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Claudio Dappiaggi, Heiko Gimperlein, Simone Murro
and Alexander Schenkel

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Claudio Dappiaggi[1,a], Heiko Gimperlein[2,3,b], Simone Murro[4,c], Alexander Schenkel[2,d]

[1] Dipartimento di Fisica, Università di Pavia & INFN, sezione di Pavia,
Via Bassi 6, 27100 Pavia, Italy.

[2] Department of Mathematics, Heriot-Watt University,
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, United Kingdom.
& Maxwell Institute for Mathematical Sciences, Edinburgh, United Kingdom.

[3]Institut für Mathematik, Universität Paderborn,
Warburger Str. 100, 33098 Paderborn, Germany.

[4] Fakultät für Mathematik, Universität Regensburg,
93040 Regensburg, Germany.

Email: [a]claudio.dappiaggi@unipv.it, [b]h.gimperlein@hw.ac.uk,
[c]simone.murro@ur.de, [d]aschenkel83@gmail.com

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Abstract

In this paper we develop the foundations for microlocal analysis on supermanifolds. Making use of pseudodifferential operators on supermanifolds as introduced by Rempel and Schmitt, we define a suitable notion of super wavefront set for superdistributions which generalizes Dencker's polarization sets for vector-valued distributions to supergeometry. In particular, our super wavefront sets detect polarization information of the singularities of superdistributions. We prove a refined pullback theorem for superdistributions along supermanifold morphisms, which as a special case establishes criteria when two superdistributions may be multiplied. As an application of our framework, we study the singularities of distributional solutions of a supersymmetric field theory.

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Contents

1 Introduction and summary

Supergeometry has its origins in theoretical physics, where it is used as a refined model of spacetime that treats Bosonic and Fermionic degrees of freedom on an equal footing. The basic concept is that of a supermanifold, which loosely speaking is a manifold with even (Bosonic) and odd (Fermionic) local coordinates. Quantum field theories on supermanifolds unify Bosonic and Fermionic quantum fields in a single entity called a super quantum field. They are very interesting from the perspective of a quantum field theorist because of their improved renormalization behavior. Such special features of supergeometric quantum field theories are collectively called non-renormalization theorems [GSR79, Sei93].

During the last decade, our mathematical understanding of perturbative quantum field theory on Lorentzian manifolds has steadily improved, mainly due to the development of perturbative algebraic quantum field theory (pAQFT), see e.g. [FR15] for a recent review. Microlocal analysis serves as one of the main techniques used in pAQFT; its role is to analyze carefully the singularities of distributions like propagators and n-point functions arising in the quantum field theory. This proves essential for performing the perturbative construction and its renormalization.

The goal of this paper is to develop the foundations of microlocal analysis on supermanifolds. Our work is based on and extends earlier investigations of Rempel and Schmitt [RS83] on pseudodifferential operators on supermanifolds. In particular, we develop a supergeometric generalization of the wavefront set, which is a suitable concept to encode polarization information about the singularities of distributions on supermanifolds. Our super wavefront sets are motivated by the polarization sets of Dencker [Den82] for vector-valued distributions. However, they are constructed in such a way that they transform in a natural way under supermanifold morphisms and not only vector bundle morphisms. The techniques which we develop in this paper are expected to play a major role in extending pAQFT to supergeometric quantum field theories [HHS15], a longer term research goal that we hope to achieve in future works. This would provide a rigorous framework to prove (and extend to curved supermanifolds) the non-renormalization theorems in [GSR79, Sei93].

The outline of the remainder of this paper is as follows: In Section 2 we fix our notations and give a brief review of some basic aspects of the theory of supermanifolds. Based on an idea of Rempel and Schmitt [RS83], we assign in Section 3 to each supermanifold $X = (\widetilde{X}, \mathcal{O}_X)$ a polarization bundle $\pi : \mathcal{P}^*X \to \mathcal{T}^*\widetilde{X}$ over the cotangent bundle of the underlying smooth manifold $\widetilde{X}$. The polarization bundle is a super vector bundle that encodes the local polarization information of superfunctions and superdistributions on X. In Section 4 we introduce super pseudodifferential operators on supermanifolds following [RS83], define their super principal symbols as bundle mappings between the polarization bundles, and develop their calculus. Examples of such operators come from the equations of motion (and the associated propagators) of the supergeometric field theories studied in [HHS15]. Crucially, our concept of super principal symbols is able to detect ellipticity (or hyperbolicity) of these operators. In Section 5 we introduce polarization sets for supermanifolds, motivated by [Den82], and thereby define the super wavefront set of a superdistribution. We analyze the transformation property of the super wavefront set under supermanifold morphisms and their compatibility with the action of super pseudodifferential operators. In Section 6 we generalize the ordinary pullback theorem for distributions on manifolds [Hör03, Theorem 8.2.4] to supermanifolds. By including the polarization information of superdistributions (and their singularities), this leads to a refinement of the ordinary pullback theorem. An important example is given by the super diagonal mapping, which provides criteria when two superdistributions may be multiplied. As an application, we analyze in Section 7 the singularities of distributional solutions to the equation of motion of the 3|2-dimensional Wess-Zumino model.

2 Preliminaries

We briefly recall some basic aspects of the theory of supermanifolds which are frequently used in our work. For a detailed introduction to this subject see, for example, [CCF11, DM99] and also [HHS15, Section 2] for a short summary.

A *superspace* is a pair $X = (\widetilde{X}, \mathcal{O}_X)$ consisting of a topological space $\widetilde{X}$ (second-countable and Hausdorff) and a sheaf of supercommutative superalgebras $\mathcal{O}_X$ on $\widetilde{X}$, called the structure sheaf. Explicitly, to each open $U \subseteq \widetilde{X}$ there is assigned a supercommutative superalgebra $\mathcal{O}_X(U)$, called the sections of $\mathcal{O}_X$ over U, and to each open $V \subseteq U \subseteq \widetilde{X}$ a superalgebra homomorphism $\mathrm{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$, called the restriction map. The restriction maps satisfy the conditions

$$\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{O}_X(U)} \ , \qquad \mathrm{res}_{V,W} \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,W} \ , \tag{2.1}$$

for all open $W \subseteq V \subseteq U \subseteq \widetilde{X}$. Moreover, given any open cover $\{U_\alpha \subseteq \widetilde{X}\}$ of $\widetilde{X}$ and any matching family of local sections, i.e.

$$\big\{ f_\alpha \in \mathcal{O}_X(U_\alpha) : \mathrm{res}_{U_\alpha, U_{\alpha\beta}}(f_\alpha) = \mathrm{res}_{U_\beta, U_{\alpha\beta}}(f_\beta) \ \ \forall \alpha, \beta \big\} \ , \tag{2.2}$$

where $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is the intersection, there exists a unique global section $f \in \mathcal{O}_X(\widetilde{X})$ such that $f_\alpha = \mathrm{res}_{\widetilde{X}, U_\alpha}(f)$. Loosely speaking, this means that a family of local sections of $\mathcal{O}_X$ which match in all overlaps can be glued to a unique global section and that any global section arises in that way.

The standard example of a superspace is $\mathbb{R}^{m|n} := (\mathbb{R}^m, C^\infty_{\mathbb{R}^m} \otimes \wedge^\bullet \mathbb{R}^n)$, where $\wedge^\bullet \mathbb{R}^n$ denotes the Grassmann algebra with n generators. The sections over any open $U \subseteq \mathbb{R}^m$ are given by $C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$. Any element $f \in C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ has an expansion

$$f = \sum_{I \in \mathbb{Z}_2^n} f_I \, \theta^I := \sum_{(i_1, \dots, i_n) \in \mathbb{Z}_2^n} f_{(i_1, \dots, i_n)} \ \theta^{1 i_1} \cdots \theta^{n i_n} \ , \tag{2.3}$$

where $\mathbb{Z}_2^n := \{0,1\}^n$, $\{\theta^a \in \mathbb{R}^n : a = 1, \dots, n\}$ is the standard basis of $\mathbb{R}^n$ and $f_I \in C^\infty(U)$.

A morphism $\chi : X \to Y$ between two superspaces $X = (\widetilde{X}, \mathcal{O}_X)$ and $Y = (\widetilde{Y}, \mathcal{O}_Y)$ is a pair $(\widetilde{\chi}, \chi^*)$ consisting of a continuous map $\widetilde{\chi} : \widetilde{X} \to \widetilde{Y}$ and a sheaf homomorphism $\chi^* : \mathcal{O}_Y \to \widetilde{\chi}_* \mathcal{O}_X$, where $\widetilde{\chi}_* \mathcal{O}_X$ is the direct image sheaf. Explicitly, to each open $U \subseteq \widetilde{Y}$ there is assigned a superalgebra homomorphism $\chi^*_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\widetilde{\chi}^{-1}(U))$, such that for all open $V \subseteq U \subseteq \widetilde{Y}$ the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\ \chi^*_U\ } & \mathcal{O}_X(\widetilde{\chi}^{-1}(U)) \\ {\scriptstyle \mathrm{res}_{U,V}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{\widetilde{\chi}^{-1}(U), \widetilde{\chi}^{-1}(V)}} \\ \mathcal{O}_Y(V) & \xrightarrow[\ \chi^*_V\ ]{} & \mathcal{O}_X(\widetilde{\chi}^{-1}(V)) \end{array} \tag{2.4}$$

commutes.

A *supermanifold* (of dimension $m|n$) is a superspace $X = (\widetilde{X}, \mathcal{O}_X)$ which is locally isomorphic to $\mathbb{R}^{m|n}$. More explicitly, this means that for any point $x \in \widetilde{X}$ there exists an open neighborhood $U \subseteq \widetilde{X}$ of x such that $X|_U := (U, \mathcal{O}_X|_U)$ is isomorphic as a superspace to $W^{m|n} := (W, C^\infty_W \otimes \wedge^\bullet \mathbb{R}^n)$, for some open subset $W \subseteq \mathbb{R}^m$. We say that $\chi : X \to Y$ is a morphism between two supermanifolds $X = (\widetilde{X}, \mathcal{O}_X)$ and $Y = (\widetilde{Y}, \mathcal{O}_Y)$ if it is a superspace morphism.

Every supermanifold $X = (\widetilde{X}, \mathcal{O}_X)$ comes together with a filtration

$$\mathcal{O}_X(U) \longleftarrow \mathcal{J}_X(U) \longleftarrow \mathcal{J}^2_X(U) \longleftarrow \cdots \quad , \tag{2.5}$$

for any open $U \subseteq \widetilde{X}$, where

$$\mathcal{J}_X(U) := \big\{ f \in \mathcal{O}_X(U) : f^N = 0\,,\ \text{for some } N \in \mathbb{N}_0 \big\} \subseteq \mathcal{O}_X(U) \tag{2.6}$$

is the superideal of nilpotents and $\mathcal{J}^k_X(U)$ is its k-th power, $k \geq 2$. Locally, i.e. for sufficiently small $U \subseteq \widetilde{X}$, by definition there exists an isomorphism $\mathcal{O}_X(U) \simeq C^\infty(W) \otimes \wedge^\bullet \mathbb{R}^n$ of superalgebras for some open $W \subseteq \mathbb{R}^m$. Applying this isomorphism to the filtration (2.5) we obtain

$$C^\infty(W) \otimes \wedge^\bullet \mathbb{R}^n \longleftarrow C^\infty(W) \otimes \wedge^{\geq 1} \mathbb{R}^n \longleftarrow C^\infty(W) \otimes \wedge^{\geq 2} \mathbb{R}^n \longleftarrow \cdots \quad , \tag{2.7}$$

which implies that locally $\mathcal{J}^k_X(U) = 0$ for all $k > n$. Indeed, in this case $C^\infty(W) \otimes \wedge^{\geq k} \mathbb{R}^n = 0$. Due to the sheaf condition the same statement holds globally, i.e. $\mathcal{J}^k_X(U) = 0$ for all $k > n$ and $U \subseteq \widetilde{X}$ open.

Let us also recall that to any $m|n$-dimensional supermanifold $X = (\widetilde{X}, \mathcal{O}_X)$ there is canonically assigned an m-dimensional manifold; it is specified by the topological space $\widetilde{X}$ together with the structure sheaf $\mathcal{O}_X / \mathcal{J}_X$. The underlying continuous map $\widetilde{\chi} : \widetilde{X} \to \widetilde{Y}$ of any supermanifold morphism $\chi : X \to Y$ is smooth with respect to this manifold structure. The supermanifold morphism $\iota_{\widetilde{X},X} : (\widetilde{X}, \mathcal{O}_X/\mathcal{J}_X) \to (\widetilde{X}, \mathcal{O}_X)$, given by $\widetilde{\iota}_{\widetilde{X},X} = \mathrm{id}_{\widetilde{X}}$ and the quotient mapping $\iota^*_{\widetilde{X},X} : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}_X$, embeds the underlying smooth manifold into the supermanifold.

3 Polarization bundles

The space of superdistributions on a supermanifold X is locally given by $\mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$, where $U \subseteq \mathbb{R}^m$ is an open subset and $\mathcal{D}'(U)$ denotes the space of distributions on U. Hence, superdistributions locally carry polarization information in the Grassmann algebra $\wedge^\bullet \mathbb{R}^n$. Following ideas of Rempel and Schmitt [RS83], we now construct a bundle over the cotangent bundle $\mathcal{T}^*\widetilde{X}$ of the underlying manifold $\widetilde{X}$, which describes the polarization information of superdistributions and their singularities.

Let us start with the case where the supermanifold is a superdomain, i.e. $U^{m|n} := (U, C^\infty_U \otimes \wedge^\bullet \mathbb{R}^n) \subseteq \mathbb{R}^{m|n}$ for some open $U \subseteq \mathbb{R}^m$. In this case the polarization bundle is defined as the trivial bundle

$$\pi : \mathcal{P}^* U^{m|n} := \mathcal{T}^* U \times \wedge^\bullet \mathbb{C}^n \longrightarrow \mathcal{T}^* U \ , \quad (x, k, \lambda) \longmapsto (x, k) \ , \tag{3.1}$$

where the fibers are the complexified Grassmann algebras and $\mathcal{T}^* U = U \times \mathbb{R}^m$ is the cotangent bundle over U.

Now consider a supermanifold morphism $\chi : U^{m|n} \to V^{m'|n'}$ between two superdomains. The underlying smooth map $\widetilde{\chi} : U \to V$ induces a fiber-wise pullback map $\mathcal{T}^*\widetilde{\chi} : \mathcal{T}^*_{\widetilde{\chi}(x)} V \to \mathcal{T}^*_x U$

of cotangent vectors, for any point $x \in U$. Our goal is to construct a suitable fiber-wise map between the polarization bundles such that the diagram

$$\begin{array}{ccc} \mathcal{P}^*V^{m'|n'}\big|_{\mathcal{T}^*_{\widetilde{\chi}(x)}V} & \xrightarrow{\ \mathcal{P}^*\chi\ } & \mathcal{P}^*U^{m|n}\big|_{\mathcal{T}^*_x U} \\ \pi\big\downarrow & & \big\downarrow\pi \\ \mathcal{T}^*_{\widetilde{\chi}(x)}V & \xrightarrow[\ \mathcal{T}^*\widetilde{\chi}\ ]{} & \mathcal{T}^*_x U \end{array} \tag{3.2}$$

commutes, for any point $x \in U$.

To approach this problem, we have to analyze in more detail the superalgebra homomorphism $\chi_V^* : C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^{n'} \to C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$. Using the (non-canonical!) $\mathbb{Z}$-gradings

$$C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^{n'} = \bigoplus_{i=0}^{n'} C^\infty(V) \otimes \wedge^i \mathbb{R}^{n'} \ , \quad C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^{n} = \bigoplus_{i=0}^{n} C^\infty(U) \otimes \wedge^i \mathbb{R}^{n} \ , \tag{3.3}$$

we decompose χ_V^* into components

$$(\chi_V^*)_j{}^i : C^\infty(V) \otimes \wedge^i \mathbb{R}^{n'} \longrightarrow C^\infty(U) \otimes \wedge^j \mathbb{R}^n \ , \tag{3.4}$$

which are linear maps by construction. Notice that $(\chi_V^*)_0{}^0 = \widetilde{\chi}^* : C^\infty(V) \to C^\infty(U)$ is the pullback of functions along the underlying smooth map $\widetilde{\chi} : U \to V$. We now show that the other components $(\chi_V^*)_j{}^i$ are relative differential operators along $\widetilde{\chi}^*$. Recall, e.g. from [CCF11, Theorem 4.1.11], that the superalgebra homomorphism χ_V^* is uniquely specified by its action on the supercoordinates $(y^{\mu'}, \zeta^{a'})$ of $V^{m'|n'}$. We have that

$$\chi_V^*(y^{\mu'}) - \widetilde{\chi}^*(y^{\mu'}) \in \mathcal{J}^2_{U^{m|n}}(U) \ , \qquad \chi_V^*(\zeta^{a'}) \in \mathcal{J}^1_{U^{m|n}}(U) \ , \tag{3.5}$$

where $\mathcal{J}^k_{U^{m|n}}(U)$ is the filtration explained in (2.5), see also (2.7). For a generic $f \in C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^{n'}$, we use the component expansion $f = \sum_{I \in \mathbb{Z}_2^{n'}} f_I \, \zeta^I$ and obtain

$$\chi_V^*(f) = \sum_{I \in \mathbb{Z}_2^{n'}} \chi_V^*(f_I) \, \chi_V^*(\zeta^I) \ . \tag{3.6}$$

Using the first property in (3.5) and Taylor expansion in the odd coordinates, we observe that

$$\chi_V^*(f_I) = \widetilde{\chi}^*(f_I) + \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \widetilde{\chi}^*\big(Q_l(f_I)\big) \, \lambda_{2l} \ , \tag{3.7}$$

where Q_l is a differential operator of order l and $\lambda_{2l} \in \wedge^{2l}\mathbb{R}^n$. Using also the second property in (3.5) and the fact that the odd coordinates θ^a on $U^{m|n}$ are nilpotent, we obtain

$$(\chi_V^*)_j{}^i = \begin{cases} \widetilde{\chi}^* \circ (D^\chi)_j{}^i & , \ \text{if } j - i \geq 0 \text{ even} \ , \\ 0 & , \ \text{else} \ . \end{cases} \tag{3.8}$$

Here $(D^\chi)_j{}^i$ are matrices of differential operators of order $\frac{j-i}{2}$. In summary, we have shown that, for any supermanifold morphism $\chi : U^{m|n} \to V^{m'|n'}$ between two superdomains, the corresponding superalgebra homomorphism χ_V^* can be factorized uniquely as

$$\chi_V^* = \widetilde{\chi}^* \circ D^\chi \ , \tag{3.9}$$

where D^χ is a matrix of differential operators.

We now define the mapping $\mathcal{P}^*\chi$ in (3.2) component-wise by

$$
\begin{aligned}
(\mathcal{P}^*\chi)_j{}^i : \mathcal{T}^*_{\widetilde{\chi}(x)}V \times \wedge^i\mathbb{C}^{n'} &\longrightarrow \mathcal{T}^*_x U \times \wedge^j \mathbb{C}^n \;,\\
\big(\widetilde{\chi}(x), k', \lambda'\big) &\longmapsto \begin{cases} \big(x, \mathcal{T}^*\widetilde{\chi}(k'), \sigma_{\frac{j-i}{2}}((D^\chi)_j{}^i)(\widetilde{\chi}(x),k')(\lambda')\big) & , \text{ if } j-i \geq 0 \text{ even} \;,\\ \big(x, \mathcal{T}^*\widetilde{\chi}(k'), 0\big) & , \text{ else} \;, \end{cases}
\end{aligned}
\tag{3.10}
$$

where σ_l denotes the principal symbol of a differential operator of order l.

Given now two supermanifold morphisms $\chi : U^{m|n} \to V^{m'|n'}$ and $\chi' : V^{m'|n'} \to W^{m''|n''}$, we can form the composition $\chi' \circ \chi : U^{m|n} \to W^{m''|n''}$. From $(\chi' \circ \chi)^*_W = \chi^*_V \circ \chi'^*_W$, it follows that the components satisfy

$$
((\chi' \circ \chi)^*_W)_j{}^i = \sum_{h=0}^{n'} (\chi^*_V)_j{}^h \circ (\chi'^*_W)_h{}^i \tag{3.11}
$$

and hence

$$
\widetilde{\chi}^* \circ \widetilde{\chi'}^* \circ (D^{\chi'\circ\chi})_{i+2l}{}^i = \sum_{j=0}^{l} \widetilde{\chi}^* \circ (D^\chi)_{i+2l}{}^{i+2j} \circ \widetilde{\chi'}^* \circ (D^{\chi'})_{i+2j}{}^i \;, \tag{3.12}
$$

for the non-vanishing components of $((\chi' \circ \chi)^*_W)_j{}^i$. Combining this with (3.10) and the multiplicativity of principal symbols, it is easy to check that the polarization mapping in (3.2) is (contravariantly) compatible with compositions, i.e.

$$
\mathcal{P}^*(\chi' \circ \chi) = (\mathcal{P}^*\chi) \circ (\mathcal{P}^*\chi') \;. \tag{3.13}
$$

Moreover, by definition it is clear that $\mathcal{P}^*\mathrm{id}_{U^{m|n}} = \mathrm{id}_{\mathcal{P}^*U^{m|n}}$.

Because of this result, the concept of polarization bundle globalizes from superdomains to supermanifolds: Let $X = (\widetilde{X}, \mathcal{O}_X)$ be any $m|n$-dimensional supermanifold and choose an open cover $\{U_\alpha \subseteq \widetilde{X}\}$ and isomorphisms

$$
\rho_\alpha : X|_{U_\alpha} \longrightarrow W_\alpha{}^{m|n} \subseteq \mathbb{R}^{m|n} \tag{3.14}
$$

to superdomains, i.e. a superatlas. In all overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ this gives rise to transition supermanifold morphisms

$$
\chi_{\alpha\beta} := \rho_\beta \circ \rho_\alpha^{-1} : W_\alpha{}^{m|n}|_{\widetilde{\rho_\alpha}(U_{\alpha\beta})} \longrightarrow W_\beta{}^{m|n}|_{\widetilde{\rho_\beta}(U_{\alpha\beta})} \;, \tag{3.15}
$$

which satisfy $\chi_{\alpha\alpha} = \mathrm{id}_{W_\alpha{}^{m|n}}$ for all α as well as the cocycle condition $\chi_{\beta\gamma} \circ \chi_{\alpha\beta} = \chi_{\alpha\gamma}$ on all triple overlaps $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$. In any superchart $W_\alpha{}^{m|n}$ we take the trivial polarization bundle $\mathcal{P}^*W_\alpha{}^{m|n}$ from (3.1). The global polarization bundle $\mathcal{P}^*X$ on the supermanifold X is then given by gluing these local bundles via the transition functions $g_{\alpha\beta} := \mathcal{P}^*\chi_{\beta\alpha}$; the cocycle condition for the $g_{\alpha\beta}$ follows from (3.13). It is important to stress that, even though the local polarization bundles (3.1) look like Grassmann algebra bundles, the transition functions $g_{\alpha\beta}$ in general *do not* preserve the product structure and the $\mathbb{Z}$-grading on the fibers – note the outer-diagonal terms in (3.10), which depend on k. However, the coarser $\mathbb{Z}_2$-grading on the fibers of the local bundles is preserved by the transition functions. Hence the polarization bundle $\pi : \mathcal{P}^*X \to \mathcal{T}^*\widetilde{X}$ is a complex super vector bundle for any supermanifold $X = (\widetilde{X}, \mathcal{O}_X)$.

4 Super pseudodifferential operators

Following Rempel and Schmitt [RS83], we introduce super pseudodifferential operators on supermanifolds and define their super principal symbols. As in the case of a manifold, the definition is local, and we first consider the case where the supermanifold is a superdomain $U^{m|n} \subseteq \mathbb{R}^{m|n}$. A linear map

$$A : C^\infty_{\mathrm{c}}(U) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \tag{4.1}$$

is called a *super pseudodifferential operator* on $U^{m|n}$ if all its components ${A_j}^i : C^\infty_{\mathrm{c}}(U) \otimes \wedge^i \mathbb{R}^n \to C^\infty(U) \otimes \wedge^j \mathbb{R}^n$ are (matrices of) pseudodifferential operators on $U \subseteq \mathbb{R}^m$. In the following all pseudodifferential operators are implicitly assumed to be properly supported and classical, see e.g. [Shu13] for the relevant definitions. Recall, in particular, that properly supported pseudodifferential operators map compactly supported functions to compactly supported functions, hence they can be composed. The composition is again a properly supported pseudodifferential operator. Given any supermanifold *isomorphism* $\chi : U^{m|n} \to V^{m|n}$ and a super pseudodifferential operator A on $U^{m|n}$, consider the linear map

$${\chi^*_V}^{-1} \circ A \circ \chi^*_V : C^\infty_{\mathrm{c}}(V) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^n \ . \tag{4.2}$$

It defines a super pseudodifferential operator on $V^{m|n}$ because the components of χ^*_V and its inverse are both (matrices of) relative differential operators, cf. (3.8).

Definition 4.1. We say that a super pseudodifferential operator A on $U^{m|n}$ is of order l if its components ${A_j}^i$ are (matrices of) pseudodifferential operators on U of order $\frac{j-i}{2} + l$, i.e.,

$$s\Psi\mathrm{DO}^l(U^{m|n}) := \Big\{ A : C^\infty_{\mathrm{c}}(U) \otimes \wedge^\bullet \mathbb{R}^n \to C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n : {A_j}^i \in \Psi\mathrm{DO}^{\frac{j-i}{2}+l}(U) \Big\} \ . \tag{4.3}$$

The super principal symbol of $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ is the super vector bundle map

$$\sigma_l(A) : \mathcal{P}^* U^{m|n} \longrightarrow \mathcal{P}^* U^{m|n} \tag{4.4}$$

with components given by

$$\begin{aligned} {(\sigma_l(A))_j}^i : \mathcal{T}^* U \times \wedge^i \mathbb{C}^n &\longrightarrow \mathcal{T}^* U \times \wedge^j \mathbb{C}^n \ , \\ (x, k, \lambda) &\longmapsto \big(x, k, \sigma_{\frac{j-i}{2}+l}({A_j}^i)(x,k)(\lambda)\big) \ , \end{aligned} \tag{4.5}$$

where $\sigma_{\frac{j-i}{2}+l}({A_j}^i)$ is the ordinary principal symbol of order $\frac{j-i}{2} + l$ of ${A_j}^i$.

Example 4.2. Let $\chi : U^{m|n} \to V^{m|n}$ be a supermanifold isomorphism between two superdomains, and consider the unique factorization $\chi^*_V = \widetilde{\chi}^* \circ D^\chi$ given in (3.8). Then D^χ is a super pseudodifferential operator of order 0, i.e. $D^\chi \in s\Psi\mathrm{DO}^0(V^{m|n})$. In the case where $U = V$ and $\widetilde{\chi} = \mathrm{id}_U$, the super principal symbol of D^χ is the polarization mapping (3.10), i.e. $\sigma_0(D^\chi) = \mathcal{P}^*\chi$.

We collect some useful properties of super pseudodifferential operators and their super principal symbols. The proofs of these statements follow easily from our definitions and are omitted.

Lemma 4.3. *Let $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ and $B \in s\Psi\mathrm{DO}^{l'}(U^{m|n})$. Then the following statements hold true:*

a) $B \circ A \in s\Psi\mathrm{DO}^{l+l'}(U^{m|n})$.

b) If $\chi : U^{m|n} \to V^{m|n}$ is a supermanifold isomorphism, then ${\chi_V^*}^{-1} \circ A \circ \chi_V^* \in s\Psi\mathrm{DO}^l(V^{m|n})$.

Lemma 4.4. *Let $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ and $B \in s\Psi\mathrm{DO}^{l'}(U^{m|n})$. Then the following statements hold true:*

a) $\sigma_{l+l'}(B \circ A) = \sigma_{l'}(B) \circ \sigma_l(A)$.

b) If $\chi : U^{m|n} \to V^{m|n}$ is a supermanifold isomorphism, then

$$\sigma_l\big({\chi_V^*}^{-1} \circ A \circ \chi_V^*\big) = (\mathcal{P}^*\chi^{-1}) \circ \sigma_l(A) \circ (\mathcal{P}^*\chi) \ . \tag{4.6}$$

Super pseudodifferential operators and their super principal symbols are easily globalized to supermanifolds by slightly adapting the globalization procedure for the pseudodifferential operators on manifolds, see e.g. [Tre80, Chapter I, Section 5]. Let $X = (\widetilde{X}, \mathcal{O}_X)$ be an $m|n$-dimensional supermanifold and $\mathcal{O}_{X,\mathrm{c}}(\widetilde{X})$ the space of compactly supported global sections of the structure sheaf. Consider a maximal superatlas $\rho_\alpha : X|_{U_\alpha} \to {W_\alpha}^{m|n}$. A super pseudodifferential operator $A \in s\Psi\mathrm{DO}^l(X)$ of order l on X is a continuous linear map $A : \mathcal{O}_{X,\mathrm{c}}(\widetilde{X}) \to \mathcal{O}_X(\widetilde{X})$ such that, for every superchart ${W_\alpha}^{m|n}$, the linear map A_α defined by the diagram

$$\begin{array}{ccccccc} C^\infty_{\mathrm{c}}(W_\alpha) \otimes \wedge^\bullet \mathbb{R}^n & & \xrightarrow{\quad A_\alpha \quad} & & & & C^\infty(W_\alpha) \otimes \wedge^\bullet \mathbb{R}^n \\ \Big\downarrow {\scriptstyle \rho_\alpha^*} & & & & & & \Big\uparrow {\scriptstyle {\rho_\alpha^*}^{-1}} \\ \mathcal{O}_{X,\mathrm{c}}(U_\alpha) & \xrightarrow[\mathrm{ext}_{\widetilde{X},U_\alpha}]{} & \mathcal{O}_{X,\mathrm{c}}(\widetilde{X}) & \xrightarrow[A]{} & \mathcal{O}_X(\widetilde{X}) & \xrightarrow[\mathrm{res}_{U_\alpha,\widetilde{X}}]{} & \mathcal{O}_X(U_\alpha) \end{array} \tag{4.7}$$

is an element in $s\Psi\mathrm{DO}^l({W_\alpha}^{m|n})$. Here ext denotes the extension (by zero) maps for compactly supported sections. To each $A \in s\Psi\mathrm{DO}^l(X)$ we associate a super principal symbol, which is a super vector bundle morphism

$$\sigma_l(A) : \mathcal{P}^*X \longrightarrow \mathcal{P}^*X \ . \tag{4.8}$$

Explicitly, the super principal symbol $\sigma_l(A)$ is constructed by gluing together the collection of all local super principal symbols $\sigma_l(A_\alpha)$ of the operators A_α in (4.7). This is consistent on account of Lemma 4.4 b).

To study the singularities of distributions, the notion of ellipticity is crucial.

Definition 4.5. We say that a super pseudodifferential operator $E \in s\Psi\mathrm{DO}^l(X)$ is *elliptic* if the super principal symbol $\sigma_l(E)$ is invertible on $\mathcal{T}^*\widetilde{X} \setminus \mathbf{0}$.

Many properties of elliptic pseudodifferential operators on ordinary manifolds are still valid in our framework. In particular, we obtain

Lemma 4.6. *Let $E \in s\Psi\mathrm{DO}^l(X)$ be an elliptic super pseudodifferential operator. Then there exists a super pseudodifferential operator $F \in s\Psi\mathrm{DO}^{-l}(X)$ such that*

$$E \circ F - \mathrm{id} \in s\Psi\mathrm{DO}^{-\infty}(X) \quad \textit{and} \quad F \circ E - \mathrm{id} \in s\Psi\mathrm{DO}^{-\infty}(X) \ , \tag{4.9}$$

where $s\Psi\mathrm{DO}^{-\infty}(X) := \bigcap_{l \in \mathbb{R}} s\Psi\mathrm{DO}^l(X)$. F is called a parametrix for E.

Proof. The proof is as in the case of ordinary manifolds, see e.g. [Shu13, Theorem 5.1]. □

We shall now give examples of super differential and super pseudodifferential operators $A \in s\Psi\mathrm{DO}^l(X)$ which have their origin in supersymmetric field theory.

Example 4.7. Let $X = \mathbb{R}^{1|1}$ be the superline. The dynamics of a superparticle on X is governed by a super differential operator, which in global supercoordinates (t,θ) on $\mathbb{R}^{1|1}$ reads as

$$\begin{aligned} P : C^\infty(\mathbb{R}) \otimes \wedge^\bullet \mathbb{R} &\longrightarrow C^\infty(\mathbb{R}) \otimes \wedge^\bullet \mathbb{R} \ , \\ f = f_0 + f_1\,\theta &\longmapsto \partial_t f_1 + \partial_t^2 f_2\,\theta \ , \end{aligned} \tag{4.10}$$

cf. [HHS15, Section 8.1]. In our component notation, the operator P is given by

$$P = \begin{pmatrix} 0 & \partial_t \\ \partial_t^2 & 0 \end{pmatrix} \ . \tag{4.11}$$

Notice that $P \in s\Psi\mathrm{DO}^{\frac{3}{2}}(\mathbb{R}^{1|1})$. Its super principal symbol

$$\sigma_{\frac{3}{2}}(P)(t,k) = \begin{pmatrix} 0 & \mathrm{i}\,k \\ -k^2 & 0 \end{pmatrix} \tag{4.12}$$

is invertible for all $(t,k) \in \mathcal{T}^*\mathbb{R} \setminus \mathbf{0}$, hence P is elliptic. Specifically, the inverse is

$$\sigma_{-\frac{3}{2}}(F)(t,k) := \sigma_{\frac{3}{2}}(P)^{-1}(t,k) = \begin{pmatrix} 0 & -\frac{1}{k^2} \\ -\frac{\mathrm{i}}{k} & 0 \end{pmatrix} \ . \tag{4.13}$$

In this case the parametrix F of P from Lemma 4.6 is explicitly given by the integral kernel

$$F(t,t') = \frac{1}{2} \begin{pmatrix} 0 & (t-t')\,\mathrm{sign}(t-t') \\ \mathrm{sign}(t-t') & 0 \end{pmatrix} \ . \tag{4.14}$$

Example 4.8. Let us consider $X = (M, C^\infty_M \otimes \wedge^\bullet \mathbb{R}^2)$, where M is a smooth 3-dimensional Lorentzian manifold. The equation of motion operator $P : \mathcal{O}_X(M) \to \mathcal{O}_X(M)$ of the 3|2-dimensional Wess-Zumino model on X is then given in component notation by

$$P = \begin{pmatrix} m & 0 & -1 \\ 0 & \mathrm{i}\,\nabla\!\!\!/ + m & 0 \\ \Box & 0 & m \end{pmatrix} \ , \tag{4.15}$$

cf. [HHS15, Section 8.2]. Here $\mathrm{i}\,\nabla\!\!\!/$ is the Dirac operator (on M), $\Box$ is the d'Alembert operator on M and $m \geq 0$ is a mass term. Notice that $P \in s\Psi\mathrm{DO}^1(X)$ is of order 1, and in local coordinates x^μ and k_μ on $\mathcal{T}^*M$ its super principal symbol is given by

$$\sigma_1(P)(x,k) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -\gamma^\mu(x)\,k_\mu & 0 \\ -k_\mu k_\nu\, g^{\mu\nu}(x) & 0 & 0 \end{pmatrix} \ . \tag{4.16}$$

Using the Clifford algebra relations $\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}$ for the gamma-matrices, it is easy to check that $\sigma_1(P)(x,k)$ is invertible for all $(x,k) \in \mathcal{T}^*M \setminus \mathbf{0}$ which are not light-like (i.e. $k_\mu k_\nu g^{\mu\nu}(x) \neq 0$). More explicitly, we have

$$\sigma_1(P)(x,k)^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{k_\mu k_\nu\, g^{\mu\nu}(x)} \\ 0 & -\frac{\gamma^\mu(x)\,k_\mu}{k_\mu k_\nu\, g^{\mu\nu}(x)} & 0 \\ -1 & 0 & 0 \end{pmatrix} \ . \tag{4.17}$$

Because $\sigma_1(P)(x,k)$ is invertible for non-light-like $(x,k) \in \mathcal{T}^*M \setminus \mathbf{0}$, we call P hyperbolic.

Remark 4.9. Our definition of orders and super principal symbols for super pseudodifferential operators on supermanifolds is well suited for the examples of super (pseudo-)differential operators arising in supersymmetric field theory. This is a consequence of our definition of the polarization bundle $\pi : \mathcal{P}^*X \to \mathcal{T}^*\widetilde{X}$ and in particular of the assignment of the polarization mapping defined in (3.10). Rempel and Schmitt [RS83] consider also more general polarization bundles (defined via polarization mappings different from (3.10)), which are classified by what they call admissible tuples. It is important to stress that all other polarization bundles in [RS83] lead to an assignment of orders and super principal symbols for super pseudodifferential operators on X which is not able to detect ellipticity and hyperbolicity in our examples above. This provides us with a motivation for our choice of polarization bundle given in (3.10).

5 Super wavefront sets

We start with the case where the supermanifold is a superdomain $U^{m|n} \subseteq \mathbb{R}^{m|n}$. Then the space of superdistributions $\mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ is the dual of $C^\infty_{\mathrm{c}}(U) \otimes \wedge^\bullet \mathbb{R}^n$, and both $C^\infty_{\mathrm{c}}(U) \otimes \wedge^\bullet \mathbb{R}^n$ and $C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ are dense sub-spaces. We say that a superdistribution $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ is smooth if it is an element of $C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$. Crucially, by duality, any (properly supported) super pseudodifferential operator A on $U^{m|n}$ admits a continuous extension to superdistributions, $A : \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n \to \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$. Global superdistributions on a supermanifold X are obtained by gluing local superdistributions in a superatlas, via the transition morphisms $\chi_{\alpha\beta}$ given in (3.15).

We define the super wavefront set of a superdistribution on X motivated by the approach of Dencker [Den82] for vector-valued distributions. The starting point is the polarization bundle $\pi : \mathcal{P}^*X \to \mathcal{T}^*\widetilde{X}$ introduced in Section 3. We denote by

$$\pi : \widehat{\mathcal{P}}^*X := \pi^{-1}\big(\mathcal{T}^*\widetilde{X} \setminus \mathbf{0}\big) \longrightarrow \mathcal{T}^*\widetilde{X} \setminus \mathbf{0} \tag{5.1}$$

the restriction of the polarization bundle to the cotangent bundle with the zero-section removed.

Definition 5.1. The super wavefront set (of order l) of a superdistribution $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ is defined as the intersection

$$s\mathrm{WF}^l(u) := \bigcap_{\substack{A \in s\Psi\mathrm{DO}^l(U^{m|n}) \\ \text{s.t. } Au \text{ smooth}}} \Big\{(x,k,\lambda) \in \widehat{\mathcal{P}}^*U^{m|n} \,:\, \sigma_l(A)(x,k)\big(\lambda\big) = 0\Big\} \subseteq \widehat{\mathcal{P}}^*U^{m|n} \ . \tag{5.2}$$

We collect some important properties of the super wavefront sets defined above.

Proposition 5.2. *For any $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$, the following properties hold true:*

a) $s\mathrm{WF}^l(u) = s\mathrm{WF}^{l'}(u)$ *for all* l, l'.

b) For $u = \sum_{I \in \mathbb{Z}_2^n} u_I\, \theta^I \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$,

$$\pi\Big(s\mathrm{WF}^l(u) \setminus \big((\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\}\big)\Big) = \bigcup_{I \in \mathbb{Z}_2^n} \mathrm{WF}(u_I) \ , \tag{5.3}$$

where $\pi : \widehat{\mathcal{P}}^*U^{m|n} \to \mathcal{T}^*U \setminus \mathbf{0}$ *is the projection* (3.1) *and* $\mathrm{WF}(u_I) \subseteq \mathcal{T}^*U \setminus \mathbf{0}$ *denotes the ordinary wavefront set of* $u_I \in \mathcal{D}'(U)$.

Proof. To show item a), take any $(x,k,\lambda) \notin s\mathrm{WF}^l(u)$. By assumption there exists $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ such that Au smooth and $\sigma_l(A)(x,k)\big(\lambda\big) \neq 0$. Composing this A with any elliptic super pseudodifferential operator $E \in s\Psi\mathrm{DO}^{l'-l}(U^{m|n})$ of order $l'-l$, we obtain $E \circ A \in$

$s\Psi\mathrm{DO}^{l'}(U^{m|n})$ such that EAu smooth and $\sigma_{l'}(E\circ A)(x,k)\big(\lambda\big) = \sigma_{l'-l}(E)(x,k)\big(\sigma_l(A)(x,k)\big(\lambda\big)\big) \neq 0$. Hence, $(x,k,\lambda) \notin s\mathrm{WF}^{l'}(u)$, which completes the proof.

Item b): We prove the inclusion "$\subseteq$" by contradiction. Suppose that there exists $(x,k,\lambda) \in s\mathrm{WF}^l(u)\setminus((\mathcal{T}^*U\setminus\mathbf{0})\times\{0\})$ such that $(x,k) \notin \bigcup_{I\in\mathbb{Z}_2^n} \mathrm{WF}(u_I)$. The latter condition implies that, for each $I \in \mathbb{Z}_2^n$, there exists $A_I \in \Psi\mathrm{DO}^l(U)$ such that $A_I u_I$ is smooth and $\sigma_l(A_I)(x,k) \neq 0$. We define $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ by placing the A_I in their corresponding diagonal entry of the matrix and setting all other entries to zero. By construction, we have that Au is smooth and that the super principal symbol $\sigma_l(A)(x,k)$ is invertible. This implies that $\lambda = 0$ and leads to a contradiction.

We prove the inclusion "$\supseteq$" by contradiction. Suppose that there exists an element $(x,k) \in \bigcup_{I\in\mathbb{Z}_2^n} \mathrm{WF}(u_I)$ such that $(x,k,\lambda) \notin s\mathrm{WF}^l(u) \setminus ((\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\})$, for any $\lambda \neq 0$. Then there exists $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ such that Au is smooth and $\sigma_l(A)(x,k)$ is invertible at (x,k). Thus, by a straightforward refinement of Lemma 4.6, as in [Tre80, Proposition 6.9] we construct a microlocal parametrix $F \in s\Psi\mathrm{DO}^{-l}(U^{m|n})$. From the existence of this microlocal parametrix F we conclude that all components u_I of u are smooth at (x,k). Hence $(x,k) \notin \bigcup_{I\in\mathbb{Z}_2^n} \mathrm{WF}(u_I)$, which is a contradiction. □

Remark 5.3. On account of item a) of the previous lemma, we drop the label l and denote the super wavefront set by $s\mathrm{WF}(u)$.

Corollary 5.4. *$u \in \mathcal{D}'(U) \otimes \wedge^\bullet\mathbb{R}^n$ is smooth if and only if $s\mathrm{WF}(u) = (\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\}$.*

Proof. The statement is a special instance of (5.3). □

Example 5.5. Let us consider the superdomain $U^{m|2}$ and the superdistribution

$$u = v + v\,\theta^1\theta^2 = \begin{pmatrix} v \\ 0 \\ v \end{pmatrix} \quad , \tag{5.4}$$

where $v \in \mathcal{D}'(\mathbb{R}^m)$ is an ordinary distribution. Then the super pseudodifferential operator

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{5.5}$$

is of order 0 and annihilates u. In particular, $Au = 0$ is smooth. The super principal symbol of order 0 of A reads as

$$\sigma_0(A)(x,k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \tag{5.6}$$

for any $(x,k) \in \mathcal{T}^*U$. Hence all polarization vectors in the super wavefront set $s\mathrm{WF}(u)$ in Definition (5.1) have necessarily a vanishing third component (i.e. highest component in the θ-expansion). Explicitly,

$$s\mathrm{WF}(u) \subseteq \big(\mathcal{T}^*U \setminus \mathbf{0}\big) \times \big\{\lambda \in \wedge^\bullet\mathbb{C}^2 \,:\, \lambda_{(1,1)} = 0\big\} \ . \tag{5.7}$$

Loosely speaking, this shows that our notion of super wavefront sets both picks out the leading singularities to determine the polarization and assigns a higher weight to the components of a superdistribution with a lower number of θ-powers. Notice that this is a direct consequence of our definition of orders and super principal symbols for super pseudodifferential operators in Definition 4.1. Hence this feature generalizes to superdomains in higher odd-dimensions $U^{m|n}$.

The super wavefront set of a superdistribution behaves well with respect to the action of super pseudodifferential operators.

Proposition 5.6. *Let* $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ *and* $A \in s\Psi\mathrm{DO}^l(U^{m|n})$*. Then*

$$s\mathrm{WF}(Au) \supseteq \sigma_l(A)\big(s\mathrm{WF}(u)\big) := \big\{\big(x,k,\sigma_l(A)(x,k)\big(\lambda\big)\big) : (x,k,\lambda) \in s\mathrm{WF}(u)\big\}\ , \tag{5.8}$$

where the equality holds true whenever A *is elliptic.*

Proof. Let $(x,k,\lambda) \in s\mathrm{WF}(u)$ and $B \in s\Psi\mathrm{DO}^{l'}(U^{m|n})$ be such that BAu is smooth. By hypothesis, we have that $\sigma_{l+l'}(B\circ A)(x,k)\big(\lambda\big) = 0$, and hence $\sigma_{l'}(B)(x,k)\big(\sigma_l(A)(x,k)\big(\lambda\big)\big) = 0$. As B was arbitrary (as long as BAu is smooth), this implies that $\big(x,k,\sigma_l(A)(x,k)\big(\lambda\big)\big) \in s\mathrm{WF}(Au)$.

If A is elliptic, we use Lemma 4.6 to obtain an elliptic $F \in s\Psi\mathrm{DO}^{-l}(U^{m|n})$, such that both $A\circ F - \mathrm{id}$ and $F \circ A - \mathrm{id}$ lie in $s\Psi\mathrm{DO}^{-\infty}(U^{m|n})$. Equality in (5.8) is then shown by replacing the role of u with Au and that of A with F. □

Remark 5.7. More generally, equality in (5.8) holds true microlocally above any point $(x,k) \in \mathcal{T}^*U \setminus \mathbf{0}$ where $\sigma_l(A)$ is invertible.

Given any supermanifold isomorphism $\chi : U^{m|n} \to V^{m|n}$, the fibre-wise polarization mapping from (3.10) defines a super vector bundle isomorphism

$$\begin{array}{ccc}
\mathcal{P}^*V^{m|n} & \xrightarrow{\ \mathcal{P}^*\chi\ } & \mathcal{P}^*U^{m|n} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathcal{T}^*V & \xrightarrow{\ \mathcal{T}^*\widetilde{\chi}\ } & \mathcal{T}^*U \\
\downarrow{\scriptstyle \pi_{\mathcal{T}^*}} & & \downarrow{\scriptstyle \pi_{\mathcal{T}^*}} \\
V & \xrightarrow[\ \widetilde{\chi}^{-1}\ ]{} & U
\end{array} \tag{5.9}$$

We now show that the super wavefront sets transform well under supermanifold isomorphisms.

Proposition 5.8. *Let* $\chi : U^{m|n} \to V^{m|n}$ *be a supermanifold isomorphism and* $u \in \mathcal{D}'(V)\otimes\wedge^\bullet\mathbb{R}^n$ *a superdistribution. Denote by* $\chi_V^*(u) \in \mathcal{D}'(U) \otimes \wedge^\bullet\mathbb{R}^n$ *the pullback of* u *along* χ*. Then*

$$s\mathrm{WF}(\chi_V^*(u)) = \mathcal{P}^*\chi\big(s\mathrm{WF}(u)\big)\ . \tag{5.10}$$

Proof. This is a direct consequence of Lemma 4.4 b). □

This transformation property of the super wavefront set under the action of all supermanifold isomorphisms allows us to globalize super wavefront sets from superdomains to supermanifolds: Let u be a superdistribution on a supermanifold $X = (\widetilde{X}, \mathcal{O}_X)$. We use a superatlas $\rho_\alpha : X|_{U_\alpha} \to {W_\alpha}^{m|n}$ and describe u in terms of a family of local superdistributions $u_\alpha \in \mathcal{D}'(W_\alpha) \otimes \wedge^\bullet\mathbb{R}^n$, which satisfy the gluing conditions

$$\mathrm{res}_{W_\beta,\widetilde{\rho_\beta}(U_{\alpha\beta})}\big(u_\beta\big) = {\chi_{\beta\alpha}}^*\big(\mathrm{res}_{W_\alpha,\widetilde{\rho_\alpha}(U_{\alpha\beta})}(u_\alpha)\big) \tag{5.11}$$

on all overlaps $U_{\alpha\beta}$. Here $\chi_{\beta\alpha}$ are the transition supermanifold morphisms. The super wavefront set of u is then obtained by gluing all subsets $s\mathrm{WF}(u_\alpha) \subseteq \mathcal{P}^*{W_\alpha}^{m|n}$ via the transition functions $g_{\alpha\beta} = \mathcal{P}^*\chi_{\beta\alpha}$ of the polarization bundle. Proposition 5.8 guarantees that this construction defines a global super wavefront set $s\mathrm{WF}(u) \subseteq \mathcal{P}^*X$.

6 Pullback and multiplication theorems

Given a *generic* supermanifold morphism $\chi : X \to Y$, we *cannot* pull back a generic superdistribution u on Y to a superdistribution on X. However, depending on the explicit form of χ, certain superdistributions u on Y may admit a (unique) pullback to X. It is the goal of this section to develop a suitable criterion to select a class of superdistributions which admit a pullback.

Before we start with supergeometric considerations, let us briefly recall the solution to the above problem in ordinary geometry, see e.g. [Hör03]: Consider a smooth map $\widetilde{\chi} : U \to V$ between two open domains $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^{m'}$. The normal set of $\widetilde{\chi}$ is the subset of $\mathcal{T}^*V$ given by

$$N_{\widetilde{\chi}} := \Big\{ \big(\widetilde{\chi}(x), k'\big) \in \mathcal{T}^*V \,:\, x \in U\,,\ \mathcal{T}^*\widetilde{\chi}(k') = 0 \Big\} \ . \tag{6.1}$$

It was shown in [Hör03, Theorem 8.2.4] that the pullback map $\widetilde{\chi}^* : C^\infty(V) \to C^\infty(U)$ admits a unique continuous extension to those distributions $u \in \mathcal{D}'(V)$ for which $\mathrm{WF}(u) \cap N_{\widetilde{\chi}} = \emptyset$ holds true.

Let us now consider a supermanifold morphism $\chi : U^{m|n} \to V^{m'|n'}$ between two superdomains. The case of a generic supermanifold morphism $\chi : X \to Y$ between two supermanifolds follows from this by localizing χ in suitable superatlases of X and Y. Recalling that χ_V^* admits a unique factorization (3.8) into a matrix of differential operators D^χ and the component-wise pullback $\widetilde{\chi}^*$ along the underlying smooth map, we analyze the pullback of superdistributions in two steps: Given any superdistribution $u \in \mathcal{D}'(V) \otimes \wedge^\bullet\mathbb{R}^{n'}$ on $V^{m'|n'}$, the first step is to act with the differential operator D^χ on u, which is *always* well-defined and results in an auxiliary superdistribution

$$D^\chi u \in \mathcal{D}'(V) \otimes \wedge^\bullet\mathbb{R}^n \ , \tag{6.2}$$

where the components are now in the Grassmann algebra $\wedge^\bullet\mathbb{R}^n$ with n generators. In the second step, we would like to pull back $D^\chi u$ along $\widetilde{\chi}^*$. However, this operation is *not always* well-defined. If we assume the condition

$$\pi\Big(s\mathrm{WF}(D^\chi u) \setminus \big((\mathcal{T}^*V \setminus \mathbf{0}) \times \{0\}\big)\Big) \cap N_{\widetilde{\chi}} = \emptyset \ , \tag{6.3}$$

then $\chi_V^* u := \widetilde{\chi}^* D^\chi u \in \mathcal{D}'(U) \otimes \wedge^\bullet\mathbb{R}^n$ exists on account of the ordinary pullback theorem [Hör03, Theorem 8.2.4]. In fact, using Proposition 5.2, the condition (6.3) is equivalent to

$$\Big(\bigcup_{I \in \mathbb{Z}_2^n} \mathrm{WF}\big((D^\chi u)_I\big) \Big) \cap N_{\widetilde{\chi}} = \emptyset \ . \tag{6.4}$$

By the ordinary pullback theorem this implies that all components $(D^\chi u)_I$ may be safely pulled back along $\widetilde{\chi}^*$, and hence also $D^\chi u$. Summing up, we have shown the following version of a pullback theorem for superdistributions.

Theorem 6.1. *Let $\chi : U^{m|n} \to V^{m'|n'}$ be a supermanifold morphism between two superdomains, and consider the unique factorization $\chi_V^* = \widetilde{\chi}^* \circ D^\chi$ given in* (3.8)*. Then the pullback map*

$$\chi_V^* : C^\infty(V) \otimes \wedge^\bullet\mathbb{R}^{n'} \longrightarrow C^\infty(U) \otimes \wedge^\bullet\mathbb{R}^n \tag{6.5}$$

has a unique continuous extension to those superdistributions $u \in \mathcal{D}'(V) \otimes \wedge^\bullet\mathbb{R}^{n'}$ which satisfy the condition (6.3).

Remark 6.2. Another condition which would guarantee the existence of $\chi_V^* u$ is given by

$$\pi\Big(s\mathrm{WF}(u) \setminus \big((\mathcal{T}^*V \setminus \mathbf{0}) \times \{0\}\big)\Big) \cap N_{\widetilde{\chi}} = \emptyset \ . \tag{6.6}$$

In fact, using Proposition 5.2, the condition (6.6) is equivalent to the strong condition $\mathrm{WF}(u_J) \cap N_{\widetilde{\chi}} = \emptyset$ for all components u_J. Because differential operators preserve wavefront sets, it follows that

$$\begin{aligned}\mathrm{WF}\big((D^\chi u)_I\big) \cap N_{\widetilde{\chi}} &= \mathrm{WF}\Big(\sum_{J\in\mathbb{Z}_2^{n'}} (D^\chi)_I{}^J u_J\Big) \cap N_{\widetilde{\chi}} \\ &\subseteq \bigcup_{J\in\mathbb{Z}_2^{n'}} \mathrm{WF}\big((D^\chi)_I{}^J u_J\big) \cap N_{\widetilde{\chi}} \subseteq \bigcup_{J\in\mathbb{Z}_2^{n'}} \mathrm{WF}(u_J) \cap N_{\widetilde{\chi}} = \emptyset \end{aligned} \tag{6.7}$$

for any I, which implies (6.3). Notice that the condition (6.6) is much coarser than our condition (6.3). Loosely speaking, it does not take into account those components of u which "vanish algebraically under pullback" due to the differential operator D^χ. Let us illustrate this important point by an example: Consider the supermanifold morphism $\chi : \{*\} \to U^{m|n}$ which maps a point into the superdomain $U^{m|n}$. Then

$$\chi_U^* : C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow \mathbb{R} \ , \quad f = \sum_{I\in\mathbb{Z}_2^n} f_I\, \theta^I \longmapsto f_{(0,\dots,0)}(\widetilde{\chi}(*)) \tag{6.8}$$

is the mapping which "forgets" all higher components in the Grassmann algebra and evaluates the lowest component at the point $\widetilde{\chi}(*) \in U$. We can clearly extend χ_U^* to *all* superdistributions $\mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ with smooth lowest component $u_{(0,\dots,0)} \in C^\infty(U)$ by setting

$$u = \sum_{I\in\mathbb{Z}_2^n} u_I\, \theta^I \longmapsto u_{(0,\dots,0)}(\widetilde{\chi}(*)) \ . \tag{6.9}$$

Because $N_{\widetilde{\chi}} = \mathcal{T}^*_{\widetilde{\chi}(*)} U$ is the cotangent space at $\widetilde{\chi}(*)$, the condition (6.6) is violated as soon as any u_I has a singularity at this point. In contrast, our condition (6.3) just involves the lowest component $u_{(0,\dots,0)}$ of the superdistribution, because the matrix of differential operators reads as $D^\chi = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ and hence $D^\chi u = u_{(0,\dots,0)}$.

In the remaining part of this section we specialize the result of Theorem 6.1 to the important case where χ is the super diagonal mapping

$$\Delta : U^{m|n} \longrightarrow U^{m|n} \times U^{m|n} \simeq (U \times U)^{2m|2n} \ . \tag{6.10}$$

The underlying smooth map $\widetilde{\Delta} : U \to U \times U\,,\ x \mapsto (x,x)$ is the diagonal map and $\Delta^*_{U\times U} : C^\infty(U \times U) \otimes \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n \to C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ factorizes as

$$\Delta^*_{U\times U} = \widetilde{\Delta}^* \circ D^\Delta = (\widetilde{\Delta}^* \otimes \mathrm{id}_{\wedge^\bullet \mathbb{R}^n}) \circ (\mathrm{id}_{C^\infty(U\times U)} \otimes \mu) \ , \tag{6.11}$$

where $\mu : \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n \to \wedge^\bullet \mathbb{R}^n$ denotes the product in the Grassmann algebra $\wedge^\bullet \mathbb{R}^n$. The normal set of $\widetilde{\Delta}$ can be characterized explicitly and it is given by

$$N_{\widetilde{\Delta}} = \Big\{ \big((x,x),(k,-k)\big) \in \mathcal{T}^*(U \times U) \ : \ (x,k) \in \mathcal{T}^*U \Big\} \ . \tag{6.12}$$

Given two superdistributions $u, v \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$, their product (if it exists) is given by $u\,v := \Delta^*_{U\times U}(u \otimes v)$. Expanding into components $u = \sum_{I\in\mathbb{Z}_2^n} u_I\,\theta^I$ and $v \in \sum_{J\in\mathbb{Z}_2^n} u_J\,\theta^J$, we obtain

$$u \otimes v = \sum_{I,J\in\mathbb{Z}_2^n} u_I \otimes v_J \ (\theta^I \otimes \theta^J) \in \mathcal{D}'(U \times U) \otimes \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n \ . \tag{6.13}$$

Due to the factorization (6.11), the product of u and v (if it exists) is computed by first multiplying in the Grassmann algebra

$$D^{\Delta}\big(u \otimes v\big) = \sum_{I,J \in \mathbb{Z}_2^n} u_I \otimes v_J \; (\theta^I \, \theta^J) \tag{6.14}$$

and then pulling back the result component-wise via $\widetilde{\Delta}^*$, i.e.

$$u\,v := \Delta^*_{U\times U}\big(u \otimes v\big) = \sum_{I,J \in \mathbb{Z}_2^n} \widetilde{\Delta}^*(u_I \otimes v_J) \; (\theta^I \, \theta^J) \; . \tag{6.15}$$

As a consequence of Theorem 6.1, we have

Corollary 6.3. *The product $u\,v \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ exists whenever*

$$\pi\Big(s\mathrm{WF}\big(D^{\Delta}(u\otimes v)\big) \setminus \big((\mathcal{T}^*(U\times U)\setminus \mathbf{0}) \times \{0\}\big)\Big) \cap N_{\widetilde{\Delta}} = \emptyset \; , \tag{6.16}$$

or equivalently, whenever all components $u_I, v_J \in \mathcal{D}'(U)$, for which $\theta^I\,\theta^J \neq 0$, can be multiplied in the sense of ordinary distributions, cf. [Hör03, Theorem 8.2.4].

Remark 6.4. It is important to stress that the condition (6.16) in the corollary above *does not* impose conditions on the components u_I and v_J which multiply trivially on account of the Grassmann algebra structure, i.e. for which $\theta^I\,\theta^J = 0$. This is a clear advantage compared to the alternative (and much coarser) condition (6.6).

7 Singularities in supergeometric field theory

In this section we apply the techniques developed in this paper to analyze the singularities of the supergeometric field theory introduced in Example 4.8. For simplifying our explicit computations, we consider only the case where $M = \mathbb{R}^3$ is the Minkowski spacetime, i.e. we take the flat Lorentzian metric $g = \mathrm{diag}(1,-1,-1)$ on M. In this case the equation of motion operator (4.15) has constant coefficients and reads as

$$P = \begin{pmatrix} m & 0 & -1 \\ 0 & \mathrm{i}\,\gamma^\mu\partial_\mu + m & 0 \\ g^{\mu\nu}\partial_\mu\partial_\nu & 0 & m \end{pmatrix} \; . \tag{7.1}$$

Let $u \in \mathcal{D}'(\mathbb{R}^3) \otimes \wedge^\bullet\mathbb{R}^2$ be any superdistribution satisfying $Pu = 0$. By Proposition 5.6, the super wavefront set $s\mathrm{WF}(u) \subseteq \widehat{\mathcal{P}}^*\mathbb{R}^{3|2}$ of u satisfies the equality

$$\sigma_1(P)\big(s\mathrm{WF}(u)\big) = (\mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0}) \times \{0\} \; , \tag{7.2}$$

where we also have used that $(\mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0}) \times \{0\}$ is the smallest possible super wavefront set, cf. Corollary 5.4. The equality (7.2) is equivalent to the inclusion

$$s\mathrm{WF}(u) \subseteq \mathcal{N}_P := \Big\{(x,k,\lambda) \in \widehat{\mathcal{P}}^*\mathbb{R}^{3|2} \; : \; \sigma_1(P)(x,k)\big(\lambda\big) = 0\Big\} \; , \tag{7.3}$$

which follows by direct inspection of the left-hand-side of (7.3) and using (5.8). Using the explicit form of the super principal symbol of (7.1), we find the inclusion

$$s\mathrm{WF}(u) \subseteq \Big((\mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0}) \times \{0\}\Big) \cup \Big\{\big(x,k,\phi + \psi\,\theta\big) \; : \; g^{\mu\nu}k_\mu k_\nu = 0 \, , \; \gamma^\mu k_\mu \psi = 0\Big\} \; , \tag{7.4}$$

where we have used the compact notation $\psi\,\theta := \psi_a\,\theta^a := \psi_1\,\theta^1 + \psi_2\,\theta^2$. In words, (7.4) tells us that all elements $(x,k,\lambda) \in s\mathrm{WF}(u)$ with nontrivial $\lambda \neq 0$ are such that k is light-like.

Moreover, $\lambda = \phi + \psi\,\theta$ does not contain a quadratic θ-term and the Fermionic polarizations ψ have to satisfy the Dirac-polarization constraint $\gamma^\mu k_\mu \psi = 0$.

We next observe that the composition $\widetilde{P} \circ P$ of (7.1) with the super (pseudo-)differential operator (of order 1)

$$\widetilde{P} = \begin{pmatrix} m & 0 & 1 \\ 0 & -\,\mathrm{i}\,\gamma^\mu \partial_\mu + m & 0 \\ -g^{\mu\nu}\partial_\mu \partial_\nu & 0 & m \end{pmatrix} \tag{7.5}$$

gives the component-wise Klein-Gordon equation

$$\widetilde{P} \circ P = (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\,\mathrm{id} =: Q\,\mathrm{id}\ . \tag{7.6}$$

In particular, each component u_I of any u satisfying $Pu = 0$ satisfies the Klein-Gordon equation $Qu_I = 0$, which entails the following inclusion

$$\mathrm{WF}(u_I) \subseteq \Omega_Q := \Big\{(x,k) \in \mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0}\ :\ g^{\mu\nu} k_\mu\, k_\nu = 0\Big\}\ , \tag{7.7}$$

for all component wavefront sets. By the standard propagation of singularities theorem (see [Hör09, Chapter 26]), this implies that all $\mathrm{WF}(u_I)$ are invariant under the flow of the Hamiltonian vector field

$$H_Q := \big\{\sigma_2(Q),\,\cdot\,\big\} = 2g^{\mu\nu}k_\mu\,\partial_\nu : C^\infty(\Omega_Q) \longrightarrow C^\infty(\Omega_Q)\ , \tag{7.8}$$

i.e. any integral curve $c:\mathbb{R}\to\Omega_Q$ of H_Q which satisfies $c(0)\in\mathrm{WF}(u_I)$ remains in $\mathrm{WF}(u_I)$. In our example, any integral curve of H_Q is of the form

$$c:\mathbb{R}\longrightarrow\Omega_Q\ ,\quad s\longmapsto \big(x^\mu + s\,2g^{\mu\nu}k_\nu, k_\nu\big)\ , \tag{7.9}$$

for some $(x^\mu, k_\nu)\in\Omega_Q$.

Following the ideas of Dencker [Den82], we now shall study the propagation of polarizations in our example. Given any integral curve $c:\mathbb{R}\to\Omega_Q$ of H_Q as in (7.9), we consider the restriction of $\mathcal{N}_P$ given in (7.3) to c, which gives rise to a vector bundle

$$\mathcal{N}_P|_c \longrightarrow \mathbb{R}\ . \tag{7.10}$$

Using (7.4), we can compute its total space

$$\mathcal{N}_P|_c = \big\{\big(s, \phi + \psi\,\theta\big)\ :\ \gamma^\mu k_\mu \psi = 0\big\}\ . \tag{7.11}$$

As the solution space of the Dirac-constraint $\gamma^\mu k_\mu\psi$ is one-dimensional (in 3 dimensions), the vector bundle $\mathcal{N}_P|_c\to\mathbb{R}$ is of rank two. A *Hamiltonian orbit* [Den82, Definition 4.1] for our operator P is a sub-line bundle $L\subseteq\mathcal{N}_P|_c$, where c is an integral curve as above and L is spanned by a section $w\in\Gamma^\infty(\mathcal{N}_P|_c)$ that satisfies $D_P w = 0$. Here $D_P := H_Q + \frac{1}{2}\{\sigma_1(\widetilde{P}),\sigma_1(P)\} + \mathrm{i}\,\sigma_1(\widetilde{P})\,\sigma_0^s(P)$ is a partial connection (cf. [Den82, Equation (4.6)]), where $\sigma_0^s(P)$ denotes the subprincipal symbol of P. Clearly, the vector bundle $\mathcal{N}_P|_c$ can be spanned by the sections $w\in\Gamma^\infty(\mathcal{N}_P|_c)$ satisfying $D_P w = 0$. In our example, we find that

$$D_P = \frac{\partial}{\partial s} + \mathrm{i}\ m\begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma^\mu\,k_\mu & 0 \\ 0 & 0 & 0\end{pmatrix} : \Gamma^\infty(\mathcal{N}_P|_c)\longrightarrow\Gamma^\infty(\mathcal{N}_P|_c)\ . \tag{7.12}$$

Notice that the connection coefficients (i.e. the second term in the expression above) act trivially on the fibers of $\mathcal{N}_P|_c$ (this follows from (7.11)), hence the expression for D_P simplifies to

$$D_P = \frac{\partial}{\partial s} : \Gamma^\infty(\mathcal{N}_P|_c)\longrightarrow\Gamma^\infty(\mathcal{N}_P|_c)\ . \tag{7.13}$$

Any Hamiltonian orbit in our example is therefore of the form

$$\mathbb{R} \times \mathrm{span}_{\mathbb{C}}\big(\phi + \psi\,\theta\big) \subseteq \mathcal{N}_P|_c \ , \tag{7.14}$$

for some $0 \neq \phi + \psi\,\theta \in \wedge^\bullet\mathbb{R}^2$ satisfying $\gamma^\mu k_\mu \psi = 0$.

Finally, we notice that $s\mathrm{WF}(u)$, for any u satisfying $Pu = 0$, is the union of such Hamiltonian orbits, i.e. the propagation of polarization result [Den82, Theorem 4.2] remains valid in our supergeometric example. This follows from the fact that $Pu = 0$ is equivalent to the component equations, for $u = \phi + \psi\,\theta + F\,\theta^1\,\theta^2 \in \mathcal{D}'(\mathbb{R}^3) \otimes \wedge^\bullet\mathbb{R}^2$,

$$m\,\phi = F \quad , \qquad \mathrm{i}\,\gamma^\mu \partial_\mu \psi + m\,\psi = 0 \quad , \qquad g^{\mu\nu}\partial_\mu\partial_\nu\phi + m\,F = 0 \ , \tag{7.15}$$

which can be decoupled into the Dirac equation $\mathrm{i}\,\gamma^\mu\partial_\mu\psi + m\,\psi = 0$ and the massive Klein-Gordon equation $g^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\,\phi = 0$. The absence of F-polarizations in the super wavefront set $s\mathrm{WF}(u)$ for F satisfying the equation $m\,\phi = F$ follows from the discussion in Example 5.5.

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