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Function Follows Form

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– Regensburg, June 5th, 2015 –

Kristof Dascher

Abstract: Urban policy visibly molds city shape. This paper's interest is in how city shape (less visibly) molds urban policy. The paper finds: A sufficiently skewed city will look after its center. That is, the more skewed a city's shape towards the city periphery, the more likely an urban majority against any policy that could take away from the city center. This, when broadly interpreted, complements Sullivan's (1896) "form follows function" view prominent in architectural theory. Function (building uses) also follows form (building contours). Ultimately combining both views may help explain further how, and when, cities sprawl.

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1 Introduction

There can be no doubt that urban policy molds city shape. Oddly, when commenting on Paris' urban development in the magazine *The New Yorker*, Alexandre Gady, a historian of Paris, appears to suggest the reverse:

> "Paris – it's beautiful. But it's a doll's house! And that's one reason the Parisian élite is so conservative. They live in the doll's house. . . . The blindness of the élites is to reproduce a model of returning to the center, always back to the center . . . " (as quoted by Gopnik (2014))

Gady argues that city form or shape molds urban policy. This paper's interest is in the urban political economy of precisely this shape-to-policy connection. We will replace the example of Paris with a monocentric city model, the notion of "doll's house" with the city's shape, and the claim of "élites' blindness" with an unbiased analysis of landlords' conscious pursuit of their interests. Then we will credit the above quote with containing more than just a grain of truth.

A city's physical shape places many constraints on urban political equilibrium. These constraints, not all of them obvious, can be brought to light by literally taking snapshots of suitable sections of the city's shape. These constraints can even be shown to vary with a simple index of the city's physical form, i.e. the city shape's skew. A city's skew puts bounds around resident interests. For example, if city skew is strong (weak) enough then a majority willing to hold on to (to shed) the traditional center is inevitable. In that sense city shape shapes urban policy. Putting fundamental urban policy decisions down to city shape, rather than tracing city shape back to urban policies, is this paper's theme – even as ultimately we must reckon with a circular relationship between city shape and urban policy.

Incidentally, the idea of reading restrictions on a city's various political interests off its shape complements a view prominent in architectural theory, originally put forward by Louis Sullivan. According to Sullivan (1896), ". . . it is the pervading law of all things organic and inorganic, . . . that the life is recognizable in its expression". Among architectural theorists, Sullivan's theme is known as the "form ever follows function"-view. Our observing the built environment impact on the polity's decisions instead provides an explanation of how: *function follows form*. I.e., if city shape drives the location of employment and retail, then city form (buildings' shapes) indeed impacts on city functions (buildings' retail, office, or residential uses).

In the urban economics literature, analyzing the causality from city policy to city shape has always been a prime concern (e.g. Brueckner (2005), Bento/Franco/Kaffine (2006), Baum-Snow (2006), de Lara et al. (2013) or Ushchev/Thisse/Sloev (2015) more recently). A general analysis of the causality from shape to policy appears to have attracted much less attention – even as specific aspects such as the role of compactness for the viability of urban public transport have been of interest (e.g. Bertaud (2003)). In part this is for

Figure 1: Munich in a 1761 painting by Canaletto

good reason. A young city's policy is unlikely to be informed by its shape. But for any mature city this seems more difficult to justify. Surely policy adapts much quicker to its surrounding physical structures than these structures do to policy? Once the city has developed its shape – subject not just to early policy but also to numerous other forces of history (e.g. hinterland safety, political institutions, past zoning) and geography (e.g. topography and soil texture) – policy should (also) be expected to follow shape.

The specific policy that is the paper's focus is the decision on whether to build a ring road. But this decision seems fundamental enough. Ignore the expense of road construction, the likely staggered development of ring road sites and the remaining cost of travelling that road. On the one hand, a ring road must create a strong rival to the traditional Central Business District (CBD). Locations along the ring road connect almost as well to one another as locations in the traditional core do. Ring road locations, moreover, and much unlike those traditional CBD locations, also serve land out of reach prior to the ring road's construction. Urban policy can clearly remake city shape. On the other hand, city shape surely also feeds into the city's decision on the ring road. A city characterized by substantial sprawl already, say, is more likely to build that road.

There are many opportunities for applying the paper's theory to urban history. We briefly sketch one of these. The 18th century painter Canaletto was known for his many detailed, even realistic cityscapes of Dresden, Warsaw and London, among others, or for his "vedutas" by art historians' terminology (e.g. Links (1999), Roeck (2004)). Figure 1, for example, shows Canaletto's 1761 veduta of Munich. The city silhouette, clearly recognizable in the painting's background, could carefully be transformed into a historical "commuting distribution", exploiting the fact that the silhouette is related to how population per unit of land varies with distance to the CBD. When entered into this paper's theory, this distribution generates policy predictions that could be backtested against the observable policies subsequently taken.

In short, to "see" (an image of) a city may even mean to "understand" that city. While city silhouettes represent an interesting field by itself (e.g. Baranow (1980)), city morphology may have uses that go beyond the descriptive.

The paper comes in five sections. Section 2 identifies bounds on resident landlord interests. This section connects the city's "physical sphere" with its "political sphere". Section 3 bounds these bounds when relating them to a novel yet also simple indicator of city shape skew. Section 3 ties the city's "political sphere" down to its "visual appearance". It is sections 2 and 3 that set out the paper's two propositions. Proposition 1 introduces two lower bounds on urban voter shares that (i) can be derived for *any* city and (ii) are almost always *useful* nonetheless. Proposition 2 shows how these two lower bounds on the political decision of where to locate traditional central city functions also follow the skew of urban form: *function follows form.* Both propositions we obtain even as the underlying model city structure remains very general. Section 4 points to various implications. Conclusions are found in section 5.

2 The Physical and the Political

Our model city's primitives are few. The city is monocentric and circular, and extends $\widetilde{r}$ units of distance out from its center. Every resident occupies one apartment, and commutes to the city center (CBD) to work and shop. Round trip commuting costs for a resident at distance r are tr. The city is closed. Its population is s, and set equal to 1. Assuming the city to be closed stacks the model's odds against relocating the CBD to the periphery ever. No additional residents can be hoped for when pushing the CBD out.

There is no agricultural hinterland. Apartments are inherited from the past, and owned by landlords who are resident, rather than absenteee. (Arguably landlords cannot both be influential and absentee.) Each landlord owns (no housing or land other than) two apartments located anywhere in the city. One of these apartments he occupies himself, the other he rents out to his tenant. Reflecting this paper's interest in the (immediate) effect of shape on policy, we ignore the (delayed) effect of policy on shape that is the dominant concern of the policy-drives-shape literature. Thus there is no apartment construction.

Following Arnott/Stiglitz (1981), we equate the city's shape with its commuting distribution, $F(r)$, such that $f(r) = F'(r)$ approximates the share of households inhabiting the unit-width ring at distance r. If $a(r)$ denotes available land and $d(r)$ captures population density at distance r then commuting density $f(r)$ may be rewritten as $f(r) = a(r)d(r)$. On the one hand, $d(r)$ surely is decreasing in r (greater distance to the CBD has rent fall, and hence buildings become smaller (Brueckner (1987))). On the other hand, in our circular city at least, land $a(r)$ surely is increasing in r. Cities differ in their topography, and hence differ in $a(r)$ at least.[1] Hence to keep the analysis as general as possible

[1] To illustrate the importance of topography, $a(r)$ might partly be constant in r in peninsulas New York or Mumbai, while for non-coastal Paris or Berlin $a(r)$ is more likely to be increasing in r. We will return to this issue at the end of section 4.

and allow for any conceivable city shape, no restriction is placed on how $f(r)$ varies with r. Commuting density may be increasing in r, or may be decreasing in r, or may even alternate between being increasing and decreasing.

Let the numbers of landlords and tenants living between 0 and r (with $r \in [0, \tilde{r}]$) be denoted by unknown c.d.f. $L(r)$ and $M(r)$, respectively. Reflecting (i) a deficit of information, (ii) the desire to start with a city model that is as general as possible and (iii) the assumption of residents being perfectly mobile across the city, we impose no restriction on the specific location that landlords and tenants inhabit other than that landlords and tenants within each ring add up, i.e. $L'(r) + M'(r) = f(r)$.

Partition the city into n equidistant rings each of width $\tilde{r}/n$ and extending from 0 out to $\tilde{r}$, such that ring i's outer annulus shows up at distance $r_i = (i/n)\,\tilde{r}$. Then the numbers of landlords, tenants, and apartments in ring i can be approximated by $L'(r_i) \equiv l_i$, $M'(r_i) \equiv m_i$, and $f(r_i) \equiv s_i$, respectively, so that $l_i + m_i = s_i$. As long as each ring is "wide", within-ring travel is assumed costless. This assumption no longer is necessary once we shrink ring width, below.

If $\tilde{r}$ denotes the city boundary then Ricardian rent at any location $r_j \leq \tilde{r}$, denoted $q(r_j)$, equals the commuting cost savings that that location makes possible vis-à-vis living at the boundary. So $q(r_j) = t(\tilde{r} - r_j)$. Imagine a landlord who resides in ring i himself yet rents out his extra, second property in ring j. This landlord's property portfolio implies a sum of commuting costs and rental income equal to $-tr_i + q(r_j)$. Let ω subsume any other benefit common to all landlords, such as the wage or some local public good that does not distance-decay. Then landlord utility is $\omega + t(\tilde{r} - r_i - r_j)$. Landlord utility is independent of whether the landlord resides in i and his tenant in j, or vice versa, and so we always will conveniently put the landlord into that one of his two properties that suits our exposition best.[2]

A ring road is proposed to the citizenry that would shift the city's center of attraction from its traditional location (the CBD) out to the urban boundary $\tilde{r}$ (the ring road), in a single instant and with t unchanged.[3] Instead of travelling to the center of the city in order to work and shop, every resident now commutes to the city's periphery to work and shop, in those office parks and shopping malls strewn along the ring road that permits its users to circle the city on it at no cost. Implementing a ring road surely must be one of the most fundamental policy decisions a city can possibly make. Vienna is one early and prominent example of a ring road decision taken. In fact its ring road is even referred to as the "Ring". In Vienna, "…characteristics of the different parts of the old, inner city were projected into the Ring" (Girouard (1989)).[4]

[2] If – within-city-mobile – tenants are indifferent across locations, so are – within-city-mobile – landlords. Neither do landlords benefit from exchanging their apartment with that of their respective tenant, nor do they benefit from renting an apartment themselves in order to receive rental incomes from both their properties.

[3] Shifting jobs from the CBD to the city periphery could also take place in a staggered fashion. Rauch (1993) points to the importance of business park developers in coordinating such successive job shifts, while the shopping center industry attests to the importance of retail space developers in coordinating movements in retail (e.g. Brueckner (1993)).

[4] According to Girouard, important city functions (i.e. substantial employment) subsequently moved

Note that tenants must be indifferent to this proposal. Their cost of living, or $tr + q(r)$, remains $t\tilde{r}$. And so it is only landlords who will vote.[5] Landlord-voters are divided over which decision to take, depending on their specific portfolio location (unknown to us). Generally, instead of commuting r_i all residents living at that distance from the CBD now commute $\tilde{r} - r_i$. Landlord utility becomes $\omega - t(\tilde{r} - r_j - r_i)$ now instead of $\omega + t(\tilde{r} - r_j - r_i)$ before. The change in utility is $2t(r_i + r_j - \tilde{r})$.[6] This expression is strictly negative if $r_i + r_j < \tilde{r}$ or

$$j \ < \ n \ + \ 1 \ - \ i. \tag{1}$$

Landlords whose property location indices satisfy inequality (1) will oppose the CBD's displacement (landlord opponents); all the other landlords can be counted on to support it (landlord proponents) as soon as indifferent landlords become negligible (which they do soon).[7]

Replacing the CBD with a string of shopping districts and office parks along the ring road is one specific, and one radical, urban transport policy scenario. Nonetheless it is meant to speak up for a wider array of urban policies. We could also, for example, have considered a marginal increase in the commuting cost parameter t as brought about by a tax on commuting (possibly even representing a carbon tax), or simply a neglect of radial roads. Suppose that tax revenues are wasted and that tenant interests could be disregarded. Taking the first derivative of the landlord utility introduced above with respect to t gives $\tilde{r} - r_i - r_j$. Thus a landlord votes *for* the tax on commuting precisely if $i + j < n + 1$. But this just repeats the familiar condition (1).

Much as we would like to add up those landlords who oppose, or support, the policy proposal, computing their totals requires prior knowledge of landlord-tenant matches. As emphasized, we do not want to count on having this information. And yet, easily accessible data on ring apartment totals s_i may be used to define lower bounds on, and hence confidence in, landlord opposition to, or support of, the policy (Proposition 1 later). It may be helpful to emphasize that the lower bounds constructed below are never meant to be *greatest* lower bounds. Paying close attention to the details of a – specific – city shape will often permit to compute lower bounds that go further than ours. Even so our

to the Ring. That's where now one could find "... the new exchange, ..., the university, ... a civic and national government section around the new town hall and parliament house, ... a museums section, ... the opera house", etc. Incidentally Victor Gruen, the inventor of the modern US style shopping mall, appears to have modeled his mall on Vienna's ring, even if "... in thinking that he could reenact the lesson of the Ringstrasse in American suburbia he was wrong" (Gladwell (2004)).

[5] An alternative (somewhat less convincing, less broadly applicable) motivation for counting landlord votes only comes from pointing to historical cities which let only those cast their vote who paid enough tax. One example is the census vote governing Prussian elections up to 1918.

[6] We have assumed that t always remains the same. If t rose as a by-product of the ring road this would actually further *magnify* landlords' implied utility changes. It would, however, not affect the threshold dividing landlord proponents and opponents. (Of course, tenants would no longer be indifferent because their cost-of-living $t\tilde{r}$ would rise.)

[7] The city is the only one (or at least expects to be the only one) to build its ring road. Then it becomes open, instead of remaining closed (Brueckner (1987)), attracting immigrants to the new urban quarters developed the ring road. In this model, selling apartments in these newly developed quarters will however not benefit our resident landlords' incomes because (as we have assumed above) these resident landlords are not the owners of the city's hinterland.

lower bounds will be useful: They can be computed for any city for which f is known, typically at least one of them must bind, and they may be estimated (as discussed in the following section) simply by calculating the city shape's skewness.

Of course, we cannot derive landlord opposition to the ring road for the given yet unknown landlord portfolios nesting into the city's shape ("shape-driven portfolios, endogenous opposition"). Yet we *can* derive those landlord portfolios consistent with the exogenous city shape that generate minimum landlord opposition to the ring road ("shape-driven opposition, endogenous portfolios"). First we identify those portfolios that give rise to the weakest conceivable opposition for each suitable "snapshot" of the city's shape. Then we select the very "snapshot" that reveals the largest of all these minimum opposition figures. Next we study the effect changing an index of city shape (i.e. the city's skewness) has for this largest minimum resistance. And at last we look for shapes robust enough to withstand (alternatively, incapable of resisting) the ring road proposal's temptation.

So consider apartments in the first ring, s_1, first. All of these are tied up in matches that point to landlords who suffer from the ring road – with the exception of those matches involving a tenant in ring n. Put differently, matches involving apartments both in rings $i = 1$ and $j = n$ fail necessary condition (1). Thus s_1 is a good first lower bound on landlord opponents were it not for the fact that every resident in ring n could be tenant to a landlord in ring 1, rather than to a landlord in any of the remaining rings. Making allowance for this observation, really only $(s_1 - s_n)$ apartments may safely be traced back to landlord opponents. Further, these latter $(s_1 - s_n)$ units might ever only involve landlords and tenants from the first ring. So ultimately we can only be certain of a mere $(s_1 - s_n)/2$ landlords to oppose the ring road.[8]

Of course, if the city shape is such that $s_1 < s_n$, then $(s_1 - s_n)/2$ is negative. In this specific case $(s_1 - s_n)/2$ is not a very good lower bound. A lower bound of zero landlord opponents is a better, and obvious, alternative choice. Yet this need not bother us given that there are many other lower bounds on offer. For example, apartments in the first two rings, $(s_1 + s_2)$, give another conservative estimate of landlord opponents if we allow for (i) $(s_{n-1} + s_n)$ tenants being matched up with some landlord in the first two rings and (ii) all remaining apartments in the first two rings to be matched up with one another. Making these two adjustments points to $((s_1 + s_2) - (s_{n-1} + s_n))/2$ as yet another lower bound. So already we have identified two "snapshots" of suitable sections of the commuting distribution that both offer some minimum opposition to the ring road consistent with the city's shape.

This idea can be generalized to the extent that *any* partial sum $l^o(j) = \sum_{i=1}^{j}(s_i - s_{n+1-i})/2$, with $j = 1, \ldots, n/2$, is a lower bound on the number of landlord opponents. Returning to our initial continuous setup, refine the city's partition into rings by increasing n. If we simultaneously increase j such that j/n remains constant, then the partial sum $l^o(j)$ tends to $l^o(b) = (\int_0^b f(r)dr - \int_{\tilde{r}-b}^{\tilde{r}} f(r)\,dr)/2$ or $(\int_0^b (f(r) - f(\tilde{r} - r))dr)/2$ which in turn

[8] It is possible that $(s_1 - s_n)/2$, or some other of the local lower bounds below, exceeds the landlord total, 1/2. We rule this out by assuming n to be sufficiently large.

can more simply be written as

$$l^o(b) \;=\; \int_0^b D(r)dr\Big/2 \qquad \text{with} \qquad b \in [0, \widetilde{r}/2] \tag{2}$$

and where $b \equiv r_j = (j/n)\widetilde{r}$.

Equation (2) sets out distance b's "true" lower bound on landlord opponents as a sum of ring differences $D(r) = (f(r) - f(\widetilde{r} - r))$. A *ring difference* is our key auxiliary concept throughout what follows. It derives naturally from our setup, juxtaposes (subtracts) commuters in ring $\widetilde{r}-r$ with (from) commuters in ring r, and is helpful in defining skewness (further below). It is clear now where we go from here. Most of all we are interested in the largest of all these lower bounds $l^o(b)$. It is that bound that is most successful at extracting information from the given city shape. To identify it, it remains to maximize $l^o(b)$ with respect to b. This last step is taken shortly, when stating Proposition 1.

Similar reasoning applies towards bounding from below the number of those landlords who are certain to benefit from, and hence *support*, the project. To see this note that all s_n units are tied up in matches that make their owners support the ring road – with the exception of those involving a resident from ring 1. Put differently, reversing the inequality in (1) and setting $i = n$ implies $j > 1$. So to assess minimum conceivable support, suppose that every resident in ring 1 is linked to someone in ring n. In this conservative scenario only $(s_n - s_1)$ apartments really point to landlords who would benefit from the policy proposal. Further, suppose that all of these $(s_n - s_1)$ apartments join landlords and tenants from ring n only. Thus $(s_n - s_1)/2$ is our first lower bound on the number of landlord *proponents*.

Here, too, there are many more lower bounds. For example, another lower bound derives from consulting both the two last and first rings, and comes to $((s_n + s_{n-1}) - (s_1 + s_2))/2$. Generally, if the last, as well as first, j rings are included, the lower bound on landlord proponents can be written as $l^p(j) = \sum_{i=1}^{j} (s_{n+1-i} - s_i)/2$, where $j = 1, \ldots, n/2$. Casting the partial sum of landlord proponents extracted from the first b ring differences in terms of arbitrarily small ring width gives

$$l^p(b) \;=\; \int_0^b \Big(-\,D(r)\Big)\,dr\Big/2 \qquad \text{with} \qquad b \in [0, \widetilde{r}/2], \tag{3}$$

or $-l^o(b)$. The largest of all these latter lower bounds is found by maximizing the integral with respect to b. Equivalently we may minimize this integral's negative, i.e. $l^o(b)$, and proceed with the negative of the minimum value obtained. This final step also is taken when stating Proposition 1, i.e. now.

Proposition 1's Part (i) summarizes most of our discussion, while also complementing it with a number of bounds properties. In essence Proposition 1 provides estimates of the impact of the city's "physical sphere" (differences between housing stocks in juxtaposed rings) on its "political sphere" (lower bounds on landlord opposition or landlord consent), an impact that operates as silently as it is fundamental. Part (ii) notes that lower bounds must be non-negative since the maximizers involved in computing the two bounds equal 0 at worst, reducing the integral to zero then. Part (iii) adds that the lower bound on landlord proponents of the ring road can also be converted into an upper bound on landlord opponents, and vice versa. Intuitively, if at least 30 out of 100 landlords support the ring road, say, then of course at most 70 landlords can be against it. So not only does $\underline{l}^p$ bound the number of proponents l^p from below, also $1/2 - \underline{l}^p$ bounds the number of opponents l^o from *above*.[9] Part (iv) emphasizes that, with the exception of the improbable case where f is symmetric, at least one of the two lower bounds must bind. For this reason alone our pair of lower bounds must be useful. Finally, and most importantly, we conclude that the ring road proposal is rejected under majority rule once $\underline{l}^o$ exceeds one fourth of the housing stock, *irrespective* of the city's specific apartment portfolio assignment (Part (iv)). Lower bounds become particularly useful once they exceed the one-fourth-of-the-residents threshold.

Proposition 1: (Lower Bounds)

(i) (Lower Bounds): Lower bounds on the number of landlord opponents, $\underline{l}^o$, and on the number of landlord proponents, $\underline{l}^p$, are identified as

$$\underline{l}^o \;=\; \max_{b\in[0,\widetilde{r}/2]} \left[\int_0^b D(r)\,dr \Big/ 2\right] \quad \textit{and} \quad \underline{l}^p \;=\; -\min_{b\in[0,\widetilde{r}/2]} \left[\int_0^b D(r)\,dr \Big/ 2\right].$$

(ii) (Nonnegative Bounds): Both lower bounds $\underline{l}^o$ and $\underline{l}^p$ are nonnegative.
(iii) (Upper Bounds): Opponents' number l^o is bounded as in $\underline{l}^o \leq l^o \leq 1/2 - \underline{l}^p$; while proponents' number, l^p, is bounded via $\underline{l}^p \leq l^p \leq 1/2 - \underline{l}^o$.
(iv) (Useful Bounds): Lower bounds $\underline{l}^o$ and $\underline{l}^p$ are both zero (i.e. useless) if and only if the density of commuting distance f is symmetric. Thus upper bounds are useful, too.
(v) (Sufficient Bounds): If $\underline{l}^o > 1/4$ (alternatively if $\underline{l}^p > 1/4$) then a majority of landlord opponents (landlord proponents) reject (push through) the ring road …whatever (!) the underlying portfolio structure.

Fig. (2) illustrates lower bounds $\underline{l}^o$ and $\underline{l}^p$. In any of the Figure's panels the horizontal axis gives commuting distance r from the CBD, while the vertical axis gives commuting density $f(r)$. (Axes are not scaled identically across panels.) The vertical line at each panel's center rises up above "midtown" $\widetilde{r}/2$, across which the graph of $f(r)$ is reflected (or "folded over") in order to obtain, and illustrate, ring differences $D(r)$ at all distances between 0 and $\widetilde{r}/2$. I.e., ring differences are represented by the vertical distances between the two graphs (lines) shown left of $\widetilde{r}/2$. Note how the commuting density is increasing over at least some subset of the support in most panels. In a circular city this easily arises whenever the increase in built-up area from adding yet another ring outweighs the diminishing population density that is typical of many (though not all) cities.

We turn to the stylized city shapes (a) through (c) first. Panel (a) shows what we might dub a "classical city". With its true distribution documented by de Lara et al. (2013, cf.

[9] Since the interval $[\underline{l}^o, 1/2 - \underline{l}^p]$ contains the true l^o, this interval's size effectively indicates the precision of our lower bound $\underline{l}^o$. (A similar point applies to $\underline{l}^p$.) For city shapes for which this interval is small our lower bound supplies a more precise estimate than for city shapes for which this interval is large.

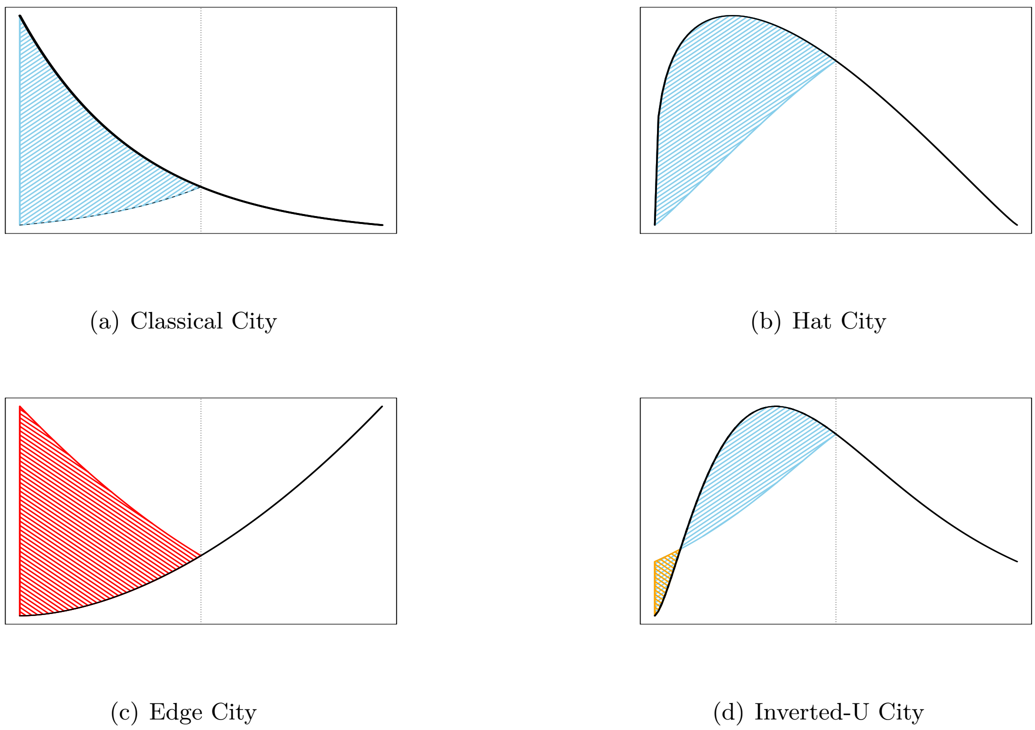

Figure 2: Lower Bounds

Fig. 3), Paris quite closely (though not perfectly) resembles this "classical city". Panel (b) depicts a "hat city" that reflects the fact that the CBD needs land, too. Panel (c) illustrates an "edge city", across which most commuters travel long distances. – Lower bounds can be inferred from consulting corresponding shaded areas. In panels (a) and (b), the lower bound on *opponents* amounts to half the shaded area *below* the density's graph (blue on screen); whereas in panel (c) the lower bound on *proponents* amounts to half the shaded area *above* the density's graph (in red).

Panel (d) adds an "inverted-U city" to our little city morphology. Moscow, for example, appears to exhibit just this shape (Bertaud/Renault (1997), cf. Fig. (1b)). The "inverted-U" city is different from the three preceding stylized city shapes. Neither is one of the two lower bounds zero, nor are lower bounds just as easily read off the shaded areas. In panel (d), and following the principles outlined above, the lower bound on opponents is equivalent to half the area obtained by subtracting the smaller, and doubly, shaded (orange) area from the larger, singly shaded, (blue) one. While early ring differences are negative, later ring differences are overwhelmingly positive. Including those later, and positive, differences in our lower bound (i.e. a cumulative sum) is preferable even if that comes at the cost of also including those earlier, and negative, differences.

Panel (d)'s "inverted-U city" even illustrates the principles underlying our lower bound on landlord *proponents*. Here this latter lower bound occurs where D vanishes, or where f

and its reflection $f(\widetilde{r}-r)$ intersect in the Figure.[10] The resulting lower bound is half the cumulative sum of all ring differences from the city center up to the intersection, or half the panel's doubly shaded (or orange) area above the density's graph. Generally, we suspect city shapes to be the more informative the more asymmetric they are. A city of symmetric shape, i.e. in which ring differences are zero always, reveals next to nothing of its politics to the observer. (The following section will revisit this idea, replacing asymmetry with skew.) Note that in panels (a) through (d) always at least one of the two lower bounds is active (i.e. strictly positive), as predicted by Proposition 1.

We also note that Figure 2's panel (a) illustrates one scenario where landlord opponents are decisive. This panel's "classical city" pictures a shaded area well in excess of half the area below the commuting density. Here the city *cannot help but* turn down the ring road proposal. Not a single constellation of landlord portfolios exists that could make collapse the anti-ring-road majority that is the inescapable consequence of that city's shape. In that sense we now really see why we *do* expect to find the interests of a majority of Parisians (or Parisian landlords at least) to always go "back to the center", as suggested by the introductory quote. Quite apparently the same cannot be said for the "hat city" in Figure 2's panel (b). Alternatively, of course, if $\underline{l}^p$ exceeds 1/4 then it is the landlord proponents of the ring road who will prevail. Figure 2 provides one example of this, too, with the "edge city" in panel (c).

3 The Political and the Visual

Very different distributions may display similarly sized bounds. It may not so much be the entire distribution that matters to urban majorities but one particular aspect of it. This aspect may even have an intuitive interpretation. This section pursues these two ideas. It suggests that the commuting distribution's *skewness* is one property that (i) itself bounds lower bounds from below and (ii) is visually appealing at that. And so really it is skewness (form) that drives decisions on the ring road proposal (function). Function follows form (Proposition 2 later).

To start us on this idea we first offer as definition of city skewness σ

$$\sigma \;=\; \int_0^{\widetilde{r}/2} D(r)\left(\widetilde{r}/2 - r\right)dr. \tag{4}$$

So σ is a weighted sum of ring differences $D(r)$, with a given ring difference's (positive) weight equal to the two underlying rings' common distance to "midtown".[11] Offering σ we justify by pointing to its visual appeal. In formula (4), early ring differences (associated with distances close to 0) receive large weights while late ring differences (associated with

[10] Minimizing $\int_0^b D(r)dr/2$ requires an interior solution, denoted $\underline{b}^{**}$, to satisfy $f(\underline{b}^{**}) = f(\widetilde{r}-\underline{b}^{**})$.

[11] In its reliance on $\tilde{r}$, the length of the commuting distribution's support, σ differs from the various definitions of skewness found in the literature. It is possible to rewrite σ as $\int_0^{\tilde{r}}(\widetilde{r}/2-r)\,dr$, or $\widetilde{r}/2-\rho$, with ρ the expected commuting distance. Thus σ also is midtown distance minus average distance, which is clearly not the same as, say, the traditional non-parametric definition of skewness.

distances close to $\widetilde{r}/2$) only benefit from small weights. That an indicator of skewness should reward early ring differences makes sense intuitively. A positive early ring difference, say, is nothing but the "first ring" offering more apartments than the "last". Yet how the very distant first and last ring compare to each other *frames* our perception of the commuting density's (i.e. city shape's) skewness by more than how the two adjacent rings on either side of midtown $\widetilde{r}/2$ compare to each other – hence the difference in weights.

Generally, for a city shape to exhibit strong positive skew, two properties contribute. First, ring differences should more often than not be positive (true for our stylized "classical city", "hat city" and "inverted-U-city" (Figure 2 again), but not true in the case of our "edge city"). And second, these positive ring differences should occur early (close to the CBD), rather than late (close to the city's boundary). The "inverted-U city" displays visibly smaller skew than our "hat city" precisely because it lacks those early positive ring differences.[12] These properties justify employing σ as an indicator of skewness.

At the same time, and as the paper's key proposition, σ also allows us to bound city politics. City skewness connects the visual with the political, by exploiting both concepts' connection with the physical. A compact statement of this idea runs through the following short sequence of inequalities:

$$\sigma \;=\; \int_0^{\widetilde{r}/2} D(r)\left(\widetilde{r}/2 - r\right) dr \quad\leq\quad \max_{b\in[0,\widetilde{r}/2]} \left[\int_0^{b} D(r)\,(\widetilde{r}/2 - r)\,dr\right] \tag{5}$$

$$\leq\quad \max_{b\in[0,\widetilde{r}/2]} \left[\int_0^{b} D(r)\,(\widetilde{r}/2)\,dr\right] \tag{6}$$

$$=\;\widetilde{r}\,\max_{b\in[0,\widetilde{r}/2]} \left[\int_0^{b} D(r)\,dr\Big/2\right] \;=\; \widetilde{r}\,\underline{l}^o, \tag{7}$$

Inequality (5) exploits the fact that the integral over weighted ring differences is greatest if the integral's upper limit is chosen freely, rather than being invariably fixed at $\widetilde{r}/2$. And equation (7) makes use of the fact that a monotonic transform of the maximand does not affect the maximization procedure's solution. This leaves us with inequality (6). Note that any further information absent, ring differences $D(r)$ cannot be signed. A ring difference may be anything: positive, zero, or even negative. So replacing $(\widetilde{r}/2 - r)$ by $\widetilde{r}/2$, as we do when going from the r.h.s. of (5) to the r.h.s. of (6), not necesssarily increases the integral.

And yet increase is precisely what that integral does. As the formal proof following shortly shows, inequality (6) is true indeed. Its proof really relies on one single important insight. By definition, the upper limit of the integral b is chosen to render the expression in square brackets on the r.h.s. of (5) as large as possible. Let r^* denote this maximizer. In the resulting integral, or in

$$\int_0^{r^*} D(r)\,(\widetilde{r}/2\,-\,r)\,dr, \tag{8}$$

[12] Additional shapes could be drawn to illustrate how σ conforms with our intuition on skewness. Symmetric distributions, for instance, are characterized by skewness being equal to zero (as they should be). For symmetric distributions, ring differences are all zero and hence so is skewness.

ring differences may well alternate in sign, but *late* ring differences at distances just short of r^* must be positive. Why else would they be included in the integral (8)? Yet these late, and positive, ring differences close to r^* are also those where replacing $(\widetilde{r}/2 - r)$ with $\widetilde{r}$ has the greatest impact. After all the change in weight applied when going from the r.h.s. of (5) to the r.h.s. of (6) is r, and hence is largest if r is close to r^*.

Intuitively then, positive ring differences come to enjoy a greater extra in weight than negative ring differences do. On balance replacing weights serves to increase the overall sum. The formal proof following in the indented paragraph below generalizes this intuitive idea to city shapes for which ring differences' signs alternate more often than just once (i.e. finitely many times). To summarize, replacing $(\widetilde{r}/2 - r)$ by $\widetilde{r}/2$ *does* contribute to raising the r.h.s. of (5), and so inequality (6) *is* true.

> Proof of inequality (6): If ring differences are positive always then there is nothing to prove. Thus suppose that the signs of ring differences $D(r)$ alternate on $[0, r^*]$. Consider all those intervals on which D retains its sign.
>
> We pair off these intervals into groups of two. That is, we divide $[0, r^*]$ into n consecutive intervals (or rings) $[0, r_1^*]$, $[r_1^*, r_2^*]$, $\ldots$, $[r_{n-1}^*, r^*]$ such that the i-th such interval (ring) decomposes into one subset on which $D < 0$, denoted $[r_{i-1}^*, \widehat{r}_i]$, and another on which $D > 0$, written $[\widehat{r}_i, r_i^*]$. We also set $r_0^* = 0$ and $r_n^* = r^*$.
>
> By the second mean value theorem of integration, there must be numbers c_i' and c_i'', satisfying $\widetilde{r}/2 \geq c_i' \geq c_i'' > 0$ as well as $c_i' \geq c_{i+1}'$ for all i, such that

$$
\begin{aligned}
\max_{b \in [0, \widetilde{r}/2]} \left[\int_0^b D(r)\left(\widetilde{r}/2 - r\right) dr \right] &= \sum_{i=1}^{n} \left[\int_{r_{i-1}^*}^{\widehat{r}_i} D(r)\left(\widetilde{r}/2 - r\right) dr + \int_{\widehat{r}_i}^{r_i^*} D(r)\left(\widetilde{r}/2 - r\right) dr \right] \\
&= \sum_{i=1}^{n} \left[c_i' \int_{r_{i-1}^*}^{\widehat{r}_i} D(r)\, dr \;+\; c_i'' \int_{\widehat{r}_i}^{r_i^*} D(r)\, dr \right] \\
&\leq \sum_{i=1}^{n} \left[c_i' \int_{r_{i-1}^*}^{\widehat{r}_i} D(r)\, dr \;+\; c_i' \int_{\widehat{r}_i}^{r_i^*} D(r)\, dr \right] \\
&= \sum_{i=1}^{n} c_i' \int_{r_{i-1}^*}^{r_i^*} D(r)\, dr. \qquad (9)
\end{aligned}
$$

> The Lemma in the Appendix shows that $\int_{r_{n-j}^*}^{r^*} D(r) dr \geq 0$ for any $j = 1, \ldots, n$. That is, summing over any j last ring differences gives a non-negative number. We continue with the r.h.s. of (9) making repeated use of this. I.e.,

$$
\begin{aligned}
\sum_{i=1}^{n} c_i' \int_{r_{i-1}^*}^{r_i^*} D(r)\, dr &= \sum_{i=1}^{n-1} c_i' \int_{r_{i-1}^*}^{r_i^*} D(r)\, dr \;+\; c_n' \int_{r_{n-1}^*}^{r^*} D(r)\, dr \\
&\leq \sum_{i=1}^{n-1} c_i' \int_{r_{i-1}^*}^{r_i^*} D(r)\, dr \;+\; c_{n-1}' \int_{r_{n-1}^*}^{r^*} D(r)\, dr \\
&= \sum_{i=1}^{n-2} c_i' \int_{r_{i-1}^*}^{r_i^*} D(r)\, dr \;+\; c_{n-1}' \int_{r_{n-2}^*}^{r^*} D(r)\, dr \;\leq\; \ldots \;\leq\; c_1 \int_0^{r^*} D(r)\, dr
\end{aligned}
$$

Note how successive inequalities repeatedly exploit the Lemma, for increasingly larger values of j. It remains to add that

$$c_1 \int_0^{r^*} D(r)\,dr \;\le\; \int_0^{r^*} D(r)(\widetilde{r}/2)\,dr \;\le\; \max_{b\in[0,\widetilde{r}/2]} \left[\int_0^{b} D(r)\left(\widetilde{r}/2\right)dr\right]$$

Putting all consecutive inequalities in this paragraph together completes the proof of inequality (6). □

We summarize the overall inequality implied by the succession of inequalities (5) through (7), and combine it with $\underline{l}^o \le l^o$ (Proposition 1), in Proposition 2's first part. Landlord support for the ring road proposal is bounded from below by skew adjusted for city size, $\sigma/\widetilde{r}$. Part (i)'s second inequality gives an independent, if related, result. The negative of adjusted city skew bounds landlord proponents from below. The more skewed the city is the more confident we are of the ring road proposal meeting landlord resistance. In sufficiently positively skewed cities (Figure 2's panel (a), (b) and (d))), part (i)'s first inequality may be useful. In sufficiently negatively skewed cities (Figure 2's panel (c)) we exploit the second inequality instead.

Proposition 2: (Function Follows Form)
(i) (Physical and Visual): City Skew σ bounds landlord opponents and proponents via

$$\sigma/\widetilde{r} \;\le\; l^o \qquad\qquad \textit{and} \qquad\qquad -\,\sigma/\widetilde{r} \;\le\; l^p.$$

(ii) (Function Follows Form): If $\sigma/\widetilde{r} > 1/4$ (or $-\sigma/\widetilde{r} > 1/4$ alternatively) the center retains its retail and employment function (the periphery takes over jobs and shops), irrespective of the city's apartment ownership structure.
(iii) (Importance of Topography): The less circular a city is, the more likely it is to exhibit positive skew. Coastal cities, or cities on a peninsula even, are more likely to hold on to their CBD than inland cities.

Proving Proposition 2's second inequality relies on a sequence of inequalities akin to that in (5) through (7). Proposition 2's Part (ii), in setting out the relationship between city shape (city form) and buildings' uses (city function), contains the paper's comparative statics. On the one hand, if $\sigma/\widetilde{r}$ exceeds 1/4 then this is not just true for $\underline{l}^o$ but *a fortiori* also for l^o. On the other hand, if $-\sigma/\widetilde{r}$ exceeds 1/4 then so does l^p. Buildings in the city traditionally house retail and office uses. If city skew is strong enough then traditional uses are preserved, while if city skew is sufficiently negative then uses are reversed, i.e. city center buildings become residential and it becomes peripheral buildings' turn to take over the retail and office function. Of course, for "amorph cities", with $\sigma/\widetilde{r}$ in the open interval $(-1/4, 1/4)$, equilibrium policy cannot be assessed further information absent. And so the relationship between σ and the electorate's policy decision cannot really be said to be monotonic.

The impact of city shape on city functions presented here reverses – or, less dogmatically, complements – Sullivan (1896)'s "form follows function". To emphasize, all of this is always true in full ignorance of the city's actual apartment portfolio assignment.[13] Part (iii) finally provides a straightforward "corollary". For coastal cities, or even cities on peninsulas, $a(r)$ is less likely to be increasing in r, and may even be constant in r. Then $f(r)$ must be decreasing in r, and hence the city is more likely to be positively skewed. From this paper's perspective, it makes sense that New York or Chicago sprawl less than Houston or Atlanta, or that Moscow sprawls more than St. Petersburg.

4 Discussion

Sprawl: As indicated in the introduction, ultimately urban policy will feed back into city shape. Not just should the city's shape be expected to impact on urban policy (this paper's overall proposition); also the city's shape naturally responds to urban policy (a well-established fact). Cities that are sufficiently skewed stick to their traditional center. Inasmuch as the existing distribution of apartments across the city already reflects the center's past attraction, nothing ever changes. A city with sufficiently positive skew is in perfect equilibrium. Both the city shape and its electorate's rejection of any ring road proposal constantly reaffirm each other, and hence persist into the future.

In great contrast, cities that are sufficiently skewed towards the center are obviously not in equilibrium. As argued above, a city with strong negative skew first sheds its traditional center. From a dynamic perspective, and after adding a city shape response at long last, the implied decentralization of employment and shopping opens up a host of new property developments beyond the ring road, on land that was out of reach previously (the city's "suburb"). This land must appear attractive also to the mobile of other cities' residents. Apartment construction beyond the ring road will make the settlement area expand far beyond its initial boundary (the city's "core").

The city's ultimate path of growth depends on a number of decisions also addressed by a large and growing literature, e.g.: Will the suburbs be incorporated into the core, to the extent of creating an agglomeration containing both suburbs and core? Will owners of suburb land be residents of that agglomeration? How do the other cities in the urban system respond, will they also shed their traditional CBD? Briefly, how the shape of the agglomeration evolves depends on the answers to all of these questions. The agglomeration's shape may be much more skewed than the city's shape ever was. Then the agglomeration settles into an equilibrium.

Just as conceivably, adding all this suburb population joint with shedding city core population may actually reinforce the initial city's negative skew. The evolving agglomeration may exhibit sufficiently negative skew, too, much like the old city core did earlier. Then

[13] An obvious and important follow-up question is how our results change if apartment ownership is relaxed. This question is left to an extension of this paper's model. Nonetheless even now we surmise that if the city is heavily skewed, relaxing apartment assignments will not overturn the result that city skewness predicts the ring road proposal's fate. This is fairly obvious if we think about two extreme alternatives, where either all landlords jointly act as a landlord class or where all residents are owner-occupiers. In both of these scenarios city skewness again can help predict equilibrium policy.

the agglomeration cannot help but build its own, and hence yet another, ring road. Much like the layers of an onion, the evolving agglomeration keeps adding ring roads and suburbs, while at the same time hollowing out further in the center. This may stop when ring road travel starts becoming expensive. We delegate a fuller analysis of such a mutual interaction between shape and policy, joint with its response to exogenous shocks, to a later paper.

Urban History: To revisit the introduction's veduta example (Fig. 1), suppose Munich's silhouette $d(r)$ were fitted a straight line $d(r) = \alpha(\widetilde{r} - r)$, with $\widetilde{r}$ and α scalars derived from the painting. (Canaletto is known for both his realism as well as his employing the at the time novel, and more precise, "camera obscura" technique, painting a projection of the cityscape onto a wall rather than the cityscape itself.) If $a(r)$ could be assumed to equal $2\pi r$ then Munich's commuting density would be $f(r) = 2\pi r\alpha(\widetilde{r} - r)$. Based on this, for each city painting we could compute adjusted skewness $\sigma/\widetilde{r}$, to then compare it with threshold 1/4 and subsequent city history.

Carbon Tax or Congestion Toll: We have pointed to the analogy between the electorate's decision on the ring road proposal and its decision on whether to introduce a tax on commuting (motivated by aiming to reduce congestion or carbon dioxide emissions), raising t marginally. If the attendant tax revenue can be ignored (e.g. because it is wasted) and if tenants' political influence can be neglected (even as tenants' costs-of-living $t\widetilde{r}$ rise) then a city that is *against* the ring road also is a city that is *for* implementing a city toll or a carbon tax. This attributes societies composed of cities of sufficient skew the political majority in favor of the carbon tax.

5 Conclusions

Urban policy molds city shape. This paper argues that city shape also molds urban policy. The more skewed a city's shape (i.e. commuting distribution), the less conceivable a majority of resident landlords that prefer replacing the traditional center at the CBD by a succession of office parks and shopping malls strewn along a city ring road. This idea connects to two long-standing themes in both architectural theory and economics. It connects to architectural theory because it provides one specific instance of where "function follows form", rather than where "form follows function" (Sullivan (1896)). As one particular architect subsequently notes himself,

> "Form follows function – that has been misunderstood. . . . Form and function should be one, joined in a spiritual union." (Frank Lloyd Wright as quoted in Saarinen (1954)

And this idea of city shape molding urban policy also connects to economics because it gives one specific instance of when drawing conclusions from an aggregate (the city form) to its component members (the city residents) is legitimate. Economics as a field has traditionally been weary of generalizing an aggregate's properties on towards the aggregate's component members lest it commit a "fallacy of division". By linking the built environment – a society aggregate – to the preferences of (at least a majority) of its inhabitants – a majority of individuals – we also provide an example at least of where inferring dominant residents' properties does seem justified. Whenever the city's shape "leans towards" the city center (a majority of) resident landlords also are inclined to maintain, and hence "lean towards", the city center. Conversely, if the city "leans towards" the periphery (a majority of) resident landlords are inclined to develop, and hence "lean towards", the periphery. For once no fallacy is involved.

6 Literature

Arnott, R. and J. Stiglitz (1981) Aggregate Land Rents and Aggregate Transport Costs, Economic Journal 91: 331-347.

Baranow, N. (1980) Die Silhouette der Stadt, Berlin: VEB (No English Translation of the Russian original).

Bento, A., S. Franco and D. Kaffine (2006) The Efficiency and Distributional Impacts of Alternative Anti-Sprawl Policies, Journal of Urban Economics 59.1: 121-141W.

Bertaud, A. (2003) The Spatial Organization of Cities. Deliberate Outcome or Unforeseen Consequence, Background Paper to World Development Report 2003.

Bertaud, A. and B. Renaud (1997) Socialist Cities without Land Markets, Journal of Urban Economics 41: 137-151.

Baum-Snow, N. (2006) Suburbanization and transportation in the monocentric model, Journal of Urban Economics 62: 405-423.

Brueckner, J. (1987) The Structure of Urban Equilibria: A Unified Treatment of the Muth-Mills-Model, in: E. Mills (ed.), Handbook of Regional and Urban Economics Vol. II: 821-845.

Brueckner, J. (1993) Inter-Store Externalities and Space Allocation in Shopping Centers, Journal of Real Estate Finance and Economics 7: 5-16.

Brueckner, J. (2005) Transport Subsidies, System Choice and Urban Sprawl, in: Regional Science and Urban Economics 35.6: 715-733.

Girouard, M. (1989) Cities and People. A Social and Architectural History, Yale University Press.

Gladwell, M. (2004) The Terrazzo Jungle. Fifty Years Ago, the Mall was Born, New Yorker March 2004 issue.

Glaeser, E. and M. Kahn (2004) Sprawl and Urban Growth, in: V. Henderson and J. Thisse (eds) Handbook of Regional and Urban Economics Vol. 4.4.

Gopnik, A. (2014) The View from a Bridge. Shopping, Tourism, and the Changing Face of Luxury. The New Yorker. Issue Dec. 8th.

de Lara et al. (2013) Congestion Pricing and Long Term Urban Form. Application to Paris Region, Regional Science and Urban Economics 43.2: 282-295.

Links, J. G. (1999) Canaletto, Phaidon Press.

Saarinen, A. (1954) Taliesin Weekend. Frank Lloyd Wright 85, Vitally Works On, New York Times on August 8, reprinted in P. Meehan (1991) (ed.) Frank Lloyd Wright Remembered, The Preservation Press: 178-182.

Rauch, J. (1993) Does History Matter Only When it Matters Little? The Case of City-Industry Location, Quarterly Journal of Economics 108: 843-867.

Roeck, B. (2004) Das historische Auge. Kunstwerke als Zeugen ihrer Zeit. Von der Renaissance zur Revolution, Göttingen.

Sullivan, L. (1896) The Tall Office Building Artistically Considered, in: Lippincott's Magazine, March issue.

Ushchev, P. I. Sloev und J.-F. Thisse (2015) Do We Go Shopping Downtown or in the 'Burbs? Journal of Urban Economics 85: 1-15.

7 Appendix

Proof of Proposition 1:

Part (i) (Lower Bounds): Consider a landlord who resides in ring i yet rents out the extra property in ring j. This is a "match" $\{i, j\}$. Let matrix B collect the frequencies with which matches $\{i, j\}$ occur. For example, $b_{1,3}$ is the number of times a landlord owning, and living in, an apartment in the first ring also owns an apartment in ring 3. Note that, with this definition, the sum of all entries in row i plus the sum of all entries in column i just yield the apartment total in ring i.

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n-2} & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,n-2} & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & b_{n-1,n-2} & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & \cdots & b_{n,n-2} & b_{n,n-1} & b_{n,n} \end{pmatrix}. \tag{10}$$

In view of condition (1), B's counter diagonal (comprising all the elements on the diagonal stretching from the bottom left corner to the top right hand corner) collects all those matches that leave landlords indifferent to the ring road. In contrast, entries above (below) B's counterdiagonal collect all those matches that involve landlord opponents (landlord proponents).

Since $(s_1 - s_2)/2$ is an obvious first lower bound, let us make precise the second lower bound discussed in the main text instead, or $(s_1 + s_2 - (s_{n-1} + s_n))/2$. In (10), s_1 and s_2 are the sums of all entries given in the first row and column and second row and column, respectively. It is clear that this sum overstates the number of landlord opponents; some of its elements are found on, or below, our counterdiagonal. The implied error amounts to

$$\Big(b_{n,1} + b_{n-1,2} + b_{n,2}\Big) \; + \; \Big(b_{1,n} + b_{2,n-1} + b_{2,n}\Big). \tag{11}$$

This error collects a subset of all the matches linking apartments in the last two rings to apartments in the first two rings, indicating landlords that do not oppose the ring road at all. We take care of these by subtracting *all* apartments in the last two rings from $(s_1 + s_2)$. Similar reasoning applies to subsequent lower bounds on landlord opponents, or to any lower bound on landlord proponents. □

Part (ii) (Nonnegative Bounds): As discussed in main text.

Part (iii) (Upper Bounds): As discussed in main text.

Part (iv) (Useful Bounds): We first show that both bounds being useless implies that f is symmetric. The proof is by contradiction. Thus suppose both lower bounds are useless. I.e., suppose

$$\max_{b \in [0, \widetilde{r}/2]} \left[\int_0^b D(r)\, dr \right] \; = \; 0 \qquad \text{and} \qquad \min_{b \in [0, \widetilde{r}/2]} \left[\int_0^b D(r)\, dr \right] \; = \; 0.$$

From the first equation we gather that $\int_0^b D(r)dr \leq 0$ for all $b \in [0, \widetilde{r}/2]$ (else $\underline{l}^o$ would need to positive, contradicting our assumption); whereas from the second equation above we infer that $\int_0^b D(r)dr \geq 0$ for all $b \in [0, \widetilde{r}/2]$ (else $\underline{l}^p$ would have to be positive, contradicting our assumption). Joining these latter two inequalities gives

$$\int_0^b D(r)\, dr = 0$$

for all $b \in [0, \widetilde{r}/2]$. This implies that the integral in the last equation is a constant function of b. Hence its derivative with respect to b, or $D(b)$, must equal zero for all b. So f is symmetric. At last, note that if f is symmetric both bounds obviously are useless (zero). □

Part (v) (Sufficient Bounds): As discussed in main text.

Lemma:

For all $j = 1, \ldots, n$,

$$\int_{r^*_{n-j}}^{r^*} D(r)\, dr \;\geq\; 0. \tag{12}$$

Ring differences in any last j rings sum to a non-negative number.[14]

Proof: The proof makes use of the notation introduced in the proof of inequality (6) in the main text. We first show that $\int_{r^*_{n-1}}^{r^*} D(r)dr \geq 0$. As our point of departure, recall that surely

$$0 \;\leq\; \int_{r^*_{n-1}}^{r^*} D(r)(\widetilde{r}/2 - r)dr$$

because otherwise r^* could not be the optimizer. (The n-th ring should not have been included in $\underline{l}^o$.) But then

$$\begin{aligned} 0 \;&\leq\; c'_n \int_{r^*_{n-1}}^{\widehat{r}_n} D(r)\, dr \;+\; c''_n \int_{\widehat{r}_n}^{r^*} D(r)dr \\ &\leq\; c'_n \int_{r^*_{n-1}}^{\widehat{r}_n} D(r)\, dr \;+\; c'_n \int_{\widehat{r}_n}^{r^*} D(r)dr \;=\; c'_n \int_{r^*_{n-1}}^{r^*} D(r)\, dr \end{aligned} \tag{13}$$

Since $c'_n > 0$, this proves $\int_{r^*_{n-1}}^{r^*} D(r)dr \geq 0$. Next we show that $\int_{r^*_{n-2}}^{r^*} D(r)dr \geq 0$. Following similar reasoning as above, clearly

$$0 \;\leq\; \int_{r^*_{n-2}}^{r^*} D(r)(\widetilde{r}/2 - r)dr$$

But then

$$\begin{aligned} 0 \;&\leq\; c'_{n-1} \int_{r^*_{n-2}}^{\widehat{r}_{n-1}} D(r)\, dr \;+\; c''_{n-1} \int_{\widehat{r}_{n-1}}^{r^*_{n-1}} D(r)dr \;+\; c'_n \int_{r^*_{n-1}}^{r^*} D(r)\, dr \\ &\leq\; c'_{n-1} \int_{r^*_{n-2}}^{\widehat{r}_{n-1}} D(r)\, dr \;+\; c'_{n-1} \int_{\widehat{r}_{n-1}}^{r^*_{n-1}} D(r)dr \;+\; c'_{n-1} \int_{r^*_{n-1}}^{r^*} D(r)\, dr \\ &\leq\; c'_{n-1} \int_{r^*_{n-2}}^{r^*} D(r)\, dr \end{aligned}$$

[14] This property also is illustrated by the simple example of Figure 2's panel (d), where positive ring differences dominate negative ring differences.

Since $c'_{n-1} > 0$, this proves $\int_{r^*_{n-2}}^{r^*} D(r)dr \geq 0$. Proceeding along these lines proves the Lemma. (A proof of induction could be given.) $\square$