Intersection Theory on Regular Schemes via Alterations and Deformation to the Normal Cone

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der Fakultät für Mathematik
der Universität Regensburg

vorgelegt von

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Regensburg

im Jahr 2015

Promotionsgesuch eingereicht am 13. April 2015.
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Chapter 1

Introduction

For a Noetherian separated regular scheme X, the Chow group $\mathrm{CH}^k_Y(X)$ of algebraic cycles of codimension k with supports in a closed subset Y of X is given as

$$\mathrm{CH}^k_Y(X) := Z^k_Y(X) \,/\, \mathrm{Rat}^k_Y(X),$$

i.e. the group of algebraic cycles $Z^k_Y(X)$ of codimension k on X with support in Y divided by a certain subgroup $\mathrm{Rat}^k_Y(X)$, called the group of algebraic cycles which are rationally equivalent to zero on Y. We set $\mathrm{CH}^k_Y(X)_Q := \mathrm{CH}^k_Y(X) \otimes_{\mathbb{Z}} Q$ for any commutative ring Q with unit element and $\mathrm{CH}^*(X)_Q := \bigoplus_{Y,k} \mathrm{CH}^k_Y(X)_Q$, where the direct sum is taken over all closed subsets Y and all $k \in \mathbb{Z}$. We are interested in giving an *intersection product with supports*

$$. \; : \mathrm{CH}^k_Y(X)_Q \otimes \mathrm{CH}^{k'}_Z(X)_Q \to \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q,$$

such that $\mathrm{CH}^*(X)_Q$ is a commutative graded Q-algebra with $[X] \in \mathrm{CH}^0_X(X)_Q$ as unit element and such that the intersection product is compatible with inclusions of supports.

In order to generalize Arakelov's arithmetic intersection theory from arithmetic surfaces to higher dimensional schemes, Gillet and Soulé [GS87] introduced an intersection product with supports for any Noetherian separated regular scheme, after tensoring the Chow groups with support by $\mathbb{Q}$. Their main tool is an isomorphism between $\mathrm{CH}^k_Y(X)_{\mathbb{Q}}$ and $\mathrm{Gr}^k_\gamma K^Y_0(X)_{\mathbb{Q}}$, the k-th graded piece of K-theory with support in Y with respect to the γ-filtration. Then the product in K-theory induces a product of Chow groups with support and coefficients in $\mathbb{Q}$.

We fix a Noetherian, excellent, regular and separated base scheme S. Throughout the text, we assume that all schemes under consideration are separated and of finite type over S and we call them *S-schemes*. The aim of this thesis is the development of a new approach to the intersection theory on regular S-schemes by combining Fulton's method of deformation to the normal cone with de Jong's result on alterations (or similar results) as a new ingredient. As it turns out, one should work very generally in a bivariant setting (cf. [Ful84, Chapter 17]) and talk about orientations (cf. [KT87, Chapter 3]): For any morphism $f : X \to Y$ of S-schemes and any ring Q, a bivariant class c in

$A^p(f)_Q$ is a collection of homomorphisms from $\mathrm{CH}^k(Y')_Q$ to $\mathrm{CH}^{k+p}(X \times_Y Y')_Q$, for all morphisms of S-schemes $Y' \to Y$ and all integers $k \in \mathbb{Z}$, compatible with proper push-forward, flat pull-back and the refined Gysin homomorphisms of regular imbeddings. There are two distinguished subgroups $B^p(f)_Q \subseteq A^p(f)_Q$ and $C^p(f)_Q \subseteq A^p(f)_Q$. The subgroup $B^p(f)_Q$ is essentially generated by systems which are induced by flat pull-backs and refined Gysin homomorphisms of regular imbeddings. The second subgroup $C^p(f)_Q$ contains $B^p(f)_Q$ and is given by all systems that commute with proper push-forward and with all systems in all $A^q(g)_Q$ for any morphism g of S-schemes and all $q \in \mathbb{Z}$. An S-scheme X is called *B_Q-orienting* (resp. *C_Q-orienting*), if all morphisms $f : Y \to X$ are equipped with one unique class $[f]$ in $B(f)_Q = \bigoplus_{p\in\mathbb{Z}} B^p(f)_Q$ (resp. $C(f)_Q = \bigoplus_{p\in\mathbb{Z}} C^p(f)_Q$), such that $[f]([Y]) = [X]$. We show, that any B_Q-orienting S-scheme is C_Q-orienting.

Kleiman and Thorup [KT87, Chapter 3] give two central results about orienting schemes:

(i) Any regular separated Noetherian universally catenary scheme is $C_{\mathbb{Q}}$-orienting.

(ii) Any regular, excellent scheme of dimension at most two is $B_{\mathbb{Z}}$-orienting.

They also show under the assumption of resolution of singularities the following generalization for the result (ii): If Y is a n-dimensional regular, excellent scheme and if every excellent scheme of dimension at most n has a desingularization, then Y is $B_{\mathbb{Z}}$-orienting. In order to generalize Kleiman and Thorup's results, we introduce the following Assumption:

> **Assumption A(S, Q, $\leq n$).**
> For a multiplicative closed subset $T \subseteq \mathbb{Z}$ with $0 \notin T$, set $Q = T^{-1}\mathbb{Z}$. Let n be an integer and let S denote a Noetherian, excellent, regular and separated scheme. We say that
>
> $$\mathrm{A}(S,\, Q,\, \leq n)$$
>
> holds, if for every integral S-scheme of Krull dimension at most n, there exists a projective alteration $\phi : X_1 \to X$, generically finite of degree $d \in T$, such that X_1 is a regular integral scheme.

This assumption allows us to include Kleiman and Thorup's assumption of resolution of singularities (i.e. $\mathrm{A}(\operatorname{Spec}\mathbb{Z},\, \mathbb{Z},\, \leq n)$) on the one hand, and results like de Jong's result on alterations on the other hand. For example if S is a Noetherian excellent regular separated scheme of Krull dimension one, then $\mathrm{A}(S,\, \mathbb{Q},\, \leq n)$ holds for all $n \in \mathbb{N}$ by [dJ97, Corollary 5.1]. With this notation, we prove the following result:

> **Theorem A** (Theorem 4.2.1, Corollary 4.2.2)**.** *Let Y be a regular S-scheme and assume that $A(S,\, Q,\, \leq \dim_S Y)$ holds. Then any S-scheme X with $X \to Y$ smooth is B_Q-orienting.*

Together with de Jong's result on alterations we immediately get the following

> **Theorem B** (Corollary 4.2.3)**.** *Let Y be a regular, separated scheme of finite type over $\mathbb{Z}$. Then Y is $B_{\mathbb{Q}}$-orienting.*

Theorem A and Theorem B improve Kleiman and Thorup's results from above.

These results allow us to define an intersection product with supports for regular S-schemes X under the Assumption $\mathrm{A}(S, Q, \leq \dim_S X)$. We first use, that B_Q-orienting schemes are B_Q-*orthocyclic* i.e. for every morphism $f : X \to Y$ of S-schemes we have $B(f)_Q = A(f)_Q$ and that the evaluation at $[Y]$ induces an isomorphism

$$E_Y : B(f)_Q = A(f)_Q \xrightarrow{\sim} \mathrm{CH}^*(X)_Q$$

of Q-modules. For closed immersion of S-schemes, we show that this is even an isomorphism of graded Q-modules. For closed subsets $Y, Z \subseteq X$ we put the reduced induced closed subscheme structure on both of them to get the closed immersions $i_Y : Y \to X$ and $i_Z : Z \to X$ and form the fiber square

$$\begin{tikzcd}
Y \times_X Z \arrow[r, "i'_Y"] \arrow[d, "i'_Z"'] & Z \arrow[d, "i_Z"'] \\
Y \arrow[r, "i_Y"] & X,
\end{tikzcd}$$

with morphisms as labeled. Then the intersection product is induced by the natural *cup-products*

$$\cup : B^p(i_Y)_Q \otimes B^q(i_Z)_Q \to B^{p+q}(i_Y \circ i'_Z)_Q.$$

> **Theorem C** (cf. Section 4.3). *This construction gives rise to the desired intersection product with supports*
>
> $$. \ : \mathrm{CH}^k_Y(X)_Q \otimes \mathrm{CH}^{k'}_Z(X)_Q \to \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q$$
>
> for B_Q-orienting schemes.

Behind this construction lies a pretty simple idea for such an intersection product: Let X be a regular S-scheme and assume that $\mathrm{A}(S, Q, \leq \dim_S X)$ holds. For a closed integral subscheme V of codimension k of X with closed immersion $i_V : V \to X$ and $[V] \in \mathrm{CH}^k_Y(X)$, we choose an alteration $\phi : V_1 \to V$ with V_1 regular as in our Assumption. Then the morphism $f : V_1 \to X$ in the diagram

$$\begin{tikzcd}
V_1 \arrow[r, "\phi"] \arrow[rr, bend left, "f"] & V \arrow[r] & X \arrow[d] \\
 & & S
\end{tikzcd}$$

is proper and l.c.i. and for $[W] \in \mathrm{CH}^{k'}_Z(X)$, we get a well-defined element

$$[V].[W] := \frac{1}{d} f_* f^! [W] \in \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_{\mathbb{Q}},$$

where d denotes the degree of the alteration and $f^!$ is Fulton's refined Gysin homomorphism for l.c.i. morphisms. Note that this construction is indeed independent of the choice of the alteration. This fact is not at all obvious and follows here from the theory of bivariant classes (cf. Section 4.3).

Under the assumption $\mathrm{A}(S,\, Q,\, \leq \dim_S X)$ we further show that our construction (Theorem C) coincides with Fulton's intersection product for divisors (cf. [Ful84, Chapter 2, Section 20.1]) in the case of regular S-schemes and with Fulton's intersection product in the case of smooth S-schemes if S is in addition one-dimensional (cf. [Ful84, Chapter 8, Section 20.2]).
This thesis is structured as follows: In Chapter 2 we recall Fulton's notion of relative dimension, which we call S-dimension, and the definition of the Chow groups of algebraic cycles on S-schemes. In Chapter 3 we provide the precise statement of our assumption on resolution of singularities and give a brief overview of cases where the assumption holds. In the first section of Chapter 4, we follow [KT87, Chapter 3] to introduce bivariant classes, orientation classes, orienting schemes, Alexander duality, exterior products and Kleiman and Thorup's fundamental results on orienting schemes. In the second section of this chapter we use these methods to prove Theorem A and Theorem B above. In the last section we will discuss how this results can be used to define an intersection product with supports on regular schemes. Chapter 5 is dedicated to the comparison of our theory to Fulton's theory. In Appendix A we summarize Fulton's theory for S-schemes for the convenience of the reader: We review the push-forward of cycles for proper morphisms, the pull-back of cycles for flat morphisms and the refined Gysin homomorphisms for regular imbeddings (and l.c.i. morphisms). Note that this is done in [Ful84, Chapters 1 - 6] for schemes X which are of finite type and separated over a ground field. In [Ful84, Section 20.1], Fulton remarks that his theorems still hold in the case of separated schemes of finite type over a Noetherian, regular and separated base scheme S. In our appendix, we formulate the theorems we need from Fulton's book for S-schemes as a summary of the statements needed in the development of the bivariant language in Section 4.1.

Notation

Throughout this thesis, all rings are assumed to be commutative with unit element. All schemes under consideration are assumed to be separated and of finite type over a fixed Noetherian, excellent, regular and separated base scheme S. We call these schemes S-schemes. For an irreducible scheme X, we denote η_X its generic point, and we sometimes write $\kappa(X)$ for the function field of X, i.e. the residue field $\kappa(\eta_X)$ of the generic point. More generally, for any $p \in X$, we set $\kappa(p) := \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$ for its residue field. The Krull dimension $\dim X$ of a S-scheme X is its dimension as a topological space (not to be confused with the S-dimension $\dim_S X$). The codimension $\operatorname{codim}(Z, X)$ of an irreducible closed subset Z of X is the supremum of integers n, such that there exists a chain

$$Z = Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

of closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \subseteq Y} \operatorname{codim}(Z, X)$$

where the infimum is taken over all closed irreducible subsets of Y. Since S-schemes are excellent and hence universally catenary, the dimension and codimension of a closed integral subscheme Z of X behave well, i.e. $\dim Z + \operatorname{codim}(Z, X) = \dim X$. We will use the definition of quasi-projective and projective morphisms as in [EGA, II 5.3.1, II 5.5.2]. In contrast to standard usage, we require a local complete intersection (l.c.i) morphism to admit a global factorization into a regular imbedding followed by a smooth morphism as in [Ful84]. A flat morphism $f : X \to Y$ of S-schemes is always assumed to have a relative S-dimension (cf. Definition A.1.2 and Remark A.1.3).

In contrast to Fulton, who denotes the Chow Groups by the letter A, we will use the more common CH, followed by a lower index if we grade by S-dimension or upper index if we grade by codimension. More specifically, $\mathrm{CH}_k X$ denotes the Chow group of algebraic cycles of S-dimension k on X. Note that codimension and S-dimension also behave well by Proposition 2.1.3 (ii). When we use the grading by codimension we will add the support of the cycles as a lower index, e.g. $\mathrm{CH}^k_Y(X)$ denotes the Chow group of algebraic cycles of codimension k on X supported in the closed subset Y. Note that neither should be confused with Chow cohomology or bivariant classes. For bivariant classes we have chosen to adapt the notation as in [KT87] and use the letters A, B and C followed by an upper index to indicate the corresponding shift in the S-dimension.

ACKNOWLEDGMENTS: The author would like to express his deepest gratitude to his advisor, Prof. Dr. Klaus Künnemann, for the continuous support, excellent guidance, and patience. The author also thanks V. Cossart, O. Piltant, P. Forré, J. Kolb, F. Wutz, B. Altmann, S. Flossmann, P. Jell, J. Sprang and P. Vollmer for helpful discussions and the collaborative research center SFB 1085 funded by the Deutsche Forschungsgemeinschaft for its support.

Chapter 2

Chow Groups of S-schemes

In this Chapter we recall the definition of the Chow groups of algebraic cycles on S-schemes i.e. separated schemes of finite type over a fixed Noetherian, excellent, regular and separated base scheme S. In [Ful84, Section 20.1], Fulton suggested a useful grading of these Chow groups by what he calls relative dimension, but we have chosen to call this S-dimension. A second way of grading the Chow groups is by codimension. A close study of the S-dimension in the first section will allow us to show, that the resulting grading of the Chow groups is compatible with the one by codimension in the usual way. Both gradings have their advantages: Proper push-forward of cycles preserves S-dimension, while flat pull-back preserves with the codimension (cf. Appendix A).

2.1 The S-Dimension

In order to introduce graded Chow groups we introduce a suitable notion of dimension, called the S-dimension. This definition goes back to Fulton [Ful84, Section 20.1].

Definition 2.1.1. Let X be an irreducible S-scheme with generic point η_X and structure morphism $p : X \to S$.

(i) For a morphism $f : X \to Y$ of S-schemes, we define

$$\dim_Y X := \operatorname{trdeg}(\kappa(X)/\kappa(f(\eta_X))) - \dim \mathcal{O}_{Y,f(\eta_X)}.$$

For $Y = S$, this gives the *S-dimension of X*:

$$\dim_S X = \operatorname{trdeg}(\kappa(X)/\kappa(p(\eta_X))) - \dim \mathcal{O}_{S,p(\eta_X)}.$$

(ii) An S-scheme X is called *S-equidimensional* of S-dimension d, if every irreducible component X_i of X satisfies

$$\dim_S X_i = d.$$

Remarks 2.1.2. (i) The S-dimension is called *relative dimension* in [Ful84, Section 20.1]. Note that Fulton works in a more general setting, namely without the assumption on the base scheme S to be separated and excellent. He proves Proposition 2.1.3 (ii), (iii) and hence (v) below in this more general setting.

(ii) As mentioned by Kleiman in [KT87, Section 2], this definition still induces a useful grading on the Chow groups, if we assume S only to be a separated, universally catenary, Noetherian scheme whose local rings are all equidimensional. Recall that a ring is called equidimensional if all its maximal ideals have the same codimension and all its minimal primes have the same dimension. Examples of schemes S whose local rings are equidimensional are Cohen-Macaulay schemes (cf. [Eis95, Proposition 18.8 and Corollary 18.11]) or schemes where every connected component is irreducible.

(iii) Note that (i) includes the case of S-schemes. So does (ii), as every regular Noetherian ring is Cohen-Macaulay (cf. [Liu02, Example 2.14]), hence such base schemes have equidimensional local rings.

The following Proposition is based on [Ful84, Lemma 20.1].

Proposition 2.1.3. *Let X, Y be irreducible S-schemes.*

(i) We have

$$\dim_S X = \dim_S X_{\text{red}}.$$

(ii) If $V \to X$ is a closed irreducible subscheme of X we have

$$\operatorname{codim}(V, X) = \dim_S X - \dim_S V.$$

(iii) For any dominant morphism $f : X \to Y$ of finite type, we have

$$\dim_S X = \dim_S Y + \operatorname{trdeg}(\kappa(X)/\kappa(Y)).$$

(iv) If $f : X \to S$ is a dominant morphism of finite type and closed, we have

$$\dim_S X = \dim X - \dim S,$$

where dim *denotes the Krull-dimension.*

(v) If $f : X \to Y$ is a morphism of S-schemes, then

$$\dim_S X = \dim_S Y + \dim_Y X.$$

(vi) If $S = \operatorname{Spec} K$ for a field K, then the S-dimension of X and the Krull dimension of X coincide. In the case that X and Y are irreducible schemes of finite type over a field K, we have

$$\dim_Y X = \dim X - \dim Y.$$

(vii) If $f : X \to Y$ is a flat morphism, for every point $y \in Y$ we have

$$\dim X_y = \dim_S X - \dim_S Y.$$

Proof. Assertion (i) follows directly from the definition, as the residue fields $\kappa(V)$ and $\kappa(p(\eta_V))$ remain the same for the reduced scheme and $\dim \mathcal{O}_{S_{\text{red}},p(\eta_X)} = \dim \mathcal{O}_{S,p(\eta_X)}$, because the Krull dimension only depends on the underlying topological spaces. Hence for (ii) - (vi) we can assume X and Y to be reduced, hence integral. Part (ii) and (iii) can be found in [Ful84, Lemma 20.1]. For the convenience of the reader, we reproduce the proof here: For (ii) let T and U be the closures of the images of X and V in S. Since S is universally catenary, the dimension formula [EGA, IV 5.6.5]

$$\operatorname{trdeg}(\kappa(X)/\kappa(T)) + \operatorname{codim}(U, T) = \operatorname{trdeg}(\kappa(V)/\kappa(U)) + \operatorname{codim}(V, X)$$

implies the claimed formula. Part (iii) follows since the transcendental degree is transitive and η_X and η_Y have the same image in S. As dominant, closed morphisms are surjective, (iv) follows from [EGA, IV 5.6.6.2]:

$$\dim_Y X = \operatorname{trdeg}(\kappa(X)/\kappa(Y)) = \dim X - \dim Y.$$

For (v), we consider the factorization of $f = g \circ h$ through the scheme theoretic image $\operatorname{Im}(f)$

$$f : X \xrightarrow{h} \operatorname{Im}(f) \xrightarrow{g} Y$$

with h dominant (cf. [GW10, Proposition 10.30]) and g a closed immersion. Then (v) follows using (ii) and (iii). The claim in (vi) follows from

$$\begin{aligned} \dim_Y X &= \operatorname{trdeg}(\kappa(X)/\kappa(f(\eta_X))) - \dim \mathcal{O}_{Y,f(\eta_X)} \\ &= \dim X - \dim \operatorname{Im}(f) - \operatorname{codim}(\operatorname{Im}(f), Y) \\ &= \dim X - \dim Y. \end{aligned}$$

For (vii), we follow the idea of [Liu02, Corollary 4.3.14]: Choose a closed point $x \in X_y$ in the fiber X_y. Let $\overline{\{x\}}^X$ denote the closure of $x \in X_y \subseteq X$ in X and $\overline{\{y\}}^Y$ the closure of y in Y. Observe that the residue field $\kappa(x)$ max either be computed as $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ or as $\mathcal{O}_{X_y,x}/\mathfrak{m}_{X_y,x}$ (as localization commutes with passage to the quotient). Since the point x is closed in X_y, $\kappa(x)$ is a finite field extension of $\kappa(y)$ (by Hilbert's Nullstellensatz) and we have $\operatorname{trdeg}(\kappa(x)/\kappa(y)) = 0$. Using (iii) for the induced morphism $\overline{\{x\}}^X_{\text{red}} \to \overline{\{y\}}^Y_{\text{red}}$ we get

$$\dim_S(\overline{\{y\}}^Y) = \dim_S(\overline{\{x\}}^X) - \operatorname{trdeg}(\kappa(x)/\kappa(y)) = \dim_S(\overline{\{x\}}^X)$$

and finally

$$\begin{aligned} \dim(X_y) &= \dim(\mathcal{O}_{X_y,x}) \\ &= \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y}) && \text{[Liu02, Theorem 4.3.12]} \\ &= \operatorname{codim}(\overline{\{x\}}, X) - \operatorname{codim}(\overline{\{y\}}, Y) \\ &= \dim_S X - \dim_S \overline{\{x\}} - \dim_S Y + \dim_S \overline{\{y\}} \\ &= \dim_S X - \dim_S Y. \end{aligned}$$

□

Examples 2.1.4. (i) Let S be a Noetherian, excellent, regular and separated base scheme S of (Krull) dimension one. A proper, flat morphism $X \to S$ with X integral is always dominant [Liu02, Proposition 3.9] and closed. Hence $\dim_S X = \dim X - 1$ by Proposition 2.1.3 (iv).

(ii) In general, the S-dimension of a scheme X is not necessarily the difference $\dim X - \dim S$ of the ordinary Krull dimensions: Let p be a prime number. For $S = \operatorname{Spec} \mathbb{Z}_p$ consider the S-schemes $X_1 = \operatorname{Spec} \mathbb{F}_p$ and $X_2 = \operatorname{Spec} \mathbb{Q}_p$. Observe that $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$ is a $\mathbb{Z}_p$-algebra of finite type. Then

$$\dim_S X_1 = -1 = \dim X_1 - \dim S.$$

However, $\dim_S X_2 = 0$, which is different from $\dim X_2 - \dim S = -1$.

2.2 Chow Groups

With the notion of S-dimension we can define the Chow group of algebraic cycles of S-dimension k on an S-scheme X.

Definition 2.2.1. Let X be an S-scheme.

(i) An *algebraic cycle of S-dimension k on X* is defined to be a finite formal sum

$$\sum_i n_i[V_i] \ ,$$

with closed integral subschemes V_i of S-dimension k of X and $n_i \in \mathbb{Z}$. Let Z_kX denote the *group of algebraic cycles of S-dimension k on X* given as the free abelian group on the closed integral subschemes V_i of S-dimension k of X.

(ii) For any closed integral subscheme W in X of S-dimension $(k+1)$ and any rational function $r \in \kappa(W)^*$ we define a cycle $[\operatorname{div}_W(r)] \in Z_kX$ by

$$[\operatorname{div}_W(r)] = \sum_{V \in W^{(1)}} \operatorname{length}_{\mathcal{O}_{W,\eta_V}}(\mathcal{O}_{W,\eta_V}/(r))[V],$$

the sum over all closed integral subschemes V of W of codimension one with generic point η_V. By the usual arguments one can deduce, that this sum is finite (cf. [Ful84, Section 1.2, Section 20.1]) and is indeed in $Z_k(X)$ by Proposition 2.1.3 (ii). Let $\operatorname{Rat}_kX \subseteq Z_kX$ be the subgroup generated by all such $[\operatorname{div}_W(r)]$ with a closed integral subscheme W in X of S-dimension $(k+1)$ and a rational function $r \in \kappa(W)^*$. We call Rat_kX the *group of algebraic cycles of S-dimension k rationally equivalent to zero on Y*. Sometimes the simplified notion $[\operatorname{div}(r)]$ or just $\operatorname{div}(r)$ is used instead of $[\operatorname{div}_W(r)]$.

(iii) Define

$$\mathrm{CH}_kX := Z_kX/\operatorname{Rat}_kX$$

to be the *Chow group of algebraic cycles of S-dimension k on X*.

(iv) Denote Z_*X (resp. CH_*X) the direct sum over the groups Z_kX (resp. CH_kX) for $k \in \mathbb{Z}$. An element in Z_*X (resp. CH_*X) is called a *cycle* on X (resp. a *cycle class* on X).

(v) Let $X_1, ..., X_t$ denote the irreducible components of an S-equidimensional scheme X of dimension d and $\eta_1, ..., \eta_t$ the corresponding generic points. Define the *(fundamental) cycle* $[X]$ of X as the cycle

$$[X] := \sum_{i=1}^{t} m_i[X_i] \quad \in Z_kX \ ,$$

with $m_i := \mathrm{length}_{\mathcal{O}_{X,\eta_i}}(\mathcal{O}_{X,\eta_i})$ denoting the *geometric multiplicity* of X_i in X. More generally we define the *(fundamental) cycle* $[X] \in \mathrm{CH}_*(X)$ of any S-scheme X in the same way.

Remark 2.2.2. A scheme and its underlying reduced scheme have the same closed integral subschemes. Therefore all the constructions above do not see whether the scheme is reduced or not, i.e. Z_kX and Z_kX_{red} are canonically isomorphic, as well as Rat_kX and $\mathrm{Rat}_kX_{\mathrm{red}}$ and hence we get $\mathrm{CH}_kX \cong \mathrm{CH}_kX_{\mathrm{red}}$.

Definition 2.2.3. Let Q be a ring and X an S-scheme.

(i) We define

$$Z_k(X)_Q := Z_kX \otimes_{\mathbb{Z}} Q$$

and

$$\mathrm{CH}_k(X)_Q := \mathrm{CH}_kX \otimes_{\mathbb{Z}} Q.$$

(ii) Denote by $Z_*(X)_Q$ (resp. $\mathrm{CH}_*(X)_Q$) the direct sum over the groups $Z_k(X)_Q$ (resp. $\mathrm{CH}_k(X)_Q$) for $k \in \mathbb{Z}$.

Remark 2.2.4. We can view a scheme X as the disjoint union of its connected components X_i. By doing this we obviously get $Z_*(X) = \bigoplus Z_*(X_i)$ and $\mathrm{CH}_*(X) = \bigoplus \mathrm{CH}_*(X_i)$. In order to give an intersection product in the sense of Theorem 4.3.3, it then suffices to give a pairing on each single $\mathrm{CH}_*(X_i)$ and set $\alpha.\beta = 0$ whenever α and β are cycle classes in different connected components.
Later we consider regular S-schemes X. Regular local rings are unique factorization domains and in particular integral domains by the Auslander-Buchsbaum-Theorem (cf. [Mat70, Theorem 48]). Thus a connected component X_i of a regular scheme X is already irreducible, since a point lying on the intersection of two different irreducible components would not have its local ring being a domain. In conclusion, we may later assume that a regular S-scheme X is irreducible, without loss of generality when we are interested in giving an intersection product

$$. \ : \mathrm{CH}_k(X)_Q \otimes \mathrm{CH}_{k'}(X)_Q \to \mathrm{CH}_{k+k'-\dim_S X}(X)_Q.$$

Chapter 3

Resolution of Singularities and Alterations

One of the main results (cf. [KT87, remarks after Proposition 3.3]) of Kleiman and Thorup about n-dimensional regular excellent schemes being $B_{\mathbb{Z}}$-orienting (in the sense of Chapter 4) is proven under the assumption that every excellent scheme of dimension at most n has a desingularization. A central idea of this thesis is to use de Jong's result on alterations to generalize Kleiman and Thorup's results. This forces us to allow algebraic cycles to have coefficients not only in $\mathbb{Z}$ but in a maybe larger ring $Q \subseteq \mathbb{Q}$. In this chapter, we introduce a suitable assumption on the existence of a suitable desingularization and we will give a brief overview of cases, where we already know that this assumption holds.

3.1 Assumption on Alterations

Definition 3.1.1. Let X denote a Noetherian integral scheme. An *alteration* X_1 of X is an integral scheme X_1 together with a morphism $\phi : X_1 \to X$, which is dominant, proper and generically finite of degree d for some $d \in \mathbb{N}_{\geq 1}$, i.e. for some nonempty open $U \subset X$, the morphism $\phi^{-1}(U) \to U$ is finite of degree d.

Assumption 3.1.2. For a multiplicative closed subset T in $\mathbb{Z}$ with $0 \notin T$, set $Q = T^{-1}\mathbb{Z}$. Let n be an integer and let S denote a Noetherian excellent regular separated scheme. We say that

$$\mathrm{A}(S, Q, \leq n)$$

holds, if the following is true:
For every integral separated scheme X of finite type over S of Krull dimension at most n, there exists an alteration $\phi : X_1 \to X$, which is generically finite of degree $d \in T$ (hence $\frac{1}{d} \in Q$) and such that X_1 is a regular integral scheme and $\phi : X_1 \to X$ is projective.

(3.1)
$$\begin{tikzcd}
X_1 \arrow[r, dotted, "\exists\phi"] & X \arrow[d] \\
& S.
\end{tikzcd}$$

3.2 State of the Art

In this section, we give a collection of to the knowledge of the author best known results, which imply that our assumption holds. For an overview see Corollary 3.2.8.

Theorem 3.2.1. *(De Jong) Let K be a field and V be a integral separated scheme of finite type over* $\operatorname{Spec} K$. *There exists an alteration $\phi_1 : V_1 \to V$ and an open immersion $j_1 : V_1 \to \overline{V_1}$ such that $\overline{V_1}$ is a projective regular integral separated scheme over* $\operatorname{Spec} K$.

Proof. For the proof see [dJ96, Theorem 3.1]. ☐

Theorem 3.2.2. *(De Jong) Let S be an excellent scheme of Krull dimension one. Assume X is an integral separated scheme of finite type over S. There exists an alteration $\phi_1 : X_1 \to X$ and an open immersion $j_1 : X_1 \to \overline{X_1}$ such that $\overline{X_1}$ is a regular integral scheme and projective over S.*

Proof. In the case that V is flat over S, this is [dJ97, Corollary 5.1]. Otherwise, $V \to S$ is not dominant (cf. [Liu02, Proposition 4.3.9]), so V is already defined over some field, in which the statement follows from Theorem 3.2.1. ☐

Theorem 3.2.3. *(Hironaka) Let X be an integral scheme of finite type over a field of characteristic zero, then there exists an algebraic subscheme D of X such that the set of points of D is exactly the singular locus of X, and if $f : \tilde{X} \to X$ is the blowing up of X with center D, then X is non-singular.*

Proof. This can be found in [Hir64, Main Theorem I]. ☐

Theorem 3.2.4. *(Abhyankar, Lipman) Let X be a reduced excellent Noetherian scheme of Krull dimension (at most) two. Then there exists a finite sequence of proper birational morphisms*

$$\pi : X' = X_n \to ... \to X_1 \to X_0 = X,$$

where $X_1 \to X$ is the normalization of X and for every $i \geq 1$,

$$X_{i+1} \to X_i$$

is the normalization of the blow-up of the singular locus of X_i endowed with the reduced scheme structure and X' is regular.

Proof. There is nothing to show in dimension zero. In dimension one it is well known, that the normalization of a curve is regular (cf. [Liu02, Example 8.3.41]). For dimension two we refer to [Liu02, Theorem 8.3.44]. ☐

Gabber obtained the following global resolution theorems, which improve de Jong's Theorems 3.2.1 and 3.2.2. We first introduce the notation of l-alteration:

Definition 3.2.5. Let S be a Noetherian scheme. Let $\mathcal{R}_S$ be the category whose objects are reduced S-schemes of finite type, whose generic points of irreducible components map to generic points of irreducible components of S with a finite residue field extension and morphisms are S-morphisms. For a prime number l, we define a *l-alteration* to be a proper surjective map in $\mathcal{R}_S$ with prime to l residue field extensions at the generic points.

Theorem 3.2.6. *(Gabber) Let V be a scheme separated and of finite type over a field K and l a prime number not equal to* $\mathrm{char(K)}$*. Then there exists a finite extension K' of K of order prime to l and an l-alteration $p : V_1 \to V$ such that V_1 is smooth and quasi-projective over K', and hence over K.*

Proof. Note that if K' is a finite extension of K, then $\operatorname{Spec} K' \to \operatorname{Spec} K$ is a finite morphism, hence quasi-projective. Then the claim follows from [ILO12, Theorem 3 (1), p. 3]. □

Theorem 3.2.7. *(Gabber) Let S be a Noetherian separated integral excellent regular scheme of dimension one. Let V be a separated scheme of finite type, l a prime number invertible on S (i.e. $l \neq \mathrm{char(K)}$ for all residue fields K). Then there exists an l-alteration $p : V_1 \to V$ such that V_1 is regular and quasi-projective over S.*

Proof. In the case that V is flat over S, this follows by [ILO12, Theorem 3 (2), p. 3]. Note, that there $S' \to S$ is finite, hence quasi-projective. Otherwise, $V \to S$ is not dominant (cf. [Liu02, Prop 4.3.9]), so V is over a field K, which is Theorem 3.2.6. □

In conclusion we have the following

Corollary 3.2.8. *The assumption holds in the following cases:*

(i) If $S = \operatorname{Spec} K$ for a field K of characteristic 0, then

$$\mathrm{A}(\operatorname{Spec} K,\ \mathbb{Z},\ \leq n)$$

holds for all $n \in \mathbb{N}$.

(ii) If $S = \operatorname{Spec} K$ for any field K, then

$$\mathrm{A}(\operatorname{Spec} K,\ \mathbb{Q},\ \leq n)$$

holds for all $n \in \mathbb{N}$. More precisely, given a prime number $l \neq \mathrm{char(K)}$, then

$$\mathrm{A}(\operatorname{Spec} K,\ \mathbb{Z}_{(l)},\ \leq n)$$

holds for all $n \in \mathbb{N}$, where $\mathbb{Z}_{(l)}$ denotes the localization of $\mathbb{Z}$ at $T = \mathbb{Z} \setminus (l)$.

(iii) For any Noetherian excellent regular separated scheme S, we have that

$$\mathrm{A}(S,\ \mathbb{Z},\ \leq 2)$$

holds.

(iv) Let S be a Noetherian excellent regular separated scheme of Krull dimension one. Then

$$\mathrm{A}(S,\, \mathbb{Q},\, \leq n)$$

holds for all $n \in \mathbb{N}$.

(v) Let S be a Noetherian separated integral excellent regular scheme of dimension one. For a prime number l invertible on S

$$\mathrm{A}(S,\, \mathbb{Z}_{(l)},\, \leq n)$$

holds for all $n \in \mathbb{N}$, where $\mathbb{Z}_{(l)}$ denotes the localization of $\mathbb{Z}$ at $T = \mathbb{Z} \setminus (l)$.

Proof. For (i) consider the diagram (3.1), where the alteration $\phi : X_1 \to X$ is given by Theorem 3.2.3 as a blowing up, hence the morphism $\phi : X_1 \to X$ is projective and X_1 is regular. The same argument works to deduce (iii) from Theorem 3.2.4: As all schemes are of finite type over S by hypothesis, they are excellent, hence the normalizations are all finite. Together with the projectivity of the blowing ups, this implies the required projectivity as well.

For (ii), (iv), and (v) consider the diagram (3.1) and note, that if X_1 is quasi-projective over S and X is of finite type and separated over S, then X_1 is also quasi-projective over X. Further we have, that if $X_1 \to X$ is quasi-projective and proper and if X is quasi-compact and quasi-separated, then $X_1 \to X$ is projective (cf. [GW10, Corollary 13.72]). Then the first claim in (ii) follows from Theorem 3.2.1, the second claim is Theorem 3.2.6, (iv) follows from Theorem 3.2.2 and (v) follows from Theorem 3.2.7. □

Remark 3.2.9. We want to remark, that there is a recent result by Cossart and Piltant, which is close to show $\mathrm{A}(S,\, \mathbb{Z},\, \leq 3)$ for any Noetherian excellent regular separated scheme S (cf. [CP14]). The only thing that is missing is the projectivity of alteration $\phi : X_1 \to X$ as in our Assumption 3.1.2 for any S-scheme of dimension at most three (this is so far only available for affine schemes).

Chapter 4

Intersection Theory with Supports on Regular Schemes

The goal of this chapter is to develop an intersection theory for regular separated schemes of finite type over a Noetherian, excellent, regular and separated base scheme S by combining de Jong's results on alterations with Fulton's method of the deformation to the normal cone. It turns out, that the results achieve their sharpest formulation in the bivariant language. In the first section we will follow [Ful84, Chapter 17] and [KT87, Chapter 3] to define bivariant classes, orientation classes, orienting schemes, and give some fundamental results on orienting schemes. This will provide the suitable framework for the second section, in which we will prove our first main result: Let S be a Noetherian, excellent, regular and separated scheme and assume $\mathrm{A}(S, Q, \leq d)$ as in Chapter 3 holds. Then we show that any regular separated scheme of dimension at most d and of finite type over S is B_Q-orienting. In the last section we will discuss how this result can be used to define an intersection product with supports on regular schemes.

Again, we fix a Noetherian, excellent, regular and separated base scheme S in this chapter. Further we assume, that all schemes under consideration are of finite type and separated over S. Recall that for a ring Q we have set

$$Z_k(X)_Q := Z_k X \otimes_{\mathbb{Z}} Q, \qquad \mathrm{CH}_k(X)_Q := \mathrm{CH}_k X \otimes_{\mathbb{Z}} Q$$

and $Z_*(X)_Q$ (resp. $\mathrm{CH}_*(X)_Q$) as the direct sum over the groups $Z_k(X)_Q$ (resp. $\mathrm{CH}_k(X)_Q$) for all $k \in \mathbb{Z}$.

4.1 Bivariant Classes and Orientations

We fix a ring Q. This section is based on [Ful84, Chapter 17] and [KT87, Chapter 3]. We merely adapt the ideas of [KT87, Chapter 3], formulate the precise statements in our situation, and sometimes give more detailed proofs. We add some corollaries, but the main ideas for the theorems of this section can already be found in Kleiman and Thorup's article. Note that for the convenience of the reader we included the needed statements from [Ful84, Chapters 1 - 6, Section 20.1, Appendix B] in Appendix A. In order to avoid set-theoretical complications, we follow Fulton's suggestion and fix a universe (cf. [Ful84, footnote on page 320] and [ML98]).

Definition 4.1.1. Let p be an integer and $f : X \to Y$ a morphism of S-schemes.

(i) A *p-system* a is a collection of homomorphisms of Q-modules

$$a^{(k)}_{Y'} : \mathrm{CH}_k(Y')_Q \to \mathrm{CH}_{k-p}(X')_Q$$

for all $k \in \mathbb{Z}$ and for all S-morphisms $Y' \to Y$ with induced fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \tag{4.1}$$

which are compatible with proper push-forward in the following sense: For all morphisms $g\colon Y' \to Y$, $h\colon Y'' \to Y'$ we form the fiber square diagram

$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "h'"'] & Y'' \arrow[d, "h"'] \\ X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"'] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.2}$$

with induced morphisms as labeled. Now assume $h : Y'' \to Y'$ to be proper and $\alpha \in \mathrm{CH}_k(Y'')_Q$, then we require

$$a^{(k)}_{Y'}(h_*\alpha) = h'_* a^{(k)}_{Y''}(\alpha) \text{ in } \mathrm{CH}_{k-p}(X')_Q. \tag{4.3}$$

The p-systems form a Q-module, which we denote by $H^p(f)_Q$. We set $H(f)_Q := \bigoplus_{p\in\mathbb{Z}} H^p(f)_Q$.

(ii) Let $A^p(f)_Q$ be the Q-sub-module of $H^p(f)_Q$ consisting of all p-systems a which commute with flat pull-backs and refined Gysin homomorphisms. More precisely: In the situation of (4.2) assume $h : Y'' \to Y'$ to be flat of relative S-dimension d and $\alpha \in \mathrm{CH}_k(Y')_Q$, then we require

$$a^{(k+d)}_{Y''}(h^*\alpha) = h'^* a^{(k)}_{Y'}(\alpha) \text{ in } \mathrm{CH}_{k+d-p}(X'')_Q.$$

If $g : Y' \to Y$, $h : Y' \to Z'$ are morphisms of S-schemes, and $i : Z'' \to Z'$ is a regular imbedding of codimension e, then for the fiber square diagram

$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "i''"'] & Y'' \arrow[r, "h'"] \arrow[d, "i'"'] & Z'' \arrow[d, "i"'] \\
X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[r, "h"] \arrow[d, "g"'] & Z' \\
X \arrow[r, "f"] & Y &
\end{tikzcd} \tag{4.4}$$

and for all $\alpha \in \mathrm{CH}_k(Y')_Q$, we require

$$a^{(k-e)}_{Y''}(i^!\alpha) = i^! a^{(k)}_{Y'}(\alpha) \ \text{ in } \mathrm{CH}_{k-e-p}(X'')_Q,$$

where $i^!$ denotes the refined Gysin homomorphisms as in Construction A.5.1. We call $A^p(f)_Q$ the *Q-module of bivariant classes* of f and set $A(f)_Q := \bigoplus_{p\in\mathbb{Z}} A^p(f)_Q$.

Remark 4.1.2. In Definition 4.1.1, a group structure is then given by

$$+ : H^p(f)_Q \times H^p(f)_Q \to H^p(f)_Q, \ (a, a') \mapsto a + a',$$

where $a + a'$ is induced by $(a^{(k)}_{Y'} + a'^{(k)}_{Y'})(\alpha) = a^{(k)}_{Y'}(\alpha) + a'^{(k)}_{Y'}(\alpha) \ \in \mathrm{CH}_{k-p}(X')_Q$ for $\alpha \in \mathrm{CH}_k(Y')_Q$. The operation

$$Q \times H^p(f)_Q \to H^p(f)_Q$$

of Q on $H^p(f)_Q$ is induced by $(q \cdot a^{(k)}_{Y'})(\alpha) = q \cdot (a^{(k)}_{Y'}(\alpha)) \in \mathrm{CH}_{k-p}(X')_Q$ for $\alpha \in \mathrm{CH}_k(Y')_Q$ and induces the structure of a Q-module on $H^p(f)_Q$.

On $H^p(f)_Q$ we have the following operations:

Lemma/Definition 4.1.3. *Let p be an integer and $f : X \to Y$ denote a morphism of S-schemes. Then the following operations are well-defined:*

(i) For a morphism $g : Y' \to Y$ of S-schemes, form the fiber square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and define the restriction, *or* pull-back, *as the homomorphism*

$$g^* : H^p(f)_Q \to H^p(f')_Q$$

given by

$$(g^*a)^{(k)}_{Y''}(\alpha) := a^{(k)}_{Y''}(\alpha) \ \text{in } \mathrm{CH}_{k-p}(X'')_Q$$

for any morphism $h : Y'' \to Y'$ of S-schemes and all $\alpha \in \mathrm{CH}_k(Y'')_Q$.

(ii) *Given a factorization* $f = g \circ h$*, such that* h *is proper, define the* push-forward

$$h_* : H^p(f)_Q \to H^p(g)_Q$$

by

$$(h_* a)^{(k)}_{Y'}(\alpha) := h'_*(a^{(k)}_{Y'}(\alpha)) \text{ in } \mathrm{CH}_{k-p}(T')_Q$$

for any morphism $Y' \to Y$ *of* S*-schemes with fiber square diagram*

$$\begin{tikzcd}
X' \arrow[r, "h'"] \arrow[d] & T' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "h"] & T \arrow[r, "g"] & Y
\end{tikzcd}$$

and all $\alpha \in \mathrm{CH}_k(Y')_Q$.

(iii) *For morphisms* $f : X \to Y$, $g : Y \to Z$ *of* S*-schemes,* $p, q \in \mathbb{Z}$*, a* p*-system* a *in* $H^p(f)_Q$ *and a* q*-system* b *in* $H^q(g)_Q$ *we define the* (intersection) product *to be the homomorphism*

$$. : H^p(f)_Q \otimes H^q(g)_Q \to H^{p+q}(g \circ f)_Q$$

given by

$$((a.b)^{(k)}_{Z'})(\alpha) := a^{(k-q)}_{Y'}(b^{(k)}_{Z'}(\alpha)) \text{ in } \mathrm{CH}_{k-p-q}(X')_Q$$

for a morphism $Z' \to Z$ *of* S*-schemes with fiber square diagram*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & Z' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

and all $\alpha \in \mathrm{CH}_k(Y'')_Q$*. This product is obviously associative.*

(iv) *Given a morphism* $g : W \to Z$ *of* S*-schemes, form the fiber square*

$$\begin{tikzcd}
X \times_S W \arrow[r] \arrow[d] & Y \times_S W \arrow[r] \arrow[d, "g'"'] & W \arrow[d, "g"'] \\
X \times_S Z \arrow[r] \arrow[d] & Y \times_S Z \arrow[r, "p'_Y"] \arrow[d, "p'_Z"'] & Z \cdot \arrow[d, "p_Z"'] \\
X \arrow[r, "f"] & Y \arrow[r, "p_Y"] & S
\end{tikzcd}$$

Then for $p, q \in \mathbb{Z}$*, a* p*-system* a *in* $H^p(f)_Q$ *and a* q*-system* b *in* $H^q(g)_Q$*, the* cartesian product

$$\times : H^p(f)_Q \otimes H^q(g)_Q \to H^{p+q}(f \times_S g)_Q$$

is defined by

$$((a \times b)^{(k)}_{Y \times_S Z})(\alpha) := (((p'_Z \circ g')^* a).((p'_Y)^* b))^{(k)}_{Y \times_S Z}(\alpha) \text{ in } \mathrm{CH}_{k-p-q}(X \times_S W)_Q$$

for $(p'_Z \circ g') : Y \times_S W \to Y$ *and* $p'_Y : Y \times_S Z \to Z$ *and for all* $\alpha \in \mathrm{CH}_k(Y \times_S Z)_Q$.

Proof. We have to show that these operations are well defined, i.e. the systems we define are compatible with proper push-forward. For the restriction in (i), given a fiber square diagrams

$$\begin{tikzcd}
X''' \arrow[r, "f'''"] \arrow[d, "h'"'] & Y''' \arrow[d, "h"'] \\
X'' \arrow[r, "f''"] \arrow[d] & Y'' \arrow[d] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

with h proper and $\alpha \in \mathrm{CH}_k(Y''')_Q$, we require

$$a^{(k)}_{Y''}(h_* \alpha) = h'_* a^{(k)}_{Y''}(\alpha) \text{ in } \mathrm{CH}_{k-p}(X'')_Q.$$

But this follows directly from the definition, as the upper fiber square is given by base-change from $f : X \to Y$ as in (4.2). For the push-forward in (ii), we consider the fiber square diagram

$$\begin{tikzcd}
X'' \arrow[r, "h''"] \arrow[d, "e''"'] & T'' \arrow[r, "g''"] \arrow[d, "e'"'] & Y'' \arrow[d, "e"'] \\
X' \arrow[r, "h'"] \arrow[d] & T' \arrow[r, "g'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "h"] & T \arrow[r, "g"] & Y
\end{tikzcd}$$

for a proper morphism $e : Y'' \to Y'$. Then we get

$$\begin{aligned}
(h_* a)^{(k)}_{Y'}(e_* \alpha) &= h'_* a^{(k)}_{Y'}(e_* \alpha) && \text{(definition)} \\
&= h'_* e''_* a^{(k)}_{Y''}(\alpha) && \text{(p-system)} \\
&= e'_* h''_* a^{(k)}_{Y''}(\alpha) && \text{(functoriality)} \\
&= e'_* (h_* a)^{(k)}_{Y''}(\alpha) \text{ in } \mathrm{CH}_{k-p}(T')_Q && \text{(definition)}
\end{aligned}$$

for all $\alpha \in \mathrm{CH}_k(Y'')_Q$, so the push-forward is well-defined. For the compatibility of the product with proper push-forward in (iii) consider the fiber square diagram

$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "h''"'] & Y'' \arrow[r, "g''"] \arrow[d, "h'"'] & Z'' \arrow[d, "h"'] \\
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & Z' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

for a proper morphism $h : Z'' \to Z'$. Then the claim follows from the calculation

$$\begin{aligned}
(h''{}_* (a.b))^{(k)}_{Z''}(\alpha) &= h''_* (a^{(k-q)}_{Y''}(b^{(k)}_{Z''}(\alpha))) && \text{(definition)} \\
&= a^{(k-q)}_{Y'} h'_* (b^{(k)}_{Z''}(\alpha)) && \text{(p-system)} \\
&= a^{(k-q)}_{Y'}(b^{(k)}_{Z'}(h_* \alpha)) && \text{(q-system)} \\
&= ((a.b)^{(k)}_{Z'})\,(h_* \alpha) \text{ in } \mathrm{CH}_{k-p-q}(X')_Q && \text{(definition)}
\end{aligned}$$

for $\alpha \in \mathrm{CH}_k(Z'')_Q$, $a \in H^p(f)_Q$ and $b \in H^q(g)_Q$. The last claim (iv) follows directly from (i) and (iii) by definition. The Q-linearity of the operations in (i) - (iv) is trivial. □

Proposition 4.1.4. *The operations in Lemma/Definition 4.1.3 restrict to bivariant classes in the following sense:*

(i) The restriction homomorphism in Lemma/Definition 4.1.3 (i) induces

$$g^* : A^p(f)_Q \to A^p(f')_Q.$$

(ii) The push-forward homomorphism in Lemma/Definition 4.1.3 (ii) induces

$$h_* : A^p(f)_Q \to A^p(g)_Q.$$

(iii) The (intersection) product in Lemma/Definition 4.1.3 (iii) induces

$$. \ : A^p(f)_Q \otimes A^q(g)_Q \to A^{p+q}(g \circ f)_Q.$$

(iv) The cartesian product in Lemma/Definition 4.1.3 (iv) induces

$$\times \ : A^p(f)_Q \otimes A^q(g)_Q \to A^{p+q}(f \times_S g)_Q.$$

Proof. Since $A^p(f)_Q$ is a Q-submodule of $H^p(f)_Q$ for all $f : X \to Y$ and all $p \in \mathbb{Z}$, we only have to check if the restriction, push-forward, intersection product and cartesian product of some given classes is again compatible with flat pull-backs and refined Gysin homomorphisms. This follows directly form the definitions in (i), (iii) and (iv). In (ii) the claim follows directly from Proposition A.1.4 (ii) and Theorem A.5.2 (i). □

Theorem 4.1.5 (Fulton)**.** *(i) Any flat morphism $f : X \to Y$ of relative S-dimension p gives a well-defined bivariant class $[f] \ \in A^{-p}(f)_{\mathbb{Z}}$.*

(ii) Any regular imbedding $f : X \to Y$ of codimension p (or more generally l.c.i. morphisms of codimension p) gives a well-defined bivariant class $[f] \ \in A^{p}(f)_{\mathbb{Z}}$.

Tensoring the Chow groups with the ring Q, these systems canonically induce systems in $A^p(f)_Q$

Proof. For (i), note that any base change $f' : X \times_Y Y' \to Y'$ is again flat and of relative S-dimension p (cf. Remark A.1.3), hence we have homomorphisms

$$f'^{*} : \mathrm{CH}_k Y' \to \mathrm{CH}_{k-p} X,$$

which are compatible with proper push-forward by A.1.4 (ii), other flat pull-backs by Proposition A.1.4 (i) and the refined Gysin homomorphisms of regular imbeddings by Theorem A.5.2 (ii). Hence the system of flat pull-backs $[f]$ is in $A^{-p}(f)_Q$. Part (ii) is exactly Construction A.5.1 (ii) and Theorem A.5.2 (i),(ii) and (iv) for regular imbeddings of codimension p and Proposition A.5.5 (iii) for l.c.i. morphisms of codimension p. □

Definition 4.1.6. We denote by $\left(B^p(f)\right)_{p,f}$ the smallest family of Q-submodules $B^p(f)$ of $H^p(f)$ for $p \in \mathbb{Z}$ and morphisms of S-schemes $f : X \to Y$ such that the following holds:

(i) If $f : X \to Y$ is flat of relative S-dimension p, the $(-p)$-system of flat pull-backs $[f]$ as in Theorem 4.1.5 is in $B^{-p}(f)_Q$.

(ii) If $f : X \to Y$ is a regular imbedding of codimension p, the p-system of refined Gysin homomorphisms $[f]$ as in Theorem 4.1.5 is in $B^p(f)_Q$.

(iii) For any morphism $g : Y' \to Y$ of S-schemes and $a \in B^p(f)_Q$, the restriction g^*a is in $B^p(f')$ (cf. Lemma/Definition 4.1.3).

(iv) Given a factorization $f = g \circ h$ with h proper and $a \in B^p(f)_Q$, the push-forward h_*a is in $B^p(g)_Q$ (cf. Lemma/Definition 4.1.3).

(v) Given a morphism $g : Y \to Z$ of S-schemes, $a \in B^p(f)_Q$ and $b \in B^q(g)_Q$, the intersection product $a.b$ is in $B^{p+q}(g \circ f)_Q$ (cf. Lemma/Definition 4.1.3).

(vi) Given any morphism $g : W \to Z$ of S-schemes, $a \in B^p(f)_Q$ and $b \in B^q(g)_Q$, the cartesian product $a \times b$ is in $B^{p+q}(f \times_S g)_Q$ (cf. Lemma/Definition 4.1.3).

We set $B(f)_Q := \bigoplus_{p\in\mathbb{Z}} B^p(f)_Q$.

Remark 4.1.7. From Definition 4.1.6, the following *induction principle* is evident: Assume we a given a graded family of Q-submodules $E(f)_Q = \bigoplus_{p\in\mathbb{Z}} E^p(f)_Q$ of $H(f)_Q$ for all morphisms $f : X \to Y$ of S-schemes which satisfies the corresponding properties (i) - (v) of Definition 4.1.6, then $B(f)_Q \subseteq E(f)_Q$, as $B(f)_Q$ is the smallest of such families of Q-modules. Note that we do not have to check (vi) since the cartesian product is by definition given as an intersection product of some restricted classes, hence the compatibility in (vi) follows from (iii) and (v).

Theorem 4.1.8. *Let $f : X \to Y$ be a morphism of S-schemes. Then $B^p(f)_Q$ is a Q-submodule of $A^p(f)_Q$.*

Proof. By Remark 4.1.7 we have to show that $A(\_)_Q$ satisfies the corresponding properties of Definition 4.1.6 (i) - (v). The $(-p)$-system of flat pull-backs $[f] \in B^{-p}(f)_Q$ for a flat morphism $f : X \to Y$ of relative S-dimension p is in $A^{-p}(f)_Q$ by Theorem 4.1.5. The p-system of refined Gysin homomorphisms $[f] \in B^p(f)_Q$ for a regular imbedding of codimension p is in $A^p(f)_Q$ by Theorem 4.1.5 as well. The rest follows from Proposition 4.1.4 . □

Definition 4.1.9. Let $f : X \to Y$ be a morphism of S-schemes. We define $C^p(f)_Q$ to be the Q-submodule consisting of all systems a in $H^p(f)_Q$, which commute with any system b in $A(t)_Q$ for any morphism $t : W \to Z$ of S-schemes by means of the intersection products of the appropriately restricted p-systems: Consider the fiber square diagram

$$\begin{tikzcd}
X \times_S W \arrow[r, "f''"] \arrow[d, "t''"'] & Y \times_S W \arrow[r, "p''_Y"] \arrow[d, "t'"'] & W \arrow[d, "t"'] \\
X \times_S Z \arrow[r, "f'"] \arrow[d, "p''_Z"'] & Y \times_S Z \arrow[r, "p'_Y"] \arrow[d, "p'_Z"'] & Z \arrow[d, "p_Z"'] \\
X \arrow[r, "f"] & Y \arrow[r, "p_Y"] & S,
\end{tikzcd}$$

then we require

$$((p'_Z \circ t')^* a).((p'_Y)^* b) = ((p'_Y \circ f')^* b).((p'_Z)^* a) \in H(f' \circ t'')_Q = H(t' \circ f'')_Q.$$

We set $C(f)_Q := \bigoplus_{p \in \mathbb{Z}} C^p(f)_Q$.

Proposition 4.1.10. *Let $f : X \to Y$ be a morphism of S-schemes. Then $B^p(f)_Q$ is a Q-submodule of $C^p(f)_Q$.*

Proof. Again we use Remark 4.1.7: We have to show that $C(\_)_Q$ satisfies the corresponding properties of Definition 4.1.6 (i) - (v): The $(-p)$-system of flat pull-backs $[f]$ for a flat morphism $f : X \to Y$ of relative S-dimension p is in $C^{-p}(f)_Q$, as any system b in $A(t)_Q$ for a morphism $t : W \to Z$ commutes with flat pull-backs by Definition 4.1.1 (ii). The p-system of refined Gysin homomorphisms $[f]$ for a regular imbedding of codimension p is in $C^p(f)_Q$ by Definition 4.1.1 (ii) as well. For any morphism $g : Y' \to Y$ of S-schemes and $a \in C^p(f)_Q$, the restriction $g^* a$ is in $C^p(f')$. This holds, as the required commutativity of the restriction $g^* a \in C^p(f')$ with systems in $A(t)_Q$ for $t : W \to Z$ follows directly from the exact same commutativity for $a \in C^p(f)$. Given a factorization $f = g \circ h$ with h proper and $a \in C^p(f)_Q$, the push-forward $h_* a$ is in $C^p(g)_Q$, because systems $b \in A(t)_Q$ for $t : W \to Z$ as in Definition 4.1.9 commute with the proper push-forward h_* by Definition 4.1.1. Given a morphism $g : Y \to Z$ of S-schemes, $a \in C^p(f)_Q$ and $b \in C^q(g)_Q$, the intersection product $a.b$ is obviously in $C^{p+q}(g \circ f)_Q$ by Definition of the intersection product (cf. Lemma/Definition 4.1.3). □

Corollary 4.1.11. *In summery, for any $f : X \to Y$ we get the following inclusions of Q-modules:*

$$H^p(f)_Q \supseteq A^p(f)_Q \supseteq B^p(f)_Q \text{ and}$$
$$H^p(f)_Q \supseteq C^p(f)_Q \supseteq B^p(f)_Q.$$

In particular, classes in $B(\_)_Q$ commute, i.e. for morphisms $f : X \to Y$, $g : W \to Z$ and classes $a \in B(f)_Q$, $b \in B(g)_Q$ consider the fiber square diagram

$$\begin{tikzcd}
X \times_S W \arrow[r, "f''"] \arrow[d, "g''"'] & Y \times_S W \arrow[r, "p''_Y"] \arrow[d, "g'"'] & W \arrow[d, "g"'] \\
X \times_S Z \arrow[r, "f'"] \arrow[d, "p''_Z"'] & Y \times_S Z \arrow[r, "p'_Y"] \arrow[d, "p'_Z"'] & Z \arrow[d, "p_Z"'] \\
X \arrow[r, "f"] & Y \arrow[r, "p_Y"] & S,
\end{tikzcd}$$

with morphisms as labeled. Then we have

$$((p'_Z \circ g')^* a).((p'_Y)^* b) = ((p'_Y \circ f')^* b).((p'_Z)^* a) \in B(f' \circ g'')_Q.$$

Proof. The inclusions follow directly from the definitions and Proposition 4.1.10. The commutativity of classes in $B(\_)_Q$ then trivially follows: We view the class $a \in B(f)_Q$ as a class in the larger Q-module $C(f)_Q$ and $b \in B(g)_Q$ as a class in $A(g)_Q$. Then the claim follows directly from Definition 4.1.9. □

Definition 4.1.12. (i) A morphism $f : X \to Y$ of S-schemes is called *B_Q-oriented*, if it is equipped with a class $[f]$ in $B(f)_Q$, such that $[f]([Y]) = [X]$. In this case, we call $[f]$ a *B_Q-orientation class.*

(ii) An S-scheme Y is called *B_Q-orienting* if every $f : X \to Y$ is B_Q-oriented in a unique way.

(iii) An S-scheme Y is called *B_Q-orthocyclic* if for every morphism $f : X \to Y$ of S-schemes we have $B(f)_Q = A(f)_Q$ and if evaluation at $[Y]$ is an isomorphism of Q-modules

$$E_Y : B(f)_Q = A(f)_Q \xrightarrow{\sim} \mathrm{CH}_*(X)_Q,\ (a_p)_{p\in\mathbb{Z}} \mapsto \sum_{p\in\mathbb{Z}} (a_p)_Y^{(k)}[Y]$$

where $k = \dim_S Y$. We will denote the inverse of E_Y by

$$F_{X/Y} : \mathrm{CH}_*(X)_Q \longrightarrow B(f)_Q.$$

For an explicit construction of $F_{X/Y}$ for B_Q-orthocyclic schemes see Remark 4.1.17.

(iv) A morphism $f : X \to Y$ of S-schemes is called *C_Q-oriented*, if it is equipped with a class $[f]$ in $C(f)_Q$, such that $[f]([Y]) = [X]$. In this case, we call $[f]$ a *C_Q-orientation class.*

(v) An S-scheme Y is called *C_Q-orienting* if every $f : X \to Y$ is C_Q-oriented in a unique way.

(vi) An S-scheme Y is called *C_Q-orthocyclic* if for every morphism $f : X \to Y$ of S-schemes we have $C(f)_Q = A(f)_Q$ and if evaluation at $[Y]$ is an isomorphism of Q-modules

$$C(f)_Q = A(f)_Q \xrightarrow{\sim} \mathrm{CH}_*(X)_Q,\ (a_p)_{p\in\mathbb{Z}} \mapsto \sum_{p\in\mathbb{Z}} (a_p)_Y^{(k)}[Y],$$

where $k = \dim_S Y$.

We will see later that (ii) and (iii) are equivalent (Lemma 4.1.15 and Corollary 4.1.17 below) and that (vi) implies (v) (Lemma 4.1.15 below).

Lemma 4.1.13. *Let $f : X \to Y$ be an S-morphism with X, Y irreducible.*

(i) If Y is B_Q-orienting, then the B_Q-orientation class

$$[f] \in B(f)_Q$$

is already in

$$B^{-\dim_Y X}(f)_Q.$$

In particular, if f is a closed immersion, the B_Q-orientation class $[f] \in B(f)_Q$ is already in $B^{\mathrm{codim}(X,Y)}(f)_Q$.

(ii) If Y is C_Q-orienting, then the C_Q-orientation class $[f] \in C(f)_Q$ is already in $C^{-\dim_Y X}(f)_Q$.

Proof. Consider a B_Q-orientation class

$$[f] = ([f]_p)_{p\in\mathbb{Z}} \in B(f)_Q = \bigoplus_{p\in\mathbb{Z}} B^p(f)_Q.$$

The part

$$[f]_{-\dim_Y X} \in B^{-\dim_Y X}(f)_Q = B^{\dim_S Y - \dim_S X}(f)_Q \subseteq B(f)_Q$$

(cf. Proposition 2.1.3 (v)) already satisfies

$$[f]_{-\dim_Y X}([Y]) = [X],$$

hence $[f]_{-\dim_Y X}$ defines a B_Q-orientation class. But now, since Y is B_Q-orienting, there is only one B_Q-orientation class, hence

$$[f] = [f]_{-\dim_Y X} \in B^{-\dim_Y X}.$$

If f is a closed immersion, the second claim in (i) follows by Proposition 2.1.3 (ii). The proof for part (ii) about the C_Q-orientation class in analogous. □

Proposition 4.1.14 (Fulton). *(i) Any flat morphism $f : X \to Y$ of S-schemes of relative S-dimension n is $B_{\mathbb{Z}}$-oriented.*

(ii) Any regular imbedding $f : X \to Y$ of S-schemes of codimension d with an S-equidimensional scheme Y is $B_{\mathbb{Z}}$-oriented.

(iii) Any l.c.i. morphism $f : X \to Y$ of S-schemes of codimension d with an S-equidimensional scheme Y is $B_{\mathbb{Z}}$-oriented.

Proof. By Theorem 4.1.5 and Definition 4.1.6, both flat morphisms of some relative S-dimension n and regular imbeddings define a system $[f]$ in $B(f)_{\mathbb{Z}}$. Then the claim in (i) follows, as $[f]([Y]) = f^*[Y] = [f^{-1}(Y)] = [X]$. For (ii), see Theorem A.5.2 (viii). The claim in (iii) follows from (i) and (ii) as we can factor an l.c.i. morphism into a regular imbedding of codimension e followed by a smooth morphism of relative S-dimension $d + e$. □

In the following, we will investigate the relations between the properties B_Q-orthocyclic, B_Q-orienting, C_Q-orthocyclic, and C_Q-orienting for S-schemes.

Lemma 4.1.15 (Kleiman-Thorup). *Suppose an S-scheme Y is B_Q-orthocyclic (resp. C_Q-orthocyclic), then Y is B_Q-orienting (resp. C_Q-orienting).*

Proof. Assume $f : X \to Y$ is any morphism of S-schemes and Y is B_Q-orthocyclic (resp. C_Q-orthocyclic). The cycle $[X] \in \mathrm{CH}_*(X)_Q$ has a unique preimage under the isomorphism $E_Y : A(f)_Q \to \mathrm{CH}_*(X)_Q$, so it determines a system $[f] := (a_p)_{p\in\mathbb{Z}}$ in $A(f)_Q = B(f)_Q$ (resp. in $A(f)_Q = C(f)_Q$) with $[f](Y) = \sum\limits_{p\in\mathbb{Z}} (a_p)_Y^{(k)}[Y] = [X]$. Assume we are given a second system $[f]' \in A(f)_Q$ with $[f]'([Y]) = [X]$, then by the injectivity of $E_Y : A(f)_Q \to \mathrm{CH}_*(X)_Q$ we have that $[f]' = [f] \in A(f)_Q$, which shows the uniqueness of the B_Q-orientation class (resp. C_Q-orientation class). □

The converse of Lemma 4.1.15 is covered by the following important criterion found by Kleiman and Thorup (cf. [KT87, Proposition 3.3]).

Proposition 4.1.16 (Kleiman-Thorup). *Let Y be an S-scheme. If every projective morphism $g : V \to Y$ of S-schemes with V integral is B_Q-oriented, then Y is B_Q-orthocyclic (and hence B_Q-orienting by Lemma 4.1.15).*

For the convenience of the reader we have reproduced the beautiful, short proof of Kleiman and Thorup:

Proof. (cf. [KT87, Proposition 3.3]) Let $g : V \to Y$ any morphism of S-schemes with V integral. First we apply Chow's lemma [EGA, II.5.6] to get a commutative diagram

$$\begin{tikzcd}
V \arrow[d, "g"'] & V' \arrow[l, hook', "h"'] \arrow[d, "i"'] \\
Y & V'' \arrow[l, hook', "g''"']
\end{tikzcd}$$

in which h is projective and birational, i is an open embedding, g'' is projective and V'' is integral. By hypothesis g'' is B_Q-oriented and we can define

$$[g] := h_*([i].[g'']) \in B_Q(g), \tag{4.5}$$

which is an B_Q-orientation class as we have

$$\begin{aligned}
[g]([Y]) &= \left(h_*([i].[g''])\right)([Y]) \\
&= h_*([i]([g'']([Y]))) \\
&= h_*([i]([V''])) \\
&= h_*[V'] \\
&= [V] \quad \in \mathrm{CH}_*(V)_Q.
\end{aligned}$$

Now, the idea is to construct the inverse $F_{X/Y} : \mathrm{CH}_*(X)_Q \to B(f)_Q$ for the isomorphism $E_Y : B(f)_Q \to \mathrm{CH}_*(X)_Q$. Given a morphism $f : X \to Y$ of S-schemes and a cycle $\alpha = \sum n_V[V] \in Z_*(X)_Q$ with induced closed immersions $j_V : V \to X$, we define

$$F_{X/Y}\alpha := \sum n_V j_{V*}[g_V] \in B(f)_Q$$

for $g_V := f \circ j_V$ and $[g_V]$ constructed as in (4.5).
Assume the cycle α is rationally equivalent to a cycle α', then $F_{X/Y}\alpha = F_{X/Y}\alpha'$: Given any morphism $\tilde{h} : Y' \to Y$ of S-schemes and a closed integral subscheme $i_W : W \to Y'$, we set $h := \tilde{h} \circ i_W : W \to Y$ and we form $[h] \in B(h)_Q$ with $[h]([Y]) = [W]$ as in (4.5). Then consider the fiber square diagram

$$\begin{tikzcd}
X \times_Y W \arrow[r] \arrow[d, "i'_W"'] & W \arrow[d, "i_W"'] \arrow[dd, bend left, "h"] \\
X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d, "\tilde{h}"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and calculate

$$\begin{aligned}
(\tilde{h}^*(F_{X/Y}\alpha))(i_{W*}[W]) &= i'_{W*}\,(h^*(F_{X/Y}\alpha))([W])\\
&= i'_{W*}\,(h^*(F_{X/Y}\alpha))([h]([Y]))\\
&= i'_{W*}\,\left(\left(h^*(F_{X/Y}\alpha)\right).[h]\right)([Y])\\
&= i'_{W*}\,((f^*[h]).(F_{X/Y}\alpha))([Y]) \qquad \text{(cf. Corollary 4.1.11)}\\
&= i'_{W*}\,(f^*[h])((F_{X/Y}\alpha)([Y]))\\
&= i'_{W*}\,(f^*[h])(\alpha)\\
&= i'_{W*}\,(f^*[h])(\alpha')\\
&= i'_{W*}\,(f^*[h])((F_{X/Y}\alpha')([Y]))\\
&= i'_{W*}\,((f^*[h]).(F_{X/Y}\alpha'))([Y])\\
&= i'_{W*}\,\left(\left(h^*(F_{X/Y}\alpha')\right).[h]\right)([Y]) \qquad \text{(cf. Corollary 4.1.11)}\\
&= i'_{W*}\,(h^*(F_{X/Y}\alpha'))([h]([Y]))\\
&= i'_{W*}\,(h^*(F_{X/Y}\alpha'))([W])\\
&= (\tilde{h}^*(F_{X/Y}\alpha'))(i_{W*}[W]) \qquad \in \mathrm{CH}_*(X \times_Y Y)_Q.
\end{aligned}$$

This means that $F_{X/Y}\alpha = F_{X/Y}\alpha' \in B(f)_Q$. Thus, we have a morphism of Q-modules

$$F_{X/Y} : \mathrm{CH}_*(X)_Q \to B(f)_Q \tag{4.6}$$

with $E_Y \circ F_{X/Y} = \mathrm{id}_{\mathrm{CH}_*(X)_Q}$ by construction.
Finally, given a class a in $A(f)_Q$, a morphism $\tilde{g} : Y' \to Y$ of S-schemes and a closed integral subscheme $i_V : V \to Y'$, we set $g := \tilde{g} \circ i_V : V \to Y$ and we form $[g] \in B(g)_Q$ with $[g]([Y]) = [V]$ as in (4.5) and denote $i'_V : X \times_Y V \to X \times_Y Y$ the base change. For $\alpha := E_Y a = a([Y])$ we have

$$\begin{aligned}
(\tilde{g}^*(F_{X/Y}E_Y a))(i_{V*}[V]) &= i'_{V*}\,g^*(F_{X/Y}E_Y a)([V])\\
&= i'_{V*}\,g^*(F_{X/Y}E_Y a)\,([g]([Y]))\\
&= i'_{V*}\,(g^*(F_{X/Y}E_Y a)).[g]\,([Y])\\
&= i'_{V*}\,(f^*[g]).(F_{X/Y}E_Y a)\,([Y]) \qquad \text{(by Corollary 4.1.11)}\\
&= i'_{V*}\,(f^*[g]).(F_{X/Y}\alpha)\,([Y])\\
&= i'_{V*}\,(f^*[g])\,(F_{X/Y}\alpha([Y]))\\
&= i'_{V*}\,(f^*[g])\,(\alpha)\\
&= i'_{V*}\,(f^*[g])\,(a([Y]))\\
&= i'_{V*}\,(f^*[g].a)\,([Y])\\
&= i'_{V*}\,g^*a.[g]\,([Y]) \qquad \text{(by Corollary 4.1.11)}\\
&= i'_{V*}\,g^*a\,([g]([Y]))\\
&= i'_{V*}\,g^*a\,([V])\\
&= \tilde{g}^*a\,(i_{V*}[V]) \qquad \in \mathrm{CH}_*(X \times_Y Y')_Q.
\end{aligned}$$

Thus $F_{X/Y}E_Y a = a$ and the class a is actually in $B(f)_Q$, i.e. $A(f)_Q = B(f)_Q$. Further we see that $F_{X/Y} \circ E_Y = \mathrm{id}_{B(f)_Q}$ and Y is B_Q-orthocyclic. □

The proof above has the following corollary:

Corollary 4.1.17. *Let Y be an S-scheme. If an S-scheme Y is B_Q-orienting, then Y is B_Q-orthocyclic. Furthermore, given a morphism $f : X \to Y$ of S-schemes, the inverse morphism for E_Y is given by*

$$\begin{aligned} F_{X/Y} : \mathrm{CH}_*(X)_Q &\to B(f)_Q, \\ \sum n_V[V] &\mapsto \sum n_V j_{V*}[g_V], \end{aligned} \tag{4.7}$$

where $j_V : V \to X$ denotes the induced closed immersions $j_V : V \to X$ and $[g_V] \in B(g_V)_Q$ is the B_Q-orientation class for $g_V := f \circ j_V : V \to Y$, which exists and is unique since Y is B_Q-orienting.

Proof. The first part follows directly from Proposition 4.1.16 and the definition of B_Q-orienting schemes. The proof, that $F_{X/Y} : \mathrm{CH}_*(X)_Q \to B(f)_Q$ can be constructed explicitly as stated, follows from the calculations in the proof of Proposition 4.1.16. □

One can compare B_Q-orienting and C_Q-orienting S-schemes in the following way. Note that this proposition is not explicitly in [KT87].

Proposition 4.1.18. *If an S-scheme Y is B_Q-orienting, then it is C_Q-orthocyclic, and hence C_Q-orienting by Lemma 4.1.15.*

Proof. Given an S-scheme Y which is B_Q-orienting, by Corollary 4.1.17 we know that Y is also B_Q-orthocyclic, i.e. for every morphism $f : X \to Y$ of S-schemes we have $B(f)_Q = A(f)_Q$ and evaluation at $[Y]$ is an isomorphism

$$E_Y : B(f)_Q \xrightarrow{\sim} \mathrm{CH}_*(X)_Q.$$

By Proposition 4.1.10 we also know that $B(f)_Q \subseteq C(f)_Q$ and we can always evaluate classes in $C(f)_Q$ at the cycle class $[Y] \in \mathrm{CH}_*(X)_Q$. This means we can form a diagram

(4.8)
$$\begin{tikzcd} A(f)_Q = B(f)_Q \arrow[r] \arrow[rr, bend right, "\sim", "E_Y"'] & C(f)_Q \arrow[r, "E_Y"] & \mathrm{CH}_*(X)_Q, \end{tikzcd}$$

and we have to show, that the second morphism $C(f)_Q \longrightarrow \mathrm{CH}_*(X)_Q$ is injective, i.e. given $a \in C(f)_Q$ with $a([Y]) = 0 \in \mathrm{CH}_*(X)_Q$, we have to show that $a = 0 \in C(f)_Q$. This means we have to show that $a_{Y'}^{(\dim_S V)}[V] = 0$ for all morphism of S-schemes $\tilde{g} : Y' \to Y$ and all closed integral subschemes $i_V : V \to Y'$ of Y'. For $g := \tilde{g} \circ i_V : V \to Y$ let $[g] \in B(g)_Q$ denote its unique B_Q-orientation class and let $i'_V : X \times_Y V \to X \times_Y Y$

denote the base change of i_V. Then we have

$$\begin{aligned} a_{Y'}^{(\dim_S V)}(V) &= (\tilde{g}^* a)\,(i_{V*}[V]) \\ &= i'_{V*}\,(\tilde{g}^* a)\,([g]([Y])) \\ &= i'_{V*}\,(\tilde{g}^* a.[g])\,([Y]) \\ &= i'_{V*}\,(f^*[g].a)\,([Y]) && \text{(by Corollary 4.1.11)} \\ &= i'_{V*}\,f^*[g]\,(a([Y])) \\ &= i'_{V*}\,f^*[g]\,(0) \\ &= 0 \qquad \in \mathrm{CH}_*(X \times_Y Y')_Q. \end{aligned}$$

This shows that $C(f)_Q \longrightarrow \mathrm{CH}_*(X)_Q$ is injective and hence all morphisms in the diagram (4.8) are isomorphisms, i.e. Y is C_Q-orthocyclic. ☐

We will now talk about Alexander duality and B_Q-orienting S-schemes. We need these results in Chapter 5.

Definition 4.1.19. Given a B_Q-oriented morphism $f : X \to Y$ of S-schemes. We say that $f : X \to Y$ satisfies *Alexander duality* if, for any morphism $g : W \to X$ of S-schemes, composition with $[f] \in B(f)_Q$

$$\_.[f] : A(g)_Q \to A(f \circ g)_Q$$

is an isomorphism.

An important class of S-morphisms that satisfy Alexander duality was found by Kleiman and Thorup:

Proposition 4.1.20 (Kleiman-Thorup)**.** *Let $f : X \to Y$ be a smooth morphism of S-schemes, then f satisfies Alexander duality and the isomorphism*

$$\_.[f] : A(g)_Q \to A(f \circ g)_Q$$

restricts to an isomorphism

$$\_.[f] : B(g)_Q \to B(f \circ g)_Q.$$

Proof. This follows from the straightforward adaptation of [KT87, Proposition 3.2], where appearances of $b \in A(f \circ g)_{\mathbb{Z}}$ resp. $b \in B(f \circ g)_{\mathbb{Z}}$ (which are denoted by $A(fg)$ resp. $B(fg)$ in loc. cit.) should be replaced by $b \in A(f \circ g)_Q$ resp. $b \in B(f \circ g)_Q$. ☐

Remark 4.1.21. The proof of Proposition 4.1.20 in [KT87, Proposition 3.2] uses a kind of slant product with the natural B_Q-orientation class $[\delta]$ as in Proposition 4.1.14 (ii) of the diagonal $\delta : X \to X \times_Y X$, which is in fact a regular imbedding by Remark A.3.25. This is an analogy to topology, hence the name Alexander duality.

Based on Proposition 4.1.20, we can prove the following

Proposition 4.1.22. *Let $f : X \to Y$ be a smooth morphism of S-schemes and let Y be B_Q-orienting. Then X is B_Q-orienting.*

Proof. Since Y is B_Q-orienting, we know by Corollary 4.1.17 that Y is B_Q-orthocyclic, i.e. for every morphism $g : W \to X$ of S-schemes we have $B(f \circ g)_Q = A(f \circ g)_Q$ and

$$E_Y : A(f \circ g)_Q \xrightarrow{\sim} \mathrm{CH}_*(W)_Q$$

is an isomorphism. By Proposition 4.1.20 the homomorphism

$$\_.[f] : A(g)_Q \to A(f \circ g)_Q$$

is an isomorphism and hence $E_X = E_Y \circ (\_.[f]) : A(g)_Q \xrightarrow{\sim} \mathrm{CH}_*(W)_Q$ is an isomorphism as well. Proposition 4.1.20 further implies, that $B(g)_Q$ is isomorphic to $B(f \circ g)_Q = A(f \circ g)_Q$, which in turn is isomorphic to $A(g)_Q$, so $B(g)_Q = A(g)_Q$ and X is indeed B_Q-orthocyclic. This implies that X is B_Q-orienting by Lemma 4.1.15. ☐

Remark 4.1.23. In [KT87, Proposition 3.4], Kleiman and Thorup propose a stronger result than what we prove in Proposition 4.1.22.

For the comparison to other approaches to intersection theory on regular schemes, we need to introduce external products:

Definition 4.1.24. Let Y be a S-scheme. We say there exist B_Q-*external products* on Y, if for all separated schemes X and Z of finite type over Y, there are maps

$$\times : \mathrm{CH}_k(X)_Q \otimes \mathrm{CH}_k(Z)_Q \to \mathrm{CH}_{k+k'}(X \times_Y Z)_Q,$$

such that the following hold:

(i) As Z varies, $[X] \times \_ : \mathrm{CH}_k(Z)_Q \to \mathrm{CH}_{k+\dim_S X}(X \times_Y Z)_Q$ is an B_Q-orientation class of $X \to Y$.

(ii) For all proper morphisms $g : W \to X$ with W separated and of finite type over X, we get a commutative square

$$\begin{tikzcd}
\mathrm{CH}_k(W)_Q \otimes \mathrm{CH}_{k'}(Z)_Q \arrow[r, "\times"] \arrow[d, "g_* \otimes \mathrm{id}_{\mathrm{CH}_{k'}(Z)}"'] & \mathrm{CH}_{k+k'}(W \times_Y Z)_Q \arrow[d, "(g \times \mathrm{id}_Z)_*"'] \\
\mathrm{CH}_k(X)_Q \otimes \mathrm{CH}_{k'}(Z)_Q \arrow[r, "\times"] & \mathrm{CH}_{k+k'}(X \times_Y Z)_Q.
\end{tikzcd}$$

If B_Q-external products exist on Y, then they are already unique. Kleiman and Thorup even showed:

Proposition 4.1.25 (Kleiman, Thorup)**.** *Unique B_Q-external products exist on Y if and only if Y is B_Q-orthocyclic. Furthermore, the external products are the same as the canonical morphisms induced by the the cartesian product of p-systems as in Lemma/Definition 4.1.3 (iv) via the isomorphism $E_Y : A(f)_Q \to \mathrm{CH}_*(X)_Q$ as in Definition 4.1.12.*

Proof. For $Q = \mathbb{Z}$ this can be found in [KT87, Proposition 3.6, discussion prior to Proposition 3.6]. The adaptions for Q-coefficients are straightforward. ☐

4.2 Alterations and B_Q-Orientations

This section gives the first central theorem of this thesis.

Theorem 4.2.1. *Let Y be a regular S-scheme of Krull dimension* $\dim Y = n$ *and assume $A(S, Q, \leq n)$ holds. Then Y is B_Q-orienting.*

We give a detailed proof, which is based on the idea by Kleiman and Thorup given in [KT87, remarks after Proposition 3.3]. For further remarks on their work also see Remark 4.2.4.

Proof. By Proposition 4.1.16 it suffices to show, that any projective morphism $g : V \to Y$ with V integral is B_Q-oriented, i.e., there is a class $[g]$ in $B_Q(g)$ such that $[g]([Y]) = [V]$. We factor such a projective morphism $g : V \to Y$ with V integral through the connected component Y_1 of Y containing the image of V. As Y is regular, this connected component Y_1 is integral (cf. Remark 2.2.4) and the inclusion $Y_1 \to Y$ is flat of relative S-dimension 0 and therefore $B_{\mathbb{Z}}$-oriented by Proposition 4.1.14. Hence without loss of generality we may assume that Y is integral.
Now the first step is to factor $g : V \to Y$ through its scheme theoretic image $g(V)$ (cf. [GW10, subsection (10.8)]), so we get a diagram

$$\begin{tikzcd} & V \arrow[dl, "g"'] \arrow[d, "\tilde{g}"] \\ Y & g(V) \arrow[l] \end{tikzcd}$$

with $\tilde{g} : V \to g(V)$ dominant and $g(V) \to Y$ a closed imbedding. As closed imbeddings are separated and of finite type and $g : V \to Y$ is projective, the morphism $\tilde{g} : V \to g(V)$ is projective as well. By [EGA, IV Theorem 6.9.1], we know that there is a nonempty open subset U of $g(V)$, such that

$$\tilde{g}|_{\tilde{g}^{-1}(U)} : \tilde{g}^{-1}(U) \to U$$

is flat. With this, we use Raynaud's and Gruson's "flattening technique" (cf. [RG71, 5.2.2]), to extend the diagram

$$\begin{tikzcd} & V \arrow[dl, "g"'] \arrow[d] & V \times_{g(V)} g(V)' \arrow[l] \arrow[d] \\ Y & g(V) \arrow[l] & g(V)' \arrow[l] \end{tikzcd}$$

where $V \times_{g(V)} g(V)' \to g(V)'$ is flat and $g(V)' \to g(V)$ is an U-admissible blow-up (i.e. $g(V)'$ is given as a blow-up $\mathrm{Bl}_Z(g(V))$ of $g(V)$ along a closed immersion $Z \to g(V)$ with $Z \cap U = \emptyset$). In particular $g(V)'$ is integral, and the morphism $g(V)' \to g(V)$ is projective and birational by Proposition A.3.18. Since $g(V)'$ is birational to a closed integral subscheme of Y, we get for its Krull dimension $\dim g(V)' \leq \dim Y = n$. Since $g(V)'$ is projective over Y, it is in particular separated and of finite type and we assumed that $\mathrm{A}(S, Q, \leq n)$ holds, hence we can find an alteration $\phi : g(V)'_1 \to g(V)'$, which is generically finite of degree $d \in T$, and such that $g(V)'_1$ is a regular integral scheme and $\phi : g(V)'_1 \to g(V)'$ is projective. Now we are left with a diagram

$$\begin{tikzcd}
 & V \arrow[dl, "g"'] \arrow[d] & V \times_{g(V)} g(V)' \arrow[l] \arrow[d] & V \times_{g(V)} g(V)'_1 \arrow[l] \arrow[d, "g'"] \arrow[ll, bend right, "h'"'] \\
Y & g(V) \arrow[l] & g(V)' \arrow[l] & g(V)'_1 \arrow[l] \arrow[lll, bend left, "h"]
\end{tikzcd}$$

with morphisms as labeled. We get that Y and $g(V)'_1$ are regular, h is projective, g' is flat and h' is proper and generically finite with degree $d \in T$ (as it is the base change of a proper, generically finite morphism with a dominant morphism). By Proposition A.3.22 (ii) the morphism $h : g(V)'_1 \to Y$ is an l.c.i. morphism, with corresponding $B_\mathbb{Z}$-orientation class $[h] \in B(h)_\mathbb{Z}$ (cf. Proposition 4.1.14 (iii) and note that Y was integral without loss of generality). The flat morphism g' the relative S-dimension $\dim_S V - \dim_S g(V)$ (cf. Remark A.1.3(v)) and hence induces an B_Q-orientation class $[g'] \in B(g')_\mathbb{Z}$ as well (cf. Proposition 4.1.14 (i)). Now consider the class

$$[g] := \frac{1}{d} {h'}_*([g'].[h]) \ \in B_Q(g).$$

Note that this class is actually in $B_Q(g)$ by Definition 4.1.6. Then we have

$$\begin{aligned}
[g]([Y]) &= (\frac{1}{d} {h'}_*([g'].[h]))\,([Y]) \\
&= \frac{1}{d} {h'}_*[g']([h]\,([Y])) \\
&= \frac{1}{d} {h'}_*[g']([g(V)'_1]) \\
&= \frac{1}{d} {h'}_*[V \times_{g(V)} g(V)'_1] \\
&= \frac{1}{d} \cdot d \cdot [V] \\
&= [V] \quad \in \mathrm{CH}_k(V)_Q,
\end{aligned}$$

so $[g]$ gives the desired B_Q-orientation class. □

This Theorem has two immediate consequences:

Corollary 4.2.2. *Let Y be a regular S-scheme of Krull dimension n and assume $A(S, Q, \leq n)$ holds. Then any S-scheme X with $X \to Y$ smooth is B_Q-orienting.*

Proof. This follows from Theorem 4.2.1 by Corollary 4.1.22. □

We now look at the special case where our base scheme $S = \operatorname{Spec} \mathbb{Z}$.

Corollary 4.2.3. *Let Y be a regular, separated scheme of finite type over $S = \operatorname{Spec} \mathbb{Z}$. Then Y is $B_{\mathbb{Q}}$-orienting.*

Proof. By Corollary 3.2.8 (iv) with $R = \mathbb{Z}$, we know that $\mathrm{A}(\operatorname{Spec} \mathbb{Z}, \mathbb{Q}, \leq d)$ holds for all $d \in \mathbb{N}$, hence the claim follows from Theorem 4.2.1. □

Remark 4.2.4. Theorem 4.2.1 and Corollary 4.2.3 are direct generalization of results found in [KT87, Section 3]. Kleiman and Thorup showed the following two results, which we will formulate in our notation here:

(i) Let Y be a regular S-scheme and assume $\mathrm{A}(S, \mathbb{Z}, \leq \dim_S Y)$ holds. Then Y is $B_{\mathbb{Z}}$-orienting ([KT87, remarks after Proposition 3.3]).

(ii) Let Y be a regular, separated scheme of finite type over $S = \operatorname{Spec} \mathbb{Z}$. Then Y is $C_{\mathbb{Q}}$-orienting ([KT87, Proposition 3.9]).

4.3 Intersection Theory with Supports on B_Q-orienting Schemes

In this section we define the intersection product with supports for B_Q-orienting S-schemes.

Definition 4.3.1. Let Q be a ring and $Y \subseteq X$ a closed subset. We can view Y as a closed subscheme of X by putting the reduced induced structure on it (cf. [Har77, II.3.2.6]). We define

$$\mathrm{CH}^k_Y(X) := \mathrm{CH}_{\dim_S X - k}(Y),$$
$$\mathrm{CH}^k_Y(X)_Q := \mathrm{CH}_{\dim_S X - k}(Y)_Q = \mathrm{CH}_{\dim_S X - k}(Y) \otimes_{\mathbb{Z}} Q$$

and

$$\mathrm{CH}^*(X)_Q := \bigoplus_{Y,k} \mathrm{CH}^k_Y(X)_Q,$$

where the direct sum is taken over all closed subschemes Y and all $k \in \mathbb{Z}$. For $Y = X$ we just write $\mathrm{CH}^k(X)_Q$ instead of $\mathrm{CH}^k_X(X)_Q$. Call $\mathrm{CH}^k_Y(X)_Q$ the *Chow group of algebraic cycles of codimension k on X supported in the closed subset Y* with Q-coefficients.

Remark 4.3.2. Let X be an S-scheme. The grading of the Chow group introduced by S-dimension as in Definition 4.3.1 is in fact compatible with the grading of algebraic cycles by codimension in the usual way. This follows directly from Proposition 2.1.3 (ii), as we have $\dim_S V = \dim_S X - \operatorname{codim}(V, X)$ for any closed integral subscheme V of X.

The next Theorem relates the approach to intersection theory by Fulton, Kleiman and Thorup to the approach by Gillet and Soulé.

Theorem 4.3.3. *(Intersection Theory with Supports) Let X be an B_Q-orienting S-scheme. Then for closed subsets $Y, Z \subseteq X$ in X there exists a morphism*

$$. : \mathrm{CH}^k_Y(X)_Q \otimes \mathrm{CH}^{k'}_Z(X)_Q \to \mathrm{CH}^{k+k'}_{Y \cap Z}(X)_Q, \tag{4.9}$$

called intersection product with supports, *such that* *(i) The Chow group* $\mathrm{CH}^*(X)_Q$ *becomes a commutative graded* Q*-algebra with* $[X] \in \mathrm{CH}^0_X(X)_Q$ *as unit element.*

(ii) The intersection product (4.9) is compatible with natural inclusions of supports $\mathrm{CH}^k_{Y'}(X)_Q \to \mathrm{CH}^k_Y(X)_Q$ *and* $\mathrm{CH}^k_{Z'}(X)_Q \to \mathrm{CH}^k_Z(X)_Q$ *for closed subsets* $Y' \subseteq Y$ *and* $Z' \subseteq Z$*.*

In this case we refer to $\mathrm{CH}^*(X)_Q$ *as the* Chow ring with Q-coefficients.

Proof. Since X is B_Q-orienting, it is also B_Q-orthocyclic by Corollary 4.1.17. This means, given the closed immersion $i_Z : Z \to X$ of S-schemes (which is given by putting the reduced induced closed subscheme structure) we have an isomorphism of Q-modules as in Definition 4.1.12 (iii):

$$E_X : B_Q(i_Z) = A_Q(i_Z) \xrightarrow{\sim} \mathrm{CH}_*(Z)_Q \cong \mathrm{CH}^*_Z(X)_Q, \ (a_p)_{p\in\mathbb{Z}} \mapsto \sum_{p\in\mathbb{Z}} (a_p)_X^{(\dim_S X)}([X]).$$

First we show that this is an isomorphism of graded Q-modules: Given a p-system $a \in B^p(i_Z)_Q$, we have

$$E_X(a) = a_X^{\dim_S X}([X]) \in \mathrm{CH}_{\dim_S X - p}(Z)_Q = \mathrm{CH}^p_Z(X)_Q$$

thus

$$E_X(B^p(i_Z)_Q) \subseteq \mathrm{CH}^p_Z(X)_Q$$

and E_X is in fact an isomorphism of graded Q-modules. Note that this implies that the inverse homomorphism of Q-modules as in Corollary 4.1.17

$$F_{Z/X} : \mathrm{CH}^*_Z(X)_Q \cong \mathrm{CH}_*(Z)_Q \to B_Q(i_Z)$$

satisfies

$$F_{Z/X}(\mathrm{CH}^p_Z(X)) \subseteq B^p(i_Z)_Q.$$

Now, for closed subsets $Y, Z \subseteq X$ we put the reduced induced closed subscheme structure on both of them to get the closed immersions $i_Y : Y \to X$ and $i_Z : Z \to X$. For the fiber square

$$\begin{tikzcd} Y \times_X Z \arrow[r, "i'_Y"] \arrow[d, "i'_Z"'] & Z \arrow[d, "i_Z"'] \\ Y \arrow[r, "i_Y"] & X, \end{tikzcd}$$

we define so called *cup-products*

$$\cup : B^p(i_Y)_Q \otimes B^q(i_Z)_Q \to B^{p+q}(i_Y \circ i'_Z)_Q, \quad a \otimes b \mapsto i_Z^*(a).b \ . \tag{4.10}$$

The cup-product is commutative by Corollary 4.1.11. By definition as a product of bivariant classes, the cup product is obviously associative and commutes with restriction of bivariant classes, i.e. for any morphism $f : X' \to X$ we have

$$f^*(a \cup b) = (f^*a) \cup (f^*b). \tag{4.11}$$

This follows as the product of bivariant classes obviously commutes with the restriction of bivariant classes. The cup-product gives rise to the desired intersection product

$$. : \mathrm{CH}^k_Y(X)_Q \otimes \mathrm{CH}^{k'}_Z(X)_Q \to \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q,\ \alpha \otimes \beta \mapsto E_X(F_{Y/X}(\alpha) \cup F_{Z/X}(\beta)).$$

The commutativity and associativity of the intersection product then follow immediately from the commutativity and associativity of the cup-product. Then for $[X] \in \mathrm{CH}^0_X(X)_Q$ and $\alpha \in \mathrm{CH}^k_Y(X)_Q$ we have

$$\begin{aligned}[X].\alpha &= E_X(F_{X/X}([X]) \cup F_{Y/X}(\alpha)) \\ &= E_X(\mathrm{id}_{\mathrm{CH}^0_X(X)_Q} \cup F_{Y/X}(\alpha)) \\ &= E_X(F_{Y/X}(\alpha)) \\ &= \alpha \quad \in \mathrm{CH}^k_Y(X)_Q,\end{aligned}$$

so $[X] \in \mathrm{CH}^0_X(X)_Q$ is the unit element. This shows (i).
Part (ii) follows directly from the definition, as for closed subsets $Z' \subseteq Z \subseteq X$ with induced closed immersions $i_{Z'} : Z' \to Z$ and $i_Z : Z \to X$ we know that

$$\begin{tikzcd}\mathrm{CH}^p_{Z'}(X)_Q \arrow[r, "F_{Z'/X}"] \arrow[d, "i_{Z'*}"] & B(i_Z \circ i_{Z'})_Q \arrow[d, "i_{Z'*}"'] \\ \mathrm{CH}^p_Z(X)_Q \arrow[r, "F_{Z/X}"] & B(i_Z)_Q\end{tikzcd}$$

commutes. □

Remark 4.3.4. We can formulate two more explicit statements of the previous Theorem in the special case that the Noetherian excellent regular separated base scheme S is in addition one-dimensional.

(i) If S is in addition one-dimensional, then by Theorem 4.2.1 and Corollary 3.2.8 (iv) any regular S-schemes X is $B_{\mathbb{Q}}$-orienting. Hence in this case Theorem 4.3.3 gives an intersection product

$$. : \mathrm{CH}^k_Y(X)_{\mathbb{Q}} \otimes \mathrm{CH}^{k'}_Z(X)_{\mathbb{Q}} \to \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_{\mathbb{Q}}.$$

(ii) If X is a smooth S-scheme and S is in addition one-dimensional, then X is $B_{\mathbb{Z}}$-orienting by Theorem 4.2.1, as S is regular of Krull dimension one and $\mathrm{A}(S, \mathbb{Z}, \leq 2)$ holds by Corollary 3.2.8 (iii). Hence by Theorem 4.3.3 we have an intersection product

$$. : \mathrm{CH}^k_Y(X) \otimes \mathrm{CH}^{k'}_Z(X) \to \mathrm{CH}^{k+k'}_{Y\cap Z}(X).$$

Lemma 4.3.5. *(Explicit Formulas of the Intersection Product) Let X be a B_Q-orienting S-scheme. Let $Y, Z \subseteq X$ denote closed subsets with induced closed immersions $i_Y : Y \to X$ and $i_Z : Z \to X$.*

(i) *For* $\alpha \in \mathrm{CH}^k_Y(X)_Q$ *and* $\beta \in \mathrm{CH}^{k'}_Z(X)_Q$*, we have*

$$\alpha.\beta = \left(i_Z^* F_{Y/X}(\alpha)\right)(\beta) \quad \in \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q,$$

for $F_{Y/X} : \mathrm{CH}^*_Y(X)_Q \to B(i_Y)_Q$ *as in Corollary 4.1.17.*

(ii) *Assume we are given closed integral subschemes* $i_V : V \to Y$ *with* $[V] \in \mathrm{CH}^k_Y(X)_Q$ *and* $i_W : W \to Z$ *with* $[W] \in \mathrm{CH}^{k'}_Z(X)_Q$ *respectively. For the closed immersions* $f := i_Y \circ i_V : V \to X$ *and* $i := i_V \times i_W : V \times_X W \to Y \times_X Z$*, we get*

$$[V].[W] = i_*[f]_W^{(\dim_S X - k')}([W]) \quad \in \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q,$$

where $[f] \in B^k(f)_Q$ *denotes the unique* B_Q*-orientation class of* $f : V \to X$*.*

(iii) *In the special case that* X *is an integral regular* S*-scheme of Krull dimension* $\dim X = n$ *and if A(S, Q,* $\leq n$*) holds (hence* X *is* B_Q*-orienting by Theorem 4.2.1), we get the following more explicit formula:*
Given a closed immersion $f : V \to X$ *of codimension* k'*, we use Assumption A(S, Q,* $\leq n$*) to get a diagram*

$$\begin{tikzcd}
V_1 \arrow[r, "\phi"'] \arrow[rr, bend left, "f_1"] & V \arrow[r, "f"'] & X \arrow[d] \\
 & & S
\end{tikzcd}$$

with an alteration $\phi : V_1 \to V$*, which is generically finite of degree* $d \in T$ *(with* $Q = T^{-1}\mathbb{Z}$*), and* V_1 *is a regular integral scheme and projective over* X*. By Proposition A.3.22 (ii) the morphism* $f_1 : V_1 \to X$ *is an l.c.i. morphism, so it determines an* B_Q*-orientation class* $[f_1] \in B^k(f_1)_{\mathbb{Z}}$ *given by the refined Gysin homomorphisms associated to* $f_1 : V_1 \to X$ *(cf. Proposition 4.1.14). Then we have*

$$[f] = \frac{1}{d}\phi_*[f_1] \in B^k(f)_Q$$

and for $\alpha \in \mathrm{CH}^k_Y(X)_Q$ *we have*

$$[V].\alpha = \frac{1}{d}\phi_* f_1^!(\alpha) \in \mathrm{CH}^{k+k'}_{V\cap Y}(X)_Q.$$

Proof. For (i), we calculate

$$\begin{aligned}
\alpha.\beta &= E_X(F_{Y/X}(\alpha) \cup F_{Z/X}(\beta)) \\
&= \left(i_Z^* F_{Y/X}(\alpha)\right).\left(F_{Z/X}(\beta)\right)([X]) && \text{(cf. Def. cup-product (4.10))} \\
&= \left(i_Z^* F_{Y/X}(\alpha)\right)\left(F_{Z/X}(\beta)([X])\right) \\
&= \left(i_Z^* F_{Y/X}(\alpha)\right)\left(E_X \circ F_{Z/X}(\beta)\right) \\
&= \left(i_Z^* F_{Y/X}(\alpha)\right)(\beta) \quad \in \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q.
\end{aligned}$$

For (ii), form the diagram

$$\begin{tikzcd}
V \times_X W \arrow[r, "i''_V"'] \arrow[d, "i''_W"'] \arrow[rd, "i"'] & Y \times_X W \arrow[r, "i''_Y"'] \arrow[d, "i'_W"'] & W \arrow[d, "i_W"'] \arrow[dd, bend left, "g"] \\
V \times_X Z \arrow[r, "i'_V"'] \arrow[d, "i''_Z"'] & Y \times_X Z \arrow[r, "i'_Y"'] \arrow[d, "i'_Z"'] & Z \arrow[d, "i_Z"'] \\
V \arrow[r, "i_V"'] \arrow[rr, bend right, "f"'] & Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

and set $g := i_Z \circ i_W$. Then we get:

$$\begin{aligned}
[V].[W] &= E_X(F_{Y/X}([V]) \cup F_{Z/X}([W])) \\
&= E_X((i_{V*}[f]) \cup (i_{W*}[g])) && \text{(cf. Corollary 4.1.17)} \\
&= (i_Z{}^*(i_{V*}[f]).(i_{W*}[g]))\,([X]) && \text{(cf. Def. cup-product (4.10))} \\
&= i_Z{}^*(i_{V*}[f])\,(i_{W*}[g]\,([X])) \\
&= i'_{V*}[f]_Z^{(\dim_S X-k')}(i_{W*}[g]_X^{(\dim_S X)}([X])) \\
&= i'_{V*}i''_{W*}[f]_W^{(\dim_S X-k')}([g]_X^{(\dim_S X)}([X])) && \text{(by (4.3))} \\
&= i_*[f]_W^{(\dim_S X-k')}([W]) \ \in \mathrm{CH}^{k+k'}_{Y\cap Z}(X)_Q,
\end{aligned}$$

where $[f] \in B^k(f)_Q$ and $[g] \in B^{k'}(g)_Q$ denote the unique B_Q-orientation classes of f and g respectively. This shows (ii).
Lastly for (iii), we have

$$\frac{1}{d}\phi_* f_1^!\,([X]) = \frac{1}{d}\phi_*[V_1] = \frac{1}{d}\cdot d \cdot [V] = [V],$$

hence $\frac{1}{d}\phi_* f_1^* \in B^k(f)_Q$ is an orientation and since X is orienting, it is unique, i.e. does in fact not depend on the choice of the alteration $\phi : V_1 \to V$. ☐

Remark 4.3.6. In the situation of Theorem 4.3.3, we may forget about the support by setting $Y = Z = X$, to get the morphism

$$.\ : \mathrm{CH}^k(X)_Q \otimes \mathrm{CH}^{k'}(X)_Q \to \mathrm{CH}^{k+k'}(X)_Q,$$

which we also call the *intersection product.*

Proposition 4.3.7. *Let $f : X \to Y$ be a morphism of B_Q-orienting S-schemes. Then for closed subsets $Z, Z' \subseteq Y$ and $W \subseteq X$ we get:*

(i) There is a Q-linear pull-back homomorphism of Q-modules

$$f^* : \mathrm{CH}^k_Z(Y)_Q \to \mathrm{CH}^k_{f^{-1}(Z)}(X)_Q,$$

and given $\alpha \in \mathrm{CH}^k_Z(Y)_Q$ and $\beta \in \mathrm{CH}^{k'}_{Z'}(Y)_Q$, one has

$$f^*(\alpha.\beta) = f^*(\alpha).f^*(\beta) \quad \in \mathrm{CH}^{k+k'-\dim_Y X}_{f^{-1}(Z\cap Z')}(X)_Q,$$

(ii) If f is proper and $n = \dim_S X - \dim_S Y$, there is a push-forward homomorphism of Q-modules

$$f_* : \mathrm{CH}^k_W(X)_Q \to \mathrm{CH}^{k-n}_{f(W)}(Y)_Q.$$

(iii) (Projection formula) Assume f is proper and $n = \dim_S X - \dim_S Y$, then for $\alpha \in \mathrm{CH}^k_Z(Y)_Q$ and $\beta \in \mathrm{CH}^{k'}_W(X)_Q$ we have

$$f_*(f^*(\alpha).\beta) = \alpha.f_*(\beta) \quad \in \mathrm{CH}^{k+k'-n}_{Z\times_Y f(W)}(Y)_Q.$$

Proof. For (i) we use that Y is B_Q-orienting, so there exists a unique class $[f] \in B^{\dim_Y X}(f)_Q$ with $[f]([Y]) = [X]$. This induces the desired morphism

$$f^* : \mathrm{CH}^k_Z(Y)_Q \to \mathrm{CH}^k_{f^{-1}(Z)}(X)_Q$$

as the composition

$$\mathrm{CH}^k_Z(Y)_Q \xrightarrow{F_{Z/Y}} B^k(i_Z)_Q \xrightarrow{f^*} B^k(i_{f^{-1}Z})_Q \xrightarrow{E_X} \mathrm{CH}^k_{f^{-1}(Z)}(X)_Q, \tag{4.12}$$

where $f^* : B^k(i_Z)_Q \to B^k(i_{f^{-1}Z})_Q$ denotes the restriction of bivariant classes as in Lemma/Definition 4.1.3 (i) for the fiber square

$$\begin{tikzcd}
f^{-1}(Z) \arrow[r, "i_{f^{-1}(Z)}"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\
Z \arrow[r, "i_Z"] & Y.
\end{tikzcd}$$

This composition is already known to be a morphism of Q-modules. Given $\alpha \in \mathrm{CH}^k_Z(Y)_Q$ and $\beta \in \mathrm{CH}^{k'}_{Z'}(Y)_Q$, one has

$$\begin{aligned}
f^*(\alpha.\beta) &= f^*\left(E_Y(F_{Z/Y}(\alpha) \cup F_{Z'/Y}(\beta))\right) && \text{(by (4.9))}\\
&= (E_X \circ f^* \circ F_{Z\times_Y Z'/Y})\left(E_Y(F_{Z/Y}(\alpha) \cup F_{Z'/Y}(\beta))\right) && \text{(by (4.12))}\\
&= (E_X \circ f^* \circ \underbrace{F_{Z\times_Y Z'/Y} \circ E_Y}_{=\mathrm{id}_{B(i_{Z\times_Y Z'})_Q}})\left(F_{Z/Y}(\alpha) \cup F_{Z'/Y}(\beta)\right) && \\
&= (E_X \circ f^*)\left(F_{Z/Y}(\alpha) \cup F_{Z'/Y}(\beta)\right) && \\
&= E_X((f^* \circ F_{Z/Y})(\alpha) \cup (f^* \circ F_{Z'/Y})(\beta)) && \text{(by (4.11))}\\
&= E_X((\underbrace{F_{f^{-1}(Z)/X} \circ E_X}_{=\mathrm{id}_{B(i_{f^{-1}(Z)})_Q}} \circ f^* \circ F_{Z/Y})(\alpha) \cup (\underbrace{F_{f^{-1}(Z')/X} \circ E_X}_{=\mathrm{id}_{B(i_{f^{-1}(Z')})_Q}} \circ f^* \circ F_{Z'/Y})(\beta)) && \\
&= E_X(F_{f^{-1}(Z)/X} f^*(\alpha) \cup F_{f^{-1}(Z')/X} f^*(\beta)) && \text{(by (4.12))}\\
&= f^*(\alpha).f^*(\beta) \qquad \in \mathrm{CH}^{k+k'-n}_{f^{-1}(Z\cap Z')}(X)_Q && \text{(by (4.9))}.
\end{aligned}$$

The push-forward in (ii) is induced from the one in Proposition A.1.1 by tensoring with Q:

$$f_* : \mathrm{CH}^k_W(X)_Q = \mathrm{CH}_{\dim_S X-k}(W)_Q \to \mathrm{CH}_{\dim_S X-k}(f(W)) = \mathrm{CH}^{k-n}_{f(W)}(Y)_Q.$$

For the projection formula assume we are given $\alpha \in \mathrm{CH}^k_Z(Y)_Q$ and $\beta \in \mathrm{CH}^{k'}_W(X)_Q$. Let $i_W : W \to X$, $i_Z : Z \to Y$ and $i_{f(W)} : f(W) \to Y$ denote the induced closed immersions by putting the reduced induced structure. Then we get

$$\begin{aligned}
f_*(f^*(\alpha).\beta) &= f_*\left(i_W^*(F_{f^{-1}(Z)/X}(f^*(\alpha)))\,(\beta)\right) && \text{(Lemma 4.3.5 (i))}\\
&= f_*\left(i_W^*(F_{f^{-1}(Z)/X}(\left(E_X \circ f^* \circ F_{Z/Y}(\alpha)\right)))(\beta)\right) && \text{(by (4.12))}\\
&= f_*\left(i_W^*\left((F_{f^{-1}(Z)/X} \circ E_X \circ f^* \circ F_{Z/Y})(\alpha)\right)(\beta)\right)\\
&= f_*\left(\left((f \circ i_W)^* F_{Z/Y}(\alpha)\right)(\beta)\right)\\
&= f_*\left(((f \circ i_W)^* a)\,(\beta)\right) && \text{for } a := F_{Z/Y}(\alpha)\\
&= f_*\left(a_W^{(k)}(\beta)\right)\\
&= a_{f(W)}^{(k)}(f_*\beta) && (*)\\
&= \left((i_{f(W)}^* F_{Z/Y})(\alpha)\right)(f_*(\beta))\\
&= \alpha.f_*(\beta) \quad \in \mathrm{CH}^{k+k'-n}_{Z\times_Y f(W)}(Y)_Q.
\end{aligned}$$

It remains to prove the equality

$$(*) \qquad f_*\left(a_W^{(k)}(\beta)\right) = a_{f(W)}^{(k)}(f_*\beta) \in \mathrm{CH}^{k+k'-n}_{Z\times_Y f(W)}(Y)_Q.$$

For this, we consider the fiber square diagram

$$\begin{tikzcd}
Z \times_Y W \arrow[r] \arrow[d] & W \arrow[d] \arrow[dd, bend left, "f \circ i_W"] \\
Z \times_Y f(W) \arrow[r] \arrow[d] & f(W) \arrow[d, "i_{f(W)}"'] \\
Z \arrow[r, "i_Z"'] & Y
\end{tikzcd}$$

with all morphisms being proper. The upper square of this diagram induces the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^{k+k'}_{Z\times_Y W}(X)_Q \arrow[d, "f_*"'] & \mathrm{CH}^{k'}_W(X)_Q \arrow[l, "a^k_W"'] \arrow[d, "f_*"] \\
\mathrm{CH}^{k+k'-n}_{Z\times_Y f(W)}(Y)_Q & \mathrm{CH}^{k'-n}_{f(W)}(Y)_Q, \arrow[l, "a^k_{f(W)}"]
\end{tikzcd}$$

which commutes as $a = F_{Z/Y}(\alpha) \in B^k(i_Z)_Q \subseteq A^k(i_Z)_Q$ is a bivariant class, i.e. it commutes with proper push-forward by definition. □

Remark 4.3.8. Note that Proposition 4.3.7 provides the same results for our definition of the intersection product with supports for regular separated schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ as Gillet and Soulé show for their's in [GS87]. However some results like the projection formula in loc. cit. rely on a version of the Riemann-Roch Theorem with supports, whose proof in [GS87] is only sketched. Furthermore, our approach enables us to easily adapt further results from Fulton's book [Ful84], e.g. the excess intersection formula (cf. [Ful84, Proposition 17.4.1]): Given a fiber square of S-schemes

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

with a regular embedding f of codimension d and a regular embedding f' of codimension d', then we get

$$g^*[f] = c_e(E).[f'] \ \in A^d(f'),$$

where $e = d - d'$ and E denotes the excess normal bundle defined by

$$E := g^* N_X Y / N_{X'} Y'$$

(cf. Theorem A.5.2 (iii)).

Chapter 5

Comparison to other Approaches to Intersection Theory

In this chapter we treat the natural question: How does the intersection theory developed in Chapter 4 relate to other existing theories? For this we again fix a Noetherian, excellent, regular and separated base scheme S. In the first section we show that our intersection product for regular S-schemes X (and if $\mathrm{A}(S,\, Q,\, \leq \dim_S X)$ holds) coincides with Fulton's intersection product for divisors (cf. Section A.2 or directly [Ful84, Chapter 2, Section 20.1]). In the second section we will consider the special case of smooth schemes over a one-dimensional scheme, and we will see that our theory coincides with Fulton's theory (cf. [Ful84, Chapter 8, Section 20.2]), and hence also with the intersection theory developed by Gillet and Soulé (cf. [GS87]) in this special case.

5.1 Intersection with Divisors

In this section let X be a regular S-scheme, i.e. a regular, separated scheme of finite type over a fixed Noetherian, excellent, regular and separated base scheme S.

Construction 5.1.1. Let X be a regular S-scheme of S-dimension $\dim_S X = n$. In order to give a pairing

$$\cdot_{\text{divisors}} \;: \mathrm{CH}^1_Y(X) \otimes \mathrm{CH}^k_Z(X) \to \mathrm{CH}^{k+1}_{Y\cap Z}(X),$$

by linearity it is enough to define

$$[V] \cdot_{\text{divisors}} [W] \in \mathrm{CH}_{n-k-1}(Y \times_X Z) = \mathrm{CH}^{k+1}_{Y\cap Z}(X)$$

for a closed imbeddings $i_V : V \to X$, $i_W : W \to X$ of integral subschemes with $[V] \in \mathrm{CH}^1_Y(X)$ and $[W] \in \mathrm{CH}^k_Z(X)$. As X is Noetherian and regular, the Chow group of algebraic cycles of codimension one and the group of Cartier divisor classes on X are isomorphic (cf. [GW10, Proposition 10.38 and Theorem 11.39])

$$\mathrm{CaCl}\,(X) \xrightarrow{\sim} \mathrm{CH}^1(X).$$

Let $D \in \mathrm{CaCl}\,(X)$ denote the corresponding Cartier divisor for the algebraic cycle $[V] \in \mathrm{CH}^1(X)$. This Cartier divisor is by definition effective (i.e. it is a closed subscheme $i : V \to X$, such that its ideal sheaf $\mathcal{J}_V \subseteq O_X$ is locally generated by one element) and its support $\mathrm{supp}(D)$ is the underlying closed subspace of the closed subscheme V (cf. [GW10, Remark 11.31]). Then Fulton (cf. Construction A.2.1 or directly [Ful84, Chapter 2, Section 20.1]) constructed a well-defined element

$$[V].{}_{\text{divisors}}[W] := D.{}_{\text{divisors}}[W] \in \mathrm{CH}_{n-k-1}(W \times_X D) = \mathrm{CH}^{k+1}_{\mathrm{supp}(D)\cap W}(X).$$

We can push-forward this class to $\mathrm{CH}^{k+1}_{Y\cap Z}(X)$ via the closed immersion $V \times_X W \to Y \times_X Z$. This construction implies the the pairing

$$\cdot_{\text{divisors}} :: \mathrm{CH}^1_Y(X)_Q \otimes \mathrm{CH}^k_Z(X)_Q \to \mathrm{CH}^{k+1}_{Y\cap Z}(X)_Q$$

after tensoring the Chow groups with Q.

Proposition 5.1.2. *Let X be a regular S-schemes X and assume that $A(S, Q, \leq \dim_S X)$ holds. Then the pairing*

$$\cdot_{\textit{divisors}} :: \mathrm{CH}^1_Y(X)_Q \otimes \mathrm{CH}^k_Z(X)_Q \to \mathrm{CH}^{k+1}_{Y\cap Z}(X)_Q$$

given in Construction 5.1.1 coincides with our intersection product in Theorem 4.3.3 (using Theorem 4.2.1, which says that X is B_Q-orienting).

Proof. Since X is regular, we may assume that X is integral (cf. Remark 2.2.4). Further it suffices to check our claim for closed immersions $f : V \to X$ (of codimension one) and $g : W \to X$ (of codimension k). Following the explicit formula of our intersection product as in Lemma 4.3.5 (iii), we take an alteration $\phi : V_1 \to V$, such that V_1 is a regular integral scheme and projective over X and hence $f_1 := f \circ \phi : V_1 \to X$ is an l.c.i. morphism with associated B_Q-orientation class $[f_1] \in B^k(f_1)_{\mathbb{Z}}$ given by the refined Gysin homomorphisms associated to $f_1 : V_1 \to X$ (cf. Proposition 4.1.14). Then we have

$$[V].[W] = f_*(\frac{1}{d}\phi_*[f_1])([W]),$$

where $\frac{1}{d}\phi_*[f_1] \in B^k(f)_Q$ is a B_Q-orientation class by Lemma 4.3.5.
On the other hand note, that $f : V \to X$ is a regular imbedding of codimension one, because V can be viewed as an effective Cartier divisor. Hence by Proposition 4.1.14 (ii), f is $B_{\mathbb{Z}}$-oriented by the system $[f]$ of the refined Gysin homomorphisms. But since regular S-schemes are B_Q-orienting by Theorem 4.2.1, we get that $\frac{1}{d}\phi_*[f_1] = [f] \in B^k(f)_Q$. Lastly we use Theorem A.5.2 (ix), which says that the construction of the Gysin homomorphism for regular imbeddings coincides with the intersection product with effective Cartier-Divisors in Construction A.2.1. In conclusion we have

$$[V].[W] = f_*(\frac{1}{d}\phi_*[f_1])([W]) = [f]([W]) = f^!([W]) = D.{}_{\text{divisors}}[W] \in \mathrm{CH}^{k+1}_{Y\cap Z}(X)_Q.$$

□

5.2 Smooth Schemes over a Dedekind scheme

In this section we fix a Noetherian, excellent, regular and separated base scheme S of **Krull dimension one**. For smooth S-schemes we want to show, that our intersection product with supports as given in Remark 4.3.4 (ii) coincides with Fulton's intersection product as in Proposition A.6.3.

As a first step, we show the following

Proposition 5.2.1. *Let S be a Noetherian, excellent, regular and separated base scheme S of (Krull) dimension one. Then Fulton's exterior products (cf. Proposition A.6.1 or directly [Ful84, Proposition 20.2]) give an explicit description of the $B_{\mathbb{Z}}$-exterior products on S (cf. Definition 4.1.24). This means, that the diagram*

$$\begin{tikzcd}
B^p(p_Y \circ i_X)_{\mathbb{Z}} \otimes B^q(p_Y \circ i_Z)_{\mathbb{Z}} \arrow[r, "\times", "\textit{Def 4.1.24}"'] \arrow[d, "\cong"', "E_S \otimes E_S"] & B^{p+q}((p_Y \circ i_X) \times (p_Y \circ i_Z))_{\mathbb{Z}} \arrow[d, "\cong"', "E_S"] \\
\mathrm{CH}_{n-p}(X) \otimes \mathrm{CH}_{n-q}(Z) \arrow[r, "\times", "\textit{Fulton}"'] & \mathrm{CH}_{2n-p-q}(X \times_S Z)
\end{tikzcd}$$

commutes.

Proof. We know that the scheme S is $B_{\mathbb{Z}}$-orienting by Theorem 4.2.1 and Corollary 3.2.8 (iii). Hence S is $B_{\mathbb{Z}}$-orthocyclic by Corollary 4.1.17. By Proposition 4.1.25 we know that there exist unique $B_{\mathbb{Z}}$-external products on S. So we only need to show that the Fulton's construction as in Proposition A.6.1 satisfies (i) and (ii) of Definition 4.1.24: For two S-schemes X and Z consider the morphism

$$a_Z^{(k)} := [X] \times \_ : \mathrm{CH}_k(Z) \to \mathrm{CH}_{k+\dim_S X}(X \times_S Z),$$

which is induced by

$$[W] \mapsto [X] \times_S [W]$$

as in formula (A.3) in Proposition A.6.1. Let $p_X : X \to S$ denote the structure morphism. For part (i) of Definition 4.1.24 we have to check, that these morphisms $a_Z^{(k)}$ induce the desired B_Q-orientation class $[p_X]$ in $B^{-\dim_S X}(p_X)_{\mathbb{Z}}$ with

$$[p_X]([S]) = [X].$$

Since we know that S is $B_{\mathbb{Z}}$-orthocyclic, we have $A^{-\dim_S X}(p_X)_{\mathbb{Z}} = B^{-\dim_S X}(p_X)_{\mathbb{Z}}$ and we only have to check that $[p_X]$ is in $A^{-\dim_S X}(p_X)_{\mathbb{Z}}$. From Proposition A.6.2 (i) we see that $[p_X]$ is in $H^{-\dim_S X}(p_X)_{\mathbb{Z}}$, Proposition A.6.2 (ii) and (iii) show $[p_X]$ is in $A^{-\dim_S X}(p_X)_{\mathbb{Z}}$ and we have

$$\begin{aligned}
[p_X]([S]) &= [X] \times_S [S] \\
&= [X \times_S S] \\
&= [X].
\end{aligned}$$

For the compatibility with proper push-forward in (ii) of Definition 4.1.24 again see Proposition A.6.2 (i). □

Using the fact that if $f : X \to Y$ is a morphism of S-schemes with Y smooth, then $\gamma_f : X \to X \times_S Y$ is a regular imbedding of codimension $n = \dim_S Y$ (cf. Remark A.3.25), Fulton constructed an intersection product on smooth S-schemes (cf. Proposition A.6.3):

Proposition 5.2.2 (Fulton)**.** *Let Y be a smooth S-scheme and let $X, Z \subseteq Y$ denote two closed subsets. Fulton's construction yields in an intersection product*

$$\cdot_{\mathit{Fulton}} : \mathrm{CH}^k_X(Y) \otimes \mathrm{CH}^{k'}_Z(Y) \to \mathrm{CH}^{k+k'}_{X\cap Z}(Y), \quad \alpha \otimes \beta \mapsto \gamma_{i_X}^!(\alpha \times_S \beta),$$

where $i_X : X \to Y$ denotes the induced closed immersion. This product makes $\mathrm{CH}^*(Y)$ into a commutative, graded $\mathbb{Z}$-algebra with unit $[Y] \in \mathrm{CH}^0(Y)$.

Proof. Identifying the terms in Proposition A.6.3 via Definition 4.3.1 directly gives the claim. ☐

We can now compare Fulton's construction to the one given in Chapter 4.

Theorem 5.2.3. *Let Y be a smooth S-scheme. Then our intersection product with supports as given in Remark 4.3.4 (ii) coincides with Fulton's theory as in Proposition 5.2.2.*

Proof. Suppose Y is a smooth S-scheme with $n = \dim_S Y$, denote the structure morphism by $p_Y : Y \to S$. We know that the scheme S is $B_{\mathbb{Z}}$-orienting and $B_{\mathbb{Z}}$-orthocyclic (Theorem 4.2.1, Corollary 3.2.8 (iii), Corollary 4.1.17). We further have that Y is $B_{\mathbb{Z}}$-orienting and $B_{\mathbb{Z}}$-orthocyclic (Proposition 4.1.22, Corollary 4.1.17). For closed subsets $X, Z \subseteq Y$ we put the reduced structure of a closed subscheme on both of them to get the closed immersions $i_X : X \to Y$ and $i_Z : Z \to Y$ and form the fiber square

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "i'_X"] \arrow[d, "i'_Z"'] & Z \arrow[d, "i_Z"'] \\
X \arrow[r, "i_X"] & Y.
\end{tikzcd}$$

Let $\gamma_{i_X} : X \to X \times_S Y$ denote the graph of $i_X : X \to Y$, which is a regular imbedding of codimension $n = \dim_S Y$ by Remark A.3.25, hence gives an $B_{\mathbb{Z}}$-orientation class $[\gamma_{i_Z}] \in B^n(\gamma_{i_Z})_{\mathbb{Z}}$ by Proposition 4.1.14. Then we have to show, that the diagram

$$\begin{tikzcd}
\mathrm{CH}^p_X(Y) \otimes \mathrm{CH}^q_Z(Y) \arrow[rr, ". \text{ as in Remark 4.3.4 (ii)}"] & & \mathrm{CH}^{p+q}_{X\cap Z}(Y) \\
B^p(i_X)_{\mathbb{Z}} \otimes B^q(i_Z)_{\mathbb{Z}} \arrow[u, "\cong"', "E_Y\otimes E_Y"] \arrow[rr, "\cup \text{ as in (4.10)}"] \arrow[d, "\_\cdot[p_Y]\otimes\_\cdot[p_Y]"'] & & B^{p+q}(i_X \circ i'_Z)_{\mathbb{Z}} \arrow[u, "\cong"', "E_Y"] \arrow[d, "\_\cdot[p_Y]"'] \\
B^p(p_Y \circ i_X)_{\mathbb{Z}} \otimes B^q(p_Y \circ i_Z)_{\mathbb{Z}} \arrow[r, "\times"] \arrow[d, "\cong"', "E_S\otimes E_S"] & B^{p+q}((p_Y \circ i_X) \times (p_Y \circ i_Z))_{\mathbb{Z}} \arrow[r, "\_\cdot[\gamma_{i_X}]"] \arrow[d, "\cong"', "E_S"] & B^{p+q}(p_Y \circ i_X \circ i'_Z)_{\mathbb{Z}} \arrow[d, "\cong"', "E_S"] \\
\mathrm{CH}_{n-p}(X) \otimes \mathrm{CH}_{n-q}(Z) \arrow[r, "\times"] & \mathrm{CH}_{2n-p-q}(X \times_S Z) \arrow[r, "{[\gamma_{i_X}]^n_{X\times_S Z}}"] & \mathrm{CH}_{n-p-q}(X \times_Y Z)
\end{tikzcd}$$

commutes. The top square of this diagram commutes by construction of the intersection product with supports in Theorem 4.3.3. The bottom left square commutes by Propositions 5.2.1 and 4.1.25, the bottom right square commutes by definition (indeed

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d] & X \times_S Z \arrow[d, "\mathrm{id}_X \times_S i_Z"] \\
X \arrow[r, "\gamma_{i_X}"] & X \times_S Y
\end{tikzcd}$$

is a fiber square). Note that the bottom line is exactly Fulton's construction as in Proposition 5.2.2. Hence the crucial point is to show the commutativity of the middle part of the diagram above. This part of our proof generalizes [KT87, Formula (3.7)]: Consider the fiber square diagram

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d] & X \times_S Z \arrow[r] \arrow[d, "t"] & Y \times_S Z \arrow[r] \arrow[d] & Z \arrow[d, "i_Z"] \\
X \arrow[r, "\gamma_{i_X}"] \arrow[dr, "\mathrm{id}_X"'] & X \times_S Y \arrow[r] \arrow[d, "s"] \arrow[rr, bend right, "u"'] & Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "p_Y"] \\
& X \arrow[r, "i_X"] & Y \arrow[r, "p_Y"] & S,
\end{tikzcd}$$

with morphisms as labeled, where s is smooth of relative S-dimension $\dim_S Y$ (hence induces $[s] \in B^{-n}(s)_{\mathbb{Z}}$ by Proposition 4.1.14). Further note that $\mathrm{id}_X : X \to X$ is flat and l.c.i. and hence we get

$$\mathrm{id}_{B^0(id_X)_{\mathbb{Z}}} = [\mathrm{id}_X] = [\gamma_{i_X}].[s] \in B^0(id_X)_{\mathbb{Z}}$$

by Proposition A.5.3. Then for $a \in B^p(i_X)_{\mathbb{Z}}$ and $b \in B^q(i_Z)_{\mathbb{Z}}$, we have

$$\begin{aligned}
(t^*[\gamma_{i_X}]).(b.[p_Y] \times_S a.[p_Y]) &= (t^*[\gamma_{i_X}]).(u^* b).[s].a.[p_Y] \\
&= (\gamma_{i_X}^* u^* b).[\gamma_{i_X}].[s].a.[p_Y] \\
&= (\gamma_{i_X}^* u^* b).[id_X].a.[p_Y] \\
&= ((u \circ \gamma_{i_X})^* b).a.[p_Y] \\
&= (b \cup a).[p_Y] \quad \in B^{p+q}(p_Y \circ i_X \circ i'_Z)_{\mathbb{Z}}.
\end{aligned}$$

□

Remark 5.2.4. In order to generalize Arakelov's arithmetic intersection theory from arithmetic surfaces to higher dimensional schemes, Gillet and Soulé [GS87] introduced an intersection product with supports for any Noetherian regular scheme, after tensoring the Chow groups with support by $\mathbb{Q}$. Their main tool is an isomorphism between $\mathrm{CH}^k_Y(X)_{\mathbb{Q}}$ and $\mathrm{Gr}^k_\gamma K_0^Y(X)_{\mathbb{Q}}$, the k-th graded piece of K-theory with support in Y with respect to the γ-filtration. Then the product in K-theory induces a product of Chow

groups with support and coefficients in $\mathbb{Q}$.
Gillet and Soulé state in [GS87, 8.4], that if Y is a smooth integral separated scheme of finite type over a separated Dedekind scheme S, then their intersection product coincides with Fulton's intersection product (also see [GS92, Section 4.5]). Therefore, by Theorem 5.2.3, we know that Gillet and Soulé's intersection product with supports coincides with the our intersection product as in Remark 4.3.4 (ii) in the case of smooth separated schemes which are of finite type over a Noetherian, excellent, regular and separated base scheme of Krull dimension one.

Corollary 5.2.5. *If X is a smooth integral S-scheme (hence $B_{\mathbb{Z}}$-orienting by Corollary 4.1.22 and Corollary 3.2.8), then we can add the following property to our intersection product with supports as in Remark 4.3.4 (ii):*

(iii) For $[Y_1] \in \mathrm{CH}^k_Y(X)$ and $[Z_1] \in \mathrm{CH}^{k'}_Z(X)$ with Y_1 and Z_1 intersecting properly (i.e. $\operatorname{codim}(Y_1 \cap Z_1, X) = \operatorname{codim}(Y_1, X) + \operatorname{codim}(Z_1, X)$), the intersection product $[Y_1].[Z_1] \in \mathrm{CH}^{k+k'}_{Y\cap Z}(X)$ is given by Serre's Tor-formula:

$$[Y_1].[Z_1] = \sum_{x \in Y_1 \cap Z_1 \cap X^{(p+q)}} \sum_{i \geq 0} (-1)^i \mathrm{length}_{\mathcal{O}_{X,x}}(\mathrm{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y_1,x}, \mathcal{O}_{Z_1,x}))[\overline{\{x\}}]$$

Proof. This follows from Remark 5.2.4 and [Sou92, Chapter I, Theorem 2 (iii)]. □

Appendix A

Fulton's Theory for S-schemes

In [Ful84, Chapters 1 - 6], Fulton developed a theory which induces the push-forward of cycles for proper morphisms, the pull-back of cycles for flat morphisms and the refined Gysin homomorphisms for regular imbeddings for schemes that are of finite type over a field. In Chapter 20 of [Ful84], Fulton states that his theory is still valid for schemes X of finite type and separated over a Noetherian, regular base scheme S. Furthermore, Fulton uses this to give an intersection product for smooth separated schemes X which are of finite type over a Noetherian one-dimensional regular base scheme S.

In this appendix, we have collected the results from [Ful84, Chapters 1 - 6] used in this thesis in the following generality: We fix a Noetherian, excellent, regular and separated base scheme S. All schemes X are assumed to be of finite type and separated over S and we call them *S-schemes*. Note that this case is already covered by [Ful84, Section 20.1]. For the intersection product for smooth S-schemes, we additionally assume that S is of Krull dimension one in the last section.

A.1 Proper push-forward and flat pull-back

Proposition A.1.1. *Let $f : X \to Y$ be a proper morphism of S-schemes. Then the* push-forward *is defined by*

$$f_* : Z_k X \to Z_k Y, \quad [V] \mapsto \deg(V/f(V))[f(V)],$$

with

$$\deg(V/f(V)) := \begin{cases} [\kappa(V) : \kappa(f(\eta_V))] & \textit{for } \dim V = \dim f(V), \\ 0 & \textit{otherwise}, \end{cases}$$

where η_V denotes the generic point of V.
If α is a k-cycle on X which is rationally equivalent to zero, then $f_*\alpha$ is rationally equivalent to zero on Y. There is therefore an induced homomorphism

$$f_* : \mathrm{CH}_k X \to \mathrm{CH}_k Y,$$

so that f_* is a covariant functor for proper morphisms.

Note that the statement $\dim V = \dim f(V)$ (where dim denotes the ordinary topological dimension) is equivalent to $\dim_{f(V)} V = 0$ and to $\dim_S V = \dim_S f(V)$ by Proposition 2.1.3 (iv) and (v).

Proof. Fulton [Ful84, bottom of page 394] claims that this still holds by using the Stein factorization [EGA, III.4.3.1] for non-excellent schemes. Since we assumed our schemes to be excellent (hence the normalizations are finite), we can simply adapt the proof given in [Ful84, Theorem 1.4 and Proposition 1.4] by replacing the Krull dimension with the S-dimension. $\square$

Definition A.1.2. (i) A morphism $f : X \to Y$ of S-Schemes has *relative dimension* n, if for all closed integral subschemes V of Y and all irreducible components V' of $f^{-1}(V)$, we have

$$\dim V' = \dim V + n. \tag{A.1}$$

By definition, $n = \dim X - \dim Y$.

(ii) A morphism $f : X \to Y$ of S-Schemes has *relative S-dimension* n, if for all closed integral subschemes V of Y and all irreducible components V' of $f^{-1}(V)$, we have

$$\dim_S V' = \dim_S V + n. \tag{A.2}$$

By definition, $n = \dim_S X - \dim_S Y$.

Remark A.1.3. (i) If the scheme Y in Definition A.1.2 is irreducible, then by Proposition 2.1.3 (v) the relative S-dimension n of the morphism f is the equal to $\dim_Y X$.

(ii) All of our schemes are universally catenary, as they are of finite type over the excellent scheme S. Hence a morphism $f : X \to Y$ of S-Schemes has relative dimension $\dim X - \dim Y$ if and only if for all V, V' as in Definition A.1.2 we have

$$\operatorname{codim}(V', X) = \operatorname{codim}(V, Y).$$

(iii) By Proposition 2.1.3 (ii), a morphism $f : X \to Y$ of S-Schemes has relative S-dimension $\dim_S X - \dim_S Y$ if and only if for all V, V' as in Definition A.1.2 we have

$$\operatorname{codim}(V', X) = \operatorname{codim}(V, Y).$$

(iv) It follows that a morphism $f : X \to Y$ of S-Schemes has relative dimension $\dim X - \dim Y$ if and only if $f : X \to Y$ relative S-dimension $\dim_S X - \dim_S Y$.

(v) Using [Ful84, B2.5], we get the following: If $f : X \to Y$ is flat, Y is irreducible and X is S-equidimensional of dimension $\dim_S Y + n$, then f has relative S-dimension n. Furthermore, any base change $f' : X \times_Y Y' \to Y'$ of a flat morphism $f : X \to Y$ of relative S-dimension n is again flat and of relative S-dimension n.

Proposition A.1.4. *(i) Let $f : X \to Y$ be a flat morphism of relative S-dimension n of S-schemes. The* pull-back

$$f^* : Z_k Y \to Z_{k+n} X$$

is defined by

$$f^*([V]) = \sum \mathrm{length}_{\mathcal{O}_{f^{-1}Z,\eta_W}}(\mathcal{O}_{f^{-1}Z,\eta_W})[W],$$

where the sum is taken over all irreducible components W of the inverse image scheme $f^{-1}Z$. If $\alpha \in Z_k X$ is rationally equivalent to zero, then $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n}X$ and there are therefore induced homomorphisms

$$f^* : \mathrm{CH}_k Y \to \mathrm{CH}_{k+n} X,$$

so that f^* is a contravariant.

(ii) Pull-back and push-forward are compatible in the following sense: Let

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a fiber square with f proper and g denoting a flat morphism of relative dimension n of S-schemes. Then for all $\alpha \in Z_k X$,

$$f'_* g'^* \alpha = g^* f_* \alpha$$

in $Z_{k+n} Y'$.

(iii) For an S-scheme X consider a closed subscheme $i : Y \to X$ and let $j : U = X - Y \to X$ be the open immersion. Then the sequence

$$\mathrm{CH}_k Y \xrightarrow{i_*} \mathrm{CH}_k X \xrightarrow{j^*} \mathrm{CH}_k U \longrightarrow 0$$

is exact for all $k \in \mathbb{Z}$.

Proof. Part (i) works like [Ful84, Theorem 1.7] by replacing $X \times \mathbb{P}^1$ with $X \times_S \mathbb{P}^1_S$ and using S-dimension. Part (ii) is [Ful84, Proposition 1.7, Section 20.1]. The last part is [Ful84, Proposition 1.8, Section 20.1] □

Remark A.1.5. By Proposition 2.1.3 (ii), flat pull-back preserves the codimension of cycles.

A.2 Intersection with Divisors

In this section we very briefly recall Fulton's definition of the intersection product for divisors, which uses the theory of pseudo-divisors. For details of the proofs we refer to [Ful84, Chapter 2, Section 20.1].

Construction A.2.1. Let D be an effective Cartier divisor on an S-scheme X (with $\dim_S X = n$), i.e. a closed subscheme $i : D \to X$, such that its ideal sheaf $\mathcal{J}_D \subseteq O_X$ is locally generated by one element, and let $i_W : W \to X$ be the closed immersion of a integral subscheme W of codimension k in X. Then Fulton defined the intersection product of $D \cdot_{\text{divisors}} [W] \in \mathrm{CH}_{n-k-1}(W \times_X D)$ by the following:
An effective Cartier divisor D induces a so called *pseudo-divisor*

$$(\mathcal{O}_X(D), \mathrm{supp}(D), s_D),$$

where $\mathcal{O}_X(D)$ denotes the associated line bundle, $\mathrm{supp}(D)$ the support of D and s_D the canonical section of $\mathcal{O}_X(D)$ (cf. [Ful84, Appendix B.4]). Take the pull-back

$$(i_W^* \mathcal{O}_X(D), i_W^{-1}(\mathrm{supp}(D)), i_W^* \, s_D)$$

of this pseudo-divisor and find a Cartier divisor $\tilde{D}$ on W which represents this pull-back (cf. [Ful84, Lemma 2.2, Section 20.1]). This Cartier divisor $\tilde{D}$ will be determined up to linear equivalence, so it determines a well-defined element

$$D \cdot_{\text{divisors}} [W] \in \mathrm{CH}_{n-k-1}(W \times_X D).$$

A.3 Cones, Chern and Segre classes

For every (geometric) vector bundle E on an S-scheme X, we will construct Chern and Segre class operations. Although the constructions do not depend on the base scheme S, we should note that we grade the Chow group $\mathrm{CH}_* X$ by S-dimension.

Proposition A.3.1. *Let L be a line bundle on X and V be an integral closed subscheme of X of S-dimension k. Then the restriction $L|_V$ of L to V is isomorphic to $\mathcal{O}_V(C)$ for some Cartier divisor C on V, determined up to linear equivalence. The associated Weil divisor determines a well defined element*

$$c_1(L) \cap [V] := [C] \text{ in } \mathrm{CH}_{k-1} X.$$

By linearity we get a homomorphism

$$c_1(L) \cap \_ : Z_k X \to \mathrm{CH}_{k-1} X, \quad \alpha \mapsto c_1(L) \cap \alpha,$$

compatible with rational equivalence, therefore inducing

$$c_1(L) \cap \_ : \mathrm{CH}_k X \to \mathrm{CH}_{k-1} X.$$

We define $c_1(L)^d \cap \alpha$ for $d \in \mathbb{N}$ inductively by $c_1(L)^d \cap \alpha = c_1(L) \cap (c_1(L)^{d-1} \cap \alpha)$.

Proof. See [Ful84, Proposition 2.5 (a), Section 20.1]. □

Definition A.3.2. The homomorphism $c_1(L) \cap \_ : \mathrm{CH}_k X \to \mathrm{CH}_{k-1} X$ as in A.3.1 is called the *operation of the first Chern class* of the line bundle L.

The following facts about the first Chern class are well known:

Proposition A.3.3. *The operation of the first Chern class satisfies the following properties:*

(i) (Commutativity) For two line bundles L, L' on an S-scheme X and $\alpha \in \mathrm{CH}_k X$, then

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha) \text{ in } \mathrm{CH}_{k-2} X.$$

(ii) (Projection formula) For a proper morphism $f : X' \to X$ of S-schemes, L a line bundle on X and $\alpha \in \mathrm{CH}_k X'$, then

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*\alpha \text{ in } \mathrm{CH}_{k-1} X.$$

(iii) (Flat pull-back) For a flat morphism $f : X' \to X$ of relative S-dimension n of S-schemes, L a line bundle on X and $\alpha \in \mathrm{CH}_k X$, then

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha) \text{ in } \mathrm{CH}_{k+n-1} X'.$$

(iv) (Additivity) If L, L' are line bundles on an S-scheme X and $\alpha \in \mathrm{CH}_k X$, then

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$

and

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha \text{ in } \mathrm{CH}_{k-1} X.$$

Proof. Compare to [Ful84, Proposition 2.5 b) - e), Section 20.1]. □

Definition A.3.4. For a vector bundle E on X of rank $e+1$ let $\mathcal{E}$ denote the associated *sheaf of sections* of E over X, i.e. the locally free $\mathcal{O}_X$-module given by $\mathcal{E}(U) = \mathrm{Hom}_U(U, E)$. Let

$$p : P(E) = \mathrm{Proj}\,(\mathrm{Sym}^\bullet \mathcal{E}^\vee) \to X$$

be the *projective bundle of lines*, with canonical line bundle $\mathcal{O}_E(1)$ on $P(E)$. Define the *operation of the ith Segre class* to be the homomorphism

$$s_i(E) \cap \_ : \mathrm{CH}_k X \to \mathrm{CH}_{k-i} X, \quad \alpha \mapsto s_i(E) \cap \alpha := p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha).$$

Remark A.3.5. Note that we use Fulton's notation [Ful84, B.5.5] for the projective bundle of lines here in contrast to standard Grothendieck's language [EGA].

Proposition A.3.6. *Let E be a vector bundle of rank $e+1$ on X.*

(i) For all $\alpha \in \mathrm{CH}_k X$,

(a) $s_i(E) \cap \alpha = 0$ *for* $i < 0$,

(b) $s_0(E) \cap \alpha = \alpha$.

(ii) If E, F *are vector bundles on an* S*-scheme* X, $\alpha \in \mathrm{CH}_k X$, *then, for all* i, j,

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha).$$

(iii) If $f : X' \to X$ *is a proper morphism of* S*-schemes,* E *a vector bundle on* X *and* $\alpha \in \mathrm{CH}_k X'$, *then, for all* i,

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

(iv) If $f : X' \to X$ *is a flat morphism of* S*-schemes of some relative* S*-dimension,* E *a vector bundle on* X *and* $\alpha \in \mathrm{CH}_k X$, *then, for all* i,

$$s_i(f^*E) \cap f^*(\alpha) = f^*(s_i(E) \cap \alpha).$$

(v) If E *is a line bundle on an* S*-scheme* X *and* $\alpha \in \mathrm{CH}_k X$, *then*

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha.$$

Proof. See [Ful84, Proposition 3.1, Section 20.1]. □

Definition A.3.7. Let E be a vector bundle on an S-scheme X. Consider the formal power series

$$s_t(E) = \sum_{i=0}^{\infty} s_i(E) t^i \in \mathcal{R}[[t]],$$

where $\mathcal{R}$ denote the sub-algebra of $\mathrm{End}\,\mathrm{CH}_* X$ generated by the Elements $s_i(E) \cap \_$. Define the *Chern polynomial* to be the inverse power series

$$c_t(E) = \sum_{i=0}^{\infty} c_i(E) t^i := s_t(E)^{-1}.$$

Every $c_i(E) \in \mathcal{R}$ defines an operation

$$c_i(E) \cap . : \mathrm{CH}_k X \to \mathrm{CH}_{k-i} X.$$

We call $c_i(E)$ the *ith Chern class* on E.

Remark A.3.8. Explicitly we get

$$\begin{aligned} c_0(E) &= 1, \\ c_1(E) &= -s_1(E), \\ c_2(E) &= s_1(E)^2 - s_2(E), \\ &\vdots \\ c_n(E) &= -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - \ldots - s_n(E). \end{aligned}$$

Theorem A.3.9. *The Chern classes satisfy the following properties:*

(i) (Vanishing) For all bundles E on an S-scheme X, all $i > \operatorname{rank}(E)$ and all $\alpha \in \mathrm{CH}_*X$,

$$c_i(E) \cap \alpha = 0.$$

So $c_t(E)$ is in fact a polynomial.

(ii) (Commutativity) For all bundles E, F on an S-scheme X, $\alpha \in \mathrm{CH}_*X$,

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha).$$

(iii) (Projection formula) Let E be a vector bundle on an S-scheme X, $f : X' \to X$ a proper morphism of S-schemes andαinCH_*X,

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*\alpha.$$

(iv) (Pull-back) Let E be a vector bundle on an S-scheme X, $f : X' \to X$ a flat morphism of S-scheme of some relative S-dimension and $\alpha \in \mathrm{CH}_*X$,

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha).$$

(v) (Whitney sum) For any exact sequence $0 \to E' \to E \to E''$ of vector bundles on an S-scheme X,

$$c_t(E) = c_t(E')c_t(E''),$$

i.e.,

$$c_k(E) = \sum_{i+j=k} c_i(E') \cdot c_j(E'').$$

(vi) (Normalization) If E is a line bundle on an integral S-scheme X (of S-dimension n), D a Cartier divisor on X with $\mathcal{O}_X(D) \cong E$, then

$$c_1(E) \cap [X] = [D],$$

where $[D]$ denotes the associated cycle class in $\mathrm{CH}_{n-1}X$ defined by the Cartier divisor D (see [Ful84, Section 2.1]).

Proof. Compare to [Ful84, Theorem 3.2, Section 20.1]. □

Theorem A.3.10. *Let $\pi : E \to X$ be a vector bundle of rank $r = e + 1$, $\mathcal{E}$ the sheaf of sections of E over X (a locally free $\mathcal{O}_X$-module),*

$$p : P(E) = \operatorname{Proj}(\operatorname{Sym}^\bullet \mathcal{E}^\vee) \to X$$

the projective bundle of lines in E and $\mathcal{O}(1) = \mathcal{O}_E(1)$ the canonical line bundle on $P(E)$.

(i) The flat pull-back

$$\pi^* : \mathrm{CH}_{k-r}X \to \mathrm{CH}_kE$$

is an isomorphism for all $k \in \mathbb{Z}$.

(ii) There is a canonical Isomorphism

$$\Theta_E : \bigoplus_{i=0}^{e} \mathrm{CH}_{k-e+i}X \xrightarrow{\sim} \mathrm{CH}_k P(E), \ \ (\alpha_{k-e+i})_{i=0,\ldots,e} \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_{k-e+i}.$$

Proof. See [Ful84, Theorem 3.3, Section 20.1]. □

Proposition A.3.11. *Let $\pi : E \to X$ be a vector bundle of rank r, $s : X \to E$ its zero section, $q : P(E \oplus 1) \to X$ the projection and ξ the universal rank r quotient bundle of $q^*(E \oplus 1)$, i.e.,*

$$0 \to O_{P(E\oplus 1)}(-1) \to q^*(E \oplus 1) \to \xi \to 0$$

is exact.
If $j : E \to P(E \oplus 1)$ is the canonical inclusion, then for all $\beta \in \mathrm{CH}_k E$ and $\bar{\beta} \in \mathrm{CH}_k P(E \oplus 1)$ with $j^*\bar{\beta} = \beta$ we have

$$s^*\beta = q_*(c_r(\xi) \cap \bar{\beta}).$$

Proof. See [Ful84, Proposition 3.3, Section 20.1]. □

Definition A.3.12. Let $s = s_E : X \to E$ be the zero section of the vector bundle $\pi : E \to X$ of rank r. Then the *Gysin morphism* is defined by

$$s^* : \mathrm{CH}_k E \to \mathrm{CH}_{k-r} X, \ s^*(\beta) = (\pi^*)^{-1}(\beta).$$

Definition A.3.13. Let X be an S-scheme, $S^\bullet = \bigoplus_{n\geq 0} S^n$ a graduated sheaf of $\mathcal{O}_X$-algebras on X with $\mathcal{O}_X \to S^0$ is an isomorphism and $S^\bullet$ is an $\mathcal{O}_X$-algebra which is locally generated by S^1. For a variable Z let $S^\bullet[Z]$ be the graded $\mathcal{O}_X$-algebra given by

$$(S^\bullet[Z])^n = S^n \oplus S^{n-1}Z \oplus ... \oplus S^1 Z^{n-1} \oplus S^0 Z^n.$$

Then the *cone* of $S^\bullet$ is defined by

$$C := C(S^\bullet) := \mathrm{Spec}\,(S^\bullet) \to X,$$

the *projective cone* of $S^\bullet$ is

$$P(C) := \mathrm{Proj}\,(S^\bullet) \to X.$$

We set

$$C \oplus 1 := C(S^\bullet[Z])$$

and

$$P(C \oplus 1) := \mathrm{Proj}\,(S^\bullet[Z])$$

the *projective closure* of C.

Remark A.3.14. Note that a vector bundle E on an S-scheme X is the cone of the graded $\mathcal{O}_X$-algebra $\mathrm{Sym}^\bullet \mathcal{E}^\vee$, where $\mathcal{E}$ denotes the sheaf of sections of E over X.

Proposition A.3.15. *Let X be an S-scheme.*

(i) *If $S^\bullet \to S'^\bullet$ is a surjective, graded homomorphism of such graded sheaves of O_X-algebras, and $C = \mathrm{Spec}\,(S^\bullet)$, $C' = \mathrm{Spec}\,(S'^\bullet)$, then there are closed imbeddings*

$$C' \to C$$

and

$$P(C') \to P(C),$$

such that $\mathcal{O}_C(1)$ restricts to $\mathcal{O}_{C'}(1)$

(ii) *The element $Z \in (S^\bullet[Z])^1$ determines a regular section*

$$s \in \Gamma(P(C \oplus 1), \mathcal{O}_{P(C\oplus 1)}(1)).$$

The zero scheme $Z(s)$ of this section is canonically isomorphic to $P(C)$ and with its complement $P(C \oplus 1)\backslash P(C)$ being canonically isomorphic to C.

Proof. See [Ful84, B5.1] or [Har77, II.7.15]. □

Example A.3.16. The zero section

$$s_E : C(\mathcal{O}_X) = X \to E = C(S^\bullet)$$

is given by the surjection

$$e : S^\bullet \to O_X, \quad e|_{S^0} = \mathrm{id}_{\mathcal{O}_X}, \; e|_{S^i} = 0 \; \forall i \geq 1.$$

Definition A.3.17. Let Y be an S-scheme and let $i : X \to Y$ be a closed imbedding with ideal sheaf J on Y. Then we define the *Normal Cone* by

$$C_X Y := \mathrm{Spec}\,(\bigoplus_{n\geq 0} J^n/J^{n+1})$$

and the *blow-up* of Y in X by

$$\mathrm{Bl}_X Y := \mathrm{Proj}\,(\bigoplus_{n\geq 0} J^n).$$

The scheme $E := X \times_Y \mathrm{Bl}_X Y$ is called the *exceptional divisor* of the blow-up $\mathrm{Bl}_X Y$.

Proposition A.3.18. *Let Y be an S-scheme.*

(i) *The morphism $Bl_X Y \to Y$ is projective.*

(ii) *If Y is integral with closed subscheme X, then $\mathrm{Bl}_X Y$ is integral.*

(iii) *The exceptional divisor E of the blow-up $\mathrm{Bl}_X Y$ is an effective Cartier divisor on $\mathrm{Bl}_X Y$ and we have $E = P(C_X Y)$.*

(iv) *If X does not contain any irreducible component of Y, then $\mathrm{Bl}_X Y \to Y$ is birational.*

(v) *If Y is relative equidimensional of dimension d, then so is C_XY.*

Proof. Part (i) is [GW10, Proposition 13.96 (i)]. Part (i) - (iv) can be found in [Liu02, 8.1.12]. For (v) consider the closed imbedding $X \to Y \to Y \times_S \mathbb{A}^1_S$ with normal cone $C_X(Y \times_S \mathbb{A}^1_S) = C_XY \oplus 1$. By (iii) $\mathrm{Bl}_X(Y \times_S \mathbb{A}^1_S)$ is birational to $Y \times_S \mathbb{A}^1_S$, so it is relative equidimensional of dimension $d+1$. Then C_XY is relative equidimensional of dimension $d+1$ as an open subscheme of the exceptional divisor $P(C_XY \oplus 1)$ of the blow-up $\mathrm{Bl}_X(Y \times_S \mathbb{A}^1_S)$. □

Definition A.3.19. Let $C = \mathrm{Spec}\,(S^\bullet) \to X$ be a cone, $q : P(C \oplus 1) \to X$ its projective closure and $\mathcal{O}_{C\oplus 1}(1)$ the canonical line bundle on $P(C \oplus 1)$. Define the *Segre class* $s(C) \in \mathrm{CH}_*X$ of the cone C as

$$s(C) = q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}_{C\oplus 1}(1))^i \cap [P(C \oplus 1)] \right) .$$

Proposition A.3.20. *If E is a vector bundle over X of rank r, then*

$$s(E) = c(E)^{-1} \cap [X] \ ,$$

where $c(E) = 1 + c_1(E) + ... + c_r(E)$ is the total Chern class *of E.*

Proof. See [Ful84, Proposition 4.1 a), Section 20.1]. □

Definition A.3.21. Let X and Y be S-schemes.

(i) A sequence of elements $a_1, ..., a_d$ in a ring A is called a *regular sequence*, if the ideal $J := (a_1, ..., a_d)$ is proper ideal of A and the image of a_i in $A/(a_1, ..., a_{i-1})$ is a non-zero-divisor, for all $i = 1, ...d$.

(ii) A closed imbedding $i : X \to Y$ is called a *regular imbedding* of codimension d, if every point of X has an affine neighborhood $U = \mathrm{Spec}\, A$ in Y, such that the ideal J in A defining $i : X \to Y$ locally is generated by a regular sequence of length d.

(iii) A morphism $f : X \to Y$ is called *local complete intersection* (*l.c.i*) morphism of codimension d, if there is a factorization $f = g \circ i$, with a regular imbedding i of codimension e and a smooth morphism g of relative S-dimension $d + e$.
So in contrast to standard usage, we require an l.c.i. morphism to admit such a global factorization.

The following Proposition is often helpful:

Proposition A.3.22. *Let X and Y be S-schemes.*

(i) If X and Y are regular, then each closed imbedding $i : X \to Y$ is a regular imbedding.

(ii) If X and Y are regular, then each projective morphism $f : X \to Y$ is an l.c.i. morphism.

Proof. The claim in (i) is proven in [EGA, IV 19.1.1]. For (ii) we note that regular schemes admit ample line bundles by [SGA, exp. II Corollaire 2.2.7.1 and Définition 2.2.5]. By [EGA, II 5.5.4 (ii)] we get that $f : X \to Y$ factors into a closed immersion $X \to \mathbb{P}^r_Y$ for some $r \in \mathbb{N}$ followed by the projection $\mathbb{P}^r_Y \to Y$. Then the claim follows directly from (i), since $\mathbb{P}^r_Y \to Y$ is smooth and hence $\mathbb{P}^r_Y$ is regular. □

Remark A.3.23. Let X and Y be S-schemes. If $i : X \to Y$ is a regular imbedding of codimension d with ideal sheaf J, J/J^2 is a locally free $\mathcal{O}_X$-module of rank d and there is a canonical isomorphism

$$\mathrm{Sym}^\bullet(J/J^2) \to \bigoplus_{n\geq 0}(J^n/J^{n+1}),$$

i.e.,

$$C_XY = \mathrm{Spec}\,(\mathrm{Sym}^\bullet(J/J^2))$$

is a vector bundle on X, with sections dual to J/J^2.

Definition A.3.24. Let X and Y be S-schemes. If $i : X \to Y$ is a regular imbedding of codimension d with ideal sheaf J, the cone C_XY is also called the *Normal Bundle* N_XY of X in Y, a vector bundle on X.

Remark A.3.25. Let S be a locally Noetherian scheme and $Y \to S$ be a smooth separated morphism. Then any section $\pi : S \to Y$ of f is a regular immersion and with normal bundle canonically isomorphic to the tangent bundle $T_{Y/X}$ of Y over X (cf [Liu02, Corollary 6.3.14]). Hence for any morphism $f : X \to Y$ of S-schemes with Y smooth, the graph morphism $\gamma_f : X \to X \times_S Y$ is a regular imbedding of codimension $n = \dim_S Y$ (as it is a section of the smooth morphism $X \times_S Y \to X$). In particular, the diagonal $\delta : Y \to Y \times_S Y$ of a smooth S-scheme Y is a regular imbedding of codimension $n = \dim_S Y$.

A.4 Deformation to the Normal bundle

Proposition A.4.1. *(Deformation to the Normal Cone) Let $i : X \to Y$ be a closed imbedding of S-schemes, with Normal Cone $C = C_XY$.*
Then there exists a uniquely determined scheme $M := M_XY$ with the following properties:

(i) There is a commutative diagram

$$\begin{tikzcd} X \times_S \mathbb{P}^1_S \arrow[rr, "\tilde{i}"] \arrow[dr, "\mathrm{pr}"'] & & M \arrow[dl, "\rho"] \\ & \mathbb{P}^1_S & \end{tikzcd} ,$$

with a closed imbedding $\tilde{i}$.

(ii) Over $\mathbb{A}^1_S = \mathbb{P}^1_S - \{\infty\}$ we have $\rho^{-1}(\mathbb{A}^1_S) = Y \times_S \mathbb{A}^1_S$.

(iii) *Over $\{\infty\}$ the Cartier divisor $M_{\{\infty\}} := \rho^{-1}(\{\infty\})$ is the sum of two effective Cartier divisors*

$$M_{\{\infty\}} = P(C \oplus 1) + \mathrm{Bl}_X Y.$$

(iv) *The closed imbedding*

$$\tilde{i}_\infty : X = X \times_S \{\infty\} \to M_{\{\infty\}} \ ,$$

induced by $\tilde{i}$, is given by the composition of the zero section of X in C with the canonical open imbedding of C in $P(C \oplus 1)$.

(v) *The intersection of the two divisors $P(C \oplus 1)$ and $\mathrm{Bl}_X Y$ is $P(C)$, regarded as the hyperplane at infinity in $P(C \oplus 1)$ resp. as the exceptional divisor of $\mathrm{Bl}_X Y$.*

(vi) *In particular, $\tilde{i}_\infty(X) \cap \mathrm{Bl}_X Y = \emptyset$. The closed imbedding $\tilde{i}$ gives a family of imbeddings of X*

$$\begin{tikzcd} X \times_S \mathbb{P}^1_S \arrow[rr, "\tilde{i}"] \arrow[dr, "\mathrm{pr}"'] & & M^0 := M - \mathrm{Bl}_X Y \arrow[dl, "\rho^0"] \\ & \mathbb{P}^1_S & \end{tikzcd} ,$$

which deforms the given imbedding i to the zero section $\tilde{i}_\infty$ of X in C.

Proof. The scheme M is given by setting

$$M := \mathrm{Bl}_{X \times_S \{\infty\}}(Y \times_S \mathbb{P}^1_S).$$

The detailed computations which show the properties follow from [Ful84, Section 5.1, Section 20.1] by replacing any appearances of $X \times \mathbb{A}^n$ and $X \times \mathbb{P}^n$ by $X \times_S \mathbb{A}^n_S$ and $X \times_S \mathbb{P}^n_S$ respectively. □

Proposition A.4.2. *Let $i_0 : X \to Y$ be a closed immersion of S-schemes with Normal Cone $C = C_X Y$ of schemes of finite type and separated over a separated regular Noetherian base scheme S. The* specialization homomorphisms

$$\sigma : Z_k Y \to Z_k C$$

are given by $\sigma[V] = [C_{V \cap X} V]$ for all $k \in \mathbb{Z}$. They are well-defined and agree with rational equivalence, i.e., these σ induce specialization homomorphisms

$$\sigma : \mathrm{CH}_k Y \to \mathrm{CH}_k C.$$

Proof. The proof follows from [Ful84, Proposition 5.2, Section 20.1]. □

A.5 Refined Gysin homomorphisms

We are now able to give the crucial construction of the refined Gysin homomorphisms for regular imbeddings of schemes.

Construction A.5.1. Let $i : X \to Y$ be a regular imbedding of codimension d (with ideal sheaf J and normal cone $N_X Y$) of S-schemes.

(i) Let $f : V \to Y$ be a morphism and

$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

a fiber square. Let $N := g^* N_X Y$ be the Normal bundle and $\pi : N \to W$ denote the projection. The ideal sheaf $J_{X/Y}$ of X in Y maps onto the ideal sheaf $J_{W/V}$ of W in V, so we get a surjection

$$\bigoplus_{n\geq 0} g^*(J^n_{X/Y}/J^{n+1}_{X/Y}) \to \bigoplus_{n\geq 0}(J^n_{W/V}/J^{n+1}_{W/V}).$$

This gives a closed imbedding $C_W V \to N$ and the diagram

$$\begin{tikzcd} C_W V \arrow[rr] \arrow[dr] & & N \arrow[dl] \\ & W & \end{tikzcd}$$

commutes. If V is relative equidimensional of dimension d, then so is $C_X Y$ by Proposition A.3.18 (v).
Define

$$X.V = s_N^*[C_W V] \in \mathrm{CH}_{k-d}(W),$$

where s_N^* is the Gysin morphism of the zero section s_N of N as in Definition A.3.12.

(ii) For a morphism $f : Y' \to Y$ form the fiber square

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd} .$$

We define the *refined Gysin homomorphisms* by

$$i^! : \mathrm{CH}_k Y' \to \mathrm{CH}_{k-d} X', \quad \sum n_i[V_i] = \sum n_i X.V_i.$$

Note, that the cone $C_{X'}Y'$ is a closed subscheme of $N = g^* N_X Y$, so $i^!$ is the composition

$$Z_k Y' \overset{\sigma}{\to} Z_k C_{X'} Y' \to \mathrm{CH}_k N \overset{s_N^*}{\to} \mathrm{CH}_{k-d} X',$$

with σ the specialization homomorphism as in Proposition A.4.2 and s_N^* the Gysin homomorphism of the zero section s_N in N as in Definition A.3.12. These agree with rational equivalence, so the refined Gysin homomorphisms are well-defined. For $Y' = Y$, $f = id_Y$, these are simply called *Gysin homomorphisms*, denoted by

$$i^* : \mathrm{CH}_k Y \to \mathrm{CH}_{k-d} X.$$

Theorem A.5.2. *Let $i : X \to Y$ be a regular imbedding of codimension d (with ideal sheaf J and normal bundle $N_X Y$) of S-schemes. Form the fiber square*

$$\begin{tikzcd}
X'' \arrow[r, "i''"] \arrow[d, "q"'] & Y'' . \arrow[d, "p"] \\
X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\
X \arrow[r, "i"] & Y
\end{tikzcd}$$

(i) (push-forward) If p is proper and $\alpha \in \mathrm{CH}_k Y''$, then

$$i^! p_*(\alpha) = q_* i^!(\alpha) \text{ in } \mathrm{CH}_{k-d} X'.$$

(ii) (pull-back) If p is flat of relative S-dimension n and $\alpha \in \mathrm{CH}_k Y'$, then

$$i^! p^*(\alpha) = q^* i^!(\alpha) \text{ in } \mathrm{CH}_{k+n-d} X''.$$

(iii) (excess formula) If i' is also a regular imbedding of codimension d', we have a canonical imbedding

$$N_{X'} Y' \to g^* N_X Y,$$

with its quotient is a vector bundle E of rank $e = d - d'$, called the excess Normal bundle. *For all $\alpha \in \mathrm{CH}_k Y''$ we have*

$$i^!(\alpha) = c_e(q^* E) \cap i'^!(\alpha) \text{ in } \mathrm{CH}_{k-d} X''.$$

(iv) (self intersection) For $\alpha \in \mathrm{CH}_k X$ we have

$$i^* i_*(\alpha) = c_d(N_X Y) \cap \alpha \text{ in } \mathrm{CH}_{k-d} X.$$

(v) (Chern class and Gysin) For a vector bundle F on Y', $\alpha \in \mathrm{CH}_k Y'$ and $m \leq 0$ we have

$$i^!(c_m(F) \cap \alpha) = c_m(i'^* F) \cap i^! \alpha \text{ in } \mathrm{CH}_{k-d-m} X'.$$

(vi) (Commutativity) Let $j : S \to T$ be a regular imbedding of codimension e of S-schemes. For the fiber square

$$\begin{tikzcd}
Y'' \arrow[r] \arrow[d, "p"'] & S \arrow[d, "j"] \\
Y' \arrow[r] & T
\end{tikzcd}$$

and all $\alpha \in \mathrm{CH}_k Y'$ *we have*

$$i^! j^!(\alpha) = j^! i^!(\alpha) \text{ in } \mathrm{CH}_{k-d-e}X''.$$

(vii) *(Functoriality) Let* $j : Y \to Z$ *be a regular imbedding of codimension* e *of* S*-schemes and*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f"'] & Z' \arrow[d, "h"] \\ Y \arrow[r, "j"] & Z \end{tikzcd}$$

a fiber square. Then $j \circ i : X \to Z$ *is a regular imbedding of codimension* $d + e$ *and for all* $\alpha \in \mathrm{CH}_k Z'$,

$$(j \circ i)^!(\alpha) = i^! j^!(\alpha) \text{ in } \mathrm{CH}_{k-d-e}(X')$$

(viii) *If* Y *is* S*-equidimensional, then* $f^![Y] = [X] \in \mathrm{CH}_{\dim_S X}(X)$.

(ix) *(effective Cartier divisors) An effective Cartier divisor* $f : D \to X$ *can be viewed as a regular immersion of codimension one. In this case the refined Gysin homomorphism* $f^!$ *coincides with the intersection product of effective Cartier divisors as in Construction A.2.1, i.e.*

$$D \cdot_{divisors} [W] = f^!([W]) \in \mathrm{CH}_{n-k-1}(W \times_X D)$$

for $[W] \in \mathrm{CH}_{n-k}(W)$

Proof. Using [Ful84, Section 20.1], the results can be found in the following places: (i) and (ii) can be found in [Ful84, Theorem 6.2], (iii) and (iv) are in [Ful84, Theorem 6.3], (v) is [Ful84, Proposition 6.3], (vi) is [Ful84, Theorem 6.4], (vii) is [Ful84, Theorem 6.5], (viii) is [Ful84, Example 6.2.1] and (ix) follows from [Ful84, Example 6.2.3 and Example 3.3.1]. □

Proposition A.5.3. *Let*

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "h"] & Y' \arrow[r, "p'"] \arrow[d, "g"] & Z' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \arrow[r, "p"] & Z \end{tikzcd}$$

be a cartesian diagram.

(i) *Let* i *be a regular imbedding of codimension* d, p *resp.* pi *flat of relative* S*-dimension* n *resp.* $n - d$*. Then* i' *is a regular imbedding of codimension* d, p' *and* $p'i'$ *are flat and for all* $\alpha \in \mathrm{CH}_k Z'$,

$$(p'i')^*(\alpha) = i'^*(p'^*\alpha) = i^! p'^*\alpha \quad \text{in } \mathrm{CH}_{k+n-d}X'.$$

(ii) *If* i *is a regular imbedding of codimension* d, p *smooth of relative* S*-dimension* n *and* pi *a regular imbedding of codimension* $d - n$.
Then for all $\alpha \in \mathrm{CH}_k Z'$,

$$(pi)^!(\alpha) = i^!(p'^*\alpha) \quad \text{in } \mathrm{CH}_{k+n-d}X'.$$

Proof. [Ful84, Proposition 6.5, Section 20.1]. □

Definition A.5.4. Let $f : X \to Y$ be an l.c.i. morphism of codimension d and

$$X \xrightarrow{i} P \xrightarrow{p} Y$$

a factorization into a regular imbedding i of codimension $d+e$ and a smooth morphism p of relative S-dimension e. For a morphism $h : Y' \to Y$ and

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

cartesian we define p' by the cartesian diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & P' \arrow[r, "p'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & P \arrow[r] & Y
\end{tikzcd}$$

and the *Gysin morphism* $f^!$ by

$$f^! : \mathrm{CH}_k Y' \to \mathrm{CH}_{k-d} X' \ , \quad f^!(\alpha) = i^!(p'^* \alpha).$$

Proposition A.5.5. *(i) If p is also separated, this definition of $f^!$ is independent of the choice of a factorization of f.*

(ii) For an flat l.c.i. morphism f we have $(f')^* = f^!$.

(iii) We get analogous properties as in Theorem A.5.2 (i), (ii), (vi) and (viii) for l.c.i. morphisms. For the functoriality in (vii) we have to assume, that all morphisms allow a factorization into a closed immersion followed by a smooth morphism, and that we can form a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "f"'] & P \arrow[r] \arrow[d] & R \arrow[d] \ , \\
 & Y \arrow[r] \arrow[dr, "g"'] & Q \arrow[d] \\
 & & Z
\end{tikzcd}$$

with the horizontal maps being regular imbeddings and the upper right square is Cartesian. The assertion in (iii) is valid as well if the excess normal bundle E for l.c.i. morphisms f and f' is defined to be

$$E = h'^* N_X P / N_{X'} P'$$

for P and P' as in Definition A.5.4.

Proof. [Ful84, Proposition 6.6, Section 20.1]. □

A.6 Intersection theory for smooth schemes over a one-dimensional base

In this section we fix a Noetherian, excellent, regular and separated base scheme S **of (Krull) dimension one**. As usual, an S-scheme is a separated scheme which is of finite type over S. In case that an S-scheme X is smooth over S, we can use Fulton definition of an intersection product as in [Ful84, Section 8, Section 20.2]. There are two ingredients to his construction, which we will discuss in the following. The first one is an explicit description of exterior products:

Proposition A.6.1. *Let X and Y be S-schemes. For closed integral subschemes $V \to Y$ with $\dim_S V = k$ and $W \to X$ with $\dim_S W = k'$ define the* product cycle

$$[V] \times_S [W] = \begin{cases} [V \times_S W] & \text{if } V \text{or } W \text{is flat over } S, \\ 0 & \text{otherwise} \end{cases} \in \mathrm{CH}_{k+k'}(X \times_S Y). \tag{A.3}$$

This passes to rational equivalence and gives the exterior product

$$\times : \mathrm{CH}_k(X) \otimes \mathrm{CH}_{k'}(Y) \to \mathrm{CH}_{k+k'}(X \times_S Y).$$

Proof. By [Ful84, Proposition 20.2] we see that the construction of the exterior product passes to rational equivalence. □

Proposition A.6.2. *The exterior products*

$$\times : \mathrm{CH}_k(X) \otimes \mathrm{CH}_{k'}(Y) \to \mathrm{CH}_{k+k'}(X \times_S Y)$$

for S-schemes X and Y as in Proposition A.6.1 are commutative and associative and are compatible with proper pushforward, flat pullback and refined Gysin homomorphisms in the following way: Let $f : X' \to X$ and $g : Y'toY$ be morphisms of S-schemes and let $f \times_S g$ be the induces morphism from $X' \times_S Y'$ to $X \times_S Y$. Then we have:

(i) If f and g are proper, so is $f \times_S g$ and we have

$$(f \times_S g)_*(\alpha \times_S \beta) = f_*\alpha \times_S g_*\beta \quad \in \mathrm{CH}_{k+k'}(X \times_S Y)$$

for $\alpha \in \mathrm{CH}_k(X')$ and $\beta \in \mathrm{CH}_{k'}(Y')$.

(ii) If f and g are flat of relative S-dimensions m and n, then $f \times_S g$ is flat of relative S-dimension $m+n$ and we have

$$(f \times_S g)^*(\alpha \times_S \beta) = f^*\alpha \times_S g^*\beta \quad \in \mathrm{CH}_{k+k+m+n}(X' \times_S Y')$$

for $\alpha \in \mathrm{CH}_k(X)$ and $\beta \in \mathrm{CH}_{k'}(Y)$.

(iii) If f and g are regular imbeddings of codimensions m and n, then $f \times_S g$ is a regular imbedding of codimension $m+n$ and we have

$$(f \times_S g)^!(\alpha \times_S \beta) = f^!\alpha \times_S g^!\beta \quad \in \mathrm{CH}_{k+k-m-n}(X' \times_S Y')$$

for $\alpha \in \mathrm{CH}_k(X)$ and $\beta \in \mathrm{CH}_{k'}(Y)$.

Proof. The exterior products are commutative and associative by definition. Part (i) and (ii) are proven in [Ful84, Proposition 1.10], part (iii) in [Ful84, Example 6.5.2] in the case of schemes over a field. In [Ful84, Section 20.2], Fulton states that the same properties hold for S-schemes (with S onedimensional). □

The second important fact for the following construction is that if $f : X \to Y$ is a morphism of S-schemes with Y smooth, then

$$\gamma_f : X \to X \times_S Y$$

is a regular imbedding of codimension $n = \dim_S Y$ (cf. Remark A.3.25). Using the refined Gysin homomorphisms, Fulton constructed an intersection product for smooth schemes over a one-dimensional base scheme as follows:

Proposition A.6.3. *Let Y be a smooth S-scheme and set $n = \dim_S Y$. Given two closed subset $X, Z \subseteq Y$, put the reduced induced structure of a closed subscheme to get separated morphisms $i_X : X \to Y$, $i_Z : Z \to Y$ of finite type. We form the fiber square*

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d] & X \times_S Z \arrow[d, "\mathrm{id}_X \times_S i_Z"] \\
X \arrow[r, "\gamma_{i_X}"] & X \times_S Y
\end{tikzcd}$$

and define the refined intersection product

$$\cdot_{\textit{Fulton}} : \mathrm{CH}_k(X) \otimes \mathrm{CH}_{k'}(Z) \to \mathrm{CH}_{k+k'-n}(X \times_Y Z), \quad \alpha \otimes \beta \mapsto \gamma_{i_X}^!(\alpha \times_S \beta),$$

where

$$\gamma_{i_X}^! : \mathrm{CH}_{k+k'}(X' \times_S Y') \to \mathrm{CH}_{k+k'-n}(X' \times_Y Y')$$

is the refined Gysin homomorphism of the graph $\gamma_{i_X} : X \to X \times_S Y$. This product is associative, commutative, satisfies the projection formula and is compatible with refined Gysin homomorphisms of regular imbeddings (cf. [Ful84, Proposition 8.1.1, Section 20.2])

Proof. This follows from [Ful84, Example 20.2.1] by setting $X' = X$. □

Remark A.6.4. We emphasize, that the results of this section work in greater generality, namely for smooth separated schemes X which are of finite type over a Noetherian one-dimensional regular base scheme S (cf. [Ful84, Section 20.2]).

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