Supplemental Material:
Two-dimensional topological insulator edge state backscattering by dephasing

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In the main manuscript, we show how dephasing affects the scattering properties of a quantum spin Hall edge state if it is scattered at a charged puddle. Here, we give more details on the model used to calculate the time-dependent probability density of the state localized in puddle. Furthermore, we give insight into how dephasing was added to the quantum time evolution used in the numerical calculations presented in the manuscript. A video visualizing the dephasing of a wave packet is attached to the Supplemental Material.

A Model Setup

In the main text, we show numerical calculations of an edge-state wave packet which is scattered by an electrostatic puddle and extract the time-dependent probability density $\rho(t)$ of the wave packet being localized in the puddle. We model the electronic structure of a HgTe/CdTe quantum well using the Bernevig-Hughes-Zhang Hamiltonian [1],

$$H = \begin{pmatrix} h(\mathbf{k}) & \begin{matrix} 0 & -\Delta \\ \Delta & 0 \end{matrix} \\ \begin{matrix} 0 & \Delta \\ -\Delta & 0 \end{matrix} & h^*(-\mathbf{k}) \end{pmatrix}, \tag{1}$$

with spin-subblock Hamiltonians

$$h(\mathbf{k}) = \begin{pmatrix} V(x) + M(x) - (B + D)\mathbf{k}^2 & Ak_+ \\ Ak_- & V(x) - M(x) + (B - D)\mathbf{k}^2 \end{pmatrix}, \tag{2}$$

where $k_\pm = k_x \pm i k_y$ and $\mathbf{k}^2 = k_x^2 + k_y^2$. Intrinsic spin-orbit coupling of the Dresselhaus-type is included by the parameter Δ in Eq. (1), which mixes the two spin blocks [2]. We use an electrostatic potential $V(x)$ to model circular and stadium shaped puddles and confine the states by the potential $M(x)$ to get a quantum spin Hall edge state at the system boundary. For the calculations presented in the manuscript, the potential strength of the puddle was set to 40 meV leading to hole-like states.

As initial state, we create a Gaussian edge-state wave packet $\psi(t_0)$, which is assembled in reciprocal space along the boundary. Throughout the manuscript, we use a Gaussian with a width in position space of $\sigma = 90$ nm. Also, we include only states propagating towards the electrostatic puddle, leading to a strongly spin-polarized wave packet. We calculate the propagation of the wavefunction $\psi(t)$ using a propagator based on an expansion of the time-evolution operator in Chebyshev Polynomials [3]. During the propagation of the wave packet $\psi(t)$, we integrate $|\psi(t)|^2$ over the puddle region resulting in the time-dependent probability density $\rho(t)$. In order to avoid effects of mesoscopic fluctuations, we average over 20 different configurations, which differ in a random impurity potential with an amplitude of 5 meV and a wall distortion of 20 nm.

B Propagation with dephasing

In the following section, we will show how dephasing is included in the numerical calculations presented in the main part of the manuscript. The dephasing algorithm described here is inspired by the concept of einselection ("environment-induced superselection") pioneered by W. Zurek [4], which is a mechanism proposed to understand the influence of dephasing on quantum systems and, more generally, to explain the quantum-to-classical transition. According to einselection, the interaction of an open system with its environment leads to decoherence[1], which causes a decay of the quantum states of the system into an incoherent mixture of so-called pointer states. This strongly suppresses quantum interference effects between different pointer states on the time scale of the dephasing time τ_ϕ. The character of the set of pointer states, the pointer basis, heavily depends on the environment and the coupling to it. For example, for very weak coupling to the environment, the pointer basis coincides with the set of energy eigenstates of the system. However, in the case of an intermediate system-environment coupling based on a local interaction, e. g., in the case of electron-phonon or electron-electron coupling, it is expected to be a set of states that is localized in phase space, i. e., in position and momentum.

As in the puddle lifetime calculations, we again employ a numerical time evolution based on a single pure state, i. e., we do not use a representation in terms of density matrices, which would drastically increase the computational effort. Still, we incorporate the dephasing-induced interference suppression by occasionally (partially) randomizing the phases of the components of the state vector in a representation that tries to faithfully mimic a decomposition in terms of the pointer basis. This randomization is done at fixed event times t_n, which are sampled from an exponential distribution with time constant τ_e, i. e., we assume the events to be fully uncorrelated (Poisson process). In between these events, the propagation is done fully coherently using the propagator based on a polynomial expansion mentioned above. The decomposition and the subsequent randomization is done in the following way: At the time of an event t_n, we extract a set of pseudo eigenstates,

$$\phi_m \propto \int_{t_n-\Delta t}^{t_n+\Delta t} \psi(t)\, e^{iE_m t/\hbar} \mathrm{d}t, \tag{3}$$

at the energies $E_m = \{0\,\mathrm{meV}, \pm 1.5\,\mathrm{meV}, ... \pm 7.5\,\mathrm{meV}\}$ using a short-time propagation of the wave packet $\psi(t)$ around the time t_n. These states fulfill the requirement that they are both local in energy as well as in position space (as they are extracted from a propagation

[1] We use decoherence and dephasing synonymously in this article.

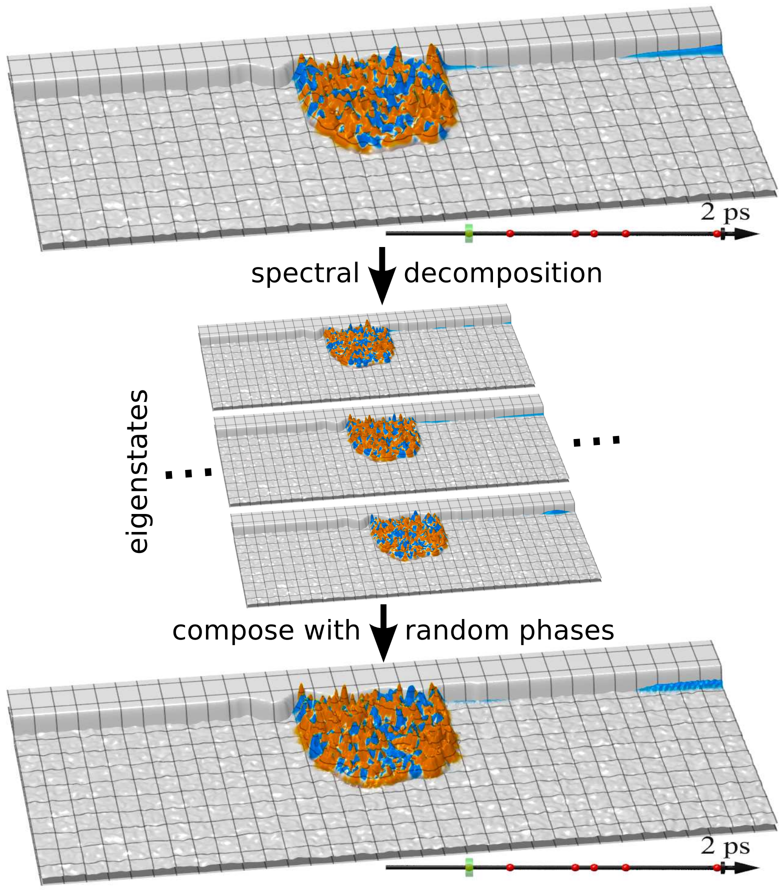

Figure 5: The figure illustrates the workings of the dephasing algorithm. At a dephasing event, the current wavefunction—here showing a wave packet spread in an electrostatic puddle (the color encodes the spin)—is spectrally decomposed in a set of pseudo eigenstates and recomposed with random phases, leading to a new dephased wavefunction. A video using this type of dephasing is available online [5].

over a finite time interval). The degree of localization in position space can by tuned by the propagation time Δt[2], which we choose $\Delta t \sim 1\,\mathrm{ps}$. We use these states ϕ_m to spectrally decompose the current state before the event $\psi(t_n)$, as sketched in Fig. 5. For this, we first orthogonalize them with a Gram-Schmidt process, leading to the set $\tilde{\phi}_m$. Then, we calculate the weights a_m, which can be used to express the current state as the decomposition

$$\psi(t_n) = \sum_m a_m \tilde{\phi}_m + \Delta\psi. \tag{4}$$

[2]For $\Delta t \to \infty$ and a matching set of energies, this approach yields the expected pointer basis in the limit of very weak coupling to the environment: the exact energy eigenstates of the system.

Since the set of 11 states is not sufficient to describe the wave packet $\psi(t_n)$ exactly, we also allow for account a small residual part $\Delta\psi$. In this decomposed representation, the dephasing can be easily added by modifying the set of amplitudes to

$$\tilde{a}_m = a_m \exp(i\pi \text{ rand}), \tag{5}$$

with the random numbers rand $\in [-Q, Q]$. Subsequently, we create the new wave packet

$$\psi(t_n + \epsilon) = \sum_m \tilde{a}_m \tilde{\phi}_m + \Delta\psi, \tag{6}$$

which is used in the following propagation. With this dephasing algorithm, we are able to calculate the time evolution in arbitrary mesoscopic systems. It roughly conserves the energy, the spatial extent, as well as the total spin of the wave packet, which is what one would expect from the interaction with a spin-unpolarized (non-magnetic) environment. The strength of a single dephasing event can be tuned by the degree of randomization of the phase in each event, i. e., by the parameter Q. This also makes the regime of weak but frequent dephasing accessible (compared to the regime of rare and very strong dephasing which allows a simpler model treatment discussed in the first part of the manuscript). However, to compare with the experimentally accessible dephasing time τ_ϕ, which is the time for full phase coherence loss, one has to find the relation between the time constant of the events τ_e and the time for full dephasing τ_ϕ. This can be done in the following way: The average correlation of a state in the pointer basis $|\psi^\mathrm{P}\rangle$ with itself after n events is given by,

$$\begin{aligned}\iiint_{-Q}^{Q} dq_1 \ldots dq_n e^{i\pi \sum_{j=1}^{n} q_j} &= \frac{1}{(2Q)^n} \iiint_{-Q}^{Q} dq_1 \cdots dq_n \cos\left(\pi \sum_{j=1}^{n} q_j\right) \\ &= \left(\frac{1}{2Q}\int_{-Q}^{Q} dq \cos(\pi q)\right)^n = \left(\frac{\sin(\pi Q)}{\pi Q}\right)^n . \end{aligned} \tag{7}$$

With the chance for n events occurring after time t in a Poisson process,

$$P(n) = \frac{t^n}{\tau_\mathrm{e}^n}\frac{1}{n!} e^{-\frac{t}{\tau_\mathrm{e}}}, \tag{8}$$

one can evaluate the average decay of autocorrelation after time t,

$$\begin{aligned}\frac{\langle\psi^{\mathrm{P}}(0)|\psi^{\mathrm{P}}(t)\rangle}{\langle\psi^{\mathrm{P}}(0)|\psi^{\mathrm{P}}(t)\rangle_{\text{coherent}}} &= \sum_{n=0}^{\infty} P(n)\left(\frac{\sin(\pi Q)}{\pi Q}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{\tau_{\mathrm{e}}}\frac{\sin(\pi Q)}{\pi Q}\right)^n e^{-\frac{t}{\tau_{\mathrm{e}}}} = \exp\left[\frac{t}{\tau_{\mathrm{e}}}\frac{\sin(\pi Q)}{\pi Q} - \frac{t}{\tau_{\mathrm{e}}}\right] \\ &= \exp\left[-\frac{t}{\tau_{\mathrm{e}}}\left(1-\frac{\sin(\pi Q)}{\pi Q}\right)\right] = \exp\left[-\frac{t}{\tau_\phi}\right], \end{aligned} \tag{9}$$

with

$$\tau_\phi = \frac{\pi Q}{\pi Q - \sin(\pi Q)}\tau_{\mathrm{e}}, \tag{10}$$

yielding the desired relation between decoherence time τ_ϕ and the time constant of the events τ_{e}. In the calculations shown in the main text, we use $Q = \frac{1}{2}$, thus, $\tau_\phi = \pi/(\pi-2)\tau_{\mathrm{e}} \approx 2.75\,\tau_{\mathrm{e}}$.

In Fig. 4 of the main manuscript, we apply the dephasing scheme on three different electrostatic puddle shapes to show how the coupling between the puddle and the edge states affects the total spin randomization of the puddle in presence of dephasing. A video of a sample propagation can be also found in the Supplemental Material [5].

References

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[4] W. H. Zurek, *Decoherence, einselection, and the quantum origins of the classical*, Rev. Mod. Phys. **75**, 715 (2003).

[5] See Supplemental Material for an example video of the wave-packet propagation used to obtain the data in Fig. 4 of the main text. The time line in the bottom denotes the times at which the dephasing events take place. The spin content of the wavefunction is color coded in blue and orange, the potential region is colored in red.