The motivic polylogarithm for smooth quasi-projective schemes and its realizations

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The significance of the polylogarithm in mathematics

The polylogarithm in its modern form is still a rather new - and yet not fully exploited - concept in mathematics. Its development started 35 years ago, and soon turned out to be a powerful technique to track down special values of zeta and L-functions via the construction of certain interesting functions and non-trivial K-classes.

Periods of the polylogarithm and their importance in mathematics

In the Hodge setting, the polylogarithm is merely a particularly nice projective system of variations of mixed Hodge structure (see section C.3.2 in the appendix). Associated to this projective system (via monodromy as explained in section C.3.2 of the appendix) is an (infinite-dimensional) matrix of functions - called "periods". The periods of the polylogarithm have so far all turned out to be highly interesting functions. The most famous of these probably are the following:

- the classical polylogarithm functions

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

  defined and studied by Euler and Spence (see [Lew81]), but already discussed in a correspondence of Leibniz with Bernoulli as early as 1696; this is shown in [BD94].
- Kronecker-Eisenstein series for a family of elliptic curves $\pi\colon E \longrightarrow S$ (see [BL94, 3.3.1, p.154]). These functions were introduced by Kronecker and Eisenstein and are treated in the book [Wei76].
- "Polylogarithmic currents" obtained by Levin in [Lev00], satisfying certain differential equations, which can be considered as a higher-dimensional analogue of the classical Kronecker-Eisenstein series above.

Associated Eisenstein series

The polylogarithm functions above give rise to certain so-called "Eisenstein classes" which have turned out to be useful tools in proofs. Kings, for example, used these classes to prove the Bloch-Kato conjecture for CM elliptic curves over an imaginary quadratic field K ([Kin01]). Apart from that, Eisenstein classes give rise to interesting Eisenstein series. Examples of such constructions are the following:

- Beilinson and Levin computed the Eisenstein classes associated to the polylogarithm of a modular elliptic curve in [BL94].
- In [BK10a], Bannai and Kings determined the Eisenstein classes associated to the syntomic polylogarithm of a modular curve in terms of p-adic Eisenstein series. Moreover, they computed the de Rham Eisenstein classes and proved that they are given by certain holomorphic Eisenstein series. These results were then used by

Bannai and Kings in [BK10a] and Niklas in [Nik10] to obtain results on the p-adic Beilinson conjecture.
- As an application of Wildeshaus' construction of the polylogarithm for Shimura varieties, Blottière constucted the Eisenstein classes associated to the polylogarithm of mixed Shimura varieties in [Blo07].

Construction of non-trivial K-classes

Another benefit of the polylogarithm is that it comes from non-trivial classes in higher K-theory via regulators. So far, the polylogarithm is basically the only method to construct such. It was first done by Beilinson and Levin in [BL94] for the case of elliptic curves, and then generalized by Kings to abelian schemes in [Kin99].

The polylogarithm and special values of L-functions

The first example of the mysterious connection between L-functions and the modern theory of the polylogarithm was found by Zagier in his papers [Zag86] and [Zag91]. He proved that for a number field K of degree $n = r_1 + 2r_2$ and discriminant d_K, the number

$$\pi^{-2(r_1+r_2)}|d_K|^{1/2}\zeta_K(2)$$

is connected to the polylogarithm in the following way: He considered a single-valued variant of Li_2, the Bloch-Wigner-function $D\colon \mathbb{P}^1 \to \mathbb{R}$, and showed that the above number is a rational linear combination of products of values of D at algebraic arguments.

This was generalized as part of Zagier's conjecture in [Zag91]: Similar to the Bloch-Wigner dilogarithm function D, Zagier introduced a single valued variant $P_m\colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{R}$ of all polylogarithm functions Li_m. If K denotes a number field, Zagier's conjecture implies that for a certain natural number $j(m)$ determined by m and K, the number

$$\pi^{-mj(m)}|d_K|^{1/2}\zeta_K(m)$$

is given by an (explicit) rational linear combination of products of values of P_m at K-algebraic arguments.

More applications

By [Oe93], the polylogarithm also occurs in the following contexts:

- volumes of polytopes in spherical and hyperbolic geometry,
- volumes of hyperbolic manifolds of dimension 3,
- geometry of configurations of points in $\mathbb{P}^1$,
- cohomology of $\mathrm{GL}_n(\mathbb{C})$,

- Chen iterated integrals,
- regulators in algebraic K-theory,
- differential equations with nilpotent monodromy and
- nilpotent completion of $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$.

Content

Main Aim: ***A new conceptional, general definition of the polylogarithm***

So far, there was no general definition of the notion of the "polylogarithm" for a wider class of schemes: Instead, polylogarithms painstakingly had to be constructed anew for every single combination of realization or theory (Hodge theory, ℓ-adic sheaves, K-theory/motives) and type of underlying scheme ($\mathbb{P}^1 \setminus \{0,1,\infty\}$, elliptic curves, general curves, abelian varieties, Shimura varieties...). A quick overview of the individual cases already dealt with are treated in the next section on the short history of the polylog. The main aim of this thesis is to finally *provide a general definition of the polylogarithm in the motivic setting and mixed realizations*. Moreover, while up to now the polylogarithm has only been considered for curves and abelian schemes, this general definition extends the notion of polylogarithms to *all noetherian, sparated, smooth and quasi-projective schemes* $\pi\colon X \longrightarrow S$ *over a reduced base-scheme* S.

Further Results:

On the way to provide a general motivic definition of the polylogarithm as well as its Hodge realization, we will have to extend basic mathematical language in several fields to fit our requirements. These results are of interest on their own and can be read individually:

- motivic generalization of the classical notion of ***bar complexes*** in Chapter I.3,
- motivic generalization of the classical notion of the ***pro-unipotent completion of the fundamental group*** in Chapter II.6.

An introduction to the polylogarithm: state of the art

A very short history of the polylogarithm

The notion of the "polylogarithm" has been around for more or less three hundred years. The classical functions were first mentioned in a correspondence of Leibniz with Bernoulli in 1696 (see [Ger71]) as a generalization of the logarithm. However, it was only in 1768 that mathematics turned towards this object again, when Euler defined the *dilogarithm* as the power series

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

which converges to a holomorphic function for all complex z in the unit disc. For $k \geq 1$ the k-th polylogarithm was defined by Spence in 1809 (see [Lew81]) as the power series

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

which converges to a holomorphic function for all complex z inside the unit disc. Here, the first polylogarithm $\mathrm{Li}_1(z)$ is just $-\log(1-z)$. Looking at the power series yields the formula

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(x) \frac{\mathrm{d}x}{x}$$

for all z with $|z| < 1$ and $k \geq 2$. By inductively defining

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(x) \frac{\mathrm{d}x}{x} \quad (k \geq 2), \quad \mathrm{Li}_1(z) = \int_0^z \frac{\mathrm{d}x}{1-x}$$

the polylogarithmic functions can be extended to multivalued functions on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The first one to note a connection of polylogarithm functions to more modern branches of mathematics was Deligne. He noted in his 1989-paper [Del89] that the dilogarithm Li_2 can be recovered in the context of variations of mixed Hodge structure, which are generally abbreviated "VMHS" (for an introduction to the formalism of VMHS, please consult section C.3.2 in the appendix). Namely, there is a certain VMHS on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose period matrix has Li_2 as an entry.

Beilinson then realized that in a similar fashion, one may obtain *all* polylogarithm functions as periods of VMHS. Details of the construction can be seen in Hain's paper [Hai94] on "Classical Polylogarithms", which contains results by Bloch, Deligne, Ramakrishnan, Suslin and Beilinson.

Later on, in his preprint "Polylogarithms and cyclotomic elements" ([Bei84]), Beilinson extended his description of the polylogarithm to the ℓ-adic setting. As noted before, we will not deal with the ℓ-adic setting, and hence I will refrain from giving any details here.

When Beilinson and Levin finally published their astounding paper "The elliptic polylogarithm" [BL94], it became obvious that the polylogarithm can be extended to a wider class of schemes and comes from the motivic world: In [BL94] they introduced a notion of the polylogarithm as a mixed sheaf on elliptic curves, that is to say as both a Hodge module and an ℓ-adic sheaf. Moreover, they showed that this construction corresponds to a certain projective limit of classes in K-theory, and calculated the periods of this elliptic polylogarithm. They turned out to be given by Kronecker-Eisenstein series.

After that, numerous publications defined a polylogarithm similar to the one of Beilinson-Levin for other varieties such as general curves of genus ≥ 1, abelian schemes or Shimura varieties, and in numerous settings, e.g. as a locally free vector bundle with connection, as a variation of mixed Hodge structure, as an ℓ-adic sheaf, or as a class in K-theory.

One might describe the development of the theory around the polylogarithm as follows:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{polylogarithm functions}\\ \mathrm{Li}_k \text{ on } \mathbb{P}^1 \setminus \{0,1,\infty\}\end{array}} \arrow[r, equal] & \boxed{\begin{array}{c}\text{periods of the VMHS}\\ \mathbb{L} \text{ on } \mathbb{P}^1 \setminus \{0,1,\infty\}\end{array}} \arrow[d, Rightarrow] \\
\boxed{\begin{array}{c}\text{Aim:}\\ \text{Gerneralize the construction}\\ \text{of } \mathbb{L} \text{ to other varieties}\end{array}} \arrow[d, Rightarrow] & \boxed{\begin{array}{c}\text{modern viewpoint:}\\ \text{"polylogarithm" of } \mathbb{P}^1 \setminus \{0,1,\infty\}\\ = \text{VMHS } \mathbb{L}\end{array}} \arrow[l, squiggly] \\
\boxed{\begin{array}{c}\text{"polylogarithm of a scheme } X\text{"}\\ =\text{a VMHS } \mathbb{L}_X \text{ on } X \text{ with similar}\\ \text{properties as } \mathbb{L}\end{array}} \arrow[r, squiggly] & \boxed{\begin{array}{c}\text{consider the periods}\\ \text{of } \mathbb{L}_X \text{ as a generalization}\\ \text{of the classical } \mathrm{Li}_k\end{array}}
\end{tikzcd}$$

To the day, the polylogarithm has not been constructed in all possible settings. Generally, however, there was a *consent that for technical reasons, the polylogarithm could* *only be constructed for curves and abelian schemes/Shimura varieties*. The following table gives an overview of some important publications in the different settings:

	$\mathbb{P}^1 \setminus \{0,1,\infty\}$	Elliptic curves	Curves genus ≥ 1	Abelian varieties	Shimura varieties
Hodge setting [BD92]	[Han97], [HW98]	[BL94]	[Kin]	[Lev00]+ [Blo07]	[Wil97]
ℓ-adic setting	[Bei89] [Bei84]	[Kin15] [BKT10],	[Kin], [Kin08]	–	[Wil97]
p-adic setting	[Ban00],	[BK10b]	–	–	–
K-theory	[BD94] [HW98]	[BL94]	–	[Kin99]	–
motivic setting	[HW98]	–	–	–	–

(0.1)

Motivation: The idea underlying the new construction of the polylogarithm - Faltings' logarithm and Gysin morphisms

As noted above, the literature on the polylogarithm up to the day considers the polylogarithm only in two cases: for $\pi\colon X \longrightarrow S$ an abelian scheme, or a family of curves. For more general schemes, a construction of the polylogarithm was deemed impossible for the following reason: The polylogarithm is construced using the so-called "logarithm sheaf". Here, the usual construction method of the polylogarithm in literature relies heavily on a calculation of the higher direct images of the logarithm, which is not possible in a more general setting. Hence, in order to define the polylogarithm in greater generality in a motivic setting, it *is futile to turn towards the already existing methods of construction* - the only way to achieve this aim is to *develop an entirely new theory of the polylogarithm in the motivic setting*. The polylogarithm is constructed from the so-called "logarithm" with heavy use of knowledge about the latter, so the crucial idea is to first find the right motivic analogue of the logarithm.

Step 1: Define a "motivic logarithm" which gives rise to the usual logarithm for curves and abelian scheme.

a.) *Inspiration: Faltings' motivic logarithm for curves*

Step 1 has already been done by Faltings in his paper [Fal12] for the special case of a smooth relative curve $\pi\colon X \longrightarrow S$, where S is an arbitrary base-scheme.
In order to get an idea how to generalize the logarithm to the motivic setting, let us quickly take a look at Faltings' construction, translated to Levine's theory of motives. From now on, we assume that the reader is vaguely familiar with Levine's theory of motives; it is summarized for the reader's convenience in Appendix B.

Let $\pi\colon X \longrightarrow S$ be a relative smooth curve with irreducible fibers, equipped with an S-point $x_0\colon S \longrightarrow X$. This section gives rise to an idempotent in $\mathrm{End}_{\mathrm{Sm}_S}(X)$ by

$$e_{x_0}\colon X \xrightarrow{\pi} S \xrightarrow{x_0} X.$$

Since $\mathrm{id}_X - e_{x_0}$ is also an idempotent, we obtain a motive

$$\mathbb{Z}_X^\circ := (\mathbb{Z}_X, (\mathrm{id}_X - e_{x_0})^*) \in \mathcal{DM}(S),$$

where $\mathbb{Z}_X$ denotes the motive of X over S. Likewise, we have motives

$$\mathbb{Z}_{X^n}^\circ := (\mathbb{Z}_{X^n}, (\mathrm{id}_X - e_{x_0})^{n*}) \in \mathcal{DM}(S).$$

Faltings then defines an inductive system of complexes in $\mathcal{DM}(S)$ as follows: The diagonal $\delta\colon X \longrightarrow X \times_S X$ satisfies $\delta \circ e_{x_0} = (e_{x_0} \otimes e_{x_0}) \circ \delta$, and hence induces a morphism

$$\delta^*\colon \mathbb{Z}_{X^2}^\circ \longrightarrow \mathbb{Z}_X^\circ.$$

For $i \geq 1$ and all $1 \leq k < i$, the morphism δ^* hence gives rise to morphisms

$$(\mathrm{id}^{k-1} \times \delta \times \mathrm{id}^{i-k})^*\colon \mathbb{Z}_{X^{i+1}}^\circ \longrightarrow \mathbb{Z}_{X^i}^\circ$$

in $\mathcal{DM}(S)$. They correspond to the morphisms $X^i \longrightarrow X^{i+1}$ doubling the argument in position k for $1 \leq k < i$. Now Faltings takes the alternating sum of these maps to obtain

$$d_i := \sum_{k=1}^{i-1} (-1)^{k-1} (\mathrm{id}^{k-1} \times \delta \times \mathrm{id}^{i-k})^*\colon \mathbb{Z}_{X^{i+1}}^\circ \longrightarrow \mathbb{Z}_{X^i}^\circ$$

and defines

$$P_n^\bullet := \{\mathbb{Z}_{X^n}^\circ \xrightarrow{d_{n-1}} \mathbb{Z}_{X^{n-1}}^\circ \xrightarrow{d_{n-2}} \ldots \xrightarrow{d_2} \mathbb{Z}_{X^2}^\circ \xrightarrow{\delta^*} \mathbb{Z}_X^\circ \xrightarrow{0} \mathbb{Z}_S)\} \in \mathcal{DM}(S).$$

as some kind of "universal n-unipotent motive" on S. Faltings' motivic logarithm, denoted by $P_n(\delta)$, is a variant of P_n where one replaces $\pi\colon X \longrightarrow S$ by the second projection $\mathrm{pr}_2\colon X \times_S X \longrightarrow X$ and slightly modifies the differential. I will not go into detail here, so see section II.6.1 for an explicit definition.

Faltings then proves that the zeroth homology group of the ℓ-adic realization of $P_n(\delta)$ is the ℓ-adic logarithm for the curve X:

Theorem (Faltings). *Let $\pi\colon X \longrightarrow S$ be a smooth morphism of quasi-projective schemes such that the prime l is invertible on S, and let $P_n^{\mathrm{et},l}(\delta)$ denote the ℓ-adic realization of $P_n(\delta)$. Then the ℓ-adic sheaf $\mathcal{H}^0(P_n^{\mathrm{et},l}(\delta))$ is the universal n-unipotent ℓ-adic sheaf on X trivialized at x_0, and therefore coincides with the étale logarithm on X as considered in literature.*

Regarding the Hodge realization, Faltings claims that similar arguments also prove the following: the zeroth homology group of the Hodge realization of $P_n(\delta)$ is the Hodge logarithm for the curve X.

b.) *Major points to note in Faltings' construction:*

- Faltings' construction can be imitated for any smooth morphism of noetherian, separated and reduced schemes $\pi\colon X \longrightarrow S$ with a section $x_0\colon S \longrightarrow X$. The reason is the following: In this setting, there is a motive $\mathbb{Z}_X \in \mathcal{DM}(S)$ and the above construction works out without any changes.
- Faltings' logarithm reminds of a well-known construction: The differentials in the sequence P_n (as well as $P_n(\delta)$) coincide with the horizontal differentials of the double complexes used to define bar complexes (see [HZ87] for a definition, as well as chapter I.1). In general, Faltings' complex P_n looks very familiar: namely, it is similar to the theory of bar resolution for groups or algebras.

c.) *Basic ideas for Step 1:*

- View Faltings' motivic logarithm as some sort of "motivic bar complex".
- Generalize Faltings' motivic logarithm and put it into a greater theoretical context, construct a theory of "motivic bar complexes".
- Define the motivic logarithm for any smooth morphism $\pi\colon X \longrightarrow S$ (where both X and S are noetherian, separated and reduced) as an immediate generalization of Faltings' logarithm, using the new language of motivic bar complexes.
- Show that like in Faltings' case, one may retrieve the classical (ℓ-adic or geometric) logarithms for curves and abelian schemes as the zeroth cohomology of the (ℓ-adic or geometric) realization of our motivic logarithm.

Step 2: View the polylogarithm as a Gysin morphism.

a.) *Inspiration: Beilinson and Levin's motivic polylogarithm for elliptic curves*

In [BL94, §6], Beilinson and Levin constructed the elliptic motivic polylogarithm in terms of classes in K-theory. This construction hints at a new interpretation of the polylogarithm as a Gysin morphism, which is in fact the basic idea of our generalization of the polylogarithm. Let us introduce this point of view by vaguely recalling the basic facts regarding Beilinson/Levin's motivic elliptic polylogarithm.

Again, we assume that the reader is vaguely familiar with Levine's theory of motives as summarized in Appendix B. Let E be an elliptic curve over some field F with zero $0 \in E$. We denote the open complement of 0 by $U := E \setminus \{0\}$ Moreover, we define $\sigma\colon U^{n+1} \to E, (x_1, \ldots, x_{n+1}) \longmapsto \sum_{i=0}^{n+1} x_i$, and put

$$U_0^{n+1} := U^{n+1} \setminus (\sigma^{n+1})^{-1}(0).$$

Now consider Beilinson's motivic cohomology groups

$$H_{\mathscr{M}}^{n+2}(X \times_S U_0^{n+1}, \mathbb{Q}(n+1)) = K_n^{(n+1)}(X \times_S U_0^{n+1})_{\mathbb{Q}},$$

where the right hand side is the $(n+1)$-st Adams eigenspace of Quillen K-theory (see section B.6 in the appendix for details). Beilinson and Levin define, for all n, certain subspaces of these cohomology groups whose definition we will not specify - let us denote them by

$$\begin{aligned} H_{\mathscr{M}}^{n+2}(X \times_S U_0^{n+1}, \mathbb{Q}(n+1))^1_{-,\mathrm{sgn}} &= K_n^{(n+1)}(X \times_S U_0^{n+1})^1_{\mathbb{Q}-,\mathrm{sgn}} \\ &\subset H_{\mathscr{M}}^{n+2}(X \times_S U_0^{n+1}, \mathbb{Q}(n+1)), \end{aligned}$$

where we stick to Beilinson/Levin's notation. For these subspaces of motivic cohomology Beilinson and Levin then prove that there is a sequence of isomorphisms

$$\begin{tikzcd} H_{\mathscr{M}}^{n+2}(X \times_S U_0^{n+1}, \mathbb{Q}(n+1))^1_{-,\mathrm{sgn}} \arrow[r, "\mathrm{res}", "\sim"'] & H_{\mathscr{M}}^{n+1}(X \times_S U_0^{n}, \mathbb{Q}(n))^1_{-,\mathrm{sgn}} \arrow[r, "\mathrm{res}", "\sim"'] & \cdots \arrow[dll, "\mathrm{res}", "\sim"'] \\ H_{\mathscr{M}}^{3}(X \times_S U, \mathbb{Q}(2))^1_{-,\mathrm{sgn}} \arrow[r, "\mathrm{res}", "\sim"'] & H_{\mathscr{M}}^{2}(X \times_S U, \mathbb{Q}(1))^1_{-,\mathrm{sgn}} \arrow[d, equal] & \\ & (\mathrm{CH}^1(X \times_S U) \otimes \mathbb{Q})^1_{-,\mathrm{sgn}} & \end{tikzcd}$$

Having established this sequence, Beilinson and Levin define the *motivic elliptic polylogarithm classes* $\mathscr{P}_{\mathscr{M}}^{(n)}$ to be the classes

$$\mathscr{P}_{\mathscr{M}}^{(n)} \in H_{\mathscr{M}}^{n+2}(X \times_S U_0^{n+1}, \mathbb{Q}(n+1))^1_{-,\mathrm{sgn}}$$

which are mapped under the above isomorphism to the class of the diagonal $\Delta \subset X \times_S U$ in $(\mathrm{CH}^1(X \times_S U) \otimes \mathbb{Q})^1_{-,\mathrm{sgn}}$.
In other words, Beilinson and Levin's motivic elliptic polylogarithm classes are determined *entirely* by

$$[\Delta] \in \mathrm{CH}^1(X \times_S U) \otimes \mathbb{Q} \cong H_{\mathscr{M}}^2(X \times_S U, \mathbb{Q}(1))$$

(for the identification with the Chow group see Theorem B.6.2 in the appendix).
Now note that by Levine's theory of mixed motives, the motivic cohomology group on the right hand side may be written as

$$H^2_{\mathscr{M}}(X \times_S U, \mathbb{Q}(1)) \cong \mathrm{Hom}_{\mathcal{DM}_{\mathbb{Q}}(U)}(\mathbb{Q}_U, \mathbb{Q}_{X\times_S U}(1)[2])$$

(see section B.4 in the appendix), where we consider $X \times_S U$ as a scheme over U via the second projection. Note that Levine introduced Gysin morphisms (see section B.2 in the appendix) in his motivic category $\mathcal{DM}_{\mathbb{Q}}(U)$, and by construction, the class of the diagonal

$$[\Delta] \in \mathrm{CH}^1(X \times_S U) \otimes \mathbb{Q} \cong H^2_{\mathscr{M}}(X \times_S U, \mathbb{Q}(1))$$

corresponds to the Gysin isomorphism $\Delta_* \colon \mathbb{Q}_U \longrightarrow \mathbb{Q}_{U\times_S X}(1)[2]$.
Following the above reasoning, the motivic polylogarithm for elliptic curves is determined by the Gysin isomorphism $\Delta_* \colon \mathbb{Q}_U \longrightarrow \mathbb{Q}_{U\times_S X}(1)[2] \in \mathcal{DM}_{\mathbb{Q}}(U)$.

b.) *Basic ideas for Step 2:* The Gysin isomorphism obviously generalizes to arbitrary smooth S-schemes $\pi \colon X \longrightarrow S$ of relative dimension d, where both X and S are noetherian, separated and reduced, and have a section $x_0 \colon S \longrightarrow X$. Putting $U := X \setminus x_0(S)$, there is a Gysin isomorphism

$$\Delta_* \colon \mathbb{Z}_U \longrightarrow \mathbb{Z}_{X\times_S U}(d)[2d] \in \mathcal{DM}(U).$$

corresponding to the diagonal $\Delta \colon U \longrightarrow X \times_S U$. Supposing the polylogarithm is a motivic object and has a general motivic definition for both curves and abelian varieties, then the obvious conclusion of Beilinson and Levin's motivic polylogarithm for elliptic curves would be that the above Gysin isomorphism basically determines the general motivic polylogarithm.

Step 3: Combining Step 1 and 2

Suppose we have finished Step 1, and are left with the following situation

- We have a motivic theory of "bar complexes" formalizing Faltings' construction in a very general setting.
- We have defined a higher-dimensional, general analogue of Faltings' motivic logarithm in terms of these motivic "bar complexes".
- We know that in realizations for curves and abelian varieties, the zeroth cohomology of our motivic logarithm yields the classical logarithm from literature.

Then the **fundamental idea** distinguishing our approach to the polylogarithm from everything in contemporary literature is the following:

In the definition of the polylogarithm, we replace the classical logarithm by our generalized motivic logarithm (whose zeroth cohomology in realizations yields the classical logarithm for curves and abelian varieties). Step 2 then yields the canonical definition of the motivic polylogarithm in the generalized setting: it will be the canonical motivic cohomology class which is determined by the Gysin isomorphism $\Delta_* \colon \mathbb{Z}_U \longrightarrow \mathbb{Z}_{X\times_S U}(d)[2d] \in \mathcal{DM}(U)$.

"Philosophy": The polylogarithm is generally believed to be of *motivic* origin - and there is no notion of zeroth cohomology in the category of motives. Hence, the above approach of replacing the zeroth cohomology by the entire complex is the natural approach to a truly motivic polylogarithm (i.e. one which goes beyond the world of K-theory). Moreover, this explains why up to now the polylogarithm was thought to only exist for curves and abelian schemes: The notion of the logarithm considered was too narrow - only for curves and abelian varieties the motivic logarithm reduces to its zeroth cohomology, i.e. the classical logarithm. For more general schemes it doesn't, thus constructions dealing with the classical logarithm are *bound* to fail.

Outline

In the preceding section, we took a look at the current landscape in the theory of polylogarithms. We want to construct a generalization of the polylogarithm for any smooth morphism of noetherian, separated schemes $\pi\colon X \longrightarrow S$ yielding the known polylogarithms in table (0.1) above. It is clear that the conventional definition of the polylogarithm does *not* generalize, as pointed out in the preceding paragraphs. Instead, we will *explicitly* construct the polylogarithm as a pro-object in Levine's triangulated category of motives, and determine its mixed realization, with a particular focus on the Hodge realization. In order to show that this object we constructed ad hoc coincides with the polylogarithms already defined in the collection of papers of the above table, we will show in chapter II.7 that the mixed realization of our construction satisfies a characterizing property of the polylogarithms as defined in literature.

This ad hoc explicit construction of the polylogarithm, however, relies on a motivic generalization of bar complexes that has not yet been introduced in literature. The first thing we have to do in order to be able to define the motivic polylogarithm is hence to introduce the notion of motivic bar complexes, motivated by the classical bar constructions. This is done in Part I as follows, where the italic parts are essentially new:

Part I:

- **Chapter I.1:** Recap of classical bar constructions and their simplicial interpretation;
- **Chapter I.2:** *Motivic generalization of the simplicial constructions to obtain a notion of motivic bar complexes;*
- **Chapter I.3:** *Computation of the geometric and ℓ-adic realization of the motivic bar complexes constructed*;

Part II:

- **Chapter II.1:**
  - Recap of Hain-Zucker's construction ([HZ87]) of the universal pro-unipotent sheaf via classical bar complexes;
  - *Interpretation of the universal pro-unipotent sheaf as the logarithm as defined in Beilinson and Levin's preprint [BL]* and
  - recollection of Beilinson/Levin's Hodge polylogarithm for curves.
- **Chapter II.2:**
  - *Motivic generalization of Hain-Zucker's and Faltings' construction using the motivic bar complexes of Part I to obtain the notion of a motivic logarithm;*
  - *Ad hoc definition of the motivic polylogarithm using the motivic logarithm of chapter II.6;*
  - *Computation of K-classes associated to the newly constructed polylogarithm;*

- **Chapter II.3:**
  - *Computation of the mixed realization of the motivic logarithm and polylogarithm;*
  - *Proof of characterizing properties of the polylogarithm in the mixed realization;*

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Notations and Conventions

- **When we talk about a "scheme", we will always mean a noetherian and separated scheme.**
- We use the abbreviation "VMHS" to mean "variation of mixed Hodge structure".
- We will always take **double-complexes** to be given by a complex of complexes, i.e. we consider the *commutative* version of double complexes (see D.1 in the appendix).
- **Good compactification of complex varieties:** In what follows, we will silently assume the following: Let X be a smooth complex variety. By Hironaka, there always exists a smooth compact algebraic variety $\bar{X}$ containing X such that the complement $D := \bar{X} - X$ is a simple normal crossing divisor. We will simply call $\bar{X}$ a *good compactification* of X. All upcoming constructions will be independent of a choice of good compactification.

Throughout the thesis, we will suppose that the reader is basically familiar with the following theories/languages:

- Hodge structures, variations of mixed Hodge structure and mixed Hodge modules (see section C.1 in the appendix),
- K-theory (see chapter B.6 in the appendix),
- Levine's theory of mixed motives (see chapter B in the appendix).

Categories:

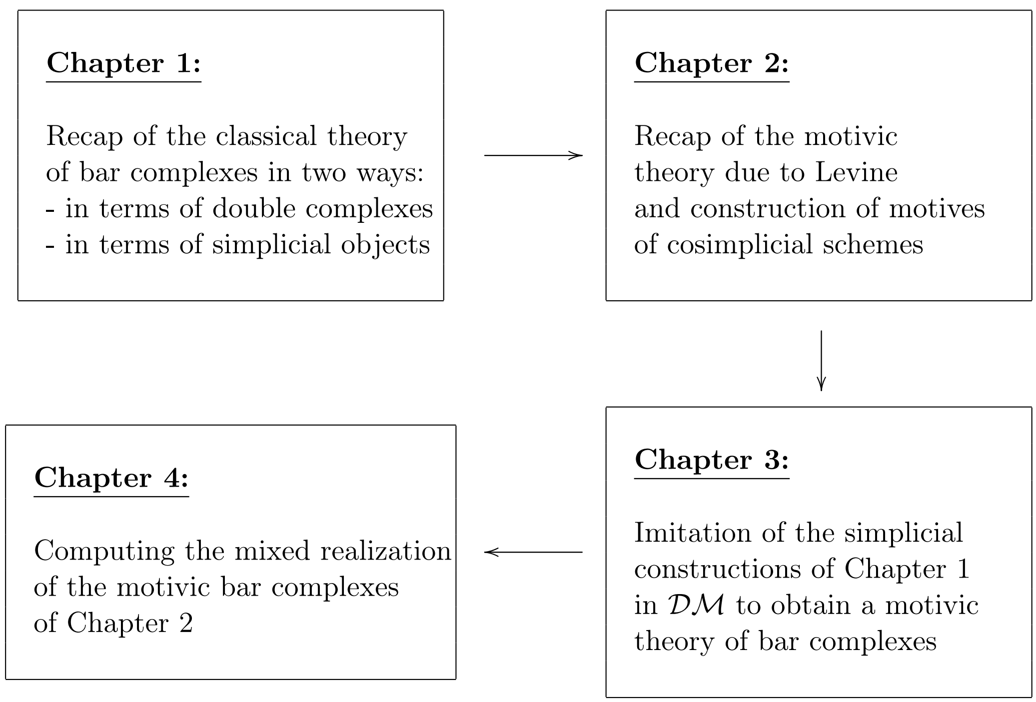
- If S is a reduced, noetherian and separated scheme, then we let Sch_S denote the category of noetherian separated S-schemes and Sm_S the full subcategory of smooth, quasi-projective S-schemes.
- Let $A \subset \mathbb{C}$ be a subring. For any (noetherian separated reduced) scheme S, we denote by $\mathcal{DM}_A(S)$ Levine's triangulated category of motives with "coefficients in A" as described in section B.1 in the appendix. If $A = \mathbb{Z}$, we simply write $\mathcal{DM}(S)$.
- If $R \subset \mathbb{C}$ is a subring, then we denote the category of mixed R-Hodge structures (see section C.3.1) by MHS_R.
- If $A \subset \mathbb{C}$ is a subfield, and S is a complex variety, we denote the category of variations of mixed A-Hodge structure on S by $\mathrm{VMHS}_A(S)$ (see section C.3.2) and the category of mixed A-Hodge modules on S by $\mathrm{MHM}_A(S)$.
- If $\mathcal{A}$ is any category, we let $\mathcal{C}^\bullet(\mathcal{A})$ ($\bullet = +, b, \emptyset$) denote the category of (bounded above, bounded, resp. not necessarily bounded) cochain complexes in $\mathcal{A}$. Sometimes, we will also drop the brackets and simply write $\mathcal{C}^\bullet\mathcal{A}$. If there is a notion of quasi-isomorphism, then the associated derived category will be denoted by $D^\bullet(\mathcal{A})$ ($\bullet = +, b, \emptyset$) or $D^\bullet\mathcal{A}$.

Part I

Motivic Bar Complexes

The overall aim of this thesis is the construction of the motivic polylogarithm in a very general setting. This works out best in a language of motivic bar complexes. Hence, this first part of the thesis does not even mention the word "polylogarithm", but is entirely dedicated to providing a theory of motivic bar complexes generalizing a certain class of classical bar complexes.

The outline of this part is very simple and straight-forward: In Chapter 1, we will recall Chen's bar constructions and list the most important properties. Here, after a down-to-earth introduction of bar complexes, we will focus on their simplicial interpretation. In Chapter 2, we simply recall the motivic theory of Levine and provide the motivic formalism we will need in the following chapter. The simplicial formalism of Chapter 1 and the motivic language of Chapter 2 will be used in Chapter 3 to carry the bar constructions over to the motivic world. A more visual summary of Part I would be the following diagram:



Chapter 1

Classical bar constructions

Classical bar constructions have been around in various fields of mathematics for a long time, and in multiple variants. The bar complex was originally introduced by Chen in order to formalize the complex of iterated integrals. When we talk about the "classical" bar complex, we will always mean Chen's bar complex - notwithstanding the fact that bar constructions can also be found in other variants, e.g. in form of the *bar resolution* of groups or algebras.

What we want to find in this part of the thesis is a motivic analogue for the classical bar complex. Naturally, we need to diligently study the classical case in order to find the correct means of motivic generalization, and the correct "setting" to work in.

In this chapter, we will proceed as follows:

- Firstly, we will consider the two types of bar constructions as introduced by Chen: the *unreduced* and *reduced bar complex*. Chen considered them as total complexes of certain double complexes, which is how we will first present them.
- Secondly, we will take a different look at the theory of classical bar complexes: bar complexes underlie the structure of simplicial objects. Since the theory of simplicial objects is a beautiful tool to work with and well understood, this approach to classical bar complexes is of major importance for us.
- Thirdly, we will apply the theory of "simplicial bar objects" underlying the bar complex to the case of smooth forms on a scheme. This application will turn out to be of motivic origin in Chapter 4.

1.1 The (unreduced) bar complex

1.1.1 Definition

In what follows we will deal with commutative double complexes as explained in section D.1 of the appendix. The bar complex constructed this way is the same as in [HZ87], only the underlying bar double complex is commuting instead of anticommuting.

Let k be a field, $R^\bullet$ be a differential graded k-algebra (the most common case is $R^\bullet = k$), and $A = \bigoplus_{p\geq 0} A^p$ a differential graded k-algebra with differential $d\colon A^k \longrightarrow A^{k+1}$

which is a differential graded R-module. Moreover, suppose $R^\bullet$ admits the structure of a differential graded $A^\bullet$-bimodule via two morphism of differential graded algebras

$$x, y\colon A^\bullet \longrightarrow R^\bullet,$$

where left-multiplication is given by x, and right-multiplication by y. Denote the degree of an element $a \in A^\bullet$ or $r \in R^\bullet$ by $|a|$ and $|r|$. Moreover, let $A^{\otimes r} := A \otimes_R A \otimes_R \ldots \otimes_R A$ be the r-fold tensor product of $A^\bullet$ over $R^\bullet$, with an element of $A^{\otimes r}$ denoted by

$$[a_1|\ldots|a_r] := a_1 \otimes \ldots \otimes a_r \qquad \text{for } a_1, \ldots, a_r \in A^\bullet.$$

We extend the degree $|\,.\,|$ of A to $A^{\otimes r}$ in the usual fashion, i.e.

$$|[a_1|\ldots|a_r]| = |a_1| + \ldots + |a_r| \qquad \text{for } a_1, \ldots, a_r \in A^\bullet.$$

Consider the array with bidegree

$$\begin{tikzcd}
\ldots \arrow[r] & (R\otimes A^{\otimes r})^2 \arrow[r, "\delta"] & (R\otimes A^{\otimes r-1})^2 \arrow[r, "\delta"] & \ldots \arrow[r, "\delta"] & (R\otimes A)^2 \arrow[r, "\delta"] & R^2 & 2 \\
\ldots \arrow[r] & (R\otimes A^{\otimes r})^1 \arrow[r, "\delta"] \arrow[u, "\partial"] & (R\otimes A^{\otimes r-1})^1 \arrow[r, "\delta"] \arrow[u, "\partial"] & \ldots \arrow[r, "\delta"] & (R\otimes A)^1 \arrow[r, "\delta"] \arrow[u, "\partial"] & R^1 & 1 \\
\ldots \arrow[r] & (R\otimes A^{\otimes r})^0 \arrow[r, "\delta"] \arrow[u, "\partial"] & (R\otimes A^{\otimes r-1})^0 \arrow[r, "\delta"] \arrow[u, "\partial"] & \ldots \arrow[r, "\delta"] & (R\otimes A)^0 \arrow[r, "\delta"] \arrow[u, "\partial"] & R^0 & 0 \\
\ldots \arrow[r] & (R\otimes A^{\otimes r})^{-1} \arrow[r, "\delta"] \arrow[u, "\partial"] & (R\otimes A^{\otimes r-1})^{-1} \arrow[r, "\delta"] \arrow[u, "\partial"] & \ldots \arrow[r, "\delta"] & (R\otimes A)^{-1} \arrow[r, "\delta"] \arrow[u, "\partial"] & 0 & -1 \\
 & -r & -(r-1) & \ldots & -1 & \deg_{\mathrm{simpl}}\backslash\deg_A &
\end{tikzcd} \tag{1.1}$$

where $(R \otimes A^{\otimes r})^j$ is the total degree j-part of $R \otimes A^{\otimes r}$, and the vertical complex $R \otimes A^{\otimes r}$ is in simplicial (i.e. horizontal) degree $-r$. Here, the differentials are given by

$$\begin{aligned}
\partial_r^s\colon b \otimes [a_1|\ldots|a_r] \longmapsto\ & (-1)^r \Bigg(db \otimes [a_1|\ldots|a_r] \\
& + \sum_{i=1}^{r} (-1)^{|b|+\sum_{k=1}^{i-1}|a_k|+i} b \otimes [a_1|\ldots|da_i|a_{i+1}|\ldots|a_r] \Bigg) \\
\delta_r'^s\colon b \otimes [a_1|\ldots|a_r] \longmapsto\ & -x(a_1) \cdot b \otimes [a_2|\ldots|a_r] \\
& + \sum_{i=1}^{r-1} (-1)^{\sum_{k=1}^{i}|a_k|+i+1} b \otimes [a_1|\ldots|a_{i-1}|a_i a_{i+1}|a_{i+2}\ldots|a_r] \\
& + (-1)^{(|b|+\sum_{k=1}^{r-1}|a_k|+r)(|a_r|+1)} y(a_r) \cdot b \otimes [a_1|\ldots|a_{r-1}].
\end{aligned}$$

where $s = |b \otimes [a_1|\dots|a_r]|$.
It is a well-known fact (which can be verified by a lengthy but simple calculation) that $\partial^2 = \delta'^2 = \delta'\partial - \partial\delta' = 0$, so the above array is a second/third quadrant double complex.

Definition 1.1.1. *We call the above double complex* (1.1) *the* <u>*(unreduced) bar double complex*</u>*, and define the* <u>*(unreduced) bar complex of the $R^\bullet$-module $A^\bullet$*</u> *$B(A^\bullet|R^\bullet)_{x,y}$ to be the associated $\oplus$-total complex of the above commutative double complex (i.e. summing the diagonals of slope (-1) and taking the differential to be $\partial_r^s + (-1)^r\delta_r'^s$.) For $R^\bullet = k$ one simply writes $B(A^\bullet)_{x,y} := B(A^\bullet|k)_{x,y}$.*

1.1.2 Properties

a.) *The bar filtration:* There is a filtration of the bar complex by $\deg_{\text{simpl}}$: Letting $B_r(A|R)_{x,y}$ denote the total complex of the subdiagram

$$\begin{array}{cccccccccc}
0 & \longrightarrow & (A^{\otimes r})^2 & \xrightarrow{\delta'} & \dots & \xrightarrow{\delta'} & (A)^2 & \xrightarrow{\delta'} & R^2 & \quad 2\\
 & & \uparrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\partial} & & & \\
0 & \longrightarrow & (A^{\otimes r})^1 & \xrightarrow{\delta'} & \dots & \xrightarrow{\delta'} & A^1 & \xrightarrow{\delta'} & R^1 & \quad 1\\
 & & \uparrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\partial} & & & \\
0 & \longrightarrow & (A^{\otimes r})^0 & \xrightarrow{\delta'} & \dots & \xrightarrow{\delta'} & A^0 & \xrightarrow{\delta'} & R^0 & \quad 0\\
 & & \uparrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\partial} & & & \\
0 & \longrightarrow & (A^{\otimes r})^{-1} & \xrightarrow{\delta'} & \dots & \xrightarrow{\delta'} & A^{-1} & \xrightarrow{\delta'} & 0 & \quad -1\\
 & & & & & & & & & \\
 & & -r & & \dots & & -1 & & \deg_{\text{simpl}} \backslash \deg_{A^{\otimes\bullet}} &
\end{array}$$

of the bar diagram in the definition, $\mathfrak{B} := \{B_r\}$ is a filtration of $B(A|R)_{x,y}$ by subcomplexes. Its graded quotients are given by the column of the bar double complex with $\deg_{\text{simpl}} = -r$ in degree $-r$, i.e. one has $\operatorname{gr}_r^{\mathfrak{B}} B(A,M)_{x,y} \cong A^{\otimes r}[r]$.

b.) *Functoriality:* The bar complex is covariantly functorial: Suppose we are given another differential graded $R^\bullet$-module A' such that $R^\bullet$ is endowed with the structure of a differential graded $A^\bullet$-bimodule via morphisms $x', y' \colon (A')^\bullet \longrightarrow R^\bullet$ of differential graded k-algebras. Suppose furthermore that we are given a morphism $\varphi\colon A \longrightarrow A'$ of differential graded R-modules such that $x = x' \circ \varphi$ and $y = y' \circ \varphi$. Then these induce morphisms $\varphi^{\otimes r}\colon A^{\otimes r} \longrightarrow A'^{\otimes r}$ of complexes of k-vector spaces (where we view A as a complex), which are obviously compatible with the morphisms of complexes δ'. Hence, φ induces morphisms of the bar double complexes associated to A and A', and hence also a morphism of the associated total complexes

$$\varphi^{\otimes\bullet}\colon B(A|R)_{x,y} \longrightarrow B(A'|R)_{x',y'}.$$

Since $\varphi^{\otimes\bullet}$ is induced by the morphism of the underlying double complexes, it is compatible with the bar filtration, i.e. induces morphisms

$$(\varphi^{\otimes\bullet})_r\colon B_r(A|R)_{x,y} \longrightarrow B_r(A'|R)_{x',y'}.$$

Proposition 1.1.2. *If φ is a quasi-isomorphism, then the induced morphisms*

$$\varphi_r^{\otimes\bullet}\colon B_r(A|R)_{x,y} \longrightarrow B_r(A'|R)_{x',y'} \text{ are quasi-isomorphisms.}$$

Proof. This is completely analogous to the reduced case, see [HZ87, 3.14, p.92]: If φ is a quasi-isomorphism, then for all i the induced morphisms

$$\varphi^{\otimes i}\colon A^{\otimes i} = \mathrm{gr}_i^{\mathfrak{B}} B(A|R)_{x,y}[-i] \longrightarrow \mathrm{gr}_i^{\mathfrak{B}} B(A'|R)_{x',y'}[-i] = A'^{\otimes i}$$

are quasi-isomorphisms. Since $\mathfrak{B}$ defines a finite filtration of $B(A|R)_{x,y}$ and $B(A'|R)_{x',y'}$, the quasi-isomorphisms

$$\mathrm{gr}_i^{\mathfrak{B}} B(A|R)_{x,y} \simeq \mathrm{gr}_i^{\mathfrak{B}} B(A'|R)_{x',y'}$$

for all i show that $B_r(A|R)_{x,y} \longrightarrow B_r(A'|R)_{x',y'}$ is a quasi-isomorphism. ■

c.) *Hodge structure:* (For the reduced case, see [HZ87, 3.15, p.92]) Assume that A and R are both cohomologically connected, i.e. graded in non-negative degrees and $H^0(A^\bullet) = k$. Moreover, suppose that both R and A underlie compatible (regarding the module-structure) mixed k-Hodge complexes with Hodge and weight filtrations (F_A, W_A) and (F_R, W_R), respectively, such that the weight filtrations of $A^\bullet$ and $R^\bullet$ are bounded below. Let us furthermore suppose that the left and right module structures x, y are morphisms of Hodge complexes. Then F_A and W_A induce filtrations on the tensor products $A^{\otimes r}[r]$ for $r > 0$ (where the shift $[r]$ induces a shift of weights), and (F_R, W_R) induces filtrations on $A^{\otimes 0} = R^\bullet$ where, again, the weight filtrations are bounded below.

Proposition 1.1.3. *The morphisms $\delta_r(x, y)$ are morphisms of mixed Hodge complexes.*

Proof. It suffices to prove that the individual summands of the morphism ∂ are compatible with Hodge and weight filtrations. Here, we may forget about signs:

(i) Claim: The morphism $[a_1|\dots|a_r] \longmapsto x(a_1)\cdot[a_2|\dots|a_r]$ is compatible with all weight and Hodge filtrations.
Suppose $a_i \in W_{k_i}A$. Then $[a_1|\dots|a_r] \in A^{\otimes r}[r]$ is in $W_{\sum_i k_i + r}$. Since x is a morphism of Hodge complexes, we know that $x(a_1)$ is an element of $W_{k_1}R$, and hence $x(a_1)\cdot[a_2|\dots|a_r]$ is in $W_{k_1+k_2}A \otimes \dots W_{k_r}A \subset W_{\sum_i k_i}(A^{\otimes r-1})$, which shows that the above morphism is indeed compatible with the weight filtrations. In exactly the same way (replace W by F everywhere) one can see that it is also filtered with respect to the Hodge filtrations.

(ii) Claim: For any $0 < i < r$, the morphism $[a_1|\dots|a_r] \longmapsto [a_1|\dots|a_i a_{i+1}|\dots|a_r]$ is filtered with respect to all weight and Hodge filtrations.
This follows from the same reasoning as above: since algebra-multiplication of A is a morphism of Hodge complexes, we know that if $[a_1|\dots|a_r]$ is as in (i), then $a_i a_{i+1}$ is in $W_{k_i+k_{i+1}}A$, and hence $[a_1|\dots|a_i a_{i+1}|\dots|a_r]$ is in $W_{k_1}A \otimes \dots \otimes W_{k_i+k_{i+1}}A \otimes \dots \otimes W_{k_r}A \subset W_{\sum_i k_i}(A^{\otimes r-1})$ as asserted. Replace W by F to see that the same holds for Hodge filtrations.

(iii) Claim: The morphism $[a_1|\dots|a_r] \longmapsto [a_1|\dots|a_{r-1}] \cdot y(a_r)$ is compatible with all weight and Hodge filtrations. This follows exactly like (ii). ∎

Since the differentials ∂_r are all morphisms of mixed Hodge complexes, the Hodge and weight filtrations of A and R induce natural Hodge and weight filtrations on the entire double complex. Given a filtration on a double complex, there is an induced filtration on the total complex given by the diagonal filtration. In our case, this is the following:

$$\delta'(W)_k B(A|R)_{x,y} := \bigoplus_{i+j=k} W_i(\text{column in}\deg_{simpl} = -j).$$

It is easy to see that it coincides with the filtration $W * \mathfrak{B}$ defined by

$$(W * \mathfrak{B})_k B(A|R)_{x,y} := \bigoplus_{i+j=k} W_i \operatorname{gr}_j^{\mathfrak{B}} B(A|R)_{x,y}$$

whose graded quotients are given by the column complexes $\bigoplus_{i+j=k} \operatorname{gr}_i^W(A^{\otimes j})[j]$.

Lemma 1.1.4. *([HZ87, 3.15, p.92] for the reduced case)*
Under the above hypotheses, $B(A|R)_{x,y}$ with filtrations $(W * \mathfrak{B}, F)$ is a mixed Hodge complex over k, filtered by $\mathfrak{B}$.

Proof. To see that $B(A|R)_{x,y}$ with the above filtrations is a mixed Hodge complex, we have to show that the weight graded quotients

$$\operatorname{gr}_k^{(W*\mathfrak{B})} \cong \bigoplus_{i+j=k} \operatorname{gr}_i^W(A^{\otimes j})[j].$$

are pure Hodge complexes. This, however, is immediate from the hypothesis that A and R are mixed Hodge complexes, and hence so is $A^{\otimes j}$ for $j \geq 0$: This means that for all i, j, the summand $\operatorname{gr}_i^W(A^{\otimes j})$ is a pure Hodge complex, and hence so is $\operatorname{gr}_k^{(W*\mathfrak{B})}$. ∎

d.) *Hopf algebra structure for $R = k$:* Again assume that A is cohomologically connected, i.e. graded in positive degrees and $H^0(A^\bullet) = k$. In the special case $R = k$ the bar complex carries additional structure:

- *Product:*
  For any three morphisms of differential graded algebras $x, y, z \colon A \longrightarrow k$, the bar complex $B(A)$ admits a product

$$\begin{aligned} B(A)_{x,z} \otimes B(A)_{z,y} &\longrightarrow B(A)_{x,y} \\ \nabla\colon [a_1|\ldots|a_r] \otimes [a_{r+1}|\ldots|a_{r+s}] &\longmapsto \sum_{\sigma\in S_{r,s}} (-1)^{\operatorname{sgn}(\sigma,a)} [a_{\sigma^{-1}(1)}|\ldots|a_{\sigma^{-1}(r+s)}], \end{aligned}$$

where $\operatorname{sgn}(\sigma, a) = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} (|a_i|-1)(|a_j|-1)$. This product is associative, graded-commutative, unital with unit given by inclusion $i\colon k \hookrightarrow B(A)_{x,y}$, and compatible with the total differential.

- *Coproduct*:
Let $x, y, z\colon A \longrightarrow k$ be as above. There is a coproduct defined by

$$\begin{aligned} \Delta\colon B(A)_{x,y} &\longrightarrow B(A)_{x,z} \otimes B(A)_{z,y} \\ [a_1|\ldots|a_r] &\longmapsto \sum_{i=0}^{r} [a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_r], \end{aligned}$$

where the right hand side is to be read as $1 \otimes [a_1|\ldots|a_r]$ for $i = 0$ and $[a_1|\ldots|a_r] \otimes 1$ for $i = r$. This coproduct is co-associative, i.e. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, compatible with the co-augmentation $\epsilon\colon B(A)_{a,b} \longrightarrow k$ given by the projection of $B(A)_{a,b}$ to k, and is a morphism of complexes. Moreover, it has a counit given by ϵ.
- *Compatibility:* The algebra and coalgebra structures are compatible with each other, i.e. the counit and coproduct are morphisms of unitary algebras.
- *Antipode:* There is an antipode defined by

$$\begin{aligned} S\colon B(A)_{a,b} &\longrightarrow B(A)_{a,b} \\ [a_1|\ldots|a_r] &\longmapsto (-1)^{r+\operatorname{sgn}(a)} [a_r|\ldots|a_1], \end{aligned}$$

where $\operatorname{sgn}(a)$ is the sign of the permutation $(a_1, \ldots, a_r) \longmapsto (a_r, \ldots, a_1)$.

For any augmentation $x\colon A^\bullet \longrightarrow k$, this gives the bar complex $B(A)_{x,x}$ the structure of a Hopf algebra, i.e the diagram

$$\begin{tikzcd} & B(A)_{x,x} \otimes B(A)_{x,x} \arrow[r, "S\otimes\mathrm{id}"] & B(A)_{x,x} \otimes B(A)_{x,x} \arrow[dr, "\nabla"] & \\ B(A)_{x,x} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & k \arrow[r, "i"] & B(A)_{x,x} \\ & B(A)_{x,x} \otimes B(A)_{x,x} \arrow[r, "\mathrm{id}\otimes S"'] & B(A)_{x,x} \otimes B(A)_{x,x} \arrow[ur, "\nabla"'] & \end{tikzcd}$$ commutes.

1.1.3 Sheaf setting

The definition of the bar complex also makes sense in the sheaf setting as follows: Let X be a scheme, k a field, $\mathscr{R}^\bullet$ a sheaf of differential graded k-algebras, and

$\mathscr{A} = \bigoplus_{p \geq 0} \mathscr{A}^p$ a sheaf of differential graded k-algebras, which moreover is a differential graded $\mathscr{R}$-module. Moreover, suppose $\mathscr{R}^\bullet$ admits the structure of a differential graded $\mathscr{A}^\bullet$-bimodule via two morphism of sheaves of differential graded k-algebras $x, y \colon \mathscr{A}^\bullet \longrightarrow \mathscr{R}^\bullet$, where left-multiplication is given by x, and right-multiplication by y. Then the same constructions as above yield a sheaf

$$B(\mathscr{A}^\bullet | \mathscr{R}^\bullet)_{x,y} \text{ on } X.$$

By local considerations, the properties of section 1.1.2 naturally carry over to the sheaf setting:

a.) *Bar filtration* Obviously, as in the non-sheaf case, there is the bar filtration $\mathfrak{B} := \{B_r(\mathscr{A}|\mathscr{R})_{x,y}\}$ of $B(\mathscr{A}|\mathscr{R})_{x,y}$, which is given by letting $B_r(\mathscr{A}|\mathscr{R})_{x,y}$ denote the total complex of

$$\begin{tikzcd}
0 \arrow[r] & (\mathscr{A}^{\otimes r})^2 \arrow[r, "\delta'"] & \dots \arrow[r, "\delta'"] & (\mathscr{A})^2 \arrow[r, "\delta'"] & \mathscr{R}^2 & 2 \\
0 \arrow[r] & (\mathscr{A}^{\otimes r})^1 \arrow[u, "\partial"] \arrow[r, "\delta'"] & \dots \arrow[r, "\delta'"] & \mathscr{A}^1 \arrow[u, "\partial"] \arrow[r, "\delta'"] & \mathscr{R}^1 & 1 \\
0 \arrow[r] & (\mathscr{A}^{\otimes r})^0 \arrow[u, "\partial"] \arrow[r, "\delta'"] & \dots \arrow[r, "\delta'"] & \mathscr{A}^0 \arrow[u, "\partial"] \arrow[r, "\delta'"] & \mathscr{R}^0 & 0 \\
0 \arrow[r] & (\mathscr{A}^{\otimes r})^{-1} \arrow[u, "\partial"] \arrow[r, "\delta'"] & \dots \arrow[r, "\delta'"] & \mathscr{A}^{-1} \arrow[u, "\partial"] \arrow[r, "\delta'"] & 0 & -1 \\
 & -r & \dots & -1 & \deg_{\mathrm{simpl}} \backslash \deg_{\mathscr{A}^{\otimes \bullet}}. &
\end{tikzcd}$$

b.) *Functoriality:* Covariant functoriality carries over verbatim from section 1.1.2, replacing R by $\mathscr{R}$, A by $\mathscr{A}$. Moreover, by local considerations, Proposition 1.1.2 immediately shows that quasi-isomorphisms of differential graded $\mathscr{R}$-modules with compatible sections yield quasi-isomorphic bar complexes.

c.) *Hodge structure:* Assume that $\mathscr{A}$ and $\mathscr{R}$ are graded in positive degrees and the stalks of $H^0(\mathscr{A}^\bullet)$ are equal to k. Moreover, suppose that both are both mixed B-Hodge complexes of sheaves (see [Bei94] for a good introduction of Hodge complexes (of sheaves)) for some subring $B \subset \mathbb{C}$, such that the $\mathscr{R}$-module structure of $\mathscr{A}$ as well as the sections x and y are morphisms of mixed Hodge complexes of sheaves. Moreover, suppose all weight filtrations are bounded below. Then considering the situation locally, it is obvious that the morphisms $\delta_r(x, y)$ are morphisms of mixed Hodge complexes. Hence, the entire bar double complex is a complex of mixed B-Hodge complexes of sheaves. As above, one can see that its total complex $B(\mathscr{A}|\mathscr{R})_{x,y}$ carries an induced structure of a mixed B-Hodge complex of sheaves.

d.) *Hopf algebra structure for $R = k$:* Assume that $\mathscr{A}$ and $\mathscr{R}$ are graded in positive degrees and the stalks of $H^0(\mathscr{A}^\bullet)$ are equal to k. Moreover, assume that $\mathscr{R}$ is a local system with stalks equal to k. Then the bar complex sheaf carries additional structure: For any three morphisms of differential graded algebras $x, y, z\colon \mathscr{A} \longrightarrow \mathscr{R}$, the bar complex $B(\mathscr{A}|\mathscr{R})$ admits a product, coproduct and antipode

$$\begin{aligned} B(\mathscr{A}|\mathscr{R})_{x,z} \otimes B(\mathscr{A}|\mathscr{R})_{z,y} &\longrightarrow B(\mathscr{A}|\mathscr{R})_{x,y} \\ \Delta\colon B(\mathscr{A}|\mathscr{R})_{x,y} &\longrightarrow B(\mathscr{A}|\mathscr{R})_{x,z} \otimes B(\mathscr{A}|\mathscr{R})_{z,y} \\ S\colon B(\mathscr{A}|\mathscr{R})_{x,y} &\longrightarrow B(\mathscr{A}|\mathscr{R})_{y,x} \end{aligned}$$

given locally by exactly the same formulae as in the non-sheaf case above. Obviously, local considerations show that the product is again associative, graded-commutative, unital with unit given by inclusion $i\colon \mathscr{R} \hookrightarrow B(\mathscr{A}|\mathscr{R})_{x,y}$, and compatible with the total differential. Likewise, the coproduct is co-associative, i.e. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, compatible with the co-augmentation $\epsilon\colon B(\mathscr{A}|\mathscr{R})_{x,y} \longrightarrow \mathscr{R}$ given by the projection of $B(\mathscr{A}|\mathscr{R})_{x,y}$ to $\mathscr{R}$, and is a morphism of complexes. Moreover, it has a counit given by ϵ. Product and coproduct are compatible. As a consequence, for any augmentation $x\colon \mathscr{A}^\bullet \longrightarrow \mathscr{R}$, this gives the bar complex $B(\mathscr{A}|\mathscr{R})_{x,x}$ the structure of a Hopf algebra, i.e the diagram

$$\begin{tikzcd} & B(\mathscr{A}|\mathscr{R})_{x,x} \otimes B(\mathscr{A}|\mathscr{R})_{x,x} \arrow[r, "S\otimes\mathrm{id}"] & B(\mathscr{A}|\mathscr{R})_{x,x} \otimes B(\mathscr{A}|\mathscr{R})_{x,x} \arrow[dr, "\nabla"] & \\ B(\mathscr{A}|\mathscr{R})_{x,x} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[r, "\epsilon"] & k \arrow[r, "i"] & & B(\mathscr{A}|\mathscr{R})_{x,x} \\ & B(\mathscr{A}|\mathscr{R})_{x,x} \otimes B(\mathscr{A}|\mathscr{R})_{x,x} \arrow[r, "\mathrm{id}\otimes S"] & B(\mathscr{A}|\mathscr{R})_{x,x} \otimes B(\mathscr{A}|\mathscr{R})_{x,x} \arrow[ur, "\nabla"'] & \end{tikzcd}$$

commutes.

1.1.4 An isomorphic definition

Let again k be a field, $R^\bullet$ be a differential graded k-algebra (the most common case is $R^\bullet = k$), and $A = \bigoplus_{p \geq 0} A^p$ a differential graded k-algebra with differential $d\colon A^k \longrightarrow A^{k+1}$ which is a differential graded R-module. Moreover, suppose $R^\bullet$ admits the structure of a differential graded $A^\bullet$-bimodule via two morphism of differential graded algebras $x, y\colon A^\bullet \longrightarrow R^\bullet$, where left-multiplication is given by x, and right-multiplication by y. However, note that everything in this section is also valid in the sheaf setting.

In this section, we want to find a different - and from a motivic perspective: more natural - definition of the bar double complex above. The first thing one notes about the bar double complex is that the columns are not just given by the complex $A^{\otimes n}$, which would seem somewhat simpler. Instead, we have the following:

Lemma 1.1.5. *The column of the bar double complex in simplicial degree $-r$*

$$0 \xrightarrow{\partial} (A^{\otimes r})^1 \xrightarrow{\partial} (A^{\otimes r})^2 \xrightarrow{\partial} (A^{\otimes r})^3 \to \dots,$$

is given by the differential graded algebra $(A[-1])^{\otimes r}[r]$.

Proof. By the ordinary conventions, the differential of $A[-1]$ is given by $-d$, and the degrees of all elements are lowered by 1. The differential of the tensor product $A \otimes B$ of two dgas (A, d_A) and (B, d_B) is defined to be $d(a \otimes b) = da \otimes b + (-1)^{\deg a} a \otimes d(b)$. Using this inductively shows that the differential of the dga $(A[-1])^{\otimes r}$ is given by

$$\begin{aligned} d([a_1|\dots|a_r]) &= \sum_{i=1}^{r} (-1)^{|a_1|-1+\dots+|a_{i-1}|-1} [a_1|\dots| - da_i|a_{i+1}|\dots|a_r] \\ &= \sum_{i=1}^{r} (-1)^{|a_1|+\dots+|a_{i-1}|+i} [a_1|\dots|da_i|a_{i+1}|\dots|a_r]. \end{aligned}$$

The dga $(A[-1])^{\otimes r}[r]$ is given in degree k by $(A^{\otimes r})^k$, and its differential is the above multiplied by $(-1)^r$ due to the $[r]$-shift. Thus, the differential graded algebra $(A[-1])^{\otimes r}[r]$ indeed coincides with the r-th row of the bar double complex. ■

Philosophically speaking, we want to "translate" the bar double complex with columns given by $(A[-1])^{\otimes r}[r]$ into a double complex with columns given by $A^{\otimes r}$ (and, if possible, nicer horizontal differentials). To do this, we have to find an explicit, natural isomorphism $(A[-1])^{\otimes r}[r] \cong A^{\otimes r}$. The first step is the following:

Proposition 1.1.6. *Let (A, d_A), B, d_B be differential graded algebras, and $n \in \mathbb{Z}$. Then we have*

$$(A[n]) \otimes B = (A \otimes B)[n] \cong A \otimes (B[n]).$$

Here, the isomorphism $A \otimes (B[n]) \longrightarrow (A \otimes B)[n]$ is given by $a \otimes b \longmapsto (-1)^{-n|a|} a \otimes b$.

Proof. Obviously, for any k, one has $(A[n] \otimes B)^k = (A \otimes B[n])^k = ((A \otimes B)[n])^k$. Hence, it suffices to check the differentials. Let $|\,.\,|$ denote the degree of an element in A. The differentials d_1 of $A[n] \otimes B$, and d_2 of $(A \otimes B)[n]$ are given by

$$d_1(a \otimes b) = (-1)^n da \otimes b + (-1)^{|a|-n} a \otimes db = (-1)^n (da \otimes b + (-1)^{|a|} a \otimes b) = d_2,$$

so we obtain $A[n] \otimes B = (A \otimes B)[n]$. Next, we use the fact that for any two differential graded algebras (A, d_A) and (B, d_B), there is an isomorphism of dgas $A \otimes B \cong B \otimes A$ given by $a \otimes b \longmapsto (-1)^{\deg(a)\deg(b)} b \otimes a$. Thus we have an isomorphism

$$\begin{array}{ccccccc} A \otimes (B[n]) & \xrightarrow{\sim} & (B[n]) \otimes A & = & (B \otimes A)[n] & \xrightarrow{\sim} & (A \otimes B)[n] \\ a \otimes b & \longmapsto & (-1)^{|a|(|b|-n)} b \otimes a & = & (-1)^{|a|(|b|-n)} b \otimes a & \mapsto & (-1)^{|a|(|b|-n)+|a||b|} a \otimes b \\ & & & & & & = (-1)^{-n|a|} a \otimes b. \end{array}$$ ■

We use this to prove the following about the columns of the bar double complex:

Proposition 1.1.7. *There is a natural isomorphism*

$$\begin{aligned}\psi_r \colon (R \otimes (A[-1])^{\otimes r})[r] &\longrightarrow A^{\otimes r}\\ b \otimes [a_1|\dots|a_r] &\longmapsto (-1)^{\mu(b,a_1,\dots,a_{r-1})} b \otimes [a_1|\dots|a_r].\end{aligned}$$

where $\mu(b, a_1, \dots, a_{r-1}) = r \cdot b + \sum_{k=1}^{r-1} (r-k) \cdot |a_k|$.

Proof. We show this by induction. For $r = 1$, we have the following:

$$\begin{array}{ccccc} (R \otimes (A[-1]))[2] & \xrightarrow{\text{flip}_1} & ((A[-1]) \otimes R)\,[1] = A \otimes R & \xrightarrow{\text{flip}_2} & R \otimes A \cong A \\ b \otimes a & \longmapsto & (-1)^{(|a|+1)|b|} b \otimes a & \longmapsto & (-1)^{|b|} b \cdot a \end{array}$$

as asserted. Suppose we have shown the assertion for $r-1$, i.e. the isomorphism $(R \otimes A[-1])^{\otimes r-1}[r-1] \cong A^{\otimes r-1}$ is given by

$$r \otimes [a_1|\dots|a_{r-1}] \longmapsto (-1)^{(r-1)\cdot b \sum_{k=1}^{r-2} (r-k-1)\cdot |a_k|} [ba_1|\dots|a_{r-1}].$$

First, note that by Proposition 1.1.6 above, we may identify

$$\left(R \otimes (A[-1])^{\otimes r}\right)[r] = \left(\left(R \otimes (A[-1])^{\otimes r-1}\right)[r-1] \otimes (A[-1])\right)[1].$$

Now consider the following composition of isomorphisms:

$$\begin{aligned}\psi_r \colon \left(\left(R \otimes (A[-1])^{\otimes r-1}\right)[r-1] \otimes (A[-1])\right)[1] &\xrightarrow[\sim]{\psi_{r-1}\otimes \mathrm{id}} \left(A^{\otimes r-1} \otimes (A[-1])\right)[1] \\ &\xrightarrow[\sim]{\text{flip}_1} \left((A[-1]) \otimes A^{\otimes r-1}\right)[1] = A \otimes A^{\otimes r-1} \xrightarrow[\sim]{\text{flip}_2} A^{\otimes r-1} \otimes A = A^{\otimes r}\end{aligned}$$

where flip_1 and flip_2 are the obvious isomorphisms. This composition is given on elements by

$$\begin{aligned}
&\psi_r(b \otimes [a_1|\dots|a_r]) \\
&= \text{flip}_2 \circ \text{flip}_1 \circ (\psi_{r-1} \otimes \mathrm{id})(b \otimes [a_1|\dots|a_{r-1}] \otimes a_r) \\
&= (\text{flip}_2 \circ \text{flip}_1)((-1)^{(r-1)\cdot b + \sum_{k=1}^{r-2}(r-1-k)\cdot|a_k|}[ba_1|\dots|a_{r-1}] \otimes a_r) \\
&= \text{flip}_1 \left((-1)^{\sum_{k=1}^{r-2}(r-k-1)\cdot|a_k| + \left(|b| + \sum_{k=1}^{r-1}|a_k|\right)(|a_r|+1)} a_r \otimes [a_1|\dots|a_{r-1}]\right) \\
&= (-1)^{\sum_{k=1}^{r-2}(r-k-1)\cdot|a_k| + \left(|b| + \sum_{k=1}^{r-1}|a_k|\right)(|a_r|+2)} [a_1|\dots|a_r] \\
&= (-1)^{r\cdot b + \sum_{k=1}^{r-1}(r-k)\cdot|a_k|}[a_1|\dots|a_r],
\end{aligned}$$

which proves the assertion. ■

What is left to do in order to find an isomorphism of the bar double complex to an "easier" double complex is to translate the differentials:

Lemma 1.1.8. *Via the isomorphism of Proposition 1.1.7, the morphism of differential graded algebras* $\delta_r'\colon R\otimes(A[-1])^{\otimes r}[r] \longrightarrow R\otimes(A[-1])^{\otimes r-1}[r-1]$ *corresponds to the morphism*

$$\begin{aligned}
A^{\otimes r} &\longrightarrow A^{\otimes r-1},\\
[a_1|\dots|a_r] &\longmapsto -[x(a_1)\cdot a_2|\dots|a_r] + \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\dots|a_ia_{i+1}|\dots|a_r]\\
&\quad +(-1)^{r+1}[a_1|\dots|a_{r-1}\cdot y(a_r)].
\end{aligned}$$

Proof. The inverse of the isomorphism ψ_r of Proposition is given by

$$\psi_r^{-1}\colon A^{\otimes r}\longrightarrow R\otimes(A[-1])^{\otimes r}[r];\ [a_1|\dots|a_r]\longmapsto(-1)^{\mu(1,a_1,\dots,a_{r-1})}1\otimes[a_1|\dots|a_r],$$

where as above $\mu(1,a_1,\dots,a_{r-1})=\sum_{k=1}^{r-1}(r-k)\cdot|a_k|$. The diagram

$$\begin{tikzcd}
R\otimes(A[-1])^{\otimes r}[r] \arrow[r, "\delta_{r-1}"] & R\otimes(A[-1])^{\otimes r-1}[r-1] \arrow[d, "\psi_{r-1}"] \\
A^{\otimes r} \arrow[u, "\psi_r^{-1}"] \arrow[r, "\delta_{r-1}'"] & A^{\otimes r-1}
\end{tikzcd}$$

then yields the result after a simple calculation which is carried out in section D.2 in the appendix. ∎

Having translated the horizontal differentials of the bar double complex into differentials of a complex of dgas, we now fix the terminology for our differentials once an for all:

Definition 1.1.9. *Let* $R^\bullet, A^\bullet, x$ *and* y *be as above. For any* $r\in\mathbb{N}$*, we denote the above morphism by*

$$\begin{aligned}
\delta_{r-1}(x,y)\colon A^{\otimes r} &\longrightarrow A^{\otimes r-1}\\
[a_1|\dots|a_r] &\longmapsto -[x(a_1)\cdot a_2|\dots|a_r] + \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\dots|a_{i-1}|a_ia_{i+1}|a_{i+2}\dots|a_r]\\
&\quad +(-1)^{r+1}[a_1|\dots|a_{r-1}\cdot y(a_r)]
\end{aligned}$$

and call it the bar complex differential.

With this, we obtain:

Lemma 1.1.10. *The bar double complex is isomorphic to the double complex*

$$\longrightarrow A^{\otimes r}\xrightarrow{\delta_{r-1}(x,y)}A^{\otimes r-1}\xrightarrow{\delta_{r-2}(x,y)}\dots\xrightarrow{\delta_2(x,y)}A^{\otimes 2}\xrightarrow{\delta_1(x,y)}A\xrightarrow{y-x}R\longrightarrow 0.$$

Proof. This is an immediate consequence of the definition of the bar double complex and Lemma 1.1.8 above. ∎

Corollary 1.1.11. *The (unreduced) bar complex is naturally isomorphic to the total complex of the double complex*

$$\longrightarrow A^{\otimes r} \xrightarrow{\delta_{r-1}(x,y)} A^{\otimes r-1} \xrightarrow{\delta_{r-2}(x,y)} \dots \xrightarrow{\delta_2(x,y)} A^{\otimes 2} \xrightarrow{\delta_1(x,y)} A \xrightarrow{y-x} R \longrightarrow 0.$$

Next, we take a look at the reduced bar complex:

1.2 The reduced bar complex

The reduced bar complex is closely related to the unreduced bar complex: it is merely a quotient.

1.2.1 Definition

Let all notation be as in the preceding section, i.e.: let k be a field, $R^\bullet$ be a differential graded k-algebra, and $A = \bigoplus_{p\geq 0} A^p$ be a differential graded R-module. Moreover, we suppose that $R^\bullet$ admits the structure of a differential graded $A^\bullet$-bimodule via two morphism of differential graded algebras $x, y\colon A^\bullet \longrightarrow R^\bullet$, where left-multiplication is given by x, and right-multiplication by y.

Definition 1.2.1. *Let $D(A^\bullet|R^\bullet)_{x,y}$ be the subcomplex of $B(A^\bullet|R^\bullet)_{x,y}$ generated in degree $-r$ by the set*

$$\left\{[a_1|\dots|a_r] \mid a_i \in A^0 \textit{ for at least one } i \in \{1,\dots,r\}\right\}.$$

Then the reduced bar complex is defined to be the quotient

$$\bar{B}(A^\bullet|R^\bullet)_{x,y} := B(A^\bullet|R^\bullet)_{a,b}/D(A^\bullet|R^\bullet)_{x,y}.$$

1.2.2 Properties

a.) Denote the quotient A/A^0 by $A^{\geq 1}$. Then the reduced bar complex is the total complex of the following sub-complex of the bar double complex:

$$\begin{array}{ccccccccccc} \dots \longrightarrow & A^{\geq 1 \otimes r} & \xrightarrow{\delta} & A^{\geq 1 \otimes r-1} & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & A^{\geq 1} & \xrightarrow{\delta} & k[0] & \\ & -r & & -(r-1) & & \dots & & -1 & & 0 & \deg_{\mathrm{simpl}} \end{array}$$

b.) Now suppose that M is concentrated in degrees ≥ 0. Then the above double complex reduces to

$$\begin{tikzcd}
(A^1)^{\otimes 4} \arrow[r, "\delta"] & (A^{\geq 1\otimes 3})^4 \arrow[r, "\delta"] & (A^{\geq 1\otimes 2})^4 \arrow[r, "\delta"] & A^4 \arrow[r] & 0 & 4 \\
 & (A^1)^{\otimes 3} \arrow[r, "\delta"] \arrow[u, "\partial"] & (A^{\geq 1\otimes 2})^3 \arrow[r, "\delta"] \arrow[u, "\partial"] & A^3 \arrow[r] \arrow[u, "\partial"] & 0 & 3 \\
 & & (A^1)^{\otimes 2} \arrow[r, "\delta"] \arrow[u, "\partial"] & A^2 \arrow[r] \arrow[u, "\partial"] & 0 & 2 \\
 & & & A^1 \arrow[r] \arrow[u, "\partial"] & 0 & 1 \\
 & & & & k \arrow[u] & 0 \\
-4 & -3 & -2 & -1 & \deg_{\mathrm{simpl}} \backslash \deg_A &
\end{tikzcd}$$

c.) *The bar filtration:* The bar filtration of the unreduced complex induces a filtration $\mathfrak{B} := \{\bar{B}_r\}$ of the reduced bar complex $\bar{B}(A|R)_{x,y}$ by subcomplexes. Naturally, the quotient map $B(A|R)_{x,y} \longrightarrow B(A|R)_{x,y}/D(A|R)_{x,y}$ is filtered with respect to $\mathfrak{B}$. One has

$$\mathrm{gr}_r^{\mathfrak{B}} B(A|R)_{a,b} \cong (A^{\geq 1}/dA^0)^{\otimes r}[r].$$

d.) *Functoriality:* [HZ87, 3.11-14, p.92] Covariant functoriality carries over from the unreduced bar complex: Let A' be another differential graded $R^\bullet$-module such that $R^\bullet$ is endowed with the structure of a differential graded $(A')^\bullet$-bimodule via morphisms $x', y' \colon (A')^\bullet \longrightarrow R^\bullet$ of differential graded k-algebras. Suppose furthermore that we are given a morphism $\varphi \colon A \longrightarrow A'$ of differential graded R-modules such that $x = x' \circ \varphi$ and $y = y' \circ \varphi$. Since φ sends elements in A^0 to elements in $(A')^0$, one has $\varphi(D) \subset D'$. Therefore φ induces morphisms

$$\begin{aligned} \bar{B}(\varphi) \colon \bar{B}(A|R)_{x,y} &\longrightarrow \bar{B}(A'|R)_{x',y'} \\ \bar{B}_n(\varphi) \colon \bar{B}_n(A|R)_{x,y} &\longrightarrow \bar{B}_n(A'|R)_{x',y'} \end{aligned}$$

just like in 1.1.2. Moreover, as in the unreduced case, we obtain ([HZ87, (3.14), p. 92]): If φ is a quasi-isomorphism, then for any r, $\bar{B}_r(\varphi) \colon \bar{B}_r(A|R)_{x,y} \longrightarrow \bar{B}_r(A'|R)_{x',y'}$ is a $\mathfrak{B}$-filtered quasi-isomorphism.

e.) *Hodge structures:* [HZ87, 3.15-17, p.92] Assume that A and R are both cohomologically connected, i.e. graded in non-negative degrees and $H^0(A^\bullet) = k$, and let $B \subset \mathbb{C}$ be a ring. Moreover, suppose that both R and A underlie compatible (regarding the module-structure) mixed B-Hodge complexes with Hodge and weight filtrations (F_A, W_A) and (F_R, W_R), respectively, such that the weight filtrations of $A^\bullet$ and $R^\bullet$ are bounded below. Then like in the unreduced case, these data induce the structure of a mixed B-Hodge complex on $\bar{B}(A|R)_{x,y}$. Moreover, given a quasi-isomorphism $\varphi\colon A \longrightarrow A'$ as above, which is compatible with respect to all filtrations and a bifiltered quasi-isomorphism, then the induced morphism

$$\bar{B}(\varphi)\colon \bar{B}(A|R)_{x,y} \longrightarrow \bar{B}(A'|R)_{x',y'}$$

is a bifiltered quasi-isomorphism with respect to the induced filtrations ([HZ87, (3.15), p.92]).

f.) *Hopf algebra structure for $R = k$:* $\bar{B}(A)_{x,x}$ inherits the structure of a Hopf algebra from $B(A)_{x,x}$: One can easily check that $D(A)_{x,x}$ forms an ideal (resp. coideal) for the multiplication (resp. comultiplication) as defined in the unreduced case. Hence, the Hopf algebra structure of the unreduced bar complex induces a Hopf algebra structure on the reduced bar complex with multiplication and comultiplication defined by the same formulae.

g.) *Sheaf setting:* Again, definition of the bar complex also makes sense in the sheaf setting, i.e. if $\mathscr{R}^\bullet$ is a sheaf of differential graded k-algebras, $\mathscr{A} = \bigoplus_{p\geq 0} \mathscr{A}^p$ a sheaf of differential graded k-algebras which are differential graded R-modules with two morphism of sheaves of differential graded k-algebras $x, y\colon \mathscr{A}^\bullet \longrightarrow \mathscr{R}^\bullet$ inducing an $\mathscr{A}$-bimodule structure on $\mathscr{R}$. Then the same constructions of above yield a sheaf

$$\bar{B}(\mathscr{A}^\bullet|\mathscr{R}^\bullet)_{x,y}.$$

The properties of section 1.1.3 naturally carry over to the reduced bar complexes in the obvious way (i.e. functoriality, Hodge structures, and Hopf algebra structure).

1.3 A simplicial view on the classical bar complex

Apart from the above straightforward definition of the bar complex, there is a different approach to bar complexes via a simplicial interpretation. It is a more conceptual view on the bar complex, and will prove to be the right tool to generalize bar complexes to the motivic setting. Thus it is necessary to take a close look at this viewpoint.

In section 1.3.1, we will first recall the essential facts of the theory of simplicial objects for the reader's convenience. In section 1.3.2, we will then take a look at the particular simplicial objects giving rise to the bar complex of the preceding sections. Finally, in section 1.4, we consider a special case of bar complexes which will be of major importance to us due to its motivic origin.

1.3.1 Notation and basic facts

For the following section we will assume basic knowledge on simplicial and cosimplicial objects. The theory needed is recalled in Appendix A, or it can be found in chapter 1.2 of the book [Lur]. First and foremost, note that while usually one takes all complexes in the theory of simplicial objects as *chain complexes in positive degrees*, we will consider them as *cochain complexes in negative degrees*. This agrees more with the setting of classical bar complexes.

Let me just recall the most important notions we will use:

For a category $\mathcal{A}$, we denote the category of simplicial objects in $\mathcal{A}$ by $\mathcal{A}^{\mathbf{\Delta}^{\mathrm{op}}}$ and the category of cosimplicial objects by $\mathcal{A}^{\mathbf{\Delta}}$.

Simplicial setting

Let $\mathcal{C}$ be an idempotent complete category, and $S_\bullet$ a simplicial object in $\mathcal{C}$ with face maps d_i and degeneracy maps s_i. The *unnormalized complex* associated to $S_\bullet$ is the complex $C_*(S)$ given by S_n in degree $-n$ with boundary maps

$$\partial = \sum_{i=0}^{n} (-1)^i d_i \colon S_n \longrightarrow S_{n-1}.$$

One defines a subcomplex of the unnormalized complex as follows: Recall that in an idempotent complete category, a priori the existence of kernels or cokernels is not guaranteed. One can prove ([Lur, 1.2.3.15, pp. 48, 49]), however, that for any simplicial object $S_\bullet \in \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ with face maps d_i and degeneracy maps s_j, the kernel/cokernel

$$N_n(S) := \ker\left((d_1, \ldots, d_n) \colon S_n \to \bigoplus_{\leq i \leq n} S_{n-1} \right) = \operatorname{coker}\left(\sum_j s_j \colon \bigoplus_{0 \leq i < n} S_{n-1} \to S_n \right)$$

actually exists. One defines the *normalized complex* of $S_\bullet$ to be the subcomplex of $C_\bullet(S)$ given in degree n by $N_n(S_\bullet)$. Sending simplicial objects $S_\bullet$ in $\mathcal{C}$ to their normalized complex yields a functor $N \colon \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \longrightarrow C(\mathcal{C})_{\leq 0}$, the *normalized cochain complex functor*. Obviously, we have a monomorphism $u \colon N_*(S) \longrightarrow C_*(S)$ and an epimorphism $v \colon C_*(S) \longrightarrow N_*(S)$ of cochain complexes arising from the description of $N_n(S)$ as a kernel, respectively cokernel. One obtains the following:

Lemma 1.3.1. *[Lur, 1.2.3.17, p.47] If the category $\mathcal{C}$ is abelian, then the canonical monomorphism $u \colon N_*(S) \hookrightarrow C_*(S)$ and epimorphism $C_*(S) \twoheadrightarrow N_*(S)$ are quasi-isomorphisms of cochain complexes.*

Cosimplicial setting

Let $S^\bullet$ be a cosimplicial object in an idempotent complete category $\mathcal{A}$. The *unnormalized cochain complex* associated to $S^\bullet$ is the complex $C_\bullet(S)$ with S^n in degree n, with boundary maps

$$\partial = \sum_{i=0}^{n} (-1)^i d^i \colon S^{n-1} \longrightarrow S^n.$$

For each $n \geq 0$, one can show that the object

$$Q(S)^n = \operatorname{coker}\left(\sum_{i=0}^{n-1} d^i \colon \bigoplus_{i=0}^{n-1} S^{n-1} \to S^n \right)$$

exists and gives rise to a subcomplex of $C_\bullet(S)$, the *normalized cochain complex*. Sending cosimplicial objects $S^\bullet$ in $\mathcal{C}$ to their normalized complex yields a functor

$$q\colon \mathcal{C}^{\boldsymbol{\Delta}} \longrightarrow \mathrm{Ch}(\mathcal{C})_{\geq 0},$$

the *normalized cochain complex functor*.

1.3.2 The simplicial bar construction

After recalling the theory of simplicial objects, we will apply it to obtain a "simplicial bar object" whose unnormalized complex will turn out to coincide with the bar complex of chapter 1.

The simplicial object

Let k be a field, and $R^\bullet$ a differential graded k-algebra with unit. We denote the category of unital differential graded k-algebras by dga_k. Let us fix a dga $A^\bullet = \bigoplus_{p\geq 0} A^p$ with differential $\partial\colon A^k \longrightarrow A^{k+1}$, which has the structure of a differential graded $R^\bullet$-module. By [vara], the category $\mathrm{Mod}_{(R^\bullet,\mathrm{d})}$ of differential graded $R^\bullet$-modules is an abelian category which has arbitrary limits and colimits. At the same time, we suppose $R^\bullet$ is endowed with the structure of a differential graded A-bimodule by virtue of two morphisms of differential graded algebras $x, y\colon A^\bullet \longrightarrow R^\bullet$.
Let all notation for degrees be as in section 1.1.1. Recall that we denote the n-fold tensor product of $A^\bullet$ with itself over $R^\bullet$ by

$$A^{\otimes n} := A^\bullet \otimes_{R^\bullet} \otimes \ldots \otimes_{R^\bullet} A^\bullet$$

and write $[a_1|\ldots|a_n] := a_1 \otimes \ldots \otimes a_n$ for an element in $A^{\otimes n}$. Recall that by section 1.1.4, the (unreduced) bar complex is naturally isomorphic to the total complex of the double complex

$$\longrightarrow A^{\otimes r} \xrightarrow{\delta_{r-1}(x,y)} A^{\otimes r-1} \xrightarrow{\delta_{r-2}(x,y)} \ldots \xrightarrow{\delta_2(x,y)} A^{\otimes 2} \xrightarrow{\delta_1(x,y)} A \xrightarrow{y-x} R \longrightarrow 0$$

with $\delta_k(x,y)$ given by

$$\begin{aligned}
&\delta_{k-1}(x,y)\colon A^{\otimes k} \longrightarrow A^{\otimes k-1}\\
&\qquad [a_1|\ldots|a_k] \longmapsto -[x(a_1)\cdot a_2|\ldots|a_k] + \sum_{i=1}^{k-1}(-1)^{i+1}[a_1|\ldots|a_{i-1}|a_ia_{i+1}|a_{i+2}\ldots|a_k]\\
&\qquad\qquad\qquad +(-1)^{k+1}[a_1|\ldots|a_{k-1}\cdot y(a_k)].
\end{aligned}$$

We now consider the following assignment:

$$\begin{aligned}
&sB^\bullet(A^\bullet|R^\bullet)_{x,y}\colon \boldsymbol{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{(R^\bullet,d)}\\
&[n] \longmapsto A^{\otimes n},\ d^j \longmapsto (d^n_j\colon A^{\otimes n+1} \longrightarrow A^{\otimes n}),\ s^j \longmapsto (s^n_j\colon A^{\otimes n} \longrightarrow A^{\otimes n+1}),
\end{aligned}$$

where the tensor product is taken over R, the maps d_j^n are given by

$$d_j^n([a_1|\dots|a_{n+1}]) = \begin{cases} -[x(a_1)a_2|\dots|a_{n+1}] & \text{for } j=0 \\ -[a_1|\dots|a_j a_{j+1}|\dots|a_{n+1}] & \text{for } j\in\{1,\dots,n\} \\ -[a_1|\dots|a_n y(a_{n+1})] & \text{for } j=n+1, \end{cases}$$

and the maps s_j^n are given by

$$s_j^n([a_1|\dots|a_{n+1}]) = -[a_1|\dots|a_j|1|a_{j+1}|\dots|a_{n+1}]$$

for $j=0,\dots,n$, where 1 is the element 1 of $k\subset A^0\subset A^\bullet$.

Proposition 1.3.2. *d_j^n and s_j^n are morphisms of complexes of differential graded R-modules for all n, j.*

Proof. This follows from a simple calculation which can be found in the appendix in E.2.1. ■

In what follows, we will often drop the upper index n in the notation of d_j^n and s_j^n when it is obvious what they should be.

Lemma 1.3.3. *$sB^\bullet(A^\bullet|R^\bullet)_{x,y}$ is a functor, i.e. comprises a simplicial object in* $\mathrm{Mod}_{(R,d)}$.

Proof. We need to show that the simplicial identities are satisfied. This follows from a simple, but long calculation which can be found in the appendix in E.2.2. ■

Definition 1.3.4. *We call the simplicial object $sB^\bullet(A^\bullet|R^\bullet)_{x,y}$ in $\mathrm{Mod}_{(R,d)}$ the classical simplicial bar object.*

The unnormalized complex

Let us first determine the unnormalized complex associated to the simplicial bar object above.

Recall that we denote the category of (bounded/bounded above) complexes of objects in a category $\mathcal{A}$ by $C\mathcal{A}$ ($C^b\mathcal{A}, C^-\mathcal{A}$).

Lemma 1.3.5. *The unnormalized complex associated to the simplicial object*

$$sB^\bullet(A^\bullet|R^\bullet)_{x,y} \quad in \quad \mathrm{Mod}_{(R,d)}$$

is given by the complex of differential graded A-modules

$$\begin{array}{ccccccccccccc} \longrightarrow A^{\otimes n} & \xrightarrow{\delta_{n-1}(x,y)} & A^{\otimes n-1} & \xrightarrow{\delta_{n-2}(x,y)} & \dots & \xrightarrow{\delta_2(x,y)} & A^{\otimes 2} & \xrightarrow{\delta_1(x,y)} & A & \xrightarrow{\delta_0(x,y)} & k & \longrightarrow & 0 \\ -n & & -n+1 & & \dots & & -2 & & -1 & & 0 & & \text{deg} \end{array}$$

in $C^-(\mathrm{Mod}_{(R,d)})$, where $A^{\otimes k}$ is in degree $-k$, the tensor product is taken over R, and the differentials are given by

$$\begin{aligned}\delta_{r-1}(x,y)\colon A^{\otimes r} &\longrightarrow A^{\otimes r-1}\\ [a_1|\dots|a_r] &\longmapsto -[x(a_1)\cdot a_2|\dots|a_r] + \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\dots|a_{i-1}|a_ia_{i+1}|a_{i+2}\dots|a_r]\\ &\qquad +(-1)^{r+1}[a_1|\dots|a_{r-1}\cdot y(a_r)]\end{aligned}$$

Hence, it coincides with the bar double complex of section 1.1.4 underlying $B(A|R)$.

Proof. This is a direct consequence of the definition of the face maps and Lemma 1.1.10. ■

Since R is a k-algebra, there is a forgetful functor $\mathrm{Mod}_{(R,d)} \longrightarrow C(\mathrm{Vect}_k)$ to the category of complexes of k-vector spaces which forgets about the R-module structure and sends a differential graded R-module $A^\bullet$ to its underlying complex

$$\dots \to A^{i-1} \xrightarrow{\partial} A^i \xrightarrow{\partial} A^{i+1} \xrightarrow{\partial} \dots$$

of k-vector spaces. This functor gives rise to a forget-functor

$$\mathrm{For}\colon C^-(\mathrm{Mod}_{(R,d)}) \longrightarrow C^*(C(\mathrm{Vect}_k))$$

sending a complex of differential graded algebras to the associated double complex. Composing it with the functor Tot sending a double complex to its associated simple complex we obtain a functor

$$\mathrm{Tot}\colon C^-(\mathrm{Mod}_{(R,d)}) \longrightarrow C(\mathrm{Vect}_k).$$

Corollary 1.3.6. *The functor* Tot *described above sends the unnormalized complex of the simplicial object $sB^\bullet(A^\bullet|R^\bullet)_{x,y}$ in $\mathrm{Mod}_{(R,d)}$ to the (unreduced) bar complex $B(A^\bullet|R^\bullet)_{x,y}$.*

The normalized complex

Now we determine the normalized complex associated to the simplicial object of section 1.3.2.

Lemma 1.3.7. *Let $R^\bullet$ and $A^\bullet$ be cohomologically connected, and let x, y be two morphisms of dgas. The normalized complex associated to the simplicial object*

$$sB^\bullet(A^\bullet|R^\bullet)_{x,y} \quad \textit{in} \quad \mathrm{Mod}_{(R,d)}$$

is the quotient of the unnormalized complex of Lemma 1.3.5 by the subcomplex generated by all elements $[a_1|\dots|a_n] \in A^{\otimes n}$ such that $a_i \in R\cdot 1_A \subset A$.

Proof. Recall that the normalized complex can be computed as a quotient of the unnormalized complex by the degeneracy subcomplex

$$D_n(sB^\bullet(A^\bullet|R^\bullet)_{x,y}) := \mathrm{Im}\left(\sum_j s_j \colon \bigoplus_{0\le i<n} sB_{n-1}(A^\bullet|R^\bullet)_{x,y} \to sB_n(A^\bullet|R^\bullet)_{x,y}\right),$$

where $s_j^n \colon A^{\otimes n} \longrightarrow A^{\otimes n+1}$, $[a_1|\dots|a_{n+1}] \longmapsto [a_1|\dots|a_j|1|a_{j+1}|\dots|a_{n+1}]$

This means that $D_\bullet(sB^\bullet(A^\bullet|R^\bullet)_{x,y})$ is indeed the subcomplex generated by all elements $[a_1|\dots|a_n] \in A^{\otimes n}$ such that $a_i \in R\cdot 1$. ■

Definition 1.3.8. *We call the total complex associated to the double complex underlying $N(sB^\bullet(A^\bullet|R^\bullet)_{x,y})$ of lemma 1.3.7 the normalized bar complex and denote it by $\widetilde{B}(A^\bullet|R^\bullet)_{x,y}$. The filtration induced by the bar filtration will be denoted by $(\widetilde{B}_r(A^\bullet|R^\bullet)_{x,y})_r$.*

Corollary 1.3.9. *The normalized bar complex $\widetilde{B}(A^\bullet|R^\bullet)_{x,y}$ is the quotient of $B(A^\bullet|R^\bullet)_{x,y}$ by the subcomplex generated by all elements $[a_1|\dots|a_n] \in A^{\otimes n}$ such that $a_i \in R\cdot 1 \subset A$.*

1.3.3 Relations between unnormalized, normalized and reduced bar complexes

Now that we have introduced the unnormalized and normalized bar complex from a simplicial set-up, we can apply the general theory of simplicial objects to deduce relations between the different versions of bar complexes: the unreduced, normalized, and reduced bar complex.

The first corollary of the theory that comes to mind is the following:

Corollary 1.3.10. *The natural maps*

$$u\colon N_\bullet(sB^\bullet(A^\bullet|R^\bullet)_{x,y}) \longrightarrow C_\bullet(sB^\bullet(A^\bullet|R^\bullet)_{x,y}) \quad \textit{and}$$
$$v\colon C_\bullet(sB^\bullet(A^\bullet|R^\bullet)_{x,y}) \longrightarrow N_\bullet(sB^\bullet(A^\bullet|R^\bullet)_{x,y})$$

of Lemma 1.3.1 are quasi-isomorphisms of chain complexes. In particular, applying Tot*, the natural projection*

$$\mathrm{pr}_{norm}\colon B(A^\bullet|R^\bullet)_{x,y} \twoheadrightarrow \widetilde{B}(A^\bullet|R^\bullet)_{x,y} \quad \textit{is a quasi-isomorphism.}$$

Proof. This is a direct consequence of 1.3.1. ■

Corollary 1.3.11. *There is a natural projection* $\mathrm{pr}_{\mathrm{red}}\colon \widetilde{B}(A^\bullet)_{x,y} \longrightarrow \bar{B}(A^\bullet)_{x,y}$.

Proof. This follows from $k \subset A^0$. ■

Lemma 1.3.12. *For $R = k \subset A^0$ the natural projections* $\mathrm{pr}_{\mathrm{red}}\colon \widetilde{B}(A^\bullet)_{x,y} \longrightarrow \bar{B}(A^\bullet)_{x,y}$ *and* $\mathrm{pr}_{norm}\, B(A^\bullet)_{x,y} \longrightarrow \bar{B}(A^\bullet)_{x,y}$ *are quasi-isomorphisms.*

Proof. We first show that $\mathrm{pr}_{\mathrm{red}}\colon \widetilde{B}(A^\bullet)_{x,y} \longrightarrow \bar{B}(A^\bullet)_{x,y}$ is a quasi-isomorphism by showing that it is a quasi-isomorphism on the finite subcomplexes $\mathrm{pr}_{\mathrm{red}}\colon \widetilde{B}_n(A^\bullet)_{x,y} \longrightarrow \bar{B}_n(A^\bullet)_{x,y}$ - the asserted result then follows by passing to the direct limit $n \to \infty$.
We consider the bar filtrations of $\widetilde{B}_n(A^\bullet)_{x,y}$ and $\bar{B}_n(A^\bullet)_{x,y}$. These filtrations give rise to spectral sequences

$$\begin{aligned} E_1(\widetilde{B}_r)^{-m,k+m} &:= H^k(\mathrm{gr}_m^{\mathfrak{B}}\, \widetilde{B}_n(A^\bullet)_{x,y}) \Rightarrow H^k(\widetilde{B}_n(A^\bullet)_{x,y}) \\ E_1(\bar{B}_r)^{-m,k+m} &:= H^k(\mathrm{gr}_m^{\mathfrak{B}}\, \bar{B}_n(A^\bullet)_{x,y}) \Rightarrow H^k(\bar{B}_n(A^\bullet)_{x,y}) \end{aligned}$$

with the differential induced by the differentials of the bar complexes. Since $A^\bullet$ is a differential graded R-module, so are $\widetilde{B}_n(A^\bullet)_{x,y}$ and $\bar{B}_n(A^\bullet)_{x,y}$. Both are filtered by the bar filtration $\mathfrak{B}$, and obviously the projection

$$\mathrm{pr}_{\mathrm{red}}\colon (\widetilde{B}_n(A^\bullet)_{x,y}, \mathfrak{B}) \longrightarrow (\bar{B}_n(A^\bullet)_{x,y}, \mathfrak{B})$$

is a morphism of fltered differential graded R-modules. Theorem 3.5 of [McC01] now says the following: $\mathrm{pr}_{\mathrm{red}}$ determines a morphism of the associated spectral sequences

$$\mathrm{pr}_{\mathrm{red}}\colon E_1(\widetilde{B}_r)^{-m,k+m} \longrightarrow E_1(\bar{B}_r)^{-m,k+m}.$$

Moreover, if $\mathrm{pr}_{\mathrm{red}}$ induces an isomorphism of bigraded R-modules on any sheet, then $\mathrm{pr}_{\mathrm{red}}$ induces an isomorphism $\mathrm{pr}_{\mathrm{red}}\colon H^*(\widetilde{B}_n(A^\bullet)_{x,y}) \longrightarrow H^*(\bar{B}_n(A^\bullet)_{x,y})$ on cohomology, since $\mathfrak{B}$ is bounded. To prove the assertion, it therefore suffices to show that

$$\mathrm{pr}_{\mathrm{red}}\colon E_1(\widetilde{B}_r)^{-m,k+m} \longrightarrow E_1(\bar{B}_r)^{-m,k+m}$$

is an isomorphism of differential bigraded R-modules. Since this morphism is obviously compatible with the grading and the R-module-structure, it suffices to prove that it is an isomorphism, i.e. that the projection

$$\mathrm{pr}_{\mathrm{red}}\colon \widetilde{B}(A^\bullet)_{x,y} = (A/k)^{\otimes r} \longrightarrow \bar{B}(A^\bullet)_{x,y} = (A^{\geq 0}/dA^0)^{\otimes r}$$

is a quasi-isomorphism columnwise. In turn, by Künneth it suffices to consider the tensor-degree-1-case, i.e. one has to show that the morphism

$$\mathrm{pr}_{\mathrm{red},1}\colon A/k \longrightarrow A^{\geq 0}/dA^0$$

is a quasi-isomorphism. However, its kernel is the complex $(A^0/k \longrightarrow dA^0)$, which is acyclic since $H^0(A) = k$ (A is cohomologically connected by assumption). This proves the first assertion. The second one then follows using Corollary 1.3.10. ∎

1.3.4 Augmentation ideals

We now consider the special case $x = y$, i.e. the right and left $A^\bullet$-module structure of $R^\bullet$ coincide. This case has some particulary nice additional structure for the following reason: The last differential of the bar complex $\delta_0^* = x^* - y^* = 0$ vanishes, and thus the double complex underlying the bar complex $B_n(A^\bullet|R^\bullet)_{x,x}$ decomposes into a sum of the double complex

$$\begin{array}{ccccccccccccccc} \dots & \xrightarrow{\partial} & A^{\otimes n} & \xrightarrow{\delta_n} & A^{\otimes n-1} & \xrightarrow{\delta_{n-1}} & \dots & \xrightarrow{\delta_3} & A^{\otimes 2} & \xrightarrow{\delta_2} & A & \xrightarrow{\delta_1} & 0 & \longrightarrow & 0 \\ & & -n & & -n+1 & & \dots & & -2 & & -1 & & 0 & & \deg_{\mathrm{simpl}} \end{array}$$

and the complex $R^\bullet[0]$ concentrated in horizontal degree zero.

Definition 1.3.13. *We define the* <u>*augmentation ideal*</u> *of* $B_n(A^\bullet|R^\bullet)_{x,x}$ *(resp.* $\widetilde{B}_n(A^\bullet|R^\bullet)_{x,x}$*) to be the complex*

$$I_n(A^\bullet|R^\bullet)_x := \mathrm{Tot}\left\{ \begin{array}{ccccccccc} A^{\otimes n} & \xrightarrow{\delta_{n-1}} & A^{\otimes n-1} & \xrightarrow{\delta_{n-2}} & \dots & \xrightarrow{\delta_1} & A & \longrightarrow & 0 \\ -n & & \dots & & -2 & & -1 & & 0 \quad \deg \end{array} \right\}$$

$$\textit{resp. } \tilde{I}_n(A^\bullet|R^\bullet)_x := \mathrm{Tot}\left\{ \begin{array}{ccccccccc} (A/R)^{\otimes n} & \xrightarrow{\delta_n} & (A/R)^{\otimes n-1} & \xrightarrow{\delta_{n-1}} & \dots & \xrightarrow{\delta_2} & A/R & \xrightarrow{\delta_1} & 0 \\ -n & & & & \dots & & -2 & -1 & 0 \end{array} \right\}$$

Similarly, the left unbounded complex

$$I(A^\bullet|R^\bullet)_x := \varinjlim_n I_n(A^\bullet|R^\bullet)_x \qquad \left(\textit{resp. } \tilde{I}(A^\bullet|R^\bullet)_x := \varinjlim_n \tilde{I}_n(A^\bullet|R^\bullet)_x\right)$$

is called the <u>*augmentation ideal*</u> *of* $B(A^\bullet|R^\bullet)_{x,x}$ *(resp.* $\widetilde{B}(A^\bullet|R^\bullet)_{x,x}$*).*

Corollary 1.3.14. *There are canonical splittings* $B_n(A^\bullet|R^\bullet)_{x,x} = I_n(A^\bullet|R^\bullet)_x \oplus R^\bullet$ *and* $\widetilde{B}_n(A^\bullet|R^\bullet)_{x,x} = \tilde{I}_n(A^\bullet|R^\bullet)_x \oplus R^\bullet$.

1.4 The simplicial bar construction for relative smooth complex varieties

Now that we have recalled the simplicial origin of bar complexes and have a nice theory to go with, we apply it to the one special case which will be of most importance for us: We consider a smooth morphism of smooth complex varieties $\pi\colon X \to S$ equipped with two sections $x, y\colon S \longrightarrow X$ of π. Denote the sheaves of complex C^∞-functions on X and S by $\mathcal{E}_X^\bullet$ and $\mathcal{E}_S^\bullet$, respectively, and the sheaf of relative C^∞-functions by $\mathcal{E}_{X|S}^\bullet := \mathcal{E}_X^\bullet/\pi^*\mathcal{E}_S^\bullet$. (That $\pi^*\mathcal{E}_S^\bullet \hookrightarrow \mathcal{E}_X^\bullet$ can be seen by looking at the situation locally, since X is locally trivial over S). We are interested in the bar complex $B(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)$. In order to work with it, we first recall some important facts on smooth forms:

1.4.1 The bar construction for $(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)$

Now consider the adjunction morphism $\mathcal{E}_S^\bullet \to \pi_*\pi^*\mathcal{E}_S^\bullet$ associated to the adjunction of functors $\pi^* \dashv \pi_*$. Since π is a continuous surjection, the functor π^* is faithful, and hence $\mathrm{id} \longrightarrow \pi_*\pi^*$ a monomorphism of functors, so the adjunction above is injective. (Another way to see this is that the adjunction $\mathcal{E}_S^\bullet \to \pi_*\pi^*\mathcal{E}_S^\bullet$ factors over $\mathcal{E}_S^\bullet \to \pi_*\pi^{-1}\mathcal{E}_S^\bullet \hookrightarrow \pi_*\pi^*\mathcal{E}_S^\bullet$, where the first arrow is a quasi-isomorphism). It yields a morphism

$$\mathcal{E}_S^\bullet \hookrightarrow \pi_*\pi^*\mathcal{E}_S^\bullet \hookrightarrow \pi_*\mathcal{E}_X^\bullet$$

of sheaves of differential graded $\mathbb{C}$-algebras, making $\pi_*\mathcal{E}_X^\bullet$ a differential graded $\mathcal{E}_S^\bullet$-module.

Conversely, the sections x and y give $\mathcal{E}_S^\bullet$ the structure of a differential graded left and right $\pi_*\mathcal{E}_X^\bullet$-module as follows: They induce morphisms $x^*, y^* \colon \pi_*\mathcal{E}_X^\bullet \longrightarrow \mathcal{E}_S^\bullet$ given for $U \subset S$ open by

$$\begin{aligned} x^*(U)\colon E^\bullet_{\pi^{-1}(S)} &\longrightarrow E^\bullet_U; \quad \varphi \longmapsto x^*\varphi \\ y^*(U)\colon E^\bullet_{\pi^{-1}(S)} &\longrightarrow E^\bullet_U; \quad \varphi \longmapsto y^*\varphi \end{aligned}$$

Here, we take x^* to define the left and y^* to define the right module structure. Note that these impose on $\mathcal{E}_S^\bullet$ the structure of a $\pi_*\mathcal{E}_X^\bullet$-bimodule. Hence, we are in the setting of the last section, and can apply our results.

Definition 1.4.1. *We call the simplicial object*

$$sB^\bullet(X|S)_{x,y} := sB^\bullet(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}$$

the simplicial bar object for $X|S$.

Corollary 1.4.2. *The unnormalized complex $C_\bullet(sB^\bullet(X|S)_{x,y})$ associated to the simplicial object $sB^\bullet(X|S)_{x,y}$ in $\mathrm{Mod}_{(\pi_*\mathcal{E}_X^\bullet,d)}$ is given by*

$$\begin{array}{ccccccccccccc} \xrightarrow{\delta_n} & (\pi_*\mathcal{E}_X^\bullet)^{\otimes n} & \xrightarrow{\delta_{n-1}} & (\pi_*\mathcal{E}_X^\bullet)^{\otimes n-1} & \xrightarrow{\delta_{n-2}} & \ldots & \xrightarrow{\delta_2} & (\pi_*\mathcal{E}_X^\bullet)^{\otimes 2} & \xrightarrow{\delta_1} & (\pi_*\mathcal{E}_X^\bullet) & \xrightarrow{\delta_0} & k & \longrightarrow 0 \\ & -n & & -n+1 & & \ldots & & -2 & & -1 & & 0 & \mathrm{deg} \end{array}$$

in $C^-(\mathrm{Mod}_{(\mathcal{E}_S^\bullet,d)}(S))$, where all tensor products are taken over $\mathcal{E}_S^\bullet$ and the differentials are given by

$$\delta_{r-1}(x,y)\colon A^{\otimes r} \longrightarrow A^{\otimes r-1}$$

$$\begin{aligned} [a_1|\ldots|a_r] \longmapsto -[x^*(a_1)\cdot a_2|\ldots|a_r] &+ \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\ldots|a_{i-1}|a_ia_{i+1}|a_{i+2}\ldots|a_r] \\ &+(-1)^{r+1}[a_1|\ldots|a_{r-1}\cdot y^*(a_r)] \end{aligned}$$

The normalized complex $N(sB^\bullet(X|S)_{x,y})$ is the quotient of $C_\bullet(sB^\bullet(X|S)_{x,y})$ by the subcomplex comprised by all elements $[a_1|\ldots|a_n] \in (\pi_*\mathcal{E}_X^\bullet)^{\otimes n}$ such that $a_i \in \mathcal{E}_S \hookrightarrow \pi_*\mathcal{E}_X$.

The total complexes associated to the above unnormalized and normalized bar complexes, are $B(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}$ and $\widetilde{B}(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}$, respectively.

Definition 1.4.3. *We call the unreduced (resp. normalized) bar complex*

$$B(X|S)_{x,y} := B(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y} \quad \Big(\textit{resp. } \widetilde{B}(X|S)_{x,y} := \widetilde{B}(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}\Big)$$

the <u>*bar complex of* $X|S$</u> *(resp.* <u>*normalized bar complex of* $X|S$</u>*).*

1.4.2 Relations between $B, \bar{B}$ and $\widetilde{B}$

Just as in the general case, we now consider relations between the different kinds of bar complexes.

Corollary 1.4.4. *There are natural identifications*

$$\widetilde{B}(X|S)_{x,y} = B(\pi_*\mathcal{E}_X^\bullet/\mathcal{E}_S^\bullet|\mathcal{E}_S^\bullet)_{x,y}, \quad \textit{and} \quad \widetilde{B}_r(X|S)_{x,y} = B_r(\pi_*\mathcal{E}_X^\bullet/\mathcal{E}_S^\bullet|\mathcal{E}_S^\bullet)_{x,y}$$

and quasi-isomorphisms

$$u\colon \widetilde{B}(X|S)_{x,y} \longrightarrow B(X|S)_{x,y} \quad \textit{and} \quad v\colon B(X|S)_{x,y} \longrightarrow \widetilde{B}(X|S)_{x,y}.$$

Proof. The first statement is due to Lemma 1.4.2 and the second one is a consequence of Lemma 1.3.1. ■

Lemma 1.4.5. *The natural projection*

$$B(X|S)_{x,y} = B(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y} \longrightarrow \bar{B}(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}$$

is a quasi-isomorphism.

Proof. It suffices to prove that the projection is a quasi-isomorphism in stalks. Let s be any point in S. The stalk of $\mathcal{E}_S^\bullet$ at s is given by $\mathbb{C}$, while $(\pi_*\mathcal{E}_X^\bullet)_s = E_{X_s}^\bullet$, where X_s denotes the fiber of $\pi\colon X \longrightarrow S$ over s. Hence, we have

$$\begin{aligned}\big(B(X|S)_{x,y}\big)_s &= \big(B(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}\big)_s = B(E_{X_s}^\bullet|\mathbb{C})_{x_s,y_s}\\ \big(\bar{B}(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}\big)_s &= \bar{B}(E_{X_s}^\bullet|\mathbb{C})_{x_s,y_s}\end{aligned}$$

where x_s, y_s are the sections x, y restricted to the point s. Hence, we only need to see that the natural projection $B(E_{X_s}^\bullet|\mathbb{C})_{x_s,y_s} \longrightarrow \bar{B}(E_{X_s}^\bullet|\mathbb{C})_{x_s,y_s}$ is a quasi-isomorphism. This, however, is Lemma 1.3.12. ■

However, note that the corresponding projections $B_n(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y} \longrightarrow \bar{B}_n(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y}$ of the truncated complexes need not be quasi-isomorphisms. Still, we have the following:

Lemma 1.4.6. *There are natural quasi-isomorphisms*

$$\begin{aligned}\widetilde{B}_n(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y} &\twoheadrightarrow \bar{B}_n(\pi_*\mathcal{E}_X^\bullet/\mathcal{E}_S^\bullet|\mathcal{E}_S^\bullet)_{x,y} \quad \textit{and}\\ \bar{B}_n(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y} &\twoheadrightarrow \bar{B}_n(\pi_*\mathcal{E}_X^\bullet/\mathcal{E}_S^\bullet|\mathcal{E}_S^\bullet)_{x,y}\end{aligned}$$

given by the obvious projections. As a consequence, there is a natural isomorphism

$$\widetilde{B}_n(\pi_*\mathcal{E}_X^\bullet|\mathcal{E}_S^\bullet)_{x,y} \cong \bar{B}_n(\pi_*\mathcal{E}_X^\bullet/\mathcal{E}_S^\bullet|\mathcal{E}_S^\bullet)_{x,y}$$

in the derived category.

Proof. First we consider the projection

$$\mathrm{pr}_{\mathrm{red}}\colon \widetilde{B}_n(\pi_*\mathcal{E}^\bullet_X|\mathcal{E}^\bullet_S)_{x,y} \twoheadrightarrow \bar{B}_n(\pi_*\mathcal{E}^\bullet_X/\mathcal{E}^\bullet_S|\mathcal{E}^\bullet_S)_{x,y}.$$

To show that this is a quasi-isomorphism, it suffices to prove that it is a quasi-isomorphism in stalks, i.e. that the morphism $\widetilde{B}_n(E^\bullet_{X_s}|\mathbb{C})_{x_s,y_s} \twoheadrightarrow \bar{B}_n(E^\bullet_{X_S}/\mathbb{C}|\mathbb{C})_{x_s,y_s}$ is a quasi-isomorphism, where X_s denotes the fiber of $X \longrightarrow S$ over $s \in S$, and $x_s = x(s), y_s = y(s)$. We consider the (bounded!) bar filtrations of $\widetilde{B}_n(E^\bullet_{X_s}|\mathbb{C})_{x_s,y_s} = B_n(E^\bullet_{X_s}/\mathbb{C}|\mathbb{C})_{x_s,y_s}$ and $\bar{B}_n(E^\bullet_{X_S}/\mathbb{C}|\mathbb{C})_{x_s,y_s}$. These filtrations give rise to E_1-page spectral sequences

$$\begin{aligned} E_1(\widetilde{B})^{-m,k+m} &:= \mathcal{H}^k(\mathrm{gr}^{\mathfrak{B}}_m \widetilde{B}_n(E^\bullet_{X_s}|\mathbb{C})_{x_s,y_s}) \Rightarrow \mathcal{H}^k(\widetilde{B}_n(E^\bullet_{X_s}|\mathbb{C})_{x_s,y_s}) \\ E_1(\bar{B})^{-m,k+m} &:= \mathcal{H}^k(\mathrm{gr}^{\mathfrak{B}}_m \bar{B}_n(E^\bullet_{X_s}/\mathbb{C}|\mathbb{C})_{x_s,y_s}) \Rightarrow \mathcal{H}^k(\bar{B}_n(E^\bullet_{X_s}/\mathbb{C}|\mathbb{C})_{x_s,y_s}) \end{aligned}$$

with the differential induced by the differentials of $B_n(E^\bullet_{X_s}/\mathbb{C}|\mathbb{C})_{x_s,y_s}$ and $\bar{B}_n(E^\bullet_{X_s}|\mathbb{C})_{x_s,y_s}$, respectively. The bar filtrations imparts the structure of a filtered morphism of differential bigraded $\mathbb{C}$-vector spaces on the projection $B_n(E^\bullet_{X_s}/\mathbb{C}|\mathbb{C})_{x_s,y_s} \longrightarrow \bar{B}_n(E^\bullet_{X_s}|\mathbb{C})_{x_s,y_s}$, and thus it induces a morphism of the respective spectral sequences

$$\mathrm{pr}_{\mathrm{red}}\colon E_1(B)^{-m,k+m} \longrightarrow E_1(\bar{B})^{-m,k+m}.$$

By Theorem 3.5 of [McC01], we have the following: If $\mathrm{pr}_{\mathrm{red}}\colon E_1(B)^{-m,k+m} \longrightarrow E_1(\bar{B})^{-m,k+m}$ is an isomorphism of bigraded $\mathbb{C}$-vector spaces, then (since the bar filtrations are bounded) $\mathrm{pr}_{\mathrm{res}}$ induces an isomorphism

$$H^*(\mathrm{pr}_{\mathrm{red}})\colon H^*\left(\widetilde{B}_n(\pi_*\mathcal{E}^\bullet_X|\mathcal{E}^\bullet_S)_{x,y}\right) \longrightarrow H^*\left(\bar{B}_n(\pi_*\mathcal{E}^\bullet_X/\mathcal{E}^\bullet_S|\mathcal{E}^\bullet_S)_{x,y}\right)$$

as claimed. Thus, it suffices to prove that

$$\mathrm{pr}_{\mathrm{red}}\colon E_1(B)^{-m,k+m} \longrightarrow E_1(\bar{B})^{-m,k+m}$$

is an isomorphism, i.e. that for all k

$$H^k\left(E^\bullet_{X_s}/\mathbb{C})^{\otimes r}\right) \cong H^k\left(\left(E_{X_s}/(E^0_{X_s} \longrightarrow dE_{X_s}/E^0_{X_s})\right)^{\otimes r}\right)$$

To show that the morphisms

$$\mathrm{pr}_{\mathrm{red},r}\colon (E^\bullet_{X_s}/\mathbb{C})^{\otimes r} \longrightarrow \left(E_{X_s}/(E^0_{X_s} \longrightarrow dE_{X_s}/E^0_{X_s})\right)^{\otimes r}$$

are quasi-isomorphisms for all $r \leq n$, it suffices to consider the tensor-degree-1-case by Künneth, i.e. one has to show that the projection

$$\mathrm{pr}_{\mathrm{red},1}\colon E^\bullet_{X_s}/\mathbb{C} \longrightarrow E_{X_s}/(E^0_{X_s} \longrightarrow dE_{X_s}/E^0_{X_s})$$

is a quasi-isomorphism. Since its kernel $E^0_{X_s}/\mathbb{C} \longrightarrow dE_{X_s}/E^0_{X_s}$ is acyclic, the assertion follows. Let us now deal with the projection

$$\bar{B}_n(\pi_*\mathcal{E}_X^\bullet, \mathcal{E}_S)_{x,y} \twoheadrightarrow \bar{B}_n(\pi_*\mathcal{E}_X^\bullet/\mathcal{E}_S^\bullet, \mathcal{E}_S)_{x,y}$$

Again, it suffices to check that this is a quasi-isomorphism in stalks, i.e. that the morphism

$$\bar{B}_n(E_{X_s}^\bullet|\mathbb{C})_{x_s,y_s} \twoheadrightarrow \bar{B}_n(E_{X_s}^\bullet/\mathbb{C}|\mathbb{C})_{x_s,y_s}$$

is a quasi-isomorphism. With the same argument as above, it suffices to see that the projection

$$\mathrm{pr}_{\mathrm{red},1}\colon E_{X_s}^\bullet/(E_{X_s}^0 \to dE_{X_s}^0) \longrightarrow \left(E_{X_s}^\bullet/(E_{X_s}^0 \to dE_{X_s}^0)\right)/\mathbb{C}$$

is a quasi-isomorphism. Since $\mathbb{C} \subset E_{X_s}^0$, the right hand side is equal to the left hand side, and $\mathrm{pr}_{\mathrm{red},1}$ is an isomorphism. ■

As in the abstract case, note that we have augmentation ideals in case $x = y$:

Definition 1.4.7. *We denote the augmentation ideal of $B_n(X|S)_{x,x}$ (resp. $\widetilde{B}_n(X|S)_{x,x}$) by*

$$I_n(X|S)_x := I_n(\pi_*\mathcal{E}_X^\bullet, \mathcal{E}_S)_x \text{ (resp. } \tilde{I}_n(X|S)_x := \tilde{I}_n(\pi_*\mathcal{E}_X^\bullet, \mathcal{E}_S)_x)$$

and similarly for the left-unbounded complexes.

Corollary 1.4.8. *There are canonical splittings*

$$B_n(X|S)_{x,x} = I_n(X|S)_x \oplus \mathcal{E}_S^\bullet \text{ and } \widetilde{B}_n(X|S)_{x,x} = \tilde{I}_n(X|S)_x \oplus \mathcal{E}_S^\bullet.$$

Chapter 2

Cosimplicial Schemes and Motives

In this chapter, we will recall all the preliminaries on motives we will need in the rest of the thesis, in particular to construct a motivic generalization of the classical bar complexes above. Most importantly, we will go into motives arising from cosimplicial schemes.
In detail, we will proceed as follows:

- First of all, we will review the construction of Levine's category of motives in as much detail as necessary. We will also recall some details of Levine's theory which will be of major importance for us: Gysin isomorphisms and relative motives.
- Then we will move on towards our main point of focus: associating motives to cosimplicial schemes. Here, we will differentiate between unnormalized and normalized motives, and study their properties.
- Last but not least, we will connect our normalized motives of cosimplicial schemes to a certain homotopy limit, and use Levine's Gysin morphism to construct a corresponding one for our normalized motives.

Note that the main points of Levine's theory of motives are outlined in greater detail in chapter B in the appendix.

2.1 Levine's triangulated category of motives

2.1.1 Construction

Let S be a reduced scheme, and let Sch_S denote the category of noetherian separated schemes, and Sm_S the full subcategory of smooth quasi-projective S-schemes. We call $\mathrm{Sm}_S^{\mathrm{ess}}$ the full subcategory of Sch_S of essentially smooth S-schemes.

The construction of the motivic category $\mathcal{DM}(S)$ of motives over S is done in several steps:

a.) [Lev98, I.1.1.1, p.9] One sets out with a category called $\mathcal{L}(\mathrm{Sm}_S)$, which is the category of equivalence classes of pairs (X, f), where X is an object of Sm_S and

$f\colon X' \longrightarrow X$ is a map in $\mathrm{Sm}_S^{\mathrm{ess}}$ which has a smooth section $s\colon X \longrightarrow X'$. Here, the equivalence is given by isomorphisms making the obvious diagram commute. Morphisms between objects $(X, f_X\colon X' \longrightarrow X)$ and $(Y, f_Y\colon Y' \longrightarrow Y)$ in $\mathcal{L}(\mathrm{Sm}_S)$ are commutative diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f_X"'] & Y' \arrow[d, "f_Y"] \\ X \arrow[r] & Y \end{tikzcd}$$

where the top horizontal morphism is flat.

b.) [Lev98, I.1.3.2, p.11] Considering the set $\mathbb{Z}$ as a symmetric monoidal category with operation $+$, one extends $\mathcal{L}(\mathrm{Sm}_S)$ to a symmetric monoidal category $\mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$. $\mathcal{L}^*(\mathrm{Sm}_S)$ is then defined to be the category obtained from $\mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$ by adjoining the morphisms $i_*\colon X(n)_f \longrightarrow (X \coprod Y)(n)_{f \coprod g}$ for any pair $(X, f), (Y, g) \in \mathcal{L}(\mathrm{Sm}_S)$, where $i\colon X \longrightarrow X \coprod Y$ is the inclusion, subject to the following relations:

- $(i \circ j)_* = i_* \circ j_*$ for $X \xrightarrow{i} X \coprod Y \xrightarrow{j} X \coprod Y \coprod Z$,
- $i_{Y_1 *} \circ p_1^* = (p_1 \coprod p_2)^* \circ i_{X_1 *}$ for a diagram

$$\begin{tikzcd} Y_1 \arrow[r, "i_{Y_1}"] \arrow[d, "p_1"'] & Y_1 \coprod Y_2 \arrow[d, "p_1 \coprod p_2"] & Y_2 \arrow[l, "i_{Y_2}"'] \arrow[d, "p_2"] \\ X_1 \arrow[r, "i_{X_1}"'] & X_1 \coprod X_2 & X_2 \arrow[l, "i_{X_2}"] \end{tikzcd}$$

- $i^* \circ i_* = \mathrm{id}$ for the canonical morphism $i\colon X \longrightarrow X \coprod \emptyset$.

By [Lev98, I.1.3.3, p.11] one may extend the symmetric monoidal structure of $\mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$ to one on $\mathcal{L}^*(\mathrm{Sm}_S)$.

c.) [Lev98, I.1.4.1, p.12] Levine then defines the category $\mathcal{A}_1(\mathrm{Sm}_S)$ to be the free additive category on $\mathcal{L}(\mathrm{Sm}_S)^*$ subject to the following list of relations. Here, we denote $X(d)_f$ as an object of $\mathcal{A}_1(\mathrm{Sm}_S)$ by $\mathbb{Z}_X(d)_f$.

- $\mathbb{Z}_\emptyset(d)_f \cong 0$,
- for any pair of objects $(X, f), (Y, g)$ in $\mathcal{L}(\mathrm{Sm}_S)$ with natural inclusions $i_X, i_Y\colon X, Y \longrightarrow X \coprod Y$, one has

$$i_{X*} \circ i_X^* + i_{Y*} \circ i_Y^* = \mathrm{id}_\Gamma,$$

where $\Gamma = \mathbb{Z}_{X \coprod Y}(0)_{(f \coprod g)}$.

The linear extension of the product on $\mathcal{L}(\mathrm{Sm}_S)^*$ makes $\mathcal{A}_1(\mathrm{Sm}_S)$ into a tensor category ([Lev98, I.1.4.2, p.12]).

d.) [Lev98, I.1.4.3/4, p.12] Given a tensor category $(\mathcal{C}, \times, t)$ without unit, one may form the *universal commutative external product* ([Lev98, Part II, I.2.4.1]) $(\mathcal{C}^\otimes, \otimes, \tau)$ by adjoining to the free tensor category on $\mathbb{C}$ the morphisms $\boxtimes_{X,Y}\colon X \otimes Y \longrightarrow X \times Y$ for each pair $X, Y \in \mathcal{C}$ subject to the relations

- (Naturality) $\boxtimes_{X',Y'} \circ (f \otimes g) = (f \times g) \circ \boxtimes_{X,Y}$ for $f\colon X \longrightarrow X', g\colon Y \longrightarrow Y'$ in $\mathcal{C}$,
- (Associativity) $\boxtimes_{X\times Y,Z} \circ (\boxtimes_{X,Y} \otimes \mathrm{id}_Z) = \boxtimes_{X,Y\times Z} \circ (\mathrm{id}_X \otimes \boxtimes_{Y,Z})$ for $X, Y, Z \in \mathcal{C}$,
- (Commutativity) $t_{X,Y} \circ \boxtimes_{X,Y} = \boxtimes_{Y,X} \circ \tau_{X,Y}$.

Levine then defines the category $(\mathcal{A}_2(\mathrm{Sm}_S), \otimes, \tau)$ to be the universal commutative external product on $\mathcal{A}_1(\mathrm{Sm}_S)$.

e.) Levine then constructs categories $\mathcal{A}_3, \mathcal{A}_4$ and $\mathcal{A}_5$ from the category $\mathcal{A}_2(\mathrm{Sm}_S)$ by adjoining some more morphisms, which are of no further importance here.

Definition 2.1.1. *a.) [Lev98, I.1.4.10, p.15] We denote the image of $X(n)_f \in \mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$ in $\mathcal{A}_5(\mathrm{Sm}_S)$ by $\mathbb{Z}_X(n)_f$. Then $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$ is defined to be the full additive subcategory of $\mathcal{A}_5(\mathrm{Sm}_S)$ generated by tensor products of objects of the form $\mathbb{Z}_X(n)_f$, or $\mathfrak{e}^{\otimes a} \otimes \mathbb{Z}_X(n)_f$. It is a DG-category.*

b.) Denote the homotopy category of $C^b_{\mathrm{mot}}(\mathrm{Sm}_S) := C^b(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))$ by $K^b_{\mathrm{mot}}(\mathrm{Sm}_S) = C^b_{\mathrm{mot}}(\mathrm{Sm}_S)/\mathrm{Htp}$.

Definition 2.1.2. *[Lev98, I.2.1.4, pp.17/18] Levine forms the triangulated tensor category $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)$ from $K^b_{\mathrm{mot}}(\mathrm{Sm}_S)$ by inverting the following morphisms:*

a.) Homotopy:

$$p^*\colon \mathbb{Z}_{Y,Z}(n)_g \longrightarrow \mathbb{Z}_{X,p^{-1}(Z)}(n)_f$$

for every map $p\colon (X,f) \longrightarrow (Y,g)$ in $\mathcal{L}(\mathrm{Sm}_S)$ such that $X \hookrightarrow Y$ is the inclusion of a closed codimension 1 subscheme, $Z \subset Y$ a closed subset such that the scheme-theoretic pull-back $p^{-1}(Z) \subset X$ is in $\mathrm{Sm}_S^{\mathrm{ess}}$, and such that there is an isomorphism $p^{-1}(Z) \times_S \mathbb{A}^1_S \cong Z$ making the obvious diagram commute.

b.) Excision:

$$j^*\colon \mathbb{Z}_{X,Z}(n)_f \longrightarrow \mathbb{Z}_{U,Z}(n)_{j^*f}$$

for every $(X,f) \in \mathcal{L}(\mathrm{Sm}_S)$, $Z \subset X$ a closed subset, and $j\colon U \longrightarrow X$ an open subscheme containing Z.

c.) Künneth isomorphism:

$$\boxtimes_{X,Y}\colon \mathbb{Z}_X \otimes \mathbb{Z}_Y \longrightarrow \mathbb{Z}_{X\times Y}$$

for $X, Y \in \mathcal{A}_1(\mathrm{Sm}_S)$.

d.) Gysin isomorphism: For the precise definition of this map see [Lev98, I.2.1.4(d), p.18].

e.) Moving lemma: the morphism induced by $\mathrm{id}\colon X \longrightarrow X$,

$$\rho_{f,g}\colon \mathbb{Z}_X(n)_{f\cup g} \longrightarrow \mathbb{Z}_X(n)_f,$$

for $(X,f) \in \mathcal{L}(\mathrm{Sm}_S)$ and $g\colon Z \longrightarrow X$ a morphism in Sm_S*, where $f \cup g$ is the morphism $f \cup g\colon X' \coprod Z \longrightarrow X$ induced by f and g.*

f.) Unit:

$$[S] \otimes \mathrm{id}\colon \mathfrak{e} \otimes \mathbb{Z}_S(0) \longrightarrow \mathbb{Z}_S(0) \otimes \mathbb{Z}_S(0).$$

Definition 2.1.3. *Let R be a commutative ring which is flat over $\mathbb{Z}$. Then Levine defines the* <u>*triangulated motivic category $\mathcal{DM}(S)_R$ with coefficients in R*</u> *to be the pseudo-abelian hull of $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$. (When R is either $\mathbb{Z}$ or understood, one drops the R in the notation.) Denote the image of $\mathbb{Z}_X(n)_f$ in $\mathcal{DM}(S)_R$ or $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$ by $R_X(n)_f$.*

2.1.2 Passage from schemes to motives

For $a, b \in \mathbb{Z}$, there is a natural pseudofunctor $\mathbb{Z}(a)[b]\colon \mathcal{K}(\mathrm{Sm}_S^{\mathrm{op}}) \longrightarrow \mathcal{DM}(S)$ given on objects by

$$\begin{aligned} \mathbb{Z}(a)[b]\colon \mathcal{K}(\mathrm{Sm}_S^{\mathrm{op}}) &\longrightarrow \mathcal{DM}(S) \\ (X; p) &\longmapsto (\mathbb{Z}_X(a)_{\mathrm{id}}[b]; p^*), \end{aligned}$$

where $\mathbb{Z}_X(a)_{\mathrm{id}}$ is the image of $(X, \mathrm{id}\colon X \longrightarrow X) \in \mathcal{L}(\mathrm{Sm}_S)$ in $C^b_{\mathrm{mot}}(\mathrm{Sm}_S)$. On morphisms $f\colon (X,p) \longrightarrow (X',p')$ in $\mathcal{K}(\mathrm{Sm}_S^{\mathrm{op}})$ it is constructed as follows: One starts off with the map

$$\begin{aligned} \mathbb{Z}(a)[b]\colon\ \mathrm{Sm}_S^{\mathrm{op}} &\longrightarrow C^b_{\mathrm{mot}}(\mathrm{Sm}_S) \\ X &\longmapsto \mathbb{Z}_X(a)_{\mathrm{id}}[b] \\ (f\colon Y \longrightarrow X) &\longmapsto (f^*\colon \mathbb{Z}_X(a)_{\mathrm{id}}[b] \longrightarrow \mathbb{Z}_Y(a)_{f\cup \mathrm{id}_X}[b]), \end{aligned}$$

where $f \cup \mathrm{id}_X$ is given by the fiber product diagram

$$\begin{tikzcd} Y \times_X X \arrow[r] \arrow[d, "f\cup\mathrm{id}_X"'] & X \arrow[d, "\mathrm{id}_X"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Due to the homotopy $\mathbb{Z}_Y(a)_{f\cup\mathrm{id}_X}[b] \simeq \mathbb{Z}_Y(a)_{\mathrm{id}_Y}[b]$ in $\mathcal{DM}(S)$, the above morphism gives rise to a pseudofunctor

$$\mathbb{Z}(a)[b]\colon\ \mathrm{Sm}_S^{\mathrm{op}} \longrightarrow D^b_{\mathrm{mot}}(\mathrm{Sm}_S); \quad X \longmapsto \mathbb{Z}_X(a)_{\mathrm{id}}[b]$$

which naturally induces a pseudofunctor

$$\mathbb{Z}(a)[b]\colon \mathcal{K}(\mathrm{Sm}_S^{\mathrm{op}}) \longrightarrow \mathcal{DM}(S); \quad (X,p) \longmapsto (\mathbb{Z}_X(a)_{\mathrm{id}}[b], p^*).$$

2.1.3 Gysin morphisms

One feature of Levine's motives we will make use of later are Gysin morphisms as constructed in [Lev98, Part I, III.2.1.2.2, p.132]: Let $i\colon Z \hookrightarrow X$ be a codimension d closed embedding in Sm_S with smooth complement. Then there is a Gysin isomorphism

$$i_*\colon R_Z(-d)[-2d] \longrightarrow R_X.$$

It has some nice properties:

a.) *Functoriality:* Given subschemes $W \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} X$ of a scheme $X \in \mathrm{Sm}_S$ with $W, Y \in \mathrm{Sm}_S$, then one has

$$(i \circ j)_* = i_* \circ j_*.$$

This is a special case of a more general version with supports (see [Lev98, III.2.2.1, p.133]).

b.) *Base-chance:* By [Lev98, III.2.4.9, p.150], the Gysin-morphism satisfies the base-change property, which will be of major use in computations later: A cartesian square

$$\begin{tikzcd} Y \times_X Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

in Sm_S is called *transverse* if $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Y) = 0$ for all $p > 0$. Then for any transverse square as above with $i \colon Y \hookrightarrow X$ a closed embedding in Sm_S, one has $f^* \circ i_* = p_{2*} \circ p_1^*$.

By local considerations, one can see that a cartesian square as above is transverse if Y and Z are closed subsets of X which intersect transversely, and i and f are the inclusions.

c.) *Compatibility with pull-backs of the base-scheme:* The Gysin-morphism is natural in the following sense: if $f \colon T \longrightarrow S$ is a map of reduced schemes, then by B.1.5. there is a pull-back functor $\mathcal{DM}(f^*) \colon \mathcal{DM}(S) \longrightarrow \mathcal{DM}(T)$, and by [Lev98, III.2.5.1, p.151], for any closed embedding $i \colon Z \longrightarrow X$ in Sm_S, one has

$$\mathcal{DM}(f^*)(i_*) = (i_{T \times_S Z})_*$$

where $i_{T \times_S Z} \colon T \times_S Z \hookrightarrow T \times_S X$ is the closed embedding in $\mathrm{Sm}(T)$ induced by i.

2.1.4 Relative motives

There is a notion of relative motives giving rise to cohomology groups which correspond to the Adams-eigenspaces of relative K-theory. The reference for the following section is - unless stated otherwise - section I.2.6 of [Lev98]. There is also a notion of relative motives which we will need in the upcoming section.

Let X be a smooth S-scheme with smooth subschemes $D_1, \ldots, D_n \subset X$. For each index $I = (1 \leq i_1 < \ldots i_s \leq n)$, denote the intersection of all subschemes D_i with $i \in I$ by $D_I := D_{i_1} \cap \ldots, D_{i_s}$.

Consider the following complex in $\mathcal{A}_{\mathrm{mot}}(S)$:

$$\mathbb{Z}_X(0)_{g_0^*} \to \bigoplus_{i=1}^{n} \mathbb{Z}_{D_i}(0)_{g_1^*} \to \ldots \to \bigoplus_{|I|=s} \mathbb{Z}_{D_I}(0)_{g_s^*} \to \bigoplus_{|I|=s+1} \mathbb{Z}_{D_I}(0)_{g_{s+1}^*} \to \ldots \to \mathbb{Z}_{D_1\cap\ldots\cap D_n}(0)_{g_n^*}$$

in degrees 0 up to n, where

$$g_s \colon \bigoplus_{|J|\geq s} D_J \to \bigoplus_{|I|=s} D_I$$

is the morphism determined by the inclusions. The differential is given in degree s as the alternating sum

$$\partial^s := \sum_{|I|=s} \sum_{i=1}^{n} (-1)^i \partial^s_{I,i},$$

where the component $\partial^s_{I,i} \colon \mathbb{Z}_{D_I} \longrightarrow \mathbb{Z}_{D_{I\cup\{i\}}}$ is defined by

$$\partial^s_{I,i} := \begin{cases} X^*_{(I\cup\{i\})\supset I} & \text{for } i \notin I \\ 0 & \text{for } i \in I \end{cases}$$

This complex in $\mathcal{A}_{\mathrm{mot}}(S)$ gives rise to an object in $\mathcal{DM}(S)$.

Definition 2.1.4. *a.) For a smooth S-scheme X with smooth subschemes $D_1, \ldots, D_n$ we define the motive of X relative to $D_1, \ldots, D_n$ as the object of $\mathcal{DM}(S)$ defined above, and denote it by $\mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)$.*

b.) For an open subscheme $j \colon U \longrightarrow W$ with complement W, the relative motive with support $\mathbb{Z}_{(X;D_1,\ldots,D_n),W}$ is defined as the cone

$$\mathbb{Z}_{(X;D_1,\ldots,D_n),W} := \mathrm{Cone}(j^* \colon \mathbb{Z}_{(X;D_1,\ldots,D_n)} \to \mathbb{Z}_{(U;D_1^U,\ldots,D_n^U)})[-1]$$

where $D_i^U := U \cap D_i$.

There is also a Gysin isomorphism for relative motives by [Lev98, III.2.6, pp.153ff. and IV. 2.3.4, p.219]: Let $i \colon Z \hookrightarrow X$ be a closed subscheme of codimension d in X, such that the D_i and Z intersect transversely. Denote the intersection of Z with the divisors D_i by $D_i^Z := Z \cap D_i$. Then there is a relative Gysin isomorphism (derived from Levine's Gysin isomorphism for diagrams in section III.2.6)

$$i_* \colon \mathbb{Z}_{(Z;D_1^Z,\ldots,D_n^Z)}(-d)[-2d] \longrightarrow \mathbb{Z}_{(X;D_1,\ldots,D_n)}.$$

2.2 Cosimplicial schemes and motives

2.2.1 Motive associated to a cosimplicial object

One may associate a motive to a truncated cosimplicial scheme as follows:

Let n be any natural number, and $X\colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathrm{Sm}_S$ be a cosimplicial scheme. As in section I.2.4.1 of [Lev98], we find a lifting of X to a cosimplicial object

$$X\colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathcal{L}(\mathrm{Sm}_S),$$

where again $\mathcal{L}(\mathrm{Sm}_S)$ is the category of equivalence classes of pairs $(X, f\colon X' \longrightarrow X)$ with $X \in \mathrm{Sm}_S$ and $f\colon X' \longrightarrow X$ in $\mathrm{Sm}_S^{\mathrm{ess}}$ equipped with a smooth section. For each $n \geq m \geq 0$, Levine lets $X^{\leq m}$ be the disjoint union

$$X^{\leq m} := \coprod_{g\colon [k] \hookrightarrow [m]} X^k$$

where the sum is over all injective, order-preserving functions g, and $f_X^{\leq m}\colon X^{\leq m} \longrightarrow X^m$ be the map which is $X(g)\colon X^k \longrightarrow X^m$ on the component indexed by g. The morphism $f_X^{\leq m}$ has an obvious smooth section given by the inclusion of X^m. Hence, this determines an object $(X^m, f_X^{\leq m}\colon X^{\leq m} \longrightarrow X^m) \in \mathcal{L}(\mathrm{Sm}_S)$.

Now let $h\colon [m_1] \longrightarrow [m_2]$ be a morphism in $\mathbf{\Delta}_{\leq n}$, giving rise to the morphism $X(h)\colon X^{m_1} \longrightarrow X^{m_2}$ in Sm_S. We need to construct a morphism

$$(X^{m_1}, f_X^{\leq m_1}\colon X^{\leq m_1} \longrightarrow X^{m_1}) \longrightarrow (X^{m_2}, f_X^{\leq m_2}\colon X^{\leq m_2} \longrightarrow X^{m_2}) \in \mathcal{L}(\mathrm{Sm}(S))$$

extending this morphism, i.e. a commutative diagram

$$\begin{tikzcd}
X^{\leq m_1} \arrow[r] \arrow[d, "f_X^{\leq m_1}"'] & X^{\leq m_2} \arrow[d, "f_X^{\leq m_2}"] \\
X^{m_1} \arrow[r, "X(h)"'] & X^{m_2}
\end{tikzcd}$$

where the top horizontal morphism is required to be flat. This is accomplished as follows: Consider the component $(X^k)_g$ for $g\colon [k] \hookrightarrow [m_1]$. The identity morphism $\mathrm{id}\colon (X^k)_g \longrightarrow (X^k)_{h\circ g}$ certainly is flat. Moreover, we have

$$f_X^{\leq m_2} \circ \mathrm{id}\Big|_{(X^k)_{h\circ g}} = X(h \circ g) = X(h) \circ X(g),$$

so it makes the above diagram commute. Letting $q(h)\colon X^{\leq m_1} \longrightarrow X^{\leq m_2}$ be the map given on the component $(X^k)_g$ for $g\colon [k] \hookrightarrow [m_1]$ by $\mathrm{id}\colon (X^k)_g \longrightarrow (X^k)_{h\circ g}$ thus makes the above diagram commute. In addition, it is a flat morphism, thus giving rise to a morphism

$$(X^{m_1}, f_X^{\leq m_1}\colon X^{\leq m_1} \longrightarrow X^{m_1}) \longrightarrow (X^{m_2}, f_X^{\leq m_2}\colon X^{\leq m_2} \longrightarrow X^{m_2}) \in \mathcal{L}(\mathrm{Sm}(S))$$

as asserted.

We thus obtain:

Proposition 2.2.1. *The above constructions yield a cosimplicial object*

$$X\colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathcal{L}(\mathrm{Sm}_S)$$

lifting the cosimplicial scheme X^*.

Composing with the natural functor

$$\mathbb{Z}(0)\colon \mathcal{L}(\mathrm{Sm}_S)^{\mathrm{op}} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S); \quad (X,f) \longmapsto \mathbb{Z}_X(0)_f,$$

now yields a simplicial object

$$\mathbb{Z}(0)_{X^*}\colon \mathbf{\Delta}^{\mathrm{op}}_{\leq n} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S); \quad [m] \longmapsto \mathbb{Z}_{X^m}(0)_{f_X^{\leq m}}.$$

The unnormalized complex $C(\mathbb{Z}_{X^*}(0))$ associated to this simplicial object is an element in $C(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))$, and its truncations $C_{\geq -n} := \sigma^{\geq -n}C(\mathbb{Z}_{X^*}(0))$ are bounded complexes in $C^b_{\mathrm{mot}}(\mathrm{Sm}_S) = C^b(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))$ and thus give rise to elements of $\mathcal{DM}(S)$.

Definition 2.2.2. *We call the image of $C_{\geq -n}(\mathbb{Z}_{X^*}(0))$ in $\mathcal{DM}(S)$ the motive associated to the cosimplicial scheme $X^{\leq n}$, and denote it by $M(X^{\leq n})$.*

Remark 2.2.3. In exactly the same way, any cosimplicial object in $\mathbb{Z}\,\mathrm{Sm}_S$ yields a motive, where $\mathbb{Z}\,\mathrm{Sm}_S$ is the additive category generated by Sm_S, i.e. objects are formal finite direct sums of objects of Sm_S and $\mathrm{Hom}_{\mathbb{Z}\,\mathrm{Sm}_S}(X,Y) := \mathbb{Z}[\mathrm{Hom}_{\mathrm{Sm}_S}(X,Y)]$.

2.2.2 Normalized motive associated to a cosimplicial object

We keep the notation we had above. The simplicial object

$$\mathbb{Z}(0)_{X^*}\colon \mathbf{\Delta}^{\mathrm{op}}_{\leq n} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S); \quad [m] \longmapsto \mathbb{Z}_{X^m}(0)_{f_X^{\leq m}}$$

of the past section also gives rise to a normalized complex, which is not an element in $C(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))$, however: Since $C(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))$ is not idempotent complete, the normalized complex $N(\mathbb{Z}(0)_{X^*})$ is an element of its idempotent completion $\mathcal{K}(C(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)))$. Again, trivial truncating of $N(\mathbb{Z}(0)_{X^*})$ yields elements $N_{\geq -n}(\mathbb{Z}(0)_{X^*}) := \sigma^{\geq -n}N(\mathbb{Z}(0)_{X^*})$ in $\mathcal{K}(C^b(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))) = \mathcal{K}(C^b_{\mathrm{mot}}(\mathrm{Sm}_S))$, which naturally yields elements in $\mathcal{DM}(S)$ by the construction of the motivic category.

Definition 2.2.4. *We call the image of $N_{\geq -n}(\mathbb{Z}_{X^*}(0))$ in $\mathcal{DM}(S)$ the normalized motive associated to the cosimplicial scheme $X^{\leq n}$, and denote it by $nM(X^{\leq n})$.*

Let X^* be a cosimplicial scheme. We denote the face, resp. degeneracy maps of X^* by d^i_n and s^i_n. For all n, let e^i_n be such that $s^ie^i = \mathrm{id}$ (one can take $e^i = d^i$, for example). Then consider the morphisms e^is^i for all i. Since $s^ie^i = \mathrm{id}$, e^is^i is an idempotent, and X^n splits into direct sum decompositions

$$X^n \cong \mathrm{Im}(e^i s^i) \oplus \ker(e^i s^i) \cong \mathrm{Im}(e^i s^i) \oplus \mathrm{Im}(\mathrm{id} - e^i s^i)$$

$$X^n \cong \mathrm{Im}\left(\sum_{i=0}^{n-1} e^i s^i \colon X^n \longrightarrow X^n\right) \oplus \mathrm{Im}\left((\mathrm{id} - e^0 s^0) \circ \ldots \circ (\mathrm{id} - e^{n-1} s^{n-1}) \colon X^n \longrightarrow X^n\right).$$

The normalized motive associated to a cosimplicial object X^* is then represented by the following diagram:

$$\begin{tikzcd}
\mathbb{Z}_{X^n}(0)_{f_X^{\leq n}} \arrow[r, "\delta^*_{n-1}"] \arrow[d, "p^*_n"] & \mathbb{Z}_{X^{n-1}}(0)_{f_X^{\leq n-1}} \arrow[r, "\delta^*_{n-2}"] \arrow[d, "p^*_{n-1}"] & \ldots \arrow[r, "\delta^*_2"] & \mathbb{Z}_{X^2}(0)_{f_X^{\leq 2}} \arrow[r, "\delta^*_1"] \arrow[d, "p^*_2"] & \mathbb{Z}_{X^1}(0)_{f_X^{\leq 1}} \arrow[r, "\delta^*_0"] \arrow[d, "p^*_1"] & \mathbb{Z}_{X^0}(0)_{\mathrm{id}} \arrow[d, "p^*_0"] \\
\mathbb{Z}_{X^n}(0)_{f_X^{\leq n}} \arrow[r, "\delta^*_{n-1}"] & \mathbb{Z}_{X^{n-1}}(0)_{f_X^{\leq n-1}} \arrow[r, "\delta^*_{n-2}"] & \ldots \arrow[r, "\delta^*_2"] & \mathbb{Z}_{X^2}(0)_{f_X^{\leq 2}} \arrow[r, "\delta^*_1"] & \mathbb{Z}_{X^1}(0)_{f_X^{\leq 1}} \arrow[r, "\delta^*_0"] & \mathbb{Z}_{X^0}(0)_{\mathrm{id}}
\end{tikzcd}$$

where $p_m := (\mathrm{id} - e^0 s^0) \circ \ldots \circ (\mathrm{id} - e^m s^m)$, the differentials $\widetilde{\delta}_k \colon X^k \longrightarrow X^{k+1}$ are the differentials induced by the differentials of the unnormalized cochain complex associated to X^* and the $f_X^{\leq k}$ are the morphisms of the preceding section.

2.2.3 Normalized motives of cosimplicial schemes vs. homotopy limits

Let n be any natural number, and $X \colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathrm{Sm}_S$ be a cosimplicial scheme. As in 2.2.1 above we have a lifting of X to a cosimplicial object

$$X \colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathcal{L}(\mathrm{Sm}_S), [k] \longmapsto \left(X^k, f_X^{\leq k} \colon \bigoplus_{g \colon [i] \hookrightarrow [k]} X^i \to X^k\right)$$

It gives rise to a simplicial object in the corresponding DG category $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$:

$$\mathbb{Z}(0)_X \colon \mathbf{\Delta}^{\mathrm{op}}_{\leq n} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S), [k] \longmapsto \mathbb{Z}_{X^k}(0)_{f_X^{\leq k}}$$

We want to compute the non-degenerate homotopy colimit $\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}, \mathrm{n.d.}} \mathbb{Z}(0)_{X^*}$ of this simplicial object. The nerve of $\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$ is the simplicial set which is given for $[m] \in \mathbf{\Delta}$ by

$$\begin{aligned}
\mathscr{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leq n})([m]) &= \mathrm{Hom}([m], \mathbf{\Delta}^{\mathrm{op}}_{\leq n}) \\
&= \left\{[j_m] \overset{\alpha_{m-1}}{\to} \ldots \overset{\alpha_0}{\to} [j_0] \mid \alpha_i \in \mathbf{\Delta}_{\leq n}\ \forall i\right\}
\end{aligned}$$

and the non-degenerate simplices are the ones not containing the identity, i.e.

$$\mathscr{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leq n})([m])_{\mathrm{n.d.}} = \{[j_m] \overset{\alpha_{m-1}}{\to} \ldots \overset{\alpha_0}{\to} [j_0] \mid \mathrm{id} \neq \alpha_i \in \mathbf{\Delta}_{\leq n}\ \forall i = 0, \ldots, m-1\}.$$

Now put

$$\mathbb{Z}(0)^{\delta}_{X^*}([m]) := \bigoplus_{\alpha \in \mathscr{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leq n})([m])_{\mathrm{n.d.}}} X^{\alpha(m)} = \bigoplus_{[j_m] \overset{\alpha_{m-1}}{\to} \ldots \overset{\alpha_0}{\to} [j_0]\ \mathrm{n.d.}} Z_{X^{j_0}}(0)_{f_X^{\leq j_0}}$$

Moreover, we define morphisms $\mathbb{Z}_{X^*}(0)^\delta([m+1]) \longrightarrow \mathbb{Z}_{X^*}(0)^\delta([m])$ as follows: Recall the coface maps given by

$$\delta^i_{m+1}\colon [m] \longrightarrow [m+1]; \quad \delta^i_{m+1}(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

for $m \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq m+1$. Then for a fixed component, i.e. a given sequence of composable morphisms $[j_m] \to \ldots \to [j_0]$ in $\mathbf{\Delta}_{\leq n}$, we put

$$\begin{aligned} d^i_{m-1}\Big|_{[j_m]\overset{\alpha_{m-1}}{\to}\ldots\overset{\alpha_0}{\to}[j_0]} &: \ (\mathbb{Z}_{X^{j_0}}(0))_{\alpha_{m-1}\ldots\alpha_0} \longrightarrow (\mathbb{Z}_{X^{j_0}}(0))_{\alpha_{m-1}\ldots(\alpha_i\circ\alpha_{i-1})\ldots\alpha_0} \\ d^i_m\Big|_{[j_m]\overset{\alpha_{m-1}}{\to}\ldots\overset{\alpha_0}{\to}[j_0]} &:= (\mathrm{id}_{\mathbb{Z}_{X^{j_0}}(0)})_{[j_m]\to\ldots\to[j_{i+1}]\overset{\alpha_i\circ\alpha_{i-1}}{\longrightarrow}[j_{i-1}]\to\ldots\to[j_0]} \end{aligned}$$

for $i = 1, \ldots, m-1$, where we dropped the $f_X^{\leq j_0}$ in the notation, and

$$\begin{aligned} d^0_{m-1}\Big|_{[j_m]\overset{\alpha_{m-1}}{\to}\ldots\overset{\alpha_0}{\to}[j_0]} &: \mathbb{Z}_{X^{j_0}}(0) \longrightarrow (\mathbb{Z}_{X^{j_1}}(0))_{\alpha_{m-1}\ldots\alpha_1} \\ d^{m+1}_m\Big|_{[j_m]\overset{\alpha_{m-1}}{\to}\ldots\overset{\alpha_0}{\to}[j_0]} &:= (\mathbb{Z}_{X^*}(0)(\alpha_0\colon [j_1] \longrightarrow [j_0]))_{[j_m]\overset{\alpha_{m-1}}{\to}\ldots\overset{\alpha_1}{\to}[j_1]} \end{aligned}$$

Then put

$$d_{m-1} := \sum_{i=0}^{m} (-1)^i d^i_{m-1}\colon \mathbb{Z}(0)^\delta_{X^*}([m]) \longrightarrow \mathbb{Z}(0)^\delta_{X^*}([m-1]).$$

Letting $\mathbb{Z}(0)^\delta_{X^*}([m])$ be in degree $-m$, $\mathbb{Z}(0)^\delta_{X^*}([*])$ together with the differentials d_* yields a complex in $C^b(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)) = C^b_{\mathrm{mot}}(\mathrm{Sm}_S)$, which is the *non-degenerate homotopy colimit* of $\mathbb{Z}(0)_{X^*}$:

$$\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n},\mathrm{n.d.}}(\mathbb{Z}(0)_{X^*}) = \mathrm{Tot}(\mathbb{Z}(0)^\delta_{X^*}([*])) = \mathbb{Z}(0)^\delta_{X^*}([*]) \in C^b_{\mathrm{mot}}(\mathrm{Sm}_S).$$

The homotopy colimit is thus equal to the complex

$$\begin{array}{ccccccccc} \ldots \xrightarrow{d_n} & \bigoplus_{[j_n]\to\ldots[j_0]} Z_{X^{j_0}}(0) & \xrightarrow{d_{n-1}} & \bigoplus_{[j_{n-1}]\to[j_0]} Z_{X^{j_0}}(0) & \xrightarrow{d_{n-2}} & \ldots & \xrightarrow{d_0} & \bigoplus_{k=0}^n Z_{X^k}(0) \\ \ldots & -n & & -(n-1) & & \ldots & & 0 \end{array}$$

where we dropped the index $f_X^{\leq 0}$ everywhere.

In comparison, recall the truncated normalized complex $N_{\geq -n}(\mathbb{Z}(0)_{X^*})$ of the simplicial object

$$\mathbb{Z}(0)_{X^*}\colon \mathbf{\Delta}^{\mathrm{op}}_{\leq n} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S); \quad [m] \longmapsto \mathbb{Z}_{X^m}(0)_{f_X^{\leq m}}$$

of the past section. It is an element in $\mathcal{K}(C^b_{\mathrm{mot}}(\mathrm{Sm}_S))$, the Karoubi envelope of $C^b_{\mathrm{mot}}(\mathrm{Sm}_S)$.

Theorem 2.2.5. *For every n, the complexes* $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n},\mathrm{n.d.}}(\mathbb{Z}(0)_{X^*})$ *and* $N_{\geq -n}(\mathbb{Z}(0)_{X^*})$ *above are homotopic in* $\mathcal{K}(C^b_{\mathrm{mot}}(\mathrm{Sm}_S))$.

The proof of this is a consequence of more general considerations in the dual setting:

Lemma 2.2.6. *Let X_* be a cosimplicial object in a DG-category $\mathscr{A}$. Then the homotopy limit* $\operatorname{holim}_{\mathbf{\Delta}_{\leq n}} X_*$ *is homotopic to the brutal truncation $\sigma^{\leq n}N(X_*)$ of the normalized complex associated to X_*.*

Proof. This proof is similar to the proof of [DG05, 3.10., p.25]. For any category $\mathscr{A}$, we denote its idempotent completion by $\mathscr{A}_{\mathrm{kar}}$. We are particularly interested in the Karoubi envelope $\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}$ of $\mathbb{Z}\mathbf{\Delta}$. Let $[*]$ denote the universal cosimplicial object of $\mathbf{\Delta}$. It is universal in the sense that for any idempotent complete category $\mathscr{A}$ the functor

$$\begin{aligned}(\text{functors } \ \mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}} \longrightarrow \mathscr{A}) &\longrightarrow (\text{cosimplicial objects of } \ \mathscr{A})\,,\\ F &\longmapsto F([*])\end{aligned}$$

is an equivalence of categories. In order to prove the statement of the lemma, it thus suffices to prove it for the cosimplicial object $X_* = [*]$ of $\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}$. The complexes $\operatorname{holim}_{\mathbf{\Delta}_{\leq n}}[*]$ and $\sigma_{\leq n}N([*])$ are both complexes in $\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}$. We will show the assertion by an application of the Yoneda lemma: The contravariant functor

$$\begin{aligned}h^*_{\mathbb{Z}}\colon \mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}} &\longrightarrow \mathcal{H}om(\mathbf{\Delta}, \mathrm{Ab})\\ X &\longmapsto h^X_{\mathbb{Z}} := \operatorname{Hom}_{\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}}(X, -),\end{aligned}$$

is fully faithful, and $\operatorname{Hom}(h^X_{\mathbb{Z}}, F) = F(X)$ for any $F \in \mathcal{H}om(\Delta, \mathrm{Ab})$. Therefore, the two complexes in question are homotopic iff they are homotopic after applying $h^*_{\mathbb{Z}}$. The advantage of applying $h^*_{\mathbb{Z}}$ is that the category $\mathcal{H}om(\mathbf{\Delta}, \mathrm{Ab})$ is abelian, and one may apply standard arguments. Let $H([*])$ denote the cosimplicial object underlying the homotopy limit $\operatorname{holim}_{\mathbf{\Delta}_{\leq n}}[*]$. We obtain the following complexes

$$\begin{aligned}h_{\mathbb{Z}}^{\operatorname{holim}_{\mathbf{\Delta}_{\leq n}}[*]} &= h_{\mathbb{Z}}^{C(H([*]))} = C\left(h_{\mathbb{Z}}^{H([*])}\right)\\ &= C\left(\operatorname{Hom}_{\mathbf{\Delta}_{\mathrm{kar}}}(H([*]),\ -\ )\right)\\ &= \operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \operatorname{Hom}_{\mathbf{\Delta}_{\mathrm{kar}}}([*],\ -\ )\\ h_{\mathbb{Z}}^{\sigma_{\leq n}N([*])} &= \sigma^{\geq -n} h_{\mathbb{Z}}^{N([*])} = \sigma^{\geq -n} N(\operatorname{Hom}_{\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}}([*],\ -\ ))\end{aligned}$$

of projective objects in the category $\mathcal{H}om(\mathbf{\Delta}, \mathrm{Ab})$, where by abuse of notation $N(X^*)$ also denotes the normalized complex of a simplicial object X^*. If we evaluate both of these complexes in $\mathcal{H}om(\mathbf{\Delta}, \mathrm{Ab})$ at the object $[p]$, we obtain the following two complexes of abelian groups:

$$\begin{aligned}h_{\mathbb{Z}}^{\operatorname{holim}_{\mathbf{\Delta}_{\leq n}}[*]}([p]) &= \operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \operatorname{Hom}_{\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}}([*],[p]) = \operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} S^*\\ h_{\mathbb{Z}}^{\sigma_{\leq n}N([*])}([p]) &= \sigma^{\geq -n} N(\operatorname{Hom}_{\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}}([*],[p])) = \sigma^{\geq -n} N(S^*)\end{aligned}$$

where $S^* = \mathrm{Hom}_{\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}}}([*],[p])$. Note that $C(S^*)$ is equal to the simplicial complex of chains of $[p]$, resp. the standard simplicial set Δ^p: The geometric realization of $[p]$ as a category is the geometric realization of its nerve, which is given by

$$\mathscr{N}([p])([k]) = \mathrm{Hom}_{\mathrm{poset}}([k],[p]) = \mathrm{Hom}_{\mathbf{\Delta}}([k],[p])$$

Due to Yoneda's lemma, we furthermore have $\mathrm{Hom}_{\mathbf{\Delta}}([k],[p]) \cong \mathrm{Hom}_{\mathrm{Ab}^{\mathbf{\Delta}}}(\Delta^k, \Delta^p)$, i.e. the geometric realization of $[p]$ is the same as that of Δ^p. As a consequence, $C(S^*)$ is indeed the simplicial complex associated to the standard simplicial set Δ^p. The normalized complex of S^*, $N(S^*)$, is the quotient of $C(S^*)$ by the image of all degeneracy maps, i.e. it is the simplicial complex of non-degenerate chains of $[p]$, resp. Δ^p.

Let us take a closer look at both complexes in question:

a.) $h_{\mathbb{Z}}^{\sigma_{\leq n} N([*])}([p]) = \sigma^{\geq -n} N(S^*)$:

As we have seen above, this complex is the truncation of the simplicial complex of non-degenerate chains of Δ^p, i.e. for all $m > p$, we have

$$N(S^*)^m = \mathrm{Hom}_{\mathbb{Z}\mathbf{\Delta}_{\mathrm{kar}},\mathrm{n.d.}}([m],[p]) = 0,$$

so $\sigma^{\geq -n} N(S^*) = N(S^*)$, and the complex in question is equal to the simplicial complex of non-degenerate chains of Δ^p computing the homology of $|\Delta^p|$. Thus, $N(S^*)$ is a projective resolution of $\mathbb{Z}$.

b.) $h_{\mathbb{Z}}^{\mathrm{holim}_{\mathbf{\Delta}_{\leq n}}[*]}([p]) = \mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} S^*$:

The homotopy colimit $\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} S^*$ is the following complex: In degree m it is given by

$$\bigoplus_{\alpha \in \mathscr{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leq n})([m])_{\mathrm{n.d.}}} S^{\alpha(m)} = \bigoplus_{[j_m] \overset{\alpha_{m-1}}{\to} \dots \overset{\alpha_0}{\to} [j_0] \ \mathrm{n.d.}} S^{j_0}$$

with $S^* = \mathrm{Hom}_{(\mathbb{Z}\mathbf{\Delta}_{\leq n})_{\mathrm{kar}}}([*],[p])$. The differentials are the following: For a fixed component $[j_m] \to \dots [j_0]$ in $\mathbf{\Delta}_{\leq n}$, we have

$$d^i_{m-1}\big|_{[j_m] \overset{\alpha_{m-1}}{\to} \dots \overset{\alpha_0}{\to} [j_0]} : \left(S^{j_0}\right)_{\alpha_{m-1}\dots\alpha_0} \longrightarrow \left(S^{j_0}\right)_{\alpha_{m-1}\dots(\alpha_i \circ \alpha_{i-1})\dots\alpha_0}$$

given by the identity in every component for $i = 1, \dots, m-1$, and

$$d^0_{m-1}\big|_{[j_m] \overset{\alpha_{m-1}}{\to} \dots \overset{\alpha_0}{\to} [j_0]} : S^{j_0} \longrightarrow (S^{j_1})_{\alpha_{m-1}\dots\alpha_1}$$

is the morphism given by $S^*(\alpha_0 \colon [j_1] \longrightarrow [j_0])$. The differential is then given on the component in question by the alternating sum $d_{m-1} := \sum_{i=0}^m (-1)^i d^i_{m-1}$. This complex can also be seen as the simplicial complex of all non-degenerate cycles

$$[q] \longrightarrow {}_{[p]/}\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$$

of the coslice category ${}_{[p]/}\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$ of all morphisms $[j_0] \longrightarrow [p]$ in $\mathbf{\Delta}$. A q-cycle of ${}_{[p]/}\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$ is given by a commutative diagram of composable morphisms

$$\begin{tikzcd}
{[j_m]} \arrow[r, "\alpha_{m-1}"] \arrow[drr, "\varphi_m"'] & {[j_{m-1}]} \arrow[r] \arrow[dr, "\varphi_{m-1}"] & \ldots \arrow[r] & {[j_1]} \arrow[r, "\alpha_0"] \arrow[dl, "\varphi_1"'] & {[j_0]} \arrow[dll, "\varphi_0"] \\
& & {[p]} & &
\end{tikzcd}$$

corresponding to the element $(\varphi_m \colon [j_m] \longrightarrow [p])_{[j_m] \overset{\alpha_{m-1}}{\to} \ldots \overset{\alpha_0}{\to} [j_0]}$. Thus, the complex in question is the simplicial complex computing the homology of the topological space

$$\left| {}_{[p]/}\mathbf{\Delta}^{\mathrm{op}}_{\leq n} \right| .$$

This topological space, however, is contractible, since ${}_{[p]/}\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$ has an initial object. As a consequence, $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} S^*$ is also a projective resolution of $\mathbb{Z}$.

Since both complexes are projective resolutions of $\mathbb{Z}$, they are homotopic. This proves the assertion. ∎

By duality, we also obtain the dual statement of the above:

Corollary 2.2.7. *Let X^* be a simplicial object in a DG-category $\mathscr{A}$, then the homotopy colimit* $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n},\mathrm{n.d.}} X^*$ *is homotopic to the brutal truncation of the normalized complex $\sigma^{\geq -n} N(X^*)$ associated to X^*. In particular, the complexes* $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n},\mathrm{n.d.}}(\mathbb{Z}(0)_{X^*})$ *and $nM(X^{\leq n})$ are homotopic.*

This immediately shows our theorem 2.2.5 above. ∎

Corollary 2.2.8. *There is an isomorphism in $\mathcal{DM}(S)$*

$$\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n},\mathrm{n.d.}}(\mathbb{Z}(0)_{X^*}) \cong \sigma^{\geq -n} N(\mathbb{Z}(0)_{X^*}).$$

2.3 Properties of motives associated to a cosimplicial object

2.3.1 Naturality

Let $\varphi \colon T \longrightarrow S$ be a morphism of schemes. Then, one has the fiber-product functor

$$- \times_S T \colon \mathbb{Z}\,\mathrm{Sm}_S \longrightarrow \mathbb{Z}\,\mathrm{Sm}_T$$

on the level of schemes, which induces a functor

$$(- \times_S T, \times_S \mathrm{id}_T) \colon \mathcal{L}(\mathbb{Z}\,\mathrm{Sm}_S) \longrightarrow \mathcal{L}(\mathbb{Z}\,\mathrm{Sm}_T).$$

By [Lev98, I.2.3., p.24], it indues a functor of DG tensor categories

$$\mathcal{A}_{\mathrm{mot}}(\varphi^*)\colon \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S) \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_T), \quad \mathbb{Z}_X(a)_f \longmapsto \mathbb{Z}_{X\times_S T}(a)_{f\times_S \mathrm{id}_T}$$

This functor again induces Levine's pull-back functor of motives

$$\mathcal{DM}(\varphi^*)\colon \mathcal{DM}(S) \longrightarrow \mathcal{DM}(T).$$

Now let X^* be a cosimplicial S-scheme with associated unnormalized and normalized S-motives $M(X^{\leq n})$ and $nM(X^{\leq n})$ in $\mathcal{DM}(S)$ for all $n \in \mathbb{N}$ as constructed in the preceding section. Likewise, the cosimplicial S-scheme $X^* \times_S T$, which is given in degree n by $X^n \times_S T$ with the obvious induced face and degeneracy maps, gives rise to associated unnormalized and normalized T-motives $M(X^{\leq n} \times_S T)$ and $nM(X^{\leq n} \times_S T)$ in $\mathcal{DM}(T)$ for all $n \in \mathbb{N}$. Let again $f_X^{\leq m}\colon X^{\leq m} = \coprod_{g\colon [k]\hookrightarrow[m]} X^k \longrightarrow X^m$ be the map which is $X(g)\colon X^k \longrightarrow X^m$ on the component indexed by g for any cosimplicial scheme X. Then

$$f^{\leq m}_{X\times_S T}\colon (X\times_S T)^{\leq m} = X^{\leq m} \times_S T \longrightarrow X^m \times_S T$$

is equal to the morphism $f_X^{\leq m} \times_S T$, so by the construction of the motivic pull-back, it is immediate that for any cosimplicial object X in Sm_S, one has

$$\begin{aligned} \mathcal{DM}(\varphi^*)(M(X^{\leq n})) &= M(X^{\leq n} \times_S T) \ \text{ as well as} \\ \mathcal{DM}(\varphi^*)(nM(X^{\leq n})) &= nM(X^{\leq n} \times_S T). \end{aligned}$$

2.3.2 Gysin morphisms for normalized motives of cosimplicial schemes

Let $Z^*, X^*\colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathrm{Sm}_S$ be two cosimplicial schemes, where $I\colon Z \hookrightarrow X$ is a codimension d closed embedding, and denote the corresponding simplicial objects in $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$ by $\mathbb{Z}_{Z^*}(0)$ and $\mathbb{Z}_{X^*}(0)$. In [Lev98, III.2.6.8., P.158], Levine defines a Gysin isomorphism

$$i_*\colon \operatorname{holim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \mathbb{Z}_{Z^*}(-d)[-2d] \longrightarrow \operatorname{holim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathbb{Z}_{X^*}(0) :$$

It is given by the morphism induced on total complexes by the following morphism of double complexes in $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$:

$$\begin{array}{ccccccc}
\bigoplus_{k=0}^n \mathbb{Z}_{Z^k}(-d)[-2d] & \xrightarrow{d_0[-2d]} & \dots & \xrightarrow{d_{n-1}[-2d]} & \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{Z^{j_n}}(-d)[-2d] & \xrightarrow{d_n[-2d]} & \dots \\
\Big\downarrow {\scriptstyle \bigoplus_{k=0}^n i_{Z^k *}} & & & & \Big\downarrow {\scriptstyle \bigoplus_{[j_n]\to\dots[j_0]} i_{Z^{j_n} *}} & & \\
\bigoplus_{k=0}^n \mathbb{Z}_{X^k}(0) & \xrightarrow{d_0} & \dots & \xrightarrow{d_{n-1}} & \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{X^{j_n}}(0) & \xrightarrow{d_n} & \dots \\
& & & & & & \\
0 & & \dots & & n & & \dots
\end{array}$$

where we dropped the index $f_X^{\leq m}$ everywhere, and moreover the complex

$$\bigoplus_{k=0}^n \mathbb{Z}_{X^k}(0) \xrightarrow{d_0} \dots \xrightarrow{d_{n-1}} \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{X^{j_n}}(0) \xrightarrow{d_n} \dots$$

is considered as the complex of the (vertical) complexes with $\bigoplus_{[j_m]\to\dots[j_0]} Z_{X^{j_m}}(0)$ concentrated in degree 0. The dual result for the homotopy colimit then yields a Gysin morphism for our normalized motives of cosimplicial schemes:

Corollary 2.3.1. *Let $Z^*, X^*\colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathrm{Sm}_S$ and d be as above. Then there is a Gysin isomorphism*

$$i_*\colon\ \mathrm{hocolim}_{\mathbf{\Delta}_{\leq n}^{\mathrm{op}}} \mathbb{Z}_{Z^*}(0)(-d)[-2d] \longrightarrow \mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathbb{Z}_{X^*}(0), \quad \textit{resp.}$$
$$i_*\colon N_{\geq -n}(\mathbb{Z}_{Z^*}(0))(-d)[-2d] \longrightarrow N_{\geq -n}(\mathbb{Z}_{X^*}(0)) \quad \textit{in } \mathcal{DM}(S).$$

It is the morphism induced on total complexes by the morphism of double complexes

$$\begin{tikzcd}
\dots \arrow[r, "{d_n[-2d]}"] & \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{Z^{j_0}}(-d)[-2d] \arrow[r, "{d_{n-1}[-2d]}"] \arrow[d, "\bigoplus_{[j_n]\to\dots[j_0]} i_{Z^{j_0}*}"] & \dots \arrow[r, "{d_0[-2d]}"] & \bigoplus_{k=0}^n \mathbb{Z}_{Z^k}(-d)[-2d] \arrow[d, "\bigoplus_{k=0}^n i_{Z^k*}"] \\
\dots \arrow[r, "d_n"] & \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{X^{j_0}}(0) \arrow[r, "d_{n-1}"] & \dots \arrow[r, "d_0"] & \bigoplus_{k=0}^n \mathbb{Z}_{X^k}(0) \\
\dots & -n & \dots & 0
\end{tikzcd}$$

in $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$, where we dropped the index $f_X^{\leq 0}$ everywhere, and the complex

$$\bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{X^{j_0}}(0) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_0} \bigoplus_{k=0}^n \mathbb{Z}_{X^k}(0)$$

is considered as the complex of the (vertical) complexes with $\bigoplus_{[j_m]\to\dots[j_0]} Z_{X^{j_0}}(0)$ concentrated in degree 0.

Chapter 3

Motivic bar constructions

In the previous chapter we recalled the classical notion of bar complexes. After introducing it explicitly as the total complex of a certain double complex, we took a look at the simplicial interpretation underlying the construction. The latter puts the theory of bar complexes into a more conceptional setting. Even more than that, it provides a natural motivic interpretation of the bar complexes of schemes as considered in section 1.4. In the upcoming chapter, we want to formalize this motivic analogue of bar complexes in the language of simplicial objects. The setting we consider is the following:
Let S be a reduced scheme, and $X \in \mathrm{Sm}_S$, where Sm_S denotes the category of smooth quasi-projective S-schemes (also, recall that we always take a "scheme" to mean "noetherian, separated scheme"). Following the usual conventions, we will often denote the i-fold fiber product of X with itself over the base S for any $i \in \mathbb{N}$ by

$$X^i := X \times_S X \times_S \ldots \times_S X.$$

When we are in the category Sm_S and write $\times$, we will always mean $\times_S$.

The main outline of this chapter:

- We first construct a cosimplicial "bar" object in the category $\mathbb{Z}\,\mathrm{Sm}_S$.
- By our considerations in the first step tying motives to cosimplicial objects, this will give rise to "bar" objects in $\mathcal{DM}(S)$.
- Finally, we will construct a motivic generalization of the classical "augmentation ideals" of section 1.3.4.

Basic idea of the chapter:

We studied classical bar complexes in Chapter 1. In particular, recall the simplicial bar object $sB(X|S)_{x,y}$ for $\pi\colon X \longrightarrow S$ a smooth S-scheme with sections $x, y\colon S \longrightarrow X$ of 1.4. It was given by

$$
\begin{aligned}
sB_n(X|S)_{x,y} &= (\pi_*\mathcal{E}_X^\bullet)^{\otimes n};\\
d_j^n([a_1|\dots|a_{n+1}]) &= \begin{cases} -[x(a_1)a_2|\dots|a_{n+1}] & \text{for } j=0\\ -[a_1|\dots|a_ja_{j+1}|\dots|a_{n+1}] & \text{for } j\in\{1,\dots,n\}\\ -[a_1|\dots|a_ny(a_{n+1})] & \text{for } j=n+1\end{cases}\\
s_j^n([a_1|\dots|a_{n+1}]) &= -[a_1|\dots|a_j|1|a_{j+1}|\dots|a_{n+1}],
\end{aligned}
$$

where all tensor products are over $\mathcal{E}_S$. One may also write the face and degeneracy maps as follows: Let $\Delta\colon X \longrightarrow X\times_S X$ denote the diagonal over S. The induced morphism $\Delta^*\colon (\pi_*\mathcal{E}_X)^{\otimes 2}\longrightarrow \pi_*\mathcal{E}_X$, is given by $[a_1|a_2]\longmapsto[a_1a_2]$, while the structure morphism and sections x,y induce morphisms

$$
\begin{aligned}
&\pi^*\colon \mathcal{E}_S\longrightarrow \pi_*\mathcal{E}_X \quad\text{and}\\
&x^*,y^*\colon \pi_*\mathcal{E}_X\longrightarrow\mathcal{E}_S.
\end{aligned}
$$

Δ^* induces morphisms of differential graded algebras

$$
\begin{aligned}
\mathrm{id}^{\otimes i-1}\otimes\Delta^*\otimes\mathrm{id}^{\otimes n-i}&\colon (\pi_*\mathcal{E}_X)^{\otimes n+1}\longrightarrow(\pi_*\mathcal{E}_X)^{\otimes n}\\
(\mathrm{id}^{\otimes i-1}\otimes\Delta^*\otimes\mathrm{id}^{\otimes n-i})([a_1|\dots|a_{n+1}]) &= [a_1|\dots|a_ia_{i+1}|\dots|a_{n+1}].
\end{aligned}
$$

Likewise, x^* and y^* induce the morphisms of differential graded algebras

$$
\begin{aligned}
&x^*\otimes\mathrm{id}^{\otimes n},\mathrm{id}^{\otimes n}\otimes y^*\colon (\pi_*\mathcal{E}_X)^{\otimes n+1}\longrightarrow(\pi_*\mathcal{E}_X)^{\otimes n}\quad\text{via}\\
&x^*\otimes\mathrm{id}^{\otimes n}([a_1|\dots|a_{n+1}]) = [x^*(a_1)a_2|\dots|a_{n+1}]\quad\text{and}\\
&\mathrm{id}^{\otimes n}\otimes y^*([a_1|\dots|a_{n+1}]) = [a_1|\dots|a_ny^*(a_{n+1})],
\end{aligned}
$$

while π^* induces, for any i, the morphisms

$$
\begin{aligned}
\mathrm{id}^{\otimes i-1}\otimes\pi^*\otimes\mathrm{id}^{\otimes n-i}\colon (\pi_*\mathcal{E}_X)^{\otimes n}&\longrightarrow(\pi_*\mathcal{E}_X)^{\otimes n+1};\\
[a_1|\dots|a_n]&\longmapsto[a_1|\dots|a_i|1|a_{i+1}|\dots|a_n].
\end{aligned}
$$

With this, we may write

$$
\begin{aligned}
d_j^n &= \begin{cases} -x^*\otimes\mathrm{id}^{\otimes n} & \text{for } j=0\\ -\mathrm{id}^{\otimes j-1}\otimes\Delta^*\otimes\mathrm{id}^{\otimes n-j} & \text{for } j\in\{1,\dots,n\}\\ -\mathrm{id}^{\otimes n}\otimes y^* & \text{for } j=n+1\end{cases}\\
s_j^n &= -\mathrm{id}^{\otimes j-1}\otimes\pi^*\otimes\mathrm{id}^{\otimes n-j}.
\end{aligned}
$$

Now note that the geometric realization of the motive $\mathbb{Q}_{X^n}(0)\in\mathcal{DM}(S)_{\mathbb{Q}}$ is given by $(\pi_*\mathcal{E}_X)^{\otimes n}$ in some sense, which we will just assume as given here. Thus we may reinterpret the above simplicial bar object in a motivic sense: Let us denote the geometric realization functor by $\mathfrak{R}$ for the time being. By the above considerations, the simplicial bar object $sB^\bullet(X|S)_{x,y}$ can be written as follows:

$$sB_n(X|S)_{x,y} = \mathfrak{R}(\mathbb{Q}_{X^n});$$
$$d_j^n = \begin{cases} -(x \times \mathrm{id}^{\otimes n})^* & \text{for } j = 0 \\ -(\mathrm{id}^{j-1} \times \Delta \times \mathrm{id}^{n-j})^* & \text{for } j \in \{1, \dots, n\} \\ -(\mathrm{id}^n \times y)^* & \text{for } j = n+1 \end{cases}$$
$$s_j^n = -(\mathrm{id}^{j-1} \times \pi \times \mathrm{id}^{n-j})^*.$$

Moreover, this construction can even be done with integer coefficients, and even on the level of cosimplicial schemes. This leads to the following definition:

3.1 Cosimplicial viewpoint in $C^b(\mathcal{K}(\mathrm{Sm}_S))$

In the preceding section we recalled the formalism we will need in what follows. The upcoming chapter will now provide a generalization of the theory of section 1.4 to motives. As mentioned above, we first construct the (co)simplicial object we want to look at in the category $C^b(\mathcal{K}(\mathrm{Sm}_S))$ of bounded complexes in the pseudo-abelian envelope of smooth S-schemes. This is done as follows:

Let $\pi\colon X \longrightarrow S$ in Sm_S be equipped with two sections $x, y\colon S \longrightarrow X$. We consider the functor

$$cB^\bullet_{\mathrm{mot}}(X|S)_{x,y}\colon \mathbf{\Delta} \longrightarrow \mathbb{Z}\,\mathrm{Sm}_S$$
$$[n] \longmapsto X^n,\ \delta^j_{n+1} \longmapsto d^j_{n+1}\colon X^n \longrightarrow X^{n+1},\ \sigma^j_n \longmapsto s^j_n\colon X^{n+1} \longrightarrow X^n$$

where the maps d^j_{n+1} and s^j_n are given by

$$d^j_{n+1} := -\begin{cases} x \times \mathrm{id}^{\times n} & \text{for } j = 0 \\ \mathrm{id}^{\times j-1} \times \Delta \times \mathrm{id}^{\times n-j} & \text{for } j \in \{1, \dots, n\} \\ \mathrm{id}^{\times n} \times y\colon X^n \to X^{n+1} & \text{for } j = n+1 \end{cases}$$
$$s^j_n := -\,\mathrm{id}^{\times j} \times \pi \otimes \mathrm{id}^{\times n-j} \ \text{ for } j = 0, \dots, n.$$

Lemma 3.1.1. *The functor $cB^\bullet_{\mathrm{mot}}(X|S)_{x,y}$ is a cosimplicial object in $\mathbb{Z}\,\mathrm{Sm}_S$.*

Proof. We need to show that the cosimplicial identities are satisfied. This is a lengthy but simple computation carried out in E.3. ■

Corollary 3.1.2. *The unnormalized complex associated to $cB^\bullet_{\mathrm{mot}}(X|S)_{x,y}$ is*

$$0 \longrightarrow S \xrightarrow{\delta_0(x,y)} X \xrightarrow{\delta_1(x,y)} X^2 \xrightarrow{\delta_2(x,y)} \dots \xrightarrow{\delta_{n-1}(x,y)} X^n \longrightarrow \dots$$

with the following differentials (where $\mathrm{id}^i\colon X^i \longrightarrow X^i$ *is the identity on* X^i*):*

$$\delta_k(x,y) := -x \times \mathrm{id}^k + \sum_{i=1}^{k} (-1)^{i-1}\,\mathrm{id}^{i-1} \times \Delta \times \mathrm{id}^{k-1} + (-1)^k\,\mathrm{id}^k \times y.$$

Proof. Direct consequence of the general theory in section 1.3.1. ■

We will not go into further detail here, but rather consider the dual situation in the category of motives in greater detail.

3.2 Simplicial viewpoint in $\mathcal{DM}(S)$

Now we may translate the above into the setting of motives using the passage from schemes to motives

$$\mathbb{Z}(0)[0]\colon \mathbb{Z}\,\mathrm{Sm}_S^{\mathrm{op}} \longrightarrow \mathcal{L}(\mathrm{Sm}_S)\times\mathbb{Z}, \quad \oplus_i X_i \longmapsto \oplus_i \mathbb{Z}_{X_i}(0)_{\mathrm{id}}.$$

Again, we let $\pi\colon X \longrightarrow S$ in Sm_S be equipped with two sections $x, y\colon S \longrightarrow X$, and denote the n-fold fiber-product of X with itself over S by $X^n = X \times_S \ldots \times_S X$. The object corresponding to X^n in $\mathcal{A}_{\mathrm{mot}}(S)$ is given by $\mathbb{Z}_{X^n}$. Moreover, we denote the diagonal in X^2 by $\Delta\colon X \longrightarrow X \times_S X$. In what follows, we will always write id for the identity morphism on $\mathbb{Z}_X$.
The cosimplicial object $cB^{\bullet}_{\mathrm{mot}}(X|S)_{x,y}$ of the preceding section now obviously gives rise to a simplicial object in $\mathcal{DM}(S)$ in the following way:

Definition 3.2.1. *We define the simplicial object $sB^{\mathrm{mot}}_{\bullet}(X|S)_{x,y}$ to be given by*

$$sB^{\mathrm{mot}}_{\bullet}(X|S)_{x,y} := \mathbb{Z}(0)[0] \circ cB^{\bullet}_{\mathrm{mot}}(X|S)_{x,y}\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$$

and call it the <u>motivic simplicial bar object</u> in the DG category $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$ underlying $\mathcal{DM}(S)$ for $X \in \mathrm{Sm}(S)$ with respect to the sections x, y.

Explicitly, the motivic simplicial bar object is given by the functor

$$\begin{aligned}&sB^{\mathrm{mot}}_{\bullet}(X|S)_{x,y}\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)\\ &[n] \longmapsto \mathbb{Z}_{X^n}(0),\ \delta^j_{n+1} \longmapsto d^{n+1}_j\colon \mathbb{Z}_{X^{n+1}}(0) \to \mathbb{Z}_{X^n}(0),\ \sigma^j_n \longmapsto s^n_j\colon \mathbb{Z}_{X^n}(0) \to \mathbb{Z}_{X^{n+1}}(0),\end{aligned}$$

where the face maps d^{n+1}_j and the degeneracy maps s^n_j are given by

$$d^{n+1}_j := -\begin{cases} (x \times \mathrm{id}^n)^* & \text{for } \ j = 0 \\ (\mathrm{id}^{j-1} \times \Delta \times \mathrm{id}^{n-j})^* & \text{for } \ j \in \{1, \ldots, n\} \\ (\mathrm{id}^n \times y)^* & \text{for } \ j = n+1 \end{cases}$$
$$s^n_j := -(\mathrm{id}^j \times \pi \times \mathrm{id}^{n-j})^* \text{ for } j = 0, \ldots, n.$$

3.3 The (unnormalized) motivic bar complex

The general theory of Section 2.2 above immediately yields results on the associated unnormalized and normalized motives. The differentials of these complexes depend on the sections x and y, and will show up throughout the thesis for varying x and y. In order not to repeat ourselves over and over again, let us once and for all fix a convenient notation for these differentials:

Definition 3.3.1. *For a smooth scheme X over S with structure morphism $\pi\colon X \longrightarrow S$ and two fixed sections $x, y\colon S \longrightarrow X$ of π we define, for any $k \in \mathbb{N}$, the morphism*

$$\delta_k^*(x,y)\colon \mathbb{Z}_{X^{k+1}} \longrightarrow \mathbb{Z}_{X^k} \in \mathcal{DM}(S)$$

by the following formula:

$$\delta_k^*(x,y) = -(x\times \mathrm{id}^k)^* + \sum_{i=1}^{k}(-1)^{i-1}(\mathrm{id}^{i-1}\times\Delta\times\mathrm{id}^{k-1})^* + (-1)^k(\mathrm{id}^k\times y)^*.$$

As shown in section 2.2, for any $n\in\mathbb{N}$, we can associate to the n-th truncated cosimplicial object $cB^\bullet_{\mathrm{mot}}(X|S)_{x,y}$ the *unnormalized motive* $M(cB^{\leq n}_{\mathrm{mot}}(X|S)_{x,y})$. It is given by the following complex in $\mathcal{L}(\mathrm{Sm}_S)\times\mathbb{Z}$:

$$\mathbb{Z}_{X^n}(0)_{f_{cB}^{\leq n*}} \xrightarrow{\delta_{n-1}^*(x,y)} \dots \xrightarrow{\delta_2^*(x,y)} \mathbb{Z}_{X^2}(0)_{f_{cB}^{\leq 2*}} \xrightarrow{\delta_1^*(x,y)} \mathbb{Z}_X(0)_{f_{cB}^{\leq 1*}} \xrightarrow{x^*-y^*} \mathbb{Z}_S(0)_{\mathrm{id}}$$

Here, the morphism $f_{cB}^{\leq k*}$ is induced by the morphism $f_{cB}^{\leq k}\colon \coprod_{g\colon [i]\hookrightarrow[k]} X^i \longrightarrow X^k$ given by $cB^*_{\mathrm{mot}}(X|S)_{x_0}(g)$ on the component indexed by $g\colon [i]\hookrightarrow [k]$.

Definition 3.3.2. *We call the motive $B_n^{\mathrm{mot}}(X|S)_{x,y} := M(cB^{\leq n}_{\mathrm{mot}}(X|S)_{x,y})$ above the* <u>*n-th motivic bar complex*</u> *of X over S with respect to the sections $x,y\colon S\longrightarrow X$.*

3.4 The normalized motivic bar complex

To find an explicit description of the normalized motives associated to $cB^\bullet_{\mathrm{mot}}(X|S)_{x,y}$, we need to determine the corresponding kernel/cokernel

$$\begin{aligned} N_n(sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}) &:= \ker\left((d_1,\dots,d_n)\colon sB_n^{\mathrm{mot}}(X|S)_{x,y} \to \bigoplus_{\leq i\leq n} sB_{n-1}^{\mathrm{mot}}(X|S)_{x,y}\right) \\ &= \operatorname{coker}\left(\sum_j s_j\colon \bigoplus_{0\leq i<n} sB_{n-1}^{\mathrm{mot}}(X|S)_{x,y} \to sB_n^{\mathrm{mot}}(X|S)_{x,y}\right) \end{aligned}$$

in the pseudo-abelian envelope $\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S))$ by expressing it in terms of the image of an idempotent. To this end, consider the idempotent $\mathrm{id}_X - x_0\pi\colon X\longrightarrow X$, and the induced idempotent

$$e^n_{X|S} := (\mathrm{id}_X - x_0\pi)^n\colon X^n \longrightarrow X^n.$$

We denote the corresponding element in $\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S))$ for all $n\in\mathbb{N}$ by

$$\mathbb{Z}^\circ_{X^n} := (\mathbb{Z}_{X^n}, (e^n_{X|S})^*).$$

Proposition 3.4.1. *In the pseudo-abelian envelope $\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S))$ of $\mathcal{A}_{\mathrm{mot}}(S)$, we have*

$$\operatorname{coker}\left(\sum_{0\leq j\leq n-1} s_j\colon \bigoplus_j sB_{n-1}^{\mathrm{mot}}(X|S)_{x,y} \to sB_n^{\mathrm{mot}}(X|S)_{x,y}\right) = \mathbb{Z}^\circ_{X^n}.$$

Proof. The morphism $e^n_{X|S}$ is an idempotent, so we have a direct sum decomposition

$$\mathbb{Z}_{X^n} \simeq \operatorname{Im}(e^{n*}_{X|S}) \oplus \operatorname{Im}\left((\mathrm{id}^n)^* - e^{n*}_{X|S}\right) \cong \mathbb{Z}^\circ_{X^n} \oplus \operatorname{Im}\left((\mathrm{id}^n)^* - e^{n*}_{X|S}\right)$$

in $\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S))$. Hence we have a natural identification of

$$\operatorname{Im}\left(e^{n*}_{X|S}\right) = \mathbb{Z}^\circ_{X^n} \cong \operatorname{coker}\left((\mathrm{id}^n)^* - e^{n*}_{X|S}\right).$$

By the categorical definition of the cokernel, we have to show the following: given a morphism $q\colon sB^{\mathrm{mot}}_n \to Z$ in $\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S))$ such that the composition

$$q \circ \left(\sum_{j=0}^{n-1} (\mathrm{id}^j \times \pi \times \mathrm{id}^{n-1-j})^*\right) : \bigoplus_{j=0}^{n-1} sB^{\mathrm{mot}}_{n-1} \longrightarrow sB^{\mathrm{mot}}_n$$

is zero (i.e $qs_j = 0$ for all j), there is a unique morphism $k\colon \operatorname{Im}\left(e^{n*}_{X|S}\right) \longrightarrow Z$ making the diagram

$$\begin{tikzcd}
\bigoplus sB^{\mathrm{mot}}_{n-1} \arrow[rr, bend left, "\sum_{j=0}^{n-1}(\mathrm{id}^j \times\pi\times\mathrm{id}^{n-1-j})^*"] \arrow[dr, "0"] \arrow[ddr, bend right, "0"'] & & sB^{\mathrm{mot}}_n \arrow[dl, "e^{n*}_{X|S}"'] \arrow[ddl, bend left, "q"] \\
& \operatorname{Im}\left(e^{n*}_{X|S}\right) \arrow[d, dashed, "\exists! k"] & \\
& Z &
\end{tikzcd}$$

commute.

The morphism $(\mathrm{id}^n)^* - e^{n*}_{X|S} = (\mathrm{id}^n - (\mathrm{id} - x_0\pi)^n)^*$ is an alternating sum of terms $(\alpha_1 \times \ldots \times \alpha_n)^*$ with $\alpha_i \in \{\mathrm{id}, x_0\pi\}$ and $\alpha_j = x_0\pi$ for at least one $j \in \{1, \ldots, n\}$. Since $q \circ (\mathrm{id}^j \times\pi \times \mathrm{id}^{n-j-1})^* = 0$ and hence also $q \circ (\mathrm{id}^j \times x_0\pi \times \mathrm{id}^{n-j-1})^* = 0$ for all $j = 0, \ldots, n-1$, it follows that

$$q \circ (\mathrm{id}^{n*} - e^{n*}_{X|S}) = q \circ (\mathrm{id}^n - (\mathrm{id} - x_0\pi)^n)^* = 0.$$

By definition of the cokernel of $\mathrm{id}^{n*} - e^{n*}_{X|S}$, the morphism q hence factors over $\operatorname{coker}\left(\mathrm{id}^{n*} - e^{n*}_{X|S}\right) = \operatorname{Im}\left(e^{n*}_{X|S}\right)$ in a unique way. The resulting unique morphism

$$k\colon \operatorname{coker}\left(\mathrm{id}^{n*} - e^{n*}_{X|S}\right) = \operatorname{Im}\left(e^{n*}_{X|S}\right) \longrightarrow Z$$

makes the above diagram commute, which proves the assertion. ■

Next, we need to determine the differentials

$$\widetilde{\delta}_k^*\colon \operatorname{coker}\left(\sum_{0\le j\le k+1} s_j^{k+1}\right) \longrightarrow \operatorname{coker}\left(\sum_{0\le j\le k} s_j^{k}\right)$$

of the normalized complex of $sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}$. These are induced by the differentials

$$\delta_k^*(x,y) := -(x\times \mathrm{id}^k)^* + \sum_{i=1}^{k}(-1)^{i-1}(\mathrm{id}^{i-1}\times\Delta\times\mathrm{id}^{k-1})^* + (-1)^k(\mathrm{id}^k\times y)^*$$

of the unnormalized complex

$$\ldots \longrightarrow \mathbb{Z}_{X^n} \xrightarrow{\delta_{n-1}^*(x,y)} \mathbb{Z}_{X^{n-1}} \xrightarrow{\delta_{n-2}^*(x,y)} \ldots \xrightarrow{\delta_1^*(x,y)} \mathbb{Z}_X \xrightarrow{x^*-y^*} \mathbb{Z}_S \longrightarrow 0.$$

Now note that the projection $\mathbb{Z}_{X^k} \longrightarrow \operatorname{coker}\left(\sum_{0\le j\le k} s_j^k\right) \cong \mathbb{Z}^\circ_{X^k}$ and the inclusion of the direct summand $\mathbb{Z}^\circ_{X^k} \longrightarrow \mathbb{Z}_{X^k}$ are given by the cartesian diagrams

$$\begin{tikzcd}
\mathbb{Z}_{X^k} \arrow[r, "e_{X|S}(x_0)^{k*}"] \arrow[d, "e_{X|S}(x_0)^{k*}"'] & \mathbb{Z}_{X^k} \arrow[d, equal] \\
\mathbb{Z}_{X^k} \arrow[r, "e_{X|S}(x_0)^{k*}"'] & \mathbb{Z}_{X^k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z}_{X^k} \arrow[d, "e_{X|S}(x_0)^{k*}"'] & \mathbb{Z}_{X^k} \arrow[l, "e_{X|S}(x_0)^{k*}"'] \arrow[d, equal] \\
\mathbb{Z}_{X^k} & \mathbb{Z}_{X^k} \arrow[l, "e_{X|S}(x_0)^{k*}"]
\end{tikzcd}$$

$$\mathbb{Z}^\circ_{X^k} \xrightarrow{e_{X|S}(x_0)^{k*}} \mathbb{Z}_{X^k} \qquad\qquad \mathbb{Z}^\circ_{X^k} \xleftarrow{e_{X|S}(x_0)^{k*}} \mathbb{Z}_{X^k}$$

where the left hand side of each diagram is the arrow representing $\mathbb{Z}^\circ_{X^k}$, i.e. the morphisms are given by

$$e_{X|S}(x_0)^{k*}\colon \mathbb{Z}_{X^k} \twoheadrightarrow \mathbb{Z}^\circ_{X^k}, \quad e_{X|S}(x_0)^{k*}\colon \mathbb{Z}^\circ_{X^k} \hookrightarrow \mathbb{Z}_{X^k}.$$

Hence the induced differential $\widetilde{\delta}_k^*$ on cokernels corresponds to the composition

$$\mathbb{Z}^\circ_{X^{k+1}} \xhookrightarrow{e_{X|S}(x_0)^{k+1}} \mathbb{Z}_{X^{k+1}} \xrightarrow{\delta_k^*(x,y)} \mathbb{Z}_{X^k} \xrightarrow{e_{X|S}(x_0)^{k*}} \mathbb{Z}^\circ_{X^k}\ , \quad \text{i.e.}$$

$$\begin{aligned}
\widetilde{\delta}_k^* &:= e_{X|S}(x_0)^{k*}\circ\delta_k^*\circ e_{X|S}(x_0)^{k+1}\\
&= -\Big((x-x_0)\times(\mathrm{id}-x_0\pi)^k\Big)^* + \sum_{i=1}^{k}(-1)^{i-1}\Big((\mathrm{id}-x_0\pi)^{i-1}\times(\Delta-\Delta x_0\pi)\times(\mathrm{id}-x_0\pi)^{k-i}\Big)^*\\
&\quad +(-1)^k\Big((\mathrm{id}-x_0\pi)^k\times(y-x_0)\Big)^*.
\end{aligned}$$

Like the original differential $\delta_k^*(x,y)$, $\widetilde{\delta}_k^*(x,y)$ strongly depends on the sections x and y, which is why we will include it in the notation, and define:

Definition 3.4.2. *For $X \in \mathrm{Sm}_S$ with structure morphism $\pi\colon X \longrightarrow S$ and three fixed sections $x, y, x_0\colon S \longrightarrow X$ of π we define, for any $k \in \mathbb{N}$, the morphisms*

$$\widetilde{\delta}_k^*(x - x_0, y - x_0)\colon \mathbb{Z}_{X^{k+1}} \longrightarrow \mathbb{Z}_{X^k}$$

in $C^-(\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))) = \mathcal{K}(C^-(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)))$ by the following formula:

$$\begin{aligned}\widetilde{\delta}_k^*(x - x_0, y - x_0) &:= e_{X|S}(x_0)^{k*} \circ \delta_k^* \circ e_{X|S}(x_0)^{k+1} \\ &= -\Big((x - x_0) \times (\mathrm{id} - x_0\pi)^k\Big)^* \\ &\quad + \sum_{i=1}^{k}(-1)^{i-1}\Big((\mathrm{id} - x_0\pi)^{i-1} \times (\Delta - \Delta x_0\pi) \times (\mathrm{id} - x_0\pi)^{k-i}\Big)^* \\ &\quad + (-1)^k\Big((\mathrm{id} - x_0\pi)^k \times (y - x_0)\Big)^*\end{aligned}$$

By the above considerations, we thus obtain the following for the normalized bar complex:

Corollary 3.4.3. *The normalized complex $N(sB_\bullet^{\mathrm{mot}}(X|S)_{x,y})$ associated to the simplicial object $sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}$ in $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$ with respect to the section x_0 is given by*

$$\ldots \longrightarrow (\mathbb{Z}_{X^n})^\circ_{f^{\leq n*}_{cB(X|S)}} \xrightarrow{\widetilde{\delta}^*_{n-1}} (\mathbb{Z}_{X^{n-1}})^\circ_{f^{\leq n-1*}_{cB}} \xrightarrow{\widetilde{\delta}^*_{n-2}} \ldots \xrightarrow{\widetilde{\delta}^*_1} (\mathbb{Z}_X)^\circ_{f^{\leq 1*}_{cB}} \xrightarrow{x^*-y^*} \mathbb{Z}_S \longrightarrow 0$$

in $C^-(\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))) = \mathcal{K}(C^-(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)))$, where $\widetilde{\delta}_k^* := \widetilde{\delta}_k^*(x - x_0, y - x_0)$, and the morphisms $f^{\leq k}_{cB} := f^{\leq k}_{cB^{\leq n}_{\mathrm{mot}}(X|S)_{x,y}}$ are given as in section 2.2.1 above.

Recall that by section 2.2 the normalized motive associated to the cosimplicial object$cB^*_{\mathrm{mot}}(X|S)_{x,y}$ is given as follows: choosing

$$e_n^i := \mathrm{id}^{i-1} \times x_0 \times \mathrm{id}^{n-i+1}\colon X^n \longrightarrow X^{n+1},$$

we obviously have $s_n^i e_n^i = \mathrm{id}$, and $e_n^i s_n^i = \mathrm{id}^{i-1} \times (x_0\pi) \times \mathrm{id}^{n-i+1}$. By construction, the normalized motive of X^* is then given by the diagram

$$\begin{tikzcd}
\mathbb{Z}_{X^n}(0)_{f^{\leq n}_{cB}} \arrow[r, "\widetilde{\delta}^*_{n-1}(x,y)"] \arrow[d, "((\mathrm{id} - x_0\pi)^n)^*"] & \ldots \arrow[r, "\widetilde{\delta}^*_2(x,y"] & \mathbb{Z}_{X^2}(0)_{f^{\leq 2}_{cB}} \arrow[r, "\widetilde{\delta}^*_1(x,y)"] \arrow[d, "((\mathrm{id} - x_0\pi)^2)^*"] & \mathbb{Z}_X(0)_{f^{\leq 1}_{cB}} \arrow[r, "x^*-y^*"] \arrow[d, "(\mathrm{id} - x_0\pi)^*"] & \mathbb{Z}_S(0)_{\mathrm{id}} \arrow[d, equal] \\
\mathbb{Z}_{X^n}(0)_{f^{\leq n}_{cB}} \arrow[r, "\widetilde{\delta}^*_{n-1}(x,y)"] & \ldots \arrow[r, "\widetilde{\delta}^*_2(x,y)"] & \mathbb{Z}_{X^2}(0)_{f^{\leq 2}_{cB}} \arrow[r, "\widetilde{\delta}^*_1(x,y)"] & \mathbb{Z}_X(0)_{f^{\leq 1}_{cB}} \arrow[r, "x^*-y^*"] & \mathbb{Z}_S(0)_{\mathrm{id}}
\end{tikzcd}$$

i.e. by the truncation of the above normalized complex of the simplicial object $sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}$. Thus, we obtain:

Definition 3.4.4. *We call the object* $\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y} := nM(cB_{\mathrm{mot}}^{\leq n}(X|S)_{x,y}))$ *in* $\mathcal{DM}(S)$ *the n-th normalized motivic bar complex of* X *over* S *with respect to the sections* $x, y\colon S \longrightarrow X$.

Remark 3.4.5. Note that we have:

$$\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y} = (B_n^{\mathrm{mot}}(X|S)_{x,y}), e_{X|S}(x_0)^*) \in \mathcal{DM}(S),$$

where $e_{X|S}(x_0)^*$ is given in degree k by $e_{X|S}(x_0)^{k*}\colon \mathbb{Z}_{X^k} \longrightarrow \mathbb{Z}_{X^k}$.

3.5 Passing to the limit

We now want to consider the "left unbounded" complexes that arise when we "pass to the limit" $n \to \infty$. Unfortunately, $\mathcal{DM}(S)$ is not a cocomplete category, so in order to do this, we need to pass to a larger category: we will consider direct systems of objects in $\mathcal{DM}(S)$. Note that left-unbounded complexes like the untruncated unnormlized and normalized bar complexes are special cases of direct systems of motives.

Definition 3.5.1. *(See section D.2 in the Appendix)*
For any two inductive systems $(A_i)_{i\in I}, (B_k)_{k\in K}$ *in* $\mathcal{DM}(S)$ *over any index sets* I, K, *we define*

$$\mathrm{Hom}_{\varinjlim \mathcal{DM}(S)}(A, B) \cong \varprojlim_i \varinjlim_k \mathrm{Hom}_{\mathcal{DM}(S)}(A_i, B_k).$$

We denote the category of all inductive systems of objects in $\mathcal{DM}(S)$ *by* $\varinjlim \mathcal{DM}(S)$.

Remark 3.5.2. a.) Any morphism in $\mathrm{Hom}_{\varinjlim \mathcal{DM}(S)}(A, B)$ is represented by a family of maps $f_i\colon A_i \to B_{k(i)}$ for a function $k\colon I \to K$, such that for any $i, j \in I$ with $i \leq j$ there is a $k \in K$ with $k \geq k(i), k(j)$ for which the diagram

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "\alpha_{ij}"'] & B_{k(i)} \arrow[dr, bend left, "\beta_{k(i)k}"] & \\ A_j \arrow[r, "f_j"'] & B_{k(j)} \arrow[r, "\beta_{k(j)k}"'] & B_k \end{tikzcd} \quad \text{commutes.}$$

b.) Since $(\mathcal{DM}(S), \otimes, 1 = \mathbb{Z}_S(0))$ is a symmetric monoidal category, $\varinjlim \mathcal{DM}(S)$ inherits a symmetric monoidal category structure, by defining

$$(A_i)_{i\in I} \otimes (B_k)_{k\in K} := (A_i \otimes B_k)_{i,k\in I\times K}.$$

By [Mey07, 1.136, p.57], this tensor product turns $\varinjlim \mathcal{DM}(S)$ into a symmetric monoidal category whose unit is the constant inductive system $\mathbb{Z}_S(0)$.

c.) On left-unbounded complexes of objects of type $\mathbb{Z}_Y$ or $\mathbb{Z}_Y^\circ$ for $Y \in \mathrm{Sm}(S)$ in $\varinjlim \mathcal{DM}(S)$ we define a pull-back as follows: Recall that for any morphism $p\colon T \longrightarrow S$ of schemes, there is an induced base-change functor $\mathcal{DM}(p^*)\colon \mathcal{DM}(S) \longrightarrow \mathcal{DM}(T)$ which is induced by sending $\mathbb{Z}_Y(a)$ to $\mathbb{Z}_{Y\times_S T}(a)$ for any $a \in \mathbb{Z}, Y \in \mathrm{Sm}(S)$. Obviously, this pull-back extends to left-unbounded complexes by applying the pull-back-functor componentwise.

Now we consider the natural inclusions

$$B_n^{\mathrm{mot}}(X|S)_{x,y} \hookrightarrow B_{n+1}^{\mathrm{mot}}(X|S)_{x,y} \quad (\text{resp. } \widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y} \hookrightarrow \widetilde{B}_{n+1}^{\mathrm{mot}}(X|S)_{x,y})$$

of complexes in $C^b(\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)))$ which give rise to a direct system $(B_n^{\mathrm{mot}}(X|S)_{x,y})_n$ (respectively $(\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y})_n$) of objects of $\mathcal{DM}(S)$.

Definition 3.5.3. *Let $\pi\colon X \longrightarrow S$ be in Sm_S with two fixed sections x, y. The ind-motives*

$$\begin{aligned} B^{\mathrm{mot}}(X|S)_{x,y} &:= \big(B_n^{\mathrm{mot}}(X|S)_{x,y}\big)_n \in \varinjlim \mathcal{DM}(S) \quad \textit{and} \\ \widetilde{B}^{\mathrm{mot}}(X|S)_{x,y} &:= \Big(\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y}\Big)_n \in \varinjlim \mathcal{DM}(S) \end{aligned}$$

are called the <u>motivic bar complex</u> of X over S with respect to x, y, respectively the <u>normalized motivic bar complex</u>.

Remark 3.5.4. Note that the ind-motive $B^{\mathrm{mot}}(X|S)_{x,y}$ can be considered as the following left-unbounded complex in $C^-(\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)))$:

$$\dots \xrightarrow{\delta_{n+1}^*(x,y)} \mathbb{Z}_{X^n} \xrightarrow{\delta_{n-1}^*(x,y)} \dots \xrightarrow{\delta_2^*(x,y)} \mathbb{Z}_{X^2} \xrightarrow{\delta_1^*(x,y)} \mathbb{Z}_X \xrightarrow{x^*-y^*} \mathbb{Z}_S.$$

and likewise for $\widetilde{B}^{\mathrm{mot}}(X|S)_{x,y}$.

3.6 Properties of the motivic bar complexes

a.) *The bar filtration:* $\mathfrak{B} := (B_r^{\mathrm{mot}}(X|S)_{x,y})_r$ (resp. $\widetilde{\mathfrak{B}} := (\widetilde{B}_r^{\mathrm{mot}}(X|S)_{x,y})_r$) is a filtration of $B^{\mathrm{mot}}(A)_{x,y}$ (resp. $\widetilde{B}^{\mathrm{mot}}(X|S)_{x,y}$) by subcomplexes. Its r-th graded quotient is given by $\mathbb{Z}_{X^r}[r]$ (resp. $\mathbb{Z}_{X^r}^\circ[r]$) concentrated in degree $-r$.

b.) *Functoriality:* Suppose we are given a morphism $\varphi\colon X' \longrightarrow X$ of smooth S-schemes, with compatible sections

$$\begin{tikzcd} X' \arrow[rr, "\varphi"] & & X \\ & S \arrow[ul, "x_0'"] \arrow[ur, "x_0"'] & \end{tikzcd} \quad \begin{tikzcd} X' \arrow[rr, "\varphi"] & & X \\ & S \arrow[ul, "x'"] \arrow[ur, "x"'] & \end{tikzcd} \quad \begin{tikzcd} X' \arrow[rr, "\varphi"] & & X \\ & S \arrow[ul, "y'"] \arrow[ur, "y"'] & \end{tikzcd}$$

Then it is easy to see that $\varphi^*\colon \mathbb{Z}_X \longrightarrow \mathbb{Z}_{X'}$ induces morphisms

$$(\varphi^k)^*\colon \mathbb{Z}_{X^k}(0)_{f^{\leq k*}_{cB(X|S)_{x,y}}} \longrightarrow \mathbb{Z}_{X'^k}(0)_{f^{\leq k*}_{cB(X'|S)_{x',y'}}},$$

for all k, where

$$f^{\leq k}_{cB(X|S)_{x,y}}\colon \coprod_{g\colon [i]\hookrightarrow[k]} X^i \longrightarrow X^k$$
$$f^{\leq k}_{cB(X'|S)_{x',y'}}\colon \coprod_{g\colon [i]\hookrightarrow[k]} (X')^i \longrightarrow (X')^k$$

are the morphisms given by $cB(X|S)_{x,y}(g)$ (respectively $cB(X'|S)_{x',y'}(g)$) on the component indexed by $g\colon [i] \hookrightarrow [k]$. Since the morphism φ^* also commutes with the differentials of the motivic bar complexes in the obvious way, φ yields a morphism of complexes

$$\begin{tikzcd}
\mathbb{Z}_{X^n} \arrow[r, "\delta^*_{n-1}"] \arrow[d, "(\varphi^n)^*"'] & \mathbb{Z}_{X^{n-1}} \arrow[r, "\delta^*_{n-2}"] \arrow[d, "(\varphi^{n-1})^*"'] & \dots \arrow[r, "\delta^*_1"] & \mathbb{Z}_X \arrow[r, "\delta^*_0"] \arrow[d, "\varphi^*"'] & \mathbb{Z}_S \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z}_{(X')^n} \arrow[r, "\delta^*_{n-1}"] & \mathbb{Z}_{(X')^{n-1}} \arrow[r, "\delta^*_{n-2}"] & \dots \arrow[r, "\delta^*_1"] & \mathbb{Z}_{X'} \arrow[r, "\delta^*_0"] & \mathbb{Z}_S \arrow[r] & 0
\end{tikzcd}$$

in $C^b_{\mathrm{mot}}(\mathrm{Sm}_S)$ for all n, an thus a morphism of motives

$$\varphi^*\colon B^{\mathrm{mot}}_n(X|S)_{x,y} \longrightarrow B^{\mathrm{mot}}_n(X'|S)_{x',y'}$$

for all n. Moreover one immediately checks that the above diagram commutes with the idempotent $e_{X|S}(x_0)^*$ of 3.4.5, giving rise to a corresponding pull-back morphism

$$\varphi^*\colon \widetilde{B}^{\mathrm{mot}}_n(X|S)_{x,y} \longrightarrow \widetilde{B}^{\mathrm{mot}}_n(X'|S)_{x',y'}$$

of the normalized motives.

c.) *Naturality:* Recall that given a morphism $f\colon T \longrightarrow S$ of schemes, there is a pull-back functor $\mathcal{DM}(f^*)\colon \mathcal{DM}(S) \longrightarrow \mathcal{DM}(T)$. Then by section 2.3.1 above, we have

$$\mathcal{DM}(f^*)B^{\mathrm{mot}}(X|S)_{x,y} = B^{\mathrm{mot}}(X\times_S T|T)_{x\times_S \mathrm{id}_T, y\times_S \mathrm{id}_T} \quad \text{and}$$
$$\mathcal{DM}(f^*)\widetilde{B}^{\mathrm{mot}}(X|S)_{x,y} = \widetilde{B}^{\mathrm{mot}}(X\times_S T|T)_{x\times_S \mathrm{id}_T, y\times_S \mathrm{id}_T}.$$

d.) *Hopf algebra structure:*

- *Product:* For any section $x\colon S \longrightarrow X$ of π, the motivic bar complex $B^{\mathrm{mot}}(X|S)_{x,x}$ admits a product

$$m\colon B^{\mathrm{mot}}(X|S)_{x,x} \otimes B^{\mathrm{mot}}(X|S)_{x,x} \longrightarrow B^{\mathrm{mot}}(X|S)_{x,x}$$

given on the component $\mathbb{Z}_{X^r} \otimes \mathbb{Z}_{X^s}$ by

$$m|_{\mathbb{Z}_{X^r} \otimes \mathbb{Z}_{X^s}} := \sum_{\sigma \in S_{r,s}} (-1)^{\mathrm{sgn}(\sigma)\sigma^*} \colon \mathbb{Z}_{X^r} \otimes \mathbb{Z}_{X^s} \longrightarrow \mathbb{Z}_{X^r} \otimes \mathbb{Z}_{X^s}$$

where σ^* is the pull-back by the morphism

$$\sigma \colon X^{r+s} \longrightarrow X^{r+s}, (x_1, \ldots, x_{r+s}) \longmapsto (x_{\sigma(1)}, \ldots, x_{\sigma(r+s)}),$$

up to Künneth isomorphism. This product is associative, graded-commutative, unital with unit given by inclusion $i \colon \mathbb{Z}_S \hookrightarrow B^{\mathrm{mot}}(X|S)_{x,x}$, and compatible with the total differential.

- *Coproduct*:
Let $x \colon S \longrightarrow X$ be a section of π. There is a coproduct defined by

$$\Delta \colon B^{\mathrm{mot}}(X|S)_{x,x} \longrightarrow B^{\mathrm{mot}}(X|S)_{x,x} \otimes B^{\mathrm{mot}}(X|S)_{x,x}$$

given on $\mathbb{Z}_{X^r}$ by

$$\Delta|_{\mathbb{Z}_{X^r}} := \sum_{i=0}^{r} k_i \colon \mathbb{Z}_{X^r} \longrightarrow \bigoplus_{i=0}^{r} \mathbb{Z}_{X^i} \otimes \mathbb{Z}_{X^{r-i}}$$

where $k_i \colon \mathbb{Z}_{X^r} \longrightarrow \mathbb{Z}_{X^i} \otimes \mathbb{Z}_{X^{r-i}}$ is the obvious isomorphism for $i = 1, \ldots, r-1$, and

$$k_0 \colon \mathbb{Z}_{X^r} \longrightarrow \mathbb{Z}_S \otimes \mathbb{Z}_{X^r} \quad \text{and } k_r \colon \mathbb{Z}_{X^r} \longrightarrow \mathbb{Z}_{X^r} \otimes \mathbb{Z}_S$$

are the natural isomorphisms. This coproduct is co-associative, i.e. $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, compatible with the co-augmentation $\epsilon \colon B^{\mathrm{mot}}(X|S)_{x,x} \longrightarrow \mathbb{Z}_S$ given by the projection of $B^{\mathrm{mot}}(X|S)_{x,x}$ to $\mathbb{Z}_S$ (note that the last differential of $B^{\mathrm{mot}}(X|S)_{x,x}$ is zero, and hence $\mathbb{Z}_S$ splits off), and is a morphism of complexes. Moreover, it has a counit given by ϵ.
- *Compatibility:* The algebra and coalgebra structures are compatible with each other, i.e. the counit and coproduct are morphisms of unitary algebras.
- *Antipode:* There is an antipode

$$S \colon B^{\mathrm{mot}}(X|S)_{x,x} \longrightarrow B^{\mathrm{mot}}(X|S)_{x,x}$$

given on $\mathbb{Z}_{X^r}$ (up to Künneth isomorphism) by the pull-back

$$s^* \colon \mathbb{Z}_{X^r} \longrightarrow \mathbb{Z}_{X^r},$$

where $s \colon X^r \longrightarrow X^r, (x_1, \ldots, x_r) \longrightarrow (x_r, \ldots, x_1)$ is the permutation of factors.

For any augmentation $x \colon A^\bullet \longrightarrow k$, this gives the bar complex $B^{\mathrm{mot}}(X|S)_{x,x}$ the structure of a Hopf algebra, i.e the diagram

$$\begin{tikzcd}
 & B(X|S)_{x,x}\otimes B(X|S)_{x,x} \arrow[r, "S\otimes\mathrm{id}"] & B(X|S)_{x,x}\otimes B(X|S)_{x,x} \arrow[dr, "\nabla"] & \\
B(X|S)_{x,x} \arrow[ur, "\Delta"] \arrow[r, "\epsilon"] \arrow[dr, "\Delta"'] & k \arrow[rr, "i"] & & B(X|S)_{x,x} \\
 & B(X|S)_{x,x}\otimes B(X|S)_{x,x} \arrow[r, "\mathrm{id}\otimes S"] & B(X|S)_{x,x}\otimes B(X|S)_{x,x} \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

commutes, where we dropped "mot" in the notation.

3.7 The normalized bar complex in terms of relative motives

One can think of this section as a generalization of section 3 of Deligne and Goncharov's famous paper [DG05] relating the fundamental group of an algebraic variety to a certain relative cohomology group (for further details, see [DG05]).

In our setting, we will relate the normalized motivic bar complex to a *relative motive*. Thus, before we start this section, let us recall the definition of relative motives as explained in more detail in section B.5 in the appendix (respectively in section 2.1.4): Let X be a smooth S-scheme with smooth subschemes $D_1, \dots, D_n \subset X$. For each index $I = (1 \leq i_1 < \dots i_s \leq n)$, denote the intersection of all subschemes D_i with $i \in I$ by $D_I := D_{i_1} \cap \dots, D_{i_s}$. The relative motive $\mathbb{Z}_{(X;D_1,\dots,D_n)}(0) \in \mathcal{DM}(S)$ given by the complex

$$(\mathbb{Z}_X(0))_{g_0} \to \bigoplus_{i=1}^n (\mathbb{Z}_{D_i})_{g_1} \to \dots \to \bigoplus_{|I|=s} (\mathbb{Z}_{D_I})_{g_s} \to \bigoplus_{|I|=s+1} (\mathbb{Z}_{D_I})_{g_{s+1}} \to \dots \to (\mathbb{Z}_{D_1\cap\dots\cap D_n})_{g_n}$$

in degrees 0 up to n, where

$$g_s \colon \bigoplus_{|J|\geq s} D_J \to \bigoplus_{|I|=s} D_I$$

is the morphism induced by the inclusions. The differential is given in degree s as the alternating sum

$$\partial^s := \sum_{|I|=s} \sum_{i=1}^n (-1)^i \partial^s_{I,i},$$

where the component $\partial^s_{I,i} \colon \mathbb{Z}_{D_I} \longrightarrow \mathbb{Z}_{D_{I\cup\{i\}}}$ is defined by

$$\partial^s_{I,i} := \begin{cases} X^*_{(I\cup\{i\})\supset I} & \text{for } i \notin I \\ 0 & \text{for } i \in I. \end{cases}$$

We need some more notation: Let us introduce the *brutal truncation* of complexes: If $C^\bullet$ is a complex and $n \in \mathbb{Z}$, then the brutal truncation from above of $C^\bullet$ is given by

$$b_{\leq n}(C^\bullet) := \{\ldots \to C^{n-1} \to C^n \to 0 \to 0\}$$

that is to say $b_{\leq n}(C^\bullet)^m = C^m$ for $m \leq n$ and $b_{\leq n}(C^\bullet)^m = 0$ for $m > n$.

Definition 3.7.1. *For an S-motive M given by the complex*

$$0 \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \ldots \longrightarrow M^{j-1} \longrightarrow M^j \longrightarrow 0$$

we define the motive $b_{\leq n}M \in \mathcal{DM}(S)$ to be the one given by the complex $b_{\leq n}(M^\bullet)^m = M^m$ for $m \leq n$ and $b_{\leq n}(M^\bullet)^m = 0$ for $m > n$. We call $b_{\leq n}M$ the brutal truncation from above after degree n.

The setting we are in is again the following:

Notation 3.7.2. Let $\pi\colon X \longrightarrow S$ be as usual, i.e. S is a reduced scheme and $X \in \mathrm{Sm}_S$, i.e. smooth and quasi-projective over S, and equipped with two sections $x, y\colon S \longrightarrow X$. In what follows, we consider the following subsets of X^n for any $n \in \mathbb{N}$:

$$\begin{aligned}
D_0^{(n)} &:= x(S) \times X^{n-1} \\
D_i^{(n)} &:= \{x_i = x_i + 1\} \subset X^n \text{ for } 1 \leq i \leq n-1 \\
D_n^{(n)} &:= X^{n-1} \times y(S),
\end{aligned}$$

In this section, we aim to prove the following:

Theorem 3.7.3. *There is a natural isomorphism*

$$\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y} \simeq b_{\leq 0}\left(\mathbb{Z}_{(X^n; D_0^{(n)}, D_1^{(n)}, \ldots, D_{n+1}^{(n)})}[n]\right) \in \mathcal{DM}(S).$$

The main tool to prove this is another complex associated to any simplicial object: the system of coefficients complex as described in [DG05].

A *system of coefficients* c over the standard simplex Δ_n with values in an additive category $\mathcal{A}$ assigns to each face τ of Δ_n an object $c(\tau) \in \mathcal{A}$, contravariantly functorial in τ with respect to inclusion morphisms of faces. Any system of coefficients c defines a chain complex

$$\mathcal{C}_p(\Delta_n, c) = \bigoplus_{|\tau|=p+1} c(\tau)$$

with differentials given by the alternating sum of the restriction morphisms: For $i \notin \tau \subset \{0, \ldots, n\}$, the differential is given by the morphism $(-1)^{|\{j \in \{\tau_0, \ldots, \tau_p\} \mid j < i\}|} c(\tau \hookrightarrow \tau \cup \{i\})$. Dually, a cosystem of coefficients $c(\tau)$, which is covariantly functorial in τ, defines a complex of cochains.

Any simplicial object $S_\bullet$ defines a system of coefficients c by putting $c(\tau) := S_\tau$ for every face τ of Δ_n, while for $\tau = \{\tau_0, \ldots, \tau_k\} \subset \{0, \ldots, n\}$ such that $\tau_0 < \ldots < \tau_k$ and any $r \in \{0, \ldots, k\}$ the differential $S_\tau \longrightarrow S_{\tau \setminus \{i_r\}}$ is given by the morphism

$$(-1)^{\tau_r - r} S(\delta_k^r \colon [k-1] \hookrightarrow [k]) = (-1)^{\tau_r - r} d_k^r \colon S_\tau \longrightarrow S_{\tau \setminus \{\tau_r\}}.$$

In particular, this means that on the component S_τ for $\tau = \{\tau_0, \dots, \tau_k\} \subset \{0, \dots, n\}$ the differential is given by the sum

$$\sum_{r=0}^{k} (-1)^{\tau_r - r} d_r^{k-1} \colon S_\tau \longrightarrow \bigoplus_{r=0}^{k} S_{\tau \setminus \{\tau_r\}},$$

where d_r^{k-1} denotes the corresponding face map.
One writes $\mathcal{C}_*(\varDelta_n, S_\bullet)$ for the resulting chain complex in $\mathcal{K}(C^b_{\mathrm{mot}}(\mathrm{Sm}_S))$. Deligne and Goncharov prove the following:

Lemma 3.7.4 (Deligne-Goncharov). *If $S_\bullet$ is a simplicial object in an additive idempotent complete category, the complexes $\mathcal{C}_*(\varDelta_n, S_\bullet)$ and the truncated normalized complex $\sigma^{\geq -n} N(S_\bullet)$ are functorially homotopic.*

Proof. [DG05, Proposition 3.10, p.25]. ■

In this chapter, we will compute and study the system of coefficients complex that arises from the simplicial bar object $sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}$ we defined. By definition, it is given by the chain complex

$$\mathcal{C}_n\left(\varDelta_n, sB^{\mathrm{mot}}_\bullet(X|S)\right) \to \ldots \to \mathcal{C}_1\left(\varDelta_n, sB^{\mathrm{mot}}_\bullet(X|S)\right) \to \mathcal{C}_0\left(\varDelta_n, sB^{\mathrm{mot}}_\bullet(X|S)\right)$$

of objects $\mathcal{C}_p\left(\varDelta_n, sB^{\mathrm{mot}}_\bullet(X|S)\right) = \bigoplus_{|\tau|=p+1} sB^{\mathrm{mot}}_\tau(X|S)_{x,y} = \bigoplus_{|\tau|=p+1} (\mathbb{Z}_{X^p})_{f \leq p}$ for $p \geq 0$, where the sums run over all faces τ, with differentials

$$d' \colon sB^{\mathrm{mot}}_\tau(X|S)_\tau \longrightarrow \bigoplus_{j \in \{\tau\}} sB^{\mathrm{mot}}_{\tau \setminus \{j\}}(X|S)$$

for a face τ induced by an alternating sum of all face maps $\oplus_{j \in \tau} d^j \colon \oplus_j \tau \setminus \{j\} \hookrightarrow \tau$. To be precise, the differential is given in components by

$$\begin{aligned}\delta'_p\big|_{(\mathbb{Z}_{X^{p+1}})_\tau} = (-1)^{\tau_0}(x \times \mathrm{id}^p)^* + \sum_{k=1}^{n-j-2} (-1)^{\tau_l - l}(\mathrm{id}^{k-1} \times \varDelta \times \mathrm{id}^{p-k})^* \\ +(-1)^{\tau_p - p}(\mathrm{id}^p \times y)^*\end{aligned}$$

if $\tau = \{\tau_0, \dots, \tau_p\}$. The resulting complex is denoted $\mathcal{C}_*\left(\varDelta_n, sB^{\mathrm{mot}}_\bullet(X|S)\right)$.

Lemma 3.7.5. *Identifying $\mathcal{C}_*\left(\varDelta_n, sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}\right) \in \mathcal{K}(C^b_{\mathrm{mot}}(\mathrm{Sm}_S))$ with an element in $\mathcal{DM}(S)$, it is naturally isomorphic to*

$$b_{\leq 0}\left(\mathbb{Z}_{(X^n; D_0^{(n)}, D_1^{(n)}, \dots, D_{n+1}^{(n)})}[n]\right) \in \mathcal{DM}(S).$$

In order to prove this, we first establish the following lemma:

Lemma 3.7.6. *The motive* $b_{\leq 0}\left(\mathbb{Z}_{(X^n;D_0^{(n)},D_1^{(n)},\ldots,D_{n+1}^{(n)})}[n]\right) \in \mathcal{DM}(S)$ *is isomorphic to the complex*

$$(\mathbb{Z}_{X^n})_{f\leq n} \xrightarrow{\delta'_{n-1}} \ldots \bigoplus_{|I|=s}(\mathbb{Z}_{X^{n-s}})_{f\leq n-s} \xrightarrow{\delta'_{n-s-1}} \bigoplus_{|I|=s+1}(\mathbb{Z}_{X^{n-s-1}})_{f\leq n-s-1} \ldots \xrightarrow{\delta'_0} \bigoplus_{i=0}^{n}\mathbb{Z}_S$$

where $\mathbb{Z}_{X^n}$ *is in degree* $-n$ *and the differentials are given on components by*

$$\delta'_p\big|_{(\mathbb{Z}_{X^{p+1}})_I} = (-1)^{\tau_0}(x\times \mathrm{id}^p)^* + \sum_{k=1}^{n-j-2}(-1)^{\tau_l - l}(\mathrm{id}^{k-1}\times\varDelta\times \mathrm{id}^{p-k})^* \\ +(-1)^{\tau_p-p}(\mathrm{id}^p\times y)^*$$

if the complement of I *in* $\{0,\ldots,n\}$ *is* $\{\tau_0,\ldots,\tau_p\}$ *with* $\tau_0<\ldots<\tau_p$.

Proof. Put $D_I^{(n)} := \bigcap_{i\in I} D_i^{(n)}$ for $I\subset\{0,\ldots,n\}$ and denote the corresponding inclusions by $i_I\colon D_I^{(n)}\hookrightarrow X^n$. Then we have natural isomorphisms $D_i^{(n)}\cong X^{n-1}$ and $D_I^{(n)}\cong X^{n-|I|}$ for $I\subset\{1,\ldots,n\}$. For $I\subset\{1,\ldots,n\}, i\notin I$, we denote the inclusion of $D_{I\cup\{i\}}$ into D_I by $\iota_{I,i}\colon D_{I\cup\{i\}} = D_I\cap D_i\hookrightarrow D_I$ and put $\iota_{I,i}:=0$ for $i\in I$. The motive $\mathbb{Z}_{(X;D_0^{(n)},\ldots,D_{n-1}^{(n)})}$ is then given by the complex

$$(\mathbb{Z}_{X^n}(0))_{g_0} \xrightarrow{\delta'_{n-1}} \ldots \bigoplus_{|I|=s}(\mathbb{Z}_{D_I}(0))_{g_s} \xrightarrow{\delta'_{n-s-1}} \bigoplus_{|I|=s+1}(\mathbb{Z}_{D_I})_{g_{s+1}} \ldots \xrightarrow{\delta'_{-1}} \mathbb{Z}_{D_0\cap\ldots\cap D_n}$$

in degrees 0 up to $n-1$, where the differential is given by the alternating sum of pull-back by the inclusions $\iota_{I,i}$, that is to say

$$\delta'_{n-s-1} = \sum_{|I|=s}\sum_{i=0}^{n}(-1)^{|\{j\in I\,|\,j<i\}|}\iota^*_{I,i}.$$

Let us determine what these differentials correspond to under the natural isomorphisms $D_I^{(n)}\cong X^{n-|I|}$: Let $I=i_1,\ldots,i_s$ and $i_1<\ldots<i_s$, and denote its complement in $\{0,1,\ldots,n\}$ by $J:=\{j_1,\ldots,j_{n+1-s}\}$, where again we suppose $j_1<\ldots<j_{n+1-s}$. By definition we have $\iota_{I,i_k}=0$ for $k=1,\ldots,s$, while for $k=1,\ldots,n+1-s$, $\iota_{I,j_k}\colon D_{I\cup\{j_k\}} = D_I\cap D_{j_k}\hookrightarrow D_I$ is the inclusion. The morphism $\iota_{I,j_1}\colon D_{I\cup\{j_1\}} = D_I\cap D_{j_1}\hookrightarrow D_I$ hence corresponds to

$$\left\{\begin{array}{ll} x\times \mathrm{id}_{X^{n-s-1}} & \text{for } k=1 \\ \mathrm{id}_{X^{n-s-1}}\times y & \text{for } k=n-s \\ \mathrm{id}_{X^{k-1}}\times\varDelta\times\mathrm{id}_{X^{n-s-k}} & \text{for } 1<k<n-s \end{array}\right\} : X^{n-s-1}\longrightarrow X^{n-s}$$

under the natural identifications $D_{I\cup\{j_1\}}\cong X^{n-s-1}$ and $D_I\cong X^{n-s}$. Via the natural identifications $\mathbb{Z}_{D_I^{(n)}}\simeq\mathbb{Z}_{X^{n-|I|}}$ g_s corresponds to $f^{\leq n-s}$, and the differential $\delta'_{n-s-1} = \sum_{|I|=s}\sum_{i=0}^{n}(-1)^{|\{j\in I\,|\,j<i\}|}\iota^*_{I,i}$ corresponds to the morphism

$$\begin{aligned}
\delta'_{n-s-1} &= \sum_{|I|=s} \left(\sum_{i \notin I} (-1)^{|\{j \in I \mid j < i\}|+1} \iota^*_{I,i} \right) \\
&= \sum_{\substack{|I| = s \\ [n] \setminus I = \{\tau_0, \ldots, \tau_{n-s-1}\} \\ \tau_0 < \ldots < \tau_{n-s-1}}} \Bigg((-1)^{|\{i \in I \mid j < \tau_0\}|} (x \times \mathrm{id}^{n-s-1})^* \\
&\quad + \sum_{k=1}^{n-s-1} (-1)^{|\{i \in I \mid j < \tau_k\}|} (\mathrm{id}^{k-1} \times \Delta \times \mathrm{id}^{n-s-1-k})^* \\
&\quad + (-1)^{|\{i \in I \mid j < \tau_{n-s-1}\}|} (\mathrm{id}^{n-s-1} \times y)^* \Bigg).
\end{aligned}$$

Since $|\{i \in I \mid j < \tau_k\}| = \tau_k - k$, the assertion follows. ■

Proof of Lemma 3.7.5. The complex $\mathcal{C}_* \left(\Delta_n, sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}\right) \in C^b(\mathcal{DM}(S))$ is given by the following complex in degree $-n$ up to 0:

$$\mathcal{C}_n \left(\Delta_n, sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}\right) \to \ldots \to \mathcal{C}_1 \left(\Delta_n, sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}\right) \to \mathcal{C}_0 \left(\Delta_n, sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}\right)$$

where the objects are given by

$$\mathcal{C}_p \left(\Delta_n, sB^{\mathrm{mot}}_\bullet(X|S)_{x,y}\right) = \bigoplus_{|\tau|=p+1} sB^{\mathrm{mot}}_\tau(X|S)_{x,y} = \bigoplus_{|\tau|=p+1} \left(\mathbb{Z}_{X^p}(0)\right)_{f \leq p}$$

for $p \geq 0$. If the image of a fixed face τ is $\{\tau_0, \ldots, \tau_p\}$, then the differential $sB^{\mathrm{mot}}_\tau(X|S)_\tau \longrightarrow \bigoplus_{j \in \{\tau\}} sB^{\mathrm{mot}}_{\tau-\{j\}}(X|S)$ is given by the alternating sum of face maps

$$\begin{aligned}
\delta'_{p-1}\big|_{(\mathbb{Z}_{X^p})_\tau} &= \sum_{l=0}^{p} (-1)^{\tau_l - l} d^p_l \\
&= (-1)^{\tau_0} (x \times \mathrm{id}^p)^* + \sum_{k=1}^{n-j-2} (-1)^{\tau_l - l} (\mathrm{id}^{k-1} \times \Delta \times \mathrm{id}^{p-k})^* \\
&\quad + (-1)^{\tau_p - p} (\mathrm{id}^p \times y)^*.
\end{aligned}$$

Comparing the resulting complex to Lemma 3.7.6 concludes the proof. ■

Now, we can finally prove the theorem we stated in the outset of this section:

Theorem 3.7.7. *There is a natural isomorphism*

$$\widetilde{B}^{\mathrm{mot}}_n(X|S)_{x,y} \simeq b_{\leq 0} \left(\mathbb{Z}_{\left(X^n; D^{(n)}_0, D^{(n)}_1, \ldots, D^{(n)}_{n+1}\right)} [n] \right) \in \mathcal{DM}(S).$$

Proof. By the Lemma of Deligne-Goncharov (Lemma 3.7.4), there is a functorial homotopy in $C^b(\mathcal{DM}(S))$:

$$\mathcal{C}_* \left(\Delta_n, sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}\right) \simeq \sigma^{\geq -n} N(sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}).$$

Considering both sides as elements in $\mathcal{DM}(S)$, we obtain

$$\mathcal{C}_* \left(\Delta_n, sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}\right) \simeq \widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y}.$$

Identifying $\mathcal{C}_* \left(\Delta_n, sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}\right)$ with an element in $\mathcal{DM}(S)$, it is naturally isomorphic to $b_{\leq 0}\left(\mathbb{Z}_{\left(X^n; D_0^{(n)}, D_1^{(n)}, \ldots, D_{n+1}^{(n)}\right)}[n]\right) \in \mathcal{DM}(S)$ by Lemma 3.7.5, which proves the assertion. ■

Remark 3.7.8. a.) If $x(S) \cap y(S) = \emptyset$, then $D_0^{(n)} \cap D_1^{(n)} \cap \ldots \cap D_n^{(n)} = \emptyset$, so the relative motive $\mathbb{Z}_{\left(X^n; D_0^{(n)}, \ldots, D_n^{(n)}\right)}$ is zero after degree n. In other words, for $x(S) \cap y(S) = \emptyset$ we have

$$\mathcal{C}_* \left(\Delta_n, B_\bullet^{\mathrm{mot}}(X|S)\right) = \mathbb{Z}_{\left(X^n; D_0^{(n)}, \ldots, D_n^{(n)}\right)}[n].$$

b.) If $x = y$, then $D_0^{(n)} \cap D_1^{(n)} \cap \ldots \cap D_n^{(n)} = x(S)^n$, so $\mathbb{Z}_{D_0^{(n)} \cap D_1^{(n)} \cap \ldots \cap D_n^{(n)}} = \mathbb{Z}_x$. Hence the last differential δ_{-1}^* of the complex $\mathbb{Z}_{\left(X^n; D_0^{(n)}, \ldots, D_n^{(n)}\right)}[n]$ gives rise to a morphism of motives $\mathcal{C}_* \left(\Delta_n, B_\bullet^{\mathrm{mot}}(X|S)\right) \longrightarrow \mathbb{Z}_{x(S)} \simeq \mathbb{Z}_S$.

3.8 Augmentation ideals

We have seen in section 3.6 that the direct systems $(B_n^{\mathrm{mot}}(X|S)_{x,x})_n$ and $(\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,x})_n$ carry the structure of Hopf algebras. As a consequence, these bar complexes have an augmentation ideal given by the kernel of the counit as defined in section 3.6. However, involving the Hopf algebra structure is more elaborate than necessary, since the direct sum-splitting of the bar complexes for $x = y$ is fairly obvious:

The last differential of the both the unnormalized and normalized bar complex $\delta_0^*(x,x) = \widetilde{\delta}_0^*(x - x_0, x - x_0) = x^* - x^* = 0$ vanishes, and thus both $B_n^{\mathrm{mot}}(X|S)_{x,x}$ and $\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,x}$ decompose as follows: $B_n^{\mathrm{mot}}(X|S)_{x,x}$ (resp. $\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,x}$) is the sum of the complex $\mathbb{Z}_S[0]$ concentrated in degree zero with the complex

$$\mathbb{Z}_{X^n} \xrightarrow{\delta_{n-1}^*(x,x)} \mathbb{Z}_{X^{n-1}} \xrightarrow{\delta_{n-2}^*(x,x)} \ldots \xrightarrow{\delta_1^*(x,x)} \mathbb{Z}_X \longrightarrow 0$$

$$\text{resp.} \quad \mathbb{Z}^\circ_{X^n} \xrightarrow{\widetilde{\delta}_{n-1}^*(x,x)} \mathbb{Z}^\circ_{X^{n-1}} \xrightarrow{\widetilde{\delta}_{n-2}^*(x,x)} \ldots \xrightarrow{\widetilde{\delta}_1^*(x,x)} \mathbb{Z}^\circ_X \longrightarrow 0$$

with $\mathbb{Z}_{X^n}$ (resp. $\mathbb{Z}^\circ_{X^n}$) in degree $-n$.

Definition 3.8.1. *The augmentation ideals of $B_n^{\mathrm{mot}}(X|S)_{x,x}$ and $\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,x}$ are defined to be the S-motives*

$$I_n^{\mathrm{mot}}(X|S)_x := \left\{ (\mathbb{Z}_{X^n})_{f\le n} \xrightarrow{\delta_{n-1}^*(x,x)} (\mathbb{Z}_{X^{n-1}})_{f\le n-1} \xrightarrow{\delta_{n-2}^*(x,x)} \dots \xrightarrow{\delta_1^*(x,x)} (\mathbb{Z}_X)_{f\le 1} \longrightarrow 0 \right\}$$

$$\tilde{I}_n^{\mathrm{mot}}(X|S)_x := \left\{ (\mathbb{Z}_{X^n})^\circ_{f\le n} \xrightarrow{\widetilde{\delta}_{n-1}^*(x,x)} (\mathbb{Z}_{X^{n-1}})^\circ_{f\le n-1} \xrightarrow{\widetilde{\delta}_{n-2}^*(x,x)} \dots \xrightarrow{\widetilde{\delta}_1^*(x,x)} (\mathbb{Z}_X)^\circ_{f\le 1} \longrightarrow 0 \right\}$$

where $\mathbb{Z}_{X^n}$ (resp. $\mathbb{Z}^\circ_{X^n}$) is in degree $-n$. The directed system with respect to inclusions

$$I^{\mathrm{mot}}(X|S)_x := (I_n^{\mathrm{mot}}(X|S)_x)_n \in \varinjlim \mathcal{DM}(S) \left(\textit{resp. } \tilde{I}^{\mathrm{mot}}(X|S)_x := (\tilde{I}_n^{\mathrm{mot}}(X|S)_x)_n\right)$$

is called the augmentation ideal of the bar complex $B^{\mathrm{mot}}(X|S)_{x,x}$ (resp. $\widetilde{B}^{\mathrm{mot}}(X|S)_{x,x}$).

Note that $\tilde{I}_n^{\mathrm{mot}}(X|S)_x = (\tilde{I}_n^{\mathrm{mot}}(X|S)_x, e^*_{X|S} \in \mathcal{DM}(S))$, where $e^*_{X|S}$ is the idempotent given in degree $-r$ by $e^{r*}_{X|S}\colon \mathbb{Z}_{X^r} \longrightarrow \mathbb{Z}_{X^r}$.

Corollary 3.8.2. *There are canonical splittings in $\mathcal{DM}(S)$:*

$$B_n^{\mathrm{mot}}(X|S)_{x,x} = I_n^{\mathrm{mot}}(X|S)_x \oplus \mathbb{Z}_S[0] \quad \textit{and} \quad \widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,x} = \tilde{I}_n^{\mathrm{mot}}(X|S)_x \oplus \mathbb{Z}_S[0].$$

Simplicial augmentation ideals for $x = y = x_0$

One would like to describe the augmentation ideal in terms of a simplicial object. However, in general this is not possible. Fortunately, in one special case, the *normalized* augmentation ideal in fact underlies a simplicial object in $\mathcal{DM}(S)$:

Proposition 3.8.3. *Put* $\mathrm{id}^{-1} = \mathrm{id}^0 = \mathrm{id}_S$. *Then the following is a cosimplicial object in the Karoubi envelope $\mathcal{K}(\mathbb{Z}(\mathrm{Sm}_S))$:*

$$\begin{aligned}
cI^*_{\mathrm{mot}}(X|S)_{x_0}\colon \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{K}(\mathbb{Z}(\mathrm{Sm}_S)\\
[n] &\longmapsto \widetilde{X^{n+1}} := \left(X^{n+1}, \mathrm{id}^n \times (\mathrm{id} - x_0\pi)\right)\\
(d^j\colon [n] \to [n+1]) &\longmapsto (d_j^n(I)\colon \widetilde{X^{n+1}} \to \widetilde{X^{n+2}},\\
(s^j\colon [n+1] \to [n]) &\longmapsto (s_j^n(I)\colon \widetilde{X^{n+2}} \to \widetilde{X^{n+1}})\\
d_j^n(I) &:= \begin{cases} x_0 \times \mathrm{id}^{n+1} & \textit{for } j = 0\\ \mathrm{id}^{j-1} \times \Delta \times \mathrm{id}^{\times n-j+1} & \textit{for } j \in \{1, \dots, n\}\\ \mathrm{id}^{\times n-1} \times (\Delta \circ (\mathrm{id} - x_0\pi)) & \textit{for } j = n+1 \end{cases}\\
s_j^{n+1}(I) &:= \mathrm{id}^j \times \pi \times \mathrm{id}^{n-j+1} \ \textit{for } j = 0, \dots, n.
\end{aligned}$$

Proof. It is easy to see that the face and degeneracy maps are in fact compatible with the idempotents in question, so it suffices to show that the simplicial identities are satisfied. This computation is carried out in the appendix. See E.4. ■

This cosimplicial object gives rise to the following simplicial object in the Karoubi envelope $\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S))$ of $\mathcal{A}_{\mathrm{mot}}(S)$:

$$\begin{aligned} sI^{\mathrm{mot}}_{\bullet}(X|S)_{x_0} \colon \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S)) \\ [n] &\longmapsto \widetilde{\mathbb{Z}}_{X^{n+1}} := \left((\mathbb{Z}_{X^{n+1}}(0))_{f^{\leq n+1}_{cB(X|S)_{x_0},x_0}}, (\mathrm{id}^n \times (\mathrm{id} - x_0\pi)^* \right) \\ (d^j \colon [n] \to [n+1]) &\longmapsto \left(d^n_j(I) \colon \widetilde{\mathbb{Z}}_{X^{n+2}} \to \widetilde{\mathbb{Z}}_{X^{n+1}} \right), \\ (s^j \colon [n+1] \to [n]) &\longmapsto \left(s^n_j(I) \colon \widetilde{\mathbb{Z}}_{X^{n+1}} \to \widetilde{\mathbb{Z}}_{X^{n+2}} \right) \end{aligned}$$

$$d^n_j(I) := \begin{cases} (x_0 \times \mathrm{id}^{n+1})^* & \text{for } j = 0 \\ (\mathrm{id}^{j-1} \times \Delta \times \mathrm{id}^{\times n-j+1})^* & \text{for } j \in \{1,\dots,n\} \\ \left(\mathrm{id}^{\times n-1} \times (\Delta \circ (\mathrm{id} - x_0\pi))\right)^* & \text{for } j = n+1 \end{cases}$$

$$s^{n+1}_j(I) := (\mathrm{id}^j \times \pi \times \mathrm{id}^{n-j+1})^* \text{ for } j = 0,\dots,n.$$

As a direct consequence of the definitions in 2.2 we obtain:

Lemma 3.8.4. *The normalized complex $N(sI_{\bullet}(X|S)_{x_0})$ associated to the simplicial object $sI_{\bullet}(X|S)_{x_0}$ is given by*

$$\begin{array}{cccccccccccc} \cdots & \xrightarrow{\widetilde{\delta}^*_{n+1}(0,0)} & (\mathbb{Z}^{\circ}_{X^{n+1}})_{f^{\leq n+1}} & \xrightarrow{\widetilde{\delta}^*_{n}(0,0)} & \cdots & \xrightarrow{\widetilde{\delta}^*_{2}(0,0)} & (\mathbb{Z}_{X^2})^{\circ}_{f^{\leq 2}} & \xrightarrow{\widetilde{\delta}^*_{1}(0,0)=\Delta^*} & (\mathbb{Z}_{X})^{\circ}_{f^{\leq 1}} & \longrightarrow & 0 \\ \cdots & & -n & & \cdots & & -1 & & 0 & & 1 \end{array}$$

in $C^-(\mathcal{K}(\mathcal{A}_{\mathrm{mot}}(S)))$, where the differentials are given by

$$\widetilde{\delta}^*_k(0,0) = e^{k*}_{X|S} \circ \left(\sum_{i=0}^{k-1} (-1)^i \,\mathrm{id}^i \times \Delta \times \mathrm{id}^{k-1-i} \right)^* .$$

Corollary 3.8.5. *The object $nM(cI^*_{\mathrm{mot}}(X|S)_{x_0})$ in $\mathcal{DM}(S)$ determined by $\sigma^{\geq -n}N(sI_{\bullet}(X|S)_{x_0})$ is equal to $\tilde{I}^{\mathrm{mot}}_n(X|S)_{x_0}[-1]$.*

The augmentation ideal in terms of relative motives

We may apply Deligne-Goncharov's lemma on the system of coefficients complex associated to $sI_{\bullet}(X|S)_{x_0}$ to find a description of the motive $\tilde{I}^{\mathrm{mot}}_{n+1}(X|S)_{x_0}[-1]$ in terms of a reduced relative motive. The system of coefficients complex was introduced in section 3.7 (in particular, see Lemma 3.7.4).

We put $D^{(n+1)}_i := \{x_i = x_i + 1\} \subset X^{n+1}$ for $1 \leq i \leq n$ and $D^{(n+1)}_0 := x_0(S) \times_S X^n$. Put $D^{(n+1)}_I := \bigcap_{i \in I} D^{(n+1)}_i$ for $I \subset \{0,\dots,n\}$ and denote the corresponding inclusions by $i_I \colon D^{(n+1)}_I \hookrightarrow X^{n+1}$. Moreover, define the idempotent

$$e_{X^{n+1}|S} := e_{X^{n+1}|S}(\mathrm{id}_{X^n} \times x_0) = (\mathrm{id}_{X^n} \times_S (\mathrm{id}_X - x_0 \circ \pi))^* : \mathbb{Z}_{X^{n+1}} \longrightarrow \mathbb{Z}_{X^{n+1}}.$$

It induces the following idempotent of the relative motive $\mathbb{Z}_{\left(X^{n+1};D_0^{(n+1)},\dots,D_{n-1}^{(n+1)}\right)}$:

$$\begin{tikzcd}
\mathbb{Z}_{X^{n+1}} \arrow[r, "\delta'_n"] \arrow[d, "e_{X^{n+1}|S}"] & \dots \arrow[r] & \bigoplus_{|I|=s} \mathbb{Z}_{D_I} \arrow[r, "\delta'_{n-s}"] \arrow[d, "\bigoplus e_{X^{n+1}|S}\big|_{D_I}"] & \bigoplus_{|I|=s+1} \mathbb{Z}_{D_I} \dots \arrow[r, "\delta'_0"] \arrow[d, "\bigoplus e_{X^{n+1}|S}\big|_{D_I}"] & \mathbb{Z}_S \arrow[d, equal] \\
\mathbb{Z}_{X^{n+1}} \arrow[r, "\delta'_n"] & \dots \arrow[r] & \bigoplus_{|I|=s} \mathbb{Z}_{X^{n+1-s}} \arrow[r, "\delta'_{n-s}"] & \bigoplus_{|I|=s+1} \mathbb{Z}_{X^{n-s}} \dots \arrow[r, "\delta'_0"] & \mathbb{Z}_S
\end{tikzcd}$$

We may thus define the S-motive

$$\mathbb{Z}^{\circ}_{\left(X^{n+1};D_0^{(n+1)},\dots,D_{n-1}^{(n+1)}\right)} := \left(\mathbb{Z}_{\left(X^{n+1};D_0^{(n+1)},\dots,D_{n-1}^{(n+1)}\right)}, e_{X^{n+1}|S}\right).$$

Lemma 3.8.6. *The object of $\mathcal{DM}(S)$ given by $\mathcal{C}_*\left(\Delta_n, sI_\bullet(X|S)_{x_0}\right)$ is naturally isomorphic to*

$$b_{\leq 0}\left(\mathbb{Z}^{\circ}_{\left(X^{n+1};D_0^{(n+1)},\dots,D_{n-1}^{(n+1)}\right)}[n]\right) \in \mathcal{DM}(S),$$

where $b_{\leq 0}$ denotes the brutal truncation from above after degree 0 (see Definition 3.7.1).

Proof. This proof is basically the same as in the case of the motivic bar complex:

- **First step:** Explicit description of $\mathbb{Z}_{\left(X^{n+1};D_0^{(n+1)},\dots,D_{n-1}^{(n+1)}\right)}$.

  We have natural isomorphisms $D_i^{(n+1)} \cong X^n$ and $D_I^{(n+1)} \cong X^{n+1-|I|}$ for $I \subset \{1,\dots,n\}$. For $I \subset \{1,\dots,n\}, i \notin I$, we denote the inclusion of $D_{I\cup\{i\}}$ into D_I by $\iota_{I,i}\colon D_{I\cup\{i\}} = D_I \cap D_i \hookrightarrow D_I$ and put $\iota_{I,i} := 0$ for $i \in I$. The motive $\mathbb{Z}_{\left(X^{n+1};D_0^{(n+1)},\dots,D_n^{(n+1)}\right)}$ is then given by the complex

$$\mathbb{Z}_{X^{n+1}}(0)_{g_0^*} \xrightarrow{\delta'_n} \dots \xrightarrow{\delta'_{n-s+1}} \bigoplus_{|I|=s} \mathbb{Z}_{D_I}(0)_{g_s^*} \xrightarrow{\delta'_{n-s}} \bigoplus_{|I|=s+1} \mathbb{Z}_{D_I}(0)_{g_{s+1}^*} \xrightarrow{\delta'_{n-s-1}} \dots \xrightarrow{\delta'_{-1}} \mathbb{Z}_{D_0\cap\dots\cap D_n}(0)_{g_n^*}$$

  in degrees 0 up to $n+1$, where all g_k are the inclusion morphisms, and the differential is given by the alternating sum of pull-back by the inclusions $\iota_{I,i}$, that is to say $\delta'_{n-s-1} = \sum_{|I|=s}\sum_{i=0}^{n}(-1)^{|\{j\in I\mid j<i\}|}\iota_{I,i}^*$. As in the proof of Lemma 3.7.6 we see that this complex is isomorphic to the complex

$$\mathbb{Z}_{X^{n+1}}(0)_{f^{\leq n+1*}} \xrightarrow{\delta'_n} \dots \xrightarrow{\delta'_{n-s+1}} \bigoplus_{|I|=s} \mathbb{Z}_{X^{n+1-s}}0()_{f^{\leq n+1-s*}} \xrightarrow{\delta'_{n-s}} \bigoplus_{|I|=s+1} \mathbb{Z}_{X^{n-s}}(0)_{f^{\leq n-s*}} \xrightarrow{\delta'_{n-s-1}} \dots \xrightarrow{\delta'_0} \mathbb{Z}_S \longrightarrow$$

  where $\mathbb{Z}_{X^{n+1}}$ is in degrees 0, and the $f^{\leq s*}$ are the usual morphisms as in the bar complex. The differentials are given by

$$\delta'_{n-s-1} = \sum_{|I|=s} \Big((-1)^{\tau_0}(x_0 \times \mathrm{id}^{n-s})^* + \sum_{i=1}^{n-s} (-1)^{\tau_i - i}(\mathrm{id}^{i-1} \times \varDelta \times \mathrm{id}^{n-s-i})^* \Big),$$

where the sum runs over all subsets I of $\{0,\ldots,n\}$ with $|I| = s$, and the complement of I in $\{0,\ldots,n\}$ is given by $\{\tau_0,\ldots,\tau_{n-s-1}\}$ where we assume $\tau_0 < \ldots < \tau_{n-s-1}$.

- **Second step:** Explicit description of the motive

$$\left(\mathbb{Z}_{\left(X^{n+1}; D_0^{(n+1)}, \ldots, D_{n-1}^{(n+1)}\right)}, (\mathrm{id}_{X^n} \times_S (\mathrm{id}_X - x_0 \circ \pi))^* \right).$$

By the first step, the reduced relative motive above is isomorphic to the motive given by the diagram:

$$\begin{tikzcd}
\mathbb{Z}_{X^{n+1}} \arrow[r, "\delta'_n"] \arrow[d, "e_{X^{n+1}|S}"] & \ldots \arrow[r] & \bigoplus_{|I|=s} \mathbb{Z}_{X^{n+1-s}} \arrow[r, "\delta'_{n-s}"] \arrow[d, "e_{X^{n-s+1}|S}"] & \bigoplus_{|I|=s+1} \mathbb{Z}_{X^{n-s}} \ldots \arrow[r, "\delta'_0"] \arrow[d, "e_{X^{n-s}|S}"] & \mathbb{Z}_S \arrow[d, equal] \\
\mathbb{Z}_{X^{n+1}} \arrow[r, "\delta'_n"] & \ldots \arrow[r] & \bigoplus_{|I|=s} \mathbb{Z}_{X^{n+1-s}} \arrow[r, "\delta'_{n-s}"] & \bigoplus_{|I|=s+1} \mathbb{Z}_{X^{n-s}} \ldots \arrow[r, "\delta'_0"] & \mathbb{Z}_S
\end{tikzcd}$$

(that is to say: it is given by the horizontal complex together with the idempotent given by the above diagram), where $\mathbb{Z}_{X^{n+1}}$ is in degree 0. This immediately shows that the reduced relative motive $\left(\mathbb{Z}_{\left(X^{n+1}; D_0^{(n+1)}, \ldots, D_{n-1}^{(n+1)}\right)}, e_{X^{n+1}|S} \right)$ is isomorphic to the complex

$$\begin{array}{ccccccccc}
(\mathbb{Z}_{X^{n+1}})^{\circ}_{f\le n+1} & \xrightarrow{\widetilde{\delta}^*_n(0,0)} & \ldots & \xrightarrow{\widetilde{\delta}^*_2(0,0)} & (\mathbb{Z}_{X^2})^{\circ}_{f\le 2} & \xrightarrow{\widetilde{\delta}^*_1(0,0)=\varDelta^*} & (\mathbb{Z}_X)^{\circ}_{f\le 1} & \longrightarrow & 0 \\
\ldots & & -n & & \ldots & & -1 & & 0
\end{array}$$

given by the truncation $\sigma^{\ge -n} N(s\boldsymbol{I}_\bullet(X|S)_{x_0})$.

- **Third step:** Explicit description of the system of coefficients complex

$$\mathcal{C}_* \left(\varDelta_n, I_\bullet^{\mathrm{mot}}(X|S)_{x_0} \right).$$

$\mathcal{C}_* \left(\varDelta_n, I_\bullet^{\mathrm{mot}}(X|S)_{x_0} \right) \in C^b(\mathcal{DM}(S))$ is given by the following complex in degrees $-n$ up to 0:

$$\mathcal{C}_n \left(\varDelta_n, sI_\bullet^{\mathrm{mot}}(X|S)_{x_0} \right) \to \ldots \to \mathcal{C}_0 \left(\varDelta_n, sI_\bullet^{\mathrm{mot}}(X|S)_{x_0} \right)$$

where the objects are given by

$$\mathcal{C}_p \left(\varDelta_n, sI_\bullet^{\mathrm{mot}}(X|S)_{x_0} \right) = \bigoplus_{|\tau|=p+1} sI_\tau^{\mathrm{mot}}(X|S)_{x,y} = \bigoplus_{|\tau|=p+1} \mathbb{Z}_{X^{p+1}}$$

for $p \geq 0$. If the image of a fixed face τ is given by $\{\tau_0, \dots, \tau_p\}$, then the differential

$$sI^{\mathrm{mot}}_{\tau}(X|S)_{\tau} \longrightarrow \bigoplus_{j \in \{\tau\}} sI^{\mathrm{mot}}_{\tau - \{j\}}(X|S)$$

for a face $\tau = \{\tau_0, \dots, \tau_p\}$ is given by the sum of face maps

$$\sum_{j \in \tau} \mathrm{id}^{|\{i \in \tau | i < j\}|} \otimes d^j_{p+1} \otimes \mathrm{id}^{\otimes p - |\{i \in \tau | i \leq j\}| + 1} = \sum_{k=0}^{p} \mathrm{id}^{\tau_k - k} \otimes d^j_{p+1} \otimes \mathrm{id}^{\otimes p - k + 1} .$$

Hence, $\mathcal{C}_p\left(\Delta_n, sI^{\mathrm{mot}}_{\bullet}(X|S)\right)$ is the complex

$$\begin{array}{ccccccccc}
(\mathbb{Z}_{X^{n+1}})^{\circ}_{f \leq n+1} & \xrightarrow{\widetilde{\delta}^*_n(0,0)} & \dots & \xrightarrow{\widetilde{\delta}^*_2(0,0)} & (\mathbb{Z}_{X^2})^{\circ}_{f \leq 2} & \xrightarrow{\widetilde{\delta}^*_1(0,0) = \Delta^*} & (\mathbb{Z}_X)^{\circ}_{f \leq 1} & \longrightarrow & 0 \\
\dots & & -n & & \dots & & -1 & & 0
\end{array}$$

in $\mathcal{K}(C^b_{\mathrm{mot}}(S))$ with δ'_k as above. Comparing the two complexes we arrived at, the motive determined by $\mathcal{C}_\bullet\left(\Delta_n, sI^{\mathrm{mot}}_{\bullet}(X|S)\right)$ is isomorphic to the brutal truncation of the motive $\left(\mathbb{Z}_{\left(X^{n+1}; D_0^{(n+1)}, \dots, D_{n-1}^{(n+1)}\right)}, (\mathrm{id}_{X^n} \times_S (\mathrm{id}_X - x_0 \circ \pi))^*\right)[n]$ after $\bigoplus_{p=0}^{n} \mathbb{Z}^{\circ}_X$ (i.e. degree 0), so the assertion follows. ∎

Corollary 3.8.7. *We keep the notation of 3.8.6. There is a canonical isomorphism of S-motives*

$$\tilde{I}^{\mathrm{mot}}_{n+1}(X|S)_{x_0}[-1] \simeq b_{\leq 0}\left(\mathbb{Z}^{\circ}_{\left(X^{n+1}; D_0^{(n+1)}, \dots, D_{n-1}^{(n+1)}\right)}[n]\right), \quad \textit{where}$$

$$\mathbb{Z}^{\circ}_{\left(X^{n+1}; D_0^{(n+1)}, \dots, D_{n-1}^{(n+1)}\right)} = \left(\mathbb{Z}_{\left(X^{n+1}; D_0^{(n+1)}, \dots, D_{n-1}^{(n+1)}\right)}, e_{X^{n+1}|S}\right)$$

Proof. This is just an application of the Lemma of Deligne-Goncharov (Lemma 3.7.4) together with Lemma 3.8.6 above. ∎

Chapter 4

The mixed realization of the motivic bar complexes

In the upcoming chapter, we will determine the mixed realization of the motivic constructions of chapter 3 and prove that the Hodge realization of our motivic simplicial bar objects indeed yields the classical simplicial bar object of section 1.4.

4.1 Preliminaries

Recall the mixed realization of motives as described in section C.4 in the appendix:

Case (i) : **The geometric case**		**Case (ii)** : **The ℓ-adic case**	
F	$=$ a subfield of $\mathbb{C}$	F	$= \mathbb{Z}\left[\frac{1}{l}\right]$
A	$= \mathbb{Q}$	A	$= \mathbb{Q}_l$
X_{top}	$:= X(\mathbb{C})$	X_{top}	$:= X \otimes_A \overline{\mathbb{Q}}$
$\mathrm{Sh}(X_{\mathrm{top}})$	$:= \mathrm{Perv}(X_{\mathrm{top}}, A)$	$\mathrm{Sh}(X_{\mathrm{top}})$	$:= \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q}_l)$
$\mathrm{Sh}(X)$	$:= \mathrm{Mod}_{rh}(D_X)$	$\mathrm{Sh}(X)$	$:= \mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$
$D^b\,\mathrm{Sh}(X)$	$:= D^b_{rh}(X)$	$D^b\,\mathrm{Sh}(X)$	$:= D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$
$f_*, f^*,$	$:= \int_f, f^\bigstar,$	$f_*, f^*,$	$:=$ the usual functors
$f_!, f^!$	$:= \int_{f!}, f^\dagger$in D^b_{rh}	$f_!, f^!$	in $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ (see [Hub97])

where

a.)
- $\mathrm{Perv}(X_{\mathrm{top}}, A)$ denotes the category of perverse sheaves on X_{top},
- $\mathrm{Mod}_{rh}(X_D)$ is the category of regular holonomic D-modules on X,

b.)
- $\mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q}_l)$ is the category of ℓ-adic perverse sheaves on X_{top} (for details see [BBD82]).
- $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ is roughly defined as follows (for details, see [Hub97]): Let $(\mathbf{S}, L)$ be a fixed pair consisting of a horizontal stratification $\mathbf{S}$ of X (see section 2 of [Hub97]) and a collection $L = \{L(S) \mid S \in \mathbf{S}\}$, where each $L(S)$ is a set of irreducible pure lisse ℓ-adic sheaves on S. For all $S \in \mathbf{S}$ and $F \in L(S)$, it is required that for the inclusion $j\colon S \hookrightarrow X$, all higher direct images $R^n j_* F$ are

$(\mathbf{S}, L)$-constructible, that is to say, when restricted to any $S \in \mathbf{S}$ they are lisse extensions of objects of $L(S)$. Denoting the derived category of ℓ-adic sheaves with constructible cohomology by $D^b_{cs}(X, \mathbb{Q}_l)$, $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ is its subcategory of complexes with $(\mathbf{S}, L)$-constructible cohomology objects.

- $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ is then defined as follows: The category $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ admits a perverse t-structure (for the notion of t-structures and their hearts, see section 8.1.1 of[HTT08], and for this particular t-structure see [Hub97]). Its heart is $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

Note that the six functor formalism of mixed sheaves satisfies the same properties as listed in section C.1.2.

4.1.1 The setting of the chapter

Let $F = \mathbb{C}$ in the geometric case, and $F = \mathbb{Z}[1/l]$ in the ℓ-adic case, A either a subfield of $\mathbb{C}$ in the geometric case or $\mathbb{Q}_l$ if $F = \mathbb{Z}[1/l]$ in the étale case. Moreover, let $S \longrightarrow \mathrm{Spec}(F)$ be a reduced scheme (recall that in this thesis, "scheme" means "noetherian and separated scheme" throughout), smooth and quasi-projective over $\mathrm{Spec}(F)$.
Let $\pi\colon X \longrightarrow S$ be in Sm_S such that π has geometrically irreducible fibers. As a consequence of the properties of S, X is also reduced, as well as smooth and quasi-projective over $\mathrm{Spec}(F)$. Moreover, we assume that π has three sections $x_0, x, y\colon S \longrightarrow X$ (these don't necessarily have to be distinct).
All other notation can be taken from the above table.

We first take a close look at the preliminaries we will use in this chapter:

4.1.2 The realization functor and base-change

By [Lev98, V.2.3.15, p.284] and [Lev98, V.2.2.9, p. 272] there is an exact realization functor

$$\mathfrak{R}_{A,\mathrm{mix}} := \left\{ \begin{matrix} \mathfrak{R}_{A,\mathrm{geo}} & \text{in case (i)} \\ \mathfrak{R}_{A,\mathrm{et},S} & \text{in case (ii)} \end{matrix} \right\} : \mathcal{DM}_A(\mathrm{Sm}_S) \longrightarrow D^b(\mathrm{Sh}(S)).$$

Lemma 4.1.1. *Let $f\colon T \longrightarrow S$ be a morphism of regular schemes. Then one has an equivalence of functors*

$$f^* \circ \mathfrak{R}_{A,\mathrm{mix}} \simeq \mathfrak{R}_{A,\mathrm{mix}} \circ \mathcal{DM}(f^*)\colon \mathcal{DM}_A(S) \longrightarrow D^b(\mathrm{Sh}(T)).$$

Proof. By the original construction of the realization functor it is sufficient to check this on objects of the kind $A_Y(m)$ for $Y \in \mathrm{Sm}(S)$. Recall that the pull-back $\mathcal{DM}(f^*)\colon \mathcal{DM}_A(S) \longrightarrow \mathcal{DM}_A(T)$ is given on motives of type $A_Y(m) \in \mathcal{DM}_A(S)$ by fiber product with T, i.e. $\mathcal{DM}(f^*)(A_Y(m)) = A_{Y\times_S T}(m) \in \mathcal{DM}_A(T)$.

(i) *geometric realization:*
As on page 284 of [Lev98], let $\bar{j}_S\colon S \hookrightarrow \bar{S}$ and $\bar{j}_Y\colon Y \hookrightarrow \overline{Y}$ be smooth compactifications of S and Y, respectively. We extend the morphism $\pi_Y\colon Y \longrightarrow S$ to a morphism $\bar{\pi}_Y$, such that we have the diagram

$$\begin{tikzcd} Y \arrow[r, hook, "\bar{j}_Y"] \arrow[d, "\pi_Y"'] & \overline{Y} \arrow[d, "\bar{\pi}_Y"] \\ S \arrow[r, hook, "\bar{j}_S"'] & \bar{S} \end{tikzcd}$$

Moreover, let $\bar{j}_T\colon T \hookrightarrow \overline{T}$ be a smooth compactification of T and extend $f\colon T \longrightarrow S$ to a morphism $\overline{f}\colon \overline{T} \longrightarrow \bar{S}$ such that the diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\bar{j}_T"'] & S \arrow[d, "\bar{j}_S"] \\ \overline{T} \arrow[r, "\overline{f}"'] & \bar{S} \end{tikzcd}$$

commutes. Then a smooth compactification of $\pi_Y \times_S \mathrm{id}_T\colon Y \times_S T \longrightarrow T$ is given by

$$\begin{tikzcd} Y \times_S T \arrow[r, hook, "\bar{j}_Y \times_{\bar{S}} \bar{j}_T"] \arrow[d, "\pi_Y \times_S \mathrm{id}_T"'] & \overline{Y} \times_{\bar{S}} \overline{T} \arrow[d, "\bar{\pi}_Y \times_{\bar{S}} \mathrm{id}_{\overline{T}}"] \\ T = S \times_S T \arrow[r, hook, "\bar{j}_T"'] & \overline{T} = \bar{S} \times_{\bar{S}} \overline{T}. \end{tikzcd}$$

By C.4.3, one has

$$\begin{aligned} f^* \circ \mathfrak{R}_{A,\mathrm{geo}} A_Y &= f^* \circ \bar{j}_S^* \bar{\pi}_{Y*} \bar{j}_{Y*} A_Y \\ \mathfrak{R}_{A,\mathrm{geo}} \circ \mathcal{DM}(f^*) &= \bar{j}_T^* (\bar{\pi}_Y \times_{\bar{S}} \mathrm{id}_{\overline{T}})_* (\bar{j}_Y \times_{\bar{S}} \bar{j}_T)_* A_{(Y \times_S T)} \\ &= \bar{j}_T^* (\bar{\pi}_{Y*} \bar{j}_{Y*} \times_{\bar{S}} \bar{j}_{T*}) A_{(Y \times_S T)} \\ &= (\mathrm{id}_{\bar{S}} \times_{\bar{S}} \bar{j}_T)^* (\bar{\pi}_{Y*} \bar{j}_{Y*} \times_{\mathrm{id}, \bar{S}, \overline{f}} \bar{j}_{T*}) A_{(Y \times_S T)} \\ &= (\bar{\pi}_{Y*} \bar{j}_{Y*} \times_{\mathrm{id}, \bar{S}, \overline{f} \circ \bar{j}_T} \bar{j}_T^* \bar{j}_{T*}) A_{(Y \times_S T)} \\ &= (\bar{j}_S^* \bar{\pi}_{Y*} \bar{j}_{Y*} \times_{\mathrm{id}, S, f} \bar{j}_T^* \bar{j}_{T*}) A_{(Y \times_S T)} \\ &\overset{(\star)}{=} (\bar{j}_S^* \bar{\pi}_{Y*} \bar{j}_{Y*} \times_{\mathrm{id}, S, f} \mathrm{id}_T) A_{(Y \times_S T)} \\ &= f^* \circ \mathfrak{R}_{A,\mathrm{geo}} A_Y \end{aligned}$$

where the equality $(\star)$ is due to the fact that $\bar{j}_T$ is an open immersion, and hence $\bar{j}_T^* \bar{j}_{T*} \simeq \mathrm{id}_T$. Now note that $A_{(Y \times_S T)} = (\mathrm{id} \times_S f)^* A_Y$, so we have

$$\begin{aligned}\mathfrak{R}_{A,\mathrm{geo}} \circ \mathcal{DM}(f^*) &= (\bar{j}_S^* \bar{\pi}_{Y*} \bar{j}_{Y*} \times_S \mathrm{id}_T)(\mathrm{id} \times_S f)^* A_{Y \times_S S} \\ &= (\bar{j}_S^* \bar{\pi}_{Y*} \bar{j}_{Y*} \times_S f^*) A_{Y \times_S S} \\ &= f^* \bar{j}_S^* \bar{\pi}_{Y*} \bar{j}_{Y*} A_Y\end{aligned}$$

and the assertion follows in the geometric case.

(ii) *ℓ-adic realization:* By Theorem C.4.1, the ℓ-adic regulator is given by

$$\mathfrak{R}_{\mathbb{Q}_l,\mathrm{et},S} \mathbb{Q}_{l,Y}(m) = R\pi_{Y*} \mathbb{Q}_{l,Y}(m),$$

for $\pi_Y : Y \longrightarrow S$ in $\mathrm{Sm}(S)$, where $\mathbb{Q}_{l,Y}(m)$ are the Tate objects in $D^b_{(\mathbf{S},L)}(Y, \mathbb{Q}_l)$. In this situation, the sides of the asserted equation are given by

$$\begin{aligned}f^* \circ \mathfrak{R}_{\mathbb{Q}_l,\mathrm{et},S} \mathbb{Q}_{l,Y}(m) &= f^* R\pi_{Y*} \mathbb{Q}_{l,Y}(m) \\ \mathfrak{R}_{\mathbb{Q}_l,\mathrm{et},S} \circ \mathcal{DM}(f^*) &= R(\pi_Y \times \mathrm{id}_T)_* \mathbb{Q}_{l\ (Y \times_S T)}(m) \\ &= R\pi_{Y*} \mathbb{Q}_{l\ (Y \times_S T)}(m) \\ &= R\pi_{Y*}(\mathrm{id}_Y \times_S f)^* \mathbb{Q}_{l\ Y}(m) = f^* R\pi_{Y*} \mathbb{Q}_{l,Y}(m).\end{aligned}$$ ■

4.1.3 Inductive systems of mixed sheaves

Recall that the motivic bar complexes are inductive systems of motives and live in the category $\varinjlim \mathcal{DM}(S)$ of inductive systems of objects in $\mathcal{DM}(S)$. Likewise, since $D^b\,\mathrm{Sh}(S)$ is not cocomplete, we need to pass to a larger category to consider this direct system of bar complexes.

Definition 4.1.2. *For any two inductive systems $(A_i)_{i \in I}, (B_k)_{k \in K}$ in $D^b\,\mathrm{Sh}(S)$ over any index sets I, K, we define*

$$\mathrm{Hom}_{\varinjlim D^b\,\mathrm{Sh}(S)}(A, B) \cong \varprojlim_i \varinjlim_k \mathrm{Hom}_{D^b\,\mathrm{Sh}(S)}(A_i, B_k).$$

We denote the category whose objects are inductive systems in $D^b\,\mathrm{Sh}(S)$ and whose morphisms are given as above by $\varinjlim D^b\,\mathrm{Sh}(S)$.

Remark 4.1.3. a.) Since $(D^b\,\mathrm{Sh}, \otimes, 1 = A_S(0))$ is a symmetric monoidal category, $\varinjlim D^b\,\mathrm{Sh}(S)$ inherits a symmetric monoidal category structure by defining

$$(A_i)_{i \in I} \otimes (B_k)_{k \in K} := (A_i \otimes B_k)_{i,k \in I \times K}.$$

By [Mey07, 1.136, p.57], this tensor product turns $\varinjlim D^b\,\mathrm{Sh}(S)$ into a symmetric monoidal category, whose unit is the constant inductive system $A_S(0)$.

b.) Recall that objects in $D^b\,\mathrm{Sh}(S)$ are given by bounded complexes of D-modules/sheaves, so left-unbounded complexes are in $\varinjlim D^b\,\mathrm{Sh}(S)$. We extend the natural pull-back functor of D-modules/ℓ-adic sheaves to left-unbounded complexes in $\varinjlim D^b\,\mathrm{Sh}(S)$ by applying the pull-back-functor of sheaves componentwise.

4.2 Basic notation and first properties

The setting for the entire chapter will be as stated in 4.1.1, i.e. $F = \mathbb{C}$ in the geometric case, and $F = \mathbb{Z}[1/l]$ in the ℓ-adic case. A is either a subfield of $\mathbb{C}$ in the geometric case or $\mathbb{Q}_l$ if $F = \mathbb{Z}[1/l]$ in the étale case. Moreover, let $S \longrightarrow \mathrm{Spec}(F)$ be a reduced scheme (recall that in this thesis, "scheme" means "noetherian and separated scheme" throughout), smooth and quasi-projective over $\mathrm{Spec}(F)$. We suppose that $\pi\colon X \longrightarrow S$ be in Sm_S such that π has geometrically irreducible fibers and three sections $x_0, x, y\colon S \longrightarrow X$. Let d designate the dimension of X over S.

Denote by $G(A_X)$ the Godement resolution of the mixed sheaf $A_X(0)$ on X. Recall that it is defined as follows: For any scheme X, let $p_X\colon \coprod_{x\in X}\{x\} \longrightarrow X$ denote the canonical continuous map from the disjoint union of points in X with discrete topology to X. Then the Godement resolution of a scheaf $\mathscr{F}$ on X is defined to be

$$G(\mathscr{F}) := p_{X*}p_X^*\mathscr{F}.$$

The complex $G(\mathscr{F})$ can also be described as follows: For any open subset $U \subset X$, the zeroth component of the complex is given on U by the product over stalks

$$G_0(\mathscr{F})(U) = \prod_{x\in U}\mathscr{F}_x$$

of $\mathscr{F}$, and inductively one has $G_i(\mathscr{F}) = G_0(G_{i-1}(\mathscr{F}))$. As is well-known, for any mixed sheaf $\mathscr{F}$ on X, $G(\mathscr{F})$ is flabby.
We now apply this to the mixed sheaf A_X on X.

Notation 4.2.1. We write $\pi_*A_X^\sharp := \pi_*G(A_X)$ and $A_S^\sharp := G(A_S)$.

Remark 4.2.2. a.) In both the ℓ-adic and the geometric case, the complex $\pi_*A_X^\sharp$ has non-vanishing cohomology only in degrees $0, \dots, 2d$ for the following reason: For $s \in S$, the stalk of the i-th cohomology sheaf is given by

$$\left(R^i\pi_*G(A_X)\right)_s = H^i_{\mathrm{mix}}(X_s, A),$$

where $X_s = \pi^{-1}(s)$ and $H^\bullet_{\mathrm{mix}}$ denotes either étale or singular cohomology. Since X_s is of dimension d, $H^i_{\mathrm{mix}}(X_s, A) = 0$ unless $0 \le i \le 2d$ in both cases. Hence, via truncations, $\pi_*G(A_X)$ is quasi-isomorphic to the complex $\tau_{\le 2d}\sigma^{\ge 0}(\pi_*G(A_X))$.

b.) The mixed realization of the morphism $\pi^*\colon A_S \longrightarrow A_X$ in $\mathcal{DM}_A(S)$ is given by the morphism $\pi^*\colon A_S^\sharp \hookrightarrow \pi_*A_X^\sharp$ defined as follows: We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] & S \\ \coprod_{x\in X}\{x\} \arrow[u, "p_X"] \arrow[r, "\coprod\pi"] & \coprod_{s\in S}\{s\} \arrow[u, "p_S"'] \end{tikzcd} \tag{4.1}$$

Hence, we have $\pi_* p_{X*} = p_{S*}(\coprod \pi)_*$. By flat base-change, $(\coprod \pi)_* p_X^* \cong p_S^* \pi_*$, i.e. $\pi_* p_{X*} p_X^* A_X = p_{S*}(\coprod \pi)_* p_X^* A_X \cong p_{S*} p_S^* \pi_* A_X$. The morphism $\mathfrak{R}_{A,\mathrm{mix}} \pi^* \colon A_S \hookrightarrow \pi_* \pi^* A_S \cong \pi_* A_X$ thus induces a natural morphism

$$\pi^* \colon A_S^\sharp = G(A_S) = p_{S*} p_S^* A_S \longrightarrow p_{S*} p_S^* \pi_* A_X \cong \pi_* p_{X*} p_X^* A_X = \pi_* G(A_X).$$

c.) If the S-scheme X is furthermore endowed with a section $x \colon S \longrightarrow X$, then this section induces an adjunction morphism $\pi_* A_X \longrightarrow \pi_* x_* x^* A_X = x^* A_X \cong A_S$. Just like above, using the diagram (4.1), this adjunction induces a natural morphism

$$x^* \colon \pi_* A_X^\sharp \longrightarrow A_S^\sharp.$$

d.) Again using the diagram (4.1), the diagonal morphism $\Delta \colon X \longrightarrow X \times_S X$ induces a natural morphism

$$\Delta^* \colon \pi_* A_X^\sharp \otimes \pi_* A_X^\sharp \longrightarrow \pi_* A_X^\sharp.$$

The following is now immediate:

Lemma 4.2.3. *The mixed realization of $A_X \in \mathcal{DM}_A(S)$ (resp. A_X°) is isomorphic to the the mixed sheaf $\pi_* A_X^\sharp$ (resp. $\pi_* A_X^\sharp / A_S^\sharp$). Also, due to Künneth, the mixed realization of $A_{X^n} \in \mathcal{DM}_A(S)$ can be computed by*

$$\begin{aligned} \pi_*(A_{X^n}(0)) &\simeq (\pi_* A_X^\sharp) \otimes_{A_S^\sharp} \dots \otimes_{A_S^\sharp} (\pi_* A_X^\sharp) \\ &=: (\pi_* A_X^\sharp)^{\otimes n}. \end{aligned}$$

In what follows, we will determine the mixed realization of the motivic bar constructions in $D^b \operatorname{Sh}(S)$:

Definition 4.2.4. *We define a simplicial object in the category $D^b \operatorname{Sh}(S)$ by putting*

$$sB_\bullet^{A,\mathrm{mix}}(X|S)_{x,y} := \mathfrak{R}_{A,\mathrm{mix}} \circ sB_\bullet^{\mathrm{mot}}(X|S)_{x,y} \otimes A \colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow D^b \operatorname{Sh}(S),$$

and call it the <u>mixed simplicial bar object</u>. Moreover, for any $n \in \mathbb{N}$ we denote the mixed realizations of the motives in the first column of the following table by the corresponding terms in the second column:

$X \in \mathcal{DM}_A(S)$	$\mathfrak{R}_{A,\mathrm{mix}} X \in D^b \operatorname{Sh}(S)$	*name of the mixed realization*
$B_n^{\mathrm{mot}}(X\vert S)_{x,y} \otimes A$	$B_n^{A,\mathrm{mix}}(X\vert S)_{x,y}$	*n-th mixed bar complex*
$\widetilde{B}_n^{\mathrm{mot}}(X\vert S)_{x,y} \otimes A$	$\widetilde{B}_n^{A,\mathrm{mix}}(X\vert S)_{x,y}$	*n-th normalized mixed bar complex*
$I_n^{\mathrm{mot}}(X\vert S)_x \otimes A$	$I_n^{A,\mathrm{mix}}(X\vert S)_x$	*augmentation ideal of* $B_n^{A,\mathrm{mix}}(X\vert S)_{x,x}$
$\tilde{I}_n^{\mathrm{mot}}(X\vert S)_x \otimes A$	$\tilde{I}_n^{A,\mathrm{mix}}(X\vert S)_x$	*augmentation ideal of* $\widetilde{B}_n^{A,\mathrm{mix}}(X\vert S)_{x,x}$

Recall that the motives in the first column of the above table give rise to inductive systems

$$\begin{aligned}&B^{\mathrm{mot}}(X|S)_{x,y} = (B_n^{\mathrm{mot}}(X|S)_{x,y})_n, \quad \widetilde{B}^{\mathrm{mot}}(X|S)_{x,y} = (\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y})_n,\\ &I^{\mathrm{mot}}(X|S)_x = (I_n^{\mathrm{mot}}(X|S)_x)_n \ \text{ and } \ \tilde{I}^{\mathrm{mot}}(X|S)_x = (\tilde{I}_n^{\mathrm{mot}}(X|S)_x)_n\end{aligned}$$

which all can be identified with left unbounded complexes as described in Remark 4.1.3. Correspondingly, their mixed realizations give rise to inductive systems which can be identified with left-unbounded complexes:

Definition 4.2.5. *Denote the mixed realizations of the motives in the first column of the following table by the corresponding terms in the second column:*

$(X_n)_n \in \varinjlim \mathcal{DM}_A(S)$	$(\mathfrak{R}_{A,\mathrm{mix}} X_n)_n$	*name of the mix. real.*
$B^{\mathrm{mot}}(X\|S)_{x,y} \otimes A$	$B^{A,\mathrm{mix}}(X\|S)_{x,y}$	*mixed bar complex*
$\widetilde{B}^{\mathrm{mot}}(X\|S)_{x,y} \otimes A$	$\widetilde{B}^{A,\mathrm{mix}}(X\|S)_{x,y}$	*norm. mixed bar complex*
$I^{\mathrm{mot}}(X\|S)_x \otimes A$	$I^{A,\mathrm{mix}}(X\|S)_x$	*augm. id. of* $B^{A,\mathrm{mix}}(X\|S)_{x,x}$
$\tilde{I}^{\mathrm{mot}}(X\|S)_x \otimes A$	$\tilde{I}^{A,\mathrm{mix}}(X\|S)_x$	*augm. id. of* $\widetilde{B}^{A,\mathrm{mix}}(X\|S)_{x,x}$

Corollary 4.2.6. *Suppose we are given a morphism $\varphi\colon X' \longrightarrow X$ of smooth S-schemes, with compatible sections*

$$\begin{tikzcd} X' \arrow[rr, "\varphi"] & & X \\ & S \arrow[ul, "x_0'"] \arrow[ur, "x_0"'] & \end{tikzcd} \quad \begin{tikzcd} X' \arrow[rr, "\varphi"] & & X \\ & S \arrow[ul, "x'"] \arrow[ur, "x"'] & \end{tikzcd} \quad \begin{tikzcd} X' \arrow[rr, "\varphi"] & & X \\ & S \arrow[ul, "y'"] \arrow[ur, "y"'] & \end{tikzcd} .$$

Then there is an induced morphism

$$\varphi^{*\otimes}\colon (-)(X|S)_{x,y} \longrightarrow (-)(X'|S)_{x',y'} \in \varinjlim D^b\,\mathrm{Sh}(S)$$

for $(-) = sB^{A,\mathrm{mix}}, B^{A,\mathrm{mix}}$ and $\widetilde{B}^{A,\mathrm{mix}}$, and in case $x = y, x' = y'$ also for $(-) = I^{A,\mathrm{mix}}$ and $\tilde{I}^{A,\mathrm{mix}}$.

Proof. This follows from the functoriality properties of 2.3.1 and 3.6. ■

Moreover, the results on naturality (i.e. functoriality with respect to the base scheme of 2.3.1 and 3.6) carry over to the mixed situation by Lemma 4.1.1:

Corollary 4.2.7. *Given a morphism $f\colon T \longrightarrow S$ of schemes, one has*

$$f^*(-)(X|S)_{x,y} = (-)(X \times_S T|T)_{x\times_S \mathrm{id}_T, y\times_S \mathrm{id}_T} \in \varinjlim D^b\,\mathrm{Sh}(T)$$

for $(-) = sB^{A,\mathrm{mix}}, B^{A,\mathrm{mix}}, \widetilde{B}^{A,\mathrm{mix}}$, and in case $x = y$ also for $(-) = I^{A,\mathrm{mix}}, \tilde{I}^{A,\mathrm{mix}}$.

4.3 The bar complexes

The face and degeneracy maps of the motivic simplicial bar object $sB^{\mathrm{mot}}(X|S)_{x,y}$ induce the face and degeneracy maps

$$d_j^{n+1} := -\left\{\begin{array}{ll}(\mathrm{id}^{j-1}\times\Delta\times\mathrm{id}^{n-j})^* & \text{for } j\in\{1,\dots,n\}\\ (x\times\mathrm{id}^n)^* & \text{for } j=0\\ (\mathrm{id}^n\times y)^* & \text{for } j=n+1\end{array}\right\}:\pi_*(A_{X^{n+1}}(0))\to\pi_*(A_{X^n}(0))$$
$$s_j^n := -\,\mathrm{id}^{\otimes j}\otimes\pi^*\otimes\mathrm{id}^{\otimes n-j}\ \text{ for } j=0,\dots,n$$

Via the quasi-isomorphism $\pi_*(A_{X^n}(0))\simeq(\pi_*A_X^\sharp)^{\otimes n}$, these maps correspond to the face and degeneracy maps

$$d_j^{n+1} := -\begin{cases}\mathrm{id}^{\otimes j-1}\otimes\Delta^*\otimes\mathrm{id}^{\otimes n-j} & \text{for } j\in\{1,\dots,n\}\\ x^*\otimes\mathrm{id}^{\otimes n} & \text{for } j=0\\ \mathrm{id}^{\otimes n}\otimes y^* & \text{for } j=n+1\end{cases}$$
$$s_j^n := -\,\mathrm{id}^{\otimes j}\otimes\pi^*\otimes\mathrm{id}^{\otimes n-j}\ \text{ for } j=0,\dots,n$$

where the morphisms on the right are those of Remark 4.2.2 above. By abuse of notation, the mixed realization of the differentials of the bar complexes are still denoted

$$\begin{aligned}\delta_n^*(x,y) &:= \mathfrak{R}_{A,\mathrm{mix}}\delta_n^*(x,y) = x^*\otimes\mathrm{id}^{\otimes n}+\\ &\quad+\sum_{i=1}^{r-1}(-1)^{i+1}\,\mathrm{id}^{\otimes i-1}\otimes\Delta^*\otimes\mathrm{id}^{\otimes n-i}+(-1)^{r+1}\,\mathrm{id}^{\otimes n}\otimes y^*\\ \widetilde{\delta}_n^*(x-x_0,y-x_0) &:= \mathfrak{R}_{A,\mathrm{mix}}\widetilde{\delta}_n^*(x-x_0,y-y_0) = e_{X|S}^{*\otimes n}\circ\delta_n^*(x,y)\circ e_{X|S}^{*\otimes n+1}\end{aligned}$$

where the morphisms on the right hand side are those of Remark 4.2.2, and $e_{X|S}^* := \mathrm{id}-\pi^*x_0^*$.
As a direct consequence of the above and the definition of the realization functors, we immediately obtain the mixed realizations of all bar complexes and augmentation ideals:

Theorem 4.3.1. *The mixed realization $B_n^{A,\mathrm{mix}}(X|S)_{x,y}$ of the unnormalized bar complex is given by the total complex of the double complex*

$$(\pi_*A_X^\sharp)^{\otimes n}\xrightarrow{\delta_n^*(x,y)}(\pi_*A_X^\sharp)^{\otimes n-1}\xrightarrow{\delta_{n-1}^*(x,y)}\dots\xrightarrow{\delta_1^*(x,y)}\pi_*A_X^\sharp\xrightarrow{x^*-y^*}A_S^\sharp\longrightarrow 0,$$

while the mixed realization $\widetilde{B}_n^{A,\mathrm{mix}}(X|S)_{x,y}$ of the normalized bar complex is given by the total complex of the double complex

$$\left(\frac{\pi_*A_X^\sharp}{A_S^\sharp}\right)^{\otimes n}\xrightarrow{\delta_n^*(x,y)}\left(\frac{\pi_*A_X^\sharp}{A_S^\sharp}\right)^{\otimes n-1}\xrightarrow{\delta_{n-1}^*(x,y)}\dots\xrightarrow{\delta_1^*(x,y)}\frac{\pi_*A_X^\sharp}{A_S^\sharp}\xrightarrow{x^*-y^*}A_S^\sharp\longrightarrow 0.$$

Here, all tensor products are taken over $A_S^\sharp$.

Corollary 4.3.2. *There are decompositions*

$$B^{A,\mathrm{mix}}(X|S)_{x,x} \simeq I^{A,\mathrm{mix}}(X|S)_x \oplus A_S \quad \text{and}$$
$$\widetilde{B}^{A,\mathrm{mix}}(X|S)_{x,x} \simeq \tilde{I}^{A,\mathrm{mix}}(X|S)_x \oplus A_S \quad \text{in } \varinjlim D^b\,\mathrm{Sh}(S).$$

Let X, S be as above, and $x = y = x_0$.
Recall that in this situation, we have the simplicial augmentation ideal $sI_\bullet(X|S)^{\mathrm{mot}}_{x_0}$ defined in section 3.8. Tensoring $sI_\bullet(X|S)^{\mathrm{mot}}_{x_0}$ with A and composing the resulting simplicial object in $\mathcal{DM}_A(S)$ with the mixed realization functor $\mathfrak{R}_{A,\mathrm{mix}}$, we obtain the simplicial object

$$sI^{A,\mathrm{mix}}_\bullet(X|S)_{x_0} := \mathfrak{R}_{A,\mathrm{mix}}(sI^{\mathrm{mot}}_\bullet(X|S)_{x_0} \otimes A)$$

in $D^b(\mathrm{Sh}(S))$. Since taking normalized complexes commutes with the geometric realization functor, we obtain as a consequence of Corollary 3.8.5 :

Corollary 4.3.3. *Considered as an element in $D^b\,\mathrm{Sh}(S)$, $N(sI^{A,\mathrm{mix}}_\bullet(X|S)_{x_0})$ is equal to $I^{A,\mathrm{mix}}_\bullet(X|S)_{x_0}[-1]$.*

4.4 Connection to the classical bar complexes

Recall the classical bar complexes of chapter 1: Let k be a field, $R^\bullet$ a differential graded k-algebra and $A = \bigoplus_{p\geq 0} A^p$ a differential graded R-module. Moreover, suppose $R^\bullet$ admits the structure of a differential graded $A^\bullet$-bimodule via two morphism of differential graded algebras

$$x, y\colon A^\bullet \longrightarrow R^\bullet,$$

where left-multiplication is given by x, and right-multiplication by y. We saw in section 1.1.4 that the classical bar complex $B_n(A|R)_{x,y}$ is naturally isomorphic to the total complex of the double complex

$$A^{\otimes n} \xrightarrow{\delta_{n-1}(x,y)} A^{\otimes n-1} \xrightarrow{\delta_{n-2}(x,y)} \dots \xrightarrow{\delta_2(x,y)} A^{\otimes 2} \xrightarrow{\delta_1(x,y)} A \xrightarrow{y-x} R \longrightarrow 0$$

With this, Theorem 4.3.1 immediately provides the connection of our motivic bar complexes to the classical ones: Note that the morphisms of complexes

$$\Delta^*\colon \pi_*A^\sharp_X \otimes \pi_*A^\sharp_X \longrightarrow \pi_*A^\sharp_X \quad \text{and}$$
$$\pi^*\colon A^\sharp_S \longrightarrow \pi_*A^\sharp_X$$

of Remark 4.2.2 provide $\pi_*A^\sharp_X$ with the structure of a sheaf of differential graded A-algebras as well as that of a differential graded $A^\sharp_S$-module. Moreover, $A^\sharp_S$ is a $\pi_*A^\sharp_X$-bimodule via the pull-back morphisms $x^*, y^*\colon \pi_*A^\sharp_X \longrightarrow A^\sharp_S$. With this, we are in the situation of the classical bar complexes, and obtain as a corollary of Theorem 4.3.1:

Corollary 4.4.1. *The mixed realization of the motivic bar complex coincides with the classical bar complex of the differential graded $A^\sharp_S$-algebra $\pi_*A^\sharp$*

$$B^{A,\mathrm{mix}}_n(X|S)_{x,y} \cong B_n(\pi_*A^\sharp_X|A^\sharp_S)_{x,y}.$$

Part II

The Motivic Logarithm and Polylogarithm

In Part I we developed a theory of motivic bar complexes. Now, we will put it to use in order to achieve our original goal: the development of a *general, motivic* notion of polylogarithms. Before we proceed to do so, however, we will first motivate our definitions by taking a look at the classical logarithm and polylogarithm in the Hodge case. It will turn out that the Hodge logarithm is closely connected to the so-called "universal pro-unipotent variation of mixed Hodge structure", which was introduced and studied by Hain and Zucker in [HZ87]. Hain and Zucker describe it in terms of classical bar complexes. Thus, the universal pro-unipotent VMHS can easily be carried over to the motivic setting by virtue of the insight we gained on bar complexes in Part I.

Using the motivic generalization of the universal pro-unipotent VMHS, we will then proceed to construct the motivic polylogarithm in a very simple way: once we have understood the nature of the logarithm, the polylogarithm turns out to be merely a Gysin morphism - in other words, the polylogarithm in itself is of an astonishingly simple nature. The main difficulty in this part will therefore *not* be the construction of this polylogarithm map in itself, but the interpretation of this new polylogarithm as the one that has been studied in the past decades. To do this, we will determine its mixed realization and study its properties in the realizations. The following table shows how we will proceed in detail in this part:

Chapter II.5:

- Recap of Hain-Zucker's universal pro-unipotent VMHS using the bar complexes of Chapter I.1
- classification of unipotent mixed Hodge modules

→

Chapter II.6:

- Imitation of the constructions of Chapter II.5 to obtain the motivic logarithm
- Definition of the motivic polylog as a Gysin morphism
- Construction of the arising polylog-class in K-theory

↓

Chapter II.7:

- Computing the mixed realization of the motivic log and pol of II.6;
- Proof of characterizing properties of the polylog and comparison with the classical definitions

Chapter 5

The classical Hodge logarithm and polylogarithm

In this chapter, we will recall the classical notions of the logarithm and polylogarithm in the setting of variations of mixed Hodge structures (VMHS). Again, we will assume the basic theory on VMHS and mixed Hodge modules. A summary of the theory can be found in section C.1 of the appendix.

As we pointed out in the introduction, up to now the polylogarithm had to be defined anew in every single realization and setting: in the setting of elliptic curves, for abelian schemes in general, for the projective line, and for curves of genus ≥ 2. Obviously, it would take up far too much space and time at this point of the thesis to recall all of them in detail. Thus, we will simply describe a select version of the logarithm and the polylogarithm which will serve best to motivate our own definition of the motivic logarithm in Chapter II.6.

This logarithm is closely connected to the so-called "universal pro-unipotent VMHS", which was introduced by Hain and Zucker in [HZ87] and is an important object of its own account. It is the variation of mixed Hodge structure on a complex variety X which arises naturally from its fundamental group $\pi_1(X, x_0)$ and is crucial in the classification of unipotent variations of mixed Hodge structure. The motivic logarithm we are about to define in the subsequent Chapter II.6 is even more closely connected to the universal pro-unipotent VMHS than the classical logarithm itself, which justifies that we take a very thorough look at the universal pro-unipotent VMHS in this chapter - we will even spend more time on it than on the logarithm itself for reasons we will also discuss later on in this chapter.

Hence, the basic outline of this chapter is the following:

- First, we recall the universal pro-unipotent VMHS as defined by Hain and Zucker in [HZ87] in as much detail as necessary: Here, we start off with a recollection of the pro-unipotent completion of the fundamental group. Then we turn towards Hain/Zucker's classification of unipotent VMHS und define the pro-unipotent VMHS in an abstract way via this classification. Last but not least, we take a look at the explicit construction of the universal pro-unipotent VMHS by means of the classical bar complexes of Chapter I.1.

- Secondly, we recall a definition of the logarithm for curves by Beilinson and Levin, and compare it to the universal pro-unipotent VMHS.
- Thirdly, we define the classical polylogarithm as introduced by Beilinson and Levin for curves in an unpublished preprint.

5.1 The universal pro-unipotent VMHS

As we pointed out above, the key to the generalization of the logarithm is a thorough understanding of the so-called "universal pro-unipotent variation of mixed Hodge structure". The universal pro-unipotent VMHS is intrinsically connected to the fundamental group of the underlying scheme.

5.1.1 The pro-unipotent completion of the fundamental group and path space

Let R be a commutative ring with unit. Recall that the group algebra of the fundamental group $\pi_1(X,x)$ over R has an augmentation $\epsilon\colon R\pi_1(X,x) \longrightarrow R$ sending every group element $g \in \pi_1(X,x)$ to 1. Its kernel is called the *augmentation ideal* and denoted by J. The homomorphism $\iota\colon R \longrightarrow R\pi_1(X,x)$ sending an element $r \in R$ to $r \cdot \mathrm{id}$, where id is the identity element in $\pi_1(X,x)$, defines an R-module splitting $R\pi_1(X,x) \cong R \oplus J$. As is well known, the group algebra of the fundamental group carries the structure of a Hopf algebra with coproduct given by

$$\Delta\colon R\pi_1(X,x) \longrightarrow R\pi_1(X,x) \otimes_R R\pi_1(X,x), \quad \Delta(g) = g \otimes g.$$

The *pro-unipotent completion of the fundamental group* is the J-adic completion of the group algebra $R\pi_1(X,x)$:

$$R\pi_1(X,x)^\wedge = \varprojlim_n R\pi_1(X,x)/J^n.$$

The structure of the group algebra of the fundamental group carries over to its completion. Again, the augmentation splits to yield an isomorphism $R\pi_1(X,x)^\wedge \cong R \oplus J^\wedge$, where $J^\wedge$ denotes the image of J in the J-adic completion. Moreover, the coproduct induces a map

$$\hat{\Delta}\colon R\pi_1(X,x)^\wedge \longrightarrow R\pi_1(X,x)^\wedge \hat{\otimes}_R R\pi_1(X,x)^\wedge,$$

imparting $R\pi_1(X,x)^\wedge$ with the structure of a complete Hopf algebra.

Let PX denote the space of piecewise-smooth paths parametrized by the unit interval I in X, together with the compact-open topology. Chen (see [Che77, (1.2)]) defined a differentiable structure on PX in the following way: If N is a smooth manifold, a mapping $h\colon N \longrightarrow PX$ is called differentiable if and only if the associated mapping $\tilde{h}\colon N \times I \longrightarrow X$ is piecewise smooth. There is an obvious map $p\colon PX \longrightarrow X \times X, p(\gamma) =$

$(\gamma(0), \gamma(1))$ down to the starting and end points of a path in PX, which is called the *free path fibration*. The fiber $P_{x,y}$ is the set of paths from x to y, and the fibers over the diagonal in $X \times X$ are the loop spaces $P_{x,x}$ of loops based at x. The basic idea for what follows is as simple as it is effective: There is a canonical identification

$$\mathbb{Z}\pi_1(X, x) \cong H_0(P_{x,x})$$

relating the group algebra of the fundamental group to the path space. The group $H_0(P_{x,x})$ can in turn be related to well-understood formalisms like Chen's iterated integrals and the classical bar constructions we considered in full generality in Chapter I.1. This is done as follows:

Theorem 5.1.1. *Let X be connected, $E_X^\bullet$ be the complex of C^∞-forms on X, and $A^\bullet$ a cohomologically connected differential graded subalgebra of $E_X^\bullet$ such that $H^1(A^\bullet) \cong H^1(E_X^\bullet)$ and $H^2(A^\bullet) \longrightarrow H^2(E_X^\bullet)$ is injective. Furthermore, for two points $x, y \in X$, we consider the morphisms $x^*, y^*\colon A^\bullet \longrightarrow \mathbb{C}$ given by pull-back of forms to the points x and y (i.e. evaluating them at x, y).*

a.) The reduced bar complex on $(A^\bullet, \mathbb{C})$, $\bar{B}(A^\bullet)_{x,y}$, is canonically isomorphic to the complex of iterated integrals on $P_{x,y}$. In other words we have canonical isomorphisms for all n

$$H^0(\bar{B}_n(A^\bullet)_{x,y}) \cong (H_0(P_{x,y})/J^n)^*.$$

b.) Put $P_x := P_{x,x}$. The reduced bar complex $\bar{B}(A^\bullet)_{x,x}$ is canonically isomorphic to the complex of iterated integrals on the loop space P_x, i.e. there are, for all n, canonical isomorphisms

$$H^0(\bar{B}_n(A)_x) \cong (H_0(P_x)/J^{n+1})^* \cong (\mathbb{Z}\pi_1(X, x)/J^{n+1})^* \quad \text{for all } n \in \mathbb{N}.$$

Proof. This is [HZ87, (3.18), p.93] together with Chen's theorem [Che77]. ■

Let now $X = \bar{X} - D$, where $\bar{X}$ is a compact Kähler manifold and D is a normal crossing divisor. For the algebra $A^\bullet$ in the above theorem we can use the C^∞ logarithmic de Rham complex $E_{\bar{X}}^\bullet(\log D) = \Omega_{\bar{X}}^\bullet(\log D) \otimes_{\Omega_{\bar{X}}^\bullet} \mathcal{E}_{\bar{X}}^\bullet$, where $\Omega_{\bar{X}}^\bullet(\log D)$ denotes the complex of holomorphic forms with logarithmic singularities along D. Recall (see section C.3.2 in the appendix for details) that the *Hodge-de Rham complex* of $(\bar{X}, D)$ gives rise to a mixed Hodge structure on the hypercohomology groups of $\mathcal{E}_{\bar{X}}^\bullet(\log D)$. It is given by

$$\mathcal{H}dg^\bullet(\bar{X} \log D) := \Big(Rj_*\underline{\mathbb{Z}}_X^\sharp, (Rj_*\underline{\mathbb{Q}}_X^\sharp, \tau), \alpha, (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty, \sigma_\infty), \beta\Big)$$

$$:= \left\{ \begin{array}{ccccccc} & & Rj_*\underline{\mathbb{Q}}_X^\sharp & & & ({}'\mathcal{E}_{\bar{X}}^\bullet(\log D), W) & \\ & \overset{\alpha_1}{\nearrow} & & \searrow\!\!\!\searrow & \overset{\beta_1}{\nearrow} & & \overset{\beta_2}{\underset{\sim}{\nwarrow}} \\ Rj_*\underline{\mathbb{Z}}_X^\sharp & & & & (Rj_*\underline{\mathbb{Q}}_X^\sharp, \tau) & & & (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty, \sigma_\infty) \end{array} \right\},$$

where

- $\beta_1'\colon Rj_*\underline{\mathbb{Q}}_X \longrightarrow Rj_*\underline{\mathbb{C}}_X$ is the natural morphism induced by the inclusion $\underline{\mathbb{Q}}_X \hookrightarrow \underline{\mathbb{C}}_X$.
- $\beta_1''\colon (Rj_*\underline{\mathbb{C}}_X, \tau) \longrightarrow (j_*\mathcal{E}_X^\bullet, \tau)$ is the natural quasi-isomorphism.
- $i\colon \mathcal{E}_{\bar{X}}^\bullet(\log D) \hookrightarrow j_*\mathcal{E}_X^\bullet$ is the natural inclusion,
- i' is the identity on the complex $\mathcal{E}_{\bar{X}}^\bullet(\log D)$, which is compatible with the filtrations τ and $\mathcal{W}^\infty$ by the following consideration: By definition, the m-th filtration subcomplexes are given by

$$\begin{aligned} \tau_m\mathcal{E}_{\bar{X}}^\bullet(\log D) &= \left\{ \ \ldots \to \mathcal{E}_{\bar{X}}^{m-1}(\log D) \to \ker(d) \to 0 \qquad \right\} \\ \mathcal{W}_m^\infty\mathcal{E}_{\bar{X}}^\bullet(\log D) &= \left\{ \ \ldots \to \mathcal{E}_{\bar{X}}^{m-1}(\log D) \to \mathcal{E}_{\bar{X}}^{m}(\log D) \to \mathcal{E}_{\bar{X}}^{m}(\log D) \otimes \mathcal{E}_{\bar{X}}^{1} \to \ \right\}, \end{aligned}$$

  so there is a natural inclusion $\tau_m\mathcal{E}_{\bar{X}}^\bullet(\log D) \hookrightarrow \mathcal{W}_m^\infty\mathcal{E}_{\bar{X}}^\bullet(\log D)$; it is easy to see that i' is a filtered quasi-isomorphism (see Lemma 4.9 in [PS08]).
- The top complex $({}'\mathcal{E}_{\bar{X}}^\bullet(\log D), W)$ is given by

$$\operatorname{Cone}\Big(i' - i\colon (\mathcal{E}_{\bar{X}}^\bullet(\log D), \tau) \longrightarrow (j_*\mathcal{E}_X^{\bullet\sharp}, \tau) \oplus (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty)\Big).$$

  Take β_1''', β_2''' to be the induced morphisms. Since both i and i' are filtered quasi-isomorphisms, there is a commutative square of quasi-isomorphisms

$$\begin{tikzcd} & ({}'\mathcal{E}_{\bar{X}}^\bullet(\log D), W) & \\ (j_*\mathcal{E}_X^\bullet, \tau) \arrow[ur, "\beta_1'''", "\sim"'] & & (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty). \arrow[ul, "\beta_2"', "\sim"] \\ & (\mathcal{E}_{\bar{X}}^\bullet(\log D), \tau) \arrow[ul, "i", "\sim"'] \arrow[ur, "i'"', "\sim"] & \end{tikzcd}$$

- Then put $\beta_1 := \beta_1''' \circ \beta_1'' \circ \beta_1'$. Since $\beta_1 \otimes \mathrm{id}_{\mathbb{C}}$, β_1'' and β_1''' are quasi-isomorphisms, so is $\beta_1 \otimes \mathrm{id}_{\mathbb{C}}$.

As in section I.1.2.2c), this induces the structure of a mixed Hodge complex on the reduced bar complex, and thus a Hodge structure on hypercohomology as follows:

Proposition 5.1.2. *[HZ87, (3.21), p.93]*

a.) The filtrations $\mathcal{W}^\infty * \mathfrak{B}$ and F of

$$\bar{B}_n(E_{\bar{X}}^\bullet(\log D))_{x,y}$$

as in lemma I.1.1.4 and section I.1.2.2 induce a $\mathfrak{B}$-filtered mixed $\mathbb{Q}$-Hodge structure on $H^i(\bar{B}_n(E_{\bar{X}}^\bullet(\log D))_{x,y})$ for all i.

b.) In case $i = 0$, these define a mixed $\mathbb{Q}$-Hodge structure on $H_0(P_{x,y}, \mathbb{C})/J^{n+1}$; in fact, we have a compatible system of mixed Hodge structures for all J^l/J^k with $l < k \in \mathbb{N}$.

c.) As a consequence, the universal pro-unipotent completion $\mathbb{C}\pi_1(X, x_0)^\wedge$ *of the fundamental group carries a mixed Hodge structure induced by the natural isomorphism*

$$\mathbb{C}\pi_1(X, x_0)^\wedge \cong H_0(\bar{B}(E^\bullet_{\bar{X}}(\log D))_{x,y}).$$

Now that we have a Hodge structure on the pro-unipotent completion of the fundamental group, we can proceed to see how it induces a classification of unipotent VMHS:

5.1.2 Classification of unipotent VMHS

Recall the classical theorem relating representations and local systems:

Theorem 5.1.3. *([Voi03, Corollary 3.10, p. 71]) Let* R *be a ring and* V *an* R*-module. If* X *is arcwise connected and locally simply connected and* x_0 *is a point of* X*, we have a natural bijection*

$$\left\{\begin{array}{l}\textit{isomorphism classes}\\ \textit{of local systems } \mathbb{V} \textit{ on } X\\ \textit{of stalk } V \textit{ at } x_0\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{conjugacy classes}\\ \textit{of representations}\\ \pi_1(X, x_0) \to \operatorname{Aut}_R V\end{array}\right\}$$

which is given by sending a local system to its monodromy representation (see definition C.3.11 in the appendix).

The classification theorem of unipotent VMHS by Hain and Zucker in [HZ87] is a generalization of this classical result:

Theorem 5.1.4. *[HZ87, (1.5),(1.6), pp.84/85] Let* k *be a subfield of* $\mathbb{C}$*,* X *be a smooth complex variety, and* $\bar{X}$ *a good compactification of* X*,* $D := \bar{X} - X$ *the corresponding normal crossing divisor. Fix any point* $x_0 \in X$*. Then the monodromy representation functor defines an equivalence of categories*

$$\left\{\begin{array}{l}\textit{unipotent VMHS}\\ \textit{satisfying condition } (\infty)\\ \textit{with index of unipotency} \leq n\\ \textit{defined over } k\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{mixed Hodge theoretic}\\ \textit{representations of}\\ k\pi_1(X, x_0)/J^{n+1}\\ \textit{defined over } k.\end{array}\right\}$$

Here, condition (∞) *on a variation of mixed Hodge structure* $\mathcal{V}$ *is the following list of properties:*

(i) The Hodge filtration bundles $\mathcal{F}^p$ *extend over* $\bar{X}$ *to sub-bundles* $\tilde{\mathcal{F}}^p$ *of the canonical extension* $\tilde{\mathcal{V}}$ *of* $\mathcal{V}$*, such that they induce the corresponding canonical extension over* $\bar{X}$ *for each pure subquotient* $\operatorname{gr}^W_k \mathbb{V}$*.*

(ii) For the nilpotent logarithm N_j *of a local monodromy transformation about a component* D_j *of* D*, the weight filtration* W *of* $\mathcal{V}$ *satisfies* $NW_k \subset W_{k-2}$*.*

Remark 5.1.5. The conditions (∞) are satisfied by all admissible variations of mixed Hodge structure by the very definition of admissibility (see [PS08, 14.49, p.363]). In fact, admissibility is *equivalent* to the conditions (∞) plus graded-polarizability, so we need not go into detail here.

For any $n \in \mathbb{N}$, Hain-Zucker's correspondence implies the existence of a universal n-unipotent VMHS with stalk $k\pi_1(X, x_0)/J^{n+1}$: Note that since multiplication in $k\pi_1(X, x_0)$ is a morphism of mixed Hodge structures and descends to a morphism of mixed Hodge structures on $k\pi_1(X, x_0)/J^{n+1}$, the left regular k-linear representation

$$\rho_l \colon k\pi_1(X, x_0)/J^{n+1} \longrightarrow \mathrm{End}_k(k\pi_1(X, x_0)/J^{n+1}), \quad \gamma \longmapsto (a \longmapsto \gamma a)$$

is in fact a Hodge theoretic representation of $k\pi_1(X, x_0)/J^{n+1}$.

Definition 5.1.6. *For any $n \in \mathbb{N}$, the variation of mixed Hodge structure corresponding to the left regular k-linear represenation of $k\pi_1(X, x_0)/J^{n+1}$ is called the universal n-unipotent VMHS on X with base-point x_0 and denoted by $\mathcal{G}^{(n)}$. It satisfies $(*)$.*

Likewise, we define:

Definition 5.1.7. *Let x_0 be a base-point of X and k a field. Denote the augmentation ideal in $k\pi_1(X, x_0)$ by J. Let $n \in \mathbb{N}$ be greater than 1.*

a.) The local system on X corresponding to the k-linear representation

$$\rho_c \colon k\pi_1(X, x_0)^\wedge \longrightarrow \mathrm{End}_k(k\pi_1(X, x_0)/J^{n+1}), \quad \gamma \longmapsto \left(a \longmapsto \gamma a \gamma^{-1}\right).$$

is denoted by $\mathcal{T}^{(n)}$ and called the n-th tautological local system on X.

b.) By abuse of notation, we will also call the augmentation ideal of $k\pi_1(X^2, (x_0, x_0))$ by J. The VMHS on $X \times X$ corresponding to the representation

$$\rho_l \colon \pi_1(X, x_0)^2 \longrightarrow \mathrm{End}_k(k\pi_1(X, x_0)^2/J^{n+1})$$

induced by the left regular k-linear representation of $\pi(X^2, (x_0, x_0)) \cong \pi_1(X, x_0)^2$ is called the n-th canonical local system and denoted by $\mathcal{C}^{(n)}$.

Remark 5.1.8. The variations of mixed Hodge structure defined above can be described in a very simple fashion:

a.) Let J_x denote the parallel transport of the augmentation ideal $J \subset H_0(P_{x_0,x_0}, k)$ to x.

Claim: *The local system underlying $\mathcal{G}^{(n)}$ (resp. $\mathcal{G}$) is $H_0(P_{x_0,x}, k)/J_x^{n+1}$ (respectively $H_0(P_{x_0,x}, k)$) with $x \in X$ varying.*

Proof: Via the isomorphism $k\pi_1(X, x_0) \cong H_0(P_{x_0,x_0}, k)$, the left regular representation of $\pi_1(X, x_0)$ corresponds to the morphism

$$\rho_l \colon k\pi_1(X, x_0) \longrightarrow \mathrm{End}_k(H_0(P_{x_0,x_0}, k))$$

induced by left-multiplication of paths in P_{x_0,x_0}. This shows that the parallel transport of the fiber $H_0(P_{x_0,x_0}, k)$ at x_0 to the point $x \in X$ along a path $\gamma \in P_{x,x_0}$ is induced by the composition of paths $P_{x,x_0} \otimes P_{x_0,x_0} \longrightarrow P_{x,x_0}$, and hence the fiber of the local system underlying $\mathcal{G}$ at x is simply given by $H_0(P_{x,x_0}, k)$, with x varying, or equivalently (left-regular and right regular representation are isomorphic) by $H_0(P_{x_0,x}, k)$. With the same reasoning, we obtain that the local system underlying $\mathcal{G}^{(n)}$ is isomorphic to the local system $H_0(P_{x_0,x}, k)/J^{n+1}$ ($\cong H_0(P_{x,x_0}, k)/J^{n+1}$) with x varying.

b.) **Claim:** *The local system underlying $\mathcal{T}$ is $H_0(P_{x,x}, k)$ with $x \in X$ varying.*

Proof: Since $\mathcal{T}^{(n)}$ corresponds to the conjugation representation, the parallel transport of the fiber at x_0 to a point $x \in X$ is induced by the conjugation

$$P_{x,x_0} \otimes P_{x_0,x_0} \longrightarrow P_{x,x}, \quad \gamma \otimes \sigma \longmapsto \gamma\sigma\gamma^{-1}$$

and hence the local system underlying the n-th tautological VMHS $\mathcal{T}^{(n)}$ is isomorphic to the local system $H_0(P_{x,x}, k)$ with varying $x \in X$.

c.) **Claim:** *The local system underlying $\mathcal{C}$ is $H_0(P_{x,y}, k)$ with $(x, y) \in X \times Y$ varying.*

Proof: Similar to a).

Remark 5.1.9. Both the tautological and the universal n-unipotent local systems can be described in terms of the canonical one:

$$\mathcal{G}^{(n)} \cong H_0(P_{x_0,x})/J^{n+1} = (x_0 \times \mathrm{id}_X)^*\mathcal{C}^{(n)}$$
$$\mathcal{T}^{(n)} \cong H_0(P_{x,x})/J^{n+1} = \Delta^*\mathcal{C}^{(n)}, \quad \text{where } \Delta\colon X \hookrightarrow X \times X \text{ is the diagonal.}$$

The reason why $\mathcal{G}^{(n)}$ is called the "universal" n-unipotent sheaf is the following:

Corollary 5.1.10. *Let F be a unipotent variation of mixed Hodge structure over k satisfying (∞) with index of unipotency $\leq n$. Then there is a natural isomorphism of k-vector spaces*

$$\mathrm{Hom}_{\mathrm{VMHS}_k(X)}(\mathcal{G}^{(n)}, \mathcal{F}) \cong \mathrm{Hom}_{k\pi(X,x_0)/J^{n+1}}(k\pi(X, x_0)/J^{n+1}, \mathcal{F}_{x_0}) \cong \mathcal{F}_{x_0}.$$

Proof. This holds by the very definition of $\mathcal{G}^{(n)}$ as being associated to the left-regular representation of $k\pi_1(X, x_0/J^{n+1})$ via Hain-Zucker's correspondence 5.1.4 above. ■

5.1.3 Explicit construction of the universal VMHS via bar complexes

The universal pro-unipotent VMHS will be our role model for the motivic logarithm. Hence, its construction is crucial for the understanding of the upcoming definitions in Chapter II.6. The important point to note here is that it will be constructed via the classical bar complex of Chapter I.5 we carried over to the motivic setting. Let us now take a close look at Hain and Zucker's constructions:

Since both the universal pro-unipotent and the tautological VMHS are just pull-backs of the canonical one, it suffices to construct the latter explicitly via bar complexes to obtain a corresponding description of the other two.

Let $p_{23}\colon X^3 \longrightarrow X^2$ denote the projection onto the last two factors. It has two obvious cross-sections $\sigma_0, \sigma_1\colon X^2 \longrightarrow X^3$ given by $\sigma_0(x, y) = (x, x, y), \sigma_1(x, y) = (y, x, y)$, which impose the structure of a right and left $\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3|X^2}$-module on the space of C^∞-functions on X^2, $\mathcal{E}^0_{X^2}$, by pull-back of forms to X^2 via σ_0 and σ_1, respectively. Hain and Zucker consider a sheaf we already encountered in section I.1.4:

$$\bar{B}_n\left(X^3|X^2\right)_{\sigma_0,\sigma_1} = \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1}$$

in which tensor products are taken over $\mathcal{E}^\bullet_{X^2}$. In Lemma 1.4.6 we saw that there is a quasi-isomorphism

$$\bar{B}_n(X^3|X^2)_{\sigma_0,\sigma_1} \simeq \widetilde{B}_n(X^3|X^2)_{\sigma_0,\sigma_1}$$

with the normalized complex associated to $sB_n(X^3|X^2)_{\sigma_0,\sigma_1}$.
The complexes $\mathcal{E}^\bullet_X, \mathcal{E}^\bullet_{X^2}$ and $\mathcal{E}^\bullet_{X^3}$ admit canonical filtrations as follows:

Definition 5.1.11. *For any morphism of smooth complex varieties $\pi\colon X \longrightarrow S$, we define the filtration $L^\bullet_{X|S}$ to be the decreasing filtration of $\mathcal{E}^\bullet_X$ induced by truncation in $\mathcal{E}^\bullet_S$ by putting*

$$L^k_{X|S}\mathcal{E}^\bullet_X := \mathrm{Im}\left(\mathcal{E}^{\bullet-k}_X \otimes_{\mathcal{E}^0_X} \pi^*\mathcal{E}^k_S \longrightarrow \mathcal{E}^\bullet_X\right).$$

This filtration will be of importance in Part II, so we gather some crucial facts:

Remark 5.1.12.

a.) *Gradeds:* The graded objects of the above filtration are given by

$$\mathrm{gr}^k_{L_{X|S}}\,\mathcal{E}^\bullet_X = \mathcal{E}^{\bullet-k}_{X|S} \otimes_{\mathcal{E}^0_X} \pi^*\mathcal{E}^k_S.$$

b.) *Base-change property:* Let $i_Z\colon Z \hookrightarrow S$ be the a complex subvariety of S, put $Y := i^{-1}(Z) = X \times_S Z$ with inclusion $i_Y\colon Y \hookrightarrow X$. Call the induced structure morphism by $\pi_Y\colon Y \longrightarrow Z$. Then one has

$$i^*_Z\pi_* L^k_{X|S}\mathcal{E}^\bullet_X \cong L^k_{Y|Z}(\pi_Y)_*\mathcal{E}^\bullet_{Y|Z} \text{ and consequently}$$
$$i^*_Z\pi_*\,\mathrm{gr}^k_{L_{X|S}}\,\mathcal{E}^\bullet_X \cong \mathrm{gr}^k_{L_{Y|Z}}(\pi_Y)_*\mathcal{E}^\bullet_{Y|Z}$$

for the following reason: By flat base-change for the diagram

$$\begin{tikzcd} Y = X \times_S Z \arrow[r, "\pi_Y"] \arrow[d, "i_Y"'] & Z \arrow[d, "i_Z"] \\ X \arrow[r, "\pi"] & S \end{tikzcd}$$

we have $i^*_Z\pi_* \cong \pi_{Y*}i^*_Y$, and hence

$$\begin{aligned} i^*_Z\pi_* L^k_{X|S}\mathcal{E}^\bullet_X &\cong \pi_{Y*}i^*_Y L^k_{X|S}\mathcal{E}^\bullet_X = \pi_{Y*}i^*_Y\,\mathrm{Im}\left(\mathcal{E}^{\bullet-k}_X \otimes_{\mathcal{E}^0_X} \pi^*\mathcal{E}^k_S \longrightarrow \mathcal{E}^\bullet_X\right) \\ &= \pi_{Y*}\,\mathrm{Im}\left(i^*_Y\mathcal{E}^{\bullet-k}_X \otimes_{i^*_Y\mathcal{E}^0_X} i^*_Y\pi^*\mathcal{E}^k_S \longrightarrow i^*_Y\mathcal{E}^\bullet_X\right) \\ &= \pi_{Y*}\,\mathrm{Im}\left(\mathcal{E}^{\bullet-k}_Y \otimes_{\mathcal{E}^0_Y} \pi^*_Y\mathcal{E}^k_S \longrightarrow \mathcal{E}^\bullet_Y\right) \\ &= L^k_{Y|Z}(\pi_Y)_*\mathcal{E}^\bullet_{Y|Z}. \end{aligned}$$

By abuse of notation, we also denote the decreasing filtration of $\widetilde{B}_n(X^3|X^2)_{\sigma_0,\sigma_1}$ induced by the filtration $L_{X^3|X^2}$ of $\mathcal{E}^\bullet_{X^3}$ by $L_{X^3|X^2}$. There is an isomorphism

$$\mathrm{gr}^p_{L_{X^3|X^2}}[\otimes^n(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3})] \cong \mathcal{E}^p_{X^2} \otimes_{\mathcal{E}^0_{X^2}} [\otimes^n(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3|X^2})],$$

which gives rise to an isomorphism

$$\mathrm{gr}^p_{L_{X^3|X^2}}\,\bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1} \cong \mathcal{E}^p_{X^2} \otimes_{\mathcal{E}^0_{X^2}} \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3|X^2}|\mathcal{E}^0_{X^2}\right)_{\sigma_0,\sigma_1}[-p]$$

of sheaves. This isomorphism is compatible with the differential, i.e. one of sheaves of differential graded algebras. Note that

$$\mathrm{gr}^0_{L_{X^3|X^2}}\,\bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1} \cong \mathcal{E}^0_{X^2} \otimes_{\mathcal{E}^0_{X^2}} \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3|X^2}|\mathcal{E}^0_{X^2}\right)_{\sigma_0,\sigma_1}$$

is endowed with a natural $\mathcal{E}^0_{X^2}$-connection. Moreover, the following filtrations impart a natural Hodge structure on $\bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^0_{X^2}\right)_{\sigma_0,\sigma_1}$ ([HZ87, (4.20), p.98]):

- *Weight filtration:* Recall the canonical filtration $\mathcal{W}^\infty$ (see Definition I.C.3.5) of

  $$\mathcal{E}^\bullet_{\bar{X}^3}(\log D^3) \simeq \mathcal{E}^\bullet_{X^3}$$

  by type of logarithmic singularities, which is already defined over $\mathbb{Q}$ (the canonical filtration of $Rj_*\underline{\mathbb{Q}}^\sharp_X$). By section 4 (Property e.)), this filtration together with the bar filtration $\mathfrak{B}$ induces a filtration $\mathcal{W}^\infty * \mathfrak{B}$ (already defined over $\mathbb{Q}$) on both

  $$\bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1} \quad \text{and} \quad \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}_{\bar{X}^3|\bar{X}^2}(\log D)^\bullet|\mathcal{E}^0_{X^2}\right)_{\sigma_0,\sigma_1}.$$

  This filtrations induces flat subbundles of the local system

  $$\mathcal{H}^0\bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}_{\bar{X}^3|\bar{X}^2}(\log D)^\bullet|\mathcal{E}^0_{X^2}\right)_{\sigma_0,\sigma_1} = \mathcal{H}^0\,\mathrm{gr}^0_{L_{X^3|X^2}}\,\bar{B}_n\left(X^3|X^2\right)_{\sigma_0,\sigma_1}.$$

- *Hodge filtration:* The natural Hodge filtrations σ_∞ (see Definition I.C.3.5) of $\mathcal{E}^\bullet_{X^2}(\log D^2)$ and $\mathcal{E}^\bullet_{X^3|X^2}$ induce a Hodge filtration on

  $$\mathcal{E}^\bullet_{X^2} \otimes \left(\otimes^\bullet\,\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3|X^2}\left(\log D^2\right)\right).$$

  This induces a filtration on the graded sheaf $\mathrm{gr}^\bullet_L\,\bar{B}_n\left(X^3|X^2\right)_{\sigma_0,\sigma_1}$ and its cohomology groups. Hain and Zucker then show in [HZ87, (4.17), p.97] that this C^∞-filtration satisfies the horizontality condition of Lemma C.3.1, such that these structures comprise the C^∞-data of a variation of mixed Hodge structure as outlined in C.3.2 in the appendix.

Putting everything together, Hain and Zucker obtain:

Corollary 5.1.13.

a.) The n-th canonical VMHS $\mathcal{C}^{(n)}$ on $X \times X$ with underlying local system $H_0(P_{x,y}, \mathbb{Q})$ is dual to the VMHS given by the following C^∞-data (see C.3.2):

- *the C^∞-vector bundle*

$$\mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n\left(X^3|X^2\right)_{\sigma_0,\sigma_1}\right) = \mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1}\right),$$

  where $L_{X^3|X^2}$ is the decreasing filtration induced by truncation in $\mathcal{E}^\bullet_{X^2}$, together with the natural $\mathcal{E}^0_{X^2}$-connection.

- *the Hodge structure induced by the natural Hodge and weight filtrations on $\mathcal{E}^\bullet_{X^3} \simeq \mathcal{E}^\bullet_{\bar{X}^3}(\log \bar{X}^3 \setminus X^3)$.*

b.) The universal n-unipotent VMHS $\mathcal{G}^{(n)}$ on X with underlying local system determined by $H_0(P_{x_0,x}, \mathbb{Q})$ is dual to the VMHS given by the following C^∞-data:

- *the C^∞-vector bundle*

$$\mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n\left(X^2|X\right)_{x_0\times\mathrm{id},\widetilde{\Delta}}\right) = \mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^2|X}} \bar{B}_n\left(\mathrm{pr}_{2*}\,\mathcal{E}^\bullet_{X^2}|\mathcal{E}_X\right)_{x_0\times\mathrm{id},\widetilde{\Delta}}\right),$$

  where $L_{X^2|X}$ is the decreasing filtration induced by truncation in $\mathcal{E}^\bullet_X$, together with the natural $\mathcal{E}^0_X$-connection.

- *the Hodge structure induced by the natural Hodge and weight filtrations on $\mathcal{E}^\bullet_{X^2} \simeq \mathcal{E}^\bullet_{\bar{X}^2}(\log \bar{X}^2 \setminus X^2)$.*

c.) The universal n-th tautological VMHS $\mathcal{T}^{(n)}$ on X with underlying local system determined by $H_0(P_{x,x}, \mathbb{Q})$ is dual to the VMHS given by the following C^∞-data:

- *the C^∞-vector bundle*

$$\mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n\left(X^2|X\right)_{\widetilde{\Delta},\widetilde{\Delta}}\right) = \mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^2|X}} \bar{B}_n\left(\mathrm{pr}_{2*}\,\mathcal{E}^\bullet_{X^2}|\mathcal{E}_X\right)_{\widetilde{\Delta},\widetilde{\Delta}}\right),$$

  where $L_{X^2|X}$ is the decreasing filtration induced by truncation in $\mathcal{E}^\bullet_X$, together with the natural $\mathcal{E}^0_X$-connection.

- *the Hodge structure induced by the natural Hodge and weight filtrations on $\mathcal{E}^\bullet_{X^2} \simeq \mathcal{E}^\bullet_{\bar{X}^2}(\log \bar{X}^2 \setminus X^2)$.*

Here, $\mathrm{pr}_2 \colon X^2 \longrightarrow X$ is the projection to the second component.

Proof. The statement about the canonical VMHS is a direct consequence of the above considerations. Using this, the statements about $\mathcal{T}^{(n)}$ and $\mathcal{G}^{(n)}$ are due to the following:

$$\begin{aligned}
\mathcal{G}^{(n)} &= (x_0\times\mathrm{id})^*\mathcal{C}^{(n)} = (x_0\times\mathrm{id})^*\mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1}\right)\\
&= \mathcal{H}^0\left((x_0\times\mathrm{id})^*\,\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1}\right)\\
&\overset{(1)}{\cong} \mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^2|X}}(x_0\times\mathrm{id})^*\bar{B}_n\left(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|\mathcal{E}^\bullet_{X^2}\right)_{\sigma_0,\sigma_1}\right)\\
&\overset{(2)}{\cong} \mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^2|X}} \bar{B}_n\left((x_0\times\mathrm{id})^*\,\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}|(x_0\times\mathrm{id})^*\mathcal{E}^\bullet_{X^2}\right)_{(x_0\times\mathrm{id})^*(\sigma_0),(x_0\times\mathrm{id})^*(\sigma_1)}\right)\\
&\overset{(3)}{\cong} \mathcal{H}^0\left(\mathrm{gr}^0_{L_{X^2|X}} \bar{B}_n\left(\mathrm{pr}_{2*}\,\mathcal{E}^\bullet_{X^2}|\mathcal{E}^\bullet_X\right)_{x_0\times\mathrm{id},\widetilde{\Delta}}\right),
\end{aligned}$$

where the isomorphism (1) is due to the base-change property of the filtration $L_{(-)}$ by Remark I.5.1.12 b), the isomorphism (2) is an immediate consequence of the definition of the bar complex, and (3) can be seen as follows:

- Due to the commuting square
$$\begin{tikzcd} X^3 \arrow[r, "\mathrm{pr}_{23}"] & X^2 \arrow[d, "x_0\times \mathrm{id}_X"] \\ X^2 \arrow[u, "\mathrm{id}_X \times x_0 \times \mathrm{id}_X"] \arrow[r, "\mathrm{pr}_2"] & X \arrow[u] \end{tikzcd}$$
we have $(x_0 \times \mathrm{id}_X)^* \, \mathrm{pr}_{23*} \, \mathcal{E}_{X^3} \simeq \mathrm{pr}_{2*}(\mathrm{id}_X \times x_0 \times \mathrm{id}_X)^* \mathcal{E}_{X^3} \cong \mathrm{pr}_2 * \mathcal{E}_{X^2}$,
- $(x_0 \times \mathrm{id})^* \mathcal{E}_{X^2} \cong \mathcal{E}_X$,
- $(x_0 \times \mathrm{id})^*(\sigma_0)(x) = \sigma_0(x_0, x) = (x_0, x_0, x)$ which is mapped to (x_0, x) by the canonical isomorphism $X \times \{x_0\} \times X \cong X^2$, so the section $(x_0 \times \mathrm{id})^*(\sigma_0)$ corresponds to $x_0 \times \mathrm{id}$ on X, and
- $(x_0 \times \mathrm{id})^*(\sigma_1)(x) = \sigma_1(x_0, x) = (x, x_0, x)$ which is mapped to (x, x) by the canonical isomorphism $X \times \{x_0\} \times X \cong X^2$, so the section $(x_0 \times \mathrm{id})^*(\sigma_1)$ corresponds to the diagonal $\widetilde{\Delta}\colon X \longrightarrow X^2$.

The assertion for the tautological VMHS follows exactly the same way, simply exchanging the term $(x_0 \times \mathrm{id})$ by $\widetilde{\Delta}$ in the above. ■

Remark 5.1.14. The above corollary might look rather messy - however, the main point to note here (and remember later on) is the following:

- $\mathcal{C}^{(n)}$ is determined by the bar complex$\bar{B}_n \left(X^3|X^2\right)_{\sigma_0,\sigma_1}$,
- $\mathcal{G}^{(n)}$ is determined by the bar complex $\bar{B}_n \left(X^2|X\right)_{x_0\times \mathrm{id},\widetilde{\Delta}}$, and
- $\mathcal{T}^{(n)}$ is determined by the bar complex $\bar{B}_n \left(X^2|X\right)_{\widetilde{\Delta},\widetilde{\Delta}}$,

and we already have motivic analogues for all of these bar complexes due to Part I.

5.2 The classical Hodge logarithm

As we pointed out in the introduction of this chapter, the classical Hodge logarithm is closely connected to the universal pro-unipotent VMHS. We will shortly see why:

5.2.1 Beilinson-Levin's logarithm for curves

Beilinson and Levin defined the logarithm for curves in their unpublished preprint [BL] as follows:

Let k be a field, $\bar{\pi}\colon \bar{X} \longrightarrow \mathrm{Spec}(\mathbb{C})$ be a projective complex curve of genus $\neq 0$, and $j\colon X \hookrightarrow \bar{X}$ an open immersion. Then the complement $D := \bar{X} \setminus X$ is a normal crossing divisor. Moreover, we fix a point $x_0 \in X$. In this case, the mixed Hodge module $\mathscr{H} := R^1\pi_! k_X(1) = H^1_c(X,k)$ satisfies $\mathscr{H} = H^1_c(X,k)(1) \cong H_1(X,k)$ via Poincaré duality. Moreover, Poincaré duality yields a map

$$c\colon k_X(1) \longrightarrow \mathscr{H} \otimes \mathscr{H}.$$

Beilinson and Levin then denote the tensor algebra of $\mathscr{H}$ by $\mathscr{T}^\bullet(\mathscr{H})$, consider the two-sided ideal $\mathfrak{c} := \mathscr{T}^\bullet(\mathscr{H})c\mathscr{T}^\bullet(\mathscr{H})$ and define the graded algebra

$$\mathscr{R}^\bullet := \mathscr{T}^\bullet(\mathscr{H})/\mathfrak{c}.$$

By theorem 1.4 in [Har04], the graded algebra $\mathscr{R}$ is isomorphic to the pro-unipotent completion $k\pi_1(X, x_0)^\wedge$ of the fundamental group as described in section 5.1.1.

By Beilinson and Levin' un-published preprint [BL], there exists a unique pair $(\mathbb{G}, \flat)$ of a pro-unipotent mixed sheaf $\mathbb{G} = \varprojlim_n \mathbb{G}/\mathbb{G}^n$ and an element $\flat \in x_0^*\mathbb{G} = \mathbb{G}_{x_0} = \mathscr{R}$ satisfying the following universal property: for any n-unipotent sheaf $\mathscr{F}$, the map

$$\pi_* \underline{\mathrm{Hom}}_X(\mathbb{G}/\mathbb{G}^n, \mathscr{F}) \longrightarrow \mathscr{F}_{x_0}, \quad \varphi \longmapsto \varphi(\flat)$$

is an isomorphism. By the universal property characterizing $\mathbb{G}/\mathbb{G}^n$, it is easy to see that it coincides with Hain-Zucker's universal n-unipotent sheaf:

Corollary 5.2.1. *There is a natural isomorphism*

$$(\mathbb{G}/\mathbb{G}^n, \flat) \cong (\mathcal{G}^{(n)}, 1 \in k\pi_1(X, x_0)^\wedge).$$

Proof. Note that $(\mathcal{G}^{(n)}, 1 \in k\pi_1(X, x_0)^\wedge)$ is n-unipotent and satisfies the defining property of $\mathbb{G}/\mathbb{G}^n$. By universality, the result is immediate. ■

5.2.2 A new view on the classical logarithm

The upcoming paragraph might seem a bit vague, but it should explain Faltings' (and our) motivic view on the logarithm that is to come up in the next chapter. We will deliver a plea for a new point of view on the logarithm: The polylogarithm is generally supposed - and in many of the classically considered cases: known - to be of motivic origin. That suggests that the logarithm should also be of motivic origin. This, however, can only be realized (as we will see) for an even more general object: the bar complex $\bar{B}(X^2|X)_{x_0\times \mathrm{id}, \widetilde{\Delta}}$ underlying $\mathcal{G}$. This is why in his paper [Fal12] Faltings calls the motivic generalization of $\bar{B}(X^2|X)_{x_0\times \mathrm{id}, \widetilde{\Delta}}$ for curves X the "motivic logarithm". We will see this in more detail at the outset of chapter II.6.

5.3 The classical polylogarithm for curves

We will now proceed to recall Beilinson and Levin's Hodge polylogarithm for curves of genus ≥ 1 as constructed in their unpublished preprint [BL]. We only deal with the case when the base-scheme is a point, though.

Let $\bar{\pi}\colon \bar{X} \longrightarrow \mathrm{Spec}(\mathbb{C})$ be an irreducible projective complex curve of genus $\neq 0$, and $j\colon X \hookrightarrow \bar{X}$ an open immersion. We denote the complement by $D := \bar{X} \setminus X$. Moreover, we fix a point $x_0 \in X$.

Recall Beilinson-Levin's logarithm $\mathbb{G}$ as introduced in section 5.2.1 above. By the universal property characterizing $\mathbb{G}/\mathbb{G}^n$, we deduced in 5.2.1 that it coincides with Hain-Zucker's universal n-unipotent sheaf $\mathcal{G}^{(n)}$. Then Beilinson and Levin consider the augmentation ideal $J := \ker(x_0^*\mathbb{G} \longrightarrow k) = \ker(k\pi_1(X, x_0)^{\wedge} \longrightarrow k)$ and prove the following

Theorem 5.3.1. *Let $\mathscr{F}$ be a mixed sheaf on S. Then with the above notations the residue map induces an isomorphism*

$$\mathrm{Ext}^1_{\bar{X}\setminus\{x_0\}}(\bar{\pi}^*\mathscr{F}, j_!\mathbb{G}(1)) \cong \mathrm{Hom}_S(\mathscr{F}, J).$$

With this, Beilinson and Levin define the (large) polylogarithm for curves as follows:

Definition 5.3.2. *The <u>(large) polylogarithm</u> is the extension class*

$$\mathscr{P}ol \in \mathrm{Ext}^1_{\bar{X}\setminus\{x_0\}}(\bar{\pi}^* J, j_!\mathbb{G}(1))$$

which corresponds to the identity map in $\mathrm{Hom}(J, J)$ *under the isomorphism of the above theorem.*

Remark 5.3.3. To be precise, what is *actually* shown in the proof of the above Theorem 5.3.1 is the following:
Denote the inclusion of the complement of $\{x_0\}$ by $h\colon \bar{X} \setminus \{x_0\} \hookrightarrow \bar{X}$. Recall that the inclusion of X into $\bar{X}$ was called j.We call the composition $j \circ x_0$ by $\bar{x}_0$. Then there is a distinguished triangle of functors $\bar{x}_{0!}\bar{x}_0^! \longrightarrow \mathrm{id} \longrightarrow h_*h^*$. Applying it to $j_!\mathbb{G}(1)$ and using the fact that $\bar{x}_{0!} \simeq \bar{x}_{0*}$ as well as the purity isomorphism $\bar{x}_0^! \simeq \bar{x}_0^*(-1)[-2]$, one obtains the distinguished triangle

$$\bar{x}_{0*}\bar{x}_0^* j_!\mathbb{G}[-2] \longrightarrow j_!\mathbb{G}(1) \longrightarrow h_*h^* j_!\mathbb{G}(1)$$

in $D^b\,\mathrm{Sh}(\bar{X})$. For any sheaf $\mathscr{F}$, the long exact sequence associated to

$$\mathrm{Hom}_{D^b\,\mathrm{Sh}(\bar{X})}(\bar{\pi}^*\mathscr{F},\ -\ ) \ \text{ yields}$$

$$\begin{aligned}
&\mathrm{Ext}^1_{D^b\,\mathrm{Sh}(\bar{X})}(\bar{\pi}^*\mathscr{F}, j_!\mathbb{G}(1)) \longrightarrow \mathrm{Ext}^1_{D^b\,\mathrm{Sh}(\bar{X}\setminus\{x_0\})}(h^*\bar{\pi}^*\mathscr{F}, h^*j_!\mathbb{G}(1)) \longrightarrow \\
&\longrightarrow \mathrm{Hom}_{D^b\,\mathrm{Sh}(\bar{X})}(\bar{\pi}^*\mathscr{F}, \bar{x}_{0*}\bar{x}_0^* j_!\mathbb{G}) \longrightarrow \mathrm{Ext}^2_{D^b\,\mathrm{Sh}(\bar{X})}(\bar{\pi}^*\mathscr{F}, j_!\mathbb{G}(1)) \longrightarrow \dots
\end{aligned}$$

Beilinson and Levin then show that the first term vanishes, while the last is isomorphic to $\mathrm{Hom}_{D^b\,\mathrm{Sh(pt)}}(\mathscr{F}, k)$, where we write $\mathrm{pt} := \mathrm{Spec}(\mathbb{C})$. Moreover, by adjunction, one has

$$\mathrm{Hom}_{D^b\,\mathrm{Sh}(\bar{X})}(\bar{\pi}^*\mathscr{F}, \bar{x}_{0*}\bar{x}_0^* j_!\mathbb{G}) \cong \mathrm{Hom}_{D^b\,\mathrm{Sh(pt)}}(\mathscr{F}, \bar{\pi}_*\bar{x}_{0*}\bar{x}_0^* j_!\mathbb{G}) = \mathrm{Hom}_{D^b\,\mathrm{Sh(pt)}}(\mathscr{F}, \bar{x}_0^* j_!\mathbb{G}),$$

and since the diagram

$$\begin{tikzcd} X \arrow[r, "j"] & \bar{X} \\ S \arrow[u, "x_0"] \arrow[r, equal] & S \arrow[u, "\bar{x}_0"'] \end{tikzcd}$$

commutes, we obtain by the usual base-change properties that $\bar{x}_0^* j_! \simeq x_0^*$, and hence the latter term is equal to $\operatorname{Hom}_{D^b \operatorname{Sh}(\mathrm{pt})}(\mathscr{F}, x_0^* \mathbb{G})$. So what is actually shown is the the following:

The distinguished triangle of functors $\bar{x}_{0!}\bar{x}_0^! \longrightarrow \mathrm{id} \longrightarrow h_* h^*$ *gives rise to an exact sequence*

$$0 \to \operatorname{Ext}^1_{D^b \operatorname{Sh}(\bar{X}\setminus\{x_0\})}(\bar{\pi}^* J, j_! \mathbb{G}(1)) \overset{\text{res}}{\to} \operatorname{Hom}_{D^b \operatorname{Sh}(\mathrm{pt})}(J, x_0^* \mathbb{G}) \to \operatorname{Hom}_{D^b \operatorname{Sh}(\mathrm{pt})}(J, k).$$

The polylogarithm is the element of $\operatorname{Ext}^1_{D^b \operatorname{Sh}(\bar{X}\setminus\{x_0\})}(\pi^* J, j_! \mathbb{G}(1))$ *which is mapped to the inclusion of* J *into* $x_0^* \mathbb{G}$ *by* res.

Chapter 6

The motivic logarithm and polylogarithm

In the preceding chapter we recalled the classical definitions of the Hodge logarithm and polylogarithm. The logarithm and polylogarithm are generally believed to be of motivic origin, but to the day there is no *general* motivic definition of either. Our aim in this chapter is to change that fact.

Recall that we discussed the term "logarithm" in section 5.2.2. We argued that - rather than the usual classical notion of the "logarithm" - one should consider the universal pro-unipotent VMHS to be more deserving of the name "logarithm". To be precise, one should consider an even more general object: its underlying bar complex. Recall that the universal pro-unipotent VMHS is the zeroth homology group of a certain bar complex, but alas, there is no notion of cohomology in the motivic setting. So would it not be more natural to consider a motivic analogue of the *entire bar complex* as a motivic generalization of the logarithm? The motivic object thus constructed would carry all the information contained in the universal pro-unipotent sheaf.

Faltings' understanding of the logarithm seems to be similar: in his paper "The motivic logarithm" ([Fal12]), Faltings defines the motivic logarithm of curves as a complex that will turn out to coincide with our motivic bar complexes of chapter I.2. Moreover, we will see that the zeroth cohomology of its Hodge realization is indeed the universal pro-unipotent VMHS. We will generalize Faltings' motivic logarithm for curves to a far more general class of schemes, and deal with it in the setting of motivic bar complexes we created in Part I.

After we have achieved this crucial step, we will define the motivic polylogarithm as follows: recall that motivic cohomology classes are, by definition, no more than just morphisms in $\mathcal{DM}(S)$ ($H^p(S,M) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S, M[i])$ for any $M \in \mathcal{DM}(S)$). In order to realize the polylogarithm as a class in motivic cohomology, we need to define a morphism of motives that turns out to yield the correct cohomology classes in realizations.

In the introduction of this thesis, we argued that the polylogarithm should be given by some sort of Gysin morphism associated to a diagonal inclusion. This is merely a vague point to start from - but trial and error have indeed led to a motivic definition of the polylogarithm that satisfies all characterizing properties of the classical polylogarithm, as we will see in the subsequent Chapter II.7. Instead of desperately trying to motivate it, we will simply give an ad hoc definition of the motivic polylogarithm in this chapter, and ask the reader to kindly wait and see. In Chapter II.7, we will go through some trouble to see that this definition actually satisfies the required properties and yields the classical polylogarithms in realizations.

Over and above, we will proceed as follows:

- Firstly, we will recall Faltings' motivic logarithm for curves, to further motivate our definition of the motivic logarithm as a direct generalization of Faltings' logarithm.
- Secondly, we present an ad hoc definition of our motivic logarithm, as well as motivic analogues of Hain/Zucker's canonical and tautological VMHS (see chapter II.5).
- With this, we finally present a candidate for a motivic generalization of the polylogarithm.

So, let us start out with Faltings' logarithm:

6.1 Faltings' motivic logarithm

Let S be an arbitrary base-scheme. Let $\mathcal{K}(\mathrm{Sm}_S)$ denote the psedo-abelian envelope of the category Sm_S of smooth quasi-projective S-schemes. It is an additive category with tensor product given by the fiber product of smooth S-schemes. The category of bounded complexes $C^b(\mathcal{K}(\mathrm{Sm}_S))$ is also an additive category with tensor product given by the usual tensor product of complexes.

In [Fal12], Faltings constructs the motivic logarithm as a pro-object in the category $C^b(\mathcal{K}(\mathrm{Sm}_S))$, which can be related to an ind-object in $\mathcal{DM}(S)$:

Let $\pi\colon X \longrightarrow S$ be a relative smooth curve with irreducible fibers, which is the complement of a normal crossing divisor in a projective variety $\bar{X}$, and is equipped with an S-point $x_0\colon S \longrightarrow X$. This section gives rise to an idempotent in $\mathrm{End}_{\mathrm{Sm}_S}(X)$ by

$$e_{x_0}\colon X \xrightarrow{\pi} S \xrightarrow{x_0} X.$$

Since $\mathrm{id}_X - e_{x_0}$ is also an idempotent, we obtain an element

$$M(X)^\circ := (X, \mathrm{id}_X - e_{x_0}) \in \mathcal{K}(\mathrm{Sm}_S).$$

Faltings then defines a projective system of complexes in $C^b(\mathcal{K}(\mathrm{Sm}_S))$ as follows: The diagonal $\delta\colon X \longrightarrow X \times X$ satisfies $\delta \circ e_{x_0} = (e_{x_0} \otimes e_{x_0}) \circ \delta$, and hence induces a morphism

$$\delta\colon M(X)^\circ \longrightarrow M(X)^\circ \otimes M(X)^\circ.$$

For $i \geq 1$ and all $1 \leq k < i$, the morphism δ hence gives rise to

$$\mathrm{id}^{\otimes k-1} \otimes \delta \otimes \mathrm{id}^{\otimes i-k} \colon M(X)^{\circ \otimes i} \longrightarrow M(X)^{\circ \otimes i+1}$$

doubling the argument in position k. Now Faltings takes the alternating sum of these maps to obtain

$$d_i := \sum_{k=1}^{i-1} (-1)^{k-1} \mathrm{id}^{\otimes k-1} \otimes \delta \otimes \mathrm{id}^{\otimes i-k} \colon M(X)^{\circ \otimes i} \longrightarrow M(X)^{\circ \otimes i+1}, \quad \text{and puts}$$

$$P_n^\bullet := \{M(S) \xrightarrow{d_0} M(X)^\circ \xrightarrow{d_2} M(X)^{\circ \otimes 2} \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} M(X)^{\circ \otimes n}\},$$

where the first map d_0 is is induced by x_0. There are compatible associative products $P_n^\bullet \otimes P_n^\bullet \longrightarrow P_n^\bullet$ induced by the maps

$$\begin{aligned} M(X)^{\circ \otimes a} \otimes M(X)^{\circ \otimes b} &\longrightarrow M(X)^{\circ \otimes (a+b)} \\ (x_1, \dots, x_a) \otimes (x_{a+1}, \dots, x_{a+b}) &\longmapsto (x_1, \dots, x_{a+b}). \end{aligned}$$

Moreover, there are graded cocommutative and coassociative shuffle coproducts $P_{m+n}^\bullet \longrightarrow P_m^\bullet \otimes P_n^\bullet$ induced by

$$(x_1, \dots, x_{a+b}) \longmapsto \sum_\sigma \mathrm{sgn}(\sigma)(x_{\sigma(1)}, \dots, x_{\sigma(a)}) \otimes (x_{\sigma(a+1)}, \dots, x_{\sigma(a+b)}),$$

where the sum is over all permutations of $\{1, \dots, a+b\}$ which are monotone on $\{1, \dots, a\}$ and on $\{a+1, \dots, a+b\}$.

Definition 6.1.1. *Let all notation be as above. Faltings' constant unipotent motive is the right unbounded complex given by the inverse limit $P^\bullet := \varprojlim_n P_n^\bullet$ of the complexes $P_n^\bullet$ defined above.*

Lemma 6.1.2. *[Fal12, 5.3, p.112] Product and coproduct on $P_n^\bullet$ are compatible and endow $P^\bullet$ with the structure of a cocommutative Hopf-algebra.*

One can think of $P^\bullet$ as the motivic analogue of the pro-unipotent completion of the fundamental group. However, it does not describe the universal pro-unipotent sheaf. For the latter, there is a minor change: We consider $X^2 := X \times_S X$ as a scheme over X via the second projection and consider the element

$$M(X^2)^\circ := (X^2, \mathrm{id}_{X^2} - \mathrm{pr}_2^*(x_0 \times_S \mathrm{id}_X)^*) \in \mathcal{K}(\mathrm{Sm}_X).$$

We consider the diagonals

$$\varDelta \colon M(X^2)^\circ \longrightarrow M(X^2)^\circ \otimes_X M(X^2)^\circ \quad \text{and} \quad \widetilde{\varDelta} \colon M(X) \longrightarrow M(X^2)^\circ.$$

Faltings now defines the complex

$$P_n^\bullet(\widetilde{\varDelta}) := \{M(X) \xrightarrow{d_0} M(X^2)^\circ \xrightarrow{d_1(\widetilde{\varDelta})} M(X^2)^{\circ \otimes 2} \xrightarrow{d_2(\widetilde{\varDelta})} \dots \longrightarrow M(X^2)^{\circ \otimes n})\}$$

with differentials

$$d_i(\widetilde{\Delta}) := -\operatorname{id}^{\otimes k} \otimes \widetilde{\Delta} + \sum_{k=1}^{i-1} (-1)^{k-1} \operatorname{id}^{\otimes k-1} \otimes \Delta \otimes \operatorname{id}^{\otimes i-k}.$$

Again there are associative products $P_n^\bullet\left(\widetilde{\Delta}\right) \otimes P_n^\bullet\left(\widetilde{\Delta}\right) \longrightarrow P_n^\bullet\left(\widetilde{\Delta}\right)$ induced by

$$\begin{aligned} M(X^2)^{\circ\otimes a} \otimes M(X^2)^{\circ\otimes b} &\longrightarrow M(X^2)^{\circ\otimes(a+b)} \\ (x_1, \dots, x_a) \otimes (x_{a+1}, \dots, x_{a+b}) &\longmapsto (x_1, \dots, x_{a+b}). \end{aligned}$$

Moreover, there are graded cocommutative and coassociative shuffle coproducts

$$\begin{aligned} P_{m+n}^\bullet(\widetilde{\Delta}) &\longrightarrow P_m^\bullet(\widetilde{\Delta}) \otimes P_n^\bullet(\widetilde{\Delta}) \text{ induced from} \\ (x_1, \dots, x_{a+b}) &\longmapsto \sum_\sigma \operatorname{sgn}(\sigma)(x_{\sigma(1)}, \dots, x_{\sigma(a)}) \otimes (x_{\sigma(a+1)}, \dots, x_{\sigma(a+b)}), \end{aligned}$$

where the sum is over all permutations of $\{1, \dots, a+b\}$ which are monotone on $\{1, \dots, a\}$ and on $\{a+1, \dots, a+b\}$.

Lemma 6.1.3. *[Fal12, 5.3, p.112] Product and coproduct on $P_n^\bullet(\widetilde{\Delta})$ endow $P^\bullet(\widetilde{\Delta})$ with the structure of a cocommutative Hopf-algebra.*

We can connect Faltings' logarithm to our motivic bar complexes in the following way: Note that Faltings' complex $P_n^\bullet(\widetilde{\Delta})$ is the normalized complex of the cosimplicial object $cB^*_{\mathrm{mot}}(X^2|X)_{x_0 \times \mathrm{id}_X, \widetilde{\Delta}}$, and similarly P_n is the normalized complex of the cosimplicial object $cB^*_{\mathrm{mot}}(X|S)_{x_0,x_0}$. In other words:

Corollary 6.1.4. *Translating Faltings' logarithm to Levine's motivic language, we obtain the motivic bar complex*

$$\widetilde{B}_n^{\mathrm{mot}}(X^2|X)_{\widetilde{\Delta}, x_0 \times \mathrm{id}_X} \simeq \widetilde{B}_n^{\mathrm{mot}}(X^2|X)_{x_0 \times \mathrm{id}_X, \widetilde{\Delta}}$$

6.2 The general motivic logarithm

We will now generalize Faltings' motivic logarithm as well as define motivic analogues of Hain/Zucker's canonical and tautological VMHS in a far more general setting. In order to do so, we will first need to recall some preliminaries and fix our notation for the entire upcoming chapter.

6.2.1 Preliminaries, setting and notation

Recall that if $p\colon T \longrightarrow S$ is a map of schemes, there is a functor $\mathcal{DM}(p^*)\colon \mathcal{DM}_S \longrightarrow \mathcal{DM}_T$ which is given on objects of the form $\mathbb{Z}_X(a)$ for $X \in \mathrm{Sm}_S$ by

$$\mathcal{DM}(p^*)(\mathbb{Z}_X(a)) := \mathbb{Z}_{p^*X}(a) = \mathbb{Z}_{X\times_S T}(a).$$

Let $X \in \mathrm{Sm}_S$ with structure morphism $\pi\colon X \longrightarrow S$ be equipped with a section $x_0\colon S \longrightarrow X$. We consider a similar setting to that of Hain and Zucker in Chapter II.5: Denote the diagonal inclusion by $\widetilde{\Delta}\colon X \longrightarrow X^2$. We consider $X^3 := X \times_S X \times_S X$ as an $X^2 := X \times_S X$-scheme via projection $\mathrm{pr}_{23}\colon X^3 \longrightarrow X^2$ to the second and third factor. We define two sections

$$\sigma_0, \sigma_1\colon X^2 \longrightarrow X^3; \quad \sigma_0((x,y)) = (x,x,y),\ \sigma_1((x,y)) = (y,x,y)$$

doubling the first, resp. second entry, and denote the section corresponding to x_0 by $x_0 \times \mathrm{id}_{X^2}\colon X^2 \longrightarrow X^3$.

For any $\pi\colon X \longrightarrow S$ in Sm_S with a section $x_0\colon S \longrightarrow X$, we denoted the idempotent induced by $\mathrm{id}_X - x_0\pi\colon X \longrightarrow X$ by

$$e_{X|S}(x_0) := \mathrm{id} - x_0\pi\colon \mathbb{Z}_X \longrightarrow \mathbb{Z}_X.$$

This way, $x_0 \times \mathrm{id}_{X^2}$ gives rise to an idempotent of X^3 by putting

$$e_{X^3|X^2}(x_0 \times \mathrm{id}_{X^2}) := (\mathrm{id}_{X^3} - x_0 \times \mathrm{id}_{X^2})\,\mathrm{pr}_{23}\colon X^3 \longrightarrow X^3.$$

Since it will usually be clear that the section considered is $x_0 \times \mathrm{id}_{X^2}$, we often simply write $e_{X^3|X^2} := e_{X^3|X^2}(x_0 \times \mathrm{id}_{X^2})$. This way we obtain the reduced X^2-motive of X^3 with respect to the idempotent $e^*_{X^3|X^2}$ in the usual way (see section B.1.2 of the appendix): we denote it by

$$\mathbb{Z}^\circ_{X^3} := (\mathbb{Z}_{X^3}, e^*_{X^3|X^2}) \quad \in \mathcal{DM}_{X^2}.$$

Similarly, we consider $X^2 := X \times_S X$ as an X-scheme via projection $\mathrm{pr}_2\colon X^2 \longrightarrow X$ to the second factor. The section $x_0 \times \mathrm{id}\colon X \longrightarrow X^2$ induces an idempotent

$$e_{X^2|X}(x_0 \times \mathrm{id}_X)^* := (\mathrm{id}_{X^2} - x_0 \times \mathrm{id}_X)\,\mathrm{pr}_2\colon X^2 \longrightarrow X^2$$

in $\mathcal{DM}(X)$. As above, we often just write $e_{X^2|X} := e_{X^2|X}(x_0 \times \mathrm{id}_X)$. Thus we obtain the reduced X-motive of X^2 with respect to $e_{X^2|X}$ as

$$\mathbb{Z}^\circ_{X^2} := (\mathbb{Z}_{X^2}, e^*_{X^2|X}) \quad \in \mathcal{DM}_X.$$

Moreover, in what follows, we will have to distinguish between two types of diagonals, so let us once and for all fix the notation to make sure there is no confusion:

$$\widetilde{\Delta}\colon X \longrightarrow X \times_S X =: X^2, \text{ while } \Delta\colon X^2 \longrightarrow (X^2) \times_X (X^2),$$

where in the second case we consider X^2 as a scheme over X via the second projection.

6.2.2 Bar complex definitions

Recall that in Hain and Zucker's construction, the canonical, universal unipotent and tautological variations of mixed Hodge structure are given as follows:

- the n-th canonical VMHS $\mathcal{C}^{(n)}$ on $X \times X$ is the one underlying the vector bundle

$$\mathcal{H}^0(\mathrm{gr}^0_{L_{X^3|X^2}} \bar{B}_n(\mathrm{pr}_{23*}\,\mathcal{E}^\bullet_{X^3}, \mathcal{E}^\bullet_{X^2})_{\sigma_0,\sigma_1})$$

- the universal n-unipotent VMHS $\mathcal{G}^{(n)}$ on X is given by $\mathcal{G}^{(n)} = (x_0 \times \mathrm{id})^* \mathcal{C}^{(n)}$.
- The n-tn tautological variation VMHS on X is given by $\mathcal{T}^{(n)} = \widetilde{\Delta}^* \mathcal{C}^{(n)}$.

Definition 6.2.1.

a.) We define the <u>*n-th canonical motive*</u> *(resp.* <u>*canonical ind-motive*</u>*) to be*

$$\begin{aligned} \mathcal{C}^{\mathrm{mot}}_n(X|S) &:= \widetilde{B}^{\mathrm{mot}}_n(X^3|X^2)_{\sigma_0,\sigma_1} \in \mathcal{DM}(X^2), \quad \textit{resp.} \\ \mathcal{C}^{\mathrm{mot}}(X|S) &:= \widetilde{B}^{\mathrm{mot}}(X^3|X^2)_{\sigma_0,\sigma_1} \in \varinjlim \mathcal{DM}(X^2). \end{aligned}$$

b.) We define the <u>*universal n-unipotent motive*</u> *(resp.* <u>*universal ind-unipotent motive*</u>*) as*

$$\begin{aligned} \mathcal{L}^{\mathrm{mot}}_n(X|S)_{x_0} &:= \mathcal{DM}((x_0 \times \mathrm{id})^*)\mathcal{C}^{\mathrm{mot}}_n(X|S) = \widetilde{B}^{\mathrm{mot}}_n(X^2|X)_{x_0\times\mathrm{id},\widetilde{\Delta}} \in \mathcal{DM}(X), \\ \mathcal{L}^{\mathrm{mot}}(X|S)_{x_0} &:= \mathcal{DM}((x_0 \times \mathrm{id})^*)\mathcal{C}^{\mathrm{mot}}(X|S) = \widetilde{B}^{\mathrm{mot}}(X^2|X)_{x_0\times\mathrm{id},\widetilde{\Delta}} \in \varinjlim \mathcal{DM}(X). \end{aligned}$$

c.) We define the <u>*n-th tautological motive*</u> *(resp.* <u>*tautological ind-motive*</u>*) as*

$$\begin{aligned} \mathcal{T}^{\mathrm{mot}}_n(X|S) &:= \mathcal{DM}(\widetilde{\Delta}^*)\mathcal{C}^{\mathrm{mot}}_n(X|S) = \widetilde{B}^{\mathrm{mot}}_n(X^2|X)_{\widetilde{\Delta},\widetilde{\Delta}} \in \mathcal{DM}(X), \textit{resp.} \\ \mathcal{T}^{\mathrm{mot}}(X|S) &:= \mathcal{DM}(\widetilde{\Delta}^*)\mathcal{C}^{\mathrm{mot}}(X|S) = \widetilde{B}^{\mathrm{mot}}(X^2|X)_{\widetilde{\Delta},\widetilde{\Delta}} \in \varinjlim \mathcal{DM}(X). \end{aligned}$$

Corollary 6.2.2. *The X-motive $\mathcal{L}^{\mathrm{mot}}_n(X|S)_{x_0}$ corresponds to Faltings' logarithm translated to Levine's motives in the case where $\pi\colon X \longrightarrow S$ is a relative smooth curve with irreducible fibers.*

Proof. This is a direct consequence of 6.1.4, since by definition

$$\mathcal{L}^{\mathrm{mot}}_n(X|S)_{x_0} = \widetilde{B}^{\mathrm{mot}}_n(X^2|X)_{x_0\times\mathrm{id}_X,\widetilde{\Delta}}. \qquad \blacksquare$$

Lemma 6.2.3. *The construction of the canonical motive is contravariantly functorial in the following sense: Given a morphism $\varphi\colon X' \longrightarrow X$ of smooth S-schemes together with sections x_0, x_0' of $\pi\colon X \longrightarrow S$ and $\pi'\colon X' \longrightarrow S$, respectively, such that $x_0 = \varphi \circ x_0$, there is an induced morphism*

$$\mathcal{C}(\varphi^*)\colon \mathcal{DM}((\varphi^2)^*)\mathcal{C}^{\mathrm{mot}}_n(X|S) \longrightarrow \mathcal{C}^{\mathrm{mot}}_n(X'|S) \quad \textit{in } \mathcal{DM}((X')^2).$$

Proof. As a direct consequence of the naturality property of bar complexes of section 3.6, the pull-back functor $\mathcal{DM}((\varphi^2)^*)\colon \mathcal{DM}(X^2) \longrightarrow \mathcal{DM}(X^3)$ yields

$$\begin{aligned}\mathcal{DM}((\varphi^2)^*)\mathcal{C}_n^{\mathrm{mot}}(X|S) &= \mathcal{DM}((\varphi^2)^*)\widetilde{B}_n^{\mathrm{mot}}(X^3|X^2)_{\sigma_0,\sigma_1}\\ &= \widetilde{B}_n^{\mathrm{mot}}(X^3 \times_{X^2} (X')^2|X'^2)_{\sigma_0\times_{X^2}\mathrm{id}_{(X')^2},\sigma_1\times_{X^2}\mathrm{id}_{(X')^2}}\\ &= \widetilde{B}_n^{\mathrm{mot}}(X \times_S (X')^2|X'^2)_{\sigma_0',\sigma_1'},\end{aligned}$$

where $\sigma_i' = (\varphi\times\mathrm{id}_{(X')^2})\circ\sigma_i\colon (X')^2 \longrightarrow (X')^3 \longrightarrow X\times_S(X')^2$ for $i = 0, 1$. Next, by the functoriality property of bar complexes of section 3.6, there is a pull-back morphism

$$(\varphi\times_S \mathrm{id}_{X^2})^*\colon \widetilde{B}_n^{\mathrm{mot}}(X\times_S (X')^2|X'^2)_{\sigma_0',\sigma_1'} \longrightarrow \widetilde{B}_n^{\mathrm{mot}}((X')^3|X'^2)_{\sigma_0(X'),\sigma_1(X')},$$

where $\sigma_i(X')\colon (X')^2 \longrightarrow (X')^3$ is defined like σ_i with X replaced by X'. The latter bar complex is equal to $\mathcal{C}_n^{\mathrm{mot}}(X'|S)$, and the assertion follows. ■

The functoriality results of motivic bar complexes enable us to deduce similar results for the above motives:

Corollary 6.2.4. *The constructions of the universal unipotent and tautological motives are contravariantly functorial in the following sense:*
Given a morphism $\varphi\colon X' \longrightarrow X$ of smooth S-schemes together with sections x_0, x_0' of X, resp. X', satisfying $x_0 = \varphi\circ x_0'$, there are induced morphisms

$$\begin{aligned}&\mathcal{L}(\varphi^*)_n\colon \mathcal{DM}(\varphi^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \longrightarrow \mathcal{L}_n^{\mathrm{mot}}(X'|S)_{x_0'}\\ &\mathcal{T}(\varphi^*)\colon \mathcal{DM}(\varphi^*)\mathcal{T}_n^{\mathrm{mot}}(X|S) \longrightarrow \mathcal{T}_n^{\mathrm{mot}}(X'|S) \quad \textit{in } \mathcal{DM}(X').\end{aligned}$$

Proof. This follows from Lemma 6.2.3 above and the following calculation: Since $\varphi^2\circ\widetilde{\Delta} = \widetilde{\Delta}\circ\varphi\colon X' \longrightarrow X^2$ and $\varphi^2\circ(\mathrm{id}\times x_0') = \mathrm{id}\times(\mathrm{id}\times x_0)\circ\varphi\colon X' \longrightarrow X^2$, we have

$$\begin{aligned}\mathcal{DM}(\widetilde{\Delta}^*)\circ\mathcal{DM}((\varphi^2)^*) &\simeq \mathcal{DM}(\widetilde{\Delta}^*\circ(\varphi^2)^*)\\ &= \mathcal{DM}(\varphi^*\circ\widetilde{\Delta}^*) = \mathcal{DM}(\varphi^*)\circ\mathcal{DM}(\widetilde{\Delta}^*)\\ \mathcal{DM}((\mathrm{id}\times x_0')^*)\circ\mathcal{DM}((\varphi^2)^*) &\simeq \mathcal{DM}((\varphi^2\circ(\mathrm{id}\times x_0'))^*) = \mathcal{DM}(((\mathrm{id}\times x_0)\circ\varphi)^*)\\ &= \mathcal{DM}(\varphi^*)\circ\mathcal{DM}((\mathrm{id}\times x_0)^*).\end{aligned}$$ ■

6.2.3 The motivic logarithm in terms of relative motives

Recall the motivic generalization of Deligne-Goncharov's result in [DG05, §3] of section I.3.7. There, we interpreted motivic bar complexes in terms of relative motives as follows: For any $\pi\colon X \longrightarrow S \in \mathcal{DM}(S)$, we showed that there is an isomorphism of S-motives

$$b_{\leq 0}\left(\mathbb{Z}_{\left(X^n; D_0^{(n)}, D_1^{(n)},\dots,D_{n+1}^{(n)}\right)}[n]\right) \cong \mathcal{C}_*\left(\Delta_n, sB_\bullet^{\mathrm{mot}}(X|S)_{x,y}\right) \cong \widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y},$$

for all n, where $b_{\leq 0}$ denotes the brutal truncation from above after degree 0 (see Definition 3.7.1), and

$$\begin{aligned} D_0^{(n)} &:= x(S) \times X^{n-1} \\ D_i^{(n)} &:= \{x_i = x_{i+1}\} \subset X^n \text{ for } 1 \leq i \leq n-1 \\ D_n^{(n)} &:= X^{n-1} \times y(S). \end{aligned}$$

Carrying this over to our situation, we obtain the following result as an immediate consequence:

Corollary 6.2.5. *Write* $D_{i,j}^{(n+2)} = \{(x_1, \ldots, x_{n+2}) \in X^{n+2} | x_i = x_j\}$ *and* $D_i^{n+2} = D_{i,i+1}^{(n+2)}$. *Then one has:*

$$\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \simeq b_{\leq 0}\left(\mathbb{Z}_{\left(X^{n+1};\, x_0(S)\times_S X^n, D_1^{(n+1)},\ldots,D_n^{(n+1)}\right)}[n]\right).$$

Proof. Using the natural isomorphism $(X^2)^{\times_X n} \cong X^{n+1}$, this is a direct consequence of the definitions and the above considerations. ■

Corollary 6.2.6. *Put* $U := X \setminus \{x_0(S)\}$, *and denote the corresponding inclusion by* $j_U \colon U \hookrightarrow X$. *By abuse of notation, we write* $D_{i,j}^{(n+1)} = \{(x_1, \ldots, x_{n+1}) \in X^n \times U | x_i = x_j\}$, $D_i^{(n+1)} = D_{i+i+1}^{(n+1)}$ *and* $D_0^{(n+1)} := x_0(S) \times_S X^{n-1} \times U$. *Then one has:*

$$\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \simeq \mathbb{Z}_{\left(X^n\times U; D_0^{(n+1)},\ldots,D_n^{(n+1)}\right)}[n].$$

Proof. One has $\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} = \widetilde{B}_n^{\mathrm{mot}}(X^2|X)_{x_0\times \mathrm{id}_X, \widetilde{\Delta}}$, so

$$\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} = \widetilde{B}_n^{\mathrm{mot}}(X \times U|U)_{x_0\times \mathrm{id}_U, \widetilde{\Delta}}$$

by the functoriality properties of normalized bar complexes of section 3.6. Similar to the proof of Corollary 6.2.5, one then has

$$\begin{aligned} \widetilde{B}_n^{\mathrm{mot}}(X \times U|U)_{x_0\times \mathrm{id}_U, \widetilde{\Delta}} &\simeq b_{\leq 0}\left(\mathbb{Z}_{\left(X^n\times U;\, x_0(S)\times_S X^{n-1}\times U, D_1^{(n+1)},\ldots,D_n^{(n+1)}\right)}[n]\right) \\ &= \mathbb{Z}_{\left(X^n\times U;\, x_0(S)\times_S X^{n-1}\times U, D_1^{(n+1)},\ldots,D_n^{(n+1)}\right)}[n] \end{aligned}$$

where the latter equality follows from the fact that $(x_0 \times \mathrm{id}_U)(U) = x_0(S) \times U$, $\widetilde{\Delta}(U) = \{(x, y) \in X \times U \mid x = y\}$ and hence $(x_0 \times \mathrm{id}_U)(U) \cap \widetilde{\Delta}(U) = \emptyset$. ■

Remark 6.2.7. Keeping the notation of Corollary 6.2.5, in a similar fashion one obtains

$$\begin{aligned} \mathcal{C}_n^{\mathrm{mot}}(X|S) &\simeq b_{\leq 0}\left(\mathbb{Z}_{\left(X^{n+2}; D_{1,n}^{(n+2)}, D_1^{(n+2)},\ldots,D_n^{(n+2)}, D_{n,n+2}^{(n+2)}\right)}[n]\right) \\ \mathcal{T}_n^{\mathrm{mot}}(X|S) &\simeq b_{\leq 0}\left(\mathbb{Z}_{\left(X^{n+1}; D_{1,n+1}^{n+1}, D_1^{(n+1)},\ldots,D_{n-1}^{(n+1)}, D_n^{(n+1)}\right)}[n]\right). \end{aligned}$$

6.3 The large motivic polylogarithm

After constructing the motivic logarithm in the preceding chapter, we finally have the basic ingredients to define the motivic polylogarithm. It will not be clear from the start that the object we will construct is indeed the polylogarithm studied in the literature to the day, but it will become clearer as we determine its mixed realization in the next chapter.

In order to define the polylogarithm, we need to consider our motives on the complement U of the section $x_0(S)$ in X with open inclusion $j_U\colon U = X \setminus x_0(S) \hookrightarrow X$ and the pull-back functor is denoted $\mathcal{DM}(j_U^*)\colon \mathcal{DM}(X) \longrightarrow \mathcal{DM}(U)$. Recall that this pull-back of motives is given on objects $\mathbb{Z}_Y$ for $Y \in \mathrm{Sm}_X$ by the fiber product $\mathbb{Z}_{Y\times_X U}$. Also, we denote the composition $\pi \circ j_U\colon U \hookrightarrow X \to S$ by $\pi_U\colon U \longrightarrow S$.

6.3.1 The definition

The polylog is defined using the following two ingredients:

- The universal n-unipotent motive $\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0}$, which underlies a cosimplicial scheme in the sense that
$$\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} = \widetilde{B}_n^{\mathrm{mot}}(X^2|X)_{x_0\times\mathrm{id}_X,\widetilde{\Delta}} = nM\left(cB_{\mathrm{mot}}^{\leq n}(X^2|X)_{x_0\times\mathrm{id}_X,\widetilde{\Delta}}\right).$$
Its restriction to U is $cB_{\mathrm{mot}}^{*}(X\times U|U)_{x_0\times\mathrm{id}_U,\widetilde{\Delta}}$.
- The augmentation ideal $\tilde{I}_n^{\mathrm{mot}}(X\times U|U)_{x_0\times\mathrm{id}_U}$. It can be constructed as the normalized motive of the cosimplicial scheme $cI_{\mathrm{mot}}^{*}(X\times U|U)_{x_0\times\mathrm{id}_U}$ in the sense that
$$nM(cI_{\mathrm{mot}}^{*}(X\times U|U)_{x_0\times\mathrm{id}_U}) \simeq \tilde{I}_{n+1}^{\mathrm{mot}}(X\times U|U)_{x_0\times\mathrm{id}_U}[-1],$$
where we consider everything in $\mathcal{DM}(U)$ (see Proposition 3.8.3 and Lemma I.3.8.6).

<u>**Aim:**</u> Construct a morphism
$$\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0\times\mathrm{id}_X} \longrightarrow \tilde{I}_{n+1}^{\mathrm{mot}}(X\times U|U)_{x_0\times\mathrm{id}_U}(d)[2d-1]$$
of motives in $\mathcal{DM}(U)$ which will eventually give rise to the polylogarithmic classes.

<u>**Approach:**</u> Construct a morphism between the underlying cosimplicial schemes which is a closed inclusion. Then one obtains the polylogarithm morphism as the resulting Gysin morphism by the following general considerations:
Let $Z^*, X^*\colon \mathbf{\Delta}_{\leq n} \longrightarrow \mathrm{Sm}_S$ be two cosimplicial schemes, where $I\colon Z \hookrightarrow X$ is a codimension d closed embedding, and denote the corresponding simplicial objects in $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$ by $\mathbb{Z}_{Z^*}(0)$ and $\mathbb{Z}_{X^*}(0)$. Recall that by section I.2.2 there is a Gysin isomorphism
$$\begin{aligned} i_*&\colon \operatorname{hocolim}_{\mathbf{\Delta}_{\leq n}^{\mathrm{op}}} \mathbb{Z}_{Z^*}(0)(-d)[-2d] \longrightarrow \operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathbb{Z}_{X^*}(0), \quad \text{resp.}\\ i_*&\colon N_{\geq -n}(\mathbb{Z}_{Z^*}(0))(-d)[-2d] \longrightarrow N_{\geq -n}(\mathbb{Z}_{X^*}(0)) \ \text{ in } \mathcal{DM}(S).\end{aligned}$$

It is the morphism induced on total complexes by the morphism of double complexes

$$\begin{tikzcd}
\dots \arrow[r, "{d_n[-2d]}"] & \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{Z^{j_0}}(-d)[-2d] \arrow[r, "{d_{n-1}[-2d]}"] \arrow[d, "\bigoplus_{[j_n]\to\dots[j_0]} i_{Z^{j_0}*}"] & \dots \arrow[r, "{d_0[-2d]}"] & \bigoplus_{k=0}^n \mathbb{Z}_{Z^k}(-d)[-2d] \arrow[d, "\bigoplus_{k=0}^n i_{Z^k*}"] \\
\dots \arrow[r, "d_n"] & \bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{X^{j_0}}(0) \arrow[r, "d_{n-1}"] & \dots \arrow[r, "d_0"] & \bigoplus_{k=0}^n \mathbb{Z}_{X^k}(0) \\
\dots & -n & \dots & 0
\end{tikzcd}$$

in $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$, where we dropped the index $f_X^{\leq 0}$ everywhere, and the complex

$$\bigoplus_{[j_n]\to\dots[j_0]} \mathbb{Z}_{X^{j_0}}(0) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_0} \bigoplus_{k=0}^n \mathbb{Z}_{X^k}(0)$$

is considered as the complex of the (vertical) complexes with $\bigoplus_{[j_m]\to\dots[j_0]} Z_{X^{j_0}}(0)$ concentrated in degree 0.

Let us hence set out with the morphism on the level of cosimplicial schemes:

Proposition 6.3.1. *There is an inclusion of cosimplicial objects*

$$c\,\mathrm{pol}\colon cB^\bullet_{\mathrm{mot}}(X\times U|U)_{x_0\times\mathrm{id}_U,\widetilde{\Delta}} \hookrightarrow cI^\bullet(X|S)_{x_0}(X\times U|U)_{x_0\times\mathrm{id}_U},$$

given on objects by the inclusion $\mathrm{id}^{n-1}\times\Big((\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ\widetilde{\Delta}\Big)$:

$$\big((X\times U)^{\times_U n},\mathrm{id}^n\big)\hookrightarrow\big((X\times U)^{\times_U n+1},\mathrm{id}^n\times(\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\big)$$

of codimension d, where $\widetilde{\Delta}\colon U\longrightarrow X\times_S U$ is the diagonal over S.

Proof. We need to show that the morphism on objects given as asserted is compatible with degeneracy and face maps on both sides:

a.) *face maps:* $d^j_{n+1}\circ c\,\mathrm{pol}=c\,\mathrm{pol}\circ d^j_{n+1}$

- $j=0$:

$$\begin{aligned}
d^0_{n+1}(I)\circ c\,\mathrm{pol} &= \big((x_0\times\mathrm{id}_U)\times_U\mathrm{id}^{\times n}\big)\circ\Big(\mathrm{id}^{n-1}\times((\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ\widetilde{\Delta})\Big)\\
&= (x_0\times\mathrm{id}_U)\times_U\mathrm{id}^{n-1}\times\Big((\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ\widetilde{\Delta}\Big)=c\,\mathrm{pol}\circ d^0_{n+1}
\end{aligned}$$

- $j=n+1$:

$$\begin{aligned}
d^{n+1}_{n+1}(I)\circ c\,\mathrm{pol} &= \mathrm{id}^{\times n}\times\Big(\Delta\circ(\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ(\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ\widetilde{\Delta}\Big)\\
&= \mathrm{id}^{\times n}\times\Big((\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ\widetilde{\Delta}\Big)\times\Big((\mathrm{id}-(x_0\times_S\mathrm{id}_U)\,\mathrm{pr}_2)\circ\widetilde{\Delta}\Big)\\
&= c\,\mathrm{pol}\circ d^{n+1}_{n+1}
\end{aligned}$$

- $j \neq 0, n+1$:

$$
\begin{aligned}
d_{n+1}^{j}(I) \circ c\,\mathrm{pol} &= \left(\mathrm{id}^{\otimes j-1} \times \Delta \times \mathrm{id}^{\otimes n+1-j}\right) \circ \left(\mathrm{id}^{n-1} \times \left((\mathrm{id} - (x_0 \times_S \mathrm{id}_U)\,\mathrm{pr}_2) \circ \widetilde{\Delta}\right)\right) \\
&= \mathrm{id}^{\otimes j-1} \times \Delta \times \mathrm{id}^{\otimes n-j} \times \left((\mathrm{id} - (x_0 \times_S \mathrm{id}_U)\,\mathrm{pr}_2) \circ \widetilde{\Delta}\right) \\
&= c\,\mathrm{pol} \circ d_{n+1}^{j}.
\end{aligned}
$$

b.) *degeneracy maps:* For all $j = 0, \ldots, n$

$$
\begin{aligned}
s_j^{n+1}(I) \circ c\,\mathrm{pol} &= \left(\mathrm{id}^{\otimes j+1} \otimes \pi \otimes \mathrm{id}^{\otimes n-j}\right) \circ \left(\mathrm{id}^{n-1} \times \left((\mathrm{id} - (x_0 \times_S \mathrm{id}_U)\,\mathrm{pr}_2) \circ \widetilde{\Delta}\right)\right) \\
&= \mathrm{id}^{\otimes j+1} \otimes \pi \otimes \mathrm{id}^{\otimes n-j-1} \times \left((\mathrm{id} - (x_0 \times_S \mathrm{id}_U)\,\mathrm{pr}_2) \circ \widetilde{\Delta}\right) \\
&= c\,\mathrm{pol} \circ s_j^{n+1}.
\end{aligned}
$$

This proves the assertion. ■

As mentioned above, this gives rise to a Gysin morphism of the associated homotopy colimits of simplicial schemes $\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} c\,\mathrm{pol}$:

$$
\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} sB(X \times U|U)_{x_0 \times \mathrm{id}_U, \widetilde{\Delta}} \longrightarrow \mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} sI(X \times U|U)_{x_0 \times \mathrm{id}_U}.
$$

Via the natural identifications

$$
\begin{aligned}
\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} sB(X \times U|U)_{x_0 \times \mathrm{id}_U, \widetilde{\Delta}} &= nM\left(sB^{\leq n}(X \times U|U)_{x_0 \times \mathrm{id}_U, \widetilde{\Delta}}\right) \\
\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} sI(X \times U|U)_{x_0 \times \mathrm{id}_U} &= nM\left(sI_\bullet(X \times U|U)_{x_0 \times \mathrm{id}_U}\right)
\end{aligned}
$$

this yields the corresponding Gysin morphism of normalized motives

$$
\mathrm{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} c\,\mathrm{pol}\colon nM\left(sB^{\leq n}(X \times U|U)_{x_0 \times \mathrm{id}_U, \widetilde{\Delta}}\right) \longrightarrow nM\left(sI_\bullet(X \times U|U)_{x_0 \times \mathrm{id}_U}\right),
$$

which is equal to a morphism

$$
\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S))_{x_0 \times \mathrm{id}_X} \longrightarrow \mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1].
$$

Definition 6.3.2. *The morphism above is denoted by*

$$
\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}\colon \mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S))_{x_0 \times \mathrm{id}_X} \longrightarrow \mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1]
$$

and called the n-th motivic polylogarithm of $X|S$. Whenever it is clear what X, S and x_0 are, we will simply write $\mathrm{pol}_n^{\mathrm{mot}} := \mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}$.

6.3.2 Functoriality of the motivic polylogarithm

Recall the functoriality properties of the universal n-unipotent motive, respectively the underlying bar complex: Given a morphism $\varphi\colon X' \longrightarrow X$ of smooth S-schemes together with sections x_0, x_0' of X, resp. X', satisfying $x_0 = \varphi \circ x_0'$, there is an induced morphism

$$\mathcal{L}(\varphi^*)_n \colon \mathcal{DM}(\varphi^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \longrightarrow \mathcal{L}_n^{\mathrm{mot}}(X'|S)_{x_0'} \quad \text{in } \mathcal{DM}(X').$$

We write $U = X \setminus x_0(S), U' = X' \setminus x_0'(S)$ with open inclusions denoted by j_U and $j_{U'}$, respectively. For the sake of simplicity, let us drop the "mot" in the notation. Likewise, we obtain a pull-back-morphism restricted to U:

$$\mathcal{L}(\varphi^*)_n \colon \mathcal{DM}(\varphi^*)\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \longrightarrow \mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X'|S)_{x_0'} \quad \text{in } \mathcal{DM}(X'),$$

as well as a pull-back morphism

$$\varphi^{*\otimes} \colon \mathcal{DM}(\varphi^*)\tilde{I}_{n+1}(X \times U|U)_{x_0 \times \mathrm{id}_U} \longrightarrow \tilde{I}_{n+1}(X' \times U'|U')_{x_0' \times \mathrm{id}_{U'}}.$$

We want to show that the polylogarithm is functorial in the obvious way, i.e. that the resulting diagram

$$\begin{tikzcd}
\mathcal{DM}(\varphi^*)\mathcal{L}_n(X|S)_{x_0} \arrow[r, "\mathcal{L}(\varphi^*)_n"] \arrow[d, "\mathcal{DM}(\varphi^*)\left(\mathrm{pol}_n(X|S)_{x_0}\right)"'] & \mathcal{L}_n(X'|S)_{x_0'} \arrow[d, "\mathrm{pol}_n(X'|S)_{x_0'}"] \\
\mathcal{DM}(\varphi^*)\tilde{I}_{n+1}(X \times U|U)_{x_0}(d)[2d-1] \arrow[r, "\mathcal{DM}(\varphi^*)(\varphi^{*\otimes})"] & \mathcal{DM}(\pi_U'^*)\tilde{I}_{n+1}(X'|S)_{x_0}(d)[2d-1]
\end{tikzcd}$$

is cartesian.

By Theorem [Lev98, III.2.6.11(iv), p.160], the Gysin morphism between motives of cosimplicial schemes is compatible with base-change as follows: Consider the commutative square of cosimplicial schemes

$$\begin{tikzcd}
cB^*_{\mathrm{mot}}(X' \times U'|U')_{x_0' \times \mathrm{id}_{U'}, \widetilde{\Delta}'} \arrow[r, "\varphi"] \arrow[d, "c\,\mathrm{pol}(X'|S)"'] & cB^*_{\mathrm{mot}}(X \times U'|U')_{x_0 \times \mathrm{id}_{U'}, \widetilde{\Delta}'} \arrow[d, "\varphi^*(c\,\mathrm{pol}(X|S))"] \\
cI^*_{\mathrm{mot}}(X' \times U'|U')_{x_0' \times \mathrm{id}_{U'}} \arrow[r, "\varphi^*(\varphi)"] & cI^*_{\mathrm{mot}}(X \times U'|U')_{x_0 \times \mathrm{id}_{U'}}
\end{tikzcd}$$

Here, $\widetilde{\Delta}' \colon U' \longrightarrow X' \times_S U'$ is the diagonal for U', $\varphi^*(c\,\mathrm{pol}(X|S))$ is the morphism obtained from $c\,\mathrm{pol}(X|S)$ after base-change from U to U' via φ, and similarly for $\varphi^*(\varphi)$. By Proposition 6.3.1 above, the vertical arrows are closed injections, and hence the diagram is transverse (see the n the appendix). Thus, by Theorem [Lev98, III.2.6.11(iv), p.160], the base-change property holds in this situation, that is to say we have:

$$\mathcal{DM}(\varphi^*)(\varphi^*) \circ \mathcal{DM}(\varphi^*)\left(\mathrm{pol}(X|S)_!\right) = \mathrm{pol}(X'|S)_! \circ \varphi^*,$$

or in our terminology:

$$\mathcal{DM}(\varphi^*)(\varphi^{*\otimes}) \circ \mathcal{DM}(\varphi^*)\left(\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}\right) = \mathrm{pol}_n^{\mathrm{mot}}(X'|S)_{x_0'} \circ \mathcal{L}(\varphi^*)_n.$$

In other words, it shows that the above diagram indeed commutes and we have an arrow

$$\varphi^*\colon \mathcal{DM}(\varphi^*)\left(\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}\right) \longrightarrow \mathrm{pol}_n^{\mathrm{mot}}(X'|S)_{x'_0}$$

of morphisms in $\mathcal{DM}(U')$ as asserted, i.e.:

Lemma 6.3.3. *Given a morphism $\varphi\colon X' \longrightarrow X$ of smooth S-schemes together with sections x_0, x'_0 of X, resp. X', satisfying $x_0 = \varphi \circ x'_0$, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{DM}(\varphi^*)\mathcal{L}_n(X|S)_{x_0} \arrow[r, "\mathcal{L}(\varphi^*)_n"] \arrow[d, "\mathcal{DM}(\varphi^*)\left(\mathrm{pol}_n(X|S)_{x_0}\right)"'] & \mathcal{L}_n(X'|S)_{x'_0} \arrow[d, "\mathrm{pol}_n(X'|S)_{x'_0}"] \\
\mathcal{DM}(\varphi^*)\tilde{I}_{n+1}(X \times U|U)_{x_0}(d)[2d-1] \arrow[r, "\mathcal{DM}(\varphi^*)(\varphi^{*\otimes})"] & \mathcal{DM}(\pi'^*_U)\tilde{I}_{n+1}(X'|S)_{x_0}(d)[2d-1].
\end{tikzcd}$$

6.3.3 The large motivic polylog in terms of relative motives

Recall that due to Deligne-Goncharov's Lemma, we were able to describe the normalized motivic bar complexes in terms of relative motives in section 6.2.3. Now we aim to know the polylogarithm in terms of this description.

Let X be of dimension d over S. Again, put $U := X \setminus \{x_0(S)\}$, and denote the corresponding inclusion by $j_U\colon U \hookrightarrow X$. As in section 6.2.3, we consider the closed subsets $D_i^{(n+1)} = \{x_i = x_{i+1}\} \subset X^n \times U$ and $D_0^{(n+1)} = x_0(S) \times X^n \times U$ in $X^n \times U$ for $n \in \mathbb{N}$, where all schemes are considered over U via the last projection. The first step is to express both sides of $\mathrm{pol}_n^{\mathrm{mot}}$ in terms of relative motives:

- By Corollary 6.2.6, we have $\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \simeq \mathbb{Z}_{\left(X^n \times U; D_0^{(n+1)}, \ldots, D_n^{(n+1)}\right)}[n]$.
- Put $e^*_{X^{n+1} \times_S U|U} := (\mathrm{id}_{X^n} \times (\mathrm{id}_{X \times_S U} - (x_0 \times_S \mathrm{id}_U)\,\mathrm{pr}_2))^*$. Moreover, write

  $$D_i^{(n+1)} := \{x_i = x_{i+1}\} \text{ for } 1 \leq i \leq n \ \text{ and } \ D_0^{(n+1)} := x_0(S) \times_S X^{n-1} \times_S U.$$

  As in Lemma I.3.8.6, we write

  $$\mathbb{Z}^\circ_{\left(X^{n+1} \times U; D_0^{(n+2)}, \ldots, D_n^{(n+2)}\right)} := \left(\mathbb{Z}_{\left(X^{n+1} \times U; D_0^{(n+2)}, \ldots, D_{n-1}^{(n+2)}\right)}; e^*_{X^{n+1} \times_S U|U}\right)$$

  Then by Lemma I.3.8.6, we have an isomorphism

  $$\tilde{I}^{\mathrm{mot}}_{n+1}(X \times U|U)_{x_0 \times \mathrm{id}_U} \simeq b_{\leq 0}\left(\mathbb{Z}^\circ_{\left(X^{n+1} \times U; D_0^{(n+2)}, \ldots, D_n^{(n+2)}\right)}[n+1]\right) \ \text{ in } \ \mathcal{DM}(U).$$

Thus, the polylog can be expressed as a morphism

$$\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}\colon \mathbb{Z}_{\left(X^n\times U;D_0^{(n+1)},\dots,D_n^{(n+1)}\right)} \longrightarrow b_{\leq n+1}\left(\mathbb{Z}^{\circ}_{\left(X^{n+1}\times_S U;D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}\right)(d)[2d].$$

Our aim in this section is to explicitly determine this morphism as a Gysin morphism of relative motives. To this end, we take a look at the following:

We consider the inclusion of the codimension d closed subscheme $\iota_D\colon D_{n+1}^{(n+2)} \hookrightarrow X^{n+1}\times U$. By section B.5.2 in the appendix, there is a Gysin isomorphism for relative motives:

$$\mathbb{Z}_{\left(D_{n+1}^{(n+2)};D_0^{(n+2)}\cap D_{n+1}^{(n+2)},\dots,D_n^{(n+2)}\cap D_{n+1}^{(n+2)}\right)} \xrightarrow{\iota_{D*}} \mathbb{Z}_{\left(X^{n+1}\times_S U;D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d].$$

Under the natural identification $D_{n+1}^{(n+2)} \cong X^n\times U$, the inclusion $\iota_D\colon D_{n+1}^{(n+2)} \hookrightarrow X^{n+1}\times U$ corresponds to $\mathrm{id}_X^{n+1}\times_X\widetilde{\Delta}\colon X^{n+1}\longrightarrow (X^n\times U)\times_U(X\times U)\cong X^{n+1}\times_S U$, and the closed subsets in $D_{n+1}^{(n+2)}$ correspond to:

$$\begin{aligned}
&D_0^{(n+2)}\cap D_{n+1}^{(n+2)} = D_0^{(n+1)}\\
&D_i^{(n+2)}\cap D_{n+1}^{(n+2)} \cong D_i^{(n+1)}\subset X^n\times U \text{ for } i=1,\dots,n-1\\
&D_n^{(n+2)}\cap D_{n+1}^{(n+2)} \cong D_n^{(n+1)}\subset X^n\times U
\end{aligned}$$

Therefore the Gysin isomorphism above can be identified with

$$\mathbb{Z}_{\left(X^n\times_S U;D_0^{(n+1)},\dots,D_n^{(n+1)}\right)} \xrightarrow{(\mathrm{id}_{X^n\times_S U}\times\widetilde{\Delta})_*} \mathbb{Z}_{\left(X^{n+1}\times_S U;D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d].$$

We consider the composition of this with the morphism

$$\mathbb{Z}_{\left(X^{n+1}\times_S U;D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d] \twoheadrightarrow \left(b_{\leq n+1}\left(\mathbb{Z}_{\left(X^{n+1}\times_S U;D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}\right)\right)(d)[2d].$$

Proposition 6.3.4. *The Gysin isomorphism* $\left(\mathrm{id}_{X^n\times_S U}\times\widetilde{\Delta}\right)_*$ *yields a morphism*

$$\mathbb{Z}_{\left(X^n\times_S U;D_0^{(n+1)},\dots,D_n^{(n+1)}\right)} \xrightarrow{e^*_{X^{n+1}\times U|U}\circ(\mathrm{id}_{X^n\times_S U}\times\widetilde{\Delta})_*} \left(b_{\leq n+1}\left(\mathbb{Z}^{\circ}_{\left(X^{n+1}\times_S U;D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}\right)\right)(d)[2d]$$

which corresponds to $\mathrm{pol}_n^{\mathrm{mot}}\colon \mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0}\longrightarrow \mathcal{DM}(\pi_U^*)\tilde{I}_n^{\mathrm{mot}}(X|S)_{x_0}$ *of Definition 6.3.2 under the natural identifications above.*

Proof. We keep the above notation and write

$$e^*_{X^{n+1}\times U|U} := \left(\mathrm{id}_{X^n}\times(\mathrm{id}_{X\times U}-(\mathrm{id}_X\times x_0)\,\mathrm{pr}_2)\right)^*.$$

For the existence of the morphism, we simply need to take care of the additional idempotent on the right hand side: note that

$$e^*_{X^{n+1}\times U|U} \circ e^*_{X^{n+1}\times U|U} \circ (\mathrm{id}_{X^n} \times \widetilde{\Delta})_* = e^*_{X^{n+1}\times U|U} \circ (\mathrm{id}_{X^n} \times \widetilde{\Delta})_*,$$

so the morphism Gysin morphism in fact induces a morphism

$$\begin{tikzcd}
\mathbb{Z}_{\left(X^n\times_S U; D_0^{(n+1)},\dots,D_n^{(n+1)}\right)} \arrow[r, "e^*_{X^{n+1}\times U|U}\circ", "\circ(\mathrm{id}_{X^n\times_S U}\times\widetilde{\Delta})_*"'] & \mathbb{Z}^\circ_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d] \arrow[d] \\
& b_{\leq n+1}\mathbb{Z}^\circ_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d].
\end{tikzcd}$$

The rest is immediate when noting that the Gysin morphism for relative motives is simply the Gysin morphism for diagrams, and hence is compatible with the construction of our Gysin morphisms for motives of cosimplicial schemes. ■

6.4 The small motivic polylogarithm

While the motivic polylogarithm is an element of

$$\mathrm{Hom}_{\mathcal{DM}(U)}\left(\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S))_{x_0\times\mathrm{id}_X}, \mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1]\right),$$

in realizations one often considers a "smaller" variant of this "large polylogarithm" by passing to a quotient of $\mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1]$. As a first step, we will define this "small" augmentation ideal:
Note that $\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}[-1]$ is given by

$$\mathbb{Z}^\circ_{X^{n+1}} \longrightarrow \dots \longrightarrow \mathbb{Z}^\circ_{X^2} \xrightarrow{e^*_{X|S}\Delta^*} \mathbb{Z}^\circ_X \longrightarrow 0$$

where $\mathbb{Z}^\circ_{X^{n+1}}$ is in degree $-n$.

Proposition 6.4.1. *The kernel (resp. the coimage) of the morphism* $\mathbb{Z}^\circ_{X^2} \xrightarrow{\Delta^*} \mathbb{Z}^\circ_X$ *exists in* $\mathcal{DM}(S)$ *and is given by the kernel (resp. coimage) of the idempotent*

$$(\pi\times\mathrm{id})^*\Delta^*\colon \mathbb{Z}^\circ_{X^2} \longrightarrow \mathbb{Z}^\circ_{X^2}$$

in the idempotent complete category $\mathcal{DM}(S)$.

Proof. It suffices to prove the assertion for kernels. Obviously, one has $\Delta^*(\pi\times\mathrm{id})^* = \mathrm{id}$ so $\pi^*\otimes\mathrm{id}$ is a monomorphism, Δ^* an epimorphism, and $(\pi\times\mathrm{id})^*\Delta^*$ is an idempotent, that is to say $(\pi\times\mathrm{id})^*\Delta^*(\pi\times\mathrm{id})^*\Delta^* = (\pi\times\mathrm{id})^*\Delta^*$. Hence, the kernel of $(\pi\times\mathrm{id})^*\Delta^*$ exists in the idempotent complete category $\mathcal{DM}(S)$.
We have to show that given any morphism $m\colon Z\to\mathbb{Z}^\circ_{X^2}$ in $\mathcal{DM}(S)$ such that $\Delta^*\circ m = 0$, there exists a unique morphism $k\colon Z\to\ker((\pi\times\mathrm{id})^*\Delta^*)$ making the diagram

$$\begin{tikzcd}
\mathbb{Z}^\circ_{X^2} \arrow[rr, bend left, "\Delta^*"] & & \mathbb{Z}^\circ_X \\
& \ker((\pi \times \mathrm{id})^* \Delta^*) \arrow[ul, hook] \arrow[ur, "0"] & \\
& Z \arrow[uul, "m"] \arrow[uur, "0"'] \arrow[u, dashed, "\exists ! k"] &
\end{tikzcd}$$

commute. The commutativity of the upper triangle is obvious, since $\Delta^* = \Delta^*(\pi \times \mathrm{id})^*\Delta^*$. Consider the morphism

$$(\mathrm{id}^2 - \Delta(\pi \times \mathrm{id}))^* \circ m \colon Z \to \mathbb{Z}^\circ_{X^2}.$$

Since $\Delta^* m = 0$, we have $(\pi \times \mathrm{id})^*\Delta^* m = 0$, so by definition of the kernel of $(\pi \times \mathrm{id})^*\Delta^*$, the morphism m factors uniquely over the kernel of $(\pi \times \mathrm{id})^*\Delta^*$. This yields the unique morphism $k \colon Z \longrightarrow \ker((\pi \times \mathrm{id})^*\Delta^*)$ in question, which proves the assertion. ■

Note that due to the above proposition and properties of idempotents, there is a direct sum decomposition

$$\begin{aligned}
\mathbb{Z}^\circ_{X^2} &\cong \ker\left((\pi \times \mathrm{id})^*\Delta^*\right) \oplus \mathrm{coim}\left((\pi \times \mathrm{id})^*\Delta^*\right) \\
&\cong \ker((\pi \times \mathrm{id})^*\Delta^*) \oplus \mathrm{Im}\left((\mathrm{id}^2)^* - (\pi \times \mathrm{id})^*\Delta^*\right)
\end{aligned}$$

Hence, the diagram

$$\begin{tikzcd}
\mathbb{Z}^\circ_{X^{n+1}} \arrow[r] & \ldots \arrow[r] & \mathbb{Z}^\circ_{X^2} \arrow[r, "e^*_{X|S}\Delta^*"] & \mathbb{Z}^\circ_X \arrow[r] & 0
\end{tikzcd}$$

comprising $\tilde{I}^{\mathrm{mot}}_{n+1}(X|S)_{x_0}[-1] \in \mathcal{DM}(S)$ splits into a direct sum of motives

$$\begin{tikzcd}
\mathbb{Z}^\circ_{X^{n+1}} \arrow[r] & \ldots \arrow[r] & \ker\left(e^*_{X|S}\Delta^*\right) \arrow[r] & 0 \arrow[r] & 0 \\
& & \oplus & \oplus & \\
& & \mathrm{coim}\left((\pi \times \mathrm{id})^*\Delta^*\right) \arrow[r, "e^*_{X|S}\Delta^*"] & \mathbb{Z}^\circ_X \arrow[r] & 0
\end{tikzcd}$$

Definition 6.4.2. *Let $U := X \setminus x_0(S)$. We denote the motive*

$$\begin{tikzcd}
\mathrm{coim}\left((\pi \times \mathrm{id})^*\Delta^*\right) \arrow[r, "e^*_{X|S}\Delta^*"] & \mathbb{Z}^\circ_X \arrow[r] & 0 \\
-2 & -1 & 0
\end{tikzcd}$$

by $i^{\mathrm{mot}}(X|S)_{x_0}$, and the projection induced by the above direct sum decomposition by

$$\mathrm{pr}_{i^{\mathrm{mot}}} \colon \tilde{I}^{\mathrm{mot}}_{n+1}(X|S)_{x_0} \longrightarrow i^{\mathrm{mot}}(X|S)_{x_0}.$$

Remark 6.4.3. Denote the restriction of $\pi\colon X \longrightarrow S$ to U by π_U. Then

$$i^{\mathrm{mot}}(X \times U|U)_{x_0 \times \mathrm{id}_U} := \mathcal{DM}(\pi_U^*) i^{\mathrm{mot}}(X|S)_{x_0}$$

is given by the diagram

$$\operatorname{coim}\left((\mathrm{pr}_2 \times \mathrm{id})^* \Delta^*\right) \xrightarrow[\circ \Delta^*]{(\mathrm{id} - \mathrm{pr}_2^*(x_0 \times \mathrm{id})^*)\circ} \mathbb{Z}^\circ_{X \times_S U} \longrightarrow 0$$

Now that we defined the "small" augmentation ideal we wanted we may define the small polylogarithm as follows:

Definition 6.4.4. *We call the composition of* $\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}$ *with* $\mathrm{pr}_{i^{\mathrm{mot}}}(d)[2d-1]$ *small motivic polylogarithm and denote it by* $p_n^{\mathrm{mot}}(X|S)_{x_0}$*. It is an element of*

$$\mathrm{Hom}_{\mathcal{DM}(U)}\left(\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S))_{x_0 \times \mathrm{id}_X}, \mathcal{DM}(\pi_U^*) i^{\mathrm{mot}}(X|S)_{x_0}\right).$$

6.5 Passing to the "limit"

Again, we pass from the category $\mathcal{DM}(U)$ of motives to the symmetric monoidal category $\varinjlim \mathcal{DM}(U)$ of direct systems of motives as introduced in Definition I.3.5.1. Moreover, recall that on left-unbounded complexes of objects of type $\mathbb{Z}_Y$ or $\mathbb{Z}_Y^\circ$ for $Y \in \mathrm{Sm}(S)$ in $\varinjlim \mathcal{DM}(S)$ we defined a pull-back by applying the ordinary pull-back in the category of motives componentwise. Thus we have inductive systems

$$\begin{aligned} \mathcal{DM}(\pi_U^*)\tilde{I}^{\mathrm{mot}}(X|S)_{x_0} &= (\mathcal{DM}(\pi_U^*)\tilde{I}_n^{\mathrm{mot}}(X|S)_{x_0})_n = \tilde{I}^{\mathrm{mot}}(X \times U|U)_{x_0 \times \mathrm{id}_U} \quad \text{and} \\ \mathcal{DM}(j_U^*)\mathcal{L}^{\mathrm{mot}}(X|S)_{x_0} &= (\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0})_n = \widetilde{B}^{\mathrm{mot}}(X \times U|U)_{x_0 \times \mathrm{id}_U, \widetilde{\Delta}}. \end{aligned}$$

Our aim is to verify that the morphisms

$$\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0} \in \mathrm{Hom}_{\mathcal{DM}(U)}\left(\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0}, \mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}\right)$$

constructed in the preceding sections give rise to a morphism of inductive systems

$$(\mathrm{pol}_n) \in \mathrm{Hom}_{\varinjlim \mathcal{DM}(S)}\left(\mathcal{DM}(j_U^*)\mathcal{L}^{\mathrm{mot}}(X|S)_{x_0}, \mathcal{DM}(\pi_U^*)\tilde{I}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1]\right).$$

To this end, note that by Definition I.3.5.1 any such morphism f of inductive system can be represented by a family of maps

$$f_i\colon \mathcal{DM}(j_U^*)\mathcal{L}_i^{\mathrm{mot}}(X|S)_{x_0} \to \mathcal{DM}(\pi_U^*)\tilde{I}_{\sigma(i)}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1]$$

for some function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that for any $i, j \in \mathbb{N}$ with $i \le j$ there is a $k \in \mathbb{N}$ with $k \ge i, j$ for which the diagram

$$\begin{tikzcd}
\mathcal{DM}(j_U^*)\mathcal{L}_i^{\mathrm{mot}}(X|S)_{x_0} \arrow[r, "f_i"] \arrow[dd, hook] & \mathcal{DM}(\pi_U^*)\tilde{I}_{\sigma(i)}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1] \arrow[d, hook] \\
& \mathcal{DM}(\pi_U^*)\tilde{I}_{k+1}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1] \\
\mathcal{DM}(j_U^*)\mathcal{L}_j^{\mathrm{mot}}(X|S)_{x_0} \arrow[r, "f_j"'] & \mathcal{DM}(\pi_U^*)\tilde{I}_{\sigma(j)}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1] \arrow[u, hook]
\end{tikzcd} \quad \text{commutes.}$$

Taking $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ to be the morphism $\sigma(i) = i+1$ and replacing f_i above by the morphisms $\mathrm{pol}_i^{\mathrm{mot}}$ obviously make the diagram commute.

Lemma 6.5.1. *The morphisms* $\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}$ *yield a morphism*

$$\mathrm{pol}^{\mathrm{mot}}(X|S)_{x_0} \in \mathrm{Hom}_{\varinjlim \mathcal{DM}(U)} \left(\mathcal{DM}(j_U^*)\mathcal{L}^{\mathrm{mot}}(X|S)_{x_0}, \mathcal{DM}(\pi_U^*)\tilde{I}^{\mathrm{mot}}(X|S)_{x_0}\right).$$

Likewise, the small polylogarithms

$$p_n^{\mathrm{mot}}(X|S)_{x_0} \in \mathrm{Hom}_{\mathcal{DM}(U)} \left(\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0}, \mathcal{DM}(\pi_U^*)i^{\mathrm{mot}}(X|S)_{x_0}\right)$$

for $n \in \mathbb{N}$ make the following diagram commute:

$$\begin{tikzcd}
\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \arrow[r, "p_n^{\mathrm{mot}}(X|S)_{x_0}"] \arrow[dd, hook] & \mathcal{DM}(\pi_U^*)i^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1] \arrow[d, equal] \\
& \mathcal{DM}(\pi_U^*)i^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1] \arrow[d, equal] \\
\mathcal{DM}(j_U^*)\mathcal{L}_{n+1}^{\mathrm{mot}}(X|S)_{x_0} \arrow[r, "p_{n+1}^{\mathrm{mot}}(X|S)_{x_0}"] & \mathcal{DM}(\pi_U^*)i^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1].
\end{tikzcd}$$

Definition 6.5.2. *We call* $\mathrm{pol}^{\mathrm{mot}}(X|S)_{x_0}$ *the* <u>*motivic polylogarithm*</u> *of* $X|S$ *for the section* x_0*, and* $\mathrm{p}^{\mathrm{mot}}(X|S)_{x_0}$ *the* <u>*small motivic polylogarithm*</u> *of* $X|S$ *for the section* x_0*.*

6.6 The polylogarithmic class in *K*-theory

The motivic polylogarithm in literature was often defined in the setting of K-theory (e.g. in section 6 of [BL94], or in Kings' paper [Kin99]). To be precise, the class in K-theory constructed in these cases is that associated to the *small polylogarithm*. In this section, we will associate a class in K-theory to our *large* motivic polylogarithm by means of duality in the category $\mathcal{DM}(S)$.

Since the construction of the polylogarithmic K-class is slightly technical, let us point out here that this upcoming section is not really of any consequence for the rest of the thesis. Skipping this section and passing straight on to Chapter 7 will therefore not impair the understanding of the rest of this thesis.

Recall that we denoted by $\mathcal{DM}(S)^{\mathrm{pr}}$ the smallest strictly full triangulated subcategory of $\mathcal{DM}(S)$ containing the objects $\mathbb{Z}_X(a)$ with X projective over S and closed under taking summands. By B.3.1 in the appendix, if X is the complement of a normal crossing divisor in some $\bar{X} \in \mathrm{Sm}(S)^{\mathrm{pr}}$, then the object $\mathbb{Z}_X$ is in $\mathcal{DM}(S)^{\mathrm{pr}}$.

For the entire section, we assume that $X \in \mathrm{Sm}(S)$ is in fact the complement of a normal crossing divisor in some $\bar{X} \in \mathrm{Sm}(S)^{\mathrm{pr}}$. This ensures that all schemes we are dealing with here are in $\mathrm{Sm}(S)^{\mathrm{pr}}$ as well. We are therefore free to apply the duality theory developed by Levine for his motives. In particular, recall the following results for relative motives (see section B.5.2 in the appendix):

Let X be a smooth equi-dimensional S-scheme of dimension d over S, $D_1, \ldots, D_n$ closed subschemes of X which form a normal crossing subscheme of X. For some $i \in \{0, \ldots, n\}$ let $U := X \setminus (D_1 \cup \ldots \cup D_i)$ and $V := X \setminus (D_{i+1} \cup \ldots \cup D_n)$, and let $\delta_{U \cap V} \colon U \cap V = X \setminus (D_1 \cup \ldots D_n) \hookrightarrow V \times_S U$ denote the diagonal inclusion. Moreover, put $D_j^V := V \cap D_j$ and $D_j^U := D_j \cap U$. We consider the relative motives

$$\mathbb{Z}_{(V; D_1^V, \ldots, D_i^V)} \quad \text{and} \quad \mathbb{Z}_{(U; D_{i+1}^U, \ldots, D_n^U)}.$$

Denote the codimension d cycle defined by the image of $\delta_{U \cap V}$ in $U \times_S V$ by $\Delta_{U \cap V}$. Let $\delta_{U,V} \colon 1 \longrightarrow \mathbb{Z}_{V; D_1^V, \ldots, D_i^V} \otimes \mathbb{Z}_{(U; D_{i+1}^U, \ldots, D_n^U)}(d)[2d]$ be the map $\mathrm{cl}(\Delta_{U \cap V})$ followed by the inverse of the Künneth isomorphism. Then one has the following:

Proposition 6.6.1. *([Lev98, IV.2.3.4, p.219]) The pair $(\mathbb{Z}_{(V; D_1^V, \ldots, D_i^V)}, \delta_{U,V})$ is the dual of $\mathbb{Z}_{(U; D_{i+1}^U, \ldots, D_n^U)}(d)[2d]$.*

Let furthermore $i_Z \colon Z \longrightarrow X$ be a closed subscheme of X of codimension $d_{Z:X}$ such that $Z, D_1, \ldots, D_n$ have transverse intersection, and let $Z_V := Z \cap V$, $Z_U := Z \cap U$, $D_{Z,i}^V := D_i \cap Z_V$ and $D_{U,i}^V := D_i \cap Z_U$. Then the collection of inclusions $D_{Z_i}^V \hookrightarrow D_i^V$ defines a morphism

$$i_{Z_V}^* \colon \mathbb{Z}_{(V; D_1^V, \ldots, D_i^V)} \longrightarrow \mathbb{Z}_{(Z_V; D_{Z,1}^V, \ldots, D_{Z,i}^V)}.$$

Similarly, there is a Gysin morphism

$$i_{Z_U *} \colon \mathbb{Z}_{(Z_U; D_{Z,1}^U, \ldots, D_{Z,i}^U)}(-d_{Z:X})[-2d_{Z:X}] \longrightarrow \mathbb{Z}_{(U; D_1^U, \ldots, D_i^U)}.$$

Proposition 6.6.2. *([Lev98, IV.2.3.4, p.219]) The map $i_{Z_U *}$ is dual to the map $i_{Z_V}^*$.*

We want to apply this to the polylogarithm. Recall that in terms of relative motives, the polylogarithm is given as follows (see section II.6.3.3):
Let X be of dimension d over S. As in section 6.2.3, we consider the closed subsets

$$\left.\begin{array}{l} D_{i,j}^{(n+1)} = \{(x_1, \ldots, x_n) \in X^n \times U | x_i = x_j\}, \\ D_i^{(n+1)} = D_{i,i+1}^{(n)} \text{ and } D_0^{(n+1)} := x_0(S) \times_S X^{n-1} \times_S U \end{array}\right\} \subset X^n \times_S U$$

for $n \in \mathbb{N}$, $i \neq j = \{1, \ldots, n\}$, where all are considered as schemes over U via the last projection.
Recall the following natural identifications of section 6.2.3:

- $\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} \simeq \mathbb{Z}_{\left(X^n \times_S U; D_0^{(n+1)}, \ldots, D_n^{(n+1)}\right)}[n]$
- $\mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0} \simeq b_{\leq n+1}\left(\mathbb{Z}^{\circ}_{(X^{n+1}\times_S U; D_0^{(n+2)}, \ldots, D_n^{(n+2)})}\right)[n+1]$,
  where the latter is the brutal truncation from above of the reduced relative motive
  $$\left(\mathbb{Z}_{\left(X^{n+1}\times_S U; D_0^{(n+2)}, \ldots, D_n^{(n+2)}\right)}, e^*_{X^{n+1}\times U|U} = (\mathrm{id}_{X^n} \times (\mathrm{id}_{X\times U} - (\mathrm{id}_X \times x_0)\,\mathrm{pr}_2))^*\right).$$

Then by section 6.3.3 the Gysin isomorphism $(\mathrm{id}_{X^n\times_S U} \times \widetilde{\Delta})_*$ yields a morphism

$$\mathbb{Z}_{\left(X^n \times_S U; D_0^{(n+1)}, \ldots, D_n^{(n+1)}\right)} \xrightarrow[(\mathrm{id}_{X^n\times_S U} \times \widetilde{\Delta})_*]{e^*_{X^{n+1}\times U|U}{}^{\circ}} b_{\leq n+1}\left(\mathbb{Z}^{\circ}_{\left(X^{n+1}\times_S U; D_0^{(n+2)}, \ldots, D_n^{(n+2)}\right)}\right)(d)[2d]$$

which corresponds to the polylogarithm under the above identifications. Via duality, we want to relate this class to a certain cycle in K-theory.

Definition 6.6.3. *Define the subset*

$$Y_{\mathcal{L}}^{(n+1)} := (X^n \times_S U) \setminus \left(\cup_{i=0}^n D_i^{(n+1)}\right) \subset X^n \times_S U$$

and consider immersions of $Y_{\mathcal{L}}^{(n+1)}$ into two spaces:

- *There is a diagonal inclusion $\Delta_{Y_{\mathcal{L}}^{(n+1)}} \colon Y_{\mathcal{L}}^{(n+1)} \hookrightarrow Y_{\mathcal{L}}^{(n+1)} \times_U (X^n \times_S U)$ given by*
  $$\Delta_{Y_{\mathcal{L}}^{(n+1)}}(x_1, \ldots, x_n, u) = (x_1, \ldots, x_n, u, x_1, \ldots, x_n, u).$$
  We denote the codimension nd-cycle given by the image of $\Delta_{Y_{\mathcal{L}}^{(n+1)}}$ by
  $$[\Delta_{Y_{\mathcal{L}}^{(n+1)}}] \in \mathrm{CH}^{nd}\left(Y_{\mathcal{L}}^{(n+1)} \times_U (X^n \times_S U)\right).$$
- *Consider the composition of closed immersions*
  $$\Delta_{\mathrm{pol}} \colon Y_{\mathcal{L}}^{(n+1)} \xrightarrow{\Delta_{Y_{\mathcal{L}}^{(n+1)}}} Y_{\mathcal{L}}^{(n+1)} \times_U (X^n \times_S U) \xrightarrow{\mathrm{id} \times_U \widetilde{\Delta}} Y_{\mathcal{L}}^{(n+1)} \times_U (X^{n+1} \times_S U)$$
  where $\widetilde{\Delta} \colon U \longrightarrow X \times_S U$ is the diagonal. We denote the $(n+1)d$-cycle given by the image of Δ_{pol_n} by
  $$\begin{aligned}[\Delta_{\mathrm{pol}_n}] &\in \mathrm{CH}^{(n+1)d}\left(Y_{\mathcal{L}}^{(n+1)} \times_U (X^{n+1} \times_S U)\right)\\ &\cong K_0^{((n+1)d)}\left(Y_{\mathcal{L}}^{(n+1)} \times_U (X^{n+1} \times_S U)\right),\end{aligned}$$
  where $K_0^{(n+1)d}$ denotes the $(n+1)d$-th Adams eigenspace (see section B.6) of K_0.

Using the duality theory of relative motives recalled in the outset of the section, we obtain:

Proposition 6.6.4. *The dual of* $\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0}$ *in* $\mathcal{DM}(S)$ *is given by the pair* $\left(\mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}}(nd)[n(2d-1)], \mathrm{cl}([\Delta_{Y_{\mathcal{L}}^{(n+1)}}])\right)$.

Proof. This is an immediate application of Proposition 6.6.2. ∎

Now we want to relate the polylogarithm - a morphism of relative motives - to relative K-groups (see Appendix B.6). This is accomplished using the Chern classes from relative motivic cohomology to relative K-groups. Let us recall them in a general setting: If $D_1, \dots, D_n$ are closed subschemes of some $Z \in \mathrm{Sm}_S$ such that all of their intersections are in Sm_S, then by [Lev98, III.1.4.8(iii), p.124], the Chern classes in K-theory (see Chapter B.6 for details) induce Chern classes for relative K-groups

$$c^{q,2q-p}_{(Z;D_1,\dots,D_n)} \colon K_p(Z; D_1, \dots, D_n) \longrightarrow H^{2q-p}_{\mathcal{M}}(Z; D_1, \dots, D_n, \mathbb{Z}(q)),$$

where $H^{2q-p}_{\mathcal{M}}(Z; D_1, \dots, D_n, \mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S, \mathbb{Z}_{(Z;D_1,\dots,D_n)}(q)[2q-p])$ are the relative motivic cohomology groups of Appendix B.5.3.

Now recall the $(n+1)d$-cycle

$$\begin{aligned}[\Delta_{\mathrm{pol}_n}] &\in \mathrm{CH}^{(n+1)d}\left(Y_{\mathcal{L}}^{(n+1)} \times_U (X^{n+1} \times_S U)\right) \\ &\cong K_0^{((n+1)d)}\left(Y_{\mathcal{L}}^{(n+1)} \times_U (X^{n+1} \times_S U)\right)\end{aligned}$$

defined in 6.6.3 above. This class in turn gives rise to a class in the relative K-group (see section B.6 in the appendix for details on relative K-theory), which by abuse of notation we also denote by

$$[\Delta_{\mathrm{pol}_n}] \in K_0^{(n+1)d}\left(Y \times_U (X^{n+1} \times U); Y \times_U D_0^{(n+2)}, \dots, Y \times_U D_n^{(n+2)}\right).$$

We now want to prove that the polylogarithm "comes from" this class in relative K-theory. We will do this in two steps in the following theorem:

Theorem 6.6.5. *a.) There is a canonical morphism* $\tilde{c}^{(n+1)d,2(n+1)d}$:

$$\begin{array}{c} K_0^{((n+1)d)}\left(Y \times_U (X^{n+1} \times U); Y \times_U D_0^{(n+2)}, \dots, Y \times_U D_n^{(n+2)}\right) \\ \Big\downarrow {\scriptstyle q_* \circ p_* \circ c^{(n+1)d,2(n+1)d}} \\ \mathrm{Hom}_{D^b(\mathrm{Sh}(U))}\left(j_U^* \mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}, \pi_U^* \tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}(d)[2d-1]\right). \end{array}$$

b.) This morphism $\tilde{c}^{(n+1)d,2(n+1)d}$ *maps* $[\Delta_{\mathrm{pol}_n}]$ *to* $\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}$.

Proof. For the sake of notation, we will sometimes write $Y := Y_{\mathcal{L}}^{(n+1)}$.

a.) This is basically due to duality: By Proposition 6.6.4 and section 6.2.3, we have

$$\mathbf{D}\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} = \left(\mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}}(nd)[n(2d-1)], \mathrm{cl}([\Delta_{Y_{\mathcal{L}}^{(n+1)}}])\right)$$
$$\mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0} \simeq \left(b_{\leq n+1}\mathbb{Z}^{\circ}_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}\right)[n+1]$$

where the latter is the brutal truncation from above of the reduced relative motive

$$\left(\mathbb{Z}_{(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)})}; e^*_{X^{n+1}\times U|U}\right) \quad \text{in } \mathcal{DM}(U)$$
for $e^*_{X^{n+1}\times U|U} := (\mathrm{id}_{X^n} \times(\mathrm{id}_{X\times U} - (\mathrm{id}_X \times x_0)\,\mathrm{pr}_2))^* : \mathbb{Z}_{X^{n+1}\times U} \longrightarrow \mathbb{Z}_{X^{n+1}\times U}$.

Consider the morphism

$$\begin{tikzcd}
\mathbb{Z}_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)} \arrow[r, two heads, "e^*_{X^{n+1}\times U|U}"] & \mathbb{Z}^{\circ}_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)} \arrow[dl, "q"] \\
b_{\leq n+1}\mathbb{Z}^{\circ}_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)} \arrow[r, equal, "\sim"] & \mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}
\end{tikzcd}$$

where q is the natural morphism. This yields a push-forward morphism

$$\begin{tikzcd}
\mathrm{Hom}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}, \mathbb{Z}_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d-1]\right) \arrow[d, "q_*\circ p_*"] \\
\mathrm{Hom}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}, \pi_U^*\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}(d)[2d-1]\right).
\end{tikzcd}$$

Using duality and

$$\mathbf{D}\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} = \left(\mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}}(nd)[n(2d-1)], \mathrm{cl}([\Delta_{Y_{\mathcal{L}}^{(n+1)}}])\right),$$

the upper term is isomorphic to

$$\begin{tikzcd}
\mathrm{Hom}_{D^b(\mathrm{Sh}(U))}\left(\mathbb{Z}_U, \mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}}(nd)[n(2d-1)] \otimes \mathbb{Z}_{\left(X^{n+1}\times_S U; D_0^{(n+2)},\dots,D_n^{(n+2)}\right)}(d)[2d-1]\right) \arrow[d, equal, "\sim"] \\
\mathrm{Hom}_{\mathcal{DM}(U)}\left(1, \mathbb{Z}_{\left(Y\times_U(X^{n+1}\times U); Y\times_U D_0^{(n+2)},\dots,Y\times_U D_n^{(n+2)}\right)}((n+1)d)[2(n+1)d]\right)
\end{tikzcd}$$

by Künneth, where the latter term is relative motivic cohomology by its very definition (Definition B.5.3 in the appendix). The relative Chern class map thus yields the asserted morphism

$$\begin{array}{c} K_0^{((n+1)d)}\left(Y \times_U (X^{n+1} \times U); Y \times_U D_0^{(n+2)}, \ldots, Y \times_U D_n^{(n+2)})\right) \\ \Big\downarrow {\scriptstyle c^{(n+1)d,2(n+1)d}} \\ \mathrm{Hom}_{\mathcal{DM}(U)}\left(1, \mathbb{Z}_{\left(Y \times_U (X^{n+1} \times U); Y \times_U D_0^{(n+2)}, \ldots, Y \times_U D_n^{(n+2)}\right)}((n+1)d)[2(n+1)d]\right) \\ \Big\downarrow {\scriptstyle \sim \circ q_* \circ p_*} \\ \mathrm{Hom}_{D^b(\mathrm{Sh}(U))}\left(j_U^* \mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}, \pi_U^* \tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}(d)[2d-1]\right). \end{array}$$

b.) Recall that $\mathrm{pol}_n^{\mathrm{mot}}$ is induced by the morphism

$$(\mathrm{id}_{X^n \times_S U} \times \widetilde{\Delta})_* \colon \mathbb{Z}_{\left(X^n \times U; D_0^{(n+1)}, \ldots, D_n^{(n+1)}\right)} \longrightarrow \mathbb{Z}^\circ_{\left(X^{n+1} \times U; D_0^{(n+2)}, \ldots, D_n^{(n+2)}\right)}(d)[2d].$$

By Proposition 6.6.4, the dual of $\mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0}[-n]$ is given by the pair $\left(\mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}}(nd)[2nd], \mathrm{cl}([\Delta_{Y_{\mathcal{L}}^{(n+1)}}]\right)$. This means that the morphism

$$\mathrm{pol}_n^{\mathrm{mot}} = (\mathrm{id}_{X^n \times_S U} \times \widetilde{\Delta})_*$$

corresponds to the composition

$$\begin{array}{l} 1 \xrightarrow{\mathrm{cl}([\Delta_{Y_{\mathcal{L}}^{(n+1)}}]} \mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}} \otimes \mathbb{Z}_{\left(X^n \times U; D_0^{(n+1)}, \ldots, D_n^{(n+1)}\right)}(nd)[2nd] \\ \xrightarrow{\mathrm{id}_{\mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}}} \otimes \left((\mathrm{id}_{X^n \times_S U} \times \widetilde{\Delta})_*\right)} \mathbb{Z}_{Y_{\mathcal{L}}^{(n+1)}} \otimes \mathbb{Z}^\circ_{(X^{n+1} \times U; D_0^{(n+2)}, \ldots, D_n^{(n+2)})}((n+1)d)[2(n+1)d] \end{array}$$

followed by the Künneth isomorphism. By the definition of the Gysin isomorphism in $\mathcal{DM}(U)$, this composition is - up to natural Künneth isomorphism, which we generally tend to ignore in the notation - equal to $[\Delta_{\mathrm{pol}_n}]$, whose preimage in

$$K_0^{((n+1)d)}\left(Y \times_U (X^{n+1} \times U); Y \times_U D_0^{(n+2)}, \ldots, Y \times_U D_n^{(n+2)})\right)$$

is the class of $[\Delta_{\mathrm{pol}_n}]$. ■

Definition 6.6.6. *We define the* <u>*n-th polylogarithmic K-class*</u> *for $X|S$ to be the class*

$$\mathrm{pol}_n^K(X|S) := [\Delta_{\mathrm{pol}_n}]$$

in $K_0^{((n+1)d)}\left(Y \times_U (X^{n+1} \times U); Y \times_U D_0^{(n+2)}, \ldots, Y \times_U D_n^{(n+2)})\right)$.

Chapter 7

The mixed realization and characteristic properties of the polylog

At the moment, the polylogarithm we introduced ad hoc in the preceding chapter is simply an abstact construction we suggested as a candidate for a "general motivic polylogarithm". We still have to justify our claim. In order to get a feeling how our new motivic polylogarithm relates to the polylogarithms which have been constructed by Beilinson, Levin, Kings, Huber, Wildeshaus and all the other mathematicians mentioned in the introduction, we will have to look into the realizations of the polylogarithm.

We will proceed as follows:

- After establishing the basic terminology, we will compute the mixed realization of both the motivic logarithm and polylogarithm.
- Once we have established the mixed logarithm and polylogarithm, it remains to show that this mixed polylogarithm satisfies properties which are characteristic of the polylogarithm and are used in the case of curves and abelian schemes to define the polylogarithm uniquely.
- In order to be able to compare the mixed polylogarithm with that of curves and abelian schemes in literature, we need establish one more fact: that in these classical cases, our mixed polylogarithm (as a morphism in the derived category of mixed sheaves) yields a morphism on zeroth cohomology. This conclusion will then suggest that our definition of the motivic polylogarithm agrees with all the definitions in literature up to now.

First, we will start off with a general picture of the mixed sheaf-setting.

General assumptions and notation for the mixed realization

The setting for this chapter is the following: Let $F = \mathbb{R}$ in the geometric case, and $F = \mathbb{Z}[1/l]$ in the ℓ-adic case, A either a subfield of $\mathbb{C}$ in the geometric case or $\mathbb{Q}_l$ if $F = \mathbb{Z}[1/l]$ in the étale case. Moreover, let $S \longrightarrow \mathrm{Spec}(F)$ be a reduced scheme (recall that in this thesis, "scheme" means "noetherian and separated scheme" throughout), smooth and quasi-projective over $\mathrm{Spec}(F)$.

Let $\pi\colon X \longrightarrow S$ be in Sm_S such that π has geometrically irreducible fibers. As a consequence of the properties of S, X is also reduced, as well as smooth and quasi-projective over $\mathrm{Spec}(F)$. Moreover, we assume that X we have three sections $x_0, x, y\colon S \longrightarrow X$ of π (these don't necessarily have to be distinct).

Recall the Godement resolution G as defined in section I.4.2, as well as the following notations:

$$\begin{aligned}\pi_* A_X^\sharp &:= \pi_* G(A_X)\\ A_S^\sharp &:= G(A_S).\end{aligned}$$

The mixed realization of $A_X \in \mathcal{DM}_A(S)$ (resp. A_X°) is then isomorphic to $\pi_* A_X^\sharp$ (resp. $\pi_* A_X^\sharp / A_S^\sharp$) by Lemma I.4.2.3. Recall the following properties and morphisms constructed in section I.4.2 in Remark I.4.2.2:

Remark 7.0.1. a.) In both the ℓ-adic and the geometric case, the complex $\pi_* A_X^\sharp$ has non-vanishing cohomology only in degrees $0, \dots, 2d$. Hence, via truncations, $\pi_* G(A_X)$ is quasi-isomorphic to the complex $\tau_{\le 2d}\sigma^{\ge 0}(\pi_* G(A_X))$.

b.) Note that due to Künneth the mixed realization of $A_{X^n \times U} \in \mathcal{DM}_A(U)$ can be computed by

$$\begin{aligned}\mathrm{pr}_{n+1*}(A_{X^n \times U}(0)) &\simeq (\mathrm{pr}_{2*}\, A^\sharp_{X\times_S U}) \otimes_{A_U^\sharp} \dots \otimes_{A_U^\sharp} (\mathrm{pr}_{2*}\, A^\sharp_{X\times_S U})\\ &=: (\mathrm{pr}_{2*}\, A^\sharp_{X\times_S U})^{\otimes n}.\end{aligned}$$

7.1 The mixed realization of the logarithm and polylogarithm

7.1.1 The mixed logarithm

Definition 7.1.1. *For any $n \in \mathbb{N}$ we denote the mixed realizations of the motives in the first column of the following table by the corresponding terms in the second column:*

$B \in \mathcal{DM}_A(S)$	$\mathfrak{R}_{A,\mathrm{mix}} B \in D^b\,\mathrm{Sh}(S)$	*name of the mixed realization*
$\mathcal{C}_n^{\mathrm{mot}}(X\|S) \otimes A$	$\mathcal{C}_n^{A,\mathrm{mix}}(X\|S)$	*n-th canonical mixed sheaf*
$\mathcal{L}_n^{\mathrm{mot}}(X\|S)_{x_0} \otimes A$	$\mathcal{L}_n^{A,\mathrm{mix}}(X\|S)$	*universal n-unipotent mixed sheaf*
$\mathcal{T}_n^{\mathrm{mot}}(X\|S) \otimes A$	$\mathcal{T}_n^{A,\mathrm{mix}}(X\|S)$	*n-th tautological mixed sheaf*
$i^{\mathrm{mot}}(X\|S)_{x_0} \otimes A$	$i^{A,\mathrm{mix}}(X\|S)_{x_0}$	

We proceed similarly with the mixed realizations of the resulting inductive systems.

Corollary 7.1.2. *We obtain the following table regarding the mixed realizations of the motivic objects considered in chapter 3:*

Motivic object	***mixed realization***
$\mathcal{C}_n^{\mathrm{mot}}(X\|S)\otimes A$	$\widetilde{B}_n^{A,\mathrm{mix}}(X^3\|X^2)_{\sigma_0,\sigma_1}$ *of Definition I.4.2.4*
$\mathcal{L}_n^{\mathrm{mot}}(X\|S)_{x_0}\otimes A$	$\widetilde{B}_n^{A,\mathrm{mix}}(X^2\|X)_{x_0\times id,\widetilde{\Delta}}$*(I.4.2.4)*
$\mathcal{T}_n^{\mathrm{mot}}(X\|S)\otimes A$	$\widetilde{B}_n^{A,\mathrm{mix}}(X^2\|X)_{\widetilde{\Delta},\widetilde{\Delta}}$*(I.4.2.4)*
$i^{\mathrm{mot}}(X\|S)_{x_0}\otimes A$	$i^{A,\mathrm{mix}}(X\|S)_{x_0}\simeq\left(\pi_*A^\sharp_X/\operatorname{Im}(e^*_{X\|S}\circ\Delta^*)\right)[-1]$,

where $e^*_{X|S}:=(\mathrm{id}_X-x_0\pi)^*\colon\mathbb{Z}_X\longrightarrow\mathbb{Z}_X$.

Proof. This is mostly a direct consequence of the definitions and section I.4.1. The only statement which doesn't go without saying is the last one, namely

$$\begin{aligned}\mathfrak{R}_{A,\mathrm{mix}}i^{\mathrm{mot}}(X|S)_{x_0}\otimes A &\simeq \pi_*A^\sharp_X/(A^\sharp_S\cup\operatorname{Im}(e^*_{X|S}\circ\Delta^*))[-1]\\ &\simeq \pi_*A^\sharp_X/\operatorname{Im}(e^*_{X|S}\circ\Delta^*))[-1].\end{aligned}$$

By definition, $i^{\mathrm{mot}}(X|S)_{x_0}$ is the diagram

$$\operatorname{coim}((\pi\times\mathrm{id})^*\Delta^*)\xrightarrow{e^*_{X|S}\Delta^*}\mathbb{Z}_X^\circ\longrightarrow 0$$

with columns in degrees $-2,-1$. Hence, the mixed realization of $i^{\mathrm{mot}}(X|S)_{x_0}\otimes A$ is the total complex of

$$\operatorname{coim}(\mathfrak{R}_{A,\mathrm{mix}}(\pi\times\mathrm{id})^*\Delta^*)\xrightarrow{\mathfrak{R}_{A,\mathrm{mix}}e^*_{X|S}\Delta^*}\pi_*A^\sharp_X/A^\sharp_S\longrightarrow 0$$

with columns in degrees $-2,-1$. In $D^b(\mathrm{Sh}(U))$, however, this complex is naturally quasi-isomorphic to $\pi_*A^\sharp_X/\operatorname{Im}(e^*_{X|S}\circ\Delta^*)[-1]$. This finishes the proof. ■

Corollary 7.1.3. *There is a direct sum decomposition*

$$x_0^*\mathcal{L}_\bullet^{A,\mathrm{mix}}(X|S)_{x_0}\cong\tilde{I}_n^{A,\mathrm{mix}}(X|S)_{x_0}\oplus A_S(0).$$

Remark 7.1.4. As a direct consequence of Lemma 6.2.3 and Corollary 6.2.4, as well as the fact that realization functors commute with pull-back according to Lemma I.4.1.1, the constructions of the canonical, universal unipotent and tautological mixed sheaves are contravariantly functorial.

7.1.2 The mixed polylogarithm

We put $U=X\setminus x_0(S)$ and again denote by $\mathrm{pr}_2\colon X\times_S U\longrightarrow U$ the projection to the second component, and $e^*_{X\times U|U}=(\mathrm{id}_{X\times U}-(x_0\times\mathrm{id}_U)\,\mathrm{pr}_2)^*$. Moreover, recall the category $\varinjlim D^b\,\mathrm{Sh}(S)$ whose objects are inductive systems in $D^b\,\mathrm{Sh}(S)$ of Definition I.3.5.1. The mixed realization functor $\mathfrak{R}_{A,\mathrm{mix}}$ naturally extends to a functor $\mathfrak{R}_{A,\mathrm{mix}}\colon\varinjlim\mathcal{DM}_A(S)\longrightarrow\varinjlim D^b\,\mathrm{Sh}(S)$. Thus we may define:

Definition 7.1.5. *We call the morphism*

$$\mathrm{pol}_n^{A,\mathrm{mix}} := \mathfrak{R}_{A,\mathrm{mix}}\,\mathrm{pol}_n^{\mathrm{mot}}(X|S)_{x_0}, \mathrm{pol}^{A,\mathrm{mix}}(X|S)_{x_0} = (\mathrm{pol}_n^{\mathrm{mix}}(X|S)_{x_0})_n$$

the mixed polylogarithm. The small mixed polylogarithm is given by

$$p_n^{A,\mathrm{mix}}(X|S)_{x_0} := \mathfrak{R}_{A,\mathrm{mix}}(p_n^{\mathrm{mot}}(X|S)_{x_0}),\ p^{A,\mathrm{mix}}(X|S)_{x_0} = (p_n^{\mathrm{mix}}(X|S)_{x_0})_n$$

7.2 The characterizing property of the mixed polylogarithm

We want to justify our claim that the motivic polylogarithm we defined is the right generalization of the polylogarithms up to date. To do so, we need to show that the mixed realization of our motivic polylogarithm satisfies properties which are characteristic of the polylogarithm, and used to define the polylogarithm uniquely in the classical cases. In order so make out this "characteristic property" we are talking of, let us take a look at an example:

7.2.1 Motivation in the geometric realization

Recall Beilinson and Levin's polylogarithm for curves:

Let $A \subset \mathbb{C}$ be a subfield. Let $S \longrightarrow \mathrm{Spec}(\mathbb{C})$ be a reduced scheme, smooth and quasi-projective over $\mathrm{Spec}(\mathbb{C})$. Moreover, we take $\pi\colon \bar{X} \longrightarrow S$ to be a family of smooth projective curves of genus $\neq 0$ in Sm_S such that π has irreducible fibers, and has three sections $x_0, x, y\colon S \longrightarrow X$. Let $j\colon X \hookrightarrow \bar{X}$ be an open immersion. We denote the complement by $D := \bar{X} \setminus X$ and call the inclusion of the complement of $x_0(S)$ by $h\colon \bar{X} \setminus x_0(S) \hookrightarrow \bar{X}$ and $\bar{x}_0 := j \circ x_0$. Denote Beilinson/Levin's polylogarithm by $\mathbb{G}$ as in chapter 5. As we have seen, it coincides with Hain/Zucker's universal pro-unipotent VMHS in the geometric case. Moreover, we denote the kernel of the augmentation $x_0^*\mathbb{G} \longrightarrow A$ by J. By Remark 5.3.3, Beilinson and Levin show the existence of an exact sequence

$$0 \to \mathrm{Ext}^1_{D^b\,\mathrm{Sh}(\bar{X}\setminus x_0(S))}(\bar{\pi}^*\mathscr{F}, j_!\mathbb{G}(1)) \stackrel{\mathrm{res}}{\to} \mathrm{Hom}_{D^b\,\mathrm{Sh}(S)}(\mathscr{F}, x_0^*\mathbb{G}) \to \mathrm{Hom}_{D^b\,\mathrm{Sh}(S)}(\mathscr{F}, A)$$

constructed from the long exact sequence associated to the distinguished triangle $\bar{x}_{0!}\bar{x}_0^! \longrightarrow \mathrm{id} \longrightarrow h_*h^*$. Beilinson and Levin's polylogarithm is then defined to be the element of $\mathrm{Ext}^1_{D^b\,\mathrm{Sh}(\bar{X}\setminus x_0(S))}(\pi^*J, j_!\mathbb{G}(1))$ which is sent to the inclusion of J into $x_0^*\mathbb{G}$ by res.

In fact, this is the most common way of defining the polylogarithm in any setting and realization. One might say that the image of the class of the polylogarithm under the boundary morphism of the long exact sequence associated to $\bar{x}_{0!}\bar{x}_0^! \longrightarrow \mathrm{id} \longrightarrow h_*h^*$ is what characterizes the polylogarithm. Let us now translate this to our setting:

Note that Beilinson-Levin's setting vaguely compares to ours as follows:

Beilinson/Levin	our situation
$(\mathbb{G})^\vee$	$\mathcal{L}^{A,\mathcal{H}dg}(X\vert S)_{x_0}$
$(J)^\vee$	$\tilde{I}^{A,\mathcal{H}dg}(X\vert S)_{x_0}$

where $(-)^\vee$ denotes the dual of VMHS (not the Verdier dual). Since we are dealing with the duals of Beilinson/Levin's sheaves here, we need to show the dual characterizing property of the polylogarithm: instead of showing that it is mapped to the inclusion of J into $x_0^*\mathbb{G}$, we want to show that the polylogarithm is sent to the projection of $x_0^*\mathcal{L}^{A,\mathcal{H}dg}(X|S)_{x_0}$ to $\tilde{I}^{A,\mathcal{H}dg}(X|S)_{x_0}$ (in the limit). On finite level, we aim to show that the polylog is mapped to the natural map

$$\mathrm{pr}_I\colon \tilde{I}_n^{A,\mathcal{H}dg}(X|S)_{x_0} \oplus A_S[0] \longrightarrow \tilde{I}_n^{A,\mathcal{H}dg}(X|S)_{x_0}$$

followed by the inclusion $\tilde{I}_n^{A,\mathcal{H}dg}(X|S)_{x_0} \hookrightarrow \tilde{I}_{n+1}^{A,\mathcal{H}dg}(X|S)_{x_0}$.

However, passing to $\bar{X}$ is not even necessary in our situation: Denote the inclusion of $U := X \setminus x_0(S)$ into X by j_U. Then like above, one can see that the distinguished triangle $\bar{x}_{0*}\bar{x}_0^! \to \mathrm{id} \to h_*h^*$ yields the long exact sequence

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^{2d-1}_{D^b\,\mathrm{MHM}(\bar{X}))}\left(j_!\mathcal{L}_n, \bar{\pi}^*\tilde{I}_{n+1}(d)\right) \arrow[r] & \mathrm{Ext}^{2d-1}_{D^b\,\mathrm{MHM}(\bar{X})}\left(j_!\mathcal{L}_n, h_*h^*\bar{\pi}^*\tilde{I}_{n+1}(d)\right) \arrow[dl, "\partial_*"] & \\
 & \mathrm{Hom}_{D^b\,\mathrm{MHM}(\bar{X})}\left(j_!\mathcal{L}_n, \bar{x}_{0*}\tilde{I}_n\right) \arrow[r] & \mathrm{Ext}^{2d}_{D^b\,\mathrm{MHM}(\bar{X})}\left(j_!\mathcal{L}_n, \bar{\pi}^*\tilde{I}_{n+1}(d)\right) \arrow[r] & {}
\end{tikzcd}$$

where we write $\mathcal{L}_n := \mathcal{L}_n^{A,\mathcal{H}dg}(X|S)_{x_0}$ and $\tilde{I}_{n+1} := \tilde{I}_{n+1}^{A,\mathcal{H}dg}(X|S)_{x_0}$. Similarly, the distinguished triangle $x_{0*}x_0^! \to \mathrm{id} \to j_{U*}j_U^*$ yields the long exact sequence

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Ext}^{2d-1}_{D^b\,\mathrm{MHM}(X)}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \arrow[r] & \mathrm{Ext}^{2d-1}_{D^b\,\mathrm{MHM}(X)}\left(\mathcal{L}_n, j_{U*}j_U^*\pi^*\tilde{I}_{n+1}(d)\right) \arrow[dl, "\partial_*"] & \\
 & \mathrm{Hom}_{D^b\,\mathrm{MHM}(X)}\left(\mathcal{L}_n, x_{0*}\tilde{I}_n\right) \arrow[r] & \mathrm{Ext}^{2d}_{D^b\,\mathrm{MHM}(X)}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \arrow[r] & {}
\end{tikzcd}$$

Using some standard six-functor-formalism-arguments, it is easy to see that these two long exact sequences are isomorphic (this is done in section E.1 of the appendix; the same works in the mixed setting). As a consequence, we will simply deal with the (simpler) distinguished triangle $x_{0*}x_0^! \to \mathrm{id} \to j_{U*}j_U^*$.

7.2.2 The defining property of the large polylogarithm

Let all notation be as above, that is to say: $F = \mathbb{C}$ in the geometric case, and $F = \mathbb{Z}[1/l]$ in the ℓ-adic case, A is either a subfield of $\mathbb{C}$ in the geometric case or $\mathbb{Q}_l$ if $F = \mathbb{Z}[1/l]$

in the étale case. Moreover, $S \longrightarrow \mathrm{Spec}(F)$ is a reduced scheme, smooth and quasi-projective over $\mathrm{Spec}(F)$. Let $\pi\colon X \longrightarrow S$ be in Sm_S such that π has geometrically irreducible fibers and three sections $x_0, x, y\colon S \longrightarrow X$.

Throughout the section, we will write $\mathcal{L}_n := \mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}$, $\tilde{I}_{n+1} := \tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}$ and $\mathrm{pol}_n := \mathrm{pol}_n^{A,\mathrm{mix}}(X|S)_{x_0}$ as well as $\widetilde{B} := \widetilde{B}^{A,\mathrm{mix}}$ since A, X, S and x_0 are fixed.

There is a well-known distinguished triangle of functors on $D^b(\mathrm{Sh}(\bar{X}))$ given by

$$x_{0*}x_0^! \to \mathrm{id} \to j_{U*}j_U^*.$$

Applied to the object $\pi^*\tilde{I}_{n+1}$ and together with the purity isomorphism

$$x_{0*}x_0^! = x_{0*}x_0^*(-d)[-2d]$$

of section C.1.2 in the appendix (which is valid more generally for mixed sheaves), we obtain the distinguished triangle

$$x_{0*}x_0^*\pi^*\tilde{I}_n[-2d](-d) \to \pi^*\tilde{I}_n \to j_{U*}j_U^*\pi^*\tilde{I}_n,$$

which (using $x_0^*\pi^* = \mathrm{id}^*$) is equal to

$$x_{0*}\tilde{I}_n[-2d](-d) \xrightarrow{i_{x_0*}} \pi^*\tilde{I}_n \longrightarrow j_{U*}\pi_U^*\tilde{I}_n. \tag{7.1}$$

Applying the derived functor $R\,\mathrm{Hom}_{D^b\,\mathrm{Sh}(X)}(\mathcal{L}_n,\ .\ )$ to the above distinguished triangle yields the following long exact sequence:

$$\begin{array}{l} \to \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \to \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, j_{U*}\pi_U^*\tilde{I}_{n+1}(d)\right) \xrightarrow{\partial_*} \\ \to \mathrm{Hom}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, x_{0*}\tilde{I}_{n+1}\right) \longrightarrow \mathrm{Ext}^{2d}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \longrightarrow \end{array} \tag{7.2}$$

Using the functorial adjunctions $x_0^* \dashv x_{0*}$ and $j_U^* \dashv j_{U*}$, this long exact sequence may be written as

$$\begin{array}{l} \to \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \to \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right) \xrightarrow{\partial_*} \\ \to \mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n \oplus A_S, \tilde{I}_{n+1}\right) \longrightarrow \mathrm{Ext}^{2d}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \longrightarrow \end{array} \tag{7.3}$$

where we used the fact that by Corollary 7.1.3 there is a direct sum decomposition $x_0^*\mathcal{L}_n \cong \tilde{I}_n \oplus A_S$.

We are interested in the element in

$$\mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n \oplus A_S, \tilde{I}_{n+1}\right)$$

the polylogarithm $\mathrm{pol}_n \in \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right)$ is mapped to by the boundary morphism ∂_*. As a main theorem, we obtain the following:

Theorem 7.2.1. *The image of the n-th mixed polylogarithm* pol_n *in*

$$\mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n \oplus A_S, \tilde{I}_{n+1}\right)$$

under the above boundary morphism ∂_* is given by the morphism

$$i_{n,n+1} \circ \mathrm{pr}_I \colon \tilde{I}_n \oplus A_S \twoheadrightarrow \tilde{I}_n \hookrightarrow \tilde{I}_{n+1}$$

where pr_I *is the projection to the summand and* $i_{n,n+1}$ *is the natural inclusion.*

<u>**Outline of Proof:**</u>

- *First step:* We translate the assertion of the theorem from a statement about the long exact sequence (7.3) to one of the isomorphic sequence (7.2) above. This way, we can present arguments using the underlying distinguished triangle (7.1)

$$x_{0*}\tilde{I}_n[-2d](-d) \xrightarrow{i_{x_0*}} \pi^*\tilde{I}_n \longrightarrow j_{U*}\pi_U^*\tilde{I}_n \xrightarrow{\partial} x_{0*}\tilde{I}_n[-2d+1](-d).$$

- *Second step:* Since the map ∂_* is given by sending a morphism f to the composition $\partial \circ f$ with boundary morphism ∂ of the distinguished triangle (7.1) above, we need to get our hands on the boundary morphism ∂. This is done via the natural quasi-isomorphism $q\colon \mathrm{Cone}(i_{x_0*}) \longrightarrow j_{U*}\pi_U^*\tilde{I}_{n+1}$. The reason for this is the following: Replacing the triangle (7.1) by the quasi-isomorphic triangle

$$x_{0*}\tilde{I}_n[-2d](-d) \xrightarrow{i_{x_0*}} \pi^*\tilde{I}_n \longrightarrow \mathrm{Cone}(i_{x_0*}) \xrightarrow{\partial} x_{0*}\tilde{I}_n[-2d+1](-d).$$

  replaces the boundary morphism ∂ by the natural projection of $\mathrm{Cone}(i_{x_0*})$ to one summand. Thus, in order to find the image of pol_n under ∂_*, we need to determine which element of

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \mathrm{Cone}(i_{x_0*})(d)\right)$$

  corresponds to

$$\mathrm{pol}_n \in \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right)$$

  under the natural quasi-isomorphism $q\colon \mathrm{Cone}(i_{x_0*}) \longrightarrow j_{U*}\pi_U^*\tilde{I}_{n+1}$. The aim of the second step is to explicitly determine this morphism.

- *Third step:* Combine Step 1 and Step 2 to prove the theorem.

Let us start with the first step:

First step

In this section, we trace the assertion of the theorem regarding sequence (7.3) back to an assertion regarding sequence (7.2). Recall that sequence (7.3) was constructed from sequence (7.2) by using adjunction isomorphisms. As a consequence, we need to reverse this procedure to obtain our aim.

This reverse-adjunction-process has to be applied to both morphisms that are compared in the assertion of the theorem:

- $\underline{\mathrm{pol}_n}$: The adjunction isomorphism

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right) \cong \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, j_{U*}\pi_U^*\tilde{I}_{n+1}(d)\right)$$

maps pol_n to the morphism

$$\widetilde{\mathrm{pol}}_n(U)\colon \quad \mathcal{L}_n \xrightarrow{\mathrm{ad}_{j_{U*}j_U^*}} j_{U*}j_U^*\mathcal{L}_n \xrightarrow{j_{U*}(\mathrm{pol}_n^{\mathrm{mot}})} j_{U*}\pi_U^*\tilde{I}_{n+1}[2d-1](d),$$

where $\mathrm{ad}_{j_{U*}j_U^*}$ is the adjunction morphism and $j_{U*}(\mathrm{pol}_n^{\mathrm{mot}})$ is induced by $\mathrm{pol}_n^{\mathrm{mot}}$.

- $\underline{i_{n,n+1}\circ \mathrm{pr}_I}$: Recall the morphism $i_{n,n+1}\circ \mathrm{pr}_I\colon \tilde{I}_n \oplus A_S \twoheadrightarrow \tilde{I}_n \hookrightarrow \tilde{I}_{n+1}$, where pr_I is the projection to the summand and $i_{n,n+1}$ is the natural inclusion. The adjunction isomorphism

$$\mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n \oplus A_S, \tilde{I}_{n+1}\right) \cong \mathrm{Hom}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, x_{0*}\tilde{I}_{n+1}\right)$$

maps $i_{n,n+1}\circ \mathrm{pr}_I$ to the morphism

$$\mathrm{pr}\circ \mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}\colon \mathcal{L}_n \longrightarrow x_{0*}\tilde{I}_{n+1},$$

where $\mathrm{ad}_{x_{0*}x_0^*}\colon \mathcal{L}_n \longrightarrow x_{0*}x_0^*\mathcal{L}_n$ is the adjunction morphism, $\mathrm{pr}\colon x_{0*}x_0^*\mathcal{L}_n \longrightarrow x_{0*}\tilde{I}_n$ is the morphism induced by the natural projection $x_0^*\mathcal{L}_n = \tilde{I}_n \oplus A_S^\sharp \twoheadrightarrow \tilde{I}_n$, and $i_{n,n+1} := x_{0*}(i_{n,n+1})\colon x_{0*}\tilde{I}_n \hookrightarrow x_{0*}\tilde{I}_{n+1}$ is induced by the natural inclusion.

The assertion of Theorem 7.2.1 is therefore equivalent to the following:

Theorem 7.2.2. *The morphism*

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, j_{U*}\pi_U^*\tilde{I}_{n+1}(d)\right) \xrightarrow{\partial_*} \mathrm{Hom}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, x_{0*}\tilde{I}_{n+1}\right)$$

of sequence (7.2) *maps* $\widetilde{\mathrm{pol}}_n(U)$ *to the morphism* $\mathrm{pr}\circ \mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}$.

Second step

Consider the distinguished triangle (7.1)

$$x_{0*}\tilde{I}_n[-2d](-d) \xrightarrow{i_{x_0*}} \pi^*\tilde{I}_n \longrightarrow j_{U*}\pi_U^*\tilde{I}_n.$$

Recall that our aim in the second step is to determine which element of

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \mathrm{Cone}(i_{x_0*})(d)\right)$$

corresponds to

$$\mathrm{pol}_n \in \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right)$$

via the natural quasi-isomorphism

$$q\colon \operatorname{Cone}(i_{x_0*}) \longrightarrow j_{U*}\pi_U^*\tilde{I}_{n+1}.$$

We will present a good candidate for such a pre-image of $\mathrm{pol}_n^{\mathrm{mot}}$ before actually proving that it is mapped to $\mathrm{pol}_n^{\mathrm{mot}}$ by q. This candidate will consist of two components, which we introduce first:

- The first is a slight variant of the mixed polylogarithm. We consider the idempotent

$$e_{X^2|X} := \mathrm{id}_{X^2} - (x_0 \times \mathrm{id}_X)\,\mathrm{pr}_2\colon X^2 \longrightarrow X^2.$$

  It induces an idempotent on $\mathbb{Z}_{X^2}$ and hence also on $\pi^*\tilde{I}_n = \tilde{I}_n(X^2|X)_{x_0\times \mathrm{id}_X}$, which we denote by $e^{n*}_{X^2|X}$. Then we define

$$\widetilde{\mathrm{pol}}_n(X) = \mathfrak{R}_{A,\mathrm{mix}}\left(e^{(n+1)*}_{X^2|X} \circ \left(\mathrm{id}_{X^n\times_S X} \times \widetilde{\Delta}\right)_*\right)\colon \mathcal{L}_n \longrightarrow \pi^*\tilde{I}_{n+1}.$$

  Note that the composition of $\widetilde{\mathrm{pol}}_n(X)$ with the adjunction morphism

$$\mathrm{ad}_{j_{U*}j_U^*}\colon \pi^*\tilde{I}_n \longrightarrow j_{U*}j_U^*\pi^*\tilde{I}_n = j_{U*}\pi_U^*\tilde{I}_n$$

  yields the morphism $\widetilde{\mathrm{pol}}_n(U)$ for the following reason:

$$\mathrm{ad}_{j_{U*}j_U^*} \circ \widetilde{\mathrm{pol}}_n(X) = \mathrm{ad}_{j_{U*}j_U^*} \circ \mathfrak{R}_{A,\mathrm{mix}}\left(e^{(n+1)*}_{X^2|X} \circ \left(\mathrm{id}_{X^n\times_S X} \times \widetilde{\Delta}\right)_*\right)$$

  is the morphism corresponding to

$$\begin{aligned} j_U^*\left(\mathfrak{R}_{A,\mathrm{mix}} e^{(n+1)*}_{X^2|X}\left(\mathrm{id}_{X^n\times_S X} \times \widetilde{\Delta}\right)_*\right) &= \mathfrak{R}_{A,\mathrm{mix}}\mathcal{DM}(j_U^*) e^{(n+1)*}_{X^2|X}\left(\mathrm{id}_{X^n\times_S X} \times \widetilde{\Delta}\right)_* \\ = \mathfrak{R}_{A,\mathrm{mix}} e^{(n+1)*}_{X\times U|U}\left(\mathrm{id}_{X^n\times_S U} \times \widetilde{\Delta}\right)_* &= \mathrm{pol}_n^{\mathrm{mix}} \end{aligned}$$

  where $e_{X\times U|U} := \mathrm{id}_{X\times_S U} - (x_0 \times \mathrm{id}_U)\,\mathrm{pr}_2\colon \mathbb{Z}_{X\times U} \longrightarrow \mathbb{Z}_{X\times U}$.
- The second is the morphism $\mathrm{pr} \circ \mathrm{ad}_{x_{0*}x_0^*} \circ i_{n,n+1}\colon \mathcal{L}_n \longrightarrow \pi^*\tilde{I}_{n+1}$ of Step 1.

With these notations, we obtain:

Proposition 7.2.3. *The morphism*

$$\left(\mathrm{pr} \circ \mathrm{ad}_{x_{0*}x_0^*} \circ i_{n,n+1}, \widetilde{\mathrm{pol}}_n(X)\right) : \mathcal{L}_n \longrightarrow \operatorname{Cone}(i_{x_0*})(d)[2d]$$

is a morphism of complexes. It is mapped to $\widetilde{\mathrm{pol}}_n(U)$ *under the isomorphism*

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{L}_n, \operatorname{Cone}(i_{x_0*})(d)\right) \cong \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{L}_n, j_{U*}\pi_U^*\tilde{I}_{n+1}(d)\right)$$

induced by the quasi-isomorphism $q\colon \operatorname{Cone}(i_{x_0*}) \longrightarrow j_{U*}\pi_U^*\tilde{I}_{n+1}$.

Proof. We have to prove that $\left(\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1},\widetilde{\mathrm{pol}}_n(X)\right)$ commutes with the differentials of both sides. To this end, let us first recall the differentials of all complexes involved, where we write $\mathfrak{R}=\mathfrak{R}_{A,\mathrm{mix}}$:

$$\begin{aligned}
&\mathcal{L}_n=\widetilde{B}_n(X|S)_{x_0\times\mathrm{id},\widetilde{\Delta}} \text{ with differential } \widetilde{\delta}_k^{\mathrm{mix}}(0,\widetilde{\Delta})\\
&x_{0*}\tilde{I}_n \text{ with differential } x_{0*}(\widetilde{\delta}_k^{\mathrm{mix}}(0,0))\\
&\pi^*\tilde{I}_n=\tilde{I}_n(X^2|X)_{x_0\times\mathrm{id}} \text{ has differential } \widetilde{\delta}_{k+1}^{\mathrm{mix}}(0,0),
\end{aligned}$$

so the differential of

$$\mathrm{Cone}(i_{x_0*})(d)[2d]=\left(x_{0*}\tilde{I}_{n+1}(d)[1]\right)\oplus\tilde{I}_{n+1}(X^2|X)_{x_0\times\mathrm{id}}(d)[2d]$$

is given by

$$d_k^C:=\begin{pmatrix}-x_{0*}\left(\delta_{k-1}^{\circ\,\mathrm{mix}}(0,0)\right) & 0\\ (-1)^{2d}i_{x_0*} & (-1)^{2d}\widetilde{\delta}_{k+1}^{\mathrm{mix}}(0,0)\end{pmatrix}.$$

We need to show that $\left(\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1},\mathrm{pol}_n(X)\right)\circ\widetilde{\delta}^{\mathrm{mix}}(0,\widetilde{\Delta})=d^C\circ\mathrm{pol}_n$, that is to say that

$$\begin{aligned}
&\left(\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1},\widetilde{\mathrm{pol}}_n(X)\right)\circ\widetilde{\delta}_k^{\mathrm{mix}}(0,\widetilde{\Delta})=\\
&=\begin{pmatrix}-x_{0*}\left(\delta_{k-1}^{\circ\,\mathrm{mix}}(0,0)\right) & 0\\ (-1)^{2d}i_{x_0*} & (-1)^{2d}\widetilde{\delta}_{k+1}^{\mathrm{mix}}(0,0)\end{pmatrix}\cdot\begin{pmatrix}\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}\\ \widetilde{\mathrm{pol}}_n(X)\end{pmatrix},
\end{aligned}$$

where we wrote $\widetilde{\delta}^{\mathrm{mix}}$ instead of $\widetilde{\delta}^{\mathrm{mix}}(0,0)$. This is equivalent to

$$\begin{aligned}
\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}\circ\widetilde{\delta}^{\mathrm{mix}}(0,\widetilde{\Delta})&=x_{0*}\left(\widetilde{\delta}^{\mathrm{mix}}(0,0)\right)\circ\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}\\
\widetilde{\mathrm{pol}}_n(X)\circ\widetilde{\delta}^{\mathrm{mix}}(0,\widetilde{\Delta})&=\widetilde{\delta}^{\mathrm{mix}}(0,0)\circ\widetilde{\mathrm{pol}}_n(X)+i_{x_0*}\circ\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}.
\end{aligned}$$

Since $\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}$ and $\widetilde{\mathrm{pol}}_n(X)$ are morphisms of complexes, one obviously has

$$\begin{aligned}
\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}\circ\widetilde{\delta}^{\mathrm{mix}}(0,\widetilde{\Delta})&=x_{0*}\left(\widetilde{\delta}^{\mathrm{mix}}(0,0)\right)\circ\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}\\
\widetilde{\mathrm{pol}}_n(X)\circ\widetilde{\delta}^{\mathrm{mix}}(0,\widetilde{\Delta})&=\widetilde{\delta}^{\mathrm{mix}}(0,0)\circ\widetilde{\mathrm{pol}}_n(X),
\end{aligned}$$

so $\left(\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1},\widetilde{\mathrm{pol}}_n(X)\right)$ is a morphism of complexes if and only if $i_{x_0*}\circ\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}$ is zero. This is a consequence of Lemma 7.2.4 below, which proves the first assertion. The second assertion is immediate by the fact that

$$j_U^*\circ\widetilde{\mathrm{pol}}_n(X)=\widetilde{\mathrm{pol}}_n(U).\qquad\blacksquare$$

Lemma 7.2.4. *The following morphism is zero in $D^b\,\mathrm{Sh}(X)$ for all n:*

$$i_{x_0*}\circ \mathrm{pr}\circ \mathrm{ad}_{x_{0*}x_0^*}\colon \mathcal{L}_n[-2d](-d)\to \pi^*\tilde{I}_n=\tilde{I}_n(X^2|X)_{x_0\times \mathrm{id}}.$$

Proof. The morphism $i_{x_0*}\circ \mathrm{pr}\circ \mathrm{ad}_{x_{0*}x_0^*}\colon \mathcal{L}_n(-d)[-2d]\longrightarrow \pi^*\tilde{I}_n$ is induced by the following morphism of double complexes by taking total complexes:

$$\begin{tikzcd}
\left(\frac{\mathrm{pr}_{2*}A^\sharp_{X^2}}{A^\sharp_X}\right)^{\otimes n}(-d)[-2d] \arrow[r] \arrow[d, "i_{x_0*}\circ\mathrm{ad}_{x_{0*}x_0^*}"] & \dots \arrow[r] & \left(\frac{\mathrm{pr}_{2*}A^\sharp_{X^2}}{A^\sharp_X}\right)^{\otimes 2}(-d)[-2d] \arrow[r] \arrow[d, "i_{x_0*}\circ\mathrm{ad}_{x_{0*}x_0^*}"] & \frac{\mathrm{pr}_{2*}A^\sharp_{X^2}}{A^\sharp_X}(-d)[-2d] \arrow[r] \arrow[d, "i_{x_0*}\circ\mathrm{ad}_{x_{0*}x_0^*}"] & 0 \\
\left(\frac{\mathrm{pr}_{2*}A^\sharp_{X^2}}{A^\sharp_X}\right)^{\otimes n} \arrow[r] & \dots \arrow[r] & \left(\frac{\mathrm{pr}_{2*}A^\sharp_{X^2}}{A^\sharp_X}\right)^{\otimes 2} \arrow[r] & \frac{\mathrm{pr}_{2*}A^\sharp_{X^2}}{A^\sharp_X} \arrow[r] & 0
\end{tikzcd}$$

where we denote the second projection by $\mathrm{pr}_2\colon X^2\longrightarrow X$, the differentials in the upper complex are $\widetilde{\delta}^*_k(0,\widetilde{\Delta})$, and those of the bottom complex are given by $\widetilde{\delta}^*_k(0,0)$. By Künneth and since

$$\mathrm{pr}_{2*}A^\sharp_{X^2}/A^\sharp_X=(\pi^*\pi_*A^\sharp_X)/A^\sharp_X,$$

it suffices to show that the morphism of complexes

$$\begin{tikzcd}
(\pi^*\pi_*A^\sharp_X)/A^\sharp_X(-d)[-2d] \arrow[r, "\mathrm{ad}_{x_{0*}x_0^*}"] & x_{0*}x_0^*(\pi^*\pi_*A^\sharp_X)/A^\sharp_X(-d)[-2d] \arrow[dl, "i_{x_0*}"] \\
(\pi^*\pi_*A^\sharp_X)/A^\sharp_X &
\end{tikzcd}$$

is zero. Note that $x_{0*}x_0^*\pi^*\pi_*A^\sharp_X=x_{0*}\pi_*A^\sharp_X$. In both the geometric and the ℓ-adic realization for a prime l, $\pi_*A^\sharp_X$ is a complex whose cohomology is concentrated in degrees $0,\dots,2d$ (see Remark 4.2.2 in section I.4.2). Thus, there is a quasi-isomorphism $\pi_*A^\sharp_X\simeq\tau_{\le 2d}\pi_*A^\sharp_X$, i.e. an isomorphism

$$\begin{array}{c}
\mathrm{Hom}_{D^b\,\mathrm{Sh}(X)}((\pi^*\pi_*A^\sharp_X)/A^\sharp_X(-d)[-2d],(\pi_*A^\sharp_X)/A_X) \\
\wr\| \\
\mathrm{Hom}_{D^b\,\mathrm{Sh}(X)}\left((\pi^*\pi_*A^\sharp_X)/A^\sharp_X(-d)[-2d],\tau_{\le 2d}(\pi^*\pi_*A^\sharp_X)/A^\sharp_X\right).
\end{array}$$

Hence, it suffices to prove that the image of $i_{x_0*}\circ\mathrm{ad}_{x_{0*}x_0^*}$ in

$$\mathrm{Hom}_{D^b\,\mathrm{Sh}(S)}((\pi_*A^\sharp_X)/A^\sharp_X(-d)[-2d],\tau_{\le 2d}(\pi_*A^\sharp_X)/A^\sharp_X)$$

is zero. Since

$$i_{x_0*}\circ\mathrm{ad}_{x_{0*}x_0^*}\colon(\pi^*\pi_*A^\sharp_X)/A^\sharp_X(-d)[-2d]\to\tau_{\le 2d}(\pi^*\pi_*A^\sharp_X)/A^\sharp_X$$

is a morphism from a complex in degrees $\geq 2d$ to a complex in degrees $\leq 2d$, all arrows are zero unless possibly in degree $2d$. Therefore, it is zero if and only if the morphism

$$i_{x_0*} \circ \mathrm{ad}_{x_{0*}x_0^*} \colon \ (\pi^*\pi_* A_X^\sharp)^0 \Big/ A_X^\sharp \to (\pi^*\pi_* A_X^\sharp)^{2d}$$

is zero. Let us consider the morphism in stalks $x \in X$: Here, by the definition of the Godement resolution we have

$$(\pi^*\pi_* A_X^\sharp)^0_x/(A_X^\sharp)_x = \left(\prod_{y\in X_{\pi(x)}} A\right) \Big/ A = \prod_{y\in X_{\pi(x)}\setminus\{x\}} A$$

since the inclusion $\pi^* \colon A \hookrightarrow \prod_{y\in X_{\pi(x)}} A$ is equal to the inclusion of the component for $x \in X_{\pi(x)}$. Let us consider the adjunction morphism at the stalk $x \in X$: For $x \neq x_0(\pi(x))$, the stalk of the adjunction at x is zero since the right hand side is. For $x = x_0(\pi(x))$, the stalk of the adjunction $\mathrm{ad}_{x_{0*}x_0^*} \colon (\pi^*\pi_* A_X^\sharp)^0 \to (\pi^*\pi_* A_X^\sharp)^0$ at x factors as

$$\mathrm{ad}_{x_{0*}x_0^*} \colon \prod_{y\in X_{\pi(x)}} A \twoheadrightarrow (A_X)_x \hookrightarrow \prod_{y\in X_{\pi(x)}} A$$

and hence induces the zero morphism on $(\pi^*\pi_* A_X^\sharp)^0_x/(A_X^\sharp)_x = \prod_{y\in X_{\pi(x)}\setminus\{x\}} A$. This proves the assertion. ∎

Third step: Proof of Theorem 7.2.1.

By Step 1, Theorem 7.2.1 is equivalent to Theorem 7.2.2, so we will prove the latter. To achieve this we use the result of Step 2: The morphism

$$\left(\mathrm{pr} \circ \mathrm{ad}_{x_{0*}x_0^*} \circ i_{n,n+1}, \widetilde{\mathrm{pol}}_n(X)\right) : \mathcal{L}_n \longrightarrow \mathrm{Cone}(i_{x_0*})(d)[2d]$$

corresponds to $\widetilde{\mathrm{pol}}_n(U)$ under the isomorphism

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{L}_n, j_{U*}\pi_U^* \tilde{I}_{n+1}(d)\right) \cong \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{L}_n, \mathrm{Cone}(i_{x_0*})(d)\right)$$

induced by the natural quasi-isomorphism $q \colon \mathrm{Cone}(i_{x_0*}) \longrightarrow j_{U*}\pi_U^*\tilde{I}_{n+1}$.
This yields the following diagram:

$$\begin{tikzcd}
x_{0*}\tilde{I}_n[-1] \arrow[r, "i_{x_0*}"] \arrow[d, equal] & \pi^*\tilde{I}_n(d)[2d-1] \arrow[r] \arrow[d, equal] & j_{U*}\pi_U^*\tilde{I}_n(d)[2d-1] \arrow[r, "\partial"] & x_{0*}\tilde{I}_n \arrow[d, equal] \\
x_{0*}\tilde{I}_n[-1] \arrow[r, "i_{x_0*}"] & \pi^*\tilde{I}_n(d)[2d-1] \arrow[r] & \mathrm{Cone}(i_{x_0*}) \arrow[r, "\mathrm{pr}"] \arrow[u, "\sim"'] & x_{0*}\tilde{I}_n
\end{tikzcd}$$

The right square can be completed to a pyramid

$$\begin{tikzcd}
 & \mathcal{L}_n \arrow[d, "\widetilde{\mathrm{pol}}_n(U)"] \arrow[ddl, "\left(\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}{,}\widetilde{\mathrm{pol}}_n(X)\right)"'] \arrow[ddr, "\mathrm{pr}\circ\mathrm{ad}_{x_{0*}x_0^*}\circ i_{n,n+1}"] \arrow[drr, dashed] & & \\
 & j_{U*}\pi_U^*\tilde{I}_n(d)[2d-1] \arrow[rr, "\partial"] & & x_{0*}\tilde{I}_n \arrow[dl, equal] \\
 \mathrm{Cone}(i_{x_0*}) \arrow[ur, "\sim"] \arrow[rr, "\mathrm{pr}"] & & x_{0*}\tilde{I}_n &
\end{tikzcd}$$

where we are interested in the discontinuous arrow. As a direct consequence,

$$\partial_*(\widetilde{\mathrm{pol}}_n(U)) = \partial \circ \widetilde{\mathrm{pol}}_n(U) = \mathrm{pr} \circ \mathrm{ad}_{x_{0*}x_0^*} \circ i_{n,n+1}.$$

This proves the assertion. ∎

7.2.3 The defining property of the small polylogarithm

Recall the small mixed polylogarithm of section 7.1.2 and the projection $\mathrm{pr}_i \colon \tilde{I}_{n+1} \twoheadrightarrow i$. The distinguished triangle of functors on $D^b(\mathrm{Sh}(X))$ given by $x_{0*}x_0^*(-d)[-2d] \to \mathrm{id} \to j_{U*}j_U^*$ thus yields a morphism of distinguished triangles

$$\begin{tikzcd}
x_{0*}\tilde{I}_{n+1}(-d)[-2d] \arrow[r] \arrow[d, "\mathrm{pr}_i"] & \pi^*\tilde{I}_{n+1} \arrow[r] \arrow[d, "\mathrm{pr}_i"] & \pi_U^*\tilde{I}_{n+1} \arrow[d, "\mathrm{pr}_i"] \\
x_{0*}i(-d)[-2d] \arrow[r] & \pi^*i \arrow[r] & \pi_U^*i
\end{tikzcd}$$

giving rise to the corresponding morphism pr_{i*} between the long exact sequences

$$\begin{tikzcd}
\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \arrow[r, "\mathrm{pr}_{i*}"] \arrow[d] & \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*i(d)\right) \arrow[d] \\
\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right) \arrow[r, "\mathrm{pr}_{i*}"] \arrow[d, "\partial_*"] & \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*i(d)\right) \arrow[d, "\partial_*"] \\
\mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(x_0^*\mathcal{L}_n, \tilde{I}_{n+1}\right) \arrow[r, "\mathrm{pr}_{i*}"] \arrow[d] & \mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(x_0^*\mathcal{L}_n, i\right) \arrow[d] \\
\mathrm{Ext}^{2d}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*\tilde{I}_{n+1}(d)\right) \arrow[r, "\mathrm{pr}_{i*}"] & \mathrm{Ext}^{2d}_{D^b(\mathrm{Sh}(X))}\left(\mathcal{L}_n, \pi^*i(d)\right).
\end{tikzcd}$$

Hence, we immediately obtain:

Theorem 7.2.5 (Universal property of the small polylogarithm). *The image of the n-th small mixed polylogarithm $p_n^{A,\mathrm{mix}}$ under ∂_* in the above short exact sequence is the morphism in* $\mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(x_0^*\mathcal{L}_n^{A,\mathrm{mix}}(X|S), i^{A,\mathrm{mix}}(X|S)\right)$ *given by the natural projection*

$$\mathrm{pr}_{i^{A,\mathrm{mix}}} \circ \mathrm{pr}_I \colon x_0^*\mathcal{L}_n^{A,\mathrm{mix}} \cong \tilde{I}_n^{A,\mathrm{mix}}(X|S) \oplus A_S \longrightarrow \tilde{I}_n^{A,\mathrm{mix}}(X|S) \longrightarrow i^{A,\mathrm{mix}},$$

where pr_I *is the natural projection.*

Proof. By construction, the mixed polylogarithm $\mathrm{pol}_n^{A,\mathrm{mix}}$ is an element in

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, j_U^*\tilde{I}_{n+1}(X^2|X)_{x_0\times\mathrm{id}}(d)\right)$$

mapping to p_n in

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, j_U^* i(d)\right)$$

via the morphism pr_{i*} of long exact sequences. By the main theorem 7.2.1 above, $\partial_* \mathrm{pol}_n$ is the morphism in

$$\mathrm{Hom}_{D^b(\mathrm{Sh}(S))}\left(\widetilde{B}_n(X|S)_{x_0,x_0}, \tilde{I}_n\right)$$

given by the natural projection $\mathrm{pr}_{\tilde{I}} \colon \widetilde{B}_n(X|S)_{x_0,x_0} \longrightarrow \tilde{I}_n$, which is obviously mapped to the natural projection

$$\mathrm{pr}_I \circ \mathrm{pr}_{\tilde{I}} \colon \widetilde{B}_n(X|S)_{x_0,x_0} \longrightarrow \tilde{I}_n \longrightarrow i \;\text{ by } \mathrm{pr}_i. \qquad \blacksquare$$

7.3 A comparison to the polylogarithm in literature

Literature on the polylogarithm up to the day considers the polylogarithm only in two cases: for $\pi\colon X \longrightarrow S$ an abelian scheme, or a family of curves. For more general schemes, a construction of the polylogarithm was deemed impossible so far, given that the usual construction method of the polylogarithm in literature heavily relies on a calculation of the higher direct images of the logarithm, which is not possible in a more general setting. In the case where $X \longrightarrow S$ is either an abelian scheme or a family of curves, literature views the mixed polylogarithm not as a class in

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{L}_n, \pi_U^*\tilde{I}_{n+1}(d)\right),$$

where again we write $\mathcal{L}_n := \mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}$ and $\tilde{I}_{n+1} = \tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}$, but as one in

$$\begin{array}{c}\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{H}^0(j_U^*\mathcal{L}_n), \mathcal{H}^0(\pi_U^*\tilde{I}_{n+1})(d)\right)\\ \parallel \\ \mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{H}^0(\mathcal{L}_n), \pi_U^*\mathcal{H}^0(\tilde{I}_{n+1})(d)\right).\end{array}$$

As explained many times before, this is due to the fact that the logarithm in literature is usually the zeroth cohomology group of our mixed logarithm $\mathcal{L}_n$, not the entire complex. The only reason why we were able to generalize the polylogrithm to arbitrary smooth quasi-projective schemes is that we did *not* a priori pass to the zeroth cohomology. However, now we are left with the task to justify that in the special case where $X \longrightarrow S$ is an abelian scheme or a family of curves, the polylogarithm we constructed indeed yields the polylogarithm as the literature of the past 20 years knows it. It is not immediately clear why $\mathrm{pol}_n^{\mathrm{mix}}(X|S)_{x_0}$ would yield an element in

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{H}^0(j_U^*\mathcal{L}_n), \mathcal{H}^0(\pi_U^*\tilde{I}_{n+1})(d)\right).$$

While it suffices to prove that the zeroth cohomology is the smallest non-zero cohomology of $j_U^*\mathcal{L}_n$ in order to get an induced morphism in

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(\mathcal{H}^0(j_U^*\mathcal{L}_n), \pi_U^*\tilde{I}_{n+1}(d)\right),$$

there is no obvious reason why it should induce a morphism to $\mathcal{H}^0(\pi_U^*\tilde{I}_{n+1})(d)$. In general, one cannot hope for such a result. However, in the case of curves of nonzero genus and abelian varieties, we know that cohomology is particularly nice. For the rest of the section, we let H^i_{mix} denote either the ℓ-adic or Betti cohomology.

7.3.1 Passing to cohomology

The one reason why our motivic polylogarithm yields the classical polylogarithm for curves and abelian schemes is the following nice cohomological property:

Proposition 7.3.1. *Let $\pi\colon X \longrightarrow S$ be either a family of smooth curves of genus $\neq 0$ or an abelian scheme, where S is an A-scheme. Then for all $q > 1$ and every fiber $X_s = \pi^{-1}(s)$ for $s \in S$, the morphism*

$$\bigoplus_{\substack{i+j=q,\\ i,j\geq 1}} H^i(X_s, A) \otimes H^j_{\mathrm{mix}}(X_s, A) \longrightarrow H^q_{\mathrm{mix}}(X_s, A) \quad \textit{is surjective.}$$

Proof. This is due to the fact that X_s is either a curve of genus $\neq 0$ or an abelian variety: If X_s a curve of genus $\neq 0$ the only thing to see is that $\varDelta^*\colon H^1_{\mathrm{mix}}(X_s, A) \otimes H^1_{\mathrm{mix}}(X_s, A) \longrightarrow H^2_{\mathrm{mix}}(X_s, A)$ is surjective, which is clear. If X_s is an abelian variety, then both its Betti and its ℓ-adic cohomology (see, for example, Lemma 11.1 in [Mil98]) satisfies $H^q_{\mathrm{mix}}(X_s, A) \cong \bigwedge^q H^1_{\mathrm{mix}}(X_s, A)$, so the arrow

$$\oplus_{\substack{i+j=q,\\ i,j\geq 1}} H^i_{\mathrm{mix}}(X_s, A) \otimes H^j_{\mathrm{mix}}(X_s, A) \longrightarrow H^q_{\mathrm{mix}}(X_s, A)$$

is obviously surjective. ■

Definition 7.3.2. *Let X be a scheme. We say that X has the property* ($\bigstar$), *if the following condition is satisfied: for all $q > 1$, the morphism*

$$\bigoplus_{\substack{i+j=q,\\ i,j\geq 1}} H^i_{\mathrm{mix}}(X_s, A) \otimes H^j_{\mathrm{mix}}(X_s, A) \longrightarrow H^q_{\mathrm{mix}}(X_s, A) \quad \textit{is surjective.}$$

The following theorem will prove to be the ingredient needed for the polylogarithm to descend to a morphism in

$$\mathrm{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\Big(\mathcal{H}^0(j_U^*\mathcal{L}_n), \mathcal{H}^0(j_U^*\tilde{I}^{A,\mathrm{mix}}_{n+1}(X^2|X)_{x_0\times\mathrm{id}})(d)\Big)$$

in the case of relative curves of genus $\neq 0$ or abelian schemes:

We stick to our usual notation: Let $F = \mathbb{C}$ in the geometric case, and $F = \mathbb{Z}[1/l]$ in the ℓ-adic case, A is either a subfield of $\mathbb{C}$ in the geometric case or $\mathbb{Q}_l$ if $F = \mathbb{Z}[1/l]$ in the étale case. $S \longrightarrow \mathrm{Spec}(F)$ is a reduced scheme, smooth and quasi-projective over $\mathrm{Spec}(F)$. $\pi\colon X \longrightarrow S$ is in Sm_S such that π has geometrically irreducible fibers and three sections $x_0, x, y\colon S \longrightarrow X$ of π. In this setting, we obtain:

Theorem 7.3.3. *Let every fiber $X_s = \pi^{-1}(s)$ have the property* ($\bigstar$) *and be the complement of a normal crossing divisor (which may also be $\emptyset$) in a projective scheme. Then the inclusion of normalized mixed bar complexes*

$$\widetilde{B}^{A,\mathrm{mix}}_n(X|S)_{x_0,x_0} \hookrightarrow \widetilde{B}^{A,\mathrm{mix}}_{n+1}(X|S)_{x_0,x_0}$$

induces the zero map on cohomology in degrees $\neq 0$.

In order to prove this, we divide the theorem into two special cases:

- First, we prove the assertion for the case that all fibers X_s are affine. This will be done using spectral sequence arguments.
- Secondly, we prove the assertion for the case that all fibers X_s are projective. Again, we use the spectral sequence of the affine case, only this time, the argument will be slightly more complicated.

This will then prove the general assertion, since the complement of a normal crossing divisor in a projective curve is affine.

7.3.2 The case of affine fibers

Theorem 7.3.4. *Suppose every fiber $X_s = \pi^{-1}(s)$ is affine and has the property* ($\bigstar$), *e.g. if $\pi\colon X \longrightarrow S$ is a family of irreducible smooth affine curves. Then the inclusion $\widetilde{B}^{\mathrm{mix}}_n(X|S)_{x_0,x_0} \hookrightarrow \widetilde{B}^{\mathrm{mix}}_{n+1}(X|S)_{x_0,x_0}$ induces the zero morphism on cohomology in degrees $\neq 0$.*

Proof. It suffices to prove the assertion in stalks, so we may reduce the Theorem to the case where X is affine, $S = \operatorname{Spec}(k) =: \mathrm{pt}$ is a point, and $x_0 \in X(k)$. Recall that $\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}$ is the total complex of the double complex

$$\begin{array}{cccccccccc}
\left(\frac{\pi_* A_X^\sharp}{A[0]}\right)^{\otimes n} & \xrightarrow{\widetilde{\delta}_{n-1}^*(0,0)} & \dots & \xrightarrow{\widetilde{\delta}_2^*(0,0)} & \left(\frac{\pi_* A_X^\sharp}{A[0]}\right)^{\otimes 2} & \xrightarrow{\Delta^*} & \frac{\pi_* A_X^\sharp}{A[0]} & \xrightarrow{0} & A[0]. & \\
-n & & \dots & & -2 & & -1 & & 0 & \deg_{\mathrm{simpl}}
\end{array}$$

Hence, there is a bounded second quadrant spectral sequence of A-modules

$$E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) := H^q\left(\left(\pi_* A_X^\sharp / A[0]\right)^{\otimes p}\right) \quad \Rightarrow H^{-p+q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$$

with morphisms $d_1^{-p,q} := \widetilde{\delta}_{n-1}^*(0,0)\colon E_1^{-p,q} \longrightarrow E_1^{-p+1,q}$. By Künneth we have

$$E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) = \left(\widetilde{H}^\bullet_{\mathrm{mix}}(X, A)^{\otimes p}\right)^q,$$

where $\widetilde{H}$ denotes reduced cohomology. Since the mixed cohomology of an affine curve is nonzero only in degrees 0 and 1, the first sheet of this spectral sequence is nonzero only for $p = q$, i.e. $E_1^{-p,q}$ looks as follows:

$$\begin{array}{ccccc|l}
 & & & & & q \\
\ddots & \vdots & \vdots & \vdots & \vdots & \\
0 & H^1(X)^{\otimes 3} \longrightarrow & 0 & 0 & 0 & 3 \\
 & & H^1(X)^{\otimes 2} \longrightarrow & 0 & 0 & 2 \\
 & & & H^1(X) \rightarrow & 0 & 1 \\
 & & & & A & 0 \\
\hline
-n & \dots & -2 & -1 & 0 & p
\end{array}$$

Thus, the spectral sequence degenerates after the first differential, and the only non-zero cohomology of $\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}$ is the zeroth. Thus, the assertion is clear. ■

7.3.3 The case of projective fibers

Theorem 7.3.5. *Let $\pi\colon X \longrightarrow S$ be as above such that every fiber $X_s = \pi^{-1}(s)$ is projective and has the property (★), e.g. if $\pi\colon X \longrightarrow S$ is a family of irreducible smooth projective curves of genus $\neq 0$ or an abelian scheme. Then the inclusion $\widetilde{B}_n^{\mathrm{mix}}(X|S)_{x_0,x_0} \hookrightarrow \widetilde{B}_{n+1}^{\mathrm{mix}}(X|S)_{x_0,x_0}$ induces the zero morphism on cohomology in degrees $\neq 0$.*

Proof. For the case of curves in the ℓ-adic realization, also see [Fal12].
It suffices to prove the assertion in stalks, so it suffices to prove the theorem for X is projective and S is a point

- *First Reduction Step:* From now on we may assume X is projective over $\mathrm{Spec}(k) =:$ pt for a field d, and $x_0 \in X(k)$. We want to prove that the inclusion of complexes $\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0} \hookrightarrow \widetilde{B}_{n+1}^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}$ induces the zero morphism in cohomology of degree ≥ 1.
  Recall the spectral sequence already introduced in the proof of Theorem 7.3.4 above:
  $$E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) := H^q\left(\left(\pi_* A_X^{\sharp}/A[0]\right)^{\otimes p}\right) \Rightarrow H^{-p+q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$$
  with morphisms $d_1^{-p,q} := \widetilde{\delta}_{n-1}^*(0,0)\colon E_1^{-p,q} \longrightarrow E_1^{-p+1,q}$ given by
  $$\widetilde{\delta}_p^*(0,0)([a_1|\ldots|a_{p+1}]) = \sum_{k=1}^{p}(-1)^{|a_1|+\ldots+|a_k|+k-1}[a_1|\ldots|a_k a_{k+1}|\ldots|a_{p+1}].$$
  By Künneth we have $E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) = \left(\widetilde{H}_{\mathrm{mix}}^{\bullet}(X,A)^{\otimes p}\right)^q$, where $\widetilde{H}$ denotes reduced cohomology. So the spectral sequence $E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ looks as follows:

$$\begin{array}{ccccccc|c}
 & & & & & & & q \\
\bigoplus_{\substack{|\alpha| = n+2,\\ \alpha_i \geq 1 \forall i}} \bigotimes_{i=1}^n H^{\alpha_i}(X) & \to & \ldots & \to & \bigoplus_{\substack{i+j=n+2,\\ i,j\geq 1}} H^i(X)\otimes H^j(X) & \to H^{n+2}(X) & \to 0 & n+2 \\
\bigoplus_{\substack{|\alpha| = n+1,\\ \alpha_i \geq 1 \forall i}} \bigotimes_{i=1}^n H^{\alpha_i}(X) & \to & \ldots & \to & \bigoplus_{\substack{i+1=n+1,\\ i,j\geq 1}} H^i(X)\otimes H^j(X) & \to H^{n+1}(X) & \to 0 & n+1 \\
H^1(X)^{\otimes n} & \longrightarrow & \ldots & \longrightarrow & \bigoplus_{\substack{i+1=n,\\ i,j\geq 1}} H^i(X)\otimes H^j(X) & \longrightarrow H^n(X) & \to 0 & n \\
 & & \ddots & & \vdots & \vdots & \vdots & \\
 & & & & H^1(X)^{\otimes 2} & \longrightarrow H^2(X) & \to 0 & 2 \\
 & & & & & H^1(X) & \to 0 & 1 \\
 & & & & & & A & 0 \\
\hline
-n & & \ldots & & -2 & -1 & 0 & p
\end{array}$$

  where we write $H^j(X)$ instead of $H^j_{\mathrm{mix}}(X,A)$. Note that since X is projective, $H^i_{\mathrm{mix}}(X,A)$ is pure of weight $-2i$, and thus

$$E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) = \left(\left(\widetilde{H}^\bullet(X,A)\right)^{\otimes p}\right)^q$$

is a pure of weight q. The spectral sequence $E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ abuts at the second sheet by the following standard argument: The differential

$$d_2^{-p,q}\colon E_2^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) \longrightarrow E_2^{-p+2,q-1}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$$

is a morphism from pure weight $-2q$ to pure weight $-2q+2$, and thus has to be zero.

Since $E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ abuts to $H^{-p+q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$, by definition there exists a filtration W of $H^\bullet(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ such that

$$\mathrm{gr}_W^m\left(H^\bullet(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})\right) \cong \bigoplus_{q-p=m} E_2^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}).$$

Since $E_2^{m-q,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ is pure of weight $-2q$ and thus the weights of all summands in the sum $\bigoplus_{q-p=m} E_2^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ differ, it follows that W coincides with the weight filtration of $H^i\left(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right)$. As we will prove in Lemma 7.3.7, we have

$$H^m\left(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right) \cong \mathrm{gr}_W^\bullet H^m\left(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right), \quad \text{and thus we obtain}$$
$$H^m\left(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right) \cong \bigoplus_{q-p=m} E_2^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}).$$

As a direct consequence of this, the assertion of the theorem is equivalent to the following assertion, which we call

(Theorem'): *Consider the morphism of spectral sequences*

$$E_1^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) \longrightarrow E_1^{-p,q}(\widetilde{B}_{n+1}^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$$

induced by the inclusion $\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0} \hookrightarrow \widetilde{B}_{n+1}^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}$. *Then the induced morphism on the abutment sheet*

$$E_2^{-p,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) \longrightarrow E_2^{-p,q}(\widetilde{B}_{n+1}^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}) \text{ is zero unless } p=q.$$

- *Second reduction step:* Now we will reduce assertion (Theorem') to an even simpler claim, which we call

  (Theorem"): *The complexes* $E_1^{\bullet,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$ *are exact unless in degrees* $(-p,p)$ *and the vertical line* $p=-n$, *i.e. on the left hand side of every complex in the diagram above.*

  (Theorem') follows from (Theorem") as follows: If (Theorem") holds, then the second sheet $E_2^{\bullet,\bullet}\left(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right)$ is zero unless in bidegrees $(-q,q)$ (which is of total degree 0) and in bidegrees $(-n, q)$ for $q > n$. Likewise, the second sheet $E_2^{\bullet,\bullet}\left(\widetilde{B}_{n+1}^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right)$ is zero unless in bidegrees $(-q, q)$ (which is of total degree 0) and in bidegrees $(-n-1, q)$ for $q > n+1$. Hence, the induced morphism of the E_2-sheets

$$E_2^{-p,q}\left(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right) \hookrightarrow E_2^{-p,q}\left(\widetilde{B}_{n+1}^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right)$$

of (Theorem') is zero apart from total degree 0, since the elements in bidegrees $(-n, q)$ for $q > n$ are mapped to zero.

- *Third step:* Taking a look at the complexes in the first sheet $E_1^{\bullet,q}(\widetilde{B}_n^{\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0})$, the assertion (Theorem"), and hence also the Theorem, follow directly from the next Lemma 7.3.6. ■

Lemma 7.3.6. *Let A be a field and X a projective variety of dimension d over a field k satisfying ($\bigstar$). Then for all nonnegative $m \leq q$, the complex*

$$\bigoplus_{\substack{\alpha_1,\ldots,\alpha_m \geq 1,\\ |\alpha| = q}} \bigotimes_{i=1}^m H^{\alpha_i}(X) \twoheadrightarrow \ldots \twoheadrightarrow \bigoplus_{\substack{i+j=q,\\ i,j\geq 1}} H^i(X)\otimes H^j(X) \twoheadrightarrow H^q(X) \twoheadrightarrow 0,$$

which we will denote by $C(X)^\bullet_{m,q}$, is exact, where the differentials are given by

$$\widetilde{\delta}_p^*(0,0)([a_1|\ldots|a_{p+1}]) = \sum_{k=1}^p (-1)^{|a_1|+\ldots+|a_k|+k-1}[a_1|\ldots|a_k a_{k+1}|\ldots|a_{p+1}].$$

Here, the complex above is taken to be in degrees $-m, \ldots, 0$.

Proof. We introduce the following sub-complex of $C(X)^\bullet_{m,q}$: Define the subset $D(X)^{-p}_{m,q} \subset C(X)^{-p}_{m,q}$ to be

$$D(X)^{-p}_{m,q} := \left\{[a_1|\ldots|a_p] \in C(X)^{-p}_{m,q} \mid a_1 \in H^d(X, A)\right\}.$$

Since X is projective, H^d is the top non-zero cohomology degree of X. Thus, if $[a_1|\ldots|a_p]$ in an element in $D(X)^{-p}_{m,q}$, i.e. $a_1 \in H^d(X,A)$, then we have

$$\begin{aligned}\widetilde{\delta}_{p-1}^*(0,0)([a_1|\ldots|a_p]) &= (-1)^{|a_1|}[a_1 a_2|\ldots|a_p] - (-1)^{|a_1|} a_1 \otimes \widetilde{\delta}_{p-2}^*(0,0))([a_2|\ldots|a_p]) \\ &= -(-1)^{|a_1|} a_1 \otimes \widetilde{\delta}_{p-2}^*(0,0))([a_2|\ldots|a_p])\end{aligned}$$

due to the fact that the cup product is zero on $a_1 \otimes a_2 \in H^d \otimes H^{\geq 1}$. This means that the element $\widetilde{\delta}_{p-1}^*(0,0)([a_1|\ldots|a_p])$ again satisfies $a_1 \in H^d(X,A)$, so $D(X)^\bullet_{m,q}$ yields a subcomplex of $C(X)^\bullet_{m,q}$ with differential

$${}_D d^{-p} = -\tau \otimes \widetilde{\delta}_{p-2}^*(0,0)\colon D(X)^{-p}_{m,q} \longrightarrow D(X)^{-p+1}_{m,q},$$

where τ acts on $H^k(X,A)$ by multiplication with $(-1)^k$. As a consequence, we note that

$$D(X)^\bullet_{m,q} \simeq H^d(X,A) \otimes C(X)_{m-1,q-2d}.$$

We can also consider the quotient complex given by $C/D(X)^\bullet_{m,q} := C(X)^\bullet_{m,q}/D(X)^\bullet_{m,q}$ whose differential is given by

$${}_{C/D}d^{-p}([a_1|\dots|a_p]) = \begin{cases} 0 & \text{if } a_1 \in H^{2d-1}(X,A) \\ \widetilde{\delta}^*_{p-1}(0,0)([a_1|\dots|a_p]) & \text{otherwise.} \end{cases}$$

The short exact sequence of complexes

$$0 \to D(X)^\bullet_{m,q} \to C(X)^\bullet_{m,q} \to C/D(X)^\bullet_{m,q} \to 0$$

induces a long exact sequence of complexes. Using the fact that $D(X)^\bullet_{m,q} \simeq H^d(X,A) \otimes C(X)_{m-1,q-2d}$, this long exact sequence reads

$$\begin{tikzcd} \dots \arrow[r] & H^{2d}(X,A)\otimes H^{-p}(C(X)^\bullet_{m-1,q-2d}) \arrow[r] & H^{-p}(C(X)^\bullet_{m,q}) \arrow[r] & H^{-p}(C/D(X)^\bullet_{m,q}) \arrow[dll, rounded corners] \\ & H^{2d}(X,A)\otimes H^{-p+1}(C(X)^\bullet_{m-1,q-2d}) \arrow[r] & H^{-p+1}(C(X)^\bullet) \arrow[r] & \dots \end{tikzcd}$$

- **First Reduction Step:** Let us put the following fact to record:

  Fact 1: *The claim follows for all (m,q) with fixed value $(m+q)$ if we know the following:*
  - $H^{-p}(C/D(X)^\bullet_{m,q}) = 0$ *for* $p = 1,\dots,m-1$, *unless* $p = q$ *and*
  - $H^{-p}(C(X)^\bullet_{m-1,q-2d}) = 0$ *for* $p = 1,\dots,m-2$.

  Note that for $p = q$, we have $C(X)^\bullet_{m-1,q-2d} = 0$ in any case, so there is no "unless $p = q$" in the second claim. This, however, does not suffice for an inductive argument over $(p+m)$ yet, however, due to the C/D-term. We will consider this term in the next step:
- **Second Reduction Step:** We proceed with C/D exactly as we did with C above: for every m,q, define the subcomplex

  $$D1(X)_{m,q} := \Big([a_1|\dots|a_p] \in C/D(X)_{m,q} \mid a_1 \in H^{2d-1}(X,A)\Big) \quad \text{of } C/D(X)_{m,q}$$

  with differential ${}_{D1}d^{-p} = -\operatorname{id}\otimes\widetilde{\delta}^*_{p-2}(0,0)\colon D1(X)^{-p}_{m,q} \longrightarrow D1(X)^{-p+1}_{m,q}$,

  i.e. $D1(X)_{m,q} \simeq H^{2d-1}(X,A)\otimes C(X)_{m-1,q-2d+1}$ and the quotient diagram

  $$C/D1(X)_{m,q} := C/D(X)_{m,q}/D1(X)_{m,q}.$$

  As above, for every q the short exact sequence of complexes

  $$0 \to D1(X)^\bullet_{m,q} \to C/D(X)^{\bullet,q}_{2d-(i-1)} \to C/D1(X)^\bullet_{m,q} \to 0$$

induces a long exact sequence

$$\longrightarrow H^{2d-1}(X)\otimes H^p(C(X)_{m-1,q-2d+1}) \longrightarrow H^p(C/D(X)^\bullet_{m,q}) \longrightarrow H^p(C/D1(X)^\bullet_{m,q}) \longrightarrow$$
$$\longrightarrow H^{2d-1}(X)\otimes H^{p+1}(C(X)_{m-1,q-2d+1}) \longrightarrow H^{p+1}(C/D(X)^\bullet_{m,q}) \longrightarrow$$

where we write $H(X) := H(X,A)$ for simplicity. We can now refine Fact 1 as follows:

Fact 2: *The claim follows for all (m,q) with fixed value $(m+q)$ if we know the following:*
- $H^{-p}(C/D1(X)^\bullet_{m,q}) = 0$ *for* $p = 1,\dots,m-1$, *unless* $p = q$ *and*
- $H^{-p}(C(X)^\bullet_{m-1,q-2d}) = 0$ *and* $H^p(C(X)_{m-1,q-2d+1}) = 0$ *for* $p = 1,\dots,m-2$.

- **Third Reduction Step:** We now proceed successively as in the second reduction step: For $i = 2,\dots,2d-1$ and every m,q, define the subcomplex

$$Di(X)_{m,q} := \Big([a_1|\dots|a_p] \in C/D(i-1)(X)_{m,q} \mid a_1 \in H^{2d-i}(X,A)\Big)$$
with differential ${}_{Di}d^{-p} = -\operatorname{id}\otimes\widetilde{\delta}^*_{p-2}(0,0)\colon Di(X)^{-p}_{m,q} \longrightarrow Di(X)^{-p+1}_{m,q}$,

i.e. $Di(X)_{m,q} \simeq H^{2d-i}(X,A)\otimes C(X)_{m-1,q-2d+i}$ and the quotient diagram

$$C/Di(X)_{m,q} := C/D(i-1)(X)_{m,q}/Di(X)_{m,q}.$$

As above, for every $m \le q$ the short exact sequence of complexes

$$0 \to Di(X)^\bullet_{m,q} \to C/D(i-1)(X)^{\bullet;q}_{m,q} \to C/Di(X)^\bullet_{m,q} \to 0$$

induces a long exact sequence

$$\longrightarrow H^{2d-i}(X)\otimes H^p(C(X)_{m-1,q-2d+i}) \longrightarrow H^p(C/D(i-1)(X)^\bullet_{m,q}) \longrightarrow H^p(C/Di(X)^\bullet_{m,q}) \longrightarrow$$
$$\longrightarrow H^{2d-i}(X)\otimes H^{p+1}(C(X)_{m-1,q-2d+i}) \longrightarrow H^{p+1}(C/D(i-1)(X)^\bullet_{m,q}) \longrightarrow$$

where we write $H(X) := H(X,A)$ for simplicity. Successively, we refine Fact 2 for $i = 2,\dots,2d-1$ as follows:

Fact (i+1): *The claim follows for all (m,q) with fixed value $(m+q)$ if we know the following:*
- $H^{-p}(C/Di(X)^\bullet_{m,q}) = 0$ *for* $p = 1,\dots,m-1$, *unless* $p = q$ *and*
- $H^{-p}(C(X)^\bullet_{m-1,q-2d}) = \dots = H^{-p}(C(X)_{m-1,q-2d+i}) = 0$ *for* $p = 1,\dots,m-2$.

- **Fourth and Last Reduction Step:** Now note that $C/D(2d-1)^\bullet_{m,q} = 0$, so for $i = 2d-1$ above, (Fact 2d) reduces to:

Fact 2d: *The claim follows for all (m,q) with fixed value $(m+q)$ if we know the following:*

$$H^{-p}(C(X)^{\bullet}_{m-1,q-i}) = 0 \ \ \textit{for} \ \ i = 1, \ldots, 2d \ \ \textit{and} \ \ p = 1, \ldots, m-2.$$

Now, the Lemma reduces to a more or less trivial induction: We show via induction over $(m+q)$ (this works since both are nonnegative numbers) that for all (m', q') $H^{-p}(C(X)^{\bullet}_{m,q}) = 0$ unless $p = q$: For $m + q = 0$, i.e. $m = q = 0$, the assertion is trivial, since the only non-trivial cohomology group is that of $p = q = 0$. Let (m, q) be any pair such that $m+q = N$. Suppose we know that for all (m', q') with $m' + q' < N = m + q$, we have $H^{-p}(C(X)^{\bullet}_{m',q'}) = 0$ unless $p = q'$. By induction hypothesis, we thus know that for $p = 1, \ldots, m-2$

$$H^{-p}(C(X)^{\bullet}_{m-1,q-2d}) = H^{-p}(C(X)_{m-1,q-2d+1}) = \ldots = H^{-p}(C(X)_{m-1,q-1}) = 0,$$

which by the above considerations is equivalent to $H^{-p}(C(X)_{m,q}) = 0$. This finishes the induction, and hence the proof of the Lemma. ■

Lemma 7.3.7. *There is a natural isomorphism*

$$H^m\left(\widetilde{B}^{\mathrm{mix}}_n(X|\,\mathrm{pt})_{x_0,x_0}\right) \cong \mathrm{gr}^{\bullet}_W H^m\left(\widetilde{B}^{\mathrm{mix}}_n(X|\,\mathrm{pt})_{x_0,x_0}\right).$$

Proof. Like in section I.3.7, we introduce the following subsets of X^n:

$$\begin{aligned} D_0^{(n)} &:= \{x_0\} \times X^{n-1}, \\ D_i^{(n)} &:= \{x_i = x_i + 1\} \subset X^n \text{ for } 1 \le i \le n-1, \\ D_n^{(n)} &:= X^{n-1} \times \{x_0\} \end{aligned}$$

and put $D_I := \bigcup_{i\in I} D_i^{(n)}$ for $I \subset \{0, \ldots, n\}$. By Lemma I.3.7.5, we have

$$\widetilde{B}^{\mathrm{mot}}_n(X|\,\mathrm{pt})_{x_0,x_0} \simeq b_{\le 0}\left(\mathbb{Z}_{\left(X^n; D_0^{(n)}, \ldots, D_n^{(n)}\right)}[n]\right) \in \mathcal{DM}(S).$$

Recall that (see section B.5.2) $\mathbb{Z}_{\left(X^n; D_0^{(n)}, \ldots, D_n^{(n)}\right)}$ is given by the complex

$$\mathbb{Z}_{X^n}(0) \to \bigoplus_{i=0}^{n} \mathbb{Z}_{D_i} \to \ldots \to \bigoplus_{|I|=s} \mathbb{Z}_{D_I} \to \bigoplus_{|I|=s+1} \mathbb{Z}_{D_I} \to \ldots \to \mathbb{Z}_{D_0^{(n)} \cap \ldots \cap D_n^{(n)}}$$

in degrees 0 up to n, where the differential in degree s is the alternating sum $\partial^s := \sum_{|I|=s} \sum_{i=1}^{n} (-1)^i \partial^s_{I,i}$, with the component $\partial^s_{I,i} \colon \mathbb{Z}_{D_I} \longrightarrow \mathbb{Z}_{D_{I\cup\{i\}}}$ given by

$$\partial^s_{I,i} := \begin{cases} X^*_{(I\cup\{i\})\supset I} & \text{for } i \notin I \\ 0 & \text{for } i \in I \end{cases}$$

(Here, we drop the additional functions g_s since they are of no consequence in the mixed realization). Its mixed realization obviously yields a complex that computes the cohomology groups of X^n relative to the divisors $D_0^{(n)}, \ldots, D_{n+1}^{(n)}$. Thus, in all degrees i apart from the zeroth (due to the truncation), we have:

$$H^i\left(\widetilde{B}_n^{A,\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right) \cong H^{i+n}_{\mathrm{mix}}(X^n; \cup_{i=0}^n D_i^{(n)}, A),$$

where the latter denotes relative ℓ-adic or singular cohomology. Since X is projective, for the usual purely formal reasons we moreover have

$$H^{i+n}_{\mathrm{mix}}(X^n; \cup_{i=0}^n D_i^{(n)}, A) \cong H^{i+n}_{\mathrm{mix}}(X^n \setminus \cup_{i=0}^n D_i^{(n)}, A).$$

On page 81 of [Del], Deligne proves that the singular (and hence, as a vector space, ℓ-adic cohomology) of any algebraic variety is isomorphic as a vector space to its associated weight graded, i.e. by the above considerations we have for all $i \neq 0$

$$\begin{aligned} H^i\left(\widetilde{B}_n^{A,\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right) &\cong H^{i+n}_{\mathrm{mix}}(X^n \setminus \cup_{i=0}^n D_i^{(n)}, A) \\ &\cong \mathrm{gr}_W^{\bullet}\, H^{i+n}_{\mathrm{mix}}(X^n \setminus \cup_{i=0}^n D_i^{(n)}, A) \\ &\cong \mathrm{gr}_W^{\bullet}\, H^i\left(\widetilde{B}_n^{A,\mathrm{mix}}(X|\,\mathrm{pt})_{x_0,x_0}\right). \qquad \blacksquare \end{aligned}$$

Finally, combining the affine and the projective cases, we obtain Theorem 7.3.3 as a direct consequence. ■

Corollary 7.3.8. *Let $\pi\colon X \longrightarrow S$ be one of the following:*

- *a smooth family of irreducible curves which are complements of normal crossing divisors in a projective curve, or*
- *an abelian scheme.*

Then the inclusion $\widetilde{B}_n^{\mathrm{mix}}(X|S)_{x_0,x_0} \hookrightarrow \widetilde{B}_{n+1}^{\mathrm{mix}}(X|S)_{x_0,x_0}$ induces the zero map on cohomology in degrees $\neq 0$.

Proof. This is a direct consequence of the above two theorems 7.3.3 and 7.3.4 together with Proposition 7.3.1 and the fact that the complement of a normal crossing divisor in a projective curve is an affine curve. ■

7.3.4 The polylogarithm class for curves and abelian schemes

We keep the above notation: Let $F = \mathbb{C}$ in the geometric case, and $F = \mathbb{Z}[1/l]$ in the ℓ-adic case, A is either a subfield of $\mathbb{C}$ in the geometric case or $\mathbb{Q}_l$ if $F = \mathbb{Z}[1/l]$ in the étale case. $S \longrightarrow \mathrm{Spec}(F)$ is a reduced scheme, smooth and quasi-projective over $\mathrm{Spec}(F)$. $\pi\colon X \longrightarrow S$ is in Sm_S such that π has geometrically irreducible fibers and three sections $x_0, x, y\colon S \longrightarrow X$ of π.

The above results now have the following impact on our motivic polylogarithm:

Theorem 7.3.9. *Suppose every fiber X_s has the property ($\bigstar$) and is either projective or an affine curve. For example, this is the case if $X \longrightarrow S$ is one of the following:*

- *a smooth family of irreducible curves which are complements of normal crossing divisors in a projective curve, or*
- *an abelian scheme.*

Then the polylogarithm $\mathrm{pol}^{A,\mathrm{mix}}\colon j_U^*\mathcal{L}^{A,\mathrm{mix}} \longrightarrow \pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1]$ *naturally induces a morphism*

$$\mathrm{pol}^{A,\mathrm{mix}}\colon \mathcal{H}^0 j_U^*\mathcal{L}^{A,\mathrm{mix}} \longrightarrow \mathcal{H}^0\pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1].$$

Proof. Recall that $\mathrm{pol}^{A,\mathrm{mix}}$ is the morphism of inductive systems $\mathrm{pol}^{A,\mathrm{mix}} = \left(\mathrm{pol}_n^{A,\mathrm{mix}}\right)_n$, where

$$\mathrm{pol}_n^{A,\mathrm{mix}}\colon j_U^*\mathcal{L}_n^{A,\mathrm{mix}} \longrightarrow \pi_U^*\tilde{I}_n^{A,\mathrm{mix}}(d)[2d-1].$$

Note that the lowest non-trivial cohomology of the complex $j_U^*\mathcal{L}_n^{A,\mathrm{mix}}$ is the zeroth, so by general theory $\mathrm{pol}_n^{A,\mathrm{mix}}$ induces a morphism

$$\mathcal{H}^0 j_U^*\mathcal{L}_n^{A,\mathrm{mix}} \longrightarrow \pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1].$$

These morphisms are compatible with the inclusions $\mathcal{L}_n^{A,\mathrm{mix}} \hookrightarrow \mathcal{L}_{n+1}^{A,\mathrm{mix}}$, thus giving rise to a morphism of inductive systems

$$\mathrm{pol}^{A,\mathrm{mix}}\colon \mathcal{H}^0 j_U^*\mathcal{L}^{A,\mathrm{mix}} \longrightarrow \pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1],$$

where $\mathcal{H}^0 j_U^*\mathcal{L}^{A,\mathrm{mix}} = \left(\mathcal{H}^0 j_U^*\mathcal{L}_n^{A,\mathrm{mix}}\right)_n$. Next, note that the inductive system $\left(\tilde{I}_n^{A,\mathrm{mix}}\right)_n$ is equal to $\left(\widetilde{B}_n^{\mathrm{mix}}(X|S)_{x_0,x_0}/A\right)_n$, and by Theorem 7.3.3 above, the morphisms of the inductive system induce the zero map in cohomoloy apart from degree zero. Hence, the morphism of inductive systems

$$\mathrm{pol}^{A,\mathrm{mix}}\colon \mathcal{H}^0 j_U^*\mathcal{L}^{A,\mathrm{mix}} \longrightarrow \pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1]$$

factors over $\mathcal{H}^0$, yielding a morphism

$$\mathrm{pol}^{A,\mathrm{mix}}\colon \mathcal{H}^0 j_U^*\mathcal{L}^{A,\mathrm{mix}} \longrightarrow \mathcal{H}^0\pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1] \quad \text{as asserted.} \quad \blacksquare$$

Of course, as a corollary we obtain a similar result for the small polylogarithm:

Corollary 7.3.10. *Let* $\pi\colon X \longrightarrow S$ *be as in Theorem 7.3.9 above. The small mixed polylogarithm* $p^{A,\mathrm{mix}}$ *yields a class in*

$$\mathrm{Hom}_{\varinjlim D^b(\mathrm{Sh}(U))}\left(j_U^*\mathcal{H}^0\mathcal{L}^{A,\mathrm{mix}}, \pi_U^*\mathcal{H}^0 i^{A,\mathrm{mix}}(d)[2d-1]\right).$$

7.4 Connection to the classical polylogarithm

Recall the connection between our motivic and the classical bar complexes of section 4.4: We saw in Corollary 4.4.1 that the mixed realization of the motivic bar complex coincides with the classical bar complex of the differential graded $A^\sharp_S$-algebra $\pi_* A^\sharp$

$$B_n^{A,\mathrm{mix}}(X|S)_{x,y} \cong B_n(\pi_* A^\sharp_X | A^\sharp_S)_{x,y}.$$

This leads to immediate consequences for our motivic logarithm and augmentation ideal:

Corollary 7.4.1. *We keep the usual notation. For all $n \in \mathbb{N}$, we have the following identifications:*

$$\begin{aligned} \mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0} &\cong B_n(\mathrm{pr}_{2*}\, A^\sharp_{X^2} | A^\sharp_X)_{x_0 \times \mathrm{id}, \widetilde{\Delta}} \\ j_U^* \mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0} &\cong B_n(\mathrm{pr}_{2*}\, A^\sharp_{X \times U} | A^\sharp_U)_{x_0 \times \mathrm{id}_U, \widetilde{\Delta}} \\ \tilde{I}_n^{A,\mathrm{mix}}(X \times U|U)_{x_0 \times \mathrm{id}_U} &\cong \tilde{I}_n(\mathrm{pr}_{2*}\, A^\sharp_{X \times U} | A^\sharp_U)_{x_0 \times \mathrm{id}_U} \end{aligned}$$

Now note that in the geometric case for $A = \mathbb{C}$, we may replace $\mathrm{pr}_{2*}\, A^\sharp_{X^2}$ by $\mathrm{pr}_{2*}\, \mathcal{E}^*_{X^2}$ and $A^\sharp_X$ by $\mathcal{E}^*_X$ - the complexes of C^∞-forms - or, equivalently, by the complexes of C^∞-forms on smooth compactifications with logarithmic poles along the complement. Recall that the D-module underlying the universal n-unipotent sheaf $\mathcal{G}^{(n)}$ of chapter 5 is given by the zeroth cohomology of the classical bar complex:

$$\mathcal{G}^{(n)} \cong \mathcal{H}^0\left(\widetilde{B}_n(\mathrm{pr}_{2*}\, A^\sharp_{X^2} | A^\sharp_X)_{x_0 \times \mathrm{id}, \widetilde{\Delta}}\right),$$

and likewise, the augmentation ideal of $x_0^* \mathcal{G}^{(n)}$ is computed by

$$\mathcal{J}^{(n)} \cong \mathcal{H}^0\left(\tilde{I}_n(\pi_* A^\sharp_X | A^\sharp_S)_{x_0}\right).$$

With this, we obtain in the geometric setting:

Corollary 7.4.2. *The geometric universal n-unipotent sheaf satisfies*

$$\mathcal{G}^{(n)} \cong \mathcal{H}^0\left(\mathcal{L}_n^{A,\mathrm{geo}}(X|S)_{x_0}\right).$$

Its augmentation ideal at x_0 is given by

$$\mathcal{J}^{(n)} \cong \mathcal{H}^0\left(\tilde{I}_n^{A,\mathrm{geo}}(X|S)_{x_0}\right)$$

while its pull-back to all of $U = X \setminus x_0(S)$ is

$$\pi_U^* \mathcal{J}^{(n)} \cong \mathcal{H}^0\left(\tilde{I}_n^{A,\mathrm{geo}}(X \times U|U)_{x_0 \times \mathrm{id}_U}\right)$$

Proof. This is an immediate consequence of the above considerations together with the fact that π_U^* is exact and thus commutes with $\mathcal{H}^0$. ■

Like in the geometric case, there is a universal n-unipotent ℓ-adic sheaf, see [Del89] or [Fal07] for a reference. Let us denote this universal n-unipotent ℓ-adic sheaf by $\mathcal{G}_\ell^{(n)}$. Faltings proved an equivalent relationship between the ℓ-adic realization of our motivic logarithm and $\mathcal{G}_\ell^{(n)}$:

Theorem 7.4.3 (Faltings). *Let $\pi\colon X \longrightarrow S$ be a smooth morphism of quasi-projective schemes such that the prime l is invertible on S. Then the ℓ-adic sheaf $\mathcal{H}^0(\mathcal{L}_n^{\mathbb{Q}_l,\mathrm{et}}(X|S)_{x_0})$ is the universal n-unipotent ℓ-adic sheaf $\mathcal{G}_\ell^{(n)}$ on X trivialized at x_0, and therefore coincides with the étale logarithm on X as considered in literature.*

Proof. Recall that our motivic logarithm coincides with construction of Falting's motivic logarithm in [Fal12]. The theorem is shown at the end of the proof of Proposition 5 in [Fal12] (actually, Faltings works in the setting where $\pi\colon X \longrightarrow S$ is a smooth curve. However, the arguments at then end of the proof of Proposition 5 do not make use of the fact that X is a family of curves, and hence generalize word by word to our motivic polylogarithm). ■

Corollary 7.4.4. *Let us denote the augmentation ideal of $x_0^*\mathcal{G}_\ell^{(n)}$ by $\mathcal{J}_\ell^{(n)}$. Then as a consequence of Faltings' theorem 7.4.3, we obtain:*

$$\pi_U^*\mathcal{J}_\ell^{(n)} \cong \mathcal{H}^0\left(\tilde{I}_n^{\mathbb{Q}_l,\ell}(X \times U|U)_{x_0\times \mathrm{id}_U}\right).$$

From now on, let us go back to the mixed situation and denote this universal n-unipotent mixed sheaf ($\mathcal{G}^{(n)}$ in the geometric situation, $\mathcal{G}_\ell^{(n)}$ in the ℓ-adic case) by $\mathcal{G}^((n))_{\mathrm{mix}}$. Likewise, we call the augmentation ideal of $x_0^*\mathcal{G}_{\mathrm{mix}}^{(n)}$ by $\mathcal{J}_{\mathrm{mix}}^{(n)}$.
We may summarize the above results as follows:

Corollary 7.4.5.

$$\mathcal{G}_{\mathrm{mix}}^{(n)} \cong \mathcal{H}^0\left(\mathcal{L}_n^{A,\mathrm{mix}}(X|S)_{x_0}\right)$$
$$\pi_U^*\mathcal{J}_{\mathrm{mix}}^{(n)} \cong \mathcal{H}^0\left(\tilde{I}_n^{A,\mathrm{mix}}(X \times U|U)_{x_0\times \mathrm{id}_U}\right).$$

Using this together with Theorem 7.3.9, we obtain:

Corollary 7.4.6. *Suppose every fiber X_s has the property (★) and is either projective or an affine curve. For example, this is the case if $X \longrightarrow S$ is one of the following:*

- *a smooth family of irreducible curves which are complements of normal crossing divisors in a projective curve, or*
- *an abelian scheme.*

Then the mixed polylogarithm $\mathrm{pol}^{A,\mathrm{mix}}\colon j_U^*\mathcal{L}^{A,\mathrm{mix}} \longrightarrow \pi_U^*\tilde{I}^{A,\mathrm{mix}}(d)[2d-1]$ *naturally induces a morphism*

$$\mathrm{pol}^{A,\mathrm{mix}}\colon \mathcal{G}_\ell^{(n)} \longrightarrow \pi_U^* * \mathcal{J}_{\mathrm{mix}}^{(n)}(d)[2d-1].$$

Remark 7.4.7. The above result implies that for nice families of curves of genus $\neq 0$, the polylogarithm we such constructed in fact *coincides* with the polylogarithms in literature.

Résumé

Finally, the time has come to evaluate our results, and try to grasp to which extent we managed to achieve the goals we set out with.

The ultimate aim of this thesis was to construct the motivic polylogarithm in a very general setting. While all previous constructions were restricted to the case where $\pi\colon X \longrightarrow S$ is a smooth curve or an abelian scheme, and the polylogarithm had to be defined anew in each and every realization and setting, we strived to find *one general* definition which would be valid in every realization and give the well-known polylogarithms in the cases of curves and abelian schemes. To be precise, we had the following aims laid out for us in the introduction:

a.) *Step 1: Define a "motivic logarithm" which gives rise to the usual logarithm for curves and abelian schemes.*

   (i) To generalize Faltings' motivic logarithm (developed in [Fal12]) and put it into a greater theoretical context, construct a theory of "motivic bar complexes".

   (ii) Define the motivic logarithm for any smooth quasi-projective morphism $\pi\colon X \longrightarrow S$ of reduced schemes using the language of "motivic bar complexes".

   (iii) Show that like in Faltings' case, one may retrieve the classical (ℓ-adic or Hodge) logarithms for curves and abelian schemes as the zeroth cohomology of our motivic logarithm.

b.) *Step 2: View the polylogarithm as a Gysin morphism.*

   By considering Beilinson and Levin's motivic polylogarithm for elliptic curves ([BL94, §6]), we deduced that the polylogarithm would essentially have to be comprised by the Gysin morphism associated to the diagonal $X \hookrightarrow X \times_S X$.

c.) *Step 3: Combining Step 1 and 2*

   (i) Define the motivic polylogarithm as an extension class of the newly defined motivic logarithm with something nice (its augmentation ideal) using the Gysin isomorphism

$$\Delta_*\colon \mathbb{Z}_U \longrightarrow \mathbb{Z}_{X\times_S U}(d)[2d] \in \mathcal{DM}(U).$$

   as the basic ingredient.

   (ii) Show that in case of curves and abelian schemes, this motivic polylogarithm yields an extension class of the zeroth cohomologies in ℓ-adic and geometric realization, which coincides with the polylogarithm in literature.

Have we achieved our aims? In order to evaluate our position, let us recollect our results:

- In Part I, we developed a theory of motivic bar complexes and its simplicial nature: For a smooth quasi-projective scheme $X \longrightarrow S$ equipped with two sections

$x, y\colon S \longrightarrow X$ we introduced the motivic bar complex as the inductive system $(B_n^{\mathrm{mot}}(X|S)_{x,y})_n \in \varinjlim \mathcal{DM}(S)$ given by the motives

$$B_n^{\mathrm{mot}}(X|S)_{x,y} = \left\{ \mathbb{Z}_{X^n} \xrightarrow{\delta^*_{n-1}(x,y)} \ldots \xrightarrow{\delta^*_2(x,y)} \mathbb{Z}_{X^2} \xrightarrow{\delta^*_1(x,y)} \mathbb{Z}_X \xrightarrow{x^*-y^*} \mathbb{Z}_S \right\}$$

with $\mathbb{Z}_{X^n}$ in degree $-n$. We also defined a "normalized" bar complex, which one might also think of as a reduced motivic bar complex, as the inductive system $(\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y})_n \in \varinjlim \mathcal{DM}(S)$ given by the motives

$$\widetilde{B}_n^{\mathrm{mot}}(X|S)_{x,y} = \left\{ \mathbb{Z}^\circ_{X^n} \xrightarrow{\widetilde{\delta}^*_{n-1}(x,y)} \ldots \xrightarrow{\delta^*_2(x,y)} \mathbb{Z}^\circ_{X^2} \xrightarrow{\widetilde{\delta}^*_1(x,y)} \mathbb{Z}^\circ_X \xrightarrow{x^*-y^*} \mathbb{Z}_S \right\},$$

with $\mathbb{Z}^\circ_{X^n}$ in degree $-n$, where $\mathbb{Z}^\circ_X$ is the reduced motive of X (if $S = \operatorname{Spec}\mathbb{Z}$, then $\mathbb{Z}^\circ_X$ can be thought of as the reduced cohomology of X).

Aim a.)(i): ✓

- In Chapter II.6, we defined a motivic logarithm generalizing Faltings' motivic logarithm for curves by putting, for $\pi\colon X \longrightarrow S$ as above and a section $x_0\colon S \longrightarrow X$:

$$\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0} := \widetilde{B}^{\mathrm{mot}}(X^2|X)_{x_0\times \mathrm{id}_X, \widetilde{\Delta}},$$

where $\widetilde{\Delta}\colon X \hookrightarrow X^2$ is the diagonal morphism over S. For curves, the inductive system $\mathcal{L}^{\mathrm{mot}}(X|S)_{x_0} := (\mathcal{L}_n^{\mathrm{mot}}(X|S)_{x_0})_n$ indeed turned out (see Corollary II.6.2.2) to be equal to Faltings' motivic logarithm.

Aim a.)(ii): ✓

Moreover, we proved in Section 7.3 that in the mixed realization, the zeroth cohomology of our logarithm yields the dual of the classical logarithm.

Aim a.)(iii): ✓

- In Chapter II.6, we defined the polylogarithm as a morphism comprised by the Gysin isomorphism

$$\widetilde{\Delta}_*\colon \mathbb{Z}_U \longrightarrow \mathbb{Z}_{X\times_S U}(d)[2d] \in \mathcal{DM}(U)$$

where $\widetilde{\Delta}\colon U \hookrightarrow X \times_S U$ denotes the diagonal. This was done as follows:
We showed in Proposition 6.3.1 that there is an inclusion of cosimplicial schemes

$$c\,\mathrm{pol}\colon cB^\bullet_{\mathrm{mot}}(X\times U|U)_{x_0\times \mathrm{id}_U, \widetilde{\Delta}} \hookrightarrow cI^\bullet(X|S)_{x_0}(X\times U|U)_{x_0\times\mathrm{id}_U},$$

given on objects by the inclusion $\mathrm{id}^{n-1} \times \Big((\mathrm{id} - (x_0 \times_S \mathrm{id}_U)\,\mathrm{pr}_2) \circ \widetilde{\Delta} \Big)$:

$$\Big((X\times U)^{\times_U n}, \mathrm{id}^n \Big) \hookrightarrow \Big((X\times U)^{\times_U n+1}, \mathrm{id}^n \times (\mathrm{id} - (x_0\times_S \mathrm{id}_U)\,\mathrm{pr}_2) \Big)$$

of codimension d, where $\widetilde{\Delta}\colon U \longrightarrow X\times_S U$ is the diagonal over S.

By our considerations on motives associated to cosimplicial schemes in section 2.3.2, this yields the corresponding Gysin morphism of normalized motives

$$\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} c\,\mathrm{pol}\colon nM\left(sB^{\leq n}(X\times U|U)_{x_0\times \mathrm{id}_U,\widetilde{\Delta}}\right)\longrightarrow nM\left(sI_\bullet(X\times U|U)_{x_0\times\mathrm{id}_U}\right),$$

which is equal to a morphism

$$\mathrm{pol}_n\colon \mathcal{DM}(j_U^*)\mathcal{L}_n^{\mathrm{mot}}(X|S))_{x_0\times\mathrm{id}_X}\longrightarrow \mathcal{DM}(\pi_U^*)\tilde{I}_{n+1}^{\mathrm{mot}}(X|S)_{x_0}(d)[2d-1].$$

This morphism is the n-th polylogarithm.
Aim b.) and c)(i): ✓

- In Chapter II.7, we determined the mixed realization of the polylogarithm, and proved that our motivic polylogarithm indeed satisfies an important characterizing property of the polylogarithm (section 7.2): We considered the following long exact sequence associated to the distinguished triangle of derived functors $x_{0*}x_0^! \to \mathrm{id}\to j_{U*}j_U^*$:

$$\begin{array}{c}
\operatorname{Ext}^{2d-1}_{D^b(\mathrm{Sh}(X))}\left(\widetilde{\mathcal{L}}_n^{A,\mathrm{mix}}(X|S)_{x_0},\pi^*\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}(d)\right)\\
\downarrow\\
\operatorname{Ext}^{2d-1}_{D^b(\mathrm{Sh}(U))}\left(j_U^*\widetilde{\mathcal{L}}_n^{A,\mathrm{mix}}(X|S)_{x_0},\pi_U^*\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}(d)\right)\\
\downarrow{\scriptstyle \partial_*}\\
\operatorname{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n^{A,\mathrm{mix}}(X|S)_{x_0}\oplus A_S,\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}\right)\\
\downarrow\\
\operatorname{Ext}^{2d}_{D^b(\mathrm{Sh}(X))}\left(\widetilde{\mathcal{L}}_n^{A,\mathrm{mix}}(X|S)_{x_0},\pi^*\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}(d)\right)
\end{array}$$

where $\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}$ is the augmentation ideal of the Hopf algebra

$$\widetilde{B}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0,x_0}=x_0^*\widetilde{\mathcal{L}}_n^{A,\mathrm{mix}}(X|S)_{x_0}.$$

In literature, the polylogarithm is usually characterized (and defined) via its image under ∂_* in

$$\operatorname{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n^{A,\mathrm{mix}}(X|S)_{x_0}\oplus A_S,\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}\right).$$

Hence, in order to see that our motivic polylogarithm coincides with the existing polylogarithms in literature, it was crucial to see that our polylogarithm is mapped to the *same* element as the previous polylogarithms. We finally managed to prove this in Theorem II.7.2.1: The image of the n-th mixed polylogarithm $\mathrm{pol}_n^{A,\mathrm{mix}}(X|S)_{x_0}$ in

$$\operatorname{Hom}_{D^b(\mathrm{Sh}(S))}\left(\tilde{I}_n^{A,\mathrm{mix}}(X|S)_{x_0}\oplus A_S[0],\tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}\right)$$

under the above boundary morphism ∂_* is given by the morphism

$$i_{n,n+1} \circ \mathrm{pr}_I \colon \tilde{I}_n^{A,\mathrm{mix}}(X|S)_{x_0} \oplus A_S[0] \twoheadrightarrow \tilde{I}_n^{A,\mathrm{mix}}(X|S)_{x_0} \hookrightarrow \tilde{I}_{n+1}^{A,\mathrm{mix}}(X|S)_{x_0}$$

where pr_I is the projection to the summand and $i_{n,n+1}$ is the natural inclusion. This is the property one would expect the motivic polylogarithm to satisfy.

Using this, we finally managed to prove in section II.7.3 that in the case of "nice schemes" (e.g. curves of genus $\neq 0$ and abelian schemes in the Hodge and ℓ-adic realization), our generalized polylogarithm gives rise to a morphism of the zeroth cohomologies. Here, "nice" means the following: We say that a smooth quasi-projective scheme X has the property ($\bigstar$), if the following condition is satisfied: for all $q > 1$, the morphism

$$\bigoplus_{\substack{i+j=q,\\ i,j\geq 1}} H^i_{\mathrm{mix}}(X) \otimes H^j_{\mathrm{mix}}(X) \longrightarrow H^q_{\mathrm{mix}}(X)$$

is surjective, where $H^\bullet_{\mathrm{mix}}$ denotes either Betti or ℓ-adic cohomology.
We proved that for schemes $\pi\colon X \longrightarrow S$ such that every fiber X_s satisfies ($\bigstar$) the mixed polylogarithm

$$\mathrm{pol}^{A,\mathrm{mix}}(X|S)_{x_0} \colon j_U^* \widetilde{\mathcal{L}}^{A,\mathrm{mix}}(X|S)_{x_0} \longrightarrow \pi_U^* \tilde{I}^{A,\mathrm{mix}}(X|S)_{x_0}(d)[2d-1]$$

naturally induces a morphism

$$\mathrm{pol}^{A,\mathrm{mix}}(X|S)_{x_0} \colon j_U^* \mathcal{H}^0 \widetilde{\mathcal{L}}^{A,\mathrm{mix}}(X|S)_{x_0} \longrightarrow \pi_U^* \mathcal{H}^0 \tilde{I}^{A,\mathrm{mix}}(X|S)_{x_0}(d)[2d-1],$$

which coincides with the classical polylogarithm in the geometric and ℓ-adic realization.
Aim b.)(ii): ✓

Conclusion:

In my opinion, the theory of polylogarithms is far from having reached its full potential. Lacking a "unified" theory, the polylogarithm seemed somewhat hidden behind its diversity, and it was only possible to progress and advance in small parts of the theory at a time, rather than developing the theory as a whole. Since this thesis provides a general theoretical background for the polylogarithm, I hope it will become easier to track down its secrets.

Appendices

Appendix A

Simplicial and cosimplicial objects

The reference for the following section is chapter 1.2 of the book [Lur].

Let $\mathbf{\Delta}$ denote the simplicial indexing category, i.e. the category with objects the ordered sets $[n] := \{0, 1, \ldots, n\}$ and arrows the order-preserving maps between them. Then a *simplicial object* in a category $\mathcal{C}$ is a functor $S_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}$, while a *cosimplicial object* is a functor $S^\bullet\colon \mathbf{\Delta} \longrightarrow \mathcal{C}$. The category of simplicial (resp. cosimplicial) objects in a category $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ (resp. $\mathcal{C}^{\mathbf{\Delta}}$). We denote by $\mathbf{\Delta}_{\leq n}$ the full subcategory of $\mathbf{\Delta}$ given by the objects $[0], \ldots, [n]$. An n-truncated (co)simplicial object of $\mathcal{C}$ is a contravariant (resp. covariant) functor from $\mathbf{\Delta}_{\leq n}$ to $\mathcal{C}$. Of course every (co)simplicial object induces an n-truncated (co)simplicial object by restriction. We denote the *coface maps* in the category $\mathbf{\Delta}$ by

$$\delta_n^i\colon [n-1] \longrightarrow [n]; \quad \delta_n^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq n$, and the *codegeneracy maps* by

$$\sigma_n^i\colon [n+1] \longrightarrow [n]; \quad \sigma_n^i(j) = \begin{cases} j & \text{if } j < i \\ j-1 & \text{if } j \geq i. \end{cases}$$

Most of the time, we will drop the n in the notation, and simply write δ^i and σ^i. The coface and codegeneracy maps generate the set of morphisms in the category $\mathbf{\Delta}^{\mathrm{op}}$ - though not freely: they are subject to the *cosimplicial identities*

(1) $\delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1}$ for $0 \leq i < j \leq n+1$,
(2) $\sigma_n^j \delta_{n+1}^j = \sigma_n^j \delta_{n+1}^{j+1} = \mathrm{id}$ for $0 \leq j \leq n$
(3) $\sigma_n^j \delta_{n+1}^i = \delta_n^i \sigma_{n-1}^{j-1}$ for $0 \leq i < j \leq n$
(4) $\sigma_n^j \delta^{i_{n+1}} = \delta_n^{i-1} \sigma_{n-1}^j$ for $0 < j+1 < i \leq n+1$
(5) $\sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j+1}$ for $0 \leq i \leq j \leq n$.

If $S_\bullet$ is a simplicial set, or more generally object, then the coface and codegeneracy maps induce the *face* and *degeneracy maps*

$$d_i^n\colon S_n \longrightarrow S_{n-1}, \quad s_i^n\colon S_n \longrightarrow S_{n+1}$$

for each n by application of $S_\bullet$ to the δ^i, σ^i.
The cosimplicial identities induce the *simplicial identities*

(1) $d_i^{n+1} d_j^{n+2} = d_{j-1}^{n+1} d_i^{n+2} \colon S_{n+2} \to S_n$ for $0 \le i < j \le n+1$
(2) $d_j^{n+1} s_j^n = d_{j+1}^{n+1} s_j^n = \mathrm{id} \colon S_n \to S_n$
(3) $d_i^{n+1} s_j^n = s_{j-1}^{n-1} d_i^n \colon S_n \to S_n$ for $i < j$
(4) $d_i^{n+1} s_j^n = s_j^{n-1} d_{i-1}^n \colon S_n \to S_n$ for $i > j+1$
(5) $s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n \colon S_n \to S_{n+2}$ for $i \le j$.

If $S^\bullet$ is a cosimplicial object, then the coface and codegeneracy maps induce the coface and codegeneracy maps for $S^\bullet$, which by abuse of notation we will also denote by d^i, s^i. They satisfy the *cosimplicial identities*

(1) $d_{n+1}^j d_n^i = d_{n+1}^i d_n^{j-1} \colon S^{n-1} \to S^{n+1}$ for $0 \le i < j \le n+1$.
(2) $s_n^j d_{n+1}^j = s_n^j d_{n+1}^{j+1} = \mathrm{id} \colon S^n \to S^n$ for $0 \le j \le n$
(3) $s_n^j d_{n+1}^i = d_n^i s_{n-1}^{j-1} \colon S^n \to S^n$ for $0 \le i < j \le n$
(4) $s_n^j d^{i_{n+1}} = d_n^{i-1} s_{n-1}^j \colon S^n \to S^n$ for $0 < j+1 < i \le n+1$
(5) $s_n^j s_{n+1}^i = s_n^i s_{n+1}^{j+1} \colon S^{n+2} \to S^n$ for $0 \le i \le j \le n$.

A *morphism of (co)simplicial objects* is a transformation of functors. The *unnormalized complex* associated to a simplicial object $S_\bullet$ in an additive category $\mathcal{C}$ is the complex $C_*(S)$ given by S_n in degree $-n$ with boundary maps $\partial = \sum_{i=0}^n (-1)^i d_i \colon S_n \longrightarrow S_{n-1}$ The simplicial identities imply that this is a chain complex in $\mathcal{C}$.
Dually, the unnormalized cochain complex associated to a cosimplicial object $S^\bullet$ is the complex $C_*(S)$ with S^n in degree n, with boundary maps $\partial = \sum_{i=0}^n (-1)^i d^i \colon S^{n-1} \longrightarrow S^n$. Again, the cosimplicial identities show that this is a complex.
On the other hand, given a negatively graded chain complex (C_*, d) with values in an additive category $\mathcal{C}$, one can associate a simplicial object of $\mathcal{C}$ to it as follows: For each $n \ge 0$, the object $\mathrm{DK}_n(C)$ is given by $\bigoplus_{\alpha\colon [n]\to[k]} C_{-k}$, where the sum is taken over all surjective maps $[n] \longrightarrow [k]$ in $\mathbf{\Delta}$. For a morphism $\beta \colon [n'] \longrightarrow [n]$ in $\mathbf{\Delta}$ there is an induced map

$$\beta^* \colon \mathrm{DK}_n(C) \simeq \bigoplus_{\alpha\colon [n]\to[k]} C_{-k} \longrightarrow \bigoplus_{\alpha'\colon [n']\to[k']} C_{-k'} \simeq \mathrm{DK}_{n'}(C)$$

given by the matrix of morphisms $\{f_{\alpha,\alpha'} \colon C_{-k} \longrightarrow C_{-k'}\}$, where the map $f_{\alpha,\alpha'}$ is the identity if $k = k'$ and the diagram

$$\begin{tikzcd}
{[n']} \arrow[r, "\beta"] \arrow[d, "\alpha'"'] & {[n]} \arrow[d, "\alpha"] \\
{[k]} \arrow[r, "\mathrm{id}"] & {[k]}
\end{tikzcd}$$

commutes, and $f_{\alpha,\alpha'}$ is given by the differential d if $k' = k-1$ and the diagram

$$\begin{tikzcd}
{[n']} \arrow[rr, "\beta"] \arrow[d, "\alpha'"'] & & {[n]} \arrow[d, "\alpha"] \\
{[k-1]} \arrow[r, "\sim"] & \{1,\dots,k\} \arrow[r] & {[k]}
\end{tikzcd}$$

commutes, and $f_{\alpha,\alpha'} = 0$ otherwise. The construction $C \longmapsto \mathrm{DK}_\bullet(C)$ yields a functor

$$\mathrm{DK}\colon \mathrm{Ch}(\mathcal{C})_{\leq 0} \longmapsto \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$$

called the *Dold-Kan construction*.
The main theorem is the Dold-Kan Correspondence:

Theorem A.0.1. *[Lur, 1.2.3.7, p.46] Let $\mathcal{C}$ be an additive category. The functor*

$$\mathrm{DK}\colon \mathrm{Ch}(\mathcal{C})_{\leq 0} \longrightarrow \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$$

if fully faithful. If $\mathcal{C}$ is idempotent complete, then DK *is an equivalence of categories.*

Of course, one can dualize the Dold Kan constructions to obtain cosimplicial objects associated to cochain complexes.
The inverse of the Dold-Kan-construction can be given as follows: Let $S_\bullet$ be a simplicial object in an abelian category $\mathcal{A}$. For each $n \geq 0$, one defines

$$N_n(S) := \ker((d_1, \dots, d_n)\colon S_n \to \bigoplus_{1\leq i\leq n} S_{n-1})$$

The map d_0 then carries $N_n(S)$ into $N_{n-1}(S)$, which gives rise to the *normalized chain complex* $N_*(S)$ of $S_\bullet$.

Theorem A.0.2. *[Lur, 1.2.3.12+13, p.47] The functor N_* sending simplicial objects to their normalized complex is inverse to the functor* DK.

A.0.1 Cosimplicial setting

Let $S^\bullet$ be a cosimplicial object in an abelian category $\mathcal{A}$.
For each $n \geq 0$, one defines

$$Q(S)^n = \operatorname{coker}\left(\sum_{i=0}^{n-1} d^i\colon \bigoplus_{i=0}^{n-1} S^{n-1} \to S^n\right)$$

The map $(-1)^{n+1}d^{n+1}$ then carries $Q(S)^n$ into $Q(S)^{n+1}$, which gives rise to the *normalized chain complex* $Q(S)^*$ of $S^\bullet$. Sending cosimplicial objects $S^\bullet$ in $\mathcal{C}$ to their normalized complex yields a functor

$$q\colon \mathcal{C}^{\mathbf{\Delta}} \longrightarrow \mathrm{Ch}(\mathcal{C})_{\geq 0},$$

the *normalized cochain complex functor*.
The dual statement of theorem A.0.2 then says that Q is an inverse to the dual Dold-Kan functor. As in the simplicial setting, one can show that the cokernel coker $\left(\sum_{i=0}^{n-1} d^i \colon \bigoplus_{i=0}^{n-1} S^{n-1} \to S^n\right)$ also exists for a cosimplicial object in an additive idempotent complete category, so even for simplicial objects in the idempotent complete setting, one may define the normalized cochain complex $Q(S)^*$ as in the abelian setting, and the generalized version of the dual Dold-Kan correspondence holds for cosimplicial objects in an additive idempotent complete category.

Appendix B

The motivic theory due to Levine

To the day, there are several definitions of a derived category of motives and motivic homology/cohomology:

(i) Beilinson defined motivic cohomology and homology as described in Definition B.6.1.
(ii) In [VSF00], Suslin, Voevosdky and Friedlander define a derived category of mixed motives by developing a theory of Nisnevich sheaves with transfer. This is probably the most common language of motives to the day. However, for reasons of citability, we will rather use the next language of motives:
(iii) In [Lev] and [Lev98], Levine gives a categorical approach to mixed motives and motivic cohomology using some sort of cycle complexes.

Each of these approaches has its own advantages: While Beilinson's K-theoretic definition allows for a direct construction of regulator maps, Levine's construction is possibly the most categorical and abstract one, but unfortunately it is not given as the hypercohomology of complexes of sheaves. The Suslin-Voevodsky motivic cohomology groups, on the other hand, fit in a good formalism, and realize motivic cohomology as the hypercohomology of a complex of sheaves. Voevodsky shows that his approach actually agrees with Beilinson's vision of a motivic cohomology theory, and Levine shows in [Lev98] that his motivic cohomology agrees both with the K-theory approach and Voevodsky's formalism. The most extensive and far-reaching exhibition of the topic in a written-up form, however, is Levine's approach, which is fully recorded in his book "Mixed Motives" [Lev98]. He also proves that his formalism agrees with Beilinson's and Voevodsky's ([Lev98, VI.2.5.5, p. 329]) in the case of motives over a field k. This is the reason why we fall back to Levine's formalism here: it lists all the properties of motives needed in this context in a beautifully citable way.

B.1 The motivic category

Let S be a reduced scheme, Sch_S denote the category of noetherian separated S-schemes, and Sm_S the full subcategory of smooth quasi-projective S-schemes.

The construction of the motivic category is rather involved and can be found in detail in Chapter I of [Lev98]. Instead of recalling all steps of its construction, we will describe the objects, morphisms, and properties we will need to be able to work with this theory, and leave the underlying constructions as a black box.

Let R be a commutative ring which is flat over $\mathbb{Z}$. The *triangulated motivic category with coefficients in* R, denoted by $\mathcal{DM}(S)_R$, is a triangulated tensor category. If $R = \mathbb{Z}$, one drops the $\mathbb{Z}$ in the notation and simply writes $\mathcal{DM}(S)$ instead of $\mathcal{DM}(X)_{\mathbb{Z}}$. $\mathcal{DM}(S)_R$ is the pseudo-abelian envelope (see [Lev98, Part II, II.2.4, pp. 427ff.]) of a tensor category $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$ described for working purposes by the following data (for details, see sections I.1-I.3 of [Lev00]):
Let S be a reduced scheme, and let Sch_S denote the category of noetherian separated schemes, and Sm_S the full subcategory of smooth quasi-projective S-schemes. We call $\mathrm{Sm}_S^{\mathrm{ess}}$ the full subcategory of Sch_S of essentially smooth S-schemes.

The construction of the motivic category $\mathcal{DM}(S)$ of motives over S is done in several steps:

a.) [Lev98, I.1.1.1, p.9] One sets out with a category called $\mathcal{L}(\mathrm{Sm}_S)$, which is the category of equivalence classes of pairs (X, f), where X is an object of Sm_S and $f\colon X' \longrightarrow X$ is a map in $\mathrm{Sm}_S^{\mathrm{ess}}$ which has a smooth section $s\colon X \longrightarrow X'$. Here, the equivalence is given by isomorphisms making the obvious diagram commute. Morphisms between objects $(X, f_X\colon X' \longrightarrow X)$ and $(Y, f_Y\colon Y' \longrightarrow Y)$ in $\mathcal{L}(\mathrm{Sm}_S)$ are commutative diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f_X"'] & Y' \arrow[d, "f_Y"] \\ X \arrow[r] & Y \end{tikzcd}$$

where the top horizontal morphism is flat.

b.) [Lev98, I.1.3.2, p.11] Considering the set $\mathbb{Z}$ as a symmetric monoidal category with operation $+$, one extends $\mathcal{L}(\mathrm{Sm}_S)$ to a symmetric monoidal category $\mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$. $\mathcal{L}^*(\mathrm{Sm}_S)$ is then defined to be the category obtained from $\mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$ by adjoining the morphisms $i_*\colon X(n)_f \longrightarrow (X \coprod Y)(n)_{f \coprod g}$ for any pair $(X, f), (Y, g) \in \mathcal{L}(\mathrm{Sm}_S)$, where $i\colon X \longrightarrow X \coprod Y$ is the inclusion, subject to the following relations:

- $(i \circ j)_* = i_* \circ j_*$ for $X \xrightarrow{i} X \coprod Y \xrightarrow{j} X \coprod Y \coprod Z$,
- $i_{Y_1 *} \circ p_1^* = (p_1 \coprod p_2)^* \circ i_{X_1 *}$ for a diagram

$$\begin{tikzcd} Y_1 \arrow[r, "i_{Y_1}"] \arrow[d, "p_1"'] & Y_1 \coprod Y_2 \arrow[d, "p_1 \coprod p_2"] & Y_2 \arrow[l, "i_{Y_2}"'] \arrow[d, "p_2"] \\ X_1 \arrow[r, "i_{X_1}"'] & X_1 \coprod X_2 & X_2 \arrow[l, "i_{X_2}"] \end{tikzcd}$$

- $i^* \circ i_* = \mathrm{id}$ for the canonical morphism $i\colon X \longrightarrow X \coprod \emptyset$.

By [Lev98, I.1.3.3, p.11] one may extend the symmetric monoidal structure of $\mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$ to one on $\mathcal{L}^*(\mathrm{Sm}_S)$.

c.) [Lev98, I.1.4.1, p.12] Levine then defines the category $\mathcal{A}_1(\mathrm{Sm}_S)$ to be the free additive category on $\mathcal{L}(\mathrm{Sm}_S)^*$ subject to the following list of relations. Here, we denote $X(d)_f$ as an object of $\mathcal{A}_1(\mathrm{Sm}_S)$ by $\mathbb{Z}_X(d)_f$.
- $\mathbb{Z}_\emptyset(d)_f \cong 0$,
- for any pair of objects $(X, f), (Y, g)$ in $\mathcal{L}(\mathrm{Sm}_S)$ with natural inclusions $i_X, i_Y\colon X, Y \longrightarrow X \coprod Y$, one has

$$i_{X*} \circ i_X^* + i_{Y*} \circ i_Y^* = \mathrm{id}_\Gamma,$$

where $\Gamma = \mathbb{Z}_{X \coprod Y}(0)_{(f \coprod g)}$.

The linear extension of the product on $\mathcal{L}(\mathrm{Sm}_S)^*$ makes $\mathcal{A}_1(\mathrm{Sm}_S)$ into a tensor category ([Lev98, I.1.4.2, p.12]).

d.) [Lev98, I.1.4.3/4, p.12] Given a tensor category $(\mathcal{C}, \times, t)$ without unit, one may form the *universal commutative external product* ([Lev98, Part II, I.2.4.1]) $(\mathcal{C}^\otimes, \otimes, \tau)$ by adjoining to the free tensor category on $\mathbb{C}$ the morphisms $\boxtimes_{X,Y}\colon X \otimes Y \longrightarrow X \times Y$ for each pair $X, Y \in \mathcal{C}$ subject to the relations
- (Naturality) $\boxtimes_{X',Y'} \circ (f \otimes g) = (f \times g) \circ \boxtimes_{X,Y}$ for $f\colon X \longrightarrow X', g\colon Y \longrightarrow Y'$ in $\mathcal{C}$,
- (Associativity) $\boxtimes_{X\times Y,Z} \circ (\boxtimes_{X,Y} \otimes \mathrm{id}_Z) = \boxtimes_{X,Y\times Z} \circ (\mathrm{id}_X \otimes \boxtimes_{Y,Z})$ for $X, Y, Z \in \mathcal{C}$,
- (Commutativity) $t_{X,Y} \circ \boxtimes_{X,Y} = \boxtimes_{Y,X} \circ \tau_{X,Y}$.

Levine then defines the category $(\mathcal{A}_2(\mathrm{Sm}_S), \otimes, \tau)$ to be the universal commutative external product on $\mathcal{A}_1(\mathrm{Sm}_S)$.

e.) Levine then constructs categories $\mathcal{A}_3, \mathcal{A}_4$ and $\mathcal{A}_5$ from the category $\mathcal{A}_2(\mathrm{Sm}_S)$ by adjoining some more morphisms, which are of no further importance here.

Definition B.1.1. *a.) [Lev98, I.1.4.10, p.15] We denote the image of $X(n)_f \in \mathcal{L}(\mathrm{Sm}_S) \times \mathbb{Z}$ in $\mathcal{A}_5(\mathrm{Sm}_S)$ by $\mathbb{Z}_X(n)_f$. Then $\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S)$ is defined to be the full additive subcategory of $\mathcal{A}_5(\mathrm{Sm}_S)$ generated by tensor products of objects of the form $\mathbb{Z}_X(n)_f$, or $\mathfrak{e}^{\otimes a} \otimes \mathbb{Z}_X(n)_f$. It is a DG-category.*

b.) Denote the homotopy category of $C^b_{\mathrm{mot}}(\mathrm{Sm}_S) := C^b(\mathcal{A}_{\mathrm{mot}}(\mathrm{Sm}_S))$ by $K^b_{\mathrm{mot}}(\mathrm{Sm}_S) = C^b_{\mathrm{mot}}(\mathrm{Sm}_S)/\mathrm{Htp}$.

Definition B.1.2. *[Lev98, I.2.1.4, pp.17/18] Levine forms the triangulated tensor category $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)$ from $K^b_{\mathrm{mot}}(\mathrm{Sm}_S)$ by inverting the following morphisms:*

a.) Homotopy:

$$p^*\colon \mathbb{Z}_{Y,Z}(n)_g \longrightarrow \mathbb{Z}_{X,p^{-1}(Z)}(n)_f$$

for every map $p\colon (X, f) \longrightarrow (Y, g)$ in $\mathcal{L}(\mathrm{Sm}_S)$ such that $X \hookrightarrow Y$ is the inclusion of a closed codimension 1 subscheme, $Z \subset Y$ a closed subset such that the scheme-theoretic pull-back $p^{-1}(Z) \subset X$ is in $\mathrm{Sm}^{\mathrm{ess}}_S$, and such that there is an isomorphism $p^{-1}(Z) \times_S \mathbb{A}^1_S \cong Z$ making the obvious diagram commute.

b.) Excision:

$$j^*: \mathbb{Z}_{X,Z}(n)_f \longrightarrow \mathbb{Z}_{U,Z}(n)_{j^*f}$$

for every $(X,f) \in \mathcal{L}(\mathrm{Sm}_S)$*,* $Z \subset X$ *a closed subset, and* $j: U \longrightarrow X$ *an open subscheme containing* Z*.*

c.) Künneth isomorphism:

$$\boxtimes_{X,Y}: \mathbb{Z}_X \otimes \mathbb{Z}_Y \longrightarrow \mathbb{Z}_{X\times Y}$$

for $X, Y \in \mathcal{A}_1(\mathrm{Sm}_S)$*.*

d.) Gysin isomorphism: For the precise definition of this map see [Lev98, I.2.1.4(d), p.18].

e.) Moving lemma: the morphism induced by $\mathrm{id}: X \longrightarrow X$*,*

$$\rho_{f,g}: \mathbb{Z}_X(n)_{f\cup g} \longrightarrow \mathbb{Z}_X(n)_f,$$

for $(X,f) \in \mathcal{L}(\mathrm{Sm}_S)$ *and* $g: Z \longrightarrow X$ *a morphism in* Sm_S*, where* $f \cup g$ *is the morphism* $f \cup g: X' \coprod Z \longrightarrow X$ *induced by* f *and* g*.*

f.) Unit:

$$[S] \otimes \mathrm{id}: \mathfrak{e} \otimes \mathbb{Z}_S(0) \longrightarrow \mathbb{Z}_S(0) \otimes \mathbb{Z}_S(0).$$

Definition B.1.3. *Let* R *be a commutative ring which is flat over* $\mathbb{Z}$*. Then Levine defines the* <u>*triangulated motivic category* $\mathcal{DM}(S)_R$ *with coefficients in* R</u> *to be the pseudo-abelian hull of* $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$*. (When* R *is either* $\mathbb{Z}$ *or understood, one drops the* R *in the notation.) Denote the image of* $\mathbb{Z}_X(n)_f$ *in* $\mathcal{DM}(S)_R$ *or* $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$ *by* $R_X(n)_f$*.*

B.1.1 List of the most important morphisms in $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R \subset \mathcal{DM}(S)_R$:

a.) *Functoriality:* Let $f: X \longrightarrow Y$ be a morphism in Sm_S, and $Z \subset X$ and $Z' \subset Y$ two closed subsets such that their open complements are in Sm_S, and such that $f(Z) \subset Z'$. Then there is a pull-back morphism

$$f^*: R_{Y,Z'} \longrightarrow R_{X,Z}.$$

b.) *Homotopy:* [Lev98, I.2.2.1,p.19] Let $p: X \times \mathbb{A}^1_S \longrightarrow X$ be the projection. Then the map

$$p^*: R_X \longrightarrow R_{X\times\mathbb{A}^1_S}$$

is an isomorphism. More generally, if $Z \subset X$ is a closed subset with $X \setminus Z \in \mathrm{Sm}_S$, then the map

$$p^*: R_{X,Z} \longrightarrow R_{X\times_S\mathbb{A}^1_S, Z\times_S\mathbb{A}^1_S}$$

is an isomorphism in $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R \subset \mathcal{DM}(S)_R$.

c.) *Products:* [Lev98, I.2.2.11,pp.12/23] For all $X, Y \in \mathrm{Sm}_S$ there are natural associative and commutative external products

$$\boxtimes_{X,Y} \colon R_X \otimes R_Y \longrightarrow R_{X \times_S Y}$$

which are isomorphisms. More generally, given closed subsets $Z_X \subset X$ and $Z_Y \subset Y$ with open complements $j_X \colon U_X \hookrightarrow X$ and $j_Y \colon U_Y \hookrightarrow Y$ in Sm_S, the external products $\boxtimes$ give an isomorphism

$$\boxtimes_{X,Y}^{Z_X,Z_Y} \colon R_{X,Z_X}(q) \otimes R_{Y,Z_Y}(q') \longrightarrow R_{X \times_S Y, Z_X \times_S Z_Y}(q + q').$$

d.) *Cycle maps:* [Lev98, I.1.4.6, p.13] Denote the group of codimension d cycles in X over S by $\mathcal{Z}^d(X|S)$. Then for any $0 \neq W \in \mathcal{Z}^d(X)$ there is a morphism

$$[W] \colon \mathfrak{e} \longrightarrow R_X(d)[2d].$$

For $W = 0$, $[0]$ is defined to be the zero map. By [Lev98, 2.1.3.3, p.17] there is also a cycle map with support: If $W \in \mathcal{Z}^d(X|S)$ is a cycle with support on a closed subset Z of X (such that $X \setminus Z \in \mathrm{Sm}_S$) (denote the subset of these cycles by $\mathcal{Z}^d_Z(X|S)$), then there is a *cycle map with support*

$$[W]_Z \colon \mathfrak{e} \longrightarrow \mathbb{Z}_{X,Z}(d)[2d].$$

These cycle maps fit together to give the *cycle class map*

$$\mathrm{cl}^d_X \colon \mathcal{Z}^d(X|S) \longrightarrow \mathrm{Hom}_{D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R}(1, R_X(d)[2d])$$

and similarly for cycles with support:

$$\mathrm{cl}^d_{X,Z} \colon \mathcal{Z}^d_Z(X|S) \longrightarrow \mathrm{Hom}_{D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R}(1, R_{X,Z}(d)[2d]).$$

Note that $\mathrm{cl}^d_X = \mathrm{cl}^d_{X,X}$. These cycle maps satisfy (see [Lev98, I.3.5.3-5, p.49]) the following properties:

- If $f \colon Y \longrightarrow X$ is a map in Sm_S, Z is a closed subset of X and Z' a closed subset of Y, both with complements in Sm_S, such that Z' contains $f^{-1}(Z)$, then

  $$f^* \circ \mathrm{cl}^d_{X,Z}(W) = \mathrm{cl}^d_{Y,Z'}(f^*(W))$$

  for $W \in \mathcal{Z}^d_Z(X|S)$.
- $\mathrm{cl}^0_S(|S|) = \mathrm{id}_1$.
- Let X, Y be in Sm_S, $Z \subset X$ and $Z' \subset Y$ be closed subsets with complements in Sm_S. Take $A \in \mathcal{Z}^d_Z(X|S)$ and $B \in \mathcal{Z}^e_{Z'}(Y|S)$. Then the product cycle $A \times_S B$ is in $\mathcal{Z}^{d+e}_{Z \times_S Z'}(X \times_S Y|S)$ and

  $$\mathrm{cl}^{d+e}_{X \times_S Y, Z \times_S Z'}(A \times_S B) = \mathrm{cl}^d_{X,Z}(A) \cup \mathrm{cl}^e_{Y,Z'}(B).$$

B.1.2 The most important morphisms of $\mathcal{DM}(S)_R$:

As we have seen, the assigment $M \longmapsto (M, \mathrm{id})$ yields a fully faithful functor from the category $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$ into $\mathcal{DM}(S)_R$. Hence, the objects of $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$ described in B.1.4 yield objects in $\mathcal{DM}(S)_R$, and the morphisms between them are in fact morphisms in the triangulated motivic category. More generally, the objects of $\mathcal{DM}(S)_R$ are given by tuples (M, p), where M is an object in $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)$ and p is an idempotent in $\mathrm{End}_{D^b_{\mathrm{mot}}(\mathrm{Sm}_S)}(M)$.

Morphisms in $\mathcal{DM}(S)_R$ are defined as follows: Let $(M,p),(N,q)$ be objects in $\mathcal{DM}(S)_R$. Then a morphism $f\colon (M,p) \longrightarrow (N,q)$ is a morphism $f\colon M \longrightarrow N$ in $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)_R$ satisfying $qf = fp$, or equivalently $qfp = f$.

The most important example for us is the following construction of a *reduced motive* of a scheme $X \in \mathrm{Sm}_S$: Unfortunately, there are no references for this, since Levine does not deal with reduced cohomology in his exposition of motives [Lev98]. However, the definitions are fairly obvious:

Let $\pi\colon X \longrightarrow S$ in Sm_S be equipped with a section $x_0\colon S \longrightarrow X$. Consider the endomorphism

$$e_{x_0}\colon X \xrightarrow{\pi} S \xrightarrow{x_0} X$$

given by the structure morphism followed up by the section x_0. It satisfies $e_{x_0}^2 = e_{x_0}$, and hence also $\mathrm{id}_X - e_{x_0} \in \mathrm{End}_{\mathrm{Sm}_S}(X)$ is an idempotent. By the functoriality property of the elements in $D^b_{\mathrm{mot}}(\mathrm{Sm}_S)$, an endomorphism of X gives rise to an endomorphism of $\mathbb{Z}_X(0) \in D^b_{\mathrm{mot}}(\mathrm{Sm}_S)$.

Definition B.1.4. *Let $X \in \mathrm{Sm}_S$ be equipped with a section $x_0\colon S \longrightarrow X$. We define the reduced motive of X with respect to the section x_0 to be the element*

$$\mathbb{Z}^\circ_X(x_0) := (\mathbb{Z}_X(0), \mathrm{id}_{\mathbb{Z}_X} - e^*_{x_0}) \in \mathcal{DM}(\mathrm{Sm}_S),$$

where $e^*_{x_0}\colon \mathbb{Z}_X(0) \longrightarrow \mathbb{Z}_X(0)$ is the composition

$$\mathbb{Z}_X \xrightarrow{x_0^*} \mathbb{Z}_S \xrightarrow{\pi^*} \mathbb{Z}_X.$$

B.1.3 Properties of the triangulated motivic category

If $p\colon T \longrightarrow S$ is a map of reduced noetherian schemes, let $p^*\colon \mathrm{Sm}_S \longrightarrow \mathrm{Sm}_T$ denote the functor $X \longmapsto X \times_S T$. Then by section I.2.3 of [Lev98], p^* induces an exact tensor functor

$$\mathcal{DM}(p^*)\colon \mathcal{DM}(S)_R \longrightarrow \mathcal{DM}(T)_R$$

such that for a sequence of morphisms of reduced noetherian schemes

$$R \xrightarrow{q} T \xrightarrow{p} S$$

there is a canonical isomorphism ([Lev98, I.2.3.6, p.26])

$$\mathcal{DM}((q \circ p)^*) \longrightarrow \mathcal{DM}(q^*) \circ \mathcal{DM}(p^*)$$

which is associative. On objects $R_X(q) \in \mathcal{DM}(S)_R$ the functor $\mathcal{DM}(p^*)$ is given by

$$\mathcal{DM}(p^*)(R_X(q)) = R_{X \times_S T}.$$

In the end, Levine obtains:

Theorem B.1.5. *[Lev98, I.2.3.7] Sending S to $\mathcal{DM}(S_{\mathrm{red}})$ and $p\colon T \longrightarrow S$ to $\mathcal{DM}(p^*_{\mathrm{red}})$ defines a pseudo-functor*

$$\mathcal{DM}\colon \mathrm{Sch}^{\mathrm{op}} \longrightarrow \mathrm{TT},$$

where Sch *is the category of noetherian schemes, and* TT *is the category of triangulated tensor categories.*

Let us now fix $R = \mathbb{Z}$. Let $\mathbf{P}\,\mathrm{Sm}_S$ denote the category of pairs (X, Z), where $X \in \mathrm{Sm}_S$ and $Z \subset X$ is a closed subset with complement in Sm_S, together with morphisms $p\colon (X, Z) \longrightarrow (Y, Z')$ such that $p \in \mathrm{Sm}_S$ and $p^{-1}(Z') \subset Z$. By [Lev98, I.2.2.9, p.22] there are functors

$$\begin{aligned} \mathbb{Z}(a)[b]\colon \mathbf{P}\,\mathrm{Sm}_S^{\mathrm{op}} &\longrightarrow D^b_{\mathrm{mot}}(\mathrm{Sm}_S) \subset \mathcal{DM}(S) \\ (X, Z) &\longmapsto \mathbb{Z}_{X,Z}(a)[b]_{\mathrm{id}} \end{aligned}$$

for all $a, b \in \mathbb{Z}$. There is a natural extension of this functor to the pseudo-abelian envelopes on both sides, which we will denote by $\mathcal{K}$:

$$\begin{aligned} \mathbb{Z}(a)[b]\colon \mathcal{K}(\mathbf{P}\,\mathrm{Sm}_S^{\mathrm{op}}) &\longrightarrow \mathcal{DM}(S) \\ (X, Z; p) &\longmapsto (\mathbb{Z}_{X,Z}(a)[b]; p^*) \end{aligned}$$

for all $a, b \in \mathbb{Z}$. For extensions of this to complexes, see 2.2 in Part I.

B.2 Gysin morphisms and pushforward

The following two version of Gysin morphisms are constructed in [Lev98]:

(I) [Lev98, I.2.2.5, p.20] Let $p\colon P \longrightarrow X$ be a smooth morphism of relative dimension d with a section $s\colon X \longrightarrow P$. Let $s_*(X)$ denote the cycle in P defined by summing up the irreducible components of $s(X)$. Then there is the Gysin isomorphism

$$R_X(-d)[-2d] \longrightarrow R_{P, s_*(X)}(0)$$

which we will denote by $\cup[s_*(X)] \circ p^*$.

(II) [Lev98, Part I, III.2.1.2.2, p.132] Let $i\colon Z \hookrightarrow X$ be a codimension d closed embedding in Sm_S with smooth complement, and W be a closed smooth subscheme of Z with $Z \setminus W \in \mathrm{Sm}_S$. Then there is a Gysin isomorphism

$$i_*\colon R_{Z,W}(-d)[-2d] \longrightarrow R_{X,W}.$$

Properties of the Gysin morphism of type (II):

a.) *Functoriality:* Given subschemes $W \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} X$ of a scheme $X \in \mathrm{Sm}_S$ with $W, Y \in \mathrm{Sm}_S$, then one has

$$(i \circ j)_* = i_* \circ j_*.$$

This is a special case of a more general version with supports (see [Lev98, III.2.2.1, p.133]).

b.) Base-chance: By [Lev98, III.2.4.9, p.150], the Gysin-morphism satisfies the base-change property, which will be of major use in computations later: A cartesian square

$$\begin{tikzcd} Y \times_X Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

in Sm_S is called *transverse* if $\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Y) = 0$ for all $p > 0$. Then for any transverse square as above with $i\colon Y \hookrightarrow X$ a closed embedding in Sm_S, one has $f^* \circ i_* = p_{2*} \circ p_1^*$.

By local considerations, one can see that a cartesian square as above is transverse if Y and Z are closed subsets of X which intersect transversely, and i and f are the inclusions.

c.) *Projection formula:* [Lev98, III.2.2.2, p.136] Let $i\colon Z \hookrightarrow X$ be a closed embedding in Sm_S. Then, (dropping the obvious Künneth isomorphism), one has:

$$i_*(\mathrm{id}_Z^* \otimes i^*) \simeq i_* \otimes \mathrm{id}\colon \mathbb{Z}_Z(-d)[-2d] \otimes \mathbb{Z}_X \longrightarrow \mathbb{Z}_X,$$

and similarly for an inclusion with supports.

d.) *Compatibility with pull-backs of the base-scheme:* The Gysin-morphism is natural in the following sense: if $f\colon T \longrightarrow S$ is a map of reduced schemes, then by B.1.5. there is a pull-back functor $\mathcal{DM}(f^*)\colon \mathcal{DM}(S) \longrightarrow \mathcal{DM}(T)$, and by [Lev98, III.2.5.1, p.151], for any closed embedding $i\colon Z \longrightarrow X$ in Sm_S, one has

$$\mathcal{DM}(f^*)(i_*) = (i_{T \times_S Z})_*$$

where $i_{T \times_S Z}\colon T \times_S Z \hookrightarrow T \times_S X$ is the closed embedding in $\mathrm{Sm}(T)$ induced by i.

Similar properties hold for Gysin morphisms of type (I), see chapter III.2.2 in [Lev98] for details.

One can use the Gysin morphism of type (I) and (II) to construct push-forward maps for projective morphisms:

Let $p\colon Y \longrightarrow X$ be a projective morphism in $\mathcal{V}$. Suppose X and Y are of pure dimension d and e over S, respectively. Let Z_X and Z_Y be closed subsets of X and Y, respectively, such that $p(Z_Y) \subset Z_X$. Choose a vector bundle $E \to X$ with associated projective bundle $q\colon \mathbb{P}(E) \to X$, and a closed embedding $i\colon Y \longrightarrow \mathbb{P}(E)$. Then there is a push-forward p_* defined as

$$p_* := q_* \circ i_* \colon \mathbb{Z}_{Y,Z_Y}(e)[2e] \longrightarrow \mathbb{Z}_{X,Z_X}(d)[2d].$$

where q_* is the Gysin morphism of type (I), and i_* is the Gysin morphism of type (II).

Properties of projective push-forward:

a.) *Functoriality:* [Lev98, III.2.4.7, p.149] For a sequence of projective morphisms in Sm_S

$$Z \xrightarrow{p'} Y \xrightarrow{p} X$$

with all schemes of pure dimension over X, one has

$$p_* \circ p'_* = (p \circ p')_*,$$

and moreover $\mathrm{id}_* = \mathrm{id}$.

b.) *B*y [Lev98, III.2.4.9, p.150], push-forward satisfies the following base-change property: for any transverse square

$$\begin{tikzcd} Y \times_X Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "f"] \\ Y \arrow[r, "p"'] & X \end{tikzcd}$$

in Sm_S with p a projective morphism, one has $f^* \circ p_* = p_{2*} \circ p_1$.

c.) *Projection formula:* The projection formula for push-forward and pull-back of projective morphisms $p\colon Y \longrightarrow X$ holds by [Lev98, III.2.4.8, p.150]. Dropping the obvious Künneth isomorphisms in the notations, one has for a projective morphism $p\colon Y \longrightarrow X$ in Sm_S

$$p_*(\mathrm{id} \otimes p^*) = i_* \otimes \mathrm{id} \colon \mathbb{Z}_Y(e)[2e] \otimes \mathbb{Z}_X \longrightarrow \mathbb{Z}_X(d)[2d]$$

where X and Y are of pure dimension e and d, respectively.

d.) *Naturality:* Also, push-forward is natural in the following sense: if $f\colon T \longrightarrow S$ is a map of reduced schemes, then by Theorem B.1.5 there is a pull-back functor

$$\mathcal{DM}(f^*) \colon \mathcal{DM}(S) \longrightarrow \mathcal{DM}(T),$$

and by [Lev98, III.2.5.1, p.151], for any projective morphism $p\colon X \longrightarrow Y$ in Sm_S, one has

$$\mathcal{DM}(f^*)(p_*) = \mathcal{DM}(f^*)(p)_*.$$

B.3 The subcategory $\mathcal{DM}(S)^{\mathrm{pr}} \subset \mathcal{DM}(S)$ and duals

Another important construction in the motivic category is the duality functor. For details, see Part I, Chapter IV, Section 1 in [Lev98]. The motivic duality is an analogue of Poincaré duality in realizations. Let $\mathcal{DM}(S)^{\mathrm{pr}}$ denote the smallest strictly full triangulated subcategory of $\mathcal{DM}(S)$ containing the objects $\mathbb{Z}_X(p)$ with X projective over S, $p \in \mathbb{Z}$, and closed under taking summands. Since $\mathbb{Z}_X(p) \otimes \mathbb{Z}_Y(q) \cong \mathbb{Z}_{X\times_S Y}(p+q)$ in $\mathcal{DM}(S)$, $\mathcal{DM}(S)^{\mathrm{pr}}$ is a triangulated tensor subcategory of $\mathcal{DM}(S)$.

Lemma B.3.1. *([Lev98, I.1.5.4, p.208]) Let X be in Sm_S. Suppose there is an open immersion $j\colon X \longrightarrow \bar{X}$ with $\bar{X}$ smooth and projective over S, such that*

(i) The complement $Z := \bar{X} \setminus X$ is a union of smooth projective S-schemes, $Z = \bigcup_{i=1}^N Z_i$ with each Z_i a union of irreducible components of Z.
(ii) For each collection of indices $i_1, \ldots, i_s$, the closed subset $Z_{i_1} \cap \ldots \cap Z_{i_s}$ of $\bar{X}$ is smooth over S

Corollary B.3.2. *The assumptions of Lemma B.3.1 are satisfied, if Z is a normal crossing subscheme of $\bar{X}$. In particular, for S a smooth complex variety, $\mathcal{DM}(S) = \mathcal{DM}(S)^{\mathrm{pr}}$.*

For X smooth and projective over S of relative dimension d, Levine then sets

$$\mathbb{Z}_X^{\mathbf{D}}(a)[b] := \mathbb{Z}_X(d-a)[2d-b].$$

Levine then shows the following:

Theorem B.3.3. *[Lev00, IV.1.4.2, p.206/7] The operation $(\ )^{\boldsymbol{D}}$ defined for projective X by the above definition extends to an exact pseudo-tensor functor*

$$(\ )^{\boldsymbol{D}}\colon (\mathcal{DM}(S)^{\mathrm{pr}})^{\mathrm{op}} \longrightarrow \mathcal{DM}(S)^{\mathrm{pr}}$$

defining an exact duality on $\mathcal{DM}(S)^{\mathrm{pr}}$, i.e. for A, B and C in $\mathcal{DM}(S)^{\mathrm{pr}}$ there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{DM}(S)}(A\otimes B^{\boldsymbol{D}}, C) &\longrightarrow \mathrm{Hom}_{\mathcal{DM}(S)}(A, C\otimes B),\\ \mathrm{Hom}_{\mathcal{DM}(S)}(A\otimes B, C) &\longrightarrow \mathrm{Hom}_{\mathcal{DM}(S)}(A, C\otimes B^{\boldsymbol{D}})\end{aligned}$$

which are exact in the variables A, B and C. In addition, there is a natural isomorphism

$$\mathrm{id} \longrightarrow ((\ )^{\boldsymbol{D}})^{\boldsymbol{D}}.$$

B.4 Cohomological and homological motives of schemes

(i) For $X \in \mathrm{Sm}_S$ the object $\mathbb{Z}_X := \mathbb{Z}_X(0)$ in $\mathcal{DM}(S)$ is called the *motive of X* .

(ii) If $Z \subset X$ is a closed subset, we call the object $\mathbb{Z}_{X,Z} := \mathbb{Z}_{X,Z}(0)$ the *motive of X with support in Z* (see "motives with support" in section B.1).

Definition B.4.1. *[Lev98, V.2.2.2, p.215] Let X be in Sm_S. The motivic cohomology $H^p(X, \mathbb{Z}(q))$ of X is defined by*

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}_X(q)[p])$$

Likewise, in $M \in \mathcal{DM}(S)$ is any motive, then the motivic cohomology of M is given by

$$H^p(X, M) := \mathrm{Hom}_{\mathcal{DM}(S)}(1, M[p]).$$

Apart from the properties listed in section B.1.1, these motives have the following properties:

Properties of motives:

a.) *Functoriality:* Let $f\colon X \longrightarrow Y$ be a morphism in Sm_S, and $Z \subset X$ and $Z' \subset Y$ two closed subsets such that $f(Z) \subset Z'$. Then there is a pull-back morphism

$$f^*\colon \mathbb{Z}_{Y,Z'} \longrightarrow \mathbb{Z}_{X,Z}.$$

b.) *Mayer-Vietoris:* [Lev98, I.2.2.6,p.21] Write X as a union of open subschemes $X = U \cup V$ with $X, U, V \in \mathrm{Sm}_S^{\mathrm{pr}}$. Then $U \cap V$ is in $\mathrm{Sm}_S^{\mathrm{pr}}$. Denoting the incusions by $j_{U\cap V,U}\colon U \cap V \hookrightarrow U, j_{U\cap V,V}\colon U \cap V \hookrightarrow V$, $j_U\colon U \hookrightarrow X$ and $j_V\colon V \hookrightarrow X$, we have the Mayer-Vietoris distinguished triangle

$$\mathbb{Z}_X(n) \longrightarrow \mathbb{Z}_U(n) \oplus \mathbb{Z}_V(n) \longrightarrow \mathbb{Z}_{U\cap V}(n) \longrightarrow Z_X(n)[1],$$

where the first arrow is given by $(j_{U\cap V,U*}, -j_{U\cap V,V*})$ and the second one by $j_U^* + j_V^*$. This gives rise to a Mayer-Vietoris sequence for motives with support ([Lev98, I.2.2.10, p.22]): If $Z = Z_1 \cup Z_2$ is a union of closed subsets in X and $Z_{12} = Z_1 \cap Z_2$ denotes their intersection, then there is a distinguished triangle

$$\mathbb{Z}_{X,Z_{12}} \longrightarrow \mathbb{Z}_{X,Z_1} \oplus \mathbb{Z}_{X,Z_2} \longrightarrow \mathbb{Z}_{X,Z} \longrightarrow \mathbb{Z}_{X,Z_{12}}[1],$$

where the first arrow is given by $(i_{Z_{12}\subset Z_1*}, -i_{Z_{12}\subset Z_2*})$ and the second arrow by $i_{Z_1\subset Z*} + i_{Z_2\subset Z*}$.

c.) *Gysin morphism:* [Lev98, Part I, III.2.1.2.2, p.132] Let $i\colon Z \hookrightarrow X$ be a codimension d closed embedding in Sm_S, and W be a closed smooth subscheme of Z. Then there is a Gysin isomorphism

$$i_*\colon \mathbb{Z}_{Z,W}(-d)[-2d] \longrightarrow \mathbb{Z}_{X,W}.$$

d.) *Localization:* [Lev98, I.2.2.10, p.22] If Z, Y are closed subsets of $X \in \mathrm{Sm}_S$, $j\colon U \hookrightarrow X$ is the complement of Z in X and $U_Y := Y \cap U$, then there is a distinguished triangle

$$\mathbb{Z}_{X,Z} \xrightarrow{i_{Z\subset Z\cup Y*}} \mathbb{Z}_{X,Z\cup Y} \xrightarrow{j^*} \mathbb{Z}_{U,U_Y} \longrightarrow \mathbb{Z}_{X,Z}.$$

In particular, taking $Y = X$, one obtains the localization sequence

$$\mathbb{Z}_{X,Z} \xrightarrow{i_{Z\subset X*}} \mathbb{Z}_X \xrightarrow{j^*} \mathbb{Z}_U \longrightarrow \mathbb{Z}_{X,F}[1].$$

Using the duality functor of sectionB.3 above, one may define homological motives in the subcategory $\mathcal{DM}(S)^{\mathrm{pr}}$ of $\mathcal{DM}(S)$:

Definition B.4.2. *[Lev98, V.2.2.2, p.215] Denote the category of smooth projective S-schemes by $\mathrm{Sm}_S^{\mathrm{pr}}$, and let X be in $\mathrm{Sm}_S^{\mathrm{pr}}$.*

a.) The homological motive of X, $\mathbb{Z}_X^h$, is the dual $\mathbb{Z}_X^D$ of $\mathbb{Z}_X$.

b.) Let $j_U\colon U \hookrightarrow X$ be a smooth open immersion, and let Z be the complement of U in X. Define the homological motive of X relative to U, $\mathbb{Z}_{X/U}^h$, as the dual $\mathbb{Z}_{X,Z}^D$ of the motive with support $\mathbb{Z}_{X,Z}$.

c.) The motivic homology $H_p(X,\mathbb{Z}(q))$ of X is defined by

$$H_p(X,\mathbb{Z}(q)) := \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}_X^h(-q)[-p]).$$

B.5 Relative motives and cohomology

There is a notion of relative motives giving rise to cohomology groups which correspond to the Adams-eigenspaces of relative K-theory. The reference for the following section is - unless stated otherwise - section I.2.6 of [Lev98].

B.5.1 Motives of n-cubes

Definition B.5.1. *a.) The n-cube is the category $\langle n\rangle$ whose objects are the subsets I of $\{1,\dots,n\}$ and morphisms given by arrows $J \longrightarrow I$ if and only if $I \subset J$.*

b.) Let $\mathcal{C}$ be a category. Then the category of n-cubes in $\mathcal{C}$, denoted by $\mathcal{C}^{\langle n\rangle}$ is the category of functors $X\colon \langle n\rangle \longrightarrow \mathcal{C}$.

We now consider n-cubes in Sm_S and lift them to n-cubes in $\mathcal{L}(\mathrm{Sm}_S)$ in the following way: Let

$$X_*\colon \langle n\rangle \longrightarrow \mathrm{Sm}_S,\ I \longmapsto X_I$$

be a functor. We want to lift the X_I to $\mathcal{L}(\mathrm{Sm}_S)$ in a compatible way from one fixed lifting via fiber product: Let $(X_\emptyset, f_\emptyset\colon X' \longrightarrow X_\emptyset)$ be a lifting of $X_\emptyset$ to an object of $\mathcal{L}(\mathrm{Sm}_S)$. For each $I \subset \{1,\dots,n\}$ form the cartesian diagram

$$\begin{tikzcd} X'_I := X' \times_{X_\emptyset} X_I \arrow[r, "\mathrm{pr}_1"] \arrow[d, "f_I:=\mathrm{pr}_2"'] & X' \arrow[d, "f_\emptyset"] \\ X_I \arrow[r, "X_{I\supset\emptyset}"'] & X_\emptyset \end{tikzcd}$$

The maps $X_{J\supset I}$ then induce maps $X'_{J\supset I}\colon X'_J \longrightarrow X'_I$, yielding an n-cube

$$X'_*\colon \langle n\rangle \longrightarrow \mathrm{Sm}_S,$$

together with a map $f_*\colon X'_* \longrightarrow X_*$ of $\langle n\rangle$-cubes. Composing the n-cube (X_*, f_*^X) with the functor $\mathbb{Z}(0)\colon \mathcal{L}(\mathrm{Sm}_S)^{op} \longrightarrow \mathcal{DM}(S)$ yields an n-cube in $\mathcal{DM}(S)$. Levine then forms a complex from this n-cube by summing up over all subsets of $\{1,\dots,n\}$ of the same cardinality, i.e. we define an element of $\mathcal{DM}(S)$ by taking $\mathbb{Z}_{X_*}(0)_{f_\emptyset}$ to denote the complex

$$\mathbb{Z}_{X_\emptyset}(0)_{f_\emptyset} \to \ldots \to \bigoplus_{|I|=s} \mathbb{Z}_{X_I}(0)_{f_I^X} \to \bigoplus_{|I|=s+1} \mathbb{Z}_{X_I}(0)_{f_I^X} \to \ldots \to \mathbb{Z}_{\{1,\dots,n\}}(0)_{f_{\{1,\dots,n\}}}$$

in degree 0 up to n. Here, the differential is given in degree s as the alternating sum

$$\partial^s := \sum_{|I|=s}\sum_{i=1}^{n}(-1)^i \partial^s_{I,i},$$

where the component $\partial^s_{I,i}\colon \mathbb{Z}_{X_I}(0)_{f_I} \longrightarrow \mathbb{Z}_{X_{I\cup\{i\}}}(0)_{f^X_{I\cup i}}$ is defined by

$$\partial^s_{I,i} := \begin{cases} X^*_{(I\cup\{i\})\supset I} & \text{for } i\notin I \\ 0 & \text{for } i \in I \end{cases}$$

There is a canonical complex $\mathbb{Z}_{X_*}(0)$ given by choosing $f_\emptyset = \mathrm{id}_{X_\emptyset}$.

B.5.2 Relative motives

We now define a relative motive by associating an n-cube to the relative setting: Let X be a smooth S-scheme with smooth subschemes $D_1,\dots,D_n \subset X$. For each index $I = (1 \le i_1 < \ldots i_s \le n)$, denote the intersection of all subschemes D_i with $i \in I$ by $D_I := D_{i_1} \cap \ldots, D_{i_s}$. Lift $X, D_1, \dots, D_n$ to $\mathcal{L}(\mathrm{Sm}_S)$ via the identity morphisms (i.e. $(X, \mathrm{id}_X\colon X \longrightarrow X)$, $(D_i, \mathrm{id}_{D_i}\colon D_i \longrightarrow D_i)$). Let

$$(X; D_1,\dots,D_n)_*\colon \langle n\rangle \longrightarrow \mathrm{Sm}_S$$

be the n-cube in Sm_S with $(X; D_1,\dots,D_n)_I = D_I$, and for $J \subset I$ take the associated morphism $(X; D_1,\dots,D_n)_{I\supset J}\colon D_I \longrightarrow D_J$ to be the inclusion. The above construction in B.5.1 then gives an object $\mathbb{Z}_{(X;D_1,\dots,D_n)}(0)$ in $\mathcal{DM}(S)$ (where we dropped the $*$ in $\mathbb{Z}_{(X;D_1,\dots,D_n)*}(0)$). It is given by the complex

$$\mathbb{Z}_X(0) \to \bigoplus_{i=1}^{n} \mathbb{Z}_{D_i} \to \ldots \to \bigoplus_{|I|=s} \mathbb{Z}_{D_I} \to \bigoplus_{|I|=s+1} \mathbb{Z}_{D_I} \to \ldots \to \mathbb{Z}_{D_1\cap\ldots\cap D_n}$$

in degrees 0 up to n. The differential is given in degree s as the alternating sum

$$\partial^s := \sum_{|I|=s} \sum_{i=1}^{n} (-1)^i \partial^s_{I,i},$$

where the component $\partial^s_{I,i} \colon \mathbb{Z}_{D_I} \longrightarrow \mathbb{Z}_{D_{I\cup\{i\}}}$ is defined by

$$\partial^s_{I,i} := \begin{cases} X^*_{(I\cup\{i\})\supset I} & \text{for } i \notin I \\ 0 & \text{for } i \in I \end{cases}$$

Definition B.5.2. *a.) For a smooth S-scheme X with smooth subschemes $D_1, \ldots, D_n$ we define the* <u>*motive of X relative to $D_1, \ldots, D_n$*</u> *as the object $\mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)$ of $\mathcal{DM}(S)$ defined above.*

b.) For an open subscheme $j \colon U \longrightarrow W$ with complement W, the <u>*relative motive with support*</u> *$\mathbb{Z}_{(X;D_1,\ldots,D_n),W}$ is defined as the cone*

$$\mathbb{Z}_{(X;D_1,\ldots,D_n),W} := \mathrm{Cone}(j^* \colon \mathbb{Z}_{(X;D_1,\ldots,D_n)} \to \mathbb{Z}_{(U;D_1^U,\ldots,D_n^U)})[-1]$$

where $D_i^U := U \cap D_i$

Properties of relative motives:

In what follows, let X be in Sm_S and $D_1, \ldots, D_n$ be closed subschemes of X.

a.) *Functoriality:* Suppose we also have $Y \in \mathrm{Sm}_S$ and smooth $E_1, \ldots, E_m \subset Y$ together with a map $f \colon X \longrightarrow Y$ such that for all i, $f(D_i) \subset E_{\alpha(i)}$ for some $\alpha(i) \in \{1, \ldots, m\}$. Then α induces a map $\alpha \colon \langle n \rangle \longrightarrow \langle m \rangle$, which in turn gives rise to maps $f^*|_{D_I} \colon \mathbb{Z}_{E_{\alpha(I)}} \longrightarrow \mathbb{Z}_{D_I}$. Putting $f^*_J := 0 \colon \mathbb{Z}_{e_J} \longrightarrow 0$ for all $J \notin \mathrm{Im}(\alpha)$, this defines a pull-back map

$$f^* \colon \mathbb{Z}_{(Y;E_1,\ldots,E_m)} \longrightarrow \mathbb{Z}_{(X;D_1,\ldots,D_n)}.$$

b.) *Relativization distinguished triangle:* There is a description of n-cubes as the shifted cone of of a morphism of $(n-1)$-cubes (see [Lev98, I.2.6.4, p.33]). This gives rise to the distinguished triangle

$$\mathbb{Z}_{(X;D_1,\ldots,D_n)}(0) \longrightarrow \mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}(0) \xrightarrow{i_n} \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{n-1,n})}(0) \longrightarrow \mathbb{Z}_{(X;D_1,\ldots,D_n)}(0)[1]$$

where $D_{i,n} := D_i \cap D_n$.

c.) *Localization:* Let moreover Z be a closed subset of X with open complement $j \colon U \hookrightarrow X$. The definition of the relative motive with support as a cone yields the localization distinguished triangle

$$\mathbb{Z}_{(X;D_1,\ldots,D_n),Z} \longrightarrow \mathbb{Z}_{(X;D_1,\ldots,D_n)} \longrightarrow \mathbb{Z}_{(U;D_1^U,\ldots,D_n^U)} \longrightarrow \mathbb{Z}_{(X;D_1,\ldots,D_n),Z}[1]$$

d.) *Gysin morphism:* [Lev98, III.2.6, pp.153ff. and IV. 2.3.4, p.219] Let $i\colon Z \hookrightarrow X$ be a closed subscheme of codimension d in X, such that the D_i and Z intersect transversely. Denote the intersection of Z with the divisors D_i by $D_i^Z := Z \cap D_i$. Then there is a relative Gysin isomorphism (derived from Levine's Gysin isomorphism for diagrams in section III.2.6)

$$i_*\colon \mathbb{Z}_{(Z;D_1^Z,\dots,D_n^Z)}(-d)[-2d] \longrightarrow \mathbb{Z}_{(X;D_1,\dots,D_n)}.$$

e.) *Relative cycle classes:* [Lev98, IV.2.3.1, p.218] Suppose the D_i intersect transversely, and let Z be a closed subset of X disjoint from all D_i. Then there is a relative cycle class map (note that there is a print error in Levine's book - compare with [Lev98, I.3.5.2.6, p.48])

$$\mathrm{cl}^q_{(X;D_1,\dots,D_n),Z}\colon \mathcal{Z}^q_Z(X/S) \longrightarrow \mathrm{Hom}(1, \mathcal{Z}_{(X;D_1,\dots,D_n),Z}(q)[2q]).$$

f.) *Duality:* [Lev98, IV.2.3.4, p.219] Let X be a smooth equi-dimensional S-scheme of dimension d over S, $D_1,\dots,D_n$ closed subschemes of X which form a normal crossing subscheme of X. For some $i \in \{0,\dots,n\}$ let $U := X \setminus (D_1 \cup \dots \cup D_i)$ and $V := X \setminus (D_{i+1} \cup \dots \cup D_n)$, and let $\delta_{U\cap V}\colon U \cap V = X \setminus (D_1 \cup \dots D_n) \hookrightarrow V \times_S U$ denote the diagonal inclusion. Moreover, put $D_j^V := V \cap D_j$ and $D_j^U := D_j \cap U$. We consider the relative motives

$$\mathbb{Z}_{(V;D_1^V,\dots,D_i^V)},\ \mathbb{Z}_{(U;D_{i+1}^U,\dots,D_n^U)}.$$

- Denote the codimension d cycle defined by the image of $\delta_{U\cap V}$ in $U \times_S V$ by $\varDelta_{U\cap V}$. By the above, this cycle defines a map

$$\mathrm{cl}(\varDelta_{U\cap V})\colon 1 \longrightarrow \mathbb{Z}_{(V\times_S U;D_1^V\times_S U,\dots,D_i^V\times_S U,V\times_S D_{i+1}^U,\dots,V\times_S D_n^U)}(d)[2d]$$

$$\text{Let } \delta_{U,V}\colon 1 \longrightarrow \mathbb{Z}_{V;D_1^V,\dots,D_i^V} \otimes \mathbb{Z}_{(U;D_{i+1}^U,\dots,D_n^U)}(d)[2d]$$

  be the map $\mathrm{cl}(\varDelta_{U\cap V})$ followed by the inverse of the Künneth isomorphism. Then the pair

$$(\mathbb{Z}_{(V;D_1^V,\dots,D_i^V)}, \delta_{U,V})$$

  is the dual of

$$\mathbb{Z}_{(U;D_{i+1}^U,\dots,D_n^U)}.$$

- Let furthermore $i_Z\colon Z \longrightarrow X$ be a closed subscheme of X of codimension $d_{Z:X}$ such that $Z, D_1,\dots,D_n$ have transverse intersection, and let $Z_V := Z \cap V$, $Z_U := Z \cap U$, $D_{Z,i}^V := D_i \cap Z_V$ and $D_{U,i}^V := D_i \cap Z_U$. Then the collection of inclusions $D_{Z_i}^V \hookrightarrow D_i^V$ defines a morphism

$$i_{Z_V}^*\colon \mathbb{Z}_{(V;D_1^V,\dots,D_i^V)} \longrightarrow \mathbb{Z}_{(Z_V;D_{Z,1}^V,\dots,D_{Z,i}^V)}.$$

  Similarly, there is a Gysin morphism

$$i_{Z_U*}\colon \mathbb{Z}_{(Z_U;D_{Z,1}^U,\dots,D_{Z,i}^U)}(-d_{Z:X})[-2d_{Z:X}] \longrightarrow \mathbb{Z}_{(U;D_1^U,\dots,D_i^U)}.$$

  Then, by [Lev98, IV.2.3.5(ii), p.219], the map i_{Z_U*} is dual to the map $i_{Z_V}^*$.

g.) *Relative Gysin distinguished triangle:* [Lev98, IV.2.3.5.1, p.220] We keep the setting of "Duality" above. Moreover, let $V' := X \setminus (D_{i+1} \cup \ldots \cup D_{n-1})$ and $D_i^{V'} := V' \cap D_i$ with inclusions $j\colon V \longrightarrow V'$ and $i\colon D_n^{V'} \hookrightarrow V'$. Denote the dimension of $D_n^{V'}$ over S by d', and $D_{n,j}^{V'} := D_n^{V'} \cap D_j$. The Gysin isomorphism

$$i_*\colon \mathbb{Z}_{(D_n^{V'};D_{n,1}^{V'},\ldots,D_{n,i}^{V'})}(d')[2d'] \longrightarrow \mathbb{Z}_{(V';D_1^{V'},\ldots,D_i^{V'})}(d)[2d]$$

together with the localization distinguished triangle for the relative motive with support gives the Gysin distinguished triangle

$$\begin{tikzcd}
\mathbb{Z}_{(V';D_1^{V'},\ldots,D_i^{V'})}(d)[2d] \arrow[r, "j^*"] & \mathbb{Z}_{(V;D_1^{V},\ldots,D_i^{V})}(d)[2d] \arrow[dll, rounded corners] & \\
\mathbb{Z}_{(D_n^{V'};D_{1,n}^{V'},\ldots,D_{i,n}^{V'})}(d')[2d'+1] \arrow[r, "i_*"] & \mathbb{Z}_{(V';D_1^{V'},\ldots,D_i^{V'})}(d)[2d+1]. &
\end{tikzcd}$$

In particular, for $i = n-1, V = X \setminus D_n, V' = X$ and denoting the codimension of D_n in X by c, one obtains the Gysin triangle

$$\begin{tikzcd}
\mathbb{Z}_{(X;D_1,\ldots,D_{n-1})} \arrow[r, "j^*"] & \mathbb{Z}_{(X\setminus D_n;D_1^V,\ldots,D_{n-1}^V)} \arrow[r] & \mathbb{Z}_{(D_n;D_{1,n},\ldots,D_{i,n})}(-c)[-2c+1] \arrow[dll, rounded corners, "i_*"] \\
\mathbb{Z}_{(X;D_1,\ldots,D_{n-1})}[1]. & &
\end{tikzcd}$$

B.5.3 Relative motivic cohomology

Definition B.5.3. *a.) For a smooth S-scheme X with smooth subschemes $D_1, \ldots, D_n$, the* <u>*motivic cohomology of X relative $D_1, \ldots, D_n$*</u> *is defined as*

$$H^p(X; D_1, \ldots, D_n, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}_{(X;D_1,\ldots,D_n)}(q)[p]).$$

b.) If moreover $j\colon U \hookrightarrow X$ is an open subscheme with complement W, the <u>*motivic cohomology of X relative $D_1, \ldots, D_n$ with support in W*</u> *is defined as*

$$H^p_W(X; D_1, \ldots, D_n, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathcal{DM}(S)}(1, \mathbb{Z}_{(X;D_1,\ldots,D_n),W}(q)[p]).$$

Properties of relative motivic cohomology:

a.) *Cycle map and compatibility with K-theory:* [Lev98, III.1.4.8-1.5, pp.123 ff.] Let X be a smooth scheme and $D_1, \ldots, D_n$ be smooth subschemes of S such that each intersection $D_I := \cap_{i\in I} D_i$ is also in Sm_S. Then there are K-groups $K_n(X; D_1, \ldots, D_n)$ of X relative to $D_1, \ldots, D_n$ defined as in [Lev98, III.1.4.8, p.123] (similarly, there are K-groups $K_n^Z(X; D_1, \ldots, D_n)$ with support in a closed subscheme $Z \subset X$). Then by [Lev98, III.1.4.8(ii). p.123/124] there are cycle maps

$$c^{q,1q-p}_{(X;D_1,\ldots,D_n)}\colon K_p(X; D_1, \ldots, D_n) \longrightarrow H^{2q-p}(X; D_1, \ldots, D_n, \mathbb{Z}(q))$$

(and similarly with support) which are compatible with pull-backs. Moreover, by [Lev98, III.1.5.2, pp.128-130], the localization and relativization exact sequences are compatible with the corresponding ones for relative K-theory.

b.) *Chern character:* [Lev98, III.3.3.10, p.171] Keeping the notation above, there is a Chern character for relative K-theory (with support):

$$\mathrm{ch}^W_{X;D_1,\dots,D_n,p}\colon K_p^W(X;D_1,\dots,D_n) \longrightarrow \prod_{q\geq 0} H_W^{2q-p}(X;D_1,\dots,D_n,\mathbb{Z}(q))$$

which is compatible with cup products on both sides.

c.) *Relative Gysin sequence:* We keep the above notation. Moreover, let $V' := X \setminus (D_{i+1}\cup\ldots\cup D_{n-1})$ and $D_i^{V'} := V'\cap D_i$ with inclusions $j\colon V \longrightarrow V'$ and $i\colon D_n^{V'} \hookrightarrow V'$. Denote the dimension of $D_n^{V'}$ over S by d', and $D_{n,j}^{V'} := D_n^{V'} \cap D_j$. There is a Gysin distinguished triangle

$$\begin{tikzcd}
H^{p+2d}(V';D_1^{V'},\dots,D_i^{V'},\mathbb{Z}(q+d)) \arrow[r, "j^*"] & H^{p+2d}(V;D_1^{V},\dots,D_i^{V}),\mathbb{Z}(q+d)) \arrow[dl] \\
H^{p+2d'+1}(D_n^{V'};D_{1,n}^{V'},\dots,D_{i,n}^{V'}),\mathbb{Z}(q+d')) \arrow[r, "i_*"] & H^{p+2d+1}(V';D_1^{V'},\dots,D_i^{V'}),\mathbb{Z}(q+d)).
\end{tikzcd}$$

In particular, for $i=n-1, V = X\setminus D_n, V' = X$ and denoting the codimension of D_n in X by c, one obtains the Gysin triangle

$$\begin{tikzcd}
H^{p}(X;D_1,\dots,D_{n-1},\mathbb{Z}(q)) \arrow[r, "j^*"] & H^{p}(X\setminus D_n;D_1^{V},\dots,D_{n-1}^{V},\mathbb{Z}(q)) \arrow[dl] \\
H^{p-2c+1}(D_n;D_{1,n},\dots,D_{i,n},\mathbb{Z}(q-c)) \arrow[r, "i_*"] & H^{p+1}(X;D_1,\dots,D_i,\mathbb{Z}(q)).
\end{tikzcd}$$

B.6 Cycle maps and comparison to K-theory:

Let me quickly recall the definition of algebraic K- and K'-theory via Quillen's $\mathcal{Q}$-construction:

For a noetherian and separated, quasi-projective scheme X we consider the following two categories:

- the category $\mathrm{Coh}(X)$ of coherent $\mathcal{O}_X$-modules, and
- its full subcategory $\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$ of locally free coherent $\mathcal{O}_X$-bundles on X (also called vector bundles).

Both are exact categories. Whenever one has an exact category $\mathcal{C}$, one may associate to it Quillen's $\mathcal{Q}$-construction: this is the category $\mathcal{QC}$ having the same objects as $\mathcal{C}$, and morphisms between two objects $A,B\in\mathcal{C}$ given by

$$\mathrm{Hom}_{\mathcal{QC}}(A,B) = \{A \overset{p}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} B \mid q(\text{resp. } i) \text{ admissible epi (resp. mono)}\}/\sim$$

where two diagrams $A \overset{p}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} B, A \overset{p'}{\twoheadleftarrow} X' \overset{i'}{\rightarrowtail} B$ are equivalent if the resulting diagram

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "i"] & \\ A & & B \\ & X \arrow[ul, "p'"] \arrow[ur, "i'"'] & \end{tikzcd}$$

commutes. With this notion, one forms the categories $\mathcal{Q}\,\mathrm{Coh}(X)$ and $\mathcal{Q}\,\mathrm{Vect}(X)$. Recall that the classifying space of a small category $\mathcal{C}$ is the geometric realization of its nerve

$$B\mathcal{C} := |N\mathcal{C}|.$$

With these notions, *Quillen's algebraic K-groups* of a noetherian, separated, quasi-projective variety X are defined as (see [Qui69, 7.1, p.116])

$$K_i(X) := K_i(\mathrm{Vect}(X)) := \pi_{i+1}(B\mathcal{Q}\,\mathrm{Vect}(X)).$$

We denote the Adams operations by $\psi_p^k\colon K_p(X) \longrightarrow K_p(X)$ for all k, p, and similarly $\psi_p^k\colon K_p(X,Y) \longrightarrow K_p(X,Y)$ on relative K-groups for a morphism $Y \longrightarrow X$, which were constructed on K-groups of quasi-projective schemes in [Sus82], for example, and by Beilinson on K-groups of regular schemes in [Bei84, 2.2, p. 2048]. Let $K_p^{(i)}(X)$ (resp. $K_p^{(i)}(X,Y)$) denote the subspace of $K_*(X) \otimes \mathbb{Q}$ (resp.$K_p(X,Y)$) on which ψ^p acts by p^i. One obtains direct sum decompositions

$$K_*(X) \otimes \mathbb{Q} \cong \bigoplus K_*^{(i)}(X)$$
$$K_*(X,Y) \otimes \mathbb{Q} \cong \bigoplus K_*^{(i)}(X,Y)$$

which is independent of p. In [Bei84, 2.2.3, p. 2048], Beilinson constructs corresponding Adams operations on K'-groups via the isomorphism of (vi): Let Y be a quasi-projective scheme over a field k, and imbed Y in a smooth scheme X. Then

$$i_*\colon K'(Y) \cong K(X, X \setminus Y).$$

Then the Adams operations ψ^p acting on the right hand side yield Adams operations on the left. This does not depend on the imbedding $Y \hookrightarrow X$, and yields a decomposition

$$K'_*(X) \otimes \mathbb{Q} \cong \bigoplus K'^{(i)}_*(X)$$

of $K'(X)$ into the p^i-eigenspaces of ψ'^p.

Definition B.6.1. *Let X be a regular scheme. Then Beilinson defines the motivic cohomology groups of X to be given by the Adams-eigenspaces*

$$H^j_{\mathcal{M}}(X, \mathbb{Q}(i)) := K^{(i)}_{2i-j}(X)$$

of the K-groups, and the motivic homology groups of X to be given by the Adams-eigenspaces

$$H_j^{\mathcal{M}}(X, \mathbb{Q}(i)) := K'^{(i)}_{2i-j}(X).$$

The relation between Beilinson's classical motivic cohomology groups and Levine's motivic cohomology are given by the following theorem:

Theorem B.6.2. *([Lev98, II.3.6.6, p.105] for $\mathcal{V} = \mathrm{Sm}_S$)*
Let S be a scheme which is a filtered projective limit of schemes S_α, such that each S_α is a smooth k_α-scheme of finite type for some field k_α, with S_α of dimension at most one over k_α. Then for X in Sm_S there is a natural isomorphism

$$K_{2q-p}(X)^{(q)} \longrightarrow H^p_{\mathcal{M}}(X, \mathbb{Q}(q)) = \mathrm{Hom}_{\mathcal{DM}_{\mathbb{Q}}(S)}(\mathbb{Q}_S, \mathbb{Q}_X(q)[p])$$

where $K_n(X)^q$ is the weight q Adams eigenspace of $K_n(X) \otimes \mathbb{Q}$.

There are similar results relating relative motives and a relative version of K-theory. The latter is defined as follows:

Let $D_1, \ldots, D_n$ be closed subschemes of $X \in \mathrm{Sm}_S$ such that all intersections of some D_i are in Sm_S. Recall that the category $\langle n \rangle$ was defined in section B.5.1 to be the opposite category of subsets of $\{1, \ldots, n\}$. Now we define the *pointed n-cube* to be $\langle n \rangle * := \langle \cup \rangle *$, where $* > J$ for each non-empty $J \subset \{1, \ldots, n\}$. As we have seen in B.5.2, $(X; D_1, \ldots, D_n)_*\colon \langle n \rangle \longrightarrow \mathrm{Sm}_S$ defines an n-cube (B.5.1) in Sm_S, where

$$(X; D_1, \ldots, D_n)_*(J) = \cap_{j \in J} D_j \subset X.$$

There is a natural extension of this n-cube to a pointed n-cube, by defining

$$(X; D_1, \ldots, D_n)\colon \langle n \rangle * \longrightarrow \mathrm{Sm}_S, J \longmapsto \cap_{j \in J} D_j, * \longmapsto *.$$

Now recall Quillen's $\mathcal{Q}$-construction introduced in Chapter B.6, as well as the notion of the classifying space $B\mathcal{C}$ of a category $\mathcal{C}$. For any $J \subset \{1, \ldots, n\}$, the category $\mathrm{Vect}_{(X;D_1,\ldots,D_n)(J)}$ of vector bundles on $(X; D_1, \ldots, D_n)(J) = \cap_{j \in J} D_j$ yields a classifying space $B\mathcal{Q}\,\mathrm{Vect}_{(X;D_1,\ldots,D_n)(J)} = B\mathcal{Q}\,\mathrm{Vect}_{\cap_{j \in J} D_j}$.v Then the K-groups of X relative to $D_1, \ldots, D_n$ are defined as

$$K_n(X; D_1, \ldots, D_n) := \pi_{n+1}(\mathrm{holim}_{\langle n \rangle *}\, J \longmapsto B\mathcal{Q}\,\mathrm{Vect}_{(X;D_1,\ldots,D_n)(J)})$$

Propertes of relative K-theory:

a.) *Chern chlasses for relative K-groups:* By [Lev98, III.1.4.8(iii), p.124], the Chern classes of Chapter B.6 induce Chern classes for relative K-groups

$$c^{q,2q-p}_{(X;D_1,\ldots,D_n)}\colon K_p(X; D_1, \ldots, D_n) \longrightarrow H^{2q-p}_{\mathcal{M}}(X; D_1, \ldots, D_n, \mathbb{Z}(q)),$$

where $H^{2q-p}_{\mathcal{M}}(X; D_1, \ldots, D_n, \mathbb{Z}(q)) = \mathrm{Hom}_{\mathcal{DM}(S)}(\mathbb{Z}_S, \mathbb{Z}_{(X;D_1,\ldots,D_n)}(q)[2q-p])$ are the relative motivic cohomology groups of B.5.3.

b.) *Localization:* [Lev98, III.1.5.2.2, p.129] We keep the setting of this section, and let moreover Z be a closed subset of X with open complement $j\colon U \hookrightarrow X$. Then there is a long exact localization sequence

$$
\begin{tikzcd}
\dots \arrow[r] & K_p^Z(X; D_1, \dots, D_n) \arrow[r] & K_p(X; D_1, \dots, D_n) \arrow[dll] \\
& K_p(U; D_1^U, \dots, D_n^U) \arrow[r] & K_{p-1}^Z(X; D_1, \dots, D_n) \arrow[r] & \dots
\end{tikzcd}
$$

Via the Chern classes, this is compatible with the corresponding long exact localization sequence of motivic cohomology

$$
\begin{tikzcd}
\dots \arrow[r] & H_Z^p(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[r] & H^p(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[dll] \\
& H^p(U; D_1^U, \dots, D_n^U, \mathbb{Z}(q)) \arrow[r] & H_Z^{p+1}(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[r] & \dots
\end{tikzcd}
$$

c.) *Relativization sequence:* Let X, D_i be as above. Then by [Lev98, III.1.5.2.1, p.129] there is a long exact relativization sequence

$$
\begin{tikzcd}
\arrow[r] & K_p(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[r] & K_p(X; D_1, \dots, D_{n-1}, \mathbb{Z}(q)) \arrow[dl] \\
& K_p(D_n; D_{1,n}, \dots, D_{n-1,n}, \mathbb{Z}(q)) \arrow[r] & K_{p-1}(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[r] & {}
\end{tikzcd}
$$

where $D_{i,n} := D_i \cap D_n$. Via Chern classes, this sequence is compatible with the relativization long exact sequence in motivic cohomology

$$
\begin{tikzcd}
\arrow[r] & H^p(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[r] & H^p(X; D_1, \dots, D_{n-1}, \mathbb{Z}(q)) \arrow[dl] \\
& H^p(D_n; D_{1,n}, \dots, D_{n-1,n}, \mathbb{Z}(q)) \arrow[r] & H^{p+1}(X; D_1, \dots, D_n, \mathbb{Z}(q)) \arrow[r] & {}
\end{tikzcd}
$$

induced by the relativization distinguished triangle of relative motives in section B.5.2 (see [Lev98, III.1.5.2.5, p.130]).

d.) [Lev98, III.3.6.3.2, p.181] Let Z be a closed subset of a smooth quasi-projective k-scheme x, and let $D_1, \dots, D_n$ be closed subschemes of S forming a normal crossing divisor. In [Lev97], Levine shows that the relative K-groups $K_*^Z(X; D_1, \dots, D_n)$ as defined above carry a lambda ring structure which is functorial and compatible with the localization and relativization sequences. Moreover, he shows that like in the case of classical K-groups, relative K-groups decompose into Adams-eigenspaces as follows: Let $K_p^Z(X; D_1, \dots, D_n)^{(q)}$ denote the weight q eigenspace of the Adams operations for the Lambda ring $K_p^Z(X; D_1, \dots, D_n)$. Then there is a finite, functorial direct sum decomposition

$$
K_p^Z(X; D_1, \dots, D_n)_{\mathbb{Q}} = \bigoplus_{q=\alpha}^{\dim_k X + p} K_p^Z(X; D_1, \dots, D_n)^{(q)}
$$

where $\alpha = 0$ for $p = 0$, $\alpha = 1$ for $p = 1$ and $\alpha = 2$ for $p \geq 2$.

Appendix C

Realizations

Motives have realizations in the ℓ-adic world and the complex analytic one. In this thesis, we will stick to the latter, even though various results might also be formulated in the setting of mixed sheaves, and are thus also valid in the ℓ-adic world. For the reader's convenience, I will recall the main properties of D-modules in the following section, and fix some notation.

C.1 Vector bundles with connection and the theory of D-modules

A very good reference for D-modules, perverse sheaves and the Riemann-Hilbert-correspondence is the wonderful book [HTT08] by Hotta, Takeuchi and Tanasaki.

Let X be a smooth algebraic variety over the complex number field $\mathbb{C}$. By GAGA, X can be considered both as an algebraic and as an analytic variety, and correspondingly there is both an analytic and an algebraic theory of D-modules. The basic definitions and results are valid in both settings. So unless specified otherwise, the following can be taken in both settings. However, the beauty of the theory is that in order to get non-trivial results, one has to combine the algebraic and analytic theory.

C.1.1 D-modules

Recall that the *sheaf of differential operators* D_X is the subalgebra of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and the sheaf of vector fields $\Theta_X = \mathcal{D}er_{\mathbb{C}_X}(\mathcal{O}_X)$ on X. Giving a D_X-module structure on a vector bundle M on X is the same as endowing M with an integrable (i.e. $\nabla^2 = 0$) connection $\nabla\colon \Theta_X \longrightarrow \mathcal{E}nd_{\mathbb{C}}(M)$ (see [HTT08, Lemma 1.2.1,p. 17]). A D_X-module M is called an *integrable connection* if it is locally free of finite rank over $\mathcal{O}_X$. Hence, integrable connections correspond to vector bundles on X with integrable connections.

There is a notion of a good class of filtrations of D-modules: Let $\pi\colon T^*X \longrightarrow X$ be the projection of the cotangent bundle down to X, and (M, F) be a filtered D_X-module. We say that F is a *good filtration* of M if F_iM is coherent over $\mathcal{O}_X$ for each i and there exists $i_0 \gg 0$ satisfying

$$(F_jD_X)(F_iM) = F_{i+j}M \text{ for } j \geq 0, i \geq i_0,$$

or equivalently (by [HTT08, Definition 2.1.1, p. 58]) if $\mathrm{gr}^F M$ is coherent over $\pi_*\mathcal{O}_{T^*X}$. In the algebraic setting such a filtration exists globally for any coherent D_X-module, while in the analytic situation, the existence of such filtrations is only granted locally.

A very nice class of coherent D_X-modules which contains all integrable connections is the class of *holonomic D-modules*. For a definition of *holonomic* D-modules, see [HTT08, Definition 2.3.6, p. 64]. The category of holonomic D_X-modules is denoted by $\mathrm{Mod}_h(D_X)$. It is abelian by [HTT08, 3.1.2, p.81], so via the usual procedures one may define the derived category $D^b_h(X)$ of holonomic D_X-modules.

The usual setting to work in in that of *regular holonomic* D-modules - however, recalling the definitions here would lead too far, so I simply refer to [HTT08, 5, pp.127 ff.] for a definition. For our purposes, it will be sufficient to think of regular holonomic D-modules just as a very nicely behaved D_X-module: all D-modules that will come up in this thesis are regular holonomic. The category of regular holonomic D_X-modules is denoted by $\mathrm{Mod}_{rh}(D_X)$. It is abelian (see [HTT08, 6.1.2, p.161]). We denote by $D^b_{rh}(D_X)$ the full triangulated subcategory of $D^b_h(D_X)$ consisting of objects $M^\bullet \in D^b_h(D_X)$ such that $H^i(M^\bullet) \in \mathrm{Mod}_{rh}(D_X)$ for any $i \in \mathbb{Z}$

The category of holonomic D-modules has a six-functor formalism, that is to say for any smooth algebraic variety X there are functors

$$\otimes^L \colon D^b_h(D_X) \otimes D^b_h(D_X) \longrightarrow D^b_h(D_X),$$

a duality functor

$$\mathbb{D}_X \colon D^b_h(D_X) \longrightarrow D^b_h(D_X)^{\mathrm{op}},$$

and, given a morphism $f \colon X \longrightarrow Y$ of smooth algebraic varieties, functors

$$\left.\begin{array}{l} \int_f \\ \int_{f!} = \mathbb{D}_Y \int_f \mathbb{D}_X \end{array}\right\} : D^b_h(D_X) \longrightarrow D^b_h(D_Y),$$
$$\left.\begin{array}{l} f^\dagger \\ f^\star = \mathbb{D}_X f^\dagger \mathbb{D}_Y \end{array}\right\} : D^b_h(D_Y) \longrightarrow D^b_h(D_X)$$

such that for $M^\bullet \in D^b_h(D_X)$ and $N^\bullet \in D^b_h(D_Y)$ there are natural isomorphisms

$$R\mathcal{H}om_{D_Y}\left(\int_{f!} M^\bullet, N^\bullet\right) \xrightarrow{\sim} Rf_*R\mathcal{H}om_{D_X}(M^\bullet, f^\dagger N^\bullet) \quad \text{and}$$
$$Rf_*R\mathcal{H}om_{D_X}(f^\star N^\bullet, M^\bullet) \xrightarrow{\sim} R\mathcal{H}om_{D_Y}\left(N^\bullet, \int_f M^\bullet\right).$$

In particular, $f^\star$ is left adjoint to $\int_f$ and $\int_{f!}$ is left adjoint to $f^\dagger$.

C.1.2 The functor formalism

The major indicator that D^b_{rh} is in fact the right category is that the functors satisfy all the adjointness properties of Grothendieck's formalism and are generally as well behaved as one could wish. There is a very nice and short overview [Vir89] providing a summary of all of these properties in the Hodge setting by R. Virk.

Push-forward and pull-back

Let $f\colon X \longrightarrow Y$ be a morphism of complex varieties.

a.) Adjointness: $f^\bigstar$ is left adjoint to $\int_f$ and $\int_{f!}$ is left adjoint to $f^\dagger$, that is to say for any $M^\bullet \in D^b_{rh}(Y), N^\bullet \in D^b_{rh}(Y)$ we have functorial isomorphisms

$$\mathrm{Hom}_{D^b_{rh}(X)}(M, f^\bigstar N) \cong \mathrm{Hom}_{D^b_{rh}(Y)}(\int_f M, N)$$
$$\mathrm{Hom}_{D^b_{rh}(X)}(M, f^\dagger N) \cong \mathrm{Hom}_{D^b_{rh}(Y)}(\int_{f!} M, N)$$

which are compatible with the adjunction of functors for constructible sheaves. (for a reference, see [Sai90b, §4.4]).

b.) Given another morphism $g\colon Y \longrightarrow Z$ of varieties, there are canonical isomorphisms

$$(gf)^* \simeq f^\bigstar g^\bigstar, \quad (gf)_* \simeq \int_g \int_f, \quad (gf)^! \simeq f^\dagger g^\dagger, \quad (gf)_! \simeq \int_{g!} \int_{f!}$$

(for a reference, see [Sai90b, §4.4])

c.) There is a natural morphism $\int_{f!} \longrightarrow \int_f$ which is an isomorphism for proper f. (see [Sai90b, 4.3.3])

d.) Purity: If f is smooth of relative dimension d, then $f^\dagger \simeq f^\bigstar[2d](d)$. (see [Sai90b, 4.4.2])

e.) Base-change: ([Sai90b, 4.4.3]) Given a cartesian diagram of complex varieties

$$\begin{tikzcd} Z \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X' \arrow[r, "g"] & Y \end{tikzcd}$$

there are natural isomorphisms of functors

$$g^\bigstar \int_{f!} \simeq \int_{f'!} g'^\bigstar, \quad g^\dagger \int_f \simeq \int_{f'} (g')^\dagger.$$

Exterior and interior tensor products

Let X and Y be complex varieties. According to [Sai90b, 4.2.13], there is an exact bifunctor

$$\boxtimes\colon D^b_{rh}(X) \times D^b_{rh}(Y) \longrightarrow D^b_{rh}(X \times Y)$$

with the following properties:

a.) $\boxtimes$ is compatible with the exterior tensor product of constructible sheaves.
b.) If Z is yet another complex variety, there is a trifunctorial isomorphism

$$(M \boxtimes N) \boxtimes L \simeq M \boxtimes (N \boxtimes L)$$

for all $M \in D^b_{rh}(X), N \in D^b_{rh}(Y)$ and $L \in D^b_{rh}A(Z)$.
c.) Let $\mathrm{flip}\colon X \times Y \longrightarrow Y \times X$ be the isomorphism of varieties given by exchanging coordinates. Then by [Sai90b, 4.4.1] there is a bifunctorial isomorphism

$$\mathrm{flip}_*(M \boxtimes N) \xrightarrow{\sim} N \boxtimes M$$

for all $M \in D^b_{rh}(X)$ and $N \in D^b_{rh}(Y)$ which is compatible with the underlying canonical isomorphisms for sheaves.

One defines a tensor product of mixed Hodge modules via the diagonal map $\varDelta\colon X \longrightarrow X \times X$ by putting

$$M \otimes N := \varDelta^\bigstar(M \boxtimes N)$$

for all $M, N \in D^b_{rh}(X)$.

Denoting the projections onto factors by $p\colon X \times Y \longrightarrow X$ and $q\colon X \times Y \longrightarrow Y$, then one has a canonical identification

$$M \boxtimes N \simeq p^\bigstar M \otimes q^\bigstar N$$

for all $M \in D^b_{rh}(X)$ and $N \in D^b_{rh}(Y)$.

Internal Hom-functor

Let $\varDelta\colon X \longrightarrow X \times X$ again denote the diagonal map. Since we have notions of tensor product and a dual, there is a natural internal $\mathcal{H}om$-functor in $D^b_{rh}(X)$ given by

$$\mathcal{H}om(M, N) := \varDelta^\dagger(\mathbf{D}M \boxtimes N).$$

Compatibilities

Let X, Y, Z be complex varieties, and $f\colon X \longrightarrow Y$ a morphism of varieties. Denote the diagonal maps by $\Delta_X\colon X \longrightarrow X \times X$ and $\Delta_Y\colon Y \longrightarrow Y \times Y$.

a.) There is a functorial isomorphism

$$(f \times \mathrm{id})^{\bigstar}(M \boxtimes N) \simeq f^{\bigstar} M \boxtimes N$$

for all $M \in D^b_{rh}(Y)$ and $M \in D^b_{rh}(Y)$.

b.) By [Sai90b, Prop 2.6 and 2.17.4] there is a bifunctorial isomorphism

$$\mathbf{D}(M \boxtimes N) \simeq \mathbf{D}M \boxtimes \mathbf{D}N$$

for all $M \in D^b_{rh}(X)$ and $N \in D^b_{rh}(Y)$.

c.) By the above, we have a functorial isomorphism

$$f^{\bigstar}(M \otimes N) = f^{\bigstar} M \otimes f^{\bigstar} N$$

for all $M, N \in D^b_{rh}(Y)$.

d.) By (iii) one obtains a bifunctorial isomorphism

$$\int_f \mathcal{H}om(f^{\bigstar} N, L) \simeq \mathcal{H}om(N, \int_f L)$$

for all $N \in D^b_{rh}(Y)$, $L \in D^b_{rh}(X)$.

e.) By [Sai90a, Cor. 2.9] there is a trifunctorial isomorphism

$$\mathrm{Hom}(L, \mathcal{H}om(M, N)) \simeq \mathrm{Hom}(L \otimes M, N)$$

for all $L, M, N \in D^b_{rh}(X)$.

f.) Since $\mathbf{D}\Delta_X^{\bigstar} = \Delta_X^{\dagger}\mathbf{D}$, we have

$$\mathcal{H}om(M, N) = \mathbf{D}(M \otimes \mathbf{D}N)$$

for all $M, N \in D^b_{rh}(X)$ which is compatible with the underlying isomorphisms of sheaves.

g.) The isomorphism in (vi) yields the functorial isomorphism

$$f^{\dagger}\mathcal{H}om(M, N) \simeq \mathcal{H}om(f^{\bigstar} M, f^{\dagger} N)$$

for all $M, N \in D^b_{rh}(X)$.

h.) Projection formula: (vii) yields a functorial isomorphism

$$\int_{f!} M \otimes N \simeq \int_{f!} (M \otimes f^{\bigstar} N).$$

C.1.3 Constructible and perverse sheaves

Constructible and in particular perverse sheaves are closely connected to D-modules and the other basic ingredient to the definition of mixed Hodge modules. A good and short reference for the theory of perverse sheaves can be found in section 8 of [HTT08]
.

Recall that an *algebraic (resp. analytic) stratification* of a complex algebraic (resp. analytic) variety X is a locally finite partition $X = \bigsqcup_{\alpha\in A} X_\alpha$ of X by locally closed subvarieties (resp., analytic subsets) such that, for any $\alpha \in A$, X_α is smooth and $\overline{X}_\alpha = \sqcup_{\beta\in B} X_\beta$ for a subset B of A. Each X_α is called a *stratum* of the stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$. Note that any algebraic stratification induces an analytic one on the associated analytic variety. An algebraic (resp. analytic) *constructible sheaf* on X is merely a sheaf on X such that for some algebraic (resp. analytic) stratification $X = \bigsqcup_{\alpha\in A} X_\alpha$ the restriction of the (associated) analytic sheaf $F|^{\mathrm{an}}_{X_\alpha^{\mathrm{an}}}$ is a local system on the (associated) analytic variety X_α^{an} for all $\alpha \in A$.

Notation C.1.1. (i) We denote the category of $\mathbb{C}$-local systems by $\mathrm{Loc}(X)$, and the category of R-local systems by $\mathrm{Loc}_R(X)$.
(ii) For an analytic variety X let us denote the bounded derived category of $\underline{\mathbb{C}}_X$-modules by $D^b(\mathbb{C}_X)$.
(iii) For an algebraic (resp. analytic) variety X, we denote by $D^b_{cs}(X)$ the full subcategory of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$ (resp. $D^b(\mathbb{C}_X)$) consisting of bounded complexes of $\mathbb{C}_{X^{\mathrm{an}}}$-modules whose cohomology groups are constructible.
(iv) Similary, one defines constructible sheaves whose restriction to strata are R-local systems for a subring $R \subset \mathbb{C}$, and denotes them by $D^b_{cs}(X, R)$

Remark C.1.2. Let X be an algebraic or analytic complex variety. Then there is a well-known equivalence of abelian categories

$$\mathrm{Conn}(X) \simeq \mathrm{Loc}(X)$$

given by associating to an integrable connection its sheaf of horizontal sections $(M, \nabla) \longmapsto M^\nabla$ and the inverse functor given by $(\mathcal{O}_X \otimes_{\mathbb{C}_X} L, d \otimes \mathrm{id}) \longleftarrow\!\!\shortmid L$.

There is also a six-functor formalism in the derived category of constructible sheaves for any analytic variety X: Apart from the functors

$$\begin{aligned} \otimes_{\mathbb{C}} \colon D^b_{cs}(X) \times D^b_{cs}(X) &\longrightarrow D^b_{cs}(X) \quad \text{and} \\ R\mathcal{H}om_{\mathbb{C}_X} \colon D^b_{cs}(X) \times D^b_{cs}(X) &\longrightarrow D^b_{cs}(X) \end{aligned}$$

there are two pull-back functors for any morphism of analytic spaces $f \colon X \longrightarrow Y$:

$$\left.\begin{matrix} f^* \\ f^! \end{matrix}\right\} \colon D^b_{cs}(Y) \longrightarrow D^b_{cs}(X).$$

There are also two push-forward functors for any *proper* morphism f of analytic spaces or just any morphism of algebraic varieties:

$$\left.\begin{matrix}\int_f \\ f_!\end{matrix}\right\} : D^b_{cs}(X) \longrightarrow D^b_{cs}(Y).$$

Moreover, there is a duality functor like in the setting of D-modules: Namely, for an analytic space X one has

$$\omega_X^\bullet := a_X^! \mathbb{C}_X \in D^b_{cs}(X)$$

where $a_X \colon X \longrightarrow \mathrm{pt}$ is the unique morphism from X to the one-point space pt. It is called the *dualizing complex* of X. It defines a functor

$$\begin{aligned} \mathbf{D}_X \colon D^b_{cs}(X) &\longrightarrow D^b_{cs}(X)^{\mathrm{op}} \\ F^\bullet &\longmapsto \mathbf{D}_X F^\bullet = R\mathcal{H}om_{\mathbb{C}_X}(F^\bullet, \omega_X^\bullet) \end{aligned}$$

called the *Verdier dual*.

As in every good six-functor-formalism, the "!-functors" and the "$*$-functors" can be related for nice f by the duality functor: Let $f \colon X \longrightarrow Y$ be a morphism of algebraic varieties of analytic spaces. Then one always has $f^! = \mathbf{D}_X \circ f^{-1} \circ \mathbf{D}_Y$. Moreover, one has $Rf_! = \mathbf{D}_Y \circ Rf_* \circ \mathbf{D}_X$ for any morphism of algebraic varieties, but only for proper morphisms of analytic spaces.

This six-functor formalism also satisfies a list of properties as the ones in section C.1.2.

$D^b_{cs}(X)$ comes equipped with a t-structure (for details on t-structures, see subsection 8.1 of [HTT08]) called the *perverse t-structure*: the full subcategories ${}^pD^{\leq 0}_{cs}(X)$ and ${}^pD^{\geq 0}_{cs}$ of $D^b_{cs}(X)$ are defined follows. For $F^\bullet \in D^b_{cs}(X)$

a.) $F^\bullet \in {}^pD^{\leq 0}_{cs}$ if and only if

$$\dim\{\operatorname{supp} H^j(F^\bullet)\} \leq -j \quad \text{ for any } j \in \mathbb{Z}, \text{ and}$$

b.) $F^\bullet \in {}^pD^{\geq 0}_{cs}$ if and only if

$$\dim\{\operatorname{supp} H^j(\mathbf{D}_X F^\bullet)\} \leq -j \quad \text{ for any } j \in \mathbb{Z}.$$

Beilinson, Bernstein and Deligne showed in [BBD82] that this in fact defines a t-structure on $D^b_{cs}(X)$.

Notation C.1.3. a.) The heart ${}^pD^{\leq 0}_{cs}(X) \cap {}^pD^{\geq 0}_{cs}$ of this t-structure is the *category of perverse sheaves* and denoted by $\mathrm{Perv}(X)$.

b.) Similarly, one has a t-structure on $D^b_{cs}(X, R)$ for $R \subset \mathbb{C}$. Its heart is the *category of perverse R-sheaves* and denoted by $\mathrm{Perv}_R(X)$.

Remark C.1.4. The heart of a t-structure is abelian ([HTT08, Theorem 8.1.9, pp.185/186]), so $\mathrm{Perv}(X)$ (resp. $\mathrm{Perv}_R(X)$) are full abelian subcategories of $D^b_{cs}(X)$ (resp. $D^b_{cs}(X, R)$).

It is important to note that perverse sheaves are in fact a generalization of local systems: For any local system $\mathbb{V}$ on the complex manifold X^{an} one can show that $\mathbb{V}[d_X] \in \mathrm{Perv}(\mathbb{C}_X)$ which follows immediately from the fact that $\mathbf{D}_X(\mathbb{V}[d_X]) = R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{V}[d_X], \mathbb{C}_X[2d_X]) \simeq \mathbb{V}^*[d_X]$, where $\mathbb{V}^*$ denotes the dual local system (see, for example, [HTT08, Proposition 8.1.31, p.198]).

Let us now quickly recall how functors on $D^b_{cs}(X)$ descend to the heart of the perverse t-structure. First of all, in view of the definition of the perverse t-structure it is obvious that the Verdier duality functor $\mathbf{D}_X$ induces an exact functor $\mathbf{D}_X\colon \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)^{\mathrm{op}}$. One may descend any functor to the category of perverse sheaves in the following way: If X,Y are algebraic varieties or analytic spaces, then for a functor $F\colon D^b_{cs}(X) \longrightarrow D^b_{cs}(Y)$ of triangulated categories one defines the functor ${}^pF\colon \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Y)$ to be the composite of the functors

$$\mathrm{Perv}(\mathbb{C}_X) \hookrightarrow D^b_{cs}(X) \xrightarrow{F} D^b_{cs}(Y) \xrightarrow{{}^pH^0} \mathrm{Perv}(\mathbb{C}_Y).$$

This way we obtain the functor formalism for perverse sheaves:
For any morphism $f\colon X \longrightarrow Y$ of algebraic varieties or analytic spaces there are functors

$${}^pf^{-1}, {}^pf^!\colon \mathrm{Perv}(\mathbb{C}_Y) \longrightarrow \mathrm{Perv}(\mathbb{C}_X).$$

If f is proper, then we also have functors

$${}^pRf_*, {}^pRf_!\colon \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_Y).$$

One sometimes denotes the functors ${}^pRf_*$ and ${}^pRf_!$ just by ${}^pf_*$ and ${}^pf_!$, respectively, to simplify the notation.

As in the case of D-modules, the list of properties in section C.1.2 corresponds to a list of properties of the six-functor formalism above when replacing the functors as follows:

holonomic D-modules	**perverse sheaves**
$f^\bigstar$	${}^pf^{-1}$
$\int_f$	${}^pf_*$
$f^\dagger$	${}^pf^!$
$\int_{f!}$	${}^pf_!$
$\mathbf{D}$	$\mathbf{D}$

C.1.4 The Riemann-Hilbert Correspondence

The bridge between regular holonomic D-modules and constructible sheaves is provided by the so-called de Rham functor.

The analytic setting

Let X be a complex manifold of dimension d_X, and $\Omega_X := \Omega_X^{d_x}$ be the sheaf of top-degree holomorphic differential forms. If $M^\bullet$ is an element of $D_h^b(D_X)$, then $\Omega_X \otimes^L M^\bullet$ are $\mathbb{C}_X$-modules by multiplication on the left. Recall that the *de Rham functor* is defined by

$$\begin{aligned} DR_X \colon D_{rh}^b(D_X) &\longrightarrow D^b(\mathbb{C}_X) \\ M^\bullet &\longmapsto \Omega_X \otimes_{\mathcal{O}_X}^L M^\bullet. \end{aligned}$$

Remark C.1.5. (i) Kashiwara's so-called constructibility theorem (see [HTT08, Theorem 4.6.3, p.116]) says that holonomic D_X-modules are mapped to $D_{cs}^b(X)$ by the de Rham functor.

(ii) DR_X is compatible with the six-functor formalism on both sides.

(iii) For a single D_X-module M concentrated in degree zero, $DR_X(M)$ can be made explicit in terms of the connection ∇ associated to M. A locally free resolution of the right D_X-module Ω_X is given by (see [HTT08, Section 4.2])

$$0 \to \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \to \Omega_X^1 \otimes_{\mathcal{O}_X} D_X \to \ldots \to \Omega_X^{d_X} \otimes_{\mathcal{O}_X} D_X \to \Omega_X \longrightarrow 0$$

One thus obtains the *de Rham complex*

$$\begin{aligned} DR_X(M) &= \Omega_X \otimes_{\mathcal{O}_X}^L M \\ &= \{0 \to M \xrightarrow{\nabla} \Omega_X^1 \otimes M \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} \Omega_X^{d_X} \otimes M\}[d_X], \end{aligned}$$

where the connection ∇ is given in higher degrees by

$$\nabla(\omega \otimes s) = d\omega \otimes s + \sum_i dx_i \wedge \omega \otimes \nabla_{\partial_i} s$$

for $\omega \in \Omega_X^p$ and $s \in M$. For an integrable connection M of rank m on X this is just the classical de Rham complex. Hence, by the holomorphic Poincaré lemma, the higher cohomology groups $H^i(\Omega_X^\bullet \otimes_{\mathcal{O}_X} M)$ vanish for $i \geq 1$, while for $i = 0$ one has $H^0(\Omega_X^\bullet \otimes_{\mathcal{O}_X} M) = M^\nabla$. In other words, the de Rham functor yields the classical equivalence

$$H^{-d_X}(DR_X(\bullet)) \colon \operatorname{Conn}(X) \xrightarrow{\sim} \operatorname{Loc}(X)$$

of categories (see C.1.2).

The general analytic Riemann-Hilbert correspondence is a generalization of the equivalence $\operatorname{Conn}(X) \simeq \operatorname{Loc}(X)$ using the de Rham functor, and was established by Kashiwara in [Kas80, Kas84], while a different proof was given later by Mebkhout in [Meb84b, Meb84a]:

Theorem C.1.6. *[RHC][HTT08, Theorem 7.2.1, p.174] For a complex manifold X the de Rham functor*

$$DR_X\colon D^b_{rh}(D_X) \longrightarrow D^b_{cs}(X^{(\mathrm{an})})$$

gives an equivalence of categories.

A result by Kashiwara states that holonomic D_X-modules are in fact mapped to perverse sheaves on X. Even more is true:

Theorem C.1.7. *[HTT08, Theorem 7.2.5, p.176] The de Rham functor induces an equivalence*

$$DR_X\colon \operatorname{Mod}_{rh}(D_X) \xrightarrow{\sim} \operatorname{Perv}(\mathbb{C}_X)$$

of categories.

The algebraic setting

Let X be a smooth algebraic variety over the complex numbers, and X^{an} the associated analytic variety. Then the *(algebraic) de Rham functor* is defined by putting

$$\begin{aligned} DR_X\colon D^b_{rh}(D_X) &\longrightarrow D^b(\mathbb{C}_{X^{\mathrm{an}}}) \\ M^\bullet &\longmapsto DR_{X^{\mathrm{an}}}(M^{\bullet\,\mathrm{an}}) \end{aligned}$$

One uses the compatibility of the de Rham functor with the six-functor-formalism to show that DR_X is fully faithful, whille essential surjectivity follows from the classical Riemann Hilbert correspondence, so one obtains:

Theorem C.1.8. *[RHC][HTT08, Theorem 7.2.2, p.174] For a smooth algebraic variety X the de Rham functor*

$$DR_X\colon D^b_{rh}(D_X) \longrightarrow D^b_c(X^{(\mathrm{an})})$$

gives an equivalence of categories.

C.2 Mixed Hodge Modules

The geometric regulator

The passage from Levine's motives to D-modules is accomplished as follows:

Theorem C.2.1. *Let $A \subset \mathbb{R}$ be a subfield. Let S be a smooth quasi-projective $\mathbb{C}$-scheme, and $p_X\colon X \longrightarrow S$ be an S-scheme. Let $\bar{p}_X\bar{X} \longrightarrow \bar{X}$ be a smooth compactification of p, and denote the inclusions by $j_X\colon X \hookrightarrow \bar{X}$ and $j_S\colon S \hookrightarrow \bar{S}$. Then sending (X, q) to the direct image of the regular holonomic D-module $j_S^*\bar{p}_{X*}j_{X*}\underline{A}_X \in D^b_{rh}(S)$ extends canonically to an exact tensor functor*

$$\mathfrak{R}_{A,\mathrm{geo}}\colon \mathcal{DM}_A(S) \longrightarrow D^b_{rh}(S),$$

called the geometric realization functor

Remark C.2.2. Under the hypotheses of the above theorem, and taking M to be in $\mathcal{DM}_A(S)$, $\mathfrak{R}_{A,\mathrm{geo}}$ induces a morphism

$$\mathfrak{R}_{A,\mathrm{geo}} \colon H^q_{\mathrm{mot}}(S, M(p)) \longrightarrow \mathrm{Hom}_{D^b_{rh}(\bar{S})}(j_{S*}\mathbb{Z}_S, j_{S*}\mathfrak{R}_{A,\mathrm{geo}} M(p)[q])$$

C.3 Hodge theory

C.3.1 Mixed Hodge structures and Hodge complexes

Let R be a subring of $\mathbb{R}$ and V an R-module of finite type. If V is a module of finite type over R, we will denote the associated rational, real and complex vector spaces by

$$V_{\mathbb{Q}} := V \otimes \mathbb{Q}, \quad V_{\mathbb{R}} := V \otimes \mathbb{R}, \quad V_{\mathbb{C}} := V \otimes \mathbb{C}.$$

Mixed Hodge structures

a.) Recall that V carries a *pure R-Hodge structure* if it is endowed with any of the two (evidently) equivalent structures:

(i) A decreasing finite filtration $F^{\cdot}$ on $V_{\mathbb{C}}$ such that $V_{\mathbb{C}}$ decomposes as

$$V_{\mathbb{C}} = \bigoplus_{p,q} F^p \cap \overline{F^q}.$$

(ii) A direct sum decomposition of the associated complex vector space $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that the $V^{p,q}$ satisfy $V^{p,q} = \overline{V^{q,p}}$.

A *polarization* of an R-Hodge structure V of weight k is a homomorphism of Hodge structures

$$S \colon V \otimes V \longrightarrow R(-k)$$

which is $(-1)^k$-symmetric and such that the real-valued symmetric bilinear form

$$Q(u, v) := (2\pi i\,)^k S(Cu, v)$$

is positive definite on $V_{\mathbb{R}}$, where C is the Weil operator. Any R-Hodge structure that admits a polarization is said to be *polarizable*.

b.) V carries a *mixed R-Hodge structure* if it is endowed with an increasing finite filtration $W_\bullet$ on $V_{\mathbb{Q}}$, called the *weight filtration* and a decreasing finite filtration $F^\bullet$ on $V_{\mathbb{C}}$, the *Hodge filtration* , such that $F^\bullet$ induces pure R-Hodge structures of weight k on the weight-graded pieces $\mathrm{gr}^W_k V_{\mathbb{Q}}$. The category of R-mixed Hodge structures is denoted by MHS_R. MHS_R is a rigid tensor category. A mixed R-Hodge structure is said to be *graded-polarizable* if the quotients $\mathrm{gr}^W_k V_{\mathbb{Q}}$ are pure, polarizable $R \otimes \mathbb{Q}$-Hodge structures. The category of graded-polarizable mixed Hodge structures is denoted by MHS^p_R. It is also a rigid tensor category.

c.) A *morphism of mixed R-Hodge structures* is a morphism $f\colon V_1 \longrightarrow V_2$ of R-modules of finite type which is compatible with the Hodge- and weight filtrations on both sides: $f(W^j V_1) \subset W^j V_2$ and $f_{\mathbb{C}}(F^k V_1) \subset F^k V_2$.

Mixed Hodge complexes

There is a natural "sheafification" of the notion of a mixed Hodge structure: the theory of Hodge complexes.

d.) A *normalized (graded-polarizable) mixed R-Hodge complex*

$$K^\bullet = \big(K_R^\bullet, (K_{\mathbb{Q}}^\bullet, W), \alpha, (K_{\mathbb{C}}^\bullet, W, F), \beta\big)$$
$$:= \left\{ \begin{tikzcd}[column sep=small] & {}'K_{\mathbb{Q}}^\bullet & & ({}'K_{\mathbb{C}}^\bullet, W) & \\ K_R^\bullet \arrow[ur, "\alpha_1"] & & (K_{\mathbb{Q}}^\bullet, W) \arrow[ul, "\alpha_2"', "\sim"] \arrow[ur, "\beta_1"] & & (K_{\mathbb{C}}^\bullet, W, F) \arrow[ul, "\beta_2"', "\sim"] \end{tikzcd} \right\}$$

consists of the following data:

- a bounded below complex $K_R^\bullet$ of sheaves of R-modules such that the cohomology groups $H^p(X, K_R^\bullet)$ are finitely generated as R-modules,
- a complex $K_{\mathbb{Q}}^\bullet$ of sheaves of $R \otimes \mathbb{Q}$-vector spaces equipped with a biregular increasing filtration W together with a diagram

$$\alpha := \left\{ \begin{tikzcd}[column sep=small] & {}'K_{\mathbb{Q}}^\bullet & \\ K_R^\bullet \arrow[ur, "\alpha_1"] & & K_{\mathbb{Q}}^\bullet \arrow[ul, "\alpha_2"', "\sim"] \end{tikzcd} \right\}$$

  representing a morphism $K_R^\bullet \dashrightarrow K_{\mathbb{Q}}^\bullet$ in the derived category of sheaves of R-modules which becomes a quasi-isomorphism after tensoring with $\mathbb{Q}$, i.e. $\alpha_1 \otimes \mathrm{id}\colon K_R^\bullet \otimes \mathbb{Q} \simeq {}'K_{\mathbb{Q}}$.
  To be precise, this choice is given by: a complex of sheaves of $R\otimes\mathbb{Q}$-vector spaces ${}'K_{\mathbb{Q}}^\bullet$ together with a quasi-isomorphism α_2, and a morphism α_1 of bounded below complexes of sheaves of R-modules which becomes a quasi-isomorphism after tensoring with $\mathbb{Q}$. The induced isomorphism $\alpha \otimes \mathrm{id}$ in the derived category of sheaves of $\mathbb{Q}$-vector spaces is called the *first comparison morphism*.
- a bifiltered complex of sheaves of $\mathbb{C}$-vector spaces $(K_{\mathbb{C}}^\bullet, W, F)$, where W is a biregular increasing filtration and F a biregular decreasing filtration, together with a diagram

$$\beta := \left\{ \begin{tikzcd}[column sep=small] & ({}'K_{\mathbb{C}}, W)^{\bullet} & \\ (K^{\bullet}_{\mathbb{Q}}, W) \arrow[ur, "\beta_1"] & & (K^{\bullet}_{\mathbb{C}}, W) \arrow[ul, "\beta_2"', "\sim"] \end{tikzcd} \right\}$$

representing a morphism $(K^{\bullet}_{\mathbb{Q}}, W) \dashrightarrow (K_{\mathbb{C}}, W)$ in the derived category of filtered sheaves of $R \otimes \mathbb{Q}$-vector spaces which becomes a quasi-isomorphism after tensoring with $\mathbb{C}$, i.e. $\beta_1 \otimes \mathrm{id}\colon (K^{\bullet}_{\mathbb{Q}} \otimes \mathbb{C}, W) \simeq ({}'K_{\mathbb{C}}, W)$. β is called the *second comparison morphism*.

- These data should satisfy the following axiom: For all $m \in \mathbb{Z}$ the triple

$$\begin{aligned} \operatorname{gr}^W_m K^{\bullet} &:= \big(\operatorname{gr}^W_m K^{\bullet}_{\mathbb{Q}}, (\operatorname{gr}^W_m K^{\bullet}_{\mathbb{C}}, F), \operatorname{gr}^W_m(\beta)\big) \\ &= \left\{ \begin{tikzcd}[column sep=small] & \operatorname{gr}^{W'}_m K^{\bullet}_{\mathbb{C}} & \\ \operatorname{gr}^W_m K^{\bullet}_{\mathbb{Q}} \arrow[ur, "\operatorname{gr}^W_m \beta_1"] & & (\operatorname{gr}^W_m K^{\bullet}_{\mathbb{C}}, \operatorname{gr}^W_m F) \arrow[ul, "\operatorname{gr}^W_m \beta_2"', "\sim"] \end{tikzcd} \right\} \end{aligned}$$

is a *normalized* (polarizable) $R \otimes \mathbb{Q}$-Hodge complex of weight m, i.e. the cohomology group $H^k(\operatorname{gr}^m K^{\bullet}_{\mathbb{C}})$ is a pure Hodge structure *of weight* $k+m$.

e.) Let X be a topological space. A *normalized (graded-polarizable) mixed R-Hodge complex of sheaves* on X

$$\begin{aligned} \mathcal{K}^{\bullet} &= \big(\mathcal{K}^{\bullet}_R, (\mathcal{K}^{\bullet}_{\mathbb{Q}}, W), \alpha, (\mathcal{K}^{\bullet}_{\mathbb{C}}, W, F), \beta\big) \\ &:= \left\{ \begin{tikzcd}[column sep=small] & {}'\mathcal{K}^{\bullet}_{\mathbb{Q}} & & ({}'\mathcal{K}^{\bullet}_{\mathbb{C}}, W) & \\ \mathcal{K}^{\bullet}_R \arrow[ur, "\alpha_1"] & & (\mathcal{K}^{\bullet}_{\mathbb{Q}}, W) \arrow[ul, "\alpha_2"', "\sim"] \arrow[ur, "\beta_1"] & & (\mathcal{K}^{\bullet}_{\mathbb{C}}, W, F) \arrow[ul, "\beta_2"', "\sim"] \end{tikzcd} \right\} \end{aligned}$$

consists of the following data:

- a bounded below complex $\mathcal{K}^{\bullet}_R$ of sheaves of R-modules such that the hypercohomology groups $\mathbb{H}^p(X, \mathcal{K}^{\bullet}_R)$ are finitely generated as R-modules,
- a complex $\mathcal{K}^{\bullet}_{\mathbb{Q}}$ of sheaves of $R \otimes \mathbb{Q}$-vector spaces equipped with a biregular increasing filtration W together with a diagram

$$\alpha := \left\{ \begin{tikzcd}[column sep=small] & {}'\mathcal{K}^{\bullet}_{\mathbb{Q}} & \\ \mathcal{K}^{\bullet}_R \arrow[ur, "\alpha_1"] & & \mathcal{K}^{\bullet}_{\mathbb{Q}} \arrow[ul, "\alpha_2"', "\sim"] \end{tikzcd} \right\}$$

representing a morphism $\mathcal{K}^{\bullet}_R \dashrightarrow \mathcal{K}^{\bullet}_{\mathbb{Q}}$ in the derived category of sheaves of R-modules which becomes a quasi-isomorphism after tensoring with $\mathbb{Q}$, i.e. $\alpha_1 \otimes$

id: $K_R^\bullet \otimes \mathbb{Q} \simeq {}'K_{\mathbb{Q}}$.
To be precise, this choice is given by: a complex of sheaves of $R \otimes \mathbb{Q}$-vector spaces ${}'K_{\mathbb{Q}}^\bullet$ together with a quasi-isomorphism α_2, and a morphism α_1 of bounded below complexes of sheaves of R-modules which becomes a quasi-isomorphism after tensoring with $\mathbb{Q}$. The induced isomorphism $\alpha \otimes \mathrm{id}$ in the derived category of sheaves of $\mathbb{Q}$-vector spaces is called the *first comparison morphism*.

- a bifiltered complex of sheaves of $\mathbb{C}$-vector spaces $(K_{\mathbb{C}}^\bullet, W, F)$, where W is a biregular increasing filtration and F a biregular decreasing filtration, together with a diagram

$$\beta := \left\{ \begin{tikzcd} & ({}'\mathcal{K}_{\mathbb{C}}, W)^\bullet & \\ (\mathcal{K}_{\mathbb{Q}}^\bullet, W) \arrow[ur, "\beta_1"] & & (\mathcal{K}_{\mathbb{C}}^\bullet, W) \arrow[ul, "\beta_2"', "\sim"] \end{tikzcd} \right\}$$

representing a morphism $(\mathcal{K}_{\mathbb{Q}}^\bullet, W) \dashrightarrow (\mathcal{K}_{\mathbb{C}}, W)$ in the derived category of filtered sheaves of $R \otimes \mathbb{Q}$-vector spaces which becomes a quasi-isomorphism after tensoring with $\mathbb{C}$, i.e. $\beta_1 \otimes \mathrm{id}\colon (K_{\mathbb{Q}}^\bullet \otimes \mathbb{C}, W) \simeq ({}'K_{\mathbb{C}}, W)$. β is called the *second comparison morphism*.

- These data should satisfy the following axiom: For all $m \in \mathbb{Z}$ the triple

$$\begin{aligned} \mathrm{gr}_m^W \mathcal{K}^\bullet &:= \left(\mathrm{gr}_m^W \mathcal{K}_{\mathbb{Q}}^\bullet, (\mathrm{gr}_m^W \mathcal{K}_{\mathbb{C}}^\bullet, F), \mathrm{gr}_m^W(\beta)\right) \\ &= \left\{ \begin{tikzcd} & \mathrm{gr}_m^{W\prime} \mathcal{K}_{\mathbb{C}}^\bullet & \\ \mathrm{gr}_m^W \mathcal{K}_{\mathbb{Q}}^\bullet \arrow[ur, "\mathrm{gr}_m^W \beta_1"] & & (\mathrm{gr}_m^W \mathcal{K}_{\mathbb{C}}^\bullet, \mathrm{gr}_m^W F) \arrow[ul, "\mathrm{gr}_m^W \beta_2"', "\sim"] \end{tikzcd} \right\} \end{aligned}$$

is a *normalized* (polarizable) $R \otimes \mathbb{Q}$-Hodge complex of sheaves of weight m, i.e. the hypercohomology group $\mathbb{H}^k(X, \mathrm{gr}^m \mathcal{K}_{\mathbb{C}}^\bullet)$ is a pure Hodge structure *of weight* $k+m$.

f.) A *morphism of normalized mixed Hodge complexes (of sheaves)* is a morphism of the corresponding diagrams respecting all the adherent structure.

g.) A *quasi-isomorphism* of mixed Hodge complexes (of sheaves) is a morphism $K^\bullet \longrightarrow L^\bullet$ (resp. $\mathcal{K}^\bullet \longrightarrow \mathcal{L}^\bullet$) inducing an isomorphism *of mixed Hodge structures* $\underline{H}^k(K_{\mathbb{C}}^\bullet) \cong \underline{H}^k(L_{\mathbb{C}}^\bullet)$ (respectively $\underline{\mathbb{H}}^k(X, \mathcal{K}^\bullet) \cong \underline{\mathbb{H}}^k(X, \mathcal{L}^\bullet)$) for all k.

C.3.2 Variations of mixed Hodge structure

For this section, let again $R \subset \mathbb{R}$ a subring.

Basic definitions

We call a locally constant sheaf of R-modules on a complex variety X an *R-local system* on X, and denote the category of R-local systems by $\mathrm{Loc}_R(X)$. If $R = \mathbb{Z}$, we just write $\mathrm{Loc}(X) := \mathrm{Loc}_{\mathbb{Z}}(X)$.

Recall that, roughly speaking, a variation of pure/mixed R-Hodge structure on X is an R-local system together with some additional data which yield pure/mixed R-Hodge structures on its fibers. To be precise, recall that a *variation of (pure) R-Hodge structure* or, shortly, *R-VHS* on a complex manifold X consists of the following data:

- a holomorphic vector bundle V equipped with an integrable connection ∇;
- a finite decreasing filtration $\{\mathcal{F}^p\}$ of $\mathcal{V} := \mathcal{O}_X(V)$ by locally free $\mathcal{O}_X$-submodules (the Hodge filtration);
- an R-local system $\mathbb{V}$ of finitely generated R-modules on X such that the sheaf of horizontal sections of ∇ is given by $\mathcal{O}_X(V)^{\nabla} = \mathbb{V} \otimes_{\mathbb{Z}} \mathbb{C}$ together with a finite increasing filtration $\{W_m\}$ of the local system $\mathbb{V}_{\mathbb{Q}} := \mathbb{V} \otimes \mathbb{Q}$ by local subsystems (the weight filtration).

These data are subject to the following conditions:

a.) the filtrations $\mathcal{F}^p(x)$ and W_m of $\mathbb{V}(x) := \mathbb{V}_x \otimes_{\mathbb{Z}} \mathbb{C}$ define a mixed $\mathbb{Q}$-Hodge structure on the stalks $\mathbb{V}_{\mathbb{Q},x}$;

b.) The connection ∇ satisfies Griffiths' transversality condition

$$\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_X.$$

A morphism of R-VHS is a morphism of holomorphic vector bundles with integrable connections compatible with both filtrations. The category of R-VHS on X is denoted by $\mathrm{VHS}_R(X)$. It is an abelian category. A *polarization* of a variation of Hodge structure of weight k on X is a morphism of variations

$$Q\colon \mathbb{V} \otimes \mathbb{V} \longrightarrow \underline{\mathbb{Z}}(-k)_X$$

which induces on each fibre a polarization of the corresponding Hodge structure of weight k. We denote the full abelian subcategories of $\mathrm{VHS}(X)$ consisting of polarizable objects by $\mathrm{VHS}(X)^p$.

Just like in the setting of Hodge structures, the mixed situation involves one additional filtration: A *variation of mixed R-Hodge structure* (or, shortly, *R-VMHS*) on a complex manifold X is given by the following data:

- an R-local system $\mathbb{V}$ on X such that the stalks are finitely generated over R;
- a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_X$ by holomorphic subbundles (the *Hodge filtration*);
- a finite increasing filtration $\{W_m\}$ of the local system $\mathbb{V}_{\mathbb{Q}} = \mathbb{V} \otimes \mathbb{Q}$ by local subsystems (the *weight filtration*).

These data are subject to the following conditions:

a.) the filtration $\mathcal{F}^p$ of $\mathbb{V}_{\mathbb{Q}}$ defines a mixed $R \otimes \mathbb{Q}$-Hodge structure on the stalks $\mathbb{V}_{\mathbb{Q},x}$, $x \in X$;
b.) the natural connection $\nabla\colon \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega_X^1$ with $\mathcal{V}^\nabla = \mathbb{V}_{\mathbb{C}}$ satisfies *Griffiths' transversality condition*
$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_X^1.$$

A *morphism of VMHS over* R is a morphism of R-local systems compatible with both filtrations. We denote the category of variations of mixed Hodge structure on X by $\mathrm{VMHS}_R(X)$. It is an abelian category. A VMHS is called *graded-polarizable* if the induced variations of pure Hodge structure $\mathrm{gr}_k^W \mathbb{V}$ are all polarizable. We denote the category of graded-polarizable variations of mixed Hodge structure on X by $\mathrm{VMHS}(X)^p$. $\mathrm{VMHS}(X)^p$ is also an abelian category.

C^∞-data for VMHS

For later use, we need an equivalent definition of R-VMHS using C^∞-data instead of holomorphic data.

Let us first recall the basic terminology. For an analytic C^∞-vector bundle V^∞ over X we will denote the sheaf of C^∞-sections of V^∞ by $\mathcal{V} := \mathcal{E}_X(V^\infty)$. A C^∞-connection ∇ on V^∞ on X is a $\mathbb{C}$-linear map $\nabla\colon \mathcal{E}_X(\mathcal{V}^\infty) \longrightarrow \mathcal{E}_X^1 \otimes \mathcal{V}^\infty$ satisfying the Leibniz rule, where $\mathcal{E}_X^1$ denotes the space of smooth 1-forms on X. Such a connection is called *integrable* if $\nabla \circ \nabla = 0$. Given such a connection, we denote by $\mathcal{V}^{\infty\nabla} := \{s \in \mathcal{V} | \nabla s = 0\}$ the sheaf of *horizontal sections* .

Let us now suppose that V is a holomorphic vector bundle on a complex manifold X, which we will now consider as a C^∞-vector bundle, with the sheaf of C^∞-sections denoted by $\mathcal{E}_X(V)$. Suppose we are given a finite decreasing filtration $\{\mathcal{F}^{\infty p}\}$ of $\mathcal{V}^\infty := \mathcal{E}_X(V)$ by $\mathcal{E}_X^0$-submodules, which correspond to the sheaves of C^∞-sections of C^∞-subbundles F^p of V. When does this filtration carry a holomorphic structure, i.e. under which conditions can we actually see F^p as a filtration by holomorphic subbundles? The answer to this question can be found in [HZ87]:

Lemma C.3.1. *[HZ87, 4.15 and 4.16, p.97] Let $\mathcal{F}^{\infty p}$ be a filtration of the holomorphic vector bundle E on a complex manifold X by C^∞-subbundles. A necessary and sufficient condition for $\mathcal{F}^{\infty p}$ to be a holomorphic subbundle is: if σ is a C^∞-section of $\mathcal{F}^{\infty p}$, then the values of the* $(0,1)$*-form $\bar{\partial}\sigma$ are also in $\mathcal{F}^{\infty p}$.*
If this condition is satisfied for all p, let $\mathcal{F}^p \subset \mathcal{F}^{\infty p}$ be the subsheaves of holomorphic sections. Then $d\mathcal{F}^p \subset \Omega_X \otimes \mathcal{F}^{p-1}$ if and only if $d\mathcal{F}^{\infty p} \subset \mathcal{E}_X^1 \otimes \mathcal{F}^{\infty p-1}$

In addition, suppose we are given an integrable C^∞-connection ∇ on V, which corresponds to the C^∞-connection $d \otimes \mathrm{id}\colon \mathcal{E}_X^0 \otimes_{\mathbb{C}_X} V^\nabla$. This connection descends to a holomorphic connection on the holomorphic bundle V by just taking ∇^{hol} to be the restriction of ∇ to the space of holomorphic sections of V, $\mathcal{O}_X \otimes_{\mathbb{C}_X} V^\nabla \cong \mathcal{O}_X(V)$, i.e. $\nabla^{hol} = d \otimes \mathrm{id}\colon \mathcal{O}_X \otimes_{\mathbb{C}_X} V^\nabla \longrightarrow \Omega_X^1 \otimes_{\mathbb{C}_X} V^\nabla$. Then the holomorphic connection ∇^{hol} together with the filtration by locally free $\mathcal{O}_X$-submodules $\mathcal{F}^p$ of the above lemma satisfy Griffiths' transversality condition iff the C^∞-connection ∇ satisfies the corresponding condition $\nabla\mathcal{F}^{\infty p} \subset \mathcal{E}^1_X \otimes \mathcal{F}^{\infty p-1}$. These considerations combined, we see that we can also define a VMHS in the following way:

Let X be a complex manifold. A *VMHS* on X consists of the following C^∞-data:

- C^∞-vector bundle V^∞ equipped with an integrable C^∞-connection ∇;
- a finite decreasing filtration $\{{}^\infty\mathcal{F}^p\}$ of $\mathcal{V}^\infty := \mathcal{E}_X(V)$ by C^∞-subbundles satisfying the following condition for all p (which is equivalent to the subbundles actually being holomorphic): if σ is a section of $\mathcal{F}^{\infty p}$, then the values of the $(0,1)$-form $\bar\partial\sigma$ are also in $\mathcal{F}^{\infty p}$.
- a local system $\mathbb{V}$ of finitely generated abelian groups on X such that the sheaf of horizontal sections of V is given by $V^\nabla = \mathbb{V} \otimes_{\mathbb{Z}} \mathbb{C}$ together with a finite increasing filtration $\{W_m\}$ of the local system $\mathbb{V}_{\mathbb{Q}} := \mathbb{V} \otimes \mathbb{Q}$ by local subsystems.

These data should satisfy the following conditions:

a.) the induced filtrations ${}^\infty\mathcal{F}^p$ and W_m of $\mathbb{V}(x) := \mathbb{V}_x \otimes_{\mathbb{Z}} \mathbb{C}$ define a mixed $\mathbb{Q}$-Hodge structure on the stalks $\mathbb{V}_{\mathbb{Q},x}$;
b.) The connection ∇ satisfies Griffiths' C^∞ transversality condition

$$\nabla^\infty\mathcal{F}^p \subset \mathcal{E}^1_X \otimes {}^\infty\mathcal{F}^{p-1}.$$

The VMHS associated to a scheme and the Hodge-de Rham complex

The de Rham cohomology of a compact complex manifold X may be computed as the hypercohomology of the constant sheaf $\underline{\mathbb{R}}_X$. Its standard Γ-acyclic resolution is the de Rham complex of C^∞-forms on X. In the case when X is not compact, one needs to find a suitable replacement for this complex. To this end, one considers a good compactification $j\colon X \hookrightarrow \bar X$ with simple normal crossing divisor $D := \bar X - X$. The de Rham cohomology of X is then computed as the hypercohomology of the sheaf $j_*\underline{\mathbb{R}}_X$ on $\bar X$. Over $\mathbb{C}$, this complex is quasi-isomorphic to the complex $j_*\mathcal{E}^\bullet_X$ of smooth forms on X extended to all of $\bar X$. However, there is one major disadvantage of $j_*\mathcal{E}^\bullet_X$: These forms may have very bad singularities along D, which makes them hard to work with. This is why instead of $j_*\mathcal{E}^\bullet_X$ one considers a different complex of smooth forms whose singularities along D are very mild: The logarithmic de Rham complex.

Definition C.3.2. *Let $j\colon X \longrightarrow \bar X$ be a good compactification with simple normal crossing divisor $D = \bar X - X$. The logarithmic de Rham complex is the subcomplex of $j_*\Omega^\bullet_X$ defined as*

$$\Omega^\bullet_{\bar X}(\log D) := \{w \in j_*\Omega^\bullet_X \mid w \textit{ and } dw \textit{ have at most a pole of order 1 along } D\}.$$

Remark C.3.3. The logarithmic de Rham complex can be determined in the neighbourhood of a point $p \in D$ as follows (see p. 449 of [GH78]): Since D is a simple normal crossing divisor, there exist local coordinates $(z_1, \dots, z_n)$ at p such that D is locally given as the zero-set of the equation $z_1, \dots, z_k = 0$. Then one can show that locally around p

$$\Omega^1_{\bar{X}}(\log D)_p = \mathcal{O}_{\bar{X},p}\frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_{\bar{X},p}\frac{dz_k}{z_k} \oplus \mathcal{O}_{\bar{X},p}dz_{k+1} \oplus \cdots \oplus \mathcal{O}_{\bar{X},p}dz_n,$$
$$\Omega^m_{\bar{X}}(\log D)_p = \bigwedge^m \Omega^1_{\bar{X}}(\log D)_p.$$

This complex will turn out to compute the cohomology of X as well as determine the mixed Hodge structures by virtue of natural filtrations on $\Omega^\bullet_{\bar{X}}(\log D)$:

Definition C.3.4. *a.) Let the descending filtration $\sigma^\bullet$ on the logarithmic de Rham complex $\Omega^\bullet_{\bar{X}}(\log D)$ be the trivial filtration, defined by*

$$\sigma^{\geq i}\Omega^\bullet_{\bar{X}}(\log D) := \Omega^{\geq i}_{\bar{X}}(\log D) = \{0 \to \Omega^i_{\bar{X}}(\log D) \to \Omega^{i+1}_{\bar{X}}(\log D) \to \dots\}.$$

b.) We define an ascending filtration $\mathcal{W}_\bullet$ on $\Omega^i_{\bar{X}}(\log D)$ by type of logarithmic singularities:

$$\mathcal{W}_m\Omega^i_{\bar{X}}(\log D) = \begin{cases} 0 & \text{for } i < 0 \\ \Omega^i_{\bar{X}}(\log D) & \text{for } m \geq i \\ \Omega^m_{\bar{X}}(\log D) \wedge \Omega^{i-m}_{\bar{X}} & \text{for } 0 \leq m \leq i. \end{cases}$$

This filtration naturally gives rise to a filtration of the complex $\Omega^\bullet_{\bar{X}}(\log D)$.

Instead of the logarithmic de Rham complex, one can use the logarithmic C^∞-complex $\mathcal{E}^\bullet_{\bar{X}}(\log D) \subset j_*\mathcal{E}^\bullet_X$:

Definition C.3.5. *[Jan88, 1.4, p.6] Let $\mathcal{E}^\bullet_{\bar{X}}$ denote the complex of smooth forms on $\bar{X}$. We define the complex of C^∞-forms on $\bar{X}$ with logarithmic singularities along D by*

$$\mathcal{E}^\bullet_{\bar{X}}(\log D) := \Omega^\bullet_{\bar{X}}(\log D) \otimes_{\Omega^\bullet_{\bar{X}}} \mathcal{E}^\bullet_{\bar{X}},$$

where the differentials are as in the tensor product of chain complexes: $d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^{\deg \alpha}\alpha \otimes d\beta$. The filtrations of Definition C.3.4 naturally induce the following filtrations of $\mathcal{E}^\bullet_{\bar{X}}(\log D)$:

a.) a decreasing filtration $\sigma^\bullet_\infty$ of $\mathcal{E}^\bullet_{\bar{X}}(\log D)$ induced by the trivial filtration σ on $\Omega^\bullet_{\bar{X}}(\log D)$:

$$\sigma^\bullet_\infty\mathcal{E}^\bullet_{\bar{X}}(\log D) := (\sigma^\bullet\Omega^\bullet_{\bar{X}}(\log D)) \otimes \mathcal{E}^\bullet_{\bar{X}},$$

b.) an increasing filtration $\mathcal{W}^{\infty\bullet}$ of $\mathcal{E}^\bullet_{\bar{X}}(\log D)$ by type of logarithmic singularities:

$$\mathcal{W}^\infty_m\mathcal{E}^i_{\bar{X}}(\log D) = \begin{cases} 0 & \text{for } i < 0 \\ \mathcal{E}^i_{\bar{X}}(\log D) & \text{for } m \geq i \\ \left(\Omega^m_{\bar{X}}(\log D) \wedge \Omega^{i-m}_{\bar{X}}\right) \otimes_{\Omega^i_{\bar{X}}} \mathcal{E}^i_{\bar{X}} & \text{for } 0 \leq m \leq i. \end{cases}$$

Remark C.3.6. Taking into account Remark C.3.3 about the local nature of the complex $\Omega^\bullet_{\bar{X}}(\log D)$, it is evident that $\mathcal{E}^i_{\bar{X}}(\log D)$ can locally be described as follows: Let $p \in D$ be a point. Since D is a simple normal crossing divisor, there exist local coordinates $(z_1, \ldots, z_n)$ at p such that D is locally given as the zero-set of the equation $z_1, \ldots, z_k = 0$. Locally around $p \in D$, the sheaf $\mathcal{E}^i_{\bar{X}}(\log D)$ is then generated by sections of the form

$$\beta \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \frac{dz_{i_2}}{z_{i_2}} \wedge \ldots \wedge \frac{dz_{i_m}}{z_{i_m}}$$

where $m \leq i$ and β is a smooth $i - m$-form .

That these two complexes - the holomorphic logarithmic de Rham complex and the C^∞-complex - are indeed quasi-isomorphic is common knowledge:

Lemma C.3.7. *[Jan88, 1.7] The natural embedding*

$$(\Omega^\bullet_{\bar{X}}(\log D), \sigma, \mathcal{W}) \hookrightarrow (\mathcal{E}^\bullet_{\bar{X}}(\log D), \sigma_\infty, \mathcal{W}^\infty) \text{ is a filtered quasi-isomorphism.}$$

Proof. $(\Omega^\bullet_{\bar{X}}(\log D), \sigma) \hookrightarrow (\mathcal{E}^\bullet_{\bar{X}}(\log D), \sigma_\infty)$ is a filtered quasi-isomorphism, see Lemma 1.7 in [Jan88]. That the embedding is also compatible with the weight filtration is obvious from the definition of σ and σ_∞. ∎

Both of these complexes can be used to compute the cohomology of X:

Proposition C.3.8. *(originally due to Deligne) [PS08, Proposition II.4.3, p.91] The inclusion $\Omega^\bullet_{\bar{X}}(\log D) \hookrightarrow j_*\Omega^\bullet_X$ is a quasi-isomorphism and induces a natural identification*

$$H^k(X; \mathbb{C}) = \mathbb{H}^k(\bar{X}, \Omega^\bullet_{\bar{X}}(\log D)).$$

Furthermore, the natural map $j_*\Omega^\bullet_X \longrightarrow Rj_*\underline{\mathbb{C}}_X$ is a quasi-isomorphism inducing

$$H^k(X; \mathbb{C}) = \mathbb{H}^k(\bar{X}, Rj_*\underline{\mathbb{C}}_X).$$

In order to impart $H^k(X; \mathbb{C})$ with a mixed $\mathbb{Z}$-Hodge structure, we need to fix an ascending and a descending filtration W and F on $\mathcal{E}^\bullet_{\bar{X}}(\log D)$, together with a "rational model" of $(\mathcal{E}^\bullet_{\bar{X}}(\log D), W)$ and an "integral model" of $\mathcal{E}^\bullet_{\bar{X}}(\log D)$.

To make the construction of the Hodge-de Rham complex of a complex variety as clear as possible, we will describe the individual complexes and comparison morphisms step by step.

Let X be a complex algebraic variety and $j\colon X \hookrightarrow \bar{X}$ be a good compactification of X with normal crossing divisor $D = \bar{X} - X$. We identify these data via GAGA with the analytic ones. Then the Hodge complex giving rise to the Hodge structure on the cohomology of X is given by the diagram

$$
\begin{aligned}
&\big(Rj_*\mathbb{Z}_X, (Rj_*\underline{\mathbb{Q}}_X, \tau), \alpha, (\mathcal{E}^\bullet_{\bar{X}}(\log D), \mathcal{W}^\infty, \sigma_\infty), \beta\big) \\
&:= \left\{
\begin{tikzcd}
 & Rj_*\underline{\mathbb{Q}}_X & & ({}'\mathcal{E}^\bullet_{\bar{X}}(\log D), W) & \\
Rj_*\underline{\mathbb{Z}}_X \arrow[ur, "\alpha_1"] & & (Rj_*\underline{\mathbb{Q}}_X, \tau) \arrow[ul, "\alpha_2"', "\sim"] \arrow[ur, "\beta_1"] & & (\mathcal{E}^\bullet_{\bar{X}}(\log D), \mathcal{W}^\infty, \sigma_\infty) \arrow[ul, "\beta_2"', "\sim"]
\end{tikzcd}
\right\},
\end{aligned}
$$

where the data are given as follows:

a.) **The complexes $Rj_*\underline{\mathbb{Z}}_X$ and $Rj_*\underline{\mathbb{Q}}_X$**: We denote the Godement resolutions of $\underline{\mathbb{Z}}_X$ and $\underline{\mathbb{Q}}_X$ by $G(\underline{\mathbb{Z}}_X)$ and $G(\underline{\mathbb{Q}}_X)$, respectively. These resolutions are flabby. We put

$$
\begin{aligned}
Rj_*\underline{\mathbb{Z}}_X &:= j_*G(\underline{\mathbb{Z}}_X) \\
Rj_*\underline{\mathbb{Q}}_X &:= j_*G(\underline{\mathbb{Q}}_X) \\
Rj_*\underline{\mathbb{C}}_X &:= j_*G(\underline{\mathbb{C}}_X).
\end{aligned}
$$

By functoriality of the Godement resolution, he natural inclusions $\underline{\mathbb{Z}}_X \hookrightarrow \underline{\mathbb{Q}}_X \hookrightarrow \underline{\mathbb{C}}_X$ give rise to arrows

$$
Rj_*\underline{\mathbb{Z}}_X \xrightarrow{\alpha_1} Rj_*\underline{\mathbb{Q}}_X \xrightarrow{\beta_1'} Rj_*\underline{\mathbb{C}}_X.
$$

b.) **The morphisms α_1, α_2**: We take α_1 to be the morphism of (a.), and $\alpha_2 := \mathrm{id}$.

c.) **The filtration τ on $Rj_*\underline{\mathbb{Q}}_X$**: Take the increasing filtration τ to be the canonical filtration given by truncation, i.e. if $K^\bullet$ is any complex, then

$$
\tau_{\leq m}K^\bullet := \{\ldots \to K^{m-2} \to K^{m-1} \to \ker(d) \to 0 \to 0 \ldots\}.
$$

d.) **The filtered complex $({}'\mathcal{E}^\bullet_{\bar{X}}(\log D), W)$ and the morphims β_1, β_2**: The right "hat" of the diagram above is given by the commutative diagram

$$
\begin{tikzcd}
 & & & ({}'\mathcal{E}^\bullet_{\bar{X}}(\log D), W) & & & \\
 & & (j_*\mathcal{E}^\bullet_X, \tau) \arrow[ur, "\beta_1'''", "\sim"'] & & (\mathcal{E}^\bullet_{\bar{X}}(\log D), \mathcal{W}^\infty) \arrow[ul, "\beta_2"', "\sim"] \arrow[ddrr, equal] & & \\
 & (Rj_*\underline{\mathbb{C}}_X, \tau) \arrow[ur, "\beta_1''", "\sim"'] & & (\mathcal{E}^\bullet_{\bar{X}}(\log D), \tau) \arrow[ul, "\sim", "i"'] \arrow[ur, "\sim"', "i'"] & & & \\
(Rj_*\underline{\mathbb{Q}}_X, \tau) \arrow[ur, "\beta_1'"] \arrow[uuurrr, bend left, "\beta_1"] & & & & & & (\mathcal{E}^\bullet_{\bar{X}}(\log D), \mathcal{W}^\infty)
\end{tikzcd}
$$

where

- $\beta_1' \colon Rj_*\underline{\mathbb{Q}}_X \longrightarrow Rj_*\underline{\mathbb{C}}_X$ is the natural morphism induced by the inclusion $\underline{\mathbb{Q}}_X \hookrightarrow \underline{\mathbb{C}}_X$.

- $\beta_1''\colon (Rj_*\underline{\mathbb{C}}_X, \tau) \longrightarrow (j_*\mathcal{E}_X^\bullet, \tau)$ is the natural quasi-isomorphism.
- $i\colon \mathcal{E}_{\bar{X}}^\bullet(\log D) \hookrightarrow j_*\mathcal{E}_X^\bullet$ is the natural inclusion,
- i' is the identity on the complex $\mathcal{E}_{\bar{X}}^\bullet(\log D)$, which is compatible with the filtrations τ and $\mathcal{W}^\infty$ by the following consideration: By definition, the m-th filtration subcomplexes are given by

$$\begin{aligned}\tau_m\mathcal{E}_{\bar{X}}^\bullet(\log D) &= \left\{ \ \dots \to \mathcal{E}_{\bar{X}}^{m-1}(\log D) \to \ker(d) \to 0 \qquad \right\}\\ \mathcal{W}_m^\infty\mathcal{E}_{\bar{X}}^\bullet(\log D) &= \left\{ \ \dots \to \mathcal{E}_{\bar{X}}^{m-1}(\log D) \to \mathcal{E}_{\bar{X}}^{m}(\log D) \to \mathcal{E}_{\bar{X}}^{m}(\log D) \otimes \mathcal{E}_{\bar{X}}^{1} \to \ \right\},\end{aligned}$$

  so there is a natural inclusion $\tau_m\mathcal{E}_{\bar{X}}^\bullet(\log D) \hookrightarrow \mathcal{W}_m^\infty\mathcal{E}_{\bar{X}}^\bullet(\log D)$; it is easy to see that i' is a filtered quasi-isomorphism (see Lemma 4.9 in [PS08]).
- The top complex $({}'\mathcal{E}_{\bar{X}}^\bullet(\log D), W)$ is given by

$$\mathrm{Cone}\left(i' - i\colon (\mathcal{E}_{\bar{X}}^\bullet(\log D), \tau) \longrightarrow (j_*\mathcal{E}_X^{\bullet\sharp}, \tau) \oplus (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty)\right).$$

  Take β_1''', β_2''' to be the induced morphisms. Since both i and i' are filtered quasi-isomorphisms, there is a commutative square of quasi-isomorphisms

$$\begin{tikzcd} & ({}'\mathcal{E}_{\bar{X}}^\bullet(\log D), W) & \\ (j_*\mathcal{E}_X^\bullet, \tau) \arrow[ur, "\beta_1'''", "\sim"'] & & (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty). \arrow[ul, "\beta_2"', "\sim"] \\ & (\mathcal{E}_{\bar{X}}^\bullet(\log D), \tau) \arrow[ul, "i", "\sim"'] \arrow[ur, "i'"', "\sim"] & \end{tikzcd}$$

- Then put $\beta_1 := \beta_1''' \circ \beta_1'' \circ \beta_1'$. Since $\beta_1 \otimes \mathrm{id}_{\mathbb{C}}$, β_1'' and β_1''' are quasi-isomorphisms, so is $\beta_1 \otimes \mathrm{id}_{\mathbb{C}}$.

Definition C.3.9. *Let X be a smooth complex algebraic variety with good compactification $\bar{X}$ and simple normal crossing divisor $D = \bar{X} - X$. The complex*

$$\mathcal{H}dg^\bullet(\bar{X}\log D) := \left(Rj_*\underline{\mathbb{Z}}_X, (Rj_*\underline{\mathbb{Q}}_X, \tau), \alpha, (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty, \sigma_\infty), \beta\right)$$

$$:= \left\{ \begin{tikzcd} & Rj_*\underline{\mathbb{Q}}_X \arrow[dr, equal] & & ({}'\mathcal{E}_{\bar{X}}^\bullet(\log D), W) & \\ Rj_*\underline{\mathbb{Z}}_X \arrow[ur, "\alpha_1"] & & (Rj_*\underline{\mathbb{Q}}_X, \tau) \arrow[ur, "\beta_1"] & & (\mathcal{E}_{\bar{X}}^\bullet(\log D), \mathcal{W}^\infty, \sigma_\infty) \arrow[ul, "\beta_2"', "\sim"] \end{tikzcd} \right\}$$

constructed above is called the <u>*Hodge-de Rham complex*</u>.

Theorem C.3.10. *[PS08, Theorem 4.2, p.90] Let X be a smooth complex algebraic variety and let $\bar{X}$ be a good compactification with simple normal crossing divisor $D = \bar{X} - X$. Then the diagram $\mathcal{H}dg^\bullet(\bar{X}\log D)$ is a Hodge complex yielding the following mixed $\mathbb{Q}$-Hodge structure on the cohomology $H^k(X;\mathbb{C}) = \mathbb{H}^k(\bar{X}, \mathcal{E}_{\bar{X}}^\bullet(\log D))$ of X:*

a.) The canonical filtration τ on the complex $Rj_*\underline{\mathbb{Q}}_X$ yields the weight filtration on the rational cohomology $H^k(X;\mathbb{Q}) = \mathbb{H}^k(\bar{X}, Rj_*\underline{\mathbb{Q}}_X)$ of X by putting

$$W_m H^k(X;\mathbb{Q}) = \operatorname{Im}\left(\mathbb{H}^k(\bar{X}, \tau_{m-k}Rj_*\underline{\mathbb{Q}}_X) \longrightarrow H^k(X;\mathbb{Q})\right).$$

b.) The filtration W defined by

$$\mathcal{W}_m^{\infty}\mathcal{E}^p_{\bar{X}}(\log D) = \begin{cases} 0 & \text{for } m < 0 \\ \mathcal{E}^p_{\bar{X}}(\log D) & \text{for } m \geq p \\ \mathcal{E}^{p-m}_{\bar{X}} \wedge \mathcal{E}^m_{\bar{X}}(\log D) & \text{for } 0 \leq m \leq p \end{cases}$$

induces in cohomology the complex weight filtration

$$W_m H^k(X;\mathbb{C}) = \operatorname{Im}\left(\mathbb{H}^k(\bar{X}, \mathcal{W}^{\infty}_{m-k}\mathcal{E}^{\bullet}_{\bar{X}}(\log D)) \longrightarrow H^k(X;\mathbb{C})\right),$$

which is compatible with the canonical filtration defined over $\mathbb{Q}$.

c.) The trivial friltration σ_∞ on the complex $\mathcal{E}^{\bullet}_{\bar{X}}(\log D)$ yields the Hodge filtration

$$F^p H^k(X;\mathbb{C}) = \operatorname{Im}\left(\mathbb{H}^k(\bar{X}, \sigma_\infty^{\geq p}\mathcal{E}^{\bullet}_{\bar{X}}(\log D)) \longrightarrow H^k(X;\mathbb{C})\right).$$

Monodromy

An important aspect in the theory of variations of Hodge structure is the associated monodromy. It is not hard to see that if X is connected, locally arcwise connected and simply connected, then every local system $\mathbb{V}$ on X is trivial, i.e. isomorphic to the constant sheaf (see [Voi03, Proposition 3.9, p. 70]). Noting this, one may associate a representation of $\pi_1(X)$ to every local system $\mathbb{V}$ on X as follows:
Let $\pi\colon \widetilde{X} \longrightarrow X$ be the universal cover of X. And let $\mathbb{V}$ be a local system on X with stalk G. Then by the above remark, the local system $\pi^{-1}\mathbb{V}$ is constant on $\widetilde{X}$ with stalk G. In particular, it is constant on any path γ on the universal cover. We fix a base-point $x \in X$, a point $\tilde{x} \in \widetilde{X}$ lying over x and an isomorphism $\alpha_x\colon \mathbb{V}_x \xrightarrow{\sim} G$. Since $\pi^{-1}\mathbb{V}$ is constant, there exists a unique isomorphism of locally constant sheaves $\beta\colon \pi^{-1}(\mathbb{V}) \cong G$ determined by the condition that the induced isomorphims $\pi^{-1}\mathbb{V}_{\tilde{x}} \cong G$ are equal to the composition $\pi^{-1}\mathbb{V}_{\tilde{x}} \cong \mathbb{V}_x \overset{\alpha}{\cong} G$. Now let the class $[\gamma] \in \pi_1(X,x)$ be represented by a path γ on $\widetilde{X}$ from $\tilde{x}$ to a point $y \in \pi^{-1}(x)$. Then γ gives rise to the isomorphism

$$\rho(\gamma) := \alpha \circ \mu_y \circ \beta_y^{-1}\colon G \longrightarrow G,$$

where $\mu_y\colon (\pi^{-1}\mathbb{V})_y \cong \mathbb{V}_x$ is the natural isomorphism. One can easily see that the map sending $\gamma \in \pi_1(X,x)$ to $\rho(\gamma) \in \operatorname{Aut}_R G$ is a representation.

Definition C.3.11. *Let $\mathbb{V}$ be a local system on a complex analytic variety X and x_0 a point of X. Then the representation*

$$\begin{aligned} \rho\colon \pi_1(X,x_0) &\longrightarrow \operatorname{Aut}_R G \\ \gamma &\longmapsto \rho(\gamma) \end{aligned}$$

constructed above is called the monodromy representation associated to $\mathbb{V}$.

This yields the following 1-1-correspondence :

Theorem C.3.12. *[Voi03, Corollary 3.10, p. 71] Let R be a ring and G an R-module. If X is arcwise connected and locally simply connected and x_0 is a point of X, we have a natural bijection between the set of isomorphism classes of local systems with stalk G and the set of representations $\pi_1(X, x_0) \longrightarrow \operatorname{Aut}_R G$, modulo the action of $\operatorname{Aut}_R G$ by conjugation.*

Unipotent VMHS

One of the most central notions in that area is that of unipotence. In this section, we consider variations of mixed Hodge structures over a subfield $k \subset \mathbb{C}$. Recall that a representation $\rho\colon \pi_1(X, x) \longrightarrow \operatorname{Aut}(V)$ of the fundamental group of X is called

- *quasi-unipotent* if for any $\gamma \in \pi_1(X, x)$ all eigenvalues of $\rho(\gamma)$ are roots of unity.
- *unipotent of index r* if for any $\gamma \in \pi_1(X, x)$ one has $\rho(\gamma)^r = \mathrm{id}$.

Let X again be a complex manifold, $x_0 \in X$ a base-point, and $(\mathbb{V}, \mathcal{F}, W)$ a k-VMHS on X. The local system $\mathbb{V}$ underlying this variation of mixed Hodge structure gives rise to a monodromy representation

$$\rho_{\mathbb{V}}\colon \pi_1(X, x_0) \longrightarrow \operatorname{Aut}(\mathbb{V}_{x_0})$$

as described above.

Let now $k\pi_1(X, x_0)$ denote the group algebra of $\pi_1(X, x)$ over the field k. Recall that it has an augmentation

$$\epsilon\colon k\pi_1(X, x) \longrightarrow k$$

sending every group element $g \in \pi_1(X, x)$ to 1. Its kernel is called the *augmentation ideal* and denoted by J. The homomorphism $\iota\colon k \longrightarrow k\pi_1(X, x)$ sending an element $r \in k$ to $r \cdot id$, where id is the identity element in $\pi_1(X, x)$, defines a k-vector space splitting

$$k\pi_1(X, x) \cong k \oplus J.$$

Lemma C.3.13. *ρ extends to an algebra homomorphism of the group ring*

$$\bar{\rho}\colon k\pi_1(X, x_0) \longrightarrow W_0 \operatorname{End}(\mathbb{V}_{x_0}).$$

Proof. It is clear that ρ extends to an algebra homomorphism of the group ring

$$\bar{\rho}\colon k\pi_1(X, x_0) \longrightarrow \operatorname{End}(\mathbb{V}_{x_0}).$$

Hence, we need to show that the image of the morphism $\bar{\rho}$ lies in $W_0 \operatorname{End}(\mathbb{V}_{x_0}) \subset \operatorname{End}(\mathbb{V}_{x_0})$. To this end, note that the canonical Hodge structure induced on $\operatorname{End}(\mathbb{V}_{x_0})$ by that of $\mathbb{V}_{x_0}$ is defined as follows:

$$W_i \operatorname{End}(\mathbb{V}_{x_0}) := \{f \in \operatorname{End}(\mathbb{V}_{x_0}) \mid f(W_k\mathbb{V}_{x_0}) \subset W_{k+i}\mathbb{V}_{x_0} \forall k\}, \quad \text{i.e.}$$
$$W_0 \operatorname{End}(\mathbb{V}_{x_0}) := \{f \in \operatorname{End}(\mathbb{V}_{x_0}) \mid f(W_k\mathbb{V}_{x_0}) \subset W_k\mathbb{V}_{x_0} \forall k\}$$

Since the subsheaves $W_k\mathbb{V} \subset \mathbb{V}$ are local subsystems of $\mathbb{V}$, the action of the fundamental group $\pi_1(X, x_0)$ on $\mathbb{V}_{x_0}$ restricts to an action on $W_k\mathbb{V}_{x_0}$ (equal to the representation giving rise to the local system $W_k\mathbb{V}$). Hence, for any $\gamma \in \pi_1(X, x_0)$, the automorphism $\rho(\gamma)$ of $\mathbb{V}_{x_0}$ satisfies $\rho(\gamma)(W_k\mathbb{V}_{x_0}) \subset W_k\mathbb{V}_{x_0}$ for all k, in other words it lies in the induced zeroth filtration subset. This proves the assertion. ■

Let us denote the kernel of the augmentation $\varepsilon\colon k\pi_1(X, x) \to k$ by J. The condition of unipotence then translates into the following equivalent conditions:

Proposition C.3.14. *[HZ87, 1.4, p. 84] Under the above assumptions and notations, the following statements are equivalent:*

a.) The representation ρ is unipotent.
b.) $\bar{\rho}$ factors through some quotient $k\pi_1(X, x)/J^{r+1}$, where J is the kernel of the augmentation of the group ring.
c.) The variations of Hodge structure of the pure quotients $\operatorname{gr}_k^W \mathbb{V}$ with the induced Hodge filtrations are constant for all k.

Definition C.3.15. *An R-VMHS $(\mathbb{V}, W, \mathcal{F})$ is called unipotent if it satisfies the equivalent conditions of proposition C.3.14. It is called quasi-unipotent, if its monodromy representation is quasi-unipotent.*

The monodromy theorem ([PS08, 11.8, p. 259]) asserts that every polarized VMHS is in fact close to being unipotent. This crucial result was proven by Landman ([Lan73]) and Clemens ([Cle69]) for geometric variations of Hodge structure, and then by Borel in the general case:

Theorem C.3.16. *[Sch73, Lemma 4.5 and Thm 6.1] Let $\mathbb{V}_{\mathbb{Z}}$ be an integer polarized variation of Hodge structure over a complex manifold X, and let $\bar{X}$ be a good compactification of X with normal crossing divisor $D = \bar{X} - X$. Then the monodromy of $\mathbb{V}_{\mathbb{Z}}$ around each local component of D is quasi-unipotent.*

Admissible VMHS

The most important class of variations of mixed Hodge structures are those with geometric origin. Suppose that $\pi\colon X \longrightarrow S$ is a morphism of complex algebraic varieties and $k \in \mathbb{N}$. Then the sheaf $R^k\pi_*\underline{\mathbb{Z}}_X$ is constructible, i.e. there exists a stratification of S such that the restriction of $R^k\pi_*\underline{\mathbb{Z}}_X$ to each stratum is a local system. Moreover, the fibers of $R^k\pi_*\underline{\mathbb{Z}}_X$ at $s \in S$ are given by the cohomology $H^k(X_s)$ of the fibers $X_s = \pi^{-1}(s)$, which carry a mixed Hodge structure. Thus, $R^k\pi_*\underline{\mathbb{Z}}_X$ restricts to a variation of mixed Hodge structure on each stratum. Now S is irreducible, and hence there exists a Zariski-open dense subset $U \subset S$ such that $R^k\pi_*\underline{\mathbb{Z}}_X|_U$ is a variation of mixed

Hodge structure. A graded-polarizable variation of mixed Hodge structure arising in this fashion is called a *geometric VMHS*.

The geometric VMHS are subset of a class of particularly well-behaved VMHS: the admissible VMHS. In fact, all VMHS we will deal with are admissible. However, recalling the precise definition would lead too far here. For the definition of admissibility, see [PS08, 14.49, p.363]. The category of admissible variations of mixed R-Hodge structures will be denoted $\mathrm{VMHS}_R(X)^{\mathrm{ad}}$. The one thing of importance for this thesis is the following theorem:

Theorem C.3.17. *[SZ85, EZ86, Kas86] Geometric variations of mixed Hodge structure are admissible.*

What makes admissible variations of mixed Hodge structure so crucial is the following theorem:

Theorem C.3.18. *[SZ85, EZ] Let $\mathbb{V}$ be an admissible variation of mixed Hodge structure on U. Then for each k the vector space $\mathbb{H}^k(U, \mathbb{V})$ carries a canonical mixed Hodge structure.*

C.3.3 Mixed Hodge Modules

The setting of mixed Hodge modules is a generalization of the notion of VMHS that makes amends for one shortcoming: there is no six-functor formalism for VMHS. For example, given a VMHS $\mathbb{V}$ on an open subset $j: U \hookrightarrow X$ of a complex manifold X, then there might not be an extension of $\mathbb{V}$ to a VMHS $j_*\mathbb{V}$ on all of X.

Any admissible VMHS $(\mathbb{V}, F, W)$ on a smooth complex variety gives rise to a holonomic D-module by putting $\mathcal{V} := \mathbb{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$. Recall that any holonomic D-module locally gives rise to a good filtration, which was defined in section C.1 as follows: If $\pi: T^*X \longrightarrow X$ is the projection of the cotangent bundle down to X, and (M, F) a filtered D_X-module, then F is a *good filtration* of M if F_iM is coherent over $\mathcal{O}_X$ for each i and there exists $i_0 \gg 0$ satisfying

$$(F_jD_X)(F_iM) = F_{i+j}M \text{ for } j \geq 0, i \geq i_0,$$

or equivalently (by [HTT08, Definition 2.1.1, p. 58]) if $\mathrm{gr}^F M$ is coherent over $\pi_*\mathcal{O}_{T^*X}$. Note that the induced D_X-module $\mathcal{V}$ already comes with two naturally induced filtrations by putting $W_k\mathcal{V} := W_k\mathbb{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$ and $\mathcal{F}_p\mathcal{V} := F^{-p}\mathcal{V}$. Then one can show that the filtration $\mathcal{F}$ of $\mathcal{V}$ is in fact a good filtration in the above sense.

Let $A \subset \mathbb{C}$ be a subfield. A *smooth mixed A-Hodge module* associated to an admissible A-VMHS $(\mathbb{V}_A, F, W)$ on a smooth complex variety X is defined to be the tuple

$$((\mathcal{V}, \mathcal{F}, W), (\mathbb{V}_A[d_U], W), \alpha \colon (\mathbb{V}_{\mathbb{Q}}[d_U], W) \simeq \mathrm{DR}(\mathcal{V}, W)))$$

comprised of

- the D_X-module $\mathcal{V} := \mathbb{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_X$ together with its good filtration $\mathcal{F}_p\mathcal{V} := F^{-p}\mathcal{V}$,
- the perverse sheaf $(\mathbb{V}_A[d_U], W)$ with its induced weight filtration, and
- the natural comparison isomorphism

$$\alpha\colon (\mathbb{V}_{\mathbb{Q}}[d_U], W) \simeq \mathrm{DR}(\mathcal{V}, W))$$

  following from remark C.1.2.

These objects live in the category $MF_{rh}W(D_X, A)$ defined as follows (see section 8.3 of [HTT08]): For a smooth complex algebraic variety X, and a subfield $A \subset \mathbb{R}$ we denote by

- $MF_{rh}(D_X)$ the category of the pairs $(\mathcal{M}, F)$, where $\mathcal{M} \in \mathrm{Mod}_{rh}(D_X)$ and F is a good filtration of $\mathcal{M}$ (see section C.1.1);
- $MF_{rh}(D_X, A)$ the category of quadruplets $(\mathcal{M}, F, K, \alpha)$ consisting of
  - $(\mathcal{M}, F) \in MF_{rh}(D_X)$ and
  - a perverse sheaf $K \in \mathrm{Perv}(\underline{A}_X)$ over A
  - together with an isomorphism $\alpha\colon \mathbb{C} \otimes_A K \xrightarrow{\sim} DR(\mathcal{M})$;
- $MF_{rh}W(D_X, A)$ the category of tuples $(\mathcal{M}, F, K, \alpha, W)$ consisting of
  - $(\mathcal{M}, F, K, \alpha) \in MF_{rh}(D_X, A)$ and
  - a finite increasing filtration $\{W_n\}$ of $(\mathcal{M}, F, K, \alpha)$ in the category $MF_{rh}(D_X, A)$.

Recall the *Tate variation of mixed Hodge structure* $\underline{A}_X(m) = (\underline{A}_X, F, W)$ with the filtrations F is given by $F_k\mathcal{O}_X = 0$ for $k < -m$ and $F_k\mathcal{O}_X = \mathcal{O}_X$ for $k \geq -m$ and W given by $W_k\mathcal{O}_X = 0$ for $k < -2m$ and $W_k\mathcal{O}_X = \mathcal{O}_X$ for $k \geq -2m$.This VMHS gives rise to the so-called *Tate objects of weight m* $A_X(m) \in MF_{rh}W(D_X, A)$ under the above correspondence between VMHS and smooth mixed Hodge modules. Using these Tate objects, one defines an *m-th Tate twist* in $MF_{rh}W(D_X, A)$ by sending an object $\mathcal{M} \in MF_{rh}W(D_X, A)$ to $A_X(m) \otimes \mathcal{M}$, where the tensor product is taken componentwise.
The category of mixed Hodge modules is an abelian subcategory of $MF_{rh}W(D_X, A)$ containing all smooth mixed Hodge modules associated to admissible variations of mixed Hodge structure, which is constructed in [Sai90b]. We will not go into the construction here, but rather describe the axioms it was built to satisfy, as well as the properties of its six-functor formalism:
Saito constructed the *category of A-mixed Hodge modules*$\mathrm{MHM}_A(X)$ as an abelian subcategory of $MF_{rh}W(D_X, A)$ with derived category denoted by $D^b\,\mathrm{MHM}_A(X)$, and called elements in the category $\mathrm{MHM}_A(X)$ *mixed A-Hodge modules*inddefmixed A-Hodge modules.

As an element of $MF_{rh}W(D_X, A)$, a general A-mixed Hodge module on X can be described in terms of a tuple

$$((\mathcal{M}, \mathcal{F}, W), (\mathcal{K}_A, W), \alpha\colon (\mathcal{K}_A, W) \simeq \mathrm{DR}(\mathcal{V}, W)))$$

comprised of

- a regular holonomic D_X-module $\mathcal{M}$ together with a good filtration $\mathcal{F}_p$ and an ascending filtration W,
- a perverse sheaf $(\mathcal{K}_A, W)$ with an ascending weight filtration, and
- a natural comparison isomorphism

$$\alpha\colon (\mathcal{K}_A, W) \simeq \mathrm{DR}(\mathcal{V}, W)$$

  in the category $W\,\mathrm{Perv}_A(X)$ of filtered perverse A-sheaves on X.

All *Tate objects*

$$A_X(m) := (\mathcal{O}_X, F, \underline{A}_X(m), W) \in MF_{rh}W(D_X, A)$$

with F and W given by

$$F_k\mathcal{O}_X = \begin{cases} 0 & \text{for } k < m \\ \mathcal{O}_X & \text{for } k \geq m \end{cases} \qquad W_k\mathcal{O}_X = \begin{cases} 0 & \text{for } k < -2m \\ \mathcal{O}_X & \text{for } k \geq -2m \end{cases}$$

are mixed A-Hodge modules. More generally, an algebraic Hodge module on a smooth variety is called *smooth* if the underlying perverse sheaf is a local system up to a shift. We denote the category of smooth mixed Hodge modules over A on a complex scheme X by $\mathrm{MHM}_A(X)^s$. It is an abelian subcategory of $\mathrm{MHM}_A(X)$. Moreover, Saito proved ([Sai90b, Theorem 3.27]) that for a smooth and separated scheme X there is an equivalence

$$\mathrm{VMHS}_A^{\mathrm{ad}}(X) \xrightarrow{\sim} \mathrm{MHM}_A(X)^s$$

between the category of admissible variations of mixed A-Hodge structures and the category of smooth algebraic A-Hodge modules on X. In case $A = \mathbb{Q}$ one usually just talks about a *mixed Hodge module* and writes $\mathrm{MHM}(X)$ instead of $\mathrm{MHM}_{\mathbb{Q}}(X)$.

In [Sai90b, 14.1.1], Saito shows that the category $\mathrm{MHM}_A(X)$ satisfies the following properties, which allow for an axiomatic definition of mixed Hodge modules:

(m1) For each complex algebraic variety X there exists an abelian category

$$\mathrm{MHM}_A(X),$$

the *category of mixed Hodge modules* on X, with the following properties:

- There is a faithful functor $\mathrm{rat}_X\colon D^b\,\mathrm{MHM}_A(X) \longrightarrow D^b_{cs}(X;A)$, such that $\mathrm{MHM}_A(X)$ corresponds to $\mathrm{Perv}(\underline{A}_X)$. We say that $\mathrm{rat}_X M$ is the *underlying perverse A-sheaf* of M. Moreover, we say that $M \in \mathrm{MHM}_A(X)$ *is supported on* Z if $\mathrm{rat}_X M$ is supported on Z.
- There is a faithful functor $\mathrm{Dmod}_X\colon D^b\,\mathrm{MHM}_A(X) \longrightarrow D^b_{rh}(D_X)$. We say that $\mathrm{Dmod}_X(M)$ is the *underlying D_X-module*.

– We demand that the triangle

$$\begin{tikzcd} D^b\,\mathrm{MHM}_A(X) \arrow[rr, "\mathrm{rat}_X\otimes\mathbb{C}"] \arrow[dr, "\mathrm{Dmod}_X"'] & & D^b_{cs}(X) \\ & D^b_{rh}(D_X) \arrow[ur, "\mathrm{DR}_X"'] & \end{tikzcd}$$

is commutative up to isomorphism, i.e. for each mixed Hodge module M there is an isomorphism $\alpha\colon\ \mathrm{rat}_X(M)\otimes\mathbb{C}\xrightarrow{\sim}\mathrm{DR}_X(\mathrm{Dmod}_X(M))$. This isomorphism is called the *comparison isomorphism*.

(m2) The category of mixed Hodge modules supported on a point is the categroy of graded polarizable mixed A-Hodge structures; the functor "rat" associates to the mixed Hodge structure the underlying A-vector space.

(m3) Each object $M\in\mathrm{MHM}_A(X)$ admits a weight filtration W such that
- morphisms preserve the weight filtration strictly;
- the object $\mathrm{gr}^W_k M$ is semisimple in $\mathrm{MHM}_A(X)$;
- if X is a point, then the W-filtration is the usual weight filtration for the mixed Hodge structure.

We say that for $M^\bullet\in D^b\,\mathrm{MHM}_A(X)$ the *weight* satisfies weight$(M^\bullet)\le n$ (respectively $\ge n$) iff $\mathrm{gr}^W_i H^j(M^\bullet)=0$ for $i>j+n$ (resp. $i<j+n$). We say $M^\bullet$ is *pure of weight* n if it has weight $\ge n$ and weight $\le n$. We say a morphism *preserves weight*, if it neither decreases or increases weights.

(m4) The duality functor $\mathbf{D}_X$ of Verdier lifts to $\mathrm{MHM}_A(X)$ as an involution (i.e. $\mathbf{D}_X\circ\mathbf{D}_X=\mathrm{id}$), also denoted $\mathbf{D}_X$, in the sense that $\mathbf{D}_X\circ\mathrm{rat}_X=\mathrm{rat}_X\circ\mathbf{D}_X$.

(m5) For each morphism $f\colon X\longrightarrow Y$ between complex algebraic varieties, there are induced functors

$$\begin{aligned} f_*, f_! = \mathbf{D}_Y\circ f_*\circ\mathbf{D}_X &\colon D^b\,\mathrm{MHM}(X)\longrightarrow D^b\,\mathrm{MHM}(Y)\\ f^*, f^! = \mathbf{D}_X\circ f^*\circ\mathbf{D}_Y &\colon D^b\,\mathrm{MHM}(Y)\longrightarrow D^b\,\mathrm{MHM}(X)\end{aligned}$$

which lift the analogous functors on the level of constructible complexes, that is to say one has

$$\begin{aligned} f_*\circ\mathrm{rat} &= \mathrm{rat}\circ f_*,\\ f_!\circ\mathrm{rat} &= \mathrm{rat}\circ f_!,\\ f^*\circ\mathrm{rat} &= \mathrm{rat}\circ f^*,\\ f^!\circ\mathrm{rat} &= \mathrm{rat}\circ f^!.\end{aligned}$$

Moreover, if f is a projective morphism, then $f_*=f_!$.

(m6) The functors $f_!, f^*$ do not increase weights in the sense that if $M^\bullet$ has weight $\le n$, the same is true for $f_!M^\bullet$ and $f^*M^\bullet$.

(m7) The functors $f^!, f_*$ do not decrease weights in the sense that if $M^\bullet$ has weight $\ge n$, the same is true for $f^!M^\bullet$ and $f_*M^\bullet$.

(m8) For proper maps between complex algebraic varieties f_* preserves weights.

While all pure variations of Hodge structure of a fixed weight are contained in the category of Hodge modules, this is not the case for variation of mixed Hodge structure. In fact, one has the following:

(m9) Let $\mathcal{H} \in \mathrm{VMHS}_A(X)^p$. Then $\phi_X(\mathcal{H}) \in MF_{rh}W(D_X, A)$ is an object of $\mathrm{MHM}_a(X)$ if and only if $\mathcal{H}$ is *admissible*. In particular, all geometric variations of mixed Hodge structure yield mixed Hodge modules (see theorem C.3.17). Moreover, this implies in particular that $\mathrm{MHM}_A(pt) = \mathrm{MHS}_A^p$ (see (m3)), where MHS_A^p is the category of polarizable mixed A-Hodge structures.

Mixed Hodge modules satisfy functoriality properties exactly as in section C.1.2.

C.4 The mixed sheaf formalism

Sometimes, the derived category of motives is too rigid: It does not have a six-functor formalism, while all its realizations do and hence allow for more flexibility. In order to deal with this, one often resorts to working in the language of mixed sheaves, which somehow is an "intermediate step" between motives and their realizations. By talking about mixed sheaves, one is able to work both in the geometric and the ℓ-adic setting at the same time. Since these theories of sheaves allow for the same formalism, it is convenient not to specify any of these theories as a setting, but to apply the abstract formalism to an abstract "sheaf"-object, bearing in mind that the computations may then be transferred to any of the theories of sheaves (with weights) bearing this formalism. Beilinson explains what he means with a "mixed sheaf theory" in the outset of his paper [BL94] - it is one of the following:

a.) the theory of mixed Hodge modules for schemes of finite type over $\mathbb{R}$ or $\mathbb{C}$ due to Saito in [Sai90b] (the category $D^b\,\mathrm{MHM}_A(X)$ of mixed A-Hodge modules), or of D-modules,
b.) $\mathbb{Q}_l$-theory for schemes of finite type over $\mathbb{F}_l$ or $\mathbb{Q}$, constructed by Beilinson-Bernstein-Deligne in [BBD82, 5.1.5, pp.126ff.] (the category $D^b_m(X_0, \bar{\mathbb{Q}}_l)$ of mixed complexes of $\bar{\mathbb{Q}}_l$-sheaves with constructible cohomology), or
c.) the theory of "mixed systems of realizations" for schemes of finite type over $\mathbb{Q}$ as defined in Saito's paper [Sai]. We will not go into further details here.

Let us take a look at cases (i) and (ii), following Chapter 1 of [HW98]: In the mixed sheaf setting, we consider two symbols

$$A \text{ and } F,$$

where a priori A could be any ring and F any field. Furthermore we consider the category $\mathrm{Sch}_{\mathrm{Spec}(A)}$ of reduced, separated, flat schemes X of finite type over $\mathrm{Spec}\,A$, and a category $(\mathrm{Sch}_{\mathrm{Spec}(A)})_{\mathrm{top}}$ which is a symbol for a category of topological spaces together with a functor

$$\begin{aligned} (\ .\ )_{\mathrm{top}} \colon \mathrm{Sch}_{\mathrm{Spec}(A)} &\longrightarrow (\mathrm{Sch}_{\mathrm{Spec}(A)})_{\mathrm{top}} \\ X &\longmapsto X_{\mathrm{top}} \end{aligned}$$

To any such scheme $X \in \mathrm{Sch}_{\mathrm{Spec}(A)}$ we specify two categories

$$\mathrm{Sh}(X_{\mathrm{top}}), \quad \mathrm{Sh}(X)$$

with subcategories

$$\mathrm{Sh}^s(X_{\mathrm{top}}) \subset \mathrm{Sh}(X_{\mathrm{top}}), \quad \mathrm{Sh}^s(X) \subset \mathrm{Sh}(X)$$

such that there exist derived categories $D^b\,\mathrm{Sh}(X)$ and $D^b\,\mathrm{Sh}^s(X)$ with functors

$$\begin{aligned} \mathrm{For} \colon D^b\,\mathrm{Sh}(X) &\longrightarrow \mathrm{Sh}(X_{\mathrm{top}}), \\ \mathrm{For}^s \colon D^b\,\mathrm{Sh}^s(X) &\longrightarrow \mathrm{Sh}^s(X_{\mathrm{top}}). \end{aligned}$$

Moreover, we assume that for any morphism $f \colon X \longrightarrow Y$ in $\mathrm{Sch}(\mathrm{Spec}(A))$ and $f_{\mathrm{top}} \colon X_{\mathrm{top}} \longrightarrow Y_{\mathrm{top}}$ in $\mathrm{Sch}(\mathrm{Spec}(A))_{\mathrm{top}}$ there exist push-forward and pull-back functors

$$\begin{array}{l|l} f_* \colon D^b\,\mathrm{Sh}(X) \longrightarrow D^b\,\mathrm{Sh}(Y) & f_{\mathrm{top}\,*} \colon \mathrm{Sh}(X_{\mathrm{top}}) \longrightarrow \mathrm{Sh}(Y_{\mathrm{top}}) \\ f_! \colon D^b\,\mathrm{Sh}(X) \longrightarrow D^b\,\mathrm{Sh}(Y) & f_{\mathrm{top}!} \colon \mathrm{Sh}(X_{\mathrm{top}}) \longrightarrow \mathrm{Sh}(Y_{\mathrm{top}}) \\ f^* \colon D^b\,\mathrm{Sh}(Y) \longrightarrow D^b\,\mathrm{Sh}(X) & f^*_{\mathrm{top}} \colon \mathrm{Sh}(Y_{\mathrm{top}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{top}}) \\ f^! \colon D^b\,\mathrm{Sh}(Y) \longrightarrow D^b\,\mathrm{Sh}(X) & f^!_{\mathrm{top}} \colon \mathrm{Sh}(Y_{\mathrm{top}}) \longrightarrow \mathrm{Sh}(X_{\mathrm{top}}) \end{array}$$

$$\overset{\mathrm{For}}{\longleftrightarrow}$$

such that the left hand functors and the right hand functors correspond to each other under For, and such that all functors satisfy the usual six-functor formalism just like listed in section C.1.2. The most important facts of this six-functor-formalism are the following adjointness properties:

$$f^* \dashv f_*, \quad f^*_{\mathrm{top}} \dashv f_{\mathrm{top}\,*}, \quad f_! \dashv f^!, \quad f_{\mathrm{top}!} \dashv f^!_{\mathrm{top}}$$

For the remaining properties, see section C.1.2. When we speak of the *mixed sheaf setting*, we refer to a collection of symbols and data as above with the additional property that these symbols are either of the following cases (i) or (ii):

Case (i) : The geometric case

F	$=$ a subfield of $\mathbb{C}$
A	$= \mathbb{Q}$
X_{top}	$:= X(\mathbb{C})$
$\mathrm{Sh}(X_{\text{top}})$	$:= \mathrm{Perv}(X_{\text{top}}, A)$
$\mathrm{Sh}(X)$	$:= \mathrm{Mod}_{rh}(D_X)$
$D^b\,\mathrm{Sh}(X)$	$:= D^b_{rh}(X)$
$(\,.\,)_{\text{top}}$	$:= \mathrm{DR}_X$
$\mathrm{Sh}^s(X_{\text{top}})$	$:= \mathrm{Loc}(X)$
$\mathrm{Sh}^s(X)$	$:= \mathrm{Conn}_{rh}(X)$
$f_*, f^*,$ $f_!, f^!$	$:= \int_f, f^{\bigstar},$ $:= \int_{f!}, f^{\dagger}$ in D^b_{rh}
$f_{\text{top}\,*}, f^*_{\text{top}},$ $f_{\text{top}!}, f^!_{\text{top}}$	$:=$ the usual functors in $\mathrm{Perv}(X_{\text{top}}, A)$ (see section C.1.3)

Case (ii) : The ℓ-adic case

F	$= \mathbb{Z}\left[\frac{1}{l}\right]$
A	$= \mathbb{Q}_l$
X_{top}	$:= X \otimes_A \overline{\mathbb{Q}}$
$\mathrm{Sh}(X_{\text{top}})$	$:= \mathrm{Perv}(X_{\text{top}}, \mathbb{Q}_l)$
$\mathrm{Sh}(X)$	$:= \mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$
$D^b\,\mathrm{Sh}(X)$	$:= D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$
$(\,.\,)_{\text{top}}$	$:=$ the forgetful functor forgetting about the $(\mathbf{S}, L)$-stratification
$\mathrm{Sh}^s(X_{\text{top}})$	$:=$ the category of lisse $\mathbb{Q}_l$-sheaves on X_{top}
$\mathrm{Sh}^s(X)$	$:= \mathrm{Et}^{l,m}_{\mathbb{Q}_l}(X)$
$f_*, f^*,$ $f_!, f^!$	$:=$ the usual functors in $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ (see [Hub97])
$f_{\text{top}\,*}, f^*_{\text{top}},$ $f_{\text{top}!}, f^!_{\text{top}}$	$:=$ the usual functors in $\mathrm{Perv}(X_{\text{top}}, \mathbb{Q}_l)$ (see [BBD82].)

where

a.)
- $\mathrm{Perv}(X_{\text{top}}, A)$ denotes the category of perverse sheaves on X_{top},
- $\mathrm{Mod}_{rh}(X_D)$ is the category of regular holonomic D-modules on X,
- $\mathrm{Loc}(X)$ denotes the category of local systems,
- $\mathrm{Conn}(X)$ is the category of integrable connections on X

b.)
- $\mathrm{Perv}(X_{\text{top}}, \mathbb{Q}_l)$ is the category of ℓ-adic perverse sheaves on X_{top} (for details see [BBD82]).
- $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ is roughly defined as follows (for details, see [Hub97]): Let $(\mathbf{S}, L)$ be a fixed pair consisting of a horizontal stratification $\mathbf{S}$ of X (see section 2 of [Hub97]) and a collection $L = \{L(S) \mid S \in \mathbf{S}\}$, where each $L(S)$ is a set of irreducible pure lisse ℓ-adic sheaves on S. For all $S \in \mathbf{S}$ and $F \in L(S)$, it is required that for the inclusion $j\colon S \hookrightarrow X$, all higher direct images $R^n j_* F$ are $(\mathbf{S}, L)$-constructible, that is to say, when restricted to any $S \in \mathbf{S}$ they are lisse extensions of objects of $L(S)$. Denoting the derived category of ℓ-adic sheaves with constructible cohomology by $D^b_{cs}(X, \mathbb{Q}_l)$, $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ is its subcategory of complexes with $(\mathbf{S}, L)$-constructible cohomology objects.
- $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ is then defined as follows: The category $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ admits a perverse t-structure (for the notion of t-structures and their hearts, see section

8.1.1 of[HTT08], and for this particular t-structure see [Hub97]). Its heart is $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

- $\mathrm{Et}^{l,m}_{\mathbb{Q}_l}(X)$ is the category of lisse mixed $\mathbb{Q}_l$-sheaves on X.

Note that the six functor formalism of mixed sheaves satisfies the same properties as listed for the category of regular holonomic D-modules in section C.1.2 above.

We encountered the Hodge realization functor for Levine's motives in section C.2 above. There is also an ℓ-adic realization functor, which is defined as follows:

Theorem C.4.1. *[Lev98, V.2.2.9, p.272] Let S be the localization of a smooth scheme over a finite, local, global, or algebraically closed field, with l invertible on S, and let $p_X\colon X \longrightarrow S$ be an S-scheme. Then sending (X, q) to the ℓ-adic perverse sheaf $Rp_{X*}\mathbb{Z}_{\mathrm{et},X,l}(q)$ extends canonically to the exact $\underline{\ell\text{-adic realization functor}}$*

$$\mathfrak{R}_{\mathrm{et},l}\colon \mathcal{DM}(\mathrm{Sm}_S) \longrightarrow D^+ \lim \mathrm{Sh}^{\mathbb{Z}_l}_{\mathrm{et}}(S).$$

Remark C.4.2. [Lev98, V.2.2.11, p.272] Tensoring the ℓ-adic realization with $\mathbb{Q}_l$, one obtains a $\mathbb{Q}_l$-realization

$$\mathfrak{R}_{\mathbb{Q}_l,\mathrm{et}}\colon \mathcal{DM}(\mathrm{Sm}_S)_{\mathbb{Q}} \longrightarrow D^+ \lim \mathrm{Sh}^{\mathbb{Z}_l}_{\mathrm{et}}(S)_{\mathbb{Q}}$$

Remark C.4.3. Including the realization functors into the mixed language, put

$$\mathfrak{R}_{A,\mathrm{mix}} := \left\{ \begin{matrix} \mathfrak{R}_{A,\mathcal{H}dg} & \text{in case (i)} \\ \mathfrak{R}_{A,\mathrm{et},S} & \text{in case (ii)} \end{matrix} \right\} : \mathcal{DM}_A(\mathrm{Sm}_S) \longrightarrow D^b(\mathrm{Sh}(S))$$

and call $\mathfrak{R}_{A,\mathrm{mix}}$ the *mixed realization functor*.

Appendix D

Various

D.1 Double complexes

The following theory of double complexes is well-known, and can be found in [varb], for example.

Definition D.1.1. *Let $\mathcal{A}$ be an additive category. A double complex in $\mathcal{A}$ is given by an element of $C^\bullet(C^\bullet(\mathcal{A}))$, where $C^\bullet(\ .\ )$ denotes the category of chain complexes. In other words, a double complex is given by an array*

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "\delta^{p-2,q+1}"] & A^{p-1,q+1} \arrow[u, "\partial^{p-1,q+1}"] \arrow[r, "\delta^{p-1,q+1}"] & A^{p,q+1} \arrow[u, "\partial^{p,q+1}"] \arrow[r, "\delta^{p,q+1}"] & A^{p+1,q+1} \arrow[u, "\partial^{p+1,q+1}"] \arrow[r, "\delta^{p+1,q+1}"] & \cdots \\
\cdots \arrow[r, "\delta^{p-2,q}"] & A^{p-1,q} \arrow[u, "\partial^{p-1,q}"] \arrow[r, "\delta^{p-1,q}"] & A^{p,q} \arrow[u, "\partial^{p,q}"] \arrow[r, "\delta^{p,q}"] & A^{p+1,q} \arrow[u, "\partial^{p+1,q}"] \arrow[r, "\delta^{p+1,q}"] & \cdots \\
\cdots \arrow[r, "\delta^{p-2,q-1}"] & A^{p-1,q-1} \arrow[u, "\partial^{p-1,q-1}"] \arrow[r, "\delta^{p-1,q-1}"] & A^{p,q-1} \arrow[u, "\partial^{p,q-1}"] \arrow[r, "\delta^{p,q-1}"] & A^{p+1,q-1} \arrow[u, "\partial^{p+1,q-1}"] \arrow[r, "\delta^{p+1,q-1}"] & \cdots \\
 & \vdots \arrow[u, "\partial^{p-1,q-2}"] & \vdots \arrow[u, "\partial^{p,q-2}"] & \vdots \arrow[u, "\partial^{p+1,q-2}"] &
\end{tikzcd}$$

of objects $A^{p,q} \in \mathcal{A}$ indexed by $(p, q) \in \mathbb{Z}$, such that one has

- *$\delta^2 = 0$, so all horizontal rows are complexes,*
- *$\partial^2 = 0$, so all vertical columns are complexes and*

• $\partial \circ \delta - \delta \circ \partial = 0$, *which means that all squares commute.*

Definition D.1.2. *Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex as in the above definition. The associated simple complex $sA^{\bullet}$, also sometimes called the associated total complex* $\mathrm{Tot}\, A^{\bullet}$ *is given by*

$$\mathrm{Tot}\, A^n = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d^n_{\mathrm{Tot}\, A} = \sum\nolimits_{n=p+q} (\delta^{p,q} + (-1)^p \partial^{p,q}).$$

Recall that if $K^{\bullet,\bullet}$ is a double complex in an additive category $\mathcal{A}$, then there are two spectral sequences associated to $K^{\bullet,\bullet}$. They have the following terms:

a.) $'E_0^{p,q} = K^{p,q}$ with $'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \longrightarrow K^{p,q+1}$,
b.) $''E_0^{p,q} = K^{q,p}$ with $''d_0^{p,q} = d_1^{q,p} : K^{q,p} \longrightarrow K^{q+1,p}$,
c.) $'E_1^{p,q} = H^q(K^{p,\bullet})$ with $'d_1^{p,q} = H^q(d_1^{p,\bullet})$,
d.) $''E_1^{p,q} = H^q(K^{\bullet,p})$ with $''d_1^{p,q} = (-1)^q H^q(d_2^{\bullet,p})$,

If the spectral sequences converge, any of the above abuts to $H^{p+q}(\mathrm{Tot}\, K^{\bullet,\bullet})$.

Remark D.1.3. In the literature one encounters a different definition where a "double complex" has the property that the squares in the diagram anti-commute, i.e. $\partial \circ \delta - \delta \circ \partial = 0$ in definition D.1.1 is replaced by $\partial \circ \delta + \delta \circ \partial = 0$. In this context, the total complex of a double complex as above is given by

$$\mathrm{Tot}\, A^n = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d^n_{\mathrm{Tot}\, A} = \sum\nolimits_{n=p+q} (\delta^{p,q} + \partial^{p,q}).$$

Note that both definitions of double complexes are equivalent: one makes a double complex $A^{\bullet,\bullet}$ with commutative squares into a double complex with anticommutative squares by using the same differential δ but taking $\partial^- \colon A^{p,q} \longrightarrow A^{p,q+1}$ to be $\partial^- := (-1)^p \partial$. The same trick can, of course, be used to make a double complex with anticommutative squares into a double complex with commutative squares. Since the signs are cancelled out by the different definitions of the total complexes in both situations, it is obvious that the cohomology of the total complex coincide both in the commutative and anticommutative version.

D.2 Categories of inductive systems

The main reference for the following section is chapter 1.4 of the book [Mey07].
Let $\mathcal{C}$ be a category, and $A = (A_i, \beta_{ij})_{i\in I}$ and $B = (B_k, \beta_{kl})_{k\in K}$ be two inductive systems in $\mathcal{C}$ with indexing sets I and K. Then the set of homomorphisms $\mathrm{Hom}_{\varinjlim \mathcal{C}}(A, B)$ of inductive systems is defined as the set of natural transformations between the associated functors

$$\varinjlim \mathrm{Hom}(\,.\,, A_i) \longrightarrow \varinjlim \mathrm{Hom}(\,.\,, B_k).$$

One defines $\varinjlim \mathcal{C}$ to be the category whose objects are inductive systems in $\mathcal{C}$ and whose morphisms are given by $\mathrm{Hom}_{\varinjlim \mathcal{C}}(A, B)$.
Explicitly, we have

$$\mathrm{Hom}_{\varinjlim \mathcal{C}}(A, B) \cong \varprojlim_i \varinjlim_k \mathrm{Hom}(A_i, B_k).$$

Hence, any morphism is represented by a family of maps $f_i \colon A_i \to B_{k(i)}$ for a function $k \colon I \to K$, such that for any $i, j \in I$ with $i \leq j$ there is a $k \in K$ with $k \geq k(i), k(j)$ for which the diagram

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[d, "\alpha_{ij}"'] & B_{k(i)} \arrow[dr, bend left, "\beta_{k(i)k}"] & \\
A_j \arrow[r, "f_j"'] & B_{k(j)} \arrow[r, "\beta_{k(j)k}"'] & B_k
\end{tikzcd}$$

Properties of the category of inductive systems:

a.) If $\mathcal{C}$ has finite coproducts, then the category of inductive systems in $\mathcal{C}$ has (possibly infinite) coproducts by [Mey07, 1.4.1, p.55].

b.) If $(\mathcal{C}, \otimes, 1)$ is a symmetric monoidal category, then $\varinjlim \mathcal{C}$ inherits a symmetric monoidal category structure, by defining

$$(A_i)_{i\in I} \otimes (B_k)_{k\in K} := (A_i \otimes B_k)_{i,k\in I\times K}.$$

By [Mey07, 1.136, p.57], this tensor product turns $\varinjlim \mathcal{C}$ into a symmetric monoidal category whose unit is the constant inductive system 1.

Appendix E

Calculations

E.1 An isomorphic definition of the classical bar complex

Let k be a field, $R^\bullet$ be a differential graded k-algebra (the most common case is $R^\bullet = k$), and $A = \bigoplus_{p\geq 0} A^p$ a differential graded k-algebra with differential $d\colon A^k \longrightarrow A^{k+1}$ which is a differential graded R-module. Moreover, suppose $R^\bullet$ admits the structure of a differential graded $A^\bullet$-bimodule via two morphism of differential graded algebras $x, y\colon A^\bullet \longrightarrow R^\bullet$, where left-multiplication is given by x, and right-multiplication by y. However, note that everything in this section is also valid in the sheaf setting.

We use this to prove the following about the columns of the bar double complex:

Proposition E.1.1. *There is a natural isomorphism*

$$\begin{aligned} \psi_r\colon (R \otimes (A[-1])^{\otimes r})[r] &\longrightarrow A^{\otimes r} \\ b \otimes [a_1|\dots|a_r] &\longmapsto (-1)^{\mu(b,a_1,\dots,a_{r-1})} b \otimes [a_1|\dots|a_r]. \end{aligned}$$

where $\mu(b, a_1, \dots, a_{r-1}) = r \cdot b + \sum_{k=1}^{r-1} (r-k)\cdot|a_k|$.

What is left to do in order to find an isomorphism of the bar double complex to an "easier" double complex is to translate the differentials:

Lemma E.1.2. *Via the isomorphism of Proposition 1.1.7, the morphism of differential graded algebras* $\delta'_r\colon R \otimes (A[-1])^{\otimes r}[r] \longrightarrow R \otimes (A[-1])^{\otimes r-1}[r-1]$ *corresponds to the morphism*

$$\begin{aligned} A^{\otimes r} &\longrightarrow A^{\otimes r-1}, \\ [a_1|\dots|a_r] &\longmapsto -[x(a_1)\cdot a_2|\dots|a_r] + \sum_{i=1}^{r-1} (-1)^{i+1} [a_1|\dots|a_i a_{i+1}|\dots|a_r] \\ &\qquad + (-1)^{r+1}[a_1|\dots|a_{r-1}\cdot y(a_r)]. \end{aligned}$$

Proof. The inverse of the isomorphism ψ_r of Proposition is given by

$$\psi_r^{-1}\colon A^{\otimes r} \longrightarrow R\otimes (A[-1])^{\otimes r}[r];\ [a_1|\dots|a_r] \longmapsto (-1)^{\mu(1,a_1,\dots,a_{r-1})} 1 \otimes [a_1|\dots|a_r],$$

where as above $\mu(1, a_1, \ldots, a_{r-1}) = \sum_{k=1}^{r-1}(r-k)\cdot|a_k|$. The diagram

$$\begin{tikzcd} R\otimes(A[-1])^{\otimes r}[r] \arrow[r, "\delta_{r-1}"] & R\otimes(A[-1])^{\otimes r-1}[r-1] \arrow[d, "\psi_{r-1}"] \\ A^{\otimes r} \arrow[u, "\psi_r^{-1}"] \arrow[r, "\delta'_{r-1}"] & A^{\otimes r-1} \end{tikzcd}$$

therefore yields

$$\begin{aligned}
&\delta_{r-1}([a_1|\ldots|a_r]) = \\
&= \psi_{r-1}\circ\delta'_{r-1}\circ\psi_r^{-1}([a_1|\ldots|a_r]) \\
&= (-1)^{\mu(1,a_1,\ldots,a_{r-1})}\psi_{r-1}\circ\delta'_{r-1}(1\otimes[a_1|\ldots|a_r]) \\
&= (-1)^{\mu(1,a_1,\ldots,a_{r-1})}\Bigg((-1)^{1+\mu(x(a_1),a_2,\ldots,a_{r-1})}[x(a_1)\cdot a_2|\ldots|a_r] \\
&\quad + \sum_{i=1}^{r-1}(-1)^{\mu(1,a_1,\ldots,a_ia_{i+1},\ldots,a_{r-1})+\sum_{k=1}^{i}|a_k|+i+1}[a_1|\ldots|a_ia_{i+1}|\ldots|a_r] \\
&\quad + (-1)^{\mu(y(a_r),a_1,\ldots,a_{r-2})+\left(\sum_{k=1}^{r-1}|a_k|+r-1\right)(|a_r|+1)}[y(a_r)\cdot a_1|\ldots|a_{r-1}]\Bigg).
\end{aligned}$$

Now we use the following calculations:

- $\mu(1, a_1, \ldots, a_{r-1}) + \mu(x(a_1), a_2, a_3 \ldots, a_{r-1}) \equiv 0 \mod 2$ since

$$\begin{aligned}
\mu(x(a_1), a_2, a_3 \ldots, a_{r-1}) &= (r-1)|x(a_1)| + \sum_{k=2}^{r-1}(r-1-(k-1))|a_k| \\
&= (r-1)|a_1| + \sum_{k=2}^{r-1}(r-k)|a_k| \\
&= \mu(1, a_1, \ldots, a_{r-1}).
\end{aligned}$$

- $\mu(1, a_1, \ldots, a_{r-1}) + \mu(1, a_1, \ldots, a_ia_{i+1}, \ldots, a_{r-1}) \equiv \sum_{k=1}^{i}|a_k| \mod 2$, since

$$\begin{aligned}
\mu(1, a_1, \ldots, a_ia_{i+1}, \ldots, a_{r-1}) &= \sum_{k=1}^{i-1}(r-1-k)\cdot|a_k| + (r-1-i)|a_ia_{i+1}| \\
&\quad + \sum_{k=i+2}^{r-1}(r-1-(k-1))\cdot|a_k| \\
&= \mu(1, a_1, \ldots, a_{r-1}) - \sum_{k=1}^{i}|a_k|.
\end{aligned}$$

-

$$\mu(1, a_1, \ldots, a_{r-1}) + \mu(y(a_r), a_1, \ldots, a_{r-2}) \equiv (r-1)|a_r| - \sum_{k=1}^{r-2} |a_k| - |a_{r-1}|$$
$$\equiv (r-1)|a_r| + \sum_{k=1}^{r-1} |a_k| \mod 2 \text{ since}$$

$$\begin{aligned}
\mu(y(a_r), a_1, a_2 \ldots, a_{r-2}) &= (r-1)|y(a_r)| + \sum_{k=1}^{r-2}(r-1-k)|a_k| \\
&= (r-1)|a_r| + \sum_{k=1}^{r-2}(r-1-k)|a_k| \\
&= (r-1)|a_r| + \sum_{k=1}^{r-2}(r-k)|a_k| - \sum_{k=1}^{r-2}|a_k| \\
&= (r-1)|a_r| - \sum_{k=1}^{r-2}|a_k| - |a_{r-1}| \\
&\quad +\mu(1, a_1, \ldots, a_{r-1}).
\end{aligned}$$

With this, we obtain:

$$\begin{aligned}
\delta_{r-1}([a_1|\ldots|a_r]) &= -[x(a_1)\cdot a_2|\ldots|a_r] \\
&\quad + \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\ldots|a_{i-1}|a_i a_{i+1}|a_{i+2}\ldots|a_r] \\
&\quad + (-1)^{\left(\sum_{k=1}^{r-1}|a_k|\right)(|a_r|+2)+(r-1)(2|a_r|+1)}[y(a_r)\cdot a_1|\ldots|a_{r-1}] \\
&= -[x(a_1)\cdot a_2|\ldots|a_r] \\
&\quad + \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\ldots|a_{i-1}|a_i a_{i+1}|a_{i+2}\ldots|a_r] \\
&\quad + (-1)^{\left(\sum_{k=1}^{r-1}|a_k|\right)|a_r|+r-1}[y(a_r)\cdot a_1|\ldots|a_{r-1}] \\
&= -[x(a_1)\cdot a_2|\ldots|a_r] \\
&\quad + \sum_{i=1}^{r-1}(-1)^{i+1}[a_1|\ldots|a_{i-1}|a_i a_{i+1}|a_{i+2}\ldots|a_r] \\
&\quad + (-1)^{r-1}[a_1|\ldots|a_{r-1}\cdot y(a_r)]
\end{aligned}$$

just as asserted. ■

E.2 Classical simplicial bar object

Let k be a field, and $R^\bullet$ a differential graded k-algebra with unit. We denote the category of unital differential graded k-algebras by dga_k. Let us fix a dga $A^\bullet = \bigoplus_{p\geq 0} A^p$

with differential $\partial\colon A^k \longrightarrow A^{k+1}$, which has the structure of a differential graded $R^\bullet$-module. By [vara], the category $\mathrm{Mod}_{(R^\bullet,\mathrm{d})}$ of differential graded $R^\bullet$-modules is an abelian category which has arbitrary limits and colimits. At the same time, we suppose $R^\bullet$ is endowed with the structure of a differential graded A-bimodule by virtue of two augmentations $x, y\colon A^\bullet \longrightarrow R^\bullet$.

Let all notation for degrees be as in section 1.1.1. Recall that we denote the n-fold tensor product of $A^\bullet$ with itself over $R^\bullet$ by

$$A^{\otimes n} := A^\bullet \otimes_{R^\bullet} \otimes \ldots \otimes_{R^\bullet} A^\bullet$$

and write $[a_1|\ldots|a_n] := a_1 \otimes \ldots \otimes a_n$ for an element in $A^{\otimes n}$. We denote by $A[1]$ the differential graded k-algebra where all grades are reduced by one. Correspondingly, we have an induced differential given by

$$\begin{aligned} d\colon (A[1])^{\otimes n} &\longrightarrow (A[1])^{\otimes n} \\ [a_1|\ldots|a_n] &\longmapsto \sum_{i=1}^{n} (-1)^{|a_1|+\ldots+|a_{i-1}|-(i-1)} [a_1|\ldots|da_i|\ldots|a_n] \end{aligned}$$

Recall that by section 1.1.4, the (unreduced) bar complex is naturally isomorphic to the total complex of the double complex

$$\longrightarrow A^{\otimes r} \ldots \xrightarrow{\delta_r \delta_2(x,y)} A^{\otimes 2} \xrightarrow{\delta_1(x,y)} A \xrightarrow{y-x} R \longrightarrow 0$$

with $\delta_k(x,y)$ given by

$$\begin{aligned} \delta_{k-1}(x,y)\colon A^{\otimes k} &\longrightarrow A^{\otimes k-1} \\ [a_1|\ldots|a_k] &\longmapsto -[x(a_1)\cdot a_2|\ldots|a_k] + \sum_{i=1}^{k-1}(-1)^{i+1}[a_1|\ldots|a_{i-1}|a_i a_{i+1}|a_{i+2}\ldots|a_k] \\ &\qquad +(-1)^{k+1}[a_1|\ldots|a_{k-1}\cdot y(a_k)]. \end{aligned}$$

We now consider the following assignment:

$$\begin{aligned} &sB^\bullet(A^\bullet|R^\bullet)_{x,y}\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{(R^\bullet,d)} \\ &[n] \longmapsto A^{\otimes n},\ d^j \longmapsto (d^n_j\colon A^{\otimes n+1} \longrightarrow A^{\otimes n}),\ s^j \longmapsto (s^n_j\colon A^{\otimes n} \longrightarrow A^{\otimes n+1}) \end{aligned}$$

where the tensor product is taken over R, the "face" maps d^n_j are given by

$$d^n_j([a_1|\ldots|a_{n+1}]) = \begin{cases} -[a_1|\ldots|a_j a_{j+1}|\ldots|a_{n+1}] & \text{for } j \in \{1,\ldots,n\} \\ -[x(a_1)a_2|\ldots|a_{n+1}] & \text{for } j = 0 \\ -[a_1|\ldots|a_n y(a_{n+1})] & \text{for } j = n+1 \end{cases}$$

and the "degeneracy" maps s^n_j are given by

$$s^n_j([a_1|\ldots|a_{n+1}]) = -[a_1|\ldots|a_j|1|a_{j+1}|\ldots|a_{n+1}]$$

for $j = 0,\ldots,n$, where 1 is the element 1 of $k \subset A^0 \subset A^\bullet$.

Proposition E.2.1. *d_j^n and s_j^n are morphisms of complexes of differential graded R-modules for all n, j.*

Proof. It is obvious that the above maps are all $R^\bullet$-linear and linear, and are compatible with degrees. So all that is left to show is that maps d_j^n, s_j^n commute with the differentials.

(i) The face maps d_j^n for $0 < j < n+1$:

$$\begin{aligned}
-d \circ d_j^n([a_1|\dots|a_{n+1}]) &= d\left(([a_1|\dots|a_j a_{j+1}|\dots|a_{n+1}]\right) \\
&= \sum_{i=1}^{j-1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_j a_{j+1}|\dots|a_{n+1}] \\
&\quad + \sum_{i=j+2}^{n+1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|a_j a_{j+1}|\dots|da_i|\dots|a_{n+1}] \\
&\quad +(-1)^{|a_1|+\dots+|a_{j-1}|}[a_1|\dots|d(a_j a_{j+1})|\dots|a_{n+1}] \\
-d_j^n \circ d([a_1|\dots|a_{n+1}]) &= d_j^n\Big(\sum_{i=1}^{n+1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_{n+1}]\Big) \\
&= \sum_{i=1}^{j-1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_j a_{j+1}|\dots|a_{n+1}] \\
&\quad + \sum_{i=j+2}^{n+1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|a_j a_{j+1}|\dots|da_i|\dots|a_{n+1}] \\
&\quad +(-1)^{|a_1|+\dots+|a_{j-1}|}[a_1|\dots|da_j a_{j+1}|\dots|a_{n+1}] \\
&\quad +(-1)^{|a_1|+\dots+|a_j|}[a_1|\dots|a_j da_{j+1}|\dots|a_{n+1}] \\
&= -d \circ d_j^n([a_1|\dots|a_{n+1}])
\end{aligned}$$

(ii) The face map d_0^n:

$$\begin{aligned}
-d \circ d_0^n([a_1|\dots|a_{n+1}]) = &= d\left([x(a_1)a_2|\dots|a_{n+1}]\right) = [d(x(a_1))a_2|\dots|a_{n+1}] \\
&\quad +(-1)^{|a_1|}\sum_{i=2}^{n+1}(-1)^{|a_2|+\dots+|a_{i-1}|}[x(a_1)a_2|\dots|da_i|\dots|a_{n+1}] \\
&= [x(d(a_1))a_2|\dots|a_{n+1}] \\
&\quad + \sum_{i=2}^{n+1}(-1)^{|a_1|+|a_2|+\dots+|a_{i-1}|}[x(a_1)a_2|\dots|da_i|\dots|a_{n+1}]\Big) \\
-d_0^n \circ d([a_1|\dots|a_{n+1}]) = &= d_0^n\left(\sum_{i=1}^{n+1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_{n+1}]\right) \\
&= [x(d(a_1))a_2|\dots|a_{n+1}] \\
&\quad + \sum_{i=2}^{n+1}(-1)^{|a_1|+\dots+|a_{i-1}|}[x(a_1)a_2|\dots|da_i|\dots|a_{n+1}]
\end{aligned}$$

(iii) The face map d_{n+1}^n:

$$
\begin{aligned}
-d \circ d^n_{n+1}([a_1|\dots|a_{n+1}]) &= d\Big((-1)^{|a_1|+\dots+|a_n|+1}\cdot[a_1|\dots|a_n y(a_{n+1})]\Big)\\
&= (-1)^{|a_1|+\dots+|a_n|}[a_1|\dots|a_n y(da_{n+1})])\\
&\quad+\sum_{i=1}^{n}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_n y(a_{n+1})]\\
-d^n_{n+1}\circ d([a_1|\dots|a_{n+1}]) &= d^n_{n+1}\left(\sum_{i=1}^{n+1}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_{n+1}]\right)\\
&= \sum_{i=1}^{n}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_n y(a_{n+1})]\\
&\quad+(-1)^{|a_1|+\dots+|a_n|}[a_1|\dots|a_n y(da_{n+1})]\\
&= \sum_{i=1}^{n}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_n y(a_{n+1})]\\
&\quad+(-1)^{|a_1|+\dots+|a_n|}[a_1|\dots|a_n dy(a_{n+1})]\\
&= -d\circ d^n_{n+1}([a_1|\dots|a_{n+1}])
\end{aligned}
$$

(iv) The degeneracy maps s^n_j:

$$
\begin{aligned}
-d\circ s^n_j([a_1|\dots|a_n]) &= = d\left([a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]\right)\\
&= \sum_{i=1}^{j}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_j|1|\dots|a_n]\\
&\quad+\sum_{i=j+1}^{n}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|a_j|1|\dots|da_i|\dots|a_n]\\
-s^n_j\circ d([a_1|\dots|a_n]) &= = s^n_j\left(\sum_{i=1}^{n}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_n]\right)\\
&= \sum_{i=1}^{j}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|da_i|\dots|a_j|1|\dots|a_n]\\
&\quad+\sum_{i=j+1}^{n}(-1)^{|a_1|+\dots+|a_{i-1}|}[a_1|\dots|a_j|1|\dots|da_i|\dots|a_n].
\end{aligned}
$$

This finishes the proof of the assertions.

■

In what follows, we will often drop the upper index n in the notation of d^n_j and s^n_j when it is obvious what they should be.

Lemma E.2.2. *$sB^\bullet(A^\bullet)_{x,y}$ is a functor, i.e. comprises a simplicial object in* dga_k.

Proof. We need to show that the simplicial identities are satisfied.

(1) *Claim:* $d^{n-1}_i d^n_j = d_{j-1}d_i\colon A^{\otimes n+1}\longrightarrow A^{\otimes n-1}$ for $i<j$

- *The case* $i\neq 0, j\neq n+1$:

$$
\begin{aligned}
d_i^{n-1}d_j^n([a_1|\dots|a_{n+1}]) &= d_i^{n-1}(-[a_1|\dots|a_ja_{j+1}|\dots|a_{n+1}]) \\
&= [a_1|\dots|a_ia_{i+1}|\dots|a_ja_{j+1}|\dots|a_{n+1}] \\
d_{j-1}^{n-1}d_i^n([a_1|\dots|a_{n-1}]) &= d_{j-1}^{n-1}(-[a_1|\dots|a_ia_{i+1}|\dots|a_{n+1}]) \\
&= [a_1|\dots|a_ia_{i+1}|\dots|a_ja_{j+1}|\dots|a_{n+1}]
\end{aligned}
$$

which proves the assertion for $i \neq 0$, $j \neq n+1$.

– *The case* $i = 0, j \neq 1, n+1$:

$$
\begin{aligned}
d_0^{n-1}d_j^n([a_1|\dots|a_{n+1}]) &= d_0^{n-1}(-[a_1|\dots|a_ja_{j+1}|\dots|a_{n+1}]) \\
&= [x(a_1)a_2|\dots|a_ja_{j+1}|\dots|a_{n+1}]) \\
d_{j-1}^{n-1}d_0^n([a_1|\dots|a_{n+1}]) &= d_{j-1}^{n-1}(-[x(a_1)a_2|\dots|a_ja_{j+1}|\dots|a_{n+1}]) \\
&= [x(a_1)a_2|\dots|a_ja_{j+1}|\dots|a_{n+1}]).
\end{aligned}
$$

– *The case* $i = 0, j = 1$:

$$
\begin{aligned}
d_0^{n-1}d_1^n([a_1|\dots|a_{n+1}]) &= d_0(-[a_1a_2|a_3|\dots|a_{n+1}]) = [x(a_1a_2)a_3|\dots|a_{n+1}] \\
d_0^{n-1}d_0^n([a_1|\dots|a_{n+1}]) &= d_0^{n-1}(-[x(a_1)a_2|a_3|\dots|a_{n+1}]) \\
&= [x(a_1)x(a_2)a_3|\dots|a_{n+1}]) = [x(a_1a_2)a_3|\dots|a_{n+1}]).
\end{aligned}
$$

– *The case* $i = 0, j = n+1$:

$$
\begin{aligned}
d_0^{n-1}d_{n+1}^n([a_1|\dots|a_{n+1}]) &= d_0^{n-1}(-[a_1|\dots|a_ny(a_{n+1})]) \\
&= [x(a_1)a_2|\dots|a_ny(a_{n+1})] \\
d_n^{n-1}d_0^n([a_1|\dots|a_{n+1}]) &= d_n^{n-1}(-[x(a_1)a_2|\dots|a_{n+1}]) \\
&= [x(a_1)a_2|\dots|a_ny(a_{n+1})]
\end{aligned}
$$

– *The case* $i \neq 0, n, j = n+1$:

$$
\begin{aligned}
d_i^{n-1}d_{n+1}^n([a_1|\dots|a_{n+1}]) &= d_i^{n-1}(-[a_1|\dots|a_ny(a_{n+1})]) \\
&= [a_1|\dots|a_ia_{i+1}|\dots|a_ny(a_{n+1})] \\
d_n^{n-1}d_i^n([a_1|\dots|a_{n+1}]) &= d_n^{n-1}(-[a_1|\dots|a_ia_{i+1}|\dots|a_{n+1}]) \\
&= [a_1|\dots|a_ia_{i+1}|\dots|a_ny(a_{n+1})].
\end{aligned}
$$

– *The case* $i = 0, j = n+1$:

$$
\begin{aligned}
d_0^{n-1}d_{n+1}^n([a_1|\dots|a_{n+1}]) &= d_0^{n-1}(-[a_1|\dots|a_ny(a_{n+1})]) \\
&= [x(a_1)a_2|\dots|a_ny(a_{n+1})] \\
d_n^{n-1}d_0^n([a_1|\dots|a_{n+1}]) &= d_n^{n-1}(-[x(a_1)a_2|\dots|a_{n+1}]) \\
&= \cdot[x(a_1)a_2|\dots|a_ny(a_{n+1})].
\end{aligned}
$$

– *The case* $i = n, j = n+1$:

$$\begin{aligned}
d_n^{n-1}d_{n+1}^n([a_1|\dots|a_{n+1}]) &= d_n^{n-1}(-[a_1|\dots|a_n y(a_{n+1})]) \\
&= [a_1|\dots|a_{n-1}y(a_n)y(a_{n+1})] \\
&= [a_1|\dots|a_{n-1}y(a_n a_{n+1})] \\
d_n^{n-1}d_n^n([a_1|\dots|a_{n+1}]) &= d_n^{n-1}(-[a_1|\dots|a_{n-1}|a_n a_{n+1}]) \\
&= [a_1|\dots|a_{n-1}y(a_n a_{n+1})]).
\end{aligned}$$

(2) *Claim:* $d_j^n s_j^n = d_{j+1}^n s_j^n = 1\colon A^{\otimes n} \longrightarrow A^{\otimes n}$ for all $j = 0, \dots, n$

– *The case* $j \neq 0, n$:

$$\begin{aligned}
d_j^n s_j^n([a_1|\dots|a_n]) &= d_j^n(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\
&= [a_1|\dots|a_j \cdot 1|a_{j+1}|\dots|a_n] = [a_1|\dots|a_n] \\
d_{j+1}^n s_j^n([a_1|\dots|a_n]) &= d_{j+1}^n(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\
&= [a_1|\dots|a_j|1 \cdot a_{j+1}|\dots|a_n] = [a_1|\dots|a_n].
\end{aligned}$$

So $d_j^n s_j^n = 1 = d_{j+1}^n s_j^n$.

– *The case* $j = 0$:

$$\begin{aligned}
d_0^n s_0^n([a_1|\dots|a_n]) &= d_0^n(-[1|a_1|\dots|a_n]) = [a_1|\dots|a_n] \\
d_1^n s_0^n([a_1|\dots|a_n]) &= d_1^n(-[1|a_1|\dots|a_n]) = [a_1|\dots|a_n]
\end{aligned}$$

since $x(1) = 1$, so $d_0^n s_0^n = 1 = d_1^n s_0^n$.

– The case $j = n$:

$$\begin{aligned}
d_n^n s_n^n([a_1|\dots|a_n]) &= d_n^n(-[a_1|\dots|a_n|1]) = [a_1|\dots|a_n \cdot 1]) \\
&= [a_1|\dots|a_n] \\
d_{n+1}^n s_n^n([a_1|\dots|a_n]) &= d_{n+1}^n(-[a_1|\dots|a_n|1]) \\
&= [a_1|\dots|a_n \cdot y(1)]) = [a_1|\dots|a_n]
\end{aligned}$$

since $y(1) = 1$ and hence $d_n^n s_n^n = 1 = d_{n+1}^n s_n^n$, which finishes the proof of (2).

(3) *Claim:* $d_i^n s_j^n = s_{j-1}^{n-1} d_i^{n-1}\colon A^{\otimes n} \longrightarrow A^{\otimes n}$ for $i < j$

– *The case* $i \neq 0$:

$$\begin{aligned}
d_i^n s_j^n([a_1|\dots|a_n]) &= d_i^n(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\
&= [a_1|\dots|a_i a_{i+1}|\dots|a_j|1|a_{j+1}|\dots|a_n] \\
s_{j-1}^{n-1} d_i^{n-1}([a_1|\dots|a_n]) &= s_{j-1}^{n-1}(-[a_1|\dots|a_i a_{i+1}|\dots|a_n]) \\
&= [a_1|\dots|a_i a_{i+1}|\dots|a_j|1|a_{j+1}|\dots|a_n]
\end{aligned}$$

so both sides agree as asserted.

– *The case* $i = 0$:

$$\begin{aligned}
d_0^n s_j^n([a_1|\dots|a_n]) &= d_0^n(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\
&= [x(a_1)a_2|\dots|a_j|1|a_{j+1}|\dots|a_n] \\
s_{j-1}^{n-1} d_0^{n-1}([a_1|\dots|a_n]) &= s_{j-1}^{n-1}(-[x(a_1)a_2|\dots|a_n]) \\
&= [x(a_1)a_2|\dots|a_j|1|a_{j+1}|\dots|a_n].
\end{aligned}$$

(4) *Claim:* $d_i^n s_j^n = s_j^{n-1} d_{i-1}^{n-1} \colon A^{\otimes n} \longrightarrow A^{\otimes n}$ for $i > j+1$

– *The case* $i \neq n+1$:

$$\begin{aligned} d_i^n s_j^n([a_1|\dots|a_n]) &= d_i^n(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\ &= [a_1|\dots|a_j|1|a_{j+1}|\dots|a_{i-1}a_i|\dots|a_n] \\ s_j^{n-1} d_{i-1}^{n-1}([a_1|\dots|a_n]) &= s_j^{n-1}(-[a_1|\dots|a_{i-1}a_i|\dots|a_n]) \\ &= [a_1|\dots|a_j|1|\dots|a_{i-1}a_i|\dots|a_n] \end{aligned}$$

so both terms agree.

– *The case* $i = n+1$:

$$\begin{aligned} d_{n+1}^n s_j^n([a_1|\dots|a_n]) &= d_{n+1}^n(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\ &= [a_1|\dots|a_j|1|a_{j+1}|\dots|a_{n-1}y(a_n)] \\ s_j^{n-1} d_n^{n-1}([a_1|\dots|a_n]) &= s_j^{n-1}(-[a_1|\dots|a_{n-1}y(a_n)]) \\ &= [a_1|\dots|a_j|1|a_{j+1}|\dots|a_{n-1}y(a_n)] \end{aligned}$$

which proves the assertion in this case and finishes the proof of (4).

(5) *Claim:* $s_i s_j = s_{j+1} s_i \colon A^{\otimes n} \longrightarrow A^{\otimes n+2}$ for $i \leq j$

$$\begin{aligned} s_i s_j([a_1|\dots|a_n]) &= s_i(-[a_1|\dots|a_j|1|a_{j+1}|\dots|a_n]) \\ &= [a_1|\dots|a_i|1|a_{i+1}|\dots|a_j|1|a_{j+1}|\dots|a_n] \\ s_{j+1} s_i([a_1|\dots|a_n]) &= s_{j+1}(-[a_1|\dots|a_i|1|a_{i+1}|\dots|a_n]) \\ &= [a_1|\dots|a_i|1|a_{i+1}|\dots|a_j|1|a_{j+1}|\dots|a_n]. \end{aligned}$$

This finishes the proof of the lemma. ■

E.3 Motivic simplicial bar object

Recall the category $C^b(\mathcal{K}(\mathrm{Sm}_S))$ of bounded complexes of elements in the pseudo-abelian envelope of Sm_S.

Let $\pi \colon X \longrightarrow S$ be in Sm_S be equipped with two sections $x, y \colon S \longrightarrow X$. We consider the functor

$$\begin{aligned} &cB^\bullet_{\mathrm{mot}}(X|S)_{x,y} \colon \mathbf{\Delta} \longrightarrow C^b(\mathcal{K}(\mathrm{Sm}_S)) \\ &[n] \longmapsto X^n,\ \delta^j_{n+1} \longmapsto d^j_{n+1} \colon X^n \to X^{n+1},\ \sigma^j_n \longmapsto s^j_n \colon X^{n+1} \to X^n \end{aligned}$$

where the maps d^j_{n+1} and s^j_n are given by

$$d^j_{n+1} := -\begin{cases} \mathrm{id}^{\times j-1} \times \Delta \times \mathrm{id}^{\times n-j} & \text{for } j \in \{1,\dots,n\} \\ x \times \mathrm{id}^{\times n} & \text{for } j = 0 \\ \mathrm{id}^{\times n} \times y \colon X^n \to X^{n+1} & \text{for } j = n+1 \end{cases}$$

$$s^j_n := -\,\mathrm{id}^{\times j} \times \pi \otimes \mathrm{id}^{\times n-j} \ \text{ for } j = 0,\dots,n$$

Lemma E.3.1. *The functor $cB^{\bullet}_{\mathrm{mot}}(X|S)_{x,y}$ is a cosimplicial object in $C^b(\mathcal{K}(\mathrm{Sm}_S))$.*

Proof. We need to show that the cosimplicial identities are satisfied. Note that all negative signs cancel out in the cosimplicial identities of $cB^{\bullet}_{\mathrm{mot}}(X|S)_{x,y}$. Moreover, to simplify things, we put

$$\alpha^i_{n+1} := \left\{ \begin{array}{ll} \Delta & \text{for } i \in \{1, \dots, n\} \\ x \times \mathrm{id} & \text{for } i = 0 \\ \mathrm{id} \times y & \text{for } i = n+1 \end{array} \right\} : X \longrightarrow X^2$$

and let $\mathrm{id}^0 = \mathrm{id}^{-1} := \mathrm{id}_S \colon S \longrightarrow S$

(1) Claim: $d^j_{n+1} d^i_n = d^i_{n+1} d^{j-1}_n \colon X^{n-1} \longrightarrow X^{n+1}$ for $i < j$

- The case $i < j-1$: One has

$$\begin{aligned} d^j_{n+1} d^i_n &= \left(\mathrm{id}^{j-1} \times \alpha^j_{n+1} \times \mathrm{id}^{n-j}\right) \circ \left(\mathrm{id}^{i-1} \times \alpha^i_n \times \mathrm{id}^{n-i-1}\right) \\ &= \mathrm{id}^{i-1} \times \alpha^i_{n+1} \times \mathrm{id}^{j-i-2} \times \alpha^j_{n+1} \times \mathrm{id}^{n-j} \\ d^i_{n+1} d^{j-1}_n &= \left(\mathrm{id}^{i-1} \times \alpha^i_{n+1} \times \mathrm{id}^{n-i}\right) \circ \left(\mathrm{id}^{j-2} \times \alpha^{j-1}_n \times \mathrm{id}^{n-j}\right) \\ &= \mathrm{id}^{i-1} \times \alpha^i_{n+1} \times \mathrm{id}^{j-i-2} \times \alpha^{j-1}_n \times \mathrm{id}^{n-j-2} \end{aligned}$$

  so the assertion follows iff $\alpha^{j-1}_n = \alpha^j_{n+1}$ for all $j \in \{1, \dots, n+1\}$ and $\alpha^i_n = \alpha^i_{n+1}$ for all $i \in \{0, \dots, n-1\}$. Both of these are immediate by the definition of the α^i.
- The case $j = i+1$ for $i = 0, \dots, n$: One has

$$\begin{aligned} d^{i+1}_{n+1} d^i_n &= \left(\mathrm{id}^i \times \alpha^{i+1}_{n+1} \times \mathrm{id}^{n-i-1}\right) \circ \left(\mathrm{id}^{i-1} \times \alpha^i_n \times \mathrm{id}^{n-i-1}\right) \\ d^i_{n+1} d^i_n &= \left(\mathrm{id}^{i-1} \times \alpha^i_{n+1} \times \mathrm{id}^{n-i}\right) \circ \left(\mathrm{id}^{i-1} \times \alpha^i_n \times \mathrm{id}^{n-i-1}\right) \end{aligned}$$

  For $i \neq 0, n$, all α occurring in the above terms are the diagonal morphism Δ, and by virtue of $(\mathrm{id} \times \Delta) \circ \Delta = (\Delta \times \mathrm{id}) \circ \Delta$ the assertion follows for $i \neq 0, n$. For $i = 0$, the asserted equation reads

$$((\Delta \times \mathrm{id}) \circ (x \times \mathrm{id})) \times \mathrm{id}^{n-2} = (x \times \mathrm{id} \times \mathrm{id}) \circ (x \times \mathrm{id})) \times \mathrm{id}^{n-2}$$

  which is also satisfied since both sides are equal to $x \times x \times \mathrm{id}^{n-1}$.
  For $i = n$, the asserted equation is equivalent to

$$(\mathrm{id}^{n-1} \times \mathrm{id} \times y) \circ (\mathrm{id}^{n-2} \times \mathrm{id} \times y) = (\mathrm{id}^{n-1} \times \Delta) \circ (\mathrm{id}^{n-2} \times \mathrm{id} \times y)$$

  which holds since both sides are equal to $\mathrm{id}^{n-1} \times y \times y$.

(2) Claim: $s^j_n d^j_{n+1} = s^j_n d^{j+1}_{n+1} = \mathrm{id} \colon X^n \longrightarrow X^n$ for all $j = 0, \dots, n$

$$s^j_n d^j_{n+1} = \left(\mathrm{id}^j \times \pi \otimes \mathrm{id}^{n-j}\right) \circ \left(\mathrm{id}^{j-1} \times \alpha^j_{n+1} \times \mathrm{id}^{n-j}\right) = \mathrm{id}^{j-1} \times \left((\mathrm{id} \times \pi) \circ \alpha^j_{n+1}\right) \times \mathrm{id}^{n-j}$$

so the assertion follows iff $(\mathrm{id} \times \pi) \circ \alpha^j_{n+1} = \mathrm{id}$ for all $j = 0, \dots, n$. For $j = 0$ one has $\pi \times \mathrm{id} \circ (x \times \mathrm{id}) = \mathrm{id}$ since x is a section of π, while for $j = 1, \dots, n$ one has $(\mathrm{id} \times \pi) \circ \Delta = \mathrm{id}$, which proves that $s^j_n d^j_{n+1} = \mathrm{id}$. On the other hand, one has

$$s_n^j d_{n+1}^{j+1} = \left(\mathrm{id}^j \times \pi \otimes \mathrm{id}^{n-j}\right) \circ \left(\mathrm{id}^j \times \alpha_{n+1}^{j+1} \times \mathrm{id}^{n-j-1}\right) = \mathrm{id}^j \times \left((\pi \times \mathrm{id}) \circ \alpha_{n+1}^{j+1}\right) \times \mathrm{id}^{n-j-1}$$

so the assertion follows iff $(\pi \times \mathrm{id}) \circ \alpha_{n+1}^{j+1} = \mathrm{id}$ for all $j = 0, \ldots, n-1$ and $(\mathrm{id} \times \pi) \circ (\mathrm{id} \times y) = \mathrm{id}$ for $j = n$. For $j = n$ one has $(\mathrm{id} \times \pi) \circ (\mathrm{id} \times y) = \mathrm{id}$ since y is a section of π, while for $j = 0, \ldots, n-1$ one has $(\pi \times \mathrm{id}) \circ \Delta = \mathrm{id}$, which proves that $s_n^j d_{n+1}^j = \mathrm{id}$.

(3) Claim: $s_n^j d_{n+1}^i = d_n^i s_{n-1}^{j-1} \colon X^n \longrightarrow X^n$ for $i < j$

- The case $i < j-1$: One has

$$\begin{aligned} s_n^j d_{n+1}^i &= \left(\mathrm{id}^j \times \pi \times \mathrm{id}^{n-j}\right) \circ \left(\mathrm{id}^{i-1} \times \alpha_{n+1}^i \times \mathrm{id}^{n-i-1}\right) \\ &= \mathrm{id}^{i-1} \times \alpha_{n+1}^i \times \mathrm{id}^{j-i-1} \times \pi \times \mathrm{id}^{n-j+1} \\ d_n^i s_{n-1}^{j-1} &= \left(\mathrm{id}^{i-1} \times \alpha_n^i \times \mathrm{id}^{n-i-2}\right) \circ \left(\mathrm{id}^{j-1} \times \pi \times \mathrm{id}^{n-j}\right) \\ &= \mathrm{id}^{i-1} \times \alpha_n^i \times \mathrm{id}^{j-i-1} \times \pi \times \mathrm{id}^{n-j+1} \end{aligned}$$

  so the claim follows iff $\alpha_n^i = \alpha_{n+1}^i$. Since we are only dealing with the cases $i = 0, \ldots, n-1$ here, this is a direct consequence of the definition of the α^i.
- The case $j = i+1$ for $i = 0, \ldots, n-1$ One has

$$\begin{aligned} s_n^{i+1} d_{n+1}^i &= \left(\mathrm{id}^{i+1} \times \pi \times \mathrm{id}^{n-i-1}\right) \circ \left(\mathrm{id}^{i-1} \times \alpha_{n+1}^i \times \mathrm{id}^{n-i}\right) \\ d_n^i s_{n-1}^i &= \left(\mathrm{id}^{i-1} \times \alpha_n^i \times \mathrm{id}^{n-i-1}\right) \circ \left(\mathrm{id}^i \times \pi \times \mathrm{id}^{n-i-1}\right) \end{aligned}$$

  Hence, for $i \neq 0, n-1$ ($\alpha_n^i = \alpha_{n+1}^i = \Delta$ in this case), the asserted equality reads

$$\mathrm{id}^{i-1} \times \Delta \times \pi \times \mathrm{id}^{n-i-1} = \mathrm{id}^{i-1} \times (\Delta \circ (\mathrm{id} \times \pi)) \times \mathrm{id}^{n-i-1}$$

  which holds since $\Delta \circ (\mathrm{id} \times \pi) = \Delta \times \pi$. For $i = 0$ the asserted equality is

$$((\mathrm{id} \times \pi) \circ (x \times \mathrm{id})) \times \mathrm{id}^{n-1} = ((x \times \mathrm{id}) \circ (\pi \times \mathrm{id})) \times \mathrm{id}^{n-2}$$

  which holds since $((\mathrm{id} \times \pi) \circ (x \times \mathrm{id})) \times \mathrm{id} = x \times \pi \times \mathrm{id} = (x \times \mathrm{id}) \circ (\pi \times \mathrm{id})$.

(4) *Claim:* $s_n^j d_{n+1}^i = d_n^{i-1} s_{n-1}^j \colon X^n \to X^n$ for $i > j+1$

$$s_n^j d_{n+1}^i = \left(\mathrm{id}^j \times \pi \times \mathrm{id}^{n-j}\right) \circ \left(\mathrm{id}^{i-1} \times \alpha_{n+1}^i \times \mathrm{id}^{n-i}\right) = \mathrm{id}^j \times \pi \times \mathrm{id}^{i-1-j} \times \alpha_{n+1}^i \times \mathrm{id}^{n-1-i}$$

since $j+1 \leq i-1$ and hence $i-1-j \geq 1$, while

$$d_n^{i-1} s_{n-1}^j = \left(\mathrm{id}^{i-2} \times \alpha_n^i \times \mathrm{id}^{n-i}\right) \circ \left(\mathrm{id}^j \times \pi \times \mathrm{id}^{n-j-1}\right)$$

For $j+1 < i-1$, the latter term is obviously equal to $\mathrm{id}^j \times \pi \times \mathrm{id}^{i-1-j} \times \alpha_{n+1}^i \times \mathrm{id}^{n-1-i}$, so the assertion holds in this case. For $j+1 = i-1$, the latter term is equal to

$$\mathrm{id}^{i-2} \times (\alpha_n^i \circ (\pi \times \mathrm{id})) \times \mathrm{id}^{n-i} = \mathrm{id}^{i-2} \times \pi \times \alpha_n^i \times \mathrm{id}^{n-i}$$

which proves the assertion.

(5) *Claim:* $s^j_{n-1}s^i_n = s^i_{n-1}s^{j+1}_n \colon X^{n+1} \to X^{n-1}$ for $i < j$.

$$\begin{aligned} s^j_{n-1}s^i_n &= \left(\mathrm{id}^j \times\pi \times \mathrm{id}^{n-j-1}\right) \circ \left(\mathrm{id}^i \times\pi \times \mathrm{id}^{n-i}\right) = \mathrm{id}^i \times\pi \times \mathrm{id}^{j-i} \times\pi \times \mathrm{id}^{n-j-1}, \text{ while} \\ s^i_{n-1}s^{j+1}_n &= \left(\mathrm{id}^i \times\pi \times \mathrm{id}^{n-i-1}\right) \circ \left(\mathrm{id}^{j+1} \times\pi \times \mathrm{id}^{n-j-1}\right) = \mathrm{id}^i \times\pi \times \mathrm{id}^{j+1-(i+1)} \times\pi \times \mathrm{id}^{n-j-1} \\ &= \mathrm{id}^i \times\pi \times \mathrm{id}^{j-i} \times\pi \times \mathrm{id}^{n-j-1} \end{aligned}$$

so the assertion is immediate.

Since all cosimplicial identities are satisfied, the claim follows. ■

E.4 Simplicial augmentation ideals

Proposition E.4.1. *Put $\mathrm{id}^{-1} = \mathrm{id}^0 = \mathrm{id}_S$. Then the following is a cosimplicial object in the Karoubi envelope $\mathcal{K}(\mathbb{Z}(\mathrm{Sm}_S))$:*

$$\begin{aligned} cI^*_{\mathrm{mot}}(X|S)_{x_0} \colon \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{K}(\mathbb{Z}(\mathrm{Sm}_S) \\ [n] &\longmapsto \widetilde{X^{n+1}} := \left(X^{n+1}, \mathrm{id}^n \times (\mathrm{id} - x_0\pi)\right) \\ (d^j \colon [n] \to [n+1]) &\longmapsto (d^n_j(I) \colon \widetilde{X^{n+1}} \to \widetilde{X^{n+2}}, \\ (s^j \colon [n+1] \to [n]) &\longmapsto (s^n_j(I) \colon \widetilde{X^{n+2}} \to \widetilde{X^{n+1}}) \end{aligned}$$

$$d^n_j(I) := \begin{cases} x_0 \times \mathrm{id}^{n+1} & \textit{for } j = 0 \\ \mathrm{id}^{j-1} \times \Delta \times \mathrm{id}^{\times n-j+1} & \textit{for } j \in \{1, \dots, n\} \\ \mathrm{id}^{\times n-1} \times (\Delta \circ (\mathrm{id} - x_0\pi)) & \textit{for } j = n+1 \end{cases}$$

$$s^{n+1}_j(I) := \mathrm{id}^j \times\pi \times \mathrm{id}^{n-j+1} \quad \textit{for } j = 0, \dots, n.$$

Proof. It is easy to see that the face and degeneracy maps are in fact compatible with the idempotents in question, so it suffices to show that the simplicial identities are satisfied. Due to the compatibility with the idempotents, we will drop them in the computation to come. To further simplify things, put

$$\alpha^{(n)}_j := \left\{ \begin{array}{ll} x_0 \times \mathrm{id} & \text{for } j = 0 \\ \Delta & \text{for } j = 1, \dots, n \\ \Delta \circ (\mathrm{id} - x_0\pi) & \text{for } j = n+1 \end{array} \right\} : \begin{cases} X \longrightarrow X^2, \text{ resp.} \\ (X, \mathrm{id} - x_0\pi) \longrightarrow (X^2, \mathrm{id} \times (\mathrm{id} - x_0\pi)) \end{cases}$$

and put $\mathrm{id}^{-1} = \mathrm{id}^0 = \mathrm{id}_S$, such that one may write $d^n_j = \mathrm{id}^j \times \alpha^{(n)}_j \times \mathrm{id}^{n-j}$.

(1) Claim: $d^n_j d^{n-1}_i = d^n_i d^{n-1}_{j-1} \colon X^n \longrightarrow X^{n+2}$ for $i < j$

– The case $i \neq j-1$:

$$\begin{aligned} d^n_j d^{n-1}_i &= (\mathrm{id}^j \times\alpha^{(n)}_j \times \mathrm{id}^{n-j}) \circ (\mathrm{id}^i \times\alpha^{(n-1)}_i \times \mathrm{id}^{n-1-i}) \\ &= \mathrm{id}^i \times\alpha^{(n-1)}_i \times \mathrm{id}^{j-i-2} \times\alpha^{(n)}_j \times \mathrm{id}^{n-j}) \\ d^n_i d^{n-1}_{j-1} &= (\mathrm{id}^i \times\alpha^{(n)}_i \times \mathrm{id}^{n-i}) \circ (\mathrm{id}^{j-1} \times\alpha^{(n-1)}_{j-1} \times \mathrm{id}^{n-j}) \\ &= \mathrm{id}^i \times\alpha^{(n)}_i \times \mathrm{id}^{j-i-2} \times\alpha^{(n-1)}_j \times \mathrm{id}^{n-j}) \end{aligned}$$

So the assertion boils down to the fact that $\alpha_i^{(n)} = \alpha_i^{(n-1)}$ for all $i = 0, \ldots, n-1$ and $\alpha_j^{(n)} = \alpha_{j-1}^{(n-1)}$ for all $j = 2, \ldots, n+1$.

– The case $i = j - 1$:

$$\begin{aligned} d_j^n d_{j-1}^{n-1} &= (\mathrm{id}^j \times \alpha_j^{(n)} \times \mathrm{id}^{n-j}) \circ (\mathrm{id}^{j-1} \times \alpha_{j-1}^{(n-1)} \times \mathrm{id}^{n-j}) \\ &= \mathrm{id}^{j-1} \times (\alpha_{j-1}^{(n-1)}(\mathrm{id} \times \alpha_j^{(n)})) \times \mathrm{id}^{n-j} \\ d_{j-1}^n d_{j-1}^{n-1} &= (\mathrm{id}^{j-1} \times \alpha_{j-1}^{(n)} \times \mathrm{id}^{n-j+1}) \circ (\mathrm{id}^{j-1} \times \alpha_{j-1}^{(n-1)} \times \mathrm{id}^{n-j}) \\ &= \mathrm{id}^{j-1} \times ((\alpha_{j-1}^{(n-1)} \times \mathrm{id})\alpha_{j-1}^{(n-1)}) \times \mathrm{id}^{n-j}) \end{aligned}$$

Hence the assertion holds iff $(\mathrm{id} \times \alpha_j^{(n)})\alpha_{j-1}^{(n-1)} = (\alpha_{j-1}^{(n-1)} \times \mathrm{id})\alpha_{j-1}^{(n-1)}$ for all $j = 1, \ldots, n+1$.

· For $j = 1$, this holds since $(\Delta \times \mathrm{id}) \circ (x_0 \times \mathrm{id}) = x_0 \times x_0 \times \mathrm{id}$ while $(x_0 \times \mathrm{id} \times \mathrm{id}) \circ (x_0 \times \mathrm{id}) = x_0 \times x_0 \times \mathrm{id}$

· For $j = 2, \ldots, n$, this holds since $(\mathrm{id} \times \Delta) \circ \Delta = \Delta_{123} = (\Delta \times \mathrm{id}) \circ \Delta$, where $\Delta_{123} \colon X \longrightarrow X^3$ is the morphism $x \longmapsto (x, x, x)$.

· For $j = n + 1$, this holds since

$$\begin{aligned} &(\mathrm{id} \times (\Delta \circ (\mathrm{id} - x_0\pi))) \circ \Delta \circ (\mathrm{id} - x_0\pi) = \Delta_{123}(\mathrm{id} - x_0\pi) \\ &= (\Delta \times \mathrm{id}) \circ \Delta \circ (\mathrm{id} - x_0\pi) \end{aligned}$$

where Δ_{123} is as above.

(2) Claim: $s_j^{n+1} d_j^n = s_j^{n+1} d_{j+1}^n = \mathrm{id} \colon X^{n+1} \longrightarrow X^{n+1}$ for all $j = 1, \ldots, n+1$

$$\begin{aligned} s_j^{n+1} d_j^n &= (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \circ (\mathrm{id}^j \times \alpha_j^{(n)} \times \mathrm{id}^{n-j}) \\ &= \mathrm{id}^j \times ((\mathrm{id} \times \pi)\alpha_j^{(n)})\, \mathrm{id}^{n-j-1} = \mathrm{id} \end{aligned}$$

since $(\mathrm{id} \times \pi) \circ \alpha_j^{(n)} = \mathrm{id}$ for all $j = 0, \ldots, n$, and

$$\begin{aligned} s_j^{n+1} d_{j+1}^n &= (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \circ (\mathrm{id}^{j+1} \times \alpha_{j+1}^{(n)} \times \mathrm{id}^{n-j-1}) \\ &= \mathrm{id}^{j+1} \times ((\mathrm{id} \times \pi) \circ \alpha_{j+1}^{(n)}) \times \mathrm{id}^{n-j-2} = \mathrm{id} \end{aligned}$$

since $(\pi \times \mathrm{id}) \circ \alpha_{j+1}^{(n)} = \mathrm{id}$ for all $j = 0, \ldots, n-1$ and $(\mathrm{id} \times \pi) \circ \alpha_{n+1}^{(n)} = \mathrm{id}$.

(3) Claim: $s_j^{n+1} d_i^n = d_i^{n-1} s_{j-1}^n \colon X^{n+1} \longrightarrow X^{n+1}$ for $i < j$

$$\begin{aligned} s_j^{n+1} d_i^n &= (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \circ (\mathrm{id}^i \times \alpha_i^{(n)} \times \mathrm{id}^{n-i}) \\ &= \mathrm{id}^i \times \alpha_i^{(n)} \times \mathrm{id}^{j-i} \times \pi \times \mathrm{id}^{n-j-1} = \mathrm{id} \\ d_i^{n-1} s_j^n &= (\mathrm{id}^i \times \alpha_i^{(n-1)} \times \mathrm{id}^{n-i-1}) \circ (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \\ &= \mathrm{id}^i \times \alpha_i^{(n-1)} \times \mathrm{id}^{j-i} \times \pi \times \mathrm{id}^{n-j-1}) \end{aligned}$$

so the claim holds.

(4) Claim: $s_j^{n+1} d_i^n = d_{i-1}^{n-1} s_j^n \colon X^{n+1} \longrightarrow X^{n+1}$ for $i > j + 1$

– For $i > j+2$, one has

$$\begin{aligned} s_j^{n+1} d_i^n &= (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \circ (\mathrm{id}^i \times \alpha_i^{(n)} \times \mathrm{id}^{n-i}) \\ &= \mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{i-j-2} \times \alpha_i^{(n)} \times \mathrm{id}^{n-i} \\ d_{i-1}^{n-1} s_j^n &= (\mathrm{id}^{i-1} \times \alpha_{i-1}^{(n-1)} \times \mathrm{id}^{n-i}) \circ (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \\ &= \mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{i-j-2} \times \alpha_{i-1}^{(n-1)} \times \mathrm{id}^{n-i} \end{aligned}$$

so both terms are equal since for all i, n one has $\alpha_i^{(n)} = \alpha_{i-1}^{(n-1)}$.

– For $i = j+2$, one has

$$\begin{aligned} s_j^{n+1} d_{j+2}^n &= (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \circ (\mathrm{id}^{j+2} \times \alpha_{j+2}^{(n)} \times \mathrm{id}^{n-j-2}) \\ &= \mathrm{id}^{j+1} \times \pi \times \alpha_{j+2}^{(n)} \times \mathrm{id}^{n-j-2} \\ d_{j+1}^{n-1} s_j^n &= (\mathrm{id}^{j+1} \times \alpha_{j+1}^{(n-1)} \times \mathrm{id}^{n-j-2}) \circ (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j-1}) \\ &= \mathrm{id}^{j+1} \times \pi \times \alpha_{j+1}^{(n-1)} \times \mathrm{id}^{n-i} \end{aligned}$$

so both terms are equal since for all i, n one has $\alpha_i^{(n)} = \alpha_{i-1}^{(n-1)}$.

(5) *Claim:* $s_j^{n+1} s_i^{n+2} = s_i^{n+1} s_{j+1}^{n+2} \colon X^{n+1} \longrightarrow X^{n+3}$ for $i \leq j$

$$\begin{aligned} s_j^{n+1} s_i^{n+2} &= (\mathrm{id}^{j+1} \times \pi \times \mathrm{id}^{n-j}) \circ (\mathrm{id}^{i+1} \times \pi \times \mathrm{id}^{n-i+1}) \\ &= \mathrm{id}^{i+1} \times \pi \times \mathrm{id}^{j-i} \times \pi \times \mathrm{id}^{n-j} \\ s_i^{n+1} s_{j+1}^{n+2} &= (\mathrm{id}^{i+1} \times \pi \times \mathrm{id}^{n-j}) \circ (\mathrm{id}^{j+2} \times \pi \times \mathrm{id}^{n-j}) \\ &= \mathrm{id}^{i+1} \times \pi \times \mathrm{id}^{j-i} \times \pi \times \mathrm{id}^{n-j} \end{aligned}$$

which finishes the proof of the proposition. ∎

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