arXiv:1406.0664v2 [hep-ph] 12 Oct 2014

Integrability of the evolution equations for heavy-light baryon distribution amplitudes

V.M. Braun[1], S.E. Derkachov[2,3] and A.N. Manashov[1,4]

[1] *Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*
[2] *St. Petersburg Department of Steklov Mathematical Institute, 191023 St.-Petersburg, Russia*
[3] *St. Petersburg State Polytechnic University, 195251 St. Petersburg, Russia*
[4] *Department of Theoretical Physics, St.-Petersburg University, 199034 St.-Petersburg, Russia*

Abstract

We consider evolution equations describing the scale dependence of the wave function of a baryon containing an infinitely heavy quark and a pair of light quarks at small transverse separations, which is the QCD analogue of the helium atom. The evolution equations depend on the relative helicity of the light quarks. For the aligned helicities, we find that the equation is completely integrable, that is it has a nontrivial integral of motion, and obtain exact analytic expressions for the eigenfunctions and the anomalous dimensions. The evolution equation for anti-aligned helicities contains an extra term that breaks integrability and creates a "bound state" with the anomalous dimension separated from the rest of the spectrum by a finite gap. The corresponding eigenfunction is found using numerical methods. It describes the momentum fraction distribution of the light quarks in, e.g., Λ_b-baryon at large scales.

1. Precision tests of the flavor sector of the Standard Model may reveal new physics and remain high on the agenda. Main attention has been so far focused on B-mesons but interest is developing to the heavy baryon decays as well. Such baryons are produced copiously at the LHC and, as more data are collected, studies of rare b-baryon decays involving flavor-changing neutral current transitions have to become quantitative in order to make an impact on the field. In particular the $\Lambda_b \to \Lambda\mu^+\mu^-$ decays are receiving a lot of attention, see e.g. Ref. [1] and references therein.

Theoretical description of the b-hadron decays is based on factorization theorems that make use of the large mass of the b-quark in order to separate calculable effects of short distances from the nonperturbative large distance physics. The corresponding formalism is similar but much less developed for baryons as compared to mesons. A recent discussion using SCET formalism can be found in Ref. [2]. For the exclusive decays involving large energy release in the final state, the relevant nonperturbative quantities are baryon wave functions at small transverse separations, dubbed light-cone distribution amplitudes (DA). Their study was started in Refs. [3, 4, 5] where the complete classification and renormalization group equations (RGE) that govern the scale-dependence are presented.

In this work we point out that these equations have a hidden symmetry and completely integrable in the case that the light quarks have the same helicity. In other words, we identify a nontrivial quantum number that distinguishes heavy baryon states with different scale dependence and obtain exact analytic solution of the evolution equations. This phenomenon is similar to integrability of RGEs for the light baryons [6, 7] and, in a more general context, to integrability in high-energy QCD [8, 9, 10] and in the $N = 4$ supersymmetric Yang-Mills theory [11, 12, 13] that attracted huge attention as a tool to check the AdS/CFT correspondence. The integrable model that we encounter in the present context is different; it has been discussed recently in [17].

The similar equation for the case that the two light quarks have opposite helicity contains an extra term that breaks integrability and creates a "bound state" with the anomalous dimension separated from the rest of the spectrum by a finite gap. The corresponding eigenfunction is found numerically. It describes the momentum fraction distribution of the light quarks in, e.g. Λ_b, at large scales and can be called "asymptotic DA" in analogy to the accepted terminology for hadrons built of light quarks.

2. Consider at first the leading-twist DA of a baryon containing an infinitely heavy quark and a transversely polarized "diquark": a pair of light quarks with aligned helicities. It can be defined as [4]

$$\langle 0|[q_1^T(z_1 n)C\not{n}\gamma_\perp^\mu q_2(z_2 n)]h_v(0)|B^{j=1}(v)\rangle \;=\; \frac{1}{\sqrt{3}}\epsilon_\perp^\mu u(v)\, f_B^{(2)}(\mu)\Psi_\perp(z_1,z_2;\mu)\,. \tag{1}$$

Here $q_{1,2}=u,d,s$ are light quarks separated by a lightlike distance, $h_v(0)$ is the effective heavy quark field with four-velocity v, C is the charge conjugation matrix, $u(v)$ is the Dirac spinor $\not{v}u(v)=u(v)$, and ϵ^μ is the diquark polarization vector, $v^\mu\epsilon_\mu=0$. The transverse projections are defined with respect to the two auxiliary light-like vectors n and $\bar n$ which we choose such that $v_\mu=(n_\mu+\bar n_\mu)/2$, $v\cdot n=1$, $n\cdot\bar n=2$:

$$\epsilon_\perp^\mu = g_\perp^{\mu\nu}\epsilon_\nu\,, \qquad\qquad g_\perp^{\mu\nu}=g^{\mu\nu}-(n^\mu\bar n^\nu+n^\nu\bar n^\mu)/(n\cdot\bar n)\,, \tag{2}$$

and similar for $\gamma_\perp^\mu$. The Wilson lines connecting the quark fields are not shown for brevity. The heavy quark field h_v can itself be related to the Wilson line as [18]

$$\langle 0|h_v(0)|h,v\rangle = \text{Pexp}\left[ig\int_{-\infty}^0 d\alpha\, v_\mu A^\mu(\alpha v)\right]\,, \tag{3}$$

so that the operator in Eq. (1) can be viewed as a pair of light quarks (a diquark), attached to the Wilson line with a cusp containing one lightlike and one timelike segment. Finally, the coupling $f_B^{(2)}$ is defined as the matrix element of the corresponding local $q_1q_2h_v$ operator; it is inserted for normalization [4]. The parameter μ is the renormalization (factorization) scale. We tacitly imply using $\overline{MS}$ scheme.

The product $\epsilon_\perp^\mu u(v)$ on the right-hand-side (r.h.s.) of Eq. (1) can be expanded in irreducible representations corresponding to physical baryon states with $J^P=1/2^+$ and $J^P=3/2^+$ using suitable projection operators, see [4]. These (ground) states form the $SU(3)_F$ multiplets (sextets), Σ_b,Ξ_b,Ω_b and $\Sigma_b^*,\Xi_b^*,\Omega_b^*$, respectively, which are degenerate in the heavy b-quark limit. The double-strange Ω_b baryon is of special interest for flavor physics as it only decays through weak interaction. The DA $\Psi_\perp(z_1,z_2;\mu)$ is written usually in terms of its Fourier transform

$$\Psi_\perp(\underline{z};\mu)=\int_0^\infty\! d\omega_1\int_0^\infty\! d\omega_2\, e^{-i(\omega_1z_1+\omega_2z_2)}\Psi_\perp(\omega_1,\omega_2;\mu)\;=\;\int_0^\infty\!\omega d\omega\int_0^1\! du e^{-i\omega(uz_2+\bar u z_1)}\,\widetilde\Psi_\perp(\omega,u;\mu)\,, \tag{4}$$

where $\underline{z}=\{z_1,z_2\}$ and in the second line we redefine $\omega_1=u\omega$, $\omega_2=\bar u\omega$ with $\bar u=1-u$. The variables ω_1 and ω_2 correspond to the energies (up to a factor two) of light quarks in the baryon rest frame. The DA (4) is the most important nonperturbative input in QCD calculations of exclusive heavy baryon decays to the leading power accuracy in the heavy quark mass.

The scale dependence of $\widetilde\Psi_\perp(\omega,u,\mu)$ or, equivalently, $\Psi_\perp(\underline{z};\mu)$, is governed by the renormalization group equation [3, 4]

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}+\frac{2\alpha_s}{3\pi}\mathbb{H}\right)f_B^{(2)}(\mu)\,\Psi_\perp(\underline{z};\mu)=0\,. \tag{5}$$

The evolution kernel $\mathbb{H}$ is an integral operator which can be decomposed as

$$\mathbb{H} = \mathcal{H}_{12} + \mathcal{H}_1^h + \mathcal{H}_2^h - 4\,. \tag{6}$$

The kernels $\mathcal{H}_k^h$ are due to heavy-light quark interactions,

$$\begin{aligned}\mathcal{H}_1^h f(\underline{z}) &= \int_0^1 \frac{d\alpha}{\alpha}\Big[f(\underline{z}) - \bar{\alpha} f(\bar{\alpha} z_1, z_2)\Big] + \ln(i z_1 \mu)\, f(\underline{z})\,,\\ \mathcal{H}_2^h f(\underline{z}) &= \int_0^1 \frac{d\alpha}{\alpha}\Big[f(\underline{z}) - \bar{\alpha} f(z_1, \bar{\alpha} z_2)\Big] + \ln(i z_2 \mu)\, f(\underline{z})\,.\end{aligned} \tag{7}$$

They are identical to the Lange-Neubert kernels [19, 20, 21]. The remaining contribution

$$\mathcal{H}_{12} f(\underline{z}) = \int_0^1 \frac{d\alpha}{\alpha}\Big[2 f(\underline{z}) - \bar{\alpha} f(z_{12}^{\alpha}, z_2) - \bar{\alpha} f(z_1, z_{21}^{\alpha}))\Big] \tag{8}$$

takes into account the interaction between the light quarks; it is similar to the standard Efremov-Radyushkin-Brodsky-Lepage evolution kernel for the pion DA. Here and below we use the notation

$$z_{12}^{\alpha} = \bar{\alpha} z_1 + \alpha z_2\,, \qquad\qquad \bar{\alpha} = 1 - \alpha\,. \tag{9}$$

The evolution kernels (7), (8) can be written in terms of the generators of the collinear subgroup of conformal transformations

$$S_+ = z^2 \partial_z + 2 j z\,, \qquad S_0 = z\partial_z + j\,, \qquad S_- = -\partial_z\,, \tag{10}$$

where $j = 1$ is the conformal spin of the light quark. The generators satisfy the standard $SL(2)$ commutation relations

$$[S_+, S_-] = 2 S_0\,, \qquad [S_0, S_\pm] = \pm S_\pm\,. \tag{11}$$

One can show that [23]

$$\mathcal{H}_1^h = \ln\big(i\mu S_+^{(1)}\big) - \psi(1)\,, \qquad\qquad \mathcal{H}_2^h = \ln\big(i\mu S_+^{(2)}\big) - \psi(1)\,,$$

where $S_+^{(1)}$, $S_+^{(2)}$ act on the first, z_1, and the second, z_2, light-cone coordinate, respectively. The last kernel (8) is written in terms of the two-particle Casimir operator S_{12}^2 [24]

$$\mathcal{H}_{12} = 2\big[\psi(J_{12}) - \psi(1)\big]\,, \tag{12}$$

where $S_{12}^2 = S_+ S_- + S_0(S_0 - 1) = J_{12}(J_{12} - 1)$, $S_+ = S_+^{(1)} + S_+^{(2)}$ etc., and $\psi(x)$ is the Euler's digamma function. Thus, the complete evolution kernel takes a very compact form

$$\mathbb{H} = \ln\big(i\mu S_+^{(1)}\big) + \ln\big(i\mu S_+^{(2)}\big) + 2\psi(J_{12}) - 4\psi(2)\,. \tag{13}$$

The evolution equation for the DA in momentum space, $\Psi_\perp(w_1, w_2; \mu)$, is given by the same expression with the $SL(2)$ generators in the momentum space representation [22]. Eigenfunctions of $\mathbb{H}$ correspond to the states that have autonomous scale dependence and the corresponding eigenvalues define anomalous dimensions.

3. Our main result is that this evolution equation can be solved explicitly. To this end we consider the following operators

$$\mathbb{Q}_1 = i\big(S_+^{(1)} + S_+^{(2)}\big), \qquad\qquad \mathbb{Q}_2 = S_0^{(1)} S_+^{(2)} - S_0^{(2)} S_+^{(1)}. \tag{14}$$

It is possible to show that $\mathbb{Q}_1$ and $\mathbb{Q}_2$ commute with each other and with the evolution kernel $\mathbb{H}$:

$$[\mathbb{Q}_1, \mathbb{Q}_2] = [\mathbb{Q}_1, \mathbb{H}] = [\mathbb{Q}_2, \mathbb{H}] = 0\,. \tag{15}$$

The first two relations are trivial, the last one can be verified using the explicit expressions for $\mathbb{Q}_2$ and $\mathbb{H}$.

If $\mathbb{H}$ is interpreted as a Hamiltonian of a certain quantum-mechanical model, the operators $\mathbb{Q}_1$ and $\mathbb{Q}_2$ correspond to the conserved charges. In the formalism of the quantum inverse scattering method (QISM) the charges $\mathbb{Q}_1, \mathbb{Q}_2$ appear in the expansion of the element $C(u)$ of the monodromy matrix,

$$C(u) = u\,\mathbb{Q}_1 + \mathbb{Q}_2.$$

The commutation relation $[C(u), \mathbb{H}] = 0$, and its generalization to more degrees of freedom then follow directly from the QISM [14, 15, 16]. Note that in classical applications of integrable models one encounters Hamiltonians that commute with the sum of diagonal elements, $A(u) + D(u)$, of the monodromy matrix. In our case the Hamiltonian commutes with $C(u)$, which corresponds to a new, nonstandard integrable model.

The conserved charges $\mathbb{Q}_1$, $\mathbb{Q}_2$ are self-adjoint operators with respect to the $SL(2, R)$ invariant scalar product

$$\langle \Phi | \Psi \rangle = \frac{1}{\pi^2} \int_{\mathbb{C}_-} d^2 z_1 \int_{\mathbb{C}_-} d^2 z_2 \left(\Phi(\underline{z})\right)^* \Psi(\underline{z})\,, \tag{16}$$

where the integration goes over the lower complex half-plane, $\operatorname{Im} z_i < 0$. The eigenfunctions of $C(u)$ provide the basis of the so-called Sklyanin's representation of Separated Variables and are known in explicit form [25]. They are labeled, for the present case, by two real numbers: $s > 0$ and $x \in \mathbb{R}$ such that

$$C(u)\phi_{s,x}(z_1, z_2) = s(u - x)\phi_{s,x}(z_1, z_2)\,, \tag{17}$$

with

$$\phi_{s,x}(\underline{z}) = \frac{s}{z_1^2 z_2^2} \int_0^1 d\alpha \left(\frac{\alpha}{\bar\alpha}\right)^{ix} \exp\left[is(\bar\alpha/z_1 + \alpha/z_2)\right] = s\rho(x)\frac{e^{is/z_1}}{z_1^2 z_2^2}\,{}_1F_1\Big(1+ix, 2, is\big(z_2^{-1} - z_1^{-1}\big)\Big), \tag{18}$$

where

$$\rho(x) = \pi x / \sinh(\pi x). \tag{19}$$

The eigenfunctions $\phi_{s,x}(\underline{z})$ form a complete system in the Hilbert space defined by the scalar product (16)

$$\left\langle \phi_{s',x'} | \phi_{s,x} \right\rangle = \frac{2\pi}{s}\,\delta(s - s')\delta(x - x')\,. \tag{20}$$

Since the conserved charges $\mathbb{Q}_1$ and $\mathbb{Q}_2$ commute with the Hamiltonian $\mathbb{H}$, they share the same set of eigenfunctions,

$$\mathbb{H}\,\phi_{s,x}(\underline{z}) = \gamma(s, x)\,\phi_{s,x}(\underline{z})\,. \tag{21}$$

The simplest way to calculate the eigenvalues is to compare the large-z asymptotics of the expressions on the both sides of this equation. In this way one obtains the anomalous dimensions

$$\gamma(s, x; \mu) = 2\ln(\mu s/s_0) + \mathcal{E}(x)\,, \qquad \mathcal{E}(x) = \psi(1 + ix) + \psi(1 - ix) + 2\gamma_E \tag{22}$$

where $s_0 = e^{2-\gamma_E}$. Going back to the RGE equation (5), we expand the DA $\Psi_\perp(\underline{z},\mu)$ over the eigenfunctions of $\mathbb{H}$

$$\Psi_\perp(\underline{z},\mu) = \int_0^\infty ds\, s \int_{-\infty}^\infty \frac{dx}{2\pi}\, \eta_\perp(s,x;\mu)\, \phi_{s,x}(\underline{z})\,. \tag{23}$$

The expansion coefficients $\eta_\perp(s,x;\mu) = \langle \phi_{s,x}|\Psi_\perp\rangle$ evolve autonomously,

$$f_B^{(2)}(\mu)\eta_\perp(s,x;\mu) = f_B^{(2)}(\mu_0)\eta_\perp(s,x;\mu_0)\left(\frac{\mu}{\mu_0}\right)^{-\frac{8}{3\beta_0}}\left(\frac{\mu_0 s}{s_0}\right)^{\frac{8}{3\beta_0}\ln L} L^{\frac{4}{3\beta_0}\left[\mathcal{E}(x)-\frac{4\pi}{\beta_0\,\alpha_s(\mu_0)}\right]}, \tag{24}$$

where $L = \alpha_s(\mu)/\alpha_s(\mu_0)$, $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$. For large scales, the coefficients $\eta_\perp(s,x;\mu)$ slowly drift towards smaller values of both parameters: $s \to 0$, thanks to the factor $s^{8\ln L/3\beta_0}$, and $|x| \to 0$, taking into account that $\psi(1+ix) \sim \ln|x|$ for $x \to \pm\infty$.

Going over to the DA in momentum space, $\widetilde{\Psi}_\perp(\omega,u;\mu)$, we define the corresponding eigenfunctions as

$$\widetilde{\phi}_{s,x}(\omega,u) = \langle e^{-i\omega(\bar{u}z_1+uz_2)}|\phi_{s,x}(z_1,z_2)\rangle\,. \tag{25}$$

Using that $\langle e^{-ikz}|z^{-2}e^{is/z}\rangle = -(1/\sqrt{ks})J_1(2\sqrt{ks})$ [23] one obtains

$$\widetilde{\phi}_{s,x}(\omega,u) \;=\; \frac{1}{\omega\sqrt{u\bar{u}}}\int_0^1 \frac{d\alpha}{\sqrt{\alpha\bar{\alpha}}}\alpha^{ix}\bar{\alpha}^{-ix}\, J_1\left(2\sqrt{ws\bar{\alpha}\bar{u}}\right) J_1\left(2\sqrt{ws\alpha u}\right)\,. \tag{26}$$

The eigenfunctions $\widetilde{\phi}_{s,x}(\omega,u)$ are orthogonal and form a complete set:

$$\frac{s}{2\pi}\int_0^\infty \omega^3 d\omega \int_0^1 du\, u\bar{u}\, \widetilde{\phi}_{s,x}(\omega,u)\, \widetilde{\phi}^*_{s',x'}(\omega,u) = \delta(s-s')\delta(x-x')\,, \tag{27}$$

$$\omega^3 u\bar{u}\int_0^\infty sds \int_{-\infty}^\infty \frac{dx}{2\pi}\, \widetilde{\phi}_{s,x}(\omega,u)\, \widetilde{\phi}^*_{s,x}(\omega',u') = \delta(\omega-\omega')\delta(u-u')\,. \tag{28}$$

Making use of the Bateman's expansion for the product of two Bessel functions we obtain a series representation

$$\widetilde{\phi}_{s,x}(\omega,u) \;=\; \frac{1}{\omega}\sum_{n=0}^\infty i^n \varkappa_n^{-1}\, C_n^{3/2}(1-2u)\, H_n(x)\, \frac{1}{\sqrt{s\omega}} J_{2n+3}(2\sqrt{s\omega})\,. \tag{29}$$

Here $C_n^{3/2}(x)$ are the Gegenbauer polynomials, $J_{2n+3}(x)$ are Bessel functions, and

$$\varkappa_n = \frac{(n+1)(n+2)}{4(2n+3)}\,. \tag{30}$$

The functions $H_n(x)$ are given by the continuous Hahn polynomials up to the prefactor $\rho(x)$:

$$H_n(x) = i^n\int_0^1 du\left(\frac{u}{\bar{u}}\right)^{ix} C_n^{3/2}(1-2u) = \frac{(n+1)(n+2)}{2} i^n \rho(x)\, {}_3F_2\left(\begin{matrix}-n, n+3, 1+ix\\ 2,2\end{matrix}\Big|1\right), \tag{31}$$

e.g.

$$\rho^{-1}(x)H_0(x) = 1\,, \qquad \rho^{-1}(x)H_1(x) = 3x\,, \qquad \rho^{-1}(x)H_2(x) = 5x^2-1\,, \tag{32}$$

etc. Hahn polynomials are real functions, have the symmetry $H_n(x) = (-1)^n H_n(-x)$, and form a complete orthogonal system. In our normalization

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} H_n(x)\, H_m(x) = \varkappa_n\, \delta_{mn}\,. \tag{33}$$

Collecting everything we obtain the final result:

$$\widetilde{\Psi}_\perp(\omega, u;\mu) = \omega^2 u\bar{u} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_0^{\infty} sds\, \widetilde{\phi}_{s,x}(\omega,u) \eta_\perp(s,x;\mu)\,, \tag{34}$$

where the scale dependence of $\eta_\perp(s,x;\mu)$ is given in Eq. (24). The expansion coefficients of $\eta_\perp(s,x;\mu)$ in Hahn polynomials are related to the expansion coefficients of $\widetilde{\Psi}_\perp(\omega,u;\mu)$ in Gegenbauer polynomials,

$$\eta_\perp(s,x;\mu) = \sum_n i^n\, \eta_n^\perp(s;\mu) H_n(x) \qquad \mapsto \qquad \widetilde{\Psi}_\perp(\omega,u;\mu) = u\bar{u} \sum_n \psi_n^\perp(\omega;\mu)\, C_n^{3/2}(2u-1)\,, \tag{35}$$

by the Bessel transform (cf. Eq. (30) in Ref. [23])

$$\psi_n^\perp(\omega;\mu) = \int_0^\infty ds \sqrt{s\omega}\, J_{2n+3}\left(2\sqrt{s\omega}\right)\, \eta_n^\perp(s;\mu)\,. \tag{36}$$

Making use of the asymptotic expansion for the Bessel function one finds that the small-s behavior $\eta_n^\perp(s) \sim s^{p_n}$ translates into the large-ω asymptotics of the function $\psi_n^\perp(\omega) \sim \omega^{-1-p_n}$ unless there is some cancellation, see below.

The expansion coefficients at the reference (low) scale can be calculated from a given model of the DA as

$$\eta_\perp(s,x;\mu_0) = \int_0^\infty \omega d\omega \int_0^1 du\, \widetilde{\phi}^*_{s,x}(\omega,u)\, \widetilde{\Psi}_\perp(\omega,u;\mu_0). \tag{37}$$

In the existing studies it is usually assumed that $\widetilde{\Psi}_\perp(\omega,u;\mu_0)$ is decreasing exponentially at large energies ω. For a rather general model of this type

$$\widetilde{\Psi}_\perp(\omega,u;\mu_0) = \omega^2 u\bar{u} \sum_n c_n \left(\frac{\omega}{\epsilon_n}\right)^{\kappa_n} \frac{e^{-\omega/\epsilon_n}}{\epsilon_n^4} C_n^{3/2}(2u-1) \tag{38}$$

one obtains

$$\eta_\perp(s,x;\mu_0) = s \sum_n i^n c_n (s\epsilon_n)^n H_n(x) \frac{\Gamma(n+4+\kappa_n)}{\Gamma(2n+4)}\, {}_1F_1\left(\begin{matrix} n+4+\kappa_n \\ 2n+4 \end{matrix}\middle| -s\epsilon_n\right). \tag{39}$$

In particular, for the simplest phenomenologically acceptable model [3, 4, 5]

$$\widetilde{\Psi}_\perp(\omega,u;\mu_0) \;=\; \omega^2 u\bar{u}\, \frac{e^{-\omega/\epsilon_0}}{\epsilon_0^4} \qquad \mapsto \qquad \eta_\perp(s,x;\mu_0) \;=\; \rho(x)\, s\, e^{-s\epsilon_0}\,. \tag{40}$$

Exponential decrease $\sim e^{-\omega/\epsilon_n}$ of each Gegenbauer harmonics in (38) amounts, from the view point of the relation in Eq. (36), to the fine tuning such that all leading power terms in the asymptotics $\omega \to \infty$ drop out. This fine tuning is, however, destroyed by the evolution so that a power-like asymptotics is always generated.

To see this, consider the simplest model in (40) corresponding to the term $n = 0$ in (38) as an example. As the result of the evolution (24) all harmonics with $n > 0$ become excited

$$\eta_n^{\perp}(s,\mu) = c_n(\mu)\, s(\mu_0 s)^{-\delta}\, e^{-s\epsilon_0} \tag{41}$$

where $\delta = -8/3\beta_0 \ln L$ and

$$c_n(\mu) \sim \int dx\, H_0(x)\, L^{4/3\beta_0 \mathcal{E}(x)}\, H_n(x)\,. \tag{42}$$

For the corresponding coefficients in the Gegenbauer expansion (35) one obtains using (36)

$$\psi_n^{\perp}(\omega,\mu) = c_n(\mu)\, \epsilon_0^{-2} \left(\frac{\epsilon_0}{\mu_0}\right)^{\delta} \left(\frac{\omega}{\epsilon_0}\right)^{n+2} \frac{\Gamma(n+4-\delta)}{\Gamma(2n+4)}\, {}_1F_1\!\left(\begin{matrix} n+4-\delta \\ 2n+4 \end{matrix}\,\middle|\, -\frac{\omega}{\epsilon_0}\right). \tag{43}$$

The confluent hypergeometric function ${}_1F_1(a,b|\omega)$ decreases as a power of ω at $\omega \to \infty$, cf. Eq. (62) below, unless $a-b$ is a nonnegative integer, in which case the asymptotic behavior is exponential. Thus, unless $\delta = 0$ and $n = 0$, we obtain

$$\psi_n^{\perp}(\omega,\mu) \sim (\omega/\epsilon_0)^{-2+\delta}\,. \tag{44}$$

Note that the asymptotic behavior is the same for any n.

4. Next, we consider heavy baryons with the light quarks having opposite helicity. The scale dependence of the leading twist DAs does not depend on the spin of the light quark pair and is the same for the $j^P = 0^+$ $SU(3)_F$ triplet and all longitudinal DAs of heavy baryons in the $j^P = 1^+$ sextets, see [4]. For definiteness, consider the Λ_b-baryon DA [3, 4] defined as

$$\langle 0|[u^T(z_1 n) C\gamma_5 \not{n} d(z_2 n)] h_v(0)|\Lambda(v)\rangle \;=\; f_\Lambda^{(1)}(\mu)\Psi_\Lambda(z_1,z_2;\mu)\, u_\Lambda(v)\,. \tag{45}$$

The evolution equation for $\Psi_\Lambda(z_1,z_2;\mu)$ contains an additional term corresponding to the gluon exchange between the light quarks (in Feynman gauge)

$$\mathcal{H}_{12} \mapsto \mathcal{H}_{12} - \delta\mathcal{H}_{12}\,, \qquad\qquad \delta\mathcal{H}_{12} f(\underline{z}) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\, f(z_{12}^{\alpha}, z_{21}^{\beta}) \tag{46}$$

that corresponds to $\mathbb{H} \mapsto \mathbb{H} - 1/J_{12}(J_{12}-1)$ in the $SL(2)$-invariant representation of the evolution kernel in Eq. (13). Expanding $\Psi_\Lambda(z_1,z_2;\mu)$ in terms of the eigenfunctions (18) of the integrable Hamiltonian (13)

$$\Psi_\Lambda(\underline{z},\mu) = \int_0^\infty ds\, s \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, \eta_\Lambda(s,x;\mu)\, \phi_{s,x}(\underline{z}) \tag{47}$$

one obtains the RGE equation for the expansion coefficients $\eta_\Lambda(s,x,\mu)$

$$\begin{aligned} \left(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \frac{2\alpha_s}{3\pi}\left[2\ln\left(\frac{\mu s}{s_0}\right) + \mathcal{E}(x)\right]\right) f_\Lambda^{(1)}(\mu)\eta_\Lambda(s,x,\mu) & \\ = \frac{2\alpha_s}{3\pi} f_\Lambda^{(1)}(\mu) \int_{-\infty}^{\infty} dx'\, V(x,x')\, \eta_\Lambda(s,x',\mu)\,, & \end{aligned} \tag{48}$$

where $\mathcal{E}(x)$ is defined in Eq. (22) and the kernel $V(x,x')$ is given by the matrix element

$$\langle \phi_{s',x'}|\delta\mathcal{H}_{12}|\phi_{s,x}\rangle = \delta(s-s')V(x,x')\,. \tag{49}$$

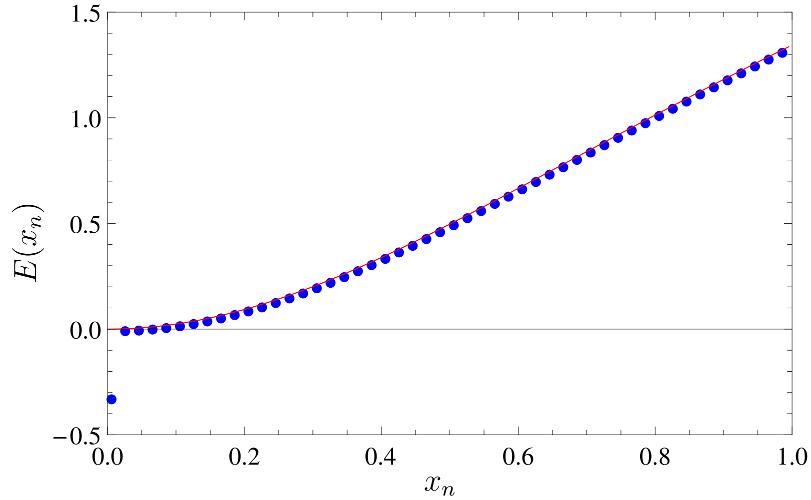

Figure 1: The spectrum of eigenvalues $E(x_n) \equiv E_n$ of the discretized version of Eq. (51) $x \mapsto x_n = (2n+1)/200$, $n = 0, 1, \ldots, 99$ (blue dots) compared to the "unperturbed" spectrum $\mathcal{E}(x_n)$ (red solid curve). In order not to overload the plot, only every second eigenvalue is shown.

We obtain

$$V(x, x') = \frac{1}{2\pi} \sum_{n=0}^{\infty} \varkappa_n^{-1} \frac{H_n^*(x) H_n(x')}{(n+1)(n+2)} = \frac{1}{2 \sinh \pi(x - x')} \left[\frac{x - x'}{xx'} - \frac{\pi \sinh \pi(x - x')}{\sinh \pi x \sinh \pi x'} \right]. \tag{50}$$

It is easy to see that $V(x, x) \sim 1/(2\pi x^2)$ for large x, and decreases exponentially in $|x - x'|$.

In order to solve (48) one needs to find the eigenfunctions of the integral equation

$$\mathcal{E}(x)\, \eta_E(x) - [V \eta_E](x) = E\, \eta_E(x)\,. \tag{51}$$

If $V \to 0$, obviously all eigenfunctions are localized in x, $\eta_a(x) \sim \delta(x - a)$. The spectrum of eigenvalues is continuous, $E_a = \mathcal{E}(a) \geq 0$, and double degenerate since $\mathcal{E}(a) = \mathcal{E}(-a)$. In order to understand the effect of the "perturbation" V we consider the discretized version of this equation: $x \to x_n = (n + 1/2)\Delta x$, $\Delta x = L/N$, $n = -N, \ldots, N-1$. The unperturbed eigenfunctions, $\eta_k(x_n) = \delta_{nk}$, correspond to local excitations at the $k - th$ site. Discretizing the integral in (48) one replaces the original eigenvalue problem (51) by the eigenvalue problem for the matrix $\mathcal{V}_{nk} = \delta_{nk}\mathcal{E}(x_n) - \Delta x\, V(x_n, x_m)$. Since $V(x, x') = V(-x, -x')$, all eigenstates have definite parity with respect to $x \to -x$; the double degeneracy is lifted and one can study x-even and x-odd eigenstates separately. Diagonalising this matrix numerically we find that the shift of eigenvalues as compared to the unperturbed spectrum is surprisingly small, $\delta E = E - \mathcal{E} \leq 0.003$, for all eigenstates except for the lowest one, cf. Fig. 1, and the corresponding eigenfunctions $\eta_k(x_n)$ remain well localized around the point x_k, see Fig. 2. At the same time the lowest x-even eigenstate changes drastically: It becomes delocalized, see Fig. 2, and separated from the rest of the spectrum by a finite gap[1]

$$\Delta E = E_0 \simeq -0.3214\,. \tag{52}$$

In the continuum limit ($\Delta x \to 0, L \to \infty$) this phenomenon can be understood as creation of a bound state in addition to the continuum spectrum that remains to be largely unperturbed. The "wave

[1]The size of the gap coincides with the gap in the spectrum of anomalous dimensions of three-light-quark operators in the large-N limit [7], indicating that these problems are related. Unravelling this connection goes beyond the tasks of this letter.

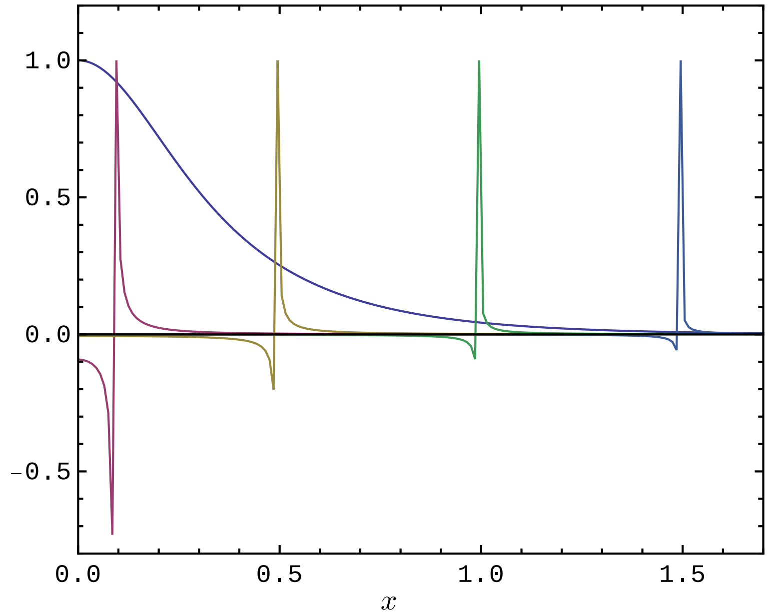

Figure 2: The x-even eigenfunctions $\eta_k^+(x_n) = \eta_k^+(-x_n)$ for $k = 0$ (the ground state), and $k = 10, 50, 100, 150$ (from left to right), for $L = 5$ and $N = 500$. Normalization is arbitrary.

function" of this (lowest) state can be approximated to a good accuracy (better that 1% for $|x| < 3$) by the following expression:

$$\eta_0(x) \simeq \frac{\sqrt{2}E_0}{\sqrt{2+x^2}} \frac{\rho(x)}{[E_0 - \mathcal{E}(x)]}, \qquad \eta_0(0) = 1. \tag{53}$$

It can be convenient to expand this function in Hahn polynomials

$$\eta_0(x) = \sum_{n=0,2,\ldots}^{\infty} \chi_n \, H_n(x), \tag{54}$$

where the first few coefficients read

$$\chi_0 \simeq 0.612, \quad \chi_2 \simeq -0.126, \quad \chi_4 \simeq 0.0574, \quad \chi_6 \simeq -0.0338, \quad \chi_8 \simeq 0.0226, \quad \chi_{10} \simeq -0.0163. \tag{55}$$

The normalization is given by

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} \, \eta_0^2(x) = \sum_{n=0}^{\infty} \varkappa_n \, \chi_n^2 \simeq 0.0758. \tag{56}$$

Coming back to the representation of the DA in the form (47) we can separate the contribution of the discrete (lowest) level as

$$\eta_\Lambda(s,x,\mu) = \xi_0(s,\mu) \, \chi_0^{-1} \eta_0(x) + \eta'_\Lambda(s,x,\mu). \tag{57}$$

where the function $\eta'_\Lambda(s,x,\mu)$ accounts for the contribution of the continuum spectrum and must be orthogonal to $\eta_0(x)$,

$$\int dx \, \eta_0(x) \, \eta'_\Lambda(s,x,\mu) = 0. \tag{58}$$

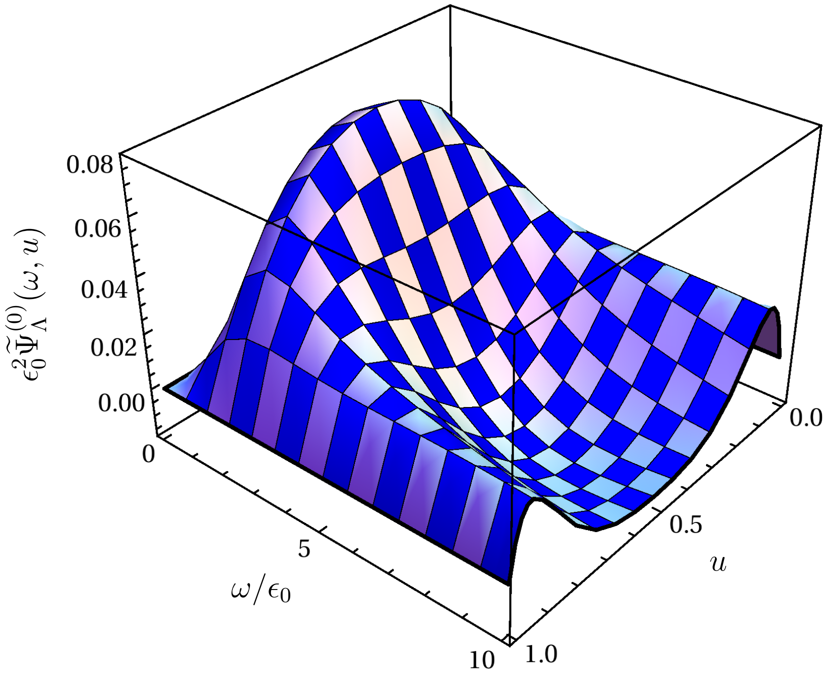

Figure 3: Asymptotic Λ_b distribution amplitude for the simplest choice (60) of the profile function $\xi_0(s,\mu_0)$.

Going over to the momentum space we obtain for the contribution of the lowest state (asymptotic DA)

$$f_\Lambda^{(1)}(\mu)\widetilde{\Psi}_\Lambda^{(0)}(\omega,u;\mu) = f_\Lambda^{(1)}(\mu_0)\chi_0^{-1}\omega^2 u\bar{u}\left(\frac{\mu}{\mu_0}\right)^{-\frac{8}{3\beta_0}} L^{\frac{4}{3\beta_0}[E_0-\frac{4\pi}{\beta_0\,\alpha_s(\mu_0)}]}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\eta_0(x)$$
$$\times\int_0^\infty sds\,\widetilde{\phi}_{s,x}(\omega,u)\xi_0(s,\mu_0)\left(\frac{\mu_0 s}{s_0}\right)^{\frac{8}{3\beta_0}\ln L}, \qquad (59)$$

cf. Eq. (24). Note that the restriction to the contribution of the discrete level implies a certain relation between the momentum fraction distribution between the two light quarks and their total momentum ω, the remaining freedom is encoded in the "profile function" $\xi_0(s,\mu_0)$ at the reference scale, which can be arbitrary. For the simplest ansatz

$$\xi_0(s,\mu_0) = se^{-s\epsilon_0}\,, \qquad (60)$$

cf. (40), we obtain (at the scale μ_0)

$$\widetilde{\Psi}_\Lambda^{(0)}(\omega,u) = \frac{1}{\epsilon_0^2}u\bar{u}\sum_{n=0,2,\ldots}^{\infty} i^n\left(\frac{\chi_n}{\chi_0}\right)\frac{\Gamma(n+4)}{\Gamma(2n+4)}C_n^{3/2}(1-2u)\left(\frac{\omega}{\epsilon_0}\right)^{n+2} {}_1F_1\left(\begin{matrix}n+4\\2n+4\end{matrix}\Big|-\frac{\omega}{\epsilon_0}\right)$$
$$= \frac{1}{\epsilon_0^2}u\bar{u}\left\{\left(\frac{\omega}{\epsilon_0}\right)^2 e^{-\omega/\epsilon_0} - \frac{1}{42}\left(\frac{\chi_2}{\chi_0}\right)\left(\frac{\omega}{\epsilon_0}\right)^4 C_2^{3/2}(1-2u)\,{}_1F_1\left(\begin{matrix}6\\8\end{matrix}\Big|-\frac{\omega}{\epsilon_0}\right)+\ldots\right\}, \qquad (61)$$

where the first few Hahn expansion coefficients χ_n are given in Eq. (55). The function $\widetilde{\Psi}_\Lambda^{(0)}(\omega,u)$ is plotted in Fig. 3. Note that the contributions of higher Gegenbauer polynomials $C_n^{3/2}(1-2u)$ in (61) are accompanied by increasing powers of ω so that the deviation from the "naive" $\sim u\bar{u}$ shape is increasing with the energy: the distribution becomes broader. Taking into account that for large ω

$$\omega^{n+2}{}_1F_1\left(\begin{matrix}n+4\\2n+4\end{matrix}\Big|-\omega\right)\simeq\omega^{-2}\frac{\Gamma(2n+4)}{\Gamma(n)}\,, \qquad n>0\,, \qquad (62)$$

we see that all terms in this expansion except for the first one have a power-like asymptotics at $\omega \to \infty$.[2] One formally gets

$$\widetilde{\Psi}^{(0)}_{\Lambda}(\omega, u) \underset{\omega\to\infty}{\simeq} \omega^{-2} u\bar{u} \sum_{n=2,4...}^{\infty} i^n \left(\frac{\chi_n}{\chi_0}\right) \frac{\Gamma(n+4)}{\Gamma(n)}\, C_n^{3/2}(1-2u)\,. \tag{63}$$

The series in (63) is divergent indicating that for large ω the function $\Psi^{(0)}_{\Lambda}(\omega, u)$ develops end-point singularities (at $u \to 0, 1$). This feature is rather robust and does not depend on the precise choice of the profile function $\xi_0(s, u)$ provided it vanishes sufficiently fast at $s \to 0$ and $s \to \infty$.

We expect that the model of the Λ_b DA in Eq. (61), or a more general one in Eq. (59), will be sufficient for phenomenological applications. If necessary, the contributions of the continuum spectrum (57) can be added, in which case the scale dependence of $\eta'_{\Lambda}(s, x, \mu)$ can be approximated using Eq. (24). In this approximation the orthogonality condition (58) will not hold at all scales, which is, however, unlikely to be numerically significant.

The evolution equation for the Λ_b DA has also been discussed in Ref. [5] using a different representation $\eta_\Lambda(s, x) \mapsto \hat{\rho}_2(w_r, \varkappa)$ where we use the notatation $\varkappa$ for the variable called u' in [5] to avoid confusion with the momentum fraction. The relation is simply $s = 1/w_r$ for the first variable, whereas going over from the x- to $\varkappa$-representation corresponds to the Fourier transform

$$\hat{\eta}(\varkappa) = \int_{-\infty}^{\infty} dx\, \eta(x) \left(\frac{\varkappa}{\bar{\varkappa}}\right)^{ix} = \int_{-\infty}^{\infty} dx\, \eta(x) e^{ipx}\,, \qquad p = \ln\frac{\varkappa}{\bar{\varkappa}}\,, \qquad \bar{\varkappa} = 1 - \varkappa\,. \tag{64}$$

In other words if our variable x (17) is interpreted as a quasimomentum, then $p = \ln(\varkappa/\bar{\varkappa})$ is the corresponding generalized coordinate[3]. For the ground state $\eta_0(x) = \eta_0(-x)$ implies $\hat{\eta}_0(\varkappa) = \hat{\eta}_0(1-\varkappa)$.

The end-point behavior of $\hat{\eta}_0(\varkappa)$ is determined by the position of the (nearest) singularity of $\eta_0(x)$ in the complex x plane. A singularity (simple pole) at $x_0 = \pm ia$ corresponds to $\hat{\eta}_0(\varkappa) \sim [\varkappa\bar{\varkappa}]^a$ for $\varkappa \to 0, \varkappa \to 1$. The position of the singularity can be related to the value of energy E_0, alias the lowest anomalous dimension. One can show that the term in V in Eq. (51) does not contribute close to the singularity so that the following exact relation holds:

$$E_0 = \mathcal{E}(x_0) = \psi(1+a) + \psi(1-a) + 2\gamma_E\,. \tag{65}$$

Using an (approximate) value $E_0 = -0.3214$, Eq. (52), we obtain, to the same accuracy

$$a = 0.3460\,. \tag{66}$$

Assuming that the asymptotic $\sim [\varkappa\bar{\varkappa}]^a$ behavior can be extrapolated to the whole interval $\varkappa \in [0, 1]$ one obtains a model for the eigenfunction of the ground state

$$\hat{\eta}_0(\varkappa) = [\varkappa\bar{\varkappa}]^a \qquad \mapsto \qquad \eta_0(x) = \Gamma[a+ix]\Gamma[a-ix]/\Gamma^2[a]\,, \tag{67}$$

which turns out to be in good agreement numerically with a more complicated parametrization in Eq. (53).

This result agrees well with the approximation for the asymptotic Λ_b DA in the $\varkappa$-space $\hat{\rho}_2(w_r, \varkappa) \sim [\varkappa\bar{\varkappa}]^{1/3}$ found in Ref. [5] (see Fig. 4 there) by expanding the eigenfunction in Gegenbauer polynomials and retaining the first few terms. Numerical convergence of this expansion (away from the

[2] The exponential falloff of the first term is tightened to the particular choice of the profile function (60) and in this sense accidental.

[3] We thank the referee for suggesting to make this comparison.

end-points) observed in [5] is in fact directly related to our result that the lowest eigenstate of the evolution equation is separated from the continuum spectrum by a finite gap (52).

The above discussion cannot be directly applied to all other eigenstates of the evolution equation that belong to the contunuum spectrum. In particular for the integrable case these solutions are given by plane waves, $\hat{\eta}(\varkappa) \sim e^{ipx}$, and possess an essential singularity at $\varkappa \to 0, \varkappa \to 1$. This singularity is only seen in extreme proximity to the end-points and cannot be found using the Gegenbauer expansion. Whether such singularities are important for phenomenological applications remains to be studied.

5. To summarize, in this letter we have studied the evolution equations that determine the scale dependence of leading-twist DAs of heavy-light baryons containing one infinitely heavy and two light quarks. The evolution equations are different for the cases that the light quarks have the same, or opposite, chirality. For the first case, which corresponds to transverse DAs of $j^P = 1^+$ sextets ($\Sigma_b, \Xi_b, \Omega_b$ and $\Sigma_b^*, \Xi_b^*, \Omega_b^*$), the evolution equation turns out to be completely integrable, that is it has a nontrivial integral of motion. The anomalous dimensions form a continuum spectrum parametrized by two real numbers, s and x (22), and the corresponding eigenfunctions are known exactly (18), (26). For the second case ($j^P = 0^+$ $SU(3)_F$ triplet and all longitudinal DAs of heavy baryons in the $j^P = 1^+$ sextets), integrability is broken by an additional contribution to the evolution kernel that effectively corresponds to an attractive interaction between the light quarks and creates a bound state. As the result, the lowest anomalous dimension becomes separated from the rest of the spectrum (that remains continuous) by a finite gap (52). The corresponding eigenfunction is delocalized in the x-space and can be found using numerical methods (53) (see also (67). It can be interpreted as the asymptotic DA at large scales (59), (61) and deviates significantly from the naive $\sim u(1-u)$ shape at large quark energies, cf. (63) and Fig. 3.

We expect that evolution equations for the higher-twist DAs of heavy baryons [3, 4] and for the three-particle quark-gluon DAs of B-mesons in the large-N_c limit [21] have similar properties and can be studied using the same methods. These can be important for practical applications since heavy hadron decay form factors for physical values of the b-quark mass are likely to be dominated by soft contributions that can be related to higher-twist DAs using light-cone sum rules, see e.g. [1, 26].

Analogous unconventional integrable models with the Hamiltonian commuting with the diagonal entry $D(u)$ of the monodromy matrix have appeared recently in the studies of high-energy scattering amplitudes in the $N = 4$ supersymmetric Yang-Mills theory [27, 28, 29, 30].

Acknowledgements

This work was supported by RFBR grants 13-01-12405, 14-01-00341 (S.D.) and by the DFG, grant BR2021/5-2 (A.M.).

References

[1] T. Feldmann and M. W. Y. Yip, Phys. Rev. D **85** (2012) 014035 [Erratum-ibid. D **86** (2012) 079901].

[2] W. Wang, Phys. Lett. B **708** (2012) 119.

[3] P. Ball, V. M. Braun and E. Gardi, Phys. Lett. B **665**, 197 (2008).

[4] A. Ali, C. Hambrock, A. Y. Parkhomenko and W. Wang, Eur. Phys. J. C **73**, 2302 (2013).

[5] G. Bell, T. Feldmann, Y. -M. Wang and M. W. Y. Yip, JHEP **1311**, 191 (2013).

[6] V. M. Braun, S. E. Derkachov and A. N. Manashov, Phys. Rev. Lett. **81**, 2020 (1998).

[7] V. M. Braun, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, Nucl. Phys. B **553**, 355 (1999).

[8] L. N. Lipatov, *"High-energy asymptotics of multicolor QCD and exactly solvable lattice models,"* hep-th/9311037.

[9] L. N. Lipatov, JETP Lett. **59**, 596 (1994) [Pisma Zh. Eksp. Teor. Fiz. **59**, 571 (1994)].

[10] L. D. Faddeev and G. P. Korchemsky, Phys. Lett. B **342** (1995) 311.

[11] J. A. Minahan and K. Zarembo, JHEP **0303**, 013 (2003).

[12] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B **664**, 131 (2003).

[13] N. Beisert and M. Staudacher, Nucl. Phys. B **670**, 439 (2003).

[14] L. D. Faddeev, *"How algebraic Bethe ansatz works for integrable model"*, In: Quantum Symmetries/Symetries Quantiques, Proc. Les-Houches summer school, LXIV. Eds. A. Connes, K. Kawedzki, J. Zinn-Justin; North-Holland, 1998, 149-211, hep-th/9605187.

[15] P. P. Kulish and E. K. Sklyanin, Lect. Notes Phys. **151** (1982) 61.

[16] E. K. Sklyanin, *Quantum Inverse Scattering Method*, in *Quantum Groups and Quantum Integrable Systems*, (Nankai lectures), ed. Mo-Lin Ge, pp. 63-97, World Scientific Publ., Singapore 1992, [hep-th/9211111]

[17] S. E. Derkachov and A. N. Manashov, *Iterative construction of eigenfunctions of the monodromy matrix for SL(2,C) magnet,* arXiv:1401.7477 [math-ph].

[18] G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. B **279**, 359 (1992).

[19] B. O. Lange and M. Neubert, Phys. Rev. Lett. **91**, 102001 (2003).

[20] V. M. Braun, D. Y. Ivanov and G. P. Korchemsky, Phys. Rev. D **69**, 034014 (2004).

[21] M. Knodlseder and N. Offen, JHEP **1110**, 069 (2011).

[22] V. M. Braun, G. P. Korchemsky and D. Mueller, Prog. Part. Nucl. Phys. **51**, 311 (2003).

[23] V. M. Braun and A. N. Manashov, Phys. Lett. B **731**, 316 (2014).

[24] A. P. Bukhvostov, G. V. Frolov, L. N. Lipatov and E. A. Kuraev, Nucl. Phys. B **258**, 601 (1985).

[25] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, JHEP **0307**, 047 (2003).

[26] V. M. Braun and A. Khodjamirian, Phys. Lett. B **718** (2013) 1014.

[27] L. N. Lipatov, J. Phys. A **42** (2009) 304020.

[28] J. Bartels, L. N. Lipatov and A. Prygarin, J. Phys. A **44** (2011) 454013.

[29] B. Basso, Nucl. Phys. B **857** (2012) 254.

[30] A. V. Belitsky, Phys. Lett. B **709** (2012) 280.