



Self-adjointness of the Dirac
Hamiltonian for a class of non-uniformly
elliptic boundary value problems

Felix Finster and Christian Röken

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**SELF-ADJOINTNESS OF THE DIRAC HAMILTONIAN
FOR A CLASS OF NON-UNIFORMLY ELLIPTIC
BOUNDARY VALUE PROBLEMS**

FELIX FINSTER AND CHRISTIAN RÖKEN

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ABSTRACT. We consider a boundary value problem for the Dirac equation in a four-dimensional, smooth, asymptotically flat Lorentzian manifold admitting a Killing field which is timelike near and tangential to the boundary. A self-adjoint extension of the Dirac Hamiltonian is constructed. Our results also apply to the situation that the space-time includes horizons, where the Hamiltonian fails to be elliptic.

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1. INTRODUCTION

Let (\mathcal{M}, g) be a four-dimensional, smooth, asymptotically flat Lorentzian manifold with boundary $\partial\mathcal{M}$. We assume that there is a *Killing field* K which is *tangential to and timelike on* $\partial\mathcal{M}$. Moreover, we assume that the integral curves of K , defined by the differential equation

$$\dot{\gamma}(t) = K(\gamma(t)),$$

exist for all $t \in \mathbb{R}$. We also assume that there exists a *spacelike hypersurface* \mathcal{N} with compact boundary $\partial\mathcal{N}$, having the property that *every* such *integral curve* γ *intersects* \mathcal{N} *exactly once*. This implies that \mathcal{M} and its boundary $\partial\mathcal{M}$ have the product structures

$$\mathcal{M} = \mathbb{R} \times \mathcal{N} \quad \text{and} \quad \partial\mathcal{M} = \mathbb{R} \times \partial\mathcal{N}. \tag{1.1}$$

Note that our assumptions also imply that the metric g is smooth up to the boundary $\partial\mathcal{M}$, thus inducing on $\partial\mathcal{N}$ a two-dimensional Riemannian metric.

For clarity, we now mention two well-known special cases. If $\partial\mathcal{N}$ is empty and \mathcal{N} is complete, the product structure (1.1) implies that (\mathcal{M}, g) is *globally hyperbolic*. Moreover, if K is timelike in the asymptotic end, the manifold is *stationary*. However, we point out that we merely assume that K is timelike on the boundary $\partial\mathcal{M}$, but it does not need to be timelike everywhere.

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In order to get a better geometric understanding of the above setting, we now construct a convenient coordinate system. Choosing the parametrization of each curve γ such that $\gamma(0) \in \mathcal{N}$, we obtain a global coordinate function T defined by

$$T : \mathcal{M} \rightarrow \mathbb{R} \quad \text{with} \quad T(\gamma(t)) = t. \quad (1.2)$$

The level sets of this time function give rise to a foliation $\mathcal{N}_t := T^{-1}(t)$ by spacelike hypersurfaces with $\mathcal{N}_0 = \mathcal{N}$. Moreover, the integral curves give rise to isometries

$$\Phi_t : \mathcal{N} \rightarrow \mathcal{N}_t, \quad \Phi_t(\gamma(0)) = \gamma(t).$$

Choosing coordinates x on \mathcal{N} , the mapping $x \circ \Phi_t^{-1}$ gives coordinates on \mathcal{N}_t . Complementing this coordinate system by the function $t = T$, we obtain coordinates (t, x) with $t \in \mathbb{R}$ and $x \in \mathcal{N}$ such that $K = \partial_t$. In these coordinates, the line element takes the form

$$ds^2 = g_{ij} dx^i dx^j = a(x) dt^2 + b_\alpha(x) dt dx^\alpha - (g_{\mathcal{N}}(x))_{\alpha\beta} dx^\alpha dx^\beta,$$

where a and b_α are smooth functions, and $g_{\mathcal{N}}$ is the induced Riemannian metric on \mathcal{N} . Here we denote the space-time indices by latin letters $i, j \in \{0, 1, 2, 3\}$, whereas spatial indices are denoted by greek letters $\alpha, \beta \in \{1, 2, 3\}$. This coordinate system can be understood as describing an observer who is co-moving along the flow lines of the Killing field. We note that in the regions where K is timelike, the function $a(x)$ is positive, and the metric is stationary. This is the case if x is near the boundary $\partial\mathcal{N}$. However, away from $\partial\mathcal{N}$, the function $a(x)$ could be negative, in which case the metric is *not stationary*, and t is *not a time coordinate*. This situation is illustrated in the following example.

Example 1.1. (Kerr geometry in Eddington-Finkelstein-type coordinates)

In the recent paper [12], horizon-penetrating Eddington-Finkelstein-type coordinates

$$(\tau, r, \theta, \phi) \quad \text{with} \quad \tau \in \mathbb{R}, \quad r \in \mathbb{R}^+, \quad \theta \in (0, \pi), \quad \phi \in (0, 2\pi)$$

are introduced in the non-extreme Kerr geometry. In these coordinates, the line element takes the form

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) d\tau^2 - \frac{4Mr}{\Sigma} (dr - a \sin^2 \theta d\phi) d\tau \\ - \left(1 + \frac{2Mr}{\Sigma}\right) (dr - a \sin^2 \theta d\phi)^2 - \Sigma d\theta^2 - \Sigma \sin^2 \theta d\phi^2,$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$. Moreover, M and aM denote the mass and the angular momentum of the black hole, respectively. The surfaces $r_\pm = M \pm \sqrt{M^2 - a^2}$ are the *event horizon* and the *Cauchy horizon* of the black hole. Note that these coordinates are regular on and across the horizons. The two Killing fields describing the stationarity and axisymmetry of the Kerr geometry are ∂_τ and ∂_ϕ .

We choose a radius $r_0 < r_-$ inside the Cauchy horizon and let

$$\mathcal{M} = \{r > r_0\}, \quad \mathcal{N} = \{\tau = 0, r > r_0\} \quad \text{and} \quad \partial\mathcal{M} = \{r = r_0\}.$$

Direct computation shows that the Killing field ∂_τ is *not* everywhere timelike on $\partial\mathcal{M}$ (due to an ergo-like region inside the Cauchy horizon; for details see [7]). But taking K as a suitable linear combination of ∂_τ and ∂_ϕ ,

$$K = \partial_\tau + b \partial_\phi$$

with a real constant $b = b(r_0) \neq 0$, it turns out that K is a Killing field which satisfies all the above assumptions. This Killing field is spacelike near spatial infinity. \diamond

Next, we shall formulate the Dirac equation. To describe spinors one introduces the *spinor bundle* $S\mathcal{M}$, a vector bundle with fibers $S_p\mathcal{M} \simeq \mathbb{C}^4$, $p \in \mathcal{M}$. Each fiber is endowed with an inner product of signature $(2, 2)$, referred to as *spin scalar product* and denoted by

$$\langle \cdot | \cdot \rangle_p : S_p\mathcal{M} \times S_p\mathcal{M} \rightarrow \mathbb{C}.$$

The geometric Dirac operator \mathcal{D} takes the form

$$\mathcal{D} = i\gamma^j \nabla_j, \quad (1.3)$$

where the Dirac matrices γ^j are related to the metric by the anti-commutation relations

$$\{\gamma^j, \gamma^k\} = 2g^{jk} \mathbf{1}_{S_p\mathcal{M}},$$

and ∇ is the metric connection on the spinor bundle (for more details see [10]). In order to allow for an external potential (like for example an electromagnetic potential), instead of (1.3) we shall consider the more general Dirac operator

$$\mathcal{D} = i\gamma^j \nabla_j + \mathcal{B}, \quad (1.4)$$

where \mathcal{B} is a smooth matrix-valued potential which we assume to be symmetric with respect to the spin scalar product, i.e. $\langle \phi | \mathcal{B}\psi \rangle = \langle \mathcal{B}\phi | \psi \rangle$.

We are interested in solutions ψ of the Dirac equation of mass m

$$(\mathcal{D} - m)\psi = 0, \quad (1.5)$$

with the Dirac operator according to (1.4). In order to analyze the dynamics of Dirac waves, it is useful to write the Dirac equation in the Hamiltonian form

$$i\partial_t \psi = H\psi, \quad (1.6)$$

where H is the Dirac Hamiltonian given by

$$H = -(\gamma^t)^{-1}(i\gamma^\alpha \nabla_\alpha + \mathcal{B} - m). \quad (1.7)$$

Taking the domain of definition

$$\mathcal{D}(H) = C_0^\infty(\overset{\circ}{\mathcal{N}}, S\mathcal{M}),$$

where $\overset{\circ}{\mathcal{N}} = \mathcal{N} \setminus \partial\mathcal{N}$ denotes the interior of \mathcal{N} , this Hamiltonian is indeed symmetric (i.e. formally self-adjoint) with respect to the scalar product

$$(\psi | \phi)_{\mathcal{N}} = \int_{\mathcal{N}} \langle \psi | \psi \phi \rangle_x d\mu_{\mathcal{N}}(x),$$

where ν is the future-directed normal on \mathcal{N} and $d\mu_{\mathcal{N}}$ is the volume form on $(\mathcal{N}, g_{\mathcal{N}})$. This can be verified with the following computation. Using current conservation together with the fact that the metric coefficients do not depend on the coordinate t , we obtain

$$\begin{aligned} 0 &= \partial_t (\psi(t) | \phi(t))_{\mathcal{N}} = (\dot{\psi} | \phi)_{\mathcal{N}} + (\psi | \dot{\phi})_{\mathcal{N}} \\ &= ((-iH\psi) | \phi)_{\mathcal{N}} + (\psi | (-iH\phi))_{\mathcal{N}} = i((H\psi | \phi)_{\mathcal{N}} - (\psi | H\phi)_{\mathcal{N}}). \end{aligned}$$

In order to pose the Cauchy problem for the Dirac equation (1.5), one needs to specify initial and boundary conditions. We choose initial data which is smooth and compactly supported,

$$\psi|_{\mathcal{N}} = \psi_0 \in C_0^\infty(\mathcal{N}, S\mathcal{M}). \quad (1.8)$$

Moreover, we impose the boundary conditions

$$(\not{n} - i) \psi|_{\partial\mathcal{M}} = 0, \quad (1.9)$$

where the slash denotes Clifford multiplication, and n is the inner normal on $\partial\mathcal{M}$ (meaning that for every $p \in \partial\mathcal{M}$ there is a curve $c : [0, \delta) \rightarrow \mathcal{M}$ with $c(0) = p$ and $\dot{c}(0) = n(p)$). Clearly, the initial data must be compatible with the boundary conditions, meaning that $(\not{n} - i) \psi_0|_{\partial\mathcal{M}} = 0$. These boundary conditions, which are very similar to those introduced in [5, Section 2], have the effect that Dirac waves are reflected on $\partial\mathcal{M}$. They can also be understood in analogy to the chiral boundary conditions in [3, 8]. The difference is that, instead of the intrinsic Dirac operator on the hypersurface, we here consider the Hamiltonian obtained from the Dirac operator in space-time by separating the t -dependence. This gives rise to the additional factor $(\gamma^t)^{-1}$ in (1.7). Our boundary conditions (1.9) can be understood as an adaptation of the chiral boundary conditions in [3, 8] to the Hamiltonian (1.7).

The boundary conditions (1.9) must be incorporated in the functional analytic setting. To this end, one extends the domain of definition to

$$\mathcal{D}(H) = \{ \psi \in C_0^\infty(\mathcal{N}, S\mathcal{M}) \quad \text{with} \quad (\not{n} - i) \psi|_{\partial\mathcal{N}} = 0 \}. \quad (1.10)$$

Then the operator H is again symmetric, as the following consideration shows. First, we rewrite the scalar product $(\psi|H\phi)_{\mathcal{N}}$ in a more convenient form. Applying the relations $(\gamma^t)^2 = g^{tt} \mathbb{1}_{S_x\mathcal{M}}$ and $\psi = \gamma^t / \sqrt{g^{tt}}$, we obtain

$$\psi(\gamma^t)^{-1} = \frac{1}{g^{tt}} \psi \gamma^t = \frac{\mathbb{1}_{S_x\mathcal{M}}}{\sqrt{g^{tt}}}.$$

Making use of the form of the Hamiltonian (1.7), this leads to

$$(\psi|H\phi)_{\mathcal{N}} = -i \int_{\mathcal{N}} \langle \psi | \gamma^\alpha \nabla_\alpha \phi \rangle_x \frac{1}{\sqrt{g^{tt}}} d\mu_{\mathcal{N}}(x) + (\text{lower order terms}).$$

Now a direct computation of the boundary terms yields

$$(\psi|H\phi)_{\mathcal{N}} - (H\psi|\phi)_{\mathcal{N}} = i \int_{\partial\mathcal{N}} \langle \psi | \not{n} \phi \rangle_x \frac{1}{\sqrt{g^{tt}}} d\mu_{\partial\mathcal{N}}(x)$$

(we note that the angular derivatives do not give rise to boundary terms because $\partial\mathcal{N}$ is compact without boundary). Using the boundary conditions in (1.10), for all $x \in \partial\mathcal{N}$ we obtain

$$i \langle \psi | \phi \rangle_x = \langle \psi | \not{n} \phi \rangle_x = \langle \not{n} \psi | \phi \rangle_x = -i \langle \psi | \phi \rangle_x,$$

proving that the boundary values indeed vanish. This shows that H is symmetric.

In order to solve the Cauchy problem and to analyze the long-time behavior of its solutions, it is of central importance to construct a self-adjoint extension of H . Namely, with such a self-adjoint extension at hand, the solution of the Cauchy problem for the Dirac equation (1.5) with initial values (1.8) and boundary conditions (1.9) can be expressed using the spectral theorem for self-adjoint operators as

$$\psi(t) = e^{-itH} \psi_0 = \int_{\sigma(H)} e^{-i\omega t} dE_\omega \psi_0.$$

This formula is also the starting point for a detailed analysis of the long-time behavior of ψ using spectral methods, similar as carried out in the exterior region of Kerr geometry in [5, 4]. In the present paper, we succeed in constructing a self-adjoint extension:

Theorem 1.2. *The Dirac Hamiltonian (1.7) with domain of definition*

$$\mathcal{D}(H) = \left\{ \psi \in C_0^\infty(\mathcal{N}, S\mathcal{M}) \quad \text{with} \quad (\not{p} - i)(H^p \psi)|_{\partial\mathcal{N}} = 0 \quad \text{for all } p \in \mathbb{N}_0 \right\} \quad (1.11)$$

is essentially self-adjoint.

Note that the domain (1.11) is smaller than (1.10). This is preferable because we want that the Cauchy problem has a global solution in $C_0^\infty(\mathcal{N}, S\mathcal{M})$ (see the strategy of our proof as described at the end of this section).

We conclude this section by putting our result into the context of previous work, and explaining the strategy of our proof. The Cauchy problem and the problem of constructing a self-adjoint extension of the Dirac Hamiltonian have been studied in several simpler situations:

- (i) If \mathcal{N} is a complete manifold without boundary, the Cauchy problem can be solved using the theory of symmetric hyperbolic systems (see for example [9, 14, 6]). In this construction, one works with local charts with local time functions. Since the resulting local solutions coincide in the regions where the charts overlap, this procedure gives rise to a unique, global smooth solution in \mathcal{M} . Then, restricting this solution to the spacelike hypersurfaces of constant t , one obtains a family of time evolution operators

$$U_{t',t} : C^\infty(\{t\} \times \mathcal{N}, S\mathcal{M}) \rightarrow C^\infty(\{t'\} \times \mathcal{N}, S\mathcal{M}),$$

which form a group. This makes it possible to apply [2] to conclude that the Hamiltonian is essentially self-adjoint on $C^\infty(\mathcal{N}, S\mathcal{M})$.

- (ii) In the ultrastatic situation

$$ds^2 = dt^2 - (g_{\mathcal{N}})_{\alpha\beta} dx^\alpha dx^\beta,$$

the Hamiltonian can be written as

$$H = \begin{pmatrix} 0 & \mathcal{D}_{\mathcal{N}} \\ \mathcal{D}_{\mathcal{N}} & 0 \end{pmatrix},$$

where $\mathcal{D}_{\mathcal{N}}$ is the intrinsic Dirac operator on \mathcal{N} . This makes it possible to apply the results in the Riemannian setting as worked out in detail in [1].

- (iii) In the static situation

$$ds^2 = a(x) dt^2 - (g_{\mathcal{N}})_{\alpha\beta} dx^\alpha dx^\beta,$$

the Hamiltonian can be written as

$$H = \sqrt{a(x)} \begin{pmatrix} 0 & \mathcal{D}_{\mathcal{N}} \\ \mathcal{D}_{\mathcal{N}} & 0 \end{pmatrix} + (\text{zero-order terms}).$$

Introducing a suitable scalar product on the Dirac wave functions, this Hamiltonian is again symmetric, making it possible to again apply the results of [1].

In the situation under consideration here, there is the major complication that the Hamiltonian is in general not uniformly elliptic, so that the methods in [1] no longer apply. In order to explain the problem, we now consider the principal symbol of the Dirac Hamiltonian. According to (1.7), the principal symbol takes the form

$$P(x, \xi) = -i(\gamma^t)^{-1} \gamma^\alpha \xi_\alpha.$$

The ellipticity condition states that the principal symbol should be bounded from below by

$$\|P(x, \xi)\| \geq \delta \|\xi\|^2$$

for a suitable constant $\delta > 0$ (for basics on the principal symbol and the connection to ellipticity see for example [13, Section 5.11]). In order to verify whether this condition holds, it is most convenient to compute the determinant of the principal symbol. Namely,

$$\det P(x, \xi) = \det ((\gamma^t)^{-1}) \det (\gamma^\alpha \xi_\alpha),$$

and using that

$$(\gamma^t)^{-1}(\gamma^t)^{-1} = \frac{\mathbb{1}_{S_x \mathcal{M}}}{g^{tt}}, \quad \gamma^\alpha \xi_\alpha \gamma^\beta \xi_\beta = g^{\alpha\beta} \xi_\alpha \xi_\beta \mathbb{1}_{S_x \mathcal{M}},$$

we obtain

$$\det P(x, \xi) = \left(\frac{g^{\alpha\beta} \xi_\alpha \xi_\beta}{g^{tt}} \right)^2.$$

This computation shows that the Hamiltonian fails to be elliptic if $g^{\alpha\beta} \xi_\alpha \xi_\beta = 0$ for a non-zero ξ . In the example of the Kerr metric in Eddington-Finkelstein-type coordinates [12], this is the case precisely on the event and Cauchy horizons. More generally, the points where the Hamiltonian fails to be elliptic can be used as the definition of the *horizons* of our space-time. Thus we face the major problem that the *Hamiltonian is not elliptic on the horizons*.

Our strategy to solve this problem is to split up the solution of the Cauchy problem into two separate problems: Near the boundary, we rewrite the problem in a form where the results in [1] apply. Away from the boundary, however, we use the theory of symmetric hyperbolic systems. Making essential use of finite propagation speed, adding the two solutions gives rise to a unique solution of our boundary value problem for small times. By iterating the procedure, we get unique global, smooth solutions, making it possible to proceed as in (i) above by applying [2].

2. A DOUBLE BOUNDARY VALUE PROBLEM

As a technical tool for the proof of Theorem 1.2, we need to show that the Cauchy problem (1.5), (1.8) with boundary conditions (1.9) has global smooth solutions. Our method is to split up the Cauchy problem into two separate problems near and away from the boundary. In preparation, we now introduce additional boundary conditions on a suitable surface Y near $\partial \mathcal{M}$. We work in Gaussian normal coordinates in a tubular neighborhood in \mathcal{N} of $\partial \mathcal{N}$. Thus for any $p \in \partial \mathcal{N}$, we let $c_p(r)$ for $0 \leq r < r_{\max}(p)$ be the geodesic in \mathcal{N} with the initial conditions

$$c_p(0) = p \quad \text{and} \quad c'_p(0) = u,$$

where $u \in T_p \mathcal{N}$ is the inner normal to $\partial \mathcal{N}$. Since $\partial \mathcal{N}$ is compact, we can choose $r_{\max} > 0$ independent of p to obtain a mapping

$$c : [0, r_{\max}) \times \partial \mathcal{N} \rightarrow \mathcal{N}, \quad c(r, p) = c_p(r).$$

Applying the implicit function theorem, possibly by decreasing r_{\max} we can arrange that c is a diffeomorphism. We introduce the sets obtained from $\partial \mathcal{N}$ by the geodesic flow by

$$\partial \mathcal{N}(r) = c(r, \partial \mathcal{N}).$$

Choosing coordinates $\Omega = (\vartheta, \varphi)$ on $\partial \mathcal{N}$ gives a corresponding coordinate system (r, Ω) on \mathcal{N} . In these coordinates, the metric on \mathcal{N} takes the form

$$(g_{\mathcal{N}})_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & g_{\partial \mathcal{N}(r)} \end{pmatrix} \quad \text{and thus} \quad (g_{\mathcal{N}})^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & (g_{\partial \mathcal{N}(r)})^{-1} \end{pmatrix}. \quad (2.1)$$

Taking again t as the time coordinate, we obtain a coordinate system (t, r, Ω) with $t \in \mathbb{R}$, $r \in [0, r_{\max})$ of \mathcal{M} which describes a neighborhood of $\partial\mathcal{M}$.

We now introduce the following new boundary value problem. Let X be the space-time region

$$X = \{(t, r, \Omega) \mid 0 \leq r \leq r_{\max}/2\}.$$

This is a Lorentzian manifold whose boundary ∂X consists of $\partial\mathcal{M}$ as well as the three-dimensional surface

$$Y := \{(t, r_{\max}/2, \Omega)\}.$$

Possibly by decreasing r_{\max} , we can arrange that K is timelike in X , implying that Y is a timelike surface. The inner normal on Y is again denoted by n . We consider the initial value problem

$$(\mathcal{D} - m)\psi = 0 \quad \text{in } X, \quad \psi|_{\mathcal{N}} = \psi_0 \in C^\infty(\mathcal{N} \cap X, S\mathcal{M}), \quad (2.2)$$

with the boundary conditions

$$(\not{n} - i)\psi|_{\partial X} = 0, \quad (2.3)$$

where $\partial X = \partial\mathcal{M} \cup Y$ now has two components. It is again useful to rewrite the Dirac equation in the Hamiltonian form (1.6) with the Hamiltonian (1.7). In order to take into account the boundary conditions, we now choose the domain of definition as the Sobolev space

$$\mathcal{D}(H) = \{\psi \in W^{1,2}(X \cap \mathcal{N}, S\mathcal{M}) \mid (\not{n} - i)\psi|_{\partial X \cap \mathcal{N}} = 0\}. \quad (2.4)$$

The next proposition gives a spectral decomposition of H .

Proposition 2.1. *There is a countable orthonormal basis $(\psi_n)_{n \in \mathbb{N}}$, $\psi_n \in \mathcal{D}(H)$, of eigenfunctions of H .*

Proof. Our method is to apply the abstract spectral theorem given in [1, Theorem 4.1]. The task is to verify the spectral conditions (C0)–(C4), which in our setting are stated as follows:

- (C0) $H : \mathcal{D}(H) \rightarrow L^2(X)$ is linear and bounded in the $W^{1,2}$ -topology on $\mathcal{D}(H)$.
- (C1) The Gårding inequality holds: There exists a constant C such that for all $\psi \in \mathcal{D}(H)$,

$$\|\psi\|_{W^{1,2}(X \cap \mathcal{N})}^2 \leq C \int_{X \cap \mathcal{N}} (\langle H\psi | \not{n} H\psi \rangle_x + \langle \psi | \not{n} \psi \rangle_x) d\mu_{\mathcal{N}}(x). \quad (2.5)$$

- (C2) Weak solutions are strong solutions (“elliptic regularity”): If $\phi \in L^2(X \cap \mathcal{N})$ satisfies

$$\int_{X \cap \mathcal{N}} \langle H\psi | \not{n} \phi \rangle_x d\mu_{\mathcal{N}}(x) = 0 \quad \text{for all } \psi \in \mathcal{D}(H),$$

then $\phi \in \mathcal{D}(H)$.

- (C3) H is symmetric, i.e. for all $\psi, \phi \in \mathcal{D}(H)$,

$$\int_{X \cap \mathcal{N}} \langle \psi | \not{n} H\phi \rangle_x d\mu_{\mathcal{N}}(x) = \int_{X \cap \mathcal{N}} \langle H\psi | \not{n} \phi \rangle_x d\mu_{\mathcal{N}}(x).$$

- (C4) $\mathcal{D}(H)$ is dense in $L^2(X \cap \mathcal{N})$.

The validity of condition (C0) follows immediately from the fact that H is a differential operator of first order. The symmetry property (C3) was verified after (1.10). The denseness property (C4) is obvious. In order to verify the Gårding inequality (C1), we exploit the specific form of the Hamiltonian (1.7). The contribution to $\langle H\psi | \psi H\psi \rangle_x$ involving first derivatives squared is estimated by

$$\langle (\gamma^t)^{-1} \gamma^\alpha \nabla_\alpha \psi | \psi (\gamma^t)^{-1} \gamma^\beta \nabla_\beta \psi \rangle_x \geq c g_{\mathcal{N}}^{\alpha\beta} \|\nabla_\alpha \psi\| \|\nabla_\beta \psi\|$$

(for a suitable constant $c > 0$), where we used that the inner product $\langle \cdot | \psi \rangle$ is positive definite and that the matrices $(\gamma^t)^{-1}$ and γ^α are uniformly bounded on X . Introducing the notation $\|\nabla\psi\|^2 = g_{\mathcal{N}}^{\alpha\beta} \|\nabla_\alpha \psi\| \|\nabla_\beta \psi\|$ and estimating the coefficients of the lower order terms by suitable constants $d_1, d_2 > 0$, we obtain the estimate

$$\begin{aligned} \langle H\psi | \psi H\psi \rangle_x &\geq c \|\nabla\psi\|^2 - d_1 \|\nabla\psi\| \|\psi\| - d_2 \|\psi\|^2 \\ &\geq \frac{c}{2} \|\nabla\psi\|^2 - \left(\frac{d_1^2}{2c} + d_2 \right) \|\psi\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\psi\|_{W^{1,2}(X \cap \mathcal{N})}^2 &= \int_{X \cap \mathcal{N}} \left(\|\nabla\psi\|^2 + \|\psi\|^2 \right) d\mu_{\mathcal{N}}(x) \\ &\leq \int_{X \cap \mathcal{N}} \left(\frac{2}{c} \langle H\psi | \psi H\psi \rangle_x + \frac{2}{c} \left(\frac{d_1^2}{2c} + d_2 \right) \|\psi\|^2 + \|\psi\|^2 \right) d\mu_{\mathcal{N}}(x). \end{aligned}$$

This shows that condition (C1) holds.

It remains to derive the regularity condition (C2). This consists of two parts: the interior regularity and the regularity at the boundary. For the interior regularity, we need to show that the operator H is uniformly elliptic (see [1, Theorem 3.7]). To this end, we make use of the fact that the Killing field K is timelike in X . As a consequence, we can use it to define a norm on the spinors by

$$\|\psi(x)\|_x^2 := \langle \psi | \gamma^t \mathbb{K} \gamma^t \psi \rangle_x.$$

Using this norm, we have

$$\begin{aligned} \|\xi_\alpha ((\gamma^t)^{-1} \gamma^\alpha \psi)(x)\|_x^2 &= \xi_\alpha \xi_\beta \langle (\gamma^t)^{-1} \gamma^\alpha \psi | \gamma^t \mathbb{K} \gamma^t (\gamma^t)^{-1} \gamma^\beta \psi \rangle_x \\ &= \xi_\alpha \xi_\beta \langle \gamma^\alpha \psi | \mathbb{K} \gamma^\beta \psi \rangle_x. \end{aligned}$$

Since $\mathbb{K} = \gamma_t$, this matrix anti-commutes with the matrices γ^β . Therefore,

$$\begin{aligned} \|\xi_\alpha ((\gamma^t)^{-1} \gamma^\alpha \psi)(x)\|_x^2 &= -\xi_\alpha \xi_\beta \langle \gamma^\beta \gamma^\alpha \psi | \mathbb{K} \psi \rangle_x \\ &= -g^{\alpha\beta} \xi_\alpha \xi_\beta \langle \psi | \mathbb{K} \psi \rangle_x = -g^{\alpha\beta} \xi_\alpha \xi_\beta \left\| (\gamma^t)^{-1} \psi \right\|_x^2, \end{aligned}$$

showing explicitly that H is uniformly elliptic. To see that the matrix $(\gamma^t)^{-1}$ is uniformly bounded, we note that, using Cramer's rule,

$$\left((\gamma^t)^{-1} \right)^2 = \frac{\mathbb{1}_{S_x \mathcal{M}}}{g^{tt}} = -\frac{\det g_{\alpha\beta}}{g_{tt}} = -\frac{\det g_{\alpha\beta}}{\langle K, K \rangle},$$

which is indeed bounded because K is timelike in X .

The remaining proof of the boundary regularity is a subtle point, which we now treat in detail. We first note that, by localizing with a test function and using the interior regularity, it suffices to consider weak solutions whose support is in a small neighborhood of $\partial\mathcal{N}$ or $Y \cap \mathcal{N}$. Since both cases can be treated in the same way, we may assume that the solution vanishes identically outside a small neighborhood of $\partial\mathcal{N}$.

Our goal is to apply [1, Theorem 5.11]. Simplifying the statement of this theorem and adapting it to our setting, this theorem gives boundary regularity for boundary value problems of the form

$$Lu = f \quad (2.6)$$

$$Pu|_{\partial\mathcal{N}} = 0. \quad (2.7)$$

Here P is a projection operator on $L^2(\partial\mathcal{N})$. Moreover, L is the differential operator

$$L = \partial_r + A + B,$$

where the operators A and B are of the following form. The operator $A : W^{1,2}(\partial\mathcal{N}) \rightarrow L^2(\partial\mathcal{N})$ is an angular differential operator which is independent of r and satisfies again the above spectral conditions (C0)–(C4). The operator $B : W^{1,2}(X \cap \mathcal{N}) \rightarrow L^2(X \cap \mathcal{N})$, on the other hand, should be such that its first-order terms vanish on $\partial\mathcal{N}$.

The first step is to rewrite the Dirac equation and the boundary conditions in the required form. Suppose that ψ is a weak solution of the inhomogeneous equation $H\psi = f$ satisfying the boundary conditions (2.3). Moreover, assume that ψ and f are supported in a small neighborhood of $\partial\mathcal{N}$. In order to implement the boundary conditions on $\partial\mathcal{N}$, we choose the projection operator P as

$$P = \frac{1}{2}(i\not{t} + 1).$$

Next, using (1.7), one can write the differential equation (2.6) as

$$\left(\partial_r + (\gamma^r)^{-1}(\gamma^\vartheta \partial_\vartheta + \gamma^\varphi \partial_\varphi) + E\right)\psi = i(\gamma^r)^{-1}\gamma^t f,$$

where ϑ and φ are coordinates on $\partial\mathcal{N}$, and E is a zero-order operator. We choose

$$A = (\gamma^r)^{-1}(\gamma^\vartheta \partial_\vartheta + \gamma^\varphi \partial_\varphi)|_{\partial\mathcal{N}} + Z \quad (2.8)$$

$$B = (\gamma^r)^{-1}(\gamma^\vartheta \partial_\vartheta + \gamma^\varphi \partial_\varphi) + E - A, \quad (2.9)$$

where Z is a zero-order operator on $\partial\mathcal{N}$ to be determined below.

The crucial point is to show that by a suitable choice of the scalar product and the zero-order operator Z , we can arrange that the operator A is symmetric. We choose the scalar product as

$$\langle \cdot | \cdot \rangle_{\partial\mathcal{N}} = \int_{\mathcal{N}} \langle \cdot | \not{K} \cdot \rangle_x d\mu_{\partial\mathcal{N}} \quad (2.10)$$

(since K is timelike near $\partial\mathcal{N}$, this inner product is indeed positive definite). Using the form of the metric (2.1) in our Gaussian normal coordinate system, the following anti-commutation relations hold,

$$\{\not{K}, \gamma^r\} = 2g_t^r = 2\delta_t^r = 0$$

$$\{\not{K}, \gamma^\vartheta\} = \{\not{K}, \gamma^\varphi\} = 0$$

$$\{\gamma^r, \gamma^\vartheta\} = \{\gamma^r, \gamma^\varphi\} = 0.$$

As a consequence, the matrices $(\gamma^r)^{-1}\gamma^\vartheta$ and $(\gamma^r)^{-1}\gamma^\varphi$ are anti-symmetric with respect to the scalar product (2.10). Thus, setting

$$Z = -\frac{1}{2}(A_0 - A_0^*) \quad \text{with} \quad A_0 := (\gamma^r)^{-1}(\gamma^\vartheta \partial_\vartheta + \gamma^\varphi \partial_\varphi)|_{\partial\mathcal{N}},$$

where the star denotes the formal adjoint with respect to the scalar product (2.10), the operator Z is indeed a multiplication operator. Moreover, using the above formulas

for Z and A_0 in (2.8), one sees that $A = (A_0 + A_0^*)/2$, which is obviously symmetric. Finally, it is clear by construction that the restriction of B to $\partial\mathcal{N}$ is a multiplication operator.

From this construction, it is obvious that the operator A has the above properties (C0), (C3) and (C4). In order to prove the Gårding inequality (C1) and the elliptic regularity (C2), we make use of the anti-commutation relations

$$\left\{ (\gamma^r)^{-1} \gamma^\alpha, (\gamma^r)^{-1} \gamma^\beta \right\} = -\frac{1}{g^{rr}} \left\{ \gamma^\alpha, \gamma^\beta \right\} = -2 \frac{g^{\alpha\beta}}{g^{rr}} \quad \text{for } \alpha, \beta \in \{\vartheta, \varphi\}.$$

Hence the operator A^2 is of the form

$$A^2 = \frac{1}{g^{rr}} \Delta_{S^2} + (\text{lower order terms}).$$

This is an elliptic operator on a bounded domain. Standard elliptic theory implies (C1) and (C2). \square

The spectral decomposition of Proposition 2.1 implies that the mixed initial/boundary value problem (2.2), (2.3) has a unique weak solution in $W^{1,2}(X \cap \mathcal{N}, S\mathcal{M})$ given by

$$\psi(t, x) = \sum_{n=1}^{\infty} c_n e^{-i\omega_n t} \psi_n(x) \quad \text{with} \quad c_n = \int_{X \cap \mathcal{N}} \langle \psi_n | \psi \psi_0 \rangle_y d\mu_{\mathcal{N}}(y), \quad (2.11)$$

where ω_n is the eigenvalue of ψ_n . In order to apply [2], we want a solution which is smooth for all times. We now state the corresponding necessary and sufficient conditions.

Lemma 2.2. *Suppose that ψ_0 satisfies the conditions*

$$(\not\partial - i)(H^p \psi_0)|_{\partial\mathcal{N}} = 0 \quad \text{for all } p \in \mathbb{N}_0. \quad (2.12)$$

Then the solution ψ of the mixed initial/boundary value problem (2.2), (2.3) is in the class $C_{sc}^\infty(\mathcal{M}, S\mathcal{M})$, where the index “sc” denotes solutions of space-like compact support (i.e. $\text{supp } \psi(t, \cdot)$ is a compact subset of \mathcal{N} for all $t \in \mathbb{R}$). Conversely, if a solution of the mixed initial/boundary value problem is smooth, then ψ_0 satisfies the conditions (2.12).

Proof. Let ψ be the solution of the mixed initial/boundary value problem (2.2), (2.3) for ψ_0 satisfying (2.12). In order to show that ψ is smooth, it clearly suffices that all time derivatives of ψ exist and are smooth in x . To this end, we consider the partial sums of (2.11)

$$\psi^N(t, x) = \sum_{n=1}^N c_n e^{-i\omega_n t} \psi_n(x)$$

for given $N \in \mathbb{N}$. Differentiating p times with respect to t gives

$$(i\partial_t)^p \psi^N(t, x) = \sum_{n=1}^N \omega_n^p c_n e^{-i\omega_n t} \psi_n(x).$$

Furthermore,

$$\omega_n^p c_n = \int_{X \cap \mathcal{N}} \langle H^p \psi_n | \psi \psi_0 \rangle_y d\mu_{\mathcal{N}}(y) = \int_{X \cap \mathcal{N}} \langle \psi_n | \psi (H^p \psi_0) \rangle_y d\mu_{\mathcal{N}}(y),$$

where we iteratively integrated by parts and used the boundary conditions (2.12). Since the function $\tilde{\psi}_0 := H^p \psi_0$ is again in $\mathcal{D}(H)$ given by (2.4), we can take the limit $N \rightarrow \infty$ to conclude that

$$(i\partial_t)^p \psi(t, x) = \sum_{n=1}^{\infty} \tilde{c}_n e^{-i\omega_n t} \psi_n(x) \quad \text{with} \quad \tilde{c}_n = \int_{X \cap \mathcal{N}} \langle \psi_n | \psi \tilde{\psi}_0 \rangle_y d\mu_{\mathcal{N}}(y).$$

This shows that ψ is indeed a smooth solution.

Assume conversely that ψ is a smooth solution to the mixed initial/boundary value problem (2.2), (2.3). Then $(\not{n} - i)\psi(t)|_{\partial\mathcal{N}} = 0$ for all t . Differentiating p times with respect to t gives

$$0 = (i\partial_t)^p \left((\not{n} - i)\psi(t)|_{\partial\mathcal{N}} \right) \Big|_{t=0} = (\not{n} - i)(H^p \psi_0)|_{\partial\mathcal{N}},$$

proving (2.12). \square

3. SOLUTION OF THE CAUCHY PROBLEM

We now return to the Cauchy problem (1.5), (1.8) with boundary conditions (1.9). Thus we seek for solutions of the Dirac equation in the Hamiltonian form

$$i\partial_t \psi = H\psi \quad \text{in } \mathcal{M}, \quad (3.1)$$

with initial and boundary values

$$\psi|_{\mathcal{N}} = \psi_0 \quad \text{and} \quad (\not{n} - i)\psi|_{\partial\mathcal{M}} = 0, \quad (3.2)$$

where the initial data is in $\mathcal{D}(H)$ as given in (1.11), i.e.

$$\psi_0 \in \left\{ \psi \in C_0^\infty(\mathcal{N}, S\mathcal{M}) \quad \text{with} \quad (\not{n} - i)(H^p \psi)|_{\partial\mathcal{N}} = 0 \quad \text{for all } p \in \mathbb{N}_0 \right\}. \quad (3.3)$$

Lemma 3.1. *There is $\varepsilon > 0$ such that the mixed initial/boundary value problem (3.2), (3.3) has a unique solution ψ in the class*

$$\left\{ \psi \in C_0^\infty([0, \varepsilon) \times \mathcal{N}, S\mathcal{M}) \quad \text{with} \quad (\not{n} - i)(H^p \psi)|_{[0, \varepsilon) \times \partial\mathcal{N}} = 0 \quad \text{for all } p \in \mathbb{N}_0 \right\}.$$

Proof. Near $\partial\mathcal{N}$, we again choose the Gaussian normal coordinate system where the metric takes the form (2.1). Moreover, we choose ε so small that the future development J^\vee of initial data sets has the properties

$$J^\vee \left(\{(0, r, \Omega) \mid r < r_{\max}/4\} \right) \cap (\{\varepsilon\} \times \mathcal{N}) \subset \{(0, r, \Omega) \mid r < r_{\max}/2\} \quad (3.4)$$

$$J^\vee \left(\{(0, r, \Omega) \mid r > r_{\max}/8\} \right) \cap (\{\varepsilon\} \times \mathcal{N}) \subset \{(0, r, \Omega) \mid r > 0\}. \quad (3.5)$$

We next decompose the initial data into a contribution ψ_0^{B} near the boundary $\partial\mathcal{N}$ and a contribution ψ_0^{I} supported in the interior of \mathcal{N} ,

$$\psi_0 = \psi_0^{\text{B}} + \psi_0^{\text{I}}.$$

To this end, we let $\eta \in C_0^\infty((-r_{\max}/4, r_{\max}/4))$ be a test function with $\eta|_{[0, r_{\max}/8]} \equiv 1$ and set (see Figure 1)

$$\psi_0^{\text{B}} := \eta(r) \psi_0 \quad \text{and} \quad \psi_0^{\text{I}} := \psi_0 - \psi_0^{\text{B}}.$$

We take ψ_0^{I} as initial value problem for the Dirac equation without boundary conditions,

$$i\partial_t \psi^{\text{I}} = H\psi^{\text{I}} \quad \text{in } X, \quad \psi^{\text{I}}|_{\mathcal{N}} = \psi_0^{\text{I}}.$$

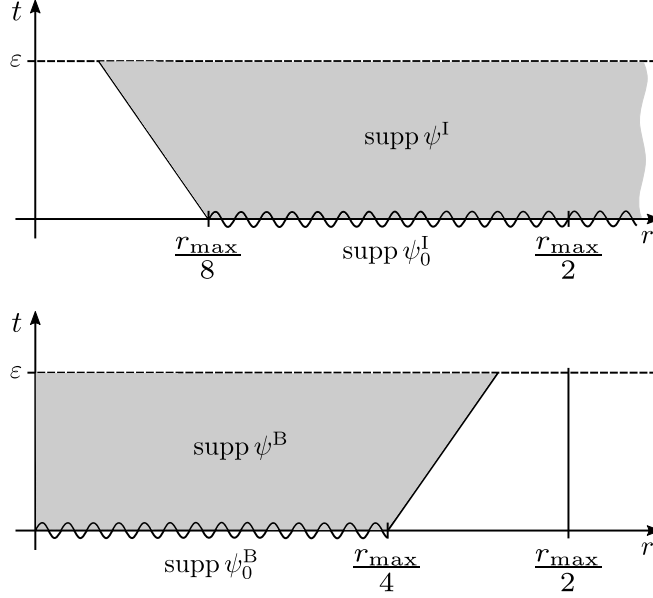


FIGURE 1. Decomposition of the solution of the Cauchy problem.

Using the theory of symmetric hyperbolic equations (see [9, Section 5.3], [14, Section 16], [11, Section 7] or [6, Chapter 5]), this initial value problem has a unique solution ψ^I in the class $C_{\text{sc}}^\infty([0, \varepsilon] \times \mathcal{N})$ (just as explained in (i) on page 5). Note that, due to finite propagation speed and (3.5), the solution vanishes identically near $\partial\mathcal{M}$ (see the top picture in Figure 1).

Next we take ψ_0^B as initial values for the double boundary value problem, i.e.

$$i\partial_t \psi^B = H\psi^B \quad \text{in } X, \quad \psi^B|_{\mathcal{N}} = \psi_0^B, \quad (\not{n} - i)\psi^B|_{\partial\mathcal{M}} = 0. \quad (3.6)$$

According to Lemma 2.2, this mixed initial/boundary value problem has a smooth solution which satisfies the initial and boundary conditions in (3.6) pointwise. Moreover, due to finite propagation speed and (3.4), we know that the solution ψ^B vanishes near the boundary $\{r = r_{\max}/2\}$, i.e.

$$\text{supp } \psi^B(t, \cdot) \subset [0, r_{\max}/2) \times \partial\mathcal{N} \quad \text{for all } t \in [0, \varepsilon]$$

(see the bottom picture in Figure 1). Therefore, extending ψ^B by zero, we obtain a global solution in all \mathcal{M} .

The function $\psi = \psi^B + \psi^I$ is the desired solution of our mixed initial/boundary value problem. Uniqueness follows immediately from standard energy estimates for symmetric hyperbolic systems (see for example [9, Section 5.3]). \square

Corollary 3.2. *The mixed initial/boundary value problem (3.2), (3.3) has a unique global solution ψ in the class of smooth wave functions with spatially compact support satisfying the boundary conditions,*

$$\left\{ \psi \in C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M}) \quad \text{with} \quad (\not{n} - i)(H^p \psi)|_{\partial\mathcal{M}} = 0 \quad \text{for all } p \in \mathbb{N}_0 \right\}.$$

The resulting time evolution operator is unitary with respect to the scalar product

$$(\psi|\phi)_{\mathcal{N}} = \int_{\mathcal{N}} \langle \psi(t, x) | \psi(t, x) \phi(t, x) \rangle_x d\mu_{\mathcal{N}}(x). \quad (3.7)$$

Proof. Since the existence time ε in Lemma 3.1 does not depend on the initial data, we can iterate the procedure to obtain smooth solutions for arbitrarily large times. Moreover, solving backwards in time, one can also obtain smooth solutions for arbitrarily large negative times. We thus obtain global smooth solutions $\psi \in C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M})$. The symmetry of H (as shown after (1.10)) implies that the scalar product (3.7) is preserved under time evolution. Therefore, the time evolution operator is unitary. \square

4. SELF-ADJOINTNESS OF THE DIRAC HAMILTONIAN

We now give the proof of Theorem 1.2. Let H be the Dirac Hamiltonian with domain $\mathcal{D}(H)$ given by (1.11). Corollary 3.2 shows that the time evolution operator for the mixed initial/boundary value problem (3.2), (3.3) defines a one-parameter group acting on $\mathcal{D}(H)$. Moreover, it is obvious that the domain is invariant under the action of H . Therefore, we can apply the result by Chernoff [2, Lemma 2.1] to conclude that H is essentially self-adjoint on $\mathcal{D}(H)$. This completes the proof of Theorem 1.2.

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