On Properties and the Application of Levin-type Sequence Transformations for the Convergence Acceleration of Fourier Series*

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Abstract

We discuss Levin-type sequence transformations $\{s_n\} \to \{s'_n\}$ that depend linearly on the sequence elements s_n, and nonlinearly on an auxiliary sequence of remainder estimates $\{\omega_n\}$. If the remainder estimates also depend on the sequence elements, non-linear transformations are obtained. The application of such transformations very often yields new sequences that are more rapidly convergent in the case of linearly and logarithmically convergent sequences. Also, divergent power series can often be summed, i.e., transformed to convergent sequences, by such transformations. The case of slowly convergent Fourier series is more difficult and many known sequence transformations are not able to accelerate the convergence of Fourier series due to the more complicated sign pattern of the terms of the series in comparison to power series. In the present work, the Levin-type $\mathcal{H}$ transformation [H.H.H. Homeier, A Levin–type algorithm for accelerating the convergence of Fourier series, Numer. Algo. **3** (1992) 245–254] is studied that involves a frequency parameter α. In particular, properties of the $\mathcal{H}$ transformation are derived, and its implementation is discussed. We also present some generalization of it to the case of several frequency parameters. Finally, it is shown how to use the $\mathcal{H}$ transformation properly in the vicinity of singularities of the Fourier series.

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1 Introduction

In almost all fields of numerical computation, there are convergence problems. In such cases, either the convergence is slow or even divergences are observed. To overcome these problems, one may use extrapolation methods that are also known as (nonlinear) sequence transformations as explained in the books of Brezinski and Redivo Zaglia [13] and Wimp [52] and also the work of Weniger [47, 48, 49, 50]. Some other important books in the field are those of Baker [1], Baker and Graves-Morris [2, 3, 4], Brezinski [8, 9, 10, 12, 11], Graves-Morris [16, 17], Graves-Morris, Saff und Varga [18], Khovanskii [31], Lorentzen and Waadeland [36], Nikishin and Sorokin [37], Petrushev and Popov [39], Ross [40], Saff and Varga [41], Wall [46], Werner and Buenger [51] and Wuytack [53].

For power series, nonlinear convergence accelerators that are derived from model sequences or by iteration of simple transformations are very successful [13, 21, 23, 24, 47].

Fourier series

$$s = s(\alpha) = a_0/2 + \sum_{j=1}^{\infty} \bigl(a_j \cos(j\alpha) + b_j \sin(j\alpha) \bigr) \tag{1}$$

with partial sums

$$s_n = s_n(\alpha) = a_0/2 + \sum_{j=1}^{n} \bigl(a_j \cos(j\alpha) + b_j \sin(j\alpha) \bigr) \tag{2}$$

can be regarded as superpositions of several power series, at least in the two variables $\exp(+i\alpha)$ and $\exp(-i\alpha)$, possibly in more variables if the coefficients a_n and b_n themselves have oscillating parts. Thus, due to the resulting more complicated sign pattern of the terms as compared to simple power series, the most usual nonlinear accelerators are not efficient on direct application to Fourier series.

In many cases, Fourier series represent functions of the argument α that are not infinitely often differentiable. This is for instance the case if the coefficients a_n and b_n decay like $n^{-\gamma}$, $\gamma > 0$ for large n [15]. Especially for small γ, convergence of the series may be so slow that the direct summation is hopeless. The remedy is the use of convergence acceleration or, equivalently, extrapolation methods.

In principle, there are several possible alternatives for solving the extrapolation problem of Fourier series: One may try to estimate the (anti)limit by a linear combination of the partial sums s_n. This leads to linear methods as exemplified by the method of Jones and Hardy [30] that was extended by various groups [5, 6, 42, 45], the method of Kiefer and Weiss [32], the methods of Longman (see [35] and references therein), and also generalizations of the Euler transformation for Fourier series as studied by Boyd [7]. Also, one may try to reformulate the problem in such a way that the usual accelerators for power series become applicable, for instance by rewriting real Fourier series as the real part of some complex power series. This approach will be discussed elsewhere [25, 26, 28]. Alternatively, one may try to develop completely new accelerators. This may be done by iteration of simple transformations that has been shown to lead to very powerful algorithms for linearly and logrithmically convergent series [21, 23, 24] and recently, also for Fourier series [24, 27]. Another method for the construction of extrapolation methods is to based on the model sequence approach. The aim is to find sequence transformations that allow to find the limit (or antilimit in the case of divergence) in a finite number of operations, and hence, are exact, for suitable model sequences. Applying these transformations to problem sequences should be successful if the latter are in some sense *close* to such model sequences. Here, we concentrate on methods that are derived within the model sequence approach.

One of the few relatively successful nonlinear accelerators for Fourier series is the ϵ algorithm of Wynn [54]. The reason is that this algorithm is exact for model sequences s_n that are finite linear combinations of powers λ_j^n, $\lambda_j \in \mathbb{C}$, $j \in M \subset \mathbb{N}$, with coefficients $c_j(n)$ that are polynomials in the variable n [13, Theorem 2.18]. Rewriting these powers as $\lambda_j^n = |\lambda_j|^n(\cos(n\phi_j) + \mathrm{i}\,\sin(n\phi_j))$, $\phi_j = \arg(\lambda_j)$ shows the relation to Fourier series.

Here, we study nonlinear transformations that are exact for model sequences that generalize the model sequences σ_n given by

$$\sigma_n = \sigma + \omega_n(c_0 + c_1(n+\beta)^{-1} + \cdots + c_{k-1}(n+\beta)^{1-k}) \tag{3}$$

that leads to the Levin transformation [33]. Here, $\omega_n \neq 0$ are called *remainder estimates*, the c_i are coefficients, $\beta > 0$ is a parameter, and σ is the (anti)limit. If the model sequences have the general form

$$\sigma_n = \sigma + \omega_n \mu_n(c_i, \pi_i) \tag{4}$$

the corresponding sequence transformations T_n that allow the calculation of the (anti)limit σ according to $\sigma = T_n(\{\sigma_n\}, \{\omega_n\}, \pi_i)$ are called *Levin-type sequence transformations*. Here the c_i are some coefficients and the π_i are some further parameters that specify the model μ_n. An example is the Levin transformation [33] itself that may be defined as

$$\mathcal{L}_k^{(n)}(\beta, s_n, \omega_n) = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \frac{s_{n+j}}{\omega_{n+j}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta+n+j)^{k-1}}{(\beta+n+k)^{k-1}} \frac{1}{\omega_{n+j}}}\,. \tag{5}$$

Another example is the $\mathcal{J}$ transformation [20, 21, 22, 23, 24, 29] that is of similar generality as the well-known E algorithm but provides a more simple algorithm in many important cases. The Levin transformation is a special case of the $\mathcal{J}$ transformation. Suitable variants of the $\mathcal{J}$ transformation belong to the most powerful nonlinear accelerators for linearly and logarithmically convergent sequences and are able to sum violently divergent power series [23].

Also the nonlinear $d^{(m)}$ transformations [34] are generalizations of the Levin transformation that are useful for the convergence acceleration of certain Fourier series [43]. However, their recursive scheme is relatively complicated for $m > 1$.

We note that the Levin transformation and several other Levin-type transformations are linear sequence transformations if the remainder estimates do not depend on the problem sequence $\{s_n\}$. However, in practical work, one chooses simple remainder estimates like $\omega_n = \triangle s_n$ or $\omega_n = (n+\beta)\triangle s_{n-1}$ that depend on the problem sequence, and thus, nonlinear sequence transformations result. Here and in the following, the symbol $\triangle$ denotes the forward difference operator with respect to the variable n acting as

$$\triangle f(n) = f(n+1) - f(n) \,, \qquad \triangle g_n = g_{n+1} - g_n \,. \tag{6}$$

In the present work, we concentrate on a further nonlinear convergence accelerator for Fourier series, the Levin-type $\mathcal{H}$ transformation [19]

$$\begin{aligned}
\mathcal{Z}_n^{(0)} &= (n+\beta)^{-1}\, s_n/\omega_n \,, \qquad \mathcal{N}_n^{(0)} = (n+\beta)^{-1}/\omega_n \,, \\
\mathcal{Z}_n^{(k)} &= (n+\beta)\mathcal{Z}_n^{(k-1)} + (n+2k+\beta)\mathcal{Z}_{n+2}^{(k-1)} - 2\cos(\alpha)(n+k+\beta)\mathcal{Z}_{n+1}^{(k-1)} \,, \\
\mathcal{N}_n^{(k)} &= (n+\beta)\mathcal{N}_n^{(k-1)} + (n+2k+\beta)\mathcal{N}_{n+2}^{(k-1)} - 2\cos(\alpha)(n+k+\beta)\mathcal{N}_{n+1}^{(k-1)} \,, \\
\frac{\mathcal{Z}_n^{(k)}}{\mathcal{N}_n^{(k)}} &= \mathcal{H}_n^{(k)}(\alpha, \beta, \{s_n\}, \{\omega_n\}) \,.
\end{aligned} \tag{7}$$

that is exact for the model sequence

$$s_n = s + \omega_n \left(\exp(\mathrm{i}\alpha n) \sum_{j=0}^{k-1} c_j (n+\beta)^{-j} + \exp(-\mathrm{i}\alpha n) \sum_{j=0}^{k-1} d_j (n+\beta)^{-j} \right) \tag{8}$$

with coefficients c_j and d_j. Here and in the following, we assume that $\cos(\alpha) \neq \pm 1$. Such a model sequence is motivated by rewriting the tail $\rho_n = s - s_n$ of a Fourier series as

$$\rho_n = \cos(n\alpha) A_n(\alpha) + \sin(n\alpha) B_n(\alpha) \tag{9}$$

with

$$\begin{aligned}
A_n(\alpha) &= -\sum_{k=1}^{\infty} \left\{ a_{k+n}\cos(k\alpha) + b_{k+n}\sin(k\alpha) \right\} \,, \\
B_n(\alpha) &= -\sum_{k=1}^{\infty} \left\{ -a_{k+n}\sin(k\alpha) + b_{k+n}\cos(k\alpha) \right\} \,.
\end{aligned} \tag{10}$$

Assuming Poincaré-type asymptotical expansions

$$\begin{aligned}
A_n(\alpha) &\sim \omega_n \sum_{j=0}^{\infty} \gamma_j (n+\beta)^{-j} \,, \\
B_n(\alpha) &\sim \omega_n \sum_{j=0}^{\infty} \delta_j (n+\beta)^{-j}
\end{aligned} \tag{11}$$

for large n, truncation to the first k terms leads to the model sequence

$$s_n = s + \omega_n \left(\cos(\alpha n) \sum_{j=0}^{k-1} \gamma_j (n+\beta)^{-j} + \sin(\alpha n) \sum_{j=0}^{k-1} \delta_j (n+\beta)^{-j} \right) \tag{12}$$

that is equivalent to the model sequence (8).

From the above, one expects that the $\mathcal{H}$ transformation is applicable if the coefficients a_n and b_n of the Fourier series are nonoscillating functions of n. Under this restriction, the $\mathcal{H}$ transformation has proven to be useful, and to be more effective than the ϵ algorithm, in the context of the Dubner-Abate-Crump approach for the inversion of the Laplace transformation [20]. The $\mathcal{H}$ transformation has been criticized [43] as being numerically less stable and less effective near singularities than the $d^{(2)}$ transformation. However, Oleksy [38] has shown that an additional preprocessing transformation followed by the $\mathcal{H}$ transformation improves the results near singularities considerably. Similarly, but conceptually much simpler, it is possible to apply the $\mathcal{H}$ transformation to the sequence $s_{\tau n}$ with $\tau \in \mathbb{N}$ with very good results even in the vicinity of singularities as will be shown in the following. But before, we discuss general properties of the $\mathcal{H}$ transformation and derive a generalization to several frequencies.

2 Properties of the $\mathcal{H}$ Transformation

Define the polynomial $P^{(2k)}$ of degree $2k$ by

$$P^{(2k)}(x) = [x^2 - 2\cos(\alpha)\, x + 1]^k = \sum_{m=0}^{2k} p_m^{(2k)}\, x^m \,. \tag{13}$$

We have $P^{(2k)}(0) = 1$. Its zeroes at $\exp(i\alpha)$ and $\exp(-i\alpha)$ are of order k. Then, one can represent the $\mathcal{H}$ transformation defined in Eq. (7) in the form [19]

$$\begin{aligned} s_n^{(k)} &= \mathcal{H}_n^{(k)}(\alpha, \beta, \{s_n\}, \{\omega_n\}) = \mathcal{Z}_n^{(k)} / \mathcal{N}_n^{(k)} \\ \mathcal{Z}_n^{(k)} &= \sum_{m=0}^{2k} p_m^{(2k)}\, (n+\beta+m)^{k-1}\, \frac{s_{n+m}}{\omega_{n+m}}; \\ \mathcal{N}_n^{(k)} &= \sum_{m=0}^{2k} p_m^{(2k)}\, (n+\beta+m)^{k-1}\, \frac{1}{\omega_{n+m}} \,. \end{aligned} \tag{14}$$

Dividing numerator and denominator by $(n+\beta+2k)^{k-1}$ shows that the transformation can also be computed using the algorithm

$$\begin{aligned} s_n^{(k)} &= \mathcal{H}_n^{(k)}(\alpha, \beta, \{s_n\}, \{\omega_n\}) = \hat{\mathcal{Z}}_n^{(k)} / \hat{\mathcal{N}}_n^{(k)} \\ \hat{\mathcal{Z}}_n^{(k)} &= \sum_{m=0}^{2k} p_m^{(2k)}\, \frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}}\, \frac{s_{n+m}}{\omega_{n+m}} \\ \hat{\mathcal{N}}_n^{(k)} &= \sum_{m=0}^{2k} p_m^{(2k)}\, \frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}}\, \frac{1}{\omega_{n+m}} \,. \end{aligned} \tag{15}$$

The quantities $\hat{\mathcal{Z}}_n^{(k)}$ and $\hat{\mathcal{N}}_n^{(k)}$ of Eq. (15) each obey the recursion relation

$$\begin{aligned}\hat{\mathcal{M}}_n^{(k)} = \hat{\mathcal{M}}_{n+2}^{(k-1)} - 2\cos(\alpha)\,\frac{n+\beta+k}{n+\beta+2k}\left[\frac{n+\beta+2k-1}{n+\beta+2k}\right]^{k-2}\hat{\mathcal{M}}_{n+1}^{(k-1)} \\ + \frac{n+\beta}{n+\beta+2k}\left[\frac{n+\beta+2k-2}{n+\beta+2k}\right]^{k-2}\hat{\mathcal{M}}_n^{(k-1)}\,.\end{aligned} \tag{16}$$

This follows directly from the definitions and the defining algorithm (7) A direct consequence of Eq. (14) are the following theorems:

Theorem 1 *The $\mathcal{H}$ transformation is quasilinear, i.e.,*

$$\mathcal{H}_n^{(k)}(\alpha,\beta,\{As_n+B\},\{\omega_n\}) = A\,\mathcal{H}_n^{(k)}(\alpha,\beta,\{s_n\},\{\omega_n\}) + B \tag{17}$$

for arbitrary constants A and B.

Theorem 2 *The $\mathcal{H}$ transformation is multiplikatively invariant in ω_n, i.e.,*

$$\mathcal{H}_n^{(k)}(\alpha,\beta,\{s_n\},\{C\omega_n\}) = \mathcal{H}_n^{(k)}(\alpha,\beta,\{s_n\},\{\omega_n\}) \tag{18}$$

for any constant $C \neq 0$.

In the following we regard the parameters α with $\cos(\alpha) \neq \pm 1$ and β as arbitrary but fixed. We are interested in conditions on the remainder estimates $\omega_n \neq 0$ that guarantee that the $\mathcal{H}$ transformation is well-defined.

Under these conditions, the transformation $\mathcal{H}_n^{(k)}$ only depends on the $(4k+2)$ numbers $\{s_{n+j}\}_{j=0}^{2k}$ and $\{\omega_{n+j}\}_{j=0}^{2k}$. Hence, one can write

$$\mathcal{H}_n^{(k)} = \gamma_n^{(k)}\Big(s_n, s_{n+1}, \ldots, s_{n+2k} \,\Big|\, \omega_n, \omega_{n+1}, \ldots, \omega_{n+2k}\Big) \tag{19}$$

and regard the transformation as a mapping

$$\begin{aligned}\gamma_n^{(k)}:\quad \mathbb{C}^{2k+1}\times\mathbb{Y}_n^{(k)} &\longrightarrow \mathbb{C}\quad, \\ (\vec{x},\vec{y}) &\longrightarrow \gamma_n^{(k)}\Big(\vec{x}\,\Big|\,\vec{y}\Big)\;.\end{aligned} \tag{20}$$

Here, $\mathbb{Y}_n^{(k)}$ is a suitable subset of $\mathbb{C}^{2k+1}$: Since the $\mathcal{H}$ transformation depends on the inverses of the remainder estimates ω_n, a necessary condition is that no component of any vector in $\mathbb{Y}_n^{(k)}$ vanishes. This implies according to the representation (15) that $\mathcal{H}_n^{(k)}$ is a continuous function of $\{\omega_{n+j}\}_{j=0}^{2k}$ if

$$\sum_{m=0}^{2k} p_m^{(2k)}\,\frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}}\,\frac{1}{\omega_{n+m}} \neq 0 \tag{21}$$

holds, i.e., if the denominator in Eq. (15) does not vanish. This is equivalent to the statement that $\gamma_n^{(k)}$ is a continuous function of $(y_1,\ldots,y_{2k+1})$ if

$$\sum_{m=0}^{2k} p_m^{(2k)}\,\frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}}\,\frac{1}{y_{m+1}} \neq 0 \tag{22}$$

holds. Hence, we define

$$\mathbb{Y}_n^{(k)} = \left\{ \vec{y} \in \mathbb{C}^{2k+1} \;\middle|\; \prod_{j=1}^{2k+1} y_j \neq 0 \text{ and (22) holds.} \right\} \tag{23}$$

This is an open set. Then $\gamma_n^{(k)}$ is defined and continuous on $\mathbb{C}^{2k+1} \times \mathbb{Y}_n^{(k)}$.

According to the representation (15), the quantities $\gamma_n^{(k)}$ can be expressed via

$$\gamma_n^{(k)} = \frac{\mathbf{L}_n^{(k)}\big((x_1/y_1, \ldots, x_{2k+1}/y_{2k+1})\big)}{\mathbf{L}_n^{(k)}\big((1/y_1, \ldots, 1/y_{2k+1})\big)} \tag{24}$$

by the linear form

$$\mathbf{L}_n^{(k)}(\vec{v}) = \sum_{j=1}^{2k+1} \lambda_{n,j}^{(k)}\, v_j, \qquad \vec{v} \in \mathbb{C}^{2k+1} \tag{25}$$

with coefficients

$$\lambda_{n,j}^{(k)} = p_{j-1}^{(2k)}\, \frac{(n+\beta+j-1)^{k-1}}{(n+\beta+2k)^{k-1}}\,. \tag{26}$$

It follows that $\gamma_n^{(k)}$ is linear in the first $(2k+1)$ variables. Further, the transformation is exact for constant sequences, i.e., we have

$$\gamma_n^{(k)}\left(c, c, \ldots, c \;\middle|\; y_1, \ldots, y_{2k+1} \right) = c\,. \tag{27}$$

The condition (22) can be rewritten as

$$\mathbf{L}_n^{(k)}\big((1/y_1, \ldots, 1/y_{2k+1})\big) \neq 0\,. \tag{28}$$

Hence, the following theorem is proved:

Theorem 3 *Assume that* $\cos(\alpha) \neq \pm 1$.

(H-0) $\mathcal{H}_n^{(k)}$ *of Eq. (14) can be regarded as a continuous mapping* $\gamma_n^{(k)}$ *on* $\mathbb{C}^{2k+1} \times \mathbb{Y}_n^{(k)}$ *where* $\mathbb{Y}_n^{(k)}$ *is defined in Eq. (23).*

(H-1) Theorems 1 and 2 imply that $\gamma_n^{(k)}$ *is a homogeneous function of the first degree in the first* $(2k+1)$ *variables and a homogeneous function of degree zero in the last* $(2k+1)$ *variables. Thus, for all vectors* $\vec{x} \in \mathbb{C}^{2k+1}$ *and* $\vec{y} \in \mathbb{Y}_n^{(k)}$ *and for all complex constants* λ *and* $\mu \neq 0$*, the relations*

$$\begin{aligned} \gamma_n^{(k)}(\lambda\vec{x}\,|\,\vec{y}) &= \lambda\gamma_n^{(k)}(\vec{x}\,|\,\vec{y})\,, \\ \gamma_n^{(k)}(\vec{x}\,|\,\mu\vec{y}) &= \gamma_n^{(k)}(\vec{x}\,|\,\vec{y}) \end{aligned} \tag{29}$$

hold.

(H-2) $\gamma_n^{(k)}$ *is linear in the first* $(2k+1)$ *variables. Thus, for all vectors* $\vec{x} \in \mathbb{C}^{2k+1}$, $\vec{x}' \in \mathbb{C}^{2k+1}$, *and* $\vec{y} \in \mathbb{Y}_n^{(k)}$, *the equation*

$$\gamma_n^{(k)}(\vec{x} + \vec{x}' \,|\, \vec{y}) = \gamma_n^{(k)}(\vec{x} \,|\, \vec{y}) + \gamma_n^{(k)}(\vec{x}' \,|\, \vec{y}) \tag{30}$$

holds.

(H-3) For all constant vectors $\vec{c} = (c, c, \ldots, c) \in \mathbb{C}^{2k+1}$ *and all vectors* $\vec{y} \in \mathbb{Y}_n^{(k)}$ *the equation*

$$\gamma_n^{(k)}(\vec{c} \,|\, \vec{y}) = c \tag{31}$$

holds.

Before leaving this section, we give a sufficient condition on the remainder estimates ω_n that guarantees that the $\mathcal{H}$ transformation is well-defined.

Theorem 4 *If the point 0 is not contained in the closure of the set* $\{\omega_n\}_{n=0}^{\infty}$, *and if the limit* $r = \lim_{n\to\infty} \omega_{n+1}/\omega_n$ *exists and satisfies* $r \notin \{\exp(i\alpha), \exp(-i\alpha)\}$ *then the* $\mathcal{H}_n^{(k)}$ *transformation is well-defined for sufficiently large* n.

PROOF. We estimate the quantity

$$\hat{\mathcal{N}}_n^{(k)} = \sum_{m=0}^{2k} p_m^{(2k)} \, \frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}} \, \frac{1}{\omega_{n+m}} \tag{32}$$

for sufficiently large n. Since

$$\omega_n \hat{\mathcal{N}}_n^{(k)} = \sum_{m=0}^{2k} p_m^{(2k)} \, \frac{(n+\beta+m)^{k-1}}{(n+\beta+2k)^{k-1}} \, \frac{\omega_n}{\omega_{n+m}} \longrightarrow \sum_{m=0}^{2k} p_m^{(2k)} \, r^{-m} = P^{(2k)}(1/r) \neq 0 \tag{33}$$

for $n \to \infty$, the denominator $\hat{\mathcal{N}}_n^{(k)}$ cannot vanish for sufficiently large n under the conditions of the theorem. This ends the proof.

We note that theorems concerning the convergence of the $\mathcal{H}$ transformation are given in [19].

3 Implementation of the $\mathcal{H}$ transformation

The algorithm (7) can be implemented with very moderate storage requirements. It seems to require two two-dimensional arrays, one for the numerators and one for the denominators. Calling the entries of such arrays $\mathcal{M}_n^{(k)}$, in fact we have an $\mathcal{M}$ table. The recursion connects table entries according to

$$\begin{array}{ll} \mathcal{M}_n^{(k-1)} & \mathcal{M}_n^{(k)} \\ \mathcal{M}_{n+1}^{(k-1)} & \\ \mathcal{M}_{n+2}^{(k-1)} & \end{array} . \tag{34}$$

Fortunately, it is not necessary to use two two-dimensional arrays. Two *one-dimensional* arrays suffice if maximal use is made of the input data $\{s_n, \omega_n\}$. Only the boundary of the table has to be stored. Formally, this can be written as

$$\mathrm{M}(n-2k) \leftarrow \mathcal{M}_{n-2k}^{(k)} \tag{35}$$

for $0 \leq k \leq [\![n/2]\!]$ where M(0:NMAX) is a (Fortran) array. The sequence transformation then is given by

$$\{s_0, \ldots, s_n\} \longmapsto s_{n-2[\![n/2]\!]}^{([\![n/2]\!])} . \tag{36}$$

This is also explained in the following diagram where k numbers the columns and n numbers the rows:

$$\begin{array}{ccccc}
x & x & x & x & o_0 \\
x & x & x & o_1 & c_5 \\
x & x & x & o_2 & \\
x & x & o_3 & c_4 & \\
x & x & o_4 & & \\
x & o_5 & c_3 & & \\
x & o_6 & & & \\
o_7 & c_2 & & & \\
o_8 & & & & \\
c_1 & & & &
\end{array} \tag{37}$$

In this example, the elements of the first column correspond to $\mathcal{M}_n^{(0)}$ for $0 \leq n \leq 9$. Between calls of the implementing subroutine, the rightmost elements of each row are stored. This means that elements of the table indicated with x have been computed and/or stored previously, but have been overwritten by elements indicated by o_j, $j = 0, \ldots, 8$. These elements are the ones stored as $o_j = \mathrm{M}(j)$ before the new call of the subroutine. During this call, the elements $c_1 = \mathrm{M}(9)$ (as the first one), $c_2 = \mathrm{M}(7)$, …, $c_5 = \mathrm{M}(1)$ are computed consecutively using the relevant recursion relation. The elements $o_0 = \mathrm{M}(0)$, $o_2 = \mathrm{M}(2)$, …, $o_8 = \mathrm{M}(8)$ are left unchanged during the example call of the subroutine. A FORTRAN 77 subroutine is given in Appendix A.

4 A Generalization of the $\mathcal{H}$ Transformation

The method for the derivation of the $\mathcal{H}$ transformation [19] can be generalized to more complicated model sequences containing more than one frequency. This will be sketched in the following. We put $e_m = \exp(\mathrm{i}\alpha_m)$ for $m = 1, \ldots, M$. A generalization of the model sequence (8) is

$$s_n = s + \omega_n \sum_{m=1}^{M} e_m^n \sum_{j=0}^{k-1} c_{m,j} (n+\beta)^{-j} . \tag{38}$$

Thus, the model sequence (8) corresponds to the special case $M = 2$, $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$. For simplicity, we assume that the model sequence (38) converges to s.

To compute s for the model sequence (38), it is necessary to find an algorithm that allows to eliminate the coefficients $c_{m,j}$ exactly. In order to do this, we introduce a generalized characteristical polynomial $P(x)$ of degree $M \cdot k$ that has k-fold zeros at all e_m and, hence, is given by

$$P(x) = \prod_{m=1}^{M} (x - e_m)^k = \sum_{\ell=0}^{M\,k} p_\ell x^\ell \, . \tag{39}$$

It is the characteristic polynomial to the recursion

$$\sum_{\ell=0}^{M\,k} p_\ell \, v_{\ell+n} = 0 \qquad (n \geq 0) \, . \tag{40}$$

Since the zeros are k-fold, the $M\,k$ linearly independent solution of this recursion with $(M\,k+1)$ terms and constant coefficients are

$$v_{n,m,l} = n^\ell e_m^n \, , \qquad \ell = 0, \ldots, k-1 \, , \quad m = 1, \ldots, M \, . \tag{41}$$

As in the case of the $\mathcal{H}$ transformation [19], the recursion is applied to $(n + \beta)^{k-1}(s_n - s)/\omega_n$ and yields zero according to

$$\sum_{\ell=0}^{M\,k} p_\ell \, (n + \beta + \ell)^{k-1} (s_{n+\ell} - s)/\omega_{n+\ell} = 0 \qquad (n \geq 0) \, . \tag{42}$$

Solving for the limit s of model sequence (38), it is given exactly by

$$s = \frac{\displaystyle\sum_{m=0}^{M\,k} p_m \, (n + \beta + m)^{k-1} s_{n+m}/\omega_{n+m}}{\displaystyle\sum_{m=0}^{M\,k} p_m \, (n + \beta + m)^{k-1}/\omega_{n+m}} \, . \tag{43}$$

The corresponding sequence transformation is

$$s_n^{(k)} = \frac{\displaystyle\sum_{m=0}^{M\,k} p_m \, (n + \beta + m)^{k-1} s_{n+m}/\omega_{n+m}}{\displaystyle\sum_{m=0}^{M\,k} p_m \, (n + \beta + m)^{k-1}/\omega_{n+m}} \, . \tag{44}$$

The coefficients p_m depend on k, M and the frequencies α_j. Proceeding as in the case of the $\mathcal{H}$ transformation, one sees after some short calculation that numerator and denominator again obey an identical recursion, that is, however, more complicated than in the case of the $\mathcal{H}$ transformation. This recursion is given by [24]

$$\mathcal{M}_n^{(k)} = \sum_{j=0}^{M} q_j\,(n+\beta+j\,k)\mathcal{M}_{n+j}^{(k-1)}\;, \tag{45}$$

where the coefficients q_j are defined by

$$\prod_{m=1}^{M}(x-e_m) = \sum_{j=0}^{M} q_j x^j\;. \tag{46}$$

In this way, the following algorithm is obtained for the generalized $\mathcal{H}$-Transformation

$$\begin{aligned}
\tilde{\mathcal{Z}}_n^{(0)} &= (n+\beta)^{-1}\,s_n/\omega_n\;, \qquad \tilde{\mathcal{N}}_n^{(0)} = (n+\beta)^{-1}/\omega_n\;,\\
\tilde{\mathcal{Z}}_n^{(k)} &= \sum_{j=0}^{M} q_j\,(n+\beta+j\,k)\tilde{\mathcal{Z}}_{n+j}^{(k-1)}\;,\\
\tilde{\mathcal{N}}_n^{(k)} &= \sum_{j=0}^{M} q_j\,(n+\beta+j\,k)\tilde{\mathcal{N}}_{n+j}^{(k-1)}\;,\\
\frac{\tilde{\mathcal{Z}}_n^{(k)}}{\tilde{\mathcal{N}}_n^{(k)}} &= \tilde{\mathcal{H}}_n^{(k)}(\{e_m\},\beta,\{s_n\},\{\omega_n\})\;.
\end{aligned} \tag{47}$$

The algorithm (7) is a special case of the algorithm (47). To see this, one observes that $M=2$, $e_1=\exp(\mathrm{i}\alpha)$ und $e_2=\exp(-\mathrm{i}\alpha)$ imply $q_0=q_2=1$ and $q_1=-2\cos(\alpha)$.

Note that the frequencies α_m can also be chosen as arbitrary distinct complex numbers. For instance, choosing $M=2$, $\alpha_1=-i\,\ln(p_1)$ and $\alpha_2=-i\,\ln(p_2)$ with $0<p_i<1$ yields $e_1=p_1$, $e_2=p_2$, $q_0=p_1\,p_2$, $q_1=-p_1-p_2$, $q_0=1$ and leads to the transformation

$$\begin{aligned}
\tilde{\mathcal{Z}}_n^{(0)} &= (n+\beta)^{-1}\,s_n/\omega_n\;, \qquad \tilde{\mathcal{N}}_n^{(0)} = (n+\beta)^{-1}/\omega_n\;,\\
\tilde{\mathcal{Z}}_n^{(k)} &= (n+\beta)\,p_1\,p_2\tilde{\mathcal{Z}}_n^{(k-1)} + (n+2k+\beta)\tilde{\mathcal{Z}}_{n+2}^{(k-1)}\\
&\quad-(p_1+p_2)(n+k+\beta)\tilde{\mathcal{Z}}_{n+1}^{(k-1)}\;,\\
\tilde{\mathcal{N}}_n^{(k)} &= (n+\beta)\,p_1\,p_2\tilde{\mathcal{N}}_n^{(k-1)} + (n+2k+\beta)\tilde{\mathcal{N}}_{n+2}^{(k-1)}\\
&\quad-(p_1+p_2)(n+k+\beta)\tilde{\mathcal{N}}_{n+1}^{(k-1)}\;,\\
\frac{\tilde{\mathcal{Z}}_n^{(k)}}{\tilde{\mathcal{N}}_n^{(k)}} &= \tilde{\mathcal{H}}_n^{(k)}(\{p_1,p_2\},\beta,\{s_n\},\{\omega_n\})\;.
\end{aligned} \tag{48}$$

that is exact for the model sequence

$$s_n = s+\omega_n\left(p_1^n\sum_{j=0}^{k-1} c_j(n+\beta)^{-j} + p_2^n\sum_{j=0}^{k-1} d_j(n+\beta)^{-j}\right) \tag{49}$$

with coefficients c_j and d_j.

5 The Application of the $\mathcal{H}$ Transformation Near Singularities

Already in the first applications of the $\mathcal{H}$ transformation it turned out that the otherwise good performance of the transformation deteriorated near discontinuities and other singularities of the Fourier series under consideration [20]. This undesirable behavior is also known [20, 43] for other convergence accelerators. It was also studied by Boyd [7] who showed that in the vicinity of singularities many standard methods perform very badly (for instance Richardson extrapolation or Chebyshev methods) or badly (e.g., Levin's u transformation, ϵ algorithm, Euler transformation).

This difficulty can be tackled by decomposing the Fourier series in a polynomial part and a much smoother Fourier series. The polynomial part can be determined by a local analysis of the Fourier series near the singularities in an analytical way [6] or by numerical procedures [14].

Alternatively, one may use related Fourier series that are either generated by considering complex Fourier series [13, 20, 43, 44] or by a somewhat complicated pretransformation [38]. We want to use a related approach.

As an example, we study convergence acceleration for the Fourier series

$$\ln(1-2q\cos(\alpha)+q^2) = -2\sum_{j=1}^{\infty}\frac{q^j}{j}\cos(j\alpha); \qquad s_n = -2\sum_{j=1}^{n+1}\frac{q^j}{j}\cos(j\alpha) \quad (50)$$

with singularities at

$$q = \exp(\pm i\alpha)\,. \quad (51)$$

For $q=1$, the function represented by the Fourier series thus is singular for $\alpha=0$. The terms of the series for $q=1$ are (up to a constant factor)

$$A_j = \frac{1}{j}\cos(j\,\alpha)\,. \quad (52)$$

They oscillate as function of the index j for $\alpha \neq 0$. The first sign change is at $j_0 = \pi/(2\alpha)$. If α is very small then j_0 becomes very large. The oscillations of the terms as a function of j then are very slow. As a consequence, the first terms of the Fourier series are difficult to distinguish numerically from the terms of the divergent series

$$\sum_{j=1}^{\infty}\frac{1}{j} \quad (53)$$

for small α.

To tackle these slow oscillations, one may consider to use subsequences of the original sequence of partial sums. In this way, the oscillations become more pronounced making an extrapolation easier.

Thus, one aims at the acceleration of a suitable subsequence. For instance, one may try to extrapolate the sequence $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ instead on the sequence $\{s_0, s_1, s_2, \ldots\}$. This possibility clearly exists for all natural numbers τ. We will concentrate below on this type of subsequences. Note that one can extrapolate instead of the sequence $\{s_0, s_1, s_2, \ldots\}$ another sequence $\{s_{R_0}, s_{R_1}, s_{R_2}, \ldots\}$ where the $\{R_l\}$ are a monotoneously increasing sequence of non-negative integers $0 \le R_0 < R_1 < R_2 \ldots$, as for instance $R_l = [\![l\,\tau]\!]$ for *real* $\tau > 1$.

The subsequence $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ can be considered as a new sequence $\{\check{s}_n\}$ with

$$\check{s}_n = s_{n\,\tau}\,, \qquad n \in \mathbb{N}_0\,. \tag{54}$$

The question arises whether the elements of this sequence can also be regarded as partial sums of a Fourier series. This is indeed the case.

By combining τ consecutive terms of the series we obtain

$$\begin{aligned}
s &= a_0/2 + \sum_{m=1}^{\infty} \left(\check{a}_m \cos(m\,\tau\,\alpha) + \check{b}_m \sin(m\,\tau\,\alpha)\right)\,, \\
\check{a}_m &= \sum_{k=1}^{\tau} \Bigg(a_{k+(m-1)\,\tau} \cos([k-\tau]\alpha) + b_{k+(m-1)\,\tau} \sin([k-\tau]\alpha) \Bigg)\,, \\
\check{b}_m &= \sum_{k=1}^{\tau} \Bigg(-a_{k+(m-1)\,\tau} \sin([k-\tau]\alpha) + b_{k+(m-1)\,\tau} \cos([k-\tau]\alpha) \Bigg)\,.
\end{aligned} \tag{55}$$

Thus, the $\check{s}_n$ are indeed the partial sums

$$\check{s}_n = a_0/2 + \sum_{m=1}^{n} \left(\check{a}_m \cos(m\,\tau\,\alpha) + \check{b}_m \sin(m\,\tau\,\alpha)\right)\,. \tag{56}$$

$$\check{s}_n = s_{n\,\tau}\,, \qquad n \in \mathbb{N}_0 \tag{57}$$

of a Fourier series with the τ-fold frequency $\tau\alpha$. It is clear that the oscillations of this τ-fold-frequency series are faster by the factor of τ, and thus, it is expected that the difficulties related to the slow oscillations disappear. This is indeed the case as we will see later.

The series (50) was also chosen as test case in [19, 43, 27]. In [19], it was shown that the $\mathcal{H}$ transformation is superior to the ϵ algorithm in this test case except in the vicinity of singularities. In [43], it was shown that the $d^{(2)}$ transformation can be applied even in the vicinity of the singularity if it operates on the partial sums s_{R_ℓ} of the real with $R_\ell = \tau\ell$. Thus, in the case of the $d^{(m)}$ transformations similar index transformations and subsequences are used.

Sidi has shown that the $d^{(1)}$ transformation with $R_\ell = \tau\ell$ for $\tau \in \mathbb{N}$ can be stabilized numerically by suitable choices of τ and in this way, good results are obtained for this transformation in the vicinity of singularities [43]. But the $d^{(1)}$ transformation with $R_\ell = \tau\ell$ for $\tau \in \mathbb{N}$ is nothing but the transformation (see [43, Eq. 4.12])

$$W_\nu^{(n)} = \frac{\triangle^{\nu+1}\left[(n+\beta/\tau)^{\nu-1} s_{\tau\,n}/(s_{\tau\,n} - s_{(\tau\,n)-1})\right]}{\triangle^{\nu+1}\left[(n+\beta/\tau)^{\nu-1}/(s_{\tau\,n} - s_{(\tau\,n)-1})\right]}\,. \tag{58}$$

Note that this is identical to the Levin transformation when applied to the sequence $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ with remainder estimates $\omega_n = (n+\beta/\tau)(s_{\tau\,n} - s_{(\tau\,n)-1})$,

Table 1: Convergence acceleration of the Fourier series (50) for $q = 1$ and $\alpha = \pi/6.1$

n	A_n	B_n	C_n	D_n	E_n
10	0.92	1.90	1.66	1.90	1.66
20	0.87	4.72	3.77	4.89	3.81
30	1.39	6.61	5.93	1.39	1.03
40	1.37	8.92	8.32	−0.56	1.62
50	1.26	10.01	9.68	0.00	−1.84

The number of exact digits is defined as the negative decadic logarithm of the relative error. We set $s'_n = \mathcal{H}^{([\![n/2]\!])}_{n-2[\![n/2]\!]}(\alpha, 1, \{s_n\}, \{q^{n+1}\})$.

A_n: Number of exact digits of s_n.

B_n: Number Δ_n of exact digits of s'_n in quadruple precision.

C_n: Predicted number δ_n of exact digits of s'_n in quadruple precision.

D_n: Number Δ_n of exact digits of s'_n in double precision.

E_n: Predicted number δ_n of exact digits of s'_n in double precision.

i.e.,

$$W^{(n)}_{\nu-1} = \mathcal{L}^{(n)}_{\nu}(\beta/\tau, s_{\tau\, n}, (n+\beta/\tau)(s_{\tau\, n} - s_{\tau\, n-1})) \,. \tag{59}$$

We remark that for $\tau \neq 1$ this is not identical to the u variant of the Levin transformation as applied to the partial sums $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ because in the case of the u variant one would have to use the remainder estimates $\omega_n = (n+\beta')(s_{\tau\, n} - s_{\tau\,(n-1)})$.

Similarly, the $d^{(m)}$ transformations may be assume to operate on the partial sums $\{s_0, s_\tau, s_{2\tau}, \ldots\}$ and corresponding terms of the series.

Note that this procedure can be *mutatis mutandis* applied to any series, i.e., also to series that are not Fourier series. Thus, for the series

$$s = \sum_{j=0}^{\infty} a_j \tag{60}$$

with partial sums

$$s_n = \sum_{j=0}^{n} a_j \tag{61}$$

collection of terms yields a new series

$$s = \sum_{j=0}^{\infty} \breve{a}_j \tag{62}$$

with terms

$$\breve{a}_0 = a_0 \,, \quad \breve{a}_j = \sum_{k=1}^{\tau} a_{k+(j-1)\tau} \qquad (j > 0) \tag{63}$$

Table 2: Convergence acceleration of the τ-fold-frequency series (55) with $\tau = 2$ for the Fourier series (50) for $q = 1$ and $\alpha = \pi/6.1$

m	n	A_n	B_n	C_n	D_n	E_n
20	10	0.87	3.89	3.43	3.89	3.43
40	20	1.37	8.28	7.69	8.28	7.69
50	25	1.26	10.63	10.00	10.69	10.05
60	30	1.97	12.72	12.05	9.15	8.72
80	40	1.50	17.21	16.46	7.30	6.96
100	50	1.58	21.58	20.84	4.85	4.35

The number of exact digits is defined as the negative decadic logarithm of the relative error. Used were $m+1$ terms of the Fourier series and $n+1$ terms of the τ-fold-frequency series. We set $s'_n = \mathcal{H}^{(\llbracket n/2 \rrbracket)}_{n-2\llbracket n/2 \rrbracket}(\tau\,\alpha, 1, \{\check{s}_n\}, \{q^{n\tau+1}\})$.

A_n: Number of exact digits of $s_{n\tau}$.

B_n: Number Δ_n of exact digits of s'_n in quadruple precision.

C_n: Predicted number δ_n of exact digits of s'_n in quadruple precision.

D_n: Number Δ_n of exact digits of s'_n in double precision.

E_n: Predicted number δ_n of exact digits of s'_n in double precision.

and partial sums

$$\check{s}_n = \sum_{j=0}^{n} \check{a}_j \,. \tag{64}$$

Also in this case, the relation $\check{s}_n = s_{\tau n}$ holds. As a matter of fact, we observe that any Levin-type transformation can be used in this way. This means that one uses as input the sequence $\{\check{s}_0, \check{s}_1, \ldots\} = \{s_0, s_\tau, s_{2\tau}, \ldots\}$ and the remainder estimates $\omega_n = (n + \beta/\tau)(s_{\tau\,n} - s_{\tau\,n-1}))$. Most Levin-type sequence transformations are multiplicatively invariant in ω_n. For these, one can equivalently use $\omega_n = (\tau\,n + \beta)(s_{\tau\,n} - s_{\tau\,n-1}))$. This simple observation will probably enlarge the range of applicability of such Levin-type as the Weniger transformations and the $\mathcal{J}$ transformation enormously.

We note that sequence transformations like the ϵ algorithm that do not make use of remainder estimates can directly be applied to the τ-fold-frequency series. It is to be expected that their performance is also improved in this way in the vicinity of singularities.

Sidi [43] has claimed that the $\mathcal{H}$ transformation is useless near singularities. He seemingly failed to notice the possibility to apply other Levin-type transformations like the $\mathcal{H}$ and $\mathcal{I}$ transformations in the way discussed above. Here, we show that in the vicinity of singularities, the $\mathcal{H}$ transformation can be applied successfully to subsequences of partial sums as discussed above, or equivalently to τ-fold-frequency Fourier series of the form (55) and then, this transformation performs similarly to the $d^{(2)}$ transformation. A corresponding result holds for

Table 3: Convergence acceleration of the τ-fold-frequency series (55) with $\tau = 3$ for the Fourier series (50) for $q = 1$ and $\alpha = \pi/6.1$

m	n	A_n	B_n	C_n	D_n	E_n
30	10	1.39	5.52	4.71	5.52	4.71
51	17	1.28	10.20	8.83	10.20	8.83
60	20	1.97	12.04	10.70	12.05	10.70
90	30	3.18	17.91	18.02	13.36	13.53
105	35	1.57	22.26	20.03	13.66	13.35
114	38	2.26	23.91	21.90	13.43	13.53
120	40	2.19	24.05	22.98	13.52	13.35
150	50	2.01	30.82	29.51	14.15	13.23

The number of exact digits is defined as the negative decadic logarithm of the relative error. Used were $m + 1$ terms of the Fourier series and $n + 1$ terms of the τ-fold-frequency series. We set $s'_n = \mathcal{H}^{([\![n/2]\!])}_{n-2[\![n/2]\!]}(\tau\,\alpha, 1, \{\check{s}_n\}, \{q^{n\tau+1}\})$. The column headers are defined as in Tab 2.

the $\mathcal{I}$ transformation [27]. The possible application to related complex series will be discussed elsewhere [26, 28].

Now, numerical examples are presented. The calculations were done using Maple with floating-point accuracies of 32 decimal digits corresponding to quadruple precision on most workstations and 15 decimal digits corresponding to double precision. This reduction of accuracy allows to judge the numerical stability of the algorithms.

We were also interested in stopping criteria and used the quantity

$$\kappa_n = \left| s^{([\![n/2]\!])}_{n-2[\![n/2]\!]} - s^{([\![(n-1)/2]\!])}_{n-1-2[\![(n-1)/2]\!]} \right| + \left| s^{([\![n/2]\!])}_{n-2[\![n/2]\!]} - s^{([\![n/2]\!])}_{n+1-2[\![(n/2]\!]} \right| \tag{65}$$

as (empirical, nonrigorous) estimate for the absolute error of the estimated limit $s^{([\![n/2]\!])}_{n-2[\![n/2]\!]}$. Thus, the question to be studied numerically is whether κ_n is a realistic measure of the error of the extrapolated limit. For the following, we regard

$$\delta_n = -\ln\left(\left| (\kappa_n / s^{([\![n/2]\!])}_{n-2[\![n/2]\!]} \right| + 10^{-2*D} \right) / \ln(10) \tag{66}$$

as the predicted number of exact decimal digits. Here, D is the number of decimal digits used in the calculation, i.e., $D = 32$ in the case of quadruple precision and $D = 15$ in the case of double precision. (Thus, D is the value of the Maple variable **Digits**.) The predicted number δ_n of exact digits is to be compared with

$$\Delta_n = -\ln\left(\left| s^{([\![n/2]\!])}_{n-2[\![n/2]\!]} / s - 1 \right| + 10^{-2*D} \right) / \ln(10)\,, \tag{67}$$

Table 4: Convergence acceleration of the τ-fold-frequency series (55) with $\tau = 10$ for the Fourier series (50) for $q = 1$ and $\alpha = \pi/50.1$

m	n	A_n	B_n	C_n	D_n	E_n
100	10	2.46	3.23	2.56	3.23	2.56
200	20	3.63	5.68	5.03	5.66	5.01
300	30	4.10	8.73	8.24	3.29	2.89
400	40	4.20	12.53	11.37	−0.88	−0.27
460	46	2.13	13.50	12.80	0.78	−0.52
470	47	1.94	14.19	13.35	−0.32	0.01
480	48	1.94	13.74	13.62	−0.43	0.49
500	50	5.46	12.77	12.38	−0.44	1.10

The number of exact digits is defined as the negative decadic logarithm of the relative error. Used were $m+1$ terms of the Fourier series and $n+1$ terms of the τ-fold-frequency series. We set $s'_n = \mathcal{H}^{([\![n/2]\!])}_{n-2[\![n/2]\!]}(\tau\,\alpha, 1, \{\check{s}_n\}, \{q^{n\tau+1}\})$. The column headers are defined as in Tab 2.

Table 5: Convergence acceleration of the τ-fold-frequency series (55) with $\tau = 20$ for the Fourier series (50) for $q = 1$ and $\alpha = \pi/50.1$

m	n	A_n	B_n	C_n	D_n	E_n
200	10	3.63	4.77	4.23	4.77	4.23
400	20	4.20	10.47	9.53	10.46	9.53
600	30	4.14	15.61	14.87	11.69	11.15
660	33	2.30	17.37	16.39	12.42	11.49
800	40	3.73	20.94	20.63	10.94	10.38
1000	50	3.57	26.25	25.82	8.92	8.41

The number of exact digits is defined as the negative decadic logarithm of the relative error. Used were $m+1$ terms of the Fourier series and $n+1$ terms of the τ-fold-frequency series. We set $s'_n = \mathcal{H}^{([\![n/2]\!])}_{n-2[\![n/2]\!]}(\tau\,\alpha, 1, \{\check{s}_n\}, \{q^{n\tau+1}\})$. The column headers are defined as in Tab 2.

i.e., with the actually achieved number of exact digits. The definitions of Δ_n and δ_n involve essentially decadic logarithms that are shifted by a very small quantity in order to avoid overflow even for vanishing arguments.

In Tables 1, 2, and 3, the $\mathcal{H}$ transformation has been applied to the Fourier series (50) for $q = 1$ and $\alpha = \pi/6.1$ and its related τ-fold-frequency series (55) with $\tau = 2$ and $\tau = 3$. For this value of α, one is relatively close to the singularity at $\alpha = 0$.

It is seen in Table 1 that in quadruple precision a pronounced convergence

Table 6: Convergence acceleration of the τ-fold-frequency series (55) with $\tau = 30$ for the Fourier series (50) for $q = 1$ and $\alpha = \pi/50.1$

m	n	A_n	B_n	C_n	D_n	E_n
300	10	4.10	5.95	5.41	5.95	5.41
600	20	4.14	15.34	13.33	12.96	13.30
900	30	3.63	22.30	20.66	12.93	13.60
1200	40	3.48	29.26	27.97	12.98	13.97
1500	50	3.40	29.92	30.74	12.93	14.14

The number of exact digits is defined as the negative decadic logarithm of the relative error. Used were $m+1$ terms of the Fourier series and $n+1$ terms of the τ-fold-frequency series. We set $s'_n = \mathcal{H}^{([\![n/2]\!])}_{n-2[\![n/2]\!]}(\tau\,\alpha, 1, \{\check{s}_n\}, \{q^{n\tau+1}\})$. The column headers are defined as in Tab 2.

acceleration is observed for $\tau = 1$, i.e., the original series, while in double precision convergence acceleration is also observed with best results of nearly five exact digits at $n = 20$ and deterioration for larger n due to rounding errors. The prediction of the error shown in columns 4 and 6 is a realistic, and in most cases conservative one, that is, the predicted accuracy is normally somewhat lower than the actual accuracy.

The data in Table 2 show using the τ-fold-frequency series for $\tau = 2$ improves the achievable accuracy and enhances the stability of the extrapolation, as shown by comparison of column 4 and 6 of this table. In double precision, best results of more than ten digits are observed for $n = 25$ corresponding to using partial sums up to $\check{s}_{25}$ and, equivalently, up to s_{50}. Thus, for $\tau = 2$ machine precision can not be reached in double precision. Again, the predicted accuracies are very close to the actual one as seen from comparing columns 4 and 5, and columns 6 and 7, respectively. The quadruple precision data in Table 2 can be compared directly to [43, Tab. 2]. The comparison shows that the $\mathcal{H}$ transformation for the τ-fold-frequency series with $\tau = 2$ performs rather similarly to the $d^{(2)}$ transformation as applied to the real Fourier series with $R_\ell = 2\ell$. For instance, the relative error of the latter using 50 terms of the original Fourier series is 1.6×10^{-10}, and using 98 terms, it is 1.5×10^{-21}. [43, Tab. 2]

Choosing $\tau = 3$ as in Table 3 produces quadruple precision results of comparable quality as in Table 2. In double precision, essentially full accuracy can now be reached.

In Tables 4, 5, and 6, the $\mathcal{H}$ transformation has been applied to the τ-fold-frequency series (55) with $\tau = 10$, $\tau = 20$ and $\tau = 30$ corresponding to the Fourier series (50) for $q = 1$ and $\alpha = \pi/50.1$. For this value of α, one is in the immediate vicinity of the singularity at $\alpha = 0$.

Similar to Table 1, also in Table 4 a pronounced convergence acceleration is observed for $\tau = 10$ in quadruple precision while in double precision best results of nearly six exact digits are obtained at $n = 20$ corresponding to using partial sums up to $\check{s}_{20}$ and, equivalently, up to s_{200} that deteriorate for larger n due to rounding errors. As in the previous cases the prediction of the error corresponding in columns 5 and 7 is realistic and usually conservative. The quadruple precision data in Table 4 can be compared directly to [43, Tab. 3]. Similarly to the case treated in Table 2, the comparison shows that the $\mathcal{H}$ transformation for the τ-fold-frequency series with $\tau = 10$ performs rather similarly to the $d^{(2)}$ transformation as applied to the real Fourier series with $R_\ell = 10\ell$. For instance, the relative error of the latter using 402 terms of the original Fourier series is 3.0×10^{-10}, using 482 terms, it is 1.2×10^{-14}, and using 562 terms, it is 2.7×10^{-13}.

As in the previous example, the data in Table 5 show that using the τ-fold-frequency series for $\tau = 20$ improves the achievable accuracy and enhances the stability of the extrapolation, as shown by comparison of column 4 and 6 of this table. In double precision, best results of more than twelve digits are observed for $n = 33$ corresponding to using partial sums up to $\check{s}_{33}$ and, equivalently, up to s_{660}. Thus, for $\tau = 20$ machine precision cannot be reached in double precision. Again, the predicted accuracies are very close to the actual one as seen from comparing columns 4 and 5, and columns 6 and 7, respectively.

Choosing $\tau = 30$ in Table 6 produces quadruple precision results of comparable quality as in Table 5. In double precision, full accuracy can now be nearly be reached.

From the data presented, it is apparent that one has to choose τ the higher, the closer one gets to the singularity. In the example (50), the numerical data show that $\tau = \gamma/d$ with $\gamma \approx 1/(2\pi)$ yields good results, where $d = \alpha$ is the distance to the singularity.

Thus, within its range of applicability, i.e., when applied to Fourier series with nonoscillating behavior of the coefficients, the $\mathcal{H}$ transformation in combination with τ-fold-frequency approach provides good results even in the vicinity of singularities. The price to pay, however, is that more terms of the original series have to be used. The quantities κ_n provide realistic estimates of the error in the examples studied, and thus they may be used in stopping criteria.

Further methods for the convergence acceleration of Fourier series, especially those relating to transformations that involve complex Fourier series will be treated elsewhere.

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A GTRLEV : A FORTRAN 77 Program Implementing the $\mathcal{H}$ Transformation

The following subroutine is a FORTRAN 77 DOUBLE PRECISION implementation of the $\mathcal{H}$ transformation. The program is based on the algorithm described in Sec. 3. The variable ALPHA corresponds to $2\cos(\alpha)$, while SOFN and OMOFN correspond to the new input data s_n and ω_n (or to c_1 of the example (37), resp.). For simplicity, we chose $\beta = 1$ in the program. The numerator and denominator sums of Eq. (14) are computed via the recursions in (7) and are stored in two one-dimensional arrays AN and AZ as described in Sec. 3. In the program, no measures are taken against a vanishing of the denominators. The program must be used in the following way: The values of s_n and ω_n with $n = 0, 1, 2, \ldots$ have to be computed in a loop in the calling program. After each calculation of a pair (s_n, ω_n) the subroutine GTRLEV has to be called It computes an estimate SEST for the limit (or anti-limit) of the series by using Eq. (36). No convergence analysis is undertaken in the subroutine GTRLEV. This has to be done in the calling program.

```
      SUBROUTINE GTRLEV(ALPHA,N,SOFN,OMOFN,NMAX,AZ,AN,EVEN,SEST)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      LOGICAL EVEN
      DIMENSION AZ(0:NMAX),AN(0:NMAX)
      IF(N.EQ.0) THEN
        EVEN = .TRUE.
      ENDIF
      AN(N) = 1.D0 / OMOFN / DBLE(N+1)
      AZ(N) = SOFN * AN(N)
      DO 100 K=1, N/2, 1
        M     = N - 2 * K
        M1    = M + 1
        M2    = M + 2
        M1K   = M1 + K
        M1K2  = M1K + K
        DM1   = DBLE(M1)
        DM1K  = DBLE(M1K)
        DM1K2 = DBLE(M1K2)
        AN(M) = DM1*AN(M) + DM1K2*AN(M2) - ALPHA*AN(M1)*DM1K
        AZ(M) = DM1*AZ(M) + DM1K2*AZ(M2) - ALPHA*AZ(M1)*DM1K
  100 CONTINUE
      IF(EVEN) THEN
        SEST = AZ(0) / AN(0)
      ELSE
        SEST = AZ(1) / AN(1)
      END IF
      EVEN = .NOT. EVEN
      RETURN
      END
```