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The fermionic projector in a time-dependent external potential: Mass oscillation property and Hadamard states

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THE FERMIONIC PROJECTOR IN A TIME-DEPENDENT EXTERNAL POTENTIAL: MASS OSCILLATION PROPERTY AND HADAMARD STATES

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Abstract. We give a non-perturbative construction of the fermionic projector in Minkowski space coupled to a time-dependent external potential which is smooth and decays faster than quadratically for large times. The weak and strong mass oscillation properties are proven. We show that the integral kernel of the fermionic projector is of Hadamard form, provided that the time integral of the spatial sup-norm of the potential satisfies a suitable bound. This gives rise to an algebraic quantum field theory of Dirac fields in an external potential with a distinguished pure quasi-free Hadamard state.

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1. Introduction

In the recent papers [20, 21], a functional analytic construction of the fermionic projector was given in a general class of globally hyperbolic space-times. In the present paper, we show that the construction in infinite lifetime in [21] applies to the Dirac equation in Minkowski space in the presence of an external potential, provided that the potential is smooth and decays suitably for large times. The main technical step is to prove that the so-called mass oscillation property holds. Assuming in addition a bound on the time integral of the spatial sup-norm of the potential, we show that the resulting fermionic projector is of Hadamard form (for an introduction to the Hadamard form see [30, 37]). These results put the previous perturbative treatment of the fermionic projector in [11, 12, 13, 16, 22] (see also the textbook [14]) on a rigorous functional analytic basis. In particular, our results show that the nonlocal low and high energy contributions as introduced in [13] by a formal power series are indeed well-defined and smooth.

Another objective of this paper is to build the bridge to algebraic quantum field theory (for an introduction see [28, 4, 3]). To this end, we construct fermionic quantum fields in the presence of a classical external potential and show that the fermionic projector gives rise to a distinguished quasi-free Hadamard state (see again [37] and the references therein).

In the remainder of the introduction, we state our results and put them into the context of the fermionic projector and of algebraic quantum field theory.

The Mass Oscillation Property. In Minkowski space without an external potential, the Dirac equation has plane-wave solutions. The sign of the frequency of these plane-wave solutions gives a splitting of the solution space into two subspaces, usually referred to as the positive and negative energy subspaces. This frequency splitting is important for the physical interpretation of the Dirac equation and for the construction of a corresponding quantum field theory. Namely, in quantum field theory one needs to construct a Fock space and a ground state therein. Choosing the vacuum state in agreement with the frequency splitting (Dirac sea vacuum), it is possible to reinterpret the negative-energy solutions in terms of anti-particle states. The plane-wave solutions of positive and negative frequencies are then identified with creation and annihilation operators, respectively, which by acting on the vacuum state generate the whole Fock space.

The above frequency splitting can still be used in static space-times (i.e. if a time-like Killing field is present). However, in *generic space-times* or in the presence of a *time-dependent* external potential, one does not have a natural frequency splitting. A common interpretation of this fact is that there is no distinguished ground state, and that the notion particles and anti-particles depend on the observer. Nonetheless, the construction of the *fermionic projector* as carried out non-perturbatively in [20, 21] does give rise to a canonical splitting of the solution space of the Dirac equation into two subspaces even in generic space-times. This also suggests that, mimicking the construction for the usual frequency splitting, there should be a canonical ground state of the corresponding quantum field theory, even without assuming a Killing symmetry. One of the goals of this paper is to construct this distinguished ground state in the presence of a time-dependent external potential in Minkowski space.

We now recall a few basic constructions and definitions from [21], always restricting attention to subsets of Minkowski space and to the Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - m)\psi_m = 0 \tag{1.1}$$

in the presence of a smooth external potential $\mathcal{B}$. Here m is the rest mass, and for clarity we add it as an index to the wave function. The construction differs considerably in the cases when space-time has finite or infinite lifetime. A typical example of a space-time of *finite lifetime* is an open subset Ω of Minkowski space contained in a strip $(-T,T)\times\mathbb{R}^3$ for some constant $T>0$ such that the surface $\{0\}\times\mathbb{R}^3$ is a Cauchy surface (for a general treatment of space-times of finite lifetime see [20]). In this case, one considers on the solution space of the Dirac equation (1.1) the usual scalar product obtained by integrating over the Cauchy surface[1]

$$(\psi_m|\phi_m)_m = 2\pi \int_{\mathbb{R}^3} \prec\!\psi_m|\gamma^0\phi_m\!\succ|_{(t=0,\vec{x})}\, d^3x \tag{1.2}$$

as well as the space-time inner product

$$<\!\psi_m|\phi_m\!> = \int_\Omega \prec\!\psi_m|\phi_m\!\succ_x d^4x \tag{1.3}$$

(here $\prec\!\psi|\phi\!\succ$ is the spin scalar product, which is often denoted by $\overline{\psi}\phi$ with the adjoint spinor $\overline{\psi}=\psi^\dagger\gamma^0$, where the dagger means complex conjugation and transposition). Finite lifetime implies that the space-time inner product is bounded in the sense that there is a constant $c>0$ such that

$$|\!<\!\phi_m|\psi_m\!>\!| \leq c\, \|\phi_m\|_m\, \|\psi_m\|_m \tag{1.4}$$

for all Dirac solutions ψ_m,ϕ_m (and $\|.\|_m$ is the norm corresponding to the scalar product (1.2)). This in turn makes it possible to represent the space-time inner product in terms of the *fermionic signature operator* $\tilde{\mathcal{S}}$, meaning that there is a unique bounded symmetric operator $\tilde{\mathcal{S}}$ such that

$$<\!\phi_m|\psi_m\!> = (\phi_m\,|\,\tilde{\mathcal{S}}\,\psi_m)_m$$

(here the tilde indicates that an external potential $\mathcal{B}$ is present, whereas the corresponding objects in the Minkowski vacuum are denoted without a tilde). Then the positive and negative spectral subspaces of $\tilde{\mathcal{S}}$ give rise to the desired splitting of the solution space.

The above construction fails in space-times of *infinite lifetime* because the time integral in (1.3) will in general diverge. The way out is to consider families of solutions $(\psi_m)_{m\in I}$ of the family of Dirac equations (1.1) with the mass parameter m varying in an open interval I. We need to assume that I does not contain the origin, because our methods for dealing with infinite lifetime do not apply in the massless case $m=0$ (this seems no physical restriction because all known fermions in nature have a non-zero rest mass). By symmetry, it suffices to consider positive masses. Thus we choose

$$I := (m_L, m_R) \subset \mathbb{R} \qquad \text{with parameters } m_L, m_R > 0\,. \tag{1.5}$$

We always choose the family of solutions $(\psi_m)_{m\in I}$ in the class $C^\infty_{\text{sc},0}(\mathscr{M}\times I, S\mathscr{M})$ of smooth solutions with spatially compact support in Minkowski space $\mathscr{M}$ which depend smoothly on m and vanish identically for m outside a compact subset of I.

[1]The factor 2π might seem unconventional. This convention was first adopted in [17] to simplify some formulas.

On such families of solutions, we can impose conditions analogous to (1.4) by suitably integrating over m. We here give the condition which is most relevant for applications (for a weaker version, which will also arise in intermediate steps of our proofs, see Definition 2.3 below).

Definition 1.1. *The Dirac operator* $i\partial\!\!\!/ + \mathcal{B}$ *has the* **strong mass oscillation property** *in the interval* I *(see* (1.5)*) if there is a constant* $c > 0$ *such that*

$$\Big| <\int_I \psi_m \, dm \,|\, \int_I \psi_{m'} \, dm'> \Big| \leq c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, dm \tag{1.6}$$

for all families of solutions $(\psi_m)_{m\in I}, (\phi_m)_{m\in I} \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$.

The point is that we integrate over the mass parameter *before* taking the space-time inner product. Intuitively speaking, integrating over the mass parameter generates a decay of the wave function, making sure that the time integral converges.

As shown in [21, Section 4], the strong mass oscillation property gives rise to the representation

$$<\int_I \psi_m \, dm \,|\, \int_I \psi_{m'} \, dm'> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \, \phi_m)_m \, dm \,,$$

which for every $m \in I$ uniquely defines the *fermionic signature operator* $\tilde{\mathcal{S}}_m$. This operator is bounded and symmetric with respect to the scalar product (1.2). Moreover, it does not depend on the choice of the interval I. Now the positive and negative spectral subspaces of the operator $\tilde{\mathcal{S}}_m$ again yield the desired splitting of the solution space. In the case of an ultrastatic space-time, the positive and negative spectral subspaces of $\tilde{\mathcal{S}}_m$ indeed coincide with the solutions of positive and negative frequencies (see [21, Theorem 5.1]).

The remaining crucial question is whether the inequality (1.6) holds in the presence of an external potential. In this paper, we give an affirmative answer, provided that the potential has suitable decay properties at infinity:

Theorem 1.2. *Assume that the external potential* $\mathcal{B}$ *is smooth and for large times decays faster than quadratically in the sense that*

$$|\mathcal{B}(t)|_{C^2} \leq \frac{c}{1 + |t|^{2+\varepsilon}} \tag{1.7}$$

for suitable constants $\varepsilon, c > 0$. *Then the strong mass oscillation property holds.*

The C^2-norm in (1.7) is defined as follows. We denote spatial derivatives by ∇ and use the notation with multi-indices, i.e. for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p)$ we set $\nabla^\alpha = \partial_{\alpha_1 \cdots \alpha_p}$ and denote the length of the multi-index by $|\alpha| = p$. For the potential $\mathcal{B}$ we work with spatial C^k-norms defined by

$$|\mathcal{B}(t)|_{C^k} := \max_{|\alpha| \leq k} \sup_{\vec{x} \in \mathbb{R}^3} |\nabla^\alpha \mathcal{B}(t, \vec{x})| \,, \tag{1.8}$$

where $|\,.\,|$ denotes any matrix norm.

The Fermionic Projector and the Hadamard Form. Using the result of the previous theorem, the fermionic projector P is defined for a fixed mass parameter $m \in I$ by (see [20, Definition 3.7] and [21, Definition 4.5])

$$P := -\chi_{(-\infty,0)}(\tilde{\mathcal{S}}_m) \, \tilde{k}_m \,, \tag{1.9}$$

where $\chi_{(-\infty,0)}(\tilde{\mathcal{S}}_m)$ is the projection onto the negative spectral subspace of the fermionic signature operator, and $\tilde{k}_m$ is the causal fundamental solution (for basic definitions see Section 2.1 below). The fermionic projector can be represented as a bi-distribution $P(x,y)$ with $x, y \in \mathscr{M}$ (see [20, Section 3.5] and [21, Section 4.3]), which satisfies the Dirac equation and is symmetric, i.e.

$$(i\partial\!\!\!/_x + \mathcal{B}(x) - m)\, P(x,y) = 0 \tag{1.10}$$

$$P(x,y)^* = P(y,x) \tag{1.11}$$

(where $P(x,y)^* = \gamma^0 P(x,y)^\dagger \gamma^0$ is the adjoint with respect to the spin scalar product $\prec .|.\succ$).

Knowing the singularity structure of the bi-distribution $P(x,y)$ is important for applications (point-splitting method, Wick polynomials, renormalization, etc.). Therefore, we shall establish that the fermionic projector is of *Hadamard form*. In our setting, this is tantamount to proving that the bi-distribution $P(x,y)$ is of the form (see [34] or [29, page 156])

$$P(x,y) = \lim_{\varepsilon \searrow 0} i\partial\!\!\!/_x \left(\frac{U(x,y)}{\sigma_\varepsilon(x,y)} + V(x,y)\, \log \sigma_\varepsilon(x,y) + W(x,y) \right), \tag{1.12}$$

where

$$\sigma_\varepsilon(x,y) := (y-x)^j\, (y-x)_j - i\varepsilon\, (y-x)^0\,, \tag{1.13}$$

and U, V and W are smooth functions on $\mathscr{M} \times \mathscr{M}$ taking values in the 4×4-matrices acting on the spinors (we always denote space-time indices by latin letters running from $0, \ldots, 3$). For clarity, we point out that on a manifold, the function σ_ε can be defined locally in a geodesically convex neighborhood of a point $x \in \mathscr{M}$, making it necessary to distinguish between the *local* Hadamard form (i.e. a local representation of the form (1.12)) and the *global* Hadamard form (implying that the singularities in (1.12) are the only singularities of the bi-distribution). For these subtle issues, we refer the reader to [27, 25, 33]. In our setting of Minkowski space, there is one global chart, and the distance function σ_ε is defined globally by (1.13). For this reason, we do not need to make a distinction between the local Hadamard form and the global Hadamard form.

In space-times of *finite lifetime*, in general the fermionic projector is *not* of Hadamard form. The first counter examples were constructed in [9], where it is shown that in ultrastatic space-times of finite lifetime, the Hadamard condition is in general violated. Other counter examples are so-called simple domains as introduced in [19, Sections 2.1 and 1.4]. In such simple domains, the fermionic projector in the massless case is an operator of finite rank with a continuous integral kernel, clearly not being of Hadamard form.

In space-times of *infinite lifetime*, the situation is better at least in the ultrastatic case. Namely, in [21, Section 5] it is shown that the fermionic projector in ultrastatic space-times is composed of all negative-energy solutions of the Dirac equation. Therefore, $P(x,y)$ coincides with the bi-distribution constructed from the frequency splitting, which in [38] was shown to be of Hadamard form. Apart from this specific result, it is unknown whether or in which space-times the fermionic projector is of Hadamard form.

The main result of this paper is to show that in a time-dependent external potential in Minkowski space, the fermionic projector is indeed of Hadamard form, provided that the potential is not too large:

Theorem 1.3. *Assume that the external potential $\mathcal{B}$ is smooth, and that its time derivatives decay at infinity in the sense that* (1.7) *holds and in addition that*

$$\int_{-\infty}^{\infty} |\partial_t^p \mathcal{B}(t)|_{C^0} \, dt < \infty \qquad \text{for all } p \in \mathbb{N}$$

(with the C^0-norm as defined in (1.8)*). Moreover, assume that the potential satisfies the bound*

$$\int_{-\infty}^{\infty} |\mathcal{B}(t)|_{C^0} \, dt < \sqrt{2} - 1 \, . \tag{1.14}$$

Then the fermionic projector $P(x,y)$ is of Hadamard form.

We note that the property of the bi-distribution $P(x,y)$ to be of Hadamard form can also be expressed in terms of the wave front set (see for example [31, 36]). Also, the smooth functions in (1.12) can be expanded in powers of the Minkowski distance $\sigma(x,y) = (y-x)^j \, (y-x)_j$,

$$U(x,y) = \sum_{n=0}^{\infty} U_n(x,y) \, \sigma^n \, , \quad V(x,y) = \sum_{n=0}^{\infty} V_n(x,y) \, \sigma^n \, , \quad W(x,y) = \sum_{n=0}^{\infty} W_n(x,y) \, \sigma^n \, .$$

The coefficients of this so-called *Hadamard expansion* can be computed iteratively using the method of *integration along characteristics* (see [30, 23] or [2]). In Minkowski space, the *light-cone expansion* [12, 13] gives a systematic procedure for computing an infinite number of Hadamard coefficients in one step. This procedure also makes it possible to compute the *smooth contributions* to $P(x,y)$, giving a connection to fermionic loop corrections in quantum field theory (see [15, §8.2 and Appendix D]).

We finally remark that, introducing an ultraviolet regularization, the fermionic projector gives rise to a corresponding *causal fermion system* (for details see [20, Section 4]). In this context, the result of Theorem 1.3 gives a justification for the formalism of the continuum limit as used in [14, 10] for the analysis of the causal action principle. We also refer the interested reader to the introduction to causal fermion systems [18].

Quantum Fields and Hadamard States. Using the standard notation in quantum field theory, the objective of the quantization of the Dirac field is to construct field operators $\Psi(x)$ and $\Psi(y)^*$ acting on a Fock space $\mathcal{H}_{\text{Fock}}$ together with a suitable ground state $|0\rangle$. The field operators should satisfy the canonical anti-commutation relations

$$\{\Psi^\alpha(x), \Psi^\beta(y)^*\} = \big(\tilde{k}_m(x,y)\big)^\alpha_\beta \qquad \text{and} \qquad \{\Psi^\alpha(x), \Psi^\beta(y)\} = 0 = \{\Psi^\alpha(x)^*, \Psi^\beta(y)^*\} \, ,$$

where the Greek indices running from $1, \ldots, 4$ denote Dirac spinor indices (we always work in natural units $\hbar = c = 1$). In the presence of a time-dependent external potential, there is no distinguished ground state. But there is common agreement that for a physically sensible theory the ground state $|0\rangle$ should be chosen such that the two-point function $\langle 0 | \, \Psi(x) \, \Psi(y)^* \, |0\rangle$ is a bi-distribution of Hadamard form. In this paper, we shall achieve this goal by arranging that the two-point function coincides with the kernel of the fermionic projector, i.e.

$$\langle 0 | \, \Psi^\alpha(x) \, \Psi^\beta(y)^* \, |0\rangle = -\big(P(x,y)\big)^\alpha_\beta \, .$$

In order to give the above formulas a mathematical meaning, one needs to "smear out" the field operators and work with operator-valued distributions (see for example [35, 7, 8, 5]). Formally, this is accomplished by setting

$$\Psi(g) = \int_{\mathscr{M}} \Psi(x)^\alpha \, g(x)_\alpha \, d^4x \qquad \text{and} \qquad \Psi^*(f) = \int_{\mathscr{M}} \big(\Psi(x)^\alpha\big)^* \, f(x)^\alpha \, d^4x \,,$$

where g and f are smooth and compactly supported co-spinors and spinors, respectively. We do not aim at defining the pointwise field operators, but instead we work exclusively with the smeared field operators $\Psi(g)$ and $\Psi^*(f)$ (for basic definitions see Section 6). Moreover, instead of considering vacuum expectation values, in the algebraic formulation of quantum field theory one prefers to work with a quasi-free state ω, making it unnecessary choose a representation of the field algebra on the Fock space. Given a state ω, a corresponding representation of the field algebra is obtained by applying the GNS construction, also making it possible to recover ω as a vacuum expectation value. A quasi-free state for which the two-point function is of Hadamard form (1.12) is called a *Hadamard state*.

Using the algebraic language, we prove the following result:

Theorem 1.4. *There is an algebra of smeared fields generated by $\Psi(g)$, $\Psi^*(f)$ together with a quasi-free state ω with the following properties:*

(a) The canonical anti-commutation relations hold:

$$\{\Psi(g), \Psi^*(f)\} = <g^* \,|\, \tilde{k}_m \, f> \,, \qquad \{\Psi(g), \Psi(g')\} = 0 = \{\Psi^*(f), \Psi^*(f')\} \,. \tag{1.15}$$

(b) The two-point function of the state is given by

$$\omega\big(\Psi(g)\, \Psi^*(f)\big) = -\iint_{\mathscr{M}\times\mathscr{M}} g(x) P(x,y) f(y) \, d^4x \, d^4y \,.$$

The main step in the proof is to use the spectral projection operators $\chi_{(-\infty,0)}(\tilde{S}_m)$ and $\chi_{[0,\infty)}(\tilde{S}_m)$ to construct a positive operator R, making it possible to apply Araki's results in [1] to obtain the desired quasi-free state.

We finally put our result into the context of other methods for constructing Hadamard states. First, there is the method of glueing the physical space-time to an ultrastatic space-time and using that the Hadamard property is preserved under time evolution (see [25, 24]). This method shows the existence of Hadamard states in every globally hyperbolic space-time and gives a constructive procedure for a class of Hadamard states. Another method is to work with pseudo-differential operators [26], again giving a whole class of Hadamard states. A method which distinguishes one specific Hadamard state using asymptotic symmetries at null infinity is given in [6]. Our method gives a *unique distinguished Hadamard state* even in the generic time-dependent setting in Minkowski space. Moreover, this method is constructive in the sense that the bi-distribution $P(x,y)$ and its Hadamard expansion can be computed explicitly (see [22, 13]). Our results exemplify that the construction of the fermionic projector in [21] is a promising method for constructing a distinguished Hadamard state without any symmetry assumptions, hopefully even in generic globally hyperbolic space-times.

2. Preliminaries

2.1. **Dirac Green's functions and the Time Evolution Operator.** Let $\mathscr{M}$ be Minkowski space, a four-dimensional real vector space endowed with an inner product of signature $(+ - - -)$. The Dirac equation in the Minkowski vacuum (i.e. without external potential) reads

$$(i\partial\!\!\!/ - m)\,\psi(x) = 0\,,$$

where we use the slash notation with the Feynman dagger $\partial\!\!\!/ := \gamma^j \partial_j$. We always work with the Dirac matrices in the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{\mathbb{C}^2} & 0 \\ 0 & -\mathbb{1}_{\mathbb{C}^2} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

(and $\vec{\sigma}$ are the three Pauli matrices). The wave functions at a space-time point x take values in the *spinor space* S_x, a four-dimensional complex vector space endowed with an indefinite scalar product of signature $(2,2)$, which we call *spin scalar product* and denote by

$$\prec\!\psi|\phi\!\succ_x = \sum_{\alpha=1}^4 s_\alpha\, \psi^\alpha(x)^\dagger \phi^\alpha(x)\,, \qquad s_1 = s_2 = 1,\;\; s_3 = s_4 = -1\,,$$

where $\psi^\dagger$ is the complex conjugate wave function (this scalar product is often written as $\overline{\psi}\phi$ with the so-called adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$). We denote the space of smooth wave functions by $C^\infty(\mathscr{M}, S\mathscr{M})$, whereas $C^\infty_0(\mathscr{M}, S\mathscr{M})$ denotes the smooth and compactly supported wave functions (here $S\mathscr{M}$ is the spinor bundle over Minkowski space with fibers $S_x\mathscr{M}$). On the spaces of wave functions, one can introduce a Lorentz-invariant pairing by integrating the spin scalar product over space-time,

$$\begin{aligned} <.|.> \;:\; C^\infty(\mathscr{M}, S\mathscr{M}) \times C^\infty_0(\mathscr{M}, S\mathscr{M}) \rightarrow \mathbb{C}\,, \\ <\psi|\phi> = \int_{\mathscr{M}} \prec\!\psi|\phi\!\succ_x d^4x\,. \end{aligned} \tag{2.1}$$

In what follows, the mass parameter of the Dirac equation m will not be fixed. It can vary in an open interval $I := (m_L, m_R)$ with $m_L, m_R > 0$. In order to make this dependence explicit, we often add the mass as an index. Moreover, we consider an *external potential* $\mathcal{B}$, which we assume to be symmetric with respect to the spin scalar product,

$$\prec\!\mathcal{B}(x)\,\psi\,|\,\phi\!\succ_x = \prec\!\psi\,|\,\mathcal{B}(x)\,\phi\!\succ_x \qquad \text{for all } x \in \mathscr{M} \text{ and } \psi, \phi \in S_x. \tag{2.2}$$

Then the Dirac equation becomes

$$(\mathcal{D} - m)\,\psi_m = 0 \qquad \text{with} \qquad \mathcal{D} := i\partial\!\!\!/ + \mathcal{B}\,. \tag{2.3}$$

Since the Dirac equation is linear and hyperbolic (meaning that it can be rewritten as a symmetric hyperbolic system), its Cauchy problem for smooth initial data is well-posed, giving rise to global smooth solutions. Moreover, due to finite propagation speed, starting with compactly supported initial data, we obtain solutions which are spatially compact at any time. We denote the space of such smooth wave functions with spatially compact support by $C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$. Using the symmetry assumption (2.2), for any solutions $\psi_m, \phi_m \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$ of the Dirac equation the vector field $\prec\!\psi_m|\gamma^j\phi_m\!\succ$ is divergence-free; this is referred to as *current conservation*. Applying Gauss' divergence theorem, this implies that the spatial integral

$$(\psi_m|\phi_m)_m\big|_t := 2\pi \int_{\mathbb{R}^3} \prec\!\psi_m|\gamma^0\phi_m\!\succ\!|_{(t,\vec{x})}\, d^3x \tag{2.4}$$

is independent of the choice of the space-like hypersurface labelled by the time parameter t. This integral defines a scalar product on the solution space corresponding to the mass m. Forming the completion, we obtain a Hilbert space, which we denote by $(\mathcal{H}_m, (.|.)_m)$. The norm on $\mathcal{H}_m$ is $\|.\|_m$.

The *retarded* and *advanced Green's operators* $\tilde{s}_m^\wedge$ and $\tilde{s}_m^\vee$ are mappings (for details see for example [2])

$$\tilde{s}_m^\wedge, \tilde{s}_m^\vee \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \rightarrow C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \,.$$

Their difference is the so-called causal fundamental solution $\tilde{k}_m$,

$$\tilde{k}_m := \frac{1}{2\pi i} \left(\tilde{s}_m^\vee - \tilde{s}_m^\wedge \right) \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \rightarrow C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m \,. \tag{2.5}$$

These operators can be represented as integral operators with a distributional kernel, for example,

$$(\tilde{k}_m \phi)(x) = \int_{\mathscr{M}} \tilde{k}_m(x,y)\, \phi(y)\, d^4y \,.$$

Leaving out the tilde always refers to the special case $\mathcal{B} \equiv 0$.

The operator $\tilde{k}_m$ can be used for constructing a solution of the Cauchy problem. To this end, we always work in the foliation $\mathscr{N}_t = \{(t, \vec{x}) \,|\, \vec{x} \in \mathbb{R}^3\}$ of constant time Cauchy hypersurfaces in a fixed reference frame $(t, \vec{x})$. For clarity, we denote the Hilbert space of square integrable spinors at time t with the scalar product (2.4) by $(\mathcal{H}_t, (.|.)|_t)$. Moreover, we denote a wave function ψ at time t by $\psi|_t$ (we use this notation both for the restriction of a wave function in space-time and for a function defined only on the hyperplane $\mathscr{N}_t$).

Proposition 2.1. *The solution of the Cauchy problem*

$$(\mathcal{D} - m)\, \psi_m = 0 \,, \qquad \psi_m\big|_{t_0} = \psi_0 \in C^\infty(\mathscr{N}_{t_0} \simeq \mathbb{R}^3, S\mathscr{M}) \tag{2.6}$$

has the representation

$$\psi_m(x) = 2\pi \int_{\mathscr{N}_{t_0}} \tilde{k}_m\big(x, (t_0, \vec{y})\big)\, \gamma^0\, \psi_0(\vec{y})\, d^3y \,.$$

For the proof see for example [20, Section 2].

Moreover, the operator $\tilde{k}_m$ can be regarded as the signature operator of the inner product (2.1) when expressed in terms of the scalar product (2.4).

Proposition 2.2. *For any $\psi_m \in \mathcal{H}_m$ and $\phi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,*

$$(\psi_m \,|\, \tilde{k}_m \phi)_m = <\!\psi_m | \phi\!> . \tag{2.7}$$

For the proof we refer to [8, Proposition 2.2] or [20, Section 3.1].

The unique solvability of the Cauchy problem allows us to introduce the group of *time evolution operators* as follows. According to Proposition 2.1, for given initial data $\psi_0 \in C_0^\infty(\mathscr{N}_{t_0}, S\mathscr{M})$, the Cauchy problem (2.6) has a unique solution $\psi_m \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$. Evaluating this solution at some other time t gives a mapping $\tilde{U}_m^{t,t_0} : \psi_0 \mapsto \psi_m|_t$. Since the scalar product (2.4) is time independent, the time evolution operator $\tilde{U}_m^{t,t_0}$ is isometric. Thus by continuity, it extends uniquely to an isometry

$$\tilde{U}_m^{t,t_0} \;:\; \mathcal{H}_{t_0} \rightarrow \mathcal{H}_t \,.$$

Since t_0 can be chosen arbitrarily and the Cauchy problem can be solved forward and backward in time, this isometry is even a unitary operator. Moreover, these operators are a representation of the group $(\mathbb{R}, +)$, meaning that

$$\tilde{U}_m^{t,t} = \mathbb{1} \qquad \text{and} \qquad \tilde{U}_m^{t'',t'}\, \tilde{U}_m^{t',t} = \tilde{U}_m^{t'',t} \,.$$

Proposition 2.1 immediately yields the following representation of $\tilde{U}_m^{t',t}$ with integral kernel,

$$\big(\tilde{U}_m^{t',t}\,\psi|_t\big)(\vec{y}) = \int_{\mathbb{R}^3} \tilde{U}_m^{t',t}(\vec{y},\vec{x})\,\psi|_t(\vec{x})\,d^3x\,, \tag{2.8}$$

$$\tilde{U}_m^{t',t}(\vec{y},\vec{x}) = 2\pi\,\tilde{k}_m\big((t',\vec{y}),(t,\vec{x})\big)\,\gamma^0\,. \tag{2.9}$$

2.2. **The Mass Oscillation Property.** We denote the families of smooth wave functions with spatially compact support, which are also compactly supported in I, by $C^\infty_{\mathrm{sc},0}(\mathscr{M}\times I, S\mathscr{M})$. The space of families of Dirac solutions within this class are denoted by $\mathcal{H}^\infty$. On $\mathcal{H}^\infty$ we introduce the scalar product

$$(\psi|\phi) = \int_I (\psi_m|\phi_m)_m\,dm\,, \tag{2.10}$$

where dm is the Lebesgue measure (and $\psi=(\psi_m)_{m\in I}$ and $\phi=(\phi_m)_{m\in I}$ are families of Dirac solutions for a variable mass parameter). Forming the completion yields the Hilbert space $(\mathcal{H},(.|.))$ with norm $\|.\|$. Then $\mathcal{H}^\infty$ can be regarded as the subspace

$$\mathcal{H}^\infty = C^\infty_{\mathrm{sc},0}(\mathscr{M}\times I, S\mathscr{M})\cap\mathcal{H}\,. \tag{2.11}$$

On $\mathcal{H}$, we introduce the operator of multiplication by m,

$$T\;:\;\mathcal{H}\to\mathcal{H}\,,\qquad (T\psi)_m = m\,\psi_m\,.$$

Obviously, this operator preserves the support properties, and thus

$$T|_{\mathcal{H}^\infty}\;:\;\mathcal{H}^\infty\to\mathcal{H}^\infty\,.$$

Moreover, it is a symmetric operator, and it is bounded because the interval I is, i.e.

$$T^* = T \in \mathrm{L}(\mathcal{H})\,.$$

Integration of ψ_m over m gives another operator

$$\mathfrak{p}\;:\;\mathcal{H}^\infty\to C^\infty_{\mathrm{sc}}(\mathscr{M},S\mathscr{M})\,,\qquad \mathfrak{p}\psi = \int_I \psi_m\,dm\,. \tag{2.12}$$

We point out for clarity that $\mathfrak{p}\psi$ no longer satisfies a Dirac equation. The following notions were introduced in [21], and we refer the reader to this paper for more details.

Definition 2.3. *The Dirac operator $\mathcal{D}=i\partial\!\!\!/+\mathcal{B}$ on Minkowski space $\mathscr{M}$ has the* **weak mass oscillation property** *in the interval $I=(m_L,m_R)$ with domain $\mathcal{H}^\infty$ if the following conditions hold:*

(a) *For every $\psi,\phi\in\mathcal{H}^\infty$, the function $\prec\!\mathfrak{p}\phi|\mathfrak{p}\psi\!\succ$ is integrable on $\mathscr{M}$. Moreover, there is a constant $c=c(\psi)$ such that*

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \le c\,\|\phi\|\qquad\text{for all }\phi\in\mathcal{H}^\infty\,. \tag{2.13}$$

(b) *For all $\psi,\phi\in\mathcal{H}^\infty$,*

$${<}\mathfrak{p}T\psi|\mathfrak{p}\phi{>} = {<}\mathfrak{p}\psi|\mathfrak{p}T\phi{>}\,. \tag{2.14}$$

Definition 2.4. *The Dirac operator $\mathcal{D}=i\partial\!\!\!/+\mathcal{B}$ on Minkowski space $\mathscr{M}$ has the* **strong mass oscillation property** *in the interval $I=(m_L,m_R)$ with domain $\mathcal{H}^\infty$ if there is a constant $c>0$ such that*

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \le c\int_I \|\phi_m\|_m\,\|\psi_m\|_m\,dm\qquad\text{for all }\psi,\phi\in\mathcal{H}^\infty\,. \tag{2.15}$$

The following theorem is proved in [21, Theorem 4.2, Proposition 4.3 and Theorem 4.7].

Theorem 2.5. *Assume that the Dirac operator $\mathcal{D}$ has the strong mass oscillation property in the interval $I = (m_L, m_R)$. Then there exists a family of linear operators $(\tilde{\mathcal{S}}_m)_{m \in I}$ with $\tilde{\mathcal{S}}_m \in \mathrm{L}(\mathcal{H}_m)$ which are uniformly bounded,*

$$\sup_{m \in I} \|\tilde{\mathcal{S}}_m\| < \infty \,,$$

such that

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \, \phi_m)_m \, dm \qquad \text{for all } \psi, \phi \in \mathcal{H}^\infty \,. \tag{2.16}$$

The operator $\tilde{\mathcal{S}}_m$ is uniquely determined for every $m \in I$ by demanding that for all $\psi, \phi \in \mathcal{H}^\infty$, the functions $(\psi_m|\tilde{\mathcal{S}}_m \phi_m)_m$ are continuous in m. Moreover, the operator $\tilde{\mathcal{S}}_m$ is the same for all choices of I containing m. Finally, there is a bi-distribution $\mathcal{P} \in \mathfrak{D}'(\mathscr{M} \times \mathscr{M})$ such that the operator P defined by

$$P := -\chi_{(-\infty,0)}(\tilde{\mathcal{S}}_m) \, \tilde{k}_m \;:\; C^\infty_0(\mathscr{M}, S\mathscr{M}) \rightarrow \mathcal{H}_m \tag{2.17}$$

has the representation

$$<\phi|P\psi> = \mathcal{P}(\overline{\phi} \otimes \psi) \qquad \text{for all } \phi, \psi \in C^\infty_0(\mathscr{M}, S\mathscr{M}) \tag{2.18}$$

(where $\overline{\phi} = \phi^\dagger \gamma^0$ is the usual adjoint spinor).

The operator P is referred to as the **fermionic projector**. We also use the standard notation with an integral kernel $P(x,y)$,

$$<\phi|P\psi> = \iint_{\mathscr{M} \times \mathscr{M}} \prec \phi(x) \,|\, P(x,y)\, \psi(y) \succ_x d^4x \, d^4y$$

$$(P\psi)(x) = \int_{\mathscr{M}} P(x,y)\, \psi(y)\, d^4y \,,$$

where $P(.,.)$ coincides with the distribution $\mathcal{P}$ in (2.18).

2.3. **The Lippmann-Schwinger Equation.** The Dirac dynamics can be rewritten in terms of a symmetric operator $\tilde{H}$. To this end, we multiply the Dirac equation (2.3) by γ^0 and bring the t-derivative separately on one side of the equation,

$$i\partial_t \psi_m = \tilde{H} \psi_m \,, \qquad \text{where} \qquad \tilde{H} := -\gamma^0 (i\vec{\gamma}\vec{\nabla} + \mathcal{B} - m) \tag{2.19}$$

(note that $\gamma^j \partial_j = \gamma^0 \partial_t + \vec{\gamma}\vec{\nabla}$). We refer to (2.19) as the Dirac equation in *Hamiltonian form.* The fact that the scalar product (2.4) is time independent implies that for any two solutions $\phi_m, \psi_m \in C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$,

$$0 = \partial_t (\phi_m \,|\, \psi_m)_m \;= i \big((\tilde{H}\phi_m \,|\, \psi_m)_m - (\phi_m \,|\, \tilde{H}\psi_m)_m \big) \,,$$

showing that the Hamiltonian is a symmetric operator on $\mathcal{H}_m$. The Lippmann-Schwinger equation can be used to compare the dynamics in the Minkowski vacuum with the dynamics in the presence of an external potential. We denote the time evolution operator in the Minkowski vacuum by U^{t,t_0}_m.

Proposition 2.6. *The Cauchy problem* (2.6) *has a solution ψ_m which satisfies the equation*

$$\psi_m|_t = U^{t,t_0}_m \psi_0 + i \int_{t_0}^t U^{t,\tau}_m \left(\gamma^0 \mathcal{B}\, \psi_m \right)\big|_\tau \, d\tau \,, \tag{2.20}$$

referred to as the **Lippmann-Schwinger equation***.*

Proof. Obviously, the wave function $\psi_m|_t$ given by (2.20) has the correct initial condition at $t = t_0$. Thus it remains to show that $\psi_m|_t$ satisfies the Dirac equation. To this end, we rewrite the Dirac equation in the Hamiltonian form (2.19), and separate the vacuum Hamiltonian H from the term involving the external potential,

$$(i\partial_t - H)\,\psi_m = -\gamma^0 \mathcal{B}\,\psi_m \qquad \text{with} \qquad H = -i\gamma^0 \vec{\gamma}\vec{\nabla} + \gamma^0 m\,. \tag{2.21}$$

Applying the operator $i\partial_t - H$ to (2.20), and observing that the time evolution operator maps to solutions of the vacuum Dirac equation, only the derivative of the upper limit of integration contributes,

$$(i\partial_t - H)\,\psi_m|_t = -U_m^{t,\tau}\, \left(\gamma^0 \mathcal{B}\,\psi_m\right)\big|_{\tau=t} = -\gamma^0 \mathcal{B}\,\psi_m|_t\,,$$

so that (2.21) is indeed satisfied. □

3. The Mass Oscillation Property in the Minkowski Vacuum

Since Minkowski space is ultrastatic, it is known from [21, Section 5] that the Dirac operator $i\partial\!\!\!/$ satisfies the weak and strong mass oscillation properties. Moreover, the decomposition of the solution space into the positive and negative spectral subspaces of the fermionic signature operator reduces to the usual frequency splitting (see [21, Theorem 5.1]). We now reproduce these results giving more explicit proofs. These explicit results and formulas will be essential for the subsequent treatment of time-dependent external potentials in Section 4.

Basically, the mass oscillation property in the Minkowski vacuum can be proved easily using Fourier methods. Here we shall give two different approaches in detail. The method of the first proof (Section 3.1) is instructive because it gives an intuitive understanding of "mass oscillations". However, this method only yields the weak mass oscillation property. The second proof (Section 3.2) is more abstract but also gives the strong mass oscillation property.

3.1. Proof of the Weak Mass Oscillation Property using Mass Derivatives.

We again consider the foliation $\mathcal{N}_t = \{(t,\vec{x}) \,|\, \vec{x} \in \mathbb{R}^3\}$ of constant time Cauchy hypersurfaces in a fixed reference frame $(t,\vec{x})$ and a variable mass parameter m in the interval $I = (m_L, m_R)$ with $m_L, m_R > 0$. The families of solutions $\psi = (\psi_m)_{m\in I}$ of the Dirac equations $(i\partial\!\!\!/ - m)\psi_m = 0$ are contained in the Hilbert space $(\mathcal{H}, (.|.))$ with the scalar product (2.10). Moreover, the subspace $\mathcal{H}^\infty \subset \mathcal{H}$ is given by (2.11).

For what follows, it is convenient to work with the Fourier transform in space, i.e.

$$\hat{\psi}(t,\vec{k}) = \int_{\mathbb{R}^3} \psi(t,\vec{x})\, e^{-i\vec{k}\vec{x}}\, d^3x\,, \qquad \psi(t,\vec{x}) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3}\, \hat{\psi}(t,\vec{k})\, e^{i\vec{k}\vec{x}}\,.$$

Then a family of solutions $\psi \in \mathcal{H}^\infty$ has the representation

$$\hat{\psi}_m(t,\vec{k}) = c_+(\vec{k},m)\, e^{-i\omega(\vec{k},m)\,t} + c_-(\vec{k},m)\, e^{i\omega(\vec{k},m)\,t} \qquad \text{for all } m \in I \tag{3.1}$$

with suitable spinor-valued coefficients $c_\pm(\vec{k},m)$ and $\omega(\vec{k},m) := \sqrt{|\vec{k}|^2 + m^2}$. Integrating over the mass parameter, we obtain a superposition of waves oscillating at different frequencies. Intuitively speaking, this leads to destructive interference for large t, giving rise to decay in time. This picture can be made precise using integration by parts in m, as we now explain. Integrating (3.1) over the mass and applying the operator $\mathfrak{p}$, (2.12), we obtain

$$\begin{aligned}
\mathfrak{p}\hat{\psi}(t,\vec{k}) &= \int_I \big(c_+ \, e^{-i\omega t} + c_- \, e^{i\omega t} \big)\, dm \\
&= \int_I \frac{i}{t\, \partial_m \omega} \big(c_+ \, \partial_m e^{-i\omega t} - c_- \, \partial_m e^{i\omega t} \big)\, dm \\
&= -\frac{i}{t} \int_I \bigg[\partial_m \Big(\frac{c_+}{\partial_m \omega} \Big)\, e^{-i\omega t} - \partial_m \Big(\frac{c_-}{\partial_m \omega} \Big)\, e^{i\omega t} \bigg]\, dm
\end{aligned}$$

(we do not get boundary terms because $\psi \in \mathcal{H}^\infty$ has compact support in m). With $\partial_m \omega = m/\omega$, we conclude that

$$\mathfrak{p}\hat{\psi}(t,\vec{k}) = -\frac{i}{t} \int_I \bigg[\partial_m \Big(\frac{\omega\, c_+}{m} \Big)\, e^{-i\omega t} - \partial_m \Big(\frac{\omega\, c_-}{m} \Big)\, e^{i\omega t} \bigg]\, dm \,.$$

Since the coefficients $c_\pm$ depend smoothly on m, the resulting integrand is bounded uniformly in time, giving a decay at least like $1/t$, i.e. $|\mathfrak{p}\hat{\psi}(t,\vec{k})| \lesssim 1/t$. Iterating this procedure, one even can prove decay rates $\lesssim 1/t^2, 1/t^3, \ldots$ The price one pays is that higher and higher powers in ω come up in the integrand, which means that in order for the spatial Fourier integral to exist, one needs a faster decay of $c_\pm$ in $|\vec{k}|$. Expressed in terms of the initial data, this means that every factor $1/t$ gives rise to an additional spatial derivative acting on the initial data. This motivates the following basic estimate.

Lemma 3.1. *For any $\psi \in \mathcal{H}^\infty$, there is a constant $C = C(m_L)$ such that*

$$\big\| (\mathfrak{p}\psi)|_t \big\|_t \leq \frac{C\,|I|}{1+t^2} \sup_{m \in I} \sum_{b=0}^{2} \big\| (\partial_m^b \psi_m)|_{t=0} \big\|_{W^{2,2}} \,, \tag{3.2}$$

where $\|.\|_t$ is the norm corresponding to the scalar product

$$(.|.)|_t := 2\pi \int_{\mathbb{R}^3} \prec . | \gamma^0 \, . \succ_{\vec{x}} \, d^3x \;:\; L^2(\mathcal{N}_t, S\mathcal{M}) \times L^2(\mathcal{N}_t, S\mathcal{M}) \rightarrow \mathbb{C}$$

(which is similar to (2.4)*, but now applied to wave functions which do not need to be solutions), and $\|.\|_{W^{2,2}}$ is the spatial Sobolev norm*

$$\|\phi\|^2_{W^{2,2}} := \sum_{\alpha \text{ with } |\alpha| \leq 2} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 \, d^3x \,, \tag{3.3}$$

where α is a multi-index.

The absolute value in (3.3) is the norm $|\,.\,| := \sqrt{\prec . | \gamma^0 . \succ}$ on the spinors. If we again identify all spinor spaces in the Dirac representation with $\mathbb{C}^4$, this simply is the standard Euclidean norm on $\mathbb{C}^4$.

The proof of this lemma will be given later in this section. Before, we infer the weak mass oscillation property.

Corollary 3.2. *The vacuum Dirac operator $i\partial\!\!\!/$ in Minkowski space has the weak mass oscillation property with domain* (2.11)*.*

Proof. For every $\psi, \phi \in \mathcal{H}^\infty$, the Schwarz inequality gives

$$|{<}\mathfrak{p}\psi | \mathfrak{p}\phi{>}| = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \big((\mathfrak{p}\psi)|_t \,\big|\, \gamma^0 \, (\mathfrak{p}\phi)|_t \big)_t \, dt \right| \leq \int_{-\infty}^{\infty} \big\| (\mathfrak{p}\psi)|_t \big\|_t \, \big\| (\mathfrak{p}\phi)|_t \big\|_t \, dt \,. \tag{3.4}$$

Applying Lemma 3.1 together with the estimate

$$\begin{aligned}\big\|(\mathfrak{p}\phi)|_t\big\|_t^2 &= \iint_{I\times I} \big(\phi_m|_t \,\big|\, \phi_{m'}|_t\big)_t \, dm\, dm' \\ &\leq \frac{1}{2} \iint_{I\times I} \Big(\|\phi_m\|_m^2 + \|\phi_{m'}\|_{m'}^2\Big)\, dm\, dm' = |I|\, \|\phi\|^2 \,,\end{aligned}$$

we obtain inequality (2.13) with

$$c = C\,|I|^{\frac{3}{2}} \sup_{m\in I} \sum_{b=0}^{2} \|\partial_m^b(\psi_m)|_{t=0}\|_{W^{2,2}} \int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt < \infty \,. \tag{3.5}$$

The identity (2.14) follows by integrating the Dirac operator by parts,

$$\begin{aligned} <\mathfrak{p}T\psi|\mathfrak{p}\phi> &= <\mathfrak{p}\mathcal{D}\psi|\mathfrak{p}\phi> = <\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_{\mathscr{M}} \prec\!\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi\!\succ_x d^4x \\ &\overset{(\star)}{=} \int_{\mathscr{M}} \prec\!\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi\!\succ_x d^4x = <\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi> \,. \end{aligned} \tag{3.6}$$

In $(\star)$, we used that the Dirac operator is formally self-adjoint with respect to the inner product $<.|.>$. Moreover, we do not get boundary terms because of the time decay in Lemma 3.1. □

The remainder of this section is devoted to the proof of Lemma 3.1. Specializing the result of Proposition 2.1 to the Minkowski vacuum, we can express the solution ψ_m of the Cauchy problem in terms of the causal fundamental solution k_m. In order to bring k_m into a more explicit form, we use (2.5) together with formulas for the advanced and retarded Green's functions. Indeed, these Green's functions are the multiplication operators in momentum space

$$s_m^\vee(k) = \lim_{\varepsilon\searrow 0} \frac{\slashed{k}+m}{k^2-m^2-i\varepsilon k^0} \qquad \text{and} \qquad s_m^\wedge(k) = \lim_{\varepsilon\searrow 0} \frac{\slashed{k}+m}{k^2-m^2+i\varepsilon k^0}$$

(with the limit $\varepsilon\searrow 0$ taken in the distributional sense, and where the vector k is the four-momentum). We thus obtain in momentum space

$$\begin{aligned} k_m(p) &= \frac{1}{2\pi i}\,(\slashed{p}+m)\, \lim_{\varepsilon\searrow 0}\left[\frac{1}{p^2-m^2-i\varepsilon p^0} - \frac{1}{p^2-m^2+i\varepsilon p^0}\right] \\ &= \frac{1}{2\pi i}\,(\slashed{p}+m)\, \lim_{\varepsilon\searrow 0}\left[\frac{1}{p^2-m^2-i\varepsilon} - \frac{1}{p^2-m^2+i\varepsilon}\right] \epsilon(p^0) \end{aligned}$$

(where ϵ denotes the step function, and for notational clarity we denoted the momentum variables by p). Employing the distributional equation

$$\lim_{\varepsilon\searrow 0}\left(\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}\right) = 2\pi i\, \delta(x)\,,$$

we obtain the simple formula

$$k_m(p) = (\slashed{p}+m)\, \delta(p^2-m^2)\, \epsilon(p^0)\,. \tag{3.7}$$

It is convenient to transform spatial coordinates of the time evolution operator to momentum space. First, in the Minkowski vacuum, the time evolution operator can be represented as in (2.8) with an integral kernel $U^{t,t'}(\vec{y},\vec{x})$ which depends only on the difference vector $\vec{y}-\vec{x}$. We set

$$U^{t,t'}(\vec{k}) := \int_{\mathbb{R}^3} U^{t,t'}(\vec{y},0)\, e^{-i\vec{k}\vec{y}}\, d^3y\,.$$

Combining (2.9) with (3.7) yields

$$U^{t,t'}(\vec{k}) = \int_{-\infty}^{\infty} (\not{k}+m)\,\gamma^0\, \delta(k^2-m^2)\big|_{k=(\omega,\vec{k})}\, \epsilon(\omega)\, e^{-i\omega(t-t')}\, d\omega\,.$$

Carrying out the ω-integral, we get

$$U^{t,t'}(\vec{k}) = \sum_{\pm} \Pi_\pm(\vec{k})\, e^{\mp i\omega(t-t')}\,, \tag{3.8}$$

where we set

$$\Pi_\pm(\vec{k}) := \pm\frac{1}{2\omega(\vec{k})}\, (\not{k}_\pm + m)\,\gamma^0 \tag{3.9}$$

$$\text{with} \qquad \omega(\vec{k}) = \sqrt{|\vec{k}|^2+m^2} \qquad \text{and} \qquad k_\pm = (\pm\omega(\vec{k}), \vec{k})\,.$$

Moreover, applying Plancherel's theorem, the scalar product (2.4) can be written in momentum space as

$$(\psi_m \,|\, \phi_m)_m = (2\pi)^{-2} \int_{\mathbb{R}^3} \prec\!\hat{\psi}_m(t,\vec{k})\,|\,\gamma^0\, \hat{\phi}_m(t,\vec{k})\!\succ d^3k\,.$$

The unitarity of the time evolution operator in position space implies that the matrix $U^{t,t'}(\vec{k})$ is unitary (with respect to the scalar product $\langle .\,,.\rangle_{\mathbb{C}^2} \equiv \prec .\,|\gamma^0\,.\succ$), meaning that its eigenvalues are on the unit circle and the corresponding eigenspaces are orthogonal. It follows that the operators $\Pi_\pm(\vec{k})$ in (3.8) are the orthogonal projection operators to the eigenspaces corresponding to the eigenvalues $e^{\mp i\omega(t-t')}$, i.e.

$$\gamma^0 \Pi_s^* \gamma^0 = \Pi_s \quad \text{and} \quad \Pi_s(\vec{k})\,\Pi_{s'}(\vec{k}) = \delta_{s,s'}\,\Pi_s(\vec{k}) \qquad \text{for } s,s' \in \{+,-\}\,.$$

(these relations can also be verified by straightforward computations using (3.9)).

The next two lemmas involve derivatives with respect to the mass parameter m. For clarity, we again denote the m-dependence of the operators by the subscript m.

Lemma 3.3. *The time evolution operator in the vacuum satisfies the relation*

$$(t-t')\, U_m^{t,t'}(\vec{k}) = \frac{\partial}{\partial m} V_m^{t,t'}(\vec{k}) + W_m^{t,t'}(\vec{k})\,, \tag{3.10}$$

where

$$V_m^{t,t'}(\vec{k}) = \sum_\pm \frac{i}{2m}\, (\not{k}_\pm + m)\gamma^0\, e^{\mp i\omega(t-t')} \tag{3.11}$$

$$W_m^{t,t'}(\vec{k}) = \sum_\pm \frac{i}{2} \Big(\frac{\not{k}_\pm \gamma^0}{m^2} \mp \frac{1}{\omega} \Big)\, e^{\mp i\omega(t-t')}\,. \tag{3.12}$$

The operators $V_m^{t,t'}$ and $W_m^{t,t'}$ are estimated uniformly by

$$\|V_m^{t,t'}(\vec{k})\| + \|W_m^{t,t'}(\vec{k})\| \leq C \left(1 + \frac{|\vec{k}|}{m}\right), \tag{3.13}$$

where the constant C is independent of m, $\vec{k}$, t and t' (and $\|\,.\,\|$ is any norm on the 2×2-matrices).

Proof. First, we generate the factor $t-t'$ by differentiating the exponential in (3.8) with respect to ω,

$$(t-t')\,U_m^{t,t'}(\vec{k}) = \sum_\pm \Pi_\pm(\vec{k})\Big(\pm i\frac{\partial}{\partial\omega}\,e^{\mp i\omega(t-t')}\Big).$$

Next, we want to rewrite the ω-derivative as a derivative with respect to m. Taking the total differential of the dispersion relation $\omega^2-|\vec{k}|^2=m^2$ for fixed $\vec{k}$, one finds that

$$\frac{\partial}{\partial\omega} = \frac{\omega}{m}\,\frac{\partial}{\partial m}\,. \tag{3.14}$$

Hence

$$\begin{aligned}(t-t')\,U_m^{t,t'} &= \sum_\pm \Pi_\pm\Big(\pm i\,\frac{\omega}{m}\frac{\partial}{\partial m}\,e^{\mp i\omega(t-t')}\Big)\\ &= \frac{\partial}{\partial m}\sum_\pm\Big(\pm i\,\frac{\omega}{m}\,\Pi_\pm\,e^{\mp i\omega(t-t')}\Big) - \sum_\pm\bigg(\frac{\partial}{\partial m}\Big[\pm i\,\frac{\omega}{m}\,\Pi_\pm\Big]\bigg)\,e^{\mp i\omega(t-t')}\,.\end{aligned}$$

Computing the operators in the round brackets using (3.9) gives the identities (3.11) and (3.12). Estimating these formulas, one obtains bounds which are at most linear in $|\vec{k}|$, proving (3.13). ☐

This method can be iterated to generate more factors of $t-t'$. In the next lemma, we prove at least quadratic decay in time. For later use, it is preferable to formulate the result in position space.

Lemma 3.4. *The time evolution operator in the vacuum has the representation*

$$U_m^{t,t'} = \frac{1}{(t-t')^2}\left(\frac{\partial^2}{\partial m^2}A_m^{t,t'} + \frac{\partial}{\partial m}B_m^{t,t'} + C_m^{t,t'}\right) \tag{3.15}$$

with operators

$$A_m^{t,t'}, B_m^{t,t'}, C_m^{t,t'} \;:\; W^{2,2}(\mathscr{N}_{t'}, S\mathscr{M}) \to L^2(\mathscr{N}_t, S\mathscr{M})\,,$$

which are bounded uniformly in time by

$$\|A_m^{t,t'}(\phi)\|_t + \|B_m^{t,t'}(\phi)\|_t + \|C_m^{t,t'}(\phi)\|_t \leq c\,\|\phi\|_{W^{2,2}}\,, \tag{3.16}$$

where c is a constant which depends only on m.

Proof. A straightforward computation using exactly the same methods as in Lemma 3.3 yields the representation

$$(t-t')^2\,U_m^{t,t'}(\vec{k}) = \frac{\partial^2}{\partial m^2}A_m^{t,t'}(\vec{k}) + \frac{\partial}{\partial m}B_m^{t,t'}(\vec{k}) + C_m^{t,t'}(\vec{k})\,, \tag{3.17}$$

where the operators $A_m^{t,t'}$, $B_m^{t,t'}$ and $C_m^{t,t'}$ are bounded by

$$\|A_m^{t,t'}(\vec{k})\| + \|B_m^{t,t'}(\vec{k})\| + \|C_m^{t,t'}(\vec{k})\| \leq \frac{C}{m}\left(1+\frac{|\vec{k}|}{m}+\frac{|\vec{k}|^2}{m^2}\right), \tag{3.18}$$

with a numerical constant $C>0$. We remark that, compared to (3.10), the right of (3.18) involves an additional $1/m$. This prefactor is necessary for dimensional reasons, because the additional factor $t-t'$ in (3.17) (compared to (3.10)) brings in an additional dimension of length (and in natural units, the factor $1/m$ also has the dimension of length). The additional summand $|\vec{k}|^2/m^2$ in (3.18) can be understood from the fact that applying (3.14) generates a factor of ω/m which for large $|\vec{k}|$ scales like $|\vec{k}|/m$.

Translating this result to position space and keeping in mind that the vector $\vec{k}$ corresponds to the derivative $-i\vec{\nabla}$, we obtain the result. ☐

Proof of Lemma 3.1. First of all, the Schwarz inequality gives

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \int_I \|\psi_m\|_m \, dm \leq \sqrt{|I|} \; \|\psi\| \,.$$

Thus it remains to show the decay for large t, i.e.

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{C\,|I|}{t^2} \sup_{m \in I} \sum_{b=0}^{2} \|\partial_m^b (\psi_m)|_{t=0}\|_{W^{2,2}} \,. \tag{3.19}$$

We apply Lemma 3.4 and integrate by parts in m to obtain

$$\begin{aligned} (\mathfrak{p}\psi)|_t &= \int_I U_m^{t,0}\, \psi_m|_{t=0}\, dm = \frac{1}{t^2} \int_I \left(\partial_m^2 A_m^{t,0} + \partial_m B_m^{t,0} + C_m^{t,0}\right) \psi_m|_{t=0}\, dm \\ &= \frac{1}{t^2} \int_I \Big(A_m^{t,0}\, (\partial_m^2 \psi_m|_{t=0}) - B_m^{t,0}\, (\partial_m \psi_m|_{t=0}) + C_m^{t,0}\, \psi_m|_{t=0} \Big)\, dm \,. \end{aligned}$$

Taking the norm and using (3.16) gives (3.19). ☐

We finally note that the previous estimates are not optimal for two reasons. First, the pointwise quadratic decay in (3.2) is more than what is needed for the convergence of the integral in (3.5). Second and more importantly, the Schwarz inequality (3.4) does not catch the optimal scaling behavior in $\vec{k}$. This is the reason why the constant in (2.13) involves derivatives of ψ_m (cf. (3.5)), making it impossible to prove inequality (2.15) which arises in the strong mass oscillation property. In order to improve the estimates, one needs to use Fourier methods both in space and time, as will be explained in the next section.

3.2. Proof of the Mass Oscillation Property using a Plancherel Method.

Theorem 3.5. *The vacuum Dirac operator in Minkowski space has the strong mass oscillation property with domain* (2.11).

Our proof relies on a Plancherel argument in space-time. It also provides an alternative method for establishing the weak mass oscillation property.

Proof of Theorem 3.5. Let $\psi = (\psi_m)_{m \in I} \in \mathcal{H}^\infty$ be a family of solutions of the Dirac equation for a varying mass parameter in the Minkowski vacuum. Using Proposition 2.1, one can express ψ_m in terms of its values at time $t=0$ by

$$\psi_m(x) = 2\pi \int_{\mathbb{R}^3} k_m(x, (0, \vec{y}))\, \gamma^0\, \psi_m|_{t=0}(\vec{y})\, d^3y \,.$$

We now take the Fourier transform, denoting the four-momentum by k. Using (3.7), we obtain

$$\begin{aligned} \psi_m(k) &= 2\pi k_m(k)\, \gamma^0 \hat{\psi}_m^0(\vec{k}) \\ &= 2\pi\, \delta(k^2 - m^2)\, \epsilon(k^0)\, (\slashed{k} + m)\, \gamma^0 \hat{\psi}_m^0(\vec{k}) \,, \end{aligned}$$

where $\hat{\psi}^0_m(\vec{k})$ denotes the spatial Fourier transform of $\psi_m|_{t=0}$. Obviously, this is a distribution supported on the mass shell. In particular, it is not square integrable over $\mathbb{R}^4$.

Integrating over m, we obtain the following function

$$(\mathfrak{p}\psi)(k) = 2\pi\, \chi_I(m)\, \frac{1}{2m}\, \epsilon(k^0)\, (\slashed{k}+m)\, \gamma^0 \hat{\psi}^0_m(\vec{k})\Big|_{m=\sqrt{k^2}}\,, \tag{3.20}$$

where m now is a function of the momentum variables. Since the function $\psi_m|_{t=0}$ is compactly supported and smooth in the spatial variables, its Fourier transform $\hat{\psi}^0_m(\vec{k})$ has rapid decay. This shows that the function (3.20) is indeed square integrable. Using Plancherel, we see that condition (a) in Definition 2.3 is satisfied. Moreover, the operator T is simply the operator of multiplication by $\sqrt{k^2}$, so that condition (b) obviously holds. This again shows the weak mass oscillation property.

In order to prove the strong mass oscillation property, we need to compute the inner product $<\!\mathfrak{p}\psi|\mathfrak{p}\phi\!>$. To this end, we first write this inner product in momentum space as

$$\begin{aligned} <\!\mathfrak{p}\psi|\mathfrak{p}\phi\!> &= \int \frac{d^4k}{(2\pi)^4} 4\pi^2\, \chi_I(m)\, \frac{1}{4m^2}\, \prec\!(\slashed{k}+m)\, \gamma^0 \hat{\psi}^0_m(\vec{k})\,|\, (\slashed{k}+m)\, \gamma^0 \hat{\phi}^0_m(\vec{k})\!\succ\Big|_{m=\sqrt{k^2}} \\ &= \int \frac{d^4k}{4\pi^2}\, \chi_I(m)\, \frac{1}{2m}\, \prec\!\gamma^0 \hat{\psi}^0_m(\vec{k})\,|\, (\slashed{k}+m)\, \gamma^0 \hat{\phi}^0_m(\vec{k})\!\succ\Big|_{m=\sqrt{k^2}}\,. \end{aligned}$$

Reparametrizing the k^0-integral as an integral over m, we obtain

$$<\!\mathfrak{p}\psi|\mathfrak{p}\phi\!> = \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \frac{d^3k}{2\,|k^0|}\, \prec\!\gamma^0 \hat{\psi}^0_m(\vec{k})\,|\, (\slashed{k}+m)\, \gamma^0 \hat{\phi}^0_m(\vec{k})\!\succ\big|_{k^0=\pm\sqrt{|\vec{k}|^2+m^2}}\,. \tag{3.21}$$

Estimating the inner product with the Schwarz inequality and applying Plancherel's theorem, one finds

$$|<\!\mathfrak{p}\psi|\mathfrak{p}\phi\!>| \leq \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \|\hat{\psi}^0_m(\vec{k})\|\, \|\hat{\phi}^0_m(\vec{k})\|\, d^3k \leq 2\pi \int_I \|\psi_m\|_m\, \|\phi_m\|_m\, dm\,.$$

Thus the inequality (2.15) holds. □

4. The Mass Oscillation Property in Minkowski Space with External Potential

4.1. Proof of the Weak Mass Oscillation Property.

In this section, we prove the following theorem.

Theorem 4.1. *Assume that the time-dependent external potential $\mathcal{B}$ is smooth and decays faster than quadratically for large times in the sense that* (1.7) *holds for suitable constants $c, \varepsilon > 0$. Then the Dirac operator $\mathcal{D} = i\slashed{\partial} + \mathcal{B}$ has the weak mass oscillation property.*

We expect that this theorem could be improved by weakening the decay assumptions on the potential. However, this would require refinements of our methods which would go beyond the scope of this paper. Also, using that Dirac solutions dissipate, the pointwise decay in time could probably be replaced or partially compensated by suitable spatial decay assumptions. Moreover, one could probably refine the result of the above theorem by working with other norms (like weighted C^k- or Sobolev norms).

The main step is the following basic estimate, which is the analog of Lemma 3.1 in the presence of an external potential.

Proposition 4.2. *Under the decay assumptions* (1.7) *on the external potential* $\mathcal{B}$, *there are constants* $c, \varepsilon > 0$ *such that for every family* $\psi \in \mathcal{H}^\infty$ *of solutions of the Dirac equation* (2.3) *with varying mass,*

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{c}{1+|t|^{1+\varepsilon}} \sup_{m \in I} \sum_{b=0}^2 \big\|(\partial_m^b \psi_m)|_{t=0}\big\|_{W^{2,2}} . \tag{4.1}$$

We first show that this proposition implies the weak mass oscillation property.

Proof of Theorem 4.1 under the assumption that Proposition 4.2 holds. In order to derive the inequality (2.13), we begin with the estimate

$$|\langle\mathfrak{p}\psi|\mathfrak{p}\phi\rangle| \leq \frac{1}{2\pi} \int_{-\infty}^\infty \Big|\big(\mathfrak{p}\psi|_t \,|\, \mathfrak{p}\phi|_t\big)|_t\Big|\, dt \leq \sup_{t\in\mathbb{R}} \big\|\mathfrak{p}\phi|_t\big\|_t \int_{-\infty}^\infty \big\|\mathfrak{p}\psi|_t\big\|_t\, dt .$$

The last integral is finite by Proposition 4.2. The supremum can be bounded by the Hilbert space norm using the Hölder inequality,

$$\|\mathfrak{p}\phi|_t\|_t = \bigg\| \int_I \phi_m|_t \, dm \bigg\|_t \leq \int_I \big\|\phi_m|_t\big\|_t \, dm \leq \sqrt{|I|} \left(\int_I \|\phi_m|_t\|_t^2 \, dm \right)^{\frac{1}{2}} = \sqrt{|I|}\, \|\phi\| ,$$

giving (2.13).

Using (2.2), the Dirac operator $i\partial\!\!\!/ + \mathcal{B}$ is formally self-adjoint with respect to the inner product $\langle .|.\rangle$. Therefore, the identity (2.14) can be obtained just as in (3.6) by integrating the Dirac operator in space-time by parts, noting that we do not get boundary terms in view of the time decay in Proposition 4.2. ◻

The remainder of this section is devoted to the proof of Proposition 4.2. One ingredient is the Lippmann-Schwinger equation (2.20),

$$\psi_m|_t = U_m^{t,0}\, \psi_m|_{t=0} + i \int_0^t U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, d\tau . \tag{4.2}$$

Since the first summand of this equation is controlled by Lemma 3.1, it remains to estimate the second summand. Again using (3.15) and integrating by parts with respect to the mass, we obtain

$$\int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, dm = \frac{1}{(t-\tau)^2} \int_I \big(A_m^{t,\tau}\, \partial_m^2 - B_m^{t,\tau} \partial_m + C_m^{t,\tau}\big) \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, dm$$

and thus

$$\begin{aligned}
\bigg\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, dm \bigg\|_t &\leq \frac{c\,|I|}{(t-\tau)^2} \sup_{m\in I} \sum_{b=0}^2 \big\|\mathcal{B}(\tau)\, (\partial_m^b \psi_m)|_\tau\big\|_{W^{2,2}} \\
&\leq \frac{c\,|I|}{(t-\tau)^2}\, |\mathcal{B}(\tau)|_{C^2} \sup_{m\in I} \sum_{b=0}^2 \big\|\partial_m^b \psi_m|_\tau\big\|_{W^{2,2}} .
\end{aligned}$$

We now bound $\mathcal{B}(\tau)$ with the help of (1.7) and estimate the Sobolev norm $\big\|\partial_m^b \psi_m|_\tau\big\|_{W^{2,2}}$ at time τ by means of Lemma A.1 proved in Appendix A. This gives rise to the inequality

$$\bigg\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, dm \bigg\|_t \leq \frac{c^2\, C\, |I|}{(t-\tau)^2}\, \frac{1+|\tau|^2}{1+|\tau|^{2+\varepsilon}} \sup_{m\in I} \sum_{b=0}^2 \big\|\partial_m^b \psi_m|_{t=0}\big\|_{W^{2,2}} ,$$

which yields the desired decay provided that τ and t are not close to each other. More precisely, we shall apply this inequality in the case $|\tau| \leq |t|/2$. Then the estimate simplifies to

$$\begin{aligned}
&\Big\| \int_I U_m^{t,\tau} \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, dm \Big\|_t \\
&\qquad\qquad \leq \frac{\tilde{C}}{t^2\,(1+|\tau|^\varepsilon)} \sup_{m\in I} \sum_{b=0}^2 \big\|\partial_m^b \psi_m|_{t=0}\big\|_{W^{2,2}} \qquad \text{if } |\tau| \leq |t|/2
\end{aligned} \tag{4.3}$$

with a new constant $\tilde{C} > 0$. In the remaining case $|\tau| > |t|/2$, we use the unitarity of $U_m^{t,\tau}$ to obtain

$$\Big\| \int_I U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)|_\tau \, dm \Big\|_t \leq |I| \, |\mathcal{B}(\tau)|_{C^0} \sup_{m\in I} \|\psi_m\| \,.$$

Applying (1.7) together with the inequality $|\tau| > |t|/2$, this gives

$$\Big\| \int_I U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)|_\tau \, dm \Big\|_t \leq \frac{\tilde{C}}{t^{2+\varepsilon}} \sup_{m\in I} \|\psi_m\| \qquad \text{if } |\tau| > |t|/2\,. \tag{4.4}$$

This again decays for large t because τ is close to t and $|\mathcal{B}(\tau)|_{C^0}$ decays for large τ.

Comparing (4.3) and (4.4), we find that the inequality in (4.3) even holds for all τ. Thus integrating this inequality over $\tau \in [0,t]$, we obtain the following estimate for the second summand in (4.2),

$$\Big\| \int_I dm \int_0^t U_m^{t,\tau} \left(\gamma^0 \mathcal{B}\, \psi_m\right)|_\tau \, d\tau \Big\| \leq \frac{C'}{t^{1+\varepsilon}} \sup_{m\in I} \sum_{b=0}^2 \big\|\partial_m^b \psi|_{t=0}\big\|_{W^{2,2}}$$

(where $C' > 0$ is a new constant). Combining this inequality with the estimate (3.2) of the first summand in (4.2), we obtain the desired inequality (4.1). This concludes the proof of Proposition 4.2.

4.2. **Proof of the Strong Mass Oscillation Property.** In this section, we prove the following result.

Theorem 4.3. *Assume that the weak mass oscillation property holds and that the external potential $\mathcal{B}$ satisfies the condition*

$$\int_{-\infty}^{\infty} |\mathcal{B}(\tau)|_{C^0} \, d\tau < \infty\,. \tag{4.5}$$

Then the Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ has the strong mass oscillation property.

Combining this theorem with Theorem 4.1, one immediately obtains Theorem 1.2.

For the proof we shall derive an explicit formula for the fermionic signature operator (Proposition 4.4). This formula is obtained by comparing the dynamics in the presence of the external potential with that in the Minkowski vacuum using the Lippmann-Schwinger equation, and by employing distributional relations for products of fundamental solutions and Green's functions (Lemma 4.7).

We first return to the formula (3.21) in the Minkowski vacuum. Applying Plancherel's theorem and using (2.4), we conclude that

$$<\!\mathfrak{p}\psi|\mathfrak{p}\phi\!> = \int_I (\psi_m^0 \,|\, \mathcal{S}_m(\vec{k})\, \phi_m^0)\, dm\,, \tag{4.6}$$

where

$$\mathcal{S}_m(\vec{k}) := \sum_{k^0=\pm\omega(\vec{k})} \frac{\slashed{k}+m}{2\,\omega(\vec{k})}\,\gamma^0 = \frac{\vec{k}\vec{\gamma}+m}{\omega(\vec{k})}\,\gamma^0\,. \tag{4.7}$$

Comparing (4.6) with (2.16), one sees that the matrix $\mathcal{S}_m(\vec{k})$ is indeed the fermionic signature operator, considered as a multiplication operator in momentum space. By direct computation, one verifies that the matrix $\mathcal{S}_m(\vec{k})$ has eigenvalues ± 1.

In order to compare the dynamics in the presence of the external potential with that in the Minkowski vacuum, we work with the Hamiltonian formulation. We decompose the Dirac Hamiltonian (2.19) into the Hamiltonian in the Minkowski vacuum (2.21) plus a potential,

$$\tilde{H} = H + \mathcal{V} \qquad \text{with} \qquad \mathcal{V} := -\gamma^0 \mathcal{B}\,.$$

Proposition 4.4. *Assume that the potential $\mathcal{B}$ satisfies the condition* (4.5)*. Then for every $\psi, \phi \in \mathcal{H}^\infty$,*

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m\, \phi_m)_m\, dm\,, \tag{4.8}$$

where $\tilde{\mathcal{S}}_m : \mathcal{H}_m \to \mathcal{H}_m$ are bounded linear operators which act on the wave functions at time t_0 by

$$\tilde{\mathcal{S}}_m = \mathcal{S}_m - \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0) \left[\mathcal{S}_m\, U_m^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}_m^{t,t_0} - \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0} \right] dt \tag{4.9}$$

$$+ \frac{1}{2} \left(\int_{t_0}^{\infty}\!\!\int_{t_0}^{\infty} + \int_{-\infty}^{t_0}\!\!\int_{-\infty}^{t_0} \right) \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t'}\, \mathcal{V}(t')\, \tilde{U}_m^{t',t_0}\, dt\, dt' \tag{4.10}$$

(and $\mathcal{S}_m$ is again the fermionic signature operator of the vacuum (4.7)*).*

Before entering the proof of this proposition, it is instructive to verify that the above formula for $\tilde{\mathcal{S}}_m$ does not depend on the choice of t_0.

Remark 4.5. (Independence of $\tilde{\mathcal{S}}_m$ on t_0) Our strategy is to differentiate the above formula for $\tilde{\mathcal{S}}_m$ with respect to t_0 and to verify that we obtain zero. We first observe that taking a solution $\phi_m \in \mathcal{H}_m$ of the Dirac equation in the presence of $\mathcal{B}$, evaluating at time t_0 and applying the time evolution operator $\tilde{U}_m^{t,t_0}$ gives ϕ_m at time t, i.e. $\tilde{U}_m^{t,t_0} \phi_m|_{t_0} = \phi_m|_t$. Differentiating with respect to t_0 yields

$$\partial_{t_0} \tilde{U}_m^{t,t_0} \phi_m|_{t_0} = 0\,.$$

The situation is different when one considers the time evolution operator of the vacuum. Namely, in the expression $U_m^{t,t_0} \phi_m|_{t_0}$, the wave function ϕ_m satisfies the Dirac equation $(i\partial_t - H)\phi_m = \mathcal{V}\phi_m$, whereas the time evolution operator solves the Dirac equation with $\mathcal{V} \equiv 0$. As a consequence,

$$\partial_{t_0} U_m^{t,t_0} \phi_m|_{t_0} = -i U_m^{t,t_0} (\mathcal{V}\phi_m)|_{t_0}\,.$$

Using these formulas together with $U^{t_0,t_0} = \mathbb{1} = \tilde{U}^{t_0,t_0}$, a straightforward computation gives

$$\begin{aligned}
\partial_{t_0} \big(\psi_m \,|\, (4.9)\, \phi_m\big)\big|_{t_0} =& -i(\psi_m \,|\, [\mathcal{S}_m, \mathcal{V}]\, \phi_m)|_{t_0} \\
&- \frac{i}{2}\,(-2)\big(\psi_m \,\big|\, \big(\mathcal{S}_m\, \mathcal{V}(t_0) - \mathcal{V}(t_0)\, \mathcal{S}_m\big)\, \phi_m\big)\big|_{t_0} \\
&- \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big((-i\mathcal{V}(t_0))\, \psi_m \,\big|\, \mathcal{S}_m\, U_m^{t_0,t}\, \mathcal{V}(t)\, \tilde{U}_m^{t,t_0}\, \phi_m\big)\big|_{t_0} dt \\
&+ \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi_m \,\big|\, \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\, (-i\mathcal{V}(t_0))\, \phi_m\big)\big|_{t_0} dt \\
\partial_{t_0} \big(\psi \,|\, (4.10)\, \phi\big)_{t_0} =& -\frac{1}{2} \int_{-\infty}^{\infty} \epsilon(t'-t_0)\, \big(\psi_m \,\big|\, \mathcal{V}(t_0)\, \mathcal{S}_m\, U_m^{t_0,t'}\, \mathcal{V}(t')\, \tilde{U}_m^{t',t_0}\, \phi_m\big)\big|_{t_0}\, dt' \\
&- \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0)\, \big(\psi_m \,\big|\, \tilde{U}_m^{t_0,t}\, \mathcal{V}(t)\, \mathcal{S}_m\, U_m^{t,t_0}\, \mathcal{V}(t_0)\, \phi_m\big)\big|_{t_0}\, dt\,,
\end{aligned}$$

where for notational simplicity we here omitted the restrictions $|_{t_0}$ for the solutions ψ_m and ϕ_m. Adding the terms gives zero. $\Diamond$

The remainder of this section is devoted to the proof of Proposition 4.4. Our strategy is to combine the Lippmann-Schwinger equation with estimates in momentum space. We begin with two technical lemmas.

Lemma 4.6. *Assume that the external potential $\mathcal{B}$ satisfies condition* (4.5). *For any $t_0 \in \mathbb{R}$, we denote the characteristic functions in the future respectively past of this hypersurface $t = t_0$ by $\chi^\pm_{t_0}(x)$ (i.e. $\chi^\pm_{t_0}(x) = \Theta(\pm(x^0 - t_0))$, where Θ is the Heaviside function). Then for any $\psi_m \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$, the wave function $k_m(\chi^\pm_{t_0} \mathcal{B}\psi_m)$ is a well-defined vector in $\mathcal{H}_{t_0}$ and*

$$\|k_m(\chi^\pm_{t_0} \mathcal{B}\psi_m)\|_{t_0} \leq \frac{1}{2\pi}\, \|\psi_m\|_m \int_{-\infty}^{\infty} \chi^\pm_{t_0}(\tau)\, |\mathcal{B}(\tau)|_{C^0}\, d\tau\,.$$

Proof. Using the integral kernel representation (2.8) and (2.9) together with the fact that the time evolution in the vacuum is unitary, we obtain

$$\begin{aligned}
&2\pi \left\| \int_{\mathbb{R}^3} k_m\big((t_0,.),(\tau,\vec{y})\big)\, \big(\chi^\pm_{t_0}\mathcal{B}\psi_m\big)(\tau,\vec{y})\, d^3y \right\|_{t_0} \\
&\quad = \big\| U_m^{t_0,\tau} \gamma^0 (\chi^\pm_{t_0}\mathcal{B}\psi_m)|_\tau \big\|_{t_0} = \big\| \gamma^0 (\chi^\pm_{t_0}\mathcal{B}\psi_m)|_\tau \big\|_\tau \leq |\mathcal{B}(\tau)|_{C^0}\, \|\psi_m\|_m\,.
\end{aligned}$$

Integrating over τ and using (4.5) gives the result. $\square$

The following lemma is proved in [16, Eqs. (2.13)–(2.17)].

Lemma 4.7. *In the Minkowski vacuum, the fundamental solution k_m and the Green's function s_m defined by*

$$s_m := \frac{1}{2}\, \big(s_m^\vee + s_m^\wedge\big) \tag{4.11}$$

satisfy the distributional relations in the mass parameters m and m'

$$\begin{aligned}
k_m\, k_{m'} &= \delta(m-m')\, p_m \\
k_m\, s_{m'} &= s_{m'}\, k_m = \frac{\text{PP}}{m-m'}\, k_m \\
s_m\, s_{m'} &= \frac{\text{PP}}{m-m'}\, (s_m - s_{m'}) + \pi^2\, \delta(m-m')\, p_m\,,
\end{aligned}$$

where PP *denotes the principal part, and* p_m *is the distribution*

$$p_m(k) = (\not{k} + m)\, \delta(k^2 - m^2) \,. \tag{4.12}$$

Proof of Proposition 4.4. Let $\psi \in \mathcal{H}^\infty$ be a family of solutions of the Dirac equation for varying mass. We denote the boundary values at time t_0 by $\psi^0_m := \psi_m|_{t_0}$. Then we can write the Lippmann-Schwinger equation (2.20) as

$$\psi_m|_t = U^{t,t_0}_m \psi^0_m + i \int_{t_0}^t U^{t,\tau}_m \big(\gamma^0 \mathcal{B}\, \psi_m\big)\big|_\tau \, d\tau \,.$$

We now bring this equation into a more useful form. Expressing the time evolution operator with the help of (2.9) in terms of the fundamental solution, we obtain

$$\begin{aligned} \psi_m(x) &= 2\pi \int_{\mathbb{R}^3} k_m\big(x, (t_0, \vec{y})\big)\, \gamma^0 \psi^0_m(t_0, \vec{y})\, d^3y \\ &\quad + 2\pi i \int_{t_0}^{x^0} dy^0 \int_{\mathbb{R}^3} d^3y\, k_m(x,y) (\mathcal{B}\, \psi_m)(y) \,. \end{aligned}$$

Applying (2.5) and using that the advanced and retarded Green's functions are supported in the future and past light cones, respectively, we can rewrite the last integral in terms of the advanced and retarded Green's functions,

$$\psi_m = 2\pi\, k_m\big(\gamma^0 \delta_{t_0} \psi^0_m\big) - s^\wedge_m\big(\chi^+_{t_0} \mathcal{B} \psi_m\big) - s^\vee_m\big(\chi^-_{t_0} \mathcal{B} \psi_m\big) \,,$$

where $\delta_{t_0}(x) := \delta(t_0 - x^0)$ is the Dirac distribution supported on the hypersurface $x^0 = t_0$. Next, we express the advanced and retarded Green's functions in terms of the Green's function (4.11): According to (2.5), we have the relations

$$s_m = s^\vee_m - i\pi k_m = s^\wedge_m + i\pi k_m$$

and thus

$$\psi_m = k_m g_m - s_m \mathcal{B} \psi_m \qquad \text{with} \qquad g_m := 2\pi\, \gamma^0 \delta_{t_0} \psi^0_m + i\pi\, \epsilon_{t_0} \mathcal{B} \psi_m \,, \tag{4.13}$$

where ϵ_{t_0} is the step function

$$\epsilon_{t_0}(x) := \epsilon(x^0 - t_0)$$

(and we omitted the brackets in expressions like $k_m g_m \equiv k_m(g_m)$). Note that the expression $k_m g_m$ is well-defined according to Lemma 4.6. We also remark that by applying the operator $(i\not{\partial} - m)$ to the distribution g_m in (4.13), one immediately verifies that ψ_m indeed satisfies the Dirac equation $(i\not{\partial} - m)\psi_m = -\mathcal{B}\psi_m$.

Now we can compute the inner product $<\mathfrak{p}\psi|\mathfrak{p}\psi>$ with the help of Lemma 4.7. Namely, using (4.13),

$$\begin{aligned} <\mathfrak{p}\psi|\mathfrak{p}\psi> &= \iint_{I\times I} <k_m g_m - s_m \mathcal{B}\psi_m \,|\, k_{m'} g_{m'} - s_{m'} \mathcal{B}\psi_{m'}>\, dm\, dm' \\ &= \int_I \Big(<g_m \,|\, p_m g_m> + \pi^2 <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> \Big) dm \\ &\quad + \iint_{I\times I} \frac{\text{PP}}{m - m'} \Big(<\mathcal{B}\psi_m \,|\, k_{m'} g_{m'}> - <k_m g_m \,|\, \mathcal{B}\psi_{m'}> \\ &\qquad\qquad + <\mathcal{B}\psi_m \,|\, (s_m - s_{m'}) \mathcal{B}\psi_{m'}> \Big)\, dm\, dm' \,. \end{aligned}$$

Note that this computation is mathematically well-defined in the distributional sense because ψ_m and g_m are smooth and compactly supported in the mass parameter m. Employing the explicit formula for g_m in (4.13), we obtain

$$<\mathfrak{p}\psi|\mathfrak{p}\psi> = \int_I \Big(<g_m \,|\, p_m g_m> + \pi^2 <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> \Big) dm \,.$$

Comparing (3.7) with (4.12) and taking into account that the operator $\mathcal{S}_m$ defined by (4.7) gives a minus sign for the states of negative frequency, we get

$$p_m = \mathcal{S}_m \, k_m \,.$$

Using this identity together with Proposition 2.2 in the vacuum yields the relations

$$\begin{aligned} <g_m \,|\, p_m g_m> &= (k_m g_m \,|\, \mathcal{S}_m \, k_m g_m)|_{t_0} \\ <\mathcal{B}\psi_m \,|\, p_m \mathcal{B}\psi_m> &= (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m \, k_m \mathcal{B}\psi_m)|_{t_0} \,. \end{aligned}$$

We finally apply Proposition 2.1 to obtain the representation

$$<\mathfrak{p}\psi|\mathfrak{p}\psi> = \int_I \Big((h_m \,|\, \mathcal{S}_m \, h_m)|_{t_0} + \pi^2 \, (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m \, k_m \mathcal{B}\psi_m)|_{t_0} \Big) \, dm \,, \tag{4.14}$$

where

$$h_m := \psi_m + i\pi \, k_m(\epsilon_{t_0} \mathcal{B}\psi_m) \,.$$

Comparing (4.8) with (4.14), we get

$$(\psi_m \,|\, \tilde{\mathcal{S}}_m \, \psi_m)_m = (h_m \,|\, \mathcal{S}_m \, h_m)|_{t_0} + \pi^2 \, (k_m \mathcal{B}\psi_m \,|\, \mathcal{S}_m \, k_m \mathcal{B}\psi_m)|_{t_0} \,.$$

Expressing the operators k_m according to (2.9) by the time evolution operator and writing ψ_m in terms of the initial data as

$$\psi_m|_t = \tilde{U}^{t,t_0} \psi|_{t_0} \,,$$

we obtain

$$\begin{aligned} (\psi_m \,|\, \tilde{\mathcal{S}}_m \, \psi_m)_m &= (\psi|\mathcal{S}_m\psi)|_{t_0} - \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0) \, \big(\psi \,\big|\, \mathcal{S}_m \, U^{t_0,t} \, \mathcal{V}(t) \, \tilde{U}^{t,t_0} \, \psi\big)\big|_{t_0} \, dt \\ &\quad + \frac{i}{2} \int_{-\infty}^{\infty} \epsilon(t-t_0) \, \big(U^{t_0,t} \, \mathcal{V}(t) \, \tilde{U}^{t,t_0} \, \psi \,\big|\, \mathcal{S}_m \, \psi\big)\big|_{t_0} \, dt \\ &\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \epsilon(t-t_0) \, \epsilon(t'-t_0) \, \big(U^{t_0,t} \, \mathcal{V}(t) \, \tilde{U}^{t,t_0} \, \psi \,\big|\, \mathcal{S}_m \, U^{t_0,t'} \, \mathcal{V}(t') \, \tilde{U}^{t',t_0} \, \psi\big)\big|_{t_0} \, dt \, dt' \\ &\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \big(U^{t_0,t} \, \mathcal{V}(t) \, \tilde{U}^{t,t_0} \, \psi \,\big|\, \mathcal{S}_m \, U^{t_0,t'} \, \mathcal{V}(t') \, \tilde{U}^{t',t_0} \, \psi\big)\big|_{t_0} \, dt \, dt' \,. \end{aligned}$$

Rearranging the terms and polarizing gives the result. ◻

Proof of Theorem 4.3. Since the time evolution operators are unitary and the operators $\mathcal{S}_m$ have norm one (see (4.7)), the representation (4.9) and (4.10) gives rise to the following estimate for the sup-norm of $\tilde{S}_m$,

$$\big\|\tilde{\mathcal{S}}_m\big\| \leq 1 + \int_{\mathbb{R}} |\mathcal{V}(t)|_{C^0} \, dt + \iint_{\mathbb{R}\times\mathbb{R}} |\mathcal{V}(t)|_{C^0} \, |\mathcal{V}(t')|_{C^0} \, dt \, dt' \,.$$

The decay assumption (4.5) implies that the sup-norm of $\tilde{S}_m$ is bounded uniformly in m. Using this fact in (4.8) gives the inequality (2.15), thereby establishing the strong mass oscillation property. ◻

We finally remark that the uniqueness statement in Theorem 2.5 implies that (4.9) and (4.10) yields an explicit representation of the fermionic signature operator in the presence of a time-dependent external potential.

5. Hadamard Form of the Fermionic Projector

In this section, we will prove Theorem 1.3. In preparation, we derive so-called frequency splitting estimates which give control of the "mixing" of the positive and negative frequencies in the solutions of the Dirac equation as caused by the time-dependent external potential (Theorem 5.1). Based on these estimates, we will complete the proof of Theorem 1.3 at the end of Section 5.2.

5.1. **Frequency Mixing Estimates.** For the following constructions, we again choose the hypersurface $\mathscr{N} := \mathscr{N}_{t_0}$ at some given time t_0. Moreover, we always fix the mass parameter $m > 0$. Since we are no longer considering families of solutions, for ease in notation we omit the index m at the Dirac wave functions, the scalar products and the corresponding norms. We also identify the solution space $\mathcal{H}_m$ with the Hilbert space $\mathcal{H}_{t_0}$ of square integrable wave functions on $\mathscr{N}$. On $\mathcal{H}_{t_0}$, we can act with the Hamiltonian H of the vacuum, and using the above identification, the operator H becomes an operator on $\mathcal{H}_m$ (which clearly depends on the choice of t_0).

We work with a so-called *frequency splitting* with respect to the vacuum dynamics. To this end, we decompose the Hilbert space $\mathcal{H}_m$ as

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^- \qquad \text{with} \qquad \mathcal{H}^\pm = \chi^\pm(H)\mathcal{H}_m \,,$$

where $\chi^\pm$ are the characteristic functions

$$\chi^+ := \chi_{[0,\infty)} \qquad \text{and} \qquad \chi^- := \chi_{(-\infty,0)} \,. \tag{5.1}$$

For convenience, we write this decomposition in components and use a block matrix notation for operators, i.e.

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} A^+_+ & A^+_- \\ A^-_+ & A^-_- \end{pmatrix} ,$$

where $A^s_{s'} = \chi^s(H) A\, \chi^{s'}(H)$ and $s, s' \in \{\pm\}$.

The representation in Proposition 4.4 makes it possible to let the fermionic signature operator $\tilde{\mathcal{S}}_m$ act on the Hilbert space $\mathcal{H}_m$ (for fixed m). We decompose this operator with respect to the above frequency splitting,

$$\tilde{\mathcal{S}}_m = \tilde{\mathcal{S}}^{\mathrm{D}} + \Delta\tilde{\mathcal{S}} \,, \qquad \text{where} \qquad \tilde{\mathcal{S}}^{\mathrm{D}} := \tilde{\mathcal{S}}^+_+ + \tilde{\mathcal{S}}^-_- \quad \text{and} \quad \Delta\tilde{\mathcal{S}} := \tilde{\mathcal{S}}^+_- + \tilde{\mathcal{S}}^-_+ \,.$$

Thus the operator $\tilde{\mathcal{S}}^{\mathrm{D}}$ maps positive to positive and negative to negative frequencies. The operator $\Delta\tilde{\mathcal{S}}$, on the other hand, mixes positive and negative frequencies. In the next theorem, it is shown under a suitable smallness assumption on $\mathcal{B}$ that the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ coincide with the projections $\chi^\pm(H)$, up to smooth contributions. The main task in the proof is to control the "frequency mixing" as described by the operator $\Delta\tilde{\mathcal{S}}$.

Theorem 5.1. *Under the assumptions of Theorem 1.3, the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ have the representations*

$$\chi^\pm(\tilde{\mathcal{S}}_m) = \chi^\pm(H) + \frac{1}{2\pi i} \oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}\, d\lambda \,, \tag{5.2}$$

where the contour integral is an integral operator with a smooth kernel.

Here $B_{\frac{1}{2}}$ denotes the open ball of radius 1/2. The operator $(\tilde{\mathcal{S}}_m - \lambda)^{-1}$ is also referred to as the *resolvent* of $\tilde{\mathcal{S}}_m$.

This theorem will be proved in several steps. We begin with a preparatory lemma.

Lemma 5.2. *Under the assumptions* (1.7) *and* (1.14)*, the spectrum of* $\tilde{\mathcal{S}}^{\mathrm{D}}$ *is located in the set*

$$\sigma(\tilde{\mathcal{S}}^{\mathrm{D}}) \subset \Big[-\frac{3}{2}, -\frac{1}{2} \Big] \cup \Big[\frac{1}{2}, \frac{3}{2} \Big] . \tag{5.3}$$

Moreover,

$$\chi^\pm(\tilde{\mathcal{S}}^{\mathrm{D}}) = \chi^\pm(H) , \tag{5.4}$$

and the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ *have the representations* (5.2).

Proof. Since the subspaces $\mathcal{H}^\pm$ are invariant under the action of $\tilde{\mathcal{S}}^{\mathrm{D}}$, our task is to show that the spectrum of $\tilde{\mathcal{S}}^{\mathrm{D}}|_{\mathcal{H}^\pm}$ is positive and negative, respectively. This statement would certainly be true if we replaced $\tilde{\mathcal{S}}^{\mathrm{D}}$ by $\mathcal{S}_m$, because the operator $\mathcal{S}_m$ has the eigenvalues ± 1 with $\mathcal{H}^\pm$ as the corresponding eigenspaces. Estimating the representation in Proposition 4.4 with the Schwarz inequality, we obtain

$$\big| (\psi | \tilde{\mathcal{S}}^{\mathrm{D}} \phi) - (\psi | \mathcal{S}_m \phi) \big| \leq \Big(c + \frac{c^2}{2} \Big) \, \|\psi\| \, \|\phi\| \quad \text{with} \quad c := \int_{-\infty}^\infty |\mathcal{B}(\tau)|_{C^0} \, d\tau \, .$$

Using the assumption (1.14), we conclude that

$$\big| (\psi | \tilde{\mathcal{S}}^{\mathrm{D}} \phi) - (\psi | \mathcal{S}_m \phi) \big| < \frac{1}{2} \, \|\psi\| \, \|\phi\| \qquad \text{for all } \psi, \phi \in \mathcal{H}_m \, .$$

Standard estimates on the continuity of the spectrum (see for example [32, §IV.3]) yield that the spectrum of $\tilde{\mathcal{S}}^{\mathrm{D}}$ differs by that of the operator $\mathcal{S}_m$ at most by 1/2. This gives (5.3) and (5.4).

In order to prove the representation (5.2), we take the resolvent identity

$$(\tilde{\mathcal{S}}_m - \lambda)^{-1} = (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1} - (\tilde{\mathcal{S}}_m - \lambda)^{-1} \, \Delta \tilde{\mathcal{S}} \, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1} \, ,$$

form the contour integral and apply (5.4). This gives the result. □

The next lemma relates the smoothness of an integral kernel to the boundedness of the product of the operator with powers of the vacuum Hamiltonian.

Lemma 5.3. *Let* $A \in \mathrm{L}(\mathcal{H}_m)$ *be an operator which maps smooth functions to smooth functions and has the property that for all* $p, q \in \mathbb{N}$*, the operator product*

$$H^q \, A \, H^p \;:\; C^\infty_0(\mathscr{N}, S\mathscr{M}) \rightarrow C^\infty(\mathscr{N}, S\mathscr{M}) \tag{5.5}$$

extends to a bounded linear operator on $\mathcal{H}_m$*. Then, considering* A *as an operator on* $\mathcal{H}_m$*, this operator can be represented as an integral operator with a smooth kernel, i.e.*

$$(A\psi)(x) = \int_{\mathscr{N}} \mathcal{A}\big(x, (t_0, \vec{y})\big) \, \gamma^0 \, \psi(t_0, \vec{y}) \, d^3y \qquad \textit{with} \qquad \mathcal{A} \in C^\infty(\mathscr{M} \times \mathscr{M}) \, .$$

Proof. Since in momentum space, the square of the Hamiltonian takes the form

$$H(\vec{k})^2 = \Big(\gamma^0 \big(\vec{\gamma} \vec{k} + m \big) \Big)^2 = \big(-\vec{\gamma}\vec{k} + m \big) \big(\vec{\gamma}\vec{k} + m \big) = |\vec{k}|^2 + m^2 \, ,$$

the wave function $\hat{\psi}$ defined by

$$\hat{\psi}(\vec{k}) := \frac{1}{|\vec{k}|^2 + m^2} \, e^{i \vec{k} \vec{x}_0} \, \Xi$$

for a constant spinor Ξ and $\vec{x}_0 \in \mathbb{R}^3$, satisfies the equation

$$H^2\, \psi(\vec{x}) = \delta^3(\vec{x}-\vec{x_0})\, \Xi\, .$$

Moreover, one verifies immediately that $\psi \in \mathcal{H}_{t_0}$ is square-integrable. Using the last equation together with (5.5), we conclude that

$$H^q A\big(\delta^3(\vec{x}-\vec{x_0})\,\Xi\big) = H^q A H^2 \psi \in \mathcal{H}_{t_0}\, .$$

Since q is arbitrary, it follows that A has an integral representation in the spatial variables,

$$(A\phi)(\vec{x}) = \int_{\mathscr{N}} \mathcal{A}(\vec{x},\vec{y})\, \gamma^0\, \phi(\vec{y})\, d^3y \qquad \text{with} \qquad \mathcal{A} \in C^\infty(\mathscr{N}\times\mathscr{N})\, .$$

We now extend this integral kernel to $\mathscr{M}\times\mathscr{M}$ by solving the Cauchy problem in the variables x and y. This preserves smoothness by the global existence and regularity results for linear hyperbolic equations, giving the result. □

Lemma 5.4. *Under the assumptions of Theorem 1.3, for all $p \in \mathbb{N}$ the iterated commutator*

$$\mathcal{S}^{(p)} := \big[\underbrace{H, \big[H, \ldots, [H}_{p \text{ factors}}, \tilde{\mathcal{S}}_m] \cdots\big]\big]$$

is a bounded operator on $\mathcal{H}_m$.

Proof. In the vacuum, the Hamiltonian clearly commutes with the time evolution operator,

$$\big[H, U_m^{t,t'}\big] = 0\, . \tag{5.6}$$

In order to derive a corresponding commutator relation in the presence of the external potential, one must take into account that $\tilde{H}$ is time-dependent. For ease in notation, we do not write out this dependence, but instead understand that the Hamiltonian is to be evaluated at the correct time, i.e.

$$\tilde{U}_m^{t,t'}\, \tilde{H} \equiv \tilde{U}_m^{t,t'}\, \tilde{H}(t') \qquad \text{and} \qquad \tilde{H}\, \tilde{U}_m^{t,t'} \equiv \tilde{H}(t)\, \tilde{U}_m^{t,t'}\, .$$

Then

$$(i\partial_t - \tilde{H})\big(\tilde{H}\, \tilde{U}_m^{t,t'} - \tilde{U}_m^{t,t'}\, \tilde{H}\big) = i\dot{\tilde{H}}\, \tilde{U}_m^{t,t'} \qquad \text{and} \qquad \tilde{H}\, \tilde{U}_m^{t,t'} - \tilde{U}_m^{t,t'}\, \tilde{H}\big|_{t=t'} = 0$$

(here and in what follows the dot denotes the partial derivative with respect to t). Solving the corresponding Cauchy problem gives

$$\big[\tilde{H}, \tilde{U}_m^{t,t'}\big] = \int_{t'}^t \tilde{U}_m^{t,\tau}\, \dot{\tilde{H}}\, \tilde{U}_m^{\tau,t'}\, d\tau\, . \tag{5.7}$$

In order to compute the commutator of H with the operator products in (4.9) and (4.10), we first differentiate the expression $U_m^{t'',t}\, \mathcal{V}\, \tilde{U}_m^{t,t'}$ with respect to t,

$$i\partial_t\big(U_m^{t'',t}\, \mathcal{V}\, \tilde{U}_m^{t,t'}\big) = iU_m^{t'',t}\, \dot{\mathcal{V}}\, \tilde{U}_m^{t,t'} + U_m^{t'',t}\, \mathcal{V}\, \tilde{H}\, \tilde{U}_m^{t,t'} - U_m^{t'',t}\, H\, \mathcal{V}\, \tilde{U}_m^{t,t'}\, . \tag{5.8}$$

Moreover, using the commutation relations (5.6) and (5.7), we obtain

$$\begin{aligned}
&H\,(U_m^{t'',t}\, \mathcal{V}\, \tilde{U}_m^{t,t'}) - (U_m^{t'',t}\, \mathcal{V}\, \tilde{U}_m^{t,t'})\, \tilde{H} \\
&\quad = U_m^{t'',t}\, H\, \mathcal{V}\, \tilde{U}_m^{t,t'} - U_m^{t'',t}\, \mathcal{V}\, \tilde{H}\, \tilde{U}_m^{t,t'} + U_m^{t'',t}\, \mathcal{V}\, [\tilde{H}, \tilde{U}_m^{t,t'}] \\
&\quad = iU_m^{t'',t}\, \dot{\mathcal{V}}\, \tilde{U}_m^{t,t'} - i\partial_t\big(U_m^{t'',t}\, \mathcal{V}\, \tilde{U}_m^{t,t'}\big) + \int_{t'}^t U_m^{t'',t}\, \mathcal{V}\, \tilde{U}_m^{t,\tau}\, \dot{\tilde{H}}\, \tilde{U}_m^{\tau,t'}\, d\tau\, ,
\end{aligned}$$

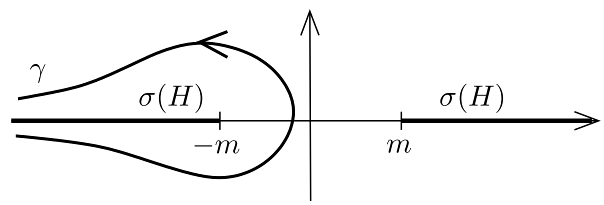

FIGURE 1. The contour γ.

where in the last step we applied (5.8). It follows that

$$\begin{aligned}\big[H, U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}\big] &= H\,(U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'}) - (U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'})\,\tilde{H} + (U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'})\,\mathcal{V} \\ &= iU_m^{t'',t}\,\dot{\mathcal{V}}\,\tilde{U}_m^{t,t'} + (U_m^{t'',t}\mathcal{V}\tilde{U}_m^{t,t'})\,\mathcal{V} - i\partial_t\big(U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,t'}\big) + \int_{t'}^t U_m^{t'',t}\,\mathcal{V}\,\tilde{U}_m^{t,\tau}\,\dot{\tilde{H}}\,\tilde{U}_m^{\tau,t'}\,d\tau\,.\end{aligned}$$

Proceeding in this way, one can calculate the commutator of H with all the terms in (4.9) and (4.10). We write the result symbolically as

$$[H, \tilde{\mathcal{S}}_m] = \mathcal{S}^{(1)}\,,$$

where $\mathcal{S}^{(1)}$ is a bounded operator. Higher commutators can be computed inductively, giving the result. ☐

We point out that this lemma only makes a statement on the iterative commutators. Expressions like $[H^p, \tilde{\mathcal{S}}_m]$ or $H^q\,\tilde{\mathcal{S}}_m\,H^p$ will not be bounded operators in general. However, the next lemma shows that the operator $\Delta\tilde{\mathcal{S}}$ has the remarkable property that multiplying by powers of H from the left and/or right again gives a bounded operator.

Lemma 5.5. *Under the assumptions of Theorem 1.3, for all $p, q \in \mathbb{N}\cup\{0\}$ the product $H^q\,\Delta\tilde{\mathcal{S}}\,H^p$ is a bounded operator on $\mathcal{H}_m$.*

Proof. We only consider the products $H^q\,\mathcal{S}^-_+\,H^p$ because the operator $\mathcal{S}^+_-$ can be treated similarly. Multiplying (5.7) from the left and right by the resolvent of H, we obtain

$$\big[(H-\mu)^{-1}, \tilde{\mathcal{S}}_m\big] = -(H-\mu)^{-1}\,\mathcal{S}^{(1)}\,(H-\mu)^{-1}\,.$$

Writing the result of Lemma 5.4 as

$$[H, \mathcal{S}^{(p)}] = \mathcal{S}^{(p+1)} \qquad \text{with} \qquad \mathcal{S}^{(p+1)} \in \mathrm{L}(\mathcal{H})$$

yields more generally the commutation relations

$$\big[(H-\mu)^{-1}, \mathcal{S}^{(p)}\big] = -(H-\mu)^{-1}\,\mathcal{S}^{(p+1)}\,(H-\mu)^{-1} \qquad \text{for } p \in \mathbb{N}\,. \tag{5.9}$$

Choosing a contour γ which encloses the interval $(-\infty, -m]$ as shown in Figure 1, one finds

$$\begin{aligned}H\mathcal{S}^-_+ &= -\frac{1}{2\pi i}\int_\gamma \mu\,(H-\mu)^{-1}\,\tilde{\mathcal{S}}_m\,\chi^+(H)\,d\mu \\ &= \mathcal{S}\,H\,\chi^-(H)\,\chi^+(H) + \frac{1}{2\pi i}\int_\gamma \mu\,(H-\mu)^{-1}\,\mathcal{S}^{(1)}(H-\mu)^{-1}\,\chi^+(H)\,d\mu \\ &= \frac{1}{2\pi i}\int_\gamma \mu\,(H-\mu)^{-1}\,\mathcal{S}^{(1)}(H-\mu)^{-1}\,\chi^+(H)\,d\mu\,,\end{aligned}$$

where in the last step we used that $\chi^-(H)\,\chi^+(H) = 0$. In order to show that this operator product is bounded, it is useful to employ the spectral theorem for H, which we write as

$$f(H) = \int_{\mathbb{R}\setminus[-m,m]} f(\lambda)\, dE_\lambda \,, \tag{5.10}$$

where dE_λ is the spectral measure of H. This gives

$$\begin{aligned} H\, \mathcal{S}^-_+ &= \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{1}{2\pi i} \int_\gamma \frac{\mu}{\lambda-\mu}\; \frac{1}{\lambda'-\mu}\, \chi^+(\lambda')\, dE_\lambda \right) \mathcal{S}^{(1)}\, dE_{\lambda'}\;\, d\mu \\ &= -\iint_{\mathbb{R}\times\mathbb{R}} \frac{\lambda}{\lambda-\lambda'}\, \chi^-(\lambda)\, \chi^+(\lambda')\, dE_\lambda\, \mathcal{S}^{(1)}\, dE_{\lambda'} \,. \end{aligned} \tag{5.11}$$

Note that the term $\lambda - \lambda'$ is bounded away from zero. Thus the factor $\lambda/(\lambda-\lambda')$ is bounded, showing that the operator $H\mathcal{S}^-_+$ is in $\mathrm{L}(\mathcal{H}_m)$.

This method can be iterated. To this end, we first rewrite the product with commutators,

$$\begin{aligned} H^q\, \mathcal{S}^-_+ &= \chi^-(H)\, \big(H^-\, \chi^-(H)\big)^p\, \tilde{\mathcal{S}}_m\, \chi^+(H) \\ &= \chi^-(H) \Big[H^-, \big[H^-, \ldots, [H^-, \mathcal{S}] \cdots\big]\Big]\; \chi^+(H) \,, \end{aligned}$$

where we used the abbreviation $H^- := H\,\chi^-(H)$. Multiplying from the right by H^p, we can commute factors $H^+ := H\,\chi^+(H)$ to the left to obtain

$$H^q\, \mathcal{S}^-_+ H^p = (-1)^p\, \chi^-(H)\, \underbrace{\big[H^+, \ldots, [H^+,}_{p \text{ factors}} \underbrace{\big[H^-, \ldots, [H^-,}_{q \text{ factors}} \tilde{\mathcal{S}}_m] \cdots]\big] \cdots\big]\, \chi^+(H) \,.$$

Representing each factor $H^\pm$ by a contour integral, one can compute the commutators inductively with the help (5.9). Applying the spectral theorem (5.10) to the left and right of the resulting factor $\mathcal{S}^{(p+q)}$ yields a constant times the expression

$$\begin{aligned} &\iint_{\mathbb{R}\times\mathbb{R}} \chi^-(\lambda)\, \chi^+(\lambda')\, dE_\lambda\, \mathcal{S}^{(p+q)}\, dE_{\lambda'} \\ &\qquad\qquad \times \oint_{\gamma_1} \frac{\mu_1\, d\mu_1}{(\lambda-\mu_1)(\lambda'-\mu_1)} \cdots \oint_{\gamma_{p+q}} \frac{\mu_{p+q}\, d\mu_{p+q}}{(\lambda-\mu_{p+q})(\lambda'-\mu_{p+q})} \,. \end{aligned}$$

Carrying out the contour integrals with residues, we obtain similar to (5.11) an expression of the form

$$H^q\, \mathcal{S}^-_+ H^p = \iint_{\mathbb{R}\times\mathbb{R}} f(\lambda, \lambda')\, \chi^-(\lambda)\, \chi^+(\lambda')\, dE_\lambda\, \mathcal{S}^{(p+q)}\, dE_{\lambda'}$$

with a bounded function f. This concludes the proof. ☐

Proof of Theorem 5.1. It remains to be shown that the contour integral in (5.2) has a smooth kernel. To this end, we multiply the integrand from the left by H^q and from the right by H^p and commute the factors H iteratively to the inside. More precisely, we use the formula

$$H^q (\tilde{\mathcal{S}}_m - \lambda)^{-1} = \sum_{a=0}^q \underbrace{\big[H, \ldots, [H,}_{a \text{ factors}} (\tilde{\mathcal{S}}_m - \lambda)^{-1}] \cdots\big]\; H^{q-a}$$

(note that the sum is telescopic; here we use the convention that the summand for $a = 0$ is simply $(\tilde{\mathcal{S}}_m - \lambda)^{-1} H^q$). Hence

$$\begin{aligned}
&H^q \, (\tilde{\mathcal{S}}_m - \lambda)^{-1} \, \Delta\tilde{\mathcal{S}} \, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1} \, H^p \\
&= \sum_{a=0}^{q} \sum_{b=0}^{p} \underbrace{\Big[H, \ldots, \big[H}_{a \text{ factors}}, (\tilde{\mathcal{S}}_m - \lambda)^{-1}\big] \cdots \Big] \, H^{q-a} \, \Delta\tilde{\mathcal{S}} \, H^{p-b} \, \Big[\cdots \big[(\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}, \underbrace{H\big], \ldots, H\Big]}_{b \text{ factors}} \,.
\end{aligned}$$

According to Lemma 5.5, the intermediate product $H^{q-a} \, \Delta\tilde{\mathcal{S}} \, H^{p-b}$ is a bounded operator. Moreover, the commutators can be computed inductively with the help of Lemma 5.4 and the formula

$$\big[H, (\tilde{\mathcal{S}}_m - \lambda^{-1})\big] = -(\tilde{\mathcal{S}}_m - \lambda^{-1}) \, \big[H, \tilde{\mathcal{S}}_m\big] \, (\tilde{\mathcal{S}}_m - \lambda^{-1})$$

(and similarly for $\tilde{\mathcal{S}}^{\mathrm{D}}$). This gives operators which are all bounded for $\lambda \in \partial B_{\frac{1}{2}}(\pm 1)$. Since the integration contour is compact, the result follows. ◻

5.2. **Proof of the Hadamard Form.** Relying on the frequency mixing estimates of the previous section, we can now give the proof of Theorem 1.3. Recall that the fermionic projector is given by (see (1.9))

$$P = -\chi^-(\tilde{\mathcal{S}}_m) \, \tilde{k}_m \,, \tag{5.12}$$

where we again used the short notation (5.1). Here again the operator $\chi^-(\tilde{\mathcal{S}}_m)$ acts on the solution space $\mathcal{H}_m$ of the Dirac equation, which can be identified with the space $\mathcal{H}_{t_0}$ of square integrable wave functions at time t_0 (see the beginning of Section 5.1). For the following arguments, it is important to note that this identification can be made at any time t_0.

In order to prove that the bi-distribution corresponding to P is of Hadamard form, we compare the fermionic projectors for three different Dirac operators and use the theorem on the propagation of singularities in [34]. More precisely, we consider the following three fermionic projectors:

(1) The fermionic projector P^{vac} in the Minkowski vacuum.

(2) The fermionic projector $\breve{P}$ in the presence of the external potential

$$\breve{\mathcal{B}}(x) := \eta\big(x^0\big) \, \mathcal{B}(x) \,,$$

where $\eta \geq 0$ is a smooth function with $\eta|_{(-\infty,0)} \equiv 0$ and $\eta|_{(1,\infty)} \equiv 1$.

(3) The fermionic projector P in the presence of the external potential $\mathcal{B}(x)$.

The potential $\breve{\mathcal{B}}$ vanishes for negative times, whereas for times $x^0 > 1$ it coincides with $\mathcal{B}$. Thus it smoothly interpolates between the dynamics with and without external potential. The specific form of the potential $\breve{\mathcal{B}}$ in the transition region $0 \leq x^0 \leq 1$ is of no relevance for our arguments.

In the Minkowski vacuum, the relation (5.12) gives the usual two-point function composed of all negative-frequency solutions of the Dirac equation. It is therefore obvious that the bi-distribution $P^{\mathrm{vac}}(x,y)$ is of Hadamard form.

We now compare P^{vac} with $\breve{P}$. To this end, we choose an arbitrary time $t_0 < 0$. Then, applying the result of Theorem 5.1 to (5.12), we get

$$P^{\mathrm{vac}} = -\chi^-(H) \, k_m \qquad \text{and} \qquad \breve{P} = -\chi^-(H) \, \breve{k}_m + (\text{smooth}) \,,$$

where $\breve{k}_m$ is the causal fundamental solution in the presence of the potential $\breve{\mathcal{B}}$. Since $\breve{\mathcal{B}}$ vanishes in a neighborhood of the Cauchy surface at time t_0, we conclude that P^{vac} and $\breve{P}$ coincide in this neighborhood up to a smooth contribution. It follows that also $\breve{P}(x,y)$ is of Hadamard form in this neighborhood. Using the theorem on the propagation of singularities [34, Theorem 5.5], we conclude that $\breve{P}(x,y)$ is of Hadamard form for all $x, y \in \mathscr{M}$.

Next, we compare $\breve{P}$ with P. Thus we choose an arbitrary time $t_0 > 1$. Using again the result of Theorem 5.1 in (5.12), we obtain

$$\breve{P} = -\chi^-(H)\, \breve{k}_m + (\text{smooth}) \qquad \text{and} \qquad P = -\chi^-(H)\, \tilde{k}_m + (\text{smooth})$$

(where the smooth contributions may of course be different). Since $\breve{\mathcal{B}}$ and $\mathcal{B}$ coincide in a neighborhood of the Cauchy surface at time t_0, we infer that $\breve{P}$ and P coincide in this neighborhood up to a smooth contribution. As a consequence, $P(x,y)$ is of Hadamard form in this neighborhood. Again applying [34, Theorem 5.5], it follows that $P(x,y)$ is of Hadamard form for all $x, y \in \mathscr{M}$. This concludes the proof of Theorem 1.3.

6. Quantum Fields and the Hadamard State

In preparation of defining the CAR algebra, we denote the co-spinor bundle by $S^*\mathscr{M}$ (thus the fiber $S^*_x\mathscr{M}$ is the dual space of $S_x\mathscr{M}$). On the smooth and compactly supported sections of $S^*\mathscr{M}$, we introduce the dual of the Dirac operator $\mathcal{D}^*$ by

$$\begin{aligned} &\mathcal{D}^* : C^\infty_0(\mathscr{M}, S^*\mathscr{M}) \rightarrow C^\infty_0(\mathscr{M}, S^*\mathscr{M})\,, \\ &\int_{\mathscr{M}} \big((\mathcal{D}^* g)(f)\big)(x)\, d^4x = \int_{\mathscr{M}} \big(g(\mathcal{D} f)\big)(x)\, d^4x \qquad \text{for all } f \in C^\infty_0(\mathscr{M}, S\mathscr{M})\,. \end{aligned}$$

Moreover, we define the space of pairs of spinorial test functions by

$$\mathfrak{D} := C^\infty_0(\mathscr{M}, S\mathscr{M}) \oplus C^\infty_0(\mathscr{M}, S^*\mathscr{M})$$

with the topology induced by the family of seminorms

$$|(f,g)|_{C^k} := \sup_{x \in \mathscr{M}} |\partial^k f(x)| + \sup_{y \in \mathscr{M}} |\partial^k g(y)|$$

(and $|.|$ is any norm on the spinors and co-spinors, respectively). Next, we define the anti-linear involution map

$$\Gamma : \mathfrak{D} \rightarrow \mathfrak{D}\,, \qquad \Gamma(f \oplus g) := g^* \oplus f^*\,, \tag{6.1}$$

where the star again denotes the adjoint with respect to the spin scalar product, i.e.

$$\begin{aligned} {}^* \;:\; C^\infty_0(\mathscr{M}, S\mathscr{M}) &\rightarrow C^\infty_0(\mathscr{M}, S^*\mathscr{M}), \qquad & f^*(\phi) &= \prec f | \phi \succ \\ {}^* \;:\; C^\infty_0(\mathscr{M}, S^*\mathscr{M}) &\rightarrow C^\infty_0(\mathscr{M}, S\mathscr{M}), \qquad & \prec g^* | \phi \succ &= g(\phi) \end{aligned}$$

(we remark that the adjoint spinor f^* can be identified with the adjoint spinor usually written as $\overline{f} = f^\dagger \gamma^0$). Finally, we introduce an inner product on $\mathfrak{D}$,

$$(.|.)_{\mathfrak{D}} \;:\; \mathfrak{D} \times \mathfrak{D} \rightarrow \mathbb{C}\,, \qquad \big(f \oplus g \,\big|\, a \oplus b\big)_{\mathfrak{D}} = <f \,|\, \tilde{k}_m a> + <b^* \,|\, \tilde{k}_m\, g^*>\,, \tag{6.2}$$

where $<.|.>$ is again the space-time inner product (2.1) (this notation is consistent with our overall convention that brackets like $(.|.)$, $(.|.)_m$ and $<.|.>$ are *sesquilinear* forms, meaning that the first argument always involves complex conjugation). Applying Proposition 2.2, we can write this inner product as

$$\big(f \oplus g \,\big|\, a \oplus b\big)_{\mathfrak{D}} = (\tilde{k}_m f \,|\, \tilde{k}_m a)_m + (\tilde{k}_m b^* \,|\, \tilde{k}_m\, g^*)_m\,,$$

showing that it is indeed positive definite. Forming the completion, we thus obtain the Hilbert space $(\mathcal{H}_{\mathfrak{D}}, (.|.)_{\mathfrak{D}})$. A short computation reveals that the involution Γ preserves the scalar product $(.|.)_{\mathfrak{D}}$ and is therefore anti-unitary, i.e.

$$\Gamma^2 = \mathbb{1} \qquad \text{and} \qquad (\Gamma h \,|\, \Gamma \tilde{h})_{\mathfrak{D}} = (\tilde{h}|h)_{\mathfrak{D}} \quad \text{for all } h, \tilde{h} \in \mathcal{H}_{\mathfrak{D}} \,. \tag{6.3}$$

Definition 6.1. *The* **field algebra** $\mathcal{F}$ *is the unital $*$-algebra generated by the abstract element $B(h)$ with $h \in \mathfrak{D}$, together with the following relations for all $f \oplus g \in \mathfrak{D}$ and $\alpha, \beta \in \mathbb{C}$:*

(i) *Linearity:* $B(\alpha f \oplus g + m \oplus \beta n) = \alpha B(f \oplus g) + \beta B(m \oplus n)$
(ii) *Hermiticity:* $B(f \oplus g)^* = B\big(\Gamma(f \oplus g)\big)$
(iii) *Dynamics:* $B\big((\mathcal{D} - m)f \oplus (\mathcal{D}^* - m)g\big) = 0$ *for all* $f \oplus g \in \mathfrak{D}$
(iv) *Canonical anti-commutation relations (CARs)*
$$\big\{B(f \oplus g)^*, B(m \oplus n)\big\} = (f \oplus g \,|\, m \oplus n)_{\mathfrak{D}} \cdot 1_{\mathcal{F}} \,.$$

We define smeared field operators by

$$\Psi(g) := B(0 \oplus g) \qquad \text{and} \qquad \Psi^*(f) := B(f \oplus 0) \,,$$

Then these field operators satisfy

$$\begin{aligned} \{\Psi(g), \Psi^*(f)\} &= \big\{B(0 \oplus g), B(f \oplus 0)\big\} \overset{\text{(ii)}}{=} \big\{B(0 \oplus g), B\big(\Gamma(f \oplus 0)\big)^*\big\} \\ &\overset{\text{(iv)}}{=} \big(\Gamma(f \oplus 0) \,\big|\, 0 \oplus g\big)_{\mathfrak{D}} \overset{(6.1)}{=} \big(0 \oplus f^* \,\big|\, 0 \oplus g\big)_{\mathfrak{D}} \overset{(6.2)}{=} <g^* \,|\, \tilde{k}_m \, f> \,, \end{aligned}$$

giving rise to the usual anti-commutation relations (1.15).

The next step is to show that the fermionic projector induces a pure, quasi-free state on the field algebra. Let us recall a few basics. By definition, a *state* ω is a linear functional on $\mathcal{F}$ which is positive and normalized, i.e.

$$\omega(B^* B) \geq 0 \quad \text{for all } B \in \mathcal{F} \qquad \text{and} \qquad \omega(1_{\mathcal{F}}) = 1 \,.$$

By linearity, it suffices to specify ω on the monomials. This gives rise to the n-point distributions $\omega_n \in (\mathfrak{D}^n)'$ defined by

$$\omega_n(h_1, \ldots, h_n) := \omega\big(B(h_1) \cdots B(h_n)\big) \,,$$

also referred to as the *n-point functions*. A state induces the usual Fock representation of the field algebra:

Theorem 6.2. (GNS construction) *Let ω be a state on a unital $*$-algebra $\mathcal{F}$. Then there exists a Hilbert space $(\mathcal{H}_{Fock}, (.\,,.)_{Fock})$ and a dense subspace $\mathcal{W} \subset \mathcal{H}_{Fock}$ as well as a representation $\pi : \mathcal{F} \to \mathrm{L}(\mathcal{W})$ and a unit vector $\Omega \in \mathcal{W}$ such that $\omega = (\Omega, \pi(.)\Omega)_{Fock}$ and $\mathcal{W} = \pi(\mathcal{F})\Omega$. The* GNS triple $(\mathcal{W}, \pi, \Omega)$ *is determined up to unitary equivalence.*

Among all possible states, a distinguished role is played by the quasi-free states, for which the n-point functions are all determined by the two-point functions:

Definition 6.3. *A state $\omega : \mathcal{F} \to \mathbb{C}$ is called* **quasi-free** *if the n-point functions vanish for odd n, while for even n, one has*

$$\omega_n(h_1, \ldots, h_n) = \sum_{\sigma \in S_n'} (-1)^{\text{sign}(\sigma)} \prod_{i=1}^{n/2} \omega_2\big(h_{\sigma(2i-1)}, h_{\sigma(2i)}\big) \,,$$

where S'_n denotes the set of ordered permutations of n elements, i.e.

$$\sigma(2i-1) < \sigma(2i) \qquad \text{for all } i = 1, \ldots, \frac{n}{2}$$
$$\sigma(2i-1) < \sigma(2i+1) \qquad \text{for all } i = 1, \ldots, \frac{n-2}{2} \,.$$

For the construction of a quasi-free state from the fermionic projector, we rely on the following result due to Araki [1]:

Lemma 6.4. *Let R be a bounded symmetric operator on $(\mathcal{H}_{\mathfrak{D}}, (.|.)_{\mathfrak{D}})$ with the following properties*

(a) $R + \Gamma R \Gamma = \mathbb{1}$,

(b) $0 \leq R = R^* \leq \mathbb{1}$.

Then there exists a unique quasi-free state ω on $\mathcal{F}$ such that

$$\omega\big(B(h)^* B(\tilde{h})\big) = (h \,|\, R \tilde{h})_{\mathfrak{D}} \qquad \text{for all } h, \tilde{h} \in \mathcal{H}_{\mathfrak{D}} \,. \tag{6.4}$$

Thus our task is to construct an operator R with the above properties using the fermionic signature operator. As will become clear below, the correct choice is to define R implicitly by

$$\big(f \oplus g \,|\, R\,(a \oplus b)\big)_{\mathfrak{D}} = \big(\tilde{k}_m f \,|\, \chi^-(\tilde{\mathcal{S}}_m)\, \tilde{k}_m a\big)_m + \big(\tilde{k}_m b^* \,|\, \chi^+(\tilde{\mathcal{S}}_m)\, \tilde{k}_m g^*\big)_m \,, \tag{6.5}$$

giving a bounded and symmetric operator. Moreover, since $\chi^\pm(\tilde{\mathcal{S}}_m)$ are projection operators, we know that

$$0 \leq \big(f \oplus g \,|\, R\,(f \oplus g)\big)_{\mathfrak{D}} \leq \big(\tilde{k}_m f \,|\, \tilde{k}_m f\big)_m + \big(\tilde{k}_m g^* \,|\, \tilde{k}_m g^*\big)_m = \big(f \oplus g \,|\, f \oplus g\big)_{\mathfrak{D}} \,,$$

showing that the condition (b) in Lemma 6.4 holds. Next, using that Γ is an anti-unitary involution (6.3) and R is symmetric, it follows that

$$\big(f \oplus g \,|\, \Gamma R \Gamma\,(a \oplus b)\big)_{\mathfrak{D}} = \big(R\Gamma\,(a \oplus b) \,|\, \Gamma\,(f \oplus g)\big)_{\mathfrak{D}} = \big(\Gamma\,(a \oplus b) \,|\, R\Gamma\,(f \oplus g)\big)_{\mathfrak{D}} \,.$$

Now we can apply (6.1) and (6.5) to obtain

$$\begin{aligned} \big(f \oplus g \,|\, \Gamma R \Gamma\,(a \oplus b)\big)_{\mathfrak{D}} &= \big(b^* \oplus a^* \,|\, R\,(g^* \oplus f^*)\big)_{\mathfrak{D}} \\ &= \big(\tilde{k}_m b^* \,|\, \chi^-(\tilde{\mathcal{S}}_m)\, \tilde{k}_m g^*\big)_m + \big(\tilde{k}_m f \,|\, \chi^+(\tilde{\mathcal{S}}_m)\, \tilde{k}_m a\big)_m \,. \end{aligned} \tag{6.6}$$

Adding (6.5) and (6.6), we get

$$\big(f \oplus g \,|\, \big(R + \Gamma R \Gamma\big)\,(a \oplus b)\big)_{\mathfrak{D}} = \big(\tilde{k}_m f \,|\, \tilde{k}_m a\big)_m + \big(\tilde{k}_m b^* \,|\, \tilde{k}_m g^*\big)_m = \big(f \oplus g \,|\, a \oplus b\big)_{\mathfrak{D}} \,,$$

proving that the condition (a) in Lemma 6.4 is satisfied. Thus Lemma 6.4 applies, giving a quasi-free state ω with the property (6.4).

We finally calculate the two-point function. Beginning with the computation

$$\begin{aligned} \omega\big(\Psi(g)\, \Psi^*(f)\big) &= \omega\big(B(0 \oplus g)\, B(f \oplus 0)\big) \overset{(\star)}{=} \omega\Big(B\big(\Gamma(0 \oplus g)\big)^*\, B(f \oplus 0)\Big) \\ &\overset{(6.1)}{=} \omega\big(B(g^* \oplus 0)\big)^*\, B(f \oplus 0)\big) \overset{(6.4)}{=} \big(g^* \oplus 0 \,|\, R\,(f \oplus 0)\big)_{\mathfrak{D}} \\ &\overset{(6.5)}{=} \big(\tilde{k}_m g^* \,|\, \chi^-(\tilde{\mathcal{S}}_m)\, \tilde{k}_m f\big)_m \overset{(2.7)}{=} \,<\! g^* \,|\, \chi^-(\tilde{\mathcal{S}}_m)\, \tilde{k}_m f \!>\end{aligned}$$

(where in $(\star)$ we used the property (ii) in Definition 6.1), we can apply the definition of the fermionic projector (2.17) to obtain

$$\omega\big(\Psi(g)\, \Psi^*(f)\big) = -\!<\! g^* | P f \!>\, = -\iint_{\mathcal{M} \times \mathcal{M}} g(x) P(x,y) f(y)\, d^4x\, d^4y \,.$$

This concludes the proof of Theorem 1.4.

Appendix A. Uniform L^2-Estimates of Derivatives of Dirac Solutions

We now derive a few estimates which will be needed for the proof of the mass oscillation property in Section 4.2. We use standard methods of the theory of partial differential equations and adapt them to the Dirac equation. In generalization of (3.3), we denote the spatial Sobolev norms by

$$\|\phi\|^2_{W^{a,2}} = \sum_{\alpha \text{ with } |\alpha| \leq a} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 \, d^3x \,.$$

Lemma A.1. *We are given two non-negative integers a and b as well as a smooth time-dependent potential $\mathcal{B}$. In the case $a > 0$ and $b \geq 0$, we assume furthermore that the spatial derivatives of $\mathcal{B}$ decay faster than linearly for large times in the sense that*

$$|\nabla \mathcal{B}(t)|_{C^{a-1}} \leq \frac{c}{1+|t|^{1+\varepsilon}} \tag{A.1}$$

for suitable constants $c, \varepsilon > 0$. Then there is a constant $C = C(c, \varepsilon, a, b)$ such that every family of solutions $\psi \in \mathcal{H}^\infty$ of the Dirac equation (2.3) for varying mass parameter can be estimated for all times in terms of the boundary values at $t = 0$ by

$$\big\| \partial_m^b \psi_m|_t \big\|_{W^{a,2}} \leq C \left(1 + |t|^b\right) \sum_{p=0}^b \big\| \partial_m^p \psi_m|_{t=0} \big\|_{W^{a,2}} \,.$$

Proof. We choose a multi-index α of length $a := |\alpha|$ and a non-negative integer b. Differentiating the Dirac equation (2.3) with respect to the mass parameter and to the spatial variables gives

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \nabla^\alpha \partial_m^b \psi_m = b\, \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha \big(\mathcal{B}\, \partial_m^b \psi_m \big) + \mathcal{B}\, \nabla^\alpha \partial_m^b \psi_m \,.$$

Introducing the abbreviations

$$\Xi := \nabla^\alpha \partial_m^b \psi_m \qquad \text{and} \qquad \phi := b\, \nabla^\alpha \partial_m^{b-1} \psi_m - \nabla^\alpha \big(\mathcal{B}\, \partial_m^b \psi_m \big) + \mathcal{B}\, \nabla^\alpha \partial_m^b \psi_m \,,$$

we rewrite this equation as the inhomogeneous Dirac equation

$$(\mathcal{D} - m)\, \Xi = \phi \,.$$

A calculation similar to current conservation yields

$$-i\partial_j \prec\! \Xi | \gamma^j \Xi \!\succ = \prec\! (\mathcal{D}-m)\Xi \,|\, \Xi \!\succ - \prec\! \Xi \,|\, (\mathcal{D}-m)\Xi \!\succ = \prec\! \phi | \Xi \!\succ - \prec\! \Xi | \phi \!\succ .$$

Integrating over the equal time hypersurfaces and using the Schwarz inequality, we obtain

$$\big| \partial_t \big(\Xi|_t | \Xi|_t \big)_t \big| \leq 2\, \big\| \Xi|_t \big\|_t\, \big\| \phi|_t \big\|_t$$

and thus

$$\Big| \partial_t \big\| \Xi|_t \big\| \Big| \leq \big\| \phi|_t \big\|_t \,.$$

Substituting the specific forms of Ξ and ϕ and using the Schwarz and triangle inequalities, we obtain the estimate

$$\Big| \partial_t \big\| \nabla^\alpha \partial_m^b \psi_m|_t \big\|_t \Big| \leq b\, \big\| \nabla^\alpha \partial_m^{b-1} \psi_m|_t \big\|_t + c\, a\, |\nabla \mathcal{B}(t)|_{C^{a-1}}\, \big\| \nabla^\alpha \partial_m^b \psi_m|_t \big\|_{W^{a-1,2}} \,, \tag{A.2}$$

where we used the notation (1.8).

We now proceed inductively in the maximal total order $a + b$ of the derivatives. In the case $a = b = 0$, the claim follows immediately from the unitarity of the time evolution. In order to prove the induction step, we note that in (A.2), the order of differentiation of the wave function on the right hand side is smaller than that on the left hand side at least by one. In the case $a = 0$ and $b \geq 0$, the induction hypothesis yields the inequality

$$\big|\partial_t \|\partial_m^b \psi_m|_t\|\big| \leq b\,\big\|\partial_m^{b-1}\psi_m|_t\big\| \leq b\,C\,\big(1+|t|^{b-1}\big)\sum_{p=0}^{b-1}\big\|\partial_m^p \psi_m|_{t=0}\big\|\,,$$

and integrating this inequality from 0 to t gives the result. In the case $a > 0$ and $b \geq 0$, we apply (A.1) together with the induction hypothesis to obtain

$$\begin{aligned}\Big|\partial_t \big\|\partial_m^b \psi_m|_t\big\|\Big|_{W^{a,2}} &\leq b\,C\,\big(1+|t|^{b-1}\big)\sum_{p=0}^{b-1}\big\|\partial_m^p \psi_m|_{t=0}\big\|_{W^{a,2}} \\ &\quad + c\,C\,\frac{1+|t|^b}{1+|t|^{1+\varepsilon}}\sum_{p=0}^{b}\big\|\partial_m^p \psi_m|_{t=0}\big\|_{W^{a-1,2}}\,.\end{aligned}$$

Again integrating over t gives the result. □

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