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ABSTRACT. The separability of the massive Dirac equation in a rotating Kerr black hole background in advanced Eddington-Finkelstein-type coordinates is shown. To this end, the Kerr spacetime is described in the framework of the Newman-Penrose formalism by a local Carter tetrad, and the Dirac wave functions are given on a spin bundle in a chiral Newman-Penrose dyad representation. Applying mode analysis techniques, the Dirac equation is separated into coupled systems of radial and angular ordinary differential equations. Asymptotic radial solutions at infinity and the event and Cauchy horizons are explicitly derived and, by means of error estimates, the decay properties are analyzed. Solutions of the angular ordinary differential equations matching the Chandrasekhar-Page equation are discussed. These solutions are used in order to study the scattering of Dirac waves by the gravitational field of a Kerr black hole for an observer described by a frame without coordinate singularities at the inner horizon boundaries such that Dirac waves can be seen to actually cross the event horizon.

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I. INTRODUCTION

Over the last five decades, the dynamics of relativistic spin-$\frac{1}{2}$ fermions (Dirac waves) in black hole spacetimes was studied extensively using different approaches. The probably most established approach is based on Chandrasekhar's mode analysis [6–8, 12, 23, 39], where the massive Dirac equation in a Kerr-Newman black hole background geometry is separated by means of time and azimuthal angle modes and rewritten in terms of 1-dimensional radial wave equations and coupled angular ordinary differential equations (ODEs). Within this framework, the asymptotic behavior of solutions of the Dirac equation can be studied, giving further rise to spectral estimates [17]. This opened up the possibility to study many physical processes like the emission and absorption of Dirac waves by black holes or the stability of black hole spacetimes under fermionic field perturbations [3, 22, 28, 30–32, 34–36]. Moreover, the dynamics of Dirac waves was analyzed in the framework of scattering theory [1, 5, 9, 10, 20]. More recently, a functional analytic construction, that is, an integral representation of the Dirac propagator in the Cauchy problem formulation of the Dirac equation [16, 18], was used.

The basis of the mode analysis approach is Chandrasekhar's famous finding that the Dirac equation in a Kerr black hole background given in Boyer-Lindquist coordinates is separable, which was worked out in his original article from 1976 [6] and led to a major breakthrough in the field. At that time, this remarkable result came a bit as a surprise because the Dirac system of coupled first-order partial differential equations (PDEs) was not expected to be separable. Despite the tremendous impact of this finding, the validity of the solutions is naturally restricted to those regions of the Kerr spacetime where the Boyer-Lindquist coordinates are well-defined, and since they have coordinate singularities at the Cauchy horizon and the event horizon, respectively, the description of Dirac waves in the vicinity of these horizons is ill-defined, i.e., Dirac waves cannot be propagated across these inner boundary surfaces. This poses a profound problem in all studies which rely on a proper description of their dynamics near and across these horizons as in the transmission and reflection of incident waves at the gravitational field of the black hole evaluated at the event horizon.

In this article, this problem is resolved by using an analytic extension of the Boyer-Lindquist coordinate frame to an advanced Eddington-Finkelstein-type coordinate system which is well-defined at the Cauchy and event horizons (for another coordinate system with similar characteristics, called Doran coordinates, see [13]). This coordinate system covers the interior and exterior black hole regions and yields a smooth transition of Dirac waves between them. Based on this description of Kerr geometry, a proper mode analysis of Dirac waves is conducted as follows.

Firstly, Kerr geometry is described in the Newman-Penrose formalism by a Carter tetrad in advanced Eddington-Finkelstein-type coordinates. Secondly, the associated spin coefficients are calculated and by their means, one formulates the Dirac equation in the chiral representation in terms of two bi-spinor equations with a Newman-Penrose dyad basis for the spinor space. In this form, the Dirac equation has no irregular singularities on the event and Cauchy horizons. Considering a factorization of the Dirac waves in coordinate time and azimuthal angle modes, separability of the Dirac equation in the advanced Eddington-Finkelstein-type coordinates into radial and polar angular ODEs is shown. Asymptotic radial solutions at infinity and at the event and Cauchy horizons are determined. Moreover, error estimates for these asymptotic solutions are given, and it is proven that the errors have suitable decay properties. The angular ODEs yield the usual Chandrasekhar-Page equation. The corresponding set of eigenfunctions and the discrete, non-degenerate spectrum of eigenvalues are briefly, however appropriately, discussed. Finally, the radial asymptotics at infinity and at the event horizon are applied to the physical problem of scattering of Dirac waves by the gravitational field of a Kerr black hole. More precisely, the net current of incident Dirac waves, which emerge from space-like infinity, expressed in terms of the reflection and transmission coefficients, is considered and evaluated at infinity and at the event horizon. Note that here, in contrast to the usual treatment in Boyer-Lindquist coordinates, where the Dirac waves are observed to approach the event horizon only asymptotically, the Dirac waves are seen to actually cross the event horizon. Furthermore, due to the boundary conditions for the scattering problem, the discontinuity in the radial Dirac current across the event horizon coming from a horizon contribution to the Dirac waves, is removed. The well-known main results of the scattering problem in a Boyer-Lindquist representation are reproduced in this context, namely that the conserved net current stays positive across the event horizon and that superradiance cannot occur.

This work provides the fundamental quantities necessary for the construction of an integral representation of the Dirac propagator in advanced Eddington-Finkelstein-type coordinates acting on Dirac waves with compact support in the black hole interior and exterior regions, which is presented in a separate article [33].

II. PRELIMINARIES

In this section, a general overview of the local Newman-Penrose null tetrad formalism of general relativity and of the local dyad Newman-Penrose formulation for spinors for a description of the general relativistic Dirac equation are given. These formulations are very well suited for the analysis of radiative transport in curved spacetimes, especially Dirac wave propagation in a (vacuum) Kerr black hole background, because they can be chosen to reflect underlying symmetries and can be adapted to certain aspects of the spacetime, which subsequently leads to a reduction in the number of conditional equations and simplified expressions for geometric quantities.

A. General Relativity in the Newman-Penrose Formalism

Let $(\mathcal{M}, \boldsymbol{g})$ be a Lorentzian 4-manifold endowed with an affine connection $\boldsymbol{\omega}$, the unique, torsion-free Levi-Civita connection, and dual basis (e_μ) and (e^μ), $\mu \in \{1,2,3,4\}$, on sections of the tangent and cotangent bundles $T\mathcal{M}$ and $T\mathcal{M}^\star$, respectively. In addition, one sets up a flat (orthonormal or null) frame bundle $F\mathcal{M}$ and a spin bundle $S\mathcal{M}$ on $\mathcal{M}$. The basis of the local frames of the fibers in $F\mathcal{M}$ at each point of spacetime consists of four vectors $\left(e_{(a)}\right)$, $a \in \{1,2,3,4\}$, and the basis of the corresponding dual frames are denoted by $\left(e^{(a)}\right)$. The basis vectors of these internal frame bundles are related to the basis vectors of the tangent and cotangent bundles via $e_{(a)} = e^\mu{}_{(a)} e_\mu$ and $e^{(a)} = e_\mu{}^{(a)} e^\mu$, where $e^\mu{}_{(a)}$ is an invertible linear map, namely a (4×4)-matrix, from $T\mathcal{M}$ to $F\mathcal{M}$. Geometrical structures in the framework of general relativity can be described in terms of these local tetrad frames since they encode the same information as the metric tensor on the tangent bundle [37].

In the Newman-Penrose formalism [26], the tetrad basis consists of two real null vectors, $\mathbf{l} = e_{(0)} = e^{(1)}$ and $\mathbf{n} = e_{(1)} = e^{(0)}$, and a complex conjugate pair, $\mathbf{m} = e_{(2)} = -e^{(3)}$ and $\overline{\mathbf{m}} = e_{(3)} = -e^{(2)}$. A Newman-Penrose frame has to fulfill the null conditions

$$\mathbf{l}\cdot\mathbf{l} = \mathbf{n}\cdot\mathbf{n} = \mathbf{m}\cdot\mathbf{m} = \overline{\mathbf{m}}\cdot\overline{\mathbf{m}} = 0, \tag{1}$$

the orthogonality conditions

$$\mathbf{l}\cdot\mathbf{m} = \mathbf{l}\cdot\overline{\mathbf{m}} = \mathbf{n}\cdot\mathbf{m} = \mathbf{n}\cdot\overline{\mathbf{m}} = 0, \tag{2}$$

Spin Coefficients		
$\kappa = \gamma_{(3)(1)(1)}$	$\varrho = \gamma_{(3)(1)(4)}$	$\epsilon = \frac{1}{2}\big(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}\big)$
$\sigma = \gamma_{(3)(1)(3)}$	$\mu = \gamma_{(2)(4)(3)}$	$\gamma = \frac{1}{2}\big(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\big)$
$\lambda = \gamma_{(2)(4)(4)}$	$\tau = \gamma_{(3)(1)(2)}$	$\alpha = \frac{1}{2}\big(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\big)$
$\nu = \gamma_{(2)(4)(2)}$	$\pi = \gamma_{(2)(4)(1)}$	$\beta = \frac{1}{2}\big(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\big)$

TABLE I: Various spin coefficients of the Newman-Penrose formalism expressed in terms of the Ricci rotation coefficients.

and the cross-normalization conditions (depending on the signature convention)

$$\mathbf{l} \cdot \mathbf{n} = -\mathbf{m} \cdot \overline{\mathbf{m}} = 1. \tag{3}$$

The metric $\boldsymbol{g}$ in terms of the Newman-Penrose null basis vectors becomes

$$\boldsymbol{g} = \mathbf{l} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{l} - \mathbf{m} \otimes \overline{\mathbf{m}} - \overline{\mathbf{m}} \otimes \mathbf{m}.$$

Acting with this metric on the dual Newman-Penrose basis vectors, one obtains the local, non-degenerate, constant metric $\boldsymbol{\eta}$ on the null frame bundle

$$\boldsymbol{\eta} = \boldsymbol{g}\left(e^{(a)}, e^{(b)}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since the connection $\boldsymbol{\omega}$ is torsion-free, the first Maurer-Cartan equation of structure reduces to

$$\mathrm{d}e^{\mu} + {\omega^{\mu}}_{\nu} \wedge e^{\nu} = 0.$$

In the tetrad formulation this equation becomes

$$\mathrm{d}e^{(a)} = {\gamma^{(a)}}_{(b)(c)} e^{(b)} \wedge e^{(c)},$$

where the symbols ${\gamma^{(a)}}_{(b)(c)}$ denote the Ricci rotation coefficients which are related to the Levi-Civita connection by means of the formula

$${\gamma^{(a)}}_{(b)(c)} e^{(c)} = {e_{\mu}}^{(a)} \mathrm{d}{e^{\mu}}_{(b)} + {e_{\mu}}^{(a)} {e^{\nu}}_{(b)} {\omega^{\mu}}_{\nu}.$$

The Ricci rotation coefficients in the framework of the Newman-Penrose formalism are called spin coefficients and designated by the twelve symbols given in TABLE I. They represent the Levi-Civita connection on the internal frame bundle described in terms of a Newman-Penrose null basis. The first Maurer-Cartan equation of structure in this formalism reads

$$\mathrm{d}\mathbf{l} = 2\Re(\epsilon)\mathbf{n} \wedge \mathbf{l} - 2\mathbf{n} \wedge \Re(\kappa\overline{\mathbf{m}}) - 2\mathbf{l} \wedge \Re\big([\tau - \overline{\alpha} - \beta]\overline{\mathbf{m}}\big) + 2\mathrm{i}\Im(\varrho)\mathbf{m} \wedge \overline{\mathbf{m}}$$

$$\mathrm{d}\mathbf{n} = 2\Re(\gamma)\mathbf{n} \wedge \mathbf{l} - 2\mathbf{n} \wedge \Re\big([\overline{\alpha} + \beta - \overline{\pi}]\overline{\mathbf{m}}\big) + 2\mathbf{l} \wedge \Re(\overline{\nu}\overline{\mathbf{m}}) + 2\mathrm{i}\Im(\mu)\mathbf{m} \wedge \overline{\mathbf{m}} \tag{4}$$

$$\mathrm{d}\mathbf{m} = \overline{(\mathrm{d}\overline{\mathbf{m}})} = (\overline{\pi} + \tau)\mathbf{n} \wedge \mathbf{l} + \big(2\mathrm{i}\Im(\epsilon) - \varrho\big)\mathbf{n} \wedge \mathbf{m} - \sigma\mathbf{n} \wedge \overline{\mathbf{m}} + \big(\overline{\mu} + 2\mathrm{i}\Im(\gamma)\big)\mathbf{l} \wedge \mathbf{m} + \overline{\lambda}\mathbf{l} \wedge \overline{\mathbf{m}} - (\overline{\alpha} - \beta)\mathbf{m} \wedge \overline{\mathbf{m}}.$$

Tetrad and Spin Coefficient Transformations
$\mathbf{l} \mapsto \mathbf{l}' = \xi\,\mathbf{l} \qquad \mathbf{n} \mapsto \mathbf{n}' = \xi^{-1}\mathbf{n} \qquad \mathbf{m} \mapsto \mathbf{m}' = \exp{(\mathrm{i}\psi)}\mathbf{m} \qquad \overline{\mathbf{m}} \mapsto \overline{\mathbf{m}}' = \exp{(-\mathrm{i}\psi)}\overline{\mathbf{m}}$
$\kappa \mapsto \kappa' = \xi^2 \exp{(\mathrm{i}\psi)}\kappa \qquad \sigma \mapsto \sigma' = \xi \exp{(2\mathrm{i}\psi)}\sigma \qquad \nu \mapsto \nu' = \xi^{-2}\exp{(-\mathrm{i}\psi)}\nu \qquad \varrho \mapsto \varrho' = \xi\varrho$
$\lambda \mapsto \lambda' = \xi^{-1}\exp{(-2\mathrm{i}\psi)}\lambda \qquad \tau \mapsto \tau' = \exp{(\mathrm{i}\psi)}\tau \qquad \pi \mapsto \pi' = \exp{(-\mathrm{i}\psi)}\pi \qquad \mu \mapsto \mu' = \xi^{-1}\mu$
$\gamma \mapsto \gamma' = \xi^{-1}\gamma + \frac{1}{2}\xi^{-2}n^\mu\partial_\mu\big(\xi\big) + \frac{\mathrm{i}}{2}\xi^{-1}n^\mu\partial_\mu\big(\psi\big) \qquad \epsilon \mapsto \epsilon' = \xi\epsilon + \frac{1}{2}l^\mu\partial_\mu\big(\xi\big) + \frac{\mathrm{i}}{2}\xi l^\mu\partial_\mu\big(\psi\big)$
$\alpha \mapsto \alpha' = \exp{(-\mathrm{i}\psi)}\alpha + \frac{\mathrm{i}}{2}\exp{(-\mathrm{i}\psi)}\overline{m}^\mu\partial_\mu\big(\psi\big) + \frac{1}{2}\xi^{-1}\exp{(-\mathrm{i}\psi)}\overline{m}^\mu\partial_\mu\big(\xi\big)$
$\beta \mapsto \beta' = \exp{(\mathrm{i}\psi)}\beta + \frac{\mathrm{i}}{2}\exp{(\mathrm{i}\psi)}m^\mu\partial_\mu\big(\psi\big) + \frac{1}{2}\xi^{-1}\exp{(\mathrm{i}\psi)}m^\mu\partial_\mu\big(\xi\big)$

TABLE II: Effect of a type III local Lorentz transformation on the Newman-Penrose basis vectors and the spin coefficients. The quantities $\xi \in \mathbb{R}\backslash\{0\}$ and $\psi \in \mathbb{R}$ are functions depending on the spacetime coordinates x^μ.

The relevant transformations that are applied on the Newman-Penrose tetrad frame in this study are elements of the 2-parameter subgroup of the 6-parameter group of local Lorentz transformations known as type III or spin-boost Lorentz transformations. These renormalize the real Newman-Penrose vectors $\mathbf{l}$ and $\mathbf{n}$, but leave their directions unchanged, and rotate the complex conjugate pair $\mathbf{m}$ and $\overline{\mathbf{m}}$ by an angle ψ in the $(\mathbf{m}, \overline{\mathbf{m}})$-plane. The effect of these transformations on the Newman-Penrose basis vectors and the spin coefficients is shown in TABLE II. There are various aspects of the Newman-Penrose formalism that are not explicitly discussed in this subsection such as the different types of local Lorentz transformations or the Weyl scalars with their algebraic classification because here, they are of no relevance. They can be found elsewhere in the literature. The interested reader may be referred to [27, 29].

B. The General Relativistic Dirac Equation in the Newman-Penrose Formalism

The general relativistic, massive Dirac equation without an external potential [19, 38] is given by the homogeneous linear first-order PDE system

$$\big(\gamma^\mu\nabla_\mu + \mathrm{i}m\big)\Psi(x^\mu) = 0,$$

where the γ^μ denote the general relativistic Dirac matrices which satisfy the anticommutator relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{id}_{\mathbb{C}^4}$, $\Psi(x^\mu)$ the Dirac 4-spinor on sections of the spin bundle $S\mathcal{M}$, ∇_μ the covariant derivative on sections of the tangent bundle $T\mathcal{M}$, and m the fermion rest mass. Using the chiral bi-spinor representation of the Dirac 4-spinors and matrices in terms of the 2-component spinors P^A and $\overline{Q}_{\dot{B}}$ and the Hermitian (2×2)-Infeld-van der Waerden symbols ${\sigma^\mu}_{A\dot{B}}$ [20]

$$\Psi = \begin{pmatrix} P^A \\ \overline{Q}_{\dot{B}} \end{pmatrix} \quad \text{and} \quad \gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \sigma^{\mu A\dot{B}} \\ \big({\sigma^\mu}_{A\dot{B}}\big)^T & 0 \end{pmatrix} \quad \text{with} \quad A \in \{1, 2\} \quad \text{and} \quad \dot{B} \in \{\dot{1}, \dot{2}\},$$

one obtains the following 2-spinor form of the Dirac equation

$$\begin{aligned} \nabla_{\dot{B}A} P^A + \mathrm{i}\mu_\star \overline{Q}_{\dot{B}} &= 0 \\ \nabla_{\dot{B}A} Q^A + \mathrm{i}\mu_\star \overline{P}_{\dot{B}} &= 0, \end{aligned} \tag{5}$$

where $\mu_\star := m/\sqrt{2}$ and $\nabla_{A\dot{B}} = \sigma^\mu{}_{A\dot{B}} \nabla_\mu$. Note that a dot over an index indicates that the index transforms via the complex conjugated transformation. Introducing a local dyad basis $\zeta^{(k)}$, $k \in \{1, 2\}$, on the spin bundle analogous to the tetrad representation of the tangent bundle, the local Dirac 2-spinors $\mathscr{O}^{(m)}$ in terms of the 2-spinors $\mathscr{O}^A$ read $\mathscr{O}^{(m)} = \zeta^{(m)}{}_A \mathscr{O}^A$, where $\zeta^{(m)}{}_A$ is an invertible (2×2)-matrix. In this dyad formalism, the spinor covariant derivative acting on the local 2-spinor $\mathscr{O}^{(m)}$ becomes

$$\nabla_{(k)(\dot{l})} \mathscr{O}^{(m)} = \zeta^A{}_{(k)} \overline{\zeta}^{\dot{B}}{}_{(\dot{l})} \zeta^{(m)}{}_C \nabla_{A\dot{B}} \mathscr{O}^C = \partial_{(k)(\dot{l})} \mathscr{O}^{(m)} + \Gamma^{(m)}{}_{(n)(k)(\dot{l})} \mathscr{O}^{(n)}$$

with the spinor partial derivative

$$\partial_{(k)(\dot{l})} = \sigma^\mu{}_{(k)(\dot{l})} \partial_\mu$$

and the spin coefficients

$$\Gamma^{(m)}{}_{(n)(k)(\dot{l})} = \Gamma^{(m)(\dot{o})}{}_{(n)(\dot{o})(k)(\dot{l})} = \sqrt{2}\, \epsilon^{(m)(q)} \epsilon^{(\dot{o})(\dot{p})} \sigma^\mu{}_{(q)(\dot{p})} \sigma^\nu{}_{(n)(\dot{o})} \sigma^\lambda{}_{(k)(\dot{l})} e_\mu{}^{(a)} e_\nu{}^{(b)} e_\lambda{}^{(c)} \gamma_{(a)(b)(c)}. \tag{6}$$

The 2-dimensional Levi-Civita symbol $\boldsymbol{\epsilon}$ acts as skew metric on the fibers of the local spin bundle and the Infeld-van der Waerden symbols in the Newman-Penrose spinor formalism yield

$$\sigma^\mu{}_{(k)(\dot{l})} = \begin{pmatrix} l^\mu & m^\mu \\ \overline{m}^\mu & n^\mu \end{pmatrix}. \tag{7}$$

With the spin coefficients of the dyad formalism (6) expressed in terms of the spin coefficients of the tetrad formalism (TABLE I), the Infeld-van der Waerden symbols (7), and the functions $\mathscr{F}_1 := P^{(0)}, \mathscr{F}_2 := P^{(1)}, \mathscr{G}_1 := \overline{Q}^{(\dot{1})}$, and $\mathscr{G}_2 := -\overline{Q}^{(\dot{0})}$, the general relativistic Dirac equation (5) in the Newman-Penrose formalism is given in the following form

$$\begin{aligned} (l^\mu \partial_\mu + \varepsilon - \varrho)\mathscr{F}_1 + (\overline{m}^\mu \partial_\mu + \pi - \alpha)\mathscr{F}_2 &= \mathrm{i}\mu_\star \mathscr{G}_1 \\ (n^\mu \partial_\mu + \mu - \gamma)\mathscr{F}_2 + (m^\mu \partial_\mu + \beta - \tau)\mathscr{F}_1 &= \mathrm{i}\mu_\star \mathscr{G}_2 \\ (l^\mu \partial_\mu + \overline{\varepsilon} - \overline{\varrho})\mathscr{G}_2 - (m^\mu \partial_\mu + \overline{\pi} - \overline{\alpha})\mathscr{G}_1 &= \mathrm{i}\mu_\star \mathscr{F}_2 \\ (n^\mu \partial_\mu + \overline{\mu} - \overline{\gamma})\mathscr{G}_1 - (\overline{m}^\mu \partial_\mu + \overline{\beta} - \overline{\tau})\mathscr{G}_2 &= \mathrm{i}\mu_\star \mathscr{F}_1. \end{aligned} \tag{8}$$

III. NEWMAN-PENROSE REPRESENTATION OF KERR GEOMETRY IN ADVANCED EDDINGTON-FINKELSTEIN-TYPE COORDINATES

Kerr geometry [24] is an asymptotically flat Lorentzian 4-manifold $(\mathcal{M}, \boldsymbol{g})$ with inner event and Cauchy horizon inner boundaries and with topology $S^2 \times \mathbb{R}^2$. It consists of a differentiable manifold $\mathcal{M}$ and a stationary, axisymmetric Lorentzian metric $\boldsymbol{g}$ with signature $(1, 3)$, which is given in Boyer-Lindquist coordinates (t, r, θ, φ) [2], where $t \in \mathbb{R}, r \in \mathbb{R}_{>0}, \theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$, by

$$\begin{aligned} \boldsymbol{g} = {} & \frac{\Delta}{\Sigma}\big(\mathrm{d}t - a\sin^2(\theta)\mathrm{d}\varphi\big) \otimes \big(\mathrm{d}t - a\sin^2(\theta)\mathrm{d}\varphi\big) - \frac{\sin^2(\theta)}{\Sigma}\big([r^2 + a^2]\mathrm{d}\varphi - a\mathrm{d}t\big) \otimes \big([r^2 + a^2]\mathrm{d}\varphi - a\mathrm{d}t\big) \\ & - \frac{\Sigma}{\Delta}\mathrm{d}r \otimes \mathrm{d}r - \Sigma\mathrm{d}\theta \otimes \mathrm{d}\theta. \end{aligned} \tag{9}$$

The horizon function is $\Delta(r) := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2$, $r_\pm := M \pm \sqrt{M^2 - a^2}$ denote the event and Cauchy horizons, respectively, M is the mass of the black hole, a the angular momentum per unit mass of the black hole with $0 \leq a \leq M$, and $\Sigma(r, \theta) := r^2 + a^2 \cos^2(\theta)$. In order to evaluate the Dirac equation in the dyadic Newman-Penrose spinor representation Eq.(8) in a Kerr black hole background, one first reformulates Kerr geometry in terms of a local Newman-Penrose null tetrad frame which is chosen to be adapted to the Kerr principal null geodesics, i.e., the tetrad coincides with the two principal null directions of the Weyl tensor. In this so-called Kinnersley frame [25], since Kerr geometry is algebraically special and of Petrov type D, one is presented with the computational advantage that the four spin coefficients κ, σ, λ, and ν vanish and only one non-zero Weyl scalar, Ψ_2, exists. In other words, the congruences formed by these two principal null directions must both be geodesic and shear-free [27]. The Kinnersley tetrad in Boyer-Lindquist coordinates can be constructed directly from the class of principal null geodesics of Kerr geometry given by their tangent vectors

$$\frac{\mathrm{d}t}{\mathrm{d}\chi} = \frac{r^2 + a^2}{\Delta} E, \quad \frac{\mathrm{d}r}{\mathrm{d}\chi} = \pm E, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\chi} = 0, \quad \text{and} \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\chi} = \frac{a}{\Delta} E, \tag{10}$$

where χ is the parametrization and E denotes a constant. The real Newman-Penrose vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ are aligned with the principal null directions and the complex conjugate pair $(\boldsymbol{m}, \overline{\boldsymbol{m}})$ is appropriately adapted such that it satisfies the Newman-Penrose conditions (1)-(3), yielding

$$\begin{aligned}
\boldsymbol{l} &= \frac{1}{\Delta}\Big(\big[r^2 + a^2\big]\partial_t + \Delta\partial_r + a\partial_\varphi\Big) \\
\boldsymbol{n} &= \frac{1}{2\Sigma}\Big(\big[r^2 + a^2\big]\partial_t - \Delta\partial_r + a\partial_\varphi\Big) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2}\big(r + \mathrm{i}a\cos(\theta)\big)}\big(\mathrm{i}a\sin(\theta)\partial_t + \partial_\theta + \mathrm{i}\csc(\theta)\partial_\varphi\big) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2}\big(r - \mathrm{i}a\cos(\theta)\big)}\big(\mathrm{i}a\sin(\theta)\partial_t - \partial_\theta + \mathrm{i}\csc(\theta)\partial_\varphi\big).
\end{aligned} \tag{11}$$

For the calculation of the corresponding spin coefficients (TABLE I), i.e., for solving the first Maurer-Cartan equation in the Newman-Penrose formalism (4), one requires the dual representation of this Newman-Penrose tetrad which is given by

$$\begin{aligned}
\boldsymbol{l} &= \mathrm{d}t - \frac{\Sigma}{\Delta}\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\varphi \\
\boldsymbol{n} &= \frac{\Delta}{2\Sigma}\left(\mathrm{d}t + \frac{\Sigma}{\Delta}\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\varphi\right) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2}\big(r + \mathrm{i}a\cos(\theta)\big)}\Big(\mathrm{i}a\sin(\theta)\mathrm{d}t - \Sigma\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\mathrm{d}\varphi\Big) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2}\big(r - \mathrm{i}a\cos(\theta)\big)}\Big(\mathrm{i}a\sin(\theta)\mathrm{d}t + \Sigma\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\mathrm{d}\varphi\Big).
\end{aligned}$$

It is possible to introduce further computational advantages by making use of the discrete time and angle reversal isometries of Kerr geometry leading to a transformation into a tetrad frame with only six independent spin coefficients, whereas in the case of a Kerr spacetime two of them are zero. From the metric (9), it can be directly seen that Kerr geometry is isometrically isomorphic under the composition of the discrete time and the discrete azimuthal angle transformations

$$t \mapsto -t \quad \text{and} \quad \varphi \mapsto -\varphi.$$

Applying the composite transformation on the tangent bundle and, in addition, a type III local Lorentz transformation (TABLE II) with parameters of the form

$$\xi = \sqrt{\frac{|\Delta|}{2\Sigma}} \quad \text{and} \quad \exp\left(\mathrm{i}\psi\right) = \frac{\sqrt{\Sigma}}{r - \mathrm{i}a\cos\left(\theta\right)} \tag{12}$$

to the Kinnersley tetrad (11), one induces the local isomorphism

$$\boldsymbol{l} \mapsto -\mathrm{sign}(\Delta)\boldsymbol{n}, \quad \boldsymbol{n} \mapsto -\mathrm{sign}(\Delta)\boldsymbol{l}, \quad \boldsymbol{m} \mapsto \overline{\boldsymbol{m}}, \quad \text{and} \quad \overline{\boldsymbol{m}} \mapsto \boldsymbol{m},$$

where

$$\mathrm{sign}(\Delta) := \begin{cases} +1 & , \Delta \geq 0 \\ -1 & , \Delta < 0 \end{cases}$$

is the signum function. This leads to the so-called Carter (symmetric) frame [4] which has spin coefficients with structure

$$\kappa = -\nu, \quad \pi = -\tau, \quad \alpha = -\beta, \quad \sigma = \mathrm{sign}(\Delta)\lambda, \quad \mu = \mathrm{sign}(\Delta)\varrho, \quad \text{and} \quad \epsilon = \mathrm{sign}(\Delta)\gamma.$$

This can be proven easily using the relation $\gamma_{(a)(b)(c)} = e_{(a)}{}^{\mu} e_{(c)}{}^{\nu} \nabla_{\nu} e_{(b)\mu}$ between the Ricci rotation coefficients and the tetrad. Thus, applying the type III local Lorentz transformation with parameters (12) to the Kinnersley tetrad (11), one obtains the Carter tetrad

$$\begin{aligned}
\boldsymbol{l} &= \frac{\mathrm{sign}(\Delta)}{\sqrt{2\Sigma|\Delta|}} \left(\left[r^2 + a^2\right]\partial_t + \Delta\partial_r + a\partial_\varphi \right) \\
\boldsymbol{n} &= \frac{1}{\sqrt{2\Sigma|\Delta|}} \left(\left[r^2 + a^2\right]\partial_t - \Delta\partial_r + a\partial_\varphi \right) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2\Sigma}} \left(\mathrm{i}a\sin\left(\theta\right)\partial_t + \partial_\theta + \mathrm{i}\csc\left(\theta\right)\partial_\varphi \right) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2\Sigma}} \left(\mathrm{i}a\sin\left(\theta\right)\partial_t - \partial_\theta + \mathrm{i}\csc\left(\theta\right)\partial_\varphi \right).
\end{aligned} \tag{13}$$

The dual of this Carter tetrad reads

$$\begin{aligned}
\boldsymbol{l} &= \sqrt{\frac{|\Delta|}{2\Sigma}} \left(\mathrm{d}t - \frac{\Sigma}{\Delta}\mathrm{d}r - a\sin^2\left(\theta\right)\mathrm{d}\varphi \right) \\
\boldsymbol{n} &= \sqrt{\frac{|\Delta|}{2\Sigma}}\, \mathrm{sign}(\Delta) \left(\mathrm{d}t + \frac{\Sigma}{\Delta}\mathrm{d}r - a\sin^2\left(\theta\right)\mathrm{d}\varphi \right) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2\Sigma}} \left(\mathrm{i}a\sin\left(\theta\right)\mathrm{d}t - \Sigma\mathrm{d}\theta - \mathrm{i}\left[r^2 + a^2\right]\sin\left(\theta\right)\mathrm{d}\varphi \right) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2\Sigma}} \left(\mathrm{i}a\sin\left(\theta\right)\mathrm{d}t + \Sigma\mathrm{d}\theta - \mathrm{i}\left[r^2 + a^2\right]\sin\left(\theta\right)\mathrm{d}\varphi \right).
\end{aligned} \tag{14}$$

Boyer-Lindquist coordinates are not well-defined at the event horizon at $r = r_+$ (as well as at the Cauchy horizon at $r = r_-$). Light cones of observers approaching the event horizon from outside the black hole close up and become degenerate (see FIG. 1). Moreover, space and time reverse their roles inside the black hole. This makes a direct study of the propagation of Dirac waves across the event horizon impossible in these coordinates. In order to have a consistent description of Dirac waves in the black hole exterior and interior that resolves this problem, advanced Eddington-Finkelstein-type coordinates (AEFTC) are used (see [14, 15] for the original Eddington-Finkelstein null coordinates). This analytically extended coordinate system covers the complete black hole region of the Kruskal manifold and allows for a smooth, well-defined transition across the event horizon. It possesses a proper coordinate time unlike in the case of the original advanced Eddington-Finkelstein null coordinates, which is relevant for a Hamiltonian formulation of the Dirac equation. The AEFTC are constructed as follows. By means of the tangent vectors (10), one can derive two relations between the time and radial coordinates along the principal null geodesics of Kerr geometry given by

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{r^2+a^2}{\Delta} \quad \Rightarrow \quad t = \pm\int \frac{r^2+a^2}{\Delta}\,\mathrm{d}r + c_\pm = \pm r_\star + c_\pm ,$$

where

$$r_\star = r + \frac{r_+^2+a^2}{r_+-r_-}\,\ln|r-r_+| - \frac{r_-^2+a^2}{r_+-r_-}\,\ln|r-r_-|$$

is the so-called Regge-Wheeler coordinate and $c_\pm$ are constants of integration. These relations motivate the transformation

$$\mathbb{R}\times\mathbb{R}_{>0}\times[0,\pi]\times[0,2\pi) \to \mathbb{R}\times\mathbb{R}_{>0}\times[0,\pi]\times[0,2\pi), \qquad (t,r,\theta,\varphi)\mapsto(\tau,r,\theta,\phi)$$

with orthonormal coordinates adapted to ingoing null geodesics

$$\begin{aligned}\tau &= t + r_\star - r = t + \frac{r_+^2+a^2}{r_+-r_-}\,\ln|r-r_+| - \frac{r_-^2+a^2}{r_+-r_-}\,\ln|r-r_-|\\ \phi &= \varphi + \int\frac{a}{\Delta}\,\mathrm{d}r = \varphi + \frac{a}{r_+-r_-}\,\ln\left|\frac{r-r_+}{r-r_-}\right|.\end{aligned}$$

This yields the two relations between the new time and radial coordinates

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -1 \quad \text{and} \quad \frac{\mathrm{d}\tau}{\mathrm{d}r} = 1 + \frac{4Mr}{\Delta}.$$

In the AEFTC system, the event horizon is located at a finite value of the radial coordinate, ingoing light rays are represented by straight lines, and the causal structure is such that the light cones are not degenerate at the event horizon (see FIG. 2). Instead, approaching the event horizon, the light cones tip over until their future light cones are aligned with the horizon, indicating the trapping property of event horizons. The metric (9) represented in these coordinates becomes

$$\begin{aligned}\boldsymbol{g} = {}&\left(1-\frac{2Mr}{\Sigma}\right)\mathrm{d}\tau\otimes\mathrm{d}\tau - \frac{2Mr}{\Sigma}\Big(\big[\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\big]\otimes\mathrm{d}\tau + \mathrm{d}\tau\otimes\big[\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\big]\Big)\\ &-\left(1+\frac{2Mr}{\Sigma}\right)\big(\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\big)\otimes\big(\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\big) - \Sigma\,\mathrm{d}\theta\otimes\mathrm{d}\theta - \Sigma\sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi.\end{aligned} \tag{15}$$

The associated dual metric tensor reads

$$\begin{aligned}\boldsymbol{g} = \frac{1}{\Sigma}\Big(&\big[\Sigma+2Mr\big]\partial_\tau\otimes\partial_\tau - 2Mr\big(\partial_\tau\otimes\partial_r + \partial_r\otimes\partial_\tau\big) - \Delta\partial_r\otimes\partial_r - a\big(\partial_r\otimes\partial_\phi + \partial_\phi\otimes\partial_r\big)\\ &-\partial_\theta\otimes\partial_\theta - \csc^2(\theta)\partial_\phi\otimes\partial_\phi\Big).\end{aligned}$$

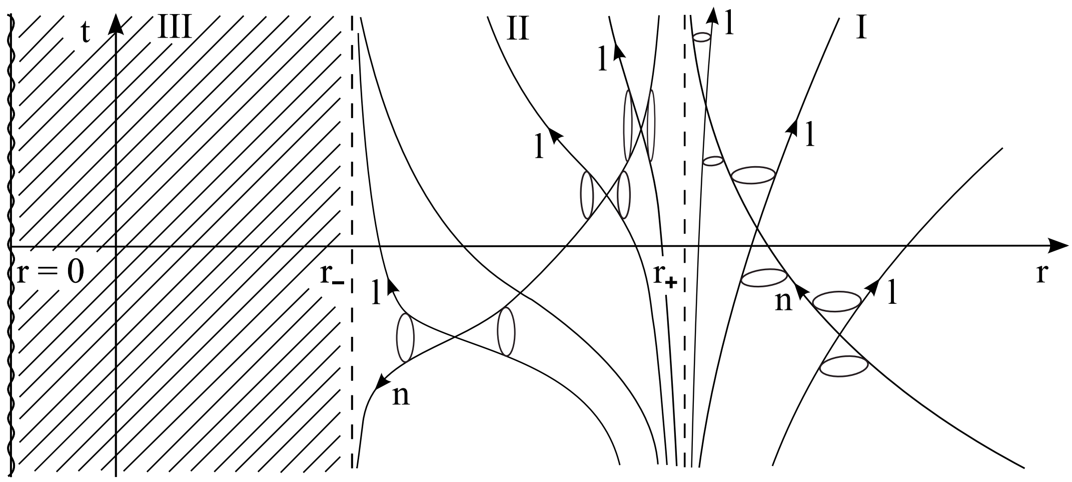

FIG. 1: Causal structure of Kerr geometry in Boyer-Lindquist coordinates. A projection onto the (t, r)-plane is presented, where every point is a 2-sphere. The real Newman-Penrose null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, pointing along the principal null directions of Kerr geometry, form the light cones. The light cones of an observer approaching the event horizon from outside the black hole ($r \searrow r_+$) close up and become degenerate at the event horizon at $r = r_+$. In contrast, they open up when the observer approaches the event horizon from inside the black hole ($r \nearrow r_+$). This stems from the fact that the roles of space and time are reversed in the black hole interior. When $r \to \infty$, the light cones become 45°-Minkowski light cones because the spacetime is asymptotically flat. In order to avoid the ring singularity, the focus is on regions I and II.

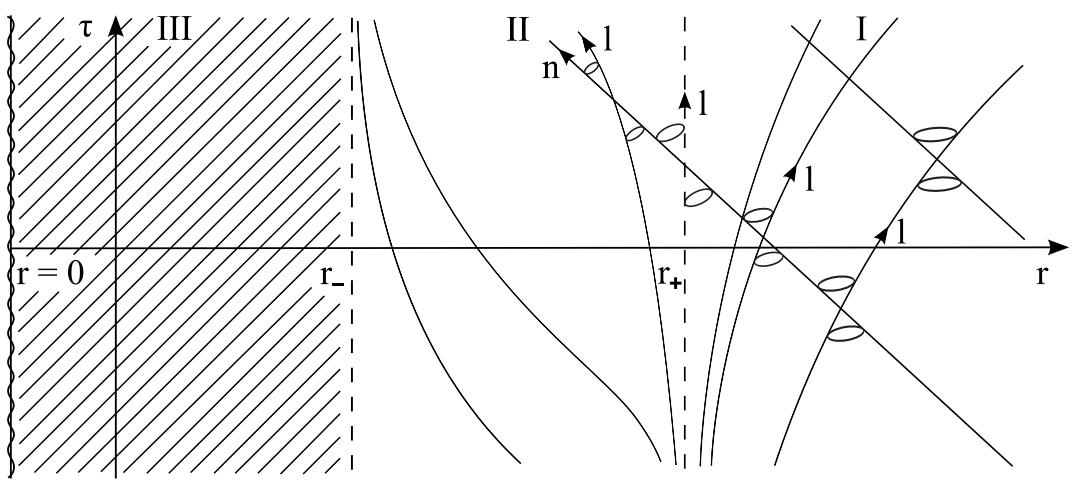

FIG. 2: Causal structure of Kerr geometry in advanced Eddington-Finkelstein-type coordinates. A projection onto the (τ, r)-plane is presented, where every point is a 2-sphere. The real Newman-Penrose null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, pointing along the principal null directions of Kerr geometry, form the light cones. Ingoing light rays are straight lines pointing in the $\boldsymbol{n}$-direction. The light cones of an observer approaching the event horizon at $r = r_+$ from outside the black hole ($r \searrow r_+$) tip over until at the event horizon the future light cone is, except from the part that overlaps with the horizon, completely in the black hole interior. This shows the trapping characteristic of the event horizon. When $r \to \infty$, the light cones become 45°-Minkowski light cones because the spacetime is asymptotically flat. In order to avoid the ring singularity, the focus is again on regions I and II.

Considering the induced metric on constant-τ hypersurfaces by restricting the metric (15) directly reveals that the constant-τ hypersurfaces are space-like and that τ is a proper coordinate time. In the conformal diagrams shown in FIG. 3 and FIG. 4, the behaviors of the constant-t and constant-r hypersurfaces in Boyer-Lindquist coordinates and of the constant-τ and constant-r hypersurfaces in AEFTC for Schwarzschild and Kerr geometries are schematically depicted. While the Boyer-Lindquist constant-t hypersurfaces become time-like inside the black hole in region II, the AEFTC constant-τ hypersurfaces are always space-like and are smoothly continued through the event horizon at $r = r_+$. The Carter tetrad (13) and its dual (14) in AEFTC read

$$\boldsymbol{l} = \frac{1}{\sqrt{2\Sigma|\Delta|}}\left([\Delta + 4Mr]\partial_\tau + \Delta\partial_r + 2a\partial_\phi\right)$$

$$\boldsymbol{n} = \sqrt{\frac{|\Delta|}{2\Sigma}}\left(\partial_\tau - \partial_r\right)$$

$$\boldsymbol{m} = \frac{1}{\sqrt{2\Sigma}}\left(\mathrm{i}a\sin(\theta)\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\partial_\phi\right)$$

$$\overline{\boldsymbol{m}} = -\frac{1}{\sqrt{2\Sigma}}\left(\mathrm{i}a\sin(\theta)\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\partial_\phi\right)$$

and

$$\begin{aligned}
\boldsymbol{l} &= \sqrt{\frac{|\Delta|}{2\Sigma}}\,\mathrm{sign}(\Delta)\left(\mathrm{d}\tau + \left[1 - \frac{2\Sigma}{\Delta}\right]\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\right) \\
\boldsymbol{n} &= \sqrt{\frac{|\Delta|}{2\Sigma}}\left(\mathrm{d}\tau + \mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\right) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2\Sigma}}\left(\mathrm{i}a\sin(\theta)[\mathrm{d}\tau + \mathrm{d}r] - \Sigma\mathrm{d}\theta - \mathrm{i}\left[r^2 + a^2\right]\sin(\theta)\mathrm{d}\phi\right) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2\Sigma}}\left(\mathrm{i}a\sin(\theta)[\mathrm{d}\tau + \mathrm{d}r] + \Sigma\mathrm{d}\theta - \mathrm{i}\left[r^2 + a^2\right]\sin(\theta)\mathrm{d}\phi\right).
\end{aligned} \tag{16}$$

Substituting the dual Carter tetrad (16) into the first Maurer-Cartan equation in the Newman-Penrose formalism Eqs.(4), one obtains the spin coefficients for Kerr geometry described by a Carter tetrad in AEFTC

$$\kappa = \sigma = \lambda = \nu = 0, \qquad \alpha = -\beta = -\frac{1}{(2\Sigma)^{3/2}}\left(\left[r^2 + a^2\right]\cot(\theta) - \mathrm{i}ra\sin(\theta)\right)$$

$$\pi = -\tau = \frac{\mathrm{i}a\sin(\theta)}{\sqrt{2\Sigma}\left(r - \mathrm{i}a\cos(\theta)\right)}, \qquad \mu = \mathrm{sign}(\Delta)\varrho = -\sqrt{\frac{|\Delta|}{2\Sigma}}\frac{1}{\left(r - \mathrm{i}a\cos(\theta)\right)}$$

$$\epsilon = \mathrm{sign}(\Delta)\gamma = \frac{1}{\sqrt{|\Delta|}(2\Sigma)^{3/2}}\left(M\left[r^2 - a^2\cos^2(\theta)\right] - ra^2\sin^2(\theta) - \mathrm{i}a\cos(\theta)\Delta\right).$$

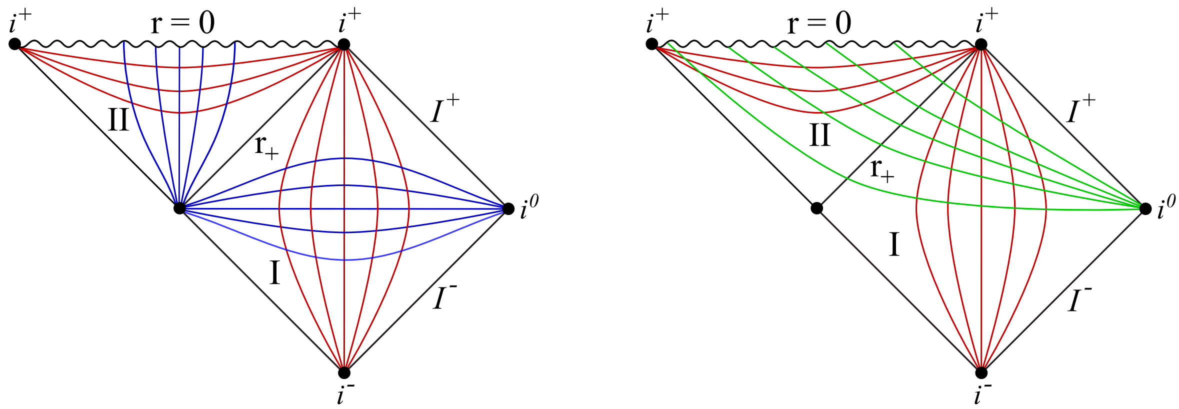

FIG. 3: Conformal diagrams for Schwarzschild geometry with $a = 0$ in Boyer-Lindquist coordinates (left) and in advanced Eddington-Finkelstein-type coordinates (right). The blue lines represent constant-t hypersurfaces, the red lines constant-r hypersurfaces, and the green lines constant-τ hypersurfaces. The constant-t and constant-r hypersurfaces are restricted to either the black hole exterior or to the black hole interior. Their characters change going from the exterior to the interior, i.e., space-like hypersurfaces become time-like and vice versa. There is no transition across the event horizon. The constant-τ hypersurfaces are space-like outside and inside the black hole, smooth across the event horizon, and end in the curvature singularity at $r = 0$. The bifurcation 2-sphere is avoided.

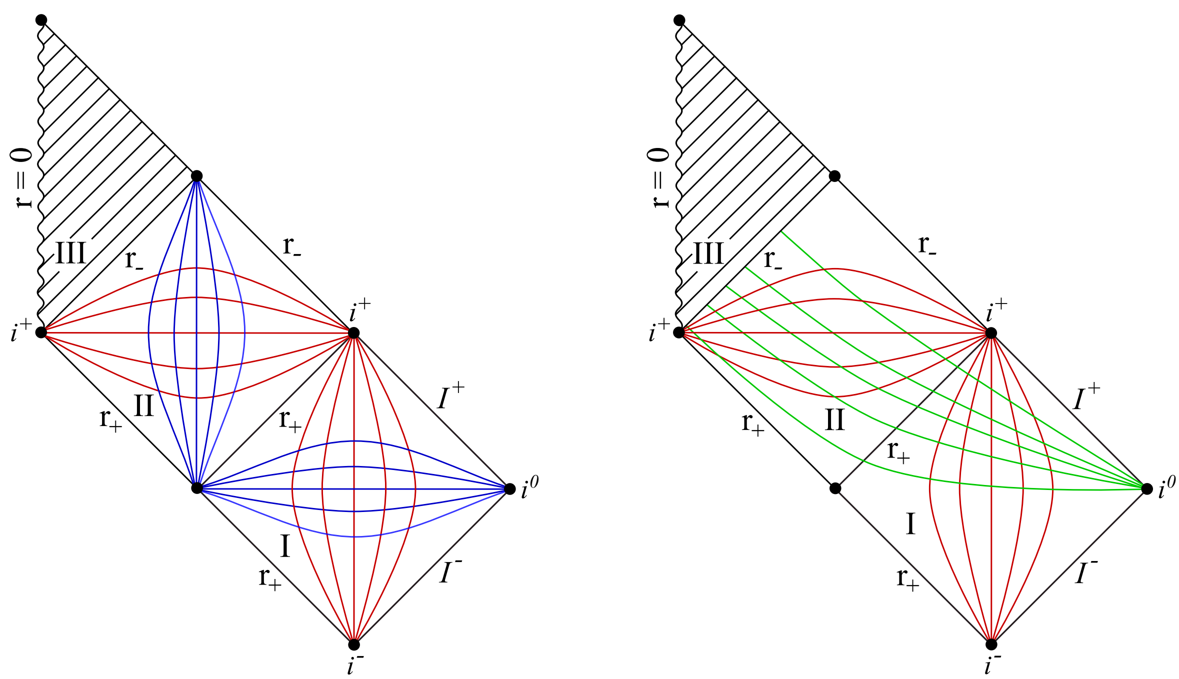

FIG. 4: Conformal diagrams for Kerr geometry with $M^2 > a^2$ in Boyer-Lindquist coordinates (left) and in advanced Eddington-Finkelstein-type coordinates (right). The blue lines represent constant-t hypersurfaces, the red lines constant-r hypersurfaces, and the green lines constant-τ hypersurfaces. As in the Schwarzschild geometry, the constant-t hypersurfaces and the constant-r hypersurfaces are restricted to either the black hole exterior or to the black hole interior, changing their characters going from the exterior to the interior without a transition across the event horizon. The constant-τ hypersurfaces (cut-off at the Cauchy horizon at $r = r_-$ in order to avoid the ring singularity region) are space-like outside and inside the black hole, smooth across the event horizon, and circumvent the bifurcation 2-sphere.

Since the real Newman-Penrose vector $\boldsymbol{l}$ in (16) and, therefore, the spin coefficients ϵ and γ are not well-defined at the event horizon, a renormalization in terms of a type III local Lorentz transformation with the parameters

$$\xi = \frac{\sqrt{|\Delta|}}{r_+} \quad \text{and} \quad \psi = 0$$

is applied, leading to a well-defined Carter tetrad

$$\begin{aligned}
\boldsymbol{l} &= \frac{1}{\sqrt{2\Sigma} r_+} \big([\Delta + 4Mr]\partial_\tau + \Delta\partial_r + 2a\partial_\phi\big) \\
\boldsymbol{n} &= \frac{r_+}{\sqrt{2\Sigma}} \big(\partial_\tau - \partial_r\big) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2\Sigma}} \big(\mathrm{i}a\sin(\theta)\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\partial_\phi\big) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2\Sigma}} \big(\mathrm{i}a\sin(\theta)\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\partial_\phi\big)
\end{aligned} \tag{17}$$

and dual representation

$$\begin{aligned}
\boldsymbol{l} &= \frac{\Delta}{\sqrt{2\Sigma}\, r_+} \left(\mathrm{d}\tau + \left[1 - \frac{2\Sigma}{\Delta}\right]\mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\right) \\
\boldsymbol{n} &= \frac{r_+}{\sqrt{2\Sigma}} \big(\mathrm{d}\tau + \mathrm{d}r - a\sin^2(\theta)\mathrm{d}\phi\big) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2\Sigma}} \Big(\mathrm{i}a\sin(\theta)[\mathrm{d}\tau + \mathrm{d}r] - \Sigma\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\mathrm{d}\phi\Big) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2\Sigma}} \Big(\mathrm{i}a\sin(\theta)[\mathrm{d}\tau + \mathrm{d}r] + \Sigma\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\mathrm{d}\phi\Big).
\end{aligned}$$

The corresponding spin coefficients are also finite at the event horizon, yielding

$$\begin{aligned}
&\kappa = \sigma = \lambda = \nu = 0, \qquad \alpha = -\beta = -\frac{1}{(2\Sigma)^{3/2}} \Big(\big[r^2 + a^2\big]\cot(\theta) - \mathrm{i}ra\sin(\theta)\Big) \\
&\pi = -\tau = \frac{\mathrm{i}a\sin(\theta)}{\sqrt{2\Sigma}\big(r - \mathrm{i}a\cos(\theta)\big)}, \quad \mu = -\frac{r_+}{\sqrt{2\Sigma}\big(r - \mathrm{i}a\cos(\theta)\big)}, \quad \varrho = -\frac{\Delta}{\sqrt{2\Sigma}\, r_+\big(r - \mathrm{i}a\cos(\theta)\big)} \\
&\gamma = -\frac{r_+}{2^{3/2}\sqrt{\Sigma}\big(r - \mathrm{i}a\cos(\theta)\big)}, \qquad \epsilon = \frac{r^2 - a^2 - 2\mathrm{i}a\cos(\theta)(r - M)}{2^{3/2}\sqrt{\Sigma}\, r_+\big(r - \mathrm{i}a\cos(\theta)\big)}.
\end{aligned} \tag{18}$$

In this specific symmetric, renormalized frame represented in AEFTC, the Dirac equation is, as shown in the next section, separable.

IV. THE DIRAC EQUATION IN THE EXTENDED KERR BLACK HOLE SPACETIME

A. Mode Ansatz and Separability

Substituting the renormalized Carter tetrad (17) and the spin coefficients (18) into the Dirac equation in the Newman-Penrose formalism (8), and employing a separation ansatz, which is adapted to the stationarity and axial symmetry of Kerr geometry, into τ- and ϕ-modes following the techniques used in Chandrasekhar's mode analysis (see, e.g., [8])

$$\mathscr{F}_i(\tau, r, \theta, \phi) = \frac{\exp\big(\mathrm{i}(\omega\tau + k\phi)\big)}{\sqrt{r - \mathrm{i}a\cos(\theta)}}\, \mathscr{H}_i(r, \theta) \quad \text{and} \quad \mathscr{G}_i(\tau, r, \theta, \phi) = \frac{\exp\big(\mathrm{i}(\omega\tau + k\phi)\big)}{\sqrt{r + \mathrm{i}a\cos(\theta)}}\, \mathscr{J}_i(r, \theta) \tag{19}$$

with the frequency $\omega \in \mathbb{R}$, $k \in \mathbb{Z} + 1/2$, and $i \in \{1, 2\}$, one obtains the coupled, linear, homogeneous first-order PDE system

$$\begin{aligned}
&\tfrac{1}{r_+}\big(\Delta\partial_r + r - M + \mathrm{i}\omega(\Delta + 4Mr) + 2\mathrm{i}ak\big)\mathscr{H}_1 + \big(\partial_\theta + \tfrac{1}{2}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{H}_2 \\
&= \sqrt{2}\mathrm{i}\mu_\star\big(r - \mathrm{i}a\cos(\theta)\big)\mathscr{J}_1 \\[2ex]
&r_+\big(\partial_r - \mathrm{i}\omega\big)\mathscr{H}_2 - \big(\partial_\theta + \tfrac{1}{2}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{H}_1 = -\sqrt{2}\mathrm{i}\mu_\star\big(r - \mathrm{i}a\cos(\theta)\big)\mathscr{J}_2 \\[2ex]
&\tfrac{1}{r_+}\big(\Delta\partial_r + r - M + \mathrm{i}\omega(\Delta + 4Mr) + 2\mathrm{i}ak\big)\mathscr{J}_2 - \big(\partial_\theta + \tfrac{1}{2}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{J}_1 \\
&= \sqrt{2}\mathrm{i}\mu_\star\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{H}_2 \\[2ex]
&r_+\big(\partial_r - \mathrm{i}\omega\big)\mathscr{J}_1 + \big(\partial_\theta + \tfrac{1}{2}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{J}_2 = -\sqrt{2}\mathrm{i}\mu_\star\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{H}_1.
\end{aligned} \tag{20}$$

The particular τ- and ϕ-dependences in the class of mode solutions given by (19) describe perturbations of the black hole background geometry in form of plane waves propagating along the directions of the translation isometries ∂_τ and ∂_ϕ of Kerr geometry in AEFTC. Along these directions, the plane wave forms are preserved. The system of Dirac PDEs (20) is separable by means of the product ansatz

$$\begin{aligned}
\mathscr{H}_1(r, \theta) &= \mathscr{R}_+(r)\mathscr{T}_+(\theta) \\
\mathscr{H}_2(r, \theta) &= \mathscr{R}_-(r)\mathscr{T}_-(\theta) \\
\mathscr{J}_1(r, \theta) &= \mathscr{R}_-(r)\mathscr{T}_+(\theta) \\
\mathscr{J}_2(r, \theta) &= \mathscr{R}_+(r)\mathscr{T}_-(\theta).
\end{aligned} \tag{21}$$

Note that the separability of the Dirac equation depends on the specific choice of the underlying coordinate systems of the fibers of the tangent bundle and of the form of the local tetrad frame. Hence, it is a peculiarity of the Carter tetrad in AEFTC (17) and the corresponding spin coefficients (18) that (20) can be separated via the ansatz (21). Applying (21) to (20) yields the quadruple of coupled radial equations written in compact form

$$\begin{aligned}
&\big(\Delta\partial_r + r - M + \mathrm{i}\omega(\Delta + 4Mr) + 2\mathrm{i}ak\big)\mathscr{R}_+ = r_+\big(\xi_{1/3} + \sqrt{2}\mathrm{i}\mu_\star r\big)\mathscr{R}_- \\[2ex]
&r_+\big(\partial_r - \mathrm{i}\omega\big)\mathscr{R}_- = \big(\xi_{2/4} - \sqrt{2}\mathrm{i}\mu_\star r\big)\mathscr{R}_+
\end{aligned} \tag{22}$$

and the quadruple of coupled angular equations

$$\begin{aligned}
\left(\partial_\theta + \tfrac{1}{2}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\right)\mathscr{T}_- &= -\left(\xi_{1/4} - \sqrt{2}\mu_\star a\cos(\theta)\right)\mathscr{T}_+ \\
\left(\partial_\theta + \tfrac{1}{2}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\right)\mathscr{T}_+ &= \left(\xi_{2/3} + \sqrt{2}\mu_\star a\cos(\theta)\right)\mathscr{T}_-,
\end{aligned} \tag{23}$$

where ξ_i, $i \in \{1,2,3,4\}$, are constants of separation. From the radial equations, it can be directly seen that the identifications $\xi_1 = \xi_3$ and $\xi_2 = \xi_4$ have to hold, while from the angular equations, one obtains the identifications $\xi_1 = \xi_4$ and $\xi_2 = \xi_3$. Thus, defining $\xi := \xi_1 = \xi_2 = \xi_3 = \xi_4$, the systems of radial and angular equations (22) and (23) reduce to

$$\begin{aligned}
\left(\Delta\partial_r + r - M + \mathrm{i}\omega(\Delta + 4Mr) + 2\mathrm{i}ak\right)\mathscr{R}_+ &= r_+\left(\xi + \sqrt{2}\mathrm{i}\mu_\star r\right)\mathscr{R}_- \\
r_+\left(\partial_r - \mathrm{i}\omega\right)\mathscr{R}_- &= \left(\xi - \sqrt{2}\mathrm{i}\mu_\star r\right)\mathscr{R}_+
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\left(\partial_\theta + \tfrac{1}{2}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\right)\mathscr{T}_- &= -\left(\xi - \sqrt{2}\mu_\star a\cos(\theta)\right)\mathscr{T}_+ \\
\left(\partial_\theta + \tfrac{1}{2}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\right)\mathscr{T}_+ &= \left(\xi + \sqrt{2}\mu_\star a\cos(\theta)\right)\mathscr{T}_-,
\end{aligned} \tag{25}$$

respectively. The system of radial ODEs (24) can be brought into a more symmetric form by means of the functions $\widetilde{\mathscr{R}}_+ = \sqrt{|\Delta|}\mathscr{R}_+$ and $\widetilde{\mathscr{R}}_- = r_+\mathscr{R}_-$, resulting in the equations

$$\begin{aligned}
\left(\Delta\partial_r + \mathrm{i}\omega(\Delta + 4Mr) + 2\mathrm{i}ak\right)\widetilde{\mathscr{R}}_+ &= \sqrt{|\Delta|}\left(\xi + \sqrt{2}\mathrm{i}\mu_\star r\right)\widetilde{\mathscr{R}}_- \\
\Delta\left(\partial_r - \mathrm{i}\omega\right)\widetilde{\mathscr{R}}_- &= \mathrm{sign}(\Delta)\sqrt{|\Delta|}\left(\xi - \sqrt{2}\mathrm{i}\mu_\star r\right)\widetilde{\mathscr{R}}_+.
\end{aligned} \tag{26}$$

In a matrix representation, where $\widetilde{\mathscr{R}} = \left(\widetilde{\mathscr{R}}_+, \widetilde{\mathscr{R}}_-\right)^T$ and the partial r-derivative is expressed in terms of the Regge-Wheeler coordinate $\partial_r = (r^2 + a^2)\Delta^{-1}\,\partial_{r_\star}$, these equations become

$$\left[\partial_{r_\star} + \frac{\mathrm{i}}{r^2 + a^2}\begin{pmatrix} \omega(\Delta + 4Mr) + 2ak & 0 \\ 0 & -\omega\Delta \end{pmatrix}\right]\widetilde{\mathscr{R}} = \frac{\sqrt{|\Delta|}}{r^2 + a^2}\begin{pmatrix} 0 & \xi + \sqrt{2}\mathrm{i}\mu_\star r \\ \mathrm{sign}(\Delta)\left(\xi - \sqrt{2}\mathrm{i}\mu_\star r\right) & 0 \end{pmatrix}\widetilde{\mathscr{R}}. \tag{27}$$

This representation is advantageous for the subsequent study of the radial asymptotics which is required for a description of the scattering process of Dirac waves by the gravitational field of a black hole or for the construction of an integral representation of the Dirac propagator. Note that in the examination of the asymptotics $r \to \infty$ and $r \to r_+$, the signum function $\mathrm{sign}(\Delta) = 1$ and for $r \to r_-$, $\mathrm{sign}(\Delta) = -1$. A matrix representation of the angular equations (25), with $\mathscr{T} = (\mathscr{T}_+, \mathscr{T}_-)^T$, is given by

$$\begin{pmatrix} \sqrt{2}\mu_\star a\cos(\theta) & -\left(\partial_\theta + \frac{1}{2}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\right) \\ \partial_\theta + \frac{1}{2}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta) & -\sqrt{2}\mu_\star a\cos(\theta) \end{pmatrix}\mathscr{T} = \xi\mathscr{T}. \tag{28}$$

B. Asymptotic Analysis of Radial Solutions at Infinity

In this subsection, following the approach of [18], asymptotic solutions of the radial ODE system (27) for $r \to \infty$ ($r_\star \to \infty$) are derived and decay properties of these solutions are examined, showing the control of the error. Rewriting (27) in the form

$$\partial_{r_\star}\widetilde{\mathscr{R}} = T(r)\widetilde{\mathscr{R}}, \tag{29}$$

where

$$T(r) := \frac{1}{r^2+a^2}\begin{pmatrix} -\mathrm{i}\big(\omega(\Delta+4Mr)+2ak\big) & \sqrt{|\Delta|}\,\left(\xi+\sqrt{2}\mathrm{i}\mu_\star r\right) \\ \mathrm{sign}(\Delta)\,\sqrt{|\Delta|}\left(\xi-\sqrt{2}\mathrm{i}\mu_\star r\right) & \mathrm{i}\omega\Delta \end{pmatrix}, \tag{30}$$

one can find asymptotic solutions at infinity by first diagonalizing the matrix T by means of the invertible matrix D, $D^{-1}TD = S$, where $S = \mathrm{diag}(\lambda_1, \lambda_2)$ is the diagonal matrix corresponding to T and $\lambda_{1/2}$ are the eigenvalues of T. Note that in this limit, $\Delta > 0$ and $\mathrm{sign}(\Delta) = 1$. In terms of the diagonal matrix S, Eq.(29) becomes

$$\partial_{r_\star}\big(D^{-1}\widetilde{\mathscr{R}}\big) = \left[S - D^{-1}\left(\partial_{r_\star}D\right)\right]\big(D^{-1}\widetilde{\mathscr{R}}\big).$$

Then, using the ansatz

$$\widetilde{\mathscr{R}}(r_\star) = D(r_\star)\begin{pmatrix}\exp\left(\mathrm{i}\phi_-(r_\star)\right)\mathfrak{f}_1(r_\star) \\ \exp\left(-\mathrm{i}\phi_+(r_\star)\right)\mathfrak{f}_2(r_\star)\end{pmatrix},$$

one obtains a linear, homogeneous, first-order ODE system for $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)^T$

$$\partial_{r_\star}\mathfrak{f} = \Big[S - W^{-1}D^{-1}\left(\partial_{r_\star}D\right)W - \mathrm{i}\,\mathrm{diag}\big(\partial_{r_\star}\phi_-, -\partial_{r_\star}\phi_+\big)\Big]\mathfrak{f}$$

with $W := \mathrm{diag}\big(\exp\left(\mathrm{i}\phi_-\right), \exp\left(-\mathrm{i}\phi_+\right)\big)$. The functions $\phi_\pm$ are fixed by demanding that $S = \mathrm{i}\,\mathrm{diag}\big(\partial_{r_\star}\phi_-, -\partial_{r_\star}\phi_+\big)$, i.e., $\partial_{r_\star}\phi_- = -\mathrm{i}\lambda_1$ and $\partial_{r_\star}\phi_+ = \mathrm{i}\lambda_2$, yielding

$$\partial_{r_\star}\mathfrak{f} = -W^{-1}D^{-1}\left(\partial_{r_\star}D\right)W\mathfrak{f}. \tag{31}$$

Lemma IV.1. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (29) is asymptotically as $r \to \infty$ $(r_\star \to \infty)$ of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_\infty(r_\star) + E_\infty(r_\star) = D_\infty\begin{pmatrix}\exp\left(\mathrm{i}\phi_-(r_\star)\right)\mathfrak{f}_\infty^{(1)} \\ \exp\left(-\mathrm{i}\phi_+(r_\star)\right)\mathfrak{f}_\infty^{(2)}\end{pmatrix} + E_\infty(r_\star) \tag{32}$$

with the asymptotic diagonalization matrix

$$D_\infty := \begin{cases} \begin{pmatrix}\cosh(\Omega) & \sinh(\Omega) \\ \sinh(\Omega) & \cosh(\Omega)\end{pmatrix} & \text{for } \omega^2 \geq 2\mu_\star^2 \\[2ex] \dfrac{1}{\sqrt{2}}\begin{pmatrix}\cosh(\Omega)+\mathrm{i}\sinh(\Omega) & \sinh(\Omega)+\mathrm{i}\cosh(\Omega) \\ \sinh(\Omega)+\mathrm{i}\cosh(\Omega) & \cosh(\Omega)+\mathrm{i}\sinh(\Omega)\end{pmatrix} & \text{for } \omega^2 < 2\mu_\star^2, \end{cases} \tag{33}$$

where

$$\Omega := \begin{cases} \dfrac{1}{4}\ln\left(\dfrac{\omega+\sqrt{2}\mu_\star}{\omega-\sqrt{2}\mu_\star}\right) & \text{for } \omega^2 \geq 2\mu_\star^2 \\[2ex] \dfrac{1}{4}\ln\left(\dfrac{\sqrt{2}\mu_\star+\omega}{\sqrt{2}\mu_\star-\omega}\right) & \text{for } \omega^2 < 2\mu_\star^2, \end{cases} \tag{34}$$

the asymptotic phases

$$\phi_\mp(r_\star) \simeq \begin{cases} -\sqrt{\omega^2-2\mu_\star^2}\;r_\star - 2M\left(\pm\omega + \dfrac{\mu_\star^2}{\sqrt{\omega^2-2\mu_\star^2}}\right)\ln(r_\star) & \text{for } \omega^2 \geq 2\mu_\star^2 \\[2ex] \sqrt{2\mu_\star^2-\omega^2}\;\mathrm{i}\,r_\star - 2M\left(\pm\omega + \dfrac{\mathrm{i}\mu_\star^2}{\sqrt{2\mu_\star^2-\omega^2}}\right)\ln(r_\star) & \text{for } \omega^2 < 2\mu_\star^2, \end{cases} \tag{35}$$

and the error

$$||E_\infty(r_\star)|| = \big|\big|\widetilde{\mathscr{R}}(r_\star) - \widetilde{\mathscr{R}}_\infty(r_\star)\big|\big| \le \frac{a}{r_\star} \tag{36}$$

for a suitable constant $a \in \mathbb{R}_{>0}$. The asymptotics of the function $\mathfrak{f}$ for large r (cf. Eq.(31)) is given by

$$\mathfrak{f}_\infty = \left(\mathfrak{f}_\infty^{(1)}, \mathfrak{f}_\infty^{(2)}\right)^T = \text{const.}$$

with an error

$$||E_\mathfrak{f}(r_\star)|| = ||\mathfrak{f}(r_\star) - \mathfrak{f}_\infty|| \le \frac{b}{r_\star}$$

for a suitable constant $b \in \mathbb{R}_{>0}$.

Proof. The matrix T defined in (30) converges for $r \to \infty$ to the matrix

$$T_\infty := \lim_{r\to\infty} T = \mathrm{i}\begin{pmatrix} -\omega & \sqrt{2}\mu_\star \\ -\sqrt{2}\mu_\star & \omega \end{pmatrix}.$$

Further, it has a regular expansion in powers of $1/r$ and, thus, in powers of $1/r_\star$, i.e., $T = T_\infty + \mathcal{O}(1/r_\star)$. The eigenvalues of T_∞ read

$$\lambda_{1/2} \simeq \begin{cases} \pm\mathrm{i}\sqrt{\omega^2 - 2\mu_\star^2} \in \mathbb{C} & \text{for } \omega^2 \ge 2\mu_\star^2 \\ \pm\sqrt{2\mu_\star^2 - \omega^2} \in \mathbb{R} & \text{for } \omega^2 < 2\mu_\star^2. \end{cases}$$

The transformation matrix D_∞, which diagonalizes T_∞, is given by (33) with arguments (34). This can be easily shown by direct calculation. Since T has a regular expansion in powers of $1/r_\star$, both the diagonal matrix S and the transformation matrix D also have regular expansions in powers of $1/r_\star$. Therefore, with the asymptotic eigenvalues of the matrix T up to first order in $1/r_\star$

$$\lambda_{1/2} \simeq \begin{cases} \mp\mathrm{i}\sqrt{\omega^2 - 2\mu_\star^2} - \dfrac{2\mathrm{i}M}{r_\star}\left(\omega \pm \dfrac{\mu_\star^2}{\sqrt{\omega^2 - 2\mu_\star^2}}\right) & \text{for } \omega^2 \ge 2\mu_\star^2 \\[2ex] \mp\sqrt{2\mu_\star^2 - \omega^2} - \dfrac{2M}{r_\star}\left(\mathrm{i}\omega \mp \dfrac{\mu_\star^2}{\sqrt{2\mu_\star^2 - \omega^2}}\right) & \text{for } \omega^2 < 2\mu_\star^2, \end{cases}$$

one solves the ODEs for the asymptotic phases stated above Eq.(31) by integration, and immediately obtains (35). With the upper bounds of the Hilbert-Schmidt norms of the inverse and of the partial $r_\star$-derivative of the transformation matrix D for $r_\star$ sufficiently close to infinity

$$||D^{-1}||_{\mathrm{HS}} \le c \quad\text{and}\quad ||\partial_{r_\star} D||_{\mathrm{HS}} \le \frac{d}{r_\star^2},$$

where c and d denote positive constants, one can estimate the $\mathbb{C}^2$-norm of Eq.(31)

$$||\partial_{r_\star}\mathfrak{f}|| \le 2\,||D^{-1}||_{\mathrm{HS}} \cdot ||\partial_{r_\star} D||_{\mathrm{HS}} \cdot ||\mathfrak{f}|| \le \frac{2cd}{r_\star^2}\,||\mathfrak{f}|| \tag{37}$$

with $||W||_{\mathrm{HS}} = ||W^{-1}||_{\mathrm{HS}} = \sqrt{2}$. Using the triangle and Cauchy-Schwarz inequalities, it can be shown that the following inequality holds

$$\big|\partial_{r_\star}||\mathfrak{f}||\big| = \frac{|\partial_{r_\star}\langle\mathfrak{f},\mathfrak{f}\rangle|}{2\,||\mathfrak{f}||} = \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle + \langle\partial_{r_\star}\mathfrak{f},\mathfrak{f}\rangle|}{2\,||\mathfrak{f}||} \le \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle| + |\langle\partial_{r_\star}\mathfrak{f},\mathfrak{f}\rangle|}{2\,||\mathfrak{f}||} = \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle|}{||\mathfrak{f}||} \le \frac{||\mathfrak{f}||\cdot||\partial_{r_\star}\mathfrak{f}||}{||\mathfrak{f}||} = ||\partial_{r_\star}\mathfrak{f}|| \tag{38}$$

and, consequently,

$$\left|\partial_{r_\star}||\mathfrak{f}||\right| \leq \frac{2cd}{r_\star^2}\,||\mathfrak{f}||. \tag{39}$$

Note that $||\mathfrak{f}|| \neq 0$ because $\widetilde{\mathscr{R}}$ has to be nontrivial. Integrating (39) over the Regge-Wheeler coordinate from r_0 to $r_\star$ and applying the triangle inequality for integrals gives

$$\left|\int_{r_0}^{r_\star} \partial_{r'_\star} \ln||\mathfrak{f}||\,\mathrm{d}r'_\star\right| \leq \int_{r_0}^{r_\star} \left|\partial_{r'_\star} \ln||\mathfrak{f}||\right|\mathrm{d}r'_\star \leq 2cd\int_{r_0}^{r_\star} \frac{\mathrm{d}r'_\star}{r'^2_\star} \tag{40}$$

for all $0 < r_0 \leq r_\star$ and, hence,

$$\left|\ln||\mathfrak{f}||\Big|_{r_0}^{r_\star}\right| \leq -\frac{2cd}{r'_\star}\Big|_{r_0}^{r_\star}. \tag{41}$$

Since $0 < 2cd/r'_\star\big|_{r_\star}^{r_0} < \infty$ for all $0 < r_0 \leq r_\star < \infty$, there exists a constant $N > 0$ such that

$$\frac{1}{N} \leq ||\mathfrak{f}|| \leq N \tag{42}$$

holds. Substituting this into (37), one finds for sufficiently large $r_\star$

$$||\partial_{r_\star}\mathfrak{f}|| \leq \frac{b}{r_\star^2} \tag{43}$$

with $b := 2cdN$, implying that $\mathfrak{f}$ is integrable and has according to (42) a finite, non-zero limit $\mathfrak{f}_\infty := \lim_{r_\star\to\infty}\mathfrak{f}(r_\star) \neq 0$ at infinity. Integrating (43) from $r_\star$ to ∞ and again using the triangle inequality for integrals, one obtains the error estimate

$$||E_{\mathfrak{f}}|| = ||\mathfrak{f} - \mathfrak{f}_\infty|| = \left|\left|\int_{r_\star}^{\infty} \partial_{r'_\star}\mathfrak{f}\,\mathrm{d}r'_\star\right|\right| \leq \int_{r_\star}^{\infty} ||\partial_{r'_\star}\mathfrak{f}||\,\mathrm{d}r'_\star \leq \frac{b}{r_\star}. \tag{44}$$

The $1/r_\star$-decay of the error E_∞ (cf. (36)) follows directly from the substitution of (44) into the $\mathbb{C}^2$-norm of E_∞ in (32). Note that the error $E_D = D - D_\infty$ in (32) is absorbed into the error E_∞. ■

C. Asymptotic Analysis of Radial Solutions at the Event Horizon

Using the solution ansatz

$$\widetilde{\mathscr{R}} = \begin{pmatrix} \exp\left(-2\mathrm{i}\left[\omega + k\Omega^{(+)}_{\mathrm{Kerr}}\right] r_\star\right)\mathfrak{g}_1(r_\star) \\ \mathfrak{g}_2(r_\star)\end{pmatrix}$$

in Eq.(29), where $\Omega^{(+)}_{\mathrm{Kerr}} := a/(2Mr_+)$ is the angular velocity of the event horizon of a Kerr black hole, yields an ODE system for the vector-valued function $\mathfrak{g} = (\mathfrak{g}_1, \mathfrak{g}_2)^T$

$$\partial_{r_\star}\mathfrak{g} = \frac{\mathrm{i}}{r^2 + a^2}\left[2k\left(2M\Omega^{(+)}_{\mathrm{Kerr}}\,r - a\right)\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} + \sqrt{\Delta}\right.$$
$$\left.\times\begin{pmatrix} \sqrt{\Delta}\left(\omega + 2k\Omega^{(+)}_{\mathrm{Kerr}}\right) & \exp\left(2\mathrm{i}\left[\omega + k\Omega^{(+)}_{\mathrm{Kerr}}\right] r_\star\right)\left[\sqrt{2}\mu_\star r - \mathrm{i}\xi\right] \\ -\exp\left(-2\mathrm{i}\left[\omega + k\Omega^{(+)}_{\mathrm{Kerr}}\right] r_\star\right)\left[\sqrt{2}\mu_\star r + \mathrm{i}\xi\right] & \sqrt{\Delta}\,\omega \end{pmatrix}\right]\mathfrak{g}. \tag{45}$$

Approaching the event horizon $r \to r_+$ ($r_\star \to -\infty$), the right-hand side vanishes and, thus, one obtains the asymptotic solution $\mathfrak{g}_{r_+} := \lim_{r\to r_+} \mathfrak{g} = \text{const.}$

Lemma IV.2. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (29) is asymptotically as $r \to r_+$ ($r_\star \to -\infty$) of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_{r_+}(r_\star) + E_{r_+}(r_\star) = \begin{pmatrix} \exp\left(-2\mathrm{i}\left[\omega + k\Omega^{(+)}_{\text{Kerr}}\right] r_\star\right) \mathfrak{g}^{(1)}_{r_+} \\ \mathfrak{g}^{(2)}_{r_+} \end{pmatrix} + E_{r_+}(r_\star) \tag{46}$$

with

$$\mathfrak{g}_{r_+} = \left(\mathfrak{g}^{(1)}_{r_+}, \mathfrak{g}^{(2)}_{r_+}\right)^T = \text{const.} \neq 0$$

and error with exponential decay

$$||E_{r_+}(r_\star)|| \leq p \exp(q r_\star) \tag{47}$$

for r sufficiently close to r_+ and suitable constants $p, q \in \mathbb{R}_{>0}$.

Proof. From Eq.(45), it follows that

$$\partial_{r_\star} \mathfrak{g} = \mathcal{O}\left(\sqrt{r - r_+}\right) \mathfrak{g} = \mathcal{O}\big(\exp(q r_\star)\big)\mathfrak{g},$$

where $r \simeq r_+ + \exp(2 q r_\star)$ and $q := (r_+ - r_-)/\big(2(r_+^2 + a^2)\big) \in \mathbb{R}_{>0}$. Thus, it exists a constant $p' \in \mathbb{R}_{>0}$ such that

$$||\partial_{r_\star} \mathfrak{g}|| \leq p' \exp(q r_\star) ||\mathfrak{g}|| \tag{48}$$

holds for $r_\star$ sufficiently close to $-\infty$. Similar to the steps (38)-(44) of the previous subsection, one can show that $||\mathfrak{g}||$ is bounded and the error $||\mathfrak{g} - \mathfrak{g}_{r_+}||$ of the asymptotic solution $\mathfrak{g}_{r_+} = \text{const.} \neq 0$ decays exponentially. Accordingly, by means of (48), one finds

$$\left| \ln ||\mathfrak{g}|| \Big|_{r_\star}^{r_0} \right| \leq \frac{p'}{q} \exp(q r'_\star)\Big|_{r_\star}^{r_0}$$

for all $r_\star \leq r_0$ and because $0 < \exp(q r'_\star)|_{r_\star}^{r_0} < \infty$ for all $-\infty < r_\star \leq r_0 < \infty$, there is a constant $N' > 0$ such that the norm $||\mathfrak{g}||$ is bounded

$$\frac{1}{N'} \leq ||\mathfrak{g}|| \leq N'. \tag{49}$$

Substituting (49) into (48) yields

$$||\partial_{r_\star} \mathfrak{g}|| \leq p \exp(q r_\star), \tag{50}$$

where $p := p' N'$, implying that $\mathfrak{g}$ is integrable and has a finite, non-zero limit for $r_\star \to -\infty$. Again integrating (50) from $-\infty$ to r_0 and applying the triangle inequality for integrals, one obtains

$$||E_{\mathfrak{g}}|| = ||\mathfrak{g} - \mathfrak{g}_{r_+}|| = \left|\left| \int_{-\infty}^{r_\star} \partial_{r'_\star} \mathfrak{g} \, \mathrm{d}r'_\star \right|\right| \leq \int_{-\infty}^{r_\star} ||\partial_{r'_\star} \mathfrak{g}|| \, \mathrm{d}r'_\star \leq p \exp(q r_\star), \tag{51}$$

proving the exponential decay of the error $E_{\mathfrak{g}}$ of the asymptotic function $\mathfrak{g}_{r_+}$ and, therefore, its control for $r_\star \to -\infty$. Subsequently, the exponential decay of the error E_{r_+} (cf. (47)) follows by using (51) in the $\mathbb{C}^2$-norm of E_{r_+} in (46). ■

D. Asymptotic Analysis of Radial Solutions at the Cauchy Horizon

Similar to the derivation of asymptotic radial solutions at the event horizon (cf. Section IV C), one begins with a solution ansatz of the form

$$\tilde{\mathscr{R}} = \begin{pmatrix} \exp\left(-2\mathrm{i}\left[\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right] r_\star\right) \mathfrak{h}_1(r_\star) \\ \mathfrak{h}_2(r_\star) \end{pmatrix},$$

with the angular velocity of a Kerr black hole at the Cauchy horizon $\Omega^{(-)}_{\mathrm{Kerr}} := a/(2Mr_-)$, and applies it to Eq.(29). This leads to a first-order ODE system for $\mathfrak{h} = (\mathfrak{h}_1, \mathfrak{h}_2)^T$

$$\partial_{r_\star}\mathfrak{h} = \frac{\mathrm{i}}{r^2+a^2}\left[2k\left(2M\Omega^{(-)}_{\mathrm{Kerr}} r - a\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{|\Delta|}\right.$$

$$\left.\times\begin{pmatrix} -\sqrt{|\Delta|}\left(\omega + 2k\Omega^{(-)}_{\mathrm{Kerr}}\right) & \exp\left(2\mathrm{i}\left[\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right] r_\star\right)\left[\sqrt{2}\mu_\star r - \mathrm{i}\xi\right] \\ \exp\left(-2\mathrm{i}\left[\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right] r_\star\right)\left[\sqrt{2}\mu_\star r + \mathrm{i}\xi\right] & -\sqrt{|\Delta|}\,\omega \end{pmatrix}\right]\mathfrak{h}.$$

In the limit $r \to r_-$ ($r_\star \to \infty$), the square bracket on the right-hand side vanishes, resulting in the asymptotic solution $\mathfrak{h}_{r_-} := \lim_{r\to r_-}\mathfrak{h} = \mathrm{const.}$

Lemma IV.3. *Every nontrivial solution $\tilde{\mathscr{R}}$ of (29) is asymptotically as $r \to r_-$ ($r_\star \to \infty$) given by*

$$\tilde{\mathscr{R}}(r_\star) = \tilde{\mathscr{R}}_{r_-}(r_\star) + E_{r_-}(r_\star) = \begin{pmatrix} \exp\left(-2\mathrm{i}\left[\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right] r_\star\right)\mathfrak{h}^{(1)}_{r_-} \\ \mathfrak{h}^{(2)}_{r_-} \end{pmatrix} + E_{r_-}(r_\star)$$

with

$$\mathfrak{h}_{r_-} = \left(\mathfrak{h}^{(1)}_{r_-}, \mathfrak{h}^{(2)}_{r_-}\right)^T = \mathrm{const.} \neq 0$$

and error with exponential decay

$$||E_{r_-}(r_\star)|| \leq u\exp(-vr_\star)$$

for r sufficiently close to r_- and suitable constants $u, v \in \mathbb{R}_{>0}$.

Proof. The proof of this lemma is analog to the proof of Lemma (IV.2). ■

E. Angular Solutions

The angular first-order ODE system (28), in its decoupled second-order form, is known as the massive Chandrasekhar-Page equation [8]. In the limit $a \searrow 0$, the solutions of this equation reduce to the spin-weighted spherical harmonics for the spin-1/2 case [21]. For non-zero angular momenta $0 < a^2 < M^2$, the solutions are usually referred to as the spin-1/2 spheroidal harmonics. For a good introduction and a compilation of some properties of these functions, the reader is referred to the recent paper [11]. For the study at hand, however, one only needs to know that the matrix-valued differential operator on the left-hand side of Eq.(28) has a spectral decomposition with discrete, non-degenerate eigenvalues and smooth eigenfunctions. This was proven in [16] and [18] and the results are restated as follows.

Proposition IV.4. *For any $\omega \in \mathbb{R}$ and $k \in \mathbb{Z} + 1/2$, the differential operator in* (28) *has a complete set of orthonormal eigenfunctions $(Y_n)_{n\in\mathbb{Z}}$ in $L^2\big((0,\pi), \sin(\theta)\,\mathrm{d}\theta\big)^2$. The corresponding eigenvalues ξ_n are real-valued and non-degenerate, and can thus be ordered as $\xi_n < \xi_{n+1}$. Moreover, the eigenfunctions are pointwise bounded and smooth away from the poles,*

$$Y_n \in L^\infty\big((0,\pi)\big)^2 \cap C^\infty\big((0,\pi)\big)^2.$$

Both the eigenfunctions Y_n and the eigenvalues ξ_n depend smoothly on ω.

V. SCATTERING OF DIRAC WAVES BY THE GRAVITATIONAL FIELD OF A KERR BLACK HOLE

In this section, the physical problem of the reflection and transmission of incident Dirac waves, emerging from space-like infinity, by the gravitational field of a Kerr black hole, from the point of view of an observer described by a well-defined (finite) AEFTC frame, is studied. To this end, the net current of incident Dirac waves is expressed by means of reflection and transmission coefficients and evaluated at infinity and at the event horizon. This approach differs from the usual treatment in Boyer-Lindquist coordinates because there, Dirac waves can be seen to reach the event horizon only asymptotically, while here, one can follow the Dirac waves beyond the event horizon. For a treatment in Boyer-Lindquist coordinates, the reader is referred to [8] and references therein.

Using the radial asymptotics at infinity (32) and at the event horizon (46) with real values of the frequency ω, boundary conditions specifying an incident wave of unit amplitude from infinity which gives rise, on the one hand, to a reflected wave of amplitude $A(\omega, \mu_\star)$ at infinity and, on the other hand, to a transmitted wave of amplitude $B(\omega, \mu_\star)$ at the event horizon can be imposed. The asymptotic ingoing and outgoing wave solutions adapted to these boundary conditions read

$$\widetilde{\mathscr{R}}_{\text{Scat.}}(r \to \infty) \simeq \begin{pmatrix} A(\omega,\mu_\star)\exp\big(\mathrm{i}\phi_-(r_\star)\big) \\ \exp\big(-\mathrm{i}\phi_+(r_\star)\big) \end{pmatrix} \tag{52}$$

and

$$\widetilde{\mathscr{R}}_{\text{Scat.}}(r \to r_+) \simeq \begin{pmatrix} 0 \\ B(\omega,\mu_\star) \end{pmatrix}. \tag{53}$$

Note that ingoing and outgoing waves can be easily identified evaluating the expectation value of the momentum operator $\widehat{p} = -\mathrm{i}\hbar\partial_{r_\star}$ on the solution space. Moreover, the boundary conditions were imposed in such a way that no waves can emerge from the event horizon, i.e., the diverging horizon contribution, namely the first component of (46), is omitted. This leads to a continuous radial Dirac current over the event horizon. Besides, only the branch of the asymptotic solution (32) with $\omega^2 \geq 2\mu_\star^2$ is considered because free particles at infinity must have energies that exceed, or at least equal, their rest energies.

Assuming the normalization condition $|\mathscr{T}_+(\theta)|^2 + |\mathscr{T}_-(\theta)|^2 = 1$ of the angular functions (cf. Eq.(28) and Section IV E), the radial Dirac current J^r yields

$$J^r = \sqrt{2}\,\sigma^r{}_{A\dot{B}}\Big(P^A\overline{P}^{\dot{B}} + Q^A\overline{Q}^{\dot{B}}\Big) = \frac{1}{\sqrt{2}\,r_+\Sigma}\Big(\text{sign}(\Delta)\big|\widetilde{\mathscr{R}}_+\big|^2 - \big|\widetilde{\mathscr{R}}_-\big|^2\Big) \tag{54}$$

with the radial Infeld-van der Waerden symbol

$$\sigma^{r}{}_{A\dot{B}} = \begin{pmatrix} l^r & m^r \\ \overline{m}^r & n^r \end{pmatrix}_{A\dot{B}} = \frac{1}{\sqrt{2\Sigma}\, r_+} \begin{pmatrix} \Delta & 0 \\ 0 & -r_+^2 \end{pmatrix}_{A\dot{B}} .$$

From the radial ODEs (26) and the corresponding complex conjugations, one can obtain the relation

$$\big|\widetilde{\mathscr{R}}_-\big|^2 - \operatorname{sign}(\Delta)\big|\widetilde{\mathscr{R}}_+\big|^2 = \text{const.}$$

by simple algebraic manipulations. Substituting this into the radial Dirac current (54), a conserved quantity, the net current, can be derived

$$\frac{\partial N}{\partial t} = -\int_0^{2\pi}\int_0^{\pi} J^r \sqrt{|\det(\boldsymbol{g})|}\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{4\pi}{\sqrt{2}r_+}\left(\big|\widetilde{\mathscr{R}}_-\big|^2 - \operatorname{sign}(\Delta)\big|\widetilde{\mathscr{R}}_+\big|^2\right) = \text{const.},$$

where $\sqrt{|\det(\boldsymbol{g})|} = \Sigma \sin(\theta)$. Both the radial Dirac current and the net current are in general discontinuous with jump discontinuities at the event and Cauchy horizons at $r = r_+$ and $r = r_-$. However, since for the scattering problem at hand, wave contributions from the event horizon are omitted, i.e., $\widetilde{\mathscr{R}}_+ \equiv 0$ at $r = r_+$, the net current becomes continuous across the event horizon. Defining the reflection and transmission coefficients

$$R(\omega,\mu_\star) := |A(\omega,\mu_\star)|^2 \quad \text{and} \quad T(\omega,\mu_\star) := |B(\omega,\mu_\star)|^2$$

and using (52) and (53), the net current at infinity and at the event horizon becomes

$$\frac{\partial N}{\partial t}\Big|_{r\to\infty} = \frac{4\pi}{\sqrt{2}r_+}\big(1 - R(\omega,\mu_\star)\big) \tag{55}$$

and

$$\frac{\partial N}{\partial t}\Big|_{r\to r_+} = \frac{4\pi}{\sqrt{2}r_+}\,T(\omega,\mu_\star), \tag{56}$$

respectively. The latter equation shows that the net current across the event horizon is always positive. From the constancy of the net current and Eqs.(55) and (56), one can further deduce that

$$R(\omega,\mu_\star) + T(\omega,\mu_\star) = 1,$$

which proofs that superradiance cannot occur because the reflection coefficient is always less than unity. These results are in agreement with those found in the analysis employing the Boyer-Lindquist representation.

VI. SUMMARY AND OUTLOOK

In this paper, the massive Dirac equation in Kerr geometry, described in terms of an advanced Eddington-Finkelstein-type coordinate frame which does not have poles at the event and Cauchy horizons and covers both the interior and exterior black hole regions, was studied. More precisely, using the Newman-Penrose formalism, Kerr geometry was represented in terms of a well-defined (finite) local Carter tetrad in advanced Eddington-Finkelstein-type coordinates (with proper coordinate time) on a null bundle and the bi-spinor form of the Dirac equation by a chiral dyad on the spin bundle over this geometry. It was shown that in this setting, applying a product ansatz with time and azimuthal angle modes for the Dirac waves, the massive Dirac equation is, as in the Boyer-Lindquist case, separable into coupled systems of radial and angular ODEs. The asymptotics of the radial solutions at infinity, the event horizon, and the Cauchy horizon, including error estimates demonstrating that these solutions have suitable decay properties, were derived. A brief discussion of the angular ODEs, namely their eigenfunctions and their eigenvalue spectrum, was given. Then, by means of the asymptotic radial solutions at infinity and at the event horizon, the scattering of Dirac waves by the gravitational field of a Kerr black hole, in a frame that does not have coordinate singularities at the event horizon such that Dirac waves can be observed to actually cross the event horizon and not only reach it asymptotically as in the case of Boyer-Lindquist coordinates, was analyzed. It was shown that the net current of Dirac waves across the event horizon is positive and that superradiant emission cannot occur. These results are, however, in agreement with those obtained for Boyer-Lindquist coordinates (see, e.g., [8]).

This work provides the basis for the Hamiltonian formulation and the spectral theory of the Dirac Hamiltonian of the massive Dirac equation in a Kerr background in a coordinate frame without poles at the inner horizon boundaries appropriate for wave propagation across the event horizon, which is derived in a subsequent article [33]. In this framework, a well-defined geometrical scalar product on the Hilbert space of Dirac wave functions and an integral representation of the Dirac propagator are constructed. Note that due to the use of the advanced Eddington-Finkelstein-type coordinates, it is not necessary to apply gluing techniques in the construction of the propagator in order to connect the interior and exterior regions of the black hole as in the Boyer-Lindquist case.

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[1] D. Batic, “Scattering for massive Dirac fields on the Kerr metric,” arXiv:gr-qc/0606051, Journal of Mathematical Physics **48**, 022502 (2007).

[2] R. H. Boyer and R. W. Lindquist, “Maximal Analytic Extension of the Kerr Metric,” Journal of Mathematical Physics **8**, 265 (1967).

[3] D. R. Brill and J. A. Wheeler, “Interaction of Neutrinos and Gravitational Fields,” Review of Modern Physics **29**, 465 (1957).

[4] B. Carter, “Black hole equilibrium states,” in Black holes/Les astres occlus, Ecole d’été Phys. Théor., Les Houches (1972).

[5] S. K. Chakrabarti and B. Mukhopadhyay, “Scattering of Dirac waves off Kerr black holes,” arXiv:astro-ph/0007277, Monthly Notices of the Royal Astronomical Society **317**, 979 (2000).

[6] S. Chandrasekhar, “The solution of Dirac’s equation in Kerr geometry,” Proceedings of the Royal Society London **349**, 571 (1976).

[7] S. Chandrasekhar and S. Detweiler, “On the reflexion and transmission of neutrino waves by a Kerr black hole,” Proceedings of the Royal Society London **352**, 325 (1977).

[8] S. Chandrasekhar, “The Mathematical Theory of Black Holes,” Oxford University Press (1983).

[9] T. Daude and F. Nicoleau, “Direct and inverse scattering at fixed energy for massless charged Dirac fields by Kerr-Newman-de Sitter black holes,” arXiv:1307.2842 [math-ph] (2013).

[10] S. R. Dolan, “Scattering and absorption of gravitational plane waves by rotating black holes,” arXiv:0801.3805 [gr-qc], Classical and Quantum Gravity **25**, 235002 (2008).

[11] S. R. Dolan and J. R. Gair, “The massive Dirac field on a rotating black hole spacetime: angular solutions,” arXiv:0905.2974 [gr-qc], Classical and Quantum Gravity **26**, 175020 (2009).

[12] S. R. Dolan, "Quasinormal mode spectrum of a Kerr black hole in the eikonal limit," arXiv:1007.5097 [gr-qc], Physical Review D **82**, 104003 (2010).

[13] C. Doran, "New form of the Kerr solution," arXiv:gr-qc/9910099, Physical Review D **61**, 067503 (2000).

[14] A. S. Eddington, "A Comparison of Whitehead's and Einstein's Formulae," Nature **113**, 192 (1924).

[15] D. Finkelstein, "Past-Future Asymmetry of the Gravitational Field of a Point Particle," Physical Review **110**, 965 (1958).

[16] F. Finster, N. Kamran, J. Smoller, and S. T. Yau, "Non-existence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry," arXiv:gr-qc/9905047, Communications in Mathematical Physics **53**, 902 (2000).

[17] F. Finster, N. Kamran, J. Smoller, and S. T. Yau, "Decay Rates and Probability Estimates for Massive Dirac Particles in the Kerr-Newman Black Hole Geometry," arXiv:gr-qc/0107094, Communications in Mathematical Physics **230**, 201 (2002).

[18] F. Finster, N. Kamran, J. Smoller, and S. T. Yau, "The Long-Time Dynamics of Dirac Particles in the Kerr-Newman Black Hole Geometry," arXiv:gr-qc/0005088, Advances in Theoretical and Mathematical Physics **7**, 25 (2003).

[19] V. Fock, "Geometrisierung der Diracschen Theorie des Elektrons," Zeitschrift für Physik **57**, 261 (1929).

[20] J. A. H. Futterman, F. A. Handler, and R. A. Matzner, "Scattering from Black Holes," Cambridge University Press (1988).

[21] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, "Spin-s Spherical Harmonics and ð," Journal of Mathematical Physics **8**, 2155 (1967).

[22] D. Häfner, "Creation of fermions by rotating charged black-holes," arXiv:math/0612501 [math.AP] (2006).

[23] J. Jing and Q. Pan, "Dirac quasinormal frequencies of the Kerr Newman black hole," arXiv:gr-qc/0506098, Nuclear Physics B **728**, 109 (2005).

[24] R. P. Kerr, "Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics," Physical Review Letters **11**, 237 (1963).

[25] W. Kinnersley, "Type D Vacuum Metrics," Journal of Mathematical Physics **10**, 1195 (1969).

[26] E. T. Newman and R. Penrose, "An approach to gravitational radiation by a method of spin coefficients," Journal of Mathematical Physics **3**, 566 (1962).

[27] B. O'Neill, "The geometry of Kerr black holes," Dover Publications (2014).

[28] D. N. Page, "Dirac equation around a charged, rotating black hole," Physical Review D **14**, 1509 (1976).

[29] R. Penrose, "A spinor approach to general relativity," Annals of Physics **10**, 171 (1960).

[30] W. H. Press and S. A. Teukolsky, "Perturbations of a Rotating Black Hole. II. Dynamical Stability of the Kerr Metric," Astrophysical Journal **185**, 649 (1973).

[31] R. H. Price, "Nonspherical Perturbations of Relativistic Gravitational Collapse. I. Scalar and Gravitational Perturbations," Physical Review D **5**, 2419 (1972).

[32] R. H. Price, "Nonspherical Perturbations of Relativistic Gravitational Collapse. II. Integer-Spin, Zero-Rest-Mass Fields," Physical Review D **5**, 2439 (1972).

[33] C. Röken and F. Finster, "An Integral Representation of the Massive Dirac Propagator in the Kerr Geometry in Eddington-Finkelstein-Type Coordinates," in preparation (2015).

[34] S. A. Teukolsky, "Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-Field Perturbations," Astrophysical Journal **185**, 635 (1973).

[35] S. A. Teukolsky and W. H. Press, "Perturbations of a rotating black hole. III - Interaction of the hole with gravitational and electromagnetic radiation," Astrophysical Journal **193**, 443 (1974).

[36] W. G. Unruh, "Separability of the Neutrino Equations in a Kerr Background," Physical Review Letters **31**, 1265 (1973).

[37] R. M. Wald, "General Relativity," University of Chicago Press (1984).

[38] H. Weyl, "Elektron und Gravitation," Zeitschrift für Physik **56**, 330 (1929).

[39] B. F. Whiting, "Mode stability of the Kerr black hole," Journal of Mathematical Physics **30**, 1301 (1989).