Universal geometrizations and the intrinsic eta-invariant

Dissertation zur Erlangung des Doktorgrades
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Die Arbeit wurde angeleitet von: Prof. Dr. Ulrich Bunke.

Prüfungsausschuss:

Vorsitzender:	Prof. Dr. Harald Garcke
Erst-Gutachter:	Prof. Dr. Ulrich Bunke
Zweit-Gutachter:	Prof. Dr. Sebastian Goette, Universität Freiburg
weiterer Prüfer:	Prof. Dr. Bernd Ammann
Ersatzprüfer:	Prof. Dr. Stefan Friedl

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Chapter 1

Introduction and motivation

Bordism theory is a central subject in algebraic topology. By now quite a few bordism invariants are known, e.g., characteristic numbers, the signature (for oriented bordism), the Todd genus (for *Spin*^{c}-bordism), Adams' e-invariant (for stably framed bordism), rho-invariants (for equivariant bordism) and Kreck-Stolz invariants (for some versions of *Spin*^{c}-bordism).

Recently, Bunke gave a unified construction for some of these invariants (e.g., Adams' e-invariant and Kreck-Stolz invariants): he defined the *universal eta-invariant*. This invariant is only defined on torsion elements in a particular bordism group. On the other hand some of the above invariants, e.g., some rho-invariants, are also defined on non-torsion elements. Hence it is natural to ask if the universal eta-invariant can be extended to non-torsion elements in a geometric way. The main goal of the present article is to construct a (non-canonical) extension of the universal eta-invariant to non-torsion elements (under suitable general conditions) and thus to answer this question in the affirmative.

The main technical tool we will use are *geometrizations*. They were introduced in [Bun11, Definition 4.5] to study the universal eta-invariant. To extend the universal eta-invariant to non-torsion elements we will introduce a new variant of geometrizations, which we call *universal geometrizations*. We motivate geometrizations and universal geometrizations in Section 1.1 below. The technical heart of the present article is to prove that such universal geometrizations exist in many cases (Existence Theorem 3.27). This is the content of Chapter 3 and Chapter 5 (the latter chapter contains the proof).

Given such a universal geometrization we can define the *intrinsic eta-invariant*. For this we use a formula given in [Bun11, Theorem 4.19] for the universal eta-invariant but we need to show that it is still well-defined. In [Bun11] this was unnecessary because Bunke shows that his formula calculates the well-defined universal eta-invariant. Since the formulas for universal eta-invariant and for the intrinsic eta-invariant agree on their common domain of definition it is clear that the intrinsic eta-invariant extends the universal eta-invariant. The construction and the proof of well-definedness are mostly straightforward and can be found in Chapter 6. Note that different universal geometrizations yield different extensions.

So far there is no homotopy theoretic interpretation of the intrinsic eta-invariant. As a first step in this direction we classify universal geometrizations (Classification Theorem 7.10). Then we apply this theorem to prove the Detection Theorem 7.17. This theorem shows that in some cases, e.g., if the bordism theory in question is the stably framed bordism theory with a background space, then the intrinsic eta-invariant can detect all non-torsion elements. More precisely, this means that the intrinsic eta-invariant evaluates non-trivially on non-torsion elements. Therefore the intrinsic eta-invariant detects all the rational information but also some torsion information. All this is the content of Chapter 7.

Moreover, we calculate some examples. These explicit computations can be found in Chapter 4 and Chapter 8.

In the last chapter we discuss the t-invariant of Crowley and Goette ([CG13]). The t-invariant is a generalized Kreck-Stolz invariant and so fits in the framework of the universal eta-invariant ([Bun11, Proposition 5.18]). We also prove this (Theorem 10.1,1.) using quite different tools compared to the ones applied by Bunke. Then we redo a particular computation of Crowley and Goette ([CG13, Example 3.5]) using the universal eta-invariant in a purely topological way. This serves to confirm the original calculation.

1.1 Connections, Chern-Weil theory and geometrizations

In this section we explain precisely in which sense geometrizations generalize connections. We also illustrate how one can extend connections from submanifolds using a "universal connection" on the classifying bundle. This serves as motivation for our definition of universal geometrizations. The first part follows [Bun12, Chapter 4.11] closely.

First we have to recall Chern-Weil theory. So let G be a compact Lie group and denote its semiring of isomorphism classes of unitary representations by $\mathrm{Rep}(G)$. We fix a smooth G-principal bundle $\pi : P \to M$, where M is a smooth compact manifold. The associated bundle construction yields a map $\mathrm{Rep}(G) \to \mathbf{KU}^0(M)$, where $\mathbf{KU}^0(M)$ denotes the complex K-theory of M (this is the abelian group of stable isomorphism classes of vector bundles on M). Now we choose a (principal bundle) connection ∇^P on P. Then the associated bundle construction gives a lift

$$\begin{tikzcd} & \widehat{\mathbf{KU}}^0(M) \arrow[d] \\ \mathrm{Rep}(G) \arrow[r] \arrow[ur, dotted, "\nabla^P"] & \mathbf{KU}^0(M) \end{tikzcd} ,$$

where $\widehat{\mathbf{KU}}^0(M)$ denotes the differential K-theory of M. It is given by stable isomorphism classes of vector bundles together with a hermitian metric and a metric connection up to an equivalence relation which incorporates the connection. We review differential K-theory in Section 2.4.

Next we observe that these constructions factor over the representation ring $R(G)$ (this is the group completion of $\mathrm{Rep}(G)$). Moreover, they also factor over $R(G)\widehat{_I}$, the completion at the dimension ideal (cf. [Bun11, Lemma 4.7]). The completed representation ring has also the following topological interpretation due to Atiyah, Hirzebruch and Segal. To this end we denote the classifying space for G-principal bundles by BG. Then there is an isomorphism ([AS69, Theorem 2.1]) $R(G)\widehat{_I} \cong \mathbf{KU}^0(BG)$ which is induced by the associated bundle construction using the universal G-principal bundle $EG \to BG$. Using this isomorphism we get the commutative diagram

$$\begin{tikzcd} & \widehat{\mathbf{KU}}^0(M) \arrow[d] \\ \mathbf{KU}^0(BG) \arrow[r, "p^*"] \arrow[ur, "\nabla^P"] & \mathbf{KU}^0(M) \end{tikzcd} .$$

Given a vector bundle together with a hermitian metric and a metric connection we can construct the Chern character form. This gives a map $R : \widehat{\mathbf{KU}^0}(M) \to \Omega^{\text{even}}_{\text{cl}}(M)$. Here $\Omega^{\text{even}}_{\text{cl}}(M)$ denotes closed differential forms in even degrees. On the other hand there is the topological Chern character $\mathbf{ch} : \mathbf{KU}^0(BG) \to \mathbf{H}^{\text{even}}(BG, \mathbb{Q})$. Putting everything in one big diagram gives

$$\begin{tikzcd} & \Omega^{\text{even}}_{\text{cl}}(M) \\ \mathbf{H}^{\text{even}}(BG,\mathbb{Q}) \arrow[ur, dotted, "c_{\nabla^P}"] & \widehat{\mathbf{KU}^0}(M) \arrow[u, "R"'] \arrow[d] \\ \mathbf{KU}^0(BG) \arrow[u, "\mathbf{ch}"] \arrow[ur, "\nabla^P"] \arrow[r, "p^*"] & \mathbf{KU}^0(M)\,. \end{tikzcd}$$

This diagram can be completed by the Chern-Weil homomorphism $c_{\nabla^P} : \mathbf{H}^{\text{even}}(BG, \mathbb{Q}) \to \Omega^{\text{even}}_{\text{cl}}(M)$ (recall that the rational cohomology $\mathbf{H}^{\text{even}}(BG, \mathbb{Q})$ can be identified with the space of invariant polynomials on the Lie algebra of G).

This is the prototypical example of a geometrization. These are roughly defined as follows (cf. Definition 3.7).

Pre-Definition 1.1. *Let M be a smooth compact manifold, B be a topological space and $p : M \to B$ be a continuous map. A **geometrization** for $p : M \to B$ consists of a map $\mathcal{G} : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(M)$ together with a map $c_{\mathcal{G}} : \mathbf{H}^{even}(B, \mathbb{Q}) \to \Omega^{even}_{cl}(M)$ such that the diagram*

$$\begin{tikzcd} & \Omega^{even}_{cl}(M) \\ \mathbf{H}^{even}(B,\mathbb{Q}) \arrow[ur, "c_{\mathcal{G}}"] & \widehat{\mathbf{KU}^0}(M) \arrow[u, "R"'] \arrow[d] \\ \mathbf{KU}^0(B) \arrow[u, "\mathbf{ch}"] \arrow[ur, "\mathcal{G}"] \arrow[r, "p^*"] & \mathbf{KU}^0(M) \end{tikzcd}$$

commutes.

*We refer to the map $c_{\mathcal{G}} : \mathbf{H}^{even}(B, \mathbb{Q}) \to \Omega^{even}_{cl}(M)$ as **cohomological character**.* ■

Remark 1.2. For later use we record that a connection on a principal bundle gives a geometrization on the classifying map. ■

Recall that connections can be extended from submanifolds (see below). This fails for geometrizations. A counterexample is given in Example 3.18. Nevertheless, we will need to extend geometrizations from submanifolds, e.g., from a manifold to a zero-bordism. Hence we need to ensure that our geometrizations are "nice". As motivation for this "niceness"-property we briefly review universal connections. For this we fix an embedding $G \hookrightarrow O(k)$, i.e., we realize G as a matrix group. Then we get G-principal bundles

$$E_n := O(n)/O(n-k) \to B_n := O(n)/\left(G \times O(n-k)\right)$$

for all $n \geq k$. The colimit over these bundles is a model for the universal bundle $EG \to BG$. On each of these bundles $E_n \to B_n$ there is a connection ∇^n induced by the projected Maurer-Cartan form (for details we refer to [Sch, §§ 1 and 2]). These connections are compatible with the restriction maps $B_n \to B_{n+1}$. Moreover, a theorem of Narasimhan and Ramanan (cf. [Sch, Theorem in §2]) states that this connection is universal in the following sense.

Given a G-principal bundle $P \to M$ with connection ∇^P there is a smooth map $p : M \to B_n$ for some n such that the pullback of the universal bundle (E_n, ∇^n) is isomorphic to (P, ∇^P), i.e., the map p classifies the bundle P together with the connection ∇^P. Now assume we are given a manifold M, a submanifold $S \subset M$, a G-principal bundle $P \to M$ and a connection ∇^Q on the restricted bundle $Q := P_{|S}$. We extend the connection to all of M as follows. Choose a smooth map $q : S \to B_n$ which classifies (Q, ∇^Q) as above. Then we can extend the map q along the inclusion $S \subset M$ to a smooth map $p : M \to B_{n+k}$ which classifies P (for some big enough k). Hence the connection $p^*\nabla^{n+k}$ on $P \to M$ extends the connection ∇^Q on $Q \to S$.

We give a rough definition of universal geometrization (cf. Definition 3.26).

Pre-Definition 1.3. *Let B be a topological space. A universal geometrization for B consists of*

- *a family of compact smooth manifolds $(B_n)_{n\in\mathbb{N}}$,*
- *smooth embeddings $f_n : B_n \hookrightarrow B_{n+1}$,*
- *continuous maps $g_n : B_n \to B$,*
- *geometrizations $\mathcal{G}_n$ for each map $g_n : B_n \to B$ and*
- *for all $n \in \mathbb{N}$ a homotopy filling the diagram*

$$\begin{tikzcd} B_n \arrow[rr, "f_n"] \arrow[dr, "g_n"'] & & B_{n+1} \arrow[dl, "g_{n+1}"] \\ & B & \end{tikzcd}$$

such that

- *B is the homotopy colimit of the system $(B_n, f_n)_{n\in\mathbb{N}}$ and*
- *the geometrizations fit together, i.e., $f_n^*\mathcal{G}_{n+1} = \mathcal{G}_n$.* ■

The main theorem of the present article proves that universal geometrizations exist for a suitable big class of spaces (Existence Theorem 3.27).

Remark 1.4. The above discussion shows that a universal connection for G-principal bundles yields a universal geometrization for the classifying space BG. ■

Now we indicate how a universal geometrization helps to extend geometrizations from submanifolds. So fix a space B together with a universal geometrization $(B_n, f_n, g_n, \mathcal{G}_n)_n$. Moreover, let M be a compact smooth manifold, $S \subset M$ be a compact submanifold and $p : M \to B$ be a continuous map. Then, since S is compact, we can find a smooth map $q_n : S \to B_n$ such that the diagram

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, "q_n"'] & M \arrow[d, "p"] \\ B_n \arrow[r, "g_n"] & B \end{tikzcd}$$

commutes for all $n \in \mathbb{N}$. Now we get a geometrization $q_n^*\mathcal{G}_n$ for the map $S \to B$ via pullback. This geometrization can be extended to all of M. By compactness of M we can find a smooth map (for some big enough k) $p_{n+k} : M \to B_{n+k}$ which extends q_n. Then the geometrization $p_{n+k}^*\mathcal{G}_{n+k}$ extends $q_n^*\mathcal{G}_n$.

Therefore we will mostly use geometrizations which arose via pullback from universal ones.

1.2 The universal eta-invariant

In this section we review the universal eta-invariant of [Bun11, Definition 2.3, Theorem 4.19]. Since it is a bordism invariant we start with a geometric model for bordism theories.

So we fix a space B together with a map $\sigma : B \to BSpin^c$. Here $BSpin^c$ denotes the classifying space for $Spin^c$-principal bundles. In this situation there is a generalized cohomology theory MB, called B-bordism. Next we describe cycles for this theory. These cycles represent homotopy classes in the stable homotopy group[1] $\pi^{\$}_{2n+1}(MB)$. Roughly, a cycle (M, f) consists of a closed Riemannian manifold M of dimension $2n+1$ together with a B-orientation, i.e., a map $f : M \to B$ such that the diagram

$$\begin{tikzcd} & B \arrow[r, "\sigma"] & BSpin^c \arrow[d] \\ M \arrow[ur, "f"] \arrow[rr, "TM"] & & BO \end{tikzcd}$$

commutes (up to fixed homotopy). Here $TM : M \to BO$ denotes a map classifying the stable tangent bundle of M. For a precise definition of cycles for B-bordism see Definition 6.6. Two cycles represent the same homotopy class if and only if they are bordant by a compact manifold (see Definition 6.8).

Observe that the map $M \to B \to BSpin^c$ endows M with a (topological) $Spin^c$-structure and hence with a Dirac operator[2] $\not{D}_M$. A (twisted) Dirac operator $\not{D}$ has an associated spectral invariant, called *eta-invariant*, which was introduced by Atiyah, Patodi and Singer ([APS75a]). To define the eta-invariant we denote the spectrum of the operator $\not{D}$ as $\mathrm{spec}(\not{D})$ and define the eta-function by (sum with multiplicity)

$$\eta(\not{D}, s) := \sum_{\lambda \in \mathrm{spec}(\not{D}) \setminus \{0\}} \mathrm{sgn}(\lambda)|\lambda|^{-s} \ .$$

The eta-function is holomorphic for $s \in \mathbb{C}$ with real part $\Re(s) \gg 0$ and has a meromorphic continuation to the complex plane. It turns out that the continuation is finite and real-valued at $s = 0$ and one defines the eta-invariant as the value at $s = 0$, i.e., $\eta(\not{D}) := \eta(\not{D}, 0)$. We will often use the reduced eta-invariant $\xi(\not{D}) := \frac{\eta(\not{D}) - \dim\ker(\not{D})}{2} \in {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$. The details are discussed in Section 6.5.

Using this reduced eta-invariant we can give a first, but wrong, definition of the universal eta-invariant. For this denote the complex K-theory of B by $\mathbf{KU}^0(B)$ and define[3]

$$\begin{aligned} \tilde{\eta}_{\mathrm{uni}}(M, f) : \mathbf{KU}^0(B) &\to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \\ [V \to B] &\mapsto -\xi(\not{D}_M \otimes f^*V) \ . \end{aligned}$$

But this is not well-defined for several reasons. One of these reasons is that we need a connection on the vector bundle f^*V to get a well-defined twisted Dirac operator $\not{D}_M \otimes f^*V$.

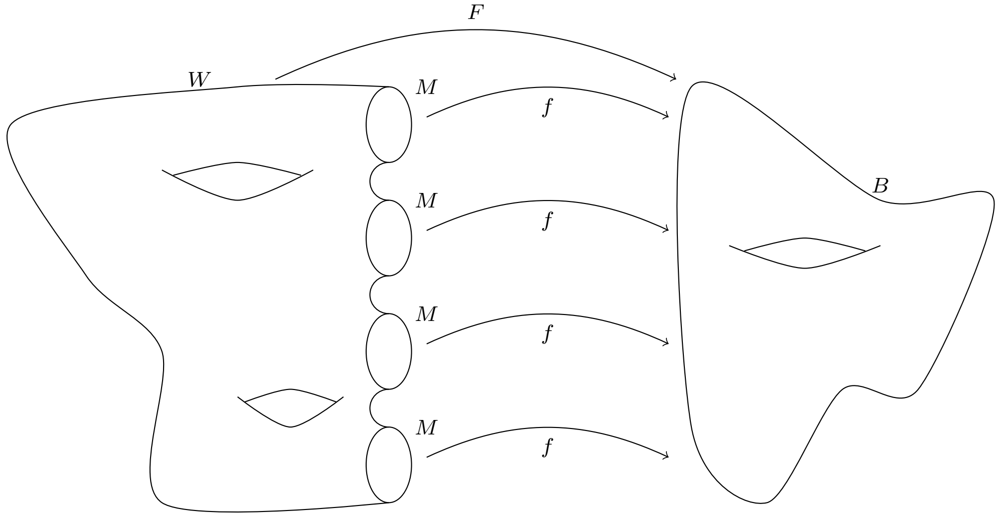
[1] In the literature the notation Ω^B_{2n+1} is also often used instead of $\pi^{\$}_{2n+1}(MB)$. To avoid confusion with $\Omega^*(M)$ (differential forms) we use the latter notation.

[2] To be precise, one has to choose geometric data to get the Dirac operator. These choices are always possible and two such choices can be connected on the cylinder. Thus the bordism class is independent of the geometric data. For the following discussion we will always fix such a geometric structure.

[3] Note that not every element $\phi \in \mathbf{KU}^0(B)$ can be represent by a (graded) vector bundle if B is not a finite CW-complex. But M is a compact manifold and thus the element $f^*\phi \in \mathbf{KU}^0(M)$ can be represented by a (graded) vector bundle. To simplify this introduction we will pretend that we can represent K-theory classes on B by vector bundles. A correct construction of both the universal eta-invariant and the intrinsic one can be found in Chapter 6.

The idea is to choose any connection ∇^{f^*V} and add a correction term to get a well-defined universal eta-invariant.

Next we give a correct definition. To this end we assume that the cycle (M, f) represents a torsion element in the bordism group $\pi^{\$}_{2n+1}(MB)$ of order l. Then there is a bordism (W, F) where W is a Riemannian manifold with Riemannian boundary $\partial W \cong \coprod_l M$ and $F : W \to B$ is a B-orientation extending the one on M (for a precise statement see Definition 6.8) .



Picture of a torsion cycle in B-bordism for $l = 4$

Note that the vector bundle F^*V extends the vector bundle f^*V. Since one can always extend connections we can find a connection ∇^{F^*V} on F^*V extending ∇^{f^*V} on each boundary component.

To define the universal eta-invariant we choose a geometrization $\mathcal{G}_W$ for $F : W \to B$ (these always exist by Example 3.12). Note that this gives a geometrization $\mathcal{G}_M := \mathrm{incl}^* \mathcal{G}_W$ for $f : M \to B$ where $\mathrm{incl} : M \to W$ is (one of) the boundary inclusions. So we get two differential K-theory classes, $\mathcal{G}_M([V \to B]), [f^*V, \nabla^{f^*V}] \in \widehat{\mathbf{KU}^0}(M)$, which refine the same ordinary K-theory class $[f^*V] \in \mathbf{KU}^0(M)$. Now we apply the long exact sequence

$$\ldots \to \mathbf{KU}^{-1}(M) \xrightarrow{\mathbf{ch}} \Omega^{\mathrm{odd}}(M, \mathbb{R})/(\mathrm{im}\, d) \xrightarrow{a} \widehat{\mathbf{KU}^0}(M) \to \mathbf{KU}^0(M) \to 0$$

to get a differential form $\gamma_V^M \in \Omega^{\mathrm{odd}}(M, \mathbb{R})/(\mathrm{im}\, d)$, called *correction form*. This form is well-defined up to the image of the Chern character $\mathbf{ch}$ and satisfies $a(\gamma_V^M) = \mathcal{G}_M([V]) - [f^*V, \nabla^{f^*V}] \in \widehat{\mathbf{KU}^0}(M)$. We set (see [Bun11, Theorem 4.19])

$$\begin{aligned} \tilde{\eta}_{\mathrm{uni}}(M, f) &: \mathbf{KU}^0(B) \to \mathbb{R}/\mathbb{Z} \\ [V \to B] &\mapsto -\xi\left(\not{D}_M \otimes (f^*V, \nabla^{f^*V})\right) - \left[\int_M \mathbf{Td}_M \wedge \gamma_V^M\right]_{\mathbb{R}/\mathbb{Z}} \end{aligned} \tag{1.1}$$

and observe that this does not depend on the differential form representative of γ_V^M because integrals of the form $\int_M \mathbf{Td}_M \wedge \mathbf{ch}([\tilde{V}])$ have values in $\mathbb{Z}$ by the odd version of the Atiyah-Singer index theorem (here $[\tilde{V} \to M] \in \mathbf{KU}^0(M)$ is arbitrary). We stress that this formula depends on the geometrization $\mathcal{G}_M$ for $f : M \to B$ via the correction form γ_V^M. Bunke shows that this formula is a well-defined bordism invariant (see [Bun11, Proposition 3.4]). Here it is used that the geometrization extends to the bordism (W, F) (this is not automatic, see Example 3.18). So we define the universal eta-invariant by

$$\begin{aligned} \eta_{\mathrm{uni}} : \operatorname{Torsion}(\pi^{\$}_{2n+1}(MB)) &\to \operatorname{Hom}(\mathbf{KU}^0(B), {}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}})/\,\mathrm{U}^{\mathbb{R}}_{2n+1}(B) \\ [M, f] &\mapsto \tilde{\eta}_{\mathrm{uni}}(M, f) \ . \end{aligned}$$

Observe that one has to pass to the quotient with respect to a subgroup $\mathrm{U}^{\mathbb{R}}_{2n+1}(B) \subset \operatorname{Hom}(\mathbf{KU}^0(B), {}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}})$ to get bordism invariance. The precise definition of this subgroup does not matter for the moment and so we refer to Section 6.2 for details.

1.3 The intrinsic eta-invariant

In the previous section we gave a formula for the universal eta-invariant. This formula has two related drawbacks:

- We need that the geometrization extends along a bordism.
- This implies that we can define the universal eta-invariant only on torsion classes in the bordism group.

To solve these problems we want to choose some additional data which yield geometrizations on all geometric cycles such that these geometrizations can be extended to bordisms. Then we will define the intrinsic eta-invariant by the above formula (1.1) where we only use these induced geometrizations.

Now we need to find a suitable class of additional data. The first idea is to just choose a geometrization for each geometric cycle (M, f) such that these extend along bordisms. But this is very complicated because there are many bordisms and thus one has to satisfy many relations. So we want to restrict to a smaller set of relations. For this we assume that we can find an approximation by manifolds for B, i.e.,

- a family of compact smooth manifolds $(B_n)_{n\in\mathbb{N}}$,
- smooth embeddings $f_n : B_n \hookrightarrow B_{n+1}$,
- continuous maps $g_n : B_n \to B$ and
- for all $n \in \mathbb{N}$ a homotopy filling the diagram

$$\begin{tikzcd} B_n \arrow[rr, hook, "f_n"] \arrow[dr, "g_n"'] & & B_{n+1} \arrow[dl, "g_{n+1}"] \\ & B & \end{tikzcd}$$

such that B is the (homotopy) colimit of the system $(B_n, f_n)_{n\in\mathbb{N}}$. Note that Proposition 3.24 gives an approximation by manifolds for a suitable big class of spaces. The idea is to construct geometrizations $\mathcal{G}_i$ for $g_i : B_i \to B$. Then we get a geometrizations $\mathcal{G}_M$ for a given geometric cycle $(M, f : M \to B)$ as follows. Since M is compact we can find a (homotopy) lift $\hat{f}_i : M \to B_i$ of f along g_i. This gives the geometrizations $\mathcal{G}_M := \hat{f}_i^* \mathcal{G}_i$ for f. Now assume that we have a zero-bordism (W, F) of (M, f). But W is also compact and so we find a lift $\hat{F}_{i+k} : W \to B_{i+k}$ of F along g_{i+k}. Thus we get a geometrization $\mathcal{G}_W := \hat{F}^*_{i+k}\mathcal{G}_{i+k}$

for F. This geometrization $\mathcal{G}_W$ extends the geometrization $\mathcal{G}_M$ if $\mathcal{G}_{i+k}$ extends $\mathcal{G}_i$. So these considerations lead us to the concept of a universal geometrization, Pre-Definition 1.3.

Finally we define the intrinsic eta-invariant. For this we fix a universal geometrization $(B_i, f_i, g_i, \mathcal{G}_i)_{i\in\mathbb{N}}$. Then we define

$$\begin{aligned} \eta_{\text{intrinsic}} : \pi^{\$}_{2n+1}(MB) &\to \operatorname{Hom}(\mathbf{KU}^0(B), {}^{\mathbb{R}}/_{\mathbb{Z}})/\, \mathrm{U}^{\mathbb{R}}_{2n+1}(B) \\ [M,f] &\mapsto \tilde{\eta}_{\text{intrinsic}}(M,f) \end{aligned}$$

by formula (1.1),

$$\begin{aligned} \tilde{\eta}_{\text{intrinsic}}(M,f) : \mathbf{KU}^0(B) &\to {}^{\mathbb{R}}/_{\mathbb{Z}} \\ [V \to B] &\mapsto -\xi\left(\not{D}_M \otimes (f^*V, \nabla^{f^*V})\right) - \left[\int_M \mathbf{Td}_M \wedge \gamma^M_V\right]_{\mathbb{R}/\mathbb{Z}} \end{aligned}$$

where we use the geometrization $\mathcal{G}_M := \hat{f}_i^*\mathcal{G}_i$ to compute the correction form γ^M_V. It turns out that this is also well-defined for non torsion classes in $\pi^{\$}_{2n+1}(MB)$ (cf. Chapter 6).

1.4 Dependence of the universal geometrizations

In the previous section we defined the intrinsic eta-invariant. This definition relies on the choice of a universal geometrization. So the next natural question is existence and uniqueness of those universal geometrization.

We have the following result regarding existence (this is a combination of Existence Theorem 3.27, Remark 3.28, Proposition 3.24 and Section 4.6).

Theorem 1.5. *Let B be a topological space which is simply-connected, has countably many connected components and has countable homotopy groups. Also assume that the complex K-theory $\mathbf{KU}^0(B)$ is (topologically) finitely generated. Then there exists a universal geometrization for B.* ◆

The conditions of this theorem are quite general. For example, this theorem applies to a situation studied by Crowley and Goette. We discuss this in Chapter 10.

We briefly comment on the proof. The proof consists of two steps. In the first step we construct the data $(B_i, f_i, g_i)_{i\in\mathbb{N}}$ (see Pre-Definition 1.3). The second step is much harder. There we have to construct geometrizations $\mathcal{G}_i$ for each g_i such that these geometrizations are compatible, i.e., $f_i^*\mathcal{G}_{i+1} = \mathcal{G}_i$. To this end we have to solve a non-trivial lifting problem. Here we use some tools from algebra, e.g, derived limits and the Smith normal form. This harder part is the content of Chapter 5.

Also, we have a complete classification of universal geometrizations for B which we give in Section 3.4. Moreover, we introduce an equivalence relation between universal geometrizations for B, and we show that the intrinsic eta-invariants associated to equivalent universal geometrizations agree (Theorem 7.13). The classification of equivalence classes of universal geometrization is given in Theorem 7.10. The result is that the set of isomorphism classes is a torsor for some group and this group is in fact a homotopy invariant for the space B. Combining these results gives an action of this group on the intrinsic eta-invariant which we make explicit in Proposition 7.15.

The above classification results are then used to prove our Detection Theorem 7.17,

Detection Theorem 7.17. *Fix a space B with $\sigma : B \to BSpin^c$ which allows universal geometrizations. Moreover, suppose $\sigma^*\,\mathbf{td} = 1 \in \mathbf{H}^*(B,\mathbb{Q})$ (e.g., σ is trivial). Fix $k \in \mathbb{N}_0$.*

Then $x \in \pi^{\$}_{2k+1}(MB)$ is torsion if and only if it satisfies $\eta^{(\mathcal{G}^1_i)}_{intrinsic}(x) = \eta^{(\mathcal{G}^2_i)}_{intrinsic}(x)$ for each pair of universal geometrizations $(\mathcal{G}^1_i)_i$ and $(\mathcal{G}^2_i)_i$ for B.

Thus the combination of all intrinsic eta-invariants for all (equivalence classes of) universal geometrizations detects all non-torsion elements in $\pi^{\$}_{odd}(MB)$. ♦

We also stress that the universal eta-invariant is not functorial in B. A counterexample is given in Chapter 9. But it is functorial for weak homotopy equivalences (Remark 7.14).

1.5 Outline

This article is structured as follows.

In Chapter 2 we introduce the basic tools which we will use often. These are Moore spectra, differential K-theory and the profinite topology. The latter is a natural topology on the (generalized) cohomology of a space. It is used to reduce some problems from infinite CW-complexes to finite ones.

Next, in Chapter 3, we recall the definition of geometrizations which were introduced in [Bun11, Definition 4.5]. To this end we also define *cohomological characters* which generalize the Chern-Weil map of Section 1.1. Then we give our definition of universal geometrizations and state our Existence Theorem 3.27. Moreover, we classify each of these structures.

Chapter 4 gives first examples of universal geometrizations. Here we also employ the classification results of Chapter 3.

The following Chapter 5 contains the proof of our Existence Theorem 3.27. This is the technical heart of the present article.

Finally we define the intrinsic eta-invariant in Chapter 6. For this we recall Thom spectra, the universal eta-invariant of [Bun11, Definition 2.3] and the Atiyah-Patodi-Singer index theorem. Then we give an adapted model for bordism which we will play an important role in the remaining part of this article. Moreover, we need a version of universal geometrizations which incorporates the $Spin^c$-structures at hand. These are called *universal $Spin^c$-geometrizations.* Once these prerequisites are completed we define the intrinsic eta-invariant and show that it is well-defined and extends the universal eta-invariant. This chapter ends with a first exemplary computation of the intrinsic eta-invariant which shows that the intrinsic eta-invariant detects strictly more information than the universal eta-invariant and hence is an honest extension of the latter.

We proceed with the classification of universal $Spin^c$-geometrizations. This classification will then allow us to prove Detection Theorem 7.17, which shows that the intrinsic eta-invariant detects in some cases, e.g., if the bordism theory in question is the stably framed bordism theory with a background space, all the rational information.

Afterwards, in Chapter 8, we explicitly compute the intrinsic eta-invariants in some examples.

Then, in Chapter 9, we discuss an example which shows that the intrinsic eta-invariant is not functorial. This is a reinterpretation of [Bun11, Remark 5.21].

The last chapter, Chapter 10, is maybe the most interesting one. There we discuss the t-invariant of Crowley and Goette ([CG13]) in an example. We show that it can be computed using the universal eta-invariant (this is also proven in [Bun11, Proposition 5.18]) and do so using only topological methods. This is, to the best of our knowledge, the first independent check of [CG13, Example 3.5].

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Chapter 2

Notations and conventions

In this chapter we give our conventions and introduce basic tools we will use later.

2.1 Conventions concerning manifolds

A manifold is always smooth but is allowed to have boundary. An embedding will always mean a neat embedding in the sense of [Hir94, page 30]. We try to avoid manifolds with corners. Hence a homotopy will be parametrized by the real line $\mathbb{R}$ instead of the interval I and be required to be locally constant on $\mathbb{R} \setminus (\varepsilon, 1-\varepsilon)$ for some $\varepsilon \in (0, 1/2)$. A similar convention will be used for paths of connections.

Finally we recall that a compact smooth manifold has always the homotopy type of a finite CW-complex ([Hir94, Chapter 6, Theorem 4.1]).

2.2 The profinite topology

In this article we are interested in the generalized cohomology of infinite spaces. We equip the cohomology groups of a space with the profinite topology as follows. Let B be a topological space and E be a (generalized) cohomology theory, e.g., complex K-theory, a bordism theory or ordinary integral cohomology. Then we equip $E^*(B)$ with the profinite topology to get a graded topological abelian group. This topology has as neighbourhood basis of the identity element the kernels of $E^*(B) \to E^*(C)$ for all continuous maps $C \to B$ whose domain is a finite CW-complex C. Basic properties can be found in [Boa95, Chapters 3 and 4]. Here we just recall the following facts.

Definition 2.1. *Let $\mathcal{C}/B$ be the category with*

- *the objects are finite CW-complexes $C \to B$ with a map to B and*
- *the morphisms are commutative triangles over B,*

and let $\mathcal{C}_{\mathrm{ho}}/B$ be the category with

- *the objects are finite CW-complexes $C \to B$ with a map to B and*
- *the morphisms are homotopy commutative triangles over B where the homotopy is only required to exist and not part of the data.* ■

Note that for a homotopy invariant functor $\mathcal{F}$ from topological spaces to some complete target category the natural map $\varprojlim_{C\to B\in\mathcal{C}_{\mathrm{ho}}/B}\mathcal{F}(C)\to\varprojlim_{C\to B\in\mathcal{C}/B}\mathcal{F}(C)$ is an isomorphism.

Proposition 2.2. *The Hausdorff completion of the topological abelian group $E^*(B)$ is $\varprojlim_{C\to B\in\mathcal{C}/B}E^*(C)$. This implies the following.*

1. *The Hausdorff completion map $E^*(B)\to\varprojlim_{C\to B\in\mathcal{C}/B}E^*(C)$ is surjective.*

2. *Let G denote a discrete group. A group homomorphism $E^*(B)\to G$ is continuous if and only if it factors over $\varprojlim_{C\to B\in\mathcal{C}/B}E^*(C)$, the Hausdorff completion.* ◆

Similarly, we define a topology on $E^*(F)$ where E,F are now both spectra (representing generalized cohomology theories). Here one filters by finite CW-spectra.

2.3 Moore spectra

We will often introduce coefficients into a generalized cohomology theory. For this we use Moore spectra. A basic reference is [Ada74, Chapter III.6].

Given an abelian group G there is a spectrum $\mathcal{M}\,G$, called **Moore spectrum**, which is characterized by its integral homology groups,

$$\mathbf{H}\,\mathbb{Z}_*(\mathcal{M}\,G)=\begin{cases} G, & *=0 \quad ,\\ 0, & \text{else} \quad .\end{cases}$$

For a spectrum E and an abelian group G we define $EG:=E\wedge\mathcal{M}\,G$ and refer to it as E-cohomology with coefficients in G. This is justified by the following version of the universal coefficient theorem.

Proposition 2.3. *Let E be a spectrum and G be an abelian group. Then there exists a short exact sequence of abelian groups*

$$0\to\pi_*^{\$}(E)\otimes G\to\pi_*^{\$}(EG)\to\mathrm{Tor}(\pi_{*-1}^{\$}(E),G)\to 0\ . \qquad ◆$$

In particular, there are isomorphisms $\mathcal{M}\,\mathbb{Z}\cong\$$, the sphere spectrum, and $\mathcal{M}\,\mathbb{Q}\cong\mathbf{H}\,\mathbb{Q}$, rational cohomology. Hence we have $E\mathbb{Z}\cong E$ and $E\mathbb{Q}\cong E\wedge\mathbf{H}\,\mathbb{Q}$. Also observe that $\mathbf{H}\,G:=\mathbf{H}\,\mathbb{Z}\wedge\mathcal{M}\,G$ is ordinary cohomology with coefficients in G.

Moreover, there is a cofibre sequence $\mathcal{M}\,\mathbb{Z}\to\mathcal{M}\,\mathbb{Q}\to\mathcal{M}\,\mathbb{Q}/\mathbb{Z}$ of spectra. Since smash products preserve cofibre sequences we get a natural cofibre sequence $E\to E\mathbb{Q}\to E\mathbb{Q}/\mathbb{Z}$ for any spectrum E. This also holds with $\mathbb{Q}$ replaced by $\mathbb{R}$.

2.4 Differential K-theory

In this section we briefly review differential K-theory.

At first we have to introduce periodic cohomology and periodic differential forms.

Periodic ordinary cohomology with coefficients in the abelian group G is defined to be the spectrum

$$\mathbf{HP}G:=\mathbf{H}G[b,b^{-1}]$$

where b has cohomological degree -2. There is a decomposition $\mathbf{HP}G\cong\vee_{i\in\mathbb{Z}}\Sigma^{2i}\mathbf{H}G$ which yields projections $\mathrm{pr}_n:\mathbf{HP}G\to\Sigma^{2n}\mathbf{H}G$ and inclusions $\mathrm{incl}_n:\Sigma^{2n}\mathbf{H}G\to\mathbf{HP}G$. We say that elements in $\Sigma^{2i}\mathbf{H}G^k(M) \subset \mathbf{HP}G^{2i+k}(M)$ have **internal degree** k. We will consider the Chern character as an isomorphism of ring spectra[1] $\mathbf{ch} : \mathbf{KU}\mathbb{Q} \xrightarrow{\cong} \mathbf{HP}\mathbb{Q}$.

Periodic differential forms are defined by

$$\Omega\mathrm{P}(M) := \Omega(M)[b, b^{-1}]$$

where b has cohomological degree -2. We consider $\Omega\mathrm{P}(M)$ as a cochain complex using the ordinary differential d (and not $b \cdot d$). So we get a graded subgroup $\Omega\mathrm{P}_{\mathrm{cl}}(M) \subset \Omega\mathrm{P}(M)$ of closed forms. As above, we say that elements in $b^i\Omega^k(M) \subset \Omega\mathrm{P}^{2i+k}(M)$ have **internal degree** k. This fits with the definition in cohomology, i.e., the **de Rham map**

$$\mathbf{Rham} : \Omega\mathrm{P}^*_{\mathrm{cl}}(M) \to \mathbf{HP}\mathbb{R}^*(M)$$

preserves internal degree.

Finally we review differential K-theory. There are many different differential refinements of K-theory. We will use a version which satisfies the uniqueness theorem of Bunke and Schick [BS10, Corollary 4.4]. An explicit construction of a model can be found in [BS09b]. In fact, since we use only the degree zero part $\widehat{\mathbf{KU}^0}$ the Simons-Sullivan model [SS10] is also appropriate.

For our purpose differential K-theory consists of a functor

$$\widehat{\mathbf{KU}^0} : \{\text{smooth manifolds}\} \to \{\text{commutative rings}\}\ ,$$

two natural transformations of commutative ring valued functors

$$I : \widehat{\mathbf{KU}^0} \to \mathbf{KU}^0 \quad \text{and} \quad R : \widehat{\mathbf{KU}^0} \to \Omega\mathrm{P}^0_{\mathrm{cl}}$$

and a natural transformation of abelian group valued functors

$$a : \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) \to \widehat{\mathbf{KU}^0}$$

such that the following diagram commutes for all smooth manifolds M,

$$\begin{tikzcd}
 & \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega\mathrm{P}^0_{\mathrm{cl}}(M) \arrow[dr, "\mathbf{Rham}"] & \\
\mathbf{HP}\mathbb{R}^{-1}(M) \arrow[ur] \arrow[dr] & & \widehat{\mathbf{KU}^0}(M) \arrow[ur, "R"] \arrow[dr, "I"] & & \mathbf{HP}\mathbb{R}^0(M) \\
 & \mathbf{KU}\mathbb{R}/\mathbb{Z}^{-1}(M) \arrow[rr, "\text{Bockstein}"] \arrow[ur] & & \mathbf{KU}^0(M) \arrow[ur, "\mathbf{ch}"] &
\end{tikzcd}\ .$$

In this diagram the lower sequence, the upper sequence and both diagonals are part of long exact sequences. For example, there is the long exact sequence

$$\ldots \to \mathbf{KU}^{-1}(M) \xrightarrow{\mathbf{ch}} \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) \xrightarrow{a} \widehat{\mathbf{KU}^0}(M) \xrightarrow{I} \mathbf{KU}^0(M) \to 0\ . \tag{2.1}$$

[1] We stress that the Chern character $\mathbf{KU}^0(B) \otimes \mathbb{Q} \to \mathbf{HP}\mathbb{Q}^0(B)$ is in general *not* an isomorphism of abelian groups (see Example 3.9).

Moreover, the map $\mathbf{KU}^{\mathbb{R}/\mathbb{Z}^{-1}}(M) \to \widehat{\mathbf{KU}}^0(M)$ is injective and identifies $\mathbf{KU}^{\mathbb{R}/\mathbb{Z}^{-1}}(M)$ with the kernel of R. We will also use the long exact sequence

$$\ldots \to \mathbf{KU}^{-1}(M) \stackrel{\mathbf{ch}}{\to} \mathbf{HP}\mathbb{R}^{-1}(M) \stackrel{a}{\to} \widehat{\mathbf{KU}}^0(M) \xrightarrow{(I,R)} \mathbf{KU}^0(M) \oplus \Omega\mathrm{P}^0_{\mathrm{cl}}(M) \xrightarrow{\mathbf{ch}-\mathbf{Rham}} \mathbf{HP}\mathbb{R}^0(M) \to 0\ . \tag{2.2}$$

So far the data we described fix only $\widehat{\mathbf{KU}}^0(M)$ but don't fix the differential refinement completely. But this is enough for our purpose and so we refrain from making the remaining data explicit. They are given, for example, in [BS09b]. Instead we recall geometric vector bundles and their relation to differential K-theory.

Definition 2.4. *A* ***geometric vector bundle*** $\mathbf{V} := (V, h^V, \nabla^V)$ *on a smooth manifold M consists of a smooth complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V \to M$ together with a hermitian metric h^V on V and a metric connection ∇^V which both are compatible with the grading.* ■

A geometric vector bundle $\mathbf{V}$ yields a class $[\mathbf{V}] \in \widehat{\mathbf{KU}}^0(M)$ such that

$$I([\mathbf{V}]) = [V] \in \mathbf{KU}^0(M) \quad \text{and} \quad R([\mathbf{V}]) = \mathbf{ch}(\nabla^V) \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$$

where $\mathbf{ch}(\nabla^V) := \mathbf{Tr}\left(\mathbf{exp}\left(\frac{-bR^{\nabla^V}}{2\pi i}\right)\right)$ is the normalized Chern character form. Here R^{∇^V} is the curvature of the connection and the Chern character form is a real form because the connection is metric.

Definition 2.5. *A geometric vector bundle (V, h^V, ∇^V) is called* ***trivial geometric vector bundle*** *if there is an isomorphism $V \cong M \times \mathbb{C}^n \oplus M \times \mathbb{C}^m$ and the metric and the connection are the trivial ones (using this isomorphism).* ■

A trivial geometric vector bundle of virtual rank zero represents the identity element in the abelian group $\widehat{\mathbf{KU}}^0(M)$.

Differential K-theory is not homotopy invariant, but there is the **homotopy formula**,

$$\mathrm{incl}_1^* x - \mathrm{incl}_0^* x = a\left(\int_{I\times M/M} R(x)\right)\ , \tag{2.3}$$

where $x \in \widehat{\mathbf{KU}}^*(M \times I)$ and incl_0 and incl_1 are the boundary inclusions $M \to M \times I$.

Chapter 3

Universal geometrizations

In this chapter we introduce (universal) geometrizations. To this end we first discuss cohomological characters. Then we study geometrizations and universal geometrizations. Our main result is Existence Theorem 3.27 which states that universal geometrizations exist for a large class of spaces. We end this chapter with the classification of (universal) geometrizations.

Note that the concepts of geometrizations and cohomological characters are due to Bunke [Bun11]. On the other hand, the concept of universal geometrizations is new.

3.1 Cohomological characters

Let $p : M \to B$ denote a continuous map where M denotes a compact smooth manifold and B a topological space.

Definition 3.1. *A* ***cohomological character*** *for $p : M \to B$ is a map*

$$c : \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$$

such that

- *it preserves the internal degree,*
- *it is a group homomorphism,*
- *it is continuous (with respect to the profinite topology on $\mathbf{HP}\mathbb{Q}^0(B)$ and the discrete topology on $\Omega\mathrm{P}^0_{\mathrm{cl}}(M)$) and*
- *the diagram*

$$\begin{tikzcd} & \Omega\mathrm{P}^0_{\mathrm{cl}}(M) \arrow[d, "\mathbf{Rham}"] \\ \mathbf{HP}\mathbb{Q}^0(B) \arrow[ur, "c"] \arrow[r, "p^*"'] & \mathbf{HP}\mathbb{R}^0(M) \end{tikzcd}$$

commutes. ■

The internal degrees of $\mathbf{HP}\mathbb{Q}^0(B)$ and $\Omega\mathrm{P}^0_{\mathrm{cl}}(M)$ have been defined in Section 2.4.

Remark 3.2. We have the following easy results regarding functoriality in M and B.

- Given a smooth map $f : N \to M$ of smooth compact manifolds and a cohomological character $c : \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$ for $p : M \to B$ we define the pullback-character f^*c as $f^*c := f^* \circ c : \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(N)$.
  Note that this construction is contravariant.

- Given a continuous map $\alpha : B \to \tilde{B}$ of topological spaces and a cohomological character $c : \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$ for $p : M \to B$ we define the pushforward-character $\alpha_* c$ as $\alpha_* c := c \circ \alpha^* : \mathbf{HP}\mathbb{Q}^0(\tilde{B}) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$.
  Note that this construction is covariant.

- Also note that the definition of a cohomological character depends only on the homotopy class of p and the weak homotopy type of B. ■

Example 3.3. The following example shows existence of cohomological characters for any map $p : M \to B$. By the remarks above it is enough to construct a cohomological character for $\mathtt{id}_M : M \to M$ (considering the right M as a topological space). Then we get a cohomological character for $p : M \to B$ via pushforward along p. But a cohomological character for $\mathtt{id}_M : M \to M$ is induced by the choice of homogeneous (w.r.t. internal degree) generators for $\mathbf{HP}\mathbb{Q}^0(M)$ and homogeneous representatives in $\Omega\mathrm{P}^0_{\mathrm{cl}}(M)$ for those. These choices always exist. Observe that continuity is automatic since M is a compact manifold and thus everything is discrete. ★

The next proposition shows that we can find compatible cohomological characters for a system of embeddings.

Proposition 3.4. *Let $(M_i)_{i\in\mathbb{N}}$ be a sequence of compact manifolds, $f_i : M_i \hookrightarrow M_{i+1}$ be smooth embeddings and $g_i : M_i \to B$ be continuous maps to a topological space B such that $g_{i+1} \circ f_i$ and g_i are homotopic ($g_{i+1} \circ f_i \simeq g_i$) for all $i \in \mathbb{N}$.*

Then we can find cohomological characters c_i for g_i satisfying the compatibility condition $f_i^* c_{i+1} = c_i$ for all $i \in \mathbb{N}$. ♦

Proof. The idea is the following: Since $\mathbf{HP}\mathbb{Q}^0(B)$ might be non-discrete we define

$$X^i := \operatorname{im}(g_i^* : \mathbf{HP}\mathbb{Q}^0(B) \to \mathbf{HP}\mathbb{R}^0(M_i))$$

which are discrete $\mathbb{Q}$-vector spaces of finite dimension. We construct group homomorphisms $c_i : X^i \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M_i)$ which preserve the internal degree such that $\mathbf{Rham} \circ c_i = \mathtt{id}_{X^i}$ and $f_i^* \circ c_{i+1} = c_i \circ f_{i+1}^*$. Then the "pushforwards" $\tilde{c}_i := c_i \circ g_i^*$ are cohomological characters for $g_i : M_i \to B$ satisfying $f_i^* \tilde{c}_{i+1} = \tilde{c}_i$.

Note that the groups $\Omega\mathrm{P}^0_{\mathrm{cl}}(M_i)$, $\Omega\mathrm{P}^{-1}(M_i)$ and $X^i \subset \mathbf{HP}\mathbb{R}^0(M_i)$ are graded by internal degree. In the following the term "homogeneous" will always refer to this grading.

To define the maps $c_i : X^i \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M_i)$ we choose homogeneous bases of each X^i and inductively choose homogeneous closed differential forms in $\Omega\mathrm{P}^0_{\mathrm{cl}}(M_i)$ representing the basis elements in a compatible way. This works as follows.

We define c_1 by just taking a homogeneous basis of X^1 and choosing homogeneous closed differential forms in $\Omega\mathrm{P}^0_{\mathrm{cl}}(M_1)$ representing the basis elements.

So assume by induction that $c_1, \dots, c_n$ are already defined and compatible. We take a homogeneous basis $e_1, \dots, e_{\dim X^{n+1}}$ of X^{n+1} and homogeneous closed differential forms $\omega_1, \dots, \omega_{\dim X^{n+1}}$ in $\Omega\mathrm{P}^0_{\mathrm{cl}}(M_{n+1})$ representing the basis elements. Then the differences $f_n^*\omega_k - c_n(f_n^* e_k)$ are exact for all $k \in \{1, \dots, \dim X^{n+1}\}$ and we fix homogeneous forms $\delta_k \in \Omega\mathrm{P}^{-1}(M_n)$ such that $d\delta_k = f_n^*\omega_k - c_n(f_n^* e_k)$.

Now we use that the maps f_i are embeddings to find homogeneous extensions $\tilde{\delta}_k \in \Omega\mathrm{P}^{-1}(M_{n+1})$ with $f_n^*\tilde{\delta}_k = \delta_k$ for all k. We define $c_{n+1} : X^{n+1} \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M_{n+1})$ by $c_{n+1}(e_k) = \omega_k - d\tilde{\delta}_k$ and extend $\mathbb{R}$-linearly.

Hence we get well-defined maps $c_i : X^i \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M_i)$ which are compatible. □

We end this section with the classification of cohomological characters. To this end we introduce the group of continuous internal degree preserving group homomorphisms $\mathrm{Hom}^{\mathrm{cont}}_{[0]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M)\big)$ where $\Omega\mathrm{P}^0_{\mathrm{ex}}(M)$ denotes the exact differential forms (with the discrete topology).

Proposition 3.5. *1. For a fixed map $p : M \to B$ the set of cohomological characters is a non-empty torsor over* $\mathrm{Hom}^{\mathrm{cont}}_{[0]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M)\big)$.

2. For a system of embeddings $\cdots \overset{f_{i-1}}{\hookrightarrow} M_i \overset{f_i}{\hookrightarrow} M_{i+1} \overset{f_{i+1}}{\hookrightarrow} \cdots$ over B as in Proposition 3.4 the set of compatible cohomological characters is a non-empty torsor over $\varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[0]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M_i)\big)$. ♦

Proof. We start with 1.

We know by Example 3.3 that there exists a cohomological characters. So assume we have two cohomological characters $c^1, c^2\colon \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$. Then the difference $\xi := c^1 - c^2$ satisfies $\mathbf{Rham} \circ \xi = 0$ and thus $\xi \in \mathrm{Hom}^{\mathrm{cont}}_{[0]}(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M))$. Clearly, for c a cohomological character and $\xi \in \mathrm{Hom}^{\mathrm{cont}}_{[0]}(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M))$ the sum $c + \xi$ is again a cohomological character.

This proves 1. and we proceed with 2.

Again, there is a compatible family of cohomological characters by Proposition 3.4. So assume we have two different compatible families $(c^1_i)_i$ and $(c^2_i)_i$ of cohomological characters. Then the difference $\xi_i := c^1_i - c^2_i$ is exact (lies in $\mathrm{Hom}^{\mathrm{cont}}_{[0]}(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M_i))$) and satisfies also the compatibility condition, i.e., $f_i^* \circ \xi_{i+1} = \xi_i$. Thus

$$\xi = (\xi_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[0]}(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(M_i)) \ .$$

The other direction is easy. □

3.2 Geometrizations

We fix a continuous map $p : M \to B$ where M is a smooth compact manifold and B is a topological space.

Definition 3.6. *A **pre-geometrization** for $p : M \to B$ is a map*

$$\mathcal{G} : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}}^0(M)$$

such that

- *it is a group homomorphism,*
- *it is continuous (with respect to the profinite topology on $\mathbf{KU}^0(B)$ and the discrete topology on $\widehat{\mathbf{KU}}^0(M)$) and*
- *the diagram*

$$\begin{tikzcd} & \widehat{\mathbf{KU}}^0(M) \arrow[d, "I"] \\ \mathbf{KU}^0(B) \arrow[ur, "\mathcal{G}"] \arrow[r, "p^*"] & \mathbf{KU}^0(M) \end{tikzcd}$$

  commutes. ■

In Remark 1.2 we have seen that a connection ∇^P for a smooth G-principal bundle $P \to M$ on a compact manifold M gives a pre-geometrization for a map $p : M \to BG$ classifying $P \to M$. But there we also discussed the Chern-Weil map $c_{\nabla^P} : \mathbf{HP}\mathbb{Q}^0(BG) \to \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$. We generalize the latter map. For this we fix an invertible (with respect to wedge product) closed form $\mathbf{Td} \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)^\times$ and an invertible class $\mathbf{td}^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$ such that both agree[1] in $\mathbf{HP}\mathbb{R}^0(M)$. We refer to these as Todd class and Todd form (because in the application these are the usual Todd class and Todd form).

Definition 3.7. *A **geometrization** is a pair $(\mathcal{G}, c_\mathcal{G})$ consisting of a pre-geometrization $\mathcal{G}$ and a cohomological character $c_\mathcal{G}$ such that the square*

$$\begin{array}{ccc} \mathbf{HP}\mathbb{Q}^0(B) & \xrightarrow{c_\mathcal{G}} & \Omega\mathrm{P}^0_{\mathrm{cl}}(M) \\ \uparrow{\scriptstyle \mathbf{td}^{-1}\cup\mathbf{ch}} & & \uparrow{\scriptstyle \mathbf{Td}\wedge R} \\ \mathbf{KU}^0(B) & \xrightarrow{\mathcal{G}} & \widehat{\mathbf{KU}}^0(M) \end{array} \tag{3.1}$$

commutes. ■

Note that the definition of a geometrization depends not only on the map $p : M \to B$ but also on the cohomology class $\mathbf{td}^{-1}$ and the closed form $\mathbf{Td}$. To stress these dependences we often call a geometrization a $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrization. On the other hand the cohomological character will be mostly suppressed from the notation. We denote geometrizations just by $\mathcal{G}$ and refer to $c_\mathcal{G}$ as the cohomological character of $\mathcal{G}$.

Example 3.8. The pre-geometrization constructed in Remark 1.2 together with the Chern-Weil map is indeed a geometrization. ★

Example 3.9. We discuss an example which shows that the map $c_\mathcal{G}$ is in general far from being unique.

Consider the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$. Its K-theory is calculated in [AH68, Theorem II]. The result is $\mathbf{KU}^0(K(\mathbb{Z}, 4)) = \mathbb{Z}$ which is the contribution of a point. On the other hand its rational cohomology ring is $\mathbf{H}^*(K(\mathbb{Z}, 4); \mathbb{Q}) = \mathbb{Q}[x]$ where x is a generator in degree 4. The 4-skeleton of $K(\mathbb{Z}, 4)$ is a sphere $\iota : S^4 \hookrightarrow K(\mathbb{Z}, 4)$ and the pullback $\iota^* x \in \mathbf{H}^4(S^4, \mathbb{Q}) \cong \mathbb{Q}$ is a generator.

Now a pre-geometrization for ι is uniquely defined by its value on the trivial rank one bundle since $\mathbf{KU}^0(K(\mathbb{Z}, 4)) = \mathbb{Z}$. We define

$$\mathcal{G}[K(\mathbb{Z}, 4) \times \mathbb{C} \to K(\mathbb{Z}, 4)] := [S^4 \times \mathbb{C} \to S^4, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}}] \in \widehat{\mathbf{KU}}^0(S^4)$$

and specialize the square (3.1) (with $\mathbf{td} = 1$ and $\mathbf{Td} = 1$)

$$\begin{array}{ccccc} 1x^0 & \qquad & \mathbb{Q}[x] & \overset{c_\mathcal{G}}{\dashrightarrow} & \Omega\mathrm{P}^0_{\mathrm{cl}}(S^4) \\ \uparrow & & \uparrow & \overset{0}{\nearrow} & \uparrow{\scriptstyle R} \\ 1 & & \mathbb{Z} & \xrightarrow[\mathcal{G}]{} & \widehat{\mathbf{KU}}^0(S^4) \end{array} .$$

Hence the map $c_\mathcal{G}$ can be chosen arbitrarily on $x\mathbb{Q}[x] \subset \mathbb{Q}[x]$. Note that it is even not unique if we require $c_\mathcal{G}$ to be a cohomological character. In this case the freedom is reduced to the choice of a closed 4-form on S^4 which represents $\iota^* x$ in $\mathbf{H}^4(S^4, \mathbb{R})$. Hence the freedom is as big as the space of exact 4-forms on S^4. ★

[1] In the application, $\mathbf{td}$ will be related to the normal bundle of a $Spin^c$-manifold and $\mathbf{Td}$ will be related to its tangent bundle. Hence we require $p^* \mathbf{td}^{-1} = \mathbf{Rham}(\mathbf{Td}) \in \mathbf{HP}\mathbb{R}^0(M)$. This confusing minus sign seems to be unavoidable.

On the other hand there is a uniqueness result for finite B.

Proposition 3.10. *Let B be a finite CW-complex and consider a continuous map $p : M \to B$ where M is a smooth compact manifold. Then, given a pre-geometrization $\mathcal{G}$ for p, there exists at most one cohomological character making the diagram* (3.1) *commute.* ♦

Proof. The Chern character induces an isomorphism $\mathbf{ch} : \mathbf{KU}^0(B) \otimes \mathbb{Q} \to \mathbf{HP}\mathbb{Q}^0(B)$ since B is a finite CW-complex. Hence we tensor the diagram (3.1) with $\mathbb{Q}$ and define $c_{\mathcal{G}} := (\mathbf{Td} \wedge R) \circ \mathcal{G} \circ \mathbf{ch}^{-1} \circ (\mathbf{td} \cup (-))$. This is the only possible candidate for a cohomological character. □

Remark 3.11. We have the following results regarding functoriality in M and B.

- Given a smooth map $f : N \to M$ of smooth compact manifolds over B, a cohomology class $\mathbf{td}^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$, a form $\mathbf{Td} \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$ representing $\mathbf{td}^{-1}$ and a $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrization $\mathcal{G} : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(M)$. Then we define the pullback-geometrization $f^*\mathcal{G}$ by
$$f^*\mathcal{G} := f^* \circ \mathcal{G} : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(N) \ .$$
Indeed, the pullback-geometrization $f^*\mathcal{G}$ is a $(p \circ f, f^* \mathbf{Td}, \mathbf{td}^{-1})$-geometrizations.
On the level of cohomological characters we have $c_{f^*\mathcal{G}} = f^*(c_{\mathcal{G}})$.
Note that this construction is contravariant.

- In the application the situation differs a bit. The details are as follows. We have a homotopy commutative triangle
$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p_M"'] & & N \arrow[dl, "p_N"] \\ & B & \end{tikzcd}$$
where M and N are compact manifolds and f is a smooth map. Moreover, we have an invertible cohomology class $\mathbf{td}^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$, closed forms $\mathbf{Td}_M \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)^\times$ and $\mathbf{Td}_N \in \Omega\mathrm{P}^0_{\mathrm{cl}}(N)^\times$ (which represent $\mathbf{td}^{-1}$) and a $(p_N, \mathbf{Td}_N, \mathbf{td}^{-1})$-geometrization $\mathcal{G}_N$. We would like to construct a $(p_M, \mathbf{Td}_M, \mathbf{td}^{-1})$-geometrization. But in general, $f^* \mathbf{Td}_N \neq \mathbf{Td}_M$.
Now suppose[2] there is a differential form $\delta \in \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$ with
$$d\delta = f^* \mathbf{Td}_N \wedge \mathbf{Td}_M^{-1} - 1 \ . \tag{3.2}$$
We will call such a form an **error form**. Then we define
$$f^!_\delta(\mathcal{G}_N) := f^* \circ \mathcal{G}_N(-) + a(\delta \wedge f^* \circ R \circ \mathcal{G}_N(-)) : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(M) \tag{3.3}$$
and one easily checks that $f^!_\delta(\mathcal{G}_N)$ is indeed a $(p_M, \mathbf{Td}_M, \mathbf{td}^{-1})$-geometrization (cf. [Bun11, Lemma 4.9]) whose cohomological character is the pullback character,
$$c_{f^!_\delta(\mathcal{G}_N)} = f^*(c_{\mathcal{G}_N}) \ . \tag{3.4}$$

[2]In the geometric situation which we study in Section 6.4 we will be able to construct such an error form (cf. Construction 6.14).

This construction is functorial in the following sense. Given two homotopy commutative triangles

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "p_M"'] & N \arrow[r, "g"] \arrow[d, "p_N"] & K \arrow[dl, "p_K"] \\ & B & \end{tikzcd}$$

with compact manifolds M, N, K and smooth maps f, g. Fix again an invertible cohomology class $\mathbf{td}^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$, closed forms $\mathbf{Td}_M \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)^\times$, $\mathbf{Td}_N \in \Omega\mathrm{P}^0_{\mathrm{cl}}(N)^\times$ and $\mathbf{Td}_K \in \Omega\mathrm{P}^0_{\mathrm{cl}}(K)^\times$ and a $(p_K, \mathbf{Td}_K, \mathbf{td}^{-1})$-geometrization $\mathcal{G}_K$. Also, suppose there are error forms $\delta \in \Omega\mathrm{P}^{-1}(N)/(\mathrm{im}\, d)$ and $\varepsilon \in \Omega\mathrm{P}^{-1}(M)/(\mathrm{im}\, d)$ with

$$d\delta = g^* \mathbf{Td}_K \wedge \mathbf{Td}_N^{-1} - 1 \quad \text{and} \quad d\varepsilon = f^* \mathbf{Td}_N \wedge \mathbf{Td}_M^{-1} - 1$$

and construct the $(p_M, \mathbf{Td}_M, \mathbf{td}^{-1})$-geometrization $f^!_\varepsilon(g^!_\delta(\mathcal{G}_K))$. One calculates that

$$f^!_\varepsilon(g^!_\delta(\mathcal{G}_K)) = (g \circ f)^!_\rho(\mathcal{G}_K) \tag{3.5}$$

holds with

$$\rho := f^*\delta + \varepsilon + \varepsilon \wedge f^* d\delta \in \Omega\mathrm{P}^{-1}(M)/(\mathrm{im}\, d)\ . \tag{3.6}$$

Note that ρ satisfies

$$d\rho = (g \circ f)^* \mathbf{Td}_K \wedge \mathbf{Td}_M^{-1} - 1$$

and thus it is an error form.

- We discuss functoriality in B. So we take a continuous map $g : B \to \tilde{B}$ and a $(p, \mathbf{Td}, \mathbf{td}_B^{-1})$-geometrization $\mathcal{G} : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(M)$. Assume we have a cohomology class $\mathbf{td}_{\tilde{B}}^{-1} \in \mathbf{HP}\mathbb{Q}^0(\tilde{B})^\times$ with $g^* \mathbf{td}_{\tilde{B}}^{-1} = \mathbf{td}_B^{-1}$.

  Then we can define the pushforward-geometrization $g_*\mathcal{G}$ by

  $$g_*\mathcal{G} := \mathcal{G} \circ g^* : \mathbf{HP}\mathbb{Q}^0(\tilde{B}) \to \widehat{\mathbf{KU}^0}(M)\ .$$

  On the level of cohomological characters we have $c_{g_*\mathcal{G}} = g_*(c_\mathcal{G})$.
  Note that this construction is covariant: For a map $\beta : \tilde{B} \to \hat{B}$ (and compatible $\mathbf{td}^{-1}$-classes) we have $\beta_*(g_*\mathcal{G}) = (g_*\mathcal{G}) \circ \beta^* = \mathcal{G} \circ g^* \circ \beta^* = \mathcal{G} \circ (\beta \circ g)^* = (\beta \circ g)_*\mathcal{G}$.

- Note that the definition of a geometrization depends only on the homotopy class of p and the weak homotopy type of B. ■

Example 3.12. As in Example 3.3 we prove existence for arbitrary maps $p : M \to B$ by constructing a geometrization for $\mathtt{id}_M : M \to M$ and then taking the pushforward of this geometrization along p to B (using the Todd class $p^* \mathbf{td}^{-1}$ on M).

By Example 3.3 we can choose a cohomological character c for $\mathtt{id}_M$. Since $\mathbf{td}^{-1}$ is invertible we get the commutative diagram (recall $\mathbf{td}^{-1} = \mathbf{Rham}(\mathbf{Td}) \in \mathbf{HP}\mathbb{R}^0(M)$)

$$\begin{tikzcd}
 & \mathbf{HP}\mathbb{R}^{-1}(M) \arrow[d, "a"] \\
 & \widehat{\mathbf{KU}^0}(M) \arrow[d, "{(I,\mathbf{Td}\wedge R)}"] \\
\mathbf{KU}^0(M) \arrow[ur, dashed, bend left, "\mathcal{G}"] \arrow[r, "{(\mathtt{id}, c\circ(\mathbf{td}^{-1}\cup\mathbf{ch}))}"] \arrow[dr, "0"'] & \mathbf{KU}^0(M) \oplus \Omega\mathrm{P}^0_{\mathrm{cl}}(M) \arrow[d, "\mathbf{ch}-\mathbf{td}\cup\mathbf{Rham}"] \\
 & \mathbf{HP}\mathbb{R}^0(M)
\end{tikzcd}$$

where the right column is part of a long exact sequence which is induced by the sequence (2.2). To construct a lift $\mathcal{G}$ we choose a splitting of the finitely generated abelian group

$$\mathbf{KU}^0(M) = \mathbf{KU}^0_{\mathtt{tors}}(M) \oplus \mathbf{KU}^0_{\mathtt{free}}(M)$$

into the torsion part and a free part. Then we choose generators $f_1, \ldots, f_m$ and $t_1, \ldots, t_n$ of the free and torsion parts such that $\mathbf{KU}^0(M) = \oplus_{i=1}^n \langle t_i \rangle \bigoplus \oplus_{i=1}^m \langle f_i \rangle$. Next we take lifts $\tilde{t}_i \in \widehat{\mathbf{KU}^0}(M)$ and $\hat{f}_i \in \widehat{\mathbf{KU}^0}(M)$ with

$$(I, \mathbf{Td} \wedge R)(\tilde{t}_i) = (\mathtt{id}, c \circ (\mathbf{td}^{-1} \cup \mathbf{ch}))(t_i) \in \mathbf{KU}^0(M) \oplus \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$$

respectively

$$(I, \mathbf{Td} \wedge R)(\hat{f}_i) = (\mathtt{id}, c \circ (\mathbf{td}^{-1} \cup \mathbf{ch}))(f_i) \in \mathbf{KU}^0(M) \oplus \Omega\mathrm{P}^0_{\mathrm{cl}}(M) .$$

Now the problem is that the map $t_i \mapsto \tilde{t}_i$ might fail to be a group homomorphism since the orders may not fit. Observe that the elements $\mathrm{ord}(t_i)\tilde{t}_i$ are in the kernel of (I, R) which is the image of a. So there exist $\rho_i \in \mathbf{HP}\mathbb{R}^{-1}(M)$ with $a(\rho_i) = \mathrm{ord}(t_i)\tilde{t}_i$ for all $i = 1, \ldots, n$. We set $\hat{t}_i := \tilde{t}_i - a\left(\frac{1}{\mathrm{ord}(t_i)}\rho_i\right)$ which satisfy

$$(I, \mathbf{Td} \wedge R)(\hat{t}_i) = (\mathtt{id}, c \circ (\mathbf{td}^{-1} \cup \mathbf{ch}))(t_i) \in \mathbf{KU}^0(M) \oplus \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$$

and have order $\mathrm{ord}(t_i) = \mathrm{ord}(\hat{t}_i)$. Finally we define the geometrization to be the map $\mathcal{G}$ with $\mathcal{G}(t_i) := \hat{t}_i$ and $\mathcal{G}(f_i) := \hat{f}_i$.

This is indeed a geometrization for $\mathtt{id}_M$ with cohomological character c. ★

Next we show that every cohomological character occurs as the cohomological character of a geometrization.

Proposition 3.13. *Let $p : M \to B$, $\mathbf{Td}$ and $\mathbf{td}^{-1}$ be as before.*

Then for each cohomological character c for p there is a $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrization $\mathcal{G}$ with cohomological character $c_{\mathcal{G}} = c$. ◆

Proof. Fix a cohomological character c^1. By Example 3.12 there exists a $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrization $\mathcal{G}^2$ with cohomological character c^2. By Proposition 3.5 we find a continuous group homomorphism $\xi : \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^0_{\mathrm{ex}}(M)$ such that $\xi = c^1 - c^2$.

We claim that there is a lift $\hat{\xi} : \mathbf{HP}\mathbb{Q}^0(B) \to \Omega\mathrm{P}^{-1}(M)/(\mathrm{im}\, d)$ such that $d\hat{\xi} = \xi$ and such that $\hat{\xi}$ is a continuous group homomorphism which shifts the internal degree by -1, i.e., maps degree k-cohomology classes to degree $(k-1)$-differential forms.

Since ξ is continuous there exists $r : C \to B$ for C a finite CW-complex such that $\ker(r^*) \subset \ker(\xi)$. But ξ factors over $\mathbf{HP}\mathbb{Q}^0(B)/\ker(\xi)$ which receives a surjection from $\operatorname{im}(r^*)$, a linear subspace of $\mathbf{HP}\mathbb{Q}^0(C)$. Since C is a finite CW-complexes we conclude that the image $\operatorname{im}(\xi) \subset \Omega\mathrm{P}^0_{\mathrm{ex}}(M)$ is a finite-dimensional $\mathbb{Q}$-subspace. We choose a section $s : \operatorname{im}(\xi) \to \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$ of $d : \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) \to \Omega\mathrm{P}^0_{\mathrm{ex}}(M)$ and define $\hat{\xi} := s \circ \xi$.

Finally we set

$$\mathcal{G}^1 := \mathcal{G}^2 + a \circ \left(\mathbf{Td}^{-1} \wedge \hat{\xi}\right) \circ \left(\mathbf{td}^{-1} \cup \mathbf{ch}\right) .$$

This is continuous as composition of continuous maps and indeed is a geometrization with cohomological character c^1. □

Definition 3.14. *We denote by* $\operatorname{Hom}^{\mathrm{cont}}_{[-1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)\right)$ *the group of continuous group homomorphisms which shift the internal degree by -1, i.e., map degree k-cohomology classes to degree $(k-1)$-differential forms.* ■

Next we give the classification of geometrizations with fixed cohomological character.

Proposition 3.15. *Consider $p : M \to B$ and $\mathbf{Td}, \mathbf{td}^{-1}$ as before. Moreover, fix a cohomological character c for p. Then the set of $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrizations with cohomological character c is a non-empty torsor over* $\operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im} \mathbf{ch}))$.

Thus the set of all $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrizations decomposes into the disjoint union of geometrizations with fixed cohomological character and each of these components is a non-empty torsor for the same abelian group. ◆

Proof. Take two $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrizations $\mathcal{G}_1$ and $\mathcal{G}_2$ with the same cohomological character c. Then the difference $\xi := \mathcal{G}_1 - \mathcal{G}_2 : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(M)$ is a continuous group homomorphism such that $I \circ \xi = 0$ and $R \circ \xi = 0$. By the long exact sequence (2.2),

$$\mathbf{KU}^{-1}(M) \xrightarrow{\mathbf{ch}} \mathbf{HP}\mathbb{R}^{-1}(M) \xrightarrow{a} \widehat{\mathbf{KU}^0}(M) \xrightarrow{(I,R)} \mathbf{KU}^0(M) \oplus \Omega\mathrm{P}^0_{\mathrm{cl}}(M) ,$$

we consider $\xi \in \operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im} \mathbf{ch}))$.

On the other hand, given some $\xi \in \operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im} \mathbf{ch}))$ and $\mathcal{G}_1$ a $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrization we set $\mathcal{G}_2 : \mathbf{KU}^0(B) \to \widehat{\mathbf{KU}^0}(M)$ to be $\mathcal{G}_2 := \mathcal{G}_1 - a \circ \xi$. This is indeed a $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrization with the same cohomological character as $\mathcal{G}_1$.

The torsor is non-empty by Proposition 3.13. □

To get a classification which is independent of a fixed cohomological character we introduce the following equivalence relation. This equivalence relation is motivated by our application Theorem 7.13.

Definition 3.16. *Two $(p : M \to B, \mathbf{Td}, \mathbf{td}^{-1})$-geometrizations $\mathcal{G}_1$ and $\mathcal{G}_2$ are called* ***equivalent****, $\mathcal{G}_1 \sim \mathcal{G}_2$, if there exists* $\hat{\xi} \in \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)\right)$ *such that* $\mathcal{G}^2 = \mathcal{G}^1 - a \circ \left(\mathbf{Td}^{-1} \wedge \hat{\xi}\right) \circ \left(\mathbf{td}^{-1} \cup \mathbf{ch}\right)$. ■

Note that the cohomological character of an equivalence class of geometrizations is not well-defined. Our next theorem classifies geometrizations up to equivalence.

Theorem 3.17. *Consider $p : M \to B$ and $\mathbf{Td}, \mathbf{td}^{-1}$ as before. Then the set of equivalence classes of $(p, \mathbf{Td}, \mathbf{td}^{-1})$-geometrizations is a non-empty torsor over the abelian group* $\operatorname{coker}(\rho)$ *with*

$$\begin{aligned} \rho : \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)\right) &\to \operatorname{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im} \mathbf{ch})\right) \\ \varepsilon &\mapsto \left(\mathbf{Td}^{-1} \wedge \varepsilon\right) \circ \left(\mathbf{td}^{-1} \cup \mathbf{ch}\right) \end{aligned}$$

where we implicitly used the projection $\mathbf{HP}\mathbb{R}^{-1}(M) \to \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im}\mathbf{ch})$. *The torsor action is induced by the action used in Proposition 3.15.* ◆

Proof. Since there exists a geometrization the set of equivalence classes is non-empty.

We make the action precise. Given an equivalence class $[\mathcal{G}]$ for some geometrization $\mathcal{G}$ and $[\varepsilon] \in \operatorname{coker}(\rho)$ for $\varepsilon \in \operatorname{Hom}^{\mathrm{cont}}\big(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im}\mathbf{ch})\big)$ the action is defined by $[\varepsilon].[\mathcal{G}] := [\mathcal{G} - a \circ \varepsilon]$. This is well-defined since $\mathcal{G} - a \circ \varepsilon$ is indeed a geometrization and because of the definition of the equivalence relation. Thus we have a well-defined action.

Next we show that the action is transitive. So take two geometrizations $\mathcal{G}^1$ and $\mathcal{G}^2$ with cohomological characters c^1 and c^2. As in the proof of Proposition 3.13 we find $\hat{\xi} \in \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)\big)$ such that $d\hat{\xi} = c_1 - c_2$ and get a third geometrizations $\hat{\mathcal{G}}^2 := \mathcal{G}^2 + a \circ \Big(\mathbf{Td}^{-1} \wedge \hat{\xi}\Big) \circ \big(\mathbf{td}^{-1} \cup \mathbf{ch}\big)$ with cohomological character c^1. Note that $\hat{\mathcal{G}}^2$ and $\mathcal{G}^2$ are equivalent. Since the cohomological characters of $\mathcal{G}^1$ and $\hat{\mathcal{G}}^2$ agree we can apply Proposition 3.15 to find $\varepsilon \in \operatorname{Hom}^{\mathrm{cont}}\big(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im}\mathbf{ch})\big)$ such that $[\varepsilon].[\mathcal{G}^1] = [\hat{\mathcal{G}}^2] = [\mathcal{G}^2]$.

It remains to show that the action is free. So suppose we are given a geometrization $\mathcal{G}$ and $\varepsilon \in \operatorname{Hom}^{\mathrm{cont}}\big(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im}\mathbf{ch})\big)$ such that $[\varepsilon].[\mathcal{G}] = [\mathcal{G}]$. Hence there exists $\hat{\xi} \in \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)\big)$ such that

$$\mathcal{G} - a \circ \varepsilon = \mathcal{G} - a \circ \Big(\mathbf{Td}^{-1} \wedge \hat{\xi}\Big) \circ \big(\mathbf{td}^{-1} \cup \mathbf{ch}\big) \ .$$

Set $\zeta := \Big(\mathbf{Td}^{-1} \wedge \hat{\xi}\Big) \circ \big(\mathbf{td}^{-1} \cup \mathbf{ch}\big)$. For $\phi \in \mathbf{KU}^0(B)$ we get $\varepsilon(\phi) \in \mathbf{HP}\mathbb{R}^{-1}(M)/\operatorname{im}(ch)$ and $\zeta(\phi) \in \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$ such that $\varepsilon(\phi) = \zeta(\phi)$ in $\Omega\mathrm{P}^{-1}(M)/(\operatorname{im}\mathbf{ch})$. We use the following diagram to show that $\zeta(\phi) \in \mathbf{HP}\mathbb{R}^{-1}(M)$,

$$\begin{tikzcd}
\mathbf{HP}\mathbb{R}^{-1}(M) \arrow[r, hook] \arrow[d, two heads, "\mathrm{pr}"] & \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) \arrow[d, two heads] & \\
\mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im}\mathbf{ch}) \arrow[r, hook] & \Omega\mathrm{P}^{-1}(M)/(\operatorname{im}\mathbf{ch}) \arrow[r, hook, "a"] & \widehat{\mathbf{KU}}^0(M) \ .
\end{tikzcd}$$

For this we choose a lift $\hat{\varepsilon}(\phi) \in \mathbf{HP}\mathbb{R}^{-1}(M)$ of $\varepsilon(\phi)$. Then $\hat{\varepsilon}(\phi)$ and $\zeta(\phi)$ differ by an element $\mathbf{ch}(\alpha)$ in $(\operatorname{im}\mathbf{ch}) \subset \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$. Hence $\zeta(\phi) = \varepsilon(\phi) + \mathbf{ch}(\alpha)$ and we conclude $\zeta(\phi) \in \mathbf{HP}\mathbb{R}^{-1}(M)$ because the map $\mathbf{HP}\mathbb{R}^{-1}(M) \hookrightarrow \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$ is injective.

There is a map $r : C \to B$ with C a finite CW-complex such that $\ker(r^*) \subset \ker(\hat{\xi})$ because $\hat{\xi}$ is continuous. This gives the diagram

$$\begin{tikzcd}
\mathbf{KU}^0(B) \arrow[r, "\mathbf{td}^{-1}\cup\mathbf{ch}"] \arrow[d, two heads, "r^*"] \arrow[rrr, bend left, "(\mathbf{Td}^{-1}\wedge\hat{\xi})\circ(\mathbf{td}^{-1}\cup\mathbf{ch})"] & \mathbf{HP}\mathbb{Q}^0(B) \arrow[r, "\mathbf{Td}^{-1}\wedge\hat{\xi}"] \arrow[d, two heads, "r^*"] & \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) & \mathbf{HP}\mathbb{R}^{-1}(M) \arrow[l, hook'] \\
\operatorname{im}(r^*) \arrow[r, "(r^*\mathbf{td}^{-1})\cup\mathbf{ch}"] \arrow[d, hook] & \operatorname{im}(r^*) \arrow[d, hook] & & \\
\mathbf{KU}^0(C) \arrow[r, "(r^*\mathbf{td}^{-1})\cup\mathbf{ch}"] & \mathbf{HP}\mathbb{Q}^0(C) & &
\end{tikzcd}$$

and we tensor with $\mathbb{Q}$ to get the diagram

$$\begin{tikzcd}
\mathbf{KU}^0(B)\otimes\mathbb{Q} \arrow[rr, "(\mathbf{Td}^{-1}\wedge\hat{\xi})\circ(\mathbf{td}^{-1}\cup\mathbf{ch})"] \arrow[d, "r^*"] & & \mathbf{HP}\mathbb{R}^{\text{-}1}(M) \\
\mathbf{KU}^0(C)\otimes\mathbb{Q} \arrow[r, "(r^*\,\mathbf{td}^{-1})\cup\mathbf{ch}"] & \mathbf{HP}\mathbb{Q}^0(C) \arrow[r, dotted, "\Phi"] & \mathbf{HP}\mathbb{R}^{\text{-}1}(M) \arrow[u, "\mathbf{Td}^{-1}\wedge(-)"']
\end{tikzcd} .$$

We can find a continuous extension Φ because $\mathbf{HP}\mathbb{Q}^0(C)$ is a discrete finite dimensional vector space. We can arrange that $\Phi \in \mathrm{Hom}^{\mathrm{cont}}_{[\text{-}1]}\big(\mathbf{HP}\mathbb{Q}^0(C), \mathbf{HP}\mathbb{R}^{\text{-}1}(M)\big)$ because $\hat{\xi}$ shifts the internal degree. This gives $\hat{\Phi} \in \mathrm{Hom}^{\mathrm{cont}}_{[\text{-}1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \mathbf{HP}\mathbb{R}^{\text{-}1}(M)\big)$ by composing with $\mathbf{HP}\mathbb{Q}^0(B) \to \mathbf{HP}\mathbb{Q}^0(C)$. By construction we have

$$\varepsilon = (\mathbf{Td}^{-1}\wedge\hat{\Phi})\circ(\mathbf{td}^{-1}\cup\mathbf{ch}) = \rho(\hat{\Phi}) \ .$$

Thus $[\varepsilon] = 0 \in \mathrm{coker}(\rho)$ and hence the action is free. This completes the proof. □

3.3 Universal Geometrizations

So far we have seen that geometrizations exist for any map $p : M \to B$ and that we can pushforward along maps $B \to \tilde{B}$ and pullback along smooth maps $N \to M$ (at least in nice situations).

Moreover, we have seen that connections on principal bundles yield geometrizations (Example 3.8). We highlight the following property of connections: Given a compact submanifold $S \hookrightarrow M$ and a principal bundle $P \to S$ which extends to $Q \to M$ we can also extend a connection ∇^P on $P \to S$ to a connection ∇^Q on $Q \to M$, i.e., ∇^Q restricts to ∇^P on S (see Section 1.1).

But geometrizations behave not so well. In [Bun11, Example 4.11] Bunke gives the following example of a geometrization on the sphere S^3 which cannot be extended to D^4.

Example 3.18. Consider the one-point space $B = \star$ and the manifolds S^3 and D^4 together with the boundary inclusion $f : S^3 \to D^4$ and the maps $p : S^3 \to \star$, $q : D^4 \to \star$. We use the trivial Todd form and the trivial Todd class $1 \in \mathbf{HP}\mathbb{Q}^0(\star)$. Note that in this situation we have $f^*\,\mathbf{Td}_{D^4} = \mathbf{Td}_{S^3} = 1$ and hence the pullback of a geometrization along the boundary inclusion is defined (cf. Remark 3.11).

Note that there is a unique cohomological character c for $q : D^4 \to \star$ by Theorem 3.5. Now we apply our classification result Proposition 3.15. We see that there is a unique $(q, 1, 1)$-geometrization with cohomological character c. On the other hand there are $\mathbf{HP}\mathbb{R}^{\text{-}1}(S^3)/(\mathrm{im}\,\mathbf{ch}) \cong {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$-many $(p, 1, 1)$-geometrizations with cohomological character f^*c. So not any geometrization on S^3 arises via pullback from a geometrization of D^4, i.e., there are geometrizations on S^3 which cannot be extended to D^4. ★

Guided by the theory of universal connections (cf. Section 1.1) we try to find geometrizations which extend over submanifolds by defining a “universal geometrization” directly on B. Given such a “universal geometrization” we can take its pullback geometrization to both the submanifold and to the manifold and hence the geometrization on the submanifold extends to the manifold.

The idea is the following. Assume that we can approximate the topological space B by a family of smooth compact manifolds B_i and that we can find compatible geometrizations $\mathcal{G}_i$ for $B_i \to B$. Then, given a continuous map $p : S \to B$, we can find a compatible family of smooth maps $p_i : S \to B_i$ such that all the pullback geometrizations $p_i^*\mathcal{G}_i$ coincide. Now assume that the map $p : S \to B$ extends to $q : M \to B$ along the embedding $S \hookrightarrow M$. Then we can find a compatible family of smooth maps $q_i : M \to B$ which extend the smooth maps p_i (for big enough i). So the pullback geometrization $q_i^*\mathcal{G}_i$ extend the geometrization $p_i^*\mathcal{G}_i$.

At first we define what we mean by an approximation of a topological space by manifolds.

Definition 3.19. *Let B be a topological space. An* ***approximation by manifolds*** *of B consists of*

- *a family of smooth compact manifolds $(B_i)_{i\in\mathbb{N}}$,*
- *a family of smooth embeddings $f_i : B_i \to B_{i+1}$,*
- *a family of continuous maps $g_i : B_i \to B$ and*
- *for all $i \in \mathbb{N}$ a homotopy filling the diagram*

$$\begin{tikzcd} B_i \arrow[rr, "f_i"] \arrow[dr, "g_i"'] & & B_{i+1} \arrow[dl, "g_{i+1}"] \\ & B & \end{tikzcd}$$

such that the space B is weakly homotopy equivalent to the homotopy colimit, i.e.,

$$(g_i)_i : \mathrm{hocolim}_{i\in\mathbb{N}} B_i \xrightarrow{\simeq} B \ . \qquad \blacksquare$$

We have the following observation.

Lemma 3.20. *Let B be a topological space and $(B_i, f_i, g_i)_i$ be an approximation by manifolds for B. Consider this as a subcategory of $\mathcal{C}_{\mathrm{ho}}/B$ (Definition 2.1).*

We claim that this subcategory is cofinal. ♦

Proof. This follows directly from the fact that B is the homotopy colimit. □

Example 3.21. For a smooth compact manifold M we have an approximation of M by manifolds given by the constant family $(M, \mathtt{id}_M, \mathtt{id}_M)_{i\in\mathbb{N}}$. ★

Example 3.22. Assume that B is a connected and simply-connected topological space with degree-wise finitely generated homotopy groups. Then existence of an approximation of B by manifolds is guaranteed by [BS10, Proposition 2.1]. ★

We can generalize the previous example since we do not assume that the maps g_i are highly connected. For this we need

Definition 3.23. *A simply-connected topological space B is of* ***countable type*** *if it is weakly homotopy equivalent to a CW-complex $\tilde{B}$ satisfying*

- *$\tilde{B}$ has countable many connected components and*
- *for each of these connected components all homotopy groups are countable.* ■

Proposition 3.24. *Let B be a simply-connected topological space of countable type. Then there exists an approximation by manifolds for B.* ♦

Proof. Clearly, we can assume that B is a CW-complex and a diagonal sequence argument reduces to B connected (and simply-connected). Since all homotopy groups are countable we can assume that B has countable many cells (all in degree 2 and above).

Now we order these cells[3] $\{C_1, C_2, C_3, \dots\}$. Then we denote by B_i the smallest subcomplex of B which contains the first i cells $\{C_1, C_2, C_3, \dots, C_{i-1}, C_i\}$. Since each cell lies in a finite subcomplex each of the spaces B_i is a finite CW-complex. This gives a system of finite CW-complex with a map to B, $(B_i, f_i, g_i)_{i\in\mathbb{N}}$ such that

$$(g_i)_i : \operatorname{hocolim}_{i\in\mathbb{N}} B_i \xrightarrow{\simeq} B \ .$$

Next we follow the proof of [BS10, Proposition 2.1] and inductively replace each of the spaces B_i by smooth compact manifolds and the transition maps $B_i \to B_{i+1}$ by smooth embeddings. This completes the proof. □

Example 3.25. Note that the existence of an approximation by manifolds for B does not imply that B is weakly homotopy equivalent to a CW-complex of finite type. A counterexample to this is an infinite disjoint union of circles, $B = \coprod_{\mathbb{N}} S^1$. This space has an approximation given by $B_i := \coprod_{n\le i} S^1$. ★

Now we define universal geometrizations.

Definition 3.26. *Let B be a topological space and let $\mathbf{td}^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$ be an invertible cohomology class. A* ***universal geometrization*** *for B consists of*

- *an approximation of B by manifolds $(B_i, f_i, g_i)_{i\in\mathbb{N}}$,*
- *a family of closed invertible forms $\mathbf{Td}_i \in \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)^\times$ representing $g_i^*\,\mathbf{td}^{-1}$,*
- *a family of error forms $\delta_i \in \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)$ with $d\delta_i = f_i^*\mathbf{Td}_{i+1} \wedge \mathbf{Td}_i^{-1} - 1$ and*
- *a family of $(g_i, \mathbf{Td}_i, \mathbf{td}^{-1})$-geometrizations $\mathcal{G}_i$*

such that $(f_i)^!_{\delta_i}\mathcal{G}_{i+1} = \mathcal{G}_i$.

We will refer to the data $(B_i, f_i, g_i, \mathbf{Td}_i, \delta_i)_i$ as ***underlying data****.* ■

Recall that the pullback geometrization $(f_i)^!_{\delta_i}\mathcal{G}_{i+1}$ was defined in Remark 3.11, formula (3.3). The main theorem of the present article is the following existence result.

Existence Theorem 3.27. *Let B be a topological space and let $\mathbf{td}^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$ be an invertible cohomology class. Fix some underlying data $(B_i, f_i, g_i, \mathbf{Td}_i, \delta_i)_i$. Then there exists a universal geometrization for B for these underlying data if the following technical condition holds:*

The split exact sequence of Proposition 5.2 has a continuous split. ♦

Proof. Since the proof is involved we postpone it to Chapter 5. □

Remark 3.28. The technical condition is satisfied if all $\mathbf{KU}^{-1}(B_i)$ are rationally trivial. Hence it holds if B rationally even.

It also holds if the profinite abelian group

$$\varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))$$

is topologically finitely generated. This is the content of Proposition 5.4.

[3]Note that this mixes the degree of the cells in general.

This is in particular the case if $\mathbf{KU}^0(B)$ is (topologically) finitely generated since then also its quotient $\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i)$ (by Proposition 2.2) and hence its topological direct summand $\varprojlim_{i\in\mathbb{N}} \mathrm{Torsion}(\mathbf{KU}^0(B_i))$ (by Proposition 5.1) are topologically finitely generated.

We also note that the technical condition seems necessary, but we have no example in this direction. ■

The name "universal" stems from the fact that a universal connection induces a universal geometrization. We discuss this in the following example.

Example 3.29. In Remark 1.4 we have discussed that a universal connection on the classifying bundle of a compact Lie group G yields a universal geometrization for the classifying space BG. ★

3.4 Classification of universal geometrizations

In this section we want to classify universal geometrizations on some space B. For simplicity we assume that $\mathbf{td}^{-1} = 1 \in \mathbf{HP}\mathbb{Q}^0(B)^\times$ and also that the Todd forms $\mathbf{Td}_i$ and the error forms δ_i are trivial. A version of this classification which incorporates non-trivial Todd classes, Todd forms and error forms is given in Chapter 7.

At first we classify universal geometrizations on a fixed approximation by manifolds. But by introducing a similar equivalence relation as in Definition 3.16 we can show that equivalence classes of geometrizations are independent of approximations by manifolds.

For the first part we fix an approximation by manifolds $(B_i, f_i, g_i)_{i\in\mathbb{N}}$ of B. Then a universal geometrization is a choice of geometrization $\mathcal{G}_i$ for each $g_i : B_i \to B$ such that $f_i^*\mathcal{G}_{i+1} = \mathcal{G}_i$. Fix such a universal geometrizations $(\mathcal{G}_i^1)_i$ with cohomological characters $(c_i^1)_i$. Now choose a second compatible family of cohomological character, $(c_i^2)_i$. The differences between these cohomological characters, $\xi_i := c_i^2 - c_i^1$, form an element $\xi = (\xi_i) \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(B_i)\big)$.

Lemma 3.30. *We can find a lift $\hat{\xi} = (\hat{\xi}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\mathrm{im}\, d)\big)$ such that $d\hat{\xi}_i = \xi_i$.* ◆

Proof. This is a family version of Proposition 3.13. Similarly to there the images of ξ_i in $\Omega\mathrm{P}^0_{\mathrm{ex}}(B_i)$ are finite-dimensional $\mathbb{Q}$-vector spaces, denoted by X_i.

Next we fix compatible bases of the X_i. To this end we choose a basis of X_1 and preimages in X_2. Then we extend the preimages to a basis of X_2 by elements in the kernel of the projection $X_2 \twoheadrightarrow X_1$. Inductively we get compatible bases for all the X_i.

For the final step note that the restrictions $f_i^* : \Omega\mathrm{P}^{-1}(B_{i+1})/(\mathrm{im}\, d) \to \Omega\mathrm{P}^{-1}(B_i)/(\mathrm{im}\, d)$ are surjective since the f_i are embeddings. Choose lifts of the basis elements of X_1 to $\Omega\mathrm{P}^{-1}(B_1)/(\mathrm{im}\, d)$ and extend them to B_2. Then proceed with the remaining basis element in X_2 and inductively construct the lift $(\hat{\xi}_i)_i$ we have been looking for. □

Using such a lift $\hat{\xi}$ we define $\mathcal{G}_i^2 := \mathcal{G}_i + a \circ \hat{\xi}_i \circ \mathbf{ch}$.

Lemma 3.31. *The $\mathcal{G}_i^2$ are indeed geometrizations with cohomological character c_i^2 for all $i \in \mathbb{N}$. We have the compatibility $f_i^*\mathcal{G}_{i+1}^2 = \mathcal{G}_i^2$ for all $i \in \mathbb{N}$ and hence $(\mathcal{G}_i^2)_i$ is also a universal geometrization.* ◆

Hence we can transfer universal geometrizations from one compatible family of cohomological characters to any other such family. Next we classify universal geometrizations on a fixed approximation with fixed cohomological character.

Proposition 3.32. *Fix an approximation by manifolds $(B_i, f_i, g_i)_i$ of B and a compatible family of cohomological characters $(c_i)_i$. Then the set of universal geometrizations with cohomological characters $(c_i)_i$ is a torsor over $\varprojlim_i \operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch}))$.* ♦

Note that we don't claim that the torsor is non-empty.

Proof. This is an immediate generalization of Proposition 3.15. □

Proposition 3.33. *The abelian group $\varprojlim_i \operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch}))$ is a weak homotopy invariant of B. In particular, it depends only on B and not on the choice of approximation.* ♦

Proof. We have an isomorphism

$$\varprojlim_i \operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch})) \cong \operatorname{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \varprojlim_i \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch}))$$

and therefore it is enough to show the claim for $\varprojlim_i \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch})$. Recall Definition 2.1. Note that the functor $X \mapsto \mathbf{HP}\mathbb{R}^{-1}(X)/(\operatorname{im}\mathbf{ch})$ is homotopy invariant. Hence we have an isomorphism $\varprojlim_{C\to B\in\mathcal{C}_{\mathrm{ho}}/B} \mathbf{HP}\mathbb{R}^{-1}(C)/(\operatorname{im}\mathbf{ch}) \xrightarrow{\cong} \varprojlim_{C\to B\in\mathcal{C}/B} \mathbf{HP}\mathbb{R}^{-1}(C)/(\operatorname{im}\mathbf{ch})$. Finally recall that the approximation is a cofinal subcategory of $\mathcal{C}_{\mathrm{ho}}/B$ (Lemma 3.20). Thus $\varprojlim_i \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch}) \cong \varprojlim_{C\to B\in\mathcal{C}/B} \mathbf{HP}\mathbb{R}^{-1}(C)/(\operatorname{im}\mathbf{ch})$. □

Our classification results will involve the following groups.

Definition and Proposition 3.34. *Let B be a topological space and let $(B_i, f_i, g_i)_i$ be an approximation by manifolds. Then we define the **classifying group** $\operatorname{coker}(\rho)$ to be the cokernel of*

$$\begin{aligned}\rho : \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)\big) &\to \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}\big(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\operatorname{im}\mathbf{ch})\big)\\ (\varepsilon_i)_i &\mapsto (\varepsilon_i \circ \mathbf{ch})_i\end{aligned}$$

where we implicitly used the projection $\mathbf{HP}\mathbb{R}^{-1}(M) \to \mathbf{HP}\mathbb{R}^{-1}(M)/(\operatorname{im}\mathbf{ch})$. Since this group is canonically isomorphic to the cokernel of

$$\begin{aligned}\hat\rho : \varprojlim_{C\to B\in\mathcal{C}/B} &\operatorname{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \mathbf{HP}\mathbb{R}^{-1}(C)\big) \to\\ &\varprojlim_{C\to B\in\mathcal{C}/B} \operatorname{Hom}^{\mathrm{cont}}\big(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(C)/(\operatorname{im}\mathbf{ch})\big)\end{aligned}$$

we also refer to the cokernel $\operatorname{coker}(\hat\rho)$ as classifying group.

Moreover, $\operatorname{coker}(\hat\rho)$ is a weak homotopy invariant of B. ♦

Proof. One proves that $\operatorname{coker}(\hat\rho)$ is a weak homotopy invariant of B along the same lines as Proposition 3.33. □

Now we introduce a family version of the equivalence relation from Definition 3.16.

Definition 3.35. *Two universal geometrizations $(\mathcal{G}^1_i)_i$ and $(\mathcal{G}^2_i)_i$ on the same approximation $(B_i, f_i, g_i)_i$ (with trivial Todd class and trivial Todd and error forms) are called **equivalent**, $(\mathcal{G}^1_i)_i \sim (\mathcal{G}^2_i)_i$, if there exists a compatible family*

$$\hat\xi = (\hat\xi_i)_i \in \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)\big)$$

such that $\mathcal{G}^2_i = \mathcal{G}^1_i - a \circ \hat\xi_i \circ \mathbf{ch}$ for all $i \in \mathbb{N}$. ■

This gives the following classification up to equivalence. To this end the following easy observation is crucial.

Proposition 3.36. *Universal geometrizations can be restricted to cofinal subsystems of a given approximation and also uniquely extended from such a cofinal subsystems.* ◆

Proof. This follows directly from the compatibility condition. □

Note that we don't require the Todd class and the Todd and error forms to be trivial.

Theorem 3.37. *Fix an approximation by manifolds of B.*

Then the set of equivalence classes of universal geometrizations on this approximation with trivial Todd class and trivial Todd and error forms is a (possible empty) torsor over the classifying group $\operatorname{coker}(\rho)$. ◆

Proof. This is a family version of the proof of Theorem 3.17. We only prove that the action is free.

For this we fix an approximation $(B_i, f_i, g_i)_i$ by manifolds, a universal geometrization $(\mathcal{G}_i)_i$ and $(\varepsilon_i)_i \in \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)/(\operatorname{im}\mathbf{ch})\right)$ such that $[(\varepsilon_i)_i].[(\mathcal{G}_i)_i] = [(\mathcal{G}_i)_i]$. Hence there exists $(\hat{\xi}_i)_i \in \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}_{[\text{-}1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{\text{-}1}(B_i)/(\operatorname{im} d)\right)$ such that

$$\mathcal{G}_i - a \circ \varepsilon_i = \mathcal{G}_i - a \circ \left(\mathbf{Td}^{-1} \wedge \hat{\xi}_i\right) \circ \left(\mathbf{td}^{-1} \cup \mathbf{ch}\right)$$

for all $i \in \mathbb{N}$. We proceed as in the proof of Theorem 3.17 to conclude that

$$\hat{\xi}_i \circ \mathbf{ch} = \varepsilon_i \in \operatorname{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)\right) .$$

Our aim is to construct a compatible family $(\hat{\Phi}_i)_i \in \operatorname{Hom}^{\mathrm{cont}}_{[\text{-}1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)\right)$ such that $\rho((\hat{\Phi}_i)_i) = (\varepsilon_i)_i$ and $\hat{\xi}_i \circ \mathbf{ch} = \hat{\Phi}_i \circ \mathbf{ch}$ for all $i \in \mathbb{N}$.

To this end we apply Proposition 2.2 and Lemma 3.20 to conclude that there is, for each $i \in \mathbb{N}$, a maximal $J(i) \in \mathbb{N}$ such that $\ker(g^*_{J(i)}) \subset \ker(\hat{\xi}_i)$. We get the diagram

$$\begin{tikzcd}
 & \mathbf{KU}^0(B) \arrow[r, "\mathbf{ch}"] \arrow[d, two heads, "g^*_{J(i)}"] & \mathbf{HP}\mathbb{Q}^0(B) \arrow[r, "\hat{\xi}_i"] \arrow[d, two heads, "g^*_{J(i)}"] & \Omega\mathrm{P}^{\text{-}1}(B_i)/(\operatorname{im} d) \arrow[r, "f^*_{i-1}"] & \Omega\mathrm{P}^{\text{-}1}(B_{i-1})/(\operatorname{im} d) \\
\mathbf{KU}^0(B_{J(i)}) \arrow[d] & \operatorname{im}(g^*_{J(i)}) \arrow[l, hook'] \arrow[r, "\mathbf{ch}"] \arrow[d, two heads] & \operatorname{im}(g^*_{J(i)}) \arrow[ur, "\hat{\xi}_i"'] \arrow[d, two heads] & & \\
\mathbf{KU}^0(B_{J(i-1)}) & \operatorname{im}(g^*_{J(i-1)}) \arrow[l, hook'] \arrow[r, "\mathbf{ch}"] & \operatorname{im}(g^*_{J(i-1)}) \arrow[uurr, "\hat{\xi}_{i-1}"'] & &
\end{tikzcd} .$$

Now we define compatible maps $\Phi_i \in \operatorname{Hom}^{\mathrm{cont}}_{[\text{-}1]}\left(\mathbf{HP}\mathbb{Q}^0(B_{J(i)}), \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)\right)$ as follows. On the image of $\mathbf{KU}^0(B) \otimes \mathbb{Q}$ in $\mathbf{HP}\mathbb{Q}^0(B_{J(i)})$ we set $\Phi_i := \hat{\xi}_i$. Denote the image of $\mathbf{HP}\mathbb{R}^{\text{-}1}(B)$ in $\mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)$ as X_i. Then we get the diagram of discrete abelian groups

$$\begin{tikzcd}
\operatorname{im}(\mathbf{KU}^0(B) \otimes \mathbb{Q}) \arrow[r, hook] \arrow[d, equal] \arrow[rr, bend left, "\hat{\xi}_i"] & \mathbf{HP}\mathbb{Q}^0(B_{J(i)}) \arrow[r, dotted, "\Phi_i"] \arrow[d, "\hat{f}^*_{J(i-1)}"] & X_i \arrow[d, two heads, "f^*_{i-1}"] \\
\operatorname{im}(\mathbf{KU}^0(B) \otimes \mathbb{Q}) \arrow[r, hook] \arrow[rr, bend right, "\hat{\xi}_{i-1}"'] & \mathbf{HP}\mathbb{Q}^0(B_{J(i-1)}) \arrow[r, dotted, "\Phi_{i-1}"] & X_{i-1}
\end{tikzcd}$$

and we want to find the dotted maps (here $\hat{f}_{J(i-1)} := f_{J(i)-1} \circ \ldots \circ f_{J(i-1)}$). Since the system $\left(\mathbf{HP}\mathbb{Q}^0(B_{J(i)}), \hat{f}^*_{J(i)}\right)_{i\in\mathbb{N}}$ consists of finite dimensional vector space it satisfies the Mittag-Leffler condition. Thus we can find a cofinal subsystem of $\mathbb{N}$ such that the image of the transition map $\hat{f}^*_{J(i)}$ is already the stable image, i.e.,

$$\mathrm{im}(\hat{f}^*_{J(i)}) = \bigcap_{k\geq i} \mathrm{im}(\hat{f}^*_{J(i)} \circ \ldots \circ \hat{f}^*_{J(k-1)}) \subset \mathbf{HP}\mathbb{Q}^0(B_{J(i)}) \ .$$

This is justified by Proposition 3.36. Next we inductively choose decompositions (as $\mathbb{Q}$-vector spaces) $\mathbf{HP}\mathbb{Q}^0(B_{J(i)}) \cong \mathrm{im}(\mathbf{KU}^0(B) \otimes \mathbb{Q}) \oplus A_i$ for all $i \in \mathbb{N}$ such that the transition maps $\hat{f}^*_{J(i-1)}$ take the form

$$\begin{pmatrix} \mathtt{id}_{\mathrm{im}(\mathbf{KU}^0(B)\otimes\mathbb{Q})} & 0 \\ 0 & \hat{f}^*_{J(i-1)} \end{pmatrix} \ .$$

Therefore we need to find maps internal degree shifting maps Φ_i such that

$$\begin{tikzcd} A_{i+1} \arrow[r, dotted, "\Phi_{i+1}"] \arrow[d, "\hat{f}^*_{J(i)}"] & X_{i+1} \arrow[r, hook] \arrow[d, two heads, "f_i^*"] & \mathbf{HP}\mathbb{R}^{\text{-}1}(B_{i+1}) \arrow[d, "f_i^*"] \\ A_i \arrow[r, dotted, "\Phi_i"] \arrow[d, "\hat{f}^*_{J(i-1)}"] & X_i \arrow[r, hook] \arrow[d, two heads, "f_{i-1}^*"] & \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i) \arrow[d, "f_{i-1}^*"] \\ A_{i-1} \arrow[r, dotted, "\Phi_{i-1}"] & X_{i-1} \arrow[r, hook] & \mathbf{HP}\mathbb{R}^{\text{-}1}(B_{i-1}) \end{tikzcd}$$

commutes. We choose Φ_0 arbitrarily and inductively construct Φ_{i+1} using that $X_{i+1} \stackrel{f_i^*}{\to} X_i$ is surjective.

This gives a compatible system of maps $\Phi_i \in \mathrm{Hom}^{\mathrm{cont}}_{[\text{-}1]}\left(\mathbf{HP}\mathbb{Q}^0(B_{J(i)}), \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)\right)$. So we get $(\hat{\Phi}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[\text{-}1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i)\right)$ such that $\rho((\hat{\Phi}_i)_i) = (\varepsilon_i)_i$ and $\hat{\xi}_i \circ \mathbf{ch} = \hat{\Phi}_i \circ \mathbf{ch}$ for all $i \in \mathbb{N}$. This completes the proof. □

Remark 3.38. The strategy of this proof will be used again in Chapter 5 to prove the Existence Theorem 3.27. ■

Now we study the dependence of the approximation by manifolds. For this we need maps between approximations.

Definition 3.39. *Take two approximations by manifolds $(B_i, f_i, g_i)_{i\in\mathbb{N}}$ and $(\tilde{B}_i, \tilde{f}_i, \tilde{g}_i)_{i\in\mathbb{N}}$ of B. A **map of approximations** consists of*

- *a non-decreasing map $J : \mathbb{N} \to \mathbb{N}$,*
- *smooth maps $h_i : B_i \to \tilde{B}_{J(i)}$ and*
- *for all $i \in \mathbb{N}$ a homotopy filling the diagram*

$$\begin{tikzcd} B_i \arrow[rr, "h_i"] \arrow[dr, "g_i"'] & & \tilde{B}_{J(i)} \arrow[dl, "\tilde{g}_{J(i)}"] \\ & B & \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} \ldots \arrow[r, hook] & B_{i-1} \arrow[r, hook, "f_{i-1}"] \arrow[d, "h_{i-1}"] & B_i \arrow[r, hook, "f_i"] \arrow[d, "h_i"] & B_{i+1} \arrow[r, hook] \arrow[d, "h_{i+1}"] & \ldots \\ \ldots \arrow[r, hook] & \tilde{B}_{J(i-1)} \arrow[r, hook, "\tilde{f}_{J(i)-1}\circ\ldots\circ\tilde{f}_{J(i-1)}"] & \tilde{B}_{J(i)} \arrow[r, hook, "\tilde{f}_{J(i+1)-1}\circ\ldots\circ\tilde{f}_{J(i)}"] & \tilde{B}_{J(i+1)} \arrow[r, hook] & \ldots \end{tikzcd} \tag{3.7}$$

commutes strictly.

Fix two such maps of approximations $(J,(h_i)_i)$ and $(\hat{J},(\hat{h}_i)_i)$ between the given approximations. A ***homotopy between maps of approximations*** *from $(J,(h_i)_i)$ to $(\hat{J},(\hat{h}_i)_i)$ consists of*

- *a non-decreasing map $K:\mathbb{N}\to\mathbb{N}$ with $K(i)\geq\max\left\{J(i),\hat{J}(i)\right\}$ for all $i\in\mathbb{N}$,*
- *smooth homotopies $H_i: B_i\times\mathbb{R}\to\tilde{B}_{K(i)}$ between $\tilde{f}_{K(i)-1}\circ\ldots\circ\tilde{f}_{J(i)}\circ h_i$ and $\tilde{f}_{K(i)-1}\circ\ldots\circ\tilde{f}_{\hat{J}(i)}\circ\hat{h}_i$ and*
- *for all $i\in\mathbb{N}$ a homotopy filling the diagram*

$$\begin{tikzcd} B_i\times\mathbb{R} \arrow[rr, "H_i"] \arrow[dr, "g_i\circ\mathrm{pr}"'] & & \tilde{B}_{K(i)} \arrow[dl, "\tilde{g}_{K(i)}"] \\ & B & \end{tikzcd}$$

such that

- *the homotopies restrict to the given ones on the boundaries $B_i\times\{0,1\}$ and*
- *the diagram*

$$\begin{tikzcd} \ldots \arrow[r, hook] & B_{i-1}\times\mathbb{R} \arrow[r, hook, "f_{i-1}"] \arrow[d, "H_{i-1}"] & B_i\times\mathbb{R} \arrow[r, hook, "f_i"] \arrow[d, "H_i"] & B_{i+1}\times\mathbb{R} \arrow[r, hook] \arrow[d, "H_{i+1}"] & \ldots \\ \ldots \arrow[r, hook] & \tilde{B}_{K(i-1)} \arrow[r, hook, "\tilde{f}_{K(i)-1}\circ\ldots\circ\tilde{f}_{K(i-1)}"] & \tilde{B}_{K(i)} \arrow[r, hook, "\tilde{f}_{K(i+1)-1}\circ\ldots\circ\tilde{f}_{K(i)}"] & \tilde{B}_{K(i+1)} \arrow[r, hook] & \ldots \end{tikzcd}$$

  commutes strictly. ■

Standard smoothening theory and the fact that B is the homotopy colimit over each of its approximations yields the following existence result.

Theorem 3.40. *Fix two approximations by manifolds of some topological space.*

1. *There exists a map of approximations between these two approximations.*
2. *Any two maps of approximations can be connected by a homotopy between maps of approximations.* ♦

We fix two approximations by manifolds $(B_i,f_i,g_i)_{i\in\mathbb{N}}$ and $(\tilde{B}_i,\tilde{f}_i,\tilde{g}_i)_{i\in\mathbb{N}}$ of B and choose a map of approximations $(J,(h_i)_i)$ between them. Given a universal geometrization $(\tilde{\mathcal{G}}_i)_i$ with trivial Todd class and trivial Todd and error forms on the approximation $(\tilde{B}_i,\tilde{f}_i,\tilde{g}_i)_i$ we get a family of geometrizations $\mathcal{G}_i := h_i^*\tilde{\mathcal{G}}_{J(i)}$ on the other approximation. We have the following easy observation.

Lemma 3.41. *The family of geometrizations $\mathcal{G}_i := h_i^* \tilde{\mathcal{G}}_{J(i)}$ is again a universal geometrization with trivial Todd class and trivial Todd and error forms.*

Thus we can pullback universal geometrizations along maps of approximations. ♦

Proof. This follows since the squares in diagram (3.7) commute strictly. □

Assume we are given $(\hat{\xi}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(\tilde{B}_i)/(\operatorname{im} d)\right)$. Then we get $(\hat{\tau}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\left(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)\right)$ defined by $\hat{\tau}_i := h_i^* \hat{\xi}_{J(i)}$. This gives two more geometrizations $\tilde{\mathcal{G}}_i^2 := \tilde{\mathcal{G}}_i - a \circ \hat{\xi}_i \circ \mathbf{ch}$ and $\mathcal{G}_i^2 := \mathcal{G}_i - a \circ \hat{\tau}_i \circ \mathbf{ch}$.

Lemma 3.42. *We have $\mathcal{G}_i^2 = h_i^* \tilde{\mathcal{G}}^2_{J(i)}$.*

Hence pullback of universal geometrizations along a map of approximations preserves equivalence classes. ♦

Now recall Theorem 3.37 which stated that the set of equivalence classes of universal geometrizations on a fixed approximation is a torsor for the classifying group $\operatorname{coker}(\hat{\rho})$ which is independent of the approximation by Definition and Proposition 3.34. A straightforward computation yields the following result.

Lemma 3.43. *Pullback of equivalence classes of universal geometrizations along a map of approximations is equivariant with respect to the action of the classifying group* $\operatorname{coker}(\hat{\rho})$. ♦

Proof. We show that the actions agree. To this end take a universal geometrization $(\tilde{\mathcal{G}}_i)_i$ on the approximation $(\tilde{B}_i, \tilde{f}_i, \tilde{g}_i)_i$ and a family

$$(\varepsilon_C)_C \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(C)/(\operatorname{im} \mathbf{ch})\right) .$$

Then we get two universal geometrizations on the approximation $(B_i, f_i, g_i)_i$, namely

$$\mathcal{G}_i^1 := \left(h_i^* \tilde{\mathcal{G}}_{J(i)}\right) - a \circ \varepsilon_{B_i} \quad \text{and} \quad \mathcal{G}_i^2 := h_i^* \left(\tilde{\mathcal{G}}_{J(i)} - a \circ \varepsilon_{\tilde{B}_{J(i)}}\right) .$$

But $\mathcal{G}_i^2 = h_i^* \tilde{\mathcal{G}}_{J(i)} - a \circ h_i^* \circ \varepsilon_{\tilde{B}_{J(i)}} = \mathcal{G}_i^1 - a \circ \left(h_i^* \circ \varepsilon_{\tilde{B}_{J(i)}} - \varepsilon_{B_i}\right) = \mathcal{G}_i^1$ since $(\varepsilon_C)_C$ is an element in the limit. □

Finally we show that the equivalence class of the pullback of a universal geometrization along a map of approximations is independent of the map of approximations.

Lemma 3.44. *Given two maps of approximations the resulting pullback universal geometrizations are equivalent.* ♦

Proof. Denote the two maps of approximations by $(J^1, (h_i^1)_i)$ and $(J^0, (h_i^0)_i)$. Then we find a homotopy $(K, (H_i)_i)$ between these two maps of approximations (Theorem 3.40, 2.). By Proposition 3.36 we can assume that $J^1 = J^0 = K = \mathtt{id} : \mathbb{N} \to \mathbb{N}$. We get two universal geometrizations on the approximation $(B_i, f_i, g_i)_i$, namely

$$\mathcal{G}_i^1 := (h_i^1)^* \tilde{\mathcal{G}}_i \quad \text{and} \quad \mathcal{G}_i^0 := (h_i^0)^* \tilde{\mathcal{G}}_i .$$

Denote the boundary inclusions $B_i \to B_i \times I$ by incl_0 and incl_1. We apply the homotopy formula (2.3) to calculate the difference ($\tilde{c}_i$ denotes the cohomological character of $\tilde{\mathcal{G}}_i$),

$$\begin{aligned}\mathcal{G}_i^1 - \mathcal{G}_i^0 =& (h_i^1)^* \tilde{\mathcal{G}}_i - (h_i^0)^* \tilde{\mathcal{G}}_i = \mathrm{incl}_1^* H_i^* \tilde{\mathcal{G}}_i - \mathrm{incl}_0^* H_i^* \tilde{\mathcal{G}}_i \overset{(2.3)}{=} \\ =& a\left(\int_{I \times B_i / B_i} R \circ H_i^* \circ \tilde{\mathcal{G}}_i(-)\right) = a\left(\int_{I \times B_i / B_i} H_i^* \circ R \circ \tilde{\mathcal{G}}_i(-)\right) =\end{aligned}$$

$$=a\left(\int_{I\times B_i/B_i} H_i^*\circ \tilde{c}_i\right)\circ \mathbf{ch}(-)\ .$$

Note that $\xi_i := \int_{I\times B_i/B_i} H_i^*\circ \tilde{c}_{J(i)}(-) \in \mathrm{Hom}^{\mathrm{cont}}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\mathrm{im}\, d)\big)$ shifts the internal degree by -1 and that $f_i^*\xi_{i+1} = \xi_i$. Hence the family $(-\xi_i)_i$ shows that the two universal geometrizations $(\mathcal{G}_i^1)_i$ and $(\mathcal{G}_i^0)_i$ are equivalent. □

The preceding lemmas give our main classification theorem.

Classification Theorem 3.45. *Let B denote a topological space.*

Then the set of universal geometrizations for B with trivial Todd class and trivial Todd and error forms is a (possible empty) torsor over the classifying group $\mathrm{coker}(\hat{\rho})$. *This group is a weak homotopy invariant of B.* ♦

This combines with our Existence Theorem 3.27 to the following uniqueness statement.

Theorem 3.46. *Let B be a topological space which is of rationally even and admits an approximation by manifolds (e.g., which is simply-connected and of countable type).*

Then there exists a unique equivalence class of universal geometrizations for B with trivial Todd class and trivial Todd and error forms. ♦

Proof. The uniqueness statement follows directly from the Classification Theorem 3.45. The existence is implied by our Existence Theorem 3.27. The technical condition there is satisfied (see Remark 3.28). □

Chapter 4

Examples of universal geometrizations

In this short chapter we discuss first examples. In each example we fix a space B and find an approximation by manifolds. Then we check that Existence Theorem 3.27 is applicable and thus that there is a universal geometrization. Then we calculate the number of universal geometrizations up to equivalence using Classification Theorem 3.45.

In most examples we consider only trivial Todd classes and trivial Todd and error forms. We postpone the explicit construction of universal geometrizations to Chapter 8. There we use them to calculate the intrinsic eta-invariant.

4.1 Spheres and tori

The sphere S^n and the torus T^n are compact manifolds. Thus there exists a universal geometrization by Example 3.21 together with Example 3.12. The classifying group $\operatorname{coker}(\hat{\rho})$ of Theorem 3.17 can be calculated by a short computation. The result is

$$\operatorname{coker}(\hat{\rho}) = \begin{cases} 0 & \text{for} \quad S^{2n}, \\ \mathbb{R}/\mathbb{Z} & \text{for} \quad S^{2n+1}, \\ \mathbb{R}/\mathbb{Z}^{I(n)} & \text{for} \quad T^n. \end{cases}$$

Here $I(n) := \left(\sum_k \binom{n}{2k}\right) \cdot \left(\sum_k \binom{n}{2k+1}\right) - \left(\sum_k \binom{n}{2k}\binom{n}{2k+1}\right) = \sum_{k \neq l} \binom{n}{2k}\binom{n}{2l+1}$.

4.2 Classifying spaces of compact Lie groups

Let G be a compact connected Lie group and BG its classifying space. Then there exists a universal geometrization by Example 3.29. Moreover, BG is of finite type and rationally even. Thus we can apply Theorem 3.46 to see that this universal geometrization is unique up to equivalence.

4.3 The classifying space of the string group

We define the classifying space for "string principal bundles", denoted by $BString$, as the homotopy fibre of $\frac{p_1}{2} : BSpin \to K(\mathbb{Z}, 4)$. Using the long exact sequence for homotopy groups of a fibration we see that the $BString$ is simply-connected and has degree-wise finitely generated homotopy groups. Thus Example 3.22 guarantees the existence of an approximation by manifolds. Moreover, $BString$ can be realized as a CW-complex of finite type. A straightforward application of the Serre spectral sequence shows that $BString$ is rationally even (cf. [BS09a, Theorem 2]). Hence we can apply Theorem 3.46 to find a unique equivalence class of universal geometrizations for $BString$ with trivial Todd class and trivial Todd and error forms.

Note that the same arguments work for all other higher covers $BO\langle n\rangle$ of BO.

This fits nicely with the theory of String connections due to Waldorf ([Wal13, Definition 1.2.2]). There Waldorf proves that any String structure on a Spin principal bundle admits a String connection and that these form a contractible space ([Wal13, Theorem 1.3.4]).

4.4 Compact Lie groups

Let G be a compact connected simply-connected semi-simple Lie group. Then G is a compact manifold and hence has a constant approximation by manifolds (Example 3.21) and a universal geometrization with trivial Todd and error forms (Example 3.12). The classifying group can be calculated from Theorem A in [Hod67]. There Hodgkin proves that $\mathbf{KU}^*(G)$ is a finitely generated exterior algebra over $\mathbb{Z}$ and gives generators. This is enough to calculate the classifying group $\operatorname{coker}(\rho)$. Since the abstract formula is not enlightening we note that the classifying group is (usually) big and discuss an example.

For the example we discuss the Lie group $SU(3)$. This compact Lie groups is semi-simple of rank 2 and therefore [Hod67, Theorem A (v)] yields that the complex K-theory is an exterior algebra on odd generators x_3 and x_5,

$$\mathbf{KU}^*(SU(3)) = \Lambda(x_3, x_5) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{for} \quad * = \text{even}, \\ \mathbb{Z}\oplus\mathbb{Z} & \text{for} \quad * = \text{odd}. \end{cases}$$

Since $SU(3)$ is compact the Chern character is a rational isomorphism and we get

$$\mathbf{HP}\mathbb{Q}^*(SU(3)) = \begin{cases} \mathbb{Q} & \text{for} \quad * = 0, \\ \mathbb{Q} & \text{for} \quad * = 3, \\ \mathbb{Q} & \text{for} \quad * = 5, \\ \mathbb{Q} & \text{for} \quad * = 8. \end{cases}$$

Again by compactness, the map $\mathtt{id} : SU(3) \to SU(3)$ is cofinal in the category $\mathcal{C}/SU(3)$. Therefore the classifying group is the cokernel of

$$\begin{aligned} \rho : \operatorname{Hom}^{\text{cont}}_{[-1]} &\left(\mathbf{HP}\mathbb{Q}^0(SU(3)), \mathbf{HP}\mathbb{R}^{-1}(SU(3))\right) \to \\ &\operatorname{Hom}^{\text{cont}}\left(\mathbf{KU}^0(SU(3)), \mathbf{HP}\mathbb{R}^{-1}(SU(3))/(\operatorname{im}\mathbf{ch})\right) \end{aligned}$$

and we conclude

$$\operatorname{coker}(\rho) \cong \operatorname{Hom}\left(\mathbb{Z}\oplus\mathbb{Z}, {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \oplus {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}\right) \cong {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}{}^4.$$

4.5 Loop spaces of Lie groups

Let G be a connected simply-connected Lie group. Set $B := \Omega G$, the based (at the identity) loop space of G.

Then B is connected, simply-connected and has degree-wise finitely generated homotopy groups. Hence there is an approximation by manifolds for B by Example 3.22. Moreover, B is of finite type. The Serre spectral sequence of the path space fibration shows that B is rationally even. Therefore there is a unique equivalence class of universal geometrizations on B by Theorem 3.46.

4.6 Countable coproducts

The aim of this section is to construct countable coproducts of universal geometrizations. So consider topological spaces $(B^k)_{k\in\mathbb{N}}$ with Todd classes $\mathbf{td}^k$ and assume we are given universal geometrizations

$$(B_i^k, f_i^k, g_i^k, \mathbf{Td}_i^k, \delta_i^k, \mathcal{G}_i^k)_{i\in\mathbb{N}}$$

for each B^k. Then one directly checks that the family $\hat{B}_i := \coprod_{k\leq i} B_k^i$, $i \in \mathbb{N}$, together with the induced maps is an approximation by manifolds for $\hat{B} := \coprod_k B^k$. We get a universal geometrization

$$(\hat{B}_i, \hat{f}_i, \hat{g}_i, \hat{\mathbf{Td}}_i, \hat{\delta}_i, \hat{\mathcal{G}}_i)_i$$

for $\hat{B}$ where $\hat{\mathbf{Td}}_i$ and $\hat{\delta}_i$ denote the obvious forms, e.g., $\hat{\mathbf{Td}}_i := \sum_{k\leq i} \mathbf{Td}_k$, and $\hat{\mathcal{G}}^i$ is the composition

$$\mathbf{KU}^0(\hat{B}) \to \mathbf{KU}^0(\hat{B}_i) \cong \oplus_{k\leq i} \mathbf{KU}^0(B_k) \stackrel{\oplus \mathcal{G}_i}{\to} \oplus_{k\leq i} \widehat{\mathbf{KU}^0}(B_k) \cong \widehat{\mathbf{KU}^0}(\hat{B}_i) \ .$$

Note that the classifying group of a coproduct is bigger than the product of the classifying groups. This example shows nicely that we don't need to assume that B is of finite type.

Chapter 5

Proof of the Existence Theorem

The proof of Existence Theorem 3.27 relies on some facts which we prove at first. In the following proposition we try to understand the K-theory of homotopy colimits.

Proposition 5.1. *Let $(B_i)_{i\in\mathbb{N}}$ be a family of finite CW-complexes and $f_i : B_i \to B_{i+1}$ be continuous maps. Denote the homotopy colimit over this system by $B := \operatorname{hocolim}_{i\in\mathbb{N}} B_i$. Then the universal coefficient theorem induces a (non-canonical) topological isomorphism*

$$\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i) \xrightarrow{\cong} \varprojlim_{i\in\mathbb{N}} \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \oplus \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))$$

such that

$$\begin{tikzcd}
\varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i)) \arrow[d, hook] \arrow[dr, "0\oplus\mathtt{id}"] & \\
\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i) \arrow[r, "\cong"] & \varprojlim_{i\in\mathbb{N}} \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \oplus \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))
\end{tikzcd}$$

commutes. Moreover, $\varprojlim_{i\in\mathbb{N}} \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \cong \prod_\kappa \mathbb{Z}$ where $\kappa \in \mathbb{N}_{\geq 0} \cup \{\infty\}$. ♦

Proof. Since **KU** is Anderson self-dual we have a natural universal coefficient theorem ([Yos75, Proposition 6 and Theorem 5]). So we get for an arbitrary map $C \to B$ a commutative diagram with short exact rows,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \operatorname{Ext}(\mathbf{KU}_{\text{-}1}(B),\mathbb{Z}) \arrow[r, hook, "\iota"] \arrow[d] & \mathbf{KU}^0(B) \arrow[r, two heads, "\pi"] \arrow[d] & \operatorname{Hom}(\mathbf{KU}_0(B),\mathbb{Z}) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Ext}(\mathbf{KU}_{\text{-}1}(C),\mathbb{Z}) \arrow[r, hook, "\iota"] & \mathbf{KU}^0(C) \arrow[r, two heads, "\pi"] & \operatorname{Hom}(\mathbf{KU}_0(C),\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}.$$

Note that, in general, the maps $\mathbf{KU}^0(B) \to \mathbf{KU}^0(B_i)$ are not surjective. Hence we define $X^i := \operatorname{im}(\mathbf{KU}^0(B) \to \mathbf{KU}^0(B_i))$ and $X_i := \operatorname{im}(\operatorname{Hom}(\mathbf{KU}_0(B),\mathbb{Z}) \to \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}))$ for all $i \in \mathbb{N}$ and one easily checks that the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \operatorname{Ext}(\mathbf{KU}_{\text{-}1}(B),\mathbb{Z}) \arrow[r, hook, "\iota"] \arrow[d] & \mathbf{KU}^0(B) \arrow[r, two heads, "\pi"] \arrow[d, two heads] & \operatorname{Hom}(\mathbf{KU}_0(B),\mathbb{Z}) \arrow[r] \arrow[d, two heads] & 0 \arrow[d, equal] \\
0 \arrow[r] \arrow[d, equal] & \operatorname{Ext}(\mathbf{KU}_{\text{-}1}(B_i),\mathbb{Z}) \cap \iota^{-1}(X^i) \arrow[r, hook, "\iota"] \arrow[d, hook] & X^i \arrow[r, two heads, "\pi"] \arrow[d, hook, "\psi_i"] & X_i \arrow[r] \arrow[d, hook] & 0 \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Ext}(\mathbf{KU}_{\text{-}1}(B_i),\mathbb{Z}) \arrow[r, hook, "\iota"] & \mathbf{KU}^0(B_i) \arrow[r, two heads, "\pi"] & \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \arrow[r] & 0
\end{tikzcd} \tag{5.1}$$

is well-defined, commutative and has exact rows. The maps $f_i : B_i \to B_{i+1}$ induce maps $f_i^* : X^{i+1} \to X^i$ and $f_i^* : X_{i+1} \to X_i$ and those maps are in fact surjective.

Since B_i is a finite CW-complex for all $i \in \mathbb{N}$ we have inverse systems $(X^i, f_i^*)_{i\in\mathbb{N}}$ and $(X_i, f_i^*)_{i\in\mathbb{N}}$ consisting of finitely generated abelian groups and surjective maps. Moreover, the X_i are free and because $\mathrm{Ext}(\mathbf{KU}_{\text{-}1}(B_i),\mathbb{Z}) \cap \iota^{-1}(X^i)$ is a torsion group it has to be the torsion subgroup, i.e., $\mathrm{Torsion}(X^i) = \mathrm{Ext}(\mathbf{KU}_{\text{-}1}(B_i),\mathbb{Z}) \cap \iota^{-1}(X^i)$. Similarly, $\mathrm{Torsion}(\mathbf{KU}^0(B_i)) = \mathrm{Ext}(\mathbf{KU}_{\text{-}1}(B_i),\mathbb{Z})$.

Since the projections $\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i) \to \mathbf{KU}^0(B_i)$ factor through X^i (by Proposition 2.2) we get a topological isomorphism $\Psi := \varprojlim_i \psi_i : \varprojlim_{i\in\mathbb{N}} X^i \xrightarrow{\cong} \varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i)$. Now we take the limit over the lower two rows of diagram (5.1) where the $\varprojlim^1$-terms vanishes because the left columns consist of finite abelian groups. This gives

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \varprojlim_i \mathrm{Torsion}(X^i) \arrow[r, hook, "\iota"] \arrow[d, hook] & \varprojlim_i X^i \arrow[r, two heads, "\pi"] \arrow[d, hook, "\Psi", "\cong"'] & \varprojlim_i X_i \arrow[r] \arrow[d, hook] & 0 \arrow[d, equal] \\
0 \arrow[r] & \varprojlim_i \mathrm{Torsion}(\mathbf{KU}^0(B_i)) \arrow[r, hook, "\iota"] & \varprojlim_i \mathbf{KU}^0(B_i) \arrow[r, two heads, "\pi"] & \varprojlim_i \mathrm{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

and we conclude that the outer vertical maps are also isomorphisms,

$$\varprojlim_i \mathrm{Torsion}(X^i) \cong \mathrm{Torsion}(\mathbf{KU}^0(B_i)) \quad \text{and} \quad \varprojlim_i X_i \cong \varprojlim_i \mathrm{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \ .$$

Our next aim is to understand $\varprojlim_{i\in\mathbb{N}} X^i$. To this end we apply [HM78, Corollary 3.8 for $G = \mathbb{Z}$] to find a split s,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(\mathbf{KU}_{\text{-}1}(B),\mathbb{Z}) \arrow[r, hook, "\iota"] & \mathbf{KU}^0(B) \arrow[r, two heads, "\pi"] & \mathrm{Hom}(\mathbf{KU}_0(B),\mathbb{Z}) \arrow[r] \arrow[l, bend right, "s"'] & 0 \ .
\end{tikzcd}$$

In the following we show that the split s induces compatible splits $s_i : X_i \to X^i$.

We choose a basis of X_0 and preimages in $\mathrm{Hom}(\mathbf{KU}_0(B),\mathbb{Z})$. Then by pushing down these preimages to X_1 we get a complement $\ker(f_0^*)^\perp$ of $\ker(f_0^*)$ in X_1, i.e., $X_1 = \ker(f_0^*) \oplus \ker(f_0^*)^\perp$. Now we repeat this procedure with $\ker(f_0^*)^\perp$ and inductively construct a compatible family of bases of all the X_i which are already lifted to $\mathrm{Hom}(\mathbf{KU}_0(B),\mathbb{Z})$. The splits s_i are defined by applying the given split s to these lifts and pushing down to X^i.

Hence we have constructed compatible splits $s_i : X_i \to X^i$ and get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Ext}(\mathbf{KU}_{\text{-}1}(B),\mathbb{Z}) \arrow[r, hook, "\iota"] \arrow[d] & \mathbf{KU}^0(B) \arrow[r, two heads, "\pi"] \arrow[d, two heads] & \mathrm{Hom}(\mathbf{KU}_0(B),\mathbb{Z}) \arrow[r] \arrow[d, two heads] & 0 \arrow[d, equal] \\
0 \arrow[r] \arrow[d, equal] & \mathrm{Torsion}(X^{i+1}) \arrow[r, hook, "\iota"] \arrow[d] & X^{i+1} \arrow[r, two heads, "\pi"] \arrow[d, two heads, "f_i^*"] & X_{i+1} \arrow[r] \arrow[d, two heads, "f_i^*"] \arrow[l, bend right, "s_{i+1}"'] & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Torsion}(X^i) \arrow[r, hook, "\iota"] & X^i \arrow[r, two heads, "\pi"] & X_i \arrow[r] \arrow[l, bend right, "s_i"'] & 0
\end{tikzcd} .$$

Thus we get compatible topological[1] isomorphisms $X^i \cong X_i \oplus \mathrm{Torsion}(X^i)$ for all $i \in \mathbb{N}$ which yield a topological isomorphism

$$\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i) \cong \varprojlim_{i\in\mathbb{N}} X^i \cong \left(\varprojlim_{i\in\mathbb{N}} X_i\right) \oplus \left(\varprojlim_{i\in\mathbb{N}} \mathrm{Torsion}(X^i)\right) \ .$$

[1]Continuity is automatic since all the X_i and the X^i are discrete.

This combines with the above isomorphisms to the claimed isomorphism

$$\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i) \cong \left(\varprojlim_{i\in\mathbb{N}} \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z})\right) \oplus \left(\varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))\right) .$$

For the last statement one just combines the compatible bases of the X_i to get an isomorphism

$$\varprojlim_{i\in\mathbb{N}} \operatorname{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \cong \varprojlim_{i\in\mathbb{N}} X_i \cong \prod_{\kappa} \mathbb{Z}$$

where the right hand side denotes a direct product and $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. □

In the next proposition we study the profinite group $\varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))$.

Proposition 5.2. *As before, let $(B_i)_{i\in\mathbb{N}}$ be a family of finite CW-complexes and $f_i : B_i \to B_{i+1}$ be continuous maps, $i \in \mathbb{N}$. Then the Bockstein sequence for $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces the short exact sequence*

$$\varprojlim_{i\in\mathbb{N}} (\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i)) \hookrightarrow \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i) \twoheadrightarrow \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))$$

and this sequence is split exact. ◆

Proof. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields the natural Bockstein long exact sequence (see Section 2.3)

$$\ldots \to \mathbf{KU}^{-1}(X) \to \mathbf{KU}\mathbb{Q}^{-1}(X) \to \mathbf{KU}\mathbb{Q}/\mathbb{Z}^{-1}(X) \to \mathbf{KU}^0(X) \to \mathbf{KU}\mathbb{Q}^0(X) \to \ldots$$

where X denotes a finite CW-complex. Hence we get a short exact sequence

$$0 \to \mathbf{KU}\mathbb{Q}^{-1}(X)/\mathbf{KU}^{-1}(X) \hookrightarrow \mathbf{KU}\mathbb{Q}/\mathbb{Z}^{-1}(X) \twoheadrightarrow \operatorname{Torsion}(\mathbf{KU}^0(X)) \to 0$$

which is natural in the finite CW-complex X.

We apply this construction to our family $(B_i, f_i)_{i\in\mathbb{N}}$ and take inverse limits. This yields the long exact sequence

$$\begin{aligned} 0 \to \varprojlim_{i\in\mathbb{N}} (\mathbf{KU}\mathbb{Q}^{-1}(B_i)/\mathbf{KU}^{-1}(B_i)) &\hookrightarrow \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\mathbb{Q}/\mathbb{Z}^{-1}(B_i) \to \\ \to \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i)) &\to \varprojlim^1_{i\in\mathbb{N}} (\mathbf{KU}\mathbb{Q}^{-1}(B_i)/\mathbf{KU}^{-1}(B_i)) \to \ldots . \end{aligned}$$

We claim that the term $\varprojlim^1_{i\in\mathbb{N}} (\mathbf{KU}\mathbb{Q}^{-1}(B_i)/\mathbf{KU}^{-1}(B_i))$ vanishes. To this end we show that the system $(\mathbf{KU}\mathbb{Q}^{-1}(B_i)/\mathbf{KU}^{-1}(B_i), f_i^*)$ satisfies the Mittag-Leffler condition by showing that the system $(\mathbf{KU}\mathbb{Q}^{-1}(B_i), f_i^*)$ satisfies it. But the latter is a system of finite-dimensional vector spaces and so the Mittag-Leffler condition holds.

Thus we get the short exact sequence

$$\varprojlim_{i\in\mathbb{N}} (\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i)) \hookrightarrow \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i) \twoheadrightarrow \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i)) .$$

Our next aim is to split this sequence. To this end we want to show that the group

$$\operatorname{Ext}\left(\varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i)), \varprojlim_{i\in\mathbb{N}} (\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i))\right)$$

is trivial and therefore that the sequence splits. But this is implied by the fact that

$$\varprojlim_{i\in\mathbb{N}} (\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i))$$

is an injective abelian group which is the statement of Proposition 5.3 below. □

Proposition 5.3. *Let $(A_n, f_n)_{n\in\mathbb{N}}$ be a family of finitely generated abelian groups. Then the limit*

$$\varprojlim_{n\in\mathbb{N}} ((A_n \otimes \mathbb{Q})/A_n)$$

is an injective abelian group. ♦

Recall that an abelian group is injective if and only if it is divisible.

Proof. Without loss of generality we can assume that the abelian groups A_n are free since we can replace the groups A_n by their quotients $A_n/\operatorname{Torsion}(A_n)$. Then we get a short exact sequence

$$0 \longrightarrow A_n \hookrightarrow A_n \otimes \mathbb{Q} \twoheadrightarrow (A_n \otimes \mathbb{Q})/A_n \longrightarrow 0$$

for all $n \in \mathbb{N}$. Note that the system $(A_n \otimes \mathbb{Q}, f_n)_n$ is Mittag-Leffler and hence we get the long exact sequence

$$0 \longrightarrow \varprojlim_{n\in\mathbb{N}} A_n \hookrightarrow \varprojlim_{n\in\mathbb{N}} (A_n \otimes \mathbb{Q}) \longrightarrow \varprojlim_{n\in\mathbb{N}} ((A_n \otimes \mathbb{Q})/A_n) \twoheadrightarrow \varprojlim^1_{n\in\mathbb{N}} A_n \longrightarrow 0\ .$$

Clearly, $\varprojlim_{n\in\mathbb{N}}(A_n \otimes \mathbb{Q})$ is a $\mathbb{Q}$-vector space and hence divisible. This gives the short exact sequence

$$0 \longrightarrow \left(\varprojlim_{n\in\mathbb{N}}(A_n \otimes \mathbb{Q})\right) / \left(\varprojlim_{n\in\mathbb{N}} A_n\right) \hookrightarrow \varprojlim_{n\in\mathbb{N}} ((A_n \otimes \mathbb{Q})/A_n) \twoheadrightarrow \varprojlim^1_{n\in\mathbb{N}} A_n \longrightarrow 0$$

where the left hand side is divisible and therefore the sequence splits. We conclude that

$$\varprojlim_{n\in\mathbb{N}} ((A_n \otimes \mathbb{Q})/A_n) \cong \varprojlim^1_{n\in\mathbb{N}} A_n \oplus \left(\varprojlim_{n\in\mathbb{N}}(A_n \otimes \mathbb{Q})\right) / \left(\varprojlim_{n\in\mathbb{N}} A_n\right)\ .$$

Therefore it remains to show that $\varprojlim^1_{n\in\mathbb{N}} A_n$ is divisible. For this we show that multiplication by $k \in \mathbb{N} \setminus \{0\}$ is surjective. Consider the short exact sequence

$$0 \longrightarrow A_n \xhookrightarrow{\cdot k} A_n \twoheadrightarrow A_n/(k \cdot A_n) \longrightarrow 0$$

and take the limit sequence,

$$0 \longrightarrow \varprojlim_{n\in\mathbb{N}} A_n \xhookrightarrow{\cdot k} \varprojlim_{n\in\mathbb{N}} A_n \longrightarrow \varprojlim_{n\in\mathbb{N}} A_n/(k \cdot A_n) \longrightarrow$$

$$\longrightarrow \varprojlim^1_{n\in\mathbb{N}} A_n \xrightarrow{\cdot k} \varprojlim^1_{n\in\mathbb{N}} A_n \twoheadrightarrow \varprojlim^1_{n\in\mathbb{N}} A_n/(k \cdot A_n) \longrightarrow 0\ .$$

But the groups $A_n/(k \cdot A_n)$ are finite and hence $\varprojlim^1_{n\in\mathbb{N}} A_n/(k \cdot A_n)$ vanishes. This implies that the map $(-) \cdot k : \varprojlim^1_{n\in\mathbb{N}} A_n \to \varprojlim^1_{n\in\mathbb{N}} A_n$ is surjective.

Thus we have shown that $\varprojlim_{n\in\mathbb{N}} ((A_n \otimes \mathbb{Q})/A_n)$ is injective. □

Now we check that the technical condition of our Existence Theorem 3.27 is satisfied under some mild finiteness condition.

Proposition 5.4. *Additionally to the assumption of Proposition 5.2 suppose that the profinite abelian group $\varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))$ is topologically finitely generated. Then the split short exact sequence of Proposition 5.2,*

$$\varprojlim_{i\in\mathbb{N}}(\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i)) \hookrightarrow \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i) \twoheadrightarrow \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i))\ ,$$

admits a split which is continuous. ♦

Proof. Our aim is to find a continuous split s of the short exact sequence

$$\begin{tikzcd} \varprojlim_{i\in\mathbb{N}}(\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i)) \arrow[r, hook] & \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i) \arrow[r, two heads] & \varprojlim_{i\in\mathbb{N}} \mathrm{Torsion}\left(\mathbf{KU}^0(B_i)\right) \arrow[l, dotted, bend right, "s"'] \end{tikzcd} .$$

Note that $\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i) \cong \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}$. Recall that the outer systems

$$(\mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}, f_i^*)_i \quad \text{and} \quad \left(\mathrm{Torsion}\left(\mathbf{KU}^0(B_i)\right), f_i^*\right)_i$$

satisfy the Mittag-Leffler condition. Therefore $\varprojlim^1_{i\in\mathbb{N}} \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i)$ vanishes by the long exact sequence

$$\begin{aligned} 0 \to \varprojlim_{i\in\mathbb{N}}(\mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}) &\hookrightarrow \varprojlim_{i\in\mathbb{N}} \mathbf{KU}{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i) \to \varprojlim_{i\in\mathbb{N}} \mathrm{Torsion}(\mathbf{KU}^0(B_i)) \to \\ \to \underbrace{\varprojlim^1_{i\in\mathbb{N}}(\mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}})}_{=0} &\to \varprojlim^1_{i\in\mathbb{N}} \mathbf{KU}{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i) \twoheadrightarrow \underbrace{\varprojlim^1_{i\in\mathbb{N}} \mathrm{Torsion}(\mathbf{KU}^0(B_i))}_{=0} \to 0 . \end{aligned}$$

This implies that the system $(\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i), f_i^*)_i$ also satisfies the Mittag-Leffler condition by [Gra66, Proposition on page 242].

Now we take stable images, i.e., we set

$$R^i := \bigcap_{k\geq i} f_i^* \circ \ldots \circ f_{k-1}^* \left(\mathbf{KU}^{\text{-}1}(B_k)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}\right) \subset \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}$$

and similarly for $S^i \subset \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i)$ and $T^i \subset \mathrm{Torsion}(\mathbf{KU}^0(B_i)$. Since the original systems satisfy the Mittag-Leffler condition the systems (R^i, f_i^*), (S^i, f_i^*) and (T^i, f_i^*) have surjective transition maps. The maps $R^i \to \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}$ induce a continuous map on the limits. But the projections $\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}} \to \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}$ factor through R^i. This gives a topological isomorphism $\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}} \xrightarrow{\cong} \varprojlim_{i\in\mathbb{N}} R^i$. This works similarly for the other two systems. Hence we want to find a continuous split of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim_{i\in\mathbb{N}} R^i \arrow[r, hook] & \varprojlim_{i\in\mathbb{N}} S^i \arrow[r, two heads] & \varprojlim_{i\in\mathbb{N}} T^i \arrow[r] \arrow[l, dotted, bend right, "s"'] & 0 \end{tikzcd} .$$

Next we recall that the abelian groups $\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}} \cong \varprojlim_{i\in\mathbb{N}} R^i$ are divisible (Proposition 5.3). Therefore the groups R^i are also divisible because the projections $\varprojlim_{i\in\mathbb{N}} R^i \to R^i$ are surjective for all $i\in\mathbb{N}$. We consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim_{i\in\mathbb{N}} R^i \arrow[r, hook, "\iota"] \arrow[d, two heads, "\pi^i"] & \varprojlim_{i\in\mathbb{N}} S^i \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\pi^i"] & \varprojlim_{i\in\mathbb{N}} T^i \arrow[r] \arrow[d, two heads, "\pi^i"] & 0 \\ & R^i \arrow[r] \arrow[d, hook] & S^i \arrow[r, "\pi"] \arrow[d, hook] & T^i \arrow[d, hook] & \\ 0 \arrow[r] & \mathbf{KU}^{\text{-}1}(B_i)\otimes {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}} \arrow[r, hook, "\iota"] & \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}^{-1}(B_i) \arrow[r, two heads, "\pi"] & \mathrm{Torsion}(\mathbf{KU}^0(B_i)) \arrow[r] & 0 \end{tikzcd} \tag{5.2}$$

where the top row and the bottom row are exact. Clearly, the maps $R^i \to S^i$ are injective, the maps $S^i \to T^i$ are surjective and their compositions are trivial for all $i\in\mathbb{N}$.

We show that the middle row is also exact. This takes all of following paragraph.

Denote the stable images of the system $(\mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}, f_i^*)_i$ by Q^i. Note that each Q^i is a $\mathbb{Q}$-vector space and that the maps $Q^i \to R^i$ are surjective. We denote the free finitely generated abelian groups $\mathbf{KU}^{-1}(B_i)/\operatorname{Torsion}(\mathbf{KU}^{-1}(B_i))$ by Z^i and set $P^i := \ker(Q^i \twoheadrightarrow R^i)$. This gives the commutative diagram (with short exact rows)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & P^i \arrow[r, hook] \arrow[d, hook] & Q^i \arrow[r, two heads] \arrow[d, hook] & R^i \arrow[r] \arrow[d, hook] & 0 \arrow[d, equal] \\
0 \arrow[r] & Z^i \arrow[r, hook] & \mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q} \arrow[r, two heads] & \mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

which implies $P^i = Z^i \times_{\mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}} Q^i$ because $Z^i \hookrightarrow \mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}$ is injective. Now we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & P^i \arrow[r, hook] \arrow[d, hook] & Z^i \arrow[r, two heads] \arrow[d, hook] & Z^i/P^i \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & Q^i \arrow[r, hook] & \mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q} \arrow[r, two heads] & \left(\mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}\right)/Q^i \arrow[r] & 0
\end{tikzcd}$$

and observe that the lower row is the rationalization of the upper one. A diagram chase yields that the right hand map is injective and thus that Z^i/P^i is a free abelian group. Therefore we conclude that there are free complements for $P^i \hookrightarrow Z^i$ for all $i \in \mathbb{N}$, i.e., decompositions $Z^i \cong P^i \oplus A^i$ where P^i and A^i are free finitely generated abelian groups. The transition maps $f_i^* : Z^{i+1} \to Z^i$ take the form

$$f_i^* = \begin{pmatrix} (f_i^*)_{|P^{i+1}} & \phi_i \\ 0 & \psi_i \end{pmatrix} .$$

Note that we get induced decompositions $\mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q} \cong Q^i \oplus (A^i\otimes\mathbb{Q})$ for all $i\in\mathbb{N}$ and that the transition maps $f_i^* : \mathbf{KU}^{-1}(B_{i+1})\otimes\mathbb{Q} \to \mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}$ become

$$f_i^* = \begin{pmatrix} (f_i^*)_{|P^{i+1}}\otimes\mathbb{Q} & \phi_i\otimes\mathbb{Q} \\ 0 & \psi_i\otimes\mathbb{Q} \end{pmatrix} .$$

But the Q^i, $i\in\mathbb{N}$, are the stable images of the system of finite dimensional vector spaces $(\mathbf{KU}^{-1}(B_i)\otimes\mathbb{Q}, f_i^*)_i$. Thus we find a cofinal subsystem of $\mathbb{N}$ such that the maps $\psi_i\otimes\mathbb{Q}$ are trivial. In particular, the maps ψ_i vanish and the transition maps of the system $(Z^i, f_i^*)_i$ have the form

$$f_i^* = \begin{pmatrix} (f_i^*)_{|P^{i+1}} & \phi_i \\ 0 & 0 \end{pmatrix} .$$

We choose (non-compatible) splits $t_i : \operatorname{Torsion}(\mathbf{KU}^0(B_i)) \to \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i)$ for all $i\in\mathbb{N}$. Then we get

$$\mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i) \cong Q^i/P^i \oplus (A^i\otimes\mathbb{Q})/A^i \oplus \operatorname{Torsion}(\mathbf{KU}^0(B_i))$$

and the transition maps $f_i^* : \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_{i+1}) \to \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i)$ take the form

$$\begin{pmatrix} \left[(f_i^*)_{|P^{i+1}}\otimes\mathbb{Q}\right] & [\phi_i\otimes\mathbb{Q}] & \alpha_i \\ 0 & 0 & \beta_i \\ 0 & 0 & (f_i^*)_{|\operatorname{Torsion}} \end{pmatrix}$$

where $\alpha_i : \mathrm{Torsion}(\mathbf{KU}^0(B_{i+1})) \to Q^i/P^i$ and $\beta_i : \mathrm{Torsion}(\mathbf{KU}^0(B_{i+1})) \to (A^i \otimes \mathbb{Q})/A^i$ are group homomorphisms. Since the maps $(f_i^*)_{|P^{i+1}} \otimes \mathbb{Q}$ are surjective we can inductively change the splits t_i such that the transition maps take the form

$$\begin{pmatrix} \left[(f_i^*)_{|P^{i+1}} \otimes \mathbb{Q}\right] & [\phi_i \otimes \mathbb{Q}] & 0 \\ 0 & 0 & \beta_i \\ 0 & 0 & (f_i^*)_{|\,\mathrm{Torsion}} \end{pmatrix}$$

Next we consider the system $(\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(B_i)/(Q^i/P^i), f_i^*)_i$. The stable images of this system can be identified with S^i/R^i. We identify

$$\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(B_i)/(Q^i/P^i) \cong (A^i \otimes \mathbb{Q})/A^i \oplus \mathrm{Torsion}(\mathbf{KU}^0(B_i))$$

to write the transition maps as

$$\begin{pmatrix} 0 & \beta_i \\ 0 & (f_i^*)_{|\,\mathrm{Torsion}} \end{pmatrix}.$$

Therefore, the projection $(\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(B_i)/(Q^i/P^i) \twoheadrightarrow \mathrm{Torsion}(\mathbf{KU}^0(B_i))$ induces an isomorphism $S^i/R^i \to T^i$. We conclude that the middle row of diagram (5.2) is exact.

Finally we construct a continuous split $s : \varprojlim_{i\in\mathbb{N}} T^i \hookrightarrow \varprojlim_{i\in\mathbb{N}} S^i$. By assumption the group $\varprojlim_{i\in\mathbb{N}} T^i$ is (topologically) finitely generated. The structure theorem for topologically finitely generated profinite abelian groups ([RZ00, Theorem 4.3.5]) gives

$$\varprojlim_{i\in\mathbb{N}} T^i \cong \left(\bigoplus_{p \text{ prime}} \bigoplus_{m_p} \mathbb{Z}_p\right) \bigoplus T$$

where each m_p is a natural number (and almost all are zero) and T denotes a discrete finite abelian group. On the discrete part T any split s is continuous. So we fix a $\mathbb{Z}_p$-summand for some prime p and take the pullback of diagram (5.2) to this summand. In detail we set $\tilde{T}^i := \pi^i(\mathbb{Z}_p) \subset T^i$ and $\tilde{S}^i := \pi^{-1}(\tilde{T}^i)$. This gives $\varprojlim_i \tilde{T}^i = \mathbb{Z}_p$ and $\varprojlim_i \tilde{S}^i = \pi^{-1}(\mathbb{Z}_p)$. Thus we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \varprojlim_{i\in\mathbb{N}} R^i \arrow[r, hook, "\iota"] \arrow[d, two heads, "\pi^i"] & \varprojlim_{i\in\mathbb{N}} \tilde{S}^i \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\pi^i"] & \mathbb{Z}_p \arrow[r] \arrow[d, two heads, "\pi^i"] & 0 \arrow[d, equal] \\
0 \arrow[r] \arrow[d, equal] & R^{i+1} \arrow[r, hook, "\iota"] \arrow[d, two heads, "f_i^*"] & \tilde{S}^{i+1} \arrow[r, two heads, "\pi"] \arrow[d, two heads, "f_i^*"] & \tilde{T}^{i+1} \arrow[r] \arrow[d, two heads, "f_i^*"] & 0 \arrow[d, equal] \\
0 \arrow[r] & R^i \arrow[r, two heads, "\iota"] & \tilde{S}^i \arrow[r, two heads, "\pi"] & \tilde{T}^i \arrow[r] & 0
\end{tikzcd}.$$

Now we choose isomorphisms $P^i \cong \mathbb{Z}^{n_i}$ and get isomorphisms $Q^i \cong \mathbb{Q}^{n_i}$ and $R^i \cong {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{n_i}$ for all $i \in \mathbb{N}$. Moreover, we take isomorphisms $\tilde{T}^i \cong \mathbb{Z}/p^{l_i}\mathbb{Z}$ and we fix decompositions $\tilde{S}^i \cong {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{n_i} \oplus \mathbb{Z}/p^{l_i}\mathbb{Z}$ (R^i is divisible). This gives

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{n_{i+1}} \arrow[r, hook, "\iota"] \arrow[d, two heads, "f_i^*"] & {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{n_{i+1}} \oplus \mathbb{Z}/p^{l_{i+1}}\mathbb{Z} \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "M_i"] & \mathbb{Z}/p^{l_{i+1}}\mathbb{Z} \arrow[r] \arrow[d, two heads, "\mathrm{pr}"] & 0 \arrow[d, equal] \\
0 \arrow[r] & {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{n_i} \arrow[r, hook, "\iota"] & {}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{n_i} \oplus \mathbb{Z}/p^{l_i}\mathbb{Z} \arrow[r, two heads, "\pi_2"] & \tilde{\mathbb{Z}}/p^{l_i}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where π_2 denotes the projection onto the second component and

$$M_i = \begin{pmatrix} f_i^* & \alpha_i \\ 0 & \mathrm{pr} \end{pmatrix}$$

for some $\alpha_i : \mathbb{Z}/p^{l_i}\mathbb{Z} \to {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{n_i}$.

Our aim is to construct p-torsion elements $(x_i, 1) \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{n_i} \oplus \mathbb{Z}/p^{l_i}\mathbb{Z}$ satisfying $M_i(x_{i+1}, 1) = (x_i, 1)$. Then we get maps $\mathbb{Z}/\operatorname{ord}(x_i, 1)\mathbb{Z} \to \tilde{S}^i$ mapping $1 \mapsto (x_i, 1)$. These combine to a continuous map $\mathbb{Z}_p \to \varprojlim_{i\in\mathbb{N}} \tilde{S}^i$ which is a split for $\varprojlim_{i\in\mathbb{N}} \tilde{S}^i \xrightarrow{\pi} \mathbb{Z}_p$.

We proceed by induction. The base case is trivial. Hence assume we found p-torsion elements $x_1, \ldots, x_k$ such that $x_i = f_i^*(x_{i+1}) + \alpha_i(1)$. Then we need a p-torsion element $x_{k+1} \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{n_{k+1}}$ satisfying $x_k = f_k^*(x_{k+1}) + \alpha_k(1)$. We apply the Smith normal form to the rational surjective map $f_k^* : P^{k+1} \to P^k$. Thus we can assume that the map $f_k^* : {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{n_{k+1}} \to {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{n_k}$ is of the form

$$f_k^* = \begin{pmatrix} \rho_1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \rho_2 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & \ldots & \rho_{n_k} & 0 & \ldots & 0 \end{pmatrix}$$

for some $\rho_j \in \mathbb{N} \setminus \{0\}$. By definition, $p^{l_k} \cdot \alpha_k(1) = 0 \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{n_k}$. Thus $\alpha_k(1) = \left(\frac{a_j}{p^{l_k}}\right)_{j=1}^{n_k}$ for some $a_j \in \mathbb{Z}$. We represent $x_k = \left(\frac{b_j}{p^\delta}\right)_{j=1}^{n_{k+1}}$ for some $b_j \in \mathbb{Z}$ and $\delta \in \mathbb{N}$ and make the ansatz $x_{k+1} = \left(\frac{c_j}{p^\varepsilon}\right)_{j=1}^{n_{k+1}}$ for some $c_j \in \mathbb{Z}$ and $\varepsilon \in \mathbb{N}$. Then we need to solve $\rho_j \frac{c_j}{p^\varepsilon} + \frac{a_j}{p^{l_k}} = \frac{b_j}{p^\delta}$ in ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ for all $j = 1, \ldots, n_k$. We decompose $\rho_j = p^{m_j} q_j$ where p and q_j are coprime and get the equations $q_j c_j + a_j \cdot p^{\varepsilon - l_k - m_j} - b_j \cdot p^{\varepsilon-\delta-m_j} = 0$ modulo $p^{\varepsilon-m_j}$ for all $j = 1, \ldots, n_k$ (we choose ε big enough). Since q_j is a unit (modulo $p^{\varepsilon-m_j}$) we get $c_j = b_j \cdot q_j^{-1} \cdot p^{\varepsilon-\delta-m_j} - a_j \cdot q_j^{-1} \cdot p^{\varepsilon-l_k-m_j}$ modulo $p^{\varepsilon-m_j}$. This gives c_j for all $j = 1, \ldots, n_k$ and we take $c_j = 0$ for all $j = n_k + 1, \ldots, n_{k+1}$.

Thus we have produced a continuous split for each $\mathbb{Z}_p$-summand and the proof is finished. □

Finally we prove our Existence Theorem 3.27.

Proof of Existence Theorem 3.27. Let $(B_i, f_i, g_i)_{i\in\mathbb{N}}$ denote an approximation of B by manifolds satisfying the technical condition.

Since every compact manifold is of the homotopy type of a finite CW-complex we can apply Proposition 5.1 to get a decomposition (as topological abelian groups)

$$\varprojlim_{i\in\mathbb{N}} \mathbf{KU}^0(B_i) \cong \varprojlim_{i\in\mathbb{N}} \operatorname{Hom}(\mathbf{KU}_0(B_i), \mathbb{Z}) \oplus \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i)) \ .$$

Moreover, by Proposition 5.2 we can find a split s of the short exact sequence,

$$\begin{tikzcd} \varprojlim_{i\in\mathbb{N}} (\mathbf{KU}\,\mathbb{Q}^{-1}(B_i)/\,\mathbf{KU}^{-1}(B_i)) \arrow[r, hook] & \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{-1}(B_i) \arrow[r, two heads] & \varprojlim_{i\in\mathbb{N}} \operatorname{Torsion}(\mathbf{KU}^0(B_i)) \arrow[l, bend right, "s"'] \end{tikzcd} \ .$$

By our technical assumption we can choose this split to be a continuous map.

Now a geometrization $\mathcal{G}_i$ for $g_i : B_i \to B$ is a lift in the following diagram.

$$\begin{tikzcd} & & & \widehat{\mathbf{KU}}^0(B_i) \arrow[d, "I"] \\ \mathbf{KU}^0(B) \arrow[r] \arrow[urrr, dashed, bend left, "\mathcal{G}_i"] & \varprojlim_i \mathbf{KU}^0(B_i) \arrow[r, "\cong"] & \varprojlim_i \operatorname{Hom}(\mathbf{KU}_0(B_i), \mathbb{Z}) \oplus \varprojlim_i \operatorname{Torsion}(\mathbf{KU}^0(B_i)) \arrow[r] \arrow[ur, dotted, "\tilde{\mathcal{G}}_i"] & \mathbf{KU}^0(B_i) \end{tikzcd}$$

The idea is to construct a lift $\tilde{\mathcal{G}}_i$ such that the induced map $\mathcal{G}_i$ is a geometrization.

To define $\tilde{\mathcal{G}}_i$ on the profinite part $\varprojlim_{i\in\mathbb{N}}\mathrm{Torsion}(\mathbf{KU}^0(B_i))$ we recall that there is an inclusion $\mathbf{KU}\,\mathbb{R}/\mathbb{Z}^{-1}(B_i) \hookrightarrow \widehat{\mathbf{KU}^0}(B_i)$. So one easily defines the lift $\tilde{\mathcal{G}}_i$ on the profinite part $\varprojlim_{i\in\mathbb{N}}\mathrm{Torsion}(\mathbf{KU}^0(B_i))$ as the composition

$$\varprojlim_{i\in\mathbb{N}}\mathrm{Torsion}(\mathbf{KU}^0(B_i)) \xrightarrow{s} \varprojlim_{i\in\mathbb{N}}\mathbf{KU}\,\mathbb{Q}/\mathbb{Z}^{-1}(B_i) \to \varprojlim_{i\in\mathbb{N}}\mathbf{KU}\,\mathbb{R}/\mathbb{Z}^{-1}(B_i) \xrightarrow{\mathrm{pr}_i} \\ \to \mathbf{KU}\,\mathbb{R}/\mathbb{Z}^{-1}(B_i) \hookrightarrow \widehat{\mathbf{KU}^0}(B_i)\ .$$

A direct calculation shows that this satisfies the compatibility condition $(f_i)^!_{\delta_i}\tilde{\mathcal{G}}_{i+1} = \tilde{\mathcal{G}}_i$. Note that, by construction, the curvature $R\circ\tilde{\mathcal{G}}_i$ vanishes on the profinite part. We check that this is compatible with the choice of any cohomological character. By continuity a cohomological character factors over $\mathbf{HP}\mathbb{Q}^0(B) \twoheadrightarrow \varprojlim_{i\in\mathbb{N}}\mathrm{Hom}(\mathbf{HP}\mathbb{Q}_0(B_i),\mathbb{Q})$. But since the diagram

$$\begin{tikzcd}
\mathbf{KU}^0(B) \arrow[r, two heads] \arrow[d, "\mathbf{ch}"] & \varprojlim_{i\in\mathbb{N}}\mathrm{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \oplus \varprojlim_{i\in\mathbb{N}}\mathrm{Torsion}(\mathbf{KU}^0(B_i)) \arrow[d, "\mathbf{ch}^*\oplus 0"] \\
\mathbf{HP}\mathbb{Q}^0(B) \arrow[r, two heads] & \varprojlim_{i\in\mathbb{N}}\mathrm{Hom}(\mathbf{HP}\mathbb{Q}_0(B_i),\mathbb{Q})
\end{tikzcd}$$

commutes any cohomological character is automatically compatible with our construction of $\mathcal{G}_i$ on the profinite part, i.e., the square (3.1) commutes after restricting to the profinite part $\varprojlim_{i\in\mathbb{N}}\mathrm{Torsion}(\mathbf{KU}^0(B_i))$.

Our next aim is to construct a map $\tilde{\mathcal{G}}_i : \varprojlim_{i\in\mathbb{N}}\mathrm{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \to \widehat{\mathbf{KU}^0}(B_i)$. To this end we use the notion of the proof of Proposition 5.1 and choose compatible bases for the X_i and compatible splits s_i for all $i\in\mathbb{N}$. As in the proof of Proposition 3.4 we construct "cohomological characters" $c_i : \mathbf{HP}\mathbb{Q}^0(B_i) \cong \mathbf{KU}^0(B_i)\otimes\mathbb{Q} \to \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)$ which satisfy $c_i\circ f_i^* = f_i^*\circ c_{i+1}$ and which preserve the internal degree. We define the maps $\tilde{s}_i$ to be the compositions

$$X_i \xrightarrow{s_i} X^i \hookrightarrow \mathbf{KU}^0(B_i) \xrightarrow{g_i^*\,\mathbf{td}^{-1}\cup\mathbf{ch}} \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i) \xrightarrow{c_i} \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)\ .$$

Then a "geometrization" $\tilde{\mathcal{G}}_i$ with "cohomological character" c_i is a lift in the diagram

$$\begin{tikzcd}
 & & \mathbf{HP}\mathbb{R}^{\text{-}1}(B_i) \arrow[d, "a"] \\
 & & \widehat{\mathbf{KU}^0}(B_i) \arrow[d, two heads, "I\times(\mathbf{Td}_i\wedge R)"] \\
\varprojlim_{i\in\mathbb{N}}\mathrm{Hom}(\mathbf{KU}_0(B_i),\mathbb{Z}) \arrow[r, two heads] & X_i \arrow[r, "s_i\times\tilde{s}_i"'] \arrow[ur, dotted, "\tilde{\mathcal{G}}_i"] & \mathbf{KU}^0(B_i)\times_{\mathbf{HP}\mathbb{R}^0(B_i)}\Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)
\end{tikzcd}$$

where the right hand column is exact by the long exact sequence (2.2). The compatibility condition for the "geometrizations" $(\tilde{\mathcal{G}}_i)_i$ is the requirement that the diagrams

$$
\begin{tikzcd}
\mathbf{HP}\mathbb{R}^{-1}(B_{i+1}) \arrow[r, "(1+d\delta_i)\wedge f_i^*"] \arrow[d, "a"] & \mathbf{HP}\mathbb{R}^{-1}(B_i) \arrow[d, "a"'] \\
\widehat{\mathbf{KU}}^0(B_{i+1}) \arrow[r, "f_i^*+a(\delta_i\wedge(R\circ f_i^*))"] \arrow[d, two heads, "I\times(\mathbf{Td}_i\wedge R)"] & \widehat{\mathbf{KU}}^0(B_i) \arrow[d, two heads, "I\times(\mathbf{Td}_i\wedge R)"'] \\
\mathbf{KU}^0(B_{i+1}) \times_{\mathbf{HP}\mathbb{R}^0(B_{i+1})} \Omega\mathrm{P}^0_{\mathrm{cl}}(B_{i+1}) \arrow[r, "f_i^*\times f_i^*"] & \mathbf{KU}^0(B_i) \times_{\mathbf{HP}\mathbb{R}^0(B_i)} \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i) \\
X_{i+1} \arrow[u, hook, "s_{i+1}\times\tilde{s}_{i+1}"'] \arrow[r, two heads, "f_i^*"] \arrow[uu, dotted, bend left=60, "\tilde{\mathcal{G}}_{i+1}"] & X_i \arrow[u, hook', "s_i\times\tilde{s}_i"] \arrow[uu, dotted, bend right=60, "\tilde{\mathcal{G}}_i"']
\end{tikzcd}
\tag{5.3}
$$

commute for all $i \in \mathbb{N}$. Note that the "cohomological characters" of the "geometrizations" $\tilde{\mathcal{G}}_i$ are the "cohomological characters" c_i which preserve the internal degree. Therefore to complete the proof of our main theorem it is enough to find compatible $(\tilde{\mathcal{G}}_i)_i$. To this end we take the pullback of the diagram (5.3) to the X_i. In detail we set $Q_i := \mathbf{HP}\mathbb{R}^{-1}(B_i)$ and define K_i to be the fibre product

$$
\begin{tikzcd}
K_i \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\mathbf{KU}}^0(B_i) \arrow[d, two heads, "I\times(\mathbf{Td}_i\wedge R)"] \\
X_i \arrow[r, hook, "s_i\times\tilde{s}_i"] & \mathbf{KU}^0(B_i) \times_{\mathbf{HP}\mathbb{R}^0(B_i)} \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)
\end{tikzcd}
\quad .
$$

This gives the following diagram where the problem is to construct compatible sections $(s_i)_i$,

$$
\begin{tikzcd}
\vdots \arrow[d, "q_2"] & \vdots \arrow[d, "k_2"] & \vdots \arrow[d, two heads, "f_2"] & \vdots \arrow[d, equal] \\
Q_2 \arrow[r] \arrow[d, "q_1"] & K_2 \arrow[r] \arrow[d, "k_1"] & X_2 \arrow[r] \arrow[l, dotted, bend right, "s_2"'] \arrow[d, two heads, "f_1"] & 0 \arrow[d, equal] \\
Q_1 \arrow[r] \arrow[d, "q_0"] & K_1 \arrow[r] \arrow[d, "k_0"] & X_1 \arrow[r] \arrow[l, dotted, bend right, "s_1"'] \arrow[d, two heads, "f_0"] & 0 \arrow[d, equal] \\
Q_0 \arrow[r] & K_0 \arrow[r] & X_0 \arrow[r] \arrow[l, dotted, bend right, "s_0"'] & 0
\end{tikzcd}
\quad .
$$

Note that the rows are parts of long exact sequences, that the Q_i are finite-dimensional $\mathbb{R}$-vector spaces and that the X_i are free finitely generated abelian groups.

Since the X_i are free we can fix a choice of (possible non-compatible) splits. We get

$$\begin{tikzcd}
\vdots \arrow[d, "q_2"] & \vdots \arrow[d, "L_2"] & \vdots \arrow[d, two heads, "f_2"] & \vdots \arrow[d, equal] \\
Q_2 \arrow[r] & \mathrm{im}(Q_2) \oplus X_2 \arrow[r] \arrow[d, "L_1"] & X_2 \arrow[r] \arrow[l, dotted, bend right, "s_2"'] \arrow[d, two heads, "f_1"] & 0 \arrow[d, equal] \\
Q_1 \arrow[r] & \mathrm{im}(Q_1) \oplus X_1 \arrow[r] \arrow[d, "L_0"] & X_1 \arrow[r] \arrow[l, dotted, bend right, "s_1"'] \arrow[d, two heads, "f_0"] & 0 \arrow[d, equal] \\
Q_0 \arrow[r] & \mathrm{im}(Q_0) \oplus X_0 \arrow[r] & X_0 \arrow[r] \arrow[l, dotted, bend right, "s_0"'] & 0
\end{tikzcd}$$

where the maps $L_i : \mathrm{im}(Q_{i+1}) \oplus X_{i+1} \to \mathrm{im}(Q_i) \oplus X_i$ are of the form $L_i = \begin{pmatrix} q_i & \alpha_i \\ 0 & f_i \end{pmatrix}$ for some $\alpha_i : X_i \to \mathrm{im}(Q_i)$. Now a split s_i is the same as a map $\tilde{s}_i : X_i \to \mathrm{im}(Q_i)$ and the compatibility condition $s_i \circ f_i = L_i \circ s_{i+1}$ translates into $\tilde{s}_i \circ f_i = q_i \circ \tilde{s}_{i+1} + \alpha_i$.

We choose compatible bases of the X_i as in the proof of Proposition 5.1. Thus we can work with one generator at a time and are reduced to the diagram

$$\begin{tikzcd}
\vdots \arrow[d, "q_2"] & \vdots \arrow[d, "L_2"] & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] \\
Q_2 \arrow[r] \arrow[d, "q_1"] & \mathrm{im}(Q_2) \oplus \mathbb{Z} \arrow[r] \arrow[d, "L_1"] & \mathbb{Z} \arrow[r] \arrow[l, dotted, bend right, "\tilde{s}_2 \oplus \mathtt{id}_{\mathbb{Z}}"'] \arrow[d, equal] & 0 \arrow[d, equal] \\
Q_1 \arrow[r] \arrow[d, "q_0"] & \mathrm{im}(Q_1) \oplus \mathbb{Z} \arrow[r] \arrow[d, "L_0"] & \mathbb{Z} \arrow[r] \arrow[l, dotted, bend right, "\tilde{s}_1 \oplus \mathtt{id}_{\mathbb{Z}}"'] \arrow[d, equal] & 0 \arrow[d, equal] \\
Q_0 \arrow[r] & \mathrm{im}(Q_0) \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, dotted, bend right, "\tilde{s}_0 \oplus \mathtt{id}_{\mathbb{Z}}"'] & 0
\end{tikzcd} \quad .$$

Now we are looking for elements $\tilde{s}_i \in \mathrm{im}(Q_i)$ satisfying $\tilde{s}_i = q_i(\tilde{s}_{i+1}) + \alpha_i(1)$. We choose lifts of $\alpha_i(1) \in \mathrm{im}(Q_i)$ to Q_i which we denote (by abuse of notation) by α_i.

Next we construct elements $\tilde{s}_i \in Q_i$ for all $i \in \mathbb{N}$ satisfying $\tilde{s}_i = q_i(\tilde{s}_{i+1}) + \alpha_i$. Their images in $\mathrm{im}(Q_i) \subset L_i$ clearly satisfy the original compatibility condition. Note that the system (Q_i, q_i) satisfies the Mittag-Leffler condition since it consists of finite-dimensional vector spaces. We take the stable images

$$\hat{Q}_i := \bigcap_{k \geq i} q_i \circ \ldots \circ q_{k-1}(Q_k) \subset Q_i$$

and get a system $(\hat{Q}_i, q_i)$ with surjective transition maps. We decompose $Q_i = \hat{Q}_i \oplus P_i$ (as $\mathbb{R}$-vector spaces) such that $q_i = \begin{pmatrix} \hat{q}_i & 0 \\ 0 & p_i \end{pmatrix}$ where $\hat{q}_i : \hat{Q}_{i+1} \twoheadrightarrow \hat{Q}_i$ is surjective and $p_i : P_{i+1} \to P_i$. Then α_i decompose into $\alpha_i = \hat{\alpha}_i + \beta_i$ with $\hat{\alpha}_i \in \hat{Q}_i$ and $\beta_i \in P_i$. We fix preimages $\hat{\alpha}_i^j \in \hat{Q}_j$ for all $j > i$ such that $\hat{q}_{j-1}(\hat{\alpha}_i^j) = \hat{\alpha}_i^{j-1}$ (where $\hat{\alpha}_i^i := \hat{\alpha}_i$). Assume that we have a family $\hat{s}_i \in Q_i$, $i \in \mathbb{N}$, satisfying $\hat{s}_i = q_i(\hat{s}_{i+1}) - \beta_i$. Then $\tilde{s}_i := \hat{s}_i - \sum_{j<i} \hat{\alpha}_j^i$ satisfies $\tilde{s}_i = q_i(\tilde{s}_{i+1}) + \alpha_i$. So it is enough to find $\hat{s}_i \in Q_i$ satisfying $\hat{s}_i = q_i(\hat{s}_{i+1}) - \beta_i$.

By taking a cofinal subsystem we can assume that $p_i = 0$ for all i (recall that the p_i are maps of $\mathbb{R}$-vector spaces). This is justified by Proposition 3.36. So set $\hat{s}_i = -\beta_i$. Then $\hat{s}_i = q_i(\hat{s}_{i+1}) - \beta_i$ is satisfied since $q_i(\hat{s}_{i+1}) = q_i(-\beta_{i+1}) = p_i(-\beta_{i+1}) = 0$.

Hence we have constructed a compatible family of "geometrizations" $\tilde{\mathcal{G}}_i$. This completes the proof.

□

Remark 5.5. The last part of the above proof simplifies if B is a CW-complex of finite type and the approximation by manifolds is (up to homotopy) the *even* CW-skeleton. This implies that the maps $\mathbf{HP}\mathbb{R}^{-1}(B_{i+1}) \xrightarrow{(1+d\delta_i)\wedge f_i^*} \mathbf{HP}\mathbb{R}^{-1}(B_i)$ are surjective by cellular cohomology. The existence of compatible "geometrizations" in diagram (5.3) is then a straightforward generalization of the argument in Example 3.12. ■

Chapter 6

The intrinsic eta-invariant

In this chapter we show that a choice of universal geometrization yields a well-defined intrinsic eta-invariant. Thus we make [Bun11, Remark 4.20] explicit.

For all of the present chapter we fix a topological space B and a map $\sigma : B \to BSpin^c$ where $BSpin^c$ is a classifying space for $Spin^c$-principal bundles. Recall that there exists a cohomology class $\mathbf{td} \in \mathbf{HP}\mathbb{Q}^0(BSpin^c)$, called the Todd class, which is invertible and which induces a class $\mathbf{td}_B := \sigma^* \mathbf{td} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$ by pullback.

In this situation we can construct a spectrum MB, called the Thom spectrum. In the first section we recall some facts about Thom spectra.
Then we review the definition of the universal eta-invariant and define abelian groups $\mathrm{Q}^{\mathbb{Q}}_*(MB)$ and $\mathrm{Q}^{\mathbb{R}}_*(MB)$ which are the targets for our eta-invariants.
Next we discuss a geometric model for B-bordism which incorporates the fact that we have a map $B \to BSpin^c$.
In the following section we study differential form representatives for the Todd class $\mathbf{td}_B$ (or rather its inverse) on a compact manifold. Moreover, we will introduce an adapted version of universal geometrizations. This concept, called universal $Spin^c$-geometrization, yields a (non-canonical) geometrization for each geometric cycle for B-bordism, which has the property that this geometrization extends along zero-bordisms.
After recalling some well-known facts about (twisted) Dirac operators and eta-invariants we finally define the intrinsic eta-invariant. This invariant will depend on the choice of a universal $Spin^c$-geometrization for $\sigma : B \to BSpin^c$. We compare the intrinsic eta-invariant with the previously defined universal eta-invariant.
We end this chapter with a first example which shows that the intrinsic eta-invariant detects strictly more information than the universal eta-invariant.

6.1 Thom spectra

We briefly recall some well-known facts about Thom spectra.

Induced by our fixed map $\sigma : B \to BSpin^c$ there is a spectrum MB, called the **Thom spectrum** associated to σ. In Section 6.3 we give a geometric model for homotopy classes $\pi^{\$}_*(MB)$. The map σ induces a canonical map $MB \to MBSpin^c$ which yields an orientation

$$MB \to MBSpin^c \xrightarrow{\mathrm{ABS}} \mathbf{KU}$$

where ABS is the Atiyah-Bott-Shapiro orientation for complex K-theory. Similarly, we can prolong to oriented bordism $MBSO$ and get an orientation

$$MB \to MBSpin^c \to MBSO \to \mathbf{HP}\mathbb{Q}$$

where the map $MBSO \to \mathbf{HP}\mathbb{Q}$ is the usual orientation for oriented bordism. The orientations yield Thom isomorphisms (which in fact are homeomorphisms)

$$\mathrm{Thom}^{\mathbf{KU}} : \mathbf{KU}^0(B) \xrightarrow{\cong} \mathbf{KU}^0(MB) \quad \text{and} \quad \mathrm{Thom}^{\mathbf{HP}\mathbb{Q}} : \mathbf{HP}\mathbb{Q}^0(B) \xrightarrow{\cong} \mathbf{HP}\mathbb{Q}^0(MB) .$$

These are connected by the Riemann-Roch Theorem,

$$\begin{tikzcd}
\mathbf{KU}^0(B) \arrow[r, "\mathrm{Thom}^{\mathbf{KU}}"] \arrow[d, "\mathbf{td}_B^{-1} \cup \mathbf{ch}"] & \mathbf{KU}^0(MB) \arrow[d, "\mathbf{ch}"] \\
\mathbf{HP}\mathbb{Q}^0(B) \arrow[r, "\mathrm{Thom}^{\mathbf{HP}\mathbb{Q}}"] & \mathbf{HP}\mathbb{Q}^0(MB) .
\end{tikzcd} \tag{6.1}$$

There are also the homological counterparts

$$\mathrm{Thom}_{\mathbf{KU}} : \mathbf{KU}_0(B) \xrightarrow{\cong} \mathbf{KU}_0(MB) \quad \text{and} \quad \mathrm{Thom}_{\mathbf{HP}\mathbb{Q}} : \mathbf{HP}\mathbb{Q}_0(B) \xrightarrow{\cong} \mathbf{HP}\mathbb{Q}_0(MB) .$$

6.2 Definition of the universal eta-invariant

We recall the definition of the universal eta-invariant [Bun11, Chapter 2.2] and of the graded abelian groups $\mathrm{Q}^{\mathbb{Q}}_*(MB)$ and $\mathrm{Q}^{\mathbb{R}}_*(MB)$.

In Section 2.3 we discussed the natural cofibre sequence of spectra $MB \to MB\mathbb{Q} \to MB\mathbb{Q}/\mathbb{Z}$. We smash with the unit $\varepsilon : \mathbb{S} \to \mathbf{KU}$ of complex K-theory and get the commutative diagram

$$\begin{tikzcd}
\Sigma^{-1}MB\mathbb{Q} \arrow[r, "\varepsilon"] \arrow[d] & \Sigma^{-1}MB\mathbb{Q} \wedge \mathbf{KU} \arrow[d] \\
\Sigma^{-1}MB\mathbb{Q}/\mathbb{Z} \arrow[r, "\varepsilon"] \arrow[d] & \Sigma^{-1}MB\mathbb{Q}/\mathbb{Z} \wedge \mathbf{KU} \arrow[d] \\
MB \arrow[r, "\varepsilon"] \arrow[d] & MB \wedge \mathbf{KU} \arrow[d] \\
MB\mathbb{Q} \arrow[r, "\varepsilon"] & MB\mathbb{Q} \wedge \mathbf{KU} .
\end{tikzcd}$$

The idea of the universal eta-invariant is as follows. Take a stable homotopy class $x \in \mathrm{Torsion}(\pi^{\mathbb{S}}_*(MB))$. Since it is torsion there exists a lift $\tilde{x} \in \pi^{\mathbb{S}}_{*+1}(MB\mathbb{Q}/\mathbb{Z})$ of x. Apply the unit ε to get $\varepsilon(\tilde{x}) \in \mathbf{KU}_{*+1}(MB\mathbb{Q}/\mathbb{Z})$ and use the evaluation pairing to get a continuous homomorphism $\langle \varepsilon(\tilde{x}), - \rangle_{\mathbb{Q}/\mathbb{Z}} : \mathbf{KU}^0(MB) \to \pi^{\mathbb{S}}_{*+1}(\mathbf{KU}\,\mathbb{Q}/\mathbb{Z})$. But because the lift $\tilde{x}$ is not well-defined we quotient out the space of choices $\mathrm{U}^{\mathbb{Q}}_*(MB)$ to get a well-defined map.

Now we give some more details. Fix an abelian group G. The evaluation pairing is a bilinear map

$$\langle -, - \rangle_G : \mathbf{KU}_*(MBG) \times \mathbf{KU}^0(MB) \to \mathbf{KU}_*(\mathcal{M}G) \cong \pi^{\mathbb{S}}_*(\mathbf{KU}G)$$

which is continuous with respect to the discrete topology on the homology and the profinite topology on the cohomology. Its adjoint is the homomorphism

$$\langle -, - \rangle_G : \mathbf{KU}_*(MBG) \to \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(MB), \pi^{\mathbb{S}}_*(\mathbf{KU}G)) .$$

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{Q}$. We consider the subspace

$$\mathrm{U}^{\mathbb{K}}_*(MB) := \left\{ [\langle \varepsilon(y), - \rangle_{\mathbb{K}}]_{\mathbb{K}/\mathbb{Z}} \mid y \in \pi^{\mathbb{S}}_{*+1}(MB\mathbb{K}) \right\} \subseteq \mathrm{Hom}^{\mathrm{cont}} \left(\mathbf{KU}^0(MB), \pi^{\mathbb{S}}_{*+1}(\mathbf{KU}\mathbb{K}/\mathbb{Z}) \right)$$

which is precisely the space of indeterminacy above. The quotient

$$\mathrm{Q}_*^{\mathbb{K}}(MB) := \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(MB), \pi_{*+1}^{\$}(\mathbf{KU}\mathbb{K}/\mathbb{Z})\right) / \mathrm{U}_*^{\mathbb{K}}(MB) \tag{6.2}$$

is the target for our eta-invariants.

Definition 6.1. *(see [Bun11, Definition 2.3])*
*The **universal eta-invariant** is the homomorphism*

$$\eta_{\mathrm{uni}} : \mathrm{Torsion}(\pi_*^{\$}(MB)) \to \mathrm{Q}_*^{\mathbb{Q}}(MB)$$

defined as follows. Take a torsion element $x \in \mathrm{Torsion}(\pi_*^{\$}(MB))$ *and choose a lift* $\tilde{x} \in \pi_{*+1}^{\$}(MB\mathbb{Q}/\mathbb{Z})$. *Then the value* $\eta_{\mathrm{uni}}(x) \in \mathrm{Q}_*^{\mathbb{Q}}(MB)$ *of the universal eta-invariant is represented by* $\langle \varepsilon(\tilde{x}), - \rangle_{\mathbb{Q}/\mathbb{Z}} \in \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(MB), \pi_{*+1}^{\$}(\mathbf{KU}\mathbb{Q}/\mathbb{Z}))$. ■

Remark 6.2. Observe that the universal eta-invariant is functorial in the spectrum MB ([Bun11, Lemma 2.4]). ■

In [Bun11, Chapter 5] it is shown that the universal eta-invariant specializes to many interesting invariants in homotopy theory, e.g., Adams e-invariant.

In the same paper the intrinsic eta-invariant is introduced as a geometric way to calculate the universal eta-invariant (see Theorem 6.28 below). Our aim is to extend this intrinsic eta-invariant to non-torsion classes in $\pi_*^{\$}(MB)$. To do this we need $\mathbb{R}$-coefficients instead of $\mathbb{Q}$-coefficients. The following lemma, proven as Lemma 4.15 in [Bun11], allows us to consider only $\mathbb{R}$-coefficients.

Lemma 6.3. *The natural map* $\mathrm{Q}_*^{\mathbb{Q}}(MB) \to \mathrm{Q}_*^{\mathbb{R}}(MB)$ *is injective.* ♦

For the construction of the intrinsic eta-invariant we will use the space B instead of its Thom spectrum MB. So we define

$$\mathrm{U}_*^{\mathbb{R}}(B) := \left\{ \mathbf{ch}^{-1}\left[\langle y, \mathbf{td}_B^{-1} \cup \mathbf{ch}(-)\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \mathbf{H}\mathbb{R}_{*+1}(B) \right\}$$

and get $\mathrm{Q}_*^{\mathbb{R}}(B) := \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \pi_{*+1}^{\$}(\mathbf{KU}\mathbb{R}/\mathbb{Z}))/\, \mathrm{U}_*^{\mathbb{R}}(B)$. Here we employed the evaluation pairing

$$\langle -, - \rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}} : \mathbf{HP}\mathbb{R}_*(B) \times \mathbf{HP}\mathbb{R}^0(B) \to \pi_*^{\$}(\mathbf{HP}\mathbb{R})$$

in $\mathbf{HP}\mathbb{R}$-theory and the fact that the Chern character induces an isomorphism

$$\mathbf{ch} : \pi_{*+1}^{\$}(\mathbf{KU}\,\mathbb{R}/\mathbb{Z}) \xrightarrow{\cong} \pi_{*+1}^{\$}(\mathbf{HP}\,\mathbb{R}/\mathbb{Z}) \,. \tag{6.3}$$

Lemma 6.4. *We have a canonical identification* $\mathrm{Q}_*^{\mathbb{R}}(B) \cong \mathrm{Q}_*^{\mathbb{R}}(MB)$. ♦

Proof. The claim would follow directly from the Thom isomorphism if we would define $\mathrm{U}_*^{\mathbb{R}}(B)$ to be the subgroup

$$\widetilde{\mathrm{U}}_*^{\mathbb{R}}(B) := \left\{ \left[\langle \varepsilon(y), \mathrm{Thom}^{\mathbf{KU}}(-)\rangle_{\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \pi_{*+1}^{\$}(MB\mathbb{R}) \right\}$$

of $\mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \pi_{*+1}^{\$}(\mathbf{KU}\mathbb{R}/\mathbb{Z}))$. Because the Chern character is multiplicative we have

$$\widetilde{\mathrm{U}}_*^{\mathbb{R}}(B) = \left\{ \mathbf{ch}^{-1}\left[\langle \mathbf{ch}(\varepsilon(y)), \mathbf{ch}(\mathrm{Thom}^{\mathbf{KU}}(-))\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \pi_{*+1}^{\$}(MB\mathbb{R}) \right\} \,.$$

We can further simplify this by the Riemann-Roch Theorem (6.1) to get

$$\widetilde{\mathrm{U}}_*^{\mathbb{R}}(B) = \left\{ \mathbf{ch}^{-1}\left[\langle \mathbf{ch}(\varepsilon(y)), \mathrm{Thom}^{\mathbf{HP}\mathbb{R}}(\mathbf{td}_B^{-1} \cup \mathbf{ch}(-))\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \pi_{*+1}^{\$}(MB\mathbb{R}) \right\}$$

and by noting that $\mathbf{ch} \circ \varepsilon$ is the usual identification $\pi_{*+1}^{\$}(MB\mathbb{R}) \cong \mathbf{H}\,\mathbb{R}_{*+1}(MB)$ we arrive at

$$\widetilde{\mathrm{U}}_*^{\mathbb{R}}(B) = \left\{ \mathbf{ch}^{-1}\left[\langle y, \mathrm{Thom}^{\mathbf{HP}\mathbb{R}}(\mathbf{td}_B^{-1} \cup \mathbf{ch}(-))\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \mathbf{H}\,\mathbb{R}_{*+1}(MB) \right\} .$$

Finally we employ $\langle \mathrm{Thom}_{\mathbf{HP}\mathbb{R}}(-), -\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}} = \langle -, \mathrm{Thom}^{\mathbf{HP}\mathbb{R}}(-)\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}$ and the Thom isomorphism for $\mathbf{HP}\mathbb{R}$-theory to conclude $\mathrm{U}_*^{\mathbb{R}}(B) = \widetilde{\mathrm{U}}_*^{\mathbb{R}}(B)$. □

We will often implicitly identify $\pi_{2k}(\mathbf{HP}\mathbb{R}) \cong \mathbb{R}$ and thus $\pi_{2k}(\mathbf{HP}\,\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. By the above isomorphism (6.3) we get $\pi_{2k}(\mathbf{KU}\,\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. Therefore we consider $\mathrm{Q}_*^{\mathbb{R}}(B)$ as a quotient of $\mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbb{R}/\mathbb{Z})$.

We will often apply the following proposition to check that an element in $\mathrm{Q}_*^{\mathbb{R}}(B)$ is trivial.

Proposition 6.5. *Let M be a smooth compact oriented manifold of even dimension $2n$ and consider $c \in \mathrm{Hom}_{[0]}^{\mathrm{cont}}(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}_{\mathrm{cl}}^0(M))$. Then the composition*

$$\mathbf{KU}^0(B) \xrightarrow{\mathbf{td}_B^{-1} \cup \mathbf{ch}} \mathbf{HP}\mathbb{Q}^0(B) \xrightarrow{c} \Omega\mathrm{P}_{\mathrm{cl}}^0(M) = \oplus_k b^k \Omega^{2k}(M) \xrightarrow{\int_M} b^n \Omega^0(\star) = \mathbb{R} \xrightarrow{[-]_{\mathbb{R}/\mathbb{Z}}} \mathbb{R}/\mathbb{Z}$$

lies in the subgroup $\mathrm{U}_{2n-1}^{\mathbb{R}}(B)$ of $\mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbb{R}/\mathbb{Z}))$. In other words, the map

$$\left[\int_M c \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(-)\right)\right]_{\mathbb{R}/\mathbb{Z}}$$

is trivial in $\mathrm{Q}_{2n-1}^{\mathbb{R}}(M)$.

The analogous statement holds if M is a smooth compact oriented manifold of odd dimension $2n-1$ and one considers $c \in \mathrm{Hom}_{[-1]}^{\mathrm{cont}}(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(M)/(\mathrm{im}\, d))$. ♦

Proof. Recall that we use the identification (6.3) and therefore we have

$$\mathrm{U}_{2n-1}^{\mathbb{R}}(B) = \left\{ \left[\langle y, \mathbf{td}_B^{-1} \cup \mathbf{ch}(-)\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \mathbf{H}\,\mathbb{R}_{2n}(B) \right\} .$$

So we need to construct a suitable class in $\mathbf{H}\,\mathbb{R}_{2n}(B)$.

To this end we observe that the diagram

$$\begin{tikzcd}
\mathbf{HP}\mathbb{Q}^0(B) \arrow[r, "c"] \arrow[d, equal] & \Omega\mathrm{P}_{\mathrm{cl}}^0(M) \arrow[r, "\int_M"] \arrow[d, equal] & \mathbb{R} \arrow[d, equal] \\
\oplus_k b^k\,\mathbf{H}\,\mathbb{Q}^{2k}(B) \arrow[r, "c"] \arrow[d, "\mathrm{pr}_n"] & \oplus_k b^k \Omega_{\mathrm{closed}}^{2k}(M) \arrow[r, "\int_M"] \arrow[d, "\mathrm{pr}_n"] & \mathbb{R} \arrow[d, equal] \\
b^n\,\mathbf{H}\,\mathbb{Q}^{2n}(B) \arrow[r, "c"] & b^n \Omega_{\mathrm{closed}}^{2n}(M) \arrow[r, "\int_M"] & \mathbb{R}
\end{tikzcd}$$

commutes because c preserves the internal degree. Hence we set $\alpha := \int_M \circ c\colon\ \mathbf{H}\,\mathbb{Q}^{2n}(B) \to \mathbb{R}$ and get the commutative diagram

$$\begin{tikzcd}
& \mathbf{H}\,\mathbb{Q}^{2n}(B) \arrow[d, "\mathrm{incl}_n"'] \arrow[dr, "\alpha"] & \\
\mathbf{HP}\mathbb{Q}^0(B) \arrow[r, equal] & \oplus_k\,\mathbf{H}\,\mathbb{Q}^{2k}(B) \arrow[r, "\int_M \circ c"'] & \mathbb{R}\,.
\end{tikzcd}$$

Note that α is continuous. This implies that there exists a finite CW-complex C and a continuous map $\iota : C \to B$ such that $\ker(\iota^*) \subseteq \ker(\alpha)$. Thus we get the commutative diagram

$$\begin{tikzcd} \mathbf{H}\,\mathbb{Q}^{2n}(B) \arrow[rr, "\alpha"] \arrow[d, two heads] & & \mathbb{R} \\ \mathbf{H}\,\mathbb{Q}^{2n}(B)/(\ker \iota^*) \arrow[urr, "\exists!"] \arrow[rr, hook, "\iota^*"] & & \mathbf{H}\,\mathbb{Q}^{2n}(C) \arrow[u, dashed, "\tilde{\alpha}"'] \end{tikzcd} .$$

We can find the dashed map $\tilde{\alpha}$ since $\iota^* : \mathbf{H}\,\mathbb{Q}^{2n}(B)/(\ker \iota^*) \to \mathbf{H}\,\mathbb{Q}^{2n}(C)$ is an inclusion of finite dimensional discrete $\mathbb{Q}$-vector spaces. By the universal coefficient theorem (UCT) we have (note that $\mathbf{H}\,\mathbb{Q}^{2n}(C)$ is discrete)

$$\mathrm{Hom}^{\mathrm{cont}}(\mathbf{H}\,\mathbb{Q}^{2n}(C), \mathbb{R}) \overset{\mathrm{UCT}}{\cong} \mathbf{H}\mathbb{R}_{2n}(C) \xrightarrow{\iota_*} \mathbf{H}\mathbb{R}_{2n}(B)$$

and we denote the image of $\tilde{\alpha}$ by $\hat{\alpha} \in \mathbf{H}\mathbb{R}_{2n}(B)$ which satisfies by construction

$$\alpha(-) = \langle \hat{\alpha}, - \rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}} \in \mathrm{Hom}^{\mathrm{cont}}(\mathbf{H}\,\mathbb{Q}^{2n}(B), \mathbb{R}) \ .$$

Hence we conclude that

$$\left[\int_M c \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(-)\right)\right]_{\mathbb{R}/\mathbb{Z}} = \left[\alpha \circ \mathrm{pr}_n \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(-)\right)\right]_{\mathbb{R}/\mathbb{Z}} = \left[\langle \hat{\alpha}, \mathbf{td}_B^{-1} \cup \mathbf{ch}(-)\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}}$$

and therefore lies in $\mathrm{U}^{\mathbb{R}}_{2n-1}(B)$.

Along the same lines one proves the second statement. $\square$

6.3 Geometric cycles for B-bordism

In this section we give a geometric model for B-bordism theory which is adapted to our situation.

To describe our model we will use several principal G-bundles for $G = O(n)$ and $G = Spin^c(n)$. To shorten the notation we write $P \otimes_G H$ for the associated bundle of a G-principal bundle P along a group homomorphism $G \to H$. The group homomorphism will always be clear from the context. Moreover, we define the tensor product of an $O(n)$-principal bundle $P \to X$ and an $O(m)$-principal bundle $Q \to X$ as the $O(n+m)$-principal bundle given by

$$P \otimes Q := (P \times_X Q) \otimes_{O(n)\times O(m)} O(n+m)$$

where the $O(n) \times O(m)$-principal bundle $P \times_X Q$ is the fibre-wise product. This construction is natural and there are the usual canonical commutativity and associativity isomorphisms of a tensor product. Analogous definitions and remarks hold for $Spin^c(n)$. Since we often use trivial G-principal bundles we denote them by $\underline{G}$, e.g., $\underline{O(n)} = X \times O(n)$ and $\underline{Spin^c(n)} = X \times Spin^c(n)$.

We fix once and for all models for the classifying spaces $BO(n)$ and $BSpin^c(n)$ (for all $n \in \mathbb{N} \cup \{\infty\}$, e.g., $O(\infty) := \varinjlim_{n\in\mathbb{N}} O(n)$) and for the maps $BO(n) \to BO(n+1)$, $BSpin^c(n) \to BSpin^c(n+1)$ and $BSpin^c(n) \to BO(n)$. We also fix $BO(n) \to BO(\infty)$ and homotopies in the diagrams (for all $n \in \mathbb{N}$)

$$\begin{tikzcd} BO(n) \arrow[rr] \arrow[dr] & & BO(n+1) \arrow[dl] \\ & BO(\infty) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} BSpin^c(n) \arrow[r] \arrow[d] & BSpin^c(n+1) \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

and similarly for $Spin^c$. We set $BO := BO(\infty)$ and $BSpin^c := BSpin^c(\infty)$.

Denote the universal principal bundles by $EO(n)$, $EO := EO(\infty)$, $ESpin^c(n)$ and $ESpin^c := ESpin^c(\infty)$. Furthermore we fix isomorphisms

$$ESpin^c(n+1)_{|BSpin^c(n)} \cong ESpin^c(n) \otimes \underline{Spin^c(1)} \tag{6.4}$$

and similarly for $O(n)$.

Now we give our geometric model for B-bordism classes. Recall that we have fixed a continuous map $\sigma : B \to BSpin^c$.

Definition 6.6. *Fix $k \in \mathbb{Z}$. A* ***k-geometric cycle for B-bordism*** *consists of*

1. *a closed Riemannian manifold M with orthonormal frame bundle $O(TM) \to M$,*
2. *a $Spin^c$-structure[1] $Spin^c(TM) \to M$ on M,*
3. *a $Spin^c$-connection $\nabla^{Spin^c(TM)}$ on $Spin^c(TM) \to M$ lifting the Levi-Civita connection $\nabla^{O(TM)}$ on $O(TM) \to M$,*
4. *a continuous map $p : M \to B$,*
5. *a continuous map $p_k : M \to BSpin^c(k)$ together with a homotopy filling*

$$\begin{tikzcd} & & BSpin^c(k) \arrow[d] \\ M \arrow[r, "p"'] \arrow[urr, "p_k"] & B \arrow[r, "\sigma"'] & BSpin^c \end{tikzcd}$$

6. *and a continuous isomorphism*

$$\phi_k : Spin^c(TM) \otimes p_k^* ESpin^c(k) \cong \underline{Spin^c(\dim(M)+k)} \ .$$

We will often denote a k-geometric cycle for B-bordism by (M, p). ■

The isomorphism ϕ_k exhibits the composition $\sigma \circ p$ as a $Spin^c$-structure on the stable normal bundle, i.e., a $Spin^c$-complement of $Spin^c(TM)$.

We can stabilize a k-geometric cycle to a $(k+1)$-geometric cycle using the fixed inclusion $BSpin^c(k) \to BSpin^c(k+1)$ and the fixed isomorphism (6.4). So we get rid of k by identify k-geometric cycles with their stabilizations.

Definition 6.7. *A* ***geometric cycle for B-bordism*** *is an equivalence class of k-geometric cycles under stabilization.* ■

By definition, a geometric cycle for B-bordism is the same data as a compatible family of k-geometric cycles for almost all k.

Note that the disjoint union of two geometric cycles yields a third geometric cycle. We want to identify two geometric cycles if their disjoint union is a boundary.

Definition 6.8. *A* ***zero bordism*** *for the k-geometric cycle (M, p) consists of*

1. *a compact Riemannian manifold W with orthonormal frame bundle $O(TW) \to W$,*
2. *a $Spin^c$-structure $Spin^c(TW) \to W$ on W,*

[1] A $Spin^c$-structure is a $Spin^c(n)$-principal bundle $Spin^c(TM) \to M$ together with a smooth isomorphism $Spin^c(TM) \otimes_{Spin^c(n)} O(n) \cong O(TM)$. Here $n := \dim(M)$.

3. *a $Spin^c$-connection $\nabla^{Spin^c(TW)}$ on $Spin^c(TW) \to W$ lifting the Levi-Civita connection $\nabla^{O(TW)}$ on $O(TW) \to W$,*

4. *a diffeomorphism $i : \partial W \cong M$ and a collar neighbourhood $U \cong M \times [0, \varepsilon)$ of M in W,*

5. *a continuous map $q : W \to B$ together with a homotopy filling the diagram*

$$\begin{tikzcd} M \arrow[rr, hook, "i"] \arrow[dr, "p"'] & & W \arrow[dl, "q"] \\ & B & \end{tikzcd},$$

6. *a continuous map $q_k : W \to BSpin^c(k)$ together with a homotopy filling*

$$\begin{tikzcd} & & BSpin^c(k) \arrow[d] \\ W \arrow[urr, "q_k"] \arrow[r, "q"'] & B \arrow[r, "\sigma"'] & BSpin \end{tikzcd}$$

7. *and a continuous isomorphism*

$$\psi_k : Spin^c(TW) \otimes q_k^* ESpin^c(k) \cong \underline{Spin^c(\dim(M) + k + 1)} \ .$$

We require these data to be compatible with the data on M in the following sense.

- *The diffeomorphism $\partial W \cong M$ is isometric and the Riemannian metric on W has product structure on the collar neighbourhood of the boundary.*

- *The outgoing unit vector field yields a decomposition*

$$O(TW)_{|U} \cong (O(TM) \times [0, \varepsilon)) \otimes \underline{O(1)} \ . \tag{6.5}$$

  Using this isomorphism the restriction $Spin^c(TW)_{|U}$ to the collar neighbourhood is a $Spin^c$-structure on $(O(TM) \times [0, \varepsilon)) \otimes \underline{O(1)}$. We require a fixed isomorphism

$$Spin^c(TW)_{|U} \cong (Spin^c(TM) \times [0, \varepsilon)) \otimes \underline{Spin^c(1)} \tag{6.6}$$

  lifting the above isomorphism (6.5) such that the restriction of $\nabla^{Spin^c(TW)}_{|U}$ to the collar neighbourhood U of ∂W coincides under this isomorphism with $(\nabla^{Spin^c(TM)} \times [0, \varepsilon)) \otimes \nabla^{\underline{Spin^c(1)}}$. Here $\nabla^{\underline{Spin^c(1)}}$ denotes the trivial connection.

- *The restriction of the continuous isomorphism ψ_k induces by*

$$\begin{aligned} Spin^c(TM) \otimes p_k^* ESpin^c(k) \otimes \underline{Spin^c(1)} &\overset{(6.6)}{\cong} (Spin^c(TW))_{|M} \otimes p_k^* ESpin^c(k) \cong \\ (Spin^c(TW) \otimes q_k^* ESpin^c(k))_{|M} &\overset{\psi_k}{\cong} \underline{Spin^c(\dim(M) + k + 1)} \cong \\ &\underline{Spin^c(\dim(M) + k)} \otimes \underline{Spin^c(1)} \end{aligned}$$

  the continuous isomorphism $\phi_k \otimes \mathtt{id}_{\underline{Spin^c(1)}}$.

*Since we can stabilize a zero bordism for a k-geometric cycle to a zero bordism for the stabilized $(k+1)$-geometric cycle we get a well-defined notion of **zero bordism** for geometric cycles.* ■

We call two geometric cycles **bordant** if there exists a zero bordism for their disjoint union. This is an equivalence relation and the set of equivalence classes is an abelian group with addition induced by disjoint union. This abelian group is graded by the dimension of the geometric cycles.

The model for B-bordism described above is equivalent to the usual model. Hence the Pontryagin-Thom Theorem yields

Theorem 6.9. *The graded abelian group constructed from geometric cycles and zero bordisms is isomorphic (as graded abelian group) to the stable homotopy groups $\pi_*^{\$}(MB)$ of the Thom spectrum MB.* ♦

6.4 $Spin^c$-structures and Todd forms

In this section our aim is to define a Todd form $\mathbf{Td}_M \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)^\times$ representing the **Todd class** $\mathbf{td}_M^{-1} := (p^* \mathbf{td}_B)^{-1} = (p \circ \sigma)^* \mathbf{td}^{-1}$ as canonical as possible (for given $p : M \to B$). Note that the Todd class $\mathbf{td} \in \mathbf{HP}\mathbb{Q}^0(BSpin^c)^\times$ is inverted.

Afterwards we discuss the behaviour of the Todd form under morphisms. Finally we define universal $Spin^c$-geometrizations and show that such a geometrization on B yields a geometrization on each geometric cycle for B-bordism that extends along zero bordisms.

At first we recall that $Spin^c$-principal bundles have complements.

Proposition 6.10. *Given a smooth (resp. continuous) $Spin^c(k)$-principal bundle $P \to M$ there exists a smooth (resp. continuous) $Spin^c(l)$-principal bundle $R \to M$ together with an isomorphism of smooth (resp. continuous) $Spin^c(k+l)$-principal bundles*

$$R \otimes P \cong \underline{Spin^c(k+l)}$$

for all big enough l.

Moreover, for $k \geq 3$, the two-out-of-three principal holds for $Spin^c$-structures on $SO(k)$-principal bundles. ♦

Proof. The continuous statements are true for $O(n)$ and $SO(n)$ instead of $Spin^c(n)$ ([BN09, Corollary 8.3]). For $Spin^c(n)$ the continuous case can be proven along the lines of Corollary 8.3 in [BN09]. Explicitly, one uses $l = 3$ and $\pi = \mathbb{Z}$ and $B(n) = BSO(n)$. To get the smooth case, as always, one just smoothes out everything.

Note that $\mathbf{H}^3(BSO(n); \mathbb{Z}) \cong \mathbf{H}^3(BSO(n+1); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ only for $n \geq 3$. Therefore we need the condition $k \geq 3$ in the second statement. □

We will often use that a complement R for a $Spin^c(k)$-principal bundle P is also a complement for the stabilization $P \otimes \underline{Spin^c(1)}$.

Now we discuss the Todd form. If (M, p) is a geometric cycle for B-bordism we have the $Spin^c$-connection $\nabla^{Spin^c(TM)}$ which yields the Todd form $\mathbf{Td}\left(\nabla^{Spin^c(TM)}\right)$. By Chern-Weil theory this form represents the Todd class $\mathbf{td}_M^{-1} := (p^* \mathbf{td}_B)^{-1} = (p \circ \sigma)^* \mathbf{td}^{-1}$ since we have the isomorphism $\phi_k : Spin^c(TM) \oplus p_k^* ESpin^c(k) \cong \underline{Spin^c(n+k)}$. This explains why we use the inverted Todd class $\mathbf{td}^{-1}$.

Next consider a compact n-dimensional manifold M and a continuous map $p : M \to B$. In this generality there might be no relation between the map p and the tangent bundle of M

(for a geometric cycle there is a relation given by Definition 6.6,4., 5. and 6.). Nevertheless we mimic this relation as follows.

Definition 6.11. *Let $k, r \in \mathbb{N}$ be natural numbers.*
*A (k, r)**-abstract** $Spin^c$**-structure** on (M, p) consists of:*

1. *A smooth $Spin^c(r)$-principal bundle $R_r \to M$ on M.*
2. *A $Spin^c$-connection ∇^{R_r} on $R_r \to M$.*
3. *A continuous map $p_k : M \to BSpin^c(k)$ together with a homotopy filling*

$$\begin{tikzcd} & & BSpin^c(k) \arrow[d] \\ M \arrow[urr, "p_k"] \arrow[r, "p"'] & B \arrow[r, "\sigma"'] & BSpin^c \end{tikzcd} .$$

4. *A continuous isomorphism $\phi_k : R_r \otimes p_k^* ESpin^c(k) \cong \underline{Spin^c(r+k)}$.*

*As before we can stabilize in k and define a **stable** r**-abstract** $Spin^c$**-structure** on (M, p) as an equivalence classes of (k, r)-abstract $Spin^c$-structures.*

We usually denote a stable r-abstract $Spin^c$-structure by $(R_r, \nabla^{R_r}, p_k)_k$. ■

For example, a geometric cycle (M, p) or a zero bordism (W, q) yield stable r-abstract $Spin^c$-structures where $r = \dim(M)$ respectively $r = \dim(W)$.

Definition 6.12. *Let (M, p) be as above and assume we are given a stable r-abstract $Spin^c$-structure $(R_r, \nabla^{R_r}, p_k, \phi_k)_k$ on (M, p). Then we get the **associated Todd form** $\mathbf{Td}_M := \mathbf{Td}(\nabla^{R_r}) \in \Omega \mathrm{P}^0_{\mathrm{cl}}(M)^\times$.* ■

Using the isomorphisms $(\phi_k)_k$ we see as before that $[\mathbf{Td}_M] = \mathbf{td}_M^{-1} \in \mathbf{HP}\mathbb{Q}^0(M)^\times$. Now we show that stable r-abstract $Spin^c$-structures always exist for all big enough r.

Proposition 6.13. *Let M be a compact manifold and $p : M \to B$ be a continuous map. Then we can find a stable r-abstract $Spin^c$-structure on (M, p) for big enough r.* ♦

Proof. Since M is compact we can find a continuous map $p_k : M \to BSpin^c(k)$ for some $k \geq 3$ such that

$$\begin{tikzcd} & & BSpin^c(k) \arrow[d] \\ M \arrow[urr, "p_k"] \arrow[r, "p"'] & B \arrow[r, "\sigma"'] & BSpin \end{tikzcd}$$

commutes up to fixed homotopy. By Proposition 6.10 there exists a continuous $Spin^c(r)$-principal bundle $\tilde{R}_r$ and an isomorphism $\tilde{R}_r \otimes p_k^* ESpin^c(k) \cong \underline{Spin^c(r+k)}$. Now one chooses a smooth structure to get R_r together with a continuous isomorphism $R_r \cong \tilde{R}_r$ yielding $R_r \otimes p_k^* ESpin^c(k) \cong \underline{Spin^c(r+k)}$. To complete the proof we choose a $Spin^c$-connection ∇^{R_r} on R_r and stabilize in k. □

Now we discuss how the associated Todd form behaves under morphisms.

Construction 6.14. Suppose we are given a homotopy-commutative triangle

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p"'] & & N \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

where M and N are compact manifolds and f is smooth. Moreover, assume there are a stable r-abstract *Spin*^{c}-structure $(R_r, \nabla^{R_r}, p_k)_k$ on p and a stable s-abstract *Spin*^{c}-structure $(S_s, \nabla^{S_s}, q_k)_k$ on q for some integers r, s.

Our aim is to understand the difference between $f^* \mathbf{Td}(\nabla^{S_s})$ and $\mathbf{Td}(\nabla^{R_r})$.

By definition, there are continuous isomorphisms

$$R_r \otimes p_k^* ESpin^c(k) \cong \underline{Spin^c(r+k)} \quad \text{and} \quad S_s \otimes q_k^* ESpin^c(k) \cong \underline{Spin^c(s+k)}$$

for some big enough k. Using the homotopy $q \circ f \simeq p$ and the homotopies of the stable abstract *Spin*^{c}-structures we get a homotopy $q_k \circ f \simeq p_k$. Therefore we get a (continuous) homotopy class of continuous isomorphisms, $[\Xi]$, represented by

$$R_r \otimes \underline{Spin^c(k+s)} \cong R_r \otimes (f^* S_s \otimes p_k^* ESpin^c(k)) \cong f^* S_s \otimes \underline{Spin^c(k+r)}$$

and we can find a smooth isomorphism

$$\Lambda : R_r \otimes \underline{Spin^c(k+s)} \cong f^* S_s \otimes \underline{Spin^c(k+r)}$$

which lies in the (continuous) homotopy class $[\Xi]$.

Using the trivial *Spin*^{c}-connections on the trivial bundles and the smooth isomorphism Λ we get the two connections

$$\Lambda^* \tilde{\nabla}^{S_s} := \Lambda^* \left(f^* \nabla^{S_s} \otimes \nabla^{\underline{Spin^c(k+r)}} \right) \quad \text{and} \quad \tilde{\nabla}^{R_r} := \nabla^{R_r} \otimes \nabla^{\underline{Spin^c(k+s)}}$$

on $R_r \otimes \underline{Spin^c(k+s)}$. We employ the transgression form $\widetilde{\mathbf{Td}}$ of these two connections to define the **error form**

$$\delta := \mathbf{Td}(\tilde{\nabla}^{R_r})^{-1} \wedge \widetilde{\mathbf{Td}}(\Lambda^* \tilde{\nabla}^{S_s}, \tilde{\nabla}^{R_r}) \in \Omega \mathrm{P}^{-1}(M)/(\operatorname{im} d)$$

which satisfies (cf. formula (3.2)) $d\delta = f^* \mathbf{Td}(\nabla^{S_s}) \wedge \mathbf{Td}(\nabla^{R_r})^{-1} - 1$ and thus

$$d\delta \wedge \mathbf{Td}(\nabla^{R_r}) = f^* \mathbf{Td}(\nabla^{S_s}) - \mathbf{Td}(\nabla^{R_r})$$

represents the difference. Hence the Todd forms agree if and only if the error form is closed.

Example 6.15. The error form δ for the boundary inclusion of a geometric cycle (M, p) into a zero bordism (W, q) is trivial, $\delta = 0 \in \Omega \mathrm{P}^{-1}(M)/(\operatorname{im} d)$. ★

Observe that we can stabilize in k to get the first part of the following result.

Proposition 6.16. *The error form δ is independent of k. Moreover, two smooth isomorphisms $\Lambda_1, \Lambda_2 : R_r \otimes \underline{Spin^c(k+s)} \cong f^* S_s \otimes \underline{Spin^c(k+r)}$ which lie in the (continuous) homotopy class $[\Xi]$ yield the same error form. In particular, the error form δ of f is independent of the choice of the smooth isomorphism Λ and thus depends only on f (and the stable abstract Spin^c-structures on p and q).* ♦

Proof. We can reduce to the following case (by considering $\Lambda_2^{-1} \circ \Lambda_1$ and connections $\nabla_1 := \Lambda_1^* \left(f^* \nabla^{S_s} \otimes \nabla^{\underline{Spin^c(k+r)}} \right)$ and $\nabla_0 := \nabla^{R_r} \otimes \nabla^{\underline{Spin^c(k+s)}}$). Set $P := R_r \otimes \underline{Spin^c(k+s)}$ and let $\Phi : P \to P$ be a smooth automorphism which is smoothly homotopic through smooth isomorphisms to $\mathtt{id}_P$. Then we have to show that the transgression forms $\widetilde{\mathbf{Td}}(\nabla_1, \nabla_0)$ and $\widetilde{\mathbf{Td}}(\Phi^* \nabla_1, \nabla_0)$ coincide in $\Omega \mathrm{P}^{-1}(M)/(\operatorname{im} d)$. By the properties of transgression we have $\widetilde{\mathbf{Td}}(\nabla_1, \nabla_0) - \widetilde{\mathbf{Td}}(\Phi^* \nabla_1, \nabla_0) = \widetilde{\mathbf{Td}}(\nabla_1, \Phi^* \nabla_1) \in \Omega \mathrm{P}^{-1}(M)/(\operatorname{im} d)$. Now we consider the homotopy as an isomorphisms $H : \mathrm{pr}^* P \to \mathrm{pr}^* P$ (here $\mathrm{pr} : M \times I \to M$ denotes the projection) of principal bundles with $H_{|M \times \{0\}} = \Phi$ and $H_{|M \times \{1\}} = \mathtt{id}_P$. Take the constant path of connections on the connection ∇_1 and pullback this path along H to get a path in the space of connections from $\Phi^*\nabla_1$ to ∇_1. Since the transgression is invariant under isomorphisms which cover the identity on M we see that

$$\widetilde{\mathbf{Td}}(\nabla_1, \Phi^*\nabla_1) = \mathtt{id}_M^* \widetilde{\mathbf{Td}}(\nabla_1, \nabla_1) = \widetilde{\mathbf{Td}}(\nabla_1, \nabla_1) = 0 \in \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$$

and hence the Proposition is proven. $\square$

We summarize: Given a homotopy-commutative triangle

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p"'] & & N \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

with compact manifolds M, N and a smooth map f. Suppose that there are fixed stable abstract $Spin^c$-structures on p and q as above. Then we get a canonical error form

$$\delta := \mathbf{Td}(\tilde{\nabla}^{R_r})^{-1} \wedge \widetilde{\mathbf{Td}}(\Lambda^*\tilde{\nabla}^{S_s}, \tilde{\nabla}^{R_r}) \in \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$$

which satisfies $d\delta \wedge \mathbf{Td}(\nabla^{R_r}) = f^*\,\mathbf{Td}(\nabla^{S_s}) - \mathbf{Td}(\nabla^{R_r})$. Thus the pullback of a geometrization along f is defined by formula (3.3). ★

We study how the error form behaves under composition (see [Bun11, Lemma 4.9]).

Proposition 6.17. *Assume we have a homotopy-commutative diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "p"'] & N \arrow[r, "g"] \arrow[d, "q"] & K \arrow[dl, "x"] \\ & B & \end{tikzcd}$$

where M, N and K are compact manifolds and f and g are smooth maps. Moreover, assume there are a stable r-abstract $Spin^c$-structure $(R_r, \nabla^{R_r}, p_k)_k$ on p, a stable s-abstract $Spin^c$-structure $(S_s, \nabla^{S_s}, q_k)_k$ on q and a stable t-abstract $Spin^c$-structure $(T_t, \nabla^{T_t}, x_k)_k$ on x for some integers r, s, t. Then the above construction yields error forms δ_g, δ_f and $\delta_{g\circ f}$. We claim that the following relation holds (cf. formula (3.6)*):*

$$\delta_{g\circ f} = f^*\delta_g + \delta_f + \delta_f \wedge f^* d\delta_g \in \Omega\mathrm{P}^{-1}(M)/((\operatorname{im} d)) \ . \qquad \blacklozenge$$

Proof. By Proposition 6.16 we can assume that all the principal bundles are identified. Thus we can consider the three connections $\nabla_1 := f^*g^*\nabla^{T_t} \otimes \nabla^{\underline{Spin^c(c)}}$, $\nabla_2 := f^*\nabla^{S_s} \otimes \nabla^{\underline{Spin^c(b)}}$ and $\nabla_3 := \nabla^{R_r} \otimes \nabla^{\underline{Spin^c(a)}}$ on $R^r \otimes \underline{Spin^c(a)}$ for big enough $a, b, c \in \mathbb{N}$. We can interpolate these on the two-simplex Δ^2 to get a connection $\hat{\nabla}$ on $R^r \otimes \underline{Spin^c(a)} \to M \times \Delta^2$. Integrating the Todd form $\mathbf{Td}(\hat{\nabla})$ fibre-wise over the two-simplex yields an explicit form

$$\varepsilon := \int_{M\times\Delta^2/M} \mathbf{Td}(\hat{\nabla}) \in \Omega\mathrm{P}^{-2}(M)/((\operatorname{im} d))$$

with $d\varepsilon = \widetilde{\mathbf{Td}}(\nabla_3, \nabla_2) + \widetilde{\mathbf{Td}}(\nabla_2, \nabla_1) - \widetilde{\mathbf{Td}}(\nabla_3, \nabla_1)$. Then one checks

$$f^*\delta_g + \delta_f + \delta_f \wedge f^*d\delta_g - \delta_{g\circ f} = d\left(\varepsilon \wedge \mathbf{Td}(\nabla^{R_r})^{-1} + f^*\delta_g \wedge \delta_f\right)$$

which proves the claim. $\square$

Now we can define the corresponding concepts for geometrizations.

Definition 6.18. *Let $p : M \to B$ be a continuous map where M is a compact manifold. A $Spin^c$-**geometrization** for p consists of a stable l-abstract $Spin^c$-structure on p (yielding the associated Todd form $\mathbf{Td}_M$) and a $(p, \mathbf{Td}_M, \sigma^* \mathbf{td}^{-1})$-geometrization.* ■

Definition 6.19. *A **universal** $Spin^c$**-geometrization for** $\sigma : B \to BSpin^c$ consists of*

- *an approximation by manifolds $(B_i, f_i, g_i)_{i\in\mathbb{N}}$ of B,*
- *a family of stable l_i-abstract $Spin^c$-structures on $g_i : B_i \to B$ (which yield associated Todd forms $\mathbf{Td}_{B_i} \in \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)$ and error forms $\delta_i \in \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)$ for the maps f_i)*
- *and a family of $(g_i, \mathbf{Td}_{B_i}, \sigma^* \mathbf{td}^{-1})$-geometrizations $\mathcal{G}_i$*

such that $f^!_{i,\delta_i}(\mathcal{G}_{i+1}) = \mathcal{G}_i$ for all $i \in \mathbb{N}$.

*We will refer to the data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$ as **underlying data**.* ■

Observe that the geometrizations fit together by formula (3.5) and Proposition 6.17. So a universal $Spin^c$-geometrization on B is a compatible system of $Spin^c$-geometrizations on an approximation $(B_i)_i$ of B. Also note that - if we can find an approximation and the technical condition is satisfied - our Existence Theorem 3.27 yields universal $Spin^c$-geometrizations by Proposition 6.13.

In the following example we show that a universal $Spin^c$-geometrization on B yields a (non-canonical) $Spin^c$-geometrization for each geometric cycle representing an element in $\pi^{\$}_*(MB)$. Moreover, we show that this $Spin^c$-geometrization extends along zero bordisms.

Example 6.20. We fix a universal $Spin^c$-geometrization for B,

$$(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i, \mathcal{G}_i)_i \ .$$

Let (M, p) be a geometric cycle for B-bordism. Then we can find a smooth lift ρ_i for p (for some big enough $i \in \mathbb{N}$) such that

$$\begin{tikzcd} M \arrow[rr, dashed, "\rho_i"] \arrow[dr, "p"'] & & B_i \arrow[dl, "g_i"] \\ & B & \end{tikzcd} \tag{6.7}$$

commutes up to fixed homotopy and we get an error form δ^i_M for ρ_i by Construction 6.14. Thus we get a $(p, \mathbf{Td}(\nabla^{Spin^c(TM)}), \sigma^* \mathbf{td}^{-1})$-geometrization $\mathcal{G}^i_M := (\rho_i)^!_{\delta^i_M}(\mathcal{G}_i)$.

We can stabilize ρ_i to $\rho_{i+1} := f_i \circ \rho_i$ and one easily checks

Lemma 6.21. *We have $(\rho_i)^!_{\delta^i_M}(\mathcal{G}_i) = (\rho_{i+1})^!_{\delta^{i+1}_M}(\mathcal{G}_{i+1})$. Hence the geometrization $\mathcal{G}^i_M$ is independent of the choice of big enough i.* ♦

Note that the geometrization depends on the choice of the lift ρ_i and the homotopy.

Next we extend along zero bordisms. So suppose we are given a zero bordism (W, q) for (M, p). As before we can find a smooth lift ξ_i of q (for some big enough $i \in \mathbb{N}$) such that in the diagram

$$\begin{tikzcd} M \arrow[rr, hook, "\mathrm{incl}"] \arrow[dr, "\rho_i"] \arrow[ddr, "p"'] & & W \arrow[dl, dashed, "\xi_i"'] \arrow[ddl, "q"] \\ & B_i \arrow[d, "g_i"] & \\ & B & \end{tikzcd}$$

the right lower triangle is filled by a homotopy and the upper triangle commutes strictly. This yields an error form δ^i_W and we get a $(q, \mathbf{Td}(\nabla^{Spin^c(TW)}), \sigma^* \mathbf{td}^{-1})$-geometrization $\mathcal{G}^i_W := (\xi_i)^!_{\delta^i_W}(\mathcal{G}_i)$.

Recall that the boundary inclusion incl : $M \hookrightarrow W$ has the trivial error form. Therefore we get a second $(p, \mathbf{Td}(\nabla^{Spin^c(TM)}), \sigma^* \mathbf{td}^{-1})$-geometrization $\hat{\mathcal{G}}^i_M := \mathrm{incl}^*(\mathcal{G}_W)$. The following proposition shows that the two geometrizations on M coincide.

Proposition 6.22. *The geometrizations $\mathcal{G}^i_W$ and $\hat{\mathcal{G}}^i_M$ are independent of the choice of big enough $i \in \mathbb{N}$. Moreover, $\delta^i_{W|M} = \delta^i_M$ and thus the geometrizations $\mathcal{G}^i_M$ and $\hat{\mathcal{G}}^i_M$ agree.* ♦

Proof. The independence of i follows by stabilization. Since transgression commutes with restriction we get $\delta^i_{W|M} = \delta^i_M$. Now the equality $\mathcal{G}^i_M = \hat{\mathcal{G}}^i_M$ follows directly from formula (3.5) and Proposition 6.17 using that the error form for the inclusion $M \hookrightarrow W$ is trivial. □

Thus we have seen that a universal *Spin*^{c}-geometrization for B yields a (non-canonical, since we have to choose the smooth lift ρ_i) *Spin*^{c}-geometrization on every geometric cycle for B-bordism which extends along zero bordisms (non-canonically) . ★

We stress again that this is non-trivial. In general geometrizations don't extend along bordisms, see [Bun11, Example 4.11] and Example 3.18.

6.5 Dirac operators and eta-invariants

In this section we recall the (reduced) eta-invariant and the Atiyah-Patodi-Singer index theorem for twisted Dirac operators on *Spin*^{c}-manifolds.

Let (M, p) be a geometric cycle for B-bordism with $n := \dim(M)$ being odd. Then we have a $Spin^c(n)$-principal bundle with connection, $\big(Spin^c(TM), \nabla^{Spin^c(TM)}\big)$. Since n is odd there is a distinguished irreducible complex representation Δ^n of $Spin^c(n)$, called the spinor representation. The associated vector bundle $\$(TM) := Spin^c(TM) \times_{Spin^c(n)} \Delta^n$ is called the **spinor bundle**. The connection $\nabla^{Spin^c(TM)}$ endows $\$(TM)$ with the structure of a Dirac bundle (cf. [LM89, Definition II.5.2]). Therefore we get a well-defined **Dirac operator** $\not{D}_M$ which acts on sections of $\$(TM)$. More generally, given a geometric vector bundle $\mathbf{V} = (V, h^V, \nabla^V)$ (see Definition 2.4) on M, we get a twisted Dirac operator $\not{D}_M \otimes \mathbf{V}$ which acts on sections of $\$(TM) \otimes V$.

Next we discuss the eta-invariant of [APS75a]. Define the eta-function of $\not{D}_M \otimes \mathbf{V}$ by

$$\eta(\not{D}_M \otimes \mathbf{V}, t) := \mathrm{Tr}_s \, |\not{D}_M \otimes \mathbf{V}\,|^{-t} \mathrm{sign}(\not{D}_M \otimes \mathbf{V})$$

where Tr_s denotes the super trace. This expression is a holomorphic function for t with real part $\Re(t) > n$ and has a meromorphic continuation to $\mathbb{C}$ with finite real value at $t = 0$ ([APS76, Theorem 4.5]). The **eta-invariant** of $\not{D}_M \otimes \mathbf{V}$ is defined as the value of the meromorphic continuation at $t = 0$,

$$\eta(\not{D}_M \otimes \mathbf{V}) := \eta(\not{D}_M \otimes \mathbf{V}, 0) \in \mathbb{R} \ .$$

This real number depends on the geometries of M and V in a possible discontinuous way. To get a continuous dependency we use the **reduced eta-invariant** ξ, defined by

$$\xi(\not{D}_M \otimes \mathbf{V}) := \left[\frac{\eta(\not{D}_M \otimes \mathbf{V}) + \dim\ker(\not{D}_M \otimes \mathbf{V})}{2} \right] \in {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \ .$$

Now we state the APS index-theorem due to Atiyah, Patodi and Singer in a form adapted to B-bordism.

Theorem 6.23. *(Atiyah-Patodi-Singer Index Theorem,[APS75a, Formula 4.3])*
Let (M,p) be a geometric cycle for B-bordism of odd dimension and (W,q) a zero bordism. Assume that $\mathbf{V} = (V, h^V, \nabla^V) \to M$ is a geometric vector bundle which extends to a geometric vector bundle $\mathbf{U} = (U, h^U, \nabla^U) \to W$ such that h^U and ∇^U have product structure on the collar neighbourhood of $\partial W \cong M$.

Then the following equation holds in ${}^{\mathbb{R}}/_{\mathbb{Z}}$,

$$\left[\int_W \mathbf{Td}(\nabla^{Spin^c(TW)}) \wedge \mathbf{ch}(\nabla^U)\right]_{\mathbb{R}/\mathbb{Z}} = \xi(\not{D}_M \otimes \mathbf{V}) \in {}^{\mathbb{R}}/_{\mathbb{Z}} \ . \qquad \blacklozenge$$

We close this section with the following easy facts.

Proposition 6.24. *The eta-invariant is additive, i.e.,*

$$\eta(\not{D}_M \otimes (\mathbf{V} \oplus \mathbf{W})) = \eta(\not{D}_M \otimes \mathbf{V}) + \eta(\not{D}_M \otimes \mathbf{W}) \ .$$

Also it is invariant under stabilization, i.e., if $\mathbf{V}$ is a trivial geometric vector bundle of virtual rank zero then $\eta(\not{D}_M \otimes \mathbf{V}) = 0$.

The analogous statements also hold for the reduced eta-invariant. $\blacklozenge$

6.6 Definition of the intrinsic eta-invariant

In this section we define the intrinsic eta-invariant as a group homomorphism

$$\eta_{\text{intrinsic}} : \pi_*^{\$}(MB) \to \mathrm{Q}_*^{\mathbb{R}}(B)$$

which restricts to η_{uni} on torsion elements.

For all what follows we fix a universal $Spin^c$-geometrization

$$(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i, \mathcal{G}_i)_i$$

for B with error forms δ_i for the maps f_i and associated Todd forms $\mathbf{Td}_i$. The $\mathcal{G}_i$ denote $(g_i, \mathbf{Td}_i, \mathbf{td}_B^{-1})$-geometrizations such that $(f_i)^!_{\delta_i}(\mathcal{G}_{i+1}) = \mathcal{G}_i$ for all $i \in \mathbb{N}$.

Let n be odd. Take an element $x \in \pi_n^{\$}(MB)$ and represent it by a geometric cycle (M,p). Then we get the associated Todd form $\mathbf{Td}_M := \mathbf{Td}(\nabla^{Spin^c(TM)})$. Example 6.20 gives a $(p, \mathbf{Td}_M, \mathbf{td}_B^{-1})$-geometrization $\mathcal{G}_M$ which is induced by one of the geometrizations $\mathcal{G}_i$ via pullback and which depends on the choice of a lift and a homotopy in diagram (6.7).

Recall that $\mathrm{Q}_n^{\mathbb{R}}(B)$ was defined as

$$\mathrm{Q}_n^{\mathbb{R}}(B) := \mathrm{Hom}^{\text{cont}}\left(\mathbf{KU}^0(B), \pi_{n+1}^{\$}(\mathbf{KU}{}^{\mathbb{R}}/_{\mathbb{Z}})\right) / \mathrm{U}_n^{\mathbb{R}}(B) \ .$$

To simplify the notation we use the Chern character as in equation (6.3) to identify

$$\pi_{n+1}^{\$}(\mathbf{KU}{}^{\mathbb{R}}/_{\mathbb{Z}}) \cong \pi_{n+1}^{\$}(\mathbf{HP}{}^{\mathbb{R}}/_{\mathbb{Z}}) \cong {}^{\mathbb{R}}/_{\mathbb{Z}} \ .$$

Using this identification we'll represent the intrinsic eta-invariant $\eta_{\text{intrinsic}}(x) \in \mathrm{Q}_n^{\mathbb{R}}(B)$ by a continuous group homomorphism $\mathbf{KU}^0(B) \to {}^{\mathbb{R}}/_{\mathbb{Z}}$. So we take $\phi \in \mathbf{KU}^0(B)$ and represent $p^*\phi$ as a ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$-graded vector bundle $V_\phi \to M$. We choose a hermitian metric h^{V_ϕ} and a metric connection ∇^{V_ϕ} to get a geometric vector bundle $\mathbf{V}_\phi := (V_\phi, h^{V_\phi}, \nabla^{V_\phi})$ which represents a lift of $p^*\phi$ to $\widehat{\mathbf{KU}}^0(M)$. But the geometrization yields another lift $\mathcal{G}_M(\phi) \in \widehat{\mathbf{KU}}^0(M)$ of $p^*\phi$. Hence, applying the long exact sequence (2.1),

$$\mathbf{KU}^{-1}(M) \xrightarrow{\mathbf{ch}} \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d) \xrightarrow{a} \widehat{\mathbf{KU}}^0(M) \xrightarrow{I} \mathbf{KU}^0(M) \to 0 \ ,$$

the difference yields the **correction form** $\gamma_\phi \in \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)$ which is unique up to $(\operatorname{im} \mathbf{ch})$ and is characterized by $a(\gamma_\phi) = \mathcal{G}_M(\phi) - [\mathbf{V}_\phi]$.

Proposition 6.25. *The map*

$$\Psi_{(M,p)} : \mathbf{KU}^0(B) \to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} ,$$
$$\phi \mapsto \Psi_{(M,p)}(\phi) := -\left[\int_M \mathbf{Td}_M \wedge \gamma_\phi\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D} \otimes \mathbf{V}_\phi)$$

is a well-defined continuous group homomorphism (where ${}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$ *is discrete).* ♦

Proof. For well-definedness we have to show that the value in ${}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$ does not depend on the choices of the correction form γ_ϕ, the vector bundle V_ϕ and its lift $\mathbf{V}_\phi$.

If we change the correction form to $\gamma_\phi + \mathbf{ch}(\omega)$ then the right hand side changes by $-\left[\int_M \mathbf{Td}_M \wedge \mathbf{ch}(\omega)\right]_{\mathbb{R}/\mathbb{Z}}$. But by the odd-version of the Atiyah-Singer index theorem the value of $\int_M \mathbf{Td}_M \wedge \mathbf{ch}(\omega)$ is an integer and hence vanishes in ${}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$.

Next suppose that we have two geometric structures $\mathbf{V}_\phi = (V_\phi, h^{V_\phi}, \nabla^{V_\phi})$ and $\tilde{\mathbf{V}}_\phi := (V_\phi, \tilde{h}^{V_\phi}, \tilde{\nabla}^{V_\phi})$ on $V_\phi \to M$. We can interpolate these structures on the cylinder to get a geometric refinement (with product structure near the boundary) $\mathbf{U} = (V_\phi \times I, h^U, \nabla^U)$ of the ${}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$-graded vector bundle $V_\phi \times I \to M \times I$. Note that the smooth lift ρ_i of $p : M \to B$ (which induced the geometrization $\mathcal{G}_M$) extends to $M \times I$ as $\rho_i \circ \mathrm{pr}$. Hence we get a geometrization $\mathcal{G}_{M\times I}$ which extends $\mathcal{G}_M$. Moreover, we have three correction forms γ_ϕ, $\tilde{\gamma}_\phi$ and $\gamma_{\mathbf{U}}$ which satisfy $a(\gamma_\phi) = \mathcal{G}_M(\phi) - \mathbf{V}_\phi$, $a(\tilde{\gamma}_\phi) = \mathcal{G}_M(\phi) - \tilde{\mathbf{V}}_\phi$ and $a(\gamma_{\mathbf{U}}) = \mathcal{G}_{M\times I}(\phi) - \mathbf{U}$, and we can arrange that $\gamma_{\mathbf{U}}$ restricts to γ_ϕ and $\tilde{\gamma}_\phi$ on the boundaries[2]. Furthermore the Todd form $\mathbf{Td}_{M\times I}$ is the constant extension of $\mathbf{Td}_M$, i.e., $\mathbf{Td}_{M\times I} = \mathrm{pr}^* \mathbf{Td}_M$.

Now the crucial observation is that the transgression $\widetilde{\mathbf{ch}}(\tilde{\nabla}^{V_\phi}, \nabla^{V_\phi})$ represents the difference $\gamma_\phi - \tilde{\gamma}_\phi$ (up to $(\operatorname{im}\mathbf{ch})$) by the homotopy formula (2.3). We apply the Atiyah-Patodi-Singer Index Theorem 6.23,

$$\left[\int_M \mathbf{Td}_M \wedge \gamma_\phi\right]_{\mathbb{R}/\mathbb{Z}} - \left[\int_M \mathbf{Td}_M \wedge \tilde{\gamma}_\phi\right]_{\mathbb{R}/\mathbb{Z}} = \left[\int_M \mathbf{Td}_M \wedge (\gamma_\phi - \tilde{\gamma}_\phi)\right]_{\mathbb{R}/\mathbb{Z}} =$$
$$\left[\int_M \mathbf{Td}_M \wedge \widetilde{\mathbf{ch}}(\tilde{\nabla}^{V_\phi}, \nabla^{V_\phi})\right]_{\mathbb{R}/\mathbb{Z}} = \left[\int_M \mathbf{Td}_M \wedge \int_{M\times I/M} \mathbf{ch}(\nabla^U)\right]_{\mathbb{R}/\mathbb{Z}} =$$
$$\left[\int_{M\times I} \mathbf{Td}_{M\times I} \wedge \mathbf{ch}(\nabla^U)\right]_{\mathbb{R}/\mathbb{Z}} \overset{\mathrm{APS}}{=} \xi(\not{D} \otimes \tilde{\mathbf{V}}_\phi) - \xi(\not{D} \otimes \mathbf{V}_\phi) ,$$

and conclude that $\Psi_{(M,p)}(\phi)$ is independent of the choice of geometric structure on V_ϕ.

Finally we show that $\Psi_{(M,p)}(\phi)$ is independent of the vector bundle V_ϕ representing $p^*\phi$. Since it is clear that only the isomorphism class of V_ϕ enters it is enough to discuss the behaviour under stabilization. So take a trivial geometric vector bundle $\mathbf{U}$ of virtual rank zero and consider $\mathbf{W}_\phi := \mathbf{V}_\phi \oplus \mathbf{U}$. We know by Proposition 6.24 that the reduced eta-invariants agree,

$$\xi(\not{D}_M \otimes \mathbf{W}_\phi) = \xi(\not{D}_M \otimes \mathbf{V}_\phi) .$$

But the correction form depends only on the differential K-theory class of $\mathbf{V}_\phi$ which is the same as the differential K-theory class of $\mathbf{W}_\phi$. Hence $\Psi_{(M,p)}(\phi)$ is independent of the choice of vector bundle V_ϕ representing $p^*\phi$.

Thus the map $\Psi_{(M,p)}$ is indeed well-defined.

One easily checks that $\Psi_{(M,p)}$ is additive.

[2]By the first step we can take $\gamma_\phi := \mathrm{incl}_0^* \gamma_{\mathbf{U}}$ and $\tilde{\gamma}_\phi := \mathrm{incl}_1^* \gamma_{\mathbf{U}}$.

For continuity one just observes that $\mathbf{KU}^0(B)$ decomposes (as a topological space) as disjoint union

$$\mathbf{KU}^0(B) = \amalg_{[V]\in \mathbf{KU}^0(M)} (p^*)^{-1}([V])$$

since p^* is continuous and $\mathbf{KU}^0(M)$ is discrete. So to check continuity it is enough to check that $\Psi_{(M,p)}(-)$ is continuous on the components $(p^*)^{-1}([V]) \subset \mathbf{KU}^0(B)$ for all $[V] \in \mathbf{KU}^0(M)$. But this is clear because we can fix a geometric vector bundle $\mathbf{V} := (V, h, \nabla)$ representing $[V]$. Then the value of

$$-\left[\int_M \mathbf{Td}_M \wedge \gamma_\phi\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D} \otimes \mathbf{V})$$

depends on ϕ only via $\mathcal{G}_M(\phi)$ and because $\mathcal{G}_M$ is continuous we completed the proof. □

Now we can define the intrinsic eta-invariant.

Definition and Proposition 6.26. *The **intrinsic eta-invariant** is the map*

$$\eta_{\text{intrinsic}} : \pi_*^{\$}(MB) \to \mathrm{Q}_*^{\mathbb{R}}(B), \quad x \mapsto [\mathbf{ch}^{-1} \circ \Psi_{(M,p)}(-)] .$$

It is a well-defined group homomorphism which respects the grading. ♦

Remark 6.27. Recall that the universal eta-invariant is functorial in B (Remark 6.2).

On the other hand, the intrinsic eta-invariant is *not* functorial in B. This is reasonable because we need a universal *Spin*^{c}-geometrization on B to define the intrinsic eta-invariant. Below we give an explicit example which shows that the universal eta-invariant depends on this choice. Moreover, in Chapter 9 we discuss an example which shows that the intrinsic eta-invariant is not functorial. Nevertheless we discuss in Remark 7.14 that the intrinsic eta-invariant is functorial for weak homotopy equivalences. ■

Proof. The intrinsic eta-invariant preserves the grading by definition and additivity is obvious.

It remains to check that it is well-defined. For this it is enough to check that $\psi_{(M,p)}$ lies in $\mathrm{U}_n^{\mathbb{R}}(B)$ whenever we can find a zero bordism (W, q) bounding the geometric cycle (M, p). Here we employ Proposition 6.5.

So suppose (M, p) is a geometric cycle for B-bordism and (W, q) is a zero bordism for it. Fix $\phi \in \mathbf{KU}^0(B)$ and choose a geometric vector bundle $\mathbf{U}_\phi = (U_\phi, h^{U_\phi}, \nabla^{U_\phi}) \to W$ representing $q^*\phi$ where h^U and ∇^U have product structure near the boundary $\partial W \cong M$. Then we get a geometric vector bundle $\mathbf{V}_\phi := \mathbf{U}_{\phi|M}$ representing $p^*\phi$.

As in Example 6.20 we find a geometrization $\mathcal{G}_W$ for q extending $\mathcal{G}_M$. Then the correction form γ_ϕ^W restricts to a correction form γ_ϕ^M (up to $(\operatorname{im} \mathbf{ch})$). Now we calculate

$$\begin{aligned}
\Psi_{(M,p)}(\phi) &= -\left[\int_M \mathbf{Td}_M \wedge \gamma_\phi^M\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_M \otimes \mathbf{V}_\phi) = \\
&= -\left[\int_W \mathbf{Td}_W \wedge d\gamma_\phi^W\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_M \otimes \mathbf{V}_\phi) = \\
&= -\left[\int_W \mathbf{Td}_W \wedge R \circ a(\gamma_\phi^W)\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_M \otimes \mathbf{V}_\phi) = \\
&= -\left[\int_W \mathbf{Td}_W \wedge R \circ (\mathcal{G}_W(\phi) - \mathbf{U}_\phi)\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_M \otimes \mathbf{V}_\phi) =
\end{aligned}$$

$$= -\left[\int_W \mathbf{Td}_W \wedge R \circ \mathcal{G}_W(\phi)\right]_{\mathbb{R}/\mathbb{Z}} + \left[\int_W \mathbf{Td}_W \wedge \mathbf{ch}(\nabla^{W_\phi})\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_M \otimes \mathbf{V}_\phi) \overset{\text{APS}}{=}$$
$$= -\left[\int_W c_{\mathcal{G}_W}(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi))\right]_{\mathbb{R}/\mathbb{Z}}$$

and conclude that

$$\Psi_{(M,p)}(-) = -\left[\int_W c_{\mathcal{G}_W}(\mathbf{td}_B^{-1} \cup \mathbf{ch}(-))\right]_{\mathbb{R}/\mathbb{Z}}$$

lies in $\mathrm{U}_n^{\mathbb{R}}(B)$ by Proposition 6.5.

Finally we have to check that the choices used in Example 6.20 to define the geometrization $\mathcal{G}_M$ don't affect the intrinsic eta-invariant. For this we fix a geometric cycle (M,p) for B-bordism. Then we have to choose a smooth lift $\rho_i : M \to B_i$ of $p : M \to B$ together with a homotopy filling diagram (6.7). But any two such choices can be connected on the cylinder. Hence we conclude the independence of these choices.

This completes the proof. □

In [Bun11] Bunke proves

Theorem 6.28. *After restricting to the torsion subgroup* $\mathrm{Torsion}(\pi_*^{\$}(MB))$ *the intrinsic eta-invariant* $\eta_{\mathrm{intrinsic}}$ *coincides with the universal eta-invariant* η_{uni}. ♦

Proof. This follows from the Secondary Index Theorem [Bun11, Theorem 3.6] together with [Bun11, Theorem 4.19]. □

As a consequence the intrinsic eta-invariant becomes independent of the choice of a universal $Spin^c$-geometrization on B after restricting to torsion elements. We can prove this directly.

Theorem 6.29. *Suppose we have two universal $Spin^c$-geometrizations on B and construct the intrinsic eta-invariants* $\eta^1_{\mathrm{intrinsic}}, \eta^2_{\mathrm{intrinsic}} : \pi_*^{\$}(MB) \to \mathrm{Q}_*^{\mathbb{R}}(B)$ *as above.*
Then the restrictions of the two intrinsic eta-invariants to torsion elements coincide. ♦

Proof. Let $x \in \pi_*^{\$}(MB)$ denote a torsion element and represent it by a geometric cycle (M,p). We can find a zero bordism (W,q) for the disjoint union $\amalg_k(M,p)$ of k copies of (M,p) where k is the order of x. Denote the boundary inclusion by $\mathrm{incl} : \amalg_k M \hookrightarrow W$.

Now we construct four different geometrizations. As in Example 6.20 we choose smooth lifts $\rho_i^1 : M \to B_i^1$ and $\rho_i^2 : M \to B_i^2$ and find extensions $\xi_i^j : W \to B_i^j$ such that $\mathrm{incl}^* \xi_i^j = \amalg_k \rho_i^j$ for $j = 1,2$ (here B_i^1 and B_i^2 denote the two approximations of B given by the two universal $Spin^c$-geometrizations). Then we can pullback the universal $Spin^c$-geometrizations and get the four geometrizations $\mathcal{G}_M^j$ and $\mathcal{G}_W^j$ satisfying $\mathrm{incl}^* \mathcal{G}_W^j = \oplus_k \mathcal{G}_M^j$ for $j = 1,2$ (we identify $\widehat{\mathbf{KU}^0}(\amalg_k M) = \oplus_k \widehat{\mathbf{KU}^0}(M)$).

Next we go through the constructing of the intrinsic eta-invariant.

For $\phi \in \mathbf{KU}^0(B)$ we choose a geometric vector bundle $\mathbf{V}_\phi \to M$ representing $p^*\phi$ and get two correction forms γ_ϕ^j, $j = 1,2$, which are characterized by $a(\gamma_\phi^j) = \mathcal{G}_M^j(\phi) - \mathbf{V}_\phi$. Since the correction form depends only on the differential K-theory class of $\mathbf{V}_\phi$ we can assume that there is a geometric vector bundle $\mathbf{W}_\phi \to W$ representing $q^*\phi$ such that $\mathrm{incl}^* \mathbf{W}_\phi = \amalg_k \mathbf{V}$ (and such that the geometric structures on $\mathbf{W}_\phi$ have product form near the boundary). Then we get two more correction forms $\tilde{\gamma}_\phi^j$ characterized by $a(\tilde{\gamma}_\phi^j) = \mathcal{G}_W^j(\phi) - \mathbf{W}_\phi$ and which satisfy $\mathrm{incl}^* \tilde{\gamma}^j_\phi = \oplus_k \gamma^j_\phi$. The values $\eta^1_{\mathrm{intrinsic}}(x)$ and $\eta^2_{\mathrm{intrinsic}}(x)$ of the intrinsic eta-invariants are by definition represented by

$$\Psi^1_{(M,p)} : \mathbf{KU}^0(B) \to \mathbb{R}/\mathbb{Z}, \quad \phi \mapsto \Psi^1_{(M,p)}(\phi) := -\left[\int_M \mathbf{Td}_M \wedge \gamma^1_\phi\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D} \otimes \mathbf{V}_\phi)$$

and

$$\Psi^2_{(M,p)} : \mathbf{KU}^0(B) \to \mathbb{R}/\mathbb{Z}, \quad \phi \mapsto \Psi^2_{(M,p)}(\phi) := -\left[\int_M \mathbf{Td}_M \wedge \gamma^2_\phi\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D} \otimes \mathbf{V}_\phi)\ .$$

Hence the difference $\varepsilon := \eta^1_{\mathrm{intrinsic}}(x) - \eta^2_{\mathrm{intrinsic}}(x)$ is represented by

$$\varepsilon : \mathbf{KU}^0(B) \to \mathbb{R}/\mathbb{Z}, \quad \phi \mapsto \left[\int_M \mathbf{Td}_M \wedge (\gamma^2_\phi - \gamma^1_\phi)\right]_{\mathbb{R}/\mathbb{Z}}\ .$$

Our aim is to show that this element $\varepsilon \in \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbb{R}/\mathbb{Z})$ lies in the subgroup $\mathrm{U}^{\mathbb{R}}_*(B)$. To this end we calculate

$$\begin{aligned}
\int_M \mathbf{Td}_M \wedge (\gamma^1_\phi - \gamma^2_\phi) =& \frac{1}{k} \int_{\amalg_k M} \mathbf{Td}_{\amalg_k M} \wedge \oplus_k (\gamma^1_\phi - \gamma^2_\phi) = \\
=& \frac{1}{k} \int_{\partial W} \mathrm{incl}^* \mathbf{Td}_W \wedge \mathrm{incl}^* (\tilde{\gamma}^1_\phi - \tilde{\gamma}^2_\phi) \overset{\text{Stokes}}{=} \\
=& \frac{1}{k} \int_W \mathbf{Td}_W \wedge d(\tilde{\gamma}^1_\phi - \tilde{\gamma}^2_\phi) \overset{d=R\circ a}{=} \\
=& \frac{1}{k} \int_W \left(c_{\mathcal{G}^1_W} - c_{\mathcal{G}^2_W}\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right)
\end{aligned}$$

and conclude that $\varepsilon \in \mathrm{U}^{\mathbb{R}}_*(B)$ by Proposition 6.5. $\square$

After studying the classification of universal *Spin*^{c}-geometrizations in Chapter 7 we will be able to give a shorter proof in Remark 7.16.

6.7 A first example

Consider the trivial map $S^1 \to \{*\} \to BSpin^c$ with associated Thom spectrum MS^1. Choose as approximation of S^1 by manifolds the constant system S^1 and as stable abstract *Spin*^{c}-structure the trivial $Spin^c(0)$-principal bundle on S^1 with the trivial connection and the trivial map $S^1 \to \{pt\} \to BSpin^c$. Observe that the associated Todd forms are the constant forms $\mathbf{Td} = 1 \in \Omega\mathrm{P}^0_{\mathrm{cl}}(S^1)$ and also that the Todd class is $\mathbf{td}_{S^1} = 1 \in \mathbf{HP}\mathbb{R}^0(S^1)$. To extend these data to a universal *Spin*^{c}-geometrization is the same as to choose a single geometrization for S^1. This in turn corresponds to choosing a differential refinement of the trivial bundle $S^1 \times \mathbb{C} \to S^1$. We can represent this refinement by a geometric vector bundle $\mathbf{V} := (S^1 \times \mathbb{C} \to S^1, h, \nabla)$.

Because our map $S^1 \to BSpin^c$ is trivial we can identify $\pi^{\$}_*(MS^1) \cong \pi^{\$}_*(S^1_+)$, the stable homotopy groups. So we see that $\pi^{\$}_1(MS^1)/\,\mathrm{Torsion} \cong \mathbb{Z}$ and a generator is given by the geometric cycle $(S^1 := \mathbb{R}/\mathbb{Z}, \mathtt{id}_{S^1})$. Moreover, $\pi_*(MS^1)$ is torsion for $* \geq 2$ and thus $\mathrm{U}^{\mathbb{R}}_*(MS^1)$ vanishes. So the intrinsic eta-invariant coincides with the universal one for $* \neq 1$. Therefore we restrict to $* = 1$. We identify $\mathrm{Q}^{\mathbb{R}}_1(S^1) \cong \mathbb{R}/\mathbb{Z}$ by evaluating a homomorphism at the trivial vector bundle of rank 1.

Now we calculate the intrinsic eta-invariant. By our identifications it is enough to evaluate the representative $\Psi_{(S^1,\mathtt{id}_{S^1})}(-) : \mathbf{KU}^0(S^1) \to \mathbb{R}$ of $\eta_{\mathrm{intrinsic}}$ on the trivial bundle. Note that we can choose the same geometric vector bundle $\mathbf{V}$ for this computation and hence get that the correction form γ is trivial. Therefore we have identified

$$\eta_{\mathrm{intrinsic}} : \mathbb{Z} \to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \ , \quad 1 \mapsto -\xi(\not{D} \otimes \mathbf{V}) \ .$$

In particular, if we choose the trivial geometric vector bundle of rank 1, $\mathbf{V} = (S^1 \times \mathbb{C} \to S^1, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}})$, we get[3] $\eta_{\mathrm{intrinsic}}(1) = -1/2 \in {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$ which is non-trivial. Thus we conclude.

> *The intrinsic eta-invariant can detect information strictly more information than the universal eta-invariant.*

Moreover, we see explicitly that the intrinsic eta-invariant depends on the universal *Spin*^{c}-geometrization. To this end note that we can produce for each $\lambda \in \mathbb{R}$ a geometric vector bundle $\mathbf{V}_\lambda = (S^1 \times \mathbb{C} \to S^1, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}} + i\lambda d\theta)$ where $d\theta$ denotes the normalized volume form on S^1. By the above discussion each of these gives a universal *Spin*^{c}-geometrization $\mathcal{G}_\lambda$ and therefore an associated intrinsic eta-invariant $\eta^{\mathcal{G}_\lambda}_{\mathrm{intrinsic}}$. We calculate the value of $\eta^{\mathcal{G}_\lambda}_{\mathrm{intrinsic}}$ at the trivial bundle. The result is[3]

$$\eta^{\mathcal{G}_\lambda}_{\mathrm{intrinsic}}(1) = -\xi(\not{D} \otimes \mathbf{V}_\lambda) = -\frac{1}{2} - \frac{\lambda}{2\pi} \ .$$

Hence we conclude that $[S^1, \mathtt{id}_{S^1}] \in \pi_1^{\$}(MS^1)$ is non-torsion.

In Theorem 7.17 we prove a general version of this statement. For each non-torsion element in $\pi^{\$}_{2k+1}(MB)$ there is a universal *Spin*^{c}-geometrization such that the intrinsic eta-invariants shows that this element is non-torsion (under some additional assumptions).

[3] The computation can be found in Section 8.1.

Chapter 7

Classification of universal $Spin^c$-geometrizations

In this chapter we classify universal $Spin^c$-geometrizations on some space B with a fixed map $\sigma : B \to BSpin^c$ (which induces $\mathbf{td}_B^{-1} \in \mathbf{HP}\mathbb{Q}^0(B)^\times$). This discussion is similar to the one in Section 3.4. In particular, we introduce an equivalence relation and show that equivalence classes of universal $Spin^c$-geometrizations are independent of a choice of underlying data. We also show that the classification up to equivalence is compatible with the intrinsic eta-invariant, i.e., the associated intrinsic eta-invariants for two equivalent universal $Spin^c$-geometrizations coincide.

For the first part we fix underlying data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$, i.e., an approximation by manifolds of B together with stable l_i-abstract $Spin^c$-structures. Then a universal $Spin^c$-geometrization with these underlying data is a choice of geometrization $\mathcal{G}_i$ for each $g_i : B_i \to B$ such that $(f_i)^!_{\delta_i}\mathcal{G}_{i+1} = \mathcal{G}_i$. Fix such a universal $Spin^c$-geometrization $(\mathcal{G}^1_i)_i$ and denote its cohomological character as $(c^1_i)_i$. Now choose a second compatible family of cohomological characters, $(c^2_i)_i$. The differences between these cohomological characters, $\xi_i := c^2_i - c^1_i$, form an element $\xi = (\xi_i)_i \in \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^0_{\mathrm{ex}}(B_i)\big)$. By Lemma 3.30 we can find a lift $\hat{\xi} = (\hat{\xi}_i)_i \in \varprojlim_i \operatorname{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)\big)$ such that $d\hat{\xi}_i = \xi_i$. Using such a lift $\hat{\xi}$ we define new geometrizations $\mathcal{G}^2_i := \mathcal{G}^1_i + a \circ \Big(\mathbf{Td}_i^{-1} \wedge \hat{\xi}_i\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big)$ for all $i \in \mathbb{N}$.

Lemma 7.1. *The $\mathcal{G}^2_i$ are indeed geometrizations with cohomological character c^2_i. We have the compatibility $(f_i)^!_{\delta_i}\mathcal{G}^2_{i+1} = \mathcal{G}^2_i$ for all i and hence the family $(\mathcal{G}^2_i)_i$ is a second universal $Spin^c$-geometrization.* ♦

Proof. These are straightforward calculations using (3.2). We just prove the compatibility:

$$
\begin{aligned}
(f_i)^!_{\delta_i}(\mathcal{G}^2_{i+1}) &= f_i^* \circ \mathcal{G}^2_{i+1} + a \circ \big(\delta_i \wedge f_i^* \circ R \circ \mathcal{G}^2_{i+1}\big) = \\
=& f_i^* \circ \mathcal{G}^1_{i+1} + a \circ \big(\delta_i \wedge f_i^* \circ R \circ \mathcal{G}^1_{i+1}\big) + f_i^* \circ a \circ \Big(\mathbf{Td}^{-1}_{i+1} \wedge \hat{\xi}_{i+1}\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) + \\
&+ a \circ \Big(\delta_i \wedge f_i^* \circ R \circ a \circ \Big(\mathbf{Td}^{-1}_{i+1} \wedge \hat{\xi}_{i+1}\Big)\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) = \\
=& \mathcal{G}^1_i + a \circ \Big(f_i^* \, \mathbf{Td}^{-1}_{i+1} \wedge f_i^* \hat{\xi}_{i+1}\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) + \\
&+ a \circ \Big(\delta_i \wedge f_i^* \, \mathbf{Td}^{-1}_{i+1} \wedge d f_i^* \hat{\xi}_{i+1}\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) \overset{\substack{a \text{ vanishes on} \\ \text{exact forms}}}{=} \\
=& \mathcal{G}^1_i + a \circ \Big(f_i^* \, \mathbf{Td}^{-1}_{i+1} \wedge f_i^* \hat{\xi}_{i+1}\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) +
\end{aligned}
$$

$$
\begin{aligned}
&+ a\circ\Big(d\delta_i\wedge f_i^*\,\mathbf{Td}_{i+1}^{-1}\wedge f_i^*\hat{\xi}_{i+1}\Big)\circ\big(\mathbf{td}_B^{-1}\cup\mathbf{ch}\big) = \\
=&\mathcal{G}_i^1 + a\circ\Big(f_i^*\,\mathbf{Td}_{i+1}^{-1}\wedge f_i^*\hat{\xi}_{i+1}\wedge(1+d\delta_i)\Big)\circ\big(\mathbf{td}_B^{-1}\cup\mathbf{ch}\big)\overset{(3.2)}{=}\\
=&\mathcal{G}_i^1 + a\circ\Big(\mathbf{Td}_i^{-1}\wedge\hat{\xi}_i\Big)\circ\big(\mathbf{td}_B^{-1}\cup\mathbf{ch}\big) = \mathcal{G}_i^2\ . \qquad\square
\end{aligned}
$$

Hence we can transfer universal $Spin^c$-geometrization from one compatible family of cohomological characters to any other such family.

Next we classify universal $Spin^c$-geometrizations for fixed underlying data together with a fixed family of cohomological characters.

Proposition 7.2. *Fix underlying data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$ for B and a compatible family of cohomological characters $(c_i)_i$.*

Then the set of universal $Spin^c$-geometrizations with these underlying data and cohomological characters is a torsor over $\varprojlim_i \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\mathrm{im}\,\mathbf{ch}))$. ♦

Note that we don't claim that the torsor is non-empty.

Proof. This is an immediate generalization of Proposition 3.15. □

Recall that the group classifying universal $Spin^c$-geometrizations with fixed underlying data and cohomological characters, $\varprojlim_i \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(B_i)/(\mathrm{im}\,\mathbf{ch}))$, is a weak homotopy invariant of B by Proposition 3.33.

Now we introduce an adapted version of the equivalence relation of Definition 3.35.

Definition 7.3. *Two universal $Spin^c$-geometrizations $(\mathcal{G}_i^1)_i$ and $(\mathcal{G}_i^2)_i$ with the same underlying data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$ are called **equivalent**, $(\mathcal{G}_i^1)_i \sim (\mathcal{G}_i^2)_i$, if there exists a compatible family $\hat{\xi} = (\hat{\xi}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\mathrm{im}\,d)\big)$ such that $\mathcal{G}_i^2 = \mathcal{G}_i^1 - a\circ\Big(\mathbf{Td}_i^{-1}\wedge\hat{\xi}_i\Big)\circ\big(\mathbf{td}_B^{-1}\cup\mathbf{ch}\big)$ for all $i\in\mathbb{N}$.* ■

This gives the following classification up to equivalence.

Theorem 7.4. *Fix underlying data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$ for B.*

Then the set of equivalence classes of universal $Spin^c$-geometrizations with these underlying data is a (possible empty) torsor over the classifying group $\mathrm{coker}(\hat{\rho})$ introduced in Definition 3.34. ♦

Note that we already showed in Definition and Proposition 3.34 that the classifying group is a weak homotopy invariant of B.

Proof. This is an immediate generalization of the proofs of Theorems 3.17 and 3.37. □

Now we study the dependence of the underlying data. We observe that the analogous statement of Proposition 3.36 holds.

Proposition 7.5. *Universal $Spin^c$-geometrizations can be restricted to cofinal subsystems of a given set of underlying data and also can be extended uniquely from such cofinal subsystems.* ♦

Proof. This follows directly from the compatibility condition. □

Recall Theorem 3.40 which yields the existence of maps of approximations and homotopies between those.

We fix two sets of underlying data,

$$
\Big(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i\Big)_i \quad\text{and}\quad \Big(\tilde{B}_i, \tilde{f}_i, \tilde{g}_i, (\tilde{T}^i_{\tilde{l}_i}, \nabla^{\tilde{T}^i_{\tilde{l}_i}}, \tilde{t}^i_{k_i})_{k_i}, \widetilde{\mathbf{Td}}_i, \tilde{\delta}_i\Big)_i\ ,
$$

and choose a map of approximations $(J,(h_i)_i)$ between them. Denote the associated error forms of the h_i by ε_i (see Construction 6.14). Given a universal *Spin*^{c}-geometrization $(\tilde{\mathcal{G}}_i)_i$ with the latter underlying data we get a family of geometrizations $\mathcal{G}_i := (h_i)^!_{\varepsilon_i}\tilde{\mathcal{G}}_{J(i)}$, $i \in \mathbb{N}$, with the first underlying data. We have the following observation.

Lemma 7.6. *The family of geometrizations $(\mathcal{G}_i)_i$ satisfies $(f_i)^!_{\delta_i}(\mathcal{G}_{i+1}) = \mathcal{G}_i$ for all $i \in \mathbb{N}$ and therefore is a universal Spin^c-geometrization.*

Thus we can pullback universal Spin^c-geometrizations along maps of approximations. ♦

Proof. This follows from Proposition 6.17 together with formulas (3.5) and (3.6) since the squares

$$\begin{tikzcd} B_i \arrow[r, hook, "f_i"] \arrow[d, "h_i"] & B_{i+1} \arrow[d, "h_{i+1}"] \\ \tilde{B}_{J(i)} \arrow[r, hook, "\tilde{f}_{J(i+1)-1}\circ\ldots\circ\tilde{f}_{J(i)}"] & \tilde{B}_{J(i+1)} \end{tikzcd}$$

commute strictly. □

Assume we are given $(\hat{\xi}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\Big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(\tilde{B}_i)/(\operatorname{im} d)\Big)$. Then we get $(\hat{\tau}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)\big)$ defined by $\hat{\tau}_i := h_i^*\hat{\xi}_{J(i)}$. This gives two more geometrizations

$$\tilde{\mathcal{G}}^2_i := \tilde{\mathcal{G}}_i - a \circ \Big(\widetilde{\mathbf{Td}}_i^{-1} \wedge \hat{\xi}_i\Big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) \text{ and } \mathcal{G}^2_i := \mathcal{G}_i - a \circ \big(\mathbf{Td}_i^{-1} \wedge \hat{\tau}_i\big) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}\big) \ .$$

Lemma 7.7. *We have: $\mathcal{G}^2_i = (h_i)^!_{\varepsilon_i}\tilde{\mathcal{G}}^2_{J(i)}$.*

Hence pullback of universal Spin^c-geometrizations along a map of approximations preserves equivalence classes. ♦

Now recall Theorem 7.4 which stated that the set of equivalence classes of universal *Spin*^{c}-geometrizations with fixed underlying data is a torsor for the abelian group $\operatorname{coker}(\hat{\rho})$ which is independent of the underlying data. The same argument as for Lemma 3.43 gives

Lemma 7.8. *The construction of the pullback of an equivalence classes of universal Spin^c-geometrizations along a map of approximations is equivariant with respect to the action of the classifying group* $\operatorname{coker}(\hat{\rho})$. ♦

Finally we show that the construction of the pullback universal *Spin*^{c}-geometrization becomes independent of the map of approximations used for the construction after passing to equivalence classes.

Lemma 7.9. *Given two maps of approximations the resulting pullback universal Spin^c-geometrizations are equivalent.* ♦

Proof. Denote the two maps of approximations by $(J^1,(h^1_i)_i)$ and $(J^0,(h^0_i)_i)$. Then we find a homotopy $(K,(H_i)_i)$ between these two maps of approximations (Theorem 3.40, 2.). By Proposition 7.5 we can assume that $J^1 = J^0 = K = \mathtt{id} : \mathbb{N} \to \mathbb{N}$. We get two universal *Spin*^{c}-geometrizations on the underlying data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$, namely

$$\mathcal{G}^1_i := (h^1_i)^!_{\varepsilon_i}\tilde{\mathcal{G}}_i \quad \text{and} \quad \mathcal{G}^0_i := (h^0_i)^!_{\varepsilon_i}\tilde{\mathcal{G}}_i \ .$$

Denote the boundary inclusions $B_i \to B_i \times I$ by incl_j, $j = 0, 1$, and the projections by $\mathrm{pr} : B_i \times I \to B_i$. Then we take the pullback of the stable l_i-abstract *Spin*^{c}-structures from B_i to $B_i \times I$ along the projection pr. Observe that the error forms for the boundary inclusions vanish and that the Todd forms satisfy $\mathbf{Td}_i^{B_i\times I} = \mathrm{pr}^* \mathbf{Td}_i$. Denote the error forms associated to the homotopies H_i by $\hat{\varepsilon}_i$ and set $\mathcal{H}_i := (H_i)^!_{\hat{\varepsilon}_i}(\tilde{\mathcal{G}}_i)$. This gives a universal $Spin^c$-geometrization

$$\Big(B_i \times I, f_i \times \mathtt{id}, g_i \circ \mathrm{pr}, \mathrm{pr}^*(T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathrm{pr}^* \mathbf{Td}_i, \hat{\varepsilon}_i, \mathcal{H}_i\Big)_i$$

for B such that $\mathrm{incl}_j^* \mathcal{H}_i = \mathcal{G}_i^j$ for $j = 0, 1$.

Now we apply the homotopy formula (2.3) to calculate the difference ($\tilde{c}_i$ denotes the cohomological character of $\tilde{\mathcal{G}}_i$)

$$\begin{aligned}
\mathcal{G}_i^1 - \mathcal{G}_i^0 &= \mathrm{incl}_1^* \mathcal{H}_i - \mathrm{incl}_0^* \mathcal{H}_i \overset{(2.3)}{=} \\
&= a\left(\int_{B_i\times I/B_i} R\circ\mathcal{H}_i\right) = a\left(\int_{B_i\times I/B_i} (\mathbf{Td}_i^{B_i\times I})^{-1} \wedge \mathbf{Td}_i^{B_i\times I} \wedge R\circ\mathcal{H}_i\right) = \\
&= a\left(\int_{B_i\times I/B_i} \mathrm{pr}^* \mathbf{Td}_i^{-1} \wedge c_{\mathcal{H}_i}\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}\right) \overset{(3.4)}{=} \\
&= a\left(\mathbf{Td}_i^{-1} \wedge \int_{B_i\times I/B_i} H_i^* \tilde{c}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}\right) \ .
\end{aligned}$$

Note that $\xi_i := \int_{B_i\times I/B_i} H_i^* \circ \tilde{c}_i \in \mathrm{Hom}^{\mathrm{cont}}\big(\mathbf{HP}\mathbb{Q}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\mathrm{im}\, d)\big)$ shifts the internal degree by -1 and that $f_i^*\xi_{i+1} = \xi_i$. Hence the family $(\xi_i)_i$ shows that the two universal geometrizations $(\mathcal{G}_i^1)_i$ and $(\mathcal{G}_i^0)_i$ are equivalent. □

The preceding lemmas combine to

Classification Theorem 7.10. *Let B denote a topological space with a fixed map $\sigma : B \to BSpin^c$. Then the set of equivalence classes of universal $Spin^c$-geometrizations for B is a (possible empty) torsor over the classifying group* $\mathrm{coker}(\hat{\rho})$. *This group is a weak homotopy invariant of B.* ◆

Observe that the classifying group is the same as in the case of Theorem 3.45.

As before this combines with our Existence Theorem 3.27 to the following uniqueness statement.

Theorem 7.11. *Let B be a topological space which is of rationally even and admits an approximation by manifolds (e.g., which is simply-connected and of countable type).*

Then there exists a unique equivalence class of universal $Spin^c$-geometrizations for B. ◆

Proof. The uniqueness statement follows directly from the Classification Theorem 7.10. The existence is implied by our Existence Theorem 3.27. The technical condition there is satisfied (see Remark 3.28). □

This completes the classification of universal $Spin^c$-geometrizations.

Our next aim is to prove that intrinsic eta-invariants associated to equivalent universal $Spin^c$-geometrizations coincide.

At first we show that the intrinsic eta-invariant is independent of the underlying data.

Proposition 7.12. *Fix a universal $Spin^c$-geometrization on some underlying data. If we find a second set of underlying data then we can choose a map of approximations and pullback the universal $Spin^c$-geometrization to get a second universal $Spin^c$-geometrization on the second set of underlying data. We claim that the intrinsic eta-invariants induced by these two universal $Spin^c$-geometrization coincide.* ◆

Proof. At first observe that the intrinsic eta-invariant depends only on a cofinal subsystem of the approximation underlying the universal *Spin*^{c}-geometrization used in its construction (Lemma 6.21).

So suppose we are given the following data.

- A universal *Spin*^{c}-geometrization for B,

$$(\tilde{B}_i, \tilde{f}_i, \tilde{g}_i, (\tilde{T}^i_{\tilde{l}_i}, \nabla^{\tilde{T}^i_{\tilde{l}_i}}, \tilde{t}^i_{k_i})_{k_i}, \widetilde{\mathbf{Td}}_i, \tilde{\delta}_i, \tilde{\mathcal{G}}_i)_i \ .$$

- A set of underlying data $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i)_i$.
- A map of approximations $(h_i : B_i \to \tilde{B}_i)_i$ (after taking a cofinal subsystem).

By Lemma 7.6 we get a second universal *Spin*^{c}-geometrization $\mathcal{G}_i := (h_i)^!_{\varepsilon_i}(\tilde{\mathcal{G}}_i)$ where ε_i denotes the error form associated to h_i.

Now we want to calculate the values of the two induced intrinsic eta-invariants, $\eta^1_{\text{intrinsic}}$ associated to $(\mathcal{G}_i)_i$ and $\eta^2_{\text{intrinsic}}$ associated to $(\tilde{\mathcal{G}}_i)_i$. To this end we choose a geometric cycle (M, p) for B-bordism of odd dimension and we choose a lift $\rho_i : M \to B_i$ of p as in Example 6.20. This yields a lift $\tilde{\rho}_i := h_i \circ \rho_i$ for the other approximation. Denote the error form associated to ρ_i (resp. $\tilde{\rho}_i$) by δ_M (resp. $\tilde{\delta}_M$). Then we get two $(p, \mathbf{Td}_M, \mathbf{td}_B^{-1})$-geometrizations $\mathcal{G}^1_M := (\rho_i)^!_{\delta_M}(\mathcal{G}_i)$ and $\mathcal{G}^2_M := (\tilde{\rho}_i)^!_{\tilde{\delta}_M}(\tilde{\mathcal{G}}_i)$. But Proposition 6.17 together with formulas (3.5) and (3.6) gives

$$\mathcal{G}^1_M = (\rho_i)^!_{\delta_M}(\mathcal{G}_i) = (\rho_i)^!_{\delta_M}(h_i)^!_{\varepsilon_i}(\tilde{\mathcal{G}}_i) = (\tilde{\rho}_i)^!_{\tilde{\delta}_M}(\tilde{\mathcal{G}}_i) = \mathcal{G}^2_M$$

and thus the geometrizations and the values of the intrinsic eta-invariant agree. □

Next we show that the intrinsic eta-invariant depends only on the equivalence class of the universal *Spin*^{c}-geometrization used in the construction of this intrinsic eta-invariant.

Theorem 7.13. *Assume we are given two equivalent universal Spin*^{c}*-geometrizations for B. Then the two associated intrinsic eta-invariants agree.* ♦

Proof. We are given a universal *Spin*^{c}-geometrization

$$(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i, \mathcal{G}^1_i)_i$$

for B and a second universal *Spin*^{c}-geometrization $(\mathcal{G}^2_i)_i$ with the same underlying data together with an element $\hat{\xi} = (\hat{\xi}_i)_i \in \varprojlim_i \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\big(\mathbf{KU}^0(B), \Omega\mathrm{P}^{-1}(B_i)/(\operatorname{im} d)\big)$ such that

$$\mathcal{G}^2_i = \mathcal{G}^1_i + a \circ (\mathbf{Td}_i^{-1} \wedge \hat{\xi}_i) \circ \big(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\big) \ .$$

Denote the two intrinsic eta-invariants associated to these universal *Spin*^{c}-geometrizations by $\eta^1_{\text{intrinsic}}$ and $\eta^2_{\text{intrinsic}}$.

Fix an odd-dimensional geometric cycle for B-bordism, (M, p), and find a lift ρ_i of p as in Example 6.20. We get two $(p, \mathbf{Td}_M, \mathbf{td}_B^{-1})$-geometrizations $\mathcal{G}^j_M := (\rho_i)^!_{\delta_M}(\mathcal{G}^j_i)$, $j = 1, 2$. Then the difference $\varepsilon := \eta^1_{\text{intrinsic}} - \eta^2_{\text{intrinsic}}$ is represented by $\varepsilon(M, p) : \mathbf{KU}^0(B) \to \mathbb{R}/\mathbb{Z}$. Fix $\phi \in \mathbf{KU}^0(B)$ and a geometric vector bundle $\mathbf{V}_\phi$ representing $p^*\phi$. Then we get two correction forms γ^j_ϕ which are characterized by $a(\gamma^j_\phi) = \mathcal{G}^j_M(\phi) - \mathbf{V}_\phi$. Hence the difference $\gamma^1_\phi - \gamma^2_\phi$ is characterized by

$$a(\gamma^1_\phi) - a(\gamma^2_\phi) = \mathcal{G}^1_M(\phi) - \mathcal{G}^2_M(\phi) = (\rho_i)^!_{\delta_M}(\mathcal{G}^1_i)(\phi) - (\rho_i)^!_{\delta_M}(\mathcal{G}^2_i)(\phi) =$$

$$
\begin{aligned}
&\overset{(3.3)}{=} \rho_i^*(\mathcal{G}_i^1)(\phi) - \rho_i^*(\mathcal{G}_i^2)(\phi) + a\left(\delta_M \wedge \rho_i^* \circ R \circ \left(\mathcal{G}_i^1(\phi) - \mathcal{G}_i^2(\phi)\right)\right) = \\
&= -\,\rho_i^* \circ a \circ \left(\mathbf{Td}_i^{-1} \wedge \hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}\right) - \\
&\quad - a\left(\delta_M \wedge \rho_i^*\, \mathbf{Td}_i^{-1} \wedge \rho_i^* \left(c_i^1 - c_i^2\right)\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) = \\
&= -\,\rho_i^* \circ a \circ \left(\mathbf{Td}_i^{-1} \wedge \hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}\right) - \\
&\quad - a\left(\delta_M \wedge \rho_i^*\, \mathbf{Td}_i^{-1} \wedge \rho_i^* d\hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) \overset{\substack{a \text{ vanishes on} \\ \text{exact forms}}}{=} \\
&= -\,a \circ \left(\rho_i^*\, \mathbf{Td}_i^{-1} \wedge \rho_i^* \hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}\right) - \\
&\quad - a\left(d\delta_M \wedge \rho_i^*\, \mathbf{Td}_i^{-1} \wedge \rho_i^* \hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) = \\
&= -\,a \circ \left((1 + d\delta_M) \wedge \rho_i^*\, \mathbf{Td}_i^{-1} \wedge \rho_i^* \hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) \overset{(3.2)}{=} \\
&= -\,a \circ \left(\mathbf{Td}_M^{-1} \wedge \rho_i^* \hat{\xi}_i\right) \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) \ .
\end{aligned}
$$

This gives us a formula for $\varepsilon(M,p)$,

$$
\begin{aligned}
\varepsilon(M,p)(\phi) &= -\left[\int_M \mathbf{Td}_M \wedge \gamma_\phi^1\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D} \otimes \mathbf{V}_\phi) + \left[\int_M \mathbf{Td}_M \wedge \gamma_\phi^2\right]_{\mathbb{R}/\mathbb{Z}} + \xi(\not{D} \otimes \mathbf{V}_\phi) \\
&= -\left[\int_M \mathbf{Td}_M \wedge \left(\gamma_\phi^1 - \gamma_\phi^2\right)\right]_{\mathbb{R}/\mathbb{Z}} = \\
&= \left[\int_M \mathbf{Td}_M \wedge \left(\mathbf{Td}_M^{-1} \wedge \rho_i^* \hat{\xi}_i\right)\right]_{\mathbb{R}/\mathbb{Z}} \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) = \\
&= \left[\int_M \rho_i^* \hat{\xi}_i\right]_{\mathbb{R}/\mathbb{Z}} \circ \left(\mathbf{td}_B^{-1} \cup \mathbf{ch}(\phi)\right) \ .
\end{aligned}
$$

Observe that $\rho_i^* \hat{\xi} \in \mathrm{Hom}^{\mathrm{cont}}_{[-1]}\left(\mathbf{HP}\mathbb{R}^0(B), \Omega\mathrm{P}^{-1}(M)/(\operatorname{im} d)\right)$. Therefore we can apply Proposition 6.5 and conclude that $\varepsilon(M,p) \in \mathrm{U}^{\mathbb{R}}_{\dim(M)}(B)$. Thus the intrinsic eta-invariants agree. □

Remark 7.14. Fix a weak homotopy equivalence $B \to \tilde{B}$ and an equivalence class of universal $Spin^c$-geometrizations on $\tilde{B}$. This gives an equivalence class of universal $Spin^c$-geometrizations on the other space B via pullback. By the above the associated intrinsic eta-invariants agree. Thus the intrinsic eta-invariant is functorial for weak homotopy equivalences. ■

By Theorem 7.10 the set of equivalence classes of universal $Spin^c$-geometrizations is a torsor for the group $\mathrm{coker}(\hat{\rho})$. We study how the intrinsic eta-invariant behaves under this action. So fix a universal $Spin^c$-geometrization $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i, \mathcal{G}_i^1)_i$ and an element $(\hat{\xi}_C)_C \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(C)/(\operatorname{im} \mathbf{ch})\right)$. Then we get a second universal $Spin^c$-geometrization $\mathcal{G}_i^2 := \mathcal{G}_i^1 + a \circ \hat{\xi}_{B_i}$ with the same underlying data. These two universal $Spin^c$-geometrization induce two intrinsic eta-invariants, $\eta^1_{\mathrm{intrinsic}}$ and $\eta^2_{\mathrm{intrinsic}}$.

Proposition 7.15. *The difference* $\varepsilon := \eta^1_{\mathrm{intrinsic}} - \eta^2_{\mathrm{intrinsic}}$ *is represented by*

$$
\begin{aligned}
\varepsilon(M,p) : \mathbf{KU}^0(B) &\to \mathbb{R}/\mathbb{Z} \\
\phi &\mapsto \left[\int_M \mathbf{Td}_M \wedge \hat{\xi}_M(\phi)\right]_{\mathbb{R}/\mathbb{Z}}
\end{aligned}
$$

for (M,p) an odd-dimensional geometric cycle for B-bordism and ρ_i a lift of p as in Example 6.20. Note that $\hat{\xi}_M$ is our notation for the value of $\hat{\xi}$ at $\rho_i : M \to B_i$. ♦

Proof. Fix an odd-dimensional geometric cycle for B-bordism (M,p) and find a lift ρ_i of p as in Example 6.20. Denote the error form associated to ρ_i by δ_M. We get two $(p, \mathbf{Td}_M, \mathbf{td}_B^{-1})$-geometrizations $\mathcal{G}_M^j := (\rho_i)^!_{\delta_M}(\mathcal{G}_i^j)$, $j = 1,2$. Take $\phi \in \mathbf{KU}^0(B)$ and a geometric vector bundle $\mathbf{V}_\phi$ representing $p^*\phi$. Then we get two correction forms γ_ϕ^j which are characterized by $a(\gamma_\phi^j) = \mathcal{G}_M^j(\phi) - \mathbf{V}_\phi$. Hence the difference $\gamma_\phi^1 - \gamma_\phi^2$ is characterized by

$$\begin{aligned} a(\gamma_\phi^1) - a(\gamma_\phi^2) =& \mathcal{G}_M^1(\phi) - \mathcal{G}_M^2(\phi) = (\rho_i)^!_{\delta_M}(\mathcal{G}_i^1)(\phi) - (\rho_i)^!_{\delta_M}(\mathcal{G}_i^2)(\phi) = \\ &\overset{(3.3)}{=} \rho_i^*(\mathcal{G}_i^1)(\phi) - \rho_i^*(\mathcal{G}_i^2)(\phi) + a\left(\delta_M \wedge \rho_i^* \circ R \circ \left(\mathcal{G}_i^1(\phi) - \mathcal{G}_i^2(\phi)\right)\right) = \\ &= -\rho_i^* \circ a \circ \hat{\xi}_{B_i}(\phi) - a\left(\delta_M \wedge \rho_i^* \circ R \circ a \circ \hat{\xi}_{B_i}(\phi)\right) = \\ &= -a\left(\rho_i^* \hat{\xi}_{B_i}(\phi) + \delta_M \wedge \rho_i^* d\hat{\xi}_{B_i}(\phi)\right) \overset{d\hat{\xi}_{B_i}(\phi)=0}{=} -a\left(\hat{\xi}_M(\phi)\right) . \end{aligned}$$

This gives us the claimed formula for $\varepsilon(M,p)$,

$$\begin{aligned} \varepsilon(M,p)(\phi) &= -\left[\int_M \mathbf{Td}_M \wedge \gamma_\phi^1\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D} \otimes \mathbf{V}_\phi) + \left[\int_M \mathbf{Td}_M \wedge \gamma_\phi^2\right]_{\mathbb{R}/\mathbb{Z}} + \xi(\not{D} \otimes \mathbf{V}_\phi) \\ &= -\left[\int_M \mathbf{Td}_M \wedge \left(\gamma_\phi^1 - \gamma_\phi^2\right)\right]_{\mathbb{R}/\mathbb{Z}} = \left[\int_M \mathbf{Td}_M \wedge \hat{\xi}_M(\phi)\right]_{\mathbb{R}/\mathbb{Z}} . \end{aligned}$$

□

Remark 7.16. Using this formula we get a simple argument for Theorem 6.29 which states that the intrinsic eta-invariant is independent of the universal $Spin^c$-geometrization for torsion elements in $\pi_*^{\$}(MB)$. So assume that (M,p) represents a torsion element in $\pi_*^{\$}(MB)$ of order k. Then we can find a zero-bordism (W,q) for $\coprod_k(M,p)$. We choose lifts of p and q to ρ_i and $\hat{\rho}_i$ as in Example 6.20. Then the difference of the two intrinsic eta-invariants is represented by

$$\begin{aligned} \varepsilon(M,p) : \mathbf{KU}^0(B) &\to \mathbb{R}/\mathbb{Z} \\ \phi \mapsto \left[\int_M \mathbf{Td}_M \wedge \rho_i^* \hat{\xi}_{B_i}(\phi)\right]_{\mathbb{R}/\mathbb{Z}} &\overset{\text{Stokes}}{=} \left[\frac{1}{k}\int_W \mathbf{Td}_W \wedge \hat{\rho}_i^* d\hat{\xi}_{B_i}(\phi)\right]_{\mathbb{R}/\mathbb{Z}} \overset{d\hat{\xi}_{B_i}(\phi)=0}{=} 0 \end{aligned}$$

and thus the two intrinsic eta-invariants agree. ■

We claim that the converse is also true.

Detection Theorem 7.17. *Fix a space B with $\sigma : B \to BSpin^c$ which allows universal $Spin^c$-geometrizations. Moreover, suppose $\mathbf{td}_B = 1$ (e.g., σ is trivial). Fix $k \in \mathbb{N}_0$. Then $x \in \pi_{2k+1}^{\$}(MB)$ is torsion if and only if it satisfies $\eta_{intrinsic}^{(\mathcal{G}_i^1)}(x) = \eta_{intrinsic}^{(\mathcal{G}_i^2)}(x)$ for each pair of universal $Spin^c$-geometrizations $(\mathcal{G}_i^1)_i$ and $(\mathcal{G}_i^2)_i$ for B.*

Thus the combination of all intrinsic eta-invariants for all (equivalence classes of) universal $Spin^c$-geometrizations detects all non-torsion elements in $\pi_{odd}^{\$}(MB)$. ♦

The proof relies on Proposition 7.15 and the following easy fact.

Remark 7.18. Assume that the Todd class is trivial, $\mathbf{td}_B = 1$. Then the definition of $\mathrm{U}_*^{\mathbb{R}}(B)$ simplifies to

$$\mathrm{U}_*^{\mathbb{R}}(B) = \left\{ \left[\langle y, \mathbf{ch}(-)\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \mid y \in \mathbf{H}\mathbb{R}_{*+1}(B) \right\} \subset \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbb{R}/\mathbb{Z}\right) .$$

In particular, for $* \neq -1$, $\left[\langle y, \mathbf{ch}(-)\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} \in \mathrm{U}^{\mathbb{R}}_{*}(B)$ evaluates on the unit $\mathbb{1} \in \mathbf{KU}^0(B)$ to zero,

$$\left[\langle y, \mathbf{ch}(\mathbb{1})\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} = \left[\langle y, \mathbb{1}\rangle_{\mathbb{R}}^{\mathbf{HP}\mathbb{R}}\right]_{\mathbb{R}/\mathbb{Z}} = 0 \ ,$$

because $y \in \mathbf{H}\,\mathbb{R}_{*+1}(B)$. ■

Proof of Theorem 7.17. Theorem 6.29 shows the "if"-part of the statement.

For the converse we fix $x \in \pi^{\$}_{2k+1}(MB)$ which is non-torsion and take a representing geometric cycle (M, p) for B-bordism. Moreover, we choose a universal $Spin^c$-geometrizations $(B_i, f_i, g_i, (T^i_{l_i}, \nabla^{T^i_{l_i}}, t^i_{k_i})_{k_i}, \mathbf{Td}_i, \delta_i, \mathcal{G}^1_i)_i$ of B and a lift $\rho_i : M \to B_i$ as in Example 6.20. By Proposition 7.15 it is enough to find

$$\hat{\xi} = (\hat{\xi}_C)_C \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(C)/(\mathrm{im}\,\mathbf{ch})\right)$$

such that the map

$$\begin{aligned} \varepsilon_{\hat{\xi}}(M,p) : \mathbf{KU}^0(B) &\to \mathbb{R}/\mathbb{Z} \\ \phi &\mapsto \left[\int_M \mathbf{Td}_M \wedge \hat{\xi}_M(\phi)\right]_{\mathbb{R}/\mathbb{Z}} \end{aligned}$$

lies not in $\mathrm{U}^{\mathbb{R}}_{2k+1}(B)$. By Remark 7.18 it is enough to check that

$$0 \neq \varepsilon_{\hat{\xi}}(M,p)(\mathbb{1}) = \left[\int_M \mathbf{Td}_M \wedge \hat{\xi}_M(\mathbb{1})\right]_{\mathbb{R}/\mathbb{Z}} \ .$$

Since we are only interested in the value of $\hat{\xi}$ at $\mathbb{1} \in \mathbf{KU}^0(B)$ we choose a base point $\star \in B$ and get a splitting $\mathbf{KU}^0(B) \cong \mathbb{Z} \oplus \widetilde{\mathbf{KU}}^0(B)$ where $\mathbb{Z}$ is generated by the unit $\mathbb{1} \in \mathbf{KU}^0(B)$. So suppose we are given $\xi = (\xi_C)_C \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathbf{HP}\mathbb{R}^{-1}(C)/(\mathrm{im}\,\mathbf{ch})$. Then we can extend each ξ_C to a continuous map $\hat{\xi}_C : \mathbf{KU}^0(B) \to \mathbf{HP}\mathbb{R}^{-1}(C)/(\mathrm{im}\,\mathbf{ch})$ which is trivial on $\widetilde{\mathbf{KU}}^0(B)$ and which maps $\mathbb{1} \in \mathbb{Z} \subset \mathbf{KU}^0(B)$ to ξ_C. This gives an element

$$\hat{\xi} = (\hat{\xi}_C)_C \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(B), \mathbf{HP}\mathbb{R}^{-1}(C)/(\mathrm{im}\,\mathbf{ch})\right) \ .$$

So we are reduced to find $\xi = (\xi_C)_C \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathbf{HP}\mathbb{R}^{-1}(C)/(\mathrm{im}\,\mathbf{ch})$ such that

$$0 \neq \left[\int_M \mathbf{Td}_M \wedge \xi_M(\mathbb{1})\right]_{\mathbb{R}/\mathbb{Z}} \ .$$

In fact we will construct $\zeta \in \mathbf{HP}\mathbb{R}^{-1}(B)$ such that

$$0 \neq \int_M \mathbf{Td}_M \wedge p^*\zeta \ .$$

This gives the element $\xi := ([p^*\zeta])_{p : C \to B} \in \varprojlim_{C \to B \in \mathcal{C}/B} \mathbf{HP}\mathbb{R}^{-1}(C)/(\mathrm{im}\,\mathbf{ch})$ and because we can multiply ζ by real numbers this is enough to prove the claim.

Now we construct ζ. Consider the rationalization map

$$\pi^{\$}_{2k+1}(MB) \to \mathbf{H}_{2k+1}(MB, \mathbb{Q}) \overset{\mathrm{Thom}}{\cong} \mathbf{H}_{2k+1}(B, \mathbb{Q})$$

which maps $x = [M, p]$ to $p_*[M]$ where $[M] \in \mathbf{H}_{2k+1}(M, \mathbb{Q})$ denotes the fundamental class of M (using the orientation induced by the $Spin^c$-structure). Recall that x is non-torsion and hence that $p_*[M]$ is also non-trivial. By the universal coefficient theorem, $\mathbf{H}^{2k+1}(B, \mathbb{Q})$ is the dual vector space to $\mathbf{H}_{2k+1}(B, \mathbb{Q})$ and we can find $\zeta \in \mathbf{H}^{2k+1}(B, \mathbb{Q})$ such that $\zeta(p_*[M]) \neq 0$. Consider $\zeta \in \mathbf{HP}\mathbb{R}^{-1}(B)$ and note that $p^*\zeta$ is in top degree. We compute

$$\int_M \mathbf{Td}_M \wedge p^*\zeta = \int_M p^*\zeta = p^*\zeta([M]) = \zeta(p_*[M]) \neq 0$$

and thus the proof is finished. $\square$

Chapter 8

Some computations of intrinsic eta-invariants

We discuss the intrinsic eta-invariant in easy examples. Our focus is on non-torsion elements in $\pi^{\$}_{\mathrm{odd}}(MB)$ since on torsion elements the intrinsic eta-invariant agrees with the universal eta-invariant by Theorem 6.28. We reduce all calculation to well-known computations. For the convenience of the reader we give lots of details.

8.1 The two-dimensional torus

We discuss the two-torus $B := T^2 := \mathbb{R}^2/\mathbb{Z}^2$ together with the trivial map to $BSpin^c$. Since T^2 is a compact manifold we approximate T^2 by the constant system as in Example 3.21. We equip it with the framing induced by $T^2 = S^1 \times S^1$ where S^1 is equipped with the non-trivial framing. This gives a $Spin^c(2)$-principal bundle which we use as abstract stable $Spin^c$-structure. Then a universal $Spin^c$-geometrization is the same as a $Spin^c$-geometrization on $\mathtt{id} : T^2 \to T^2$.

Before we construct such a $Spin^c$-geometrization we recall the classifying group which we calculated in Section 4.1,

$$\operatorname{coker}(\hat{\rho}) = {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \oplus {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \ .$$

Let $\gamma_i : S^1 := \mathbb{R}/\mathbb{Z} \hookrightarrow T^2$, $i = 1, 2$, denote the inclusions and let $\gamma_i^* \in \mathbf{HP}\mathbb{Q}^{-1}(T^2)$ be the corresponding dual basis in cohomology. Then the maps

$$\alpha_i : \mathbf{KU}^0(T^2) \to \mathbf{KU}^0(\star) \to \mathbf{HP}\mathbb{R}^{-1}(T^2),$$

which map the trivial rank 1 bundle to $\pi \cdot \gamma_i^*$ represent non-trivial elements in the two summands of $\operatorname{coker}(\hat{\rho})$. Observe that all elements in $\mathrm{U}^{\mathbb{R}}_1(T^2)$ are of the form

$$\begin{aligned} \mathbf{KU}^0(T^2) &\to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \\ \phi &\mapsto \left[\lambda \int_{T^2} \mathbf{ch}(\phi)\right]_{\mathbb{R}/\mathbb{Z}} \end{aligned}$$

for some $\lambda \in \mathbb{R}$ and thus vanish on trivial vector bundles. Hence

$$\mathrm{U}^{\mathbb{R}}_*(T^2) \cong \begin{cases} {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}, & * = 1 \quad , \\ 0, & * \geq 2 \quad . \end{cases}$$

Now we construct a $Spin^c$-geometrization on $\mathtt{id} : T^2 \to T^2$. At first we define a map $\mathcal{G} : \mathbf{KU}^0(T^2) \to \widehat{\mathbf{KU}}^0(T^2)$. Since $\mathbf{H}^*(T^2, \mathbb{Z})$ is a free abelian group so is $\mathbf{KU}^0(T^2)$. Also, the

Chern character is integral because we are in dimension 2. Hence we have $\mathbf{KU}^0(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, where one generator is the trivial rank 1-bundle, $T^2 \times \mathbb{C} \xrightarrow{\mathrm{pr}} T^2$. The other generator is the Poincaré-bundle which can be constructed as follows. Start with the trivial bundle $\mathbb{R}^2 \times \mathbb{C} \xrightarrow{\mathrm{pr}} \mathbb{R}^2$ and consider the action of $\mathbb{Z}^2$ from the right given by

$$(x, y, v).(a, b) := (x + a, y + b, v \cdot e^{2\pi i y a}) \ .$$

Then the quotient $\mathcal{P} := (\mathbb{R}^2 \times \mathbb{C})/\mathbb{Z}^2 \to T^2$ is the Poincaré-bundle. Note that the connection $\nabla := d + 2\pi i x dy$ is $\mathbb{Z}^2$-equivariant and metric (for the standard metric) and hence gives a metric connection $\nabla^{\mathcal{P}}$ on the Poincaré-bundle with respect to the induced hermitian metric $h^{\mathcal{P}}$. The curvature is $R^{\nabla} = d(2\pi i x dy) = 2\pi i dx \wedge dy$ and thus the Poincaré-bundle indeed is the second generator. Now we define

$$\begin{aligned} \mathcal{G} : \mathbf{KU}^0(T^2) &\to \widehat{\mathbf{KU}^0}(T^2) \\ [T^2 \times \mathbb{C}] &\mapsto \left[T^2 \times \mathbb{C}, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}}\right] \\ [\mathcal{P}] &\mapsto \left[\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}}\right] \end{aligned}$$

which is clearly a pre-geometrizations. To define a cohomological character we note that $\mathbf{HP}\mathbb{Q}^0(T^2) \cong \mathbb{Q} \oplus \mathbb{Q}$ with generators $1 \in \mathbf{H}^0(T^2, \mathbb{Q}) \subset \mathbf{HP}\mathbb{Q}^0(T^2)$ and $[dx \wedge dy] \in \mathbf{H}^2(T^2, \mathbb{Q}) \subset \mathbf{HP}\mathbb{Q}^0(T^2)$. Then set

$$\begin{aligned} c_{\mathcal{G}} : \mathbf{HP}\mathbb{Q}^0(T^2) &\to \Omega\mathrm{P}^0_{\mathrm{cl}}(T^2) \\ 1 &\mapsto 1 \\ [dx \wedge dy] &\mapsto dx \wedge dy \end{aligned}$$

which clearly is a cohomological character compatible with the pre-geometrization $\mathcal{G}$. Hence we get a geometrization $(\mathcal{G}, c_{\mathcal{G}})$.

The Pontryagin-Thom Theorem together with the Künneth isomorphism (see [Boa95, Theorem 4.2]) gives $\pi^{\$}_*(MT^2) \cong \pi^{\$}_*(T^2_+) \cong \pi^{\$}_*(S^1_+) \otimes \pi^{\$}_*(S^1_+)$. The free part of $\pi^{\$}_{\mathrm{odd}}(MT^2)$ is $\left(\pi^{\$}_{\mathrm{odd}}(MT^2)\right) / \,\mathrm{Torsion} \cong \mathbb{Z}[1] \oplus \mathbb{Z}[1]$ where $[-]$ denotes the degree. Geometric cycles for the generators are the two inclusions $\gamma_i : S^1 \hookrightarrow T^2$, $i = 1, 2$, discussed above. We equip each of these cycles with the flat metric (i.e., $S^1 := \mathbb{R}/\mathbb{Z}$) and the $Spin^c$-structure induced by the non-trivial framing. Note that the error forms for the maps γ_i are trivial.

Next we discuss the intrinsic eta-invariant. To calculate the correction form we will use the pullback of the geometric vector bundle which we used to define the geometrization. For example, to calculate the value of $\eta_{\mathrm{intrinsic}}(S^1, \gamma_i) : \mathbf{KU}^0(T^2) \to {}^{\mathbb{R}}/_{\mathbb{Z}}$ at $[\mathcal{P}] \in \mathbf{KU}^0(T^2)$ we use the geometric vector bundle $\gamma_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})$. Hence the correction form vanishes. The value of $\eta_{\mathrm{intrinsic}}(S^1, \gamma_i) \in \mathrm{Q}^{\mathbb{R}}_1(T^2)$ is represented by

$$\begin{aligned} \eta_{\mathrm{intrinsic}}(S^1, \gamma_i) : \mathbf{KU}^0(T^2) &\to {}^{\mathbb{R}}/_{\mathbb{Z}} \\ [T^2 \times \mathbb{C}] &\mapsto -\xi(\not{D}_{S^1}) \\ [\mathcal{P}] &\mapsto -\xi\left(\not{D}_{S^1} \otimes \gamma_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) \ . \end{aligned}$$

Now we calculate this eta-invariants. For this we note that the map of trivial vector bundles covering the inclusion $\mathrm{incl}_i : \mathbb{R} \hookrightarrow \mathbb{R}^2$, $i = 1, 2$,

$$\begin{aligned} \mathbb{R} \times \mathbb{C} &\to \mathbb{R}^2 \times \mathbb{C} \\ (x, v) &\mapsto (\mathrm{incl}_i(x), v) \end{aligned}$$

induces a trivialization $\gamma_i^*\mathcal{P} \cong S^1 \times \mathbb{C}$. In this trivialization the metric and the connection are trivial. Hence $\xi\left(D\!\!\!/_{S^1} \otimes \gamma_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) = \xi\left(D\!\!\!/_{S^1}\right)$. We can identify the Dirac operator $D\!\!\!/_{S^1}$ as the operator $i\frac{d}{dt}$ acting on 1-periodic functions. The eigenfunctions are $\exp(2\pi ikt)$ for all $k \in \mathbb{Z}$ and the spectrum is $\operatorname{spec}(D\!\!\!/_{S^1}) = 2\pi\mathbb{Z}$ (with multiplicity one). Since the spectrum is symmetric around 0 the eta-function vanishes and only the kernel contributes to the reduced eta-invariant of $D\!\!\!/_{S^1}$. But since the kernel is 1-dimensional, $\xi\left(D\!\!\!/_{S^1}\right) = {}^1\!/\!_2 \in {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$. This gives

$$
\begin{aligned}
\eta_{\text{intrinsic}}(S^1, \gamma_i) : \mathbf{KU}^0(T^2) &\to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \\
[T^2 \times \mathbb{C}] &\mapsto -{}^1\!/\!_2 \\
[\mathcal{P}] &\mapsto -{}^1\!/\!_2 \ .
\end{aligned}
$$

Since this map is non-trivial on the trivial bundle it cannot be in $\mathrm{U}_1^{\mathbb{R}}(T^2)$. Therefore this intrinsic eta-invariant shows that $[S^1, \gamma_i] \in \pi_1^{\$}(MT^2)$ is a non-trivial element of order at least 2.

On the other hand, Detection Theorem 7.17 yields that there has to be a universal $Spin^c$-geometrization such that the associated intrinsic eta-invariant shows that $[S^1, \gamma_i] \in \pi_1^{\$}(MT^2)$ is non-torsion. Motivated by the explicitly determined representatives α_1 and α_2 for the classifying group $\operatorname{coker}(\hat{\rho})$ we consider the pre-geometrization (for $j \in \{1, 2\}$ and $\lambda \in \mathbb{R}$)

$$
\begin{aligned}
\mathcal{G}_{j,\lambda} : \mathbf{KU}^0(T^2) &\to \widehat{\mathbf{KU}^0}(T^2) \\
[T^2 \times \mathbb{C}] &\mapsto \left[T^2 \times \mathbb{C}, h^{\text{triv}}, \nabla^{j,\lambda}\right] \\
[\mathcal{P}] &\mapsto \left[\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}}\right]
\end{aligned}
$$

where we set $\nabla^{1,\lambda} := d + i\lambda dx$ and $\nabla^{2,\lambda} := d + i\lambda dy$. These are indeed geometrizations with the same cohomological character as $\mathcal{G}$. Then the associated intrinsic eta-invariant is[1] (using appropriate correction forms as before)

$$
\begin{aligned}
\eta^{j,\lambda}_{\text{intrinsic}}(S^1, \gamma_i) : \mathbf{KU}^0(T^2) &\to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \\
[T^2 \times \mathbb{C}] &\mapsto -\xi\left(D\!\!\!/_{S^1} \otimes \gamma_i^*(T^2 \times \mathbb{C}, h^{\text{triv}}, \nabla^{j,\lambda})\right) \\
[\mathcal{P}] &\mapsto -\xi\left(D\!\!\!/_{S^1} \otimes \gamma_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) \ .
\end{aligned}
$$

Note that $\gamma_i^*(T^2 \times \mathbb{C}, h^{\text{triv}}, \nabla^{j,\lambda}) \cong (S^1 \times \mathbb{C}, h^{\text{triv}}, \nabla^{triv})$ for $i \neq j$. So we consider only the case $i = j$. Then $\gamma_i^*(T^2 \times \mathbb{C}, h^{\text{triv}}, \nabla^{i,\lambda}) \cong (S^1 \times \mathbb{C}, h^{\text{triv}}, \nabla^{1,\lambda})$. Here we can identify the twisted Dirac operator $D\!\!\!/_{S^1} \otimes (S^1 \times \mathbb{C}, h^{\text{triv}}, \nabla^{1,\lambda})$ with the operator $i\frac{d}{dt} - \lambda$ acting on 1-periodic functions. This is only a shift of the operator discussed before, thus $\operatorname{spec}\left(D\!\!\!/_{S^1} \otimes (S^1 \times \mathbb{C}, h^{\text{triv}}, \nabla^{1,\lambda})\right) = \{2\pi k - \lambda | k \in \mathbb{Z}\}$ (with multiplicity one). Clearly, the spectra for λ and $\lambda \pm 2\pi$ agree. Thus we assume that $\lambda \in (-2\pi, 0)$.

Now the idea is to consider instead the family $D\!\!\!/_\mu := D\!\!\!/_{S^1} \otimes (S^1 \times \mathbb{C}, h^{\text{triv}}, \nabla^{1,-\mu})$ for $\mu \in (0, 2\pi)$. Note that the kernel dimension is constant, $\dim(\ker(D\!\!\!/_\mu)) = 0$ on this interval. Applying [GR12, Proposition 1] we get that the function $\mu \mapsto \eta(D\!\!\!/_\mu, s)$ is smooth on the interval $(0, 2\pi)$ with derivative

$$
\frac{\partial}{\partial \mu}\eta(D\!\!\!/_\mu, s) = -s \cdot \zeta_{D\!\!\!/_\mu^2}\left(\frac{s+1}{2}\right)
$$

[1] The calculation of this intrinsic eta-invariant can be simplified by using Proposition 7.15. We will do so in the following section. Here we proceed with a hands-on spectral computation.

where $\zeta_{\not{D}^2_\mu}(s) := \sum_{\chi \in \operatorname{spec}(\not{D}^2_\mu)\setminus\{0\}} \frac{1}{\chi^s}$ (sum with multiplicity) is the zeta-function of $\not{D}^2_\mu$. Since the kernel is constant $\mu \mapsto \xi(\not{D}_\mu)$ is also smooth and the derivative is

$$\frac{\partial}{\partial\mu}\xi(\not{D}_\mu) = -\operatorname{res}_{s=1/2}\left(\zeta_{\not{D}^2_\mu}(s)\right) .$$

Recall that the zeta function can be expressed in terms of the heat kernel expansion of $\not{D}^2_\mu$ ([Ros97, Theorem 5.2]). For this we denote the asymptotic expansion of the trace of the heat kernel as

$$\operatorname{Tr}\left(\exp(-t\not{D}^2_\mu)\right) \sim (4\pi t)^{-1/2}\sum_k t^k \int_{S^1} u^\mu_k$$

with $u^\mu_0 = dx$ the Riemannian volume form. Then the formula for the zeta-function is

$$\zeta_{\not{D}^2_\mu}(s) = \frac{1}{(4\pi)^{1/2}\Gamma(s)} \cdot \left(\sum_{k=0}^N \frac{1}{s+k-\frac{1}{2}}\int_{S^1} u^\mu_k\right) + \text{ terms holomorphic near } s = 1/2 ,$$

where $\Gamma(s)$ denotes the Gamma-function. This gives

$$\begin{aligned}\operatorname{res}_{s=1/2}\left(\zeta_{\not{D}^2_\mu}(s)\right) &= \operatorname{res}_{s=1/2}\left(\frac{1}{(4\pi)^{1/2}\Gamma(s)}\cdot\left(\frac{1}{s-1/2}\int_{\mathbb{R}/\mathbb{Z}} dx\right)\right) = \\ &= \frac{1}{2\pi}\operatorname{res}_{s=1/2}\left(\frac{1}{s-1/2}\right) = \frac{1}{2\pi}\end{aligned}$$

and thus $\frac{\partial}{\partial\mu}\xi(\not{D}_\mu) = -\frac{1}{2\pi}$. So we need the value of $\xi(\not{D}_\mu)$ at a single $\mu \in (0, 2\pi)$. The value $\mu = \pi$ is accessible. There the spectrum is symmetric (with multiplicity) and we get $\xi(\not{D}_\pi) = 0$. Hence, by the fundamental theorem of calculus, $\xi(\not{D}_\mu) = \frac{\pi-\mu}{2\pi} = \frac{1}{2} - \frac{\mu}{2\pi} \in \mathbb{R}/\mathbb{Z}$ for all $\mu \in (0, 2\pi)$. This implies $\xi\left(\not{D}_{S^1} \otimes (S^1 \times \mathbb{C}, h^{\mathrm{triv}}, \nabla^{1,\lambda})\right) = \frac{1}{2} + \frac{\lambda}{2\pi}$ and we conclude our final formula for the intrinsic eta-invariant,

$$\begin{aligned}\eta^{i,\lambda}_{\mathrm{intrinsic}}(S^1, \gamma_i) : \mathbf{KU}^0(T^2) &\to \mathbb{R}/\mathbb{Z} \\ [T^2 \times \mathbb{C}] &\mapsto -\frac{1}{2} - \frac{\lambda}{2\pi} \\ [\mathcal{P}] &\mapsto -1/2 .\end{aligned}$$

This shows that $[S^1, \gamma_i] \in \pi^{\$}_1(MT^2)$, $i = 1, 2$, are non-torsion elements. Moreover, we can even see that they are linearly independent if we consider the case $i \neq j$.

For illustration we repeat these calculations for different geometric cycles. This also serves to check our calculations.

So consider the manifolds $\mathbb{R}/2\pi\mathbb{Z}$ together with the induced framing, $Spin^c$-structure and metric. We take the maps

$$\begin{aligned}\hat{\gamma}_i : \mathbb{R}/2\pi\mathbb{Z} &\to S^1 \to T^2 \\ [x] &\mapsto \left[\frac{x}{2\pi}\right] \mapsto \gamma_i\left(\left[\frac{x}{2\pi}\right]\right)\end{aligned}$$

and we get geometric cycles $(\mathbb{R}/2\pi\mathbb{Z}, \hat{\gamma}_i)$, $i = 1, 2$. Clearly, $[\mathbb{R}/2\pi\mathbb{Z}, \hat{\gamma}_i] = [\mathbb{R}/\mathbb{Z}, \gamma_i] \in \pi^{\$}_1(MT^2)$. The intrinsic eta-invariant associated to the geometrizations $\mathcal{G}_{j,\lambda}$ (using appropriate correction forms as before) is represented by

$$\eta^{j,\lambda}_{\mathrm{intrinsic}}(\mathbb{R}/2\pi\mathbb{Z}, \hat{\gamma}_i) : \mathbf{KU}^0(T^2) \to \mathbb{R}/\mathbb{Z}$$

$$[T^2 \times \mathbb{C}] \mapsto -\xi\left(\not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes \hat{\gamma}_i^*(T^2 \times \mathbb{C}, h^{\mathrm{triv}}, \nabla^{j,\lambda})\right)$$
$$[\mathcal{P}] \mapsto -\xi\left(\not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes \hat{\gamma}_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) \ .$$

As above we can trivialize $\hat{\gamma}_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})) \cong \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{C}$ such that the metric and the connection become trivial. We can identify the Dirac operator $\not{D}_{\mathbb{R}/2\pi\mathbb{Z}}$ as the operator $i\frac{d}{dt}$ acting on 2π-periodic functions. The eigenfunctions are $\exp(ikt)$ for all $k \in \mathbb{Z}$ and the spectrum is $\mathrm{spec}(\not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes \hat{\gamma}_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})) = 2\pi\mathbb{Z}$ (with multiplicity one). Since the spectrum is symmetric around 0 the eta-function vanishes and only the kernel contributes to the reduced eta-invariant of $\not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes \hat{\gamma}_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})$. But since the kernel is 1-dimensional, $\xi\left(\not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes \hat{\gamma}_i^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) = {}^1\!/_2 \in {}^{\mathbb{R}}\!/_{\mathbb{Z}}$.

Recall that $\hat{\gamma}_i$ is defined as a composition. Thus we can also trivialize

$$\hat{\gamma}_i^*(T^2 \times \mathbb{C}, h^{\mathrm{triv}}, \nabla^{i,\lambda}) \cong (\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{C}, h^{\mathrm{triv}}, d + i\frac{\lambda}{2\pi}dx)$$

and as before we consider the family

$$\hat{\not{D}}_\mu := \not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes (\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{C}, h^{\mathrm{triv}}, d - i\mu dx)$$

for $\mu \in (0,1)$. These operators can be identified with the operators $i\frac{d}{dt} + \mu$ acting on 2π-periodic functions. The above arguments give $\frac{\partial}{\partial\mu}\xi(\hat{\not{D}}_\mu) = -\operatorname{res}_{s={}^1\!/_2}\left(\zeta_{\hat{\not{D}}_\mu^2}(s)\right)$ on the interval $\mu \in (0,1)$. We compute

$$\begin{aligned}\operatorname{res}_{s={}^1\!/_2}\left(\zeta_{\hat{\not{D}}_\mu^2}(s)\right) &= \operatorname{res}_{s={}^1\!/_2}\left(\frac{1}{(4\pi)^{1/2}\Gamma(s)} \cdot \left(\frac{1}{s-{}^1\!/_2}\int_{\mathbb{R}/2\pi\mathbb{Z}} dx\right)\right) = \\ &= \frac{1}{2\pi}\operatorname{res}_{s={}^1\!/_2}\left(\frac{2\pi}{s-{}^1\!/_2}\right) = 1\end{aligned}$$

and thus $\frac{\partial}{\partial\mu}\xi(\hat{\not{D}}_\mu) = -1$. Next we consider $\mu = {}^1\!/_2$. The spectrum of $\xi(\hat{\not{D}}_{1/2})$ is symmetric (with multiplicity) and we get $\xi(\hat{\not{D}}_{1/2}) = 0$. Hence, $\xi(\hat{\not{D}}_\mu) = {}^1\!/_2 - \mu \in {}^{\mathbb{R}}\!/_{\mathbb{Z}}$ for all $\mu \in (0,1)$. This implies $\xi\left(\not{D}_{\mathbb{R}/2\pi\mathbb{Z}} \otimes (\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{C}, h^{\mathrm{triv}}, d + i\frac{\lambda}{2\pi}dx)\right) = \frac{1}{2} + \frac{\lambda}{2\pi}$ and we conclude our final formula for the intrinsic eta-invariant,

$$\begin{aligned}\eta^{i,\lambda}_{\mathrm{intrinsic}}(\mathbb{R}/2\pi\mathbb{Z}, \hat{\gamma}_i) : \mathbf{KU}^0(T^2) &\to {}^{\mathbb{R}}\!/_{\mathbb{Z}} \\ [T^2 \times \mathbb{C}] &\mapsto -\frac{1}{2} - \frac{\lambda}{2\pi} \\ [\mathcal{P}] &\mapsto -{}^1\!/_2 \ .\end{aligned}$$

This is the same formula as for $\eta^{i,\lambda}_{\mathrm{intrinsic}}(S^1, \gamma_i)$ and so our calculation seems to be correct.

8.2 The three-dimensional torus

We study a second explicit example, the three-torus $B := T^3 := \mathbb{R}^3/\mathbb{Z}^3$ together with the trivial map to $BSpin^c$. Since T^3 is a compact manifold we approximate T^3 by the constant system as in Example 3.21. We equip it with the framing induced by $T^3 = S^1 \times S^1 \times S^1$ where S^1 is equipped with the non-trivial framing. This gives a $Spin^c(3)$-principal bundle which we use as abstract stable $Spin^c$-structure. Then a universal $Spin^c$-geometrization is the same as a $Spin^c$-geometrization on $\mathtt{id} : T^3 \to T^3$.

Before we construct such a *Spin*^{c}-geometrization we recall the classifying group which we calculated in Section 4.1,

$$\operatorname{coker}(\rho) = {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}{}^{\oplus 7} .$$

Observe that all elements in $\mathrm{U}_1^{\mathbb{R}}(T^3)$ vanish on trivial vector bundles (Remark 7.18).

Now we construct a *Spin*^{c}-geometrization on `id` $: T^3 \to T^3$. At first we define a map $\mathcal{G} : \mathbf{KU}^0(T^3) \to \widehat{\mathbf{KU}^0}(T^3)$. Since $\mathbf{H}^*(T^3, \mathbb{Z})$ is a free abelian group so is $\mathbf{KU}^0(T^3)$. Also, the Chern character is integral because we are in dimension 3. Hence we have $\mathbf{KU}^0(T^3) \cong \mathbb{Z} \oplus \mathbb{Z}^3$, where one generator is the trivial rank 1-bundle, $T^3 \times \mathbb{C} \xrightarrow{\mathrm{pr}} T^3$. The other generators are $\mathrm{proj}_i^* \mathcal{P}$ where we denoted the projection $T^3 \to T^2$ which omits the i-th entry by proj_i, $i = 1, 2, 3$ (and $\mathcal{P}$ denotes the Poincaré-bundle, see Section 8.1). Now we define

$$\begin{aligned}
\mathcal{G} : \mathbf{KU}^0(T^3) &\to \widehat{\mathbf{KU}^0}(T^3) \\
[T^3 \times \mathbb{C}] &\mapsto \left[T^3 \times \mathbb{C}, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}}\right] \\
\mathrm{proj}_i^*[\mathcal{P}] &\mapsto \mathrm{proj}_i^* \left[\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}}\right]
\end{aligned}$$

which is clearly a pre-geometrizations. A cohomological character is given by

$$\begin{aligned}
c_{\mathcal{G}} : \mathbf{HP}\mathbb{Q}^0(T^3) &\to \Omega\mathrm{P}^0_{\mathrm{cl}}(T^3) \\
1 &\mapsto 1 \\
\mathrm{proj}_i^*[dx \wedge dy] &\mapsto \mathrm{proj}_i^*(dx \wedge dy)
\end{aligned}$$

and it is compatible with the pre-geometrization $\mathcal{G}$. Hence we get a geometrization $(\mathcal{G}, c_{\mathcal{G}})$.

The Pontryagin-Thom Theorem together with the Künneth isomorphism (see [Boa95, Theorem 4.2]) (applied twice) gives $\pi_*^{\$}(MT^3) \cong \pi_*^{\$}(T^3_+) \cong \pi_*^{\$}(S^1_+) \otimes \pi_*^{\$}(S^1_+) \otimes \pi_*^{\$}(S^1_+)$. The free part of $\pi_{\mathrm{odd}}^{\$}(MT^3)$ is

$$\left(\pi_{\mathrm{odd}}^{\$}(MT^3)\right) / \,\mathrm{Torsion} \cong \mathbb{Z}[1] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[3]$$

where $[-]$ denotes the degree. Geometric cycles for the generators are the three inclusions $\rho_i : S^1 \hookrightarrow T^3$, $i = 1, 2, 3$, in degree 1 and the identity `id` $: T^3 \to T^3$ in degree 3. We equip each of these cycles with the flat metric and *Spin*^{c}-structure induced by the non-trivial framing, i.e., we set $S^1 := \mathbb{R}/\mathbb{Z}$ and $T^3 := \mathbb{R}^3/\mathbb{Z}^3$. In particular, all error forms vanish.

Next we discuss the intrinsic eta-invariant. The value of $\eta_{\mathrm{intrinsic}}(S^1, \gamma_i) \in \mathrm{Q}_1^{\mathbb{R}}(T^3)$ is represented by (using appropriate correction forms as in Section 8.1)

$$\begin{aligned}
\eta_{\mathrm{intrinsic}}(S^1, \rho_i) : \mathbf{KU}^0(T^3) &\to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \\
[T^3 \times \mathbb{C}] &\mapsto -\xi(\not{D}_{S^1}) \\
\mathrm{proj}_j^*[\mathcal{P}] &\mapsto -\xi\left(\not{D}_{S^1} \otimes \rho_i^* \,\mathrm{proj}_j^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) .
\end{aligned}$$

and because $\mathrm{proj}_j \circ \rho_i$ is either constant or one of the maps γ_k we used in Section 8.1 all calculations there apply. The result is (for all $i \in \{1, 2, 3\}$)

$$\begin{aligned}
\eta_{\mathrm{intrinsic}}(S^1, \rho_i) : \mathbf{KU}^0(T^3) &\to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}} \\
[T^3 \times \mathbb{C}] &\mapsto -{}^1\!/\!_2 \\
\mathrm{proj}_j^*[\mathcal{P}] &\mapsto -{}^1\!/\!_2 .
\end{aligned}$$

Since it is non-trivial on a trivial bundle it represents a non-trivial element in $\mathrm{Q}_1^{\mathbb{R}}(T^3)$.

The value of $\eta_{\text{intrinsic}}(T^3, \mathtt{id}) \in \mathrm{Q}_3^{\mathbb{R}}(T^3) = \mathrm{Hom}(\mathbf{KU}^0(T^3), {}^{\mathbb{R}}/_{\mathbb{Z}})$ is represented by (using appropriate correction forms)

$$\begin{aligned}\eta_{\text{intrinsic}}(T^3, \mathtt{id}) : \mathbf{KU}^0(T^3) &\to {}^{\mathbb{R}}/_{\mathbb{Z}} \\ [T^3 \times \mathbb{C}] &\mapsto -\xi(\not{D}_{T^3}) \\ \mathrm{proj}_j^*[\mathcal{P}] &\mapsto -\xi\left(\not{D}_{T^3} \otimes \mathrm{proj}_j^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right) .\end{aligned}$$

We compute $\xi(\not{D}_{T^3})$. For this recall that sections of the complex spinor bundle[2] $\not{S}(TT^3)$ can be identified with the space of 1-periodic functions,

$$C_1^\infty(\mathbb{R}^3, \mathbb{C}^2) := \left\{f \in C^\infty(\mathbb{R}^3, \mathbb{C}^2) | f(x+a, y+b, z+c) = f(x,y,z) \ \forall (a,b,c) \in \mathbb{Z}^3\right\}$$

and the action of the Clifford algebra is given by left multiplication with the Pauli matrices on the spinors,

$$\alpha_1 \cdot = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \alpha_2 \cdot = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \alpha_3 \cdot = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} .$$

Therefore we have identified the Dirac operator $\not{D}_{T^3}$ with

$$\begin{aligned}\hat{\not{D}}_{T^3} : C_1^\infty(\mathbb{R}^3, \mathbb{C}^2) \to& C_1^\infty(\mathbb{R}^3, \mathbb{C}^2) \\ f \mapsto& \alpha_1 \cdot \partial_x f + \alpha_2 \cdot \partial_y f + \alpha_3 \cdot \partial_z f .\end{aligned}$$

Let $f(x,y,z)$ be an eigenfunction of $\hat{\not{D}}_{T^3}$ with eigenvalue λ. Then $f(-x,-y,-z)$ is also an eigenfunction with eigenvalue $-\lambda$. Thus the spectrum $\mathrm{spec}(\not{D}_{T^3}) = \mathrm{spec}(\hat{\not{D}}_{T^3})$ is symmetric (with multiplicity) and the eta-invariant $\eta(\not{D}_{T^3})$ vanishes. Note that $\hat{\not{D}}_{T^3}^2 = -(\partial_x^2 + \partial_y^2 + \partial_z^2)\mathtt{id}_{\mathbb{C}^2}$ and hence the kernel $\ker(\not{D}_{T^3})$ consists of locally constant functions. Therefore it is 2-dimensional and the reduced eta-invariant $\xi(\not{D}_{T^3})$ also vanishes.

Next we compute $\xi\left(\not{D}_{T^3} \otimes \mathrm{proj}_j^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})\right)$. To simplify the notation we assume that $j = 3$, i.e., we consider the pullback bundle $\mathrm{proj}_3^*\mathcal{P}$ where $\mathrm{proj}_3([x,y,z]) := [x,y]$ and set $\not{D}_{T^3}^{\mathcal{P}} := \not{D}_{T^3} \otimes \mathrm{proj}_3^*(\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})$.

The idea is to employ that the Dirac bundle $\not{S}(TT^3) \otimes \mathrm{proj}_3^*\mathcal{P} \to T^3$ is the product of the Dirac bundle[3] $\not{S}(TT^2) \otimes \mathcal{P} \to T^2$ and the Dirac bundle $\not{S}(TS^1) \to S^1$. So we get a decomposition

$$\not{D}_{T^3}^{\mathcal{P}} = \not{D}_{T^2}^{\mathcal{P}} \otimes \mathtt{id}_{\not{S}(TS^1)} + (\sigma \otimes \mathtt{id}_{\mathcal{P}}) \otimes \not{D}_{S^1}$$

as operators acting on sections of $\left(\not{S}(TT^2) \otimes \mathcal{P}\right) \otimes \not{S}(TS^1) \to T^3$. Here σ denotes the grading of Dirac bundle $\not{S}(TT^2) \to T^2$ on the even dimensional manifold T^2 and $\not{D}_{T^2}^{\mathcal{P}} := \not{D}_{T^2} \otimes (\mathcal{P}, h^{\mathcal{P}}, \nabla^{\mathcal{P}})$ is the twisted Dirac operator on the 2-torus. We denote the spaces of eigenfunctions for $\not{D}_{T^2}^{\mathcal{P}}$ (resp. $\not{D}_{S^1}$) to the eigenvalue $\lambda \in \mathbb{R}$ by $E_\lambda^{T^2,\mathcal{P}}$ (resp. $E_\lambda^{S^1}$). Then the space $\bigoplus_{\lambda \in \mathbb{R}, k \in \mathbb{Z}} \left(E_\lambda^{T^2,\mathcal{P}} \otimes E_{2\pi k}^{S^1}\right)$ is dense in the domain of $\not{D}_{T^3}^{\mathcal{P}}$.

Observe that multiplication by σ induces an isomorphism $E_\lambda^{T^2,\mathcal{P}} \xrightarrow{\sigma} E_{-\lambda}^{T^2,\mathcal{P}}$ for all $\lambda \in \mathbb{R}$. So $\not{D}_{T^3}^{\mathcal{P}}$ preserves the subspaces $E_0^{T^2,\mathcal{P}} \otimes E_{2\pi k}^{S^1}$ for all $k \in \mathbb{Z}$. The subspaces $\left(E_\lambda^{T^2,\mathcal{P}} \oplus E_{-\lambda}^{T^2,\mathcal{P}}\right) \otimes$

[2]Here TT^3 denotes the tangent bundle of the torus.

[3]Note that we did *not* explain the spinor bundle $\not{S}(TT^2)$ in Section 6.5 because T^2 is even dimensional. The theory works similarly, but the bundles and the Dirac operator are now ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$-graded. See [LM89, Chapters §II.5 and §II.6].

$E^{S^1}_{2\pi k}$ are also preserved for all $\lambda \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}$. We study the restriction of $D\!\!\!/^{\mathcal{P}}_{T^3}$ to each of these subspaces. To simplify the notation we always identifying $E^{S^1}_{2\pi k} \cong \mathbb{C}$ using the eigenfunctions $z \mapsto \exp(-2\pi i k z)$ and $E^{T^2,\mathcal{P}}_{-\lambda} \cong E^{T^2,\mathcal{P}}_{\lambda}$ using σ (for $\lambda > 0$).

For $\lambda \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}$, i.e., on a summand of the form $\left(E^{T^2,\mathcal{P}}_{\lambda} \oplus E^{T^2,\mathcal{P}}_{-\lambda}\right) \otimes E^{S^1}_{2\pi k}$, we get

$$\begin{aligned} D\!\!\!/^{\mathcal{P}}_{T^3} : E^{T^2,\mathcal{P}}_{\lambda} \oplus E^{T^2,\mathcal{P}}_{\lambda} &\mapsto E^{T^2,\mathcal{P}}_{\lambda} \oplus E^{T^2,\mathcal{P}}_{\lambda} \\ \begin{pmatrix} f \\ g \end{pmatrix} &\mapsto \begin{pmatrix} \lambda \mathtt{id}_{E^{T^2,\mathcal{P}}_{\lambda}} & 2\pi k \mathtt{id}_{E^{T^2,\mathcal{P}}_{\lambda}} \\ 2\pi k \mathtt{id}_{E^{T^2,\mathcal{P}}_{\lambda}} & -\lambda \mathtt{id}_{E^{T^2,\mathcal{P}}_{\lambda}} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \end{aligned}$$

which has eigenvalues $\mu = \pm\sqrt{\lambda^2 + 4\pi^2 k^2}$ and thus has symmetric spectrum (with multiplicity) and no kernel. For $\lambda = 0$ and $k \in \mathbb{Z}$, i.e., on a summand of the form $E^{T^2,\mathcal{P}}_0 \otimes E^{S^1}_{2\pi k}$, we get

$$\begin{aligned} D\!\!\!/^{\mathcal{P}}_{T^3} : E^{T^2,\mathcal{P}}_0 &\to E^{T^2,\mathcal{P}}_0 \\ f &\mapsto 2\pi k \sigma f \end{aligned}$$

which has only a kernel if $k = 0$. In this case the kernel is all of $E^{T^2,\mathcal{P}}_0 \otimes E^{S^1}_0$. We diagonalize $\sigma : E^{T^2,\mathcal{P}}_0 \to E^{T^2,\mathcal{P}}_0$. Since $\sigma^2 = \mathtt{id}$ we get $E^{T^2,\mathcal{P}}_0 \cong \mathbb{C}^a \oplus \mathbb{C}^b$ and

$$\sigma_{|E^{T^2,\mathcal{P}}_0} \mathrel{\hat{=}} \begin{pmatrix} \mathtt{id}_a & 0 \\ 0 & -\mathtt{id}_b \end{pmatrix}$$

where $a, b \in \mathbb{N}_0$. But $E^{T^2,\mathcal{P}}_0$ is the kernel of the twisted, ${}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}$-graded Dirac operator $D\!\!\!/^{\mathcal{P}}_{T^2}$. Thus $a - b = \operatorname{index}\left(D\!\!\!/^{\mathcal{P}}_{T^2}\right)$. So the spectrum of $\left(D\!\!\!/^{\mathcal{P}}_{T^3}\right)_{|E^{T^2,\mathcal{P}}_0 \otimes E^{S^1}_{2\pi k}}$ consists of $+2\pi k$ with multiplicity a and of $-2\pi k$ with multiplicity b. Therefore the spectrum on summands of the form $E^{T^2,\mathcal{P}}_0 \otimes \left(E^{S^1}_{-2\pi k} \oplus E^{S^1}_{2\pi k}\right)$ (for $k > 0$) is symmetric (with multiplicity).

We conclude that the kernel of $D\!\!\!/^{\mathcal{P}}_{T^3}$ has the same dimension as the kernel of $D\!\!\!/^{\mathcal{P}}_{T^2}$ and that the spectrum $\operatorname{spec}(D\!\!\!/^{\mathcal{P}}_{T^3})$ is symmetric around 0 (with multiplicity). Hence the eta-function vanishes and the reduced eta-invariant is $\xi\left(D\!\!\!/^{\mathcal{P}}_{T^3}\right) = {}^1\!/\!{}_2 \dim\left(\ker D\!\!\!/^{\mathcal{P}}_{T^2}\right) \in {}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}}$.

Note that

$${}^1\!/\!{}_2 \dim\left(\ker D\!\!\!/^{\mathcal{P}}_{T^2}\right) = {}^1\!/\!{}_2(a+b) = {}^1\!/\!{}_2(a-b) + b = {}^1\!/\!{}_2(a-b) = {}^1\!/\!{}_2 \operatorname{index}\left(D\!\!\!/^{\mathcal{P}}_{T^2}\right) \in {}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}} \ .$$

To compute the latter we use the Atiyah-Singer Index Theorem [LM89, Theorem 13.10],

$$\operatorname{index}\left(D\!\!\!/^{\mathcal{P}}_{T^2}\right) = \int_{T^2} \mathbf{ch}\left(\nabla^{\mathcal{P}}\right) = \int_{T^2} (1 + dx \wedge dy) = 1 \in \mathbb{Z} \ ,$$

and we conclude $\xi\left(D\!\!\!/^{\mathcal{P}}_{T^3}\right) = {}^1\!/\!{}_2 \operatorname{index}\left(D\!\!\!/^{\mathcal{P}}_{T^2}\right) = {}^1\!/\!{}_2 \in {}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}}$. This gives the intrinsic eta-invariant,

$$\begin{aligned} \eta_{\text{intrinsic}}(T^3, \mathtt{id}) : \mathbf{KU}^0(T^3) &\to {}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}} \\ [T^3 \times \mathbb{C}] &\mapsto 0 \\ \operatorname{proj}_j^*[\mathcal{P}] &\mapsto -{}^1\!/\!{}_2 \ . \end{aligned}$$

So this intrinsic eta-invariant shows only that the cycle $(T^3, \mathtt{id})$ represent a non-trivial element in $\pi_3^{\$}(MT^3)$ but note that it is non-torsion.

Next we construct a different geometrization such that the associated intrinsic eta-invariant shows that this element is non-torsion.

For this we choose $\hat{\xi} \in \mathrm{Hom}\big(\mathbf{KU}^0(T^3), \mathbf{HP}\mathbb{R}^{-1}(T^3)/(\mathrm{im}\,\mathbf{ch})\big)$ (which represents an element in the classifying group $\mathrm{coker}(\hat{\rho})$) and get the geometrization $\mathcal{G}^{\hat{\xi}} := \mathcal{G} + a \circ \hat{\xi}$. Proposition 7.15 gives

$$\begin{aligned} \eta^{\mathcal{G}^{\hat{\xi}}}_{\mathrm{intrinsic}}(T^3, \mathtt{id}) : \mathbf{KU}^0(T^3) &\to \mathbb{R}/\mathbb{Z} \\ [T^3 \times \mathbb{C}] &\mapsto -\left[\int_{T^3} \mathbf{Td}_{T^3} \wedge \hat{\xi}([T^3 \times \mathbb{C}])\right]_{\mathbb{R}/\mathbb{Z}} \\ \mathrm{proj}_j^*[\mathcal{P}] &\mapsto -1/2 - \left[\int_{T^3} \mathbf{Td}_{T^3} \wedge \hat{\xi}(\mathrm{proj}_j^*[\mathcal{P}])\right]_{\mathbb{R}/\mathbb{Z}} . \end{aligned}$$

Since we equipped the torus with a flat metric the Todd form is trivial, $\mathbf{Td}_{T^3} = 1$. So we can take for example

$$\begin{aligned} \hat{\xi} : \mathbf{KU}^0(T^3) &\to \mathbf{HP}\mathbb{R}^{-1}(T^3)/(\mathrm{im}\,\mathbf{ch}) \\ [T^3 \times \mathbb{C}] &\mapsto \pi\,[dx \wedge dy \wedge dz] \\ \mathrm{proj}_j^*[\mathcal{P}] &\mapsto 0 \end{aligned}$$

and get the associated intrinsic eta-invariant

$$\begin{aligned} \eta^{\mathcal{G}^{\hat{\xi}}}_{\mathrm{intrinsic}}(T^3, \mathtt{id}) : \mathbf{KU}^0(T^3) &\to \mathbb{R}/\mathbb{Z} \\ [T^3 \times \mathbb{C}] &\mapsto -[\pi]_{\mathbb{R}/\mathbb{Z}} \\ \mathrm{proj}_j^*[\mathcal{P}] &\mapsto -[1/2]_{\mathbb{R}/\mathbb{Z}} . \end{aligned}$$

Therefore the cycle $(T^3, \mathtt{id})$ represents a non-torsion element in $\pi_3^{\$}(MT^3)$.

8.3 Spheres

We generalize Section 6.7 to higher dimensional spheres. So let $B := S^{2n+1}$, $n \geq 1$, be an odd-dimensional sphere and consider the trivial map to $BSpin^c$. We approximate S^{2n+1} by the constant family $S^{2n+1} \subset \mathbb{R}^{2n+2}$ and take the trivial $Spin^c(0)$-bundles as abstract $Spin^c$-structures. There are $\mathbb{R}/\mathbb{Z}$-many equivalence classes of universal $Spin^c$-geometrizations by Classification Theorem 7.10. Each of them is characterized by its value on the trivial rank 1 vector bundle.

We calculate the intrinsic eta-invariant for the geometrization $\mathcal{G}$ which maps the trivial rank 1 vector bundle $S^{2n+1} \times \mathbb{C} \to S^{2n+1}$ to the trivial rank 1 vector bundle together with the trivial connection and the trivial metric, denoted by $\mathbf{V}$. Since the map $S^{2n+1} \to BSpin^c$ is trivial the homotopy groups of the bordism spectrum are the stable homotopy groups of S^{2n+1}, $\pi_*^{\$}(S_+^{2n+1})$ (by the Pontryagin-Thom Theorem). These are all finite except in degree[4] $2n+1$ where its is free on one generator. This generator can be represented by the geometric cycle $(S^{2n+1}, \mathtt{id})$ which we equip with the geometric structures induced from $\mathbb{R}^{2n+2}$. We choose the stable framing such that the bordism class $[S^{2n+1}] \in \pi_{2n+1}^{\$} = \pi_{2n+1}^{\$}(\star_+)$

[4] There is also a $\mathbb{Z}$ in degree 0. But since we are only interest in odd degrees we ignore this $\mathbb{Z}$.

vanish (i.e., we take the bounding structure). Note that $\mathrm{U}^{\mathbb{R}}_{2n+1}(S^{2n+1})$ is trivial. Hence we can evaluate the intrinsic eta-invariant on the trivial rank 1 vector bundle. To calculate the correction form $\gamma_{S^{2n+1}\times\mathbb{C}}$ we choose the same geometric refinement $\mathbf{V}$. We get $a(\gamma_{S^{2n+1}\times\mathbb{C}}) = \mathtt{id}^!_\delta\mathcal{G}(S^{2n+1}\times\mathbb{C}) - \mathbf{V} = \mathbf{V} + a(\delta\wedge R(\mathbf{V})) - \mathbf{V} = a(\delta)$ where δ denotes the error form for the identity $\mathtt{id} : S^{2n+1} \to S^{2n+1}$ (the left hand S^{2n+1} is equipped with the geometric structures induced from $\mathbb{R}^{2n+2}$ whereas the right hand S^{2n+1} carries the trivial abstract *Spin*^{c}-structure). Recall that the error form is

$$\delta = \mathbf{Td}^{-1}_{S^{2n+1}} \wedge \widetilde{\mathbf{Td}}(\nabla^{\mathrm{triv}}, \nabla^{S^{2n+1}}) \ .$$

Therefore the intrinsic eta-invariant is

$$\begin{aligned}\eta_{\mathrm{intrinsic}}(S^{2n+1}, \mathtt{id}) &= -\left[\int_{S^{2n+1}} \mathbf{Td}_{S^{2n+1}} \wedge \gamma_{S^{2n+1}}\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_{S^{2n+1}} \otimes \mathbf{V}) = \\ &= -\left[\int_{S^{2n+1}} \widetilde{\mathbf{Td}}(\nabla^{\mathrm{triv}}, \nabla^{S^{2n+1}})\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_{S^{2n+1}}) \ .\end{aligned}$$

This is precisely the formula for Adams e-invariant, evaluated on $[S^{2n+1}] \in \pi^{\$}_{2n+1}$, by [APS75b, Theorem 4.14]. But the bordism class of $[S^{2n+1}]$ is trivial because it is the boundary of the (framed) disk. Thus $\eta_{\mathrm{intrinsic}}(S^{2n+1}, \mathtt{id}) = 0 \in \mathbb{R}/\mathbb{Z}$.

Hence this intrinsic eta-invariant doesn't detect the generator. By Detection Theorem 7.17 there is a universal *Spin*^{c}-geometrization such that the associated intrinsic eta-invariant detects the generator. It can be constructed similarly as at the end of the previous Section. Here one uses

$$\begin{aligned}\hat{\xi} : \mathbf{KU}^0(S^{2n+1}) &\to \mathbf{HP}\mathbb{R}^{-1}(S^{2n+1})/(\mathrm{im}\,\mathbf{ch}) \\ [S^{2n+1}\times\mathbb{C}] &\mapsto \pi\,[\mathrm{vol}_{S^{2n+1}}]\end{aligned}$$

where $\mathrm{vol}_{S^{2n+1}}$ denotes a normalized volume form.

8.4 Classifying spaces of compact Lie groups

Let G be a compact Lie group and consider its classifying space BG together with the trivial map to $BSpin^c$. We apply Theorem 7.11 and see that there is a unique equivalence class of universal *Spin*^{c}-geometrizations. Note that $\pi^{\$}_{\mathrm{odd}}(MBG) \cong \pi^{\$}_{\mathrm{odd}}(BG_+)$ since the map $BG \to BSpin^c$ is trivial. Since BG is rationally even these are torsion groups. Theorem 6.28 yields that the intrinsic eta-invariant coincides with the universal one.

8.5 The classifying space of the string group

Let $BString$ denote the classifying space of the string group (see Section 4.2) and equip it with the trivial map to $BSpin^c$. We can apply Theorem 7.11 and see that there is a unique equivalence class of universal *Spin*^{c}-geometrizations. By the Pontryagin-Thom Theorem, $\pi^{\$}_{\mathrm{odd}}(MBString) \cong \pi^{\$}_{\mathrm{odd}}(BString_+)$ since the map $BString \to BSpin^c$ is trivial. Again, $BString$ is rationally even and thus the intrinsic eta-invariant and the universal eta-invariant coincide.

8.6 Eilenberg-MacLane spaces

Let $B := K(\mathbb{Z}, n)$ denote an Eilenberg-MacLane space where $n \geq 3$ (we already discussed $K(\mathbb{Z}, 1) \simeq S^1$ and $K(\mathbb{Z}, 2) \simeq BS^1$) together with the trivial map to $BSpin^c$. Then B is of finite type, connected and simply-connected. Thus there is an approximation by manifolds $(B_i, f_i, g_i)_i$ of B by Example 3.22 and we endow it with the trivial abstract $Spin^c$-structure. The K-theory of B has been calculated by Anderson and Hodgkin ([AH68, Theorem II]). The result is that $\widetilde{\mathbf{KU}}^0(B)$ is a divisible abelian group and therefore maps only trivially to finitely generated abelian groups. Therefore, by Proposition 2.2, a continuous map $\mathbf{KU}^0(B) \to G$ where G is some discrete group factors over $\mathbf{KU}^0(\star)$. Thus a universal $Spin^c$-geometrization for B is given by a compatible family of geometric refinements of the trivial rank 1 vector bundles over the approximating manifolds B_i. We accomplish this by taking the trivial connection and the trivial metric.

Note that these are only pre-geometrizations. But in this case the cohomological characters are unproblematic. Using Proposition 3.4 we can choose a compatible family of cohomological characters $(c_i)_i$. Now we have to check whether the square (3.1) commutes,

$$\begin{tikzcd} \mathbf{HP}\mathbb{Q}^0(B) \arrow[r, "c_i"] & \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i) \\ \mathbf{KU}^0(B) \arrow[u, "\mathbf{ch}"] \arrow[r, "\mathcal{G}"', "i"] & \widehat{\mathbf{KU}}^0(B_i) \arrow[u, "R"'] \end{tikzcd} .$$

We have $\mathbf{HP}\mathbb{Q}^0(B) \xrightarrow{\cong} \varprojlim_{i\in\mathbb{N}} \mathbf{HP}\mathbb{Q}^0(B_i)$ and thus we conclude that the Chern character also factors over $\mathbf{KU}^0(\star)$. Therefore it remains to check that

$$\begin{tikzcd} \mathbf{HP}\mathbb{Q}^0(B) \arrow[r, "c_i"] & \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i) \\ \mathbf{KU}^0(\star) \arrow[u, "\mathbf{ch}"] \arrow[r, "\mathcal{G}", "i"'] & \widehat{\mathbf{KU}}^0(B_i) \arrow[u, "R"'] \end{tikzcd}$$

commutes. But because a cohomological character preserves the internal degree and is compatible with **Rham** it is uniquely determined on the subgroup $\mathbf{H}^0(B, \mathbb{Q}) \subset \mathbf{HP}\mathbb{Q}^0(B)$. In particular, a cohomological character always sends the unit $1 \in \mathbf{H}^0(B, \mathbb{Q})$ to the unit $1 \in \Omega\mathrm{P}^0_{\mathrm{cl}}(B_i)$. Therefore the square (3.1) commutes for all cohomological characters.

A straightforward calculation gives the classifying group. The result is

$$\operatorname{coker}(\rho) = \begin{cases} 0, & n \text{ even} \\ \mathbb{R}/\mathbb{Z}, & n \text{ odd} . \end{cases}$$

The group $\pi^{\$}_{odd}(MK(\mathbb{Z}, n))$ is torsion for n even. So we restrict to odd $n = 2k+1$. By the Pontryagin-Thom Theorem we have $\pi^{\$}_*(MK(\mathbb{Z}, 2k+1)) \cong \pi^{\$}_*(K(\mathbb{Z}, 2k+1)_+)$. Since the rational cohomology of $K(\mathbb{Z}, 2k+1)$ (and thus of $MK(\mathbb{Z}, 2k+1)$ by the Thom isomorphism) is an exterior algebra on one generator in degree $2k+1$ we are interested in $\pi^{\$}_{2k+1}(MK(\mathbb{Z}, 2k+1))$. The Freudenthal Suspension Theorem gives $\pi^{\$}_{2k+1}(K(\mathbb{Z}, 2k+1)_+)/\,\mathrm{Torsion} \cong \pi_{2k+1}(K(\mathbb{Z}, 2k+1)_+)/\,\mathrm{Torsion} \cong \mathbb{Z}$.

A generator is given by the $(2k+1)$-skeleton[5] $g : S^{2k+1} \to K(\mathbb{Z}, 2k+1)$. Therefore we can calculate the intrinsic eta-invariant of the universal $Spin^c$-geometrization constructed above.

[5] In fact, we can choose an approximation such that this is the map $g_1 : B_1 \to B$.

Because $\mathrm{U}^{\mathbb{R}}_{2k+1}(MK(\mathbb{Z}, 2k+1))$ is trivial the intrinsic eta-invariant can be evaluated at this geometric cycle. We equip S^{2k+1} with a geometric structure and framing as in Section 8.3 and get

$$\eta_{\mathrm{intrinsic}}(S^{2k+1}, g) = -\left[\int_{S^{2n+1}} \widetilde{\mathbf{Td}}(\nabla^{\mathrm{triv}}, \nabla^{S^{2n+1}})\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}_{S^{2n+1}})$$

which vanishes by the discussion in Section 8.3.

Hence the intrinsic eta-invariant associated to the universal $Spin^c$-geometrization construct above does not detect the generator. But Detection Theorem 7.17 shows that there is a universal $Spin^c$-geometrization which can do this.

Chapter 9

Functoriality defect

In this section we review [Bun11, Remark 5.21] which gives an explicit example that the intrinsic eta-invariant is not functorial.

To this end we recall that there is a fibration (which is a path space fibration)

$$K(\mathbb{Q},3) \to \star \to K(\mathbb{Q},4)\ .$$

We study the intrinsic eta-invariants for the spaces A and B which are defined as homotopy pullbacks,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"] \arrow[dr, phantom, "\lrcorner"] & B \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner"] & K(\mathbb{Q}/\mathbb{Z},3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \star \arrow[d] \\ S^4 \arrow[r, hook] & \mathbb{H}\mathrm{P}^2 \arrow[r, "c_2"] & K(\mathbb{Z},4) \arrow[r] & K(\mathbb{Q},4) \end{tikzcd}$$

where $c_2 : \mathbb{H}\mathrm{P}^2 \to \mathbb{H}\mathrm{P}^\infty \to K(\mathbb{Z},4)$ denotes the second Chern class ($\mathbb{H}\mathrm{P}^\infty \simeq BSU(2)$).

The completion theorem ([AS69, Theorem 2.1]) gives $\mathbf{KU}^0(BSU(2)) \cong \mathbb{Z}[|H|]$ where

$$H := ESU(2) \times_{SU(2)} \mathbb{C}^2 - (BSU(2) \times \mathbb{C}^2)$$

has virtual rank 0. Here we use the standard representation of $SU(2)$ on $\mathbb{C}^2$. The induced elements in the complex K-theory of $\mathbb{H}\mathrm{P}^2$, S^4, B and A are denoted by $H_{\mathbb{H}\mathrm{P}^2}$, H_{S^4}, H_B and H_A. We remark that the induced K-theory class on S^4 can explicitly be described as

$$H_{S^4} = S^7 \times_{SU(2)} \mathbb{C}^2 - (S^4 \times \mathbb{C}^2)$$

where $S^7 \to S^4$ denotes the Hopf bundle.

We use that A and B are total spaces of $K(\mathbb{Q},3)$-fibrations over connected and simply-connected base spaces to calculate the rational cohomology via the Leray-Serre spectral sequence. As input we use

$$\mathbf{H}^*(K(\mathbb{Q},3),\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,3 \\ 0, & * = \text{ else} \end{cases}$$

and we compare with $\star \to K(\mathbb{Q},4)$ to compute the differential. The results are

$$\mathbf{H}^*(S^4,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,4 \\ 0, & * = \text{ else} \end{cases} \quad \text{and} \quad \mathbf{H}^*(A,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,7 \\ 0, & * = \text{ else} \end{cases} \quad \text{and}$$

$$\mathbf{H}^*(\mathbb{H}\mathrm{P}^2,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,4,8 \\ 0, & * = \text{ else} \end{cases} \quad \text{and} \quad \mathbf{H}^*(B,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,11 \\ 0, & * = \text{ else} \end{cases}\ .$$

In particular, the groups $\mathrm{U}^{\mathbb{R}}_7(S^4)$, $\mathrm{U}^{\mathbb{R}}_7(A)$ and $\mathrm{U}^{\mathbb{R}}_7(B)$ vanish and the groups $\pi^{\$}_7(B)$ and $\pi^{\$}_7(S^4)$ are torsion groups.

We equip the spaces A, B and S^4 with the trivial map to $BSpin^c$ and get well-defined universal eta-invariants for B and S^4. The space A is simply-connected and of countable type. So there exists an approximation by manifolds for A by Proposition 3.24. But A is also rationally even and so we can apply our Existence Theorem 3.27 to find a universal $Spin^c$-geometrization for A (cf. Remark 3.28).

To prove that the intrinsic eta-invariant is not functorial it is enough to show that the diagram[1]

$$\begin{array}{ccccc}
\pi^{\$}_7(B) & \xrightarrow{\eta_{\mathrm{uni}}} & \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbb{R}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}_{H_B}} & \mathbb{R}/\mathbb{Z} \\
{\scriptstyle f_*}\uparrow & & & & \| \\
\pi^{\$}_7(A) & \xrightarrow{\eta_{\mathrm{intrinsic}}} & \mathrm{Q}^{\mathbb{R}}_7(A) & \xrightarrow{\mathrm{ev}_{H_A}} & \\
{\scriptstyle h_*}\downarrow & & & & \| \\
\pi^{\$}_7(S^4) & \xrightarrow{\eta_{\mathrm{uni}}} & \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(S^4), \mathbb{R}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}_{H_{S^4}}} & \mathbb{R}/\mathbb{Z}
\end{array}$$

does *not* commute. To this end we show that the outer square

$$\begin{array}{ccccc}
\pi^{\$}_7(B) & \xrightarrow{\eta_{\mathrm{uni}}} & \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(B), \mathbb{R}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}_{H_B}} & \mathbb{R}/\mathbb{Z} \\
{\scriptstyle f_*}\uparrow & & & & \| \\
\pi^{\$}_7(A) & & & & \\
{\scriptstyle h_*}\downarrow & & & & \| \\
\pi^{\$}_7(S^4) & \xrightarrow{\eta_{\mathrm{uni}}} & \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(S^4), \mathbb{R}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}_{H^4_S}} & \mathbb{R}/\mathbb{Z}
\end{array}$$

does not commute.

For this we compute the stable homotopy groups using the (Leray-Serre-)Atiyah-Hirzebruch spectral sequence. Here our input is

$$\pi^{\$}_*(K(\mathbb{Q},3)) = \left\{ \begin{array}{ll} 0, & * \neq 3 \\ \mathbb{Q}, & * = 3 \end{array} \right. .$$

The result is $\pi^{\$}_7(A) \cong \pi^{\$}_3 \oplus \mathbb{Q}$ and the induced map h_* is the projection, i.e.,

$$h_* = \mathrm{pr}_{\pi^{\$}_3} : \pi^{\$}_7(A) \cong \pi^{\$}_3 \oplus \mathbb{Q} \twoheadrightarrow \pi^{\$}_3 \cong \pi^{\$}_7(S^4) \ .$$

The spectral sequence also gives an exact sequence

$$\pi^{\$}_0 \xrightarrow{d} \pi^{\$}_7(A) \xrightarrow{f_*} \pi^{\$}_7(B) \to 0 \ .$$

We decompose $\pi^{\$}_7(A) \cong \pi^{\$}_3 \oplus \mathbb{Q}$ and $\pi^{\$}_0 \cong \mathbb{Z}$ to identify the differential with $d = (d_1, d_2)$ for $d_1 \in \pi^{\$}_3$ and $d_2 \in \mathbb{Q}$. To compute the differential d_2 we compare with the spectral sequence for $\star \to K(\mathbb{Q},4)$ and get that $d_2 = 1$. For d_1 we compare with the spectral sequence for $\mathtt{id} : \mathbb{H}\mathrm{P}^2 \to \mathbb{H}\mathrm{P}^2$. But this spectral sequence is just the long exact sequence (in stable homotopy) of the homotopy cofibre sequence $S^7 \xrightarrow{\Phi} S^4 \to \mathbb{H}\mathrm{P}^2$. So $d_1 \in \pi^{\$}_3$ is given by the image of the Hopf map $\Phi \in \pi_7(S^4)$ in $\pi^{\$}_7(S^4) \cong \pi^{\$}_3$.

Note that the stabilization map $\Sigma^\infty : \pi_7(S^4) \to \pi^{\$}_7(S^4)$ maps the Hopf fibration to a generator of order 24 and thus gives an identification $\pi^{\$}_7(S^4) \cong \mathbb{Z}/24\mathbb{Z}$ ([Hu59, Theorem 16.4]).

[1] By the Pontryagin-Thom Theorem, $\pi^{\$}_*(MX) \cong \pi^{\$}_*(X_+) \cong \pi^{\$}_* \oplus \pi^{\$}_*(X)$ for any space X together with the trivial map to $BSpin^c$. But because $\pi^{\$}_*$ is a torsion group for $*$ odd the intrinsic eta-invariant is functorial on this summand. Therefore we ignore this summands for all what follows.

We consider the element $(1/24, 1) \in \pi_7^{\$}(A)$. It maps to the generator $1/24 \in \pi_7^{\$}(S^4)$ and vanishes in $\pi_7^{\$}(B)$. In particular, the composition $\mathrm{ev}_{H_B} \circ \eta_{\mathrm{uni}} \circ f_*(1/24, 1) = 0$ is trivial. Hence we have to compute $\mathrm{ev}_{H_{S^4}} \circ \eta_{\mathrm{uni}} \circ h_*(1/24, 1) = \mathrm{ev}_{H_{S^4}} \circ \eta_{\mathrm{uni}}(1/24) \in \mathbb{R}/\mathbb{Z}$. For this we introduce $\mathbb{Q}/\mathbb{Z}$-coefficients. Note that $\pi^{\$}\mathbb{Q}/\mathbb{Z}_8(S^4) \xrightarrow{\cong} \widetilde{\pi_7^{\$}(S^4)}$ is an isomorphism. We apply the unit ε of complex K-theory and get $\varepsilon(1/24) \in \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}_8(S^4)$. Now we use that $H_{S^4} \in \widetilde{\mathbf{KU}}^0(S^4) \cong \widetilde{\mathbb{Z} \text{ is a}}$ generator and the suspension isomorphism. Then we have to pair $\Sigma^{-4}\varepsilon(1/24) \in \mathbf{KU}\,\mathbb{Q}/\mathbb{Z}_4 \cong \widetilde{\mathbf{KU}\,\mathbb{Q}/\mathbb{Z}}^{-4}(S^4)$ with $H_{S^4} \in \widetilde{\mathbf{KU}}^4(S^4)$. By Adams work on the J-homomorphism [Ada66, Theorem 7.15] it is known that $\Sigma^{-4}\varepsilon(1/24) = \varepsilon(\Sigma^{-4}1/24) \in \widetilde{\mathbf{KU}\,\mathbb{Q}/\mathbb{Z}}_4(S^4)$ has order 12 (note that we use the complex version of the e-invariant since we work with complex K-theory). Thus we conclude from the multiplicative structure of complex K-theory that $\mathrm{ev}_{H_{S^4}} \circ \eta_{\mathrm{uni}} \circ h_*(1/24, 1) \in \mathbb{R}/\mathbb{Z}$ has order 12, too.

So the above diagram doesn't commute. This implies that the intrinsic eta-invariant is *not* functorial.

Chapter 10

A computation due to Crowley and Goette

Crowley and Goette ([CG13]) defined the t-invariant as a generalized Kreck-Stolz invariant and computed it in examples via methods relying on characteristic classes. Bunke ([Bun11, Proposition 5.18]) showed that the universal eta-invariant generalizes the complex version of the t-invariant. In this chapter we redo a particular computation of Crowley and Goette using the universal eta-invariant in a purely topological way. This serves to confirm the original calculation.

This chapter has some overlap with the previous one. For the convenience of the reader we repeat all definitions and argument so that these two chapters can be read independently.

10.1 The setup of Crowley and Goette

We recall the setup of [CG13, Example 3.5]. For this we equip the sphere S^7 with the Riemannian structure and stable framing induced from $S^7 \subset \mathbb{R}^8$. In particular, we get a *Spin*-manifold with Dirac operator $\not{D}_{S^7}$.

Now take a $SU(2)$-principal bundle with connection $(E \to S^7, \nabla^E)$. We associate along the standard representation of $SU(2)$ on $\mathbb{C}^2$ and get a geometric vector bundle $\mathcal{E} = (E \times_{SU(2)} \mathbb{C}^2 \to S^7, h^{\mathcal{E}}, \nabla^{\mathcal{E}})$. Then the t-invariant is[1] ([CG13, formula (1.9)])

$$t_{S^7}(E) := \frac{1}{2}\left[\xi(\not{D}^{\mathcal{E}}_{S^7}) - \operatorname{rank}(E)\xi(\not{D}_{S^7}) + \int_{S^7} \widehat{\mathbf{A}}_{S^7} \wedge \hat{c}_2(\nabla^{\mathcal{E}}) \wedge \left(1 - \frac{1}{12}c_2(\nabla^{\mathcal{E}})\right)\right] \in \mathbb{Q}/\mathbb{Z}\ .$$

Here $\hat{c}_2(\nabla^{\mathcal{E}}) \in \Omega^3(S^7)$ is any differential form such that $d\hat{c}_2(\nabla^{\mathcal{E}}) = c_2(\nabla^{\mathcal{E}})$. Denote the trivial geometric vector bundle of rank 2 as $\underline{\mathbb{C}^2} := (S^7 \times \mathbb{C}^2 \xrightarrow{\mathrm{pr}} S^7, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}})$ and set $\mathbf{V}_{\mathcal{E}} := \mathcal{E} - \underline{\mathbb{C}^2}$. This gives (since $\underline{\mathbb{C}^2}$ is flat)

$$t_{S^7}(E) = \frac{1}{2}\left[\xi(\not{D}^{\mathbf{V}_{\mathcal{E}}}_{S^7}) + \int_{S^7} \widehat{\mathbf{A}}_{S^7} \wedge \hat{c}_2(\nabla^{\mathbf{V}_{\mathcal{E}}}) \wedge \left(1 - \frac{1}{12}c_2(\nabla^{\mathbf{V}_{\mathcal{E}}})\right)\right] \in \mathbb{Q}/\mathbb{Z}\ .$$

Crowley and Goette proved that this number (in $\mathbb{Q}/\mathbb{Z}$) is independent of the geometric structures on S^7 and $E \to S^7$ ([CG13, remarks following Proposition 1.3]).

Now we construct a $SU(2)$-principal bundle on S^7 for each $n \in \mathbb{Z}$. We start with the Hopf bundle $\Phi : S^7 \to S^4$ which is a $SU(2)$-principal bundle. Then we fix a smooth degree

[1] For this formula and the following one we consider the reduced eta-invariant as a real number.

n-map $\deg_n : S^4 \to S^4$ and define a $SU(2)$-principal bundle $E_n \to SU(2)$ via pullback,

$$\begin{tikzcd} E_n \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & S^7 \arrow[d, "\Phi"] \\ S^7 \arrow[r, "\Phi"] & S^4 \arrow[r, "\deg_n"] & S^4 \end{tikzcd} .$$

Crowley and Goette computed the t-invariant for these $SU(2)$-principal bundles ([CG13, Example 3.5]). The result is $t_{S^7}(E_n) = \left[\frac{n(n-1)}{24}\right] \in \mathbb{Q}/\mathbb{Z}$.

Our aim is to redo this calculation topologically. To this end we use the universal eta-invariant. But we defined the universal eta-invariant using complex K-theory. So we need a complex version of the t-invariant. A natural guess for this is

$$t^{\mathbb{C}}_{S^7}(E) := \xi(\not{D}^{\mathbf{V}_\varepsilon}_{S^7}) + \left[\int_{S^7} \mathbf{Td}_{S^7} \wedge \hat{c}_2(\nabla^{\mathbf{V}_\varepsilon}) \wedge \left(1 - \frac{1}{12} c_2(\nabla^{\mathbf{V}_\varepsilon})\right)\right]_{\mathbb{R}/\mathbb{Z}} \in \mathbb{Q}/\mathbb{Z} .$$

Note that $\frac{\dim(S^7)-1}{2}$ is odd. So we expect to be off by a factor of two. Indeed, we will show below that we can compute $t^{\mathbb{C}}_{S^7}(E_n)$ using the universal eta-invariant and that the result is $t^{\mathbb{C}}_{S^7}(E_n) = \left[\frac{n(n-1)}{12}\right] \in \mathbb{Q}/\mathbb{Z}$ (at least up to a generator of $\mathbb{Z}/24\mathbb{Z}$).

10.2 The topological setup

Recall that $\mathbb{HP}^\infty \simeq BSU(2)$ and that $\mathbb{HP}^1 \cong S^4$. We use the path space fibration

$$K(\mathbb{Q}, 3) \to \star \to K(\mathbb{Q}, 4)$$

and consider the diagram of homotopy pullbacks

$$\begin{tikzcd} A \arrow[r, "\hat{\iota}"] \arrow[d, "h"] \arrow[dr, phantom, "\lrcorner"] & X \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & K(\mathbb{Q}/\mathbb{Z}, 3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \star \arrow[d] \\ S^4 \arrow[r, hook, "\iota"] & \mathbb{HP}^\infty \arrow[r, "c_2"] & K(\mathbb{Z}, 4) \arrow[r] & K(\mathbb{Q}, 4) \end{tikzcd}$$

where $c_2 : \mathbb{HP}^\infty \to K(\mathbb{Z}, 4)$ denotes the second Chern class. Note that each vertical map is a fibration with fibre $K(\mathbb{Q}, 3)$.

Our aim is to study the universal eta-invariant of the space X. For this we equip the spaces $Z = A, X, S^4$ in the above diagram with the trivial map to $BSpin^c$. Therefore we have isomorphisms $\pi^{\$}_*(MZ) \cong \pi^{\$}_*(Z_+)$ which we use to consider the universal eta-invariant as a map

$$\eta_{\mathrm{uni}} : \mathrm{Torsion}\left(\pi^{\$}_7(X_+)\right) \to \mathrm{Q}^{\mathbb{Q}}_7(X) .$$

We construct geometric cycles in $\pi^{\$}_7(MX) \cong \pi^{\$}_7(X_+)$ as follows. As manifold we take the sphere S^7 with the Riemannian structure and stable framing induced from $S^7 \subset \mathbb{R}^8$. In particular, $[S^7, S^7 \to \star] = 0 \in \pi^{\$}_7 := \pi^{\$}_7(\star_+)$. We choose a continuous map $p : S^7 \to S^4$. Since $h : A \to S^4$ is a $K(\mathbb{Q}, 3)$ fibration there exists a unique (up to homotopy) lift $\hat{p} : S^7 \to A$. This gives a unique bordism class $[S^7, \hat{\iota} \circ \hat{p}] \in \pi^{\$}_7(X)$.

We use that A and X are total spaces of $K(\mathbb{Q},3)$-fibrations over connected and simply-connected base spaces to calculate the rational cohomology via the Leray-Serre spectral sequence. As input we use

$$\mathbf{H}^*(K(\mathbb{Q},3),\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,3 \\ 0, & * = \text{ else} \end{cases} \qquad \text{and} \qquad \mathbf{H}^*(\mathbb{HP}^\infty,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,4,8,\dots \\ 0, & * = \text{ else} \end{cases}$$

and we compare with $\star \to K(\mathbb{Q},4)$ to compute the differential. The results are

$$\mathbf{H}^*(A,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0,7 \\ 0, & * = \text{ else} \end{cases} \qquad \text{and} \qquad \mathbf{H}^*(X,\mathbb{Q}) = \begin{cases} \mathbb{Q}, & * = 0 \\ 0, & * = \text{ else} \end{cases} .$$

In particular, the groups $\mathrm{U}_7^{\mathbb{R}}(S^4)$, $\mathrm{U}_7^{\mathbb{R}}(A)$ and $\mathrm{U}_7^{\mathbb{R}}(X)$ vanish and the group $\pi_7^{\$}(X)$ is a torsion group. So the universal eta-invariant is a map

$$\eta_{\mathrm{uni}} : \pi_7^{\$}(X_+) \to \mathrm{Hom}^{\mathrm{cont}}\left(\mathbf{KU}^0(X), {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}\right) .$$

We use $\mathbb{HP}^\infty \simeq BSU(2)$ and the Completion Theorem ([AS69, Theorem 2.1]) to conclude that $\mathbf{KU}^0(\mathbb{HP}^\infty) \cong \mathbb{Z}[|H|]$ where

$$H := ESU(2) \times_{SU(2)} \mathbb{C}^2 - (\mathbb{HP}^\infty \times \mathbb{C}^2) \in \mathbf{KU}^0(\mathbb{HP}^\infty)$$

has virtual rank 0 (we use the standard representation to associate a vector bundle). This gives induced K-theory classes H_X, H_{S^4} and H_A via pullback. We remark that the induced K-theory class on $S^4 = \mathbb{HP}^1$ can explicitly be described as

$$H_{S^4} = S^7 \times_{SU(2)} \mathbb{C}^2 - (S^4 \times \mathbb{C}^2) \in \mathbf{KU}^0(S^4)$$

where $\Phi : S^7 \to S^4$ is the Hopf bundle.

Next we construct a map $p_n : S^7 \to S^4$ for each $n \in \mathbb{Z}$. For this we denote the Hopf fibration by $\Phi : S^7 \to S^4$ and the degree n-map by $\deg_n : S^4 \to S^4$. Then we define p_n as the composition

$$p_n : S^7 \xrightarrow{\Phi} S^4 \xrightarrow{\deg_n} S^4 .$$

This gives bordism classes $x_n := [S^7, \hat{\iota} \circ \hat{p}_n] \in \pi_7^{\$}(X)$ for all $n \in \mathbb{Z}$.

Since $\mathrm{U}_7^{\mathbb{R}}(X)$ vanishes and since $\mathrm{Torsion}\left(\pi_7^{\$}(X_+)\right) = \pi_7^{\$}(X_+)$ we get well-defined numbers $\eta_{\mathrm{uni}}(x_n)(H_X) \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$. The remaining part of this chapter is devoted to prove

Theorem 10.1.

1. *The intrinsic formula for* $-\eta_{\mathrm{uni}}(x_n)(H_X)$ *coincides with the formula for* $t^{\mathbb{C}}_{S^7}(E_n)$.
2. *We have* $\eta_{\mathrm{uni}}(x_n)(H_X) = \varepsilon\left[\frac{n-n^2}{12}\right] \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ *where* $\varepsilon \in {}^{\mathbb{Z}}\!/\!_{24\mathbb{Z}}$ *is some generator.* ◆

Note that the first part is a reformulation of [Bun11, Proposition 5.18]. We give a quite different proof in Section 10.4.

Before we start with honest computations we do the easy cases $n = 0$ and $n = 1$. The map p_0 is null-homotopic and thus $[S^7, \hat{\iota} \circ \hat{p}_0] = 0 \in \pi_7^{\$}(X)$. Recall that $\mathbb{HP}^2$ is defined as the (homotopy) pushout

$$\begin{tikzcd} S^7 \arrow[r, hook] \arrow[d, "\Phi"] & D^8 \arrow[d] \\ S^4 \arrow[r, hook] & \mathbb{HP}^2 \end{tikzcd} .$$

So the map $\iota \circ p_1$ is also null-homotopic (because $D^8 \simeq \star$) and thus $[S^7, \hat{\iota} \circ \hat{p}_1] = 0 \in \pi_7^{\$}(X)$. We conclude that $\eta_{\mathrm{uni}}(x_0)(H_X) = \eta_{\mathrm{uni}}(x_1)(H_X) = 0 \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$.

10.3 The topological computation

In this section we compute $\eta_{\mathrm{uni}}(x_n)(H_X) = \varepsilon\left[\frac{n-n^2}{12}\right] \in \mathbb{Q}/\mathbb{Z}$.

For this we compute at first the stable homotopy groups of A and X using the (Leray-Serre-)Atiyah-Hirzebruch spectral sequence with input

$$\pi_*^{\$}(K(\mathbb{Q},3)) = \begin{cases} 0, & * \neq 3 \\ \mathbb{Q}, & * = 3 \end{cases} .$$

The result is $\pi_7^{\$}(A) \cong \pi_3^{\$} \oplus \mathbb{Q}$ and the induced map h_* is the projection, i.e.,

$$h_* = \mathrm{pr}_{\pi_3^{\$}} : \pi_7^{\$}(A) \cong \pi_3^{\$} \oplus \mathbb{Q} \twoheadrightarrow \pi_3^{\$} \cong \pi_7^{\$}(S^4) .$$

The spectral sequence also gives an exact sequence

$$\pi_0^{\$} \xrightarrow{d} \pi_7^{\$}(A) \overset{f_*}{\twoheadrightarrow} \pi_7^{\$}(X) \to 0 .$$

We decompose $\pi_7^{\$}(A) \cong \pi_3^{\$} \oplus \mathbb{Q}$ and $\pi_0^{\$} \cong \mathbb{Z}$ to identify the differential with $d = (d_1, d_2)$ for $d_1 \in \pi_3^{\$}$ and $d_2 \in \mathbb{Q}$. To compute the differential d_2 we compare with the spectral sequence for $\star \to K(\mathbb{Q},4)$ and get that $d_2 = 1$. For d_1 we compare with the spectral sequence for $\mathtt{id} : \mathbb{HP}^\infty \to \mathbb{HP}^\infty$. But this spectral sequence, in degrees up to 10, is just the long exact sequence (in stable homotopy) of the homotopy cofibre sequence $S^7 \xrightarrow{\Phi} S^4 \to \mathbb{HP}^2$. So $d_1 \in \pi_3^{\$}$ is given by the image of the Hopf map $\Phi \in \pi_7(S^4)$ in $\pi_7^{\$}(S^4) \cong \pi_3^{\$}$.

Note that the stabilization map $\Sigma^\infty : \pi_7(S^4) \to \pi_7^{\$}(S^4)$ maps the Hopf fibration to a generator of order 24 and thus gives an identification $\pi_7^{\$}(S^4) \cong \mathbb{Z}/24\mathbb{Z}$ ([Hu59, Theorem 16.4]). Next we employ that the universal eta-invariant is functorial ([Bun11, Lemma 2.4]) which gives the commutative diagram (recall that $\pi_7^{\$}(X)$ is a torsion group)

$$\begin{tikzcd}
(\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Q})/(1/24, 1)\mathbb{Z} \arrow[r, equal] & \pi_7^{\$}(X) \arrow[r, "\eta_{\mathrm{uni}}"] & \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(X), \mathbb{Q}/\mathbb{Z}) \arrow[r, "\mathrm{ev}_{H_X}"] & \mathbb{Q}/\mathbb{Z} \\
\mathbb{Z}/24\mathbb{Z} \arrow[u, hook] \arrow[r, equal] \arrow[d, equal] & \mathrm{Torsion}(\pi_7^{\$}(A)) \arrow[u, hook, "\hat\iota_*"] \arrow[r, "\eta_{\mathrm{uni}}"] \arrow[d, two heads, "h_*", "\cong"'] & Q_7^{\mathbb{Q}}(A) \arrow[u, "\hat\iota_*"] \arrow[r, "\mathrm{ev}_{H_A}"] \arrow[d, "h_*"] & \mathbb{Q}/\mathbb{Z} \arrow[u, equal] \arrow[d, equal] \\
\mathbb{Z}/24\mathbb{Z} \arrow[r, equal] & \pi_7^{\$}(S^4) \arrow[r, "\eta_{\mathrm{uni}}"] & \mathrm{Hom}^{\mathrm{cont}}(\mathbf{KU}^0(S^4), \mathbb{Q}/\mathbb{Z}) \arrow[r, "\mathrm{ev}_{H_{S^4}}"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

and we stress that it is not clear (and also wrong) that the map $\hat p_n : S^7 \to A$ represents a torsion element in $\pi_7^{\$}(A)$. Nevertheless, we will show that the image $\hat\iota_*[\hat p_n]$ is in the image of $\hat\iota_* : \mathrm{Torsion}(\pi_7^{\$}(A)) \to \pi_7^{\$}(X)$. To this end we use the composition

$$\pi_7(S^4) \xrightarrow{\iota_*} \pi_7(\mathbb{HP}^\infty) \cong \pi_7(X) \xrightarrow{\Sigma^\infty} \pi_7^{\$}(X) .$$

The long exact sequence of the Hopf fibration gives the split short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi_*(S^7) \arrow[r, hook, "\Phi_*"] & \pi_*(S^4) \arrow[r, two heads, "\partial"] & \pi_{*-1}(S^3) \arrow[r] \arrow[l, bend right, dashed, "\Sigma"'] & 0
\end{tikzcd}$$

and thus that the suspension map $\Sigma : \pi_{*-1}(S^3) \hookrightarrow \pi_*(S^4)$ is injective. Also recall that $\pi_6(S^3)$ is isomorphic to[2] $\mathbb{Z}/12\mathbb{Z}$ and therefore that $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$. We compare with the long

[2]We use the generator given in [Hu59, Chapter XI.16].

exact sequence of the universal bundle $ESU(2) \to BSU(2)$ and get a factorization

$$\begin{tikzcd} \pi_7(S^4) \arrow[r, two heads, "\partial"] \arrow[rr, bend left, dashed, "\iota_*"] & \pi_6(S^3) \arrow[r, "\cong"] & \pi_7(\mathbb{H}\mathrm{P}^\infty) \end{tikzcd} .$$

This gives the equation $\hat{\iota}_*[\hat{p}_n] = \hat{\iota}_* \left[\widehat{\Sigma \circ \partial(p_n)}\right]$ where $\left[\widehat{\Sigma \circ \partial(p_n)}\right] \in \pi_7^\$(A)$ is torsion since $\pi_6(S^3)$ is a torsion group. We conclude

$$\begin{aligned} \eta_{\mathrm{uni}}(x_n)(H_X) &= \eta_{\mathrm{uni}}(\hat{\iota}_*[\hat{p}_n])(H_X) = \eta_{\mathrm{uni}}(\hat{\iota}_*[\widehat{\Sigma \circ \partial(p_n)}])(H_X) = \\ &= \eta_{\mathrm{uni}}([\widehat{\Sigma \circ \partial(p_n)}])(H_A) = \eta_{\mathrm{uni}}(h_*[\widehat{\Sigma \circ \partial(p_n)}])(H_{S^4}) = \\ &= \eta_{\mathrm{uni}}([\Sigma \circ \partial(p_n)])(H_{S^4}) = \eta_{\mathrm{uni}}([\Sigma \circ \partial(\deg_n \circ \Phi)])(H_{S^4}) \ . \end{aligned}$$

The universal eta-invariant for S^4 together with the trivial map to $BSpin^c$ coincides with Adams e-invariant (cf. [Bun11, Chapter 5.1] and Chapter 9) and we conclude from [Ada66, Theorem 7.15] that (we get a factor two since we use the complex version of Adams e-invariant)

$$\begin{aligned} \eta_{\mathrm{uni}}(-)(H_{X_{S^4}}) : \mathbb{Z}/24\mathbb{Z} &\to \mathbb{Q}/\mathbb{Z} \\ \left[\frac{1}{24}\right] &\mapsto \varepsilon \left[\frac{2}{24}\right] \ . \end{aligned}$$

where $\varepsilon \in \mathbb{Z}/24\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ is some generator. So we have to identify the image of $[\Sigma \circ \partial(\deg_n \circ \Phi)] \in \pi_7(S^4)$ in $\pi_7^\$(S^4)$.

Using the decomposition $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$ the suspension map is given by ([Hu59, Theorem 16.4])

$$\begin{aligned} \Sigma^\infty : \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} &\to \mathbb{Z}/24\mathbb{Z} \\ \left(a, \left[\frac{b}{12}\right]\right) &\mapsto \left[\frac{a+2b}{24}\right] \ . \end{aligned}$$

To study the degree n map $\deg_n$ we use the Hopf invariant. It is a group homomorphism $\mathcal{H} : \pi_7(S^4) \to \mathbb{Z}$ which maps $[\Phi]$ to 1 and which satisfies $\mathcal{H}([deg_n \circ p]) = n^2\mathcal{H}(p)$ (the latter property follows immediately from the definition). This gives the commutative diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "n^2\cdot"] & \pi_7(S^4) \arrow[l, "\mathcal{H}"'] \arrow[r, "\Sigma^\infty"] \arrow[d, "\deg_n"] & \pi_7^\$(S^4) \arrow[d, "\deg_n"] \\ \mathbb{Z} & \pi_7(S^4) \arrow[l, "\mathcal{H}"'] \arrow[r, "\Sigma^\infty"] & \pi_7^\$(S^4) \end{tikzcd}$$

and since stably $\deg_n \circ(-) = n \cdot (-)$ we have

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "n^2\cdot"] & \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \arrow[l, "\mathrm{pr}_\mathbb{Z}"'] \arrow[r, "\Sigma^\infty"] \arrow[d, "\begin{pmatrix} \alpha_n & 0 \\ \beta_n & \gamma_n \end{pmatrix}"] & \mathbb{Z}/24\mathbb{Z} \arrow[d, "n\cdot"] \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \arrow[l, "\mathrm{pr}_\mathbb{Z}"'] \arrow[r, "\Sigma^\infty"] & \mathbb{Z}/24\mathbb{Z} \end{tikzcd} .$$

Since we know Σ^∞ explicit we get $\alpha_n = n^2$, $\gamma_n = \left[\frac{n}{12}\right]$ and $\beta_n = \left[\frac{n-n^2}{12}\right]$. This gives that $[\deg_n \circ \Phi] \in \pi_7(S^4)$ corresponds to $\left(n^2, \left[\frac{n-n^2}{12}\right]\right) \in \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$. Hence $[\Sigma \circ \partial(\deg_n \circ \Phi)] \in \pi_7(S^4)$

corresponds to $\left(0, \left[\frac{n-n^2}{12}\right]\right) \in \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$ and $\Sigma^\infty([\Sigma \circ \partial(\deg_n \circ \Phi)]) \in \pi_7^{\$}(S^4)$ corresponds to $\left[\frac{n-n^2}{24}\right] \in \mathbb{Z}/24\mathbb{Z}$.

At the end of the day we arrive at $\eta_{\mathrm{uni}}(x_n)(H_X) = \varepsilon \left[\frac{n-n^2}{12}\right] \in \mathbb{Z}/24\mathbb{Z}$. Note that this fits with our previous computations for $n = 0, 1$. So we have proven Theorem 10.1, 2.

Remark 10.2. We remark that the result is $\varepsilon \left[\frac{n-n^2}{24}\right] \in \mathbb{Z}/24\mathbb{Z}$ if we do the computation with real K-theory instead of complex K-theory. ■

10.4 Comparing the t-invariant with the universal eta-invariant

In this section we compare the formula for the complex t-invariant,

$$t^{\mathbb{C}}_{S^7}(E_n) := \xi(\not{D}^{\mathbf{V}\mathcal{E}_n}_{S^7}) - \left[\int_{S^7} \mathbf{Td}_{S^7} \wedge \hat{c}_2(\nabla^{\mathbf{V}\mathcal{E}_n}) \wedge \left(1 - \frac{1}{12} c_2(\nabla^{\mathbf{V}\mathcal{E}_n})\right)\right]_{\mathbb{R}/\mathbb{Z}} \in \mathbb{Q}/\mathbb{Z} \ ,$$

with the intrinsic formula of the universal eta-invariant (cf. Proposition 6.25)

$$\eta_{\mathrm{uni}}(x_n)(H_X) = \eta_{\mathrm{uni}}(\hat{\iota}_*[\hat{p}_n])(H_X) = -\left[\int_{S^7} \mathbf{Td}_{S^7} \wedge \gamma_{H_X}\right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}^{\mathbf{V}_{H_X}}) \in \mathbb{Q}/\mathbb{Z} \ .$$

This is also discussed in [Bun11, Chapter 5.5] but we give a different perspective. For example, in our proof continuity is automatic. On the other hand, Bunke has to check this by hand

Note that there exists a unique equivalence class of universal *Spin*^{c}-geometrization for X by Theorem 7.11. In the following we construct such a universal *Spin*^{c}-geometrization explicitly.

To this end we need an approximation by manifolds for X. Consider the spaces X_l, $l \in \mathbb{N}$, which are defined as homotopy pullbacks,

$$\begin{tikzcd} X_l \arrow[r] \arrow[d, "\pi_l"] \arrow[dr, phantom, "\lrcorner"] & K(\mathbb{Z}/l\mathbb{Z}, 3) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \star \arrow[d] \\ \mathbb{H}\mathrm{P}^\infty \arrow[r, "c_2"] & K(\mathbb{Z}, 4) \arrow[r, "l\cdot"] & K(\mathbb{Z}, 4) \end{tikzcd} .$$

Note that $\pi_l : X_l \to \mathbb{H}\mathrm{P}^\infty$ is a $K(\mathbb{Z}, 3)$-fibration. By construction we have maps $X_l \to X$. We order $\mathbb{N}$ by divisibility and get $\mathrm{hocolim}_{l\in\mathbb{N}} X_l \simeq X$. Since $l \cdot c_2 : \mathbb{H}\mathrm{P}^\infty \to K(\mathbb{Z}, 4)$ is a rationally isomorphism the space X_l is rational trivial for all $l \in \mathbb{N}$. Also note that the spaces X_l are connected, simply-connected and have degree-wise finitely generated homotopy groups. So we can assume that each X_l is a CW-complex with finite skeletons $(X^i_l)_{i\in\mathbb{N}}$. Denote a suitable diagonal sequence of $(X^i_l)_{l,i\in\mathbb{N}}$ by $(\tilde{B}_i)_{i\in\mathbb{N}}$. We replace the latter system by compact smooth manifolds and neat embeddings (see the proof of [BS10, Proposition 2.1]) to get an approximation by manifolds of X, denoted as $(B_i, f_i, g_i)_{i\in\mathbb{N}}$. Since all the spaces X_l are rationally trivial we can assume that $\mathbf{H}^{\leq 4}(B_i, \mathbb{Q}) = 0$ for all $i \in \mathbb{N}$ by discarding the low-degree skeletons.

Next we construct a $(p, \mathbf{Td}_M, 1)$-geometrization $\mathcal{G}_p$ for any map $p : M \to X$ together with $\mathbf{Td}_M \in \Omega\mathrm{P}^0_{\mathrm{cl}}(M)$. Note that there is a unique cohomological character since $\widetilde{\mathbf{HP}\mathbb{Q}}^0(X) = 0$

(by Theorem 3.5). We choose a base point $\star \in X$ and get a splitting $\mathbf{KU}^0(X) \cong \mathbb{Z} \oplus \widetilde{\mathbf{KU}}^0(X)$. Since $\mathbb{Z}$ is discrete and free we easily define the geometrization on this summand. So we restrict to reduced complex K-theory. For this we apply the long exact sequence

$$\ldots \to \mathbf{KU}^{-1}(X) \to \mathbf{KU}\,\mathbb{Q}^{-1}(X) \to \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(X) \to \mathbf{KU}^0(X) \to \mathbf{KU}\,\mathbb{Q}^0(X) \to \ldots$$

and that $\widetilde{\mathbf{KU}\,\mathbb{Q}}^*(X) \overset{\mathbf{ch}}{\cong} \widetilde{\mathbf{HP}\mathbb{Q}}^*(X) = 0$. This gives $\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(X) \xrightarrow{\cong} \widetilde{\mathbf{KU}}^0(X)$. We define a continuous map $\mathcal{G}_p : \mathbf{KU}^0(X) \to \widehat{\mathbf{KU}^0}(M)$ as the composition

$$\mathcal{G}_p : \widetilde{\mathbf{KU}}^0(X) \cong \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(X) \xrightarrow{p^*} \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(M) \hookrightarrow \widehat{\mathbf{KU}^0}(M)$$

which satisfies $R \circ \mathcal{G}_p = 0$ and thus is a geometrization. Note that this construction is natural in $p : M \to B$: Given a smooth map $f : M \to N$ we have $f^*\mathcal{G}_p = \mathcal{G}_{f \circ p}$. In particular, homotopic maps give the same pullback geometrizations. We choose arbitrary abstract stable $Spin^c$-structures on the approximation[3] and note that $(f_i)^!_{\delta_i}(\mathcal{G}_{g_{i+1}}) = \mathcal{G}_{g_i}$ for all $i \in \mathbb{N}$ since $R \circ \mathcal{G}_p = 0$ (see formula (3.3)). This finishes the construction of the universal $Spin^c$-geometrization for X.

We take the pullback of the universal $SU(2)$-principal bundle to B_i along the maps $B_i \xrightarrow{g_i} X \xrightarrow{\pi} \mathbb{H}\mathrm{P}^\infty \simeq BSU(2)$. By the discussion of Section 1.1 we can assume that each of these bundles $E_i \to B_i$ is smooth and has a connection ∇^{E_i} and that these are compatible with the maps $f_i : B_i \to B_{i+1}$. We associate along the standard representation of $SU(2)$ and get geometric vector bundles $\mathcal{E}_i = (E_i \times_{SU(2)} \mathbb{C}^2 \to B_i, h^{\mathcal{E}_i}, \nabla^{\mathcal{E}_i})$. Next we study the Chern character $\mathbf{ch}(\nabla^{\mathcal{E}_i})$. Since the bundles are constructed via pullback from X and X is rationally trivial the Chern character vanishes, $\mathbf{ch}(\nabla^{\mathcal{E}_i}) = 0$. In particular, the second Chern class $c_2(\nabla^{\mathcal{E}_i}) \in \Omega^4_{\mathrm{cl}}(B_i)$ is exact. But $\mathbf{H}^3(B_i, \mathbb{Q}) = 0$ and so there exist unique forms $\hat{c}_2(\nabla^{\mathcal{E}_i}) \in \Omega^3(B_i)/(\operatorname{im} d)$ for all $i \in \mathbb{N}$ such that $d\hat{c}_2(\nabla^{\mathcal{E}_i}) = c_2(\nabla^{\mathcal{E}_i})$.

The Chern character $\mathbf{ch}(ESU(2) \times_{SU(2)} \mathbb{C}^2) \in \mathbf{H}^*(\mathbb{H}\mathrm{P}^\infty, \mathbb{Q}) \cong \mathbb{Q}[c_2]$ is a polynomial in the second Chern class with constant coefficient $2 = \operatorname{rank}(ESU(2) \times_{SU(2)} \mathbb{C}^2)$. So there exists a unique polynomial $\widetilde{\mathbf{ch}} \in \mathbb{Q}[c_2]$ such that $c_2 \cdot \widetilde{\mathbf{ch}} = \mathbf{ch}(ESU(2) \times_{SU(2)} \mathbb{C}^2) - 2$. The first terms are

$$\widetilde{\mathbf{ch}} = -1 + \frac{1}{12}c_2 - \frac{1}{60}c_2^2 + \ldots\ .$$

We define $\widetilde{\mathbf{ch}}(\nabla^{\mathcal{E}_i}) \in \Omega^*_{\mathrm{cl}}(B_i)$ by replacing c_2 with $c_2(\nabla^{\mathcal{E}_i})$ in the polynomial $\widetilde{\mathbf{ch}}$. Then $d\hat{c}_2(\nabla^{\mathcal{E}_i}) \wedge \widetilde{\mathbf{ch}}(\nabla^{\mathcal{E}_i}) = \mathbf{ch}(\nabla^{\mathcal{E}_i}) - 2$. Recall that $\underline{\mathbb{C}^2}$ denotes the trivial geometric vector bundle of rank 2. We set $\mathbf{V}_i := \mathcal{E}_i - \underline{\mathbb{C}^2} \in \widehat{\mathbf{KU}^0}(B_i)$. Note that $I(\mathbf{V}_i) = H_{B_i} := g_i^*(H_X)$ and $d\hat{c}_2(\nabla^{\mathbf{V}_i}) \wedge \widetilde{\mathbf{ch}}(\nabla^{\mathbf{V}_i}) = \mathbf{ch}(\nabla^{\mathbf{V}_i})$. This gives flat differential K-theory classes $\mathbf{W}_i := \mathbf{V}_i - a\left(\hat{c}_2(\nabla^{\mathbf{V}_i}) \wedge \widetilde{\mathbf{ch}}(\nabla^{\mathbf{V}_i})\right) \in \widehat{\mathbf{KU}^0}(B_i)$ which are compatible, i.e., satisfy $f_i^*\,\mathbf{W}_{i+1} = \mathbf{W}_i$ (by uniqueness of $\hat{c}_2(\nabla^{\mathcal{E}_i})$). In particular, $\mathbf{W}_i \in \mathbf{KU}\,{}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}}{}^{-1}(B_i)$. So we get $(\mathbf{W}_i)_{i\in\mathbb{N}} \in \varprojlim_{i\in\mathbb{N}} \mathbf{KU}\,{}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}}{}^{-1}(B_i)$. Also observe that $I(\mathcal{G}_{g_i}(H_X)) = g_i^* H_X = I(\mathbf{W}_i)$.

Note that the topological abelian groups $\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!{}_{\mathbb{Z}}{}^{-1}(Z)$ and $\mathbf{KU}\,{}^{\mathbb{R}}\!/\!{}_{\mathbb{Z}}{}^{-1}(Z)$ are Hausdorff and complete for all spaces Z. To see this we combine Proposition 2.2, [Yos75, Proposition 5], [HM78, Theorem 1.1(2)] and [Sch03, Proposition 5.6]. Moreover, the Bockstein sequences

[3]To be explicit we choose the trivial $Spin^c(0)$-bundles as abstract stable $Spin^c$-structures. Then the error forms δ_i vanish. We set $\mathcal{G}_{g_i}(X \times \mathbb{C} \xrightarrow{\mathrm{pr}_X} X) := [B_i \times \mathbb{C}, h^{\mathrm{triv}}, \nabla^{\mathrm{triv}}]$ on the $\mathbb{Z}$-summand. This is compatible with the transition maps $f_i : B_i \to B_{i+1}$. We remark that the geometrization $\mathcal{G}_{S^7}$ (see below) has a less trivial value on the trivial bundle.

for $\mathbb{Z} \to \mathbb{Q} \to {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ and for $\mathbb{Z} \to \mathbb{R} \to {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$ give an isomorphism $\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{-1}(X) \xrightarrow{\cong} \mathbf{KU}\,{}^{\mathbb{R}}\!/\!_{\mathbb{Z}}{}^{-1}(X)$ because X is rationally trivial. This gives $(\mathbf{W}_i)_{i\in\mathbb{N}} \in \mathbf{KU}\,{}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{-1}(X)$. But, $(\mathbf{W}_i)_{i\in\mathbb{N}}$ corresponds to H_X under the isomorphism $\mathbf{KU}\,{}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}{}^{-1}(X) \xrightarrow{\cong} \widetilde{\mathbf{KU}}^0(X)$. This implies that $\mathcal{G}_{g_i}(H_X) = \mathbf{W}_i \in \widehat{\mathbf{KU}^0}(B_i)$ for all $i \in \mathbb{N}$.

Now we evaluate the intrinsic formula for the universal eta-invariant. So we choose a smooth lift $\hat{p}^i_n : S^7 \to B_i$ of $\hat{\imath} \circ \hat{p}_n : S^7 \to X$ with error form δ_{S^7} (cf. Example 6.20). Then we get a geometrization[4] $\mathcal{G}_{S^7} := (\hat{p}^i_n)^!_{\delta_{S^7}}(\mathcal{G}_{g_i}) = \mathcal{G}_{\hat{p}^i_n}$. To compute a suitable correction form $\gamma_{H_X} \in \Omega\mathrm{P}^{-1}(S^7)/(\operatorname{im} d)$ we need a geometric vector bundle $\mathbf{V}_{H_X} \to S^7$ representing $(\hat{\imath} \circ \hat{p}_n)^* H_X \in \widetilde{\mathbf{KU}}^0(S^7)$. We choose $\mathbf{V}_{H_X} := (\hat{p}^i_n)^*\,\mathbf{V}_i$. Then the correction form satisfies

$$\begin{aligned}
a(\gamma_{H_X}) =& \mathcal{G}_{S^7}(H_X) - [\mathbf{V}_{H_X}] = (\hat{p}^i_n)^*[\mathbf{W}_i] - (\hat{p}^i_n)^*[\mathbf{V}_i] = \\
=& (\hat{p}^i_n)^* \left([\mathbf{V}_i] - a\left(\hat{c}_2(\nabla^{\mathbf{V}_i}) \wedge \widetilde{\mathbf{ch}}(\nabla^{\mathbf{V}_i}) \right) - [\mathbf{V}_i] \right) = \\
=& - (\hat{p}^i_n)^* a\left(\hat{c}_2(\nabla^{\mathbf{V}_i}) \wedge \widetilde{\mathbf{ch}}(\nabla^{\mathbf{V}_i}) \right) = \\
=& - a\left(\hat{c}_2(\nabla^{\mathbf{V}_{H_X}}) \wedge \widetilde{\mathbf{ch}}(\nabla^{\mathbf{V}_{H_X}}) \right) = \\
=& a\left(\hat{c}_2(\nabla^{\mathbf{V}_{H_X}}) \wedge (1 - \frac{1}{12} c_2(\nabla^{\mathbf{V}_{H_X}})) \right) .
\end{aligned}$$

So we take $\gamma_{H_X} := \hat{c}_2(\nabla^{\mathbf{V}_{H_X}}) \wedge (1 - \frac{1}{12} c_2^2(\nabla^{\mathbf{V}_{H_X}})) \in \Omega\mathrm{P}^{-1}(S^7)/(\operatorname{im} d)$. Therefore, the intrinsic formula for the universal eta-invariant is

$$\eta_{\mathrm{uni}}(x_n)(H_X) = -\left[\int_{S^7} \mathbf{Td}_{S^7} \wedge \hat{c}_2(\nabla^{\mathbf{V}_{H_X}}) \wedge (1 - \frac{1}{12} c_2(\nabla^{\mathbf{V}_{H_X}})) \right]_{\mathbb{R}/\mathbb{Z}} - \xi(\not{D}^{\mathbf{V}_{H_X}}) \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}} .$$

To evaluate the formula for the t-invariant we use the $SU(2)$-principal bundle $E_n := (\hat{p}^i_n)^* E_i$ together with the connection $(\hat{p}^i_n)^* \nabla^{E_i}$. Then we get $\mathbf{V}_{\mathcal{E}_n} = \mathbf{V}_{H_X}$ and thus

$$t^{\mathbb{C}}_{S^7}(E_n) := \xi(\not{D}^{\mathbf{V}_{H_X}}_{S^7}) + \left[\int_{S^7} \mathbf{Td}_{S^7} \wedge \hat{c}_2(\nabla^{\mathbf{V}_{H_X}}) \wedge \left(1 - \frac{1}{12} c_2(\nabla^{\mathbf{V}_{H_X}}) \right) \right]_{\mathbb{R}/\mathbb{Z}} \in {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}} .$$

So we have proven Theorem 10.1, 1.

Remark 10.3. Note that all arguments work fine with real K-theory instead of complex K-theory. Also, the definitions of cohomological characters and (universal) geometrizations generalize immediately to real K-theory. So we believe that the Secondary Index Theorem [Bun11, Theorem 3.6] holds for real K-theory. This would allow us to get rid of the factor 2 in Theorem 10.1. ■

[4] Note that the equality $(\hat{p}^i_n)^!_{\delta_{S^7}}(\mathcal{G}_{g_i}) = \mathcal{G}_{\hat{p}^i_n}$ only holds on the reduced part $\widetilde{\mathbf{KU}}^0(X)$.

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