Remark 3.8 there are constants $C_{\alpha_1,\alpha_2,\beta_1,\beta_2}$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta T_\varepsilon u = \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha \\ \beta_1+\beta_2+\beta_3=\beta}} C_{\alpha_1,\alpha_2,\beta_1,\beta_2} R^{\beta_1,\beta_2,\beta_3}_{\alpha_1,\alpha_2,\alpha_3} u.$$

Additionally we get $\mathrm{ad}(D_x)^\gamma P_\varepsilon \in \mathrm{OP}S^0_{1,0}$ and $\mathrm{ad}(-ix)^\delta Q_\varepsilon \in \mathrm{OP}S^{-|\delta|}_{1,0} \subseteq \mathrm{OP}S^0_{1,0}$ for each $\gamma, \delta \in \mathbb{N}_0^n$ due to Remark 3.7. On account of Theorem 3.18 we obtain constants $C_{q,k,\gamma}$, $C_{q,l,\delta}$ and $k, l \in \mathbb{N}_0$, independent of $0 < \varepsilon \leq 1$ and $u \in L^q(\mathbb{R}^n)$, such that

$$\| \operatorname{ad}(D_x)^\gamma P_\varepsilon u\|_{L^q} \leq C_q |D_x^\gamma p_\varepsilon|_k^{(0)} \|u\|_{L^q} \leq C_q |p_\varepsilon|_{k+|\gamma|}^{(0)} \|u\|_{L^q} \leq C_{q,k,\gamma} \|u\|_{L^q}, \quad (5.53)$$

$$\| \operatorname{ad}(-ix)^\delta Q_\varepsilon u\|_{L^q} \leq C_q |\partial_\xi^\delta q_\varepsilon|_l^{(0)} \|u\|_{L^q} \leq C_q |q_\varepsilon|_{l+|\delta|}^{(0)} \|u\|_{L^q} \leq C_{q,l,\delta} \|u\|_{L^q}, \quad (5.54)$$

for every $u \in L^q(\mathbb{R}^n)$. Here the last inequality holds because of the boundedness of $\{p_\varepsilon : 0 < \varepsilon \leq 1\}$ and $\{q_\varepsilon : 0 < \varepsilon \leq 1\}$ in $S^0_{1,0}$, which we have checked in Lemma 3.4. Since $T \in \mathcal{A}^{0,M}_{0,0}(\tilde{m}, q)$, we also have a constant C, independent of $u \in L^q(\mathbb{R}^n)$, such that for each $\alpha_2, \beta_2 \in \mathbb{N}_0^n$ with $|\beta_2| \leq \tilde{m}$ and $|\alpha_2| \leq M$ the following inequality is fulfilled:

$$\| \operatorname{ad}(-ix)^{\alpha_2} \operatorname{ad}(D_x)^{\beta_2} T u\|_{L^q} \leq C \|u\|_{L^q} \qquad \text{for all } u \in L^q(\mathbb{R}^n). \quad (5.55)$$

Combining (5.53), (5.54) and (5.55) provides the existence of a constant C, independent of $0 < \varepsilon \leq 1$ and $u \in L^q(\mathbb{R}^n)$, such that $\|R^{\beta_1,\beta_2,\beta_3}_{\alpha_1,\alpha_2,\alpha_3} u\|_{L^q} \leq C\|u\|_{L^q}$ for arbitrary $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0^n$, $\beta_1, \beta_2, \beta_3 \in \mathbb{N}_0^n$ with $|\beta_2| \leq \tilde{m}$ and $|\alpha_2| \leq M$ and $u \in L^q(\mathbb{R}^n)$. In particular there is a constant $C_{\alpha,\beta,q}$, which is independent of $0 < \varepsilon \leq 1$ and $u \in \mathcal{S}(\mathbb{R}^n)$, with the property

$$\| \operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta T_\varepsilon u\|_{L^q} \leq C_{\alpha,\beta,q} \|u\|_{L^q} \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

At last it remains to apply Remark 2.56 in order to obtain the lemma. □

One statement we also will need for the characterization of the non-smooth pseudodifferential operators whose coefficients are in a Hölder space $C^s(\mathbb{R}^n)$ is formulated in the next proposition:

Proposition 5.31. *Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $0 < \varepsilon \leq 1$. For each $y \in \mathbb{R}^n$ we define $g_y := \tau_y(g)$. Moreover, we define the function $p_{\varepsilon,0} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ as*

$$p_{\varepsilon,0}(x, \xi, y) := e^{-ix\cdot\xi} T_\varepsilon(e_\xi g_y)(x) \qquad \textit{for all } (x, \xi, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n.$$

Then $p_{\varepsilon,0} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

We develop some further tools needed in the proof of the proposition, first. We start with the definition of the following normed space:

Definition 5.32. For $k \in \mathbb{N}_0$ we define the normed space $L^q_k(\mathbb{R}^n)$ as

$$L^q_k(\mathbb{R}^n) := \left\{ f \in L^q(\mathbb{R}^n) : \|f\|_{L^q_k} := \|\langle x \rangle^{k+1} f(x)\|_{L^q(\mathbb{R}^n_x)} < \infty \right\}.$$

A first technical result for the space $L^q_k(\mathbb{R}^n)$ is shown in the next proposition:

Proposition 5.33. *Let $k \in \mathbb{N}_0$ be arbitrary. For every $\xi \in \mathbb{R}^n$ we define the function $M_\xi : L^q_k(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ by $M_\xi(f) := e_\xi f$ for all $f \in L^q_k(\mathbb{R}^n)$. Moreover, we define $M : \mathbb{R}^n \to \mathscr{L}(L^q_k(\mathbb{R}^n), L^q(\mathbb{R}^n))$ by*

$$M(\xi) := M_\xi \qquad \textit{for each } \xi \in \mathbb{R}^n.$$

Then $M \in C^k(\mathbb{R}^n, \mathscr{L}(L^q_k(\mathbb{R}^n), L^q(\mathbb{R}^n)))$ and

$$\left[\partial^\alpha_\xi M_\xi(f)\right](x) = \partial^\alpha_\xi e^{ix\cdot\xi} f(x) = (ix)^\alpha e^{ix\cdot\xi} f(x)$$

for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le k$, $f \in L^q_k(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Proof: First of all note, that Lemma 2.12 yields for each $x, \xi, \eta \in \mathbb{R}^n$:

$$\begin{aligned}
|e^{ix\cdot\xi} - e^{ix\cdot\eta}| &= \sup_{0\le t\le 1} \left| D_{\tilde\xi} \left\{ e^{ix\cdot\tilde\xi} \right\} \Big|_{\tilde\xi = t\xi + (1-t)\eta} \right| |\xi - \eta| \\
&\le \sup_{0\le t\le 1} \left| ixe^{ix\cdot[t\xi+(1-t)\eta]} \right| |\xi - \eta| \\
&\le |x||\xi - \eta| \le \langle x\rangle^{k+1} |\xi - \eta|. \qquad (5.56)
\end{aligned}$$

In order to prove the proposition by mathematical induction with respect to l, $l \le k$, we show the continuity of M, now. Therefore let $\xi \in \mathbb{R}^n$ be arbitrary. Then we get for every $f \in L^q_k(\mathbb{R}^n)$ and for all $\eta \in \mathbb{R}^n$ if we use (5.56):

$$\begin{aligned}
\|e_\xi f - e_\eta f\|^q_{L^q} &= \int_{\mathbb{R}^n} \left(|e^{ix\cdot\xi} - e^{ix\cdot\eta}||f(x)| \right)^q dx \le |\xi - \eta|^q \int_{\mathbb{R}^n} \left(\langle x\rangle^{k+1} |f(x)| \right)^q dx \\
&= |\xi - \eta|^q \|f\|^q_{L^q_k}.
\end{aligned}$$

Applying this inequality, we get the continuity of M at the point $\xi \in \mathbb{R}^n$:

$$\|M(\xi) - M(\eta)\|_{\mathscr{L}(L^q_k(\mathbb{R}^n), L^q(\mathbb{R}^n))} = \sup_{\|f\|_{L^q_k} \le 1} \|e_\xi f - e_\eta f\|_{L^q} \le |\xi - \eta| \xrightarrow{\eta \to \xi} 0.$$

It remains to prove the induction step. To this end let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| = l < k$ and $j \in \{1, \ldots, n\}$ be arbitrary. Furthermore, we choose an arbitrary $\xi \in \mathbb{R}^n$ and we define $\Phi : L^q_k(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ by

$$\Phi(f)(x) := i^{|\alpha|+1} x_j x^\alpha e^{ix\cdot\xi} f(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } f \in L^q_k(\mathbb{R}^n).$$

The Taylor series provides for $h \in \mathbb{R}$ the existence of a constant C, independent of $h \in \mathbb{R}$, such that

$$|e^{ix\cdot(\xi+he_j)} - e^{ix\cdot\xi} - ix_j e^{ix\cdot\xi} h| = \left| \int_0^h (h-t) i x_j e^{ix\cdot(\xi+te_j)} dt \right| \le \int_0^h |h-t||x_j|d|t|$$

$$\leq \int_0^h |h-t|\langle x\rangle d|t| \leq C|h|^2\langle x\rangle$$

for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}$. So we obtain for all $f \in L^q_k(\mathbb{R}^n)$:

$$\begin{aligned}
&\|\partial_\xi^\alpha M_{\xi+he_j}(f) - \partial_\xi^\alpha M_\xi(f) - h\Phi(f)\|_{L^q}^q \\
&\quad = \int |(ix)^\alpha e^{ix\cdot(\xi+he_j)} f(x) - (ix)^\alpha e^{ix\cdot\xi} f(x) - h i^{|\alpha|+1} x_j x^\alpha e^{ix\cdot\xi} f(x)|^q dx \\
&\quad \leq \int \left[|x|^{|\alpha|} |e^{ix\cdot(\xi+he_j)} - e^{ix\cdot\xi} - ix_j e^{ix\cdot\xi} h||f(x)|\right]^q dx \\
&\quad \leq C|h|^{2q} \int |\langle x\rangle^{k+1} f(x)|^q dx = C|h|^{2q} \|f\|_{L^q_k}^q,
\end{aligned}$$

where C is independent of $h \in \mathbb{R}$. Therefore we get the existence of $\partial_{\xi_j}\partial_\xi^\alpha M$ and $\partial_{\xi_j}\partial_\xi^\alpha M_\xi(f) = \Phi(f)$ at the point ξ for each $f \in L^q_k(\mathbb{R}^n)$:

$$\left\| \frac{\partial_\xi^\alpha M_{\xi+he_j} - \partial_\xi^\alpha M_\xi}{h} - \Phi \right\|_{\mathscr{L}(L^q_k(\mathbb{R}^n), L^q(\mathbb{R}^n))} \leq C|h| \xrightarrow{h\to 0} 0.$$

We only have not verified the continuity of $\partial_{\xi_j}\partial_\xi^\alpha M$, yet. Thus let $\xi \in \mathbb{R}^n$ be arbitrary. Then we conclude for every $f \in L^q_k(\mathbb{R}^n)$, if we use (5.56):

$$\begin{aligned}
&\|\partial_{\xi_j}\partial_\xi^\alpha M_\xi(f) - \partial_{\xi_j}\partial_\xi^\alpha M_\eta(f)\|_{L^q}^q = \int \left(|x_j x^\alpha f(x)||e^{ix\cdot\xi} - e^{ix\cdot\eta}|\right)^q dx \\
&\quad \leq \int \left(\langle x\rangle^{|\alpha|+2} |f(x)||\xi-\eta|\right)^q dx \leq |\xi-\eta|^q \int \left(\langle x\rangle^{k+1}|f(x)|\right)^q dx \leq |\xi-\eta|^q \|f\|_{L^q_k}^q.
\end{aligned}$$

Finally, using the previous inequality provides

$$\|\partial_{\xi_j}\partial_\xi^\alpha M(\xi) - \partial_{\xi_j}\partial_\xi^\alpha M(\eta)\|_{\mathscr{L}(L^q_k(\mathbb{R}^n), L^q(\mathbb{R}^n))} \leq |\xi-\eta| \xrightarrow{\eta\to\xi} 0,$$

which implies $M \in C^{l+1}(\mathbb{R}^n, \mathscr{L}(L^q_k(\mathbb{R}^n), L^q(\mathbb{R}^n)))$. □

Similarly one can prove the following proposition:

Proposition 5.34. *Let $k \in \mathbb{N}_0$ and $g \in \mathcal{S}(\mathbb{R}^n)$. We define $G : \mathbb{R}^n \to L^q_k(\mathbb{R}^n)$ and $\delta : \mathbb{R}^n \to \mathscr{L}(C_b^{k+1}(\mathbb{R}^n), \mathbb{C})$ by*

- *$G(y) := \tau_y(g)$ for all $y \in \mathbb{R}^n$,*
- *$(\delta(x))(f) := f(x)$ for all $x \in \mathbb{R}^n$ and $f \in C_b^{k+1}(\mathbb{R}^n)$.*

Then $G \in C^\infty(\mathbb{R}^n, L^q_k(\mathbb{R}^n))$ and $\delta \in C^k(\mathbb{R}^n, \mathscr{L}(C_b^{k+1}(\mathbb{R}^n), \mathbb{C}))$ with

$$\partial_y^\alpha G(y) = (-1)^{|\alpha|}\tau_y(\partial_x^\alpha g) \qquad \text{for all } y \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}_0^n,$$

$$[\partial_x^\alpha \delta(x)](f) = (\partial_x^\alpha f)(x) \qquad \text{for all } x \in \mathbb{R}^n, f \in C_b^{k+1}(\mathbb{R}^n), \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k.$$

Now we have checked all utilities to show the smoothness of the function $p_{\varepsilon,0}$, defined in Proposition 5.31:

Proof of Proposition 5.31: Let $k \in \mathbb{N}_0$ be arbitrary but fixed. We define the functions $\delta : \mathbb{R}^n \to \mathscr{L}(C_b^{k+1}(\mathbb{R}^n), \mathbb{C})$, $G : \mathbb{R}^n \to L_k^q(\mathbb{R}^n)$ and the function $M : \mathbb{R}^n \to \mathscr{L}(L_k^q(\mathbb{R}^n), L^q(\mathbb{R}^n))$ as in Proposition 5.34 and in Proposition 5.33. Furthermore, we define the functions $\tilde{\delta} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathscr{L}(C_b^{k+1}(\mathbb{R}^n), \mathbb{C})$, $\tilde{G} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to L_k^q(\mathbb{R}^n)$ and $\tilde{M} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathscr{L}(L_k^q(\mathbb{R}^n), L^q(\mathbb{R}^n))$ by

$$\tilde{\delta}(x, y, \xi) := \delta(x),\ \tilde{G}(x, y, \xi) := G(y),\ \tilde{M}(x, y, \xi) := M(\xi) \qquad \text{for all } x, y, \xi \in \mathbb{R}^n.$$

Because of Proposition 5.34 and of Proposition 5.33 we know, that $\tilde{G}$ is a smooth function and that $\tilde{\delta}, \tilde{M}$ are k-times continuous differentiable. Now we define the bilinear operator

$$H : \mathscr{L}(L_k^q(\mathbb{R}^n), L^q(\mathbb{R}^n)) \times L_k^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \quad \text{by} \quad H(C, f) := Cf.$$

H is bounded since

$$\|H(C, f)\|_{L^q} = \|Cf\|_{L^q} = \|C\|_{\mathscr{L}(L_k^q(\mathbb{R}^n), L^q(\mathbb{R}^n))} \|f\|_{L_k^q} \leq 1$$

for all $(C, f) \in \mathscr{L}(L_k^q(\mathbb{R}^n), L^q(\mathbb{R}^n)) \times L_k^q(\mathbb{R}^n) =: X$ with $\|(C, f)\|_X \leq 1$. Therefore the product rule of higher derivatives, cf. e.g. [78], p.193, Exercise 4.1g, yields:

$$\begin{aligned} \tilde{M}(x, y, \xi) \circ \tilde{G}(x, y, \xi) &= H(\tilde{M}(x, y, \xi), \tilde{G}(x, y, \xi)) \\ &\in C^k(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n, L^q(\mathbb{R}^n)). \end{aligned} \tag{5.57}$$

We have shown in Lemma 5.29 that $T_\varepsilon \in \mathscr{L}(L^q(\mathbb{R}^n), C_b^{k+1}(\mathbb{R}^n))$. Together with (5.57) this provides

$$T_\varepsilon(\tilde{M}(x, y, \xi) \circ \tilde{G}(x, y, \xi)) \in C^k(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n, C_b^{k+1}(\mathbb{R}^n)).$$

Analogous to (5.57), we obtain the following result by an application of the product rule of higher derivatives:

$$\tilde{\delta}(x, y, \xi) \circ T_\varepsilon(\tilde{M}(x, y, \xi) \circ \tilde{G}(x, y, \xi)) \in C^k(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n).$$

So it remains to note that $e^{-ix\cdot\xi} \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n)$ in order to obtain that

$$p_{\varepsilon,0}(x, y, \xi) = e^{-ix\cdot\xi} T_\varepsilon(e_\xi g_y)(x) = e^{-ix\cdot\xi} \tilde{\delta}(x, y, \xi) \circ T_\varepsilon(\tilde{M}(x, y, \xi) \circ \tilde{G}(x, y, \xi))$$

is an element of $C^k(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n)$. Since $k \in \mathbb{N}_0$ was chosen arbitrary, we conclude the proposition. □

5.4 Characterization of Pseudodifferential Operators with Symbols in $C^s S^m_{0,0}$

Having proved all needed auxiliary tools, we are now in the position to verify the characterization of pseudodifferential operators with symbols of the symbol-class $C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. This result is extended to non-smooth pseudodifferential operators of the class $C^s S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ of the order m. In the non-smooth case, one is confronted with the following problem: In general we do not have the continuity of non-smooth pseudodifferential operators with coefficients in a Hölder space as a map from $H^m_q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. But every element of the set $\mathcal{A}^{m,M}_{0,0}(\tilde{m}, q)$ is a linear and bounded map from $H^m_q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Hence this ansatz just provides a characterization of those non-smooth pseudodifferential operators which are linear and bounded as maps from $H^m_q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. As already mentioned, above said proof relies on the main idea of the proof in the smooth case by J. Ueberberg [74].

We now start with a technical statement needed for this proof.

Proposition 5.35. *Let $f, \chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$ and $u, g \in \mathcal{S}(\mathbb{R}^n)$ with $g(0) = 1$ and $g(-x) = g(x)$ for all $x \in \mathbb{R}^n$. For an arbitrary $y \in \mathbb{R}^n$ we define the translation function g_y by $g_y := \tau_y(g)$. Additionally the functions $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ and $r_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ are defined as*

$$r(x,z) := f(x,z) \cdot Os\text{-}\iint e^{-iy\cdot\xi} e^{iz\cdot\xi} u(y) g_y(z) dy đ\xi,$$
$$r_\alpha(x,z) := f(x,z) \cdot \iint e^{-iy\cdot\xi} e^{iz\cdot\xi} u(y) g_y(z) \chi(\alpha y, \alpha\xi) dy đ\xi$$

for all $\alpha > 0$. Then

$$\lim_{\alpha\to 0} \int r_\alpha(x,z) dz = \int r(x,z) dz \qquad \text{for all } x \in \mathbb{R}^n.$$

Proof: Let $x \in \mathbb{R}^n$ be arbitrary but fixed. Furthermore, we choose $l \in \mathbb{N}$ with $-2l < -n$. Since $g \in \mathcal{S}(\mathbb{R}^n) \subseteq C^\infty_b(\mathbb{R}^n)$, we obtain for each $\gamma_2 \in \mathbb{N}^n_0$

$$|D^{\gamma_2}_y \{g_y(z)\}| = |(\partial^{\gamma_2}_y g)(z-y)| \leq C_{\gamma_2} \qquad \text{for all } z, y \in \mathbb{R}^n, \tag{5.58}$$

where C_{γ_2} is independent of $z, y \in \mathbb{R}^n$. Applying Lemma 2.22 with $A = \alpha I$ and $b = 0$, we get $\chi(\alpha y, \alpha\xi) \in \mathcal{S}(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$ for each fixed $\alpha > 0$. Now let $z \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ be fixed, too. Note that we have $\chi(\alpha y, \alpha\xi) \in \mathcal{S}(\mathbb{R}^n_y)$ because of Remark 2.20 for each fixed $\alpha > 0$. Choosing $A = -I$ and $b = z$, Lemma 2.22 yields $g_y(z) = g(z-y) \in \mathcal{S}(\mathbb{R}^n_y)$. Collecting these results provides

$$u(y) g_y(z) \chi(\alpha y, \alpha\xi) \in \mathcal{S}(\mathbb{R}^n_y).$$

Using $\langle D_y\rangle^{2l} = \sum_{|\gamma|\leq l} a_{\gamma,l} D_y^{2\gamma}$ and the Leibniz-rule, we get due to (5.58) and $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n\times\mathbb{R}^n)$:

$$\begin{aligned}|\langle D_y\rangle^{2l}[u(y)g_y(z)\chi(\alpha y,\alpha\xi)]| &\leq C_l \sum_{|\gamma|\leq l}\sum_{\gamma_1+\gamma_2+\gamma_3=2\gamma} |D_y^{\gamma_1}u(y)||D_y^{\gamma_2}g_y(z)||D_y^{\gamma_3}\chi(\alpha y,\alpha\xi)|\\ &\leq C_l \sum_{|\gamma|\leq l}\sum_{\gamma_1+\gamma_2+\gamma_3=2\gamma} |D_y^{\gamma_1}u(y)| \leq C_l\langle y\rangle^{-n-1}, \qquad (5.59)\end{aligned}$$

where C_l is independent of $y, z, \xi \in \mathbb{R}^n$ and $\alpha > 0$. The last inequality holds because of $u \in \mathcal{S}(\mathbb{R}^n)$. Since $u(y)g_y(z)\chi(\alpha y,\alpha\xi) \in \mathcal{S}(\mathbb{R}^n_y)$ and since we know $e^{-iy\cdot\xi} = \langle\xi\rangle^{-2l}\langle D_y\rangle^{2l}e^{-iy\cdot\xi}$ due to Remark 2.10, we can integrate by parts and obtain

$$\begin{aligned}&\int e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)\chi(\alpha y,\alpha\xi)dy\\ &\quad= \int e^{-iy\cdot\xi}e^{iz\cdot\xi}\langle\xi\rangle^{-2l}\langle D_y\rangle^{2l}\left[u(y)g_y(z)\chi(\alpha y,\alpha\xi)\right]dy. \qquad (5.60)\end{aligned}$$

Combining (5.59), (5.60) and Theorem 2.11, we get for all $z \in \mathbb{R}^n$:

$$\begin{aligned}&\left|\iint e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)\chi(\alpha y,\alpha\xi)dy đ\xi\right|\\ &\quad\leq \iint \langle\xi\rangle^{-2l}\left|\langle D_y\rangle^{2l}\left[u(y)g_y(z)\chi(\alpha y,\alpha\xi)\right]\right| dy đ\xi \leq C_l \iint \langle\xi\rangle^{-2l}\langle y\rangle^{-n-1}dy đ\xi\\ &\quad\leq C_l,\end{aligned}$$

where C_l is independent of $z \in \mathbb{R}^n$ and $\alpha > 0$. On account of Remark 2.20 we have $f(x,.) \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. By means of the previous estimate and of $f(x,.) \in L^1(\mathbb{R}^n)$ we can prove existence of a L^1-majorant of $\{r_\alpha(x,z) : \alpha > 0\}$:

$$|r_\alpha(x,z)| = |f(x,z)|\cdot\left|\iint e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)\chi(\alpha y,\alpha\xi)dy đ\xi\right| \leq C_l|f(x,z)|$$

for all $x, z \in \mathbb{R}^n$. In order to use the theorem of Lebesgue, it remains to show the pointwise convergence $r_\alpha \to r$ for $\alpha \to 0$. This follows directly from the definition of the oscillatory integral:

$$\begin{aligned}\lim_{\alpha\to 0} r_\alpha(x,z) &= f(x,z)\lim_{\alpha\to 0}\iint e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)\chi(\alpha y,\alpha\xi)dy đ\xi\\ &= f(x,z)\ \mathrm{Os}\text{-}\iint e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)dy đ\xi = r(x,z)\end{aligned}$$

for all $z \in \mathbb{R}^n$. Hence all assumptions of Lebesgue's theorem hold. An application of Lebesgue's theorem arises

$$\lim_{\alpha\to 0}\int r_\alpha(x,z)dz = \int r(x,z)dz.$$

□

Now we have checked all utilities to verify the characterization of non-smooth pseudodifferential operators with coefficients in the Hölder space $C^\tau(\mathbb{R}^n)$ which are smooth enough with respect to ξ:

Theorem 5.36. *Let $1 < q < \infty$ and $m \in \mathbb{N}_0$ with $m > n/q$. Additionally let $M \in \mathbb{N} \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}_0$ with $M \geq 2l > n$. Moreover, we define $\tilde{M} := M - (n+2)$ if n is even and $\tilde{M} := M - (n+1)$ else. Considering $T \in \mathcal{A}^{0,M}_{0,0}(m,q)$ and $\tilde{M} \geq 1$, we get for all $0 < \tau \leq m - n/q$ with $\tau \notin \mathbb{N}_0$*

$$T \in OPC^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(L^q(\mathbb{R}^n)).$$

Proof: Let $\tau \in (0, m - n/q]$ with $\tau \notin \mathbb{N}$ be arbitrary but fixed. Note that such a τ exists since $m > n/q$. We choose an arbitrary $\varphi \in C^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ for all $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ for all $|x| \geq 1$. Additionally we define for $\varepsilon \in (0,1]$ the pseudodifferential operators

$$P_\varepsilon := \tilde{p}_\varepsilon(x, D_x) \qquad \text{and} \qquad Q_\varepsilon := q_\varepsilon(x, D_x),$$

where the symbols $\tilde{p}_\varepsilon$ and q_ε are defined as $\tilde{p}_\varepsilon(x,\xi) := \varphi(\varepsilon x)$ and $q_\varepsilon(x,\xi) := \varphi(\varepsilon \xi)$. Moreover, we define for every $\varepsilon \in (0,1]$ the linear operator T_ε by

$$T_\varepsilon := P_\varepsilon Q_\varepsilon T P_\varepsilon Q_\varepsilon.$$

The proof of this theorem is divided into three different parts. First we write T_ε as a pseudodifferential operator with a double symbol. In step two we reduce the double symbol to an ordinary symbol p_ε of T_ε. Finally, we conclude the proof in part three. Here we use the pointwise convergence of a subsequence of $(p_\varepsilon)_{\varepsilon > 0}$ to get a symbol p with the property $p(x, D_x)u = Tu$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

We begin with step one: Applying Lemma 5.28 provides the continuity of $T_\varepsilon : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Therefore Theorem 2.63 gives us the existence of a Schwartz-kernel, denoted by $t_\varepsilon(x,y)$, which is associated to T_ε. Thus

$$T_\varepsilon u(x) = \int t_\varepsilon(x,y) u(y) dy \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n) \text{ and all } x \in \mathbb{R}^n. \tag{5.61}$$

In particular, we know that $t_\varepsilon \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ due to Theorem 2.63. Now we choose $u, g \in \mathcal{S}(\mathbb{R}^n)$ with $g(0) = 1$ and $g(-x) = g(x)$ for all $x \in \mathbb{R}$. We define the translated function $g_y : \mathbb{R}^n \to \mathbb{C}$ for an arbitrary $y \in \mathbb{R}^n$ by $g_y := \tau_y(g)$. Next let $x \in \mathbb{R}^n$ be arbitrary, but fixed. Then we define $h : \mathbb{R}^n \to \mathbb{C}$ as

$$h(z) := u(z) g_z(x) \qquad \text{for all } z \in \mathbb{R}^n.$$

This leads to $h(x) = u(x) g_x(x) = u(x)$. Using the inversion formula, cf. Remark 3.12, we obtain

$$u(x) = h(x) = \text{Os-}\iint e^{i(x-y)\cdot\xi} h(y) dy đ\xi = \text{Os-}\iint e^{i(x-y)\cdot\xi} u(y) g_y(x) dy đ\xi.$$

Consequently (5.61) provides

$$\begin{aligned} T_\varepsilon u(x) &= \int t_\varepsilon(x,z)u(z)dz \\ &= \int t_\varepsilon(x,z)\left[\text{Os-}\iint e^{i(z-y)\cdot\xi}u(y)g_y(z)dy\,đ\xi\right]dz. \end{aligned} \tag{5.62}$$

Now we choose $\tilde{\chi} \in C_c^\infty(\mathbb{R}^n)$ with $\tilde{\chi}(0) = 1$ and set

$$\chi(y,\xi) := \tilde{\chi}(y)\tilde{\chi}(\xi) \qquad \text{for all } y,\xi \in \mathbb{R}^n.$$

Therefore $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\chi(0,0) = 1$. Applying Proposition 5.35 and Fubini's theorem to (5.62), we get

$$\begin{aligned} T_\varepsilon u(x) &= \lim_{\alpha\to 0}\int t_\varepsilon(x,z)\cdot\iint e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)\chi(\alpha y,\alpha\xi)dy\,đ\xi\,dz \\ &= \lim_{\alpha\to 0}\iint e^{-iy\cdot\xi}\chi(\alpha y,\alpha\xi)\int t_\varepsilon(x,z)e^{iz\cdot\xi}g_y(z)dz\,u(y)\,dy\,đ\xi \\ &= \lim_{\alpha\to 0}\iint e^{-iy\cdot\xi}\chi(\alpha y,\alpha\xi)\left[T_\varepsilon(e_\xi g_y)(x)\right]u(y)dy\,đ\xi \end{aligned}$$

Here we were able to use Fubini's theorem since

$$t_\varepsilon(x,z)e^{-iy\cdot\xi}e^{iz\cdot\xi}u(y)g_y(z)\chi(\alpha y,\alpha\xi) \in \mathcal{S}(\mathbb{R}^n_y\times\mathbb{R}^n_\xi\times\mathbb{R}^n_z) \subseteq L^1(\mathbb{R}^n_y\times\mathbb{R}^n_\xi\times\mathbb{R}^n_z)$$

for all $x \in \mathbb{R}^n$ and $\alpha > 0$. Defining $p_{\varepsilon,0}: \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{C}$ by

$$p_{\varepsilon,0}(x,\xi,y) := e^{-ix\cdot\xi}T_\varepsilon(e_\xi g_y)(x) \qquad \text{for all } x,\xi,y \in \mathbb{R}^n,$$

we conclude part one by means of the previous equality:

$$\begin{aligned} T_\varepsilon u(x) &= \lim_{\alpha\to 0}\iint e^{i(x-y)\cdot\xi}\chi(\alpha y,\alpha\xi)p_{\varepsilon,0}(x,\xi,y)u(y)dy\,đ\xi \\ &= \text{Os-}\iint e^{i(x-y)\cdot\xi}p_{\varepsilon,0}(x,\xi,y)u(y)dy\,đ\xi. \end{aligned}$$

Here $p_{\varepsilon,0}$ is the double symbol of T_ε, cf. Lemma 4.68, as we will see in step two.

Secondly we want to construct for all $0 < \varepsilon \leq 1$ pseudodifferential symbols $p_\varepsilon \in C^\tau S^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M})$, with the properties

i) $T_\varepsilon u = p_\varepsilon(x,D_x)u$ for all $u \in \mathcal{S}(\mathbb{R}^n)$,

ii) $(p_\varepsilon)_{0<\varepsilon\leq 1}$ is a bounded sequence of $C^\tau S^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M})$.

In view of Lemma 5.30, there is a constant $C_{\alpha,\beta}$, independent of $0 < \varepsilon \le 1$ such that

$$\sup_{0<\varepsilon\le 1} \left\| \mathrm{ad}(-ix)^\alpha \, \mathrm{ad}(D_x)^\beta T_\varepsilon \right\|_{\mathscr{L}(L^q(\mathbb{R}^n))} \le C_{\alpha,\beta} \tag{5.63}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\beta| \le m$ and $|\alpha| \le M$. Since $g \in \mathcal{S}(\mathbb{R}^n)$, we derive from Lemma 2.21 that $D_x^\beta g \in \mathcal{S}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$. Consequently there is a constant C_β, independent of $y, \xi \in \mathbb{R}^n$, with the property

$$\int_{\mathbb{R}^n} \left| \left(D_x^\beta g \right)(x) \right|^q dx := C_\beta < \infty.$$

Because of Lemma 5.28 and Proposition 5.31, $T_\varepsilon : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear and continuous operator and $p_{\varepsilon,0}$ is an element of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Hence all assumptions of Lemma 2.65 are fulfilled. If we use Lemma 2.65 and (5.63) first, and substitude $\tilde{x} := x - y$ afterwards, we can estimate for arbitrary $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\alpha| \le M$:

$$\begin{aligned}
\left\| \partial_\xi^\alpha D_y^\gamma p_{\varepsilon,0}(.,\xi,y) \right\|_{H_q^m}^q &= \sum_{|\beta|\le m} \left\| \partial_\xi^\alpha D_x^\beta D_y^\gamma p_{\varepsilon,0}(x,\xi,y) \right\|_{L^q(\mathbb{R}_x^n)}^q \\
&\le \sum_{|\beta|\le m} \sum_{\beta_1+\beta_2=\beta} \left\| C_{\beta_1,\beta_2} \left[\mathrm{ad}(-ix)^\alpha \, \mathrm{ad}(D_x)^{\beta_1} T_\varepsilon \right] \left(e^{ix\cdot\xi} D_x^{\beta_2+\gamma} g_y \right)(x) \right\|_{L^q(\mathbb{R}_x^n)}^q \\
&\le \sum_{|\beta|\le m} \sum_{\beta_1+\beta_2=\beta} C_{\alpha,\beta_1,\beta_2} \left\| e^{ix\cdot\xi} D_x^{\beta_2+\gamma} g_y(x) \right\|_{L^q(\mathbb{R}_x^n)}^q \\
&= \sum_{|\beta|\le m} \sum_{\beta_1+\beta_2=\beta} C_{\alpha,\beta_1,\beta_2} \int_{\mathbb{R}^n} \left| (D_x^{\beta_2+\gamma} g)(x-y) \right|^q dx \\
&= \sum_{|\beta|\le m} \sum_{\beta_1+\beta_2=\beta} C_{\alpha,\beta_1,\beta_2} \int_{\mathbb{R}^n} \left| (D_{\tilde{x}}^{\beta_2+\gamma} g)(\tilde{x}) \right|^q d\tilde{x} \le C_{\alpha,m,\gamma} < \infty
\end{aligned}$$

for all $\xi, y \in \mathbb{R}^n$, where $C_{\alpha,m,\gamma}$ is independent of $y, \xi \in \mathbb{R}^n$ and $0 < \varepsilon \le 1$. Applying the continuous embedding $H_q^m(\mathbb{R}^n) \hookrightarrow C^\tau(\mathbb{R}^n)$ we obtain for arbitrary $\alpha, \gamma \in \mathbb{N}_0^n$ with $|\alpha| \le M$:

$$\left\| \partial_\xi^\alpha D_y^\gamma p_{\varepsilon,0}(.,\xi,y) \right\|_{C^\tau} \le C \left\| \partial_\xi^\alpha D_y^\gamma p_{\varepsilon,0}(.,\xi,y) \right\|_{H_q^m} \le C_{\alpha,m,\gamma} \qquad \text{for all } \xi, y \in \mathbb{R}^n.$$

Here $C_{\alpha,m,\gamma}$ is independent of $\xi, y \in \mathbb{R}^n$ and $0 < \varepsilon \le 1$. Thus we have shown the boundedness of the subset $\{p_{\varepsilon,0} : 0 < \varepsilon \le 1\} \subseteq C^\tau S_{0,0}^0(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M)$. Now we define

$$p_\varepsilon(x,\xi) := \mathrm{Os}\text{-}\iint e^{-iy\cdot\eta} p_{\varepsilon,0}(x, \xi+\eta, x+y) dy \,đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n.$$

Having verified the assumptions, we can use Theorem 5.26 which implies i)

$$T_\varepsilon u = p_\varepsilon(x, D_x)u \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

Therefore it remains to apply Theorem 5.24. This provides the boundedness of $(p_\varepsilon)_{0<\varepsilon\le 1}$ as a subset of $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M})$. So we can turn to step three now.

On account of ii) it is possible to apply Lemma 5.11 which yields the existence of a subsequence $(p_{\varepsilon_k})_{k\in\mathbb{N}}$ of $(p_\varepsilon)_{0<\varepsilon\le 1}$ with $\varepsilon_k \to 0$ if k converges to ∞ such that

$$p_{\varepsilon_k} \xrightarrow{k\to\infty} p \qquad \text{pointwise},$$

where $p \in C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$. Let $u \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary. For all $\xi \in \mathbb{R}^n$ we have the convergence

$$e^{ix\cdot\xi} p_{\varepsilon_k}(x,\xi)\hat{u}(\xi) \xrightarrow{k\to\infty} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi). \tag{5.64}$$

Note that $u \in \mathcal{S}(\mathbb{R}^n)$ implies $\hat{u} \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Using the fact, that $(p_{\varepsilon_k})_{k\in\mathbb{N}}$ is a bounded subset of $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M})$, we obtain

$$|e^{ix\cdot\xi} p_{\varepsilon_k}(x,\xi)\hat{u}(\xi)| = |p_{\varepsilon_k}(x,\xi)||\hat{u}(\xi)| \le C|\hat{u}(\xi)| \in L^1(\mathbb{R}^n_\xi), \tag{5.65}$$

where C is independent of $x, \xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Due to (5.64) and (5.65), an application of Lebesgue's Theorem yields the convergence

$$\begin{aligned} p_{\varepsilon_k}(x, D_x)u(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} p_{\varepsilon_k}(x,\xi)\hat{u}(\xi)\, đ\xi \\ &\xrightarrow{k\to\infty} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi = p(x, D_x)u(x) \end{aligned} \tag{5.66}$$

for all $x \in \mathbb{R}^n$. Choosing $N \in \mathbb{N}$ with the property $n < 2N \le M$ we get by Lemma 5.14 the existence of a constant $C_{N,n}$, independent of $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, such that

$$|p_{\varepsilon_k}(x, D_x)u(x)| \le C_{N,n}\langle x\rangle^{-2N} \qquad \text{for all } x \in \mathbb{R}^n. \tag{5.67}$$

The independence of the $C_{N,n}$ of $k \in \mathbb{N}$ enables us to verify the next inequality:

$$|p(x, D_x)u(x)| = \lim_{k\to\infty} |p_{\varepsilon_k}(x, D_x)u(x)| \le C_{N,n}\langle x\rangle^{-2N} \quad \text{for all } x \in \mathbb{R}^n. \tag{5.68}$$

Using (5.67) and (5.68) first and Theorem 2.11 afterwards gives us

$$\begin{aligned} |p_{\varepsilon_k}(x, D_x)u(x) - p(x, D_x)u(x)|^q &\le (|p_{\varepsilon_k}(x, D_x)u(x)| + |p(x, D_x)u(x)|)^q \\ &\le C_{N,n}\langle x\rangle^{-2Nq} \in L^1(\mathbb{R}^n_x). \end{aligned}$$

Thus we have found an L^1-majorant of $\{|p_{\varepsilon_k}(x,D_x)u - p(x,D_x)u|^q : k \in \mathbb{N}\}$, since $C_{N,n}$ is independent of $k \in \mathbb{N}$. Therefore Lebesgue's theorem on dominated convergence provides

$$\begin{aligned} &\|p_{\varepsilon_k}(x,D_x)u - p(x,D_x)u\|^q_{L^q(\mathbb{R}^n)} \\ &\quad = \int_{\mathbb{R}^n} |p_{\varepsilon_k}(x,D_x)u(x) - p(x,D_x)u(x)|^q dx \xrightarrow{k\to\infty} 0 \end{aligned}$$

because we have proved (5.66) yet. In particular we have

$$\lim_{k\to\infty} \|p_{\varepsilon_k}(x,D_x)u - p(x,D_x)u\|_{L^q(\mathbb{R}^n)} = 0,$$

which implies

$$L^q - \lim_{k\to\infty} p_{\varepsilon_k}(x,D_x)u = p(x,D_x)u.$$

Finally, an application of i) and Lemma 5.27 finishes the proof:

$$p(x,D_x)u = L^q - \lim_{k\to\infty} p_{\varepsilon_k}(x,D_x)u = L^q - \lim_{k\to\infty} T_{\varepsilon_k}u = Tu.$$

Here the last equality holds due to Lemma 5.27. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$ the previous equality implies the boundedness of $p(x,D_x)$ as a map from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Therefore we even have $Tu = p(x,D_x)u$ for all $u \in L^q(\mathbb{R}^n)$. Consequently T is an element of $\mathrm{OP}C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(L^q(\mathbb{R}^n))$. □

Let us have a critical look at the previous characterization: In general we do not have the continuity of non-smooth pseudodifferential operators as maps from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. But with the previous theorem we characterized only pseudodifferential operators which are continuous as maps from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

The next goal of this section is to construct a characterization of pseudodifferential operators of the same symbol-class, but of arbitrary order m. The proof is based on an application of order reducing invertible operators to get the case $m = 0$ we already know. Having solved the problem in the case $m = 0$ we just have to verify that an application of the inverse of the order reducing operator provides the claim:

Proposition 5.37. *Let $m \in \mathbb{R}$, $s > 0$, $1 < q < \infty$ and $M \in \mathbb{N}_0 \cup \{\infty\}$. For every linear map $P : H^m_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ we get the following result: If $P\Lambda^{-m}$ is an element of $OPC^sS^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap \mathscr{L}(L^q(\mathbb{R}^n))$, we have*

$$P \in OPC^sS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Proof: We define the operator $\tilde{P} := P\Lambda^{-m} \in \mathrm{OP}C^sS^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. On account of $\tilde{P} \in \mathrm{OP}C^sS^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ and $\mathrm{OP}\Lambda^m \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we arise from Remark 4.44:

$$P = P\Lambda^{-m}\Lambda^m = \tilde{P}\Lambda^m \in \mathrm{OP}C^sS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M).$$

Hence it remains to prove that P is an element of $\mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n))$. An application of Theorem 3.18 to $\Lambda^m \in \mathrm{OP}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ yields $\Lambda^m \in \mathcal{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n))$. Together with $P\Lambda^{-m} \in \mathscr{L}(L^q(\mathbb{R}^n))$ we obtain

$$P = P\Lambda^{-m}\Lambda^m \in \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

□

The previous proposition allows us to show the characterization of non-smooth pseudodifferential operators of the class $C^sS^m_{0,0}$ of arbitrary order m:

Lemma 5.38. *Let $m \in \mathbb{R}$, $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}_0$ with $M \geq 2l > n$. We define $\tilde{M} := M - (n+2)$ if n is even and $\tilde{M} := M - (n+1)$ else. Considering an operator $T \in \mathcal{A}^{m,M}_{0,0}(\tilde{m}, q)$ and an $\tilde{M} \geq 1$ we have for $s \in (0, \tilde{m} - n/q]$ with $s \notin \mathbb{N}_0$:*

$$T \in OPC^sS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Proof: First of all we choose an $s \in (0, \tilde{m} - n/q]$ with $s \notin \mathbb{N}_0$. Additionally let $\delta \in \mathbb{N}^n_0$ be arbitrary. Due to Remark 3.7 we know that $\mathrm{ad}(-ix)^\delta\Lambda^{-m}$ is an element of $\mathrm{OP}S^{-m-|\delta|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence Theorem 3.18 provides that

$$\mathrm{ad}(-ix)^\delta\Lambda^{-m} : L^q(\mathbb{R}^n) \to H^{m+|\delta|}_q(\mathbb{R}^n) \subseteq H^m_q(\mathbb{R}^n) \text{ is continuous.} \tag{5.69}$$

The last inclusion holds because of Lemma 2.45. Let $l \in \mathbb{N}_0$, $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ and $\gamma_1, \ldots, \gamma_l \in \mathbb{N}^n_0$ with $|\beta| \leq \tilde{m}$ and $|\gamma| \leq M$ be arbitrary. Here we define $\beta := \beta_1 + \ldots + \beta_l$ and $\gamma := \gamma_1 + \ldots + \gamma_l$. Applying $T \in \mathcal{A}^{m,M}_{0,0}(\tilde{m}, q)$ we have the continuity of

$$\mathrm{ad}(-ix)^{\gamma_1}\,\mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\gamma_l}\,\mathrm{ad}(D_x)^{\beta_l}T : H^m_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n). \tag{5.70}$$

Combining (5.69) and (5.70) we obtain

$$[\mathrm{ad}(-ix)^{\gamma_1}\,\mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\gamma_l}\,\mathrm{ad}(D_x)^{\beta_l}T][\mathrm{ad}(-ix)^\delta\Lambda^{-m}] \in \mathscr{L}(L^q) \tag{5.71}$$

Now let $l \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ such that $|\beta| \leq \tilde{m}$ and $|\alpha| \leq M$ be arbitrary, where we define $\beta := \beta_1 + \ldots + \beta_l$ and $\alpha := \alpha_1 + \ldots + \alpha_l$. Since $\partial^{\tau_2}_\xi D^\delta_x \lambda^{-m}(\xi) \equiv 0$ for every $\tau_2, \delta \in \mathbb{N}^n_0$ with $|\delta| \neq 0$, Remark 3.7 yields $\mathrm{ad}(-ix)^{\tau_2}\,\mathrm{ad}(D_x)^\delta\Lambda^{-m} \equiv 0$. Therefore we get due to Proposition 2.55:

$$\mathrm{ad}(-ix)^\tau\,\mathrm{ad}(D_x)^\upsilon(T\Lambda^{-m}) = \sum_{\tau_1+\tau_2=\tau} C_{\tau_1}[\mathrm{ad}(-ix)^{\tau_1}\,\mathrm{ad}(D_x)^\upsilon T][\mathrm{ad}(-ix)^{\tau_2}\Lambda^{-m}]$$

for all $\tau, \upsilon \in \mathbb{N}_0^n$ with $|\tau| \le M$ and $|\upsilon| \le \tilde{m}$. Hence we obtain

$$\begin{aligned}
&\operatorname{ad}(-ix)^{\alpha_1}\operatorname{ad}(D_x)^{\beta_1}\ldots\operatorname{ad}(-ix)^{\alpha_l}\operatorname{ad}(D_x)^{\beta_l}(T\Lambda^{-m})\\
&\quad=\sum_{\substack{\gamma_1+\delta_1=\alpha_1\\ \vdots\\ \gamma_l+\delta_l=\alpha_l}} C_{\gamma_1,\ldots,\gamma_l}[\operatorname{ad}(-ix)^{\gamma_1}\operatorname{ad}(D_x)^{\beta_1}\ldots\operatorname{ad}(-ix)^{\gamma_l}\operatorname{ad}(D_x)^{\beta_l}T][\operatorname{ad}(-ix)^{\delta}\Lambda^{-m}].
\end{aligned}$$

Here δ is defined by $\delta := \delta_1 + \ldots + \delta_l$. Together with (5.71) this implies the continuity of

$$\operatorname{ad}(-ix)^{\alpha_1}\operatorname{ad}(D_x)^{\beta_1}\ldots\operatorname{ad}(-ix)^{\alpha_l}\operatorname{ad}(D_x)^{\beta_l}(T\Lambda^{-m}) : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n).$$

Therefore $T\Lambda^{-m} \in \mathcal{A}^{0,M}_{0,0}(\tilde{m}, q)$. If we use Theorem 5.36, we get

$$T\Lambda^{-m} \in \mathrm{OP}C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(L^q(\mathbb{R}^n)).$$

Applying Proposition 5.37 we conclude the claim of the lemma:

$$T \in \mathrm{OP}C^s S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

□

5.5 Characterization of Pseudodifferential Operators with Symbols in $C^s S^m_{1,0}$

In applications to partial differential equations the pseudodifferential operators are predominantly of the class $C^s S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. As we have seen in Example 5.2, these operators are elements of the set $\mathcal{A}^m_{1,0}(\lfloor s \rfloor, q)$ with $1 < q < \infty$. In the present section we show that operators being in the set $\mathcal{A}^{m,M}_{1,0}(\tilde{m}, q)$ for sufficiently large $\tilde{m}$ are also non-smooth pseudodifferential operators of the order m whose coefficients are in a Hölder space. As an ingredient we use that $\mathcal{A}^{m,M}_{1,0}(\tilde{m}, q)$ is a subset of $\mathcal{A}^{m,M}_{0,0}(\tilde{m}, q)$. Consequently we may apply the characterization of the pseudodifferential operators of the class $C^s S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n, M)$ in order to obtain the following main result of this chapter:

Theorem 5.39. *Let $m \in \mathbb{R}$, $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $M \in \mathbb{N}_0$ be such that there is an $l \in \mathbb{N}$ with $M \ge 2l > n$. We define $\tilde{M}$ by $\tilde{M} := M - (n + 2)$ if n is even and $\tilde{M} := M - (n + 1)$ else. Assuming $P \in \mathcal{A}^{m,M}_{1,0}(\tilde{m}, q)$ and $\tilde{M} \ge 1$, we obtain for all $\tau \in (0, \tilde{m} - n/q]$ with $\tau \notin \mathbb{N}_0$:*

$$P \in OPC^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Proof: Let $\tilde{m} - n/q \geq \tau > 0$ with $\tau \notin \mathbb{N}_0$ and $P \in \mathcal{A}^{m,M}_{1,0}(\tilde{m}, q)$ be arbitrary. On account of Lemma 5.3 we know that $P \in \mathcal{A}^{m,M}_{1,0}(\tilde{m}, q) \subseteq \mathcal{A}^{m,M}_{0,0}(\tilde{m}, q)$. Consequently we get due to Lemma 5.38:

$$P \in \mathrm{OP}C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Let $p \in C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$ be the symbol of P. Moreover, let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{M} - 1$ be arbitrary. Because of Remark 4.45, $\mathrm{ad}(-ix)^\alpha P$ is a pseudodifferential operator with symbol $\partial^\alpha_\xi p(x, \xi)$. Additionally the definition of $P \in \mathcal{A}^{m,M}_{1,0}(\tilde{m}, q)$ yields for all $l \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\beta| \leq \tilde{m}$ and $|\gamma| \leq M - |\alpha|$

- $P_\alpha := (\mathrm{ad}(-ix)^\alpha P) : H^{m-|\alpha|}_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous,
- $\mathrm{ad}(-ix)^{\gamma_1} \mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\gamma_l} \mathrm{ad}(D_x)^{\beta_l} P_\alpha : H^{m-|\alpha|-|\gamma|}_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous,

where $\beta := \beta_1 + \ldots + \beta_l$ and $\gamma := \gamma_1 + \ldots + \gamma_l$. Therefore $\mathrm{ad}(-ix)^\alpha P$ is an element of $\mathcal{A}^{m-|\alpha|,M-|\alpha|}_{1,0}(\tilde{m}, q)$. By means of Lemma 5.3, we get that $\mathrm{ad}(-ix)^\alpha P$ is an element of $\mathcal{A}^{m-|\alpha|,M-|\alpha|}_{0,0}(\tilde{m}, q)$. Thus we obtain with Lemma 5.38

$$\mathrm{ad}(-ix)^\alpha P \in \mathrm{OP}C^\tau S^{m-|\alpha|}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - |\alpha| - 1).$$

Since the symbol of $\mathrm{ad}(-ix)^\alpha P$ is $\partial^\alpha_\xi p(x, \xi)$, as remarked before, the next estimate holds:

$$\|\partial^\alpha_\xi p(., \xi)\|_{C^\tau(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Here C_α is independent of $\xi \in \mathbb{R}^n$. Choosing $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{M} - 1$ arbitrary, the last inequality implies $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$. □

In the case $\tilde{M} - 1 > \max\{n/2, n/q\}$, $1 < q < \infty$, every pseudodifferential operator whose symbol is in the class $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$, where $\tau > 0$ and $m \in \mathbb{R}$, is an element of $\mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n))$ due to Theorem 4.51. Consequently we have in this case

$$\mathrm{OP}C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)) = \mathrm{OP}C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

5.6 Improvement of the Characterization

In the previous section we have derived the main theorem of this chapter: Assuming $T \in \mathcal{A}^m_{1,0}(\tilde{m}, q)$ for sufficient large M, Theorem 5.39 provides that T is a non-smooth pseudodifferential operator whose coefficient is in a Hölder space C^τ where $\tau \in (0, \tilde{m} - n/q]$. Therefore we loose some regularity with respect to $\tilde{m}$. Hence the question arises, whether we are able to get a better result. In this section we see that T is even an element of $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$. The proof of this statement is essentially the same as the one of the characterization in Section 5.5. We just have to replace the results for pseudodifferential operators with coefficients in a Hölder space with analogous ones for pseudodifferential operators with coefficients in an uniformly local Sobolev space.

The main difficulty comes along with the symbol reduction of non-smooth double symbols of the class $XS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M)$ to non-smooth single symbols with coefficients in X, where $X = W^{\tilde{m},q}_{uloc}$. Both cases, $X = C^{\tilde{m},\tau}$ and $X = W^{\tilde{m},q}_{uloc}$ make use of the estimate

$$\sup_{y \in \mathbb{R}^n} \|\partial_y^\gamma \partial_\xi^\delta a(.,\xi,.+y)\|_X \leq C\langle\xi\rangle^m,$$

where $a \in XS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; M)$ and $\gamma, \delta \in \mathbb{N}_0^n$ are multi-indices with $|\delta| \leq M$. While this estimate directly follows from the definition of the symbol-class in the case $X = C^{\tilde{m},\tau}$, this proof turned out to be rather tedious for the case $X = W^{\tilde{m},q}_{uloc}$ in Section 4.1.2. The symbol reduction for uniformly local Sobolev spaces is subject of Subsection 5.6.2. Considering a bounded sequence $(p_\varepsilon)_{\varepsilon>0}$ of the symbol-class $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ provides the boundedness of $(p_\varepsilon)_{\varepsilon>0}$ as a subset of $C^sS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for $s \in (0, \tilde{m} - n/q]$ due to Remark 4.27. Hence we know from Theorem 5.11 that there is a subsequence of $(p_\varepsilon)_{\varepsilon>0}$ which converges pointwise to a symbol p of the class $C^sS^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. Consequently we just have to verify whether p is even an element of $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. This fact is treated in Subsection 5.6.1.

Then we have all utilities at hand to show the characterization of non-smooth pseudodifferential operators of the class $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ in Subsection 5.6.3 for sufficiently large M. As before we first check the case $m = 0$. Next we generalize the obtained result to non-smooth pseudodifferential operators of arbitrary order by reducing the arbitrary case to the case $m = 0$ by means of a order reducing operator. In the same manner as in Section 5.5 we use the result of Subsection 5.6.3 in order to show the characterization of non-smooth pseudodifferential operators of the class $W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. M again has to be chosen sufficiently large. We present this proof in detail in Subsection 5.6.4.

5.6.1 Pointwise Convergence in $W^{\tilde{m},q}_{uloc}S^0_{0,0}$

Assuming a bounded sequence $(p_\varepsilon)_{\varepsilon>0} \subseteq W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ we show the existence of a subsequence of $(p_\varepsilon)_{\varepsilon>0}$ which converges pointwise in the symbol-class $W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. So far we have seen in Theorem 5.11 that there is a sequence in $(p_\varepsilon)_{\varepsilon>0}$ which converges pointwise to $p \in C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$

because $W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$ is a subset of $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$ for all $\tau \in (0, \tilde{m} - n/q]$. Hence we just have to verify that p is even an element of $W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. This is the topic of this subsection.

Theorem 5.40. *Let $M \in \mathbb{N}_0 \cup \{\infty\}$, $\tilde{m} \in \mathbb{N}_0$ and $1 < q < \infty$. Additionally let $(B_1(z_j) \times B_1(\eta_k))_{j,k\in\mathbb{N}}$ be a countable open cover of bounded sets of $\mathbb{R}^n \times \mathbb{R}^n$ where $z_j, \eta_k \in \left\{n^{-1/2}z : z \in \mathbb{Z}^n\right\}$ for each $j,k \in \mathbb{N}$. Furthermore, let $(p_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Then there is a subsequence $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_l \to 0$ for $l \to \infty$ and a function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that*

i) $p(x,.) \in C^{M-1}(\mathbb{R}^n)$ *for all* $x \in \mathbb{R}^n$,

ii) $\partial_x^\beta \partial_\xi^\alpha p \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$,

iii) $\partial_x^\beta \partial_\xi^\alpha p_{\varepsilon_l} \xrightarrow{l\to\infty} \partial_x^\beta \partial_\xi^\alpha p$ *uniformly on each* $\overline{B_1(z_j) \times B_1(\eta_k)}$, $j,k \in \mathbb{N}$

for every $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \le M-1$ and $|\beta| < \tilde{m} - n/q$. Moreover,

$$p \in W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1).$$

Proof: Let $\tau \le \tilde{m} - n/q$ with $\tau \notin \mathbb{N}$ be arbitrary. From the continuous embedding $W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \subseteq C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, cf. Remark 4.27, we get the boundedness of $(p_\varepsilon)_{\varepsilon>0}$ in $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. Hence an application of Theorem 5.11 provides the existence of a subsequence $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq (p_\varepsilon)_{\varepsilon>0}$ with $\varepsilon_l \to 0$ for $l \to \infty$ which fulfilles the properties i), ii) and iii) for a $p \in C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. Thus it remains to show that p is even an element of $W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M-1)$. Since i) and ii) already hold, we just have to check $\partial_\xi^\alpha p(.,\xi) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ for all $\xi \in \mathbb{R}^n$ and the existence of a constant C_α, independent of $\xi \in \mathbb{R}^n$, such that

$$\|\partial_\xi^\alpha p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \le C_\alpha \qquad \text{for all } \xi \in \mathbb{R}^n,$$

for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le M-1$. Let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \le M-1$ and $\xi \in \mathbb{R}^n$ be arbitrary but fixed. The boundedness of $(p_{\varepsilon_l})_{l\in\mathbb{N}} \subseteq W^{\tilde{m},q}_{uloc}S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ yields for all $j,l \in \mathbb{N}$

$$\|\partial_\xi^\alpha p_{\varepsilon_l}(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))} \le \|\partial_\xi^\alpha p_{\varepsilon_l}(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \le C_\alpha, \tag{5.72}$$

where C_α is independent of $j,l \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$. This implies the boundedness of $(\partial_\xi^\alpha p_{\varepsilon_l}(.,\xi))_{l\in\mathbb{N}}$ in $H^{\tilde{m}}_q(B_1(z_j))$ for all $j \in \mathbb{N}$. Let $j \in \mathbb{N}$ be arbitrary but fixed. Because of the reflexivity of $H^{\tilde{m}}_q(B_1(z_j))$, cf. e.g. [8], Example 6.11.3), there exists a weakly convergent subsequence $(\partial_\xi^\alpha p_{\varepsilon_{l_m}})_{m\in\mathbb{N}}$ of $(\partial_\xi^\alpha p_{\varepsilon_l})_{l\in\mathbb{N}}$ such that

$$\partial_\xi^\alpha p_{\varepsilon_{l_m}}(.,\xi) \rightharpoonup q_{\alpha,\xi,j} \qquad \text{in } H^{\tilde{m}}_q(B_1(z_j))$$

for $m \to \infty$. The compact embedding $H^{\tilde{m}}_q(B_1(z_j)) \hookrightarrow\hookrightarrow C^0(\overline{B_1(z_j)})$ even gives us the strong convergence

$$\partial^\alpha_\xi p_{\varepsilon_{l_m}}(.,\xi) \xrightarrow{m\to\infty} q_{\alpha,\xi,j} \qquad \text{in } C^0(\overline{B_1(z_j)}).$$

Together with *iii*) the uniqueness of the limit provides $q_{\alpha,\xi,j} = \partial^\alpha_\xi p(.,\xi)$. Consequently every arbitrary weak convergent subsequence of $(\partial^\alpha_\xi p_{\varepsilon_l})_{l\in\mathbb{N}}$ has the same weakly limit. Hence an application of [60], Chapter 3, Lemma 0.3 obtains the weak convergence of

$$\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi) \rightharpoonup \partial^\alpha_\xi p(.,\xi) \qquad \text{in } H^{\tilde{m}}_q(B_1(z_j))$$

for $m \to \infty$. Using the previous weak convergence first and inequality (5.72) afterwards, we get for all $j \in \mathbb{N}$ the existence of a constant C_α, independent of $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, such that

$$\begin{aligned}\|\partial^\alpha_\xi p(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))} &\leq \liminf_{l\to\infty} \|\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))} \leq \sup_{l\in\mathbb{N}} \|\partial^\alpha_\xi p_{\varepsilon_l}(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_j))}\\ &\leq C_\alpha. \end{aligned} \tag{5.73}$$

Due to the choice of the cover there is an $N \in \mathbb{N}$, independent of $x_0 \in \mathbb{R}^n$, where

$$B_1(x_0) \subseteq \bigcup_{k=1}^{N} B_1(z_{j_k}) \tag{5.74}$$

for $j_1, \ldots, j_N \in \mathbb{N}$. A combination of inequality (5.73) and (5.74) yields for arbitrary $x_0 \in \mathbb{R}^n$:

$$\|\partial^\alpha_\xi p(.,\xi)\|_{H^{\tilde{m}}_q(B_1(x_0))} \leq \sum_{k=1}^{N} \|\partial^\alpha_\xi p(.,\xi)\|_{H^{\tilde{m}}_q(B_1(z_{j_k}))} \leq C_\alpha.$$

Since C_α is independent of $x_0, \xi \in \mathbb{R}^n$, the previous inequality concludes the claim:

$$\sup_{\xi\in\mathbb{R}^n} \|\partial^\alpha_\xi p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} = \sup_{\xi\in\mathbb{R}^n} \sup_{x_0\in\mathbb{R}^n} \|\partial^\alpha_\xi p(.,\xi)\|_{H^{\tilde{m}}_q(B_1(x_0))} \leq C_\alpha.$$

□

5.6.2 Symbol Reduction of Double Symbols in $W^{\tilde{m},q}_{uloc}S^m_{0,0}$

The last missing piece towards a better characterization is the improvement of the results of Section 5.2. We start with a first technical estimate we need later on:

Proposition 5.41. *Let $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $n < 2l \leq N$. Moreover, let $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be a bounded subset and $a \in \mathscr{B}$. Considering $l_0 \in \mathbb{N}_0$ with $n < 2l_0 \leq N$, we define the function $r : \mathbb{R}^{4n} \to \mathbb{C}$ by*

$$r(x,\xi,\eta,y) := \langle y \rangle^{-2l_0} \langle D_\eta \rangle^{2l_0} a(x, \xi+\eta, x+y) \qquad \text{for all } x,\xi,\eta,y \in \mathbb{R}^n.$$

Then $\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \in L^1(\mathbb{R}^n_\eta)$ for all $x,\xi \in \mathbb{R}^n$. If we define

$$I(x,\xi) := \int \left[\int e^{-iy\cdot\eta} r(x,\xi,\eta,y)dy \right] đ\eta$$

for each $x,\xi \in \mathbb{R}^n$, then there is a constant C, independent of $\xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that

$$\|I(.,\xi)\|_{W^{\tilde{m},q}_{uloc}} \leq C\langle \xi \rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Proof: Let $\xi \in \mathbb{R}^n$ be arbitrary. For every $\tilde{\gamma} \in \mathbb{N}^n_0$ we get due to the boundedness of $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ and because of Lemma 4.73:

$$\begin{aligned}
&\left\| \partial_y^{\tilde{\gamma}} \left\{ \langle D_\eta \rangle^{2l_0} a(x,\xi+\eta,x+y) \right\} \right\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \\
&\leq \sum_{|\alpha| \leq l_0} C_\alpha \left\| \partial_y^{\tilde{\gamma}} D_\eta^{2\alpha} a(x,\xi+\eta,x+y) \right\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \\
&\leq \sum_{|\alpha| \leq l_0} C_\alpha \sup_{y \in \mathbb{R}^n} \left\| \partial_y^{\tilde{\gamma}} D_\eta^{2\alpha} a(x,\xi+\eta,x+y) \right\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C_{\tilde{\gamma}} \langle \xi+\eta \rangle^m
\end{aligned}$$

for all $y,\eta \in \mathbb{R}^n$. Here $C_{\tilde{\gamma}}$ is independent of $y,\xi,\eta \in \mathbb{R}^n$ and $a \in \mathscr{B}$. An application of the Leibniz rule, the last inequality and Remark 2.8 provides the existence of a constant C_l, independent of $x,y,\xi,\eta \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that

$$\|\langle D_y \rangle^{2l} r(x,\xi,\eta,y)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C_l \langle y \rangle^{-2l_0} \langle \xi+\eta \rangle^m$$

for all $x,y,\eta \in \mathbb{R}^n, l \in \mathbb{N}_0$ and $a \in \mathscr{B}$. Hence all assumptions of Proposition 5.15 are fulfilled. We conclude the proof by means of Proposition 5.15. □

In the same manner as in the proof of Lemma 5.19 the previous result enables us to prove the next lemma:

Lemma 5.42. *Let $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $n < 2l \leq N$. We define $\tilde{N} := N - (n+2)$ if n is even and $\tilde{N} := N - (n+1)$ else. Moreover, let $\mathscr{B} \subseteq W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ be bounded. If we define for each $a \in \mathscr{B}$ the function $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x,\eta+\xi,x+y)dyđ\eta \qquad \text{for all } x,\xi \in \mathbb{R}^n,$$

there is a constant C_γ, independent of $\xi \in \mathbb{R}^n$ and $a \in \mathscr{B}$, such that we have for each $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \tilde{N}$

$$\|\partial_\xi^\gamma a_L(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\gamma \langle \xi \rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Proof: Let $\tau > 0$ with the properties $\tau \notin \mathbb{N}$ and $\tau \leq \tilde{m} - n/q$ be arbitrary. Since $W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N) \subseteq C^\tau_* S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, Lemma 5.19 yields the existence of $a_L(x,\xi)$ for each $x, \xi \in \mathbb{R}^n$ and all $a \in \mathscr{B}$. We show the claim for the case $\gamma = 0$ first: Due to Remark 5.18 we have $a \in \mathscr{A}_0^{m,N}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$ for all $a \in \mathscr{B}$. Moreover, $N - \tilde{N} = 2k > n$ for a $k \in \mathbb{N}_0$. Therefore the assumptions of Theorem 4.41 are fulfilled. Using Theorem 4.41 we can write for each $x, \xi \in \mathbb{R}^n$, $a \in \mathscr{B}$ and $l_0 \in \mathbb{N}_0$ with $n < 2l_0 \leq N$:

$$\begin{aligned} a_L(x,\xi) &= \mathrm{Os}\text{-}\iint e^{-iy\cdot\eta} \langle y \rangle^{-2l_0} \langle D_\eta \rangle^{2l_0} a(x, \xi+\eta, x+y) dy đ\eta \\ &= \iint e^{-iy\cdot\eta} \langle y \rangle^{-2l_0} \langle D_\eta \rangle^{2l_0} a(x, \xi+\eta, x+y) dy đ\eta \qquad \text{for all } x, \xi \in \mathbb{R}^n. \end{aligned}$$

The last equality holds because of Proposition 5.16. While applying Proposition 5.41 we obtain the existence of a constant C, which is independent of $\xi \in \mathbb{R}^n$ and of $a \in \mathscr{B}$, such that

$$\begin{aligned} \|a_L(.,\xi)\|_{W^{\tilde{m},q}_{uloc}} &= \left\| \iint e^{-iy\cdot\eta} \langle y \rangle^{-2l_0} \langle D_\eta \rangle^{2l_0} a(x, \xi+\eta, x+y) dy đ\eta \right\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \\ &\leq C \langle \xi \rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n. \end{aligned}$$

Therefore we have checked the theorem in the case $\delta = 0$. Now we assume an arbitrary $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \tilde{N}$. Using $a \in \mathscr{A}_0^{m,N}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta)$ again all assumptions of Theorem 4.39 are fulfilled because of $N - \tilde{N} = 2k > n$ for a $k \in \mathbb{N}_0$. Thus an application of Theorem 4.39 yields

$$\partial_\xi^\gamma a_L(x,\xi) = \mathrm{Os}\text{-}\iint e^{-iy\cdot\eta} \partial_\xi^\gamma a(x, \eta+\xi, x+y) dy đ\eta.$$

We know that $\mathscr{B}^\gamma := \left\{ \partial_\xi^\gamma a : a \in \mathscr{B} \right\} \subseteq W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N - |\gamma|)$ is bounded because of Remark 4.72. Hence the first case, applied on the set $\mathscr{B}^\gamma$, gives us the existence of a constant C_γ, which is independent of $x, \xi \in \mathbb{R}^n$ and of $a \in \mathscr{B}$, such that

$$\|\partial_\xi^\gamma a_L(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\gamma \langle \xi \rangle^m \qquad \text{for all } \xi \in \mathbb{R}^n.$$

□

Making use of the previous lemma we are able to show the improvement of the symbol reduction in the non-smooth case:

Theorem 5.43. *Let $1 < q < \infty$, $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$ and $m \in \mathbb{R}$. Additionally let $N \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $N \geq 2l > n$. We define $\tilde{N} := N - (n+2)$ if n is even and $\tilde{N} := N - (n+1)$ else. Furthermore, let $\mathscr{B}$ be a bounded subset of $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$. Assuming $a \in \mathscr{B}$, we define $a_L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by*

$$a_L(x,\xi) := Os\text{-}\iint e^{-iy\cdot\eta} a(x, \eta+\xi, x+y) dy đ\eta$$

for all $x, \xi \in \mathbb{R}^n$. Then $\{a_L : a \in \mathscr{B}\} \subseteq W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; \tilde{N})$ is bounded and we have for every $a \in \mathscr{B}$ and for each $u \in \mathcal{S}(\mathbb{R}^n)$:

$$a(x, D_x, x')u = a_L(x, D_x)u. \tag{5.75}$$

Proof: Let $a \in \mathscr{B}$ be arbitrary. Our first goal is to verify that a_L is an element of $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; \tilde{N})$. Lemma 5.42 provides the existence of $a_L(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$ and the existence of a constant C_β, independent of $a \in \mathscr{B}$ and of $\xi \in \mathbb{R}^n$, such that

$$\|\partial^\beta_\xi a_L(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\beta \langle\xi\rangle^m \quad \text{for all } \xi \in \mathbb{R}^n, \beta \in \mathbb{N}^n_0 \text{ with } |\beta| \leq \tilde{N}. \tag{5.76}$$

Now let τ be such that $\tilde{m} - n/q - 1 < \tau < \tilde{m} - n/q$ in the case $n/q \in \mathbb{N}$ and $\lfloor \tilde{m} - n/q \rfloor < \tau < \tilde{m} - n/q$ otherwise. Since $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ is a subset of $C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$, Theorem 5.24 yields $a_L \in C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$. Therefore $\partial^\alpha_\xi \partial^\beta_x a_L \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$ and $\partial^\beta_x a_L(x,.) \in C^{\tilde{N}}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{N}$ and $|\beta| \leq \lfloor\tau\rfloor$. Together with inequality (5.76) we obtain our first purpose: $\{a_L : a \in \mathscr{B}\} \subseteq W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{N})$ is bounded.

Due to $a \in C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; N)$ we are able to apply Theorem 5.26. An application of Theorem 5.26 provides (5.75). Hence we have checked the claim. □

5.6.3 Characterization of Pseudodifferential Operators with Symbols in $W^{\tilde{m},q}_{uloc}S^m_{0,0}$

With all the work done in the last sections we are now in the position to verify the characterization of non-smooth pseudodifferential operators of the class $W^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for sufficiently large M. In the same manner as in Section 5.4, we prove the case $m = 0$ first. After that we generalize the obtained result to non-smooth pseudodifferential operators of arbitrary order by bringing it back to the case $m = 0$ by means of a order reducing operator. Our first goal is reached after the proof of the next theorem:

Theorem 5.44. *Let $1 < q < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $M \in \mathbb{N} \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}_0$ with $M \geq 2l > n$. Furthermore, we* *define $\tilde{M} := M - (n+2)$ if n is even and $\tilde{M} := M - (n+1)$ else. Considering $T \in \mathscr{A}^0_{0,0}(\tilde{m}, q)$ and $\tilde{M} \geq 1$, we get*

$$T \in OPW^{\tilde{m},q}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(L^q(\mathbb{R}^n)).$$

Proof: The proof of the theorem is essentially the same as that one of Theorem 5.36. We just have to replace the results for pseudodifferential operators with coefficients in the Hölder spaces with analog ones for pseudodifferential operators with coefficients in the uniformly local Sobolev spaces.
Therefore we have to use the continuous embedding $H^{\tilde{m}}_q(\mathbb{R}^n) \hookrightarrow W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ instead of the continuous embedding $H^{\tilde{m}}_q(\mathbb{R}^n) \hookrightarrow C^\tau(\mathbb{R}^n)$ in step two. Additionally we have to apply Theorem 5.43 instead of Theorem 5.26 and Theorem 5.24 in the second step.
In step three we have to replace Theorem 5.11 with Theorem 5.40. We also use the fact that $W^{\tilde{m},q}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M})$ is a subset of $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M})$ for every $\tau \in (0, \tilde{m} - n/q]$ with $\tau \notin \mathbb{N}$, cf. Remark 4.27. □

As before the previous theorem provides just a characterization for non-smooth pseudodifferential operators which are linear and bounded as maps from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Next we want to extend the improvement of the characterization for pseudodifferential operators of the same symbol-class, but of arbitrary order m. As in Section 5.4 the proof is based on the next statement:

Proposition 5.45. *Let $m \in \mathbb{R}$, $1 < q < \infty$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > n/q$. For every linear operator $P : H^m_q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ we get the following result: If $P\Lambda^{-m}$ is an element of $OPW^{\tilde{m},q}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap \mathscr{L}(L^q(\mathbb{R}^n))$, we have*

$$P \in OPW^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Proof: In order to check the claim we choose an arbitrary $0 < \tau \leq \tilde{m} - n/q$ such that $\tau \notin \mathbb{N}_0$. Since $W^{\tilde{m},q}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \subseteq C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, cf. Remark 4.27, we are able to apply Proposition 5.37 and get

$$P \in \mathrm{OP}C^\tau S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

On account of Remark 4.44 we know that the symbol of P is $\tilde{p}(x,\xi)\langle\xi\rangle^m$, if $\tilde{p}(x,\xi)$ is the symbol of $P\Lambda^{-m}$. Using $\tilde{p}(x,\xi) \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; M)$, we obtain $\tilde{p}(x,\xi)\langle\xi\rangle^m \in W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; M)$. Consequently P is even an element of $\mathrm{OP}W^{\tilde{m},q}_{uloc} S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$. □

The previous proposition allows us to show the characterization of non-smooth pseudodifferential operators with coefficients in the space $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ of arbitrary order.

Lemma 5.46. *Let $m \in \mathbb{R}$, $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $M \in \mathbb{N}_0 \cup \{\infty\}$ be such that there is an $l \in \mathbb{N}$ with $M \geq 2l > n$. We define $\tilde{M} := M - (n+2)$ if n is even and $\tilde{M} := M - (n+1)$ else. Considering $T \in \mathcal{A}^{m,M}_{0,0}(\tilde{m}, q)$ and $\tilde{M} \geq 1$ we have*

$$T \in OPW^{\tilde{m},q}_{uloc}S^m_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Proof: We obtain the claim in the same way as the claim of Lemma 5.38. We merely have to use Theorem 5.44 and Proposition 5.45 instead of Theorem 5.36 and Proposition 5.37. □

5.6.4 Characterization of Pseudodifferential Operators with Symbols in $W^{\tilde{m},q}_{uloc}S^m_{1,0}$

The last missing piece towards the improvement of the main theorem of this chapter is to combine the results of the last subsection with those ones of Section 5.5. Then we obtain the characterization of non-smooth pseudodifferential operators whose symbols are in the class $W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for sufficiently large M.

Theorem 5.47. *Let $m \in \mathbb{R}$, $1 < q < \infty$, $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/q$. Additionally let $M \in \mathbb{N}_0$ be such that there is an $l \in \mathbb{N}$ with $M \geq 2l > n$. We define $\tilde{M} := M - (n+2)$ if n is even and $\tilde{M} := M - (n+1)$ else. Considering $P \in \mathcal{A}^{m,M}_{1,0}(\tilde{m}, q)$ and $\tilde{M} \geq 1$ we obtain*

$$P \in OPW^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Proof: Let $\tau \in (0, \tilde{m} - n/q]$. Since all assumptions of Theorem 5.39 hold, an application of this theorem yields

$$P \in \mathrm{OP}C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1) \cap \mathscr{L}(H^m_q(\mathbb{R}^n), L^q(\mathbb{R}^n)).$$

Therefore it remains to show that P is an element of $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$. Thus let $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq \tilde{M} - 1$ be arbitrary. In the proof of Theorem 5.39 we have seen that $\mathrm{ad}(-ix)^\alpha P$ is an element of $\mathcal{A}^{m-|\alpha|, M-|\alpha|}_{1,0}(\tilde{m}, q)$. Hence an application of Lemma 5.46 provides

$$\mathrm{ad}(-ix)^\alpha P \in \mathrm{OP}W^{\tilde{m},q}_{uloc}S^{m-|\alpha|}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - |\alpha| - 1).$$

Since the symbol of $\mathrm{ad}(-ix)^\alpha P$ is $\partial^\alpha_\xi p(x,\xi)$, as we know from the proof of Theorem 5.39 we get

$$\|\partial^\alpha_\xi p(.,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

This implies that P is an element of $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$. □

5.7 Composition of Pseudodifferential Operators Revised

The present section is devoted to an application of the characterization of non-smooth pseudodifferential operators. In Section 4.3 we already have treated the composition of non-smooth pseudodifferential operators. Just in some specific cases we were able to show that the composition PQ of two non-smooth pseudodifferential operators P and Q is a non-smooth pseudodifferential operator again. It was the task of Remark 4.44 to verify this statement for P being a non-smooth pseudodifferential operator of the symbol-class $C^{\tilde{m},\tau}S^{m_1}_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n;M)$ and a smooth pseudodifferential operator Q having the symbol $q(\xi)\in S^{m_2}_{\rho,0}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$ which is independent of x for suitable $\tilde{m},\tau,m_1,m_2,M$ and ρ. With the characterization of non-smooth pseudodifferential operators at hand, we are in the position to prove a similar result if the non-smooth pseudodifferential operators $P\in \mathrm{OP}C^{\tilde{m}_1,\tau_1}S^{m_1}_{\rho_1,0}(\mathbb{R}^n\times\mathbb{R}^n;M_1)$ and $Q\in \mathrm{OP}C^{\tilde{m}_2,\tau_2}S^{m_2}_{\rho_2,0}(\mathbb{R}^n\times\mathbb{R}^n;M_2)$ are smooth enough: For instance a sufficient condition for PQ being a non-smooth pseudodifferential operator is $\tilde{m}_2+\tau_2>\max\{m_1;n+2-m_1\}$ if $\rho_1=\rho_2=1$ and $\tilde{m}_1$, M_1 and M_2 are large enough. This is the topic of the next theorem:

Theorem 5.48. *Let $m_i\in\mathbb{R}$, $M_i\in\mathbb{N}\cup\{\infty\}$ and $\rho_i\in\{0,1\}$ for $i\in\{1,2\}$. Additionally let $0<\tau_i<1$ and $\tilde{m}_i\in\mathbb{N}_0$ be such that $\tau_i+\tilde{m}_i>(1-\rho_i)n/2$ for $i\in\{1,2\}$. We define $k_i:=(1-\rho_i)n/2$ for $i\in\{1,2\}$, $\rho:=\min\{\rho_1;\rho_2\}$ and $m:=m_1+m_2+k_1+k_2$. Moreover, let $\tilde{m},M\in\mathbb{N}$ and $1<q<\infty$ be such that the next inequalities hold:*

i) $M\leq \min\{M_i-\max\{n/q;n/2\}:i\in\{1,2\}\}$,

ii) $n/q<\tilde{m}\leq\min\{\tilde{m}_1;\tilde{m}_2\}$,

iii) $\tilde{m}<\tilde{m}_2+\tau_2-m_1-k_1$,

iv) $\rho M+\tilde{m}<\tilde{m}_2+\tau_2+m_1+k_1$,

v) $\tilde{M}\geq 1$, *where* $\tilde{M}:=M-(n+2)$ *if n is even and* $\tilde{M}:=M-(n+1)$ *else,*

vi) $q=2$ *in the case* $(\rho_1,\rho_2)\neq(1,1)$.

Considering two symbols $p_i\in C^{\tilde{m}_i,\tau_i}S^{m_i}_{\rho_i,0}(\mathbb{R}^n\times\mathbb{R}^n;M_i)$, $i\in\{1,2\}$, we obtain

$$p_1(x,D_x)p_2(x,D_x)\in OPW^{\tilde{m},q}_{uloc}S^m_{\rho,0}(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-1).$$

Proof: Let $l\in\mathbb{N}$, $\tilde{\alpha}_1,\ldots,\tilde{\alpha}_l\in\mathbb{N}_0^n$ and $\tilde{\beta}_1,\ldots,\tilde{\beta}_l\in\mathbb{N}_0^n$ with $|\tilde{\alpha}|\leq M$, $|\tilde{\beta}|\leq\tilde{m}$ and $|\tilde{\alpha}_1|+|\tilde{\beta}_1|=\ldots=|\tilde{\alpha}_l|+|\tilde{\beta}_l|=1$ be arbitrary. Here $\tilde{\alpha}:=\tilde{\alpha}_1+\ldots+\tilde{\alpha}_l$ and $\tilde{\beta}:=\tilde{\beta}_1+\ldots+\tilde{\beta}_l$. Due to Remark 4.45, *i)* and *ii)* we know that

$$\mathrm{ad}(-ix)^{\tilde{\alpha}_l}\,\mathrm{ad}(D_x)^{\tilde{\beta}_l}\ldots\mathrm{ad}(-ix)^{\tilde{\alpha}_1}\,\mathrm{ad}(D_x)^{\tilde{\beta}_1}p_i(x,D_x)$$

is a pseudodifferential operator with symbol

$$\partial_\xi^{\tilde{\alpha}} D_x^{\tilde{\beta}} p_i \in C^{\tilde{m}_i - |\tilde{\beta}|, \tau_i} S^{m_i - \rho_i|\tilde{\alpha}|}_{\rho_i, 0}(\mathbb{R}^n \times \mathbb{R}^n; M_i - |\tilde{\alpha}|)$$

for $i \in \{1, 2\}$. Since $0 < \tilde{m}_1 - |\tilde{\beta}| + \tau_1$ because of *ii*), an application of Theorem 4.51 if $\rho_1 = 1$ and Theorem 4.53 else provides for all elements u of $H_q^{m_1 - \rho|\tilde{\alpha}| + k_1}(\mathbb{R}^n)$:

$$\begin{aligned} \| \operatorname{ad}(-ix)^{\tilde{\alpha}_l} \operatorname{ad}(D_x)^{\tilde{\beta}_l} \ldots \operatorname{ad}(-ix)^{\tilde{\alpha}_1} \operatorname{ad}(D_x)^{\tilde{\beta}_1} p_1(x, D_x) u \|_{L^q} &\leq C \|u\|_{H_q^{m_1 - \rho_1|\tilde{\alpha}| + k_1}} \\ &\leq C \|u\|_{H_q^{m_1 - \rho|\tilde{\alpha}| + k_1}}. \end{aligned} \tag{5.77}$$

Now let $k \in \mathbb{N}_0$ with $k \leq M$ be arbitrary. Making use of estimate *iii*) yields

$$m_1 - \rho k + k_1 \leq m_1 + k_1 < \tilde{m}_2 - \tilde{m} + \tau_2 \leq \tilde{m}_2 - |\tilde{\beta}| + \tau_2. \tag{5.78}$$

On account of inequality *iv*) we have

$$\begin{aligned} -(\tilde{m}_2 - |\tilde{\beta}| + \tau_2) \leq -(\tilde{m}_2 - \tilde{m} + \tau_2) &< m_1 - \rho M + k_1 \\ &\leq m_1 - \rho k + k_1. \end{aligned} \tag{5.79}$$

Because of (5.78) and (5.79) we are able to apply Theorem 4.51 if $\rho_1 = 1$ and Theorem 4.53 else and get for all $u \in H_q^{m - \rho(k + |\tilde{\alpha}|)}(\mathbb{R}^n)$:

$$\begin{aligned} \| \operatorname{ad}(-ix)^{\tilde{\alpha}_l} \operatorname{ad}(D_x)^{\tilde{\beta}_l} \ldots \operatorname{ad}(-ix)^{\tilde{\alpha}_1} \operatorname{ad}(D_x)^{\tilde{\beta}_1} p_2(x, D_x) u \|_{H_q^{m_1 - \rho k + k_1}} \\ \leq C \|u\|_{H_q^{m - \rho k - \rho_2|\tilde{\alpha}|}} \leq C \|u\|_{H_q^{m - \rho(|\tilde{\alpha}| + k)}}. \end{aligned} \tag{5.80}$$

We assume arbitrary $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha| \leq M$, $|\beta| \leq \tilde{m}$ and $|\alpha_1| + |\beta_1| = \ldots = |\alpha_l| + |\beta_l| = 1$. Here $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Using Proposition 2.55 first and (5.77) and (5.80) afterwards we obtain for all $u \in H_q^{m - \rho|\alpha|}(\mathbb{R}^n)$:

$$\begin{aligned} &\left\| \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} \ldots \operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} [p_1(x, D_x) p_2(x, D_x)] u \right\|_{L^q} \\ &\quad \leq C \sum_{\substack{\tilde{\alpha}_j + \gamma_j = \alpha_j \\ \tilde{\beta}_j + \delta_j = \beta_j}} \| [\operatorname{ad}(-ix)^{\tilde{\alpha}_l} \operatorname{ad}(D_x)^{\tilde{\beta}_l} \ldots \operatorname{ad}(-ix)^{\tilde{\alpha}_1} \operatorname{ad}(D_x)^{\tilde{\beta}_1} p_1(x, D_x)] \\ &\qquad\qquad [\operatorname{ad}(-ix)^{\gamma_l} \operatorname{ad}(D_x)^{\delta_l} \ldots \operatorname{ad}(-ix)^{\gamma_1} \operatorname{ad}(D_x)^{\delta_1} p_2(x, D_x)] u \right\|_{L^q} \\ &\quad \leq C \|u\|_{H_q^{m - \rho|\alpha|}}, \end{aligned}$$

where the constant C is independent of $u \in H_q^{m - \rho|\alpha|}(\mathbb{R}^n)$. Consequently the operator $p_1(x, D_x) p_2(x, D_x) \in \mathcal{A}^{m, M}_{\rho, 0}(\tilde{m}, q)$. Due to *ii*) and *v*) all assumptions of Theorem 5.47 and Lemma 5.46 hold. Hence an application of Theorem 5.47 in the case $\rho = 1$ and of Lemma 5.46 if $\rho = 0$ provides the claim. □

We are even able to improve this result for non-smooth pseudodifferential operators with coefficients in the uniformly local Sobolev Spaces. Here we consider merely symbols, which are smooth in ξ. But this is not a big disadvantage because in applications the symbols usually are smooth in ξ.

Theorem 5.49. *Let $m_i \in \mathbb{R}$ and $1 < q_i < \infty$ for $i \in \{1,2\}$. Additionally let $\tilde{m}_i \in \mathbb{N}_0$ with $\tilde{m}_i > n/q_i$ for $i \in \{1,2\}$. We define $m := m_1 + m_2$. Moreover let $\tilde{m}, M \in \mathbb{N}$ and $1 < q < \infty$ be such that the next inequalities hold:*

i) $n/q < \tilde{m} < \min\{\tilde{m}_1 - n/q_1; \tilde{m}_2 - n/q_2\}$,

ii) $\tilde{m} \leq \tilde{m}_2 - n(1/q_2 - 1/q)^+ - m_1$,

iii) $M + \tilde{m} < \tilde{m}_2 - n/q_2 + m_1$,

iv) $\tilde{M} \geq 1$, *where* $\tilde{M} := M - (n+2)$ *if n is even and* $\tilde{M} := M - (n+1)$ *else.*

Considering two symbols $p_i \in W^{\tilde{m}_i,q_i}_{uloc} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $i \in \{1,2\}$, we obtain

$$p_1(x,D_x)p_2(x,D_x) \in OPW^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

Proof: The proof of the theorem is essentially the same as that one of Theorem 5.48. We just have to replace Remark 4.45 with Remark 4.28 and Theorem 4.51 with Theorem 4.56. □

The previous theorem in fact is an improvement of Theorem 5.48 for non-smooth pseudodifferential operators $p_i(x,D_x)$ whose symbols $p_i(x,\xi)$ are in the symbol-class $W^{\tilde{m}_i,q_i}_{uloc} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ where $\tilde{m}_i, q_i$ and m_i are chosen as in Theorem 5.49 for $i = \{1,2\}$. In order to show this we assume, that all assumptions of Theorem 5.49 hold for $p_i(x,D_x)$, $i \in \{1,2\}$. Since $W^{\tilde{m}_i,q_i}_{uloc} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a subset of $C^{s_i} S^{m_i}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $s_i \in (0, \tilde{m}_i - n/q_i]$, $s_i \notin \mathbb{N}$, where $i \in \{1,2\}$, all assumptions of Theorem 5.48 hold if the s_i are sufficiently close to $\tilde{m}_i - n/q_i$ for $i \in \{1,2\}$. An application of Theorem 5.48 yields

$$p_1(x,D_x)p_2(x,D_x) \in \mathrm{OP}W^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1), \tag{5.81}$$

where in particular $\tilde{m} \in \mathbb{N}$ has to fulfill the next estimate:

$$\tilde{m} < s_2 - m_1 \leq \tilde{m}_2 - n/q_2 - m_1. \tag{5.82}$$

Due to Theorem 5.49, (5.81) holds for all $\tilde{m}$, where

$$\tilde{m} \leq \tilde{m}_2 - n(1/q_2 - 1/q)^+ - m_1 \tag{5.83}$$

holds instead of inequality (5.82). Since $q \in (1,\infty)$, inequality (5.83) is a less strict condition for $\tilde{m}$ than estimate (5.82). Hence Theorem 5.49 provides a better result than Theorem 5.48 in this case.

Note that results provide better mapping properties of classical pseudodifferential operators of the class $W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ than those ones of pseudodifferential operators whose symbols are in the class $W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, cf. Theorem 4.56 and Theorem 4.62. Consequently we certainly are able to improve the previous result for pseudodifferential operators of the symbol-class $W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ if one uses Theorem 4.62 and Remark 4.35 instead of Theorem 4.56 and Remark 4.28 in the proof of Theorem 5.49.

In the same way as the statement of Theorem 5.48 it should be possible to verify a similar result for the composition of two pseudodifferential operators of the symbol-class $H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ by using Remark 4.31 and Theorem 4.54 instead of Remark 4.45 and Theorem 4.51.

We also could have a look at the composition of two non-smooth pseudodifferential operators whose coefficients are either in a Hölder space, in a Bessel potential space or in an uniformly local Sobolev spaces, however in different spaces. For sure we obtain a result which is similar to that one of Theorem 5.48.

In order to illustrate the usefulness of the characterization of non-smooth pseudodifferential operators we will introduce another application in the next chapter.

Chapter 6

The Inverse of a Pseudodifferential Operator

A basic result in the theory of pseudodifferential operators allows one to directly implicate that the inverse of a pseudodifferential operator with a symbol in the Hörmander class $S^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, which is invertible as an operator on $L^2(\mathbb{R}^n)$, is again a pseudodifferential operator in the same symbol-class. This important statement was derived by R. Beals [16] and J. Ueberberg [74]. Their proof even showed that same holds for all Bessel potential spaces $H^s_2(\mathbb{R}^n)$, $s \in \mathbb{R}$, and that the spectrum is independent of the choice of the space.

E. Schrohe extended this result for weighted L^p-Sobolev-spaces in [64] and together with H.G. Leopold even for Besov spaces of variable order of differentiation $B^{s;a}_{p,q}(\mathbb{R}^n)$ in [46]. They verified that the spectrum of smooth pseudodifferential operators in certain symbol-classes is independent of the choice of the weighted L^p-Sobolev space and of the choice of the Besov space of variable order of differentiation respectively, cf. [46] and [64].

There are several other results for spectral invariance of smooth pseudodifferential operators in the literature, cf. e.g [33], [41] and [49].

We show the spectral invariance for non-smooth pseudodifferential operators whose symbols are in the symbol-class $C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ in this chapter. In analogy to the proof of J. Ueberberg in the smooth case, one uses the characterization of pseudodifferential operators via iterated commutators.

In this chapter we proceed as follows: Section 6.1 is devoted to the inverse of a non-smooth pseudodifferential operator P in the symbol-class $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$. We show that P^{-1} is also a non-smooth pseudodifferential operator of the symbol-class $C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$, where $s < \tau$. Unfortunately, in contrast to the smooth case, we loose some smoothness of the coefficients. Our next goal is to prove the spectral invariance of non-smooth pseudodifferential operators of the class $C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$ for sufficiently large N. To be more precise, we arrive at the following statement: The inverse of a non-smooth pseudodifferential operator of the order zero with coefficients in the Hölder space $C^{\tilde{m},\tau}(\mathbb{R}^n)$ is also a non-smooth pseudodifferential operator if its inverse is an element of $\mathscr{L}(H^r_q(\mathbb{R}^n))$ for one $|r| < \tilde{m}+\tau$. This is the topic of Section 6.3. Beyond the characterization of non-smooth pseudodifferential operators we also use the technique of difference quotients for the proof of the above mentioned statement. We introduce this technique in Section 6.2. We are able to improve the results of Section 6.3 in Section 6.4 for non-smooth pseudodifferential operators of the order zero with coefficients in the space $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$. Here we consider merely pseudodifferential operators with symbols which are smooth in ξ. But this should not be a big disadvantage because in applications to the regularity theory for partial differential equations the symbols usually are polynomials in ξ. It turns out that we even get a better result for the subsets $\mathrm{OP}H^{\tilde{m}}_q S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathrm{OP}W^{\tilde{m},q}_{uloc} S^0_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ of $\mathrm{OP}W^{\tilde{m},q}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In applications most pseudodifferential operators are classical ones. Hence the restriction to classical pseudodifferential operators is not that a big disadvantage.

6.1 The Inverse of a Pseudodifferential Operator in the Symbol-Class $C^\tau S^0_{0,0}$

In the present section we prove the following statement: The inverse of a pseudodifferential operator belonging to the symbol-class $C^{\tilde{m},\tau} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ is a pseudodifferential operator of the same symbol-class but with less regularity with respect to their coefficients. Hence we show the validity of the following statement:

Theorem 6.1. *Let $\tilde{m} \in \mathbb{N}_0$ and $0 < \tau < 1$ be arbitrary. We assume that $\hat{m} := \max\{k \in \mathbb{N}_0 : \tilde{m} + \tau - k > n/2\} > n/2$. For every non-smooth symbol $p \in C^{\tilde{m},\tau} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $p(x,D_x)^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we get*

$$p(x,D_x)^{-1} \in OPW^{\hat{m},2}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq OPC^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $s \in (0, \hat{m} - n/2]$ with $s \notin \mathbb{N}$.

J. Ueberberg proved a similar result for the smooth case, cf. [74], Theorem 4.3: the inverse of a smooth pseudodifferential operator of the order zero is a pseudodifferential operator of the same symbol-class if its inverse is an element of $\mathscr{L}(L^2(\mathbb{R}^n))$. To be more precise:

Theorem 6.2. *Let $1 < q < \infty$ and $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$.*

i) Considering a symbol $p \in S^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ where $p(x,D_x)^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we obtain $p(x,D_x)^{-1} \in OPS^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

ii) Assuming a symbol $p \in S^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ *where* $p(x, D_x)^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ *we get* $p(x, D_x)^{-1} \in OPS^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

In order to verify Theorem 6.1, we use the main idea of the proof in the smooth case: We want to apply the characterization of pseudodifferential operators. Thus we just have to show the boundedness of certain iterated commutators of $p(x, D_x)^{-1}$. Since we already know that the iterated commutators of $p(x, D_x)$ have these mapping properties, we try to write the iterated commutators of $p(x, D_x)^{-1}$ as a sum and compositions of $p(x, D_x)^{-1}$ and the iterated commutators of $p(x, D_x)$. Unfortunately, non-smooth pseudodifferential operators are in general not bounded as operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ like the smooth ones. Therefore we only are able to calculate the iterated operators formally first:

Remark 6.3 (Formal identities for the iterated commutators).
Let $m, s \in \mathbb{R}$, $1 < q < \infty$ and $M, \tilde{m} \in \mathbb{N}_0$ with $\tilde{m} + M \geq 1$. We assume the operator $P \in \mathscr{L}(H^{s+m}_q, H^s_q)$ which has an inverse $P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$ and the property

$$\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} \ldots \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} P \in \mathscr{L}(H^{s+m}_q, H^s_q)$$

for all $l \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, l\}$, $|\alpha_1| + \ldots + |\alpha_l| \leq M$ and $|\beta_1| + \ldots + |\beta_l| \leq \tilde{m}$. For two arbitrary multi-indices $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha + \beta| = 1$ the definition of the iterated commutators provides $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. We consider $|\beta| = 0$ and $\alpha = e_j$ for an arbitrary $j \in \{1, \ldots, n\}$ first. On account of $\operatorname{ad}(-ix_j)P$ being an element of $\mathscr{L}(H^{s+m}_q, H^s_q)$, we know that

$$\operatorname{ad}(-ix_j)Pu = -ix_j Pu + P(ix_j u) \in H^s_q(\mathbb{R}^n) \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n). \tag{6.1}$$

If $u \in \mathcal{S}(\mathbb{R}^n) \subseteq H^{m+s}_q(\mathbb{R}^n)$ we obtain $P(ix_j u) \in H^s_q(\mathbb{R}^n)$ due to the choice of P. Together with (6.1) this implies

$$-ix_j Pu \in H^s_q(\mathbb{R}^n) \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n). \tag{6.2}$$

Now we define $\mathscr{D} := \{Pu : u \in \mathcal{S}(\mathbb{R}^n)\} \subseteq H^s_q(\mathbb{R}^n)$. In order to show the density of $\mathscr{D}$ in $H^s_q(\mathbb{R}^n)$ we choose an arbitrary $v \in H^s_q(\mathbb{R}^n)$. On account of P^{-1} being an element of $\mathscr{L}(H^s_q, H^{s+m}_q)$ we have $u := P^{-1}v \in H^{s+m}_q(\mathbb{R}^n)$ and therefore $v = Pu$. Considering a sequence $(u_j)_{j \in \mathbb{N}_0} \subseteq \mathcal{S}(\mathbb{R}^n)$, which converges against u in $H^{s+m}_q(\mathbb{R}^n)$, we define $v_j := Pu_j$ for each $j \in \mathbb{N}_0$. Due to the assumptions P is an element of $\mathscr{L}(H^{s+m}_q, H^s_q)$. Thus we get the following convergence:

$$\lim_{j\to\infty} v_j = P \lim_{j\to\infty} u_j = Pu = v.$$

Since $v \in H^s_q(\mathbb{R}^n)$ was chosen arbitrary, this implies the density of $\mathscr{D}$ in $H^s_q(\mathbb{R}^n)$. Next we define the operator $Q : \mathscr{D} \to H^{s+m}_q(\mathbb{R}^n)$ by $Qu := -ix_j P^{-1}u + P^{-1}(ix_j u)$ for all $u \in \mathscr{D}$. Due to (6.2) we have

$$Q(Pu) = -ix_j P^{-1}Pu + P^{-1}(ix_j Pu) = -ix_j u + P^{-1}(ix_j Pu) \in H^{s+m}_q(\mathbb{R}^n)$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. Consequently Q is well-defined. For all $u \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\begin{aligned} Q(Pu) &= -ix_j u + P^{-1}(ix_j Pu) = P^{-1}P\{-ix_j u\} + P^{-1}(ix_j Pu) \\ &= P^{-1}\{P(-ix_j u) + ix_j Pu\} = -P^{-1}[\mathrm{ad}(-ix_j)P]u. \end{aligned} \tag{6.3}$$

With $\mathrm{ad}(-ix_j)P \in \mathscr{L}(H^{s+m}_q, H^s_q)$ and $P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$ we are able to prove the next inequality:

$$\begin{aligned} \|Q(Pu)\|_{H^{s+m}_q} &= \|P^{-1}[\mathrm{ad}(-ix_j)P]u\|_{H^{s+m}_q} \leq C\|\,\mathrm{ad}(-ix_j)Pu\|_{H^s_q} \leq C\|u\|_{H^{s+m}_q} \\ &= C\|P^{-1}Pu\|_{H^{s+m}_q} \leq C\|Pu\|_{H^s_q} \qquad \text{for all } u \in H^s_q(\mathbb{R}^n). \end{aligned}$$

Due to the density of $\mathscr{D}$ in $H^s_q(\mathbb{R}^n)$ this implies $Q \in \mathscr{L}(H^s_q, H^{s+m}_q)$. As a direct consequence we obtain

$$\mathrm{ad}(-ix_j)P^{-1} \in \mathscr{L}(H^s_q, H^{s+m}_q)$$

since $Qu = \mathrm{ad}(-ix_j)P^{-1}u$ for all $u \in \mathcal{S}(\mathbb{R}^n)$. Together with $\mathscr{D} \subseteq H^s_q(\mathbb{R}^n)$ and (6.3) we get

$$[\mathrm{ad}(-ix_j)P^{-1}]Pu = -P^{-1}[\mathrm{ad}(-ix_j)P]u \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

On account of $[\mathrm{ad}(-ix_j)P^{-1}]P \in \mathscr{L}(H^{s+m}_q)$ and $P^{-1}[\mathrm{ad}(-ix_j)P] \in \mathscr{L}(H^{s+m}_q)$ the previous equality provides

$$[\mathrm{ad}(-ix_j)P^{-1}]Pu = -P^{-1}[\mathrm{ad}(-ix_j)P]u \qquad \text{for all } u \in H^{s+m}_q(\mathbb{R}^n).$$

The surjectivity of $P \in \mathscr{L}(H^{s+m}_q; H^s_q)$ yields for all $v \in H^s_q(\mathbb{R}^n)$:

$$\begin{aligned} \mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta P^{-1}v &= [\mathrm{ad}(-ix_j)P^{-1}]v = -P^{-1}[\mathrm{ad}(-ix_j)P]P^{-1}v \\ &= -P^{-1}[\mathrm{ad}(-ix)^\alpha \,\mathrm{ad}(D_x)^\beta P]P^{-1}v. \end{aligned} \tag{6.4}$$

In the case $\beta = e_j$, $j \in \{1, \dots, n\}$ and $|\alpha| = 0$ we get the formula (6.4) for all $u \in \mathcal{S}(\mathbb{R}^n)$ in the same way as before. Moreover, let $l \in \mathbb{N}$, $\alpha_1, \dots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \dots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \dots, l\}$, $|\alpha_1| + \ldots + |\alpha_l| \leq M$ and $|\beta_1| + \ldots + |\beta_l| \leq \tilde{m}$. Denoting $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$ we get by mathematical induction with respect to l:

$$\mathrm{ad}(-ix)^{\alpha_1} \,\mathrm{ad}(D_x)^{\beta_1} \ldots \mathrm{ad}(-ix)^{\alpha_l} \,\mathrm{ad}(D_x)^{\beta_l} P^{-1} = \sum_{\substack{(\alpha^1_1+\ldots+\alpha^1_l)+\ldots+(\alpha^l_1+\ldots+\alpha^l_l)=\alpha \\ (\beta^1_1+\ldots+\beta^1_l)+\ldots+(\beta^l_1+\ldots+\beta^l_l)=\beta}} R_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l}$$

where

$$\begin{aligned} R_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l} :=& C_{\alpha^1_1,\ldots,\alpha^l_l,\beta^1_1,\ldots,\beta^l_l} P^{-1} \\ & \circ \left[\mathrm{ad}(-ix)^{\alpha^1_l} \,\mathrm{ad}(D_x)^{\beta^1_l} \ldots \mathrm{ad}(-ix)^{\alpha^1_1} \,\mathrm{ad}(D_x)^{\beta^1_1} P\right] P^{-1} \\ & \circ \ldots \circ \left[\mathrm{ad}(-ix)^{\alpha^l_l} \,\mathrm{ad}(D_x)^{\beta^l_l} \ldots \mathrm{ad}(-ix)^{\alpha^l_1} \,\mathrm{ad}(D_x)^{\beta^l_1} P\right] P^{-1}. \end{aligned}$$

For $l = 1$ we already checked the claim. The induction step can be shown in a similar way.

With this remark at hand, we now are able to show Theorem 6.1:

Proof of Theorem 6.1: First of all note that $p(x, D_x) \in \mathcal{A}^0_{0,0}(\hat{m}, 2)$ due to Example 5.2 *ii*) and $P^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$. Therefore the assumptions of Remark 6.3 are fulfilled. Let $l \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}_0^n$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}_0^n$ with $|\alpha_j + \beta_j| = 1$ for all $j \in \{1, \ldots, n\}$ and $|\beta_1 + \ldots + \beta_l| \leq \hat{m}$ be arbitrary. Denoting $P := p(x, D_x)$, an application of Remark 6.3 provides:

$$\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} \ldots \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} P^{-1} = \sum_{\substack{(\alpha_1^1+\ldots+\alpha_l^1)+\ldots+(\alpha_1^l+\ldots+\alpha_l^l)=\alpha \\ (\beta_1^1+\ldots+\beta_l^1)+\ldots+(\beta_1^l+\ldots+\beta_l^l)=\beta}} R_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l}$$

where $R_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l}$ are defined as in Remark 6.3. Since $P \in \mathcal{A}^0_{0,0}(\hat{m}, 2)$ and $P^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we obtain

$$\operatorname{ad}(-ix)^{\alpha_1} \operatorname{ad}(D_x)^{\beta_1} \ldots \operatorname{ad}(-ix)^{\alpha_l} \operatorname{ad}(D_x)^{\beta_l} P^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n)).$$

Hence $P^{-1} \in \mathcal{A}^0_{0,0}(\hat{m}, 2)$. Considering $0 < s \leq \hat{m} - n/2$, $s \notin \mathbb{N}$, Theorem 5.44 and Remark 4.27 yields

$$P^{-1} \in \mathrm{OP}W^{\hat{m},2}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq \mathrm{OP}C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

□

Using Example 5.2 *i*) instead of Example 5.2 *ii*) in the proof of Theorem 6.1 provides a similar result:

Lemma 6.4. *Let $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > n/2$ and $0 < \tau < 1$. For every non-smooth symbol $p \in C^{\tilde{m},\tau} S^{-n/2}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $p(x, D_x)^{-1} \in \mathscr{L}(L^2(\mathbb{R}^n))$ we get*

$$p(x, D_x)^{-1} \in OPW^{\tilde{m},2}_{uloc} S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n) \subseteq OPC^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $s \in (0, \tilde{m} - n/2]$ with $s \notin \mathbb{N}$.

6.2 Properties of Difference Quotients

Our next aim is to prove the spectral invariance for pseudodifferential operators P whose symbols are in the symbol-class $C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $\tau > 0$. The proof is again based on the formal identities for the iterated commutators of P^{-1}, cf. Remark 6.3. In this case $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P$, $|\alpha| \neq 0$ are pseudodifferenial operators of negative order $-|\alpha|$. Hence the order of the Bessel potential space increases by applying $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P$, $|\alpha| \neq 0$. Therefore $P^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ is not sufficient. We even need $P^{-1} \in \mathscr{L}(H^{-s}_q(\mathbb{R}^n))$ for certain $s \in \mathbb{N}_0$. As we always try to restrict the assumptions to a minimal, we use the tools of difference quotients in order to get $P^{-1} \in \mathscr{L}(H^{-s}_q(\mathbb{R}^n))$ if $P^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ is assumed.

First of all, we define difference quotients:

Definition 6.5. Let $h \in \mathbb{R}\backslash\{0\}$ and $j \in \{1, \ldots, n\}$. For $u \in H_p^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $1 < p < \infty$ we define the *difference quotient* of u by

$$\partial_{x_j}^h u(x) := h^{-1}\{u(x + he_j) - u(x)\} \qquad \text{for all } x \in \mathbb{R}^n.$$

Note that the definition of the difference quotient implies $\partial_{x_j}^h u \in H_p^s(\mathbb{R}^n)$ for each $u \in H_p^s(\mathbb{R}^n)$ and all $h \in \mathbb{R}\backslash\{0\}$, $j \in \{1, \ldots, n\}$.

The difference quotient has the following useful properties:

Remark 6.6. Let $h \in \mathbb{R}\backslash\{0\}$ and $j \in \{1, \ldots, n\}$.

i) For $u \in H_p^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $1 < p < \infty$ we have

$$\frac{e^{ihe_j \cdot \xi} - 1}{h} \hat{u}(\xi) = \widehat{\partial_{x_j}^h u}(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

ii) $(\partial_{x_j}^h)^* = -\partial_{x_j}^{-h}$.

Proof: The claims follow directly from the definition of the Fourier transformation and of the adjoint operator on account of variable transformation. $\square$

Difference quotients of non-smooth symbols with coefficients in the Hölder spaces are non-smooth symbols, too. We just loose some smoothness of the coefficients:

Lemma 6.7. *Let $m \in \mathbb{R}$, $\tilde{m} \in \mathbb{N}$, $0 < \tau < 1$ and $M \in \mathbb{N}_0 \cup \{\infty\}$. Considering $p \in C^{\tilde{m},\tau} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M)$, we get the boundedness of*

$$\left\{\partial_{x_j}^h p(x, \xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq C^{\tilde{m}-1,\tau} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M)$$

for all $j \in \{1, \ldots, n\}$.

Proof: Let $j \in \{1, \ldots, n\}$ be arbitrary. Using the fundamental theorem of calculus and $p \in C^{\tilde{m},\tau} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n; M)$ first and Remark 2.37 afterwards, we get for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq M$:

$$\begin{aligned}
\|\partial_\xi^\alpha \partial_{x_j}^h p(x, \xi)\|_{C^{\tilde{m}-1,\tau}(\mathbb{R}_x^n)} &= |h|^{-1} \|\partial_\xi^\alpha p(x + he_j, \xi) - \partial_\xi^\alpha p(x, \xi)\|_{C^{\tilde{m}-1,\tau}(\mathbb{R}_x^n)} \\
&= |h|^{-1} \left\| \int_0^1 \frac{d}{dt} \left[\partial_\xi^\alpha p(x + the_j, \xi)\right] dt \right\|_{C^{\tilde{m}-1,\tau}(\mathbb{R}_x^n)} \\
&\leq \int_0^1 \left\|D_{x_j} \partial_\xi^\alpha p(x + the_j, \xi)\right\|_{C^{\tilde{m}-1,\tau}(\mathbb{R}_x^n)} dt \\
&\leq \int_0^1 \left\|\partial_\xi^\alpha p(x + the_j, \xi)\right\|_{C^{\tilde{m},\tau}(\mathbb{R}_x^n)} dt \leq C_{\alpha,j} \langle \xi \rangle^{m-|\alpha|}.
\end{aligned}$$

$\square$

The previous lemma enables us to show that the commutator of a non-smooth pseudodifferential operator with a difference quotient is also a non-smooth pseudodifferential operator:

Lemma 6.8. *Let* $1 < q < \infty$, $m \in \mathbb{R}$, $\tilde{m} \in \mathbb{N}$, $0 < \tau < 1$. *Additionally let* $M \in \mathbb{N}_0 \cup \{\infty\}$ *with* $M > n/2$ *for* $q \geq 2$ *and* $M > n/q$ *else. For a non-smooth symbol* $p \in C^{\tilde{m},\tau} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ *we get for every* $j \in \{1, \ldots, n\}$ *and* $h \in \mathbb{R}\backslash\{0\}$*:*

$$[\partial^h_{x_j}, p(x, D_x)]u(x) = \left[\left(\partial^{-h}_{x_j} p\right)(x, D_x)u\right](x + he_j) \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n.$$

Moreover, for all $|s| < \tilde{m} - 1 + \tau$ *there is a constant* C*, independent of* $h \in \mathbb{R}\backslash\{0\}$*, such that*

$$\|[\partial^h_{x_j}, p(x, D_x)]u\|_{H^s_q} \leq C\|u\|_{H^{s+m}_q} \qquad \text{for all } u \in H^{s+m}_q(\mathbb{R}^n),$$

where $j \in \{1, \ldots, n\}$ *and* $h \in \mathbb{R}\backslash\{0\}$.

Proof: Let $u \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \{1, \ldots, n\}$ be arbitrary. Then an application of Remark 6.6 *i)* yields for all $x \in \mathbb{R}^n$:

$$\begin{aligned}
\partial^h_{x_j}[p(x, D_x)u(x)] &= h^{-1}\{p(x, D_x)u(x + he_j) - p(x, D_x)u(x)\} \\
&= h^{-1}\left\{\int e^{i(x+he_j)\cdot\xi} p(x + he_j, \xi)\hat{u}(\xi) đ\xi - \int e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi) đ\xi\right\} \\
&= \int e^{i(x+he_j)\cdot\xi} \frac{p(x + he_j, \xi) - p(x, \xi)}{h}\hat{u}(\xi) đ\xi + \int e^{ix\cdot\xi} \frac{e^{ihe_j\cdot\xi} - 1}{h} p(x, \xi)\hat{u}(\xi) đ\xi \\
&= \int e^{i(x+he_j)\cdot\xi} \left(\partial^{-h}_{x_j} p\right)(x + he_j, \xi)\hat{u}(\xi) đ\xi + \int e^{ix\cdot\xi} p(x, \xi)\widehat{\partial^h_{x_j} u}(\xi) đ\xi \\
&= \left[\left(\partial^{-h}_{x_j} p\right)(x, D_x)u\right](x + he_j) + \left[p(x, D_x)\left(\partial^h_{x_j} u\right)\right](x).
\end{aligned}$$

Hence we obtain the first part of the lemma:

$$\begin{aligned}
[\partial^h_{x_j}, p(x, D_x)]u(x) &= \partial^h_{x_j}[p(x, D_x)u(x)] - p(x, D_x)[\partial^h_{x_j} u(x)] \\
&= \left[\left(\partial^{-h}_{x_j} p\right)(x, D_x)u\right](x + he_j) \qquad (6.5)
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Since Lemma 6.7 provides the boundedness of

$$\left\{\partial^{-h}_{x_j} p(x, \xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq C^{\tilde{m}-1,\tau} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M),$$

we arise for all $|s| < \tilde{m} - 1 + \tau$ from Theorem 4.51 the existence of a constant C, independent of $h \in \mathbb{R}\backslash\{0\}$, such that

$$\left\|\left(\partial^{-h}_{x_j} p\right)(x, D_x)u\right\|_{H^s_q} \leq C\|u\|_{H^{s+m}_q} \qquad \text{for all } u \in H^{s+m}_q(\mathbb{R}^n). \qquad (6.6)$$

Using (6.5) and (6.6) we get for all $u \in \mathcal{S}(\mathbb{R}^n)$:

$$\left\| [\partial_{x_j}^h, p(x, D_x)]u \right\|_{H_q^s} = \left\| \left[\left(\partial_{x_j}^{-h} p \right) (x, D_x) \right] \tau_{-he_j}(u) \right\|_{H_q^s} \leq C \|u\|_{H_q^{s+m}}.$$

Here C is independent of $h \in \mathbb{R}\backslash\{0\}$. Due to the density of $\mathcal{S}(\mathbb{R}^n)$ in $H_q^{s+m}(\mathbb{R}^n)$ the claim is true. □

Difference quotients are very useful tools in order to check the existence of weak derivatives. This is the topic of the next theorem:

Theorem 6.9. *(Difference quotients and weak derivatives)*

i) We suppose $1 < p < \infty$ and $u \in H_p^{s+1}(\mathbb{R}^n)$ with $s \in \mathbb{R}$. Then there is a constant C, independent of $h \in \mathbb{R}\backslash\{0\}$ and $u \in H_p^{s+1}(\mathbb{R}^n)$, such that

$$\|\partial_{x_j}^h u\|_{H_p^s} \leq C \|\partial_{x_j} u\|_{H_p^s}$$

for all $j \in \{1, \ldots, n\}$ and all $h \in \mathbb{R}\backslash\{0\}$.

ii) Let $1 < p < \infty$ and $u \in H_p^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$. Additionally there exists a constant C, independent of $h \in \mathbb{R}\backslash\{0\}$, such that

$$\|\partial_{x_j}^h u\|_{H_p^s} \leq C$$

for all $j \in \{1, \ldots, n\}$ and all $h \in \mathbb{R}\backslash\{0\}$. Then $u \in H_p^{s+1}(\mathbb{R}^n)$ and $\|\partial_{x_j} u\|_{H_p^s} \leq C$.

Note that assertion ii) is false for $p = 1$ while i) also holds for $p = 1$.

This result is an extension of Theorem 5.8.3 in [29], where L.C. Evans proved the case $s = 0$ for a compact subset of $\mathbb{R}^n$.

Proof of Theorem 6.9. The proof of the case $s = 0$ is essentially the same as that one of Theorem 5.8.3 in [29] given by L.C. Evans. Now we show the case $s \neq 0$. Assuming an arbitrary $s \in \mathbb{R}\backslash\{0\}$ and $u \in H_p^{s+1}(\mathbb{R}^n)$, $\langle \xi \rangle^s \in S_{1,0}^s(\mathbb{R}^n \times \mathbb{R}^n)$ yields $\langle D_x \rangle^s u \in W_p^1(\mathbb{R}^n)$. Since $\partial_{x_j}^{-h} \langle \xi \rangle^s = 0$, Lemma 6.8 provides $[\partial_{x_j}^h, \langle D_x \rangle^s] = 0$ for all $h \in \mathbb{R}\backslash\{0\}$ and $j \in \{1, \ldots, n\}$. Hence we are able to commute $\partial_{x_j}^h$ and $\langle D_x \rangle^s$. An application of the case $s = 0$ provides for each $j \in \{1, \ldots, n\}$:

$$\begin{aligned}\|\partial_{x_j}^h u\|_{H_p^s} &= \|\langle D_x \rangle^s \partial_{x_j}^h u\|_{L^p} = \|\partial_{x_j}^h \langle D_x \rangle^s u\|_{L^p} \leq C \|\partial_{x_j} \langle D_x \rangle^s u\|_{L^p} \\ &= C \|\langle D_x \rangle^s \partial_{x_j} u\|_{L^p} = C \|\partial_{x_j} u\|_{H_p^s},\end{aligned}$$

where C is independent of $h \in \mathbb{R}\backslash\{0\}$ and of $u \in H_p^{s+1}(\mathbb{R}^n)$. Therefore (i) holds for all $s \in \mathbb{R}$. Similary to (i) we obtain the case $s \in \mathbb{R}$ of (ii) as a consequence of case $s = 0$ and Lemma 6.8. □

The previous theorem allows us to verify the following proposition:

Proposition 6.10. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{R}$ and $1 < q < \infty$. Moreover, let P be an operator, which fulfills for all $s \in \{r, r+1, \dots, r+k\}$ the properties*

i) $P \in \mathscr{L}(H_q^s, H_q^s)$,

ii) $P \in \mathscr{L}(H_q^{r+k+1}, H_q^{r+k+1})$,

iii) $\{[P, \partial_{x_j}^h] : h \in \mathbb{R}\backslash\{0\}\} \subseteq \mathscr{L}(H_q^s, H_q^s)$ *is bounded for all* $j \in \{1, \dots, n\}$,

iv) $P^{-1} \in \mathscr{L}(H_q^r, H_q^r)$.

Then $P^{-1} \in \mathscr{L}(H_q^s, H_q^s)$ *for each* $s \in \{r, r+1, \dots, r+k+1\}$.

Proof: We prove the claim by mathematical induction with respect to s. In the case $s = r$ there is nothing to show. Let $s \in \{r, r+1, \dots, r+k\}$ be arbitrary. Furthermore, we choose an arbitrary $j \in \{1, \dots, n\}$ and $f \in H_q^{s+1}(\mathbb{R}^n) \subseteq H_q^s(\mathbb{R}^n)$. The induction hypothesis provides the existence of a $u \in H_q^s(\mathbb{R}^n)$ such that $u = P^{-1}f$. Due to $P \in \mathscr{L}(H_q^s, H_q^s)$, we get $Pu \in H_q^s(\mathbb{R}^n)$. Therefore the definition of $\partial_{x_j}^h$ provides $\partial_{x_j}^h(Pu) \in H_q^s(\mathbb{R}^n)$. In a similar way we get that $P(\partial_{x_j}^h u)$ is an element of $H_q^s(\mathbb{R}^n)$. An application of P^{-1} to $P(\partial_{x_j}^h u) = [P, \partial_{x_j}^h]u + \partial_{x_j}^h(Pu)$ yields

$$\partial_{x_j}^h u = P^{-1}\{[P, \partial_{x_j}^h]u + \partial_{x_j}^h(Pu)\}. \tag{6.7}$$

We already know that P^{-1} is an element of $\mathscr{L}(H_q^s, H_q^s)$ because of the induction hypothesis. Together with (6.7) and the assumptions the next estimate can be verified:

$$\begin{aligned}\|\partial_{x_j}^h u\|_{H_q^s} &= \|P^{-1}\{[P, \partial_{x_j}^h]u + \partial_{x_j}^h(Pu)\}\|_{H_q^s} \leq C\|[P, \partial_{x_j}^h]u + \partial_{x_j}^h(Pu)\|_{H_q^s} \\ &\leq C\|[P, \partial_{x_j}^h]u\|_{H_q^s} + C\|\partial_{x_j}^h f\|_{H_q^s}.\end{aligned}$$

On account of *iii*) and Theorem 6.9 *i*) we get the existence of a constant C, independent of $h \in \mathbb{R}\backslash\{0\}$ and $u \in H_q^s(\mathbb{R}^n)$, such that

$$\|\partial_{x_j}^h u\|_{H_q^s} \leq C\|u\|_{H_q^s} + C\|\partial_{x_j} f\|_{H_q^s} \leq C\|u\|_{H_q^s} + C\|f\|_{H_q^{s+1}} \leq C.$$

An application of Theorem 6.9 *ii*) provides $u \in H_q^{s+1}(\mathbb{R}^n)$. Therefore P is linear, bounded and bijective as map from $H_q^{s+1}(\mathbb{R}^n)$ to $H_q^{s+1}(\mathbb{R}^n)$. Then P^{-1} is an element of $\mathscr{L}(H_q^{s+1}, H_q^{s+1})$ by means of the bounded inverse theorem, cf. [8], Satz 5.8. □

As a direct consequence of the previous proposition we obtain the central result of this section: If the inverse of a non-smooth pseudodifferential operator with coefficients in the Hölder spaces of the order zero is an element of $\mathscr{L}(H_q^r, H_q^r)$ for one r, then this is even true in a neighbourhood of r:

Lemma 6.11. *Let* $1 < q < \infty$, $0 < \tau < 1$, $\tilde{m} \in \mathbb{N}$ *and* $N \in \mathbb{N}_0 \cup \{\infty\}$ *with* $N > n/2$ *for* $q \geq 2$ *and* $N > n/q$ *else. We define* $k := \max\{l \in \mathbb{N}_0 : r+l < \tilde{m}+\tau\}$ *for one* $r \in \mathbb{R}$ *with* $|r| < \tilde{m}+\tau$. *Considering a symbol* $p \in C^{\tilde{m},\tau}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$, *where* $p(x, D_x)^{-1} \in \mathscr{L}(H^r_q, H^r_q)$ *we obtain*

$$p(x, D_x)^{-1} \in \mathscr{L}(H^s_q; H^s_q) \qquad \text{for all } s \in [-r-k, r+k]. \tag{6.8}$$

Proof: In order to check the theorem, we want to apply Proposition 6.10 to $P := p(x, D_x)$ and its adjoint Operator P^*. Therefore we have to verify the assumptions of Proposition 6.10. Because of Theorem 4.51 we know that

$$P \in \mathscr{L}(H^s_q; H^s_q) \qquad \text{for all } |s| < \tilde{m} + \tau. \tag{6.9}$$

Additionally Lemma 6.8 provides for every $j \in \{1, \ldots, n\}$ the boundedness of

$$\left\{[P, \partial^h_{x_j}] : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq \mathscr{L}(H^s_q; H^s_q) \qquad \text{for all } |s| < \tilde{m} - 1 + \tau. \tag{6.10}$$

Due to (6.9) and (6.10) the assumptions of Proposition 6.10 are fulfilled, which implies

$$P^{-1} \in \mathscr{L}(H^s_q, H^s_q) \qquad \text{for all } s \in \{r, \ldots, r+k\}. \tag{6.11}$$

Thus it remains to prove (6.8) for all $s \in \{-r-k, \ldots, r-1\}$. Then we get the claim by means of Theorem 2.53. Since we want to use a duality argument we define $1 < q' < \infty$ by $1/q' + 1/q = 1$. An application of Lemma 2.51 to (6.9) and (6.11) gives us:

$$P^* \in \mathscr{L}(H^s_{q'}, H^s_{q'}) \qquad \text{for all } |s| < \tilde{m} + \tau, \tag{6.12}$$

$$(P^*)^{-1} = (P^{-1})^* \in \mathscr{L}(H^s_{q'}, H^s_{q'}) \qquad \text{for all } s \in \{-r-k, \ldots, -r\}. \tag{6.13}$$

On account of Remark 6.6 we have for all $j \in \{1, \ldots, n\}$ and $h \in \mathbb{R}\backslash\{0\}$:

$$\begin{aligned}[P^*, \partial^h_{x_j}] &= P^*\partial^h_{x_j} - \partial^h_{x_j}P^* = (\partial^{-h}_{x_j})^*P^* - P^*(\partial^{-h}_{x_j})^* = (P\partial^{-h}_{x_j} - \partial^{-h}_{x_j}P)^* \\ &= [P, \partial^{-h}_{x_j}]^*.\end{aligned}$$

In the same way as in the proof of Lemma 2.51, we get for each $j \in \{1, \ldots, n\}$ by duality from (6.10) the boundedness of

$$\left\{[P^*, \partial^h_{x_j}] = [P, \partial^{-h}_{x_j}]^* : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq \mathscr{L}(H^s_{q'}; H^s_{q'}) \tag{6.14}$$

for all $|s| < \tilde{m} - 1 + \tau$. Now we are able to apply Proposition 6.10 again since (6.12), (6.13) and (6.14) hold. Hence Proposition 6.10 provides

$$(P^{-1})^* = (P^*)^{-1} \in \mathscr{L}(H^s_{q'}, H^s_{q'}) \qquad \text{for all } s \in \{-r, \ldots, r+k\}.$$

Consequently the claim holds for all $s \in \{-r-k, \ldots, r\}$ by duality, cf. Lemma 2.51. □

6.3 Spectral Invariance of Pseudodifferential Operators in the Symbol-Class $C^{\tilde{m},\tau}S^0_{1,0}$

With all the work done in the last section, we are now in the position to show the spectral invariance of pseudodifferential operators in the symbol-class $C^{\tilde{m},\tau}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for sufficiently large M. As in Theorem 6.1 the proof is based on the formal identity for the iterated commutators of P^{-1} calculated in Remark 6.3. But in contrast to Theorem 6.1, the order of the iterated commutators of P are not always zero. The order also can be negative. Hence the property that P^{-1} is a bounded map from $H^{-s}_p(\mathbb{R}^n)$ to $H^{-s}_p(\mathbb{R}^n)$, is needed in the proof for certain $s \in \mathbb{N}_0$ and not only for $s = 0$. However, Lemma 6.11 enables us to limit this assumption on P^{-1} to $P^{-1} \in \mathscr{L}(H^{-s}_p, H^{-s}_p)$ for just one s.

Theorem 6.12. *Let $1 < q_0 < \infty$, $0 < \tau < 1$ and $\tilde{m}, \hat{m} \in \mathbb{N}_0$ with $\tilde{m} \geq \hat{m} > n/q_0$. Additionally let $M \in \mathbb{N}_0$ be such that $n < 2l \leq M \leq \tilde{m} - \hat{m}$ for some $l \in \mathbb{N}$. We define $\tilde{M} := M-(n+2)$ if n is even and $\tilde{M} := M-(n+1)$ else. Furthermore, let $N \in \mathbb{N} \cup \{\infty\}$ with $N - M > n/2$ if $q_0 \geq 2$ and $N - M > n/q_0$ else. Considering a symbol $p \in C^{\tilde{m},\tau}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$, where $p(x, D_x)^{-1} \in \mathscr{L}(H^r_{q_0}, H^r_{q_0})$ for one $|r| < \tilde{m} + \tau$ we get*

$$p(x, D_x)^{-1} \in OPW^{\hat{m},q_0}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

In the case $\tilde{M} - 1 > n/\tilde{q}$ for one $1 < \tilde{q} \leq 2$, we even have

$$p(x, D_x)^{-1} \in \mathscr{L}(L^q, L^q) \qquad \text{for all } q \in [\tilde{q}; \infty) \cup \{q_0\}.$$

Proof: First of all note that $|M| = M \leq \tilde{m} - 1 \leq r + k$, where k is defined as in Lemma 6.11. Therefore an application of Lemma 6.11 provides the boundedness of

$$p(x, D_x)^{-1} \in \mathscr{L}(H^{-s}_{q_0}, H^{-s}_{q_0}) \qquad \text{for all } s \in \{0, \ldots, M\}. \tag{6.15}$$

Let $l \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_l \in \mathbb{N}^n_0$ and $\beta_1, \ldots, \beta_l \in \mathbb{N}^n_0$ with $|\alpha_j| + |\beta_j| = 1$ for all $j \in \{1, \ldots, l\}$, $|\alpha| \leq M$ and $|\beta| \leq \hat{m}$ where $\alpha := \alpha_1 + \ldots + \alpha_l$ and $\beta := \beta_1 + \ldots + \beta_l$. Then Remark 4.45 yields that the operator

$$\mathrm{ad}(-ix)^{\alpha_l}\, \mathrm{ad}(D_x)^{\beta_l} \ldots \mathrm{ad}(-ix)^{\alpha_1}\, \mathrm{ad}(D_x)^{\beta_1} p(x, D_x)$$

is an element of $\mathrm{OP}C^{\tilde{m}-|\beta|,\tau}S^{-|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N - |\alpha|)$ and therefore also an element of $\mathrm{OP}C^{\tilde{m}-\hat{m},\tau}S^{-|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N - M)$. Because of Theorem 4.51 we obtain for all $|s| < \tilde{m} - \hat{m} + \tau$

$$\mathrm{ad}(-ix)^{\alpha_l}\, \mathrm{ad}(D_x)^{\beta_l} \ldots \mathrm{ad}(-ix)^{\alpha_1}\, \mathrm{ad}(D_x)^{\beta_1} p(x, D_x) \in \mathscr{L}(H^{-s-|\alpha|}_{q_0}, H^{-s}_{q_0}). \tag{6.16}$$

Since $M-|\alpha| \leq M \leq \tilde{m}-\hat{m} < \tilde{m}-\hat{m}+\tau$ the boundedness of (6.16) holds in particular for each $s \in \{0,\dots,M-|\alpha|\}$. Setting $P := p(x,D_x)$ the assumptions of Remark 6.3 are fulfilled because of (6.15) and (6.16). Hence we get due to Remark 6.3 the equality

$$\operatorname{ad}(-ix)^{\alpha_1}\operatorname{ad}(D_x)^{\beta_1}\dots\operatorname{ad}(-ix)^{\alpha_l}\operatorname{ad}(D_x)^{\beta_l}P^{-1} = \sum_{\substack{(\alpha_1^1+\ldots+\alpha_l^1)+\ldots+(\alpha_1^l+\ldots+\alpha_l^l)=\alpha \\ (\beta_1^1+\ldots+\beta_l^1)+\ldots+(\beta_1^l+\ldots+\beta_l^l)=\beta}} R_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l}$$

where

$$\begin{aligned} R_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l} :=& C_{\alpha_1^1,\ldots,\alpha_l^l,\beta_1^1,\ldots,\beta_l^l}P^{-1} \\ &\circ \left[\operatorname{ad}(-ix)^{\alpha_l^1}\operatorname{ad}(D_x)^{\beta_l^1}\dots\operatorname{ad}(-ix)^{\alpha_1^1}\operatorname{ad}(D_x)^{\beta_1^1}P\right]P^{-1} \\ &\circ\ldots\circ \left[\operatorname{ad}(-ix)^{\alpha_l^l}\operatorname{ad}(D_x)^{\beta_l^l}\dots\operatorname{ad}(-ix)^{\alpha_1^l}\operatorname{ad}(D_x)^{\beta_1^l}P\right]P^{-1}. \end{aligned}$$

Hence (6.15) and (6.16) imply the boundedness of

$$\operatorname{ad}(-ix)^{\alpha_1}\operatorname{ad}(D_x)^{\beta_1}\dots\operatorname{ad}(-ix)^{\alpha_l}\operatorname{ad}(D_x)^{\beta_l}P^{-1} : H_{q_0}^{-|\alpha|}(\mathbb{R}^n) \to L^{q_0}(\mathbb{R}^n).$$

Consequently P^{-1} is an element of $\mathcal{A}_{1,0}^{0,M}(\hat{m},q_0)$. An application of Theorem 5.47 and Remark 4.27 provides for each $0<\tilde{\tau}\leq \hat{m}-n/q_0$ with $\tilde{\tau}\notin\mathbb{N}$:

$$\begin{aligned} p(x,D_x)^{-1} &\in \mathrm{OP}W_{uloc}^{\hat{m},q_0}S_{1,0}^0(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-1)\cap\mathscr{L}(L^{q_0}(\mathbb{R}^n)) \\ &\subseteq C^{\tilde{\tau}}S_{1,0}^0(\mathbb{R}^n\times\mathbb{R}^n;\tilde{M}-1). \end{aligned}$$

Finally, considering $\tilde{M}-1>n/\tilde{q}$ for one $1<\tilde{q}\leq 2$ we obtain for every $q\in[\tilde{q},\infty)$ due to Theorem 4.51 the boundedness of

$$P^{-1} : L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n).$$

□

Now we know that the following statement holds for a pseudodifferential operator with symbol $p\in C^{\tilde{m},\tau}S_{1,0}^0(\mathbb{R}^n\times\mathbb{R}^n;N)$: Its inverse is also a pseudodifferential operator of the same symbol-class but with less smoothness in x and ξ. Here $\tilde{m}$, τ and N are chosen as in the previous theorem. Thus the question arises whether this statement is related to the spectral invariance. This is the topic of the next corollary:

Corollary 6.13. *We assume that all assumptions of Theorem 6.12 hold. Additionally we choose an arbitrary but fixed $\tilde{q}\in(1,2]$ fulfilling the conditions of Theorem 6.12 and denote*

$$P_{L^q} := p(x,D_x) : L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n) \qquad \text{for all } \tilde{q}\leq q<\infty.$$

Then we obtain the spectral invariance of these operators:

$$\sigma(P_{L^q}) = \sigma(P_{L^r}) \qquad \text{for all } \tilde{q}\leq q,r<\infty.$$

Proof: Let $q \in [\tilde{q}, \infty)$ be arbitrary. We assume that λ is not in the spectrum of P_{L^q}. Therefore $\lambda - p(x, D_x)$ is invertible in $\mathscr{L}(L^q(\mathbb{R}^n))$. On account of Example 3.2 *ii*) and Lemma 4.7, $\lambda - p(x, D_x)$ is an element of $\mathrm{OP}C^{\tilde{m},\tau}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; N)$. Hence all assumptions of Theorem 6.12 are fulfilled. This implies

$$(\lambda - p(x, D_x))^{-1} \in \mathscr{L}(L^r(\mathbb{R}^n)) \qquad \text{for all } \tilde{q} \le r < \infty.$$

Hence λ is not in the spectrum $\sigma(P_{L^r})$ of P_{L^r}. Therefore we have

$$\mathbb{C}\backslash\sigma(P_{L^q}) \subseteq \mathbb{C}\backslash\sigma(P_{L^r}) \qquad \text{for all } \tilde{q} \le q, r < \infty.$$

In particular this implies $\mathbb{C}\backslash\sigma(P_{L^r}) \subseteq \mathbb{C}\backslash\sigma(P_{L^q})$. This gives us the spectral invariance of $p(x, D_x)$:

$$\sigma(P_{L^q}) = \sigma(P_{L^r}) \qquad \text{for all } \tilde{q} \le q, r < \infty.$$

□

Now one may wonder whether it is possible to prove that $p(x, D_x)^{-1}$ is even an element of $\mathrm{OP}W^{\hat{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ in the case that all assumptions of the previous theorem are fulfilled and additionally the symbol of $p(x, D_x)$ is in the symbol-class $C^{\tilde{m},\tau}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Unfortunately in general this is not the case as we can see in the next example:

Example 6.14. Let $s > 0$, $1 < q_0 < \infty$ and $\tau > s + \lfloor n/q_0 \rfloor + n + 4$. Additionally let $p(\xi) \in S^0_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ be a symbol which is not constantly equal to zero and where $p(D_x)^{-1} \in \mathscr{L}(L^{q_0}(\mathbb{R}^n), L^{q_0}(\mathbb{R}^n))$. Moreover, we choose a function $a \in C^\tau(\mathbb{R}^n)$ with the following two properties: there is no open set $U \subseteq \mathbb{R}^n$, $U \neq \varnothing$ such that $a|_U \in C^\infty(U)$ and there are two constants $c, C > 0$ such that $C > a(x) > c$ for all $x \in \mathbb{R}^n$. Then $T := a(x)p(D_x) \in C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ fulfills all assumptions of Theorem 6.12 for $M = n + 3$ and $\hat{m} := \lfloor \tau \rfloor - (n + 3)$. Consequently T^{-1} is an element of $\mathrm{OP}W^{\hat{m},q_0}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1)$, where $\tilde{M}$ is defined as in Theorem 6.12, but $T^{-1} \notin \mathrm{OP}C^{\tilde{\tau}}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\tilde{\tau} \in (0, \hat{m} - n/q_0]$. In particular $T^{-1} \notin \mathrm{OP}W^{\hat{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ due to Remark 4.27.

Proof: First of all note that $T \in C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ because of Example 4.4. We define $b : \mathbb{R}^n \to \mathbb{R}$ by $b(x) := (a(x))^{-1}$ for all $x \in \mathbb{R}^n$. Then $b \in C^\tau(\mathbb{R}^n)$. Together with the fact that $p(D_x)^{-1} \in \mathrm{OP}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ due to Theorem 6.2, we are able to write $T^{-1} = p(D_x)^{-1}b(x)$. In particular the boundedness of b and $p(D_x)^{-1} \in \mathscr{L}(L^{q_0}(\mathbb{R}^n), L^{q_0}(\mathbb{R}^n))$ imply $T^{-1} \in \mathscr{L}(L^{q_0}(\mathbb{R}^n), L^{q_0}(\mathbb{R}^n))$. Therefore all assumptions of Theorem 6.12 are fulfilled for $M = n+3$ and $\hat{m} := \lfloor \tau \rfloor - (n+3)$. Let $\tilde{\tau} \in (0, \hat{m} - n/q_0]$ be arbitrary but fixed. Assuming $T^{-1} \in C^{\tilde{\tau}}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ there is a kernel $\tilde{k} : \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \to \mathbb{C}$ such that $\tilde{k}(x, .) \in C^\infty(\mathbb{R}^n\backslash\{0\})$ for each $x \in \mathbb{R}^n$ and

$$T^{-1}f(x) = \int \tilde{k}(x, x - y)f(y)dy \tag{6.17}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$ due to Theorem 4.63. An application of Theorem 3.14 provides the existence of a kernel $k \in C^\infty(\mathbb{R}^n \backslash \{0\})$ such that

$$p(D_x)^{-1}u(x) = \int k(x-y)u(y)dy \qquad \text{for all } x \notin \operatorname{supp} u \tag{6.18}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. Now let $(\delta_\varepsilon)_{\varepsilon>0} \subseteq C_c^\infty(\mathbb{R}^n)$ be a Dirac family, i.e. for all $\varepsilon > 0$ we have $\delta_\varepsilon \geq 0$, $\int \delta_\varepsilon(x)dx = 1$ and $\lim_{\varepsilon\to 0} \int_{|x|\geq d} \delta_\varepsilon(x)dx = 0$ for every $d > 0$. The properties of the convolution, cf. e.g [38], Theorem 10.7, imply $\delta_\varepsilon * b \in C^\infty(\mathbb{R}^n)$ for each $\varepsilon > 0$. Since b is bounded we even get the boundedness of every derivative of $\delta_\varepsilon * b$: Let $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ be arbitrary. Then

$$\begin{aligned} |\partial_x^\alpha(\delta_\varepsilon * b)(x)| = |((\partial_x^\alpha \delta_\varepsilon) * b)(x)| &\leq \int |(\partial_x^\alpha \delta_\varepsilon)(y)|\, |b(x-y)|\, dy \leq \|b\|_{L^\infty} \|\partial_x^\alpha \delta_\varepsilon\|_{L^1} \\ &\leq C_{\alpha,\varepsilon}, \end{aligned}$$

where $C_{\alpha,\varepsilon}$ is independent of $x \in \mathbb{R}^n$. In the case $|\alpha| = 0$ the constant $C_{\alpha,\varepsilon}$ is even independent of $\varepsilon > 0$. In order to verify the last estimate we have used [38], Theorem 10.7 and $\partial_x^\alpha \delta_\varepsilon \subseteq C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. In particular we know, that $\delta_\varepsilon * b \in C_b^\infty(\mathbb{R}^n) \subseteq C_{poly}^\infty(\mathbb{R}^n)$ for every $\varepsilon > 0$. On account of Lemma 2.17 we get $(\delta_\varepsilon * b)f \in \mathcal{S}(\mathbb{R}^n)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Additionally we obtain for all $f \in \mathcal{S}(\mathbb{R}^n)$ with $x \notin \operatorname{supp} f$ the existence of a constant C, independent of $\varepsilon > 0$, such that

$$|k(x-y)(\delta_\varepsilon * b)(y)f(y)| \leq C|k(x-y)f(y)| \in L^1(\mathbb{R}_y^n). \tag{6.19}$$

Using the properties of the convolution with a Dirac family, cf. e.g [38], Remark 10.12, yields the pointwise convergence of

$$(\delta_\varepsilon * b)(y)f(y) \xrightarrow{\varepsilon\to 0} b(y)f(y) \tag{6.20}$$

for all $y \in \mathbb{R}^n$. This implies the pointwise convergence of

$$k(x-y)(\delta_\varepsilon * b)(y)f(y) \xrightarrow{\varepsilon\to 0} k(x-y)b(y)f(y) \tag{6.21}$$

for all $y \in \mathbb{R}^n$. In view of (6.19) and (6.21) we can apply Lebesgue's theorem. Therefore we get for all $x \in \mathbb{R}^n$ and each $f \in \mathcal{S}(\mathbb{R}^n)$ with $x \notin \operatorname{supp} f$:

$$\lim_{\varepsilon\to 0} \int k(x-y)(\delta_\varepsilon * b)(y)f(y)dy = \int k(x-y)b(y)f(y)dy. \tag{6.22}$$

In Theorem 3.18 we have shown the boundedness of $p(D_x)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Note that $f \in \mathcal{S}(\mathbb{R}^n)$ and $b \in C_b^0(\mathbb{R}^n)$ implies $bf \in L^2(\mathbb{R}^n)$. Using this boundedness and (6.20) first we obtain together with (6.18) and (6.22) for all $f \in \mathcal{S}(\mathbb{R}^n)$ the equality

$$T^{-1}f(x) = p(D_x)^{-1}\left(b(x)f(x)\right) = p(D_x)^{-1}\left[\lim_{\varepsilon\to 0}(\delta_\varepsilon * b)(x)f(x)\right]$$

$$= \lim_{\varepsilon\to 0} p(D_x)^{-1}\left[(\delta_\varepsilon * b)(x)f(x)\right] = \lim_{\varepsilon\to 0}\int k(x-y)(\delta_\varepsilon * b)(y)f(y)dy$$
$$= \int k(x-y)b(y)f(y)dy \qquad \text{for all } x\notin \operatorname{supp} f. \tag{6.23}$$

Here we are able to apply equality (6.18) because $(\delta_\varepsilon * b)f \in \mathcal{S}(\mathbb{R}^n)$. Now we fix $x \in \mathbb{R}^n$ such that $\tilde{k}(x,.)$ is not constantly equal to zero. Such a choice is possible, otherwise we would have $p \equiv 0$. A combination of (6.17) and (6.23) provides for all $f \in \mathcal{S}(\mathbb{R}^n)$ with $x \notin \operatorname{supp} f$:

$$\int k(x-y)b(y)f(y)dy = T^{-1}f(x) = \int \tilde{k}(x,x-y)f(y)dy.$$

An application of the fundamental lemma of calculus of variations, cf. e.g [5], Theorem A.7, yields

$$k(x-y)b(y) = \tilde{k}(x,x-y) \qquad \text{for all } y \in \mathbb{R}^n\backslash\{x\}$$

since $k(x-y)$, $\tilde{k}(x,x-y)$ and $b(y)$ are continuous with respect to $y \in \mathbb{R}^n\backslash\{x\}$. By means of variable transformation we obtain

$$k(z) = a(x-z)\tilde{k}(x,z) \in C^\infty(\mathbb{R}^n_z\backslash\{0\}). \tag{6.24}$$

Now we choose a $z \in \mathbb{R}^n\backslash\{0\}$ with $\tilde{k}(x,z) \neq 0$. Due to $\tilde{k}(x,.) \in C^\infty(\mathbb{R}^n\backslash\{0\})$, there is a $\delta > 0$ such that $\tilde{k}(x,\tilde{z}) \neq 0$ for all $\tilde{z} \in B_\delta(z)$ and $0 \notin B_\delta(z)$. Together with $\tilde{k}(x,.) \in C^\infty(\mathbb{R}^n\backslash\{0\})$ and (6.24) we obtain $a \in C^\infty(B_\delta(x-z))$. This is a contradiction to the choice of a. Therefore $T^{-1} \notin C^{\tilde{\tau}}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. □

6.4 Spectral Invariance of Pseudodifferential Operators in the Symbol-Class $W^{\tilde{m},q}_{uloc}S^0_{1,0}$

The present section serves to improve Theorem 6.12 for non-smooth pseudodifferential operators of the order zero with coefficients in the uniformly local Sobolev spaces.

Verifying the proof of Theorem 6.12 we see that we need similar results for pseudodifferential operators whose symbols are in $W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ instead of Lemma 6.11 and Remark 4.45. We will check these results in the same way as those ones for pseudodifferential operators with coefficients in the Hölder spaces. The proof of the following lemma is one step in this direction.

Lemma 6.15. *Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1+n/q$. Considering a symbol $p \in W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we get the boundedness of*

$$\left\{\partial^h_{x_j} p(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq W^{\tilde{m}-1,q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $j \in \{1,\ldots,n\}$.

Proof: Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that supp $\psi \subseteq \overline{B_2(0)}$ and $\psi(x) = 1$ for all $x \in B_1(0)$. Assuming an arbitrary $\alpha \in \mathbb{N}_0^n$ we get for all $\xi \in \mathbb{R}^n$ due to Theorem 6.9 and $p \in W^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\begin{aligned}
\|\partial_\xi^\alpha \partial_{x_j}^h p(x,\xi)\|_{W^{\tilde{m}-1,q}_{uloc}(\mathbb{R}^n_x)} &= \sum_{|\beta|\leq \tilde{m}-1} \sup_{y\in\mathbb{R}^n} \|\partial_{x_j}^h \partial_x^\beta \partial_\xi^\alpha p(.,\xi)\|_{L^q(B_1(y))} \\
&\leq \sum_{|\beta|\leq \tilde{m}-1} \sup_{y\in\mathbb{R}^n} \|\partial_{x_j}^h \partial_x^\beta \partial_\xi^\alpha [p(x,\xi)\psi(x-y)]\|_{L^q(\mathbb{R}^n_x)} \\
&\leq C \sum_{|\beta|\leq \tilde{m}-1} \sup_{y\in\mathbb{R}^n} \|\partial_{x_j} \partial_x^\beta \partial_\xi^\alpha [p(x,\xi)\psi(x-y)]\|_{L^q(\mathbb{R}^n_x)} \\
&\leq C \sum_{|\beta|\leq \tilde{m}} \sup_{y\in\mathbb{R}^n} \|\partial_x^\beta \partial_\xi^\alpha p(.,\xi)\|_{L^q(B_2(y))} \\
&\leq C \|\partial_\xi^\alpha p(x,\xi)\|_{W^{\tilde{m},q}_{uloc}(\mathbb{R}^n_x)} \leq C\langle\xi\rangle^{m-|\alpha|}.
\end{aligned}$$

Here C is independent of $\xi \in \mathbb{R}^n$ and $h \in \mathbb{R}\backslash\{0\}$. This implies the boundedness of

$$\left\{\partial_{x_j}^h p(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq W^{\tilde{m}-1,q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n). \qquad \square$$

Making use of the previous lemma, enables us to estimate the commutator of a difference quotient and a non-smooth pseudodifferential operator whose coefficient is in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$.

Lemma 6.16. *Let $1 < \tilde{q}, q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1 + n/q$. Assuming a symbol $p \in W^{\tilde{m},q}_{uloc} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we get for every $j \in \{1, \ldots, n\}$ and all $h \in \mathbb{R}\backslash\{0\}$:*

$$[\partial_{x_j}^h, p(x,D_x)]u(x) = \left(\partial_{x_j}^{-h} p\right)(x,D_x)u(x+he_j) \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n.$$

Moreover, for all $-\tilde{m}+1+n/q < s \leq \tilde{m}-1-n(1/q-1/\tilde{q})^+$ there is a constant C, independent of $h \in \mathbb{R}\backslash\{0\}$, such that

$$\|[\partial_{x_j}^h, p(x,D_x)]u\|_{H^s_{\tilde{q}}} \leq C\|u\|_{H^{s+m}_{\tilde{q}}} \qquad \text{for all } u \in H^{s+m}_{\tilde{q}}(\mathbb{R}^n),$$

where $j \in \{1, \ldots, n\}$.

Proof: The proof of the lemma is essentially the same as that one of Lemma 6.8. We just have to replace Lemma 6.7 with Lemma 6.15 and Theorem 4.51 with Theorem 4.56. $\square$

The last missing piece towards the improvement of the spectral invariance result is to verify that whenever the inverse of a non-smooth pseudodifferential operator whose coefficient is in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$ is an element of $\mathscr{L}(H^r_{\tilde{q}}, H^r_{\tilde{q}})$ for one r, then this is even true in a neighbourhood of r.

Lemma 6.17. *Let* $1 < q, \tilde{q} < \infty$ *and* $\tilde{m} \in \mathbb{N}$ *with* $\tilde{m} > 1 + n/q$. *Considering a non-smooth symbol* $p \in W^{\tilde{m},q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, *where the inverse operator* $p(x, D_x)^{-1} \in \mathscr{L}(H^r_{\tilde{q}}, H^r_{\tilde{q}})$ *for one* $-\tilde{m} + n/q < r \leq \tilde{m} - n(1/q - 1/\tilde{q})^+$, *we obtain*

$$p(x, D_x)^{-1} \in \mathscr{L}(H^s_{\tilde{q}}, H^s_{\tilde{q}}) \qquad \text{for all } s \in [r - l, r + k].$$

Here k *and* l *are defined by* $k := \max\{\tilde{k} \in \mathbb{N}_0 : r + \tilde{k} \leq \tilde{m} - n(1/q - 1/\tilde{q})^+\}$ *and* $l := \max\{\tilde{l} \in \mathbb{N}_0 : -\tilde{m} + n/q < r - \tilde{l}\}$.

Proof: Using Theorem 4.56 instead of Theorem 4.51 and Lemma 6.16 instead of Lemma 6.8 the statement follows in the same way as that one of Lemma 6.11. □

Comparing the previous result with that one of Lemma 6.11 the difference lies in the choice of the neighbourhood of r. The previous lemma allows us to show the improvement of the spectral invariance for non-smooth pseudodifferential operators with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$, now:

Theorem 6.18. *Let* $1 < q, q_0 < \infty$ *and* $\tilde{m} \in \mathbb{N}_0$ *with* $\tilde{m} > \max\{1 + n/q, n/q_0\}$. *Additionally let* $\hat{m} \in \mathbb{N}_0$ *with* $n/q_0 < \hat{m} \leq \max\{r \in \mathbb{N}_0 : r < \tilde{m} - n/q\}$. *Moreover, let* $M \in \mathbb{N}_0$ *be such that* $n < 2l \leq M < \tilde{m} - \hat{m} - n/q$ *for one* $l \in \mathbb{N}$. *We define* $\tilde{M} := M - (n + 2)$ *if* n *is even and* $\tilde{M} := M - (n + 1)$ *else. Considering a symbol* $p \in W^{\tilde{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, *where* $p(x, D_x)^{-1} \in \mathscr{L}(H^r_{q_0}, H^r_{q_0})$ *for one* $-\tilde{m} + n/q < r \leq \tilde{m} - n(1/q - 1/q_0)^+$ *we get*

$$p(x, D_x)^{-1} \in OPW^{\hat{m},q_0}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

In the case $\tilde{M} - 1 > n/\tilde{q}$ *for one* $1 < \tilde{q} \leq 2$, *we even have*

$$p(x, D_x)^{-1} \in \mathscr{L}(L^{\tilde{q}}, L^{\tilde{q}}) \qquad \text{for all } \tilde{q} \in [\tilde{q}; \infty) \cup \{q_0\}.$$

Proof: As mentioned before we get the statement in the same way as that one of Theorem 6.12. We just have to replace Lemma 6.11 with Lemma 6.17 and Remark 4.45 with Remark 4.28. Moreover, we have to use Theorem 4.56 instead of Theorem 4.51. □

The previous theorem in fact is an improvement for a non-smooth pseudodifferential operator P of the symbol-class $W^{\tilde{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ because Theorem 6.18 holds for the less strict assumption $-\tilde{m} + n/q < r \leq \tilde{m} - n(1/q - 1/q_0)^+$. Since $W^{\tilde{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a subset of $C^s S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $s \in (0, \tilde{m} - n/q]$ with $s \notin \mathbb{N}$, cf. Remark 4.27, all assumptions of Theorem 6.12 hold if s is sufficiently close to $\tilde{m} - n/q$ and r sufficiently small. An application of Theorem 6.12 yields

$$P^{-1} \in \mathrm{OP}W^{\hat{m},q_0}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1), \tag{6.25}$$

where $\hat{m} \in \mathbb{N}_0$ with the properties *i)* $\lfloor s\rfloor \geq \hat{m} > n/q_0$,

ii) $n < 2l \leq M \leq \lfloor s\rfloor - \hat{m}$ for one $l \in \mathbb{N}$.

Due to Theorem 6.18, (6.25) holds for all $\hat{m}$ with the properties

iii) $n/q_0 < \hat{m} \leq \max\{r \in \mathbb{N}_0 : r < \tilde{m} - n/q\}$,

iv) $n < 2l \leq M < \tilde{m} - \hat{m} - n/q$ for one $l \in \mathbb{N}$.

Assuming $n/q \in \mathbb{N}$ we have

$$\lfloor s\rfloor < s \leq \tilde{m} - n/q = \lfloor \tilde{m} - n/q\rfloor = \max\{r \in \mathbb{N}_0 : r < \tilde{m} - n/q\}.$$

Hence *i)* and *ii)* are stronger conditions for $\hat{m}$ than *iii)* and *iv)*. Therefore Theorem 6.18 provides a more general result than Theorem 6.12 in this case.

It turns out that we even get a better result for the subsets $H^{\tilde{m}}_q S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $W^{\tilde{m},q}_{uloc} S^0_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ of $W^{\tilde{m},q}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In the same manner as for non-smooth pseudodifferential operators of the symbol-class $W^{\tilde{m},q}_{uloc} S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we show the spectral invariance of non-smooth pseudodifferential operators of the symbol-classes $H^{\tilde{m}}_q S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ or $W^{\tilde{m},q}_{uloc} S^0_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence we have to show that difference quotients of symbols in the symbol-classes $H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ or $W^{\tilde{m},q}_{uloc} S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$ are again symbols of the same symbol-class, but of lower order. This is done in the next two lemmas:

Lemma 6.19. *Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{R}$ with $\tilde{m} > 1 + n/q$. Considering $p \in H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we get the boundedness of*

$$\left\{\partial^h_{x_j} p(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq H^{\tilde{m}-1}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $j \in \{1,\ldots,n\}$.

Proof: Let $\alpha \in \mathbb{N}^n_0$ be arbitrary. Since $p \in H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ we know that $\partial^\alpha_\xi p(x,\xi) \in H^{\tilde{m}}_q(\mathbb{R}^n_x)$ for all $\xi \in \mathbb{R}^n$. An application of Theorem 6.9 provides for each $\xi \in \mathbb{R}^n$:

$$\begin{aligned}\|\partial^\alpha_\xi \partial^h_{x_j} p(x,\xi)\|_{H^{\tilde{m}-1}_q(\mathbb{R}^n_x)} = \|\partial^h_{x_j} \partial^\alpha_\xi p(x,\xi)\|_{H^{\tilde{m}-1}_q(\mathbb{R}^n_x)} &\leq C\|\partial_{x_j} \partial^\alpha_\xi p(x,\xi)\|_{H^{\tilde{m}-1}_q(\mathbb{R}^n_x)}\\ &\leq C\|\partial^\alpha_\xi p(x,\xi)\|_{H^{\tilde{m}}_q(\mathbb{R}^n_x)} \leq C\langle\xi\rangle^{m-|\alpha|},\end{aligned}$$

where C is independent of $\xi \in \mathbb{R}^n$ and $h \in \mathbb{R}\backslash\{0\}$. Hence the claim holds. □

Lemma 6.20. *Let $1 < q < \infty$, $m \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1 + n/q$. Considering $p \in W^{\tilde{m},q}_{uloc} S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$, we get the boundedness of*

$$\left\{\partial^h_{x_j} p(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq W^{\tilde{m}-1,q}_{uloc} S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $j \in \{1,\ldots,n\}$.

Proof: We know from Lemma 6.15 the boundedness of the set

$$\left\{\partial^h_{x_j}p(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq W^{\tilde{m}-1,q}_{uloc}S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Therefore it remains to show for all $h \in \mathbb{R}\backslash\{0\}$ that $\partial^h_{x_j}p(x,\xi)$ is even an element of $W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence let $h \in \mathbb{R}\backslash\{0\}$ and $j \in \{1,\ldots,n\}$ be arbitrary. Since $p \in W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$, there exists an expansion where p_k are homogeneous of degree $m-k$ in ξ (for $|\xi| \geq 1$) such that we have for all $N \in \mathbb{N}_0$

$$p(x,\xi) - \sum_{k=0}^{N} p_k(x,\xi) \in W^{\tilde{m},q}_{uloc}S^{m-N-1}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

Let $N \in \mathbb{N}_0$ be arbitrary. In the same way as in the proof of Lemma 6.15 we get the boundedness of:

$$\begin{aligned}&\left\{\partial^h_{x_j}p(x,\xi) - \sum_{k=0}^{N}\partial^h_{x_j}p_k(x,\xi) : h \in \mathbb{R}\backslash\{0\}\right\}\\ &\quad = \left\{\partial^h_{x_j}\left[p(x,\xi) - \sum_{k=0}^{N}p_k(x,\xi)\right] : h \in \mathbb{R}\backslash\{0\}\right\} \subseteq W^{\tilde{m}-1,q}_{uloc}S^{m-N-1}_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).\end{aligned}$$

Additionally $\partial^h_{x_j}p_k(x,\xi)$ is homogeneous of degree $m-k$ in ξ (for $|\xi| \geq 1$) for every $k \in \mathbb{N}_0$ because of

$$\begin{aligned}\partial^h_{x_j}p_k(x,r\xi) &= h^{-1}\left[p_k(x+he_j,r\xi) - p_k(x,r\xi)\right]\\ &= h^{-1}r^{m-k}\left[p_k(x+he_j,\xi) - p_k(x,\xi)\right]\\ &= r^{m-k}\partial^h_{x_j}p_k(x,\xi)\end{aligned}$$

for all $r \geq 1, |\xi| \geq 1$ and $x \in \mathbb{R}^n$. □

With the previous two lemmas at hand, we are able to prove a similar statement as that one of Lemma 6.8.

Lemma 6.21. *Let* $1 < \tilde{q}, q < \infty$, $m \in \mathbb{R}$ *and* $\tilde{m} \in \mathbb{N}$ *with* $\tilde{m} > 1 + n/q$. *Additionally let* p *be either an element of the symbol-class* $H^{\tilde{m}}_q S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *or of* $W^{\tilde{m},q}_{uloc}S^m_{cl}(\mathbb{R}^n \times \mathbb{R}^n)$. *Then we get for every* $j \in \{1,\ldots,n\}$ *and* $h \in \mathbb{R}\backslash\{0\}$*:*

$$[\partial^h_{x_j}, p(x,D_x)]u(x) = \left(\partial^{-h}_{x_j}p\right)(x,D_x)u(x+he_j) \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n.$$

Moreover, for all $s \in \mathbb{R}$ *with* $n(1/\tilde{q}+1/q-1)^+ - \tilde{m}+1 < s \leq \tilde{m}-1-n(1/q-1/\tilde{q})^+$ *there is a constant* C*, independent of* $h \in \mathbb{R}\backslash\{0\}$*, such that*

$$\|[\partial^h_{x_j}, p(x,D_x)]u\|_{H^s_{\tilde{q}}} \leq C\|u\|_{H^{s+m}_{\tilde{q}}} \qquad \text{for all } u \in H^{s+m}_{\tilde{q}}(\mathbb{R}^n),$$

where $j \in \{1,\ldots,n\}$.

Proof: The proof of the lemma is essentially the same as that one of Lemma 6.8. At first we take a look at the case $p \in H_q^{\tilde{m}} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$: We just have to replace Lemma 6.7 with Lemma 6.19 and Theorem 4.51 with Theorem 4.54. Finally, it remains to verify the case $p \in W_{uloc}^{\tilde{m},q} S_{cl}^m(\mathbb{R}^n \times \mathbb{R}^n)$: Using Lemma 6.20 instead of Lemma 6.7 and Theorem 4.62 instead of Theorem 4.51 provides the claim. □

The previous lemma enables us to verify

Lemma 6.22. *Let $1 < q, \tilde{q} < \infty$ and $\tilde{m} \in \mathbb{N}$ with $\tilde{m} > 1 + n/q$. We consider either $p \in H_q^{\tilde{m}} S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$ or $p \in W_{uloc}^{\tilde{m},q} S_{cl}^0(\mathbb{R}^n \times \mathbb{R}^n)$ with the following property: The inverse operator $p(x, D_x)^{-1}$ is an element of $\mathscr{L}(H_{\tilde{q}}^r, H_{\tilde{q}}^r)$ for one $r \in \mathbb{R}$ with $n(1/\tilde{q} + 1/q - 1)^+ - \tilde{m} < r \leq \tilde{m} - n(1/q - 1/\tilde{q})^+$. Then we obtain*

$$p(x, D_x)^{-1} \in \mathscr{L}(H_{\tilde{q}}^s, H_{\tilde{q}}^s) \qquad \text{for all } s \in [r - l, r + k].$$

Here k and l are defined by $k := \max\{\tilde{k} \in \mathbb{N}_0 : r + \tilde{k} \leq \tilde{m} - n(1/q - 1/\tilde{q})^+\}$ and $l := \max\{\tilde{l} \in \mathbb{N}_0 : n(1/\tilde{q} + 1/q - 1)^+ - \tilde{m} < r - \tilde{l}\}$.

Proof: The statement follows in the same way as that one of Lemma 6.11. We merely have to use Lemma 6.21 instead of Lemma 6.8. Moreover, we have to apply Theorem 4.54 instead of Theorem 4.51 in the case $p \in H_q^{\tilde{m}} S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$ and Theorem 4.61 otherwise. □

Now that we have checked Lemma 6.22, we are in the position to prove the spectral invariance of non-smooth pseudodifferential operators whose symbols are in $H_q^{\tilde{m}} S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$ or in $W_{uloc}^{\tilde{m},q} S_{cl}^0(\mathbb{R}^n \times \mathbb{R}^n)$:

Theorem 6.23. *Let $1 < q, q_0 < \infty$ and $\tilde{m} \in \mathbb{N}_0$ with $\tilde{m} > \max\{1 + n/q, n/q_0\}$. Furthermore, let $\hat{m} \in \mathbb{N}_0$ with $n/q_0 < \hat{m} \leq \max\{r \in \mathbb{N}_0 : r < \tilde{m} - n/q\}$. Additionally let $M \in \mathbb{N}_0$ be such that $n < 2l \leq M < \tilde{m} - \hat{m} - n(1/q + 1/q_0 - 1)^+$ for some $l \in \mathbb{N}$. We define $\tilde{M} := M - (n+2)$ if n is even and $\tilde{M} := M - (n+1)$ else. Considering a symbol $p \in H_q^{\tilde{m}} S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n)$ or $p \in W_{uloc}^{\tilde{m},q} S_{cl}^0(\mathbb{R}^n \times \mathbb{R}^n)$, where $p(x, D_x)^{-1} \in \mathscr{L}(H_{q_0}^r, H_{q_0}^r)$ for one $r \in \mathbb{R}$ with*

$$n(1/q_0 + 1/q - 1)^+ - \tilde{m} < r \leq \tilde{m} - n(1/q - 1/q_0)^+$$

we get

$$p(x, D_x)^{-1} \in OPW_{uloc}^{\hat{m},q_0} S_{1,0}^0(\mathbb{R}^n \times \mathbb{R}^n; \tilde{M} - 1).$$

In the case $\tilde{M} - 1 > n/\tilde{q}$ for one $1 < \tilde{q} \leq 2$, we even have

$$p(x, D_x)^{-1} \in \mathscr{L}(L^{\tilde{q}}, L^{\tilde{q}}) \qquad \text{for all } \tilde{q} \in [\tilde{q}; \infty) \cup \{q_0\}.$$

Proof: We get the statement in the same way as that one of Theorem 6.12. We just have to replace Lemma 6.11 with Lemma 6.22 and Remark 4.45 with Remark 4.31 in the case $p \in H_q^{\tilde{m}} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ and with Remark 4.35 else. Moreover, we have to use Theorem 4.54 instead of Theorem 4.51 in the case $p \in H_q^{\tilde{m}} S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ and Theorem 4.61 otherwise. □

Chapter 7

Summary and Conclusions

One of the main goals of this dissertation was to establish a characterization for non-smooth pseudodifferential operators with coefficients in the Hölder spaces via iterated commutators. The starting point of this thesis was the smooth case. In this case such a characterization was shown by R. Beals in [16] and J. Ueberberg in [74]: Every linear operator $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a pseudodifferential operator of the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$, if

$$\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P : H^{s+m-\rho|\alpha|+\delta|\beta|}_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n) \tag{7.1}$$

is linear and bounded for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, all $s \in \mathbb{R}$ and for one $q \in (1, \infty)$. Having a close look at the proof of this statement provides that (7.1) just has to hold for $s = 0$. Now one may wonder which conditions a linear operator $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ has to fulfill in order to be a non-smooth pseudodifferential operator. We know that the Hörmander class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is equal to the intersection of all non-smooth symbol-classes $C^{\tilde{m},s} S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with respect to $\tilde{m}$ and s with $\tilde{m} \in \mathbb{N}_0$ and $0 < s \leq 1$ due to Remark 4.6. Consequently, operators fulfilling similar but less strict assumptions than (7.1) seemed to be a good choice for being non-smooth pseudodifferential operators of the symbol-class $C^{\tilde{m},s} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, $\rho \in \{0, 1\}$. But which of these conditions are redundant for being a non-smooth pseudodifferential operator of the symbol-class $C^{\tilde{m},s} S^m_{\rho,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, $\rho \in \{0, 1\}$? Pseudodifferential operators of the symbol-class $C^{\tilde{m},s} S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$, just fulfill the assumption (7.1) for suitable small multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, as we have seen in Remark 4.45 and Theorem 4.51. Hence, a natural choice for the characterization set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$ was to take all linear operators $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ fulfilling (7.1) for not all, but sufficiently many multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. Proving that each element of the set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$, $\rho \in \{0, 1\}$ is a non-smooth pseudodifferential operator with coefficients in a Hölder space was the task of Chapter 5. Hereby M had to be sufficiently large. For the proof we used to main ideas of the smooth case. The main new difficulties, we were confronted with, were the limited mapping properties of pseudodifferential operators with non-smooth symbols. We even were able to improve this result: In Section 5.6 we showed in detail that each element of the set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$, $\rho \in \{0, 1\}$ is even a non-smooth pseudodifferential operator with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$.

Such a characterization may be another piece in the puzzle of non-smooth pseudodifferential operators. For instance, we have seen in Section 4.3 that the composition of two pseudodifferential operators with coefficients in a Hölder space in general is not a pseudodifferential operator. We just were able to show an asymptotic expansion of this composition. Making use of the characterization we were able to show for certain cases in Section 5.7 that the composition of two non-smooth pseudodifferential operators is a pseudodifferential operator again.

In the theory of pseudodifferential operators it is of particular interest, whether the inverse of a pseudodifferential operator is a pseudodifferential operator again. Such a result already exists in the smooth case, cf. [16] and [74]. Section 6.1 was devoted to verify a similar result for a non-smooth pseudodifferential operator P of the symbol-class $C^\tau S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Analogous to the smooth case in [74] we proved that P^{-1} is also a non-smooth pseudodifferential operator of the symbol-class $C^s S^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ where $s < \tau$ if P^{-1} is linear and bounded as a map on $L^2(\mathbb{R}^n)$. For the proof of this statement we used the main idea of the proof in the smooth case: The application of the characterization of pseudodifferential operators. Thus, we just had to show the boundedness of certain iterated commutators of $p(x, D_x)^{-1}$. Since we already knew that the iterated commutators of $p(x, D_x)$ have these mapping properties, we wrote the iterated commutators of $p(x, D_x)^{-1}$ as a sum and compositions of $p(x, D_x)^{-1}$ and the iterated commutators of $p(x, D_x)$. The main new difficulty, we were confronted with, was the following: non-smooth pseudodifferential operators are not linear and bounded as operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ like the smooth ones. In order to treat this difficulty, we used the linearity and boundedness of $p(x, D_x)^{-1}$ and of the iterated commutators of P as maps from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. This turned out to be the solution of this problem, since that allows the above-mentioned sum and compositions to be well-defined.

Deriving some spectral invariance results in the non-smooth case was the second main purpose of this dissertation: In Section 6.3 such a result was verified for non-smooth pseudodifferential operators of the symbol-class $C^\tau S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n; M)$ for sufficiently large M . Similarly as in Section 6.1 we used the characterization of non-smooth pseudodifferential operators for proving the spectral invariance. However, in this case $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P$, $|\alpha| \neq 0$ are pseudodifferenial operators of negative order. Hence, the order of the Bessel potential space increases by applying the operator $\operatorname{ad}(-ix)^\alpha \operatorname{ad}(D_x)^\beta P$, $|\alpha| \neq 0$. Therefore, $P^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$ was not sufficient. We even needed $P^{-1} \in \mathscr{L}(H^{-s}_q(\mathbb{R}^n))$ for certain $s \in \mathbb{N}_0$. As we always tried to restrict the assumptions to a minimal, we used the tools of difference quotients in order to derive $P^{-1} \in \mathscr{L}(H_q^{-s}(\mathbb{R}^n))$ from $P^{-1} \in \mathscr{L}(L^q(\mathbb{R}^n))$.

We were able to improve the results obtained in Section 6.3 for pseudodifferential operators with coefficients in $W^{\tilde{m},q}_{uloc}(\mathbb{R}^n)$. It turned out that we even got, once again, better results for some subsets of $\mathrm{OP}W^{\tilde{m},q}_{uloc}S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In Theorem 6.23 we have just verified the spectral invariance of pseudodifferential operators of the symbol-class $H^{\tilde{m}}_q S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ in the case $\tilde{m} \in \mathbb{N}_0$. But is the restriction of $\tilde{m}$ to natural numbers really necessary? Answering this question would be a very interesting task.

Altogether, we have proved some statements in this dissertation, which may help solving problems in the field of partial differential equations. In applications the L^p-spectrum for $p > 2$ is often of particular interest. Being equipped with a Hilbert space structure it is mostly easier to calculate the L^2-spectrum. However, in the field of nonlinear partial differential equations there are still some partial differential operators being non-smooth pseudodifferential operators with coefficients in a Hölder space. In these cases we get the spectral invariance under certain conditions by means of the results of Chapter 6.

In applications it is often much easier to verify whether an operator P is an element of the set $\mathcal{A}^{m,M}_{\rho,0}(\tilde{m}, q)$ than to prove directly that P is a pseudodifferential operator. Hence, the characterization of non-smooth pseudodifferential operators may serve as an important tool to simplify such proofs. A good candidate for an application of this theory is the following problem: We consider the elliptic partial differential operator

$$a(x, D_x) := \sum_{|\alpha| \leq m} c_\alpha(x) D_x^\alpha$$

for $m \in \mathbb{N}_0$ and $c_\alpha \in W^{\tilde{m},q}_{uloc}(\mathbb{R}^{n+1}_+)$ for all $|\alpha| \leq m$. Here $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. Furthermore, we assume that the partial differential equation

$$\begin{cases} a(x, D_x)v = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v|_{x_n=0} = u \end{cases} \tag{7.2}$$

has a unique solution $v \in W^{\tilde{m}}_q(\mathbb{R}^{n+1}_+)$ for all $u \in \{w|_{x_n=0} : w \in W^{\tilde{m}}_q(\mathbb{R}^{n+1}_+)\}$. Then it is of interest, in which cases the Dirichlet-to-Neumann operator N is a pseudodifferential operator. Here N is defined by

$$Nu := \partial_{x_n} v|_{x_n=0} \qquad \text{for all } u \in \{w|_{x_n=0} : w \in W^{\tilde{m}}_q(\mathbb{R}^{n+1}_+)\},$$

where v is the solution of (7.2) with respect to u. Solving this problem is an attractive task for a future project.
However, there still remain many other interesting unresolved problems regarding non-smooth pseudodifferential operators which will require further effort to shed light on.

Appendix A

Basic Results of Measure Theory

This chapter serves to prove an easy consequence of the basic results in the measure theory needed in this thesis. For an introduction to the measure theory we refer to e.g. [28].

Proposition A.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^m \to \mathbb{C}$ be measurable. If we denote the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ by $F(x, y) := f(x)g(y)$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then F is measurable, too.*

Proof: First of all we verify the measurability of $\tilde{f}, \tilde{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$, which are defined by

$$\tilde{f}(x, y) := f(x) \quad \text{and} \quad \tilde{g}(x, y) := g(y) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^m.$$

Because of Corollary 4.6 in Chapter 3 of [28], we can assume, that $\tilde{f}$ is real-valued. Let $\alpha \in \mathbb{R}$ be arbitrary. The measurability of f yields the measurability of the set $A := \{x \in \mathbb{R}^n : f(x) > \alpha\}$. Since A and $\mathbb{R}^m$ are measurable, we know, that $A \times \mathbb{R}^m = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \tilde{f}(x, y) = f(x) > \alpha\}$ is measurable with respect to the product measure of $\mathbb{R}^n \times \mathbb{R}^m$. This provides the measurability of the function $\tilde{f}$. In the same way one gets the measurability of $\tilde{g}$. On account of Corollary 4.8, Chapter 3 of [28] we obtain the measurability of $F = \tilde{f} \cdot \tilde{g}$. $\square$

Appendix B

Banach Space Valued Function Spaces

In the present chapter we define the Banach space valued Sobolev and Hölder spaces and present those properties of these spaces, we need in this work. For more details we refer to e.g. [11], [12] and [63].

B.1 Banach Space Valued Sobolev Spaces

The purpose of this section is to define Banach space valued Sobolev spaces. Hence, we consider some general assumptions for this chapter: M is always a subset of $\mathbb{R}^n$ and X is an arbitrary Banach space. The first step to reach the aim of the present section is the definition of the Banach space valued L^q-spaces for $1 \leq q < \infty$. Since we will make use of the Bochner integral, we start with the definition of simple and strongly measurable functions. This section is based on [77].

Definition B.1.

- A function $f : M \to X$ is called *simple function*, if $f(M)$ is a finite set and for all $y \in f(M)$ the set $f^{-1}(y)$ is measurable with a finite measure.
- A function $f : M \to X$ is called *strongly measurable*, if there is a sequence $(f_k)_{k\in\mathbb{N}}$ of simple functions such that

$$f_k(x) \xrightarrow{k\to\infty} f(x) \qquad \text{in } X \text{ for almost every } x \in M.$$

Now we use the previous definition in order to define the Bochner-integral, cf. e.g. [77], Section V.5:

Definition B.2.

- Assuming a simple function $f : M \to X$ the *Bochner-integral* is defined by

$$\int_M f(x)dx := \sum_{y \in f(M)} y\mu(f^{-1}(y)) \in X,$$

where μ is the Lebesgue measure.

- A function $f : M \to X$ is called *Bochner-integrable*, if there is a sequence $(f_k)_{k\in\mathbb{N}}$ of simple functions such that

  i) $f_k(x) \xrightarrow{k\to\infty} f(x)$ in X for almost every $x \in M$.

  ii) $\int_M \|f_k(x) - f(x)\|_X dx \xrightarrow{k\to\infty} 0$.

  Then the *Bochner-integal of f* is defined by

$$\int_M f(x)dx := \lim_{k\to\infty} \int_M f_k(x)dx.$$

The norm of the Bochner-integral can be estimated in the following way:

Lemma B.3. *A function $f : M \to X$ is Bochner-integrable if and only if f is strongly measurable and $\int_M \|f(x)\|_X dx < \infty$. In this case we have*

$$\left\| \int_M f(x)dx \right\|_X \leq \int_M \|f(x)\|_X dx.$$

We refer to [77], Theorem V.5.1 and Corollary V.5.1 for the proof. With this lemma at hand, we are able to define for all $1 \leq q < \infty$ the spaces:

$$\begin{aligned} L^q(M;X) &:= \{f : M \to X \text{ strongly measurable} : \|f\|_{L^q(M;X)} < \infty\}, \\ L^q_{loc}(M;X) &:= \{f : M \to X \text{ strongly measurable} : \\ &\qquad\qquad f|_K \in L^q(K;X) \text{ for all } K \subseteq M \text{ compact}\}, \end{aligned}$$

where $\|f\|_{L^q(M;X)} := \left(\int_M \|f(x)\|_X^q dx\right)^{1/q}$.

Next we list some important facts about the Bochner-integral, cf. e.g. [4], Theorem 1.9:

Lemma B.4.

i) Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Considering $f \in L^p(M;X)$ and $g \in L^q(M;X')$ we obtain

$$\left| \int_M \langle f(x), g(x)\rangle_{X;X'} dx \right| \leq \|f\|_{L^p(M;X)} \|g\|_{L^q(M;X')}.$$

ii) *For all $f \in L^1(M;X)$ and $A \in \mathscr{L}(X,Y)$, where Y is a Banach space, we have $Af \in L^1(M;Y)$ and*

$$A \int_M f(x)dx = \int_M Af(x)dx.$$

iii) *Assuming $f_k(x) \xrightarrow{k\to\infty} f(x)$ in X for nearly all $x \in M$ and $f_k \in L^1(M;X)$ for all $k \in \mathbb{N}$, we get $f \in L^1(M;X)$ and*

$$\int_M f(x)dx = \lim_{k\to\infty} \int_M f_k(x)dx \qquad \text{in } X.$$

Proof: In the same way as in [79], Chapter 23 an application of the Hölder inequality and of the estimate $|\langle f(x), g(x)\rangle_{X;X'}| \leq \|f(x)\|_X \|g(x)\|_{X'}$ yields claim *i)*. For the proof of claim *ii)* we refer to [77], Corollary V.5.2. Claim *iii)* has been shown in [5], Theorem A.15. □

Additionally we have the following convergence of functions in $L^p(\mathbb{R}^n;X)$:

Lemma B.5. *Let $f \in L^p(\mathbb{R}^n;X)$ with $1 \leq p < \infty$. Moreover, let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence fulfilling the following properties:*

i) *$(\varphi_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^1(\mathbb{R}^n)$.*

ii) *$\int \varphi_k(x)dx = 1$ for all $k \in \mathbb{N}$.*

iii) *For each $r > 0$ we have $\int\limits_{\mathbb{R}^n \setminus B_r(0)} \varphi_k(x)dx \xrightarrow{k\to\infty} 0$.*

Then we obtain the following convergence:

$$\varphi_k * f \xrightarrow{k\to\infty} f \qquad \text{in } L^p(\mathbb{R}^n;X).$$

Proof: If the additional assumption $\varphi_k \geq 0$ holds for all $k \in \mathbb{N}$, this statement was shown e.g. in [8], Lemma 2.14. Verifying the proof of the just mentioned lemma, we see that the statement is also true if we skip the assumption $\varphi_k \geq 0$ for all $k \in \mathbb{N}$. This implies the claim. □

We mention one example for a bounded sequence $(\varphi_k)_{k\in\mathbb{N}}$ which fulfills the assumptions of the previous lemma:

Example B.6. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ for all $|x| \leq \frac{1}{2}$ and $\varphi(x) = 0$ for all $|x| \geq 1$. We define for each $\varepsilon \in (0,1]$ the functions $\psi_\varepsilon : \mathbb{R}^n \to \mathbb{C}$ by

$$\psi_\varepsilon(x) := \varepsilon^{-n} \mathscr{F}^{-1}[\varphi](x/\varepsilon) \qquad \text{for all } x \in \mathbb{R}^n.$$

Then $(\psi_\varepsilon)_{\varepsilon\in(0,1]} \subseteq L^1(\mathbb{R}^n)$ is bounded and the assumptions *ii)* and *iii)* of Lemma B.5 hold.

Proof: On account of Lemma 2.18 we know that $\mathscr{F}^{-1}[\varphi] \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Together with variable transformation we get the existence of a constant $C > 0$, independent of $\varepsilon \in (0,1]$, such that

$$\|\psi_\varepsilon\|_{L^1(\mathbb{R}^n)} = \int |\mathscr{F}^{-1}[\varphi](x/\varepsilon)|\varepsilon^{-n}dx = \int |\mathscr{F}^{-1}[\varphi](y)|dy < C$$

for all $\varepsilon \in (0,1]$. Assumption *ii*) of Lemma B.5 can be checked in the same way as the previous estimate. Hence it remains to show assumption *i*) of Lemma B.5 by means of variable transformation:

$$\begin{aligned}\int \psi_\varepsilon(x)dx &= \int \mathscr{F}^{-1}[\varphi](x/\varepsilon)\varepsilon^{-n}dx = \int \mathscr{F}^{-1}[\varphi](y)dy = \int e^{-i0\cdot y}\mathscr{F}^{-1}[\varphi](y)dy \\ &= \mathscr{F}\left[\mathscr{F}^{-1}[\varphi]\right](0) = \varphi(0) = 1 \qquad \text{for all } \varepsilon \in (0,1]. \end{aligned}$$

□

The last missing piece towards the definition of the Banach space valued Sobolev spaces is the introduction of Banach space valued regular distributions and of the distributional derivatives:

Definition B.7. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Denoting $\mathscr{D}(\Omega) := C_c^\infty(\Omega)$, the set of all *X-valued distributions* is defined by

$$\mathscr{D}'(\Omega; X) := \{F : \mathscr{D}(\Omega) \to X : F \text{ is linear and continuous}\}.$$

$F \in \mathscr{D}'(\Omega; X)$ is called *regular distribution* if there is an $f \in L^1_{loc}(\Omega; X)$ with

$$\langle F, \varphi\rangle_{\mathscr{D}'(\Omega;X);\mathscr{D}(\Omega)} = \int_\Omega f(x)\varphi(x)dx \qquad \text{for all } \varphi \in \mathscr{D}(\Omega).$$

Then $F_f := F$. Moreover, we define for $\alpha \in \mathbb{N}_0^n$ the *α-th distributional derivative* of $F \in \mathscr{D}'(\Omega; X)$, if

$$\langle \partial_x^\alpha F, \varphi\rangle_{\mathscr{D}'(\Omega;X);\mathscr{D}(\Omega)} = (-1)^{|\alpha|}\langle F, \partial_x^\alpha \varphi\rangle_{\mathscr{D}'(\Omega;X);\mathscr{D}(\Omega)} \qquad \text{for all } \varphi \in \mathscr{D}(\Omega).$$

For an X-valued distribution $F \in \mathscr{D}'(\Omega; X)$ we say $F \in L^q(\Omega; X)$, if there is an $f \in L^q(\Omega; X)$ with $F = F_f$. Here $\Omega \subseteq \mathbb{R}^n$ is an open set and $1 \le q < \infty$. With all the work done in this section we now are in the position to define the Banach space valued Sobolev spaces, cf. e.g. [12], Chapter 2:

Definition B.8. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and X be an arbitrary Banach space. Considering $m \in \mathbb{N}_0$ and $1 \le q < \infty$ we define the *X-valued Sobolev space* by

$$W_q^m(\Omega; X) := \{f \in L^q(\Omega; X) : \partial_x^\alpha f \in L^q(\Omega; X) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le m\}.$$

$W_q^m(\Omega; X)$ is a Banach space equipped with the norm

$$\|f\|_{W_q^m(\Omega;X)} := \left(\sum_{|\alpha|\le m} \|\partial_x^\alpha f\|^q_{L^q(\Omega;X)}\right)^{1/q} \qquad \text{for all } f \in W_q^m(\Omega; X).$$

Having defined the Banach space valued Sobolev spaces, we now turn to the Banach space valued Hölder spaces.

B.2 Banach Space Valued Hölder Spaces

This section serves to introduce Banach space valued Hölder spaces and investigate the relationship to the Banach space valued Sobolev spaces. We assume the general assumptions $\Omega \subseteq \mathbb{R}^n$ being an open set and X being a Banach space for the whole section. Then we define the Banach space valued space of all bounded m-times continuously differentiable functions in the following way:

Definition B.9. For $m \in \mathbb{N}_0$ the X-valued space of all bounded m-times continuously differentiable functions $C_b^m(\overline{\Omega}; X)$ is defined by

$$\begin{aligned} C^m(\overline{\Omega}; X) &:= \{f : \Omega \to X : f \text{ is } m\text{-times continuously differentiable and for all} \\ &\qquad \alpha \in \mathbb{N}_0^n, |\alpha| \leq m, \text{ there is a continuous extension of } \partial_x^\alpha f \text{ on } \overline{\Omega}\}, \\ C_b^m(\overline{\Omega}; X) &:= \left\{ f \in C^m(\overline{\Omega}; X) : \|f\|_{C_b^m(\overline{\Omega};X)} := \max_{|\alpha|\leq m} \sup_{x\in\overline{\Omega}} \|\partial_x^\alpha f(x)\|_X < \infty \right\}. \end{aligned}$$

Assuming a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$, the α-th distributional derivative of $f \in C_b^m(\overline{\Omega}; X)$ turns out to be the well-known partial derivative $\partial_x^\alpha f := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f$, where

$$\partial_{x_j} f := \lim_{h\to 0} \frac{f(x + he_j) - f(x)}{h} \in X$$

for all $j \in \{1, \ldots, n\}$.

In the same way as in Section 2.4 we define the X-valued Hölder spaces, now:

Definition B.10. Let $0 < s \leq 1$ and $m \in \mathbb{N}_0$. Then the *X-valued Hölder space* $C^{m,s}(\overline{\Omega}; X)$ is defined by

$$\begin{aligned} C^{0,s}(\overline{\Omega}; X) &:= \left\{ f \in C_b^0(\overline{\Omega}; X) : \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{\|f(x) - f(y)\|_X}{|x - y|^s} < \infty \right\}, \\ C^{m,s}(\overline{\Omega}; X) &:= \left\{ f \in C_b^m(\overline{\Omega}; X) : \partial_x^\alpha f \in C^{0,s}(\overline{\Omega}; X) \text{ for all } \alpha \in \mathbb{N}_0^n, |\alpha| \leq m \right\}. \end{aligned}$$

We define the norm of the X-valued Hölder spaces via

$$\|f\|_{C^{m,s}(\overline{\Omega};X)} := \max_{|\alpha|\leq m} \|\partial_x^\alpha f\|_{C_b^m(\overline{\Omega};X)} + \max_{|\alpha|\leq m} \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{\|\partial_x^\alpha f(x) - \partial_x^\alpha f(y)\|_X}{|x - y|^s}$$

for all $f \in C^{m,s}(\overline{\Omega}; X)$. If $\tau := m + s \notin \mathbb{N}$, we denote $C^{m,s}(\overline{\Omega}; X)$ by $C^\tau(\overline{\Omega}; X)$.

As in the case $X = \mathbb{C}$, the next continuous embedding statement holds:

Theorem B.11. *Let* $m \in \mathbb{N}_0$*,* $1 < q < \infty$ *and* $\tau \notin \mathbb{N}$ *with* $0 < \tau \leq m - n/q$*. Additionally, let* $\Omega \subseteq \mathbb{R}^n$ *be an open, bounded set with smooth boundary. Then we obtain the continuous embedding*

$$W_q^m(\Omega; X) \hookrightarrow C^\tau(\overline{\Omega}; X).$$

Proof: Corollary 4.3 in [12] and the embeddings (3.1)-(3.3) and (3.6) in [12] provide the continuous embedding $W_q^m(\Omega; X) \hookrightarrow C^\tau(\overline{\Omega}; X)$. □

Appendix C

Proof of an Interpolation Result

The goal of this section is to prove the second statement of Lemma 2.41 *ii*). There we mentioned the following interpolation result for Hölder spaces:

Lemma C.1. *Let $k, m \in \mathbb{N}$ with $k \leq m$ and $0 < \tau < 1$. Setting $\theta := \frac{k}{m+\tau}$ we have*

$$\|f\|_{C_b^k(\mathbb{R}^n)} \leq C\|f\|_{C_b^0(\mathbb{R}^n)}^{1-\theta}\|f\|_{C^{m,\tau}(\mathbb{R}^n)}^{\theta} \qquad \text{for all } f \in C^{m,\tau}(\mathbb{R}^n).$$

Proof: For $\theta \in (0,1)$ the sets $J_\theta(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))$ and $K_\theta(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))$ are assumed as the usual classes of intermediate spaces of the interpolation theory, cf. e.g. [51], Definition 1.19. The definitions of these spaces immediately provide

$$C_b^0(\mathbb{R}^n) \in J_0(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)) \cap K_0(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)). \tag{C.1}$$

Additionally we define for all $\theta \in (0,1)$ and $p \in [1,\infty]$ the interpolation spaces $(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))_{\theta,p}$ as e.g. in [51], Definition 1.2. Then we know due to [51], Proposition 1.3 and [50], Theorem 1.2.17 that

$$(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))_{\frac{m+\tau}{m+1},1} \subseteq (C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))_{\frac{m+\tau}{m+1},\infty} = C^{m,\tau}(\mathbb{R}^n).$$

As a consequence of the previous embedding we obtain

$$C^{m,\tau}(\mathbb{R}^n) \in J_{\frac{m+\tau}{m+1}}(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)) \cap K_{\frac{m+\tau}{m+1}}(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n)), \tag{C.2}$$

cf. e.g. [51], p.31. Now we set $\theta := \frac{k}{m+\tau}$. On account of (C.1) and (C.2) an application of the reiteration theorem, cf. e.g. [50], Theorem 1.2.15, yields

$$(C_b^0(\mathbb{R}^n), C^{m,\tau}(\mathbb{R}^n))_{\theta,1} \subseteq (C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))_{\frac{k}{m+1},1}. \tag{C.3}$$

We arise from [51], Remark 1.22 that $C_b^k(\mathbb{R}^n) \in J_{\frac{k}{m+1}}(C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))$. Hence the assumptions of [51], Proposition 1.20 hold. Applying (C.3) and [51], Proposition 1.20 we get

$$(C_b^0(\mathbb{R}^n), C^{m,\tau}(\mathbb{R}^n))_{\theta,1} \subseteq (C_b^0(\mathbb{R}^n), C_b^{m+1}(\mathbb{R}^n))_{\frac{k}{m+1},1} \subseteq C_b^k(\mathbb{R}^n). \tag{C.4}$$

Finally, it remains to use the previous embedding and [71], Theorem 1.3.3 in order to show the claim:

$$\|f\|_{C_b^k(\mathbb{R}^n)} \leq C\|f\|_{(C_b^0(\mathbb{R}^n),C^{m,\tau}(\mathbb{R}^n))_{\theta,1}} \leq C\|f\|_{C_b^0(\mathbb{R}^n)}^{1-\theta}\|f\|_{C^{m,\tau}(\mathbb{R}^n)}^{\theta}$$

for all $f \in C^{m,\tau}(\mathbb{R}^n)$. $\square$

List of Symbols

General Symbols

Functions and Operators

Spaces and Sets

Norms

Semi-Norms

Symbol-Classes

Sets of Pseudodifferential Operators

Bibliography

[1] H. Abels. *(2009) Fourieranalysis [online]*. lecture notes, University Regensburg, available in internet: http://www.uni-r.de/Fakultaeten/nat_Fak_I/abels/FourieranalysisSkript.pdf [23.10.2014, 12:23].

[2] H. Abels. *(2011) Interpolationstheorie und Funktionenräume [online]*. lecture notes, Regensburg, available in internet: http://www.uni-r.de/Fakultaeten/nat_Fak_I/abels/SkriptInterpolationstheorieSoSe11.pdf [23.10.2014, 12:30].

[3] H. Abels. Pseudodifferential boundary value problems with non-smooth coefficients. *Comm. Partial Differential Equations*, 30(10-12):1463–1503, 2005.

[4] H. Abels. *Partielle Differentialgleichungen II.* lecture notes, University Regensburg, 2010/2011.

[5] H. Abels. *Pseudodifferential and singular integral operators: an introduction with applications.* De Gruyter, Berlin/Boston, 2012.

[6] H. Abels, G. Grubb, and I.G. Wood. Extension theory and Kreĭn-type resolvent formulas for nonsmooth boundary value problems. *J. Funct. Anal.*, 266(7):4037–4100, 2014.

[7] H. Abels and M. Kassmann. The Cauchy problem and the martingale problem for integro-differential operators with non-smooth kernels. *Osaka J. Math.*, 46(3):661–683, 2009.

[8] H.W. Alt. *Lineare Funktionalanalysis.* Springer Berlin Heidelberg New York, 5. Auflage, 2006.

[9] J. Alvarez and J. Hounie. Spectral invariance and tameness of pseudo-differential operators on weighted Sobolev spaces. *J. Oper. Theory*, 30(1):41–67, 1993.

[10] J.D. Alvarez Alonso and A.P. Calderón. Functional calculi for pseudo-differential operators. I. Fourier analysis, Proc. Semin., El Escorial 1979, 1-61, 1980.

[11] H. Amann. *(2003). Vector-Valued Distributions and Fourier Multipliers [online].* available in internet: http://user.math.uzh.ch/amann/files/distri butions.pdf [29.02.2012, 13:15].

[12] H. Amann. Compact embeddings of vector-valued Sobolev and Besov spaces. *Glas. Mat., III. Ser.*, 35(1):161–177, 2000.

[13] H. Amann and J. Escher. *Analysis III.* Birkhäuser Verlag Basel Boston Berlin, 2008.

[14] M. Beals and M. Reed. Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems. *Trans. Am. Math. Soc.*, 285:159–184, 1984.

[15] R. Beals. A general calculus of pseudodifferential operators. *Duke Math. J.*, 42:1–42, 1975.

[16] R. Beals. Characterization of pseudodifferential operators and applications. *Duke Math. J.*, 44(1):45–57, 1977.

[17] R. Beals. Correction to: "Characterization of pseudodifferential operators and applications" (Duke Math. J. **44** (1977), no. 1, 45–57). *Duke Math. J.*, 46(1):215, 1979.

[18] R. Beals. Weighted distribution spaces and pseudodifferential operators. *J. Analyse Math.*, 39:131–187, 1981.

[19] J. Bergh and J. Löfström. Interpolation spaces. An introduction. Grundlehren der mathematischen Wissenschaften. 223. Berlin-Heidelberg-New York: Springer-Verlag. X, 1976.

[20] J.-M. Bony. Caractérisations des opérateurs pseudo-différentiels. *Sémin. Équ. Dériv. Partielles, Éc. Polytech., Cent. Math. Laurent Schwartz, Palaiseau*, 1997.

[21] B. Booss and D.D. Bleecker. Topology and analysis. The Atiyah-Singer index formula and gauge- theoretic physics. Transl. from the German by D. D. Bleecker and A. Mader. Universitext. New York etc.: Springer-Verlag. XVI., 1985.

[22] E. Buzano and A. Ziggioto. On the essential spectrum of hypoelliptic pseudo-differential operators. *Math. Nachr.*, 281(1):5–24, 2008.

[23] R.R. Coifman and Y. Meyer. Au dela des opérateurs pseudo-différentiels. Asterisque 57, 1978.

[24] H.O. Cordes. On pseudodifferential operators and smoothness of special Lie-group representations. *Manuscripta Math.*, 28(1-3):51–69, 1979.

[25] H.O. Cordes. On some C^*-algebras and Fréchet*-algebras of pseudodifferential operators. Pseudodifferential operators and applications, Proc. Symp., Notre Dame/Indiana 1984, Proc. Symp. Pure Math. 43, 79-104, 1985.

[26] E.B. Davies, B. Simon, and M. Taylor. L^p spectral theory of Kleinian groups. *J. Funct. Anal.*, 78(1):116–136, 1988.

[27] R. Denk and J. Seiler. On the maximal L_p-regularity of parabolic mixed-order systems. *J. Evol. Equ.*, 11(2):371–404, 2011.

[28] J. Elstrodt. *Maß- und Integrationstheorie.* Springer-Verlag Berlin [u.a.], 2011.

[29] L.C. Evans. *Partial Differential Equations.* Americal Mathematical Society, Second Edition, 2010.

[30] I. Gohberg and N. Krupnik. Einführung in die Theorie der eindimensionalen singulären Integraloperatoren. In deutscher Sprache hrsg. von E. Meister. Deutsche Übersetzung von B. Schüppel. Mathematische Reihe, Band 63. Lehrbücher und Monographen aus dem Gebiete der exakten Wissenschaften. Basel - Boston - Stuttgart: Birkhäuser Verlag, 1979.

[31] B. Gramsch. Relative Inversion in der Störungstheorie von Operatoren und Ψ- Algebren. (Relative inversions in perturbation theory of operators and Ψ-algebras). *Math. Ann.*, 269:27–71, 1984.

[32] B. Gramsch, J. Ueberberg, and K. Wagner. Spectral invariance and submultiplicativity for Fréchet algebras with applications to pseudo-differential operators and Ψ^*-quantization. In *Operator calculus and spectral theory. Proceedings of a symposium, held in Lambrecht, Germany, December 9-14, 1991*, pages 71–98. Basel: Birkhäuser Verlag, 1992.

[33] G. Grubb. Parameter-elliptic and parabolic pseudodifferential boundary problems in global L_p Sobolev spaces. *Math. Z.*, 218(1):43–90, 1995.

[34] G. Grubb. Spectral asymptotics for nonsmooth singular Green operators. *ArXiv e-prints*, May 2012.

[35] A. Heye, S. Becker, A. Mang, T.A. Schütz, A. Toma, and T.M. Buzug. Ein kontinuierlicher ansatz zur modellierung von tumorwachstum und strahlentherapie. In *Bildverarbeitung für die Medizin'11*, pages 384–388, 2011.

[36] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer-Verlag Berlin Heidelberg New York Tokyo, 1983.

[37] H. Jarchow. Locally convex spaces. Mathematische Leitfäden. Stuttgart: B. G. Teubner, 1981.

[38] W. Kaballo. *Einführung in die Analysis III.* Heidelberg: Spektrum Akademischer Verlag, 1999.

[39] W. Kaballo. *Grundkurs Funktionalanalysis.* Heidelberg: Spektrum Akademischer Verlag, 2011.

[40] J.U. Kim. On the local regularity of solutions in linear viscoelasticity of several space dimensions. *Trans. Am. Math. Soc.*, 346(2):359–398, 1994.

[41] V.D. Kryakvin. Characterization of pseudodifferential operators in Hölder-Zygmund spaces. *Differ. Equ.*, 49(3):306–312, 2013.

[42] H. Kumano-Go. *Pseudo-Differential Operators.* MIT Press, Cambridge, Massachusetts, and London, 1974.

[43] H. Kumano-Go and M. Nagase. Pseudo-differential operators with non-regular symbols and applications. *Funkc. Ekvacioj, Ser. Int.*, 21:151–192, 1978.

[44] K. Königsberger. *Analysis 2.* Springer Verlag, Berlin, 5. korrigierte Auflage, 2004.

[45] D. Köppl. *Pseudodifferentialoperatoren mit nichtglatten Koeffizienten auf Mannigfaltigkeiten.* Diploma thesis, University Regensburg, 2011.

[46] H.-G. Leopold and E. Schrohe. Spectral invariance for algebras of pseudo-differential operators on Besov spaces of variable order of differentiation. *Math. Nachr.*, 156:7–23, 1992.

[47] H.-G. Leopold and E. Schrohe. Spectral invariance for algebras of pseudo-differential operators on Besov-Triebel-Lizorkin spaces. *Manuscr. Math.*, 78(1):99–110, 1993.

[48] H.-G. Leopold and E. Schrohe. Invariance of the L_p spectrum for hypoelliptic operators. *Proc. Am. Math. Soc.*, 125(12):3679–3687, 1997.

[49] H.-G. Leopold and H. Triebel. Spectral invariance for pseudodifferential operators on weighted function spaces. *Manuscr. Math.*, 83(3-4):315–325, 1994.

[50] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Basel: Birkhäuser, 1995.

[51] A. Lunardi. *Interpolation theory.* Pisa: Scuola Normale Superiore, 1999.

[52] J. Marschall. *Pseudo-differential operators with non regular symbols.* Berlin: Freie Univ. Berlin, FB Math., 1985.

[53] J. Marschall. Pseudodifferential operators with nonregular symbols of the class $S^m_{\rho\delta}$. *Comm. Partial Differential Equations*, 12(8):921–965, 1987.

[54] J. Marschall. Pseudodifferential operators with coefficients in Sobolev spaces. *Trans. Amer. Math. Soc.*, 307(1):335–361, 1988.

[55] Y. Meyer. Remarques sur un théorème de J. M. Bony. *Suppl. Rend. Circ. Mat. Palermo (2)*, 1:1–20, 1981.

[56] S. Molahajloo and M.W. Wong. Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $\mathbb{S}^1$. *J. Pseudo-Differ. Oper. Appl.*, 1(2):183–205, 2010.

[57] M. Pascu. On the essential spectrum of the relativistic magnetic Schrödinger operator. *Osaka J. Math.*, 39(4):963–978, 2002.

[58] V.S. Rabinovich. Local exponential estimates for h-pseudodifferential operators and tunneling for Schrödinger, Dirac, and square root Klein-Gordon operators. *Russ. J. Math. Phys.*, 16(2):300–308, 2009.

[59] X.S. Raymond. *Elementary Introduction to the Theory of Pseudodifferential Operators.* CRC Press, Boca Raton, 1991.

[60] M. Růžička. *Nichtlineare Funktionalanalysis. Eine Einführung.* Berlin: Springer, 2004.

[61] W. Rudin. Functional analysis. McGraw-Hill Series in Higher Mathematics. New York etc.: McGraw-Hill Book Comp. XIII, 1973.

[62] H. H. Schaefer and M.P. Wolff. *Topological vector spaces. 2nd ed.* New York, NY: Springer, 2nd ed. edition, 1999.

[63] B. Scharf, H.-J. Schmeißer, and W. Sickel. Traces of vector-valued Sobolev spaces. *Math. Nachr.*, 285(8-9):1082–1106, 2012.

[64] E. Schrohe. Boundedness and spectral invariance for standard pseudodifferential operators on anisotropically weighted L^p-Sobolev spaces. *Integral Equations Oper. Theory*, 13(2):271–284, 1990.

[65] E. Schrohe. Spectral invariance, ellipticity, and the Fredholm property for pseudodifferential operators on weighted Sobolev spaces. *Ann. Global Anal. Geom.*, 10(3):237–254, 1992.

[66] E.M. Stein. *Harmonic Analysis.* Princeton University Press, Princeton, New York Jersey, 1993.

[67] M.E. Taylor. *Pseudodifferential Operators and Nonlinear PDE.* Birkhäuser Boston, Basel, Berlin, 1991.

[68] M.E. Taylor. Beals-Cordes-type characterizations of pseudodifferential operators. *Proc. Amer. Math. Soc.*, 125(6):1711–1716, 1997.

[69] M.E. Taylor. *Tools for PDE.* Mathematical Surveys and Monographs, AMS, 2000.

[70] F. Treves. *Topological Vector Spaces and Distributions and Kernels.* Academic Press New York, San Francisco, London, 1967.

[71] H. Triebel. *Interpolation theory, function spaces, differential operators.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[72] H. Triebel. *Theory of Function Spaces.* Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.

[73] V. Turunen. Commutator characterization of periodic pseudodifferential operators. *Z. Anal. Anwend.*, 19(1):95–108, 2000.

[74] J. Ueberberg. Zur Spektralinvarianz von Algebren von Pseudodifferentialoperatoren in der L^p-Theorie. *Manuscripta Math.*, 61(4):459–475, 1988.

[75] R.L. Wheeden and A. Zygmund. Measure and integral. An introduction to real analysis. Monographs and Textbooks in Pure and Applied Mathematics. 43. New York - Basel: Marcel Dekker, Inc. X, 1977.

[76] H. Widom. Singular integral equations in L_p. *Trans. Am. Math. Soc.*, 97:131–160, 1960.

[77] K. Yosida. Functional analysis. 4th ed. Die Grundlehren der mathematischen Wissenschaften. Band 123. Berlin-Heidelberg-New York: Springer-Verlag. XI, 1974.

[78] E. Zeidler. *Nonlinear Functional Analysis and its Applications I.* Springer-Verlag New York Inc., 1986.

[79] E. Zeidler. *Nonlinear functional analysis and its applications. II/A: Linear monotone operators. Transl. from the German by the author and by Leo F. Boron.* New York etc.: Springer-Verlag, 1990.