Surfaces of minimal complexity in low-dimensional topology

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der Fakultät für Mathematik
der Universität Regensburg.

vorgelegt von **Matthias Nagel**
aus Stuttgart im Jahr 2015.

Promotionsgesuch eingereicht am 20.05.2015.

Die Arbeit wurde angeleitet von Prof. Stefan Friedl.

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1 Introduction

Let us consider a knot $K \subset S^3$, i.e. a smooth embedded 1-manifold diffeomorphic to the circle S^1. We can choose an open tubular neighbourhood $\nu(K)$ of K and look at the complement $N(K) := S^3 \setminus \nu(K)$, which is a 3-manifold with a torus as boundary. Recall that every knot admits a Seifert surface, which is an embedded oriented connected surface bounded by the knot. There are many Seifert surfaces for a given knot and we define the genus $g(K)$ of a knot K to be the minimal genus among all Seifert surfaces. The genus of a knot K exerts a subtle influence on the topology of the knot. For example, if K bounds a disc, so $g(K) = 0$, then K is the unknot.

We will review the considerations above in order to make sense of them in a general 3-manifold. By Alexander duality the homology group $\mathrm{H}_2(N(K), \partial N(K); \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Furthermore, the fundamental class of a Seifert surface is a generator of the above group. Conversely, given any embedded oriented surface whose fundamental class is a generator, then we can transform the surface to a Seifert surface without increasing its genus. Therefore, asking about the genus of a knot K is the same as the following representability question: What is the minimal genus of all the oriented embedded surfaces whose fundamental class is a given class, say a generator of $\mathrm{H}_2(N(K), \partial N(K); \mathbb{Z})$?

In a general 3-manifold M there is no preferred class in $\mathrm{H}_2(M, \partial M; \mathbb{Z})$ so we consider every class in $\mathrm{H}_2(M, \partial M; \mathbb{Z})$. For technical reasons we use the following invariant to measure the complexity of a surface Σ instead of the genus $g(\Sigma)$. Denote the components of Σ by Σ_i. Furthermore, we denote the Euler characteristic of the component Σ_i by $\chi(\Sigma_i)$. Then we associate to Σ the complexity $\chi_-(\Sigma) := \sum_i \max(0, -\chi(\Sigma_i))$. Thurston [Thu86] considered for each class $\sigma \in \mathrm{H}_2(M, \partial M; \mathbb{Z})$ the number

$$\|\sigma\|_T := \min\{\chi_-(\Sigma) : \Sigma \text{ surface representing } \sigma\}$$

and noticed that this function defines a semi-norm on $\mathrm{H}_2(M, \partial M; \mathbb{Z})$, i.e. the relations $\|k\sigma\|_T = |k|\|\sigma\|_T$ and $\|\alpha + \beta\|_T \leq \|\alpha\|_T + \|\beta\|_T$ hold for $k \in \mathbb{Z}$ and homology classes $\alpha, \beta \in \mathrm{H}_2(M, \partial M; \mathbb{Z})$. Further details are explained in Chapter 3.

Let us have a closer look at the inequality $\|k\sigma\|_T \leq |k|\|\sigma\|_T$ in an orientable 3-manifold M. This inequality can be obtained by considering push-offs. Let Σ be a surface which represents its fundamental class while having minimal $\chi_-(\Sigma)$. The normal bundle of Σ will be trivial and thus a neighbourhood of Σ is diffeomorphic to $\Sigma \times [0,1]$ mapping $\Sigma \times \{0\}$ to Σ. Now consider k parallel copies of Σ in $\Sigma \times [0,1]$ and obtain a surface in M representing $k[\Sigma]$ with complexity $k\chi_-(\Sigma)$.

One would like to find an effective way to compute the Thurston norm or at least find good lower bounds. A method for this will be described in Section 1.2.

In contrast to the Thurston norm in 3-manifolds the related invariant in 4-manifolds, which we describe in the next section, is much more mysterious. For example the inequality above will not hold in general.

1.1 Circle bundles over 3-manifolds

Consider a closed 4-manifold W. Completely analogously to the 3-dimensional case, we assign to every class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ the number

$$x(\sigma) := \min\{\chi_-(\Sigma) : \Sigma \text{ surface representing } \sigma\},$$

using the same measure of complexity $\chi_-(\Sigma)$ for an embedded surface $\Sigma \subset W$ as above. In W such a surface Σ has codimension 2. In contrast to the case where the ambient manifold has dimension 3, the normal bundle of Σ can be non-trivial. Therefore we cannot construct multiples of Σ by push-offs. This is more than just an inconvenience. In general the function $x(\sigma)$ will fail to be linear in σ, which is witnessed by the adjunction inequality [KM94] and also reflected in the theorem below.

The determination of $x(\sigma)$ for general 4-manifolds is not just an interesting challenge on its own, it has implications for our understanding of smooth structures in dimension 4 [GS99, Section 2]. Furthermore, questions, which seem on the first view unrelated, can be rephrased as minimal genus questions. On example for this is the Milnor conjecture, which claims that the (p,q)-torus knot has unknotting number $\frac{1}{2}(p-1)(q-1)$. The conjecture was confirmed by Kronheimer-Mrowka [KM93, Corollary 1.3] by considerations of the minimal genus in the K3-surface.

We restrict ourselves to the cases where W is a circle bundle over a 3-manifold M. Due to the fact that the topology of W is to a great extent controlled by M, the hope of relating the complexity $x(\sigma)$ to the Thurston norm $\|\sigma\|_T$ is not unreasonable. In Chapter 5 we obtain the theorem:

Theorem 1.1. *Let M be an irreducible 3-manifold which is neither covered by S^3 nor a torus bundle. Let $p\colon W \to M$ be a circle bundle over M.*

Then the complexity $x(\sigma)$ of every class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ satisfies the inequality

$$x(\sigma) \geq |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

In the theorem above the expression $\sigma \cdot \sigma$ denotes the self-intersection number. Recall that using Poincaré duality map PD and the cap product, it can be defined as follows:

$$\sigma \cdot \sigma = \sigma \frown \mathrm{PD}\,\sigma \in \mathrm{H}_0(W;\mathbb{Z}) = \mathbb{Z}.$$

For many 3-manifolds M and every circle bundle $p\colon W \to M$ over M and every class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$, Friedl-Vidussi [FV14, Corollary 1.3] have constructed a surface Σ with $[\Sigma] = \sigma$ which realises the above lower bound

$$\chi_-(\Sigma) = |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

Usually one certifies the minimality of $\chi_-(\Sigma)$ for the surface Σ with the help of the adjunction inequality, see Theorem 5.9. Here a problem arises. There is often a gap between $\chi_-(\Sigma)$ and the lower bound obtained by the adjunction inequality. This is remedied in the above theorem by also taking finite covers into account. Now in many cases there is no gap between the constructed surfaces and the lower bounds and so the complexity $x(\sigma)$ is determined exactly for all classes $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$.

First results in this direction were obtained by Kronheimer [Kro99]. He proved the estimate for the trivial circle bundle $W = M \times S^1$ and also gave the construction for realising the lower bound in this case.

Later Friedl-Vidussi [FV14, Theorem 1.1] generalised Kronheimer's result. They obtained the inequality for all but finitely many circle bundles over irreducible 3-manifolds with virtual RFRS fundamental group. The theorem above improves the result of Friedl-Vidussi in two ways.

Friedl-Vidussi rely on Agol's result [Ago08, Theorem 5.1] that a 3-manifold with a virtually RFRS fundamental group fibres over S^1 in many ways. By results due to Wise [Wis11], Przytycki-Wise [PW12] and Agol [Ago13] this holds for a large class of 3-manifolds. However, there are closed graph manifolds which do not virtually fibre, i.e. they have no finite cover which fibres. We refer to Example 2.20 for such a graph manifold. So for these cases a different approach is needed.

Secondly, our result holds for all circle bundles over M. This makes the theorem much stronger. To check whether a given circle bundle is among the finitely many circle bundles, which Friedl-Vidussi excluded, one has to know the Seiberg-Witten invariants of M. Nevertheless, the estimate holds for all circle bundles as they have conjectured.

1.2 Estimating the Thurston norm

Let us go back to the example of a knot K. Upper bounds for the genus of a knot can be found by constructing Seifert surfaces. Lower bounds are harder to come up with. Alexander [Ale28] introduced the Alexander polynomial which is an algebraic invariant of the knot. We call the difference between the highest and the lowest power the *width* of a polynomial. Seifert [Sei35, Satz 3] realised that the width of the Alexander polynomial gives a way to bound the genus from below:

Theorem 1.2 (Seifert)**.** *Suppose $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$ is an Alexander polynomial of a knot K and Σ a Seifert surface. Then the Euler characteristic of Σ satisfies the inequality*

$$-\chi(\Sigma) \geq \operatorname{width} \Delta_K - 1 = \operatorname{width}\left(\frac{\Delta_K}{t-1}\right).$$

How can we make sense of this theorem for a 3-manifold which is not a knot complement? Milnor [Mil62a, Theorem 4] expressed the Alexander polynomial of a knot K in terms of Reidemeister torsion: He calculated that the Reidemeister torsion of the knot complement $N(K)$ is exactly

$$\tau(N(K);\mathbb{Q}(t)) = \frac{\Delta_K}{t-1},$$

see Theorem 4.16. Kitano [Kit96] realised that also the twisted Alexander polynomial introduced by Lin [Lin01] and Wada [Wad94] can be phrased in terms of Reidemeister torsion. The benefit of the twisted Alexander polynomial is that we have the freedom to twist with a representation V of the fundamental group, see Definition 4.8. As a result it also contains information about the finite covers.

So is there also a result analogue to the theorem of Seifert above? Indeed there is. To a representation V over a field $\mathbb{K}$ and a class $\sigma \in \mathrm{H}_2(M,\partial M;\mathbb{Z})$, we can associate a representation V_σ over $\mathbb{K}(t)$, see Definition 4.27. Now the twisted Reidemeister torsion $\tau(M;V_\sigma)$ is represented by elements in the quotient field $\mathrm{Quot}\left(\mathbb{K}[t^{\pm 1}]\right)$ and in fact width $\tau(M;V_\sigma)$ is well-defined. An estimate similar to the one given by Seifert's theorem for 1-dimensional representations restricted from the free quotient of $\mathrm{H}_2(M,\partial M;\mathbb{Z})$ has been obtained by McMullen [McM02]. This was generalised to all 1-dimensional representations by Turaev [Tur02, Theorem 2.2]. For a general representation Friedl-Kim [FK06, Theorem 1.1] obtained the following theorem:

Theorem 1.3 (Friedl-Kim). *Let M be an irreducible 3-manifold and suppose M is not $D^2 \times S^1$. Assume that V is a representation of $\pi_1(M)$. Then all $\sigma \in \mathrm{H}_2(M,\partial M;\mathbb{Z})$ fulfil the inequality*

$$(\dim V)\,\|\sigma\|_T \geq \operatorname{width} \tau(M;V_\sigma).$$

A natural question arising from the above theorem is whether there is always a representation V such that equality holds in the inequality above. This would imply that one can recover the Thurston norm from twisted Reidemeister torsion. In Chapter 6 we obtain the theorem below. The definition of the various properties of representations can be found in that chapter as well.

Theorem 1.4. *Let M be an irreducible 3-manifold which is not $D^2 \times S^1$. For every homology class $\sigma \in \mathrm{H}_2(M,\partial M;\mathbb{Z})$ both statements hold:*

1. *There exists an integral representation V over the complex numbers, factoring through a finite group, such that*

$$(\dim V)\,\|\sigma\|_T = \operatorname{width} \tau(M;V_\sigma).$$

2. *For all but finitely many primes $p \in \mathbb{N}$, there exists a representation V over $\mathbb{F}_p$ of $\pi_1(M)$ such that*

$$(\dim V)\,\|\sigma\|_T = \operatorname{width} \tau(M;V_\sigma).$$

We say a 3-manifold M fibres if there exists a map $\pi \colon M \to S^1$ turning M into the total space of a fibre bundle. With this structure in mind we consider the class $[F] \in \mathrm{H}_2(M;\mathbb{Z})$ of an arbitrary fibre F. Friedl-Kim [FK06, Theorem 1.2] noted that for the class $[F]$ the inequality will be sharp for any representation. Using this fact Friedl-Vidussi [FV12, Theorem 1.2] showed that for 3-manifolds with virtually RFRS fundamental group, there always exists a representation V making the inequality strict as in the theorem above. Again Agol's theorem [Ago08, Theorem 5.1] was key to ensure that there are enough finite covers which fibre in the right ways. We extend this result to all irreducible 3-manifolds by including graph manifolds.

One benefit of the result above is that it shows that we have the freedom to do the computations of twisted Reidemeister torsion over finite fields. This is important for determining the Thurston norm computationally, see [FV12, Section 6].

Beside this practical benefit, the theorem also has implications of theoretical nature: The fact that one can detect the Thurston norm with representations over finite fields has been used by Boileau-Friedl [FB15] to show that the knot genus is determined by the profinite completion of the knot group.

In the discussion of twisted Reidemeister torsion so far we have suppressed that the invariant $\tau(M;V)$ exists only if the representation V has the property that the chain complex $V \otimes_{\mathbb{Z}[\pi_1(M)]} \mathrm{C}(M)$ is acyclic, where $\mathrm{C}(M)$ is the cellular chain complex of the universal cover of M. If this chain complex is acyclic we say that the representation V is M-acyclic. Another result we cover in this thesis is the following characterisation of 3-manifolds M which admit an M-acyclic representation:

Theorem 1.5. *Let M be a 3-manifold not diffeomorphic to S^3. Let $M \cong P_1 \# \ldots \# P_k$ be its prime decomposition. The following statements are equivalent:*

1. *There is a non-trivial unitary representation which is M-acyclic.*

2. *The boundary of M is toroidal (possibly empty). Furthermore, at most one of the P_i is not a rational homology sphere.*

1.3 Organisation of the thesis

In Chapter 2 we recall the definition of graph manifolds and introduce various constructions. We show how they can be simplified by taking finite covers. The key result, which is Theorem 2.24, shows the existence of a finite cover M and a character $\pi_1(M) \to \mathbb{Z}/p\mathbb{Z}$ which does not vanish on any Seifert fibre.

We introduce the Thurston norm and collect its relevant properties for the rest of this thesis in Chapter 3. We then proceed with the calculation of the norm in the 3-manifolds which are relevant to us. The results will be used to see that the lower bounds obtained from twisted Reidemeister torsion are sharp.

In Chapter 4 we proceed by describing the theory of Reidemeister torsion. We define Turaev's maximal Abelian torsion and use the character of Theorem 2.24 to calculate the twisted Reidemeister torsion for graph manifolds.

In Chapter 5 the relation with Seiberg-Witten theory is explored and we prove the advertised theorem on the complexity in circle bundles in Theorem 5.1.

We proceed by introducing various properties of representations in Chapter 6. After that we explain how Theorem 1.4 follows from the results obtained so far.

We conclude with Chapter 7 in which we characterise the 3-manifolds M which admit a non-trivial M-acyclic representation.

The discussion in Chapter 2, 4 and 5 is based on the article [Nag14]. The content of Chapter 6 and 7 is based on joint work with Stefan Friedl [FN15a, FN15b].

1.4 Conventions

We only consider smooth manifolds with boundary. A surface is a compact oriented 2-manifold with boundary. An irreducible 3-manifold is also connected, compact, oriented and has toroidal boundary.

By circle bundle we refer to an S^1-fibre bundle with oriented fibres.

1.5 Acknowledgements

I am most grateful to my advisor Stefan Friedl. He introduced me to the wonderful world of 3-manifolds and I owe much of my insights to his splendid explanations. He gave me the freedom to pursue my ideas and always encouraged me to travel and exchange mathematics.

I thank Raphael Zentner for the many enlightening discussions mathematical and non-mathematical alike.

I also would like to thank Hansjörg Geiges for teaching me all I know about contact topology and the guidance he provided during my first year in Cologne.

I thank Brendan Owens and the University of Glasgow for the two splendid visits.

Thanks to Michael Völkl for the many lively discussions in his office and all his preachings. Furthermore, I would like to thank Peter Arndt, Kilian Barth, Matthias Blank, Stephan Gareis, Dominic Jänichen, Mark Powell, Martin Ruderer and Johannes Sprang for many stimulating mathematical discussions. All of them and many unmentioned friends made my graduate time as enjoyable as it was.

I thank Matthias Blank, Corinna Lange and Michael Völkl for taking the time to read my drafts.

I am grateful to Corinna Lange for all the love, support and still bearing with me. I thank my parents for their unconditional and continuous support.

I was funded by GK 1269 of the University of Cologne, employed by the University of Regensburg and funded by the SFB 1085 of the University of Regensburg. Both the GK and the SFB were funded by the Deutsche Forschungsgemeinschaft.

2 Graph manifolds

First we introduce language for working with graph manifolds. Then we describe how graph manifolds can be simplified by taking finite covers. More precisely, Theorem 2.19 describes three classes of 3-manifolds such that every graph manifold is finitely covered by a manifold in one of the classes. This is of great help if we are allowed to reduce the situation to finite covers to prove a theorem. Then we proceed to prove Theorem 2.24, which is the key result of this chapter and allows us to calculate the Reidemeister torsion of graph manifolds by splitting it into blocks.

The chapter is based on the article [Nag14].

2.1 Seifert fibred spaces

We review the notion of a Seifert fibred manifold. Much of the content of this section goes back to investigations due to Seifert [Sei33]. The main purpose of this section is to fix notations and collect results we will use at a later point.

We construct multiple decompositions of the solid torus into circles. They will be prototypes for neighbourhoods of Seifert fibres.

Example 2.1 (Fibred solid torus)**.** Let $p, q \geq 1$ be coprime integers. We consider the quotient $T_{p,q}$ of $\mathbb{R} \times D^2$ by the relation

$$(t+1, z) \sim (t, \exp(2\pi i p/q)z) \text{ for } t \in \mathbb{R}, z \in D^2.$$

The map $z \mapsto \exp(2\pi i p/q)z$ is isotopic to the identity and thus $T_{p,q}$ is diffeomorphic to a solid torus. For $z \in D^2$ the image C_z of the line $\mathbb{R} \times \{z\}$ under the quotient map is a circle and the collection $\{C_z\}_{z \in D^2}$ is a decomposition of $T_{p,q}$ in the sense below.

Definition 2.2. For a 3-manifold M we define the following notions:

1. A collection of embedded circles C_M forms a *decomposition* of M if each point $x \in M$ is contained in exactly one circle. Given such a decomposition C_M, a subset of M is *saturated* if it is a union of elements of C_M.

2. A *Seifert fibred structure* on M is a decomposition C_M of $M = \bigcup C_M$ such that each circle $C \in C_M$ has a saturated neighbourhood U_C which is diffeomorphic to a fibred torus $T_{p,q}$ preserving the decomposition.

3. A *Seifert manifold* is a 3-manifold which admits a Seifert fibred structure.

Remark 2.3. From the definition it follows that the boundary ∂M will have the structure of an S^1-fibre bundle over a compact 1-manifold with fibres the circles of the decomposition. Thus ∂M will be a union of tori with the decomposition coming from a choice of trivialisation $\partial M \cong \bigcup S^1 \times S^1$.

Example 2.4. Let $p\colon M \to B$ be a circle bundle, i.e. a fibre bundle over a surface B with circles as fibres. The collection $\{p^{-1}(b)\}_{b\in B}$ defines a Seifert fibred structure on the total space M.

The example above is central as every Seifert manifold is finitely covered by a circle bundle. As we need additional control on the covering of the boundary to work with graph manifolds later, we cover this fact in more detail in Theorem 2.6 below.

We call a surface *toroidal* if every component is diffeomorphic to a torus. Note that a finite cover of a toroidal surface is again a toroidal surface.

Definition 2.5.

1. A cover $\pi\colon \widetilde{T} \to T$ of a toroidal surface is called *k-characteristic* if the cover restricted to any component C is a cover induced by the subgroup

$$\{g^k : g \in \pi_1(C)\} \subset \pi_1(C).$$

2. A cover $\pi\colon M \to N$ of a 3-manifold N with toroidal boundary is *k-characteristic* if the induced cover on the boundary $\pi_\partial\colon \partial M \to \partial N$ is k-characteristic.

The theorem below will be deduced by considering the orbifolds which are associated to Seifert fibred manifolds. This will be the only place, where orbifolds are used and we refer to Thurston's lecture notes for an introduction [Thu80, Chapter 13].

Theorem 2.6 (Thurston)**.** *Let N be a Seifert manifold. For each $k \geq 2$, there exists a circle bundle M and a finite k-characteristic cover $\pi\colon M \to N$.*

Proof. To every Seifert fibred space N, we can associate a 2-dimensional orbifold S such that N is diffeomorphic to the total space of an S^1-bundle over S. As we only allow fibred solid tori as a local model, we only have elliptic points as singularities.

Suppose S is finitely covered by a surface F, i.e. the orbifold S is good. Then we can pull back the bundle and obtain a finite cover M of N. The 3-manifold M will be a circle bundle as F is a surface and therefore has no orbifold singularities.

In the case where N has boundary, we have to ensure that the resulting cover will be k-characteristic. This is done by attaching disc orbifolds with a $1/k$ elliptic singularity to the boundary of S. In the cases were this operation results in a good orbifold, we proceed as described above. After that we remove the preimage of the disc orbifolds. As the circle bundle M has non-empty boundary, we can choose a trivialisation $M \cong F \times S^1$. The cover $M \to N$ is not yet k-characteristic. However, with a further cover induced from the k-fold cover along the S^1-factor, this can be arranged.

Now consider the cases where we end up with bad orbifolds. Thurston [Thu80, Theorem 13.3.6] classified all 2-orbifolds including a list of the bad ones. The closed bad 2-orbifolds with only elliptic points have underlying surface S^2 and at most two elliptic points. If N is closed, then the orbifold S can only be bad if N is a lens space. Therefore N is covered by S^3 which is the total space of the Hopf bundle. If the manifold N has boundary and after attaching a disc orbifold the associated orbifold is still bad, then N has to be a solid fibred tori. Here we used that for $k \geq 2$ adding the disc orbifolds adds at least an extra elliptic point. We conclude that as a Seifert manifold N is diffeomorphic to $D^2 \times S^1$ which is itself the total space of a circle bundle. □

A circle bundle over a surface with non-empty boundary is a trivial circle bundle. From this fact together with the theorem above, we immediately deduce the corollary:

Corollary 2.7. *A Seifert manifold with non-empty boundary has a finite connected k-characteristic cover which is diffeomorphic to the product $\Sigma \times S^1$ for a surface Σ.*

2.2 Graph manifolds

Now we will make the transition to graph manifolds, which have been introduced by Waldhausen [Wal67]. Roughly speaking, they are 3-manifolds which are obtained by gluing together Seifert fibred spaces along tori. We have seen that Seifert fibred spaces have very simple finite covers and this fact can also be used to greatly simplify graph manifolds.

Definition 2.8. An embedded surface $\Sigma \subset M$ in a 3-manifold is *incompressible* if for every component C of Σ the induced homomorphism $\pi_1(C) \to \pi_1(M)$ is injective.

We describe the process of cutting a 3-manifold M along an embedded surface Σ in more detail. Choose a map $f\colon M \to S^1$ such that $-1 \in S^1$ is a regular value and the property that $f^{-1}(-1) = \Sigma$. Such a map can for example be defined in a tubular neighbourhood of Σ so that it is constant to 1 in a neighbourhood of the boundary and then we can extend it by the constant map to all of M. Now we define $M|\Sigma$ as the fibre product of the diagram below.

$$\begin{array}{ccc} M|\Sigma & \xrightarrow{\;f_\Sigma\;} & [-\pi,\pi] \\ {\scriptstyle \mathrm{gl}_\Sigma}\downarrow & \lrcorner & \downarrow{\scriptstyle \exp} \\ M & \xrightarrow[\;f\;]{} & S^1 \end{array}$$

This fibre product exists in manifolds as $\exp\colon [-\pi,\pi] \to S^1$ and f are transverse. Conversely, the map gl_Σ identifies the boundary components $f_\Sigma^{-1}(-\pi) \subset M|\Sigma$ with the preimage $f_\Sigma^{-1}(\pi)$. Gluing together with this identification, we obtain again a manifold naturally diffeomorphic to M, see [BJ73, Section 13].

Definition 2.9. For an irreducible 3-manifold M a *graph structure* consists of an incompressible embedded toroidal surface $S \subset M$ and a Seifert fibred structure on $M|S$. An irreducible 3-manifold M admitting a graph structure is called a *graph manifold.*

Consider a graph structure on M with toroidal surface S. To each component C of S correspond two boundary components $C_\pm$ of $M|S$. Let C_+ denote the boundary component whose orientation agrees with the boundary orientation. Furthermore, the component of $M|S$ which contains C_+ is said to fill C *on the negative side.* In the opposite case we say that the component C_- fills C *on the positive side.*

Definition 2.10. Let M be an irreducible 3-manifold with a graph structure with toroidal surface S. The *Bass-Serre graph* is the following graph $(V(M), E(M), s, t)$:

$$\begin{aligned} V(M) &:= \{\text{components of } M|S\} \\ E(M) &:= \{\text{components of } S\}. \end{aligned}$$

The map $s\colon E(M) \to V(M)$ associates to a component T of S the component in $M|S$ which fills T on the negative side. The map $t\colon E(M) \to V(M)$ associates the component which fills T on the positive side. The components of $M|S$ are called *blocks.* We refer to the components of S as *graph tori.*

We construct an invariant of graph structures describing how the Seifert fibres sit in the toroidal surface. Each component T of S is part of the boundary of two Seifert fibred pieces, namely $t(T)$ and $s(T)$. The Seifert fibred structures on these pieces give rise to two embedded loops in T: one loop $\gamma_t(T)$ is the Seifert fibre coming from the Seifert fibred structure of $t(T)$ and one loop $\gamma_s(T)$ is the one coming from the fibred structure of $s(T)$. Orient the loops arbitrarily. Denote their intersection number in T by $c(T) := |\gamma_t(T) \cdot \gamma_s(T)|$.

Definition 2.11. The number $c(T)$ defined above is called the *fibre-intersection number* in the torus T. A graph structure whose fibre-intersection numbers are all non-zero is called *reduced.*

Lemma 2.12. *Given a graph structure on M with toroidal surface T, there exists a reduced graph structure which has a subset of the components of T as graph tori.*

Proof. Denote by S the surface consisting of the components C of T with $c(C) \neq 0$. The surface S is toroidal and again incompressible.

Two embedded loops which have the same homology class in a torus are isotopic, see e.g. [Rol90, Theorem 2.C.16]. So for all components C with $c(C) = 0$, we can make the Seifert structure on $M|T$ locally near the boundaries $C_\pm$ compatible and glue them together along the tori C. This gives us a Seifert fibred structure on $M|S$. □

2.3 Simplifying graph manifolds

Before addressing how to construct finite covers of a graph manifold. We make a detour and collect results on the covering theory of surfaces.

Definition 2.13. A cover $\pi\colon \Sigma' \to \Sigma$ of a surface Σ is called a *k-characteristic cover* if π restricted to each boundary component is a k-fold cover of the circle.

Lemma 2.14. *Let Σ be a connected surface of negative Euler characteristic $\chi(\Sigma) < 0$. Then for every natural number $d \geq 3$ there is a connected d-characteristic finite cover $\pi\colon \Sigma' \to \Sigma$ such that Σ' has positive genus.*

Proof. If Σ is closed, then we can choose π to be the identity. This follows from the equality below, which is relating the genus $g(\Sigma)$ and number of boundary components $b_0(\partial\Sigma)$ to the Euler characteristic $\chi(\Sigma)$:

$$\chi(\Sigma) = 2 - 2g(\Sigma) - b_0(\partial\Sigma).$$

In the case where Σ has only one boundary component, the cover $q\colon \Sigma_b \to \Sigma$ induced by the Hurewicz homomorphism $\pi_1(\Sigma) \to \mathrm{H}_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ is 1-characteristic. It is at least of degree 2 as Σ has negative Euler characteristic. Thus Σ_b has at least two boundary components and we compose the cover q with the cover described below for the surface Σ_b.

If Σ has two or more boundary components, consider the cover $\pi\colon \Sigma' \to \Sigma$ induced by the Hurewicz homomorphism

$$\pi_1(\Sigma) \to \mathrm{H}_1(\Sigma; \mathbb{Z}/d\mathbb{Z}).$$

This cover is d-characteristic. Let n denote the degree of the cover. Note that n has to be larger than 3. Using that the Euler characteristic is multiplicative, we check that the genus $g(\Sigma')$ is indeed positive:

$$\begin{aligned} 2 - 2g(\Sigma') - b_0(\partial\Sigma') &= n\,(2 - 2g(\Sigma) - b_0(\partial\Sigma)) \\ \Rightarrow 2 - 2g(\Sigma') - (n/d)\, b_0(\partial\Sigma) &= n\,(2 - 2g(\Sigma) - b_0(\partial\Sigma)) \\ \Rightarrow 2 - 2g(\Sigma') &= n\left(2 - 2g(\Sigma) - \frac{d-1}{d} \cdot b_0(\partial\Sigma)\right). \end{aligned}$$

If Σ has positive genus, then we have the inequality $g(\Sigma') \geq g(\Sigma)$ and the lemma holds. If $g(\Sigma) = 0$, then Σ has at least 3 boundary components. Thus we again deduce the estimate $g(\Sigma') > 0$. ☐

We describe ways to simplify graph manifolds by going up to finite covers. The first procedure targets the Bass-Serre graph. As we see later, having a bipartite Bass-Serre graph facilitates arguments involving Mayer-Vietoris sequences. This will be helpful in the calculation of twisted Reidemeister torsions. In the next lemma we show that this condition can always be achieved by a finite cover.

Lemma 2.15. *Let N be a manifold with a graph structure with toroidal surface S. Denote the components of S by $C_1, \dots, C_k$. Let $\pi\colon M \to N$ be the cover induced by the kernel of the map*

$$\begin{aligned} \pi_1(N) &\to \mathbb{Z}/2\mathbb{Z} \\ \gamma &\mapsto \sum_{i=1}^{k} \gamma \cdot [C_i]. \end{aligned}$$

The surface $\pi^{-1}(S) \subset M$ is an embedded toroidal surface and the induced Seifert fibred structure on $M|\pi^{-1}(S)$ defines a graph structure on M which has a bipartite Bass-Serre graph.

Proof. We can construct the cover M by cutting along S and gluing together $\mathbb{Z}/2\mathbb{Z}$-labelled copies of the components [Rol90, Section 5.C]. On the level of Bass-Serre graphs this yields exactly the bipartite double cover. □

Remark 2.16.

1. Having a bipartite Bass-Serre graph implies no self-pastings, i.e. each graph torus bounds two different blocks.
2. A finite cover of a manifold with a graph structure which has a bipartite Bass-Serre graph will again have a bipartite Bass-Serre graph.

We describe a way to construct a cover of a manifold with graph structure from finite covers of its blocks. This allows us to apply Theorem 2.6 to simplify the manifold via a finite cover. This was used by Hempel [Hem87, Section 4] to show that 3-manifolds with a geometric decomposition have a residually finite fundamental group. Usually these covers will not be normal. The next theorem describes how a k-characteristic cover of $N|S$ can be glued together to a cover of N.

Theorem 2.17 (Hempel)**.** *Let N be a manifold with a graph structure with toroidal surface S. Let $\pi_X\colon X \to N|S$ be a finite cover which is k-characteristic and of degree d.*

Then there is a finite cover $\pi\colon M \to N$ of degree d such that there a diffeomorphism $g\colon M|\pi^{-1}(S) \to X$ making the diagram below commutative:

$$\begin{array}{ccc} M|\pi^{-1}(S) & \xrightarrow{\quad g \quad} & X \\ & \searrow^{\pi} \quad \swarrow_{\pi_X} & \\ & N|S & \end{array}.$$

Sketch. Recall the construction of $N|S$ in Section 2.2. With this construction, we obtain a map $f_\Sigma\colon N|\Sigma \to [-\pi, \pi]$ and an identification $F\colon f_\Sigma^{-1}(-\pi) \to f_\Sigma^{-1}(\pi)$. As the cover π_X is characteristic, we can lift F to an identification

$$F_X\colon (\pi_X \circ f)^{-1}(-\pi) \to (\pi_X \circ f)^{-1}(\pi).$$

There is no canonical choice for the lift. We pick collars and glue with F_X the corresponding boundary components together. This gives rise to the manifold M. $\square$

Definition 2.18. A *composite graph structure* of an irreducible 3-manifold M is a graph structure on M with toroidal surface T and a Seifert fibre preserving diffeomorphism $M|T \cong \bigcup_{v \in V(M)} \Sigma_v \times S^1$ where all the surfaces Σ_v have positive genus.

We proceed by describing a collection of 3-manifolds such that every graph manifold is finitely covered by a manifold in the collection.

Theorem 2.19. *Let N be a connected graph manifold. Then there exists a finite cover $\pi\colon M \to N$ such that at least one of the statements below holds.*

1. *M is a Seifert manifold.*
2. *M is a torus bundle.*
3. *M admits a composite graph structure.*

Proof. Equip N with a graph structure with toroidal surface S such that the number of components of S is minimal. If S is empty, then N itself is a Seifert manifold.

So we assume that S is non-empty. By Theorem 2.7, we can find a 2-characteristic cover $X \to N|S$ such that each component of X is a trivial circle bundle. By taking copies of each component, we can arrange that $X \to N|S$ is globally of degree d. Using Theorem 2.17 we obtain a cover N' of N. As every component of X was a trivial circle bundle, there is a graph structure on N' with toroidal surface S' and a diffeomorphism $N'|S' \cong \bigcup_{v \in V(N')} \Sigma_v \times S^1$.

As S' is incompressible no surface Σ_v is a disc. We show that we can also remove the Σ_v which are annuli. Let Σ_v be an annulus. Assume that the corresponding block $B_v \cong \Sigma_v \times S^1$ bounds two different components of S'. Let C be a component of the surface S' bounding the block $\Sigma_v \times S^1$. By changing the Seifert fibred structure on this block $I \times T^2 \cong \Sigma_v \times S^1$ we can arrange that the Seifert fibres on C coming from $C_\pm$ agree. Therefore we can remove the component C from S'. Now there are two possibilities: either we can remove all annuli or there is a block $\Sigma_v \times S^1$ which is glued together along its two boundaries. In latter case N' is a torus bundle.

So we may assume that N' has a graph structure such that the Euler characteristic of each Σ_v is negative. Using Lemma 2.14 and Theorem 2.17 we find a cover $M \to N$ and a graph structure on M such that all blocks B_v are diffeomorphic to $\Sigma_v \times S^1$ with Σ_v a surface of positive genus. This is a composite graph structure for M. $\square$

It follows from combining works of Wise [Wis11], Przytycki-Wise [PW12] and Agol [Ago13], that the fundamental group $\pi_1(M)$ of an irreducible 3-manifold which is not a graph manifold is virtually special. By Agol [Ago08, Corollary 2.3] and Haglund-Wise [HW08] these fundamental groups are virtually RFRS. Agol [Ago08, Theorem 5.1] proved that aspherical 3-manifolds with virtually RFRS fundamental group admit a finite cover which fibres. In contrast to this there are graph manifolds which do not virtually fibre. Neumann [Neu97, Theorem E] characterised these graph manifolds. We give concrete examples of two such graph manifolds below.

Example 2.20. 1. Let $p\colon M \to \Sigma$ be a circle bundle with Euler number $e(M) \neq 0$ and whose base surface Σ has negative Euler characteristic $\chi(\Sigma) < 0$. The manifold M is not the total space of a surface bundle over the circle S^1 [Gab86, Theorem 1.2]. Any finite cover of M will inherit the structure of a circle bundle and the Euler number will remain non-zero. We deduce that M does not virtually fibre.

2. The above example is a Seifert fibred manifold. We also want to mention the following example which is a graph manifold M with a composite graph structure. Let Σ be a torus with an open disc removed. The product structure of $\Sigma \times S^1$ also induces a product structure on the boundary $\partial\Sigma \times S^1 = S^1 \times S^1$. Denote by ϕ the following orientation reversing diffeomorphism:

$$\begin{aligned} \phi\colon S^1 \times S^1 &\to S^1 \times S^1 \\ (m, l) &\mapsto (2m + 5l, m + 2l) \end{aligned}$$

Consider the manifold $M = \Sigma \times S^1 \cup_\phi \Sigma \times S^1$. This manifold does not fibre in any finite cover, see [LW97, Section 3 - Examples].

2.4 Composite graph manifolds

Let N be a graph manifold with a composite graph structure with toroidal surface S. Recall that we have fixed a diffeomorphism $N|S \cong \bigcup_{B \in V(N)} \Sigma_B \times S^1$ where all surfaces Σ_B have positive genus and are connected. The class $\{x\} \times [S^1] \in \mathrm{H}_1(\Sigma \times S^1; \mathbb{Z})$ is independent of $x \in \Sigma$. We denote this class by $t_B \in \mathrm{H}_1(N|S; \mathbb{Z})$. We also refer by the same name to the corresponding class in $\mathrm{H}_1(N; \mathbb{Z})$. The main result of this chapter is to show the existence of the following characters in a finite cover of N.

Definition 2.21. A character $\alpha\colon \pi_1(N) \to \mathbb{Z}/k\mathbb{Z}$ is called *Seifert non-vanishing* if $\langle \alpha, t_B \rangle \neq 0$ for every $B \in V(N)$.

In the case where a surface Σ has positive genus, the following lemma shows that by pulling back a cohomology class in $\mathrm{H}^1(\partial\Sigma; \mathbb{Z}/k\mathbb{Z})$ along a suitable finite cover, we can extend it to all of the cover. This is not possible for classes defined over the integers.

Lemma 2.22. *Let Σ be a connected surface of positive genus. For every $k \geq 2$ and $\beta \in \mathrm{H}^1(\partial\Sigma; \mathbb{Z}/k\mathbb{Z})$ there exists a finite 1-characteristic connected cover $\pi\colon \Sigma' \to \Sigma$ and a class $\beta' \in \mathrm{H}^1(\Sigma'; \mathbb{Z}/k\mathbb{Z})$ such that*

$$(\pi_\partial)^*\beta = i^*\beta' \in \mathrm{H}^1(\partial\Sigma'; \mathbb{Z}/k\mathbb{Z}),$$

where π_∂ is the restriction of π to the boundary $\partial\Sigma'$ and $i\colon \partial\Sigma' \to \Sigma'$ the inclusion.

Proof. Pick a non-zero element in $\gamma \in \mathrm{H}_1(\Sigma; \mathbb{Z})$ and consider the cover $\pi\colon \Sigma' \to \Sigma$ given by the kernel of the homomorphism

$$\begin{aligned} \pi_1(\Sigma) &\to \mathbb{Z}/k\mathbb{Z} \\ g &\mapsto g \cdot_\Sigma \gamma \end{aligned}$$

This is a 1-characteristic cover of Σ of degree k. Note that $\langle (\pi_\partial)^*\beta, \partial[\Sigma']\rangle \in \mathbb{Z}/k\mathbb{Z}$ vanishes by the equality:

$$\langle (\pi_\partial)^*\beta, \partial[\Sigma']\rangle = k \cdot \langle \beta, \partial[\Sigma]\rangle = 0.$$

Consider the long exact sequence of the pair $(\Sigma', \partial\Sigma')$, i.e.

$$\mathrm{H}^2(\Sigma', \partial\Sigma'; \mathbb{Z}/k\mathbb{Z}) \xleftarrow{\delta} \mathrm{H}^1(\partial\Sigma'; \mathbb{Z}/k\mathbb{Z}) \xleftarrow{i^*} \mathrm{H}^1(\Sigma'; \mathbb{Z}/k\mathbb{Z}).$$

As $\langle (\pi_\partial)^*\beta, \partial[\Sigma']\rangle = 0$ holds, we have the equality $\delta(\pi_\partial^*\beta) = 0$. Thus there exists a class $\beta' \in \mathrm{H}^1(\Sigma', \mathbb{Z}/k\mathbb{Z})$ with $i^*\beta' = (\pi_\partial)^*\beta$. ☐

Lemma 2.23. *Let Σ be a connected positive genus surface. For every $k \geq 2$ and every class $\alpha \in \mathrm{H}_1(\partial\Sigma \times S^1; \mathbb{Z}/k\mathbb{Z})$ defined on the boundary $\partial\Sigma \times S^1$ of $\Sigma \times S^1$, there exists a finite 1-characteristic connected cover $\pi\colon \Sigma' \times S^1 \to \Sigma \times S^1$ and a class $\alpha' \in \mathrm{H}_1(\Sigma' \times S^1; \mathbb{Z}/k\mathbb{Z})$ such that*

$$(\pi_\partial)^*\alpha = i^*\alpha',$$

where π_∂ is the restriction of π to the boundary $\partial\Sigma'$ and $i\colon \partial\Sigma' \to \Sigma'$ the inclusion.

Proof. We can express the homology group $\mathrm{H}_1(\Sigma \times S^1; \mathbb{Z}/k\mathbb{Z})$ with the Künneth isomorphism as a direct sum:

$$\begin{aligned} \mathrm{H}_1(\Sigma \times S^1; \mathbb{Z}/k\mathbb{Z}) &\cong \mathrm{H}_1(\Sigma; \mathbb{Z}/k\mathbb{Z}) \oplus \mathrm{H}_1(S^1; \mathbb{Z}/k\mathbb{Z}) \\ \alpha &\mapsto \beta + \theta. \end{aligned}$$

Now apply Lemma 2.22 above to obtain a cover $\pi_\Sigma\colon \Sigma' \to \Sigma$ and a class $\beta' \in \mathrm{H}_1(\Sigma; \mathbb{Z}/k\mathbb{Z})$. Applying the Künneth isomorphism in the reverse direction, we obtain an element $\alpha' \in \mathrm{H}_1(\Sigma \times S^1; \mathbb{Z}/k\mathbb{Z})$ corresponding to $\beta' + \theta$. The element α' together with the 1-characteristic cover $\pi := \pi_\Sigma \times \mathrm{Id}_{S^1}$ fulfils the assertions. ☐

Now we can state the key result of this chapter on the existence of a Seifert non-vanishing character. These characters will play an important role in Chapter 4. There in Theorem 4.34 we will see that if we twist with such a character, then the twisted Reidemeister of a composite graph manifold is of a simple form.

Theorem 2.24. *Let N be a graph manifold with a reduced composite graph structure with toroidal surface S. Let $k \geq 2$ be a natural number coprime to every fibre-intersection number $c(T)$ of every component T of S.*

Then there exists a finite cover $\pi\colon M \to N$ such that M admits a Seifert non-vanishing character $\alpha\colon M \to \mathbb{Z}/k\mathbb{Z}$.

Proof. By going up to a finite cover we may assume that N has a bipartite Bass-Serre graph, see Lemma 2.15. By the cutting and pasting constructing we see that this cover preserves the fibre-intersection numbers.

We define a class $\alpha_S \in \mathrm{H}^1(S;\mathbb{Z}/k\mathbb{Z})$ as follows. A component T bounds two distinct blocks $B_+ := t(T)$ and $B_- := s(T)$. We trivialise the component $T \cong S^1 \times S^1$ by the product structure of its positive side $T \subset \partial\Sigma_{B_+} \times S^1$. Consequently, we obtain an identification $\mathrm{H}^1(T;\mathbb{Z}/k\mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}\langle e_1, t_{B_+}\rangle$. Note that we can express the class t_{B_-} in terms of e_1 and t_{B_+}:

$$t_{B_-} = -c(T)e_1 + mt_{B_+},$$

for a suitable element $m \in \mathbb{Z}/k\mathbb{Z}$. Knowing that k is coprime to $c(T)$ and thus $c(T)$ is invertible in $\mathbb{Z}/k\mathbb{Z}$, we see that the two classes $t_{B_\pm}$ span a basis of $\mathrm{H}^1(T;\mathbb{Z}/k\mathbb{Z})$. Define a class $\alpha_T \in \mathrm{H}^1(T;\mathbb{Z}/k\mathbb{Z})$ by declaring $\alpha_T(t_{B_\pm}) = 1$. Their sum defines an element $\alpha_S \in \mathrm{H}^1(S;\mathbb{Z}/k\mathbb{Z})$.

Using Lemma 2.23 we find a 1-characteristic cover $\pi_X\colon X \to N|S$ and a cohomology class $\alpha_X \in \mathrm{H}^1(X;\mathbb{Z}/k\mathbb{Z})$ such that $(\pi_X|_\partial)^*\alpha_S = i^*\alpha_X$ holds for the inclusion $i\colon \partial X \to X$ and $\pi_X|_\partial$ the restriction of the cover π_X to the boundary. With Theorem 2.17 we can glue the components of X together to a cover $\pi\colon M \to N$.

The manifold M inherits a composite graph structure with toroidal surface $\pi^{-1}(S)$. Also the Bass-Serre graph will stay bipartite. This means that there is a decomposition $V(M) = V_+ \cup V_-$ of the set of blocks such that each torus of the toroidal surface sits between a block in V_+ and a block in V_-. Pick such a decomposition. Denote the inclusion $\coprod_{B\in V_\pm} B \subset M$ by $\phi_\pm$. The diagram below is a push-out diagram:

$$\begin{array}{ccc} \coprod_{B\in V_+} B & \xrightarrow{\phi_+} & M \\ {\scriptstyle i^+}\uparrow & \lrcorner & \uparrow{\scriptstyle \phi_-} \\ \coprod_{T\in E(M)} T & \xrightarrow[i^-]{} & \coprod_{B\in V_-} B \end{array}.$$

The maps $i_\pm$ are inclusions of subcomplexes and so cofibrations. As a consequence, we obtain the Mayer-Vietoris sequence

$$\bigoplus_{T\in E(M)} \mathrm{H}^1(T;\mathbb{Z}/k\mathbb{Z}) \xleftarrow{i_+^* - i_-^*} \bigoplus_{B\in V(M)} \mathrm{H}^1(B;\mathbb{Z}/k\mathbb{Z}) \leftarrow \mathrm{H}^1(M;\mathbb{Z}/k\mathbb{Z}).$$

Note that $\bigoplus_{B\in V(M)} \mathrm{H}^1(B;\mathbb{Z}/k\mathbb{Z}) = \mathrm{H}^1(X;\mathbb{Z}/k\mathbb{Z})$. Thus there exists a cohomology class $\alpha \in \mathrm{H}^1(M;\mathbb{Z}/k\mathbb{Z})$ which restricts to α_X if $\left(i_+^* - i_-^*\right)\alpha_X = 0$. This can be checked as follows:

$$\left(i_+^* - i_-^*\right)\alpha_X = (\pi_X|_\partial)^*\alpha_S - (\pi_X|_\partial)^*\alpha_S = 0.$$

As the cover $\pi_X\colon X \to N|S$ is 1-characteristic, we get for every block B' covering B that $\pi_{X*}t_{B'} = t_B$. Thus for each block B' of M, we have

$$\langle \alpha, t_{B'}\rangle = \langle \alpha_X, t_{B'}\rangle = \langle i^*\alpha_X, t_{B'}\rangle = \langle (\pi_X|_\partial)^*\alpha_S, t_{B'}\rangle = \langle \alpha_S, t_B\rangle = 1.$$

Therefore M with toroidal surface $\pi^{-1}(S)$ and the character $\alpha\colon M \to \mathbb{Z}/k\mathbb{Z}$ fulfils the assertions. □

Remark 2.25. Depending on the manifold N it might be that no Seifert non-vanishing character α in any finite cover M of N lifts to the integers $\mathbb{Z}$:

$$\begin{array}{ccc} \pi_1(M) & \stackrel{\not}{\longrightarrow} & \mathbb{Z} \\ & \searrow^{\alpha} & \downarrow \\ & & \mathbb{Z}/k\mathbb{Z} \end{array}.$$

First note that we have the isomorphism $\mathrm{H}^1(M;\mathbb{Z}) \cong \operatorname{Hom}(\pi_1(M), \mathbb{Z})$ given by the universal coefficient theorem, which associates a cohomology class $\beta \in \mathrm{H}^1(M;\mathbb{Z})$ to such a lift $\pi_1(M) \to \mathbb{Z}$. We can pull back the cohomology class β to a class β_B in every block B.

Recall that by definition the class $\beta \in \mathrm{H}^1(M;\mathbb{Z})$ fibres if M admits a fibre bundle structure $f\colon M \to S^1$ such that the class is a pull-back of a class in S^1. The class β_B fibres: Tischler [Tis70, Theorem 1] proved that a class fibres if we can represent it by a nowhere-vanishing 1-form. Pick a 1-form representing β_B. As the class β_B comes from a Seifert non-vanishing character, it evaluates to non-zero on each Seifert fibre. Averaging over the Seifert fibres gives a non-vanishing 1-form proving that the class β_B fibres.

The class β fibres if and only if each class β_B fibres [EN85, Theorem 4.2]. This implies that M is the total space of a fibre bundle.

Nevertheless there are graph manifolds N which do not virtually fibre. We gave a construction of such a manifold in Example 2.20. In these manifolds any Seifert non-vanishing character cannot lift as we have seen above.

3 The Thurston norm

In this chapter we define the Thurston norm. We describe various properties of it: it is a semi-norm on $\mathrm{H}_2(M, \partial M; \mathbb{Z})$, it is multiplicative with respect to taking finite covers and additive under gluing along incompressible tori. In the second part, we compute it in some examples. In the later sections we will be able to reduce arguments to these cases.

3.1 Definition and properties

Similar to the Euler characteristic the following complexity is multiplicative under finite covers.

Definition 3.1. Let Σ be a surface with components Σ_i. Define the complexity $\chi_-(\Sigma)$ in terms of the Euler characteristic $\chi(\Sigma_i)$ of the components by

$$\chi_-(\Sigma) := \sum_i \max(-\chi(\Sigma_i), 0).$$

Using the complexity above, Thurston [Thu86] introduced the following semi-norm. Recall that with our convention a surface Σ is not just orientable but oriented and we denote its fundamental class by $[\Sigma]$.

Definition 3.2 (Thurston)**.** Let M an irreducible 3-manifold. The *Thurston norm* of a homology class $\sigma \in \mathrm{H}_2(M, \partial M; \mathbb{Z})$ is

$$\|\sigma\|_T := \min\{\chi_-(\Sigma) : \Sigma \text{ an embedded surface with } [\Sigma] = \sigma\}.$$

Using Poincaré duality this semi-norm is extended to a semi-norm on $\mathrm{H}^1(M; \mathbb{Z})$, i.e. we define $\|\theta\|_T := \|\operatorname{PD}\theta\|_T$ for all $\theta \in \mathrm{H}^1(M; \mathbb{Z})$.

Remark 3.3.

1. Let $\Sigma \subset M$ be an embedded surface. We can always add a fillable 2-sphere or a boundary compressible disc to Σ without changing its fundamental class. Because of this, we use $\chi_-(\Sigma)$ instead of the negative Euler characteristic $-\chi(\Sigma)$ in the definition above. The genus has the drawbacks that it is not multiplicative under finite covers and does not behave well with cut-and-paste operations.
2. Every class $\theta \in \mathrm{H}^1(M; \mathbb{Z})$ is Poincaré dual to an embedded surface. Indeed, denote by $\tau \in \mathrm{H}^1(S^1; \mathbb{Z})$ the class associated to the orientation of S^1 as the boundary of the disc in the complex plane. As S^1 is an Eilenberg-Mac Lane

space, there exists a continuous map $f\colon M \to S^1$ with $f^*\tau = \theta$. We pick a smooth map F which is transverse to $1 \in S^1$ and homotopic to f. The surface $F^{-1}(1)$ is an embedded surface representing the Poincaré dual of θ.

Theorem 3.4 (Thurston)**.** *The Thurston norm is a semi-norm, i.e. for every $k \in \mathbb{Z}$ and for every $\alpha, \beta \in \mathrm{H}_2(M, \partial M; \mathbb{Z})$ the Thurston norm fulfils the relations*

$$\|k\alpha\|_T = |k|\|\alpha\|_T$$
$$\|\alpha + \beta\|_T \leq \|\alpha\|_T + \|\beta\|_T.$$

Proof. See [Thu86, Theorem 1]. □

Let $\Sigma \subset N$ be an embedded surface and $\pi\colon M \to N$ be a finite cover. We can consider the preimage of Σ and obtain a surface $\pi^{-1}(\Sigma) \subset M$ which has the fundamental class $\pi^![\Sigma]$, where $\pi^!$ denotes the umkehr map $\mathrm{PD} \circ (\pi^*) \circ \mathrm{PD}$. It is a natural question whether a minimal representative of the class $\pi^![\Sigma]$ can always be realised by the preimage of an embedded surface. Surprisingly, there is an affirmative answers. The theorem below is rather involved and uses Gabai's deep insights in sutured manifolds.

Theorem 3.5 (Gabai)**.** *Let $\pi\colon M \to N$ be a k-fold cover of an irreducible 3-manifold N. Then $\|\pi^*\theta\|_T = k\|\theta\|_T$ holds for every $\theta \in \mathrm{H}^1(N; \mathbb{Z})$.*

Proof. See [Gab83, Corollary 6.13]. □

Many 3-manifolds can be obtained by gluing other 3-manifolds together along incompressible boundary tori, e.g. the composite graph manifolds we have discussed in Chapter 2. Eisenbud-Neumann used cut and paste arguments to calculate the Thurston norm for these gluings.

Theorem 3.6 (Eisenbud-Neumann)**.** *Let M be a 3-manifold and S an incompressible embedded toroidal surface. For the inclusion $i\colon M|S \to M$ of M cut along S and every class $\theta \in \mathrm{H}^1(M)$, the equality*

$$\|\theta\|_T = \|i^*\theta\|_T$$

holds.

Proof. Note that the argument stated in the article [EN85, Proposition 3.5] uses merely that the tori are incompressible and not tori of a JSJ-decomposition. □

3.2 Examples

We calculate the Thurston norm in the key situations, which we need later. A more complete list can be found in the article [McM02, Section 7].

First we consider fibre bundles. In this case Thurston [Thu86, Section 3] noted the following minimality property of the fibre:

Lemma 3.7 (Thurston)**.** *Let $p\colon M \to S^1$ be a fibre bundle with connected fibre F. Any inclusion of a fibre F gives a Thurston norm minimising surface of class $[F]$, i.e. the equality $\|[F]\|_T = \chi_-(F)$ holds.*

We give a more expanded version of Thurston's original argument.

Proof. Let Σ be an embedded surface such that its fundamental class $[\Sigma]$ agrees with the class $[F]$ of a fibre. We prove that $\chi_-(\Sigma) \geq \chi_-(F)$.

We pull back the bundle along the universal cover $\mathbb{R} \to S^1$ obtaining a trivial bundle $\overline{M}$ with fibre F. We pick a trivialisation $\overline{M} \cong \mathbb{R} \times F$ and denote the projection onto the fibre by $\pi_F\colon \overline{M} \to F$.

We first prove that the embedding $i\colon \Sigma \to M$ lifts, i.e. there is map $\bar{i}$ such that the diagram below is commutative:

$$\begin{array}{ccc} & & \overline{M} \\ & \overset{\bar{i}}{\nearrow} & \downarrow \\ \Sigma & \xrightarrow[i]{} & M \end{array} .$$

The map $\overline{M} \to M$ is the cover which is induced by the kernel of the map

$$\begin{aligned} \pi_1(M) &\to \mathbb{Z} \\ g &\mapsto g \cdot [F]. \end{aligned}$$

Recall that $g \cdot [F]$ denotes the intersection product between classes in $\mathrm{H}_1(M;\mathbb{Z})$ and in $\mathrm{H}_2(M;\mathbb{Z})$. So let γ be a loop in Σ. As the normal bundle of Σ in M is orientable, we have $\gamma \cdot [\Sigma] = 0$ and so $\gamma \cdot [F] = 0$. Thus the obstruction for lifting [Bre93, Theorem 4.1] vanishes and we obtain an embedding $\bar{i}\colon \Sigma \to \overline{M}$. Now the equality $[\Sigma] = [F]$ holds in the cover $\overline{M}$ as well. Therefore the map $\pi_F \circ \bar{i}\colon \Sigma \to F$ is of degree 1.

As a degree 1 map cannot factor through a cover, we deduce that the map above induces a surjection of $\pi_1(\Sigma)$ onto $\pi_1(F)$. By the classification of surfaces we know that $\chi_-(\Sigma) \geq \chi_-(F)$. ☐

Lemma 3.8. *Let $p\colon M \to S^1$ be a torus bundle. Then the Thurston norm vanishes.*

Proof. The 3-manifold M is diffeomorphic to a mapping torus $\mathrm{M}(T^2, \phi)$ with monodromy $\phi\colon T^2 \to T^2$ a diffeomorphism of the 2-torus. The mapping torus is obtained by the following quotient

$$\mathrm{M}(T^2, \phi) := T^2 \times \mathbb{R}/(\phi(x), t) \sim (x, t+1) .$$

An application of the Mayer-Vietoris sequence [Hat02, Example 2.48] gives the short exact sequence

$$\ldots \to \mathrm{H}_{k+1}(\mathrm{M}(T^2,\phi);\mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_k(T^2;\mathbb{Z}) \xrightarrow{\mathrm{Id} - f_*} \mathrm{H}_k(T^2;\mathbb{Z}) \to \mathrm{H}_k(\mathrm{M}(T^2,\phi);\mathbb{Z}) \to \ldots .$$

Now let $\sigma \in \mathrm{H}_2(\mathrm{M}(T^2, \phi); \mathbb{Z})$ be an arbitrary class. Consequently, we have the equality $(\mathrm{Id} - f_*)\partial\sigma = 0$. There exists an embedded loop $c \subset T^2$ such that $\partial\sigma = a[c]$ for an $a \in \mathbb{Z}$. As $(\mathrm{Id} - f_*)[c] = 0$ we can even arrange that c is fixed by the diffeomorphism f, see [Rol90, Chapter 2.D Theorem 4]. Thus the image of $c \times \mathbb{R} \subset T^2 \times \mathbb{R}$ in $\mathrm{M}(T^2, \phi)$ gives rise to an embedded torus $S \subset \mathrm{M}(T^2, \phi)$ with $\partial[S] = [c]$.

We claim that $\|\sigma - a[S]\|_T = 0$. First note that the boundary $\partial(\sigma - a[S])$ is the zero class. By the exact sequence above we know that $\sigma - a[S]$ is a multiple of a fibre. As the fibres are all tori, we obtain $\|\sigma - a[S]\|_T = 0$.

The Thurston norm satisfies the triangle inequality and so we obtain the estimate

$$\|\sigma\|_T \leq \|\sigma - a[S]\|_T + \|a[S]\|_T = 0.$$

□

We have seen that many Seifert fibred spaces are covered by circle bundles. The next lemma computes the Thurston norm for these cases. Again, it is also contained in the article [Nag14, Proposition 3.4].

Lemma 3.9. *Let $\pi\colon M \to \Sigma$ be a circle bundle over a connected surface Σ.*

1. *If the circle bundle is non-trivial, then the Thurston norm vanishes.*
2. *If the circle bundle is trivialised $M = \Sigma \times S^1$, then for all $\theta \in \mathrm{H}^1(M; \mathbb{Z})$ we have the equality*
$$\|\theta\|_T := \chi_-(\Sigma)\,|\langle\theta, t\rangle|,$$
where $t := [\{x\} \times S^1]$ is the class of a fibre.

Proof. In the case of a non-trivial circle bundle, the Euler class of the circle bundle $\pi\colon M \to \Sigma$ is a non-zero element $e \in \mathrm{H}^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$. Recall the Gysin sequence [Bre93, Theorem 13.2]:

$$\ldots \to \mathrm{H}^k(\Sigma; \mathbb{Q}) \xrightarrow{\pi^*} \mathrm{H}^k(M; \mathbb{Q}) \xrightarrow{\pi_*} \mathrm{H}^{k-1}(\Sigma; \mathbb{Q}) \xrightarrow{\cup e} \mathrm{H}^{k+1}(\Sigma; \mathbb{Q}) \to \ldots.$$

As the Euler class e is non-zero, the map $\cup e\colon \mathrm{H}^0(\Sigma; \mathbb{Q}) \to \mathrm{H}^2(\Sigma; \mathbb{Q})$ is injective. Therefore the induced homomorphism $\pi^*\colon \mathrm{H}^1(\Sigma; \mathbb{Q}) \to \mathrm{H}^1(M; \mathbb{Q})$ has to be surjective. To show that the Thurston norm on M vanishes, it will enough to prove the equality $\|\pi^*\theta\|_T = 0$ for an arbitrary class $\theta \in \mathrm{H}^1(\Sigma; \mathbb{Q})$. Represent the Poincaré dual of the class θ by embedded loops in Σ. The preimage of these loops under π will be a collection of tori, which represent $\pi^*\theta$. Therefore we obtain the equality $\|\pi^*\theta\|_T = 0$.

Note that in the case of a trivial circle bundle $\Sigma \times S^1$, we can make the identifications below using the Künneth isomorphism:

$$\mathrm{H}^1(\Sigma \times S^1) \cong \mathrm{H}^1(\Sigma) \oplus \mathrm{H}^1(S^1) = \mathrm{H}^1(\Sigma) \oplus \mathbb{Z}\langle\theta\rangle.$$

The inclusion $\mathrm{H}^1(\Sigma; \mathbb{Z}) \subset \mathrm{H}^1(\Sigma \times S^1; \mathbb{Z})$ is just the map π^*, where π is the projection $\pi\colon \Sigma \times S^1 \to \Sigma$. As above one shows that each class $\alpha \in \mathrm{H}^1(\Sigma; \mathbb{Z}) \subset \mathrm{H}^1(\Sigma \times S^1; \mathbb{Z})$ is Poincaré dual to embedded tori. Consequently, the Thurston norm $\|\alpha\|_T = 0$ vanishes.

The class θ is Poincaré dual to the class of a fibre $[\Sigma]$. By Lemma 3.7, we get the equality $\|\theta\|_T = \chi_-(\Sigma)$. Using the reverse triangle inequality we deduce

$$\|m\theta + \alpha\|_T = |m|\|\theta\|_T,$$

for every $m \in \mathbb{Z}$ and every class $\alpha \in \mathrm{H}^1(\Sigma) \subset \mathrm{H}^1(\Sigma \times S^1)$. Let $t \in \mathrm{H}_1(\Sigma \times S^1; \mathbb{Z})$ denote the class of a fibre $\{x\} \times [S^1]$. By the considerations above we obtain the equality

$$\|m\theta + \alpha\|_T = |m| \cdot \chi_-(\Sigma) = |\langle m\theta + \alpha, t\rangle|.$$

□

4 Twisted Reidemeister torsion

We define twisted Reidemeister torsion and review the necessary foundation of CW-structures and cellular chain complexes. Then we recall the relation of Reidemeister torsion with the Alexander polynomial and introduce Turaev's maximal Abelian torsion [Tur76]. After that we focus on the calculation of twisted Reidemeister torsion of graph manifolds. For completeness sake we conclude with a quick overview of the situation of fibred 3-manifolds.

The discussion of the key results of this chapter follows the article [Nag14].

4.1 CW-structures and cellular complexes

We recall the notion of a CW-complex due to Whitehead [Whi49, Section 5] and the associated cell complex, see e.g. [Tur01, II.5.6-7].

Definition 4.1.

1. A topological space $X \supset Y$ is obtained from a subspace Y by *attaching k-cells*, if there exists an index set Z and a push-out diagram:

$$\begin{array}{ccc} \coprod_Z D^k & \longrightarrow & X \\ \uparrow & \lrcorner & \uparrow \\ \coprod_Z S^{k-1} & \longrightarrow & Y \end{array}.$$

2. A *CW-structure* for a space X is a filtration

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset X^k \subset \ldots$$

such that $X = \bigcup_k X^k$, and $X^{k-1} \subset X^k$ is obtained by attaching k-cells, and X carries the colimit topology.

3. A path component of $X^k \setminus X^{k-1}$ is called a *k-cell* and a closed subset A which is a union of cells is called a *subcomplex*.

4. A map $\Phi_e \colon D^k \to X$ which identifies the interior $\operatorname{Int} D^k$ with a k-cell e is called a *characteristic map* for the cell e.

Fix a connected finite CW-complex X. Denote the set of k-cells with Z^k. Pick a characteristic map $\Phi_e \colon D^k \to X^k$ for each cell $e \in Z^k$. With the help of the characteristic maps, the k-cells define a basis for the homology group $\mathrm{H}_k(X^k, X^{k-1}; \mathbb{Z})$ by the following isomorphism [tD08, Proposition 12.1.1]

$$\begin{aligned} \bigoplus_{e \in Z^k} \mathbb{Z}\langle e \rangle &\to \mathrm{H}_k(X^k, X^{k-1}; \mathbb{Z}) \\ \Phi_e &\mapsto \Phi_{e*}[D^k, S^k]. \end{aligned}$$

Furthermore, we fix a universal cover $\pi\colon \widetilde{X} \to X$. The filtration $\widetilde{X} = \pi^{-1}(X)$ defines a CW-structure on $\widetilde{X}$. The filtration is invariant under the deck transformations and so $\mathrm{H}_k(\widetilde{X}^k, \widetilde{X}^{k-1}; \mathbb{Z})$ is a left $\mathbb{Z}[\pi_1(X)]$-module. The deck transformations also act on the cells by permutation.

Definition 4.2. Let X be a finite connected CW-complex and a fixed universal cover $\pi\colon \widetilde{X} \to X$. Let $A \subset X$ be subcomplex.

1. A *fundamental family* $\mathfrak{e}$ consists of the following datum:
   a) an ordering on the set of k-cells Z^k,
   b) for every cell e of X a cell $\widetilde{e}$ of $\widetilde{X}$ covering e, and
   c) a characteristic map $\Phi_{\widetilde{e}}$ for these lifted cells $\widetilde{e}$ of $\widetilde{X}$.

2. The *cellular chain complex* $\mathrm{C}(A \subset X)$ of the CW-subcomplex $A \subset X$ is the chain complex of $\mathbb{Z}[\pi_1(X)]$-modules having chain modules

   $$\mathrm{C}_k(A \subset X) := \mathrm{H}_k(\pi^{-1}(A^k), \pi^{-1}(A^{k-1}); \mathbb{Z})$$

   with the boundary morphisms being induced from the exact sequence of the triple $(\pi^{-1}(A^k), \pi^{-1}(A^{k-1}), \pi^{-1}(A^{k-2}))$. We abbreviate $\mathrm{C}(X \subset X)$ with $\mathrm{C}(X)$.

The cellular complex $\mathrm{C}(A \subset X)$ is a finite free chain complex of left $\mathbb{Z}[\pi_1(X)]$-modules. Let Z^k denote the cells of X and let $\mathfrak{e}$ be a fundamental family of X. We write $Z^k(A)$ for the subset of Z^k consisting of the k-cells shared with A. As above, the following is an isomorphism of $\mathbb{Z}[\pi_1(X)]$-modules:

$$\begin{aligned} \bigoplus_{e \in Z^k(A)} \mathbb{Z}[\pi_1(X)]\, \langle \widetilde{e} \rangle &\to \mathrm{C}_k(A \subset X) \\ \widetilde{e} &\mapsto \Phi_{\widetilde{e}*}[D^k, S^{k-1}]. \end{aligned}$$

We see that having a fundamental family, defines a basis in each of the chain modules $\mathrm{C}_k(A \subset X)$.

Definition 4.3. A *based* chain complex C is a chain complex together with a basis of each chain module C_k. The cellular chain complex inherits from a fundamental family the structure of a based chain complex as above.

4.2 Reidemeister torsion

We recall the notion of Reidemeister torsion following Milnor [Mil62a] and Turaev [Tur01]. For this section fix a field $\mathbb{K}$ and let C be a finite chain complex over $\mathbb{K}$ having a preferred (ordered) finite basis c_k for each chain-module C_k. Furthermore, assume that C is acyclic.

Definition 4.4. Let $a = \{a_1, \ldots, a_n\}$ and $b = \{b_1, \ldots, b_n\}$ be two bases of the vector space V. Define $[a/b] \in \mathrm{GL}(n, \mathbb{K})$ to be the matrix m_{ij} fulfilling

$$a_i = \sum_j m_{ij} b_j.$$

For each k we fix a basis b_k of the image $\operatorname{Im} \partial_k$ of the k-th boundary operator of C. As C is acyclic we have for every natural number k a short exact sequence

$$0 \to \operatorname{Im} \partial_{k+1} \to C_k \xrightarrow{\partial_k} \operatorname{Im} \partial_k \to 0.$$

Pick a collection of lifts $\widetilde{b}_k$ under ∂_k of b_k. The concatenated collection $b_{k+1}\widetilde{b}_k$ is again a basis of C_k and the determinant $\det [b_{k+1}\widetilde{b}_k/c]$ is independent of the choice of lifts.

Definition 4.5. The *Reidemeister torsion* of C is the unit

$$\tau(C) := \prod_k \det [b_{k+1}\widetilde{b}_k/c]^{(-1)^k} \in \mathbb{K}^*$$

Remark 4.6. The Reidemeister torsion is independent of the choice of basis for the $\operatorname{Im} \partial_k$, but depends on the choice of basis for each chain module C_k.

Definition 4.7. A *representation* of a group G is a $(\mathbb{K}, \mathbb{Z}[G])$-bimodule. For a representation V, the $\mathbb{K}$-linear map corresponding to right multiplication with a group element $g \in G$ is denoted by $g_V : V \to V$.

For an irreducible 3-manifold M with a CW-structure and a fundamental family $\mathfrak{e}$, the chain complex $\mathrm{C}(M)$ will not be acyclic. This can be remedied by tensoring $\mathrm{C}(M)$ with a suitable representation. So let V be such a bimodule. We then proceed by picking a basis $v := \{v_1, \ldots, v_n\}$ of the $\mathbb{K}$-vector space V. The chain complex $V \otimes \mathrm{C}(M, \mathfrak{e})$ has chain modules $V \otimes_{\mathbb{Z}[\pi_1(M)]} \mathrm{C}_k(M)$ with basis

$$v_1 \otimes c_1, v_2 \otimes c_1, \ldots, v_n \otimes c_1, v_1 \otimes c_2, \ldots, \ldots.$$

Definition 4.8.

1. A representation V is called *M-acyclic* if the twisted cellular chain complex $V \otimes \mathrm{C}(M)$ is acyclic.
2. For a M-acyclic representation V, define the *twisted Reidemeister torsion* of M by
$$\tau(M, \mathfrak{e}; V) := \tau(V \otimes \mathrm{C}(M, \mathfrak{e})) \in \mathbb{K}^*.$$
Its image in the quotient $\mathrm{Wh}(V) := \mathbb{K}^*/\langle \pm \det g_V : g \in \pi_1(M)\rangle$ is denoted by $\tau(M; V) \in \mathrm{Wh}(V)$.

Remark 4.9. 1. Changing from one fundamental family into another multiplies the twisted Reidemeister torsion by a group element of the form $\pm \det g_V$ for $g \in \pi_1(M)$. Thus $\tau(M;V) \in \mathrm{Wh}(V)$ is really independent of the choice of fundamental family.

2. The twisted Reidemeister torsion $\tau(M;V) \in \mathrm{Wh}(V)$ also does not depend on the choice of CW-structure. This follows from a result due to Kirby-Siebenmann [KS69, Theorem IV] or more generally by Chapman [Cha74, Theorem 1].

The following lemmas are helpful for calculating Reidemeister torsion. Let A, B, C be based cell complexes with bases a, b, c, respectively. Suppose that there is an exact sequence in chain complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Let $\widetilde{c}$ be a lift of the basis c under the chain map g.

Definition 4.10. The short exact sequence above is called *based compatibly* if the base change matrix has determinant $\det[a_k\widetilde{c}_k/b_k] = \pm 1$ for every $k \in \mathbb{Z}$.

Example 4.11. Let Z be a finite set. Let $\{Z_i\}_{i\in I}$ be a collection of subsets $Z_i \subset Z$ with I a finite ordered index set. Suppose that all triple intersections are empty, i.e. $Z_i \cap Z_j \cap Z_k = \emptyset$ for all distinct $i, j, k \in I$. We abbreviate $Z_i \cap Z_j$ with Z_{ij}. We denote the free vector space over a set A by $\mathbb{K}[A]$. We obtain induced maps $q_i\colon \mathbb{K}[Z_i] \to \mathbb{K}[Z]$. The inclusion $Z_{ij} \subset Z_j$ induces a map $\mathbb{K}[Z_{ij}] \to \mathbb{K}[Z_i]$. In the case $i < j$ we refer to this map by q_+. For $j < i$ the map is called q_-. The following sequence is short exact:

$$0 \to \bigoplus_{i<j} \mathbb{K}[Z_{ij}] \xrightarrow{q_+ - q_-} \bigoplus_{i\in I} \mathbb{K}[Z_i] \to \mathbb{K}[Z] \to 0,$$

where the second map is induced by the inclusions $Z_i \subset Z$. Additionally, it is also based compatibly with regards to their natural basis.

Lemma 4.12. *In the situation above, the following statements hold.*

1. *If two of the chain complexes A, B, C are acyclic, then so is the third.*

2. *If all three chain complexes are acyclic and they are based compatibly, then*

$$\tau(B) = \tau(A) \cdot \tau(C).$$

Proof. See Turaev [Tur01, Theorem 3.4]. □

Often the topological setting suggest a splitting $C_k = A_k \oplus B_k$ of the chain modules. The lemma below facilitates the calculation of the torsion. We denote the inclusion $B_k \subset C_k$ by i_k and the projection $C_k \to A_k$ by p_k. Define the map s_k by

$$s_k\colon B_k \to A_{k-1}$$
$$s_k := p_{k-1} \circ \partial_k \circ i_k.$$

Lemma 4.13. *Let C be a chain complex over $\mathbb{K}$. Suppose a splitting of each chain module $C_k = A_k \oplus B_k$ as based $\mathbb{K}$-vector spaces is given. Let s_k denote the map defined above. Suppose that s_k is invertible for every k.*

Then C is acyclic and the Reidemeister torsion of C is

$$\tau(C) = \pm \prod_{i \in \mathbb{N}} (\det s_i)^{(-1)^i}.$$

Proof. See [Tur01, Theorem 2.2]. □

4.3 Alexander polynomial

We motivate the constructions made so far by an observation due to Milnor [Mil62a, Theorem 4] relating the Alexander polynomial of a knot to the Reidemeister torsion of the knot complement. We follow the treatment of Turaev [Tur86].

Let K be a knot in S^3. Pick an orientation of K and an open tubular neighbourhood $\nu(K)$. Abbreviate the knot complement of K with $N(K) := S^3 \setminus \nu(K)$. A maximal Abelian cover $\overline{N(K)}$ is a connected cover of $N(K)$ induced by the kernel of the Hurewicz map $\pi_1(N(K)) \to \mathrm{H}_1(N(K);\mathbb{Z}) = \mathbb{Z}$. The last equality is obtained by sending $1 \in \mathbb{Z}$ to a positive meridian. We identify the group ring $\mathbb{Z}[\mathbb{Z}]$ with the ring of Laurent polynomials $\mathbb{Z}[t^{\pm 1}]$. Recall that $\mathbb{Z}[t^{\pm 1}]$ is not principal but a unique factorisation domain.

For a commutative ring R and a finitely presented R-module M, we define the order ideal $E_0 M$ as follows: pick a presentation

$$0 \to R^s \xrightarrow{A} R^r \to M \to 0.$$

Now $E_0 M$ is the ideal which is generated by all determinants of $r \times r$-minors of A. We denote a greatest common divisor of the elements of $E_0 M$ by $\operatorname{ord} M$. The element $\operatorname{ord} M$ only depends on the module M, see e.g. [CF63, Theorem VII.4.5]. The $\mathbb{Z}[t^{\pm 1}]$-module $\mathrm{H}_1(\overline{N(K)};\mathbb{Z})$ has a square presentation and therefore the ideal $E_0\,\mathrm{H}_1(\overline{N(K)};\mathbb{Z})$ is principal [Rol90, Corollary 8.C.4].

Definition 4.14. The *Alexander module* $A(K)$ is the $\mathbb{Z}[t^{\pm 1}]$-module $\mathrm{H}_1(\overline{N(K)};\mathbb{Z})$. An element $\Delta_K := \operatorname{ord} A(K)$ as defined above is called an *Alexander polynomial* of the knot K.

The next lemma allows us to express the Reidemeister torsion in terms of orders. We denote the quotient field of the polynomial ring $\mathbb{Q}[t]$ by $\mathbb{Q}(t)$.

Lemma 4.15 (Turaev)**.** *Let C be a based chain complex of $\mathbb{Z}[t^{\pm 1}]$-modules. If $\mathbb{Q}(t) \otimes C$ is acyclic, then its Reidemeister torsion is*

$$\tau(\mathbb{Q}(t) \otimes C) = \prod_{i \in \mathbb{N}} \operatorname{ord} \mathrm{H}_i(C)^{(-1)^{i+1}}$$

Proof. See [Tur86, Lemma 2.1.1] □

Using the description above, we proceed with calculating the Reidemeister torsion of a knot complement. We only need to determine the orders of $\mathrm{H}_k(\overline{N(K)};\mathbb{Z})$ for $k = 0, 2$, which is straight forward:

Theorem 4.16 (Milnor)**.** *Let K be a knot with complement $N(K)$. Let Δ_K be an Alexander polynomial for K. Then the twisted Reidemeister torsion of $N(K)$ is*

$$\tau(N(K);\mathbb{Q}(t)) = \frac{\Delta(K)}{t-1}.$$

The argument is sketched below. We refer to Turaev [Tur01, SectionII.11.4] for more details.

Sketch. We pick a triangulation of $N(K)$, which has boundary. We can remove all 3-cells by elementary collapses [Tur01, Section II.8]. This procedure leaves the homotopy type and Reidemeister torsions invariant. As there are no 3-cells, we immediately obtain that $\operatorname{ord}\mathrm{H}_k(\mathbb{Z}[t^{\pm 1}] \otimes \mathrm{C}(N(K))) = 1$ for $k \geq 3$.

The group $\mathrm{H}_2(\mathbb{Z}[t^{\pm 1}] \otimes \mathrm{C}(N(K)))$ is a subgroup of $\mathrm{C}_2(N(K))$ which is free. Therefore, we have $\operatorname{ord}\mathrm{H}_2(\mathbb{Z}[t^{\pm 1}] \otimes \mathrm{C}(N(K))) = 1$ as well.

By Lemma 4.15 we are left to compute $\operatorname{ord}\mathrm{H}_0(\mathbb{Z}[t^{\pm 1}] \otimes \mathrm{C}(N(K))) = t - 1$. The complex $N(K)$ consist of a single component. Therefore, the deck transformations leave this component invariant and we have $\mathrm{H}_0(\mathbb{Z}[t^{\pm 1}] \otimes \mathrm{C}(N(K))) \cong \mathbb{Z}[t^{\pm 1}]/\langle t-1\rangle$. We obtain the following equation

$$\tau(N(K);\mathbb{Q}(t)) = \prod_{i\in\mathbb{N}} \operatorname{ord}\mathrm{H}_i(\mathbb{Z}[t^{\pm 1}] \otimes C)^{(-1)^{i+1}} = \operatorname{ord}\mathrm{H}_1(\mathbb{Z}[t^{\pm 1}] \otimes C)/t-1 = \frac{\Delta_K}{t-1}.$$

□

4.4 Maximal Abelian torsion

We proceed by introducing the maximal Abelian torsion due to Turaev [Tur76]. Our approach follows [Tur01, Section II.13].

Fix an irreducible 3-manifold M with $b_1(M) \geq 2$. The maximal Abelian torsion of M encodes the twisted Reidemeister torsion over all 1-dimensional representations in an element of $\mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})]$. First we describe the group ring $\mathbb{Q}[T]$ of a finite Abelian group T in terms of cyclotomic fields. Associated to a group homomorphism $\sigma\colon T \to \mathbb{C}^*$, also called a *character*, is the ring homomorphism

$$\begin{aligned} \phi_\sigma\colon \mathbb{Q}[T] &\to \mathbb{C} \\ a_g g &\mapsto a_g\sigma(g). \end{aligned}$$

There is a unique smallest cyclotomic field C_σ containing the image of ϕ_σ.

Lemma 4.17. *Let T be a finite Abelian group. There exists a finite set S of characters such that the map below is an isomorphism:*

$$\prod_{\sigma \in S} \phi_\sigma \colon \mathbb{Q}[T] \to \prod_{\sigma \in S} C_\sigma.$$

Proof. [Tur01, Lemma II.12.6] □

Let $\widetilde{M}$ be an universal cover for the manifold M and a fundamental family $\mathfrak{e}$. We can pick a splitting of $\mathrm{H}_1(M;\mathbb{Z}) = T \oplus F$ into the torsion part T and a free complement F. Additionally, we pick a set of characters S for T as in the lemma above. We obtain the following description of the ring of quotients $\mathrm{Quot}\,(\mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})])$:

$$\mathrm{Quot}\,(\mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})]) = \mathrm{Quot}\,(\mathbb{Q}[T] \otimes_{\mathbb{Q}} \mathbb{Q}[F]) = \prod_{\sigma \in S} C_\sigma(F).$$

Denote the set of characters $\sigma \in S$ for which C_σ is M-acyclic by S_{acy}. Summing over S_{acy} we obtain the element

$$\Delta := \sum_{\sigma \in S_{acy}} \tau(M, \mathfrak{e}; C_\sigma(F)) \in \prod_{\sigma \in S} C_\sigma(F).$$

By the above identification this defines an element $\Delta \in \mathrm{Quot}\,(\mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})])$. For manifolds M with $b_1(M) \geq 2$, the element Δ lies in fact in $\mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})]$, see [Tur02, Corollary II.1.4, Corollary II.4.3].

Definition 4.18 (Turaev)**.** The element $\Delta \in \mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})]$ constructed above is called the *maximal Abelian torsion.*

In the next lemma, we see that the maximal Abelian torsion is universal for all the Reidemeister torsions of 1-dimensional representations. Let $\mathbb{K}$ be a field of characteristic 0 and $\phi \colon \mathbb{Q}[\mathrm{H}_1(M;\mathbb{Z})] \to \mathbb{K}$ a ring homomorphism. We turn $\mathbb{K}$ into a representation of $\pi_1(M)$ by defining the right action to be

$$\begin{aligned} \mathbb{K} \times \pi_1(M) &\to \mathbb{K} \\ (z, g) &\mapsto z\phi(g) \end{aligned}$$

Lemma 4.19. *Let $\mathbb{K}$ be as above. The maximal Abelian torsion has the following properties:*

1. *If $\mathbb{K}$ is M-acyclic, then the equality $\tau(M, \mathfrak{e}; \mathbb{K}) = \phi(\Delta)$ holds.*
2. *If $\phi(\Delta) \neq 0$, then $\mathbb{K}$ is M-acyclic and consequently $\tau(M, \mathfrak{e}; \mathbb{K}) = \phi(\Delta)$.*

Proof. For the first property see [Tur01, Theorem 13.3]. The second property follows from functoriality of twisted Reidemeister torsion [Tur01, Proposition 3.6]. □

Corollary 4.20. *The maximal Abelian torsion is independent of the choice of splitting $\mathrm{H}_1(M;\mathbb{Z}) = T \oplus F$ and the choice of characters S.*

Having constructed the maximal Abelian torsion Δ, we introduce a new semi-norm on $\mathrm{H}_1(M;\mathbb{Z})$. Denote the torsion subgroup of $\mathrm{H}_1(M;\mathbb{Z})$ by T. For the construction expand the maximal Abelian torsion as follows:

$$\Delta = \sum_{h \in \mathrm{H}_1(M;\mathbb{Z})} a_h h \in \mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})] \text{ with } a_h \in \mathbb{Z}.$$

We also define the Alexander norm introduced by McMullen [McM02]. For this we consider the image Δ_{fr} of Δ in $\mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})/T]$, which we call *free Abelian torsion*. Again we introduce names for the coefficients:

$$\Delta_{fr} = \sum_{h \in \mathrm{H}_1(M;\mathbb{Z})/T} b_h h \in \mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})/T] \text{ with } b_h \in \mathbb{Z}.$$

Definition 4.21 (McMullen, Turaev)**.** Let M be an irreducible 3-manifold such that $b_1(M) \geq 2$. Let $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ be a cohomology class.

1. The *torsion norm* of θ is defined by

   $$\|\theta\|_{tr} := \max\{\theta(h) - \theta(h') : h, h' \in \mathrm{H}_1(M;\mathbb{Z}) \text{ with } a_h \neq 0 \text{ and } a_{h'} \neq 0\},$$

   where the $a_h \in \mathbb{Z}$ are the coefficients of the maximal Abelian torsion of M as above.

2. The *Alexander norm* of a class θ is defined by

   $$\|\theta\|_A := \max\{\theta(h) - \theta(h') : h, h' \in \mathrm{H}_1(M;\mathbb{Z})/T \text{ with } b_h \neq 0 \text{ and } b_{h'} \neq 0\},$$

   where the $b_h \in \mathbb{Z}$ are the coefficients of Δ_{fr} as above.

Here, in both cases we adopt the convention that the maximum over the empty set is zero.

Remark 4.22.
1. Although the maximal Abelian torsion depends on the choice of fundamental family, the torsion norm itself does not.
2. Clearly, the inequality $\|\theta\|_{tr} \geq \|\theta\|_A$ holds for all classes $\theta \in \mathrm{H}^1(M;\mathbb{Z})$.

The following inequality will play a key part in estimating the Thurston norm. We will focus on these estimates in more detail in Chapter 6.

Theorem 4.23. *Let M be an irreducible 3-manifold with $b_1(M) \geq 2$. Then every class $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ satisfies the inequality $\|\theta\|_T \geq \|\theta\|_{tr}$.*

Proof. See [Tur02, Theorem 2.2]. □

The next theorem will be the heart of this chapter.

Theorem 4.24. *Let N be a graph manifold with $b_1(N) \geq 2$. Then there is a finite cover $\pi\colon M \to N$ such that for every class $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ the equality $\|\theta\|_T = \|\theta\|_{tr}$ holds.*

Proof. By Lemma 2.19 we can pick a cover $\pi\colon M \to N$ such that M is either a torus bundle, or a Seifert fibred space, or a manifold which admits a composite graph structure. We consider the cases one by one.

If M is a torus bundle, then we know that the Thurston norm vanishes by Lemma 3.8. By Theorem 4.23 we immediately get the equality.

If M is a Seifert fibred space, we may assume that M is circle bundle by taking a further cover, see Theorem 2.6. If the circle bundle is non-trivial, then the Thurston norm vanishes, see Lemma 3.9. As above we obtain equality. So let M be a trivial circle bundle, i.e. $M \cong \Sigma \times S^1$ with Σ a surface with $\chi(\Sigma) < 0$. In this case the equality is shown by the calculation in Proposition 4.25 below.

The case where M admits a composite graph structure is dealt with in Theorem 4.36 below. □

4.5 Torsion of composite graph manifolds

First we illustrate the calculation of Reidemeister torsion for the product $\Sigma \times S^1$ with Σ being a surface. Then we proceed with the computation of Reidemeister torsion for general composite graph manifolds. We conclude with the calculation for arbitrary circle bundles.

The product structure of $\Sigma \times S^1$ gives an identification $\pi_1(\Sigma \times S^1) \cong \pi_1(\Sigma) \times \mathbb{Z}\langle t\rangle$, where $t \in \pi_1(S^1)$ is the generator traversing S^1 in positive direction.

Proposition 4.25. *Let V be a representation of $\mathbb{Z}[\pi_1(\Sigma \times S^1)]$ and suppose the endomorphism $(t-1)_V\colon V \to V$ is invertible. Then V is $(\Sigma \times S^1)$-acyclic and the twisted Reidemeister torsion is*

$$\tau(\Sigma \times S^1; V) = (\det{(t-1)_V})^{-\chi(\Sigma)} \in \mathrm{Wh}(V).$$

Proof.

1. First we set up the fundamental cover and a CW-structure. By Remark 4.9 the result will be independent of this choice. The cover $\exp\colon \mathbb{R} \to S^1$ is a universal cover. We equip S^1 with the CW-structure $\emptyset \subset \{1\} \subset S^1$. As a fundamental family we pick the cells $e_0 := \{0\} \subset \mathbb{R}$ and $e_1 := (0,1) \subset \mathbb{R}$.

   The characteristic map of the 1-cell gives rise to an element $t \in \pi_1(S^1)$. We choose an arbitrary CW-structure on Σ and give $\Sigma \times S^1$ the product CW-structure. Let $\widetilde{\Sigma}$ be the universal cover of Σ and $\{b^k_\lambda\}$ the k-cells of a fundamental family. We denote by n_k the number of k-cells of Σ. The Euler characteristic of Σ can be expressed in terms of the numbers n_k by $\chi(\Sigma) = \sum_k (-1)^k n_k$.

   We pick $\widetilde{\Sigma} \times \mathbb{R}$ as the universal cover of $\Sigma \times S^1$. We take the k-cells $b^k_\lambda \times e^0$ and $b^{k-1}_\lambda \times e^1$ as a fundamental family for $\widetilde{\Sigma} \times \mathbb{R}$.

2. Note that the k-cells of the fundamental family above are already divided into two classes. Namely, the ones made out of k-cells of $\widetilde{\Sigma}$ and the ones obtained by crossing a $k-1$-cell of $\widetilde{\Sigma}$ with the 1-cell e^1. We calculate the torsion using

Lemma 4.13. We split $\mathrm{C}_k(\Sigma \times S^1)$ as a based $\mathbb{Z}[\pi_1(\Sigma \times S^1)]$-module

$$\mathrm{C}_k(\Sigma \times S^1) = \underbrace{\bigoplus_{\lambda} \mathbb{Z}[\pi_1(\Sigma \times S^1)]\langle b_\lambda^k \times e^0\rangle}_{=:A_k} \oplus \underbrace{\bigoplus_{\lambda} \mathbb{Z}[\pi_1(\Sigma \times S^1)]\langle b_\lambda^{k-1} \times e^1\rangle}_{=:B_k}.$$

The boundary operator is of the following form:

$$\partial\left(b_\lambda^{k-1} \times e^1\right) = (-1)^{k-1}(t-1)\left(b_\lambda^{k-1} \times e^0\right) + (-1)^{k-1}\partial b^{k-1} \times e^1.$$

After tensoring let denote s_k the induced map $s_k \colon V \otimes B_k \to V \otimes A_k$. From the formula above we calculate $\det s_k = \pm\,(\det(t-1)_V)^{n_{k-1}}$. By Lemma 4.13, the twisted Reidemeister torsion of $\Sigma \times S^1$ is

$$\tau(\Sigma \times S^1; V) = \prod_{k=0}^{3} (\det(t-1)_V)^{-(-1)^k n_k} = (\det(t-1)_V)^{-\chi(\Sigma)} \in \mathrm{Wh}(V).$$

□

Remark 4.26. The proposition above also holds for the torus $S^1 \times S^1$ [Tur01, Lemma 11.11]. The twisted Reidemeister torsion is $\tau(S^1 \times S^1; V) = 1$.

We make a small excursion into representations, the ring of Laurent polynomials and how to recover information from Reidemeister torsion defined over this ring. This will be further expanded in Chapter 6.

Definition 4.27. Let M be a 3-manifold and $\mathbb{K}$ a field.

1. For each class $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ the representation $\mathbb{K}_\theta$ of $\pi_1(M)$ over $\mathbb{K}(t)$ has underlying $\mathbb{K}(t)$-vector space $\mathbb{K}(t)$. The right action of $\pi_1(M)$ is given by

$$\begin{aligned} \mathbb{K}_\theta \times \pi_1(M) &\to \mathbb{K}_\theta \\ (v, g) &\mapsto v t^{\langle \theta, g\rangle}, \end{aligned}$$

   where $\langle \theta, g\rangle$ is the evaluation of θ on the class corresponding to g in $\mathrm{H}_1(M;\mathbb{Z})$.

2. Let V be a representation of $\pi_1(M)$ over $\mathbb{K}$. Define V_θ to be the representation

$$V_\theta := \mathbb{K}_\theta \otimes_{\mathbb{K}} V,$$

   where the fundamental group acts diagonally from the right on the factors.

As hinted in the introduction, the function width will be our main tool to extract information from the twisted Reidemeister torsion. It is defined as follows:

Definition 4.28. Let $\mathbb{K}$ be a field.

1. The function width: $\mathbb{K}[t^{\pm 1}] \to \mathbb{N}$ is defined by

$$\operatorname{width}\left(\sum_{i=a}^{b} c_i t^i\right) = b - a,$$

where $c_a, c_b \neq 0$ and $a \leq b$.

2. It is extended to a function width: $\mathbb{K}(t) \to \mathbb{Z}$ by

$$\operatorname{width}(f/g) = \operatorname{width} f - \operatorname{width} g,$$

for $f, g \in \mathbb{K}[t^{\pm 1}]$ with $g \neq 0$.

Remark 4.29. Note that for $f, g \neq 0$, the equality $\operatorname{width}(fg) = \operatorname{width} f + \operatorname{width} g$ holds and so width: $\mathbb{K}(t) \to \mathbb{Z}$ is well-defined.

Let $k \geq 2$ be a natural number. The Abelian group $\mathbb{Z}/k\mathbb{Z}$ acts on the complex numbers $\mathbb{C}$ by multiplication with $\exp(2\pi i/k)$. Given a character $\alpha\colon \pi_1(M) \to \mathbb{Z}/k\mathbb{Z}$, we denote by $\mathbb{C}^\alpha$ the representation with right action

$$\begin{aligned} \mathbb{C}^\alpha \times \pi_1(M) &\to \mathbb{C}^\alpha \\ (v, g) &\mapsto v\alpha(g). \end{aligned}$$

These representations will be analysed in more detail in Chapter 6. The following lemma connects the Thurston norm and twisted Reidemeister torsion.

Lemma 4.30. *Let M be an irreducible 3-manifold with $b_1(M) \geq 2$ and a character $\alpha\colon M \to \mathbb{Z}/k\mathbb{Z}$ with $k \geq 2$. Denote the maximal Abelian torsion by $\Delta \in \mathbb{Z}[\mathrm{H}_1(M;\mathbb{Z})]$.*

Then every class $\theta \in \mathrm{H}_1(M;\mathbb{Z})$ such that $\mathbb{C}^\alpha_\theta$ is M-acyclic satisfies the inequality

$$\|\theta\|_T \geq \operatorname{width} \tau(M; \mathbb{C}^\alpha_\theta).$$

Proof. Note that the representation $\mathbb{C}^\alpha_\theta$ is obtained from the ring homomorphism

$$\begin{aligned} \phi\colon \mathbb{Q}[\mathrm{H}_1(M;\mathbb{Z})] &\to \mathbb{C}(t) \\ qh &\mapsto q\alpha(h)t^{\langle \theta, h\rangle} \text{ for } h \in \mathrm{H}_1(M;\mathbb{Z}), q \in \mathbb{Q} \end{aligned}$$

as in Lemma 4.19. Therefore, we have $\operatorname{width}\phi(\Delta) = \operatorname{width}\tau(M;\mathbb{C}^\alpha_\theta)$. From the definitions, we deduce:

$$\|\theta\|_{tr} \geq \operatorname{width}\phi(\Delta) = \operatorname{width}\tau(M;\mathbb{C}^\alpha_\theta).$$

The claim follows from the inequality $\|\theta\|_T \geq \|\theta\|_{tr}$, see Theorem 4.23. □

Lemma 4.31. *Let Σ be a connected surface with $\chi(\Sigma) < 0$. Let α denote the character*

$$\alpha\colon \pi_1(\Sigma \times S^1) \to \pi_1(S^1) = \mathbb{Z}\langle s\rangle \to \mathbb{Z}/k\mathbb{Z}.$$

for a natural number $k \geq 2$. For every class $\theta \in \mathrm{H}^1(\Sigma \times S^1; \mathbb{Z})$, the twisted Reidemeister torsion with respect to the representation $\mathbb{C}^\alpha_\theta$ is

$$\tau(\Sigma \times S^1; \mathbb{C}^\alpha_\theta) = \left(\alpha(s)t^{\theta(s)} - 1\right)^{-\chi(\Sigma)}.$$

Consequently, the equality $\operatorname{width} \tau(\Sigma \times S^1; \mathbb{C}^\alpha_\theta) = -\chi(\Sigma)|\langle \theta, s\rangle|$ *holds and so*

$$\|\theta\|_T = \|\theta\|_{tr}.$$

Proof. Note that $(s-1)$ acts as follows: $z \cdot (s-1) = \left(\alpha(s)t^{\theta(s)} - 1\right) z$ for all $z \in \mathbb{C}(t)$. Therefore $\det(s-1)_V = \left(\alpha(s)t^{\theta(s)} - 1\right)$ and by Proposition 4.25 we know the equality

$$\tau(\Sigma \times S^1; \mathbb{C}^\alpha_\theta) = \left(\alpha(s)t^{\theta(s)} - 1\right)^{-\chi(\Sigma)}.$$

Using the multiplicativity of the function width, we obtain the equalities

$$\operatorname{width} \tau(\Sigma \times S^1; \mathbb{C}^\alpha_\theta) = -\chi(\Sigma)|\langle \theta, s\rangle| = \chi_-(\Sigma)\,|\langle \theta, s\rangle|.$$

Recall the calculation of the Thurston norm of a product from Lemma 3.9. We obtain $\|\theta\|_{tr} \geq \operatorname{width} \tau(\Sigma \times S^1; \mathbb{C}^\alpha_\theta) = \|\theta\|_T$ and by the estimate from Lemma 4.30 it is an equality. □

Now we want to calculate the twisted Reidemeister torsion for a composite graph manifold. Fix a composite graph manifold M with toroidal surface S. Suppose that M admits a Seifert non-vanishing character $\alpha\colon M \to \mathbb{Z}/k\mathbb{Z}$ and has a bipartite Bass-Serre graph. Recall the construction of $M|S$. We equip each block B, a component of $M|S$, with the the CW-structure induced by the inclusion $\mathrm{gl}_S\colon B \to M$. Now the restriction $\mathrm{gl}_S\colon B \to M$ is the inclusion of a subcomplex. Pick a bipartition of the blocks $V(M) = V_+(M) \cup V_-(M)$. First we consider the chain complex $\mathrm{C}(B \subset M)$ for a block B.

Lemma 4.32. *Let B be a connected subcomplex of the connected CW-complex M. Let $\pi\colon \widetilde{M} \to M$ a universal cover and $\widetilde{B}$ a universal cover of B. For every lift $\phi\colon \widetilde{B} \to \widetilde{M}$ of $\widetilde{B}$ to the universal cover $\pi\colon \widetilde{M} \to M$, the following map is an isomorphism*

$$\begin{aligned}\phi\colon \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(B)]} \mathrm{C}(B) &\to \mathrm{C}(B \subset M)\\ g \otimes e &\mapsto g \cdot \phi_* e.\end{aligned}$$

Proof. Pick a component $\overline{B}$ of $\pi^{-1}(B)$ and a covering $\widetilde{B} \to \overline{B}$. Furthermore, we fix lifts $\widetilde{e} \subset \widetilde{B}$ for every cell $\overline{e} \subset \overline{B}$. We define an inverse on the cells of $\overline{B}$ and extend it equivariantly:

$$\begin{aligned}\psi\colon \mathrm{C}(B \subset M) &\to \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(B)]} \mathrm{C}(B)\\ g \cdot \overline{e} &\mapsto g \cdot \widetilde{e}.\end{aligned}$$

This is well-defined: if a cell can be translated to both $\overline{e}_1 \in \overline{B}$ and $\overline{e}_2 \in \overline{B}$, then there is a $h \in \pi_1(B)$ such that $h \cdot \overline{e}_1 = \overline{e}_2$. For the two choices the image of ψ is either $gh \otimes \widetilde{e}_1$ or $g \otimes \widetilde{h \cdot e_2}$. As $\widetilde{h \cdot e_2}$ and $h \cdot \widetilde{e}_2$ cover the same cell in $\overline{B}$, there is an element k in the kernel of $\pi_1(B) \to \pi_1(M)$ such that $\widetilde{h \cdot e_2} = kh \cdot \widetilde{e}_2$. As a result we obtain that the two constructed images are equal $gh \otimes \widetilde{e}_1 = g \otimes \widetilde{h \cdot e_2}$.

Now one can check that $\psi \circ \phi$ and $\phi \circ \psi$ are the corresponding identities. □

We have already seen that a fundamental family for M gives rise to a preferred basis of $\mathrm{C}(B \subset M)$. Via the above isomorphism we obtain a preferred basis for $\mathrm{C}(B)$ as well.

Recall that M is obtained by gluing the blocks along S as described in the diagram below:

$$\begin{array}{ccc} \coprod_{B \in V_+} B & \xrightarrow{\phi_+} & M \\ {\scriptstyle i^+}\uparrow & & \uparrow{\scriptstyle \phi_-} \\ S & \xrightarrow[i^-]{} & \coprod_{B \in V_-} B \end{array} .$$

Consequently, there is a short exact sequence:

$$0 \to \mathrm{C}(S \subset M) \xrightarrow{i_+ - i_-} \sum_{B \in V(M)} \mathrm{C}(B \subset M) \to \mathrm{C}(M) \to 0.$$

Lemma 4.33. *The short exact sequence above is based compatibly.*

Proof. Note that $S \subset M$ and $B \subset M$ are inclusions of subcomplexes for the toroidal surface S and every block B of M. Recall the way how we based the chain complexes $\mathrm{C}(S \subset M)$ and $\mathrm{C}(B \subset M)$ with a fundamental family in Definition 4.3. The cellular chain modules are freely generated by the cells. Note that the k-cells of a component $T \subset S$ are exactly the k-cells shared by the blocks $B_\pm$ filling T on the positive and negative side, i.e. $Z(T) = Z(B_+) \cap Z(B_-)$. Now we see that we have already studied the situation in Example 4.11. □

Let B be a block of M. The identification $B \cong \Sigma_B \times S^1$ is part of a composite graph structure. The class $[\{x\} \times S^1] \in \mathrm{H}_1(B; \mathbb{Z})$ is independent of the point $x \in \Sigma_B$ and is denoted by s_B. We have already calculated the twisted Reidemeister torsion for B in Proposition 4.25. In the theorem below we combine these calculations.

Theorem 4.34. *Let V be a representation of $\pi_1(M)$ such that $(1 - s_B)_V$ is invertible for all blocks $B \in V(M)$ of M. Then the following statements hold:*

1. *The representation V is M-acyclic.*

2. *The twisted Reidemeister torsion is*

$$\tau(M;V) = \prod_{B\in V(M)} (\det(s_B - 1)_V)^{-\chi(\Sigma_B)}.$$

Proof. The following sequence of chain complexes is short exact

$$0 \to V\otimes \mathrm{C}(S\subset M) \xrightarrow{i_+ - i_-} \sum_{B\in V(M)} V\otimes \mathrm{C}(B\subset M) \to V\otimes \mathrm{C}(M) \to 0.$$

and also based compatibly by Lemma 4.33. Recall that we constructed an isomorphism $V\otimes \mathrm{C}(B\subset M)\cong V\otimes \mathrm{C}(B)$ in Lemma 4.32. In Proposition 4.25 we have seen that the chain complex $V\otimes \mathrm{C}(B)$ is acyclic. By Remark 4.26 the same holds for the chain complex $V\otimes \mathrm{C}(S\subset M)$. With Lemma 4.12 we conclude that $V\otimes \mathrm{C}(M)$ is acyclic.

Using the same lemma, we obtain the equality

$$\tau(S;V)\cdot\tau(M;V) = \prod_{B\in V(M)} \tau(B;V) \in \mathrm{Wh}(V).$$

As $\tau(S;V)=1$ the claim follows. □

Specialising the theorem above to the representation $\mathbb{C}^{\alpha}_{\theta}$ we obtain the following corollary.

Corollary 4.35. *For every class $\theta\in \mathrm{H}^1(M;\mathbb{Z})$ the statements below hold.*

1. *$\mathbb{C}^{\alpha}_{\theta}$ is M-acyclic.*
2. *The maximal Abelian torsion of M is non-zero.*
3. *The twisted Reidemeister torsion of M is*

$$\tau(M;\mathbb{C}^{\alpha}_{\theta}) = \prod_{B\in V(M)} \left(\zeta_k t^{\langle\theta,s_B\rangle} - 1\right)^{-\chi(\Sigma_B)} \in \mathrm{Wh}(\mathbb{C}^{\alpha}_{\theta}),$$

where ζ_k is the complex number $\exp(2\pi i/k)\in\mathbb{C}$.

Theorem 4.36. *Let M be a composite graph manifold with a Seifert non-vanishing character. Suppose $b_1(M)\geq 2$. Then the Thurston norm and the torsion norm agree on M.*

Proof. Let $\theta\in \mathrm{H}^1(M;\mathbb{Z})$ be a cohomology class. In Corollary 4.35, we saw that the representation $\mathbb{C}^{\alpha}_{\theta}$ is M-acyclic and the twisted Reidemeister torsion is

$$\tau(M;\mathbb{C}^{\alpha}_{\theta}) = \prod_{B\in V(M)} \left(\zeta_k t^{\langle\theta,s_B\rangle} - 1\right)^{-\chi(\Sigma_B)} \in \mathrm{Wh}(\mathbb{C}^{\alpha}_{\theta}).$$

Using Lemma 4.30, we see that the torsion norm $\|\theta\|_{tr}$ fulfils the inequality

$$\begin{aligned}\|\theta\|_{tr} \geq \operatorname{width} \tau(M;\mathbb{C}^{\alpha}_{\theta}) &= \sum_{B\in V(M)} \operatorname{width}\left(\zeta_k t^{\langle\theta,s_B\rangle} - 1\right)^{-\chi(\Sigma_B)} \\ &= \sum_{B\in V(M)} \chi_-(\Sigma_B)|\langle\theta,s_B\rangle| = \sum_{B\in V(M)} \|i_B^*\theta\|_T,\end{aligned}$$

where the last equality follows from the calculation of the Thurston norm in Lemma 3.9. Theorem 3.6 tells us that the last sum is $\|\theta\|_T$. With Theorem 4.23 we conclude that $\|\theta\|_T = \|\theta\|_{tr}$ holds. □

We conclude this section with an application to circle bundles.

Lemma 4.37. *Let $p\colon M \to \Sigma$ be a non-trivial circle bundle over a closed surface Σ with non-positive Euler characteristic $\chi(\Sigma) \leq 0$. Let $e \in \mathbb{Z}$ be the Euler number of the bundle M. Suppose the Euler number $|e| > 1$. Then the maximal Abelian torsion of M is non-zero.*

Proof. Let $\overline{\Sigma}$ be the surface Σ with an open disc removed. Note that we construct a manifold diffeomorphic to M by gluing $\overline{\Sigma} \times S^1 \cup_\phi D^2 \times S^1$ along the map

$$\begin{aligned}\phi\colon \partial\overline{\Sigma} \times S^1 &\to \partial D^2 \times S^1 \\ (a,b) &\mapsto (a, b + e\cdot a)\end{aligned}$$

We consider the Mayer-Vietoris sequence associated to this decomposition

$$\mathrm{H}^1(S^1 \times S^1;\mathbb{Z}/e\mathbb{Z}) \leftarrow \mathrm{H}^1(\overline{\Sigma}\times S^1;\mathbb{Z}/e\mathbb{Z}) \oplus \mathrm{H}^1(D^2\times S^1;\mathbb{Z}/e\mathbb{Z}) \leftarrow \mathrm{H}^1(M;\mathbb{Z}/e\mathbb{Z}).$$

We conclude that there exists an element $\beta \in \mathrm{H}^1(M)$ which maps the class of a fibre to $1 \in \mathbb{Z}/e\mathbb{Z}$. We claim that the representation $\mathbb{C}^\beta$ is M-acyclic.

As above we have the exact sequence of chain complexes, which is based compatibly,

$$\begin{aligned}0 \to \mathrm{C}(S^1\times S^1 \subset M;\mathbb{C}^\beta) \to \mathrm{C}(\overline{\Sigma}\times S^1 \subset M;\mathbb{C}^\beta) \oplus \mathrm{C}(D^2\times S^1 \subset M;\mathbb{C}^\beta) \\ \to \mathrm{C}(M;\mathbb{C}^\beta) \to 0\end{aligned}$$

Using Lemma 4.32 and the calculations in this section, we obtain that $\mathbb{C}^\beta$ is M-acyclic. We abbreviate $\exp(2\pi i/e) \in \mathbb{C}$ with ζ_e. We obtain

$$\tau(M;\mathbb{C}^\beta) = (1-\zeta_e)^{-\chi(\Sigma)}.$$

By Lemma 4.19 the maximal Abelian torsion is non-zero. □

Remark 4.38. For a non-trivial circle bundle, we can arrange that the condition $|e| > 1$ holds by taking a finite cover.

4.6 Fibred 3-manifolds

Recall that a 3-manifold M fibres if there exists a map $p\colon M \to S^1$ turning M into a surface bundle. This implies that M is diffeomorphic to the mapping torus of a surface.

Definition 4.39. Let Σ be a surface and $\phi\colon \Sigma \to \Sigma$ an orientation preserving diffeomorphism. The *mapping torus* $\mathrm{M}(\Sigma, \phi)$ is the quotient

$$\mathrm{M}(\Sigma, \phi) := \Sigma \times \mathbb{R}/(\phi(x), t) \sim (x, t+1) \, .$$

Definition 4.40. A class $\theta \in \mathrm{H}^1(M; \mathbb{Q})$ is *fibred* if there exists a map $p\colon M \to S^1$ giving M the structure of a fibre bundle and a class $\tau \in \mathrm{H}^1(S^1; \mathbb{Q})$ such that $p^*\tau = \theta$.

Agol realised that being able to approximate classes by fibred classes in finite covers is related to the following property of the fundamental group. We use the formulation of the survey article [AFW12, Section 6 E.4].

Definition 4.41 (Agol)**.** A group G is called *RFRS* if there exists a filtration

$$G = G_0 \supset G_1 \supset G_2 \supset \dots$$

by normal finite-index subgroups $G_i \subset G$ such that

1. their intersection $\bigcap_{i \in \mathbb{N}} G_i$ is trivial, and
2. for each $i \in \mathbb{N}$ and $H_i := \mathrm{H}_1(G_i; \mathbb{Z})/\text{torsion}$ the quotient map $G_i \to G_i/G_{i+1}$ factors through the quotient map $G_i \to H_i$:

$$\begin{array}{ccc} G_i & \longrightarrow & G_i/G_{i+1} \\ & \searrow \quad \nearrow & \\ & H_i & \end{array} .$$

Theorem 4.42 (Agol)**.** *Let N be an aspherical 3-manifold with virtually RFRS fundamental group. Let $\theta \in \mathrm{H}^1(N; \mathbb{Q})$ be a non-zero cohomology class. Then there exists a finite cover $\pi\colon M \to N$ and fibred classes θ_n converging to $\pi^*\theta$.*

Proof. [Ago08, Theorem 5.1] □

Remark 4.43. A possible first impression that having a virtually RFRS fundamental group is rare condition is deceptive. By results due to Wise [Wis11], Przytycki-Wise [PW12] and Agol [Ago13] an irreducible 3-manifold M whose JSJ decomposition has at least one hyperbolic piece has a virtually RFRS fundamental group $\pi_1(M)$. As a consequence the manifolds which do not have such a fundamental group are all graph manifolds.

A version of Theorem 4.24 has also been proven for aspherical 3-manifolds with virtually RFRS fundamental group.

Theorem 4.44 (Friedl-Vidussi)**.** *Let N be an aspherical 3-manifold with virtually RFRS fundamental group and $b_1(N) \geq 2$. Let $\theta \in \mathrm{H}^1(N;\mathbb{Z})$ be a cohomology class. There exists a finite cover $\pi\colon M \to N$ such that both statements hold:*

1. *the maximal Abelian torsion of M is non-zero, and*
2. *the torsion norm satisfies the equality $\|\pi^*\theta\|_A = \|\pi^*\theta\|_{tr} = \|\pi^*\theta\|_T$.*

We refer to the article [FV12, Theorem 5.9] and sketch a proof.

Sketch. Let $\mathrm{M}(\Sigma,\phi)$ be a mapping torus with Σ not diffeomorphic to S^2 or D^2. Denote the torsion subgroup of $\mathrm{H}_1(\mathrm{M}(\Sigma,\phi);\mathbb{Z})$ by T. A calculation using an explicit CW-structure establishes the following: for a fibred class θ the representation $\mathbb{C}_\theta$ is $\mathrm{M}(\Sigma,\phi)$-acyclic [FK06, Theorem 1.2]. This implies that the maximal Abelian torsion is non-zero. Furthermore, the twisted Reidemeister torsion fulfils the equation

$$\operatorname{width} \tau(M;\mathbb{C}_\theta) = \|\alpha\|_T.$$

If $b_1(\mathrm{M}(\Sigma,\phi)) \geq 2$, then this implies that $\|\alpha\|_A = \|\alpha\|_{tr} = \|\alpha\|_T$ for fibred classes α by Lemma 4.19 and the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{H}_1(\mathrm{M}(\Sigma,\phi);\mathbb{Z})] & \longrightarrow & \mathbb{Z}[\mathrm{H}_1(\mathrm{M}(\Sigma,\phi);\mathbb{Z})/T] \\ & \searrow & \downarrow \\ & & \mathbb{C}_\theta \end{array}.$$

As all three semi-norms are continuous the equality also holds for classes in the closure of fibred classes.

By Agol's virtual fibring theorem, which is Theorem 4.42, there is a finite cover $\pi\colon M \to N$ such that M fibres and $\pi^*\theta$ is in the closure of fibred classes. As M fibres we have that the maximal Abelian torsion is non-zero. By the argument above, we obtain the equalities

$$\|\pi^*\theta\|_A = \|\pi^*\theta\|_{tr} = \|\pi^*\theta\|_T.$$

□

5 Circle bundles over 3-manifolds

The main result of this chapter is the theorem below. The discussion follows along the lines of the author's article [Nag14]. Recall that by convention circle bundles have a orientation of the fibres and so are principal S^1-bundles.

Theorem 5.1. *Let M be a closed irreducible 3-manifold neither covered by S^3 nor a torus bundle. Let $p\colon W \to M$ be a circle bundle over M.*

Then the complexity $x(\sigma)$ of every class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ satisfies the inequality

$$x(\sigma) \geq |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

Proof. See Proposition 5.15 and Corollary 5.20. □

Let us first review the notions which are used in the theorem. On a 4-manifold, one defines the complexity $x(\sigma)$ of a class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ similar to the definition of the Thurston norm on a 3-manifold, see Definition 5.2.

As we have discussed in the introduction, the complexity $x(\sigma)$ is in general not linear in the class σ. This is reflected in the above lower bound by the term $|\sigma \cdot \sigma|$, the self-intersection of the class σ in the manifold W. Let $\Sigma \subset W$ be an embedded surface with fundamental class σ. The normal bundle $\nu(\Sigma)$ of Σ has this number as its Euler number $\langle e(\nu(\Sigma)), [\Sigma]\rangle = \sigma \cdot \sigma$. Then the self-intersection has the following topological interpretation. By the tubular neighbourhood theorem $\nu(\Sigma)$ describes a neighbourhood of Σ. This implies that if $\sigma \cdot \sigma$ is non-zero, then we cannot realise multiples of σ by push-offs of Σ.

For the case of circle bundles over irreducible 3-manifolds where the 3-manifolds have virtually RFRS fundamental group, we rely on prior work of Friedl-Vidussi [FV14]. We sketch some of their arguments to give the reader a complete picture. The theorem above supersedes their result in two ways: we include all graph manifolds apart from the mentioned exceptions and we do not have to exclude any circle bundle.

5.1 The genus function in 4-manifolds

Similar to the Thurston norm for 3-manifolds, defined in Definition 3.2, one can define a corresponding invariant for 4-manifolds:

Definition 5.2. Let W be a closed 4-manifold and $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ a class. The *complexity* of σ is the number

$$x(\sigma) := \min\{\chi_-(\Sigma) : \Sigma \text{ an embedded surface with } [\Sigma] = \sigma\},$$

where $\chi_-(\Sigma)$ is the invariant defined in Definition 3.1.

Remark 5.3. 1. We have seen in Remark 3.3 that in a 3-manifold M every class in $\mathrm{H}_2(M;\mathbb{Z})$ is the fundamental class of an embedded surface. This also holds in dimension 4. We consider a cohomology class $\theta \in \mathrm{H}^2(W;\mathbb{Z})$. To this class corresponds a map into $\mathbb{CP}^\infty$, the Eilenberg-Mac Lane space $K(\mathbb{Z},2)$. Any such map can be perturbed to a smooth map into a $\mathbb{CP}^N$. Furthermore, we isotope it to a map $f\colon W \to \mathbb{CP}^N$ transverse to $\mathbb{CP}^{N-1} \subset \mathbb{CP}^N$. The preimage $f^{-1}(\mathbb{CP}^{N-1})$ will be a surface Poincaré dual to the class θ.

2. We use $\chi_-(\Sigma)$ to measure the complexity of an embedded surface Σ. Traditionally the genus $g(\Sigma)$ is used. Analogously to the definition above, one also obtains an invariant for each class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$. This invariant is called the genus function. Note that the genus directly translates into the Euler characteristic by $\chi(\Sigma) = 2 - 2g(\Sigma)$. If W has the property that every embedding $f\colon S^2 \to W$ of a 2-sphere extends to a map of $D^3 \to W$, then we are free to remove the spherical components of Σ and obtain the equality $\chi_-(\Sigma) = 2g(\Sigma) - 2$. This is the case for the circle bundles we consider in Theorem 5.1.

The genus function in a 4-manifold depends on its smooth structure. This can be seen as follows: Denote the K3-surface by S_4. The smooth 4-manifolds $S_4\#\overline{\mathbb{CP}^2}$ and $\#3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ are homeomorphic, but not diffeomorphic [GS99, Exercise 2.4.13]. This is also reflected in the genus functions which are different. To calculate these and to see that they are not diffeomorphic one relies on Seiberg-Witten invariants.

5.2 Tangential structures

The Lie group $\mathrm{SO}(n)$ admits a two-fold connected cover called $\mathrm{Spin}(n)$ for $n \geq 2$. The Lie group $\mathrm{Spin}^c(n)$ is then defined to be the quotient

$$\mathrm{Spin}^c(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} S^1.$$

This Lie group $\mathrm{Spin}^c(n)$ comes with maps $\det\colon \mathrm{Spin}^c(n) \to S^1$ and $\mu\colon \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ induced by the projection on the factors.

Given a smooth oriented n-manifold X with a Riemannian metric, we can form its orthonormal frame bundle $\mathrm{Fr}(TX)$, which is an $\mathrm{SO}(n)$-principal bundle.

Recall that given a principal G-bundle P over X and a Lie group homomorphism $f\colon G \to S^1$, we can construct a complex line bundle by clamping $\mathbb{C}$ to the fibres:

$$P \times_f \mathbb{C} := P \times \mathbb{C}/(pg, z) \sim (p, f(g)z) \text{ for all } p \in P, g \in G, z \in \mathbb{C}.$$

Definition 5.4. A Spin^c-*structure* $\xi = (P,\kappa)$ on a smooth oriented n-manifold X is a $\mathrm{Spin}^c(n)$-principal bundle P together with a two-fold covering map $\kappa\colon P \to \mathrm{Fr}(TX)$

such that the following diagram is commutative:

$$\begin{array}{ccc} P \times \mathrm{Spin}^c(n) & \longrightarrow & P \\ \Big\downarrow \kappa \times \mu & & \Big\downarrow \kappa \\ \mathrm{Fr}(TX) \times \mathrm{SO}(n) & \longrightarrow & \mathrm{Fr}(TX) \end{array} .$$

The horizontal maps are the right actions of the respective principal bundle structures. The *determinant line bundle* L_ξ is the complex line bundle $P \times_{\det} \mathbb{C}$. Its Chern class is denoted by $c_1(\xi) \in \mathrm{H}^2(X;\mathbb{Z})$.

Remark 5.5. Orientable 3-manifolds are parallelizable [Sti35, Satz F] and so they admit a Spin^c-structure. Also orientable 4-manifolds admit a Spin^c-structure [HH58, Section 4.1 iv)] .

If $\pi\colon \widetilde{X} \to X$ is a covering map, then the differential $T\pi$ induces a bundle isomorphism $T\widetilde{X} \cong \pi^*TX$. Given a Spin^c-structure $\xi := (P,\kappa)$ on X, we can pull the principal bundle P back to a bundle π^*P over $\pi^*\,\mathrm{Fr}(TX) \cong \mathrm{Fr}(T\widetilde{X})$. The bundle π^*P together with the map $\pi^*\kappa$ is again a Spin^c-structure on $\widetilde{X}$, which we denote by $\pi^*\xi$. Its Chern class is $c_1(\pi^*\xi) = \pi^*c_1(\xi)$.

The set of Spin^c-structures on X is an $\mathrm{H}^2(X;\mathbb{Z})$-torsor, i.e. it has a transitive free action of the group $\mathrm{H}^2(X;\mathbb{Z})$ [Mor96, Chapter 3.1]. Under this action the Chern class changes by $c_1(\xi + e) = c_1(\xi) + 2e$ for every $e \in \mathrm{H}^2(X;\mathbb{Z})$.

For the rest of the chapter we will consider a circle bundle $p\colon W \to M$ over a closed irreducible 3-manifold M. A connection on this fibre bundle gives a splitting of the bundle $TW = p^*TM \oplus \underline{\mathbb{R}}$. Note that we used here that the fibres of W are oriented. Via this splitting we can add to a frame of TM the positive unit vector in $\mathbb{R}$ and obtain a frame in TW. This exposes the pull-back of the frame bundle $p^*\,\mathrm{Fr}(TM)$ as a reduction of the structure group to $\mathrm{SO}(3)$ of the bundle $\mathrm{Fr}(TW)$, i.e. we have $p^*\,\mathrm{Fr}(TM) \times_{\mathrm{SO}(3)} \mathrm{SO}(4) = \mathrm{Fr}(TW)$. Consequently, if $\xi := (P,\kappa)$ is a Spin^c-structure on M, then the bundle $p^*P \times_{\mathrm{Spin}^c(3)} \mathrm{Spin}^c(4)$ and the map $\kappa \times \mu$ form a Spin^c-structure on W. We denote this Spin^c-structure by $p^*\xi$. Its Chern class is $c_1(p^*\xi) = p^*c_1(\xi)$.

5.3 Adjunction inequality

Often the only way to obtain estimates on the complexity $x(\sigma)$ for a class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ is through adjunction inequalities, which we will describe below. To know which adjunction inequalities hold, one has to determine the Seiberg-Witten basic classes.

Seiberg-Witten theory is a gauge theory, whose equations were suggested by Witten [Wit94] and further enriched by Kronheimer-Mrowka [KM94]. We refer to Morgan's book [Mor96] for an introduction to this theory.

In the cases we consider later on, we will always have $b_2^+(W) > 1$. From the moduli space of solutions of the Seiberg-Witten equations a function $\mathrm{SW}\colon \mathrm{Spin}^c(W) \to \mathbb{Z}$ is constructed. It is non-zero only on finitely many Spin^c-structures.

Recall that $g^!$ denotes the umkehr map $\mathrm{PD} \circ (g^*) \circ \mathrm{PD}$ corresponding to a map g between two closed manifolds.

Definition 5.6. The set of *basic classes* of the 4-manifold W is the set

$$\mathrm{bas}\,(W) := \{c_1(\xi) \in \mathrm{H}^2(W;\mathbb{Z}) : \xi \in \mathrm{Spin}^c\,(W) \text{ with } \mathrm{SW}\,\xi \neq 0\}.$$

For a 3-manifold M with $b_1(M) \geq 3$ and a Spin^c-structure $\xi \in \mathrm{Spin}^c\,(M)$, we define its Seiberg-Witten invariant $\mathrm{SW}\,\xi$ by $\mathrm{SW}\,\xi := \mathrm{SW}\,p^*\xi$, where $p\colon M \times S^1 \to M$ denotes the projection. This is merely convenience, there are more intrinsic formulation of the Seiberg-Witten invariant in dimension 3, see e.g. [Auc96].

Definition 5.7. A closed irreducible 3-manifold N has *enough basic classes* if for every finite cover $\pi\colon M \to N$ and every class $\sigma \in \mathrm{H}_2(M;\mathbb{Z})$, there is a further cover $g\colon P \to M$ with $b_1(P) \geq 3$ and a basic class $s \in \mathrm{bas}\,(P)$ with

$$\langle s, g^!\sigma\rangle = \|g^!\sigma\|_T.$$

Remark 5.8. The condition $b_1(M) \geq 3$ ensures that any circle bundle $p\colon W \to M$ over the 3-manifold M fulfils $b_2^+(W) > 1$. This follows from the Gysin sequence. The technical condition $b_2^+(W) > 1$ is important for the moduli space considerations in Seiberg-Witten theory.

Kronheimer-Mrowka [KM94, Section 6] realised that Seiberg-Witten basic classes give a way to bound the complexity of a homology 2-class from below. They obtain the adjunction inequality for positive self-intersection. Later Ozsváth-Szabó [OS00, Corollary 1.7] also included the case of negative self-intersection. In the special case of a circle bundle this inequality can be formulated as follows:

Theorem 5.9 (Adjunction inequality)**.** *Let $p\colon W \to M$ be a circle bundle over an closed irreducible 3-manifold M with $b_1(M) \geq 3$.*

Every basic class $s \in \mathrm{bas}\,(W)$ and every homology class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ satisfies the inequality

$$x(\sigma) \geq |\sigma \cdot \sigma| + \langle s, \sigma\rangle.$$

Proof. See [FV14, Theorem 3.1]. The argument only uses the adjunction inequality for positive self-intersections. Furthermore, irreducibility of M is used to show that there is always a minimal surface representing the class σ without any spherical components. □

We see that it will be key to identify the Seiberg-Witten basic classes of the circle bundle. This happens in two steps. Theorem 5.11 below relates the Seiberg-Witten invariants of the base manifold to ones of the total space.

The second step is to describe the Seiberg-Witten invariants of the base manifold. Meng-Taubes [MT96] discovered that the Seiberg-Witten invariant is related to Milnor torsion. This was refined by Turaev [Tur98, Theorem 1] leading to a combinatorial description of the Seiberg-Witten invariant in dimension 3. We will use the following formulation, which is very convenient for the use with the adjunction inequality.

Theorem 5.10 (Turaev)**.** *Let M be a closed 3-manifold with $b_1(M) \geq 2$. If the maximal Abelian torsion is non-zero, then* $\operatorname{bas}(M)$ *is non-empty. In this case the equality below holds for every class $\sigma \in \mathrm{H}_2(M;\mathbb{Z})$:*

$$\|\sigma\|_{tr} = \max_{s \in \operatorname{bas}(M)} \langle s, \sigma \rangle.$$

Proof. See [Tur02, Chapter IX.1.2, XI.2.3]. □

The following theorem relates the Seiberg-Witten invariants of the total space of a circle bundle to the Seiberg-Witten invariants of the basis.

Theorem 5.11 (Baldrige)**.** *Let $p \colon W \to M$ be a circle bundle over a closed irreducible 3-manifold M with $b_1(M) \geq 3$. Let $\xi \in \mathrm{Spin}^c(M)$ be a Spin^c-structure on M.*

1. *If the Euler class $e \in \mathrm{H}^2(M;\mathbb{Z})$ is non-torsion, then the equality*

$$\mathrm{SW}(p^*\xi) = \sum_{k \in \mathbb{Z}} \mathrm{SW}(\xi + ke)$$

holds.

2. *If the circle bundle is trivial, then the Seiberg-Witten invariant satisfies the equality $\mathrm{SW}(p^*\xi) = \mathrm{SW}(\xi)$.*

Proof. See [Bal01, Theorem 1, Proposition 7]. □

We conclude this section with a description of the closed irreducible 3-manifolds with enough basic classes.

Proposition 5.12. *Let N be a closed irreducible 3-manifold which is not covered by S^3 nor a torus bundle. Then N has enough basic classes.*

Proof. Note that N is a graph manifold or N is aspherical with a virtually RFRS fundamental group. Recall that the Betti numbers can only increase in a finite cover.

Claim. In both cases the manifold N admits a finite cover M with $b_1(M) \geq 3$.

By assumption the group $\pi_1(N)$ cannot be either finite nor solvable [AFW12, Theorem 1.20]. In this case N has to have infinite virtual Betti number: this holds for manifolds N with virtually RFRS fundamental group [AFW12, Diagram 4] and graph manifolds [AFW12, Diagram 1]. Thus the claim holds.

We proceed with establishing the the proposition. Let $\pi \colon M \to N$ be a finite cover. Without loss of generality we may assume that $b_1(M) \geq 3$ as we can consider the fibre product of two covers and the claim above. Furthermore, let $\sigma \in \mathrm{H}_2(M;\mathbb{Z})$ be a given class. We want to find a cover $g \colon P \to M$ such that the maximal Abelian torsion of P is non-zero and the Thurston norm and the torsion norm agrees. Then by Theorem 5.10 there is a class $s \in \operatorname{bas}(P)$ with $\|g^!\sigma\|_T = \langle s, \sigma \rangle$.

In the case of composite graph manifolds this can be arranged by Theorem 4.24 and Corollary 4.35. For non-trivial circle bundles we have seen that these manifolds have vanishing Thurston norm in Lemma 3.9. Their maximal Abelian torsion is non-zero by Lemma 4.37.

If M is aspherical and has virtually RFRS fundamental group, then this can be deduced with Theorem 4.44. $\square$

5.4 Finite covers

Let $p\colon W \to N$ a circle bundle and $\pi\colon M \to N$ a finite cover of closed 3-manifolds. We can pull back the bundle along the cover π and obtain a circle bundle $p'\colon W' \to M$. The diagram below describes the situation.

$$\begin{array}{ccc} \pi^*W & \xrightarrow{\pi'} & W \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ M & \xrightarrow[\pi]{} & N \end{array} \tag{5.1}$$

Definition 5.13. 1. An embedded surface $\Sigma \subset W$ with fundamental class σ is called *non-degenerate* if Σ fulfils the inequality

$$\chi_-(\Sigma) \geq |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

2. A class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ is called *non-degenerate* if σ fulfils the inequality

$$x(\sigma) \geq |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

The lemma below shows that the property of being non-degenerate is compatible with taking finite covers.

Lemma 5.14. *Let Σ be an embedded surface and $\sigma \in \mathrm{H}_2(W;\mathbb{Q})$ a homology class. Then the three statements below hold:*

1. *On rational homology the equality $\pi^! p_* = (p')_*(\pi')^!$ holds.*
2. *If the surface $(p')^{-1}(\Sigma)$ is non-degenerate, then so is Σ.*
3. *If the class $(p')^!\sigma$ is non-degenerate, then so is σ.*

Proof. We abbreviate $(p')^{-1}(\Sigma)$ with Σ' and respectively $(p')^!\sigma$ with σ'.

1. First we establish the equality $\pi^! p_* = (p')_*(\pi')^!$ on rational homology. As π and π' are finite covers, we have that $\pi_* \circ \pi^! = \deg \pi$ and $\pi'_* \circ (\pi')^! = \deg(\pi') = \deg \pi$ [Bre93, Proposition 14.1 (6)]. The commutativity of the diagram above amounts to the equality $p_* \circ \pi'_* = \pi_* \circ p'_*$. Therefore, we can make the following deduction:

$$\begin{aligned} \left(\pi_* \circ \pi^!\right) \circ p_* &= p_* \circ \left(\pi'_* \circ (\pi')^!\right) \\ \Rightarrow \pi_* \circ \pi^! \circ p_* &= \pi_* \circ {p'}_* \circ (\pi')^! \\ \Rightarrow \pi^! \circ p_* &= (p')_* \circ (\pi')^!. \end{aligned}$$

2. The complexity $\chi_-(\Sigma)$ is multiplicative, so $(\deg \pi)\ \chi_-(\Sigma) = \chi_-(\Sigma')$. Recall that one can calculate $\sigma \cdot \sigma$ by counting self-intersections. Thus we obtain the equality $(\deg \pi)\ \sigma \cdot \sigma = \sigma' \cdot \sigma'$. As Σ' is non-degenerate, the complexity $\chi_-(\Sigma)$ satisfies the inequality $\chi_-(\Sigma') \geq |\sigma' \cdot \sigma'| + \|p'_*\sigma'\|_T$. By the calculation above, we know that $p'_*\sigma' = \pi^! p_*\sigma$ holds. Using Gabai's Theorem 3.5 we obtain $\|\pi^! p_*\sigma\|_T = \deg \pi \|\sigma\|_T$ and therefore also the inequality

$$\chi_-(\Sigma) \geq |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

This establishes the second statement.

3. We proceed to prove the third statement. By the above considerations it will be enough to show the inequality $(\deg \pi)\ x(\sigma) \geq x(\sigma')$. Denote by Θ the set of surfaces which are embedded into N and have fundamental class σ. Then we have the following estimate

$$(\deg \pi)\ x(\sigma) = \min_{S \in \Theta} \chi_-\left((\pi')^{-1}(S)\right) \geq x(\sigma').$$

This proves the statement.

□

Now we can prove Theorem 5.1 for circle bundles with torsion Euler class.

Proposition 5.15. *Let N be a closed irreducible 3-manifold with enough basic classes and $p\colon W \to N$ a circle bundle whose Euler class is torsion. Then each class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ is non-degenerate.*

Proof. First we prove the statement for a trivial circle bundle $p\colon W \to N$. We use the notation from Diagram 5.1. Pick a class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$. As N has enough basic classes there is a cover $\pi\colon M \to N$ such that $b_1(M) \geq 3$ and a basic class $s \in \mathrm{bas}\,(M)$ such that $\langle s, \pi^! p_*\sigma\rangle = \|\pi^! p_*\sigma\|_T$. By the lemma above we have $\pi^! p_*\sigma = p'_*(\pi')^!\sigma$. By Theorem 5.11 the class $p^*s \in \mathrm{H}^2(\pi^*W;\mathbb{Z})$ is basic if $s \in \mathrm{H}^2(M;\mathbb{Z})$ is basic. Using the adjunction inequality from Theorem 5.9 we obtain that the class $\sigma' := (\pi')^!\sigma$ satisfies the equality

$$x(\sigma') \geq |\sigma' \cdot \sigma'| + \langle p^*s, \sigma'\rangle = |\sigma' \cdot \sigma'| + \|(p')_*\sigma'\|_T.$$

Therefore the class σ' is non-degenerate and hence so is the class σ by Lemma 5.14.

Now the general statement follows from Lemma 5.14 and the fact that N has a finite cover $\pi\colon M \to N$ such that the pulled-back circle bundle π^*W is trivial, see Bowden [Bow09, Proposition 3]. □

5.5 Drilling circle bundles

In this section M denotes a closed irreducible 3-manifold with enough basic classes.

Lemma 5.16. *Let $\gamma \subset M$ be an embedded loop. Let $p\colon W \to M$ be a circle bundle with Euler class $e \in \mathrm{H}^2(M;\mathbb{Z})$. Furthermore, let $p_k\colon W_k \to M$ be the circle bundle with Euler class $e + k\,\mathrm{PD}[\gamma]$ for a $k \in \mathbb{Z}$. Then the two circle bundles become isomorphic when restricted to $M \setminus \nu(\gamma)$, i.e.*

$$W\,|_{M\setminus\nu(\gamma)} \cong W_k\,|_{M\setminus\nu(\gamma)}\;.$$

Proof. Denote the inclusion $M \setminus \nu(\gamma) \subset M$ by i_ν. First we calculate that $i_\nu^*\,\mathrm{PD}[\gamma] = 0$. Consider the diagram

$$\begin{array}{ccc}
\mathrm{H}^2(M;\mathbb{Z}) & \xrightarrow{\;i_\nu^*\;} & \mathrm{H}^2(M\setminus\nu(\gamma);\mathbb{Z}) \\
 & & \downarrow \mathrm{PD} \\
\mathrm{PD}\downarrow & & \mathrm{H}_1(M\setminus\nu(\gamma),\partial\nu(\gamma);\mathbb{Z}) \\
 & & \downarrow \cong \\
\mathrm{H}_1(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(M,\nu(\gamma);\mathbb{Z})
\end{array}\;.$$

It is commutative [Bre93, Corollary VI.8.4] and by comparing the two ways to go around we obtain the equality $i_\nu^*\,\mathrm{PD}[\gamma] = 0$.

As the Euler class is natural, we have the equality

$$e\left(W\,|_{M\setminus\nu(\gamma)}\right) = i_\nu^* e = e\left(W_k\,|_{M\setminus\nu(\gamma)}\right).$$

Recall that the classifying space of principal circle bundles BS^1 is an Eilenberg-Mac Lane space $K(\mathbb{Z},2)$. Consequently, we have natural bijections

$$\{S^1\text{-principal bundles over } M\}/\text{iso.} \cong [M, BS^1] \cong \mathrm{H}^2(M;\mathbb{Z}),$$

whose composition assigns to a bundle its Euler class. So the Euler class classifies circle bundles up to bundle isomorphisms and the claim follows. □

To apply the adjunction inequality we need to use Baldrige's theorem to translate knowledge on the 3-dimensional Seiberg-Witten invariant to the 4-dimensional one. But in this formula it is very well possible that the summands cancel. Nevertheless, this can only happen for very special Euler classes. Furthermore, the Euler class is very sensitive to local changes near essential loops. This is used to prove the following theorem.

Lemma 5.17. *Let $\Sigma \subset W$ be an embedded surface in the circle bundle $p\colon W \to M$ and γ a curve embedded into M which represents a non-torsion class $[\gamma] \in \mathrm{H}_1(M;\mathbb{Z})$. Suppose that Σ and the torus $p^{-1}(\gamma)$ are disjoint. Then Σ is non-degenerate.*

Proof. Denote the fundamental class of the surface Σ by σ. As a first case we consider the situation where there is a basic class $s \in \operatorname{bas}(M)$ such that $\langle s, p_*\sigma\rangle = \|p_*\sigma\|_T$. Pick a Spin^c-structure $\xi \in \operatorname{Spin}^c(M)$ with Chern class $c_1(\xi) = s$ and $\operatorname{SW}\xi \neq 0$.

Abbreviate the Euler class of the circle bundle W with $e \in \mathrm{H}^2(M;\mathbb{Z})$. Let the bundle $p_k\colon W_k \to M$ be a circle bundle with Euler class $e_k = e + k\,\mathrm{PD}[\gamma]$. In Lemma 5.16 we have seen that the total spaces of these two bundles are diffeomorphic away from a neighbourhood $\nu\left(p^{-1}(\gamma)\right)$. As the surfaces Σ and $p^{-1}(\gamma)$ are disjoint, we can consider Σ also as embedded into circle bundle W_k

$$\Sigma \subset W \setminus \nu\left(p^{-1}(\gamma)\right) \subset W_k.$$

Its fundamental class will be denoted by $\sigma_k \in \mathrm{H}_2(W_k;\mathbb{Z})$. As both $\sigma_k \cdot \sigma_k$ and $\sigma \cdot \sigma$ can be calculated from the normal bundle of Σ, they will be equal.

As the Seiberg-Witten function SW has finite support and $[\gamma]$ is non-torsion, there exists a $k \in \mathbb{Z}$ such that for all $l \neq 0$ the invariant $\operatorname{SW}(\xi + le_k)$ vanishes. For such a k, Baldrige's formula from Theorem 5.11 shows that $\operatorname{SW}(p_k^*\xi)$ is non-zero and therefore p_k^*s is a basic class of W_k. Using Theorem 5.9 we obtain the inequality

$$\chi_-(\Sigma) \geq x(\sigma_k) \geq |\sigma_k \cdot \sigma_k| + \langle p^*s, \sigma_k\rangle = |\sigma\cdot\sigma| + \langle s, p_*\sigma\rangle = |\sigma\cdot\sigma| + \|p_*\sigma\|_T.$$

Thus Σ is non-degenerate, which concludes the first case.

Now we reduce the general case to the one above. The manifold M has enough basic classes and thus there is a cover $\pi\colon P \to M$ and basic class $s \in \operatorname{bas}(P)$ such that $\langle s, \pi^!\sigma\rangle = \|\pi^!\sigma\|_T$. We pull-back the circle bundle along π and denote the cover of the total spaces with $g\colon \pi^*W \to W$. Abbreviate $\sigma' := g^!\sigma$ and $\Sigma' := g^{-1}(\Sigma)$ accordingly. A component of $\pi^{-1}(\gamma)$ will project onto γ and thus also the class of this component will not be torsion. Applying the case above to Σ' and a component of $\pi^{-1}(\gamma)$ in the pull-back bundle π^*W, we obtain the inequality

$$\chi_-\left(\Sigma'\right) \geq |\sigma'\cdot\sigma'| + \|p'_*\sigma'\|_T.$$

Thus Σ' is non-degenerate and so is Σ by Lemma 5.14. □

The existence of such a curve γ as required by the lemma above is by no means guaranteed. In fact, it does not always exist. The key idea will be to attach 1-handles to the embedded surface Σ to make enough space so that there will be a curve γ with the required properties. Of course, this increases the complexity of the surface which we compensate by considering the procedure in finite covers.

Lemma 5.18. *Let $p\colon W \to M$ be a circle bundle. Let $\Sigma \subset W$ be an embedded surface and γ a curve in M such that $\Gamma := p^{-1}(\gamma)$ and Σ intersect transversely. Suppose that their algebraic intersection $\gamma \cdot p_*[\Sigma]$ vanishes. Denote by $k \in \mathbb{N}$ the cardinality of the intersection $\Gamma \cap \Sigma$. Then there exits a surface $\Sigma_\gamma \subset W$ such that*

1. *the two surfaces share the same fundamental class $[\Sigma] = [\Sigma_\gamma]$,*
2. *the surface Σ_γ does not intersect Γ, and*

3. the complexities $\chi_-(\Sigma)$ and $\chi_-(\Sigma_\gamma)$ satisfy the inequality

$$\chi_-(\Sigma_\gamma) \leq \chi_-(\Sigma) + k.$$

Proof. We show that by adding a tube we can obtain an embedded surface $\Sigma' \subset W$ such that $[\Sigma] = [\Sigma']$, the cardinality of $\Sigma' \cap \Gamma$ is $k-2$ and $\chi_-(\Sigma') \leq \chi_-(\Sigma) + 2$.

By a small perturbation of Σ, we may assume that no two intersection points in $\Gamma \cap \Sigma$ lie in the same fibre of the bundle W. Thus to each intersection point corresponds via p a unique point on γ. There exists an arc $\gamma_{a,b} \subset \gamma$ with endpoints $a, b \in p(\Gamma \cap \Sigma)$ such that the corresponding intersection points have opposite sign and no other intersection point lies over $\gamma_{a,b}$.

Pick a neighbourhood U of $\gamma_{a,b}$ in M small enough so that there is a trivialisation $W|_U \cong U \times S^1$. All our changes to Σ will be contained in $p^{-1}(U)$, the total space of $W|_U$. Let us denote $\Sigma \cap p^{-1}(U)$ by Σ_U.

As Γ and Σ intersect transversely the map $\Sigma_U \xrightarrow{p} U$ will have full rank close to the intersections. Thus after shrinking U this map becomes an embedding $p\colon \Sigma_U \to U$. Its image is a submanifold S and there exists a function $s\colon S \to S^1$ such that

$$\{(x, s(x)) : x \in S\} \subset U \times S^1$$

corresponds exactly to Σ. Note that after shrinking U further the function s is homotopic to a constant function.

The surface S and $\gamma_{a,b}$ intersect exactly in the points a, b. There exists a tubular neighbourhood $\pi\colon V \to \gamma_{a,b}$ such that $S \cap V \subset \pi^{-1}(\{a, b\})$. This follows from the fact that a tubular neighbourhood of $\gamma_{a,b}$ near the points a, b can be extended to a tubular neighbourhood of $\gamma_{a,b}$ [Hir88, Section 4.6]. The tubular neighbourhood V is a solid cylinder whose discs at the ends lie in S. As s is homotopic to a constant function there is no obstruction extending s to a map $s\colon V \to S^1$. We can now lift the solid cylinder V to W via $\{(x, s(x)) : x \in V\}$.

Doing surgery along the lifted cylinder we obtain the embedded surface Σ'. We have removed two discs of Σ and attached an annulus. Therefore, $\chi_-(\Sigma') \leq \chi_-(\Sigma) + 2$. Furthermore, there is a map from the trace of the surgery to W which agrees with the embeddings of Σ and Σ' at the boundary, so we obtain the equality for their fundamental classes: $[\Sigma] = [\Sigma']$. Also Σ' does not intersect γ in a, b anymore and we have not introduced new intersection points. □

Proposition 5.19. *Let $p\colon W \to M$ be a circle bundle with non-torsion Euler class $e \in \mathrm{H}_2(W;\mathbb{Z})$. Let $\Sigma \subset W$ be an embedded surface. Then Σ is non-degenerate.*

Proof. Using Lemma 5.14 and the fact that M has enough basic classes, we may assume that $b_1(N) \geq 3$. The fundamental class of Σ is denoted by σ. Pick a non-torsion element in the kernel of the map

$$I_{p_*\sigma}\colon \mathrm{H}_1(N;\mathbb{Z}) \to \mathbb{Z}$$
$$\omega \mapsto \omega \cdot p_*\sigma.$$

This element can be represented by an embedded loop $\gamma \subset M$. We may assume that Σ intersects $\Gamma := p^{-1}(\gamma)$ transversely and denote the cardinality of the intersection by m.

Now for each $k \geq 2$ let $\pi \colon M_k \to M$ be the cover induced from the kernel of the homomorphism

$$I_{p_*\sigma} \colon \pi_1 N \to \mathbb{Z}/k\mathbb{Z}$$
$$\alpha \mapsto [\alpha] \cdot p_*\sigma.$$

We can pick a lift $\gamma_k \subset M_k$ of γ. As γ is in the kernel, the lift γ_k is again an embedded loop. We denote the pull-back bundle of W along π by $p_k \colon W_k \to M_k$ and the associated cover by $\pi' \colon W_k \to W$. Abbreviate the surface $(\pi')^{-1}(\Sigma) \subset W_k$ with Σ_k and the preimage of γ_k under the map p_k with Γ_k. The fundamental class of Σ_k is denoted by $\sigma_k = (\pi')^! \sigma$.

The cover π restricts to a diffeomorphism in a neighbourhood of γ_k. We observe that $\gamma_k \cdot p_{k*}\Sigma_k = 0$ in W_k and the cardinality of the set $\Gamma_k \cap \Sigma_k$ is again m.

Now we apply Lemma 5.18 to Σ_k and γ_k. We obtain a surface $\Sigma_{k,\gamma}$, which is disjoint from γ_k. By Lemma 5.17 the surface $\Sigma_{k,\gamma}$ is non-degenerate and so $\chi_-(\Sigma_k)$ satisfies

$$\chi_-(\Sigma_k) + m \geq \chi_-(\Sigma)_{k,\gamma} \geq |\sigma_k \cdot \sigma_k| + \|p_{k*}\sigma_k\|_T.$$

By the multiplicativity of the left and right hand side, we have the inequality

$$\chi_-(\Sigma) + m/k \geq |\sigma \cdot \sigma| + \|p_*\sigma\|_T$$

for all $k \geq 2$. Because m is a fixed number independent of k, we see that Σ has to be non-degenerate. □

Corollary 5.20. *Let $p \colon W \to M$ be a circle bundle over a closed irreducible 3-manifold M with enough basic classes and non-torsion Euler class $e \in \mathrm{H}_2(W;\mathbb{Z})$.*

Then each class $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ is non-degenerate.

5.6 Realising the lower bound

In this section we construct surfaces for many classes in $\mathrm{H}_2(W;\mathbb{Z})$ which realise the lower bound given by Theorem 5.1. This construction is due to Friedl-Vidussi [FV14, Lemma 4.1, Lemma 4.2] and is given for the convenience of the reader.

If a class $\alpha \in \mathrm{H}_2(M;\mathbb{Z})$ is not primitive, then it is never the fundamental class of a connected surface [Thu86, Lemma 1]. We say a class α is *connected* if there exists a connected surface $S \subset M$ and a natural number $k \in \mathbb{N}$ such that $k[S] = \alpha$ and S is minimising the Thurston norm, i.e. $\|[S]\|_T = \chi_-(S)$.

Proposition 5.21 (Friedl-Vidussi)**.** *Let $\sigma \in \mathrm{H}_2(W;\mathbb{Z})$ such that $p_*\sigma$ is connected. Then there exists a surface $\Sigma \subset W$ representing σ with*

$$\chi_-(\Sigma) = |\sigma \cdot \sigma| + \|p_*\sigma\|_T.$$

Proof. Pick an S and a k such that the equality $p_*\sigma = k[S]$ holds and so that S is Thurston norm minimising. As M is irreducible we assume that S has no spherical components. We check that we can lift the surface S along the bundle projection $p\colon W \to M$. For this we consider the Gysin sequence:

$$\begin{array}{ccccccccc} \longrightarrow & \mathrm{H}^k(M) & \xrightarrow{\;p^*\;} & \mathrm{H}^k(W) & \longrightarrow & \mathrm{H}^{k-1}(M) & \xrightarrow{\;\cup e\;} & \mathrm{H}^{k+1}(M) & \longrightarrow \\ & \big\downarrow \mathrm{PD} & & \big\downarrow \mathrm{PD} & & \big\downarrow \mathrm{PD} & & \big\downarrow \mathrm{PD} & \\ \longrightarrow & \mathrm{H}_{3-k}(M) & \xrightarrow{\;p^!\;} & \mathrm{H}_{4-k}(W) & \xrightarrow{\;p_*\;} & \mathrm{H}_{4-k}(M) & \xrightarrow{\;\cap e\;} & \mathrm{H}_{2-k}(M) & \longrightarrow \end{array},$$

where $e \in \mathrm{H}^2(M;\mathbb{Z})$ denotes the Euler class of the circle bundle. From the exactness of the bottom row we obtain $\langle e, k\cdot[S]\rangle = \langle e, p_*\sigma\rangle = 0$. Therefore restricting the bundle W to S gives the trivial circle bundle and we can lift k copies of S to W. Denote the embedded surface consisting of these k copies by Σ_k.

Note that the equality $p_*[\Sigma_k] - p_*\sigma = 0$ holds and by the exact sequence above there exists a union of embedded loops $\gamma \subset M$ such that $\sigma = [\Sigma_k] + [p^{-1}(\gamma)]$. We may assume that γ and the surface S intersect transversely. Furthermore, as the surface S is connected we can cancel intersection points of opposite signs between S and γ. Accordingly, we assume that the algebraic intersection number agrees with the geometric one, i.e. the set of intersection points $S \cap \gamma$ has cardinality exactly $|S \cdot \gamma|$.

We express the intersection number $|\sigma \cdot \sigma|$ as

$$|\sigma \cdot \sigma| = 2\,|[\Sigma_k] \cdot [p^{-1}(\gamma)]| = 2k\,|[S] \cdot [\gamma]| = 2k\,(\#S \cap \gamma),$$

where $\#S \cap \gamma$ denotes the cardinality of the set $S \cap \gamma$. The surfaces Σ_k and $p^{-1}(\gamma)$ intersect transversely in $k \cdot (\#S \cap \gamma)$ points. Each of these intersections can be resolved by removing two discs and gluing back an annulus. This procedure does not change the fundamental class and so after resolving every intersection point we obtain a surface Σ with fundamental class $\sigma = [\Sigma_k] + [p^{-1}(\gamma)]$.

The complexity $\chi_-(\Sigma)$ can be calculated by keeping track of the Euler characteristic throughout the construction of Σ:

$$\chi_-(\Sigma) = \chi_-(\Sigma_k) + \chi_-\left(p^{-1}(\gamma)\right) + 2k \cdot (\#S \cap \gamma) = \chi_-(\Sigma_k) + |\sigma \cdot \sigma|.$$

Recall that the surface Σ has fundamental class σ and we have the equality

$$\chi_-(\Sigma) = k\chi_-(S) + |\sigma \cdot \sigma| = |\sigma \cdot \sigma| + \|p_*\sigma\|_T,$$

which proves the claim. □

Remark 5.22. McMullen [McM02, Section 4 & Proposition 6.1] showed that if M has a sufficiently generic Alexander polynomial, then every class in $\mathrm{H}_2(M;\mathbb{Z})$ is connected.

6 Determining the Thurston norm

In Chapter 4 we used twisted Reidemeister torsion to estimate the Thurston norm. Note that we only used one dimensional representations and we had to go up to finite covers to make them effective. In this chapter we focus on recovering the Thurston norm from such estimates. Surprisingly, we will see that it is possible to recover it completely.

The results of this chapter are also covered in the article [FN15b].

6.1 Representations of the fundamental group

Let $\mathbb{K}$ be a field and M an irreducible 3-manifold. We have defined a representation of a group G to be a $(\mathbb{K}, \mathbb{Z}[G])$-bimodule which is finite-dimensional as a $\mathbb{K}$-vector space. In Section 4.2 we introduced twisted Reidemeister torsion. We tensored the cellular complex with an M-acyclic representation and then took its torsion, see Definition 4.8. If the representation is defined over the field $\mathbb{K}(t) = \mathrm{Quot}\left(\mathbb{K}[t^{\pm 1}]\right)$ we can apply the function width to the torsion of the twisted chain complex, see Definition 4.28

In Definition 4.27 we introduced the representation $\mathbb{K}_\theta$ of $\pi_1(M)$ for a cohomology class $\theta \in \mathrm{H}^1(M; \mathbb{Z})$. We have also seen that we can transform a representation V over $\mathbb{K}$ of $\pi_1(M)$ to a representation

$$V_\theta := \mathbb{K}_\theta \otimes_{\mathbb{K}} V$$

of $\pi_1(M)$ over $\mathbb{K}(t)$ by letting $\pi_1(M)$ act diagonally. The theorem below is a generalisation of Lemma 4.30.

Theorem 6.1 (Friedl-Kim)**.** *Let M be an irreducible 3-manifold which is not diffeomorphic to $D^2 \times S^1$. Let $\theta \in \mathrm{H}^1(M; \mathbb{Z})$ be a cohomology class and V an M-acyclic representation over $\mathbb{K}$.*

Then the Thurston norm $\|\theta\|_T$ is bounded by the inequality

$$\dim V \|\theta\|_T \geq \mathrm{width}\, \tau(M; V_\theta).$$

Proof. See [FK06, Theorem 1.1]. □

Naturally we are interested in finding representations such that the above inequality is sharp. McMullen [McM02] considered this question for many examples. For a hyperbolic knot K numerical evidence suggests that a lift of the holonomy representation $\pi_1(N(K)) \to \mathrm{SL}(2, \mathbb{C})$ detects the Thurston norm [DFJ12, Conjecture 1.7] in the following sense:

Definition 6.2. A representation V *detects the Thurston norm* of the cohomology class $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ if the equality $\|\theta\|_T = 0$ holds, or V_θ is M-acyclic and the Thurston norm satisfies the equality

$$\dim V \|\theta\|_T = \operatorname{width} \tau(M; V_\theta).$$

Further properties of representations are defined below. They are especially of interest for algorithmic considerations. Given a representation V of a group H and a group homomorphism $\phi\colon G \to H$, we can let G act on V through this homomorphism and so obtain a representation $\operatorname{res}_\phi V$ of G.

Definition 6.3. 1. A representation V of G over $\mathbb{C}$ is *integral* if there exists a $(\mathbb{Z}, \mathbb{Z}[G])$-bimodule W such that $V \cong \mathbb{C} \otimes_{\mathbb{Z}} W$.

2. A representation V of G *factors through a finite group* if there is finite group H, a homomorphism $\phi\colon G \to H$ and representation W of H such that $V \cong \operatorname{res}_\phi W$.

6.2 Induced representations

We have seen that many irreducible 3-manifolds N have a finite cover $\pi\colon M \to N$ which much simpler than the original manifold N. For example, if N has a virtually RFRS fundamental group, then it is covered by a manifold M which fibres, see Section 4.6. For graph manifolds we have discussed various simplifications in Chapter 2. We discuss how one can obtain a representation of $\pi_1(N)$ from a representation of $\pi_1(M)$. Furthermore, we will see that the properties introduced above will be preserved.

Let us fix an irreducible 3-manifold N and a finite cover $p\colon M \to N$ of degree d. Also we pick left coset representatives $g_1, \ldots, g_d$, so $\bigcup_i \pi_1(M) g_i = \pi_1(N)$.

Definition 6.4. Let V be a representation of $\pi_1(M)$. The *induced representation* $\operatorname{ind}_{\pi_1(N)} V$ is the representation $V \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}[\pi_1(N)]$.

Remark 6.5. The induced representation $\operatorname{ind}_{\pi_1(N)} V$ is also finite dimensional. If V has a basis $\{v_i\}$, then $\{v_i \otimes g_j\}$ is a basis of $\operatorname{ind}_{\pi_1(N)} V$. Therefore, we have the equality

$$d \cdot \dim V = \dim \left(\operatorname{ind}_{\pi_1(N)} V \right).$$

We have introduced the properties of being integral and of factoring through a finite group in Definition 6.3. The lemma below shows that these two properties are preserved under inducing representations.

Lemma 6.6. *1. If V is integral, then so is $\operatorname{ind}_{\pi_1(N)} V$.*

2. If V factors through a finite group, the so does $\operatorname{ind}_{\pi_1(N)} V$.

Proof. If V is integral, then there is a representation W with $V \cong \mathbb{C} \otimes W$. There is an isomorphism

$$V \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}[\pi_1(N)] \cong \mathbb{C} \otimes \left(W \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}[\pi_1(N)] \right)$$

and so $\operatorname{ind}_{\pi_1(N)} V$ is integral.

Suppose V factors through a finite group. There exists a finite group H, a homomorphism $\phi\colon \pi_1(M) \to H$ and a representation W of H with $V \cong \operatorname{res}_\phi W$. We consider the induced representation

$$\operatorname{ind}_{\pi_1(N)} V \cong \operatorname{res}_\phi W \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}[\pi_1(N)].$$

A representation factors through a finite group if and only if the orbit of every vector is finite. Recall that we have picked left coset representatives g_i of the group extension $\pi_1(M) \subset \pi_1(N)$. The orbit of every vector is finite as the pure tensors $w \otimes g$ have the finite orbit

$$\{w \cdot h \otimes g_i : h \in \pi_1(M)\}.$$

□

The main reason for us to consider induced representations is that they inherit the property of detecting the Thurston norm.

Proposition 6.7. *Let $p\colon M \to N$ be a finite cover of degree d. Let V be a representation which detects the Thurston norm of the class $p^*\theta \in \mathrm{H}^1(M;\mathbb{Z})$. Then $\operatorname{ind}_{\pi_1(N)} V$ detects the Thurston norm of θ.*

Proof. First note that we have the equalities

$$(\operatorname{ind}_{\pi_1(N)} V)_\theta = \mathbb{K}(t) \otimes_{\mathbb{K}} (V \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{Z}[\pi_1(N)]) = \operatorname{ind}_{\pi_1(N)} V_{p^*\theta}.$$

Furthermore, the map below is an isomorphism of chain complexes:

$$\begin{aligned} \operatorname{ind}_{\pi_1(N)} V \otimes_{\mathbb{Z}[\pi_1(N)]} \mathrm{C}(N) &\to V \otimes_{\mathbb{Z}[\pi_1(M)]} \mathrm{C}(M) \\ (v \otimes g) \otimes e &\mapsto v \otimes g \cdot e. \end{aligned}$$

By assumption V is M-acyclic, i.e. the chain complex $V \otimes \mathrm{C}(M)$ is acyclic. Therefore also $\operatorname{ind}_{\pi_1(N)} V$ is N-acyclic. The representation V detects the Thurston norm, so

$$\dim V \cdot \|p^*\theta\|_T = \operatorname{width} \tau(M; V_{p^*\theta}).$$

We construct a fundamental family for M from a fundamental family of N. Let $\{\widetilde{e}\}$ be a collection of lifts of k-cells, then $\{g_i \cdot \widetilde{e}\}$ are lifts of the k-cells of M. We can translate the characteristic maps in the same way. Let $\{v_i\}$ be a basis for V. If we choose $\{v_i \otimes g_j\}$ as a basis for the representation $\operatorname{ind}_{\pi_1(N)} V_{p^*\theta}$, then the above isomorphism $V_{p^*\theta} \otimes \mathrm{C}(M) \cong \operatorname{ind}_{\pi_1(N)} V_{p^*\theta} \otimes \mathrm{C}(N)$ preserves the basis. Therefore we obtain the equality $\operatorname{width} \tau(M; V_{p^*\theta}) = \operatorname{width} \tau(N; \operatorname{ind}_{\mathbb{Z}[\pi_1(N)]} V_{p^*\theta})$.

By Theorem 3.5 we have the equality $\|p^*\theta\|_T = d\|\theta\|_T$, where d is the degree of the cover $\pi\colon M \to N$. We put the results together and obtain the equalities

$$\begin{aligned} \dim \left(\operatorname{ind}_{\pi_1(N)} V\right) \|\theta\|_T = \dim V \|p^*\theta\|_T &= \operatorname{width} \tau(M; V_{p^*\theta}) \\ &= \operatorname{width} \tau(N; (\operatorname{ind}_{\mathbb{Z}[\pi_1(N)]} V)_\theta). \end{aligned}$$

□

6.3 Detecting the Thurston norm

We proceed with the proof of the following theorem. Note that we give a construction of the representations whose existence is claimed.

Theorem 6.8. *Let M be an irreducible 3-manifold which is not diffeomorphic to $D^2 \times S^1$. Let $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ be a cohomology class. Then both statements hold:*

1. *There is an integral representation V defined over the complex numbers factoring through a finite group which detects the Thurston norm of θ.*
2. *For all but finitely many primes p, there is a representation V over the finite field $\mathbb{F}_p$ which detects the Thurston norm of θ.*

Proof. See Theorem 6.14 and Theorem 6.16 below. □

Remark 6.9. Friedl-Vidussi describe an algorithm to compute the Thurston norm using twisted Reidemeister torsion [FV12, Section 6]. Essentially it runs through all representations and compares the obtained lower estimates with surfaces constructed by normal surface theory. Although this approach seems inefficient, in practice this is a very good way to calculate the Thurston norm algorithmically.

By Theorem 6.8 we can restrict ourselves to computations over a finite field. This accelerates computations and lowers space complexity significantly.

We already made contact with some representations. For example, the representation below was important for many of the considerations in Chapter 4.

Example 6.10. Let M be a composite graph manifold with a Seifert non-vanishing character $\alpha\colon M \to \mathbb{Z}/k\mathbb{Z}$. The representation $\mathbb{C}^\alpha$ detects the Thurston norm of every class $\theta \in \mathrm{H}^1(M;\mathbb{Z})$, see Corollary 4.35. Unfortunately, the representation is in general not integral.

First we focus on graph manifolds. Let M be a graph manifold with a composite graph structure and a Seifert non-vanishing character $\alpha\colon \pi_1(M) \to \mathbb{Z}/k\mathbb{Z}$. Let V be a representation of $\pi_1(N)$ such that the linear map $(t_B - 1)_V$ is invertible for every Seifert fibre t_B of every block B. For these representations the twisted Reidemeister torsion of M can be calculated directly from the blocks without taking the gluing maps into account, see Theorem 4.34. We can construct various examples of these representations by restricting along the Seifert non-vanishing character α.

Definition 6.11. Let $R[G]$ denote the group ring of a group G over a commutative ring R. The *augmentation ideal* $\operatorname{aug}(R[G])$ is the kernel of the augmentation homomorphism

$$\begin{aligned} R[G] &\to R \\ \sum_{g\in G} a_g g &\mapsto \sum_{g\in G} a_g. \end{aligned}$$

Lemma 6.12. *Let $p \in \mathbb{N}$ be a prime number and n a natural number, which is coprime to p. Let $\mathbb{K}$ be either a field of characteristic 0 or the field $\mathbb{F}_p$ of characteristic p. The $(\mathbb{K}, \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}])$-bimodule $V := \operatorname{aug}(\mathbb{K}[\mathbb{Z}/n\mathbb{Z}])$ has the property that $(1-h)_V$ is invertible for all units $h \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. We check that the kernel of $(1-h)_V$ is the trivial vector space. Let the vector $v = \sum_{g \in \mathbb{Z}/n\mathbb{Z}} a_g g \in V$ be in the kernel. We have the equalities

$$a_e = a_h = a_{h^2} = \dots a_{h^k} = \dots .$$

As h generates $\mathbb{Z}/n\mathbb{Z}$ we obtain $a_e = a_g$ for all $g \in \mathbb{Z}/n\mathbb{Z}$.

If $\mathbb{K}$ has characteristic 0, then we have $a_e = 0$ and so $v = 0$. On the other hand if $\mathbb{K} = \mathbb{F}_p$, then we apply the augmentation map to v and obtain that the equality $n \cdot a_e = 0$ holds in $\mathbb{F}_p$. As n and p are coprime, this implies $a_e = 0$ and so $v = 0$. □

Proposition 6.13. *Let M be a composite graph manifold with Seifert non-vanishing character $\alpha\colon M \to \mathbb{Z}/k\mathbb{Z}$. Let $\mathbb{K}$ be either a field of characteristic 0 or the field $\mathbb{F}_p$ with prime characteristic p which is coprime to k.*

Then the representation $V := \operatorname{res}_\alpha \operatorname{aug}(\mathbb{K}[\mathbb{Z}/k\mathbb{Z}])$ detects the Thurston norm of every class $\theta \in \mathrm{H}^1(M;\mathbb{Z})$.

Proof. By Theorem 4.34 we know that V_θ is M-acyclic and has twisted Reidemeister torsion

$$\tau(M; V_\theta) = \prod_{B \in V(M)} (\det(t_B - 1)_{V_\theta})^{-\chi(\Sigma_B)} .$$

The determinant on the tensor product can be calculated as follows:

$$\det(t_B - 1)_{V_\theta} = \det(t_B - 1)_V \cdot (\det(t_B - 1)_{\mathbb{K}_\theta})^{\dim V}$$

We deduce the equality using $\operatorname{width}\det(t_B - 1)_{V_\theta} = \dim V \cdot |\langle \theta, [\Sigma_B] \rangle|$. As in the proof of Theorem 4.36, we express the width of $\tau(M; V_\theta)$ as

$$\begin{aligned} \operatorname{width} \tau(M; V_\theta) &= \sum_{B \in V(M)} (\dim V)\, |\langle \theta, [\Sigma_B] \rangle| \\ &= (\dim V) \sum_{B \in V(M)} \|i_B^* \theta\|_T = (\dim V) \|\theta\|_T. \end{aligned}$$

□

We complete the discussion of graph manifolds with the following theorem:

Theorem 6.14. *Let N be a graph manifold which is not diffeomorphic to $D^2 \times S^1$. Let $\theta \in \mathrm{H}^1(N;\mathbb{Z})$ be a cohomology class. Then both statements hold:*

1. *There is an integral representation V defined over the complex numbers factoring through a finite group which detects the Thurston norm of θ.*

2. *For all but finitely many primes p, there is a representation V over the finite field $\mathbb{F}_p$ which detects the Thurston norm θ.*

Proof. Suppose N is a graph manifold which is covered by a graph manifold $\pi\colon M \to N$ with a composite graph structure and a Seifert non-vanishing character $\alpha\colon \pi_1(M) \to \mathbb{Z}/k\mathbb{Z}$ with k prime. Pick any prime number p coprime to k. On M the representation $V := \operatorname{res}_\alpha \operatorname{aug}(\mathbb{C}[\mathbb{Z}/k\mathbb{Z}])$ and $W := \operatorname{res}_\alpha \operatorname{aug}(\mathbb{F}_p[\mathbb{Z}/k\mathbb{Z}])$ detect the Thurston norm of $\pi^*\theta$ by Proposition 6.13. The representation V is an integral representation, and W is defined over $\mathbb{F}_p$. Both factor through a finite group.

By Proposition 6.7 the Thurston norm of θ is detected by the representations $\operatorname{ind}_{\pi_1(N)} V$ and $\operatorname{ind}_{\pi_1(N)} W$. Using Lemma 6.6 we see that the induced representations have the above properties as well.

Suppose N is covered by S^3, a non-trivial circle bundle, or a torus bundle. By the discussion in Chapter 3 their Thurston norms vanish. So by convention there is nothing to show. These are all cases we had to consider by Theorem 2.19. □

We conclude with a sketch of the case of aspherical 3-manifold with virtually RFRS fundamental groups. Further references and more details can be found in the article [FN15b].

Lemma 6.15. *Let M be an irreducible 3-manifold with $b_1(M) \geq 2$. Let $\theta \in \mathrm{H}^1(M;\mathbb{Z})$ be a class such that $\|\theta\|_T = \|\theta\|_A$.*

Then there exists an integral representation V defined over the complex numbers factoring through a finite group which detects the Thurston norm of θ.

Sketch. We suppose that $\|\theta\|_T > 0$. We denote the torsion subgroup of $\mathrm{H}_1(M;\mathbb{Z})$ by T. Let Δ_{fr} be the free Abelian torsion, see Section 4.4. We introduce names for the coefficients:

$$\Delta_{fr} = \sum_{h \in \mathrm{H}_1(M;\mathbb{Z})/T} a_h h \text{ with } a_h \in \mathbb{Z}.$$

Let H_θ denote the set of $h \in \mathrm{H}_1(M;\mathbb{Z})/T$ with $a_h \neq 0$ and $\theta(h)$ maximal. The set L_θ consists of $h \in \mathrm{H}_1(M;\mathbb{Z})/T$ with $a_h \neq 0$ and $\theta(h)$ minimal. Furthermore, we make the following abbreviations:

$$\Delta_H := \sum_{h \in H_\theta} a_h h \text{ and } \Delta_L := \sum_{h \in L_\theta} a_h h.$$

Fact. For a large enough prime $q \geq 2$, the following holds: there is a group homomorphism $\alpha\colon \mathrm{H}_1(M;\mathbb{Z})/T \to \mathbb{Z}/q\mathbb{Z}$ such that every non-trivial character $\rho\colon \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^*$ has the property that

$$\rho \circ \alpha(\Delta_H) \neq 0 \text{ and } \rho \circ \alpha(\Delta_L) \neq 0.$$

Pick such a q and α. From Lemma 4.19 we deduce that for every such character ρ the representation $\mathbb{C}^{\rho\circ\alpha}$ is M-acyclic and that $\operatorname{width} \tau(M;\mathbb{C}^{\rho\circ\alpha}) = \|\theta\|_T$.

Fact. There exists a finite collection S of non-trivial characters $\rho\colon \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^*$ such that there is an isomorphism of the $\mathbb{Z}/q\mathbb{Z}$-representations

$$\mathbb{C}[\mathbb{Z}/q\mathbb{Z}] \cong \prod_{\rho \in S} \mathbb{C}^{\rho}.$$

Pick such a set S and denote the representation $\operatorname{res}_\alpha \mathbb{C}[\mathbb{Z}/q\mathbb{Z}]$ by V. The proposition follows from the equalities:

$$\operatorname{width} \tau(M; \operatorname{res}_\alpha \mathbb{C}[\mathbb{Z}/q\mathbb{Z}]) = \sum_{\rho \in S} \operatorname{width} \tau(M; \mathbb{C}^{\rho \circ \alpha}) = \dim\left(\mathbb{C}[\mathbb{Z}/q\mathbb{Z}]\right) \|\theta\|_T.$$

□

Theorem 6.16. *Let N be an aspherical 3-manifold with virtually RFRS fundamental group which is not diffeomorphic to $D^2 \times S^1$. Let $\theta \in \mathrm{H}^1(N; \mathbb{Z})$ be a cohomology class. Then both statements hold:*

1. *There is an integral representation V defined over the complex numbers factoring through a finite group which detects the Thurston norm of θ.*
2. *For all but finitely many primes p, there is a representation V over the finite field $\mathbb{F}_p$ which detects the Thurston norm θ.*

We give a sketch and refer to the article [FN15b, Theorem 4.1] for more details.

Proof. By Theorem 4.44 there is a finite cover $\pi\colon M \to N$ such that $\|p^*\theta\|_T = \|p^*\theta\|_A$. Applying the lemma above and Proposition 6.7, we obtain a representation $V^{\mathbb{C}}$ fulfilling the first statement. As the representation is integral there is a $(\mathbb{Z}, \mathbb{Z}[\pi_1(M)])$-bimodule $V^{\mathbb{Z}}$ which has the property that $V^{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{Z}} V^{\mathbb{Z}}$.

By Lemma [FN15b, Lemma 4.5] for all but finitely many primes p, the representation $\mathbb{F}_p \otimes_{\mathbb{Z}} V^{\mathbb{Z}}$ is M-acyclic and fulfils the equality

$$\operatorname{width} \tau(M; V^{\mathbb{C}}) = \operatorname{width} \tau(M; \mathbb{F}_p \otimes_{\mathbb{Z}} V^{\mathbb{Z}}).$$

□

7 Acyclic representations

In the chapters before we restricted ourselves exclusively to irreducible 3-manifolds. One reason for this is that for defining twisted Reidemeister torsion on a 3-manifold M we have to find an M-acyclic representation, see Definition 4.8. For most irreducible 3-manifolds we have seen in Chapter 6 that there is such a representation. Now we proceed to characterise the compact 3-manifolds M which admit a unitary M-acyclic representation. The characterisation can be found in Theorem 7.5.

We follow article [FN15a] and put it in context with rest of the thesis.

7.1 Characterisation

First, we introduce another property of a representation. It is weaker than factoring through a finite group.

Definition 7.1. A representation V of $\pi_1(M)$ over $\mathbb{C}$ is *unitary* if there exists an inner product on the $\mathbb{C}$-vector space V which is preserved by the right action of $\pi_1(M)$.

Remark 7.2. Note that if a representation factors through a finite group, then it will be a unitary representation.

Considering the remark above and Theorem 6.8, we immediately obtain the corollary below. Non-trivial means that the representation is not the zero-dimensional vector space.

Corollary 7.3. *Let M be an irreducible 3-manifold with non-vanishing Thurston norm. Then M admits a non-trivial unitary representation which is M-acyclic.*

Recall that any 3-manifold is diffeomorphic to the connected sum of prime 3-manifolds and the prime summands are unique, see [Kne29, p.257], [Mil62b, Theorem 1] and [AFW12, Section 1.1] for reference also in the case with non-empty boundary.

Furthermore, recall that prime oriented 3-manifolds are either irreducible or $S^1 \times S^2$ [Hem04, Lemma 3.13]. A 3-manifold M is called a *rational homology sphere* if its rational homology groups are

$$\mathrm{H}_k(M;\mathbb{Q}) = \mathrm{H}_k(S^3;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } k = 0, 3 \\ 0 & \text{otherwise} \end{cases}.$$

Example 7.4. The group $\mathrm{SO}(4)$ acts on $S^3 \subset \mathbb{R}^4$ through linear transformations. Therefore, a discrete subgroup $\Gamma \subset \mathrm{SO}(4)$ which acts freely on S^3 gives rise to a manifold S^3/Γ. These spherical manifolds are all rational homology spheres. But note that not all rational homology spheres arise this way.

The next theorem classifies which compact 3-manifold M admits a non-trivial unitary and M-acyclic representation V.

Theorem 7.5. *Let M be a 3-manifold not diffeomorphic to S^3. Let $M \cong P_1 \# \ldots \# P_k$ be its prime decomposition. The following statements are equivalent:*

1. *There is a non-trivial unitary representation which is M-acyclic.*

2. *The boundary of M is toroidal (possibly empty). Furthermore, at most one of the P_i is not a rational homology sphere.*

7.2 Boundary

We will see in this section that if a 3-manifold M has boundary ∂M which is not toroidal, then we are not able to make it acyclic using any non-trivial representation.

The next lemma gives a helpful obstruction. Recall that the Euler characteristic $\chi(C)$ of a chain complex C over the complex numbers $\mathbb{C}$ is defined to be

$$\chi(C) := \sum_k (-1)^k \dim C_k.$$

Furthermore, this invariant can be expressed in terms of the dimension of the homology groups by

$$\chi(C) = \sum_k (-1)^k \dim \mathrm{H}_k(C).$$

For this equality it is essential that the cell complex C was defined over a field. For a manifold M we abbreviate $\chi(V \otimes \mathrm{C}(M))$ with $\chi(M; V)$.

Lemma 7.6. *Let V be a representation of $\pi_1(M)$ over $\mathbb{C}$. The Euler characteristic $\chi(M; V)$ can also be expressed as*

$$\chi(M; V) = \dim V \cdot \chi(M; \mathbb{C}) = \dim V \cdot \chi(M) = \frac{\dim V \cdot \chi(\partial M)}{2},$$

where $\mathbb{C}$ is the trivial representation of $\pi_1(M)$.

Proof. For the first equality, note that the following holds:

$$\begin{aligned} \chi(V \otimes \mathrm{C}(M)) &= \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{C}_k(M; V) \\ &= (\dim V) \cdot \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{C}_k(M; \mathbb{C}) \\ &= \dim V \cdot \chi(\mathbb{C} \otimes \mathrm{C}(M)). \end{aligned}$$

The dimension $\dim \mathbb{C} \otimes \mathrm{C}_k(M)$ is exactly the number of k-cells of M. We immediately obtain the second equality.

The last equality follows from the equality $\chi(M) = \frac{1}{2}\chi(\partial M)$. We see that this equality holds in every 3-manifold M. As the Euler characteristic is multiplicative with respect to finite covers, we may assume that M is orientable. By Poincaré duality, we have $\dim \mathrm{H}_k(M, \partial M; \mathbb{C}) = \dim \mathrm{H}_{3-k}(M; \mathbb{C})$ for every $0 \leq k \leq 3$. We write $\chi(M, \partial M)$ for the Euler characteristic of the chain complex $\mathbb{C} \otimes \mathrm{C}(M, \partial M)$. From Poincaré duality, we obtain the equation $\chi(M, \partial M) = -\chi(M)$. Considering the cells of M, we also get

$$\chi(M) = \chi(M, \partial M) + \chi(\partial M).$$

Combining the last two equalities, we have $\chi(M) = \frac{1}{2}\chi(\partial M)$. □

Proposition 7.7. *Let M be a 3-manifold. If there is a non-trivial unitary representation which is M-acyclic, then ∂M is toroidal.*

Proof. Assume the representation V is M-acyclic. By Lemma 7.2 we know that $\chi(M) = \chi(\partial M) = 0$. We pick a CW-structure for M. Let $\widehat{M}$ denote the 3-manifold obtained by filling all spherical components of M. This operation does not change the fundamental group.

By definition we have $\chi(\partial \widehat{M}) \leq 0$. Also we may assume that $\partial \widehat{M}$ is non-empty. This implies that the homology group $\mathrm{H}_3(\widehat{M}; V) = 0$ vanishes. The 3-manifold $\widehat{M}$ inherits a CW-structure with the same k-cells for $k \leq 2$. This yields an isomorphism of the homology groups $\mathrm{H}_1(\widehat{M}; V) = \mathrm{H}_1(M; V) = 0$. We obtain that

$$0 \leq \sum_{i=0}^{2} (-1)^i \dim \mathrm{H}_i(\widehat{M}; V) = \dim V \cdot \chi(\widehat{M}).$$

By Lemma 7.2 we have $\chi(\partial \widehat{M}) \geq 0$. Together with the bound above, we conclude that the equality $\chi(\partial \widehat{M}) = 0$ holds, which ultimately implies that ∂M has to be toroidal. □

7.3 The proof

The proposition below is rather technical so we will refer to the article [FN15a, Proposition 4.2] for full details.

Proposition 7.8. *Let $M \cong N_1 \# N_2$ be a connected sum decomposition of a given 3-manifold M. If M admits a unitary M-acyclic representation V, then either N_1 or N_2 is a rational homology sphere.*

Sketch. By definition there is a separating 2-sphere $S^2 \subset N_1 \# N_2$ witnessing the connect sum. We can find open neighbourhoods A, B covering the parts of the two summands respectively and intersecting in a tubular neighbourhood $A \cap B$ of S^2. The Mayer-Vietoris sequence corresponding to this cover is

$$\to \mathrm{H}_k(S^2; V) \to \mathrm{H}_k(N_1 \setminus D^3; V) \oplus \mathrm{H}_k(N_1 \setminus D^3; V) \to \mathrm{H}_k(M; V) \to$$

For $k = 0, 1$ the groups $\mathrm{H}_k(N_i \backslash D^3; V)$ only depend on the 2-skeleton of N_i. Therefore we have the equality $\mathrm{H}_k(N_i \setminus D^3; V) = \mathrm{H}_k(N_i)$. The homology groups of S^2 in these low degrees are $\mathrm{H}_0(S^2; V) = V$ and $\mathrm{H}_1(S^2; V) = 0$. From the exact sequence above we see that one of the N_i has to fulfil $\dim \mathrm{H}_0(N_i; V) \geq \frac{\dim V}{2}$. Suppose this holds for N_1.

A more detailed but rather technical analysis of the exact sequences shows that N_1 has to be a rational homology sphere [FN15a, Proposition 4.2]. Here we also use the assumption that V is a unitary representation. □

In Example 7.4 we have seen how to obtain rational homology spheres from S^3 by taking a finite quotient. In fact all 3-manifolds covered by S^3 arise this way. If the manifold is not covered trivially by S^3, then we can find an acyclic representation.

Lemma 7.9. *Let $\pi\colon S^3 \to M$ be a non-trivial cover. Then there exists a representation V which is M-acyclic.*

Proof. Note that the deck transformations form a finite subgroup $\Gamma \subset \mathrm{SO}(4)$ such that $M \cong S^3/\Gamma$. Again, we let $\mathrm{SO}(4)$ act by linear transformations on S^3. We identify $\pi_1(M)$ with Γ to obtain an action of $\pi_1(M)$ on $\mathbb{R}^4$. We complexify this representation to a representation on $V := \mathbb{C}^4$.

As Γ acts freely and is non-trivial, every non-trivial element of the group Γ does not fix any non-zero vector. For a non-trivial $g \in \Gamma$ this implies that the linear map $D_g\colon V \to V$ with $D_g(v) := g \cdot v - v$ has trivial kernel and thus is surjective. From this and the following description

$$\mathrm{H}_0(M; V) \cong V/\langle g \cdot v - v : g \in \Gamma, v \in V\rangle = 0,$$

we deduce that $\mathrm{H}_0(M; V) = 0$, see e.g. [FN15a, Lemma 2.5].

By the representation theory of finite groups, V is a direct summand of $\mathbb{C}[\Gamma]$. This implies that for all $k \in \mathbb{N}$ the homology group $\mathrm{H}_k(M; V)$ is a summand of $\mathrm{H}_k(M; \mathbb{C}[\Gamma]) = \mathrm{H}_k(S^3; \mathbb{C})$. Therefore we also know that the equalities

$$\mathrm{H}_1(M; V) = \mathrm{H}_2(M; V) = 0$$

hold. The vanishing of the last group $\mathrm{H}_3(M; V)$ follows from the fact that the Euler characteristic vanishes: $\chi(M; V) = \chi(M) = 0$, see Lemma 7.6. □

We have already seen many 3-manifolds where a representation exists which is M-acyclic. By the proposition below this holds for all prime manifolds which are not the 3-sphere.

Proposition 7.10. *Let M be a prime 3-manifold with toroidal boundary and M not diffeomorphic to S^3. Then there exists a non-trivial unitary representation V which is M-acyclic.*

Proof. First note that the proposition holds for all irreducible 3-manifolds with non-zero Thurston norm by Theorem 6.8. As the case for the circle bundles $D^2 \times S^1$ and

$S^2 \times S^1$ over D^2 and S^2 is clear, we can deduce the claim for circle bundles from Lemma 4.37.

We are left to prove the claim for torus bundles. This follows from general arguments of the twisted homology of fibred manifolds [FN15a, Proof of Proposition 4.6]. $\square$

The last point we have to consider is that we can freely take connected sums with rational homology spheres. This is the content of the next lemma.

Let V be a M-acyclic representation. The Seifert-van Kampen theorem gives us an identification $\pi_1(M\#S) \cong \pi_1(M)*\pi_1(S)$, where $*$ denotes the coproduct in groups. We see that $\pi_1(M)$ is a quotient of $\pi_1(M\#S)$ and that we can restrict the representation V along this quotient map $q\colon \pi_1(M\#S) \to \pi_1(M)$. Denote this representation $\operatorname{res}_q V$ by $V^{\#}$.

Lemma 7.11. *Let M be a 3-manifold with a M-acyclic representation V and S a rational homology sphere. Then the representation $V^{\#}$ constructed above is $M\#S$-acyclic.*

Proof. Consider a connected sum $M\#S$ of a 3-manifold M and a rational homology sphere S. Let $D \subset M$ be the 3-ball which is the attaching region in M for the 1-handle of the connected sum and D' the corresponding ball in S. There is a continuous map $q\colon M\#S \to M$ which collapses the summand S to a point and restrict to the identity on $M \setminus D$. Note that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(M\#S) & \xrightarrow{\text{SvK}} & \pi_1(M) * \pi_1(S) \\ & {\scriptstyle \pi_1(q)} \searrow \quad \swarrow & \\ & \pi_1(M) & \end{array}.$$

This shows that the local coefficient systems $V^{\#}$ and V over S^2 and over $M \setminus D$ agree and that $V^{\#}$ is trivial over S. We have a morphism of the Mayer-Vietoris sequences

$$\begin{array}{ccccccc} \to \mathrm{H}_k(S^2;V^{\#}) & \to & \mathrm{H}_k(M \setminus D;V^{\#}) \oplus \mathrm{H}_k(S \setminus D';V^{\#}) & \to & \mathrm{H}_k(M\#S;V^{\#}) \to \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \oplus q_*} & & \downarrow{\scriptstyle q_*} \\ \to \mathrm{H}_k(S^2;V) & \to & \mathrm{H}_k(M \setminus D;V) \oplus \mathrm{H}_k(D;V) & \to & \mathrm{H}_k(M;V) \to \end{array}.$$

We want to apply the five lemma to prove that the homomorphism q induces an isomorphism $\mathrm{H}_k(M\#S;V^{\#}) \cong \mathrm{H}_k(M;V) = 0$. What is left to check is that the homomorphism

$$q_*\colon \mathrm{H}_k(S \setminus D';V^{\#}) \to \mathrm{H}_k(D;V)$$

is an isomorphism. The coefficient systems are in both cases trivial and all we have to check is that $\mathrm{H}_k(S \setminus D;\mathbb{C}) = 0$ for $k \geq 1$. This is immediately deduced from the fact that S is a rational homology sphere and so $S \setminus D$ is rational homology ball. $\square$

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