Reduction to Directrix-near points in resolution of singularities of schemes

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES DER NATURWISSENSCHAFTEN (DR. RER. NAT.) DER FAKULTÄT FÜR MATHEMATIK DER UNIVERSITÄT REGENSBURG

vorgelegt von

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aus Brest, Weißrussland,

im Jahr 2015

Promotionsgesuch eingereicht am: 07. Oktober 2015

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Introduction

Abstract

By a result from [CJS], for a blow-up $X' \to X$ of a locally noetherian scheme X in a permissible center D every point x' of X', which is near to its image x in X with $x \in D$, lies in $\mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$, if $\dim X \leq 2$. We show that this holds for $\dim X \leq 5$ under the additional assumption

$$\dim \mathrm{Rid}_{X',x'} + \mathrm{trdeg}(\kappa(x')/\kappa(x)) = \dim \mathrm{Rid}_{X,x}$$

and get an application to resolution of singularities.

Resolution of the singularities

A resolution of singularities of a locally noetherian scheme X is a proper birational morphism $\pi : Y \to X$ with Y regular, i.e., such that Y has no singular points. Then π induces an isomorphism between open dense subschemes of X and Y. Thus X and Y share many properties. For example, if X is an integral scheme, then $\dim X = \dim Y$ and X, Y have isomorphic function fields. Therefore sometimes a resolution of singularities makes it possible to reduce a problem to the case of a regular scheme. For instance, the Riemann-Roch theorem for smooth projective algebraic surfaces over $\mathbb{C}$ can be generalized to proper schemes with rational singularities which admit a resolution of singularities. This raises the question if a given locally noetherian scheme X admits a resolution of singularities.

Brief historical overview

The theory of resolution of singularities is rather old. In 1676 Newton resolved singularities of plane curves over $\mathbb{C}$. The biggest influence to the theory came from Zariski and his students Abhyankar and Hironaka. For three-dimensional varieties there is a resolution of singularities, if the ground field has characteristic zero, [Za], or the characteristic is greater than six, [Ab]. In his celebrated paper [Hi1] Hironaka proved the existence of a resolution of singularities for reduced excellent schemes X (see definition (4.1.1)) with residue fields of characteristic zero (e.g. reduced schemes of finite type over a field of characteristic zero). He proved that there is a finite sequence of permissible blow-ups (see definition (2.1.6))

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

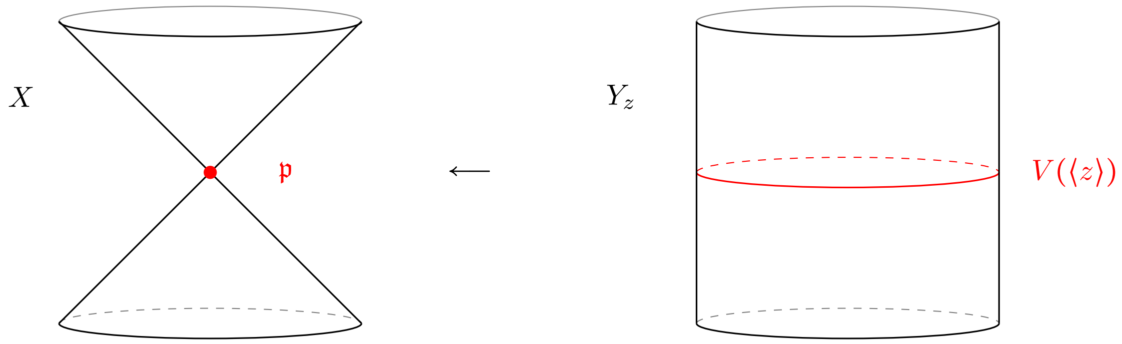
with X_n regular. For arbitrary characteristic of the residue fields there is not that much known. By the paper [CJS] there is a functorial (see definition (4.3.3)) resolution of singularities for excellent surfaces. Every three-dimensional separated noetherian quasi-excellent scheme admits a resolution of singularities by the recent paper [CP]. For dimensions greater than three the problem is open, at least in the form stated above. In [dJ] de Jong proves a weaker form of resolution of singularities $\pi : Y \to X$ for a integral separated scheme X of finite type over a field, where π is not necessarily birational.

An example for a resolution

Let us look at the following example. Let X be the spectrum of the ring $k[x,y,z]/\langle x^2+y^2-z^2\rangle$ for a field k with $\mathrm{char} k \neq 2$. It has a singularity at the closed point given by the maximal ideal $\mathfrak{p} = \langle x,y,z\rangle$, cf. the picture below. Blowing up X in the closed subscheme $\overline{\{\mathfrak{p}\}}$ of X we get a morphism $Y \to X$ for a scheme Y covered by the open affine subschemes

$$\mathrm{Spec}\Big(\frac{k[x,y,z]}{\langle 1+y^2-z^2\rangle}\Big) =: Y_x, \quad \mathrm{Spec}\Big(\frac{k[x,y,z]}{\langle x^2+1-z^2\rangle}\Big) =: Y_y, \quad \mathrm{Spec}\Big(\frac{k[x,y,z]}{\langle x^2+y^2-1\rangle}\Big) =: Y_z$$

(for more details see example (2.1.4)). The fiber of $\mathfrak{p}$ in Y_z is $V(\langle z\rangle)$. The blow-up pulls apart the point and leaves the complement of the center unchanged (up to isomorphism). As Y_x, Y_y and Y_z are regular, the morphism $Y \to X$ is a resolution of the singularities of X.



The invariant $H^{(m)}_{X,x}$

We come back to Hironaka's method. Assume, to resolve the singularities of X, one has constructed a sequence of blow-ups $\ldots \to X_2 \to X_1 \to X_0 = X$. How do we show that X_n is regular for some n? It is common to study the behavior of local invariants which measure the complexity of the singularities, as also Hironaka did. One invariant is the m-th Hilbert-Samuel-function $H^{(m)}_{X,x}$ of the graded algebra $\mathrm{gr}\mathcal{O}_{X,x}$ of a point $x \in X$, for $m \in \mathbb{N}$. If X is a hypersurface of a regular scheme Z, then $H^{(m)}_{X,x}$ contains the same information as the multiplicity of X in Z at x (see lemma (2.2.2)). Thus it can be seen as a generalization of the multiplicity. The function has values in the partially ordered set $\mathbb{N}^{\mathbb{N}}$ with the product order. It takes its minimal value (depending on $\dim \mathcal{O}_{X,x}$ and m) if and only if x is non-singular. Further, for a permissible blow-up $X' \to X$ and a point x' over $x \in X$, one has $H^{(m+d)}_{X',x'} \leq H^{(m)}_{X,x}$ for the transcendence degree $d = \mathrm{trdeg}(\kappa(x')/\kappa(x))$. Thus, if the inequality is strict, one sees an improvement of the singularity. If we have equality, we say x' is near to x.

Ridge, directrix and the invariant $i^{(m)}_{X,x}$

Hironaka's method of maximal contact implies that the points near to x all lie in a hypersurface of X. But this works only if the residue field $\kappa(x)$ of x has characteristic zero. For positive characteristic one has a weaker form of maximal contact: There is an additive group scheme $\mathrm{Rid}(C_{X,D,x})$, called ridge, naturally associated to the normal cone $C_{X,D,x}$ over $\kappa(x)$ (see definitions (2.1.6) and (2.5.12)). The near points all lie on the associated projective bundle $\mathbb{P}(\mathrm{Rid}(C_{X,D,x})) \subseteq \pi^{-1}(\{x\})$ (see remark (3.1.3)). Under additional assumptions all near points lie in $\mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$ (see below). Here $\mathrm{Dir}(C_{X,D,x})$ is the directrix. This is a vector group scheme, i.e., as an additive group scheme, it is isomorphic to $\mathbb{G}_a^m$ for some $m \in \mathbb{N}$. The directrix is also naturally associated to $C_{X,D,x}$ and it is contained in the ridge. Usually, it is easier to calculate the directrix then to calculate the ridge.
If x' is near to x, the ridge $\mathrm{Rid}_{X,x}$, associated to the cone $\mathrm{Spec}(\mathrm{gr}\mathcal{O}_{X,x})$ over $\kappa(x)$, is a second invariant. One has $\dim \mathrm{Rid}_{X',x'} + d \leq \dim \mathrm{Rid}_{X,x}$ if $H^{(m+d)}_{X',x'} = H^{(m)}_{X,x}$. Then the invariant

$$i^{(m)}_{X,x} = (H^{(m)}_{X,x}, \dim \mathrm{Rid}_{X,x} + m),$$

with values in the partially ordered set $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ with the lexicographical order, is finer than $H^{(m)}_{X,x}$. If $i^{(m+d)}_{X',x'} = i^{(m)}_{X,x}$ for all m, we say x' is i-near to x.

Main theorem

By a theorem from [CJS], if x' is near to x and $\dim X \leq 2$, then x' is Dir-near to x, i.e. x' lies on $\mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$ (see theorem (3.1.2)). This fact was crucial in [CJS] to successfully resolve the singularities of two-dimensional noetherian excellent reduced schemes. The proof of the cited theorem uses a result of Hironaka about Hironaka schemes, special additive group schemes defined in [Hi3] (see definition (2.5.7)), that each Hironaka scheme of dimension at most two is a vector group. If $\kappa(x)$ is a perfect field the statement is true for arbitrary dimension of X (see remark (3.1.3)). In general, for $\dim X \geq 3$ (already for $\dim X = 3$, see example (3.1.4)) the point x' can be near without beeing Dir-near to x. This can be repaired for $\dim X \leq 5$ if one replaces 'near' by 'i-near'. Our main result is (cf. theorem (3.2.1))

Main theorem. *Let $\pi : X' \to X$ be a blow-up of a locally noetherian scheme X with $\dim X \leq 5$ in a permissible center D and let x' be a point of X' i-near to a point $x \in X$ with $x \in D$. Then x' is* Dir*-near to x.*

In the proof we show that there is a Hironaka scheme $\mathcal{B}$ with $\dim \mathcal{B} \leq X$, associated to the point x', which is not a vector group scheme, if x' is near but not Dir-near to x. Then by Oda's characterization of non-vector group Hironaka schemes $\mathcal{B}$ with $\dim \mathcal{B} \leq 5$, see [Od], $\mathcal{B}$ has an explicit form. A calculation yields $\dim \mathrm{Rid}_{X',x'} + d < \dim \mathrm{Rid}_{X,x}$. The theorem does not hold for arbitrary dimensions of X. In fact, for $\dim X = 7$ there is a counterexample (see (3.1.5)). For $\dim X = 6$ the question is still open.

Blow-up strategies and their i^N-iterated variation

As an application we modify existing blow-up strategies to resolve singularities and give a criterion for the modified strategy to be a resolution of singularities. To be more precise let $\mathcal{C}$ be a subcategory of the category $\mathcal{S}_N$ of all noetherian excellent reduced schemes with dimension at most N for some bound $N \in \mathbb{N}$ where the morphisms of S_N are arbitrary scheme morphisms. A strategy s on $\mathcal{C}$ (to resolve singularities) is the datum of a sequence of permissible blow-ups

$$s(X) = (X = s(X)_0 \leftarrow s(X)_1 \leftarrow s(X)_2 \leftarrow \dots)$$

for each scheme X of $\mathcal{C}$. For example one can take the strategy constructed in [CJS] for $\mathcal{C} = \mathcal{S}_N$ and for an arbitrary N. Assume that a strategy s on $\mathcal{C}$ is given. Depending on s and N we define a new strategy $i^N(s)$ with the property

$$i^N(s)(X) = (s(X)_0 \leftarrow s(X)_1 \leftarrow \dots \leftarrow s(X)_n \leftarrow i^N(s)(Y)_1 \leftarrow i^N(s)(Y)_2 \leftarrow \dots)$$

for $Y := s(X)_n$ if the sequence $s(X)_0 \leftarrow \dots \leftarrow s(X)_n$ is a short i^N-decrease. Here we call $X = s(X)_0 \leftarrow \dots \leftarrow s(X)_n = Y$ an i^N-decrease if for each singularity x of X with $x \in \{i_X^N = \max\}$ (see definition (4.1.8)) there is no point y of Y i-near to x, and the sequence $X \leftarrow \dots \leftarrow Y$ is a short i^N-decrease if additionally $X = s(X)_0 \leftarrow \dots \leftarrow s(X)_{n-1}$ is not an i^N-decrease. If for each n the sequence $s(X)_0 \leftarrow \dots \leftarrow s(X)_n$ is not an i^N-decrease, we set $i(s)(X) := s(X)$.

A criterion for $i^N(s)$ to be a resolution of singularities

We show that if the given strategy s is a desingularization, i.e. for each scheme X of $\mathcal{C}$ in the sequence $s(X)_0 \leftarrow s(X)_1 \leftarrow \dots$ some $s(X)_n$ is regular, then $i^N(s)$ is a desingularization. Further we show that $i^N(s)$ is a desingularization if and only if for each scheme X of $\mathcal{C}$ for some n the sequence $s(X)_0 \leftarrow \dots \leftarrow s(X)_n$ is an i^N-decrease. To verify that a given sequence $X \overset{\pi_1}{\leftarrow} X_1 \overset{\pi_2}{\leftarrow} \dots \overset{\pi_n}{\leftarrow} X_n$ of permissible blow-ups is an i^N-decrease it is enough to study pairs of singularities $x_j \in X_j$, $x_{j+1} \in X_{j+1}$ with x_{j+1} i-near to x_j. Our main theorem implies that for such a pair the point x_{j+1} is Dir-near to x_j, provided $\dim X \leq 5$. Thus we have the following criterion for $i^N(s)$ to be a desingularization (cf. corollary (4.2.6)).

Assume that all schemes of $\mathcal{C}$ have dimension at most five. The strategy $i^N(s)$ is a desingularization if and only if for each scheme X of $\mathcal{C}$ there is some $n \in \mathbb{N}$ such that there is no point of $i^N(s)(X)_n$ which is Dir*-near (see definition (4.2.4)) and i-near to a singularity x of X with $x \in \{i_X^N = \max\}$.*

Functoriality of $i^N(s)$

The strategy $i^N(s)$ inherits functoriality of s, at least with respect to regular (e.g. smooth) surjective morphisms. We call a strategy s on $\mathcal{C}$ functorial in E, where E is a class of scheme morphisms, if for each pair of scheme X, Y of $\mathcal{C}$ and each morphism $Y \to X$ of E the sequences $s(X) \times_X Y$ and $s(Y)$ are equal up to 'cutting out isomorphisms' (cf. definition (4.3.3)). We show (cf. corollary (4.3.8))

Assume that each morphism of E is surjective and regular, that E contains isomorphisms and that E is stable under base change and compositions. Then $i^N(s)$ is functorial in E if s in functorial in E.

We do not think that the surjectivity assumption can be dropped, as we expect problems with the functoriality with respect to open immersions, see remark (4.3.9).

Structure of the thesis

In the first chapter we recall of the definition and some properties of blow-ups, the Hilbert-Samuel-function and group schemes. We focus on additive group schemes, examples of which are the ridge, the directrix and the Hironaka scheme. In the last section of this chapter we cite some results by Hironaka about blow-ups and near points.
In chapter two the objective is the proof of the main theorem. We reduce the problem to the case of a point blow-up in the origin of a cone over a field. Then we reformulate the problem into an inequality of dimensions of rings of invariants $\mathcal{U}$ of homogeneous ideals I of a polynomial ring, see definition (2.5.15) and theorem (3.2.6). We achieve this with a case analysis using Oda's characterization of Hironaka schemes of dimension at most five. The ring of invariants $\mathcal{U}$ is generated by elements Df for elements f of a Giraud basis F of I (see section 3.7) and differential operators D associated to multi-indices. Since a reduced Gröbner basis is a Giraud basis we can find F via the Buchberger algorithm. Keeping track of the operations which appear in the Buchberger algorithm, the poof of the main theorem is completed at the end of chapter two in several technical steps.
In chapter three we cite results from [CJS] about blow-ups of finite-dimensional excellent schemes and a variant of the Hilbert-Samuel-function. As a corollary we get that, for a noetherian reduced finite-dimensional excellent scheme X and a sequence $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$ of i^N-decreases, some X_n is regular. In the second section, for a given strategy s, we define the i^N-iterated variation $i^N(s)$. We show that $i^N(s)$ is a desingularization if s has this property and we reformulate our main theorem as a criterion for $i^N(s)$ to be a desingularization. In the last section we discuss the functoriality of $i^N(s)$.

Comparison with the Ph.D. thesis of Bernhard Dietel

We should mention the Ph.D. thesis of Bernhard Dietel, [Di], which considers topics related to the present thesis. His theorem C is our main theorem but he proved it with a completely different approach. Dietel defined a refined version of Hironaka schemes, in short by replacing the invariant $H^{(m)}_{X,x}$ by $i^{(m)}_{X,x}$. His main aim is to show results about the refined Hironaka scheme in analogy to Hironaka's results about the original Hironaka scheme. In our approach we just use the classical notions of ridge, directrix and Hironaka schemes. We both use Oda's characterization of Hironaka schemes with dimension at most five.

Acknowledgments

I wish to express my gratitude to my advisor Uwe Jannsen for inviting me to his working group. He had always time for my frequent questions, helped me a lot and I learned much during the writing of this thesis. Several times he was able to encourage me to attack my problems from another point of view.
Further I thank Bernhard Dietel for the endless talks about commutative algebra, group schemes, differential operators and blow-ups.
Vincent Cossart and Olivier Piltant brought my work into the right direction during the fall school about resolution of threefolds in positive characteristic 2013 in Regensburg, for which I am thankful.
There are others I want to thank for their help and the answers to my questions: Christian Dahlhausen, Julius Hertel, Timo Keller, Bernd Schober and Jascha Smacka.
Special thank goes to my wife Maria for her warm support and for her patience, when I wanted to tell my mathematical problems to someone.
This project was supported by the SFB 1085 - Higher Invariants.

Conventions and notations

Zero is a natural number, i.e. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. A regular scheme is locally noetherian and a regular ring is noetherian. By a symmetric algebra over field k we mean the graded k-algebra $A = \oplus_{i\geq 0} A_i = \mathrm{Sym}_k(A_1)$ over k, i.e. A is a polynomial ring over k in $\dim_k A_1$ variables and A has a grading by setting $\deg v = 1$ for each variable v. For a scheme X we write ΓX for the ring of global sections $\Gamma(X, \mathcal{O}_X)$. For a point x of a scheme X we write $\mathcal{O}_{X,x}$, $\mathfrak{m}_{X,x}$ and $\kappa(x)$ for the local ring at x, the maximal ideal of $\mathcal{O}_{X,x}$ and the residue field of $\mathcal{O}_{X,x}$. For a local ring A with maximal ideal $\mathfrak{m}$ we write $\mathrm{gr}A$ for the graded $A/\mathfrak{m}$-algebra $\oplus_{n\in\mathbb{N}} \mathfrak{m}^n \otimes_A A/\mathfrak{m}$.

Chapter 2

Preliminaries

2.1 Blow-ups I

We recall the definition of blow-ups $X' \to X$ and list some of their properties. Further under some assumptions we can give a description of the local rings of X'.
In this section we fix a scheme X and we fix a closed subscheme D of X. We denote the quasi-coherent ideal sheaf of $\mathcal{O}_X$ which is associated to the closed immersion $D \to X$ by $\mathcal{I}$. We say that D is an effective Cartier divisor on X if $D = \varnothing$ or if $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module, see [GW], (13.19).

Definition (2.1.1). *A* ***blow-up of X in the center D*** *is a morphism of schemes $\pi : X' \to X$ such that $\pi^{-1}D$ is an effective Cartier divisor and such that π is universal with this property, i.e. for each morphism of schemes $\overline{\pi} : \overline{X}' \to X$ such that $\overline{\pi}^{-1}D$ is an effective Cartier divisor there is a unique scheme morphism $f : \overline{X}' \to X'$ with $\pi \circ f = \overline{\pi}$. We write $\mathrm{Bl}_D X$ for X'. We call $\pi^{-1}D$* ***the exceptional divisor****.*

By the universal property a blow-up is unique up to a unique isomorphism.

Proposition (2.1.2). *a) Let $\mathcal{G}$ denote the graded quasi-coherent $\mathcal{O}_X$-algebra $\oplus_{n\in\mathbb{N}}\mathcal{I}^n$ where we set $\mathcal{I}^0 := \mathcal{O}_X$. Then the projective spectrum $\mathrm{Proj}\mathcal{G} \to X$ of $\mathcal{G}$ is the blow-up of X in D.*

b) For a X-scheme Y there is a unique scheme morphism $\mathrm{Bl}_{Y\times_X D}(Y) \to \mathrm{Bl}_D X$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Bl}_{Y\times_X D}Y & \longrightarrow & \mathrm{Bl}_D X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

c) For a flat X-scheme Y the diagram in b) is cartesian. In particular the blow-up of X in D is a gluing of blow-ups $\mathrm{Bl}_{D_i}X_i \to X_i$ of open affine subschemes X_i of X in $D_i = X_i \times_X D$.

d) For a closed immersion $Y \to X$ the morphism $\mathrm{Bl}_{Y\times_X D}Y \to \mathrm{Bl}_D X$ from b) is a closed immersion.

e) For the open immersion $Y = X\backslash D \to X$ *the scheme morphism* $\pi^{-1}Y = \mathrm{Bl}_{\varnothing}Y \to Y$ *is an isomorphism.*

f) The open subscheme $\pi^{-1}(X\backslash D)$ *of* X' *is dense.*

g) If X *is locally noetherian, then a blow-up* $X' \to X$ *is proper.*

Proof. a)-e) [GW], Propositions 13.91, 13.92, 13.96; f) [GW], remarks 11.25 and 9.24; g) [Li], Proposition 8.1.12. □

Remark (2.1.3). *We have the following affine description of the blow-up of* X *in* D*. For a affine open subscheme* $Y = \mathrm{Spec}A$ *of* X*, for the ideal* I *of* A *with* $Y \times_X D = \mathrm{Spec}(A/I)$ *and for the graded* A*-algebra* $G = \oplus_{n\in\mathbb{N}} G_n := \oplus_{n\in\mathbb{N}} I^n$ *(where we set* $I^0 =: A$*) the base change of* $\mathrm{Bl}_D X \to X$ *by* Y *is the morphism* $\mathrm{Proj}G \to \mathrm{Spec}A$*. The scheme* $\mathrm{Proj}G$ *is covered by the affine open subschemes* $\mathrm{D}_+(f) = \mathrm{Spec}(G_{(f)})$ *for homogeneous elements* $f \in G$ *of degree one. There is a unique* A*-algebra morphism* $G \to A$ *such that the map* $G_1 \to G \to A$ *is the inclusion* $I \subseteq A$*. For each element* $f \in I = G_1$ *the composition* $G_{(f)} \to G_f \to A_f$ *is injective and the image is the* A*-subalgebra* $A[I/f]$ *of* A_f *generated by elements* i/f*,* $i \in I$*. We get an isomorphism of* Y*-schemes* $\mathrm{Spec}(A[I/f]) \cong D_+(f)$*. We conclude that the blow-up of* X *in* D *is locally of the form* $\mathrm{Spec}(A[I/f]) \to \mathrm{Spec}A$*. We have* $f \cdot A[I/f] = I \cdot A[I/f]$ *which induces an isomorphism*

$$D_+(f) \times_X D \cong \mathrm{Spec}(A[I/f] \oplus_A A/I) \cong \mathrm{Spec}\big(A[I/f]/(f \cdot A[I/f])\big).$$

Thus the preimage of D *under the morphism* $\mathrm{Spec}(A[I/f]) \to \mathrm{Spec}A$ *is the closed subscheme* $V(f)$ *of* $\mathrm{Spec}(A[I/f])$*.*

Example (2.1.4). In the following example the scheme X has a singular point. Blowing-up the point resolves the singularity (cf. the introduction of the thesis). Let X be the closed subscheme $V(g)$ of the affine scheme $Z = \mathrm{Spec}(B)$ for the three-dimensional polynomial ring $B = k[x,y,z]$ over a field k with $\mathrm{char}k \neq 2$ and for the polynomial $g = x^2 + y^2 - z^2$. Let s be the point of X corresponding to the maximal ideal $I := \langle x,y,z\rangle$ of B. The $k = \kappa(s)$-vector space $\mathfrak{m}_{X,s}/\mathfrak{m}_{X,s}^2 \cong xk \oplus yk \oplus zk$ has dimension $3 > 2 = \dim B/\langle g\rangle = \dim \mathcal{O}_{X,s}$. Thus s is a singularity of X. The open subscheme $X\backslash\{s\}$ of X is smooth over k and therefore there is no singular point of X other than s. To see the smoothness, for $A := B/\langle g\rangle$, cover $X\backslash\{s\}$ with the standard open subschemes $\mathrm{Spec}(A_x)$, $\mathrm{Spec}(A_y)$, $\mathrm{Spec}(A_z)$ of X. We have $A_y = k[w,x,y,z]/\langle P,Q\rangle$ for $P = x^2 + y^2 - z^2$, $Q = wy - 1$. W.r.t. the polynomials (P,Q) and the variables (w,x,y,z) the Jacobian matrix is

$$\mathrm{Jac} = \begin{pmatrix} 0 & 2x & 2y & -2z \\ y & 0 & w & 0 \end{pmatrix}.$$

Since $y, 2y$ are units of A_y the matrix Jac has rank two. Thus $k \to B_y$ is a smooth morphism. Similarly one sees that A_x and A_z are smooth k-algebras. Let $Y \to X$ denote the blow-up of X in the center $D = \{s\}$. For $f \in \{x,y,z\}$ write $Y_f := \mathrm{Spec}(A[I/f])$. By remark (2.1.3) the schemes Y_x, Y_y and Y_z cover Y. The isomorphism $\phi : B \cong B[I/x]$ of k-algebras with $(\phi(x), \phi(y), \phi(z)) = (x, y/x, z/x)$ induces an isomorphism of k-schemes $Y_x \cong \mathrm{Spec}(B/\langle 1+y^2-z^2\rangle)$. Similarly one gets

$$Y_y \cong \mathrm{Spec}(k[x,y,z]/\langle x^2 + 1 - z^2\rangle), \quad Y_z \cong \mathrm{Spec}(k[x,y,z]/\langle x^2 + y^2 - 1\rangle).$$

The schemes Y_x, Y_y, Y_z are smooth over k. For example $(0 \;\; 2y \;\; -2z)$ is the Jacobian matrix w.r.t. the polynomial $1 + y^2 - z^2$ and the variables (x, y, z). It has rank one since it is left invers to the 3×1-matrix $(0, -y/2, z/2)$. Then Y is regular and $Y \to X$ is a resolution of the singularities of X (by propositions (2.1.2) and (2.1.5) $Y \to X$ is proper and birational). We determine the preimage of D = the fiber of s. For $f \in \{x, y, z\}$ the isomorphism $B \to B[I/f]$ from above induces an isomorphisms $B/\langle f \rangle \to (B[I/f])/\langle f \rangle$. Then, by remark (2.1.3) for $f \in \{x, y, z\}$ the preimage of D in Y_f is the closed subscheme $V(f)$ of Y_f.

Proposition (2.1.5). *Let $\pi : X' \to X$ be the blow-up in D.*

a) If X is locally noetherian, then π is locally of finite type and X' is locally noetherian.

b) If X is reduced, then X' is reduced.

c) The by π induced morphism $(X')_{\mathrm{red}} \to X_{\mathrm{red}}$ is the blow-up of X_{red} in $X_{\mathrm{red}} \times_X D$.

Assume additionally that D contains no generic points of X. Then

d) π is birational, if X is reduced,

e) π induces a bijection between the generic points of X' and X,

f) for each irreducible component Z of X the closed subscheme $\mathrm{Bl}_{Z \times_X D} Z$ is an irreducible component of X' and

g) $\dim X = \dim X'$, if X is locally noetherian.

Proof. a) By remark (2.1.3) π is locally given by morphisms of the form $A \to A[I/f]$ for a finitely generated ideal I of A. Thus π is locally of finite type, which implies that X' is locally noetherian.

b) By remark (2.1.3) X' is covered by open affine schemes $\mathrm{Spec}(A[I/f])$ where $\mathrm{Spec}(A)$ is an open affine subscheme of X. Then A is reduced. Thus $A[I/f]$ is reduced as a subring of the reduced ring A_f. ring of the reduced ring A_f.

c) We have a commutative diagram

$$\begin{array}{ccc} (X_{\mathrm{red}})' & \xrightarrow{\;i\;} & X' \\ {\scriptstyle \pi_{\mathrm{red}}}\downarrow & & \downarrow{\scriptstyle \pi} \\ X_{\mathrm{red}} & \longrightarrow & X \end{array}$$

where π_{red} denotes the blow-up in $X_{\mathrm{red}} \times_X D$. Since $(X_{\mathrm{red}})'$ is reduced, it is enough to show that the closed imm8ersion i is a homeomorphism. This follows form the fact that $\pi^{-1}(X \backslash D)$ resp. $\pi_{\mathrm{red}}^{-1}(X_{\mathrm{red}} \backslash X_{\mathrm{red}} \times_X D)$ is dense in X' resp. $(X_{\mathrm{red}})'$.

d) Since X', X are reduced it is enough to show that there are open dense subschemes $U' \subseteq X'$, $U \subseteq X$ such that π induces an isomorphism $U' \to U$. This follows form proposition (2.1.2)e),f) for $U = X \backslash D$, $U' = \pi^{-1} U$.

e) Follows form c), d).

f) Let $\pi_Z : Z' \to Z$ denote the blow-up of Z in $Z \times_X D =: D_Z$. The closed immersion $Z \to X$ induces a closed immersion $Z' \to X'$. By e) Z' is irreducible. The blow-up π_Z induces an isomorphism $\pi_Z^{-1}(Z \backslash D_Z) = Z' \times_Z (Z \backslash D_Z) \cong Z \backslash D_Z$. On the other hand the blow-up π induces an isomorphism

$$\pi^{-1}(Z \backslash D_Z) = X' \times_X (Z \backslash D_Z) = X' \times_X (X \backslash D) \times_X Z \cong (X \backslash D) \times_X Z = Z \backslash D_Z .$$

Thus the by $Z' \to X'$ induced closed immersion $\pi_Z^{-1}(Z \backslash D_Z) \to \pi^{-1}(Z \backslash D_Z)$ is an isomorphism. In particular the preimage for the generic point of Z under π lies in $\pi_Z^{-1}(Z \backslash D_Z) \subseteq Z'$. Then Z' is closed in X', irreducible and contains a generic point of X'.

g) By proposition (2.1.2)g), π is proper. By c),f) we may assume that X is an integral scheme. Then π is birational. With [Li], corollary 8.2.7, we get $\dim X = \dim Y$.

□

Definition (2.1.6). *Let x be a point of X.*

a) Let $\mathrm{gr}_{\mathcal{I}}\mathcal{O}_X$ denote the graded $\mathcal{O}_D$-algebra sheaf $\oplus_{n \in \mathbb{N}} \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{O}_D$. ***The normal cone $C_{X,D}$ of X along D*** *is the D-scheme $\mathrm{Spec}(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_X)$. For $x \in D$ we denote the $\mathrm{Spec}(\kappa(x))$-scheme $C_{X,D} \times_D \kappa(x)$ by $C_{X,D,x}$.*

b) ***The tangential cone $C_{X,x}$ of X at x*** *is the $\mathrm{Spec}(\kappa(x))$-scheme $\mathrm{Spec}(\mathrm{gr}\mathcal{O}_{X,x})$.*

c) We say ***X is normally flat along D*** *if $C_{X,D} \to D$ is flat and we call D* ***permissible*** *if additionally D is regular. For $x \in D$ we say* ***X is normally flat along D at x*** *if $C_{X,D} \times_D \mathrm{Spec}(\mathcal{O}_{D,x}) \to \mathrm{Spec}(\mathcal{O}_{D,x})$ is flat and we call D* ***permissible at x*** *if additionally $\mathcal{O}_{D,x}$ is a regular ring. The blow-up of X in a center D is permissible if D is permissible.*

Remark (2.1.7). *a) For a point $x \in D$ we have a commutative diagram with cartesian squares*

$$\begin{array}{ccccc} \mathrm{Proj}(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_X \otimes_{\mathcal{O}_D} \kappa(x)) & \longrightarrow & \mathrm{Proj}(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_X) & \longrightarrow & \mathrm{Bl}_D X \\ \downarrow & & \downarrow & & \downarrow \\ x & \longrightarrow & D & \longrightarrow & X \end{array}$$

b) Assume that x is a closed point and assume $D = \{x\}$. Let π denote the blow-up $X' \to X$ in D. Then we have $\pi^{-1}(\{x\}) = \mathrm{Proj}(\mathrm{gr}\mathcal{O}_{X,x}) =: E$. Let A be a symmetric algebra over $\kappa(x)$ and let I be a homogeneous ideal of A such that we have an isomorphism of graded $\kappa(x)$-algebras $\mathrm{gr}\mathcal{O}_{X,x} \cong A/I$. Let $\phi_1, \ldots, \phi_m$ be non-zero homogeneous generators of I. Let x' be a point of E. Let v an element of A_1 such that x' lies in the open subscheme $\mathrm{Spec}((A/I)_{(v)}) =: U$ of E. We have $(A/I)_{(v)} = A_{(v)}/I_{(v)}$ and the ideal $I_{(v)}$ of $A_{(v)}$ is generated by $\psi_1, \ldots, \psi_m$ where for each $j \in \{1, \ldots, m\}$ we set $\psi_j := \phi_j \cdot v^{-\deg \phi_j}$. The ring $\mathcal{O}_{E,x'} = \mathcal{O}_{U,x'}$ is the localization of $A_{(v)}/I_{(v)}$ by a prime ideal. Let $\mathfrak{p}$ denote the induced prime ideal of $A_{(v)}$ and identify $\psi_1, \ldots, \psi_m$ with their image in $(A_{(v)})_{\mathfrak{p}}$. Then we have

$$\mathcal{O}_{E,x'} \cong (A_{(v)})_{\mathfrak{p}} / \langle \psi_1, \ldots, \psi_m \rangle.$$

In particular, if $\mathcal{O}_{X,x} =: R$ is a regular ring, then we can take $A = \mathrm{gr}R$ and we have $\mathcal{O}_{E,x'} = (\mathrm{gr}R_{(v)})_{\mathfrak{p}}$.

c) *Assume that* $\mathcal{O}_{X,x}$ *is noetherian and assume* $I_x := \ker(\mathcal{O}_{X,x} \to \mathcal{O}_{D,x}) \neq \mathcal{O}_{X,x}$. *By [Ma], theorem 15.7, the graded* $\mathcal{O}_{D,x}$*-algebra* $\mathrm{gr}_{I_x}\mathcal{O}_{X,x} = \oplus_{n\in\mathbb{N}} I_x^n \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{D,x}$ *has the same Krull dimension as* $\mathcal{O}_{X,x}$. *In particular we get*

$$\dim C_{X,D,x} \leq \dim \mathrm{gr}_{I_x}\mathcal{O}_{X,x} \leq \dim X.$$

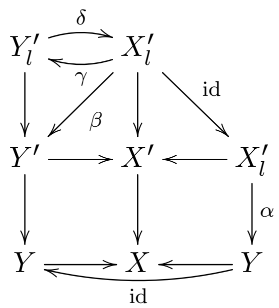
In the case $I_x = \mathcal{O}_{X,x}$ *one has* $C_{X,D,x} = \varnothing$. *The same argument yields*

$$\dim C_{X,x} = \dim \mathrm{gr} O_{X,x} = \dim \mathcal{O}_{X,x} \leq \dim X.$$

In the remark we described the local ring $\mathcal{O}_{E,x'}$. The following three lemmata give a description of the local ring $\mathcal{O}_{X',x'}$ if D is regular at x and $\mathcal{O}_{X,x}$ is a quotient of a regular ring.

Lemma (2.1.8). *Let* $\pi : X' \to X$ *be the blow-up of* X *in* D. *Let* x *resp.* x' *be a point of* D *resp.* $x' \in \pi^{-1}(\{x\})$. *Let* $Y' \to Y$ *denote the blow-up of* $Y := \mathrm{Spec}(\mathcal{O}_{X,x})$ *in* $Y \times_X D$. *Let* y *denote the closed point of* Y. *Then there is a unique point* y' *of* Y' *which lies over* x' *and* y. *Further* $\mathcal{O}_{X',x'}$ *and* $\mathcal{O}_{Y',y'}$ *are isomorphic as* $\mathcal{O}_{X,x}$*-algebras.*

Proof. Since $Y \to X$ is flat, $Y' \to Y$ is the base change of π with Y. We have $\kappa(y) \times_X \kappa(x') \cong \kappa(x')$. Thus there is a unique point $y' \in Y'$ which lies over x' and y. Write $Y'_l := \mathrm{Spec}(\mathcal{O}_{Y',y'})$, $X'_l := \mathrm{Spec}(\mathcal{O}_{X',x'})$. Consider following diagram of schemes



We define the morphisms $\alpha, \beta, \gamma, \delta$ below. Without these four morphisms the diagram commutes. The morphism $X' \to X$ induces α. The morphisms $X'_l \to X'$ and α induce β which induces $\gamma : X'_l = \mathrm{Spec}(\mathcal{O}_{X'_l,x'}) \to \mathrm{Spec}(\mathcal{O}_{Y',y'}) = Y'_l$. The resulting diagram commutes. The morphism $Y' \to X'$ induces a morphism $\delta : Y'_l \to X'_l$ of Y-schemes which is a morphism of Y'-schemes by the universal property of the fiber product $Y' = Y \times_X X'$. Thus the whole diagram commutes. Then we have a morphism $X'_l \to Y'_l \to X'_l$ of X'-schemes and a morphism $Y'_l \to X'_l \to Y'_l$ of Y'-schemes. Both are the identity because for a scheme Z and a point $z \in Z$ the only morphism of Z-schemes $\mathrm{Spec}(\mathcal{O}_{Z,z}) \to \mathrm{Spec}(\mathcal{O}_{Z,z})$ is the identity. Then δ is an isomorphism of Y-schemes. □

Lemma (2.1.9). *Let* x *be a point of* $X =: Z$, *such that* Z *and* D *are regular at* x. *Let* $\pi : Z' \to Z$ *be the blow-up of* Z *in* D. *Then for every point* x' *of* $\pi^{-1}(\{x\})$ *there is a regular parameter* v *of* $\mathcal{O}_{Z,x} =: R$ *with* $v \in \mathfrak{p} := \ker(\mathcal{O}_{Z,x} \to \mathcal{O}_{D,x})$ *and there is a prime ideal* q *of* $R[\mathfrak{p}/v]$ *which contains the maximal ideal* $\mathfrak{m}$ *of the subring* R *of* $R[\mathfrak{p}/v]$ *such that* $(R[\mathfrak{p}/v])_q$ *and* $\mathcal{O}_{Z',x'}$ *are isomorphic as* $\mathcal{O}_{Z,x} = R$*-algebras.*

Here $R[\mathfrak{p}/v]$ denotes the R-subalgebra of R_v generated by the elements p/v, $p \in \mathfrak{p}$.

Proof. By lemma (2.1.8) we may assume $Z = \mathrm{Spec}(\mathcal{O}_{Z,x})$. Since $Z = \mathrm{Spec}R$ and $D = \mathrm{Spec}(R/\mathfrak{p})$ are regular $\mathfrak{p}$ is a prime ideal of R generated by regular parameters $v_1, \ldots, v_n$ of R. Then Z' is covered by the affine open subschemes $D_+(v_1), \ldots, D_+(v_n)$. Choose a $v \in \{v_1, \ldots, v_n\}$ with $x' \in D_+(v)$. By remark (2.1.3) $D_+(v)$ and $\mathrm{Spec}R[\mathfrak{p}/v]$ are isomorphic as $Z = \mathrm{Spec}R$-schemes. Let q the to x' corresponding prime ideal of $\mathrm{Spec}R[\mathfrak{p}/v]$. Since q maps to x under $\mathrm{Spec}R[\mathfrak{p}/v] \to Z$ we have $q \supseteq \mathfrak{m}$. Then the isomorphism $D_+(v) \cong \mathrm{Spec}R[\mathfrak{p}/v]$ induces an isomorphism of $\mathcal{O}_{Z,x} = R$-algebras $(R[\mathfrak{p}/v])_q \cong \mathcal{O}_{Z',x'}$. □

Now we study blow-ups of schemes X which are imbedded in a regular scheme Z. Until the end of this section we are in the following situation. Let X be a closed subscheme of a regular scheme Z and let D be a regular closed subscheme of X. We have a commutative diagram of schemes

$$\begin{array}{ccc} X' & \longrightarrow & Z' \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\ X & \longrightarrow & Z \end{array}$$

where π_X resp. π_Z denotes the blow-up of X resp. Z in D. Let $x \in D \subseteq X \subseteq Z$, $x' \in X' \subseteq Z'$ be points with $\pi_X(x') = x$. Choose a regular parameter v of $\mathcal{O}_{Z,z} =: R$ with $v \in \mathfrak{p} := \ker(R \to \mathcal{O}_{D,x})$ and a prime ideal q of $R[\mathfrak{p}/v]$ which contains the maximal ideal $\mathfrak{m}$ of R such that $(R[\mathfrak{p}/v])_q =: R'$ and $\mathcal{O}_{Z',x'}$ are isomorphic as $\mathcal{O}_{Z,x} = R$-algebras (see lemma (2.1.9)). Write $J := \ker(R \to \mathcal{O}_{X,x})$.

Definition (2.1.10). *a) For a non-zero element f of R we denote the number $\nu \in \mathbb{N}$ with $f \in \mathfrak{p}^{\nu} \backslash \mathfrak{p}^{\nu+1}$ by $\nu_{\mathfrak{p}} f$ (where $\mathfrak{p}^0 = R$).*

*b) **The strict transform J' of J in R'** is the ideal of R' generated by all elements $f/v^n \in R'$ for non-zero elements $f \in J$ with $v_{\mathfrak{p}} f \geq n \geq 0$.*

*c) For a non-zero element of R **the initial form $\mathrm{in}_{\mathfrak{p}} f$ of f in $\mathrm{gr}_{\mathfrak{p}} R$** is the image of f under*

$$\mathfrak{p}^{\nu_{\mathfrak{p}} f} \to \mathfrak{p}^{\nu_{\mathfrak{p}} f} \otimes_R R/\mathfrak{p} \to \oplus_{n \in \mathbb{N}} \mathfrak{p}^n \otimes_R R/\mathfrak{p} = \mathrm{gr}_{\mathfrak{p}} R.$$

d) For the ideal $\overline{\mathfrak{p}} = \mathfrak{p} R/J$ we define the homogeneous ideal $\mathrm{In}_{\mathfrak{p}} J := \ker(\mathrm{gr}_{\mathfrak{p}} R \to \mathrm{gr}_{\overline{\mathfrak{p}}}(R/J))$.

e) If $\mathfrak{p} =: \mathfrak{m}$ is the maximal ideal of R than $\nu_{\mathfrak{m}} f$, $\mathrm{in}_{\mathfrak{m}} f$, $\mathrm{In}_{\mathfrak{m}} J$ can be written without "'$\mathfrak{m}$"'.

*f) Assume that $\mathfrak{p} =: \mathfrak{m}$ is the maximal ideal of R. A **standard basis of J** is a finite tuple $(f_1, \ldots, f_m)$ of non-zero elements of J such that*

i) the ideal $\langle \mathrm{in} f_1, \ldots, \mathrm{in} f_m \rangle$ of $\mathrm{gr} R$ is equal to $\mathrm{In} J$,

ii) $\nu f_1 \leq \nu f_2 \leq \ldots \leq \nu f_m$ and

iii) for all $j \in \{2, \ldots, m\}$ the element $\mathrm{in} f_j$ lies not in the ideal $\langle \mathrm{in} f_1, \ldots, \mathrm{in} f_{j-1} \rangle$ of $\mathrm{gr} R$.

Remark (2.1.11). *By Krull's intersection theorem we have $\cap_{n \in \mathbb{N}} \mathfrak{p}^n = \{0\}$. Thus $\nu_{\mathfrak{p}} f$ exists.*

Lemma (2.1.12). *Assume the situation of definition (2.1.10)*

a) The ideal $\langle \mathrm{in}_{\mathfrak{p}} f \mid f \in J \rangle$ of $\mathrm{gr}_{\mathfrak{p}} R$ is equal to $\mathrm{In}_{\mathfrak{p}} J$.

b) Let F be a subset of J with $\langle \mathrm{in}_{\mathfrak{p}} f \,|\, f \in F\rangle = \mathrm{In}_{\mathfrak{p}} J$. Then the ideal $\langle f/v^n \,|\, n \in \mathbb{N},\, f \in F\backslash\{0\},\, \nu_{\mathfrak{p}} f \geq n\rangle$ of R' is the strict transform of J in R'.

c) There is a morphism of rings α and an isomorphism of rings β such that the following diagram of rings commutes

$$\begin{array}{ccccc}
\mathcal{O}_{X',x'} & \xrightarrow{\beta} & R'/J' & \longleftarrow & R' \\
\uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow \\
\mathcal{O}_{X,x} & \xrightarrow[\cong]{} & R/J & \longleftarrow & R
\end{array}$$

d) For two non-zero elements f, f' of R one has $\nu_{\mathfrak{p}}(f \cdot f') = \nu_{\mathfrak{p}} f + \nu_{\mathfrak{p}} f'$.

Proof. a) For each $n \in \mathbb{N}$ the kernel of the R/J-module morphism $\mathfrak{p}^n/\mathfrak{p}^{n+1} \to \mathfrak{p}^n + J/(\mathfrak{p}^{n+1} + J)$ is generated by the images in $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ of all elements $f \in R$ with $f \in (\mathfrak{p}^n\backslash\mathfrak{p}^{n+1}) \cap J$. This is equivalent to the claim since $\mathrm{gr}_{\mathfrak{p}} R \to \mathrm{gr}_{\overline{\mathfrak{p}}}(R/J)$ is a morphism of graded rings.

b) [Hi1], chapter III, lemma 6 on page 216.

c) By lemma (2.1.8) we may assume $X = \mathrm{Spec}(R/J)$, $Z = \mathrm{Spec}(R)$. By definition we have $J' \supseteq JR'$. Thus $R \to R'$ induces α. Let $\overline{v}$, $\overline{\mathfrak{p}}$ denote the image of $v \in R$, $\mathfrak{p} \subseteq R$ in R/J and let $\overline{\mathfrak{q}}$ denote the image of $\mathfrak{q} \subseteq R[\mathfrak{p}/v]$ in $R/J[\overline{\mathfrak{p}}/\overline{v}]$. Let $\overline{v}$ resp. $\overline{q}$ denote the induced element of $R/J = \mathcal{O}_{X,x}$ resp. prime ideal of $R[\overline{\mathfrak{p}}/\overline{v}]$ induced by $v \in R$ resp. $q \in \mathrm{Spec} R[\mathfrak{p}/v]$. We have an isomorphism of $R/J = \mathcal{O}_{X,x}$-algebras $\big((R/J)[\overline{\mathfrak{p}}/\overline{v}]\big)_{\overline{q}} \cong \mathcal{O}_{X',x'}$ and we have a commutative diagram of rings

$$\begin{array}{ccccc}
 & & \mathcal{O}_{X',x'} & \twoheadleftarrow & \mathcal{O}_{Z',x'} \\
 & & {\scriptstyle\cong}\uparrow & & \uparrow{\scriptstyle\cong} \\
 & & \big((R/J)[\overline{\mathfrak{p}}/\overline{v}]\big)_{\overline{q}} & \twoheadleftarrow & (R[\mathfrak{p}/v])_{\mathfrak{q}} = R' \\
 & & \uparrow & & \uparrow \\
R_v/J_v = (R/J)_{\overline{v}} & \hookleftarrow & (R/J)[\overline{\mathfrak{p}}/\overline{v}] & \twoheadleftarrow & R[\mathfrak{p}/v]
\end{array}$$

We show that the morphism $R' \to \mathcal{O}_{X',x'}$ induces an isomorphism $R'/I' \cong \mathcal{O}_{X',x'}$. It is enough to show that the kernel of $(R[\mathfrak{p}/v] \to R_v/J_v) =: \gamma$ is the ideal $\mathfrak{a}$ generated by all elements $f/v^n \in R[\mathfrak{p}/v]$ for non-zero elements $f \in J$ with $v_{\mathfrak{p}} f \geq n \geq 0$. The inclusion $\mathfrak{a} \subseteq \ker\gamma$ follows from $\mathfrak{a}(R_v/J_v) = J(R_v/J_v) = \{0\}$. An element g of $R[\mathfrak{p}/v]$ has the form $g = h/v^m$ for suitable $m \in \mathbb{N}$, $h \in \mathfrak{p}^m$. If we have $\gamma(g) = 0$ then there is some $f \in J$ and some $n \in \mathbb{N}$ with $g = f/v^n$. This implies the equality in $v^n h = f v^m$ in R and with d) we get $v_{\mathfrak{p}} f = v_{\mathfrak{p}} h + n - m \geq n$ which implies $g \in \mathfrak{a}$.

d) Write $\nu := \nu_{\mathfrak{p}}(f)$, $\nu' := \nu_{\mathfrak{p}}(f')$. Since $R/\mathfrak{p}$ is regular, there are regular parameters $x_1, \ldots, x_m$ of R, i.e. $\mathrm{in}_{\mathfrak{p}} x_1, \ldots, \mathrm{in}_{\mathfrak{p}} x_n$ is a $k := R/\mathfrak{m}$-basis of $\mathfrak{m}/\mathfrak{m}^2$, such that $\mathfrak{p} = \langle x_1, \ldots, x_m\rangle$ for some $m \in \{1, \ldots, n\}$. There are homogeneous polynomials P, P' with degrees ν, ν' in m variables and with coefficients in $R^\times \cup \{0\}$ such that

$$f - P(x_1, \ldots, x_m) \in \mathfrak{m}^{\nu+1}, \quad f' - P'(x_1, \ldots, x_m) \in \mathfrak{m}^{\nu'+1}.$$

Then we have $f \cdot f' - (P \cdot P')(x_1, \ldots, x_m) \in \mathfrak{m}^{\nu+\nu'+1}$. Since R is regular, the morphism of k-algebras $\mathrm{Sym}_k(\mathfrak{m}/\mathfrak{m}^2) \to \mathrm{gr}R$ is an isomorphism. Thus $(P \cdot P')(x_1, \ldots, x_m) \notin \mathfrak{m}^{\nu+\nu'}$ which implies $\nu(f \cdot f') = \nu + \nu'$.

□

2.2 Hilbert-Samuel-function

We define the Hilbert-Samuel-function at a point x of a locally noetherian scheme X and cite a result (theorem (2.2.6)) about its behavior for permissible blow-ups.
For two partially ordered sets I, J let I^J denote the set of maps $J \to I$. For two maps $f, g : J \to I$ we write $f \leq g$ if it is true pointwisely. The set of all maps $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ becomes a partially ordered set. For a graded ring A and a natural number $m \in \mathbb{N}^n$

$$A[T_1, \ldots, T_m] = \oplus_{n \geq 0} A[T_1, \ldots, T_m]_n$$

denotes the graded polynomial ring over A in m homogeneous degree one variables which has A as a graded subring. A field k becomes a graded k-algebra by setting $k_n := 0$ for $n > 0$.

Definition (2.2.1). *a) For a graded Algebra $A = \oplus_{n\geq 0} A_n$ over a field k with finite-dimensional k-vector spaces A_n, $n \geq 0$,* ***the Hilbert-Samuel-function $H(A)$ of A*** *is the map*

$$H(A) : \mathbb{N} \to \mathbb{N}^{\mathbb{N}} : m \to H^{(m)}(A), \quad H^{(m)}(A)(n) = \dim_k(A[T_1, \ldots, T_m]_n).$$

We say $H^{(m)}(A)$ is the m-th Hilbert-Samuel-function of A.

b) Let x be a point of a locally noetherian scheme X. ***The Hilbert-Samuel-function $H_{X,x}$ of X at x*** *is the Hilbert-Samuel-function of the graded $\kappa(x)$-algebra $\mathrm{gr}\mathcal{O}_{X,x} = \oplus_{n\geq 0} \mathfrak{m}_{X,x}^n / \mathfrak{m}_{X,x}^{n+1}$.*

The following properties are easily verified.

Lemma (2.2.2). *Let A resp. A' be a graded algebra over a field k reps. k' with finite-dimensional homogeneous parts.*

a) One has the equivalences

$$H(A) = H(A') \quad \Leftrightarrow \quad \exists m \in \mathbb{N} : \quad H^{(m)}(A) = H^{(m)}(A'),$$

$$H(A) \leq H(A') \quad \Leftrightarrow \quad H^{(0)}(A) \leq H^{(0)}(A').$$

b) For all $s, m \in \mathbb{N}$ one has $H^{(m)}(A[T_1, \ldots, T_s]) = H^{(m+s)}(A)$.

c) For all $n, m \in \mathbb{N}$ one has $H^{(m+1)}(A)(n) = \sum_{n'=0}^{n} H^{(m)}(A)(n')$.

d) For all $n \in \mathbb{N}_{\geq 1}$, $m \in \mathbb{N}$ one has

$$H^{(m)}(k)(n) = H^{(0)}(k[T_1, \ldots, T_m])(n) = \binom{m+n-1}{n}.$$

e) Let x be a point of a locally noetherian scheme X and write $d := \dim \mathcal{O}_{X,x}$. Then one has $H_{X,x} \geq H(\kappa(x)[T_1, \ldots, T_d])$. Equality holds if and only if X is regular at x.

f) For a homogeneous non-zero element f of $A = k[T_1, \ldots, T_m]$ of degree d one has

$$H^{(0)}(A/\langle f\rangle)(n) = \left\{ \begin{array}{ll} H^{(0)}(A)(n) & \text{if } n < d \\ H^{(0)}(A)(n) - H^{(0)}(n-d) & \text{if } n \geq d \end{array} \right\} = \binom{m+n-1}{n} - \binom{m+n-d-1}{n-d}$$

where we set $\binom{a}{b} := 0$ if $a < b$.

Proof of e) and f). e) For a noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue field κ one has the inequality $\dim A \leq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. The Equality holds if and only if A is regular if and only if there is an isomorphism of graded κ-algebras $\kappa[T_1, \ldots, T_{\dim A}] \to \mathrm{gr}A$.

f) Let $\langle f\rangle_n$ denote the n-th homogeneous part $\langle f\rangle \cap A_n$ of the ideal $\langle f\rangle$ of A. For $n < d$ we have an isomorphism of k-vector spaces $(A/\langle f\rangle)_n \cong A_n$ and therefore $H^{(0)}(A/\langle f\rangle)(n) = H^{(0)}(A)(n)$. Assume $n \geq d$. Write $f = \sum_{\alpha\in\mathbb{N}^n} \lambda_\alpha \underline{T}^\alpha$ where we write $\underline{T} = (T_1, \ldots, T_m)$, $\underline{T}^\alpha = T_1^{\alpha_1} \cdot \ldots \cdot T_m^{\alpha_m}$ and where $\lambda_\alpha \in k$ are coefficients. Let multideg(f) denote the maximal $\alpha \in \mathbb{N}^m$ with $\lambda \neq 0$ w.r.t. the lexicographical order on $\mathbb{N}^m$ and let LTf denote the monomial $\lambda_\alpha \underline{T}^\alpha$ for $\alpha = \mathrm{multideg}(f)$ (cf. section 3.7). Let $\mathcal{B}$ denote the set of monomials of A_n and define the subset $\mathcal{B}_f := \mathcal{B}\backslash\langle \mathrm{LT}f\rangle \subseteq \mathcal{B}$ where $\langle \mathrm{LT}f\rangle$ is the ideal of A generated by LTf. We show that the k-linear map

$$\phi : \bigoplus_{\mathcal{B}_f} k \to A_n/\langle f\rangle_n, \qquad \sum_{m\in\mathcal{B}_f} \lambda_m \mapsto \sum_{m\in\mathcal{B}_f} \lambda_m \cdot m \bmod \langle f\rangle$$

is an isomorphism. With the isomorphism we get

$$H^{(0)}(A/\langle f\rangle)(n) = \dim_k \Big(\bigoplus_{\mathcal{B}_f} k\Big) = \#\mathcal{B} - \#(\mathcal{B} \cap \langle \mathrm{LT}f\rangle) = H^{(0)}(A)(n) - H^{(0)}(A)(n-d).$$

Assume that there is an element $\sum_{m\in\mathcal{B}_f} \lambda_m \in \ker(\phi)\backslash\{0\}$. Then there is some homogeneous polynomial $g \in A$ with $\sum_{m\in\mathcal{B}_f} \lambda_m m = gf$ in A. Then we have $\lambda_{m_0} m_0 = \mathrm{LT}(\sum_{m\in\mathcal{B}_f} \lambda_m m) = \mathrm{LT}(gf) = \mathrm{LT}(g)\mathrm{LT}(f)$ (see remark (3.7.7)) for a suitable $m_0 \in \mathcal{B}_f$ with $\lambda_{m_0} \neq 0$. This is a contradiction to the definition of B_f. Thus ϕ is injective. Let h be a non-zero element of A_n and write $\beta := \mathrm{multideg}(h)$. We show that $h \bmod \langle f\rangle$ lies in the image of ϕ by induction on β. Write $h' := h - \mathrm{LT}h$. If $h' = 0$ then $h' \bmod \langle f\rangle = \phi(0)$. If $h' \neq 0$, we have multideg$(h') < \beta$ and by induction hypothesis $h' \bmod \langle f\rangle$ lies in the image of ϕ. If $\mathrm{LT}h \notin \langle \mathrm{LT}f\rangle$ then we have $\mathrm{LT}h \bmod \langle f\rangle \in \mathrm{im}\phi$ and we are done. Assume $\mathrm{LT}h = g \cdot \mathrm{LT}f$ for some monomial $g \in A$. Then we have $\mathrm{LT}h = g \cdot \mathrm{LT}f = \mathrm{LT}(gf)$. Then for $h'' := \mathrm{LT}h - gf$ we have $\mathrm{LT}h \bmod \langle f\rangle = h'' \bmod \langle f\rangle$ and $h'' = 0$ of multideg$(h'') < \beta$. By induction hypothesis we get $h'' \bmod \langle f\rangle \in \mathrm{im}\phi$. Thus we have

$$h \bmod \langle f\rangle = (\mathrm{LT}h + h') \bmod \langle f\rangle = h'' \bmod \langle f\rangle + h' \bmod \langle f\rangle \in \mathrm{im}\phi.$$

Then ϕ is an isomorphism and the proof of f) is complete.

□

Remark (2.2.3). *By lemma (2.2.2) for two graded algebras A, A' over fields k, k' one has the implication $H^{(0)}(A) \leq H^{(0)}(A') \Rightarrow H^{(1)}(A) \leq H^{(1)}(A')$. This is not an equivalence in general. For example for $k = k'$, $A = k[X]/\langle X^3\rangle$, $A' = k[X,Y]/\langle X^2, XY, Y^2\rangle$ one has*

$$H^{(0)}(A) = (1,1,1,0,0,\ldots) \nleq H^{(0)}(A') = (1,2,0,0,0,\ldots),$$

$$H^{(1)}(A) = (1,2,3,3,3,\ldots) \leq H^{(1)}(A') = (1,3,3,3,3,\ldots).$$

We will need the following proposition in sections 2.6 and 3.4.

Proposition (2.2.4). *Let A be a noetherian local ring, let z be an element of the maximal ideal of A and define $B := A/zA$. Then one has $\mathrm{H}^{(2)}(\mathrm{gr}B) \geq \mathrm{H}^{(1)}(\mathrm{gr}A)$. The equality holds if and only if the image Z of z in $\mathrm{gr}^1 A$ is not a zero-divisor in $\mathrm{gr}A$ and the morphism of graded rings $\mathrm{gr}A \to \mathrm{gr}B$ induces an isomorphism $\mathrm{gr}A/\langle Z\rangle \cong \mathrm{gr}B$.*

Proof. [Hi4], Proposition 5. □

Remark (2.2.5). *Note that $Z \neq 0$ implies that Z is the initial form* $\mathrm{in}z$ *of z.*

Theorem (2.2.6). *Let $X' \to X$ be a permissible blow-up of a locally noetherian scheme X. Let x' be a point of X' and denote its image in X by x. Write $d := \mathrm{trdeg}(\kappa(x')/\kappa(x))$. Then one has*

$$H^{(d)}_{X',x'} \leq H^{(0)}_{X,x}$$

Proof. We may assume that x lies in the blow-up center. Then the claim follows form [Si], main theorem. □

Definition (2.2.7). *In the situation of (2.2.6) we say* ***x' is near to x*** *if $H^{(d)}_{X',x'} = H^{(0)}_{X,x}$.*

2.3 Additive elements

In this section we want to prove proposition (2.3.9). In the language of group schemes it says that the ideal of an additive subgroup scheme of a vector group scheme is generated by additive polynomials (cf. section 2.5). Further we define the ring of invariants (see definition (2.3.11)). The ring of invariants $\mathcal{U}$ of an additive group scheme G carries the whole information about G. If G is the ridge of a cone (see definition (2.5.12)) one calculate $\mathcal{U}$ with differential operators (cf. section 3.7).
In the whole section we fix a field k and denote its characteristic exponent by p, i.e. one has $p = 1$, if $\mathrm{char}k = 0$, or $p = \mathrm{char}k$, otherwise. We fix a noetherian symmetric algebra $A = \oplus_{i\geq 0} A_i = \mathrm{Sym}_k(A_1)$ over k. Let m denote the morphism $A \to A \otimes_k A$ of k-algebras which is induced by the morphism of k-vector spaces $A_1 \to A \otimes_k A : x \to x \otimes 1 + 1 \otimes x$. We fix a homogeneous ideal $I \neq A$ of A.

Definition (2.3.1). *An element f of A is called* ***additive*** *if m maps f to $f \otimes 1 + 1 \otimes f$.*

Example (2.3.2). a) Let $x_1, \ldots, x_n$ be a choice of a k-basis of A_1. We show that a homogeneous element f of A of is additive if and only if it has the form

(2.3.2.A) $$f = \lambda_1 x_1^q + \ldots + \lambda_n x_n^q$$

for coefficients $\lambda_1, \ldots, \lambda_n \in k$ and for a p-power q. Then we see that an element of A is additive if and only if it lies in the k-subvector space of A generated by all elements x_i^q for $i \in \{1, \ldots, n\}$, q a p-power.

The choice of a k-basis of A_1 defines an isomorphism of k-algebras from $A \otimes_k A$ to the polynomial ring $k[x_1, \ldots, x_n, x_1', \ldots, x_n']$ in $2n$ variables. Write $\underline{x}$ resp. $\underline{x'}$ for $(x_1, \ldots, x_n)$ resp. $(x_1', \ldots, x_n')$. The morphism m identifies with the k-algebra morphism $k[\underline{x}] \to k[\underline{x}, \underline{x'}]$ which sends x_i to $x_i + x_i'$. A polynomial $f(\underline{x})$ is additive if and only if one has $f(\underline{x} + \underline{x'}) = f(\underline{x}) + f(\underline{x'})$. Clearly a polynomial like (2.3.2.A) is additive.

Let us show that every homogeneous additive element f of A has the form (2.3.2.A). For a multi-index $\alpha \in \mathbb{N}^n$ let V_α denote the k-subvector space of $k[\underline{x}, \underline{x'}]$ generated by all monomials $\underline{x}^\beta \underline{x'}^{\beta'}$ for $\beta + \beta' = \alpha$. Then we have a decomposition $k[\underline{x}, \underline{x'}] = \oplus_{\alpha \in \mathbb{N}^n} V_\alpha$. Further we have $m(\underline{x}^\alpha) \in V_\alpha$. Thus we may assume $f = \underline{x}^\alpha$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. If for some $i \neq j$ one has $\alpha_i, \alpha_j > 0$ then $m(\underline{x}^\alpha)$ has at least four monomials in $k[\underline{x}, \underline{x'}]$ and $\underline{x}^\alpha$ is not additive. Thus $f = x_i^e$ for suitable $i \in \{1, \ldots, n\}$ and $e \in \mathbb{N}_{\geq 1}$. We have $e > 0$ because $1 \in A$ is not additive. For $p = 1$ one sees that x_i^e is additive if and only if $e = 1$. Assume $p > 0$. Write $e = qs$ for some p-power q and for some element $s \in \mathbb{N}_{\geq 1} \backslash p\mathbb{Z}$. Then we have

$$m(f) = (x_i + x_i')^e = (x_i^q + x_i'^q)^s = \sum_{t=0}^{s} \binom{s}{t} (x_i^q)^t (x_i'^q)^{s-t}.$$

For $s \neq 1$ we would get a monomial $s \cdot x_i^q x_i'^{q(s-1)}$ not appearing in $m(f) = x_i^e + x_i'^e$. Thus $f = x_i^q$.

b) Let k be a perfect field. Then every homogeneous additive element f of A is a power of an element of A_1. To see this write

$$f = \lambda_1 x_1^q + \ldots + \lambda_n x_n^q$$

as in (2.3.2.A) for a choice of a k-basis $x_1, \ldots, x_n$ of A_1. For the unique qth roots $\lambda_1^{1/q}, \ldots, \lambda_n^{1/q} \in k$ of $\lambda_1, \ldots, \lambda_n$ we have

$$f = (\lambda_1^{1/q} x_1 + \ldots + \lambda_n^{1/q} x_n)^q.$$

Lemma (2.3.3). *Assume that I is generated by additive elements. Then the k-algebra morphism $m : A \to A \otimes_k A$ induces a k-algebra morphism $A/I \to A/I \otimes_k A/I$.*

Proof. To show: $I \subseteq \ker(A \to A \otimes_k A \to A/I \otimes_k A/I)$. Let $f_1, \ldots, f_m$ be additive elements which generate I. Write an arbitrary element f of I in the form $f = \sum_i f_i g_i$ for suitable $g_i \in A$. Then we have $m(f) = \sum_i [(f_i \otimes 1) \cdot m(g_i) + (1 \otimes f_i) \cdot m(g_i)]$. Thus $m(f)$ lies in the kernel of $A \otimes_k A \to A/I \otimes_k A/I$ because $(f_i \otimes 1)$, $(1 \otimes f_i)$ do. □

We want to show the opposite direction of lemma (2.3.3). For this we use a homogeneous additive basis of I (see the definition below).

Lemma (2.3.4). *Let $\mathcal{A}$ be a subset of A of homogeneous additive elements.*

a) Let $\mathcal{B}$ be a subset of $\mathcal{A}$. Then $\mathcal{A}$ lies in the k-subalgebra $k[\mathcal{B}]$ generated by $\mathcal{B}$ if and only if $\mathcal{A}$ lies in the ideal $\langle \mathcal{B} \rangle$ of A.

b) There is a k-algebraically independent subset $\mathcal{B}$ of $\mathcal{A}$ with $\mathcal{A} \subseteq k[\mathcal{B}]$.

Definition (2.3.5). *a) In the situation of lemma (2.3.4)b) we call $\mathcal{B}$ a **homogeneous additive basis of $\mathcal{A}$**.*

*b) A **homogeneous additive basis of I** is a homogeneous additive basis of the set of all homogeneous additive elements of A lying in I.*

Remark (2.3.6). *A k-algebraically subset of A has at most $\dim A$ elements because for finitely many k-algebraically independent elements $e_1, \ldots, e_m$ of A one has*

$$m = \dim k[e_1, \ldots, e_m] = \operatorname{trdeg}(\operatorname{Quot}(k[e_1, \ldots, e_m])/k) \leq \operatorname{trdeg}(\operatorname{Quot}(A)/k) = \dim A.$$

Proof of lemma (2.3.4). a) One implication follows from $k[\mathcal{B}] \cap A \backslash A_0 \subseteq \langle \mathcal{B} \rangle$. Let us show the other implication. Assume $\mathcal{A} \subseteq \langle \mathcal{B} \rangle$. Choose a k-basis $x_1, \ldots, x_n$ of A_1. It is enough to show $k[\mathcal{A}] \subseteq k[\mathcal{B}]$. We show this by induction on n. For $n = 0$ the claim is empty. For $n = 1$ every homogeneous additive element has the form λx_1^q for $\lambda \in k$ and a p-power q. For two p-powers q, q' we have $x_1^q \in \langle x_1^{q'} \rangle$ if and only if $x_1^q = (x_1^{q'})^e$ for some $e \in \mathbb{N}_{\geq 1}$. Assume $n > 1$. If $\mathcal{B}$ lies in $k[x_1, \ldots, x_{n-1}]$ then this also true for $\mathcal{A}$ and we can apply the induction hypothesis. Assume that there is some element $b \in \mathcal{B}$ which has a monomial λx_n^q for some $\lambda \in k \backslash \{0\}$. Choose such b such that q is minimal. We have $k[x_1, \ldots, x_{n-1}] + k[b] \supseteq \mathcal{A}$. For every element $a \in \mathcal{A}$ and every element $P \in k[b]$ we have $k[a, b] = k[a - P, b]$. Thus, replacing the elements of $a \in \mathcal{A}$ (in particular of $\mathcal{B}$) by elements $a - P_a$ for suitable $P_a \in k[b]$, we may assume, that all elements of $\mathcal{A}$, except for b, lie in $k[x_1, \ldots, x_{n-1}]$. Then the claim follows by induction.

b) Choose a k-basis $x_1, \ldots, x_n$ on A_1. We construct $\mathcal{B}$ by induction on n. For $n = 1$ and $\mathcal{A} \neq \varnothing$ we have $\langle b \rangle \supseteq \mathcal{A}$ for the element b of A with the smallest degree. Assume $n > 1$. If $\mathcal{A}$ lies in $k[x_1, \ldots, x_{n-1}]$ we can apply the induction hypothesis. Assume that there is some element $b \in \mathcal{A}$ which has a monomial λx_n^q for some $\lambda \in k \backslash \{0\}$. Choose such b such that q is minimal. We have $k[x_1, \ldots, x_{n-1}] + k[b] \supseteq \mathcal{A}$. Thus for every element $a \in \mathcal{A}$ there is some homogeneous element $P_a \in k[b]$ with $a - P_a \in k[x_1, \ldots, x_{n-1}]$. By induction hypothesis there is a k-algebraically independent subset $\mathcal{B}'$ of $\{a - P_a \,|\, a \in \mathcal{A}\} =: \mathcal{A}'$ with $k[\mathcal{B}'] \subseteq \mathcal{A}'$. Then $\mathcal{B}' \cup \{b\}$ is k-algebraically independent because we have $b \notin k[x_1, \ldots, x_{n-1}] \supseteq \mathcal{B}'$. Then $\mathcal{B} := \{b + P_b \,|\, b \in \mathcal{B}'\} \cup \{b\}$ is k-algebraically independent and we have $k[\mathcal{B}] \supseteq \mathcal{A} \supseteq \mathcal{B}$.

□

Remark (2.3.7). *By definition I is generated by homogeneous additive elements if and only if I is generated by a homogeneous additive basis of I.*

Lemma (2.3.8). *Let $\mathcal{A}$ be a subset of A of homogeneous additive elements and let $\mathcal{B}$ be a homogeneous additive basis of $\mathcal{A}$. Then each homogeneous additive element a of S with $a \in k[\mathcal{A}]$ has the form $a = \sum_{b \in \mathcal{B}} \lambda_b b^{e_b}$ for coefficients $\lambda_b \in k$, almost all zero, and p-powers $e_b \in \mathbb{N}$.*

Proof. Let $K|k$ be an extension of k by a perfect field. Then there are K-linearly independent elements l_b, for $b \in \mathcal{B}$, of $(K \otimes_k A)_1$ with $b = l_b^{\deg b}$ for all $b \in \mathcal{B}$. Let a be an arbitrary homogeneous additive element of $k[\mathcal{A}] \backslash \{0\}$. Set $\mathcal{B}' := \{b \in \mathcal{B} \,|\, \deg b \leq \deg a\}$. Then for suitable $\lambda_b \in K$, $b \in \mathcal{B}'$, we have

$$a = \sum_{b \in \mathcal{B}'} \lambda_b l_b^{\deg a} = \sum_{b \in \mathcal{B}'} \lambda_b b^{e_b}$$

for $e_b := \deg a / \deg b$. The coefficients λ_b lie in k because $a \in A$ and because $\mathcal{B}$ is K-linearly independent where we identify $\mathcal{B}$ with its image in $K \otimes_k A$. □

Proposition (2.3.9). *a) The ideal I lies in the kernel of $A \overset{m}{\to} A \otimes_k A \to A/I \otimes_k A/I$ if and only if I is generated by homogeneous additive elements. Is this the case then m induces a k-algebra morphism $A/I \to A/I \otimes_k A/I$.*

b) Assume that I is generated by homogeneous additive elements. The equalizer

$$A \rightrightarrows A/I \otimes_k A$$

of the morphisms of rings $A \overset{m}{\to} A \otimes_k A \to A/I \otimes_k A$ and $a \mapsto 1 \otimes a$ is the graded k-subalgebra of A generated by a homogeneous additive basis of I.

Proof. Choose a homogeneous additive basis $\mathcal{B}$ of I. Let $\overline{k}$ be an algebraic closure of k and let $\overline{A}$ resp. $\overline{I}$ denote the $\overline{k}$-vector space $A \otimes_k \overline{k}$ resp. $I \otimes_k \overline{k}$. We have commutative diagrams with vertical injective maps

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes_k A & \longrightarrow & A/I \otimes_k A/I \\ \downarrow & & \downarrow & & \downarrow \\ \overline{A} & \longrightarrow & \overline{A} \otimes_{\overline{k}} \overline{A} & \longrightarrow & \overline{A}/\overline{I} \otimes_{\overline{k}} \overline{A}/\overline{I} \end{array} \qquad \begin{array}{ccccc} A & \longrightarrow & A \otimes_k A & \longrightarrow & A/I \otimes_k A \\ \downarrow & & \downarrow & & \downarrow \\ \overline{A} & \longrightarrow & \overline{A} \otimes_{\overline{k}} \overline{A} & \longrightarrow & \overline{A}/\overline{I} \otimes_{\overline{k}} \overline{A} \end{array}$$

The elements of $\mathcal{B}$ are powers of homogeneous elements of $\overline{A}$ of degree one. Since $\mathcal{B}$ is k-algebraically independent there is a $\overline{k}$-basis $x_1, \dots, x_n$ of $\overline{A}_1$ such that, as a subset of $\overline{A}$, $\mathcal{B}$ is equal to $\{x_1^{q_1}, \dots, x_s^{q_s}\}$ for a suitable $s \in \{1, \dots, n\}$ and for suitable p-powers $q_1, \dots, q_s$.

a) Assume I is not generated by homogeneous additive elements but lies in the kernel of $A \to A/I \otimes_k A/I$. Choose a homogeneous element $f \in I \backslash \langle \mathcal{B} \rangle$ with the smallest degree. By assumption the image of f in $A/I \otimes_k A/I$ is zero. Since $A/\langle \mathcal{B} \rangle \to A/\langle \mathcal{B} \rangle \otimes_k \overline{k}$ is injective the element f is not generated by $x_1^{q_1}, \dots, x_s^{q_s}$ in $\overline{A}$. Then f has a non-additive monomial h in the variables $x_1, \dots, x_n$ which is not divisible by any of the $x_i^{q_i}$. We can see $\overline{A} \times_{\overline{k}} \overline{A}$ as a polynomial ring over $\overline{k}$ in the variables $x_1 \otimes 1, \dots, x_n \otimes 1, 1 \otimes x_1, \dots, 1 \otimes x_n$. Then the monomials of $m(h) - h \otimes 1 - 1 \otimes h$ are monomials of $g := m(f) - f \otimes 1 - 1 \otimes f$. The element g lies in the ideal $I \otimes_k A + A \otimes_k I$ of $A \otimes_k A$ and it is therefore a finite sum of elements of the two forms $i \otimes a$, $a' \otimes i'$ for homogeneous elements $i, i' \in I$, $a, a' \in A$. By the minimality of f the elements i, i' lie in $\langle \mathcal{B} \rangle$. Then, as an element of $\overline{A} \otimes_{\overline{k}} \overline{A}$, g lies in the ideal J generated by $x_1^{q_1} \otimes 1, \dots, x_s^{q_s} \otimes 1, 1 \otimes x_1^{q_1}, \dots, 1 \otimes x_s^{q_s}$. Since these elements do not generate the monomials of $m(h) - h \otimes 1 - 1 \otimes h$, which appear in g, g can not lie in J. This is a contradiction.

b) Let α resp. β denote the morphism $A \overset{m}{\to} A \otimes_k A \to A/I \otimes_k A$ resp. $a \mapsto 1 \otimes a$. Then the equalizer of α and β is the set $\mathcal{U} := \{f \in A \,|\, \alpha(f) = \beta(f)\}$. Since α and β are morphisms of graded k-algebras, $\mathcal{U}$ is a graded k-subalgebra of A. For an element $b \in \mathcal{B}$ one has $\alpha(b) = b \otimes 1 + 1 \otimes b = 0 + 1 \otimes b = \beta(b)$. Thus we have $k[\mathcal{B}] \subseteq \mathcal{U}$. We show the other inclusion. Let $\overline{\mathcal{U}}$ denote the $\overline{k}$-subalgebra $\mathcal{U} \otimes_k \overline{k}$ of $\overline{A}$. Assume there is an element $f \in \mathcal{U} \backslash k[\mathcal{B}] \subseteq \overline{\mathcal{U}} \backslash \overline{k}[\mathcal{B}]$. The element $\alpha(f) - \beta(f)$ lies in the ideal $\overline{I} \otimes_{\overline{k}} \overline{A}$ of $\overline{A} \otimes_{\overline{k}} \overline{A}$. Since $\overline{I}$ is generated by the monomials $x_1^{q_1}, \dots, x_s^{q_s}$, the elements $x_1^{q_1} \otimes 1, \dots, x_s^{q_s} \otimes 1$ generate the ideal

$\overline{I} \otimes_{\overline{k}} \overline{A}$. For a multi-index $\gamma \in \mathbb{N}^n$ let V_γ denote the k-subvector space of $\overline{A} \otimes_{\overline{k}} \overline{A}$ generated by all monomials $\underline{x}^\delta \otimes \underline{x}^{\delta'}$ for $\delta + \delta' = \gamma$. We have a decomposition $\overline{A} \otimes_{\overline{k}} \overline{A} = \oplus_{\gamma \in \mathbb{N}^n} V_\gamma$ and we have $\alpha(\underline{x}^\gamma), \beta(\underline{x}^\gamma) \in V_\gamma$. Thus we can assume that f is a monomial in $x_1, \dots, x_n$. Write $f = f_1 f_2$ for monomials $f_1 \in \overline{k}[\mathcal{B}]$ and $f_2 \in \overline{A} \backslash \langle \mathcal{B} \rangle = \overline{A} \backslash \overline{I}$. We have

$$\overline{I} \otimes_{\overline{k}} \overline{A} \ni \alpha(f) - \beta(f) = (\alpha(f_1) - \beta(f_1))\alpha(f_2) + \beta(f_1)(\alpha(f_2) - \beta(f_2)).$$

Since $\alpha(f_1) - \beta(f_1) \in \overline{I} \otimes_{\overline{k}} \overline{A}$ and $\beta(f_1) = 1 \otimes f_1$, we get $g := \alpha(f_2) - \beta(f_2) \in \overline{I} \otimes_{\overline{k}} \overline{A}$. The monomial $f_2 \otimes_k 1 \in \overline{A} \otimes_{\overline{k}} \overline{A}$ is a monomial of g and lies not in $\overline{I} \otimes_{\overline{k}} \overline{A}$. This is a contradiction.

□

Remark (2.3.10). *Assume that k is perfect. Then by example (2.3.2)b) the homogeneous additive elements of A are powers of elements of A_1. Thus, if I is generated by homogeneous additive elements and I is equal to its radical, then $I = \langle I_1 \rangle$.*

Definition (2.3.11). *Assume that I is generated by homogeneous additive elements.* ***The ring of invariants of (I,A)*** *is the graded k-subalgebra* $\operatorname{diffker}(A \rightrightarrows A/I \otimes_k A)$ *of A from proposition (2.3.9).*

Remark (2.3.12). *a) If I is generated by homogeneous additive elements by definition for a homogeneous additive basis $\mathcal{B}$ of I the graded k-algebra $k[\mathcal{B}]$ is the ring of invariants of (I, A).*

b) For I not necessarily generated by homogeneous additive elements we will define the ring of invariants of (I, A) via the ridge in (2.5.15).

Lemma (2.3.13). *Every graded k-subalgebra $\mathcal{U}$ of A which is generated by homogeneous additive elements is the ring of invariants of $(\langle \mathcal{U}_+ \rangle, A)$ where $\langle \mathcal{U}_+ \rangle$ is the ideal of A generated by the homogeneous elements of $\mathcal{U}$ of positive degree.*

Proof. Let $\mathcal{A}$ be the set of all additive elements of A lying in $\mathcal{U}$. Let $\mathcal{B}$ be a homogeneous additive basis of $\mathcal{A}$. Then we have $k[\mathcal{B}] = \mathcal{U}$ and $\langle \mathcal{B} \rangle = \langle \mathcal{U}_+ \rangle$. Thus $\mathcal{B}$ is a homogeneous additive basis of $\langle \mathcal{U}_+ \rangle$. Then the claim follows with remark (2.3.12). □

Lemma (2.3.14). *Assume that I is generated by homogeneous additive elements and let $\mathcal{U}$ denote the invariant ring (I, A). Then we have* $\dim A/I = \dim A - \dim \mathcal{U}$.

Proof. Let $\mathcal{B}$ be a homogeneous additive basis of I. Then $\mathcal{B}$ consists of finitely many k-algebraically independent homogeneous additive elements $b_1, \dots, b_t$ of A. Then we have $\mathcal{U} = k[b_1, \dots, b_t]$ and $\dim \mathcal{U} = t$. We show $\dim A/I = \dim A - t$. For every field extension $K|k$ the set $\{b_1 \otimes 1, \dots, b_t \otimes 1\}$ is a homogeneous additive basis of the ideal $I \otimes_k K$ of $A \otimes_k K$ and one has $\dim A/I = \dim A/I \otimes_k K$. Thus we may assume that k is algebraically closed. Then every b_i is a power $y_i^{e_i}$ of an element $y_i \in A_1$. Then for the nilradical $\mathcal{N}$ of the ring A/I we have $\dim A/I = \dim (A/I)/\mathcal{N}$ and $(A/I)/\mathcal{N} \cong A/\langle y_1, \dots, y_t \rangle$. Since $y_1, \dots, y_t$ are k-algebraically independent, we have $\dim A/\langle y_1, \dots, y_t \rangle = \dim A - t$. □

2.4 Group schemes

We define group schemes and actions by group schemes and we give some examples.
Let S be a scheme and let $\mathcal{C}$ denote the category of S-schemes. Let (Grp) resp. (Set) denote the category of groups resp. sets. Let

$$F : (\mathrm{Grp})^{C^{\mathrm{opp}}} \to (\mathrm{Set})^{C^{\mathrm{opp}}}$$

be the forgetful functor form the category of contravariant functors form $\mathcal{C}$ to groups to the category of contravariant functors form $\mathcal{C}$ to sets. For each scheme Y over S let $Y(\_)$ denote the functor

$$C^{\mathrm{opp}} \to (\mathrm{Set}) : X \mapsto Y(X) = \mathrm{Hom}_S(X, Y).$$

Definition (2.4.1). *a) A* ***group scheme over S*** *is a group object in $\mathcal{C}$, i.e. it is a S-scheme G together with a functor $G_{(\mathrm{Grp})}(\_) : \mathcal{C}^{\mathrm{opp}} \to (\mathrm{Grp})$ with $F(G_{(\mathrm{Grp})}(\_)) = G(\_)$. Often we write $G(\_)$ for $G_{(\mathrm{Grp})}(\_)$ and G for $(G, G_{(\mathrm{Grp})}(\_))$.*

b) A ***morphism of group schemes (over S)*** *$(G, G_{(\mathrm{Grp})}(\_)) \to (G', G'_{(\mathrm{Grp})}(\_))$ is a morphism $\phi : G \to G'$ in $\mathcal{C}$ together with a morphism $\phi(\_) : G_{(\mathrm{Grp})}(\_) \to G'_{(\mathrm{Grp})}(\_)$ in $(\mathrm{Grp})^{C^{\mathrm{opp}}}$ with $F(\phi(\_)) = G(\phi)$. Often we write $G \to G'$ for $(G, G_{(\mathrm{Grp})}(\_)) \to (G', G'_{(\mathrm{Grp})}(\_))$.*

c) Let G be a group scheme over S. A ***subgroup scheme of G*** *is a morphism of group schemes $(H, H_{(\mathrm{Grp})}(\_)) \to (G, G_{(\mathrm{Grp})}(\_))$ such that $H \to G$ is a inclusion of a closed subscheme of G.*

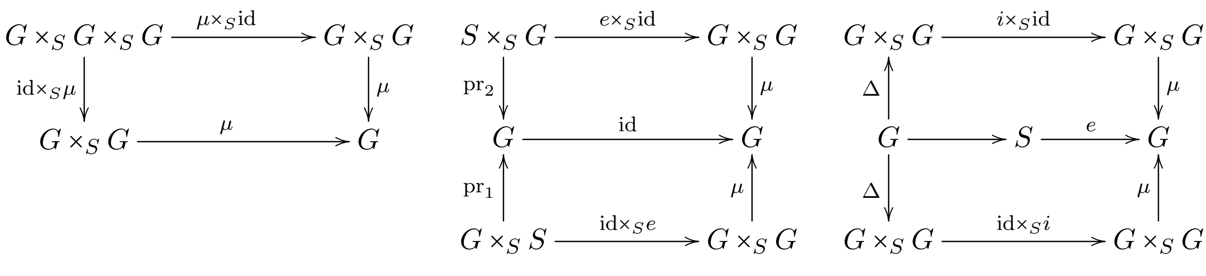
Remark (2.4.2). *a) The contravariant yoneda functor*

$$\mathcal{C} :\to (\mathrm{Set})^{C^{\mathrm{opp}}} : Y \to \mathrm{Hom}_S(\_, Y) = Y(\_)$$

is faithfully flat. Therefore by the yoneda lemma the datum of the functor $G_{(\mathrm{Grp})}(\_) : \mathcal{C} \to (\mathrm{Grp})^{C^{\mathrm{opp}}}$ is equivalent to the datum of three morphisms of S-schemes

$$\mu : G \times_S G \to G, \quad e : S \to G, \quad i : G \to G$$

such that one has commutative diagrams



where Δ denotes the diagonal morphism. We call $e : S \to G$ ***the neutral element morphism*** *and $\mu : G \times_S G \to G$* ***the group law morphism****. Note that e is the neutral element of the group $G(S)$.*

b) *The datum of a morphism of group schemes* $(G, G_{(\mathrm{Grp})}(\_)) \to (G', G'_{(\mathrm{Grp})}(\_))$ *is equivalent to the datum of a morphism of schemes* $G \to G'$ *such that one of the following equivalent conditions is true.*

i) *For all* S*-schemes* X *the induced map* $G(X) \to G'(X)$ *of sets is a group homomorphism.*

ii) *The diagram*

$$\begin{array}{ccc} G' \times_S G' & \longrightarrow & G' \\ \uparrow & & \uparrow \\ G \times_S G & \longrightarrow & G \end{array}$$

commutes where the horizontal maps are the group law morphisms.

Example (2.4.3). Let n be a natural number.

a) Let $S' \to S$ a scheme morphism and let G be a group scheme over S. Set $G' := G \times_S S'$. For every S'-scheme X the universal property of the fiber product induces a bijection

$$\mathrm{Hom}_S(X, G) \cong \mathrm{Hom}_{S'}(X, G')$$

which is functorial in X. Thus the group scheme structure of G over S defines a group scheme G' over S'.

b) Assume that S is affine and let V be a S-scheme isomorphic to $\mathbb{A}^n_S$ (as a S-scheme). Each of the following data are equivalent

i) A group scheme structure on V.

ii) A morphism of S-schemes $S \to V$.

iii) A graded ΓS-algebra structure $\Gamma = \oplus_{j \in \mathbb{N}} (\Gamma V)_j$ on ΓV and an isomorphism of graded ΓS-algebras $\Gamma V \cong \mathrm{Sym}_{\Gamma S}(\Gamma V)_1$.

i) $\Rightarrow$ *ii)* Just take the neutral element morphism $e : S \to V$.

ii) $\Rightarrow$ *iii)* Define the ΓS-module $M := \ker(\Gamma V \to \Gamma S) \otimes_{\Gamma V} \Gamma S$. Then one has an isomorphism of ΓS-algebras $\Gamma V \cong \mathrm{Sym}_{\Gamma S} M$ which induces a graded ΓS-algebra structure on ΓV with $\Gamma V = \mathrm{Sym}_{\Gamma S}(\Gamma V)_1$.

iii) $\Rightarrow$ *i)* For every S-scheme X we have bijections

$$V(X) \cong \mathrm{Hom}_{\Gamma S-\mathrm{algebras}}(\Gamma V, \Gamma X) \cong \mathrm{Hom}_{\Gamma S-\mathrm{modules}}((\Gamma V)_1, \Gamma X),$$

functorial in X. By pointwise addition and scalar multiplication $V(X)$ becomes a ΓX-module which is free of rank n. This defines a group scheme V. The scheme morphisms $\mu : V \times_S V \to V$ and $i : V \to V$ from the remark above are induced by the ΓS-module morphisms

$$(\Gamma V)_1 \to \Gamma V \otimes_{\Gamma S} \Gamma V : m \mapsto m \otimes 1 + 1 \otimes m \quad \text{and} \quad (\Gamma V)_1 \to \Gamma V : m \mapsto -m.$$

Note that $\Gamma V \to \Gamma V \otimes_{\Gamma S} \Gamma V$ is an injective morphism of graded ΓS-algebras.

We call this group scheme **the $\boldsymbol{n}$-dimensional vector group scheme over $\boldsymbol{S}$** and denote it by $\mathbb{G}^{\boldsymbol{n}}_{\boldsymbol{a,S}}$. Write $\mathbb{G}^n_{a,S} =: \mathbb{G}^n_{a,k}$ if $S = \operatorname{Spec} k$ for a field k.

c) Let $V \cong \mathbb{G}^n_{a,S}$ and $V' \cong \mathbb{G}^{n'}_{a,S}$ be two vector group schemes over S. A scheme morphism $V \to V'$ is a morphism of groups schemes if and only if the diagram of ΓS-algebras

$$\begin{array}{ccc} \Gamma V' & \longrightarrow & \Gamma V \\ \downarrow & & \downarrow \\ \Gamma V' \otimes_{\Gamma S} \Gamma V' & \longrightarrow & \Gamma V \otimes_{\Gamma S} \Gamma V \end{array}$$

is commutative where the vertical morphisms are induced by the group law and the horizontal morphisms are induced by $V \to V'$. By the definition of the vertical maps (see b) above) the diagram commutes if $\Gamma V' \to \Gamma V$ is graded. The opposite direction is not true in general. For example the diagram commutes if $S = \operatorname{Spec} k$ for a field k of characteristic two, ΓV and $\Gamma V'$ are both a polynomial ring $k[x]$ in one variable over k and $\Gamma V \to \Gamma V'$ is the morphism of k-algebras which maps x to x^2.

d) Assume $S = \operatorname{Spec} k$ for a field k. Let G be a closed subscheme of a n-dimensional vector group scheme V over k give by an homogeneous ideal I of $\Gamma V =: A$. We show that the following are equivalent.

i) G is a subgroup scheme of V.

ii) the ideal I is generated by homogeneous additive elements of A.

i) $\Rightarrow$ *ii)* The morphism $G \to V$ is a group morphism. Thus the diagram of k-algebras

(2.4.3.A)
$$\begin{array}{ccc} A & \xrightarrow{m} & A \otimes_k A \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & A/I \otimes_k A/I \end{array}$$

commutes, where by b) above m is induced by the k-linear map $A_1 \to A \oplus_k A : x \to x \otimes_1 +1 \oplus x$ and where the horizontal morphism is induced/given by the projection $A \to A/I$. Then by proposition (2.3.9) I is generated by additive elements.

ii) $\Rightarrow$ *i)* By lemma (2.3.3) the diagram (2.4.3.A) commutes. Let X be an arbitrary S-scheme. Then (2.4.3.A) induces a commutative diagram

$$\begin{array}{ccc} V(X) & \longleftarrow & V(X) \times V(X) \\ \uparrow & & \uparrow \\ G(X) & \longleftarrow & G(X) \times G(X) \end{array}$$

where the upper horizontal map is the group law on $V(X)$. Since I is homogeneous we have a factorization of k-algebras $A \to A/I \to k$ which assures that the neutral element of $V(X)$ lies in $G(X)$. Further we have a commutative diagram of k-algebras

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & A/I \end{array}$$

where $A \to A$ is induced by the k-linear map $A_1 \to A : x \to -x$. This shows that $G(X)$ is closed under $(\_) \to (\_)^{-1}$ in $V(X)$. Then $G(X)$ is a subgroup of $V(X)$.

e) Let S be a scheme, let $n \in \mathbb{N}_{\geq 1}$ by a natural number and let $\mathrm{GL}_{n,S}$ denote the open subscheme of $\mathbb{A}_S^{n^2}$

$$S \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[T_{ij} \,|\, i,j \in \{1,\dots,n\}][\det{}^{-1}]$$

where det denotes the polynomial

$$\det = \sum_{\sigma \in \mathrm{S}_n} \mathrm{sign}\sigma T_{1\sigma(1)} \cdot \ldots \cdot T_{n\sigma(n)}.$$

Then for every S-scheme X we have a bijection from $\mathrm{GL}_{n,S}(X)$ to the invertible matrices $\mathrm{GL_n}(\Gamma X)$ over ΓX, functorial in X. The matrix multiplication defines a group scheme $\mathrm{GL}_{n,S}$ over S, **the general linear group over S**. For $n = 1$ the group $\mathrm{Gl}_{n,S}(X)$ is the group of units $(\Gamma X)^{\times}$. We call $\mathrm{GL}_{1,S} =: \mathbb{G}_{m,S}$ **the multiplicative group over S**. Write $\mathbb{G}_{m,S} =: \mathbb{G}_{m,k}$ if $S = \mathrm{Spec}k$ for a field k.

f) For a S-scheme X the set $S(X)$ has only one element. This defines the trivial group scheme S over S. For a group scheme G over S the neutral element morphism $e : S \to G$ is a morphism of group schemes over S. If e is a closed immersion the trivial group becomes a subgroup scheme of G.

g) Let $\phi : G' \to G$ be a morphism of group schemes over S and let H be a subgroup scheme of G. Then for every S-scheme X we have

$$G' \times_G H(X) = \{f \in G'(X) \,|\, \phi \circ f \text{ factors through } H\} = (\phi(X))^{-1}(H(X)).$$

Then $G' \times_G H$ becomes a subgroup scheme of G'. We denote it by $\phi^{-1}(H)$.

h) Let G be a group scheme over S and let $e : S \to G$ be the neutral element morphism. Let $\phi : G' \to G$ be a morphism of group schemes over S. For every S-scheme X we have

$$G' \times_G S(X) = \{f \in G'(X) \,|\, \phi \circ f = e \circ (X \to S)\} = \ker(\phi(X)).$$

Then the group structure of G' defines a group scheme structure on $G' \times_G S$. We call this group scheme the kernel of ϕ and denote it by $\ker\phi$. The projection $G' \times_G S \to G'$ defines a morphism of S-group schemes $\ker\phi \to G'$. If the morphism e is a closed immersion than $\ker\phi$ is a subgroup scheme of G'.

Definition (2.4.4). *Let S be a scheme and let G be a group scheme over S.*

a) For a S-scheme Y an ***(left) action of G on Y*** *is a morphism of S-schemes $G \times_S Y \to Y$ such that for each S-scheme X the map $G(X) \times Y(X) = (G \times_S Y)(X) \to Y(X)$ is an action of the group $G(X)$ on the set $Y(X)$.*

b) A morphism of $\psi : Y \to Y'$ of S-schemes with G-action ***respects the action by G*** *if the following diagram of schemes commutes.*

$$\begin{array}{ccc} G \times_S Y & \longrightarrow & Y \\ {\scriptstyle \mathrm{id}\times\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ G \times_S Y' & \longrightarrow & Y' \end{array}$$

Example (2.4.5). a) Let V be a vector group scheme over S (see example (2.4.3)). The ΓS-module morphism $(\Gamma V)_1 \to \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V : x \mapsto T \cdot x$ induces a morphism of S-schemes $\mathbb{G}_{m,S} \times_S V \to V$. For each S-scheme X the induced map $(\Gamma X)^\times \times V(X) = (\mathbb{G}_{m,S} \times_S V)(X) \to V(X)$ is the scalar multiplication in the ΓX-module $V(X)$. Thus $\mathbb{G}_{m,S} \times_S V \to V$ is an action of $\mathbb{G}_{m,S}$ on V. By an action of $\mathbb{G}_{m,S}$ on a vector group scheme we always mean the action just defined.

For two vector groups V, V' and a morphism of schemes $V \to V'$ the diagram

$$\begin{array}{ccc} \Gamma V & \longrightarrow & \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V \\ \uparrow & & \uparrow \\ \Gamma V' & \longrightarrow & \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V' \end{array}$$

commutes if $\Gamma V' \to \Gamma V$ is graded. The opposite direction is also true: Let f' be an element of $(\Gamma V')_1$. Denote its image in ΓV by f. Let $f_0, f_1, f_2, \ldots$ be the homogeneous components of f. Since the diagram commutes, we have $T^1 \cdot f = \sum_{d \in \mathbb{N}} T^d \cdot f_d$ which implies $f = f_1$. Thus $\Gamma V' \to \Gamma V$ is graded. Note that by example (2.4.3)c) $V \to V'$ is a morphism of group schemes if $\Gamma V' \to \Gamma V$ is graded.

b) Let V be an vector group scheme over S. Let C be a closed subscheme of V and let I denote the ideal $\ker(\Gamma V \to \Gamma C)$. We show that the following are equivalent.

i) the ideal I is homogenous.

ii) There is a $\mathbb{G}_{m,S}$-action on C such that $C \to V$ respects the $\mathbb{G}_{m,S}$-action.

i) $\Rightarrow$ *ii)* Let $\Gamma V \to \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V$ be the ΓS-algebra morphism from a). For a homogeneous non-zero element f of ΓV of degree d the image of f under $\Gamma V \to \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V$ is $T^d \otimes f$. We get commutative diagrams of ΓS-algebras resp. S-schemes

(2.4.5.A)
$$\begin{array}{ccc} \Gamma V & \longrightarrow & \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V \\ \downarrow & & \downarrow \\ \Gamma C & \longrightarrow & \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma C \end{array}$$

(2.4.5.B)
$$\begin{array}{ccc} V & \longleftarrow & \mathbb{G}_{m,S} \times_S V \\ \uparrow & & \uparrow \\ C & \longleftarrow & \mathbb{G}_{m,S} \times_S C \end{array}$$

ii) $\Rightarrow$ *i)* We have a commutative diagram of S-schemes as in (2.4.5.B). This corresponds to the commutative diagram (2.4.5.A) of ΓS-algebras. Let f be an element of I and let $f_0, f_1, f_2, \ldots$ denote the homogeneous components. The image of f under $\Gamma V \to \Gamma S[T,T^{-1}] \otimes_{\Gamma S} \Gamma V$ is $\sum_{d \in \mathbb{N}} T^d \cdot f_d$. It lies in $\Gamma S[T,T^{-1}] \otimes_{\Gamma S} I$, i.e. $f_0, f_1, f_2, \ldots \in I$. Thus I is homogeneous.

Note that the $\mathbb{G}_{m,S}$-action on C is uniquely determined by the $\mathbb{G}_{m,S}$-action on V.

c) For a group scheme G over a scheme S the group law of G defines a (left) group action of G on G.

d) For a group scheme G over S and a S-scheme Y the projection $\mathrm{pr}_2 : G \times_S Y \to Y$ is an G-action on Y. For each S-scheme X the induced map $G(X) \times Y(X) \to Y(X)$ is the trivial action.

e) For a group scheme G over S and two S-schemes Y, Y' with G-action the morphism $G \times_S (Y \times_S Y) \to Y \times_S Y'$ defined by the diagram of schemes

$$\begin{array}{ccccc} G \times_S (Y \times_S Y') & \xrightarrow{\mathrm{id}\times\mathrm{pr}_1} & G \times_S Y & \longrightarrow & Y \\ \downarrow{\scriptstyle \mathrm{pr}_1\times\mathrm{id}} & & & & \\ G \times_S Y' & & & & \\ \downarrow & & & & \\ Y' & & & & \end{array}$$

is an action of G on $Y \times_S Y'$. For each S-scheme X the induced map

$$G(X) \times Y(X) \times Y'(X) \to Y(X) \times Y'(X)$$

is the componentwise action.

2.5 Additive group schemes over a field

The aim of the section is the definition of the additive group schemes ridge and directrix of a given cone over a field (definition (2.5.12)). An other important additive group scheme is the Hironaka scheme (see (2.5.7)). In can be described with its ring of invariants (see (2.5.5)). At the end of the section we study quotients of cones by vector groups.
Let k be a field. In this section all schemes and group schemes are over k.

Definition (2.5.1). *a) For a vector group scheme V a* ***subcone of V*** *is a closed non-empty subscheme C of V with a (unique) $\mathbb{G}_{m,a}$-action such that $C \to V$ respects the action. A* ***cone*** *is a subcone of a vector group scheme. An* ***additive group scheme*** *is a subcone G of a vector group scheme V such that $G \to V$ makes G to a subgroup scheme of V.*

b) A ***morphism of cones*** *$C \to C'$ is a morphism of schemes $C \to C'$ which respects the $\mathbb{G}_{m,k}$-action. A* ***morphism of additive group schemes*** *$G \to G'$ is a morphism of cones $G \to G'$ which is a morphism of group schemes. If $C \to C'$ resp. $G \to G'$ is the inclusion of a closed subscheme then we call C a* ***subcone of C'*** *resp. G an* ***additive subgroup scheme of*** *G'. A* ***subvector group scheme*** *of a group scheme G is an additive subgroup scheme V of G which is a vector group scheme.*

From now on we omit ”‘scheme”’ in expressions like ”‘group scheme”’ or ”‘morphism of additive group schemes”’.

Remark (2.5.2). *a) For a cone C by example (2.4.5)b) ΓC is a graded k-algebra with $(\Gamma C)_0 = k$ and, as a k-algebra, ΓC is generated by $(\Gamma C)_1$. We call the maximal ideal $(\Gamma C)_+ = \oplus_{n\geq 1}(\Gamma C)_n$ of ΓC **the origin** and denote it by 0.*

b) Let C be a subcone of a vector group V. With a similar argument as in example (2.4.3)d) we see that for each k-scheme X the neutral element of $V(X)$ lies in $C(X)$ and $C(X)$ is closed under $(\_) \mapsto (\_)^{-1}$.

c) For every cone C there is a smallest additive vector group V_C which has C as a subcone. By example (2.4.3)b) the graded ring $\mathrm{Sym}_k(\Gamma C)_1$ defines a vector group $\mathrm{Spec}(\mathrm{Sym}_k(\Gamma C)_1) =: V_C$. We have an epimorphism of k-algebras $\mathrm{Sym}_k(\Gamma V)_1 \to \Gamma C$. By (2.4.5)b) C is a subcone of V_C. Let C be a subcone of a vector group V. The the epimorphism $(\Gamma V)_1 \to (\Gamma C)_1 = (\Gamma V_C)_1$ makes V_C to a subvector group of V. Assume that $C =: G$ is an additive subgroup of V. Write $I := \ker(\Gamma V \to \Gamma G)$, $I_G := \ker(\Gamma V_G \to \Gamma G)$. We have a commutative diagram of k-vector spaces with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \Gamma V & \longrightarrow & \Gamma G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & I_G & \longrightarrow & \Gamma V_G & \longrightarrow & \Gamma G & \longrightarrow & 0 \end{array}$$

Then by snake lemma $I \to I_G$ is surjective. Since I is generated by homogenous additive elements, this is holds also for I_G. Thus G is an additive subgroup of V_G.

d) For two cones C, C' the following data are equivalent.

α) A morphism of cones $C \to C'$.

β) A k-algebra morphism $\Gamma C' \to \Gamma C$ such that the following diagram commutes

$$\begin{array}{ccc} k[T,T^{-1}] \otimes_k \Gamma C & \longleftarrow & \Gamma C \\ \uparrow & & \uparrow \\ k[T,T^{-1}] \otimes_k \Gamma C' & \longleftarrow & \Gamma C' \end{array}$$

γ) A morphism of graded k-algebras $\Gamma C' \to \Gamma C$.

The equivalence $\alpha) \Leftrightarrow \beta)$ and the implication $\beta) \Leftarrow \gamma)$ are easy. To show $\beta) \Rightarrow \gamma)$ we argument as in example (2.4.5)a). Let f' be an element of $(\Gamma C')_1$. Denote its image in ΓC by f and let $f_0, f_1, f_2, \ldots$ be the homogeneous components of f. Since the diagram in ii) commutes, we have $T^1 \cdot f = \sum_{d\in\mathbb{N}} T^d \cdot f_d$ which implies $f = f_1$. This shows $\beta) \Rightarrow \gamma)$.

A morphism of cones $C \to C'$ induces a k-linear map $(\Gamma C')_1 \to (\Gamma C)_1$. This induces a morphism of graded k-algebras $\mathrm{Sym}_k(\Gamma C')_1 \to \mathrm{Sym}_k(\Gamma C)_1$ and a morphism of vector groups $V_C \to V_{C'}$ which respects the $\mathbb{G}_{m,k}$-action.

e) In the situation of d), if $C = G$, $C' = G$ are additive group schemes, the data $\alpha) - \gamma)$ are equivalent to the datum of

δ) a group morphism $G \to G'$.

The implication $\delta) \Rightarrow \alpha)$ *is clear. Assume we have a morphism of cones* $G \to G'$*. Then for each* k*-scheme* X *we have a commutative diagram*

$$\begin{array}{ccc} V_G(X) & \longrightarrow & V_{G'}(X) \\ \uparrow & & \uparrow \\ G(X) & \longrightarrow & G'(X) \end{array}$$

where $V_G(X) \to V_{G'}(X)$ *is a group homomorphism and the vertical maps are inclusions of subgroups. Then* $G(X) \to G'(X)$ *is a group homomorphism. This implies* δ*).*

f) Let G *be an additive subgroup of a vector group* V*. Then by examples (2.4.3)d) and (2.4.5)b) the ideal* $I := \ker(\Gamma V \to \Gamma G)$ *of* ΓV *is generated by homogeneous additive elements of* ΓV*. Assume that* k *is perfect. Then homogeneous additive elements are powers of elements of* $(\Gamma V)_1$*, see remark (2.3.10). Thus the reduced scheme* G_{red} *associated to* G *is a vector group.*

Definition (2.5.3). *Let* C *be a cone over a field. Define* $\mathbb{P}(\boldsymbol{C}) := \mathbf{Proj}(\boldsymbol{\Gamma C})$*.*

Remark (2.5.4). *Let* $\pi : X' \to X$ *be the blow-up of a scheme* X *in a center* D*. Let* $\mathcal{I}$ *the to* $D \subseteq X$ *associated quasi-coherent ideal sheaf of* $\mathcal{O}_X$*. Let* x *be a point of* D*. Then the grading on* $\mathrm{gr}_{\mathcal{I}}\mathcal{O}_X$ *defines a cone* $C_{X,D,x} = \mathrm{Spec}(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_X) \times_D \kappa(x)$ *over* $\kappa(x)$*. By remark (2.1.7) we have* $\pi^{-1}(\{x\}) = \mathbb{P}(C_{X,D,x})$*.*

Definition (2.5.5). *a) Let* G *be an additive subgroup of a vector group* V*. Write* $A := \Gamma V$*,* $I := \ker(\Gamma V \to \Gamma G)$*.* ***The ring of invariants of*** $\boldsymbol{(G, V)}$ *is the ring of invariants of* (I, A) *(see definition (2.3.11)).*

b) For an additive group G ***the ring of invariants of G*** *is the ring of invariants of* (G, V_G) *where* V_G *is the smallest vector group which contains* G *as an additive subgroup (see remark (2.5.2)).*

Remark (2.5.6). *a) By lemma (2.3.13) for a vector group* V*, building the ring of invariants defines a bijection between the additive subgroups of* V *and the* k*-subalgebras of* ΓV *generated by homogeneous additive elements of* A*.*

b) Let G *be an additive subgroup of a vector group* V *and let* $\mathcal{U} \subseteq \Gamma V$ *be the invariant ring of* (G, V)*. Then by lemma (2.3.14) we have* $\dim G = \dim V - \dim \mathcal{U}$*.*

For a given vector group V and a point $y \in \mathbb{P}(V)$ in [Hi3], page 1, Hironaka defines a special additive subgroup of V. With the remark after that definition we get the following equivalent definition.

Definition (2.5.7). *Let* V *be a vector group. Let* y *be an element of* $\mathrm{Proj}(\Gamma V) = \mathbb{P}(V) =: \mathbb{P}$*. Also denote the induced element of* $\mathrm{Spec}(\Gamma V) = V$ *by* y*. Let* $\mathfrak{m}$ *denote the maximal ideal of the local ring* $\mathcal{O}_{V,y}$*. Let* M *denote the subset of* $\Gamma V =: A$ *of all homogeneous additive non-zero elements* h *of* A *whose associated element in* $\mathcal{O}_{V,y}$ *lies in* $\mathfrak{m}^{\deg h} \backslash \mathfrak{m}^{\deg h+1}$*.* ***The Hironaka scheme at the point y of*** $\mathbb{P}(\boldsymbol{V})$ *is the additive subgroup* $\mathcal{B}_{\mathbb{P},y}$ *of* V *which has* $k[M]$ *as a ring of invariants of* $(\mathcal{B}_{\mathbb{P},y}, V)$*. In particular one has* $\mathcal{B}_{\mathbb{P},y} = \mathrm{Spec}(A/\langle M \rangle)$*.*

Remark (2.5.8). *a) Let $\mathfrak{q}$ denote the homogenous prime ideal of ΓV associated to $y \in \mathbb{P}(V)$. The set M is a subset of $\mathfrak{q}$ which implies $\mathfrak{q} \in \operatorname{Spec}(A/\langle M\rangle) = \mathcal{B}_{\mathbb{P},y}$ and $y \in \mathbb{P}(\mathcal{B}_{\mathbb{P},y})$.*

b) Assume that k is perfect. Then by remark (2.5.6)f) every element of M is a power of some element $v \in A_1$. By definition of M such a v lies also in M. Thus $\langle M\rangle$ is generated by elements of A_1 and therefore $\mathcal{B}_{\mathbb{P},y}$ is a vector group.

c) For every homogeneous element u of $k[M]$ the induced element of $\mathcal{O}_{\mathbb{P},y}$ lies in $\mathfrak{m}^{\deg u}$.

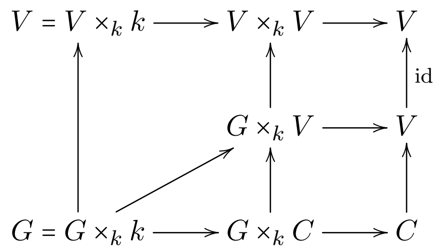
Remark (2.5.9). *Let C be a subcone of a vector group V. For an additive subgroup G of V we write $G + C \subseteq C$ if and only if there is a (unique) G-action on C such that $C \to V$ respects the G-action, i.e. one has a commutative diagram of k-schemes*

$$\begin{array}{ccc} G \times_k V & \longrightarrow & V \\ \uparrow & & \uparrow \\ G \times_k C & \longrightarrow & C \end{array}$$

where $G \times_k V \to V$ is the G-action on V and the vertical morphisms are the obvious inclusions. Equivalently for all k-schemes X one has a commutative diagram of sets

$$\begin{array}{ccc} G(X) \times V(X) & \longrightarrow & V(X) \\ \uparrow & & \uparrow \\ G(X) \times C(X) & \longrightarrow & C(X) \end{array}$$

Assume $G + C \subseteq G$. We show that the closed immersion $G \to V$ factors through the closed immersion $C \to V$ which makes G to a closed subscheme of C. We have the following commutative diagram



Here the vertical morphisms are closed immersions induced by $C \to V$, $G \to V$, $\mathrm{id} : V \to V$. The three left non-vertical morphisms are induced by the neutral element morphisms $k \to V$, $k \to C$. Note that for each k-scheme X the neutral element of $V(X)$ lies in $C(X)$ which induces a morphism $k \to C$ such the triangle with the vertices $G, G \times_k C, G \times_k V$ commutes. The composition of the upper horizontal morphisms is the identity. Thus $G \to V$ factors through $C \to V$.

Theorem (2.5.10). *Let C be a subcone of a vector group V. Write $I := \ker(\Gamma V \to \Gamma C)$ and $A := \Gamma V$.*

a) There is a smallest k-subalgebra $\mathcal{U}_{\mathrm{Rid}}$ of A generated by homogeneous additive elements such that I is the ideal $\langle \mathcal{U}_{\mathrm{Rid}} \cap I\rangle$ of A generated by $\mathcal{U}_{\mathrm{Rid}} \cap I$. The associated additive subgroup G_{Rid} of V is the biggest additive subgroup of V with $G_{\mathrm{Rid}} + C \subseteq C$.

b) There is a smallest k-subalgebra $\mathcal{U}_{\text{Dir}}$ of A generated by elements of A_1 such that I is the ideal $\langle \mathcal{U}_{\text{Dir}} \cap I \rangle$ of A. The associated subvector group V_{Dir} of V is the biggest subvector group of V with $V_{\text{Dir}} + C \subseteq C$.

Proof. a) [BHM]

b) Let W be a subvector group of V and let $\mathcal{U}$ denote the ring of invariants of (W, V). Let $m : A \to A \otimes_k A$ be the k-algebra morphism with $m(x) = x \otimes 1 + 1 \otimes x$ for all $x \in A_1$, which induces the group law morphism $V \times_k V \to V$. We have $W + C \subseteq C$ if and only if m induces a k-algebra morphism $\Gamma C \to \Gamma C \otimes_k \Gamma W$, or equivalently, if $m(I) \subseteq I \otimes_k A + A \otimes_k \langle \mathcal{U}_1 \rangle$. Below we show

Claim 1. $m(I) \subseteq I \otimes_k A + A \otimes_k \langle \mathcal{U}_1 \rangle \Leftrightarrow I = \langle I \cap \mathcal{U} \rangle$.

Then with claim 1 the existence of $\mathcal{U}_{\text{Dir}}$ follows form the existence of V_{Dir}. Let W, W' be two subvector groups of V an let $\mathcal{U}, \mathcal{U}' \subseteq A$ denote their rings of invariants. Define $\mathcal{S} := \mathcal{U} \cap \mathcal{U}'$ and define the subvector group $S = \operatorname{Spec}(A/\langle \mathcal{S}_1 \rangle)$ of V. Below we show

Claim 2. For all k-schemes X we have $W(X) + W(X') = S(X)$.

With claim 2 we see that $W + W' := S$ is a subvector group of V, and for all k-schemes X one has $(W + W')(X) + C(X) = W(X) + W'(X) + C(X) \subseteq C(X)$ if $W + C \subseteq C$, $W' + C \subseteq C$ holds. This guarantees the existence of V_{Dir}.

Proof of claim 1. For each element f of $I \cap \mathcal{U}$ the element $m(f) - f \otimes 1$ lies in $A \otimes_k \langle \mathcal{U}_1 \rangle$. This shows "'$\Leftarrow$"'. We show "'$\Rightarrow$"'. Assume $I \neq \langle I \cap \mathcal{U} \rangle$. Choose a k-basis $\underline{y} = (y_1, \ldots, y_n)$ of $\mathcal{U}_1$ and extend it to a basis $(\underline{y}, \underline{z}) = (y_1, \ldots, y_n, z_1, \ldots, z_l)$ of A_1. Every element f of A has the form $\sum_{\beta \in \mathbb{N}^l} f_\beta \underline{z}^\beta$ for unique coefficients $f_\beta \in \mathcal{U}$. We equip $\mathbb{N}^l$ with the (total) graded lexicographical order (see example (3.7.4)). For $f \neq 0$ define $\operatorname{multideg} f := \max\{\beta \in \mathbb{N}^l \mid f_\beta \neq 0\}$. Define

$$\mathcal{A} := \{f \in A \mid f \text{ homogeneous and } f \in I \backslash \langle I \cap \mathcal{U} \rangle\}, \quad \delta := \min\{\operatorname{multideg} f \mid f \in \mathcal{A}\}.$$

Consider the k-linear map $\psi : A \otimes_k A \to A$ with

$$\psi(\underline{y}^\alpha \underline{z}^\beta \otimes \underline{y}^{\alpha'} \underline{z}^{\beta'}) = \left\{ \begin{array}{ll} \underline{y}^\alpha \underline{z}^\beta & \text{if } \alpha' = 0 \text{ and } \beta' = \delta \\ 0 & \text{else} \end{array} \right\} \quad \text{for} \quad \alpha, \alpha' \in \mathbb{N}^n, \ \beta, \beta' \in \mathbb{N}^l.$$

Choose some $f \in \mathcal{A}$ with $\operatorname{multideg} f = \delta$. We have

$$m(f) = \sum_{\beta \in \mathbb{N}^l} m(f_\beta) \sum_{\beta_1 \leq_c \beta} \underline{z}^{\beta - \beta_1} \otimes \underline{z}^{\beta_1}, \quad \psi(m(f)) = \sum_{\beta \in \mathbb{N}^l, \, \delta \leq_c \beta} f_\beta \underline{z}^{\beta - \delta} = f_\delta$$

where $\leq_c$ denotes the product order on $\mathbb{N}^l$, i.e. $\beta' \leq_c \beta \Leftrightarrow \beta - \beta' \in \mathbb{N}^l$. We get $f_\delta = \psi(m(f)) \in \psi(I \otimes_k A + A \otimes_k \langle \mathcal{U}_1 \rangle) \subseteq I$. This yields $f_\delta \underline{z}^\delta \in \langle I \cap \mathcal{U} \rangle$ (we have $f_\delta \in \mathcal{U}$). We have $\operatorname{multideg} r < \operatorname{multideg} f = \delta$ for $r := f - f_\delta \underline{z}^\delta$. This implies $r \notin \mathcal{A}$, i.e. $r \in \langle I \cap \mathcal{U} \rangle$. We get $f = r + f_\delta \underline{z}^\delta \in \langle I \cap \mathcal{U} \rangle$ in contradiction to $f \in \mathcal{A}$. This completes the proof of claim 1.

Proof of claim 2. We have $V(X) = \operatorname{Hom}_{k-\text{linear}}(A_1, \Gamma X)$ and

$$W(X) = \{f \in V(X) \mid f(\mathcal{U}_1) = 0\}, \quad W'(X) = \{f \in V(X) \mid f(\mathcal{U}'_1) = 0\},$$

$$S(X) = \{f \in V(X) \mid f(\mathcal{S}_1) = 0\}.$$

We see $W(X), W'(X) \subseteq S(X)$, i.e. $W(X) + W'(X) \subseteq S(X)$. For the other inclusion let f be an element of $S(X)$. Choose a k-subvector space T of $\mathcal{U}_1$ such that the k-linear map $T \oplus \mathcal{S}_1 \to \mathcal{U}_1$ is an isomorphism. Further choose k-linear maps $g, g' : A_1 \to A_1$ with $g(T) = 0$, $g'(\mathcal{U}'_1) = 0$, $g + g' = \mathrm{id}$. Then we have

$$(f \circ g)(\mathcal{U}_1) = (f \circ g)(T) + (f \circ g)(\mathcal{S}_1) = 0, \quad (f \circ g')(\mathcal{U}'_1) = 0.$$

Thus $f = f \circ g + f \circ g' \in W(X) + W'(X)$. This completes the proof of claim 2 and the poof of the theorem.

□

Remark (2.5.11). *Let V_C be the smallest vector group which has C as a subcone, see remark (2.5.2). Then V_C is a subvector group of V. Since $G_{\mathrm{Rid}} \subseteq C \subseteq V_C$, the group G_{Rid} is the biggest additive subgroup of V_C with $G_{\mathrm{Rid}} + C \subseteq C$. This shows that the group G_{Rid} depends not the choice of a imbedding of C in a vector group. The same argument shows that G_{Dir} is independent of the choice of a imbedding $C \subseteq V$.*

Definition (2.5.12). *In the situation of theorem (2.5.10) the subgroup G_{Rid} resp. V_{Dir} of V is called* ***the ridge of C*** *resp.* ***the directrix of C*** *and is denoted by* $\mathbf{Rid(C)}$ *resp.* $\mathbf{Dir(C)}$.

Definition (2.5.13). *Let x be a point of a locally noetherian scheme X.* ***The ridge $\mathbf{Rid}_{X,x}$ at x*** *resp.* ***the directrix $\mathbf{Dir}_{X,x}$ at x*** *is the ridge resp. directrix of the cone $C_{X,x} = \mathrm{Spec}(\mathrm{gr}\mathcal{O}_{X,x})$ over $\kappa(x)$.*

Remark (2.5.14). *a) Assume that k is a perfect field. Then for a cone C over k we have*

$$\mathrm{Dir}(C) = \mathrm{Dir}(C)_{\mathrm{red}} \subseteq \mathrm{Rid}(C)_{\mathrm{red}} \subseteq \mathrm{Rid}(C),$$

where $(\_)_{\mathrm{red}}$ denotes the associated reduced scheme. By remark (2.5.2)f) $\mathrm{Rid}(C)_{\mathrm{red}}$ is a vector group. By definition of the directrix we get $\mathrm{Dir}(C) = \mathrm{Rid}(C)_{\mathrm{red}}$.

b) One can calculate the ring of invariants of the ridge by applying differential operators on a Giraud basis of I (see section 3.7). This implies the following result (see corollary (3.7.17)). Let C be a subcone of a vector group V over k. Let $K|k$ be a field extension. Then we have a equality of additive subgroups of $V \times_k K$ over K

$$\mathrm{Rid}(C) \times_k K = \mathrm{Rid}(C \times_k K).$$

c) Let $K|k$ be a field extension of k by a perfect field K. By a) we have $\mathrm{Dir}(C \times_k K) = \mathrm{Rid}(C \times_k K)_{\mathrm{red}}$. With b) we get $\dim \mathrm{Dir}(C \times_k K) = \dim \mathrm{Rid}(C)$.

Definition (2.5.15). *a) For a subcone C of a vector group V* ***the ring of invariants of (C, V)*** *is the ring of invariants of $(\mathrm{Rid}(C), V)$.*

b) For a noetherian symmetric algebra A over k and a homogeneous ideal I of A ***the ring of invariants of (I, A)*** *is the ring of invariants of (C, V) where C is the subcone $\mathrm{Spec}(A/I)$ of the vector group $V = \mathrm{Spec}(A)$.*

Remark (2.5.16). *If C is an additive group then $C = \mathrm{Rid}(C)$. Thus definition (2.5.15) generalizes definitions (2.3.11) and (2.5.5).*

Lemma (2.5.17). *Let V be a vector group over a field k, let C be a subcone of V and let V' be a subvector group of V. Let C' denote the the subcone $C \cap V'$ of V'. Then we have*

$$\dim V' - \dim \mathrm{Rid}(C') \leq \dim V - \dim \mathrm{Rid}(C).$$

Proof. We have $\mathrm{Rid}(C) + C \subseteq C$. Then the additive subgroup $S := \mathrm{Rid}(C) \cap V'$ of V' has the property $S + C' \subseteq C'$. This implies $S \subseteq \mathrm{Rid}(C')$. Let $\mathcal{R}$ resp. $\mathcal{S}$ resp. $\mathcal{V}$ denote the ring of invariants $\subseteq \Gamma V$ of $(\mathrm{Rid}(C), V)$ resp. (S, V) resp. (V', V). By lemma (2.3.13) $\mathcal{S}$ is uniquely determined by the property that it is a k-subalgebra of ΓV generated by homogeneous additive elements and $\Gamma V/\langle \mathcal{S}_+ \rangle = \Gamma S$. We have $\Gamma S = \Gamma(\mathrm{Rid}(C)) \otimes_{\Gamma V} \Gamma V' = \Gamma V/\langle \mathcal{R}_+ \rangle + \langle \mathcal{V}_+ \rangle$. Thus $\mathcal{S}$ is the image of the k-algebra morphism $\mathcal{R} \otimes_k \mathcal{V} \to \Gamma V$, $a \otimes b \mapsto a \cdot b$. With remark (2.5.6) we get

$$\dim S = \dim V - \dim \mathcal{S} \geq \dim V - \dim \mathcal{R} - \dim \mathcal{V} = \dim \mathrm{Rid}(C) + \dim V' - \dim V.$$

Then $S \subseteq \mathrm{Rid}(C')$ implies $\dim V' - \dim \mathrm{Rid}(C') \leq \dim V' - \dim S \leq \dim V - \dim \mathrm{Rid}(C)$. □

For a subcone C of a vector group V' and for a subvector group V of V', whose action on V' induces an action of C, we can define a quotient C/V (see the proposition below and definition (2.5.20)). Our aim is to show that Rid and Dir commute with $(\_)/V$ (see lemma (2.5.23)).

Proposition (2.5.18). *Let C be a subcone of a vector group V'. Let V be a subvector group of V' with $V + C \subseteq C$. Write $I := \ker(\Gamma V' \to \Gamma C)$ and let $\mathcal{V} \subseteq \Gamma V'$ denote the ring of invariants of (V, V'). Define the graded k-subalgebra $\mathcal{U} := \mathcal{V}/(I \cap \mathcal{V})$ of ΓC. Let Q denote the subcone* $\mathrm{Spec}(\mathcal{U})$ *of the vector group* $\mathrm{Spec}(\mathcal{V})$*. The k-scheme $Q \times_k V$ has a V-action induced by the trivial V-action on Q and the group law-action on V (see remark (2.4.5)e)). There is a (non-canonical) isomorphism of cones $\phi : C \to Q \times_k V$ with the following properties.*

a) It respects the V-action.

b) The composition $\mathrm{pr}_1 \circ \phi : C \to Q$ is induced by the morphism $\mathcal{U} \subseteq \Gamma C$.

c) The morphism $\mathrm{pr}_1 \circ \phi$ is the quotient of C by V in the sense that $\mathrm{pr}_1 \circ \phi : C \to Q$ is the coequalizer of the projection $V \times_k C \to C$ and the V-action $V \times_k C \to C$.

Proof. We have a commutative diagram of k-vector spaces an upper exact row

(2.5.18.A)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}_1 & \longrightarrow & (\Gamma V')_1 & \longrightarrow & (\Gamma V)_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ & & \Gamma V \otimes_k \mathcal{V} & \longrightarrow & \Gamma V \otimes_k \Gamma V' & \longrightarrow & \Gamma V \otimes_k \Gamma V & & \end{array}$$

where the vertical morphisms are defined by

$$\alpha(x) = 1 \otimes x, \quad \beta(x) = (x \bmod \langle \mathcal{V}_+ \rangle) \otimes 1 + 1 \otimes x, \quad \gamma(y) = y \otimes 1 + 1 \otimes y.$$

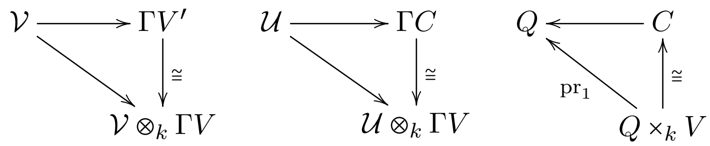
Choose an isomorphism of k-vector spaces $\mathcal{V}_1 \oplus (\Gamma V)_1 \cong (\Gamma V')_1$ such that

$$\begin{array}{ccccc} \mathcal{V}_1 & \longrightarrow & (\Gamma V')_1 & \longrightarrow & (\Gamma V)_1 \\ & \searrow & \downarrow{\scriptstyle \cong} & \nearrow & \\ & & \mathcal{V}_1 \oplus (\Gamma V)_1 & & \end{array}$$

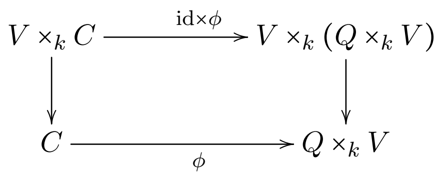
commutes. This induces an isomorphism of k-algebras

$$\mathcal{V} \otimes_k \Gamma V \cong \mathrm{Sym}_k(\mathcal{V}_1 \oplus (\Gamma V)_1) \cong \mathrm{Sym}_k((\Gamma V')_1) = \Gamma V'.$$

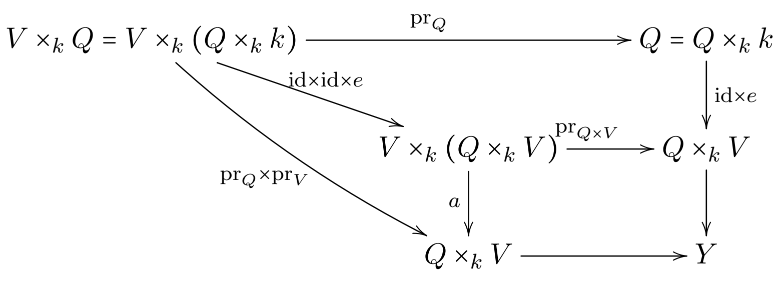
By claim 1 in the proof of theorem (2.5.10) the set $I \cap \mathcal{V}$ generates the ideal I of $\Gamma V'$. Thus the last isomorphism induces an isomorphism of k-algebras $\mathcal{U} \otimes_k \Gamma V \cong \Gamma C$, an isomorphism of cones $Q \times_k V \cong C$ and commutative diagrams



This defines ϕ and shows b). The commutativity of the first two diagrams yields the commutativity of the diagram



where the vertical morphisms are the actions on C, $Q \times_k V$ as defined above, which shows a). Let a denote the V-action $V \times_k (Q \times_k V) \to Q \times_k V$. For c) it is enough to show that $\mathrm{pr}_1 : Q \times_k V \to Q$ is the coequalizer of the pair $(\mathrm{pr}_{Q\times V}, a)$, i.e. one has $\mathrm{pr}_1 \circ \mathrm{pr}_{Q\times V} = \mathrm{pr}_1 \circ a$ and pr_1 is universal with this property. The last equality holds since both sides are the projection on Q. Let $V \times_k (Q \times_k V) \to Y$ be a morphism of k-schemes such that the compositions with $\mathrm{pr}_{Q\times V}$ and a are equal. Then we have a commutative diagram



where e is the neutral element morphism $k \to V$. The composition of the two right vertical morphisms gives us a morphism $Q \to Y$. The commutativity of the diagram yields that $Q \times_k V \to Y$ is the composition $\mathrm{pr}_{\mathrm{Q}} \circ (Q \to Y)$. On the other hand every morphism $Q \to Y$ with $\mathrm{pr}_Q \circ (Q \to Y) = V \times_k Q \to Y$ is unique, since by the commutativity of the last diagram it is the composition $Q \to Q \times_k V \to Y$. This shows c). □

Remark (2.5.19). *a) If we replace V' by the vector group V_C then we get the same morphism $C \to Q$.*

b) If we have $V' = C$ then $\mathcal{U}$ is the ring of invariants of (V, V') and Q is a vector group.

c) For every k-Scheme X the map $C(X) \to Q(X)$ is the quotient of the set $C(X)$ by the group $V(X)$, i.e. $Q(X) = C(X)/V(X)$.

Definition (2.5.20). *We call the tuple $(Q, C \to Q)$ form proposition (2.5.18)* ***the quotient of C by V*** *and we write C/V for Q.*

Remark (2.5.21). *a) Let x be a point of a scheme X and let D be an at x permissible closed subscheme of X. Assume that X is a closed subscheme of a regular scheme Z. Then $C_{Z,x}$, $C_{D,x}$ are vector groups over $\kappa(x)$. For the ideal $I := \ker(\mathcal{O}_{Z,x} \to \mathcal{O}_{D,x})$ the inclusions $I^i \subseteq \mathfrak{m}_{Z,x}^i$, $i \in \mathbb{N}$, define a morphism of graded $\kappa(x)$-algebras*

$$\alpha : \Gamma(C_{X,D,x}) = \mathrm{gr}_I \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{D,x}} \kappa(x) \to \mathrm{gr}\mathcal{O}_{X,x}$$

with image $\Gamma(C_{X,x}/C_{D,x})$. By [Hi1], pages 194,195, the morphism α is injective. Thus we get isomorphisms of cones over $\kappa(x)$

$$C_{X,D,x} \cong C_{X,x}/C_{D,x}\,, \qquad C_{X,x} \cong C_{X,D,x} \times_{\kappa(x)} C_{D,x}\,.$$

b) Assume the situation of proposition (2.5.18). The epimorphism of graded k-algebras $\mathcal{V} \to \mathcal{U}$ makes C/V to a subcone of V'/V. Assume $C = G$ is an additive subgroup of V'. Then G/V is an additive subgroup of V'/V, because for all k-schemes X the set $G(X)/V(X)$ is a subgroup of $V'(X)/V(X)$. Since $G \to G/V \times_k V$ is an isomorphism of cones, by remark (2.5.2)e) $G \to G/V \times_k V$ is an isomorphism of groups. In particular the composition $G \to G/V \times_k V \to G/V$ is a morphism of groups. If G is an additive subgroup of an additive group G', then G/V is an additive subgroup of G'/V. On the other hand a subgroup $\overline{G}$ of G'/V defines a subgroup $\phi^{-1}(\overline{G})$ of G' (for the quotient morphism $\phi : G' \to G'/V$), see example (2.4.3)g).

c) In the situation of proposition (2.5.18), for a subgroup G of V' with $V \subseteq G$, one has

$$G + C \subseteq C \Leftrightarrow G/V + C/V \subseteq C/V,$$

since for a k-scheme X one has

$$G(X) + C(X) \subseteq C(X) \quad \Leftrightarrow \quad G(X)/V(X) + C(X)/V(X) \subseteq C(X)/V(X).$$

Lemma (2.5.22). *Let V be a subvector group of an additive group G'. Let ϕ denote the morphism of groups $G' \to G'/V$. The mapping $G \mapsto G/V$ is a bijection from the set of all additive subgroups of G', which have V as a subgroup, to the set of all additive subgroups of G/V. The inverse map is $\overline{G} \mapsto \phi^{-1}(\overline{G})$. Further G is a vector group if and only if G/V is a vector group.*

Proof. Let X be a k-scheme. The mapping $H \mapsto H/(V(X))$ is a bijection from the set of all subgroups H of $G'(X)$ with $V(X) \subseteq H$ to the set of all subgroups of $G(X)/V(X)$, with inverse map $\overline{H} \mapsto \phi(X)^{-1}(\overline{H})$. The first claim follows with the yoneda lemma. Let us show the second claim. If G/V is a vector group then G is isomorphic to the product $G/V \times_k V$ of vector groups and therefore it is a vector group. If G is a vector group, then by remark (2.5.19) G/V is a vector group. □

Lemma (2.5.23). *Let C be a subcone of a vector group V'. Let V be a subvector group of V' with $V + C \subseteq C$. Then V is an additive subgroup of $\mathrm{Dir}(C)$ and $\mathrm{Rid}(C)$ and we have equalities of subgroups of V'/V*

$$(\mathrm{Dir}(C))/V = \mathrm{Dir}(C/V), \qquad (\mathrm{Rid}(C))/V = \mathrm{Rid}(C/V).$$

Proof. Remark (2.5.21)c) + lemma (2.5.22). □

2.6 Blow-ups II

In this section we recall some results by Hironaka about permissible blow-ups $X' \to X$ of closed subschemes X of regular schemes. We will need this later. More precisely we have an estimation for the behavior of the dimension of the ridge (theorem (2.6.2)), a comparison of the the Hilbert-Samuel-functions at points of X, X' and at the fibers of $X' \to X$ (proposition (2.6.6)), and a statement for Hironaka schemes associated to points of X' (lemma (2.6.7)).
For the whole section we assume the following situation. Let

$$\begin{array}{ccc} X' & \overset{i'}{\hookrightarrow} & Z' \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\ X & \underset{i}{\hookrightarrow} & Z \end{array}$$

be a commutative diagram of schemes, where X is a locally noetherian scheme, i is a closed immersion, π_X resp. π_Z is the blow-up of X resp. Z in a center $D \subseteq X \subseteq Z$ and i' is the induced closed immersion. Let $x' \in X' \subseteq Z'$ be a point such that its image $x \in X \subseteq Z$ lies in D. Assume that D is regular and that X is normally flat along D at x. Write $d = \operatorname{trdeg}(\kappa(x')/\kappa(x))$. Choose a regular parameter v of $\mathcal{O}_{Z,x} =: R$ with $v \in \mathfrak{p} := \ker(\mathcal{O}_{Z,x} \to \mathcal{O}_{D,x})$ and a prime ideal q of $R[\mathfrak{p}/v]$ which contains the maximal ideal $\mathfrak{m}$ of R such that $(R[\mathfrak{p}/v])_q =: R'$ and $\mathcal{O}_{Z',x'}$ are isomorphic as $\mathcal{O}_{Z,x} = R$-algebras (see lemma (2.1.9)). Write

$$J := \ker(R \to \mathcal{O}_{X,x}), \quad \overline{R} := \mathcal{O}_{X,x}, \quad \overline{R}' := \mathcal{O}_{X',x'}, \quad R'_E := \mathcal{O}_{\pi_Z^{-1}(\{x\}),x'}, \quad \overline{R}'_E := \mathcal{O}_{\pi_X^{-1}(\{x\}),x'} \; .$$

For the proof of theorem (2.6.2) below we need the following remark.

Remark (2.6.1). *There is a numerical character $\nu^*_x(X,Z)$, introduced in [Hi1], which measures the singularities $x \in X$. The Hilbert-Samuel-function can be seen as an alternative invariant to $\nu^*_x(X,Z)$. By [Hi4], Theorem III, one has $\nu^*_{x'}(X',Z') = \nu^*_x(X,Z)$ if and only if x' is near to x.*

Theorem (2.6.2). *If x' is near to x, one has the inequality*

$$\dim \operatorname{Rid}_{X',x'} + d \le \dim \operatorname{Rid}_{X,x}.$$

Proof. By lemma (2.1.8) we may assume $X = \operatorname{Spec}(\overline{R})$. Let $K|\kappa(x')$ be a field extension of $\kappa(x')$ by a perfect field K. Define the cones resp. vector groups over K

$$V_K := C_{Z,x} \times_{\kappa(x)} K, \quad C_K := C_{X,x} \times_{\kappa(x)} K, \quad V'_K := C_{Z',x'} \times_{\kappa(x')} K, \quad C'_K := C_{X',x'} \times_{\kappa(x')} K.$$

Since x' is near to x, we have $\nu^*_{x'}(X',Z') = \nu^*_x(X,Z)$. Then by [Hi2], theorem (1,A), we get the inequality

$$\tau^{(t)}_x(X/Z)_K \le \tau^{(t)}_{x'}(X'/Z')_K$$

where, translated in our notation, $\tau^{(t)}_x(X/Z)_K = \dim V_K - \dim \operatorname{Dir}(C_K)$ and $\tau^{(t)}_{x'}(X'/Z')_K = \dim V'_K - \dim \operatorname{Dir}(C'_K)$. With remark (2.5.14) we get

$$\dim V_K - \dim V'_K \le \dim \operatorname{Rid}_{X,x} - \dim \operatorname{Rid}_{X',x'}$$

Since Z' is not empty we have $D \neq Z$. Then by proposition (2.1.5) $Z' \to Z$ is a birational morphism locally of finite type of noetherian integral schemes. Since Z is regular, it is universally catenary (see definition (4.1.1) and see [Li], corollary 8.2.16). Then by [Li], theorem 8.2.5, we have

$$d = \dim \mathcal{O}_{Z,x} - \dim \mathcal{O}_{Z',x'} = \dim V_K - \dim V'_K.$$

With the last inequality we get the claim. □

Remark (2.6.3). *The assumption that X is embedded into a regular scheme is not necessary, since with lemma (3.3.2) below the non-embedded case can be reduced the embedded case.*

Definition (2.6.4). *Define the map*

$$i : \mathbb{N} \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N} : \quad m \to i^{(m)}_{X,x} = (H^{(m)}_{X,x}, \dim \mathrm{Rid}_{X,x} + m).$$

Let $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ have the lexicographical order, i.e. for $\nu, \nu' \in \mathbb{N}^{\mathbb{N}}$ and $r, r' \in \mathbb{N}$ one has

$$(\nu, r) \leq (\nu', r') \quad \Leftrightarrow \quad \nu = \nu' \text{ and } r \leq r' \quad \text{or} \quad \nu < \nu' \text{ in } \mathbb{N}^{\mathbb{N}}.$$

The theorems (2.2.6) and (2.6.2) yield

Corollary (2.6.5). $i^{(d)}_{X',x'} \leq i^{(0)}_{X,x}$.

Proposition (2.6.6). *For $s := \dim \mathcal{O}_{D,x}$ one has* $H^{(1+d)}_{X',x'} \leq H^{(2+d+s)}_{\pi_X^{-1}(\{x\}),x'} \leq H^{(1+s)}_{C_{X,D,x}} = H^{(1)}_{X,x}$.

Proof. [Hi4], inequality (4.1). □

In section 3.6 we will need the following result about point blow-ups. So assume additionally that x is a closed point and that $D = \{x\}$, in particular $\mathfrak{p} = \mathfrak{m}$ is the maximal ideal of R. The grading of $\mathrm{gr}R$ defines a vector group structure on $V := C_{Z,x} = \mathrm{Spec}(\mathrm{gr}R)$ over $\kappa(x)$. By remark (2.1.7) we have $\pi_Z^{-1}(\{x\}) = \mathrm{Proj}(\mathrm{gr}R) = \mathbb{P}(V) =: \mathbb{P}$. Identify x' with its image in $\mathbb{P}$. Let $B_{\mathbb{P},x'}$ denote the Hironaka scheme at $x' \in \mathbb{P}$. Let $\mathcal{U}_{\mathbb{P},x'} \subseteq \mathrm{gr}R$ denote the ring of invariants of $(B_{\mathbb{P},x'}, V)$.

Lemma (2.6.7). *Assume that x' is near to x.*

a) There is a standard basis $(f_1, \dots, f_m)$ of J with $\mathrm{in}(f_1), \dots, \mathrm{in}(f_m) \in \mathcal{U}_{\mathbb{P},x'}$.

For $j \in \{1, \dots, m\}$ let ψ_j denote the element $\mathrm{in} f_j \cdot (\mathrm{inv})^{-\nu(f_j)}$ of $\mathrm{gr}R_{(\mathrm{inv})} \subseteq R'_E$ (by remark (2.1.7) R'_E is a localization of $\mathrm{gr}R_{(\mathrm{inv})}$ by a prime ideal).

b) The tuple $(\psi_1, \dots, \psi_m)$ is a standard basis of $\ker(R'_E \to \overline{R}'_E)$ and for each $j \in \{1, \dots, m\}$ one has $\nu(\psi_j) = \nu(f_j)$.

Proof. By lemma (2.1.8) we may assume $X = \mathrm{Spec}(\mathcal{O}_{X,x})$ and $Z = \mathrm{Spec}(\mathcal{O}_{Z,x})$. The point x' lies in the open affine subscheme $\mathrm{Spec}(R[\mathfrak{m}/v])$ of Z'. By remark (2.1.3) we have $\mathrm{Spec}(R[\mathfrak{m}/v]) \times_X D = \mathrm{Spec}\big(R[\mathfrak{m}/v]/(v \cdot R[\mathfrak{m}/v])\big)$ and therefore we have $R'_E = R'/vR'$. Similarly one gets $\overline{R}'_E = \overline{R}'/v\overline{R}'$. By [Hi4], Theorem III, we have $\nu^*_{x'}(X', Z') = \nu^*_x(X, Z)$. Then by [Hi4], proposition 21,

1) there is a standard basis $(f_1, \dots, f_m)$ of J with $\mathrm{in}(f_1), \dots, \mathrm{in}(f_m) \in U_{g,x'}$,

where $U_{g,x'} \subseteq \mathrm{gr}R$ is the ring of invariants of $(B_{g,x'}, C_{Z,x})$ for a certain additive subgroup $B_{g,x'}$ of $C_{Z,x}$. In the case $D = \{x\}$ one has $B_{g,x'} = B_{\mathbb{P},x'}$ and therefore $U_{g,x'} = \mathcal{U}_{\mathbb{P},x'}$. This shows a). Further by that proposition and the remark to the proof of this proposition on the same page

2) there is a standard basis $(g_1'', \dots, g_m'')$ of the strict transform $J' \subseteq R'$ of J such that for each $j \in \{1, \dots, m\}$ one has $(g_j'' \bmod \langle v \rangle) = \psi_j$ under the identification $R'/vR' = R_E'$ and one has $\nu(g_j'') = \nu(f_j) = \nu(\psi_j)$,

3) the tuple $(\mathrm{in}\psi_1, \dots, \mathrm{in}\psi_m)$ in $\mathrm{gr}R_E'$ of the initial forms of $\psi_1, \dots, \psi_m$ is a standard basis of the ideal it generates in $\mathrm{gr}R_E'$,

where 3) means that $(\psi_1, \dots, \psi_m)$ of R_E' satisfies conditions $ii), iii)$ in the definition (2.1.10)f). Since x' is near to x, by proposition (2.6.6) one has

$$H^{(1+d)}(\mathrm{gr}\overline{R}') = H^{(1+d)}_{X',x'} = H^{(2+d)}_{\pi_X^{-1}(\{x\}),x'} = H^{(2+d)}(\mathrm{gr}\overline{R}_E').$$

Then by proposition (2.2.4) one has $\mathrm{gr}\overline{R}_E' = \mathrm{gr}\overline{R}'/\langle \mathrm{in}v \rangle$. We have $\overline{R}' = R'/J'$. Since $(g_1'', \dots, g_m'')$ is a standard basis of J' one has $\mathrm{gr}\overline{R}' = \mathrm{gr}R/\langle \mathrm{in}g_1'', \dots, \mathrm{in}g_m'' \rangle$. Let j be an element of $\{1, \dots, m\}$, set $d_j := \nu(f_j)$ and let $\mathfrak{m}_{R'}$ denote the maximal ideal of R'. In the commutative diagram

$$\begin{array}{ccc} {\mathfrak{m}_{R'}}^{d_j} & \longrightarrow & {\mathfrak{m}_{R'}}^{d_j}/{\mathfrak{m}_{R'}}^{d_j+1} \\ \downarrow & & \downarrow \\ {\mathfrak{m}_{R'}}^{d_j} + \langle v \rangle/\langle v \rangle & \longrightarrow & {\mathfrak{m}_{R'}}^{d_j} + \langle v \rangle/{\mathfrak{m}_{R'}}^{d_j+1} + \langle v \rangle \end{array}$$

we have the mappings

$$\begin{array}{ccc} g_j'' & \longmapsto & \mathrm{in}g_j'' \\ \downarrow & & \downarrow \\ \psi_j & \longmapsto & \mathrm{in}\psi_j = \mathrm{in}g_j'' \bmod \langle \mathrm{in}v \rangle \end{array}$$

Note that the images of g_j'' and ψ_j under the horizontal maps are not zero because one has $\nu(g_j'') = d_j = \nu(\psi_j)$. Then one has

$$\langle \mathrm{in}g_1'', \dots, \mathrm{in}g_m'' \rangle/\langle \mathrm{in}v \rangle = \langle \mathrm{in}\psi_1, \dots, \mathrm{in}\psi_m \rangle$$

which implies $\mathrm{gr}\overline{R}_E' = \mathrm{gr}R_E'/\langle \mathrm{in}\psi_1, \dots, \mathrm{in}\psi_m \rangle$ and $\langle \mathrm{in}\psi_1, \dots, \mathrm{in}\psi_m \rangle = \mathrm{In}J_E'$ for the ideal $J_E' := \ker(R_E' \to \overline{R}_E')$. Then with 3) the tuple $(\psi_1, \dots, \psi_m)$ is a standard basis of J_E'. □

Chapter 3

The main theorem

In this chapter we formulate the main theorem and prove it. First we show that the main theorem follows from a fact about graded algebras over a field (theorem (3.2.6)). This is done in three reduction steps in the sections 3.3, 3.4 and 3.6. For the last reduction we need the explicit description (up to isomorphism) of Hironaka schemes of dimension at most five from Oda's characterization (section 3.5). We prove theorem (3.2.6) in section 3.8. For this we use Giraud bases to calculate ridges. More precisely our Giraud bases are reduced Gröbner bases (see section 3.7).

3.1 Introduction

Let X be a locally noetherian scheme, let $\pi : X' \to X$ be the blow-up of X in a closed subscheme D and let x be a point of D.

Definition (3.1.1). *A point x' of X' is* **Directrix-*near to x*** *or* **Dir-*near to x*** *if x' is near to x and x' lies in the closed subscheme $\mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$ of $\mathbb{P}(C_{X,D,x}) = \pi^{-1}(\{x\})$ (see remark (2.5.4)).*

We recall the following theorem and its proof, see [CJS], Theorem 2.14.

Theorem (3.1.2). *If $\dim X \leq 2$, X is a closed subscheme of a regular scheme Z and D is permissible at x then a point of X' is near to x if and only if it is* Dir-*near to x.*

Proof. Let x' be a point of X' near to x. By remark (2.5.21) we have an isomorphism $C_{X,D,x} \cong C_{X,x}/C_{D,x}$ of cones over $\kappa(x)$. We have $\pi^{-1}(\{x\}) = \mathbb{P}(C_{X,D,x}) \subseteq \mathbb{P}(C_{Z,D,x}) =: \mathbb{P}$. Identify x' with its image in $\mathbb{P}$. Let $\mathcal{B}_{\mathbb{P},x'}$ be the Hironaka scheme at x'. We have $x' \in \mathbb{P}(\mathcal{B}_{\mathbb{P},x'})$ by remark (2.5.8). There is an additive subgroup scheme $B_{g,x'}$ of $C_{Z,x}$ (defined in [Hi4], §2) with $B_{g,x'} \supseteq C_{D,x}$ and $B_{g,x'}/C_{D,x} \cong \mathcal{B}_{\mathbb{P},x'}$. By theorem IV from [Hi4] one has $B_{g,x'} + C_{X,x} \subseteq C_{X,x}$, which implies $\mathcal{B}_{\mathbb{P},x'} + C_{X,D,x} \subseteq C_{X,D,x}$ (see remark (2.5.21)). This implies $\mathcal{B}_{\mathbb{P},x'} \subseteq C_{X,D,x}$ and $\dim \mathcal{B}_{\mathbb{P},x'} \leq \dim C_{X,D,x} \leq \dim X \leq 2$ where the second inequality holds by remark (2.1.7). Then by [Hi3] $\mathcal{B}_{\mathbb{P},x'}$ is a vector group. By definition of the directrix this yields $\mathcal{B}_{\mathbb{P},x'} \subseteq \mathrm{Dir}(C_{X,D,x}) \subseteq C_{X,D,x}$ which implies $x' \in \mathbb{P}(\mathcal{B}_{\mathbb{P},x'}) \subseteq \mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$. □

Remark (3.1.3). *a) The proof of theorem (3.1.2) shows that, if we omit the assumption* $\dim X \le 2$*, the theorem still holds, if* $\mathcal{B}_{\mathbb{P},x'}$ *is a vector group. This is satisfied, if* $\kappa(x)$ *is a perfect field (see remark (2.5.8)) or if* $\dim X \le 2\cdot\mathrm{char}(\kappa(x)) - 2$ *(see [Mi]).*

b) Further we see that the inclusion $\mathcal{B}_{\mathbb{P},x'} + C_{X,D,x} \subseteq C_{X,D,x}$ *implies* $\mathcal{B}_{\mathbb{P},x'} \subseteq \mathrm{Rid}(C_{X,D,x})$*, by the definition of the ridge. In particular we get* $x' \in \mathbb{P}(\mathcal{B}_{\mathbb{P},x'}) \subseteq \mathbb{P}(\mathrm{Rid}(C_{X,D,x}))$*. For this conclusion we do not need the assumption* $\dim X \le 2$*.*

The theorem does not hold for higher dimension of X. For $\dim X = 3$, $\mathrm{char}\,\kappa(x) = 2$ we give a negative example below. Hironaka showed in [Hi3] that up to isomorphism there is a unique Hironaka scheme $\mathcal{B}$ of dimension three which is not a vector group (cf. also the Hironaka scheme of type 3 in theorem (3.5.5)). In the following example the underlying scheme of $\mathcal{B}$ is X.

Example (3.1.4). Let X be the closed subscheme $V(\tau)$ of $\mathrm{Spec}(k[y_1,y_2,y_3,y_4]) = \mathbb{A}_k^4$ where k is a field of characteristic two and where τ is the polynomial $\tau = y_1^2 + a_3y_2^2 + a_2y_3^2 + a_2a_3y_4^2$ for coefficients $a_2, a_3 \in k$ with $[k^2(a_2,a_3) : k^2] = 4$. Let x denote the closed point $\langle y_1,\dots,y_4\rangle$ of X and let D denote the closed subscheme $D = \{x\}$ of X. We have $C_{X,D,x} = \mathrm{Spec}(G)$ for the $\kappa(x) = k$-algebra $G := k[y_1,\dots,y_4]/\langle\tau\rangle = \mathrm{gr}\mathcal{O}_{X,x}$. We show

a) that $\mathrm{Dir}(C_{X,D,x})$ is the closed point $V(\langle y_1,\dots,y_4\rangle)$ of $\mathbb{A}_k^4$ and

b) that there is a point $x' \in X' = \mathrm{Bl}_D X$ near to x.

Then we have $\mathbb{P}(\mathrm{Dir}(C_{X,D,x})) = \mathrm{Proj}(k) = \varnothing$, in particular x' is not Dir-near to x.

a) Assume the contrary. Then we have $\dim \mathrm{Dir}(C_{X,D,x}) \ge 1$. Then there is a subvector space $W \subseteq k[y_1,\dots,y_4]_1$ with $\dim W \le 3$ and with $\tau \in k[W]$. Let b_1,b_2,b_3 be k-linearly independent elements of $k[y_1,\dots,y_4]_1$ with $k[b_1,b_2,b_3]_1 \supseteq W$. Choose some $b_4 \in \{y_1,\dots,y_4\}$ such that b_1,b_2,b_3,b_4 is a k-basis of $k[y_1,\dots,y_4]_1$. Write $y_i = \sum_{j=1}^4 \lambda_{ij}b_j$ for $\lambda_{ij} \in k$, $i,j \in \{1,\dots,4\}$. Then we have $\tau = w + \lambda b_4^2$ for $\lambda := \lambda_{14}^2 + a_3\lambda_{24}^2 + a_2\lambda_{34}^2 + a_2a_3\lambda_{44}^2$ and for a suitable $w \in k[b_1,b_2,b_3]$. By the assumption on a_2,a_3 we have $\lambda \ne 0$. Then $\lambda b_4^2 \notin k[b_1,b_2,b_3]$ implies $\tau \notin k[b_1,b_2,b_3] \supseteq k[W]$, which is a contradiction. This shows a).

b) Let U denote the open subscheme $D_+(y_4) = \mathrm{Spec}\big((k[y_1,\dots,y_4]/\langle\tau\rangle)[y_1',\dots,y_4']\big)$, for $y_i' := y_i/y_4$, of X' (see remark (2.1.3)). The inclusion $k[y_1',y_2',y_3',y_4] \subseteq k[y_1,\dots,y_4][y_1',\dots,y_4']$ is an isomorphism of k-algebras. It induces an isomorphism of k-schemes

$$\mathrm{Spec}(k[y_1',y_2',y_3',y_4]/\langle\tau'\rangle) \cong U \quad \text{for} \quad \tau' := {y_1'}^2 + a_3{y_2'}^2 + a_2{y_3'}^2 + a_2a_3.$$

Consider the point x' of U with

$$x' = \langle\xi_1,\xi_2,\xi_3,\xi_4\rangle, \quad \text{for} \quad \xi_1 = y_1' + y_2'y_3', \quad \xi_2 = {y_2'}^2 + a_2, \quad \xi_3 = {y_3'}^2 + a_3, \quad \xi_4 = y_4.$$

Then we have $x' \in \pi^{-1}(D) = \pi^{-1}(\{x\})$ by remark (2.1.3). The point x' has the residue field $\kappa(x') = k(\sqrt{a_2},\sqrt{a_3})$. The scheme U is a closed subscheme of $\mathrm{Spec}(k[y_1',y_2',y_3',y_4]) =: V$. Identify $\tau',\xi_1,\dots,\xi_4$ with their image in $\mathcal{O}_{V,x'}$. We have $\tau' = \xi_1^2 + \xi_2\xi_3$. The elements $\mathcal{X}_1 := \mathrm{in}\xi_1,\dots,\mathcal{X}_4 := \mathrm{in}\xi_4$ are $\kappa(x')$-algebraically independent generators of the $\kappa(x')$-algebra $\mathrm{gr}\mathcal{O}_{V,x'}$ and we have $\mathrm{in}\tau_1' = \mathcal{X}_1^2 + \mathcal{X}_2\mathcal{X}_3$. Then we have $\mathrm{gr}\mathcal{O}_{X,x} = k[y_1,\dots,y_4]/\langle\tau\rangle$ with $\deg\tau = 2$ and $\mathrm{gr}\mathcal{O}_{X',x'} = \kappa(x')[\mathcal{X}_1,\dots,\mathcal{X}_4]/\langle\mathrm{in}\tau'\rangle$ with $\deg\tau' = 2$. Thus by lemma (2.2.2) we have $H_{X,x}^{(0)} = H_{X',x'}^{(0)}$. Further we have $\mathrm{trdeg}(\kappa(x')/\kappa(x)) = \mathrm{trdeg}(k[\sqrt{a_2},\sqrt{a_3}]/k) = 0$. This shows b).

In the example above we see the following improvement of the singularity x'. Since $\tau \in k[y_1, \dots, y_4]$ is an homogeneous additive element we have $\mathrm{Rid}_{X,x} = \mathrm{Rid}(C_{X,x}) = C_{X,x}$ which has dimension three. On the other hand $\tau' \in \kappa(x')[\mathcal{X}_1, \dots, \mathcal{X}_4]$ is not an additive element which implies $\mathrm{Rid}_{X',x'} \subsetneq C_{X',x'}$ and $\dim \mathrm{Rid}_{X',x'} < \dim C_{X',x'} = 3$.
Our main theorem (3.2.1) below shows that for $\dim X \le 5$ and for a point x' of X' near but not Dir-near to x one always have the inequality $\dim \mathrm{Rid}_{X',x'} + d < \dim \mathrm{Rid}_{X,x}$ for $d = \mathrm{trdeg}(\kappa(x')/\kappa(x))$. Unfortunately our main theorem holds not for arbitrary dimensions of X. For $\dim X = 7$ there is the following negative example.

Example (3.1.5). Let X be the closed subscheme $V(\tau)$ of $\mathrm{Spec}(k[x_1, \dots, x_4, y_1, \dots, y_4]) = \mathbb{A}^8_k$ for a field k of characteristic two where τ is the polynomial

$$\tau = y_4^2 + a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + a_2 a_3 x_1^2 + a_1 a_3 x_2^2 + a_1 a_2 x_3^2 + a_1 a_2 a_3 x_4^2$$

for coefficients $a_1, a_2, a_3 \in k$ with $[k^2(a_1, a_2, a_3) : k^2] = 8$. Let x denote the closed point $\langle x_1, \dots, x_4, y_1, \dots, y_4 \rangle$ of X and let D denote the closed subscheme $\{x\}$ of X. We have $C_{X,D,x} = \mathrm{Spec}(G)$ for the graded k-algebra $G := k[x_1, \dots, x_4, y_1, \dots, y_4]/\langle \tau \rangle = \mathrm{gr}\mathcal{O}_{X,x}$. As in the example above one shows

$$\mathbb{P}(\mathrm{Dir}(C_{X,D,x})) = \mathrm{Proj}(k) = \varnothing.$$

Define the point $x' = \langle \mathcal{X}_1, \dots, \mathcal{X}_4, \mathcal{Y}_1, \dots, \mathcal{Y}_4 \rangle$ for

$$\mathcal{X}_1 = {x'_1}^2 + a_1, \quad \mathcal{X}_2^2 = {x'_2}^2 + a_2, \quad \mathcal{X}_3 = {x'_3}^2 + a_3, \quad \mathcal{X}_4 = x_4,$$

$$\mathcal{Y}_1 = y'_1 + x'_2 x'_3, \quad \mathcal{Y}_2 = y'_2 + x'_1 x'_3, \quad \mathcal{Y}_3 = y'_3 + x'_1 x'_2, \quad \mathcal{Y}_4 = y'_4 + x'_1 x'_2 x'_3$$

of the open subscheme $D_+(x_4) = \mathrm{Spec}(k[x'_1, x'_2, x'_3, x_4, y'_1, y'_2, y'_3, y'_4]/\langle \tau' \rangle) =: U$ of X' for

$$\tau' = {y'_4}^2 + a_1 {y'_1}^2 + a_2 {y'_2}^2 + a_3 {y'_3}^2 + a_2 a_3 {x'_1}^2 + a_1 a_3 {x'_2}^2 + a_1 a_2 {x'_3}^2 + a_1 a_2 a_3.$$

Then we have $x' \in \pi^{-1}(\{x\})$ and $\kappa(x') = k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$. The scheme U is a closed subscheme of $\mathrm{Spec}(k[x'_1, x'_2, x'_3, x_4, y'_1, y'_2, y'_3, y'_4]) =: V$. Identify $\tau', \mathcal{X}_1, \dots, \mathcal{X}_4, \mathcal{Y}_1, \dots, \mathcal{Y}_4$ with their images in $\mathcal{O}_{V,x'}$. We have $\tau' = \mathcal{Y}_4^2 + a_1 \mathcal{Y}_1^2 + a_2 \mathcal{Y}_2^2 + a_3 \mathcal{Y}_3^2 + \mathcal{X}_1 \mathcal{X}_2 \mathcal{X}_3$. The element

$$(\mathrm{in}\mathcal{Y}_4 + \sqrt{a_1}\mathrm{in}\mathcal{Y}_1 + \sqrt{a_2}\mathrm{in}\mathcal{Y}_2 + \sqrt{a_3}\mathrm{in}\mathcal{Y}_3)^2 \in \mathrm{gr}\mathcal{O}_{V,x'}$$

is the initial form of τ'. As in the example above we see that x' is near to x. Since $\mathrm{in}\tau'$ is an homogeneous additive element of $\mathrm{gr}\mathcal{O}_{V,x'}$ we have $\dim \mathrm{Rid}_{X',x'} + d = \dim \mathrm{Rid}_{X,x}$ (for $d = \mathrm{trdeg}(\kappa(x')/\kappa(x)) = 0$).

So we have a theorem for $\dim X \le 5$ and a negative example for $\dim X \ge 7$. What about dimension six? We have neither a positive nor a negative answer. To prove our main result we use Oda's characterization of Hironaka schemes $\mathcal{B}$ with $\dim \mathcal{B} \le 5$, see theorem (3.5.5). If one wants to verify our main theorem for $\dim X = 6$ one has to handle six-dimensional Hironaka schemes. The author could not manage this.

3.2 The main theorem and the strategy of the proof

Our main theorem is

Theorem (3.2.1). *Let X be a locally noetherian scheme with* $\dim X \le 5$*, let $x \in X$ be a point, and $D \subseteq X$ be a closed subscheme with $x \in D$ such that D is permissible at $x \in D$. Let $\pi : X' \to X$ be the blow-up of X in D and let x' be a point of X' near but not* Dir*-near to x. Then we have*

$$\dim \mathrm{Rid}_{X',x'} + \mathrm{trdeg}(\kappa(x')/\kappa(x)) < \dim \mathrm{Rid}_{X,x}.$$

Definition (3.2.2). *Let $\pi : X' \to X$ be a permissible blow-up of a locally noetherian scheme X. For points $x' \in X'$ and $x \in X$ we say **x' is i-near to x** if $\pi(x') = x$ and $i^{(d)}_{X',x'} = i^{(0)}_{X,x}$ for $d := \mathrm{trdeg}(\kappa(x')/\kappa(x))$.*

Remark (3.2.3). *a) Note that $i^{(d)}_{X',x'} = i^{(0)}_{X,x}$ implies $i^{(d+m)}_{X',x'} = i^{(m)}_{X,x}$ for all $m \in \mathbb{N}$.*

b) With the notion of i-near points we can formulate our main theorem as follows.

Let $\pi : X' \to X$ be a blow-up of a locally noetherian scheme X with $\dim X \le 5$ in a center D and let x' be a point of X' i-near to a point $x \in X$ with $x \in D$ such that D is permissible at x. Then x' is Dir*-near to x.*

In the section 3.3, "'Reduction to the embedded local case"' we prove that the main theorem holds if the following theorem holds.

Theorem (3.2.4). *Let Z be the spectrum of a regular local ring, let X be closed subscheme of Z with $\dim X \le 5$, let $x \in X$ be the closed point and let $D \subseteq X$ be a closed subscheme such that D is permissible at $x \in D$. Let $\pi : X' \to X$ be the blow-up of X in D and let x' be a point of X' near but not* Dir*-near to x. Then we have*

$$\dim \mathrm{Rid}_{X',x'} + \mathrm{trdeg}(\kappa(x')/\kappa(x)) < \dim \mathrm{Rid}_{X,x}.$$

In the section 3.4, "'Reduction to cones"', we prove that theorem (3.2.4) follows form the following theorem.

Theorem (3.2.5). *Let C be a cone over a field k with $\dim C \le 5$ and let $\pi : C' \to C$ be the point blow up of C in the origin $0 =: x$ (see remark (2.5.2)a)). Let $x' \in C'$ be a point near to x not beeing* Dir*-near to x. Then we have*

$$\dim \mathrm{Rid}_{C',x'} + \mathrm{trdeg}(\kappa(x')/k) < \dim \mathrm{Rid}_{C,x}.$$

In the section 3.6, "'Reduction to algebra"', we prove that the theorem (3.2.5) holds if the following theorem holds.

Theorem (3.2.6). *Let $\mathcal{S} = \oplus_{n\in\mathbb{N}}\mathcal{S}_0$ be a symmetric algebra over a perfect field K of characteristic $p = 2$ or $= 3$. Assume $\dim_K \mathcal{S}_1 \ge 3$. For natural numbers $n \ge 1$, $m \ge 2$, $l \ge 0$ let $(x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_l)$ be a K-basis of $\mathcal{S}_1$. Let $h_1, \dots, h_n$ be homogeneous elements of $\mathcal{S}$ of degree p, not all zero. For $p = 2$ assume $h_1, \dots, h_n \in K[x_i x_j \,|\, 1 \le i < j \le m]$, for $p = 3$ assume $n = 1$ and $h_1 = x_1 \cdot x_2^2$. Let F be a finite subset of $K[y_1^p, \dots, y_n^p, z_1, \dots, z_l]$ of homogeneous* *elements of $\mathcal{S}$ such that $\mathcal{S}/\langle F\rangle$ has Krull dimension m. Let ψ denote the $K[z_1,\dots,z_l]$-algebra homomorphism*

$$\psi : K[y_1^p,\dots,y_n^p,z_1,\dots,z_l] \to K[y_1^p+h_1,\dots,y_n^p+h_n,z_1,\dots,z_l]$$

with $\psi(y_i^p) = y_i^p + h_i$ for $i = 1,\dots,n$. Let $\mathcal{U}_{\langle F\rangle} \subseteq \mathcal{S}$ resp. $\mathcal{U}_{\langle \psi F\rangle} \subseteq \mathcal{S}$ denote the ring of invariants of $(\langle F\rangle,\mathcal{S})$ resp. $(\langle \psi F\rangle,\mathcal{S})$ (see definition (2.5.15)). Then we have $\dim\mathcal{U}_{\langle F\rangle} + 2 \leq \dim\mathcal{U}_{\langle \psi F\rangle}$.

The key ingredient for the proof of the implication (3.2.6)⇒(3.2.5) is Oda's characterization of Hironaka schemes of dimension ≤ 5 from [Od] which will be discussed in the section 3.5.
We prove theorem (3.2.6) in the section 3.8. In the proof we use Gröbner bases, see section 3.7.

3.3 Reduction to the embedded local case

We show that for the proof of the main theorem (3.2.1) we may assume that X is embedded into a regular scheme. More precisely we show that the main theorem follows from theorem (3.2.4). The argument will use the following lemma.

Lemma (3.3.1). *Let $f : S_1 \to S$ be a morphism of schemes and let $s_1 \in S_1$, $s \in S$ be points with $f(s_1) = s$. We assume that f* ***quasi-equal at s_1****, i.e. the induced morphism of local rings*

$$A := \mathcal{O}_{S,s} \to \mathcal{O}_{S_1,s_1} =: A_1$$

is flat, and the image of the maximal ideal $\mathfrak{m}$ of A generates the maximal ideal of A_1 and the induced morphism of residue fields $\kappa := \kappa(s) \to \kappa(s_1)$ is an isomorphism. Let $f' : S_1' \to S'$ be a base change of f by a S-scheme S' and let s' be a point of S' over $s \in S$.

a) The morphism of graded κ-algebras $\mathrm{gr}A \to \mathrm{gr}A_1$ *is an isomorphism. In particular the κ-cones C_{S_1,s_1} and $C_{S,s}$ are isomorphic.*

b) There is a unique point s_1' of S_1' over $s_1 \in S_1$ and over $s' \in S'$.

c) The morphism f' is quasi-equal at s_1'.

d) Assume that S, S_1 are locally noetherian.

i) The rings A and A_1 have the same Krull dimension.

ii) A closed subscheme D of S with $s \in D$ is permissible at s if and only if $D \times_S S_1 =: D_1$ is permissible at s_1.

Proof. a) Let $\mathrm{gr}^n A$ denote the nth homogeneous part of $\mathrm{gr}A$ for $n \in \mathbb{N}$. The morphism $\mathrm{gr}^n A \to \mathrm{gr}^n A_1$ is the composition

$$\mathrm{gr}^n A = \mathfrak{m}^n \otimes_A \kappa(s) \cong \mathfrak{m}^n \otimes_A \kappa(s_1) \cong \mathfrak{m}^n \otimes_A A_1 \otimes_{A_1} \kappa(s_1) \cong (\mathfrak{m}A_1)^n \otimes_{A_1} \kappa(s_1) = \mathrm{gr}^n A_1.$$

b) The set-theoretical image of the morphism of schemes $s_1 \times_s s' \to S_1 \times_S S' = S_1'$ is the set of all points of S_1' with images s_1 in S_1 and s' in S'. The isomorphism $\kappa(s) \to \kappa(s_1)$ induces an isomorphism $s_1 \times_s s' \cong s'$ of schemes.

c) Let A' (resp. A'_1) denote the ring $\mathcal{O}_{S',s'}$ (resp. $\mathcal{O}_{S'_1,s'_1}$). Then the by f' induced morphism of local rings $A' \to A'_1$ is the composition

$$A' \xrightarrow{\alpha} A' \otimes_A A_1 \xrightarrow{\beta} A'_1$$

where α is the base change of $A \to A_1$ and β is the localization of $A' \otimes_A A_1$ by the prime ideal which corresponds to s'_1. Then α and β are flat. Let $\mathfrak{m}'$ resp. κ' denote the maximal ideal resp. residue field of A'. We have

$$(A' \otimes_A A_1)/(\mathfrak{m}' \otimes_A A_1) \cong A'/\mathfrak{m}' \otimes_A A_1 \cong \kappa' \otimes_A A_1 \cong \kappa' \otimes_\kappa \kappa \otimes_A A_1 \cong \kappa'.$$

Thus $\mathfrak{m}' \otimes_A A_1$ is a maximal ideal of $A' \otimes_A A_1$. Since $\beta(\alpha(\mathfrak{m}'))$ lies in the maximal ideal of A'_1 we get that β is the localization by $\mathfrak{m}' \otimes_A A_1$. Then the image of $\mathfrak{m}'$ in A'_1 generates the maximal ideal of A'_1 and the residue field of A'_1 is isomorphic to $(A' \otimes_A A_1)/(\mathfrak{m}' \otimes_A A_1) \cong \kappa'$. Thus f' is quasi-equal at s'_1.

d) i) Since $A \to A_1$ is flat we have $\dim A_1 = \dim A + \dim A_1 \otimes_A A/\mathfrak{m}$, see [Li], theorem 4.3.12. As $A_1 \otimes_A A/\mathfrak{m}$ is a field, we get $\dim A = \dim A_1$.

ii) First we show that D is regular at s if and only if D_1 is regular at s_1. As a base change of f the morphism $D_1 \to D$ is quasi-equal at s_1. Thus it is enough to show that S is regular at s if and only if S_1 is regular at s_1. This follows from a) and i).

Second we show that D is normally flat at s along S if and only if D_1 is normally flat at s_1 along S_1. Write

$$\overline{A} := \mathcal{O}_{D,s}, \quad I := \ker(A \to \overline{A}), \quad G := \oplus_{i \in \mathbb{N}} I^i/I^{i+1},$$

$$\overline{A}_1 := \overline{A} \otimes_A A_1, \quad I_1 := I \otimes_A A_1, \quad G_1 := G \otimes_A A_1.$$

Since $A \to A_1$ is flat we have $G_1 \cong \oplus_{i \in \mathbb{N}} I_1^i/I_1^{i+1}$. Then D (resp. D_1) is normally flat at s (resp. s_1) along S (resp. S_1) if and only if G (resp. G_1) is a flat over $\overline{A}$ (resp. $\overline{A}_1$).

- Assume that G is flat over $\overline{A}$. Then for every short exact sequence $\mathcal{E}'$ of $\overline{A}_1$-modules the sequence

$$\mathcal{E}' \otimes_{\overline{A}_1} G_1 \cong \mathcal{E}' \otimes_{\overline{A}_1} (A_1 \otimes_A \overline{A} \otimes_{\overline{A}} G) \cong \mathcal{E}' \otimes_{\overline{A}} G$$

is exact.

- Assume that G_1 is flat over $\overline{A}_1$. Then for every short exact sequence $\mathcal{E}$ of $\overline{A}$-modules we have

$$A_1 \otimes_A (\mathcal{E} \otimes_{\overline{A}} G) \cong (A_1 \otimes_A \mathcal{E}) \otimes_{\overline{A}} G \cong (A_1 \otimes_A \mathcal{E}) \otimes_{\overline{A}_1} G_1$$

which implies the exactness of $\mathcal{E} \otimes_{\overline{A}} G$, because, as a flat morphism of local rings, $A \to A_1$ is faithfully flat.

□

Lemma (3.3.2). *Let $\pi : X' \to X$ be a blow-up of a locally noetherian scheme X in a center D. Let x', x be points of X', X with $\pi(x') = x$. Let $\widehat{X}$ denote the spectrum of the completion of the local noetherian ring $\mathcal{O}_{X,x}$ and let $\widehat{x}$ be the closed point of $\widehat{X}$. Let $\widehat{\pi} : \widehat{X}' \to \widehat{X}$ denote the blow-up of $\widehat{X}$ in $\widehat{D} := D \times_X \widehat{X}$. Then* *a)* $\widehat{D}$ *is permissible at* $\widehat{x} \in \widehat{X}$ *if and only if* D *is permissible at* $x \in X$*,*

b) there is a unique point $\widehat{x}' \in \widehat{X}'$ *with image* x' *in* X' *and image* $\widehat{x}$ *in* $\widehat{X}$*,*

c) $\kappa(\widehat{x}') = \kappa(x')$, $\kappa(\widehat{x}) = \kappa(x)$*,*

d) $H_{\widehat{X}',\widehat{x}'} = H_{X',x'}$, $H_{\widehat{X},\widehat{x}} = H_{X,x}$*,*

e) $\widehat{x}'$ *is near to* $\widehat{x}$ *if and only if* x' *is near to* x*,*

f) $\widehat{x}' \in \mathbb{P}(\mathrm{Dir}(C_{\widehat{X},\widehat{D},\widehat{x}}))$ *if and only if* $x' \in \mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$*,*

g) $\dim \mathrm{Rid}_{\widehat{X}',\widehat{x}'} = \dim \mathrm{Rid}_{X',x'}$, $\dim \mathrm{Rid}_{\widehat{X},\widehat{x}} = \dim \mathrm{Rid}_{X,x}$ *and*

h) $\widehat{X}$ *is a closed subscheme of a regular scheme.*

Proof. Let κ, κ' denote the residue fields at x, x'. We have $\kappa(\widehat{x}) = \kappa$. The composition $f : \widehat{X} \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ maps $\widehat{x}$ maps to x and f is quasi-equal at $\widehat{x}$ (see lemma (3.3.1)). Since f is flat, $\widehat{\pi}$ is the base change of π with $\widehat{X}$. By lemma (3.3.1) we have

1. the by $\widehat{X} \to X$ induced morphism $C_{\widehat{X},\widehat{x}} \to C_{X,x}$ is an isomorphism of κ-cones,
2. claims a), b) and c),
3. the morphism $\widehat{D} \to D$ is quasi-equal at $\widehat{x}$ and the induced morphism $C_{\widehat{D},\widehat{x}} \to C_{D,x}$ is an isomorphism of κ-cones,
4. there is a unique point $\widehat{x}' \in \widehat{X}'$ which maps to x' and to $\widehat{x}$, and we have $\kappa(\widehat{x}) = \kappa'$,
5. the morphism $\widehat{X}' \to X'$ is quasi-equal at $\widehat{x}'$ and
6. the κ'-cones $C_{X',x'}$ and $C_{\widehat{X}',\widehat{x}'}$ are isomorphic.

Then we get claim d), we have isomorphisms of additive groups $\mathrm{Rid}_{X,x} \cong \mathrm{Rid}_{\widehat{X},\widehat{x}}$ resp. $\mathrm{Rid}_{X',x'} \cong \mathrm{Rid}_{\widehat{X}',\widehat{x}'}$ over κ resp. κ' and we have a commutative diagram

$$\begin{array}{ccc}
\widehat{X}' & \longrightarrow & X' \\
\uparrow & & \uparrow \\
\widehat{X}' \times_{\widehat{X}} \widehat{x} & \longrightarrow & X' \times_X x \\
\uparrow \cong & & \cong \uparrow \\
\mathbb{P}(C_{\widehat{X},\widehat{D},\widehat{x}}) & \xrightarrow{\cong} & \mathbb{P}(C_{X,D,x}) \\
\hookuparrow & & \hookuparrow \\
\mathbb{P}(\mathrm{Dir}(C_{\widehat{X},\widehat{D},\widehat{x}})) & \xrightarrow{\cong} & \mathbb{P}(\mathrm{Dir}(C_{X,D,x}))
\end{array}$$

This implies f). Claim d) implies e). By Cohen structure theorem, [Co], every complete noetherian local ring is a quotient of a regular local ring. Then there is a regular local ring R such that $\widehat{X}$ is a closed subscheme of $\mathrm{Spec} R$. □

Proof that theorem (3.2.4) implies theorem (3.2.1). Let $x' \in X' \overset{\pi}{\to} X \supseteq D \ni x$ as in theorem (3.2.1) and let $\widehat{x}' \in \widehat{X}' \overset{\widehat{\pi}}{\to} \widehat{X} \supseteq \widehat{D} \ni \widehat{x}$ as in lemma (3.3.2). Then by theorem (3.2.4) and lemma (3.3.2) we have $\dim \mathrm{Rid}_{\widehat{X}',\widehat{x}'} + \mathrm{trdeg}(\kappa(\widehat{x}')/\kappa(\widehat{x})) < \dim \mathrm{Rid}_{\widehat{X},\widehat{x}}$. Again by lemma (3.3.2) we get $\dim \mathrm{Rid}_{X',x'} + \mathrm{trdeg}(\kappa(x')/\kappa(x)) < \dim \mathrm{Rid}_{X,x}$. □

3.4 Reduction to cones

In this section we show that for the proof of the main theorem one can assume that X is the underlying scheme of a cone over a field. More precisely we prove that theorem (3.2.4) follows from theorem (3.2.5).

Lemma (3.4.1). *Let $\pi_C : C' \to C$ be the point blow-up in the origin x of a cone C over a field κ. Let x' be a point of $\pi_C^{-1}(\{x\}) =: E$ and let κ' denote its residue field. Then there is an isomorphism $C_{C',x'} \cong C_{E,x'} \times_{\kappa'} \mathbb{A}^1_{\kappa'}$ of cones over κ'.*

Proof. Let C be a subcone of a vector group V over κ. Let S denote the graded κ-algebra ΓV and let $I \subseteq S$ denote the homogeneous ideal $\ker(\Gamma V \to \Gamma C)$. The local ring $R := \mathcal{O}_{V,x}$ is the localization of S by the maximal ideal $S_+ = \oplus_{n\geq 1} S = M$. Let $\mathfrak{m}$ denote the maximal ideal of R. We have a commutative diagram

$$\begin{array}{ccc} V' & \xrightarrow{\pi_V} & V \\ \uparrow & & \uparrow \\ C' & \xrightarrow{\pi_C} & C \end{array}$$

where π_V denotes the point blow-up of V in x and the left vertical morphism is the induced closed immersion. We have $V' = \mathrm{Proj}(\oplus_{n\geq 0} M^n)$. Let T be a non-zero element of S_1 such that $x' \in C' \subseteq V'$ lies in the chart $\mathrm{Spec}((\oplus_{n\geq 0} M^n)_{(T)}) \subseteq V'$. The ring $(\oplus_{n\geq 0} M^n)_{(T)}$ is the S-subalgebra $\kappa[T, T_1/T, \ldots, T_s/T] =: S'$ of S_T where $T_1, \ldots, T_s$ are elements of S_1 such that $T, T_1, \ldots, T_s$ is a κ-basis of S_1. The local ring $\mathcal{O}_{V',x'} =: R'$ is the localization of S' by a prime ideal q of S' with $T \in q$ (see Lemma (2.1.9)). Define the subring $S'_E := \kappa[T_1/T, \ldots, T_s/T]$ of S'. Let η denote the ideal of S' generated by the image of q under

$$S' \overset{\alpha}{\to} S'/\langle T\rangle \cong S'_E \subseteq S'.$$

Then we have $q = \eta + \gamma$ for the by T generated ideal γ of S'. Let q_E denote the ideal $\alpha(q)$ of $S'/\langle T\rangle$. The κ-module morphisms $\langle T\rangle^k \otimes_\kappa q_E^l \to q^{k+l} : a \otimes b \mapsto a \cdot b$, for $k, l \in \mathbb{N}$, induce a morphism of graded rings

$$\beta : \mathrm{gr}_{\langle T\rangle} \kappa[T] \otimes_\kappa \mathrm{gr}_{q_E} S'_E \to \mathrm{gr}_q S'.$$

It is surjective, because we have

$$\eta^k = \bigoplus_{i\in\mathbb{N}} q_E^k \cdot T^i \quad \text{and} \quad q^k = \sum_{i=0}^{k} \gamma^i \cdot \eta^{k-i} = \bigoplus_{j\in\mathbb{N}} q_E^{\max\{0,k-j\}} \cdot T^j\,.$$

Every element f of q^k can be written $f = \sum_{i=0}^k f_i T^i + r$ for some $f_i \in q_E^{k-i}$ and some $r \in q^{k+1}$, i.e. $r = \sum_{j\in\mathbb{N}} g_j T^j$ for suitable $g_j \in q_E^{\max\{0,k+1-j\}}$. For each i the element $f_i \bmod q_E^{k+1-i}$ is unique since it is the image under the composition of morphisms of k-vector spaces $S' \to S'_E \to S'_E/q_E^{k+1-i}$ where the morphism $S' \to S'_E$ is given by $\sum_{k\in\mathbb{N}} h_k T^k \mapsto h_i$. Thus β is injective. Let $f_1, \dots, f_m \in S\backslash\{0\}$ be homogeneous generators of I with degrees $d_1, \dots, d_m$ and identify them with their image under the injective map $S \to R$. For $i \in \{1, \dots, m\}$ let f'_i denote the element $T^{-d_i} f_i$ of S' and identify it with its image under the injective map $S' \to R'$. Let I' denote the ideal of S' generated by $f'_1, \dots, f'_m$. By lemma (2.1.12) $I'R'$ is the kernel of $R' \to \mathcal{O}_{C',x'}$ and by remark (2.1.3) $T \cdot R' + I' \cdot R'$ is the kernel of $R' \to \mathcal{O}_{E,x'}$. The elements $f'_1, \dots, f'_m$ lie in S'_E because $f_1, \dots, f_l \in S$ are homogeneous. Let I'_E be the ideal of S'_E generated by $f'_1, \dots, f'_m$. We have $I' = I'_E S' = \kappa[T] \otimes_\kappa I'_E$. We determine the preimage $\beta^{-1}\mathrm{In}_q I'$ (see definition (2.1.10)). The ideal $\mathrm{In}_q I'$ is generated by initial forms $\mathrm{in}_q g$ of elements g of I'. An element $g \in I'$ can be written in the form $g = \sum_{i\in\mathbb{N}} g_i T^i$ for suitable elements $g_i \in I'_E$. If g lies in $q^n \backslash q^{n+1}$ then there is a non-empty subset $\Theta \subseteq \{0, 1, \dots, n\}$ such that for all $i \in \Theta$ we have $g_i \in q_E^{n-i} \backslash q_E^{n+1-i}$ and for all $j \in \mathbb{N}\backslash\Theta$ we have $g_j \in q_E^{\max\{0,n+1-j\}}$. Then by the definition of β we have

$$\beta^{-1}(\mathrm{in}_q g) = \sum_{i\in\Theta} \mathrm{in}_{\langle T\rangle} T^i \otimes_\kappa \mathrm{in}_{q_E} g_i \in \mathrm{gr}_{\langle T\rangle}\kappa[T] \otimes_\kappa \mathrm{In}_{q_E} I'_E =: J.$$

Thus we have $\beta(J) \supseteq \mathrm{In}_q I'$. The other inclusion also holds. Thus β induces an isomorphism

$$\epsilon : \mathrm{gr}_{\langle T\rangle}\kappa[T] \otimes_\kappa \mathrm{gr}_{q_E} S'_E / \mathrm{In}_{q_E} I'_E \cong \mathrm{gr}_q S' / \mathrm{In}_q I'.$$

Let $\bar{q}$ resp. $\bar{q}_E$ denote the ideal $q(S'/I')$ resp. $q_E(S'_E/I'_E)$ of S'/I' resp. S'_E/I'_E. We have isomorphisms of graded κ'-algebras

$$\mathrm{gr}\mathcal{O}_{C',c'} \cong \mathrm{gr}(S'/I' \otimes_{S'} R') \cong \mathrm{gr}_{\bar{q}}(S'/I') \otimes_{S'/q} \kappa' \cong (\mathrm{gr}_q S'/\mathrm{In}_q I') \otimes_{S'/q} \kappa',$$

$$\mathrm{gr}\mathcal{O}_{E,x'} \cong \mathrm{gr}(S'/(T\cdot S' + I') \otimes_{S'} R') \cong \mathrm{gr}_{\bar{q}_E}(S'_E/I'_E) \otimes_{S'/q} \kappa' \cong (\mathrm{gr}_{q_E} S'_E/\mathrm{In}_{q_E} I'_E) \otimes_{S'/q} \kappa',$$

$$\kappa'[T] \otimes_{\kappa'} \mathrm{gr}\mathcal{O}_{E,x'} \cong \kappa[T] \otimes_\kappa \mathrm{gr}\mathcal{O}_{E,x'} \cong \mathrm{gr}_{\langle T\rangle}\kappa[T] \otimes_\kappa (\mathrm{gr}_{q_E} S'_E/\mathrm{In}_{q_E} I'_E) \otimes_{S'/q} \kappa' \overset{\epsilon}{\cong} \mathrm{gr}\mathcal{O}_{C',c'}.$$

This completes the proof. □

Proof that theorem (3.2.4) follows from theorem (3.2.5). Assume the situation of theorem (3.2.4). Set $C := C_{X,D,x}$. Then the point blow-up $\pi_C : C' \to C$ in the origin $c := 0$ is permissible. Write $k := \kappa(x)$. Note that we have $C = C_{C,\overline{\{c\}},c}$. We have an isomorphisms of projective k-schemes

$$\pi_X^{-1}(\{x\}) \cong \mathbb{P}(C_{X,D,x}) = \mathbb{P}(C) \cong \pi_C^{-1}(\{c\}).$$

Let c' denote the point of $\pi^{-1}(\{c\}) \subseteq C'$ which corresponds to $x' \in \pi_X^{-1}(\{x\})$. Then we have $c' \in \pi_C^{-1}(\{c\})\backslash\mathbb{P}(\mathrm{Dir}(C))$, since $x' \in \pi_X^{-1}(\{x\})\backslash\mathbb{P}(\mathrm{Dir}(C_{X,D,x}))$. By lemma (3.4.1) we have

$$H^{(d)}_{C',c'} = H^{(d+1)}_{\pi_C^{-1}(\{c\}),c'}$$

for $d := \mathrm{trdeg}(\kappa(x')/k)$. Since x' is near to x the inequalities in (2.6.6) are equalities. Thus we have

$$H^{(d)}_{C',c'} = H^{(d+1)}_{\pi_C^{-1}(\{c\}),c'} = H^{(d+1)}_{\pi_X^{-1}(\{x\}),x'} = H^{(0)}_{C_{X,D,x}} = H^{(0)}_{C,c}.$$

Thus c' is near to c. Then by Theorem (3.2.5) we have

$$\dim \mathrm{Rid}_{C',c'} + d < \dim \mathrm{Rid}_{C,c}.$$

Using the inequalities in (2.6.6) again and that x' is near to x we get $H^{(0)}_{X',x'} = H^{(1+s)}_{\pi_X^{-1}(\{x\}),x'}$ for $s := \dim \mathcal{O}_{D,x}$. There is some element u_0 of $\mathcal{O}_{X',x'}$ with $\mathcal{O}_{\pi_X^{-1}(D),x'} = \mathcal{O}_{X',x'}/\langle u_0\rangle$. Choose s elements $u_1,\dots,u_s$ in $\mathcal{O}_{X,x}$ such that their images in $\mathcal{O}_{D,x}$ generate the maximal ideal of $\mathcal{O}_{D,x}$. Then we have

$$\mathcal{O}_{\pi_X^{-1}(\{x\}),x'} = \mathcal{O}_{\pi_X^{-1}(D),x'} \otimes_{\mathcal{O}_{D,x}} k = \mathcal{O}_{X',x'}/\langle u_0,u_1,\dots,u_s\rangle.$$

With proposition (2.2.4) we have

$$H^{(2+s)}(\mathrm{gr}(\mathcal{O}_{X',x'}/\langle u_0,\dots,u_s\rangle)) \geq H^{(1+s)}(\mathrm{gr}(\mathcal{O}_{X',x'}/\langle u_0,\dots,u_{s-1}\rangle)) \geq \dots \geq H^{(1)}(\mathrm{gr}\mathcal{O}_{X',x'}).$$

All inequalities are equalities since $H^{(1)}_{X',x'} = H^{(2+s)}_{\pi_X^{-1}(\{x\}),x'}$. Again with proposition (2.2.4) we get $\mathrm{gr}\mathcal{O}_{\pi_X^{-1}(\{x\}),x'} \cong \mathrm{gr}\mathcal{O}_{X',x'}/\langle U_0,\dots,U_s\rangle$ and for all $i\in\{0,1,\dots,s\}$ U_i is not a zero-divisor in $\mathrm{gr}\mathcal{O}_{X',x'}/\langle U_0,\dots,U_{i-1}\rangle$, where U_i denotes the image of u_i in $\mathrm{gr}^1\mathcal{O}_{X',x'}$. This implies

$$\mathrm{gr}\mathcal{O}_{\pi_X^{-1}(\{x\}),x'} \cong \mathrm{gr}\mathcal{O}_{X',x'}/\langle \mathrm{in}u_0,\mathrm{in}u_1,\dots,\mathrm{in}u_s\rangle$$

and that the elements $\mathrm{in}u_0,\mathrm{in}u_1,\dots,\mathrm{in}u_s$ are $\kappa(x')$-linearly independent. Choose a vector group V over $\kappa(x')$ which has $C_{X',x'}$ as a subcone. Let $u_0^V,\dots,u_s^V$ be homogeneous preimages of $\mathrm{in}u_0,\mathrm{in}u_1,\dots,\mathrm{in}u_s$ under $\Gamma V\to\Gamma C_{X',x'}$. Then $u_0^V,\dots,u_s^V$ are $\kappa(x')$-linearly independent elements of $(\Gamma V)_1$. Thus $V' := \mathrm{Spec}(\Gamma V/\langle u_0^V,\dots,u_s^V\rangle)$ is a subvector group of V. The intersection $C_{X',x'}\cap V'$ of subcones of V is the cone $C_{\pi_X^{-1}(\{x\}),x'}$. Then by lemma (2.5.17) we have

$$\dim\mathrm{Rid}_{X',x'} = \dim\mathrm{Rid}(C_{X',x'}) \leq \dim\mathrm{Rid}(C_{\pi_x^{-1}(\{x\}),x'}) + \dim V - \dim V' = \dim\mathrm{Rid}_{\pi_X^{-1}(\{x\}),x'} + s + 1.$$

By lemma (3.4.1) we have $\dim\mathrm{Rid}_{C',c'} = \dim\mathrm{Rid}_{\pi_C^{-1}(\{c\}),c'} + 1$. Further by remark (2.5.21) there is an isomorphism of k-cones $C_{X,x}\cong C\times_k C_{D,x} = C_{C,c}\times_k C_{D,x}$. Since $C_{D,x}$ is a vector group over k of dimension s we get $\dim\mathrm{Rid}_{C,c} + s = \dim\mathrm{Rid}_{X,x}$. Altogether we get

$$\begin{aligned}\dim\mathrm{Rid}_{X',x'} + d &\leq \dim\mathrm{Rid}_{\pi_X^{-1}(\{x\}),x'} + d + s + 1 = \mathrm{Rid}_{\pi_C^{-1}(\{c\}),c'} + d + s + 1\\ &= \dim\mathrm{Rid}_{C',c'} + d + s < \dim\mathrm{Rid}_{C,c} + s = \dim\mathrm{Rid}_{X,x}\end{aligned}$$

which completes the proof. □

3.5 Hironaka schemes with dimension at most five

Let k be a field of positive characteristic p. We denote the k-module of Kähler differentials $\Omega^1_{k/\mathbb{Z}} = \Omega^1_{k/\mathbb{F}_p} = \Omega^1_{k/k^p}$ by $\Omega^1(k)$. By a derivation we mean an element of $\mathrm{Der}_{\mathbb{Z}}(k,k) = \mathrm{Der}_{\mathbb{F}_p}(k,k) = \mathrm{Der}_{k^p}(k,k)$.

Let $\mathcal{B}$ be a Hironaka scheme over k of dimension ≤ 5. If $\mathcal{B}$ is a vector group then, as an additive group, it is isomorphic to $\mathbb{G}_{a,k}^m$ for $m = \dim\mathcal{B}$. If $\mathcal{B}$ is not a vector group than we have a characterization by Oda, [Od], see theorem (3.5.5) below. We get some corollaries which will be needed in section 3.6.

Definition (3.5.1). *Let q be a prime number. A family $(x_i)_{i\in I}$ of elements of k is **q-independent** if $q = \mathrm{char}\,k = p$ and the family of elements of k*

$$\Big(\prod_{i\in J} x_i^{e_i} \mid J\subseteq I \text{ finite, } (e_i)_{i\in J}\in\{0,1,\dots,p-1\}^J\Big)$$

is k^p-linearly independent.

Remark (3.5.2). *a) By [Stacks], Tag 07P0, a family $(x_i)_{i\in I}$ in k is p-independent if and only if the family $(\mathrm{d}x_i)_{i\in I}$ in $\Omega^1(k)$ is k-linearly independent.*

b) By a) for a p-independent family $(x_i)_{i\in I}$ in k there are derivations $\partial_i : k \to k$, $i \in I$, such that for $i \neq j$ one has $\partial_i(x_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta.

It turns out that for the classification of Hironaka schemes one should consider types instead of isomorphism classes, cf. the introduction of [Od].

Definition (3.5.3). *Two additive groups G and G' over k are* ***of the same type*** *if there is field automorphism $k \to k$ and a isomorphism $G \times_k k \cong G'$ of additive groups over k.*

Remark (3.5.4). *By remark (2.5.2)d) two additive groups G and G' over k are of the same type if and only if there is an isomorphism of graded rings $\Gamma G \cong \Gamma G'$.*

Theorem (3.5.5). *A Hironaka scheme over k of dimension ≤ 5, which is not a vector group, is of the same type as one of the following non-isomorphic additive groups.*

Type 3. *The additive subgroup* $\mathrm{Spec}(k[X_1, X_2, X_3, Y_1]/\langle\tilde{\tau}_1\rangle)$ *of* $\mathbb{G}^4_{a,k}$*, where*

$$\tilde{\tau}_1 = Y_1^2 + a_3X_2^2 + a_2X_3^2 + a_2a_3X_1^2$$

for 2-independent elements a_2, a_3 of k,

Type 4-1. $\mathrm{Spec}(k[X_1, \ldots, X_4, Y_1]/\langle\tau_1\rangle) \subseteq \mathbb{G}^5_{a,k}$,

Type 4-2. $\mathrm{Spec}(k[X_1, \ldots, X_4, Y_1, Y_2]/\langle\tau_1, \tau_2\rangle) \subseteq \mathbb{G}^6_{a,k}$,

Type 4-3. $\mathrm{Spec}(k[X_1, \ldots, X_4, Y_1, Y_2, Y_3]/\langle\tau_1, \tau_2, \tau_3\rangle) \subseteq \mathbb{G}^7_{a,k}$

Type 4-4. $\mathrm{Spec}(k[X_1, \ldots, X_4, Y_1, Y_2, Y_3, Y_4]/\langle\tau_1, \tau_2, \tau_3, \tilde{\tau}_4\rangle) \subseteq \mathbb{G}^8_{a,k}$*, or, equivalently,* $\mathrm{Spec}(k[X_1, \ldots, X_4, Y_1, Y_2, Y_3, Y_4]/\langle\tau_1, \tau_2, \tau_3, \tau_4\rangle) \subseteq \mathbb{G}^8_{a,k}$*, where*

$$\begin{aligned}
\tau_1 &= Y_1^2 + a_3X_2^2 + a_2X_3^2 + a_2a_3X_4^2,\\
\tau_2 &= Y_2^2 + a_3X_1^2 + a_1X_3^2 + a_1a_3X_4^2,\\
\tau_3 &= Y_3^2 + a_2X_1^2 + a_1X_2^2 + a_1a_2X_4^2,\\
\tilde{\tau}_4 &= Y_4^2 + a_2a_3X_1^2 + a_1a_3X_2^2 + a_1a_2X_3^2,\\
\tau_4 &= Y_4^2 + a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 + a_2a_3X_1^2 + a_1a_3X_2^2 + a_1a_2X_3^2 + a_1a_2a_3X_4^2
\end{aligned}$$

for 2-independent elements a_2, a_3 (resp. a_1, a_2, a_3) of k in the case of ***type 4-1*** *(resp. in the case of* ***type 4-2****,* ***type 4-3****,* ***type 4-4****),*

Type 5. $\mathrm{Spec}(k[X_1, \ldots, X_5, Y_1]/\langle\tau_0\rangle) \subseteq \mathbb{G}^6_{a,k}$*, where*

$$\tau_0 = Y_1^3 + a_1X_1^3 + a_2X_2^3 + a_1^2X_3^3 + a_1a_2X_4^3 + a_1^2a_2X_5^3$$

for 3-independent elements a_1, a_2 of k,

Type 5-1. *The product of the one-dimensional vector group $\mathbb{G}_{a,k}$ with a Hironaka scheme of type 4-1,*

Type 5-2. $\mathbb{G}_{a,k}\times$*type 4-2,*

Type 5-3. $\mathbb{G}_{a,k}\times$*type 4-3,*

Type 5-4. $\mathbb{G}_{a,k}\times$*type 4-4,*

Type 5-5. $\operatorname{Spec}(k[X_1,\dots,X_5,Y_1]/\langle\tau_5\rangle) \subseteq \mathbb{G}_{a,k}^6$, *where*

$$\tau_5 = Y_1^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_2a_3X_4^2 + a_1a_3X_5^2$$

for 2*-independent elements* a_1, a_2, a_3 *of* k*,*

Type 5-*. $\operatorname{Spec}(k[X_1,\dots,X_5,Y_1,\dots,Y_\nu]/\langle\tau_1,\dots,\tau_\nu\rangle) \subseteq \mathbb{G}_{a,k}^{5+\nu}$, $\nu \in \mathbb{N}_{\geq 1}$, *with*

$$\tau_j = Y_j^2 + \sum_{i=1}^{4}(\partial_i g_j)X_i^2 + \big(g_j + \sum_{i=1}^{4} a_i\partial_i g_j\big)X_5^2 \quad (j = 1,\dots,\nu)$$

where $a_1,\dots,a_4$ *are* 2*-independent elements of* k*,* $\partial_1,\dots,\partial_4 : k \to k$ *are derivations with* $\partial_i(a_j) = \delta_{ij}$ *(see remark (3.5.2)) and* $g_1,\dots,g_\nu$ *are elements of* $k^2(a_1,\dots,a_4)$ *such that the elements* $1, a_1,\dots,a_4, g_1,\dots,g_\nu$ *of* k *are* k^2*-linearly independent, and that the matrix*

$$A = (\partial_i g_j)_{i=1,\dots,4;\, j=1,\dots,\nu}$$

has the property that the rows of $\mathrm{d}A$*, as elements of* $\Omega^1(k)^{\oplus\nu}$*, are* k*-linearly independent.*

Proof. [Od], Theorem 3.14. □

Remark (3.5.6). *a) A Hironaka scheme of type 4-1 is of the same type as the product of* $\mathbb{G}_{a,k}$ *with a Hironaka scheme of type 3.*

b) To be more precise, Oda shows that two Hironaka schemes of type 5- given by choices*

$$a_1,\dots,a_4,g_1,\dots,g_\nu \quad \text{resp.} \quad a_1',\dots,a_4',g_1',\dots,g_{\nu'}'$$

are of the same type if and only if there is some non-zero element u *of the by the elements* $1, a_1,\dots,a_4,g_1,\dots,g_\nu$ *generated* k^2*-subvector space* U *of* k *such that* $u^{-1}\cdot U$ *is the by* $1, a_1',\dots,a_4',g_1',\dots,g_{\nu'}'$ *generated* k^2*-subvector space of* k*.*

We collect some corollaries which will be used in the proof (see next section) that theorem (3.2.5) holds if the theorem (3.2.6) holds.

Corollary (3.5.7). *Let* V *be a vector group over* k*. Let* $\mathcal{B} = \mathcal{B}_{\mathbb{P},y}$ *be the Hironaka scheme at a point* y *of* $\mathbb{P}(V) =: \mathbb{P}$ *such that* $\mathcal{B}$ *is not a vector group with* $m := \dim\mathcal{B} \leq 5$*. Let* $S = \oplus_{k\in\mathbb{N}}S_k$ *denote the symmetric* k*-algebra* ΓV *and let* $\mathcal{U} \subseteq S$ *denote the ring of invariants of* $(\mathcal{B}, V)$*. Then there is a* k*-basis* $(X_1,\dots,X_m,Y_1,\dots,Y_n,Z_1,\dots,Z_l)$ *of* S_1 *with*

$$\mathcal{U} = k[\eta_1,\dots,\eta_n,Z_1,\dots,Z_l]$$

for homogeneous elements $\eta_1,\dots,\eta_n$ *of* S *of degree* p*, which are, depending on the type of the Hironaka scheme* $\mathcal{B}$*, as follows (with the notations of theorem (3.5.5)).*

(Type 3) $n = 1,\ m = 3,\ \eta_1 = \tilde{\tau}_1,$

(Type 4-**1**) $n = 1,\ m = 4,\ \eta_1 = \tau_1,$

(Type 5-**1**) $n = 1,\ m = 5,\ \eta_1 = \tau_1,$

(Type 4-n for 2 ≤ n ≤ 4) $2 \le n \le 4,\ m = 4,\ \eta_1 = \tau_1, \dots, \eta_n = \tau_n,$

(Type 5-n for 2 ≤ n ≤ 4) $2 \le n \le 4,\ m = 5,\ \eta_1 = \tau_1, \dots, \eta_n = \tau_n,$

(Type 5) $n = 1,\ m = 5,\ \eta_1 = \tau_0,$

(Type 5-5) $n = 1,\ m = 5,\ \eta_1 = \tau_5,$

(Type 5-*) $n = \nu \ge 1,\ m = 5,\ \eta_1 = \tau_1, \dots, \eta_n = \tau_n.$

Proof. Write $\underline{X}$, $\underline{Y}$, $\underline{Z}$, $\underline{\eta}$ for $(X_1, \dots, X_m)$, $(Y_1, \dots, Y_n)$, $(Z_1, \dots, Z_l)$, $(\eta_1, \dots, \eta_n)$. Fix a type of the Hironaka scheme. By theorem (3.5.5) $\mathcal{B}$ is of the same type as the additive subgroup $\mathcal{B}' := \operatorname{Spec}(k[\underline{X}, \underline{Y}]/\langle \underline{\eta} \rangle)$ of $\mathbb{G}_{a,k}^{m+n}$. We may assume that $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic as additive group schemes over k by replacing p-independent elements by the images under a field automorphism. Let $V_{\mathcal{B}}$ be the smallest vector group which has $\mathcal{B}$ as an additive subgroup, see remark (2.5.2)c). Then $V_{\mathcal{B}}$ is a subvector group of V. We get commutative diagrams of additive groups over k resp. graded k-algebras resp. k-vector spaces

$$\begin{array}{ccccc} \mathcal{B}' & \hookrightarrow & \mathbb{G}_{a,k}^{n+m} & & \\ \downarrow \cong & & \downarrow \cong & & \\ \mathcal{B} & \hookrightarrow & V_{\mathcal{B}} & \hookrightarrow & V \end{array} \qquad \begin{array}{ccccc} k[\underline{X}, \underline{Y}]/\langle \underline{\eta} \rangle & \twoheadleftarrow & k[\underline{X}, \underline{Y}] & & \\ \uparrow \cong & & \uparrow \cong & & \\ \Gamma\mathcal{B} & \twoheadleftarrow & \Gamma(V_{\mathcal{B}}) & \twoheadleftarrow & S \end{array} \qquad \begin{array}{ccc} k[\underline{X}, \underline{Y}]_1 & & \\ \uparrow \cong & & \\ (\Gamma(V_{\mathcal{B}}))_1 & \overset{}{\underset{e}{\twoheadleftarrow}} & S_1 \end{array}$$

Choose a section $(\Gamma(V_{\mathcal{B}}))_1 \to S_1$ of e. This induces a morphism of k-algebras $k[\underline{x}, \underline{y}] \cong \Gamma(V_{\mathcal{B}}) \to S$. Identify $\underline{X}, \underline{Y}, \underline{\eta}$ with their images in S. Choose elements $Z_1, \dots, Z_l$ of S_1 such that $(\underline{X}, \underline{Y}, \underline{Z})$ is a k-basis of S_1. Then the ideal $\ker(S \to \Gamma\mathcal{B})$ of S is generated by the tuple of homogeneous additive elements $(\underline{\eta}, \underline{Z})$. By lemma (2.3.13) the graded k-subalgebra $\mathcal{U}' := k[\underline{\eta}, \underline{Z}]$ of S is the ring of invariants of $(\langle \mathcal{U}'_+ \rangle, S) = (\langle \underline{\eta}, \underline{Z} \rangle, S)$. Thus by definition of the ring of invariants of $(\mathcal{B}, V)$ (see definition (2.5.5)) we have $\mathcal{U}' = \mathcal{U}$. □

In the situation of corollary (3.5.7) let us define a homogeneous element v of S and a family σ of elements of $S_{(v)}$, depending on the type of the Hironaka scheme, as follows:

(**Type 3**) $v = X_1, \quad \sigma = (\mathcal{Z}_1, \dots, \mathcal{Z}_l, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1),$

(**Type 4-1 or type 5-1**) $v = X_4, \quad \sigma = (\mathcal{Z}_1, \dots, \mathcal{Z}_l, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1),$

(**Type 4-*n* or type 5-*n* for 2 ≤ *n* ≤ 4**) $v = X_4, \quad \sigma = (\mathcal{Z}_1, \dots, \mathcal{Z}_l, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1, \dots, \mathcal{Y}_n),$

(**Type 5**) $v = X_5, \quad \sigma = (\mathcal{Z}_1, \dots, \mathcal{Z}_l, \mathcal{W}_1, \dots, \mathcal{W}_5),$

(**Type 5-5**) we distinguish two cases,

(**Type 5-5, case 1**) $v = X_4, \quad \sigma = (\mathcal{Z}_1, \dots, \mathcal{Z}_l, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, X_1/v, X_5/v),$

(**Type 5-5, case 2**) $v = X_5, \quad \sigma = (\mathcal{Z}_1, \dots, \mathcal{Z}_l, \mathcal{V}_1, \dots, \mathcal{V}_4),$

(Type 5-*) $v = X_5, \quad \sigma = (\mathcal{Z}_1, \ldots, \mathcal{Z}_l, \mathcal{X}_1, \ldots, \mathcal{X}_4, \mathcal{Y}_{r_1}, \ldots, \mathcal{Y}_{r_\nu}),$

where we define the following elements of $S_{(v)}$

$\mathcal{Z}_i = Z_i/v$ for $i = 1, \ldots, l$,

$\mathcal{X}_i = (X_i/v)^2 + a_i$ for $i = 1, \ldots, 4$,

$\mathcal{Y}_i = Y_i/v + X_j X_k/v^2$ for $\{i, j, k\} = \{1, 2, 3\}$,

$\mathcal{Y}_4 = Y_4/v + X_1 X_2 X_3/v^3$,

$\mathcal{W}_1 = (X_4/v)^3 - a_1$, $\mathcal{W}_2 = (X_3/v)^3 + a_2$, $\mathcal{W}_3 = Y_1/v - X_3 \cdot X_4^2/v^3$,

$\mathcal{W}_4 = X_1/v - X_3 X_4/v^2$, $\mathcal{W}_5 = X_2/v - (X_4/v)^2$,

$\mathcal{U}_1 = Y_1/v + X_2 X_3/v^2$, $\mathcal{U}_2 = (X_2/v)^2 + a_3$, $\mathcal{U}_3 = (X_3/v)^2 + a_2$,

$\mathcal{V}_1 = Y_1/v + X_1 X_3/v^2$, $\mathcal{V}_2 = X_2/v + X_1 X_4/v^2$,

$\mathcal{V}_3 = (X_1/v)^2 + a_3$, $\mathcal{V}_4 = (X_3/v)^2 + a_2 (X_4/v)^2 + a_1$,

$\mathcal{Y}_{r_i} = Y_j/v + r_j$ for $j = 1, \ldots, \nu$ where r_j is the square root $\sqrt{\rho(g_j)}$ in $k[X_1/v, \ldots, X_4/v]$ of the image $\rho(g_j)$ under the k^2-linear map

(3.5.7.A) $$\rho : k^2(a_1, \ldots, a_4) \to k^2[(X_1/v)^2, \ldots, (X_4/v)^2] \subseteq k[X_1/v, \ldots, X_4/v]$$

which is given by $\rho(\prod_{i \in I} a_i) = \prod_{i \in I} (X_i/v)^2$ for all subsets $I \subseteq \{1, 2, 3, 4\}$.

Corollary (3.5.8). *In the situation of corollary (3.5.7) let $\mathfrak{q} \in \mathrm{Spec}(S) = V$ denote the to $y \in \mathbb{P} = \mathrm{Proj}(S)$ associated homogeneous prime ideal of S. For the element v of S and the family of elements σ of $S_{(v)}$, as above, there is a prime ideal $\mathfrak{p}$ of $S_{(v)}$ with $\mathfrak{p} \supseteq \langle \sigma \rangle$ such that $S_{(v)} \to S_{(\mathfrak{q})}$ induces an isomorphism $R'_E := \left(S_{(v)}\right)_{\mathfrak{p}} \to S_{(\mathfrak{q})}$ and such that the by σ induced family in R'_E is a part of a system of regular parameters of R'_E.*

Proof. By remark (2.5.8) we have $y \in \mathcal{B}_{\mathbb{P},y}$. In particular the elements $Z_1, \ldots, Z_l$ lie in $\mathfrak{q}$. Choose derivations $\partial_i : k \to k$ with $\partial_i(a_j) = \delta_{ij}$ (see remark (3.5.2)). Let ∂_i^y denote the induced derivations $S_q \to S_q$ with $\partial_i^y(\{X_1, \ldots, X_m, Y_1, \ldots, Y_n, Z_1, \ldots, Z_l\}) = \{0\}$. By remark (2.5.8) we have $\eta_1, \ldots, \eta_n \in \mathfrak{m}^p$ for the maximal ideal $\mathfrak{m} := qA_q$ of A_q. For all i, j we have $\partial_i^y \eta_j \in \mathfrak{m}^{p-1}$. Inductively every element of the form $\partial_{i_1}^y \cdot \ldots \cdot \partial_{i_\mu}^y$, for $\mu < p$, lies in $\mathfrak{m}^{p-\mu}$ and therefore it lies in $\mathfrak{m} \cap S = q$. Then, in the respective cases, we get

(Type 3) $\eta_1,\ X_2^2 + a_2 X_1^2\ , X_3^2 + a_3 X_1^2 \in \mathfrak{q}$,

(Type 4-1 or type 5-1) $\eta_1,\ X_2^2 + a_2 X_4^2\ , X_3^2 + a_3 X_4^2 \in \mathfrak{q}$,

(Type 4-*n* or type 5-*n* for 2 ≤ *n* ≤ 4) $\eta_1, \ldots, \eta_n,\ X_1^2 + a_1 X_4^2,\ X_2^2 + a_2 X_4^2,\ X_3^2 + a_3 X_4^2 \in \mathfrak{q}$,

(Type 5) $\eta_1,\ X_1^3 + 2a_1 X_3^3 + a_2 X_4^3 + 2a_1 a_2 X_5^3,\ X_4^3 + 2a_1 X_5^3,\ X_2^3 + a_1 X_4^3 + a_1^2 X_5^3,\ 2X_3^3 + 2a_2 X_5^3 \in \mathfrak{q}$,

(Type 5-5) $\eta_1,\ X_1^2 + a_3 X_5^2,\ X_2^2 + a_3 X_4^2,\ X_3^2 + a_2 X_4^2 + a_1 X_5^2 \in \mathfrak{q}$, (**Type 5-***) $\eta_1,\dots,\eta_n,\ \sum_{i=1}^n(\partial_{i'}\partial_i g_j)(X_i^2+a_iX_5^2)\in\mathfrak{q}$ for all $1\le i'\le 4$, $1\le j\le n$.

We have $\mathrm{d}\partial_i g_j=\sum_{i'=1}^4\partial_{i'}\partial_i g_j\mathrm{d}a_{i'}$ in $\Omega^1(k)$ and by remark (3.5.2) the elements $\mathrm{d}a_{i'}$ are k-linearly independent. Thus by assumption the four elements $(\partial_{i'}\partial_1 g_j)_{i',j},\dots,(\partial_{i'}\partial_4 g_j)_{i',j}$ of the k-vector space k^{4n} are k-linearly independent. Thus we get in the case of type 5-*

$$\eta_1,\dots,\eta_4,\ X_1^2+a_1X_5^2,\dots,X_4^2+a_4X_5^2\in\mathfrak{q}.$$

For the further argumentation in the case of type 5-5 we distinguish the cases $X_5\in\mathfrak{q}$ (case 1) and $X_5\notin\mathfrak{q}$ (case 2).
In every case the element $v\in S$ lies not in $\mathfrak{q}$. This follows, in all cases except the case 2 of type 5-5, from $\mathfrak{q}\neq S_+$ because $v\in\mathfrak{q}$ would imply

$$\mathfrak{q}\supseteq\langle Z_1,\dots,Z_l,\eta_1,\dots,\eta_n,X_1^p,\dots,X_m^p,v\rangle=\langle Z_1,\dots,Z_l,Y_1^p,\dots,Y_n^p,X_1^p,\dots,X_m^p,v\rangle$$

in contradiction to $\mathfrak{q}\neq S_+$. Let $\mathfrak{p}$ denote the by $\mathfrak{q}$ induced prime ideal of $S_{(v)}$. Then we have $S_{(\mathfrak{q})}=(S_{(v)})_{\mathfrak{p}}$ and we have $\mathfrak{b}+\mathfrak{c}\subseteq\mathfrak{p}$ for the ideals of $S_{(v)}$

$$\mathfrak{b}=\langle Z_1/v,\dots,Z_l/v\rangle=\langle\mathcal{Z}_1,\dots,\mathcal{Z}_l\rangle\quad\text{and}$$

(**Type 3 or type 4-1 or type 5-1**)

$$\mathfrak{c}=\langle\eta_1/v^2,(X_2^2+a_2v^2)/v^2,(X_3^2+a_3v^2)/v^2\rangle=\langle\mathcal{Y}_1^2,\mathcal{X}_2,\mathcal{X}_3\rangle,$$

(**Type 4-n or type 5-n for $2\le n\le 4$**) $\mathfrak{c}=\langle\mathcal{Y}_1^2,\dots,\mathcal{Y}_n^2,\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3\rangle$,

(**Type 5**) $\mathfrak{c}=\langle\mathcal{W}_3^3,\mathcal{W}_4^3,\mathcal{W}_5^3,\mathcal{W}_1,\mathcal{W}_2\rangle$,

(**Type 5-5, case 1**)

$$\mathfrak{c}=\langle\eta_1/v^2,(X_2^2+a_3X_4^2)/v^2,(X_3^2+a_2X_4^2)/v^2,X_1/v,X_5/v\rangle=\langle\mathcal{U}_1^2,\mathcal{U}_2,\mathcal{U}_3,X_1/v,X_5/v\rangle,$$

(**Type 5-5, case 2**)

$$\mathfrak{c}=\langle\eta_1/v^2,(X_1^2+a_3X_5)/v^2,(X_2^2+a_3X_4^2)/v^2,(X_3^2+a_2X_4^2+a_1X_5^2)/v^2\rangle=\langle\mathcal{V}_1^2,\mathcal{V}_2^2,\mathcal{V}_3,\mathcal{V}_4\rangle,$$

(**Type 5-***)

$$\mathfrak{c}=\langle(Y_1^2+g_1X_5^2)/v^2,\dots,(Y_n^2+g_nX_5^2)/v^2,\mathcal{X}_1,\dots,\mathcal{X}_4\rangle=\langle\mathcal{Y}_{r_1}^2,\dots,\mathcal{Y}_{r_n}^2,\mathcal{X}_1,\dots,\mathcal{X}_4\rangle.$$

One sees that the ideal $\langle\sigma\rangle$ of $S_{(v)}$ generated by the family σ is the radical ideal of $\mathfrak{b}+\mathfrak{c}$. Thus $\mathfrak{b}+\mathfrak{c}\subseteq\mathfrak{p}$ implies $\langle\sigma\rangle\subseteq\mathfrak{p}$.
It remains to show that, as a family of elements of R'_E, σ can be extended to a system of regular parameters of R'_E. Let t be the number of elements of σ. We show below that $R'_E/\langle\sigma\rangle R'_E=:C$ is a regular ring of dimension $\dim R'_E-t$. Then $R'_E/\langle\sigma\rangle R'_E$ has a system ρ of regular parameters of length $\dim R'_E-t$. Let ρ_l be a lift of ρ to a system of elements of R'_E. Then (ρ_l,σ) has $\dim R'_E$ elements and it generates the maximal ideal of R'_E. Therefore σ has the wished property.
The ring $C=R'_E/\langle\sigma\rangle R'_E$ is a localization of the ring $B:=S_{(v)}/\langle\sigma\rangle$ by the prime ideal $\mathfrak{p}B$. We calculate B and $\dim B$ for each type of the Hironaka scheme.

(**Type 3**) $B \cong k(\sqrt{a_2}, \sqrt{a_3})$, $\dim B = 0$,

(**Type 4-1**) $B \cong k(\sqrt{a_2}, \sqrt{a_3})[X_4/v]$, $\dim B = 1$,

(**Type 4-*n* for 2 ≤ *n* ≤ 4**) $B \cong k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$, $\dim B = 0$,

(**Type 5-1**) $B \cong k(\sqrt{a_2}, \sqrt{a_3})[X_4/v, X_5/v]$, $\dim B = 2$,

(**Type 5-*n* for 2 ≤ *n* ≤ 4**) $B \cong k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})[X_5/v]$, $\dim B = 1$,

(**Type 5**) $B \cong k(a_1^{1/3}, a_2^{1/3})$, $\dim B = 0$,

(**Type 5-5, case 1**)) $B \cong k(\sqrt{a_2}, \sqrt{a_3})$, $\dim B = 0$,

(**Type 5-5, case 2**)) $B \cong k(\sqrt{a_3})[X_3/v, X_4/v]/\langle (X_3/v)^2 + a_2(X_4/v)^2 + a_1\rangle$, $\dim B = 1$,

(**Type 5-***) $B \cong k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4})$, $\dim B = 0$.

We see that in the case of every type of the Hironaka scheme, except case 2 of type 5-5, the ring B is a polynomial ring over a field in 0, 1 or 2 variables. Thus in these cases C is a regular ring. In case 2 of type 5-5, if C would not be regular, than the by $(X_3/v)^2+a_2(X_4/v)^2+a_1$ induced element r of $D_{\mathfrak{p}D}$ for $D := k(\sqrt{a_3})[X_3/v, X_4/v]$ would lie in the second power $\mathfrak{n}^2$ of the maximal ideal $\mathfrak{n}$ of $D_{\mathfrak{p}D}$. The elements a_1, a_2 of $k(\sqrt{a_3}) =: K$ are 2-independent. Choose a derivation $\partial : K \to K$ over K^2 with $\partial(a_1) = 1$ and $\partial(a_2) = 0$. Let ∂_D denote the induced derivation $D_{\mathfrak{p}D} \to D_{\mathfrak{p}D}$ with $\partial_D(\{X_3/v, X_4/v\}) = \{0\}$. Then the assumption would imply $1 = \partial_D(r) \in \partial_D(\mathfrak{n}^2) \subseteq \mathfrak{n}$ which is a contradiction. Thus in all cases C is regular. Further we have

$$\begin{aligned}\dim R'_E - \dim R'_E/\langle\sigma\rangle R'_E &= \dim \left(S_{(v)}\right)_{\mathfrak{p}} - \dim B_{\mathfrak{p}B}\\ &= \dim S_{(v)} - \dim S_{(v)}/\mathfrak{p} - \dim B + \dim B/\mathfrak{p}B\\ &= n + m + l - 1 - \dim B = t\end{aligned}$$

As explained above now we find a system of elements ρ_l of R'_E such that (ρ_l, σ) is a system of regular parameters of R'_E. □

Corollary (3.5.9). *Let $\eta_1, \dots, \eta_n \in \mathcal{U} \subseteq S$ be the homogeneous additive elements from corollary (3.5.7) of degree $p = \mathrm{char}(k)$ ($p = 3$ in the case of type 5 and $p = 2$ else). The induced elements $\eta'_i := v^{-p}\eta_i \in \left(S_{(v)}\right)_{\mathfrak{p}} = R'_E$ (for $i = 1, \dots, n$) can be written, depending on the type of the Hironaka scheme, in terms of elements of σ as follows.*

(Type 3 or type 4-1 or type 5-1) $\eta'_1 = \mathcal{Y}_1^2 + \mathcal{X}_2\mathcal{X}_3$,

(Type 4-n or type 5-n for 2 ≤ n ≤ 4) $\eta'_i = \mathcal{Y}_i^2 + \mathcal{X}_j\mathcal{X}_k$ *for* $\{i, j, k\} = \{1, 2, 3\}$ *and* $\eta'_4 = \mathcal{Y}_4^2 + a_1\mathcal{Y}_1^2 + a_2\mathcal{Y}_2^2 + a_3\mathcal{Y}_3^2 + \mathcal{X}_1\mathcal{X}_2\mathcal{X}_3$,

(Type 5) $\eta'_1 = \mathcal{W}_3^3 + a_1\mathcal{W}_4^3 + a_2\mathcal{W}_5^3 + \mathcal{W}_1^2\mathcal{W}_2$,

(Type 5-5, case 1) $\eta'_1 = \mathcal{U}_1^2 + a_1(X_1/v)^2 + a_1a_3(X_5/v)^2 + \mathcal{U}_2\mathcal{U}_3$,

(Type 5-5, case 2) $\eta'_1 = \mathcal{V}_1^2 + a_2 \cdot \mathcal{V}_2^2 + \mathcal{V}_3\mathcal{V}_4$, *(**Type 5-***)* $\eta_i' = \mathcal{Y}_{r_i}^2 + P_i + \mathcal{R}_i$ *for* $1 \le i \le n$ *where the elements* P_i *are non-zero* k*-linear combinations of elements of the form* $\mathcal{X}_i\mathcal{X}_j$*, for* $1 \le i < j \le 4$*, and the elements* $\mathcal{R}_i$ *lie in the third power* $\mathfrak{m}_{R_E'}^3$ *of the maximal ideal* $\mathfrak{m}_{R_E'}$ *of* R_E'*.*

Proof. In every case except the case of type 5-* the claim is verified by a short calculation. For the identity in the case of type 5-* we set $N := \{1,2,3,4\}$ and for every subset $I \subseteq N$ we set

$$X_I' := \prod_{i\in I}(X_i/v), \quad \mathcal{X}_I := \prod_{i\in I}\mathcal{X}_i, \quad a_I := \prod_{i\in I} a_i, \quad \partial_I := \prod_{i\in I}\partial_i.$$

Then we have $g_j = \sum_{I\subseteq N}\lambda_{j,I}a_I$ for suitable coefficients $\lambda_{j,I} \in k^2$ (for $j \in \{1,\dots,n\}$). For the elements $\rho(g_j)$ (see definition of $\mathcal{Y}_{r_i}$, (3.5.7.A)) we have

$$\begin{aligned}\rho(g_j) &= \sum_{I\subseteq N}\lambda_{j,I}(X_I')^2 = \sum_{I\subseteq N}\lambda_{j,I}\sum_{J\subseteq I}\mathcal{X}_J a_{I\setminus J} = \sum_{I\subseteq N}\lambda_{j,I}\sum_{J\subseteq N}\mathcal{X}_J\cdot\partial_J a_I = \sum_{J\subseteq N}(\partial_J g_j)\mathcal{X}_J \\ &= g_j + \sum_{i=1}^{4}(\partial_i g_j)\mathcal{X}_i + \sum_{J\subseteq N,\,\#J\ge 2}(\partial_J g_j)\mathcal{X}_J.\end{aligned}$$

Thus we have $\eta_j' = (Y_j/v)^2 + \sum_{i=1}^4(\partial_i g_j)\big((X_i/v)^2 + a_i\big) + g_j = \mathcal{Y}_{r_j}^2 + P_j + \mathcal{R}_j$ for

$$P_j := \sum_{J\subseteq N,\,\#J=2}(\partial_J g_j)\mathcal{X}_J, \quad \mathcal{R}_j := \sum_{J\subseteq N,\,\#J>2}(\partial_J g_j)\mathcal{X}_J.$$

For coefficients $c_1,\dots,c_n \in k^2$ we have following equivalences

$$\begin{aligned}\sum_j c_jP_j = 0 \quad &\Leftrightarrow \quad \sum_j c_j \sum_{J\subseteq I\subseteq N,\,\#J=2}\lambda_{j,I}\mathcal{X}_J a_{I\setminus J} = 0 \\ &\Leftrightarrow \quad \text{for all subsets } I\subseteq N \text{ with } \#I\ge 2 \text{ one has } \sum_j c_j\lambda_{j,I} = 0 \\ &\Leftrightarrow \quad \sum_j c_jg_j = \sum_j c_j\sum_{I\subseteq N,\,\#I\le 1}\lambda_{j,I}a_I\end{aligned}$$

Thus $P_1,\dots,P_n$ are k^2-linearly independent, since $1, a_1, a_2, a_3, a_4, g_1,\dots,g_n$ are k^2-linearly independent. In particular the P_i are non-zero. This completes the proof of corollary (3.5.9). ☐

In the proof of theorem (3.6.1) below we study the behavior of the dimension of the ridge. Since the dimension of the ridge is invariant under base change with field extensions, at some point of the proof one can assume that the field is perfect. This simplifies the situation as follows.

Corollary (3.5.10). *Let* $\eta_1,\dots,\eta_n \in S$ *be as in corollary (3.5.7) and let* $\eta_1',\dots,\eta_n' \in R_E'$ *be as in corollary (3.5.9). Let* $K|\kappa'$ *be the extension of the residue field* κ' *of* R_E' *by a perfect field* K*. There are*

- *a symmetric algebra* $\mathcal{S} = \oplus_{i\in\mathbb{N}}\mathcal{S}_i$ *over* K*,*
- *a graded* K*-subalgebra* $\mathcal{S}'$ *of* S*,*
- *a* K*-basis* $(x_1,\dots,x_m,y_1,\dots,y_n,z_1,\dots,z_l)$ *of* $\mathcal{S}_1$ *and*
- *isomorphisms of graded* K*-algebras* $\alpha : S\otimes_k K \to \mathcal{S}$*,* $\alpha' : \mathrm{gr}R_E'\otimes_{\kappa'} K \to \mathcal{S}'$*,*

such that we have

$$(3.5.10.A) \qquad (\alpha(\eta_1),\ldots,\alpha(\eta_n),\alpha(Z_1),\ldots,\alpha(Z_l)) = (y_1^p,\ldots,y_n^p,z_1,\ldots,z_l),$$
$$(3.5.10.B) \qquad (\alpha'(\mathrm{in}\eta_1'),\ldots,\alpha'(\mathrm{in}\eta_n'),\alpha'(\mathcal{Z}_1),\ldots,\alpha'(\mathcal{Z}_l)) = (y_1^p+h_1,\ldots,y_n^p+h_n,z_1,\ldots,z_l)$$

for $h_1,\ldots,h_n \in \mathcal{S}$, depending on the type of the Hironaka scheme, as follows.

(Type 3 or type 4-1 or type 5-2 or type 5-5) $h_1 = x_1x_2$,

(Type 4-n or type 5-n for $2 \le n \le 4$) $h_i = x_jx_k$ *for* $\{i,j,k\} = \{1,2,3\}$, $h_4 = 0$,

(Type 5) $h_1 = x_1 \cdot x_2^2$,

(Type 5-*) $h_1,\ldots,h_n \in k[x_ix_j \,|\, 1 \le i < j \le 4]\backslash\{0\}$.

Proof. Let $\mathcal{S}$ be a symmetric algebra over K with $\dim_K \mathcal{S}_1 = n+m+l$. We choose a K-basis $(x_1,\ldots,x_m,y_1,\ldots,y_n,z_1,\ldots,z_l)$ of $\mathcal{S}_1$ for natural numbers n,m,l as above. We identify $X_1,\ldots,X_m,Y_1,\ldots,Y_n,Z_1,\ldots,Z_l$ with their image in the symmetric algebra $S\otimes_k K = \mathrm{Sym}_K(S_1\otimes_k K)$ over K. As K is perfect, for every $j\in\{1,\ldots,n\}$ there is a unique homogeneous element $\epsilon_j \in S_1\otimes_k K$ with $\epsilon_j^p = \eta_j$. The family

$$(X_1,\ldots,X_m,\epsilon_1,\ldots,\epsilon_n,Z_1,\ldots,Z_l)$$

is a K-basis of $S_1\otimes_k K$. There is a unique morphism $\alpha : S\otimes_k K \to \mathcal{S}$ of graded k-algebras with

$$(\alpha(X_1),\ldots,\alpha(X_m),\alpha(\epsilon_1),\ldots,\alpha(\epsilon_n),\alpha(Z_1),\ldots,\alpha(Z_l)) = (x_1,\ldots,x_m,y_1,\ldots,y_n,z_1,\ldots,z_l).$$

Then α is an isomorphism and satisfies (3.5.10.A). Let $\mathfrak{m}_{R_E'}$ denote the maximal ideal of R_E'. For a system $(\rho_1,\ldots,\rho_t)$ of regular parameters of R_E', its image $(\mathrm{in}\rho_1,\ldots,\mathrm{in}\rho_t)$ in the symmetric algebra $\mathrm{gr}R_E'\otimes_{\kappa'}K = \mathrm{Sym}_K(\mathfrak{m}_{R_E'}\otimes_{R_E'}K)$ over K is a K-basis of $\mathfrak{m}_{R_E'}\otimes_{R_E'}K$. Every homogeneous additive element P of $\mathrm{gr}R_E'\otimes_{\kappa'}K$ of degree p has a unique element Q of $\mathfrak{m}_{R_E'}\otimes_{R_E'}K$ with $Q^p = P$. Thus, depending on the type of the Hironaka scheme, there are elements $\epsilon_1',\ldots,\epsilon_n'$ of $\mathfrak{m}_{R_E'}\otimes_{\kappa'}K$ with

(**Type 3 or type 4-1 or type 5-1**) $\mathrm{in}\eta_1' = {\epsilon_1'}^2 + \mathrm{in}\mathcal{X}_2\cdot\mathrm{in}\mathcal{X}_3$,

(**Type 4-*n* or type 5-*n* for $2\le n\le 4$**) $\mathrm{in}\eta_i' = {\epsilon_i'}^2 + \mathrm{in}\mathcal{X}_j\mathrm{in}\mathcal{X}_k$, for $\{i,j,k\} = \{1,2,3\}$, and $\mathrm{in}\eta_4' = \mathrm{in}(\eta_4' + \mathcal{X}_1\mathcal{X}_2\mathcal{X}_3) = {\epsilon_4'}^2$,

(**Type 5**) $\mathrm{in}\eta_1' = {\epsilon_1'}^3 + \mathrm{in}\mathcal{W}_1^2\mathrm{in}\mathcal{W}_2$,

(**Type 5-5, case 1**) $\mathrm{in}\eta_1' = {\epsilon_1'}^2 + \mathrm{in}\mathcal{U}_2\mathrm{in}\mathcal{U}_3$,

(**Type 5-5, case 2**) $\mathrm{in}\eta_1' = {\epsilon_1'}^2 + \mathrm{in}\mathcal{V}_3\mathrm{in}\mathcal{V}_4$,

(**Type 5-***) $\mathrm{in}\eta_i' = \mathrm{in}(\eta_i' + \mathcal{R}_i) = {\epsilon_i'}^2 + \mathrm{in}P_i$ for $1\le i\le n$.

By corollary (3.5.8) we can extend σ, as a family of elements of R_E', by suitable elements $\mathcal{T}_1,\ldots,\mathcal{T}_s$ to a system $(\sigma,\mathcal{T}_1,\ldots,\mathcal{T}_s)$ of regular parameters of R_E'. Then the family

$$(p_1,\ldots,p_{m'},\mathrm{in}\mathcal{T}_1,\ldots,\mathrm{in}\mathcal{T}_s,\epsilon_1',\ldots,\epsilon_n',\mathrm{in}\mathcal{Z}_1,\ldots,\mathrm{in}\mathcal{Z}_l)$$

is a K-basis of $\mathfrak{m}_{R_E'}\otimes_{\kappa'}K$ where, depending on the type of the Hironaka scheme, the $p_1,\ldots,p_{m'}$ are as follows.

(**Type 3 or type 4-**1 **or type 5-**1) $(p_1, \dots, p_{m'}) = (\mathrm{in}\mathcal{X}_2, \mathrm{in}\mathcal{X}_3)$,

(**Type 4-*n* or type 5-*n* for 2 ≤ *n* ≤ 4**) $(p_1, \dots, p_{m'}) = (\mathrm{in}\mathcal{X}_1, \mathrm{in}\mathcal{X}_2, \mathrm{in}\mathcal{X}_3)$,

(**Type 5**) $(p_1, \dots, p_{m'}) = (\mathrm{in}\mathcal{W}_2, \mathrm{in}\mathcal{W}_1, \mathrm{in}\mathcal{W}_4, \mathrm{in}\mathcal{W}_5)$,

(**Type 5-5, case 1**) $(p_1, \dots, p_{m'}) = (\mathrm{in}\mathcal{U}_2, \mathrm{in}\mathcal{U}_3, \mathrm{in}(X_1/v), \mathrm{in}(X_5/v))$,

(**Type 5-5, case 2**) $(p_1, \dots, p_{m'}) = (\mathrm{in}\mathcal{V}_3, \mathrm{in}\mathcal{V}_4, \mathrm{in}\mathcal{V}_2)$,

(**Type 5-***) $(p_1, \dots, p_{m'}) = (\mathrm{in}\mathcal{X}_1, \dots, \mathrm{in}\mathcal{X}_4)$.

Then there is a unique morphism $\alpha' : \mathrm{gr}R'_E \otimes_{\kappa'} K \to \mathcal{S}$ of graded K-algebras with

$$(\alpha'(p_1), \dots, \alpha'(p_{m'}), \alpha'(\mathrm{in}\mathcal{T}_1), \dots, \alpha'(\mathrm{in}\mathcal{T}_s), \alpha'(\epsilon'_1), \dots, \alpha'(\epsilon'_n), \alpha'(\mathrm{in}\mathcal{Z}_1), \dots, \alpha'(\mathrm{in}\mathcal{Z}_l)) =$$

$$= (x_1, \dots, x_{m'}, x_{m'+1}, \dots, x_{m'+s}, y_1, \dots, y_n, z_1, \dots, z_l).$$

Then α' is an isomorphism on its image=: $\mathcal{S}'$ and satisfies (3.5.10.B). □

Remark (3.5.11). *In all cases except the case* $5-5$ *we have* $\kappa' = k(a_i^{1/p} \mid i)$ *and therefore the elements* $\epsilon'_1, \dots, \epsilon'_n$ *lie* $\mathfrak{m}_{R'_E} \otimes_{\kappa'} \kappa' = \mathfrak{m}_{R'_E}$*. Thus the proof shows that in these cases corollary (3.5.10) holds if we set* $\kappa' =: K$ *(then* K *is not necessarily perfect).*

3.6 Reduction to algebra

Using the results from the last section we prove

Theorem (3.6.1). *The theorem (3.2.5) holds if the theorem (3.2.6) holds.*

Proof. Assume the situation of theorem (3.2.5). Let C be the subcone of a vector group V over k. Let S denote the symmetric algebra ΓV over k and let I denote the homogenous ideal $\ker(S \to \Gamma C)$ of S. We have a commutative diagram of schemes

$$\begin{array}{ccccc} E_V := \pi_V^{-1}(\{x\}) & \hookrightarrow & V' & \xrightarrow{\pi_V} & V \\ \uparrow & & \uparrow & & \uparrow \\ E_C := \pi_C^{-1}(\{x\}) & \hookrightarrow & C' & \xrightarrow{\pi_C} & C \end{array}$$

where π_C, π_V are point blow-ups in x and the two left vertical morphism are the induced closed immersions. We get a commutative diagram local rings at x resp. x'

$$\begin{array}{ccccc} R'_E & \longleftarrow & R' & \longleftarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ \overline{R}'_E & \longleftarrow & \overline{R}' & \longleftarrow & \overline{R} \end{array}$$

where R resp. $\overline{R}$ is the localization of S resp. S/I by the maximal ideal $S_+ \subseteq S$ resp. $\overline{S}_+ \subseteq \overline{S}$. The morphism of rings $S \to R$ induces an isomorphism of k-vector spaces resp. k-algebras

$S_1 \cong \mathfrak{m}_R/\mathfrak{m}_R^2$ resp. $S \cong \mathrm{gr}R$ where $\mathfrak{m}_R$ denotes the maximal ideal of R. By remark (2.1.7) we have $E_V = \mathrm{Proj}(\mathrm{gr}R) = \mathbb{P}(V) =: \mathbb{P}$. Identify x' with its image in $\mathbb{P}$. Let $B_{\mathbb{P},x'}$ denote the Hironaka scheme at $x' \in \mathbb{P}$. Let $\mathcal{U}_{\mathbb{P},x'} \subseteq S$ denote the ring of invariants of $(B_{\mathbb{P},x'}, V)$. By lemma (2.1.9) there is a regular parameter $v \in R$ and a prime ideal q of $R[\mathfrak{m}_R/v]$ such that $(R[\mathfrak{m}_R/v])_q$ and R' are isomorphic as R-algebras. By lemma (2.6.7) there is a standard basis $(f_1, \dots, f_t)$ of $J := \ker(R \to \overline{R}) = IR$ with $\mathrm{in}(f_1), \dots, \mathrm{in}(f_t) \in \mathcal{U}_{\mathbb{P},x'}$. Copying the argument in the proof of theorem (3.1.2) we get the inequality $\dim \mathcal{B}_{\mathbb{P},x'} \le \dim C \le 5$ and we get that $\mathcal{B}_{\mathbb{P},x'}$ is not a vector group. Then Oda's result gives us a characterization of $\mathcal{B}_{\mathbb{P},x'}$ as cited in theorem (3.5.5). Then the characteristic p of k is two or three. By corollary (3.5.7) there are a k-basis $(X_1, \dots, X_m, Y_1, \dots, Y_n, Z_1, \dots, Z_l)$ of S_1 and certain homogeneous elements $\eta_1, \dots, \eta_n$ of S of degree p with

$$\mathcal{U}_{\mathbb{P},x'} = k[\eta_1, \dots, \eta_n, Z_1, \dots, Z_l].$$

Write $\kappa' := \kappa(x')$. We have $k = \kappa(x)$. We have to show

$$\dim \mathrm{Rid}_{C',x'} + \mathrm{trdeg}(\kappa'/k) < \dim \mathrm{Rid}_{C,x} \tag{3.6.1.A}$$

Let C'_E denote the cone $\mathrm{Spec}(\mathrm{gr}\overline{R}'_E)$ over κ'. By definition we have

$$\mathrm{Rid}_{C,x} = \mathrm{Rid}(C), \quad \mathrm{Rid}_{E_C,x'} = \mathrm{Rid}(C'_E). \tag{3.6.1.B}$$

Let $K|\kappa'$ be a extension of κ' by a perfect field K. By remark (2.5.14) or by corollary (3.7.17) we have

$$\mathrm{Rid}(C \times_k K) \cong \mathrm{Rid}(C) \times_k K, \quad \mathrm{Rid}(C'_E \times_{\kappa'} K) \cong \mathrm{Rid}(C'_E) \times_{\kappa'} K. \tag{3.6.1.C}$$

By corollary (3.5.10) there is a symmetric K-algebra $\mathcal{S}$, a graded K-subalgebra $\mathcal{S}'$ of $\mathcal{S}$, a K-basis

$$(x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_l) =: (\underline{x}, \underline{y}, \underline{z})$$

of $\mathcal{S}_1$ and isomorphisms of graded K-algebras $\alpha : S \times_k K \to \mathcal{S}$, $\alpha' : \mathrm{gr}R'_E \otimes_{\kappa'} K \to \mathcal{S}'$ such that the following diagram of graded K-algebras commutes

$$\begin{array}{ccc}
S \times_k K & \xrightarrow{\quad\alpha\quad} & \mathcal{S} \\
\uparrow & & \uparrow \\
K[\eta_1, \dots, \eta_n, Z_1, \dots, Z_l] & \longrightarrow & K[y_1^p, \dots, y_n^p, \underline{z}] \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} \\
K[\mathrm{in}\eta'_1, \dots, \mathrm{in}\eta'_n, \mathrm{in}\mathcal{Z}_1, \dots, \mathrm{in}\mathcal{Z}_l] & \longrightarrow & K[y_1^p + h_1, \dots, y_n^p + h_n, \underline{z}] \\
\downarrow & & \downarrow \\
\mathrm{gr}R'_E \otimes_{\kappa'} K & \xrightarrow[\quad\alpha'\quad]{} & \mathcal{S}'
\end{array}$$

where we write $\eta'_i = \eta_i \cdot (\mathrm{in}v)^{-p}$ and $\mathcal{Z}_j = Z_j \cdot (\mathrm{in}v)^{-1} \in \mathrm{gr}R_{(\mathrm{in}v)} \subseteq R'_E$, where $h_1, \dots, h_n \in \mathcal{S}$ are some homogeneous elements of degree p specified in corollary (3.5.10), and where for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, l\}$ we have

$$\alpha(\eta_i) = y_i^p, \quad \alpha'(\mathrm{in}\eta'_i) = y_i^p + h_i, \quad \theta(\eta_i) = \mathrm{in}\eta'_i, \quad \psi(y_i^p) = y_i^p + h_i,$$

$$\alpha(Z_j) = z_j, \quad \alpha'(\mathrm{in}\mathcal{Z}_j) = z_j, \quad \theta(Z_j) = \mathrm{in}\mathcal{Z}_j, \quad \psi(z_j) = z_j.$$

Define the finite sets of homogeneous elements

$$F := \alpha(\{\mathrm{in}f_1, \dots, \mathrm{in}f_t\}) \subseteq K[y_1^p, \dots, y_n^p, \underline{z}], \quad F' := \psi(F) \subseteq K[y_1^p + h_1, \dots, y_n^p + h_n, \underline{z}]$$

By lemma (2.6.7) we may assume that the elements $f_1, \dots, f_t$ have the property that, for the elements $\psi_j := \mathrm{in}f_j \cdot (\mathrm{in}v)^{-\nu(f_j)} \in \mathrm{gr}R_{(\mathrm{in}v)} \subseteq R'_E$, the tuple $(\psi_1, \dots, \psi_t)$ is a standard basis of the ideal $\ker(R'_E \to \overline{R}'_E)$ of R'_E and for each $j \in \{1, \dots, t\}$ one has $\nu(\psi_j) = \nu(f_j)$. Then we get an isomorphism of graded κ'-algebras

$$\mathrm{gr}\overline{R}'_E \cong (\mathrm{gr}R'_E)/\langle \mathrm{in}\psi_1, \dots, \mathrm{in}\psi_t\rangle.$$

The morphism θ maps $\mathrm{in}f_j$ to $\mathrm{in}\psi_j$. Thus α' induces an isomorphism of graded K-algebras $\mathrm{gr}\overline{R}'_E \times_{\kappa'} K \cong \mathcal{S}'/\langle F'\rangle$. The last diagram induces commutative diagrams of cones over K

$$\begin{array}{ccc} V \times_k K & \xrightarrow[\beta]{\cong} & \mathrm{Spec}(\mathcal{S}) \\ \uparrow & & \uparrow \\ C \times_k K & \xrightarrow{\cong} & \mathrm{Spec}(\mathcal{S}/\langle F\rangle) \\ \uparrow & & \uparrow \\ \mathrm{Rid}(C \times_k K) & \xrightarrow[\gamma]{\cong} & \mathrm{Rid}(\mathrm{Spec}(\mathcal{S}/\langle F\rangle)) \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}(\mathrm{gr}R'_E) \times_{\kappa'} K & \xrightarrow[\delta]{\cong} & \mathrm{Spec}(\mathcal{S}') \\ \uparrow & & \uparrow \\ C'_E \times_{\kappa'} K & \xrightarrow{\cong} & \mathrm{Spec}(\mathcal{S}'/\langle F'\rangle) \\ \uparrow & & \uparrow \\ \mathrm{Rid}(C'_E \times_{\kappa'} K) & \xrightarrow[\epsilon]{\cong} & \mathrm{Rid}(\mathrm{Spec}(\mathcal{S}'/\langle F'\rangle)) \end{array}$$

Where $\beta, \gamma, \delta, \epsilon$ are isomorphisms of additive groups over K. Let $\mathcal{U}_{\langle F\rangle} \subseteq \mathcal{S}$ resp. $\mathcal{U}_{\langle F'\rangle} \subseteq \mathcal{S}'$ denote the ring of invariants of $(\mathrm{Rid}(\mathrm{Spec}(\mathcal{S}/\langle F\rangle)), \mathrm{Spec}(\mathcal{S}))$ resp. $(\mathrm{Rid}(\mathrm{Spec}(\mathcal{S}'/\langle F'\rangle)), \mathrm{Spec}(\mathcal{S}'))$. By remark (2.5.6) we have

$$\dim \mathrm{Rid}(\mathrm{Spec}(\mathcal{S}/\langle F\rangle)) = \dim \mathcal{S} - \dim \mathcal{U}_{\langle F\rangle}, \quad \dim \mathrm{Rid}(\mathrm{Spec}(\mathcal{S}'/\langle F'\rangle)) = \dim \mathcal{S}' - \dim \mathcal{U}_{\langle F'\rangle}.$$

With (3.6.1.B) and (3.6.1.C) we get

(3.6.1.D) $$\dim \mathrm{Rid}_{C,x} = \dim \mathcal{S} - \dim \mathcal{U}_{\langle F\rangle}, \quad \dim \mathrm{Rid}_{E_C,x'} = \dim \mathcal{S}' - \dim \mathcal{U}_{\langle F'\rangle}.$$

By lemma (3.4.1) we have $\dim \mathrm{Rid}_{C',x'} = \dim \mathrm{Rid}_{E_C,x'} + 1$. We get

(3.6.1.E) $$\dim \mathrm{Rid}_{C',x'} = \dim \mathcal{S}' - \dim \mathcal{U}_{\langle F'\rangle} + 1.$$

Let $\mathfrak{q}$ denote the to $x' \in \mathbb{P}$ associated homogeneous prime ideal of S. Then R'_E and $S_{(\mathfrak{q})}$ are isomorphic. For the homogeneous element $w := v \in S$ and the prime ideal $\mathfrak{p}$ of $S_{(w)}$ from corollary (3.5.8) we have an isomorphism of rings $(S_{(w)})_{\mathfrak{p}} \cong S_{(\mathfrak{q})}$. Thus the residue field κ' of R'_E is the quotient field of $S_{(w)}/\mathfrak{p}$. Then we have

$$\dim S_{(w)} = \dim(S_{(w)})_{\mathfrak{p}} + \dim S_{(w)}/\mathfrak{p} = \dim R'_E + \mathrm{trdeg}(\kappa'/k).$$

Combining this with

$$\dim S_{(w)} = \dim S - 1, \quad S \otimes_k K \cong \mathcal{S}, \quad \mathrm{gr}R'_E \otimes_{\kappa'} K \cong \mathcal{S}'$$

we get

$$\dim \mathcal{S} - 1 = \dim \mathcal{S}' + \operatorname{trdeg}(\kappa'/k). \tag{3.6.1.F}$$

Combining (3.6.1.D),(3.6.1.E) and (3.6.1.F) we get

$$\dim \operatorname{Rid}_{C',x'} + \operatorname{trdeg}(\kappa'/k) = \dim \operatorname{Rid}_{C,x} + (\dim \mathcal{U}_{\langle F\rangle} - \dim \mathcal{U}_{\langle F'\rangle}).$$

Thus the claimed inequality (3.6.1.A) follows if we show the inequality $\dim \mathcal{U}_{\langle F\rangle} - \dim \mathcal{U}_{\langle F'\rangle} < 0$. By theorem (3.2.6) we have the stronger inequality $\dim \mathcal{U}_{\langle F\rangle} + 2 \le \dim \mathcal{U}_{\langle F'\rangle}$. Note that the assumption $\dim(\mathcal{S}/\langle F\rangle) = m$ from theorem (3.2.6) is satisfied because we have

$$m = \dim \mathcal{B}_{\mathbb{P},x'} \le \dim C = \dim(\mathcal{S}/\langle F\rangle) \le \dim(\mathcal{S}/\langle y_1,\dots,y_n,z_1,\dots,z_l\rangle) = m.$$

□

3.7 On Giraud bases and computation of the ridge

We recall the Buchberger algorithm for calculating (reduced) Gröbner bases. A reduced Gröbner basis is a Giraud basis (lemma (3.7.14)). The ridge of a cone can be calculated by applying differential operators on a Giraud basis, see theorem (3.7.16).
Let k be a field. For the whole section let $S := k[X_1,\dots,X_n]$ be a graded polynomial ring over k, i.e. $S = \oplus_{n\in\mathbb{N}} S_n$ is a symmetric algebra over k and $(X_1,\dots,X_n)$ is a k-basis of S_1. Let $I \subseteq S$ be an non-zero ideal (not necessarily homogeneous). A finite subset of $\cup_{n\in\mathbb{N}} S_n$ will be called homogeneous finite subset of S. For a multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ we write $X^\alpha := X_1^{\alpha_1}\cdot\ldots\cdot X_n^{\alpha_n}$. For a polynomial $f = \sum_{\alpha\in\mathbb{N}^n} \lambda_\alpha X^\alpha$, $\lambda_\alpha \in k$, of S and for a multi-index $\beta \in \mathbb{N}^n$ we write $f_\beta := \lambda_\beta X^\beta$. For multi-indices $\alpha = (\alpha_1,\dots,\alpha_n), \beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$ we write $\alpha!$ resp. $\binom{\beta}{\alpha}$ for the product of factorials resp. of binomial coefficients

$$\alpha_1!\cdot\ldots\cdot\alpha_n! \quad \text{resp.} \quad \binom{\beta_1}{\alpha_1}\cdot\ldots\cdot\binom{\beta_n}{\alpha_n}$$

where we set $\binom{\beta_i}{\alpha_i} := 0$ if $\beta_i < \alpha_i$. For a multi-index $\alpha \in \mathbb{N}^n$ let D_α denote the k-linear map $S \to S$ with

$$D_\alpha(X^\beta) = \binom{\beta}{\alpha} X^{\beta-\alpha}$$

for all $\beta \in \mathbb{N}^n$. The maps D_α, $\alpha \in \mathbb{N}$, are differential operators on S and they are also called Hasse-Schmidt derivations.

Definition (3.7.1). *For a finite subset $F = \{f_1,\dots,f_m\} \subseteq S$ we define the k-subalgebra of S*

$$\mathcal{U}_S(f_1,\dots,f_m) := \mathcal{U}_S(F) := k[D_\alpha f \mid f \in F,\, \alpha \in \mathbb{N}^n].$$

Remark (3.7.2). *a) For arbitrary $\alpha,\beta \in \mathbb{N}$ one has $D_\alpha \circ D_\beta = \binom{\alpha+\beta}{\alpha} D_{\alpha+\beta}$.*

b) For $\alpha \in \mathbb{N}^n$ define the $\mathbb{Z}$-linear map $\partial^\alpha_{\mathbb{Z}} : \mathbb{Z}[X_1,\dots,X_n] =: \mathbb{Z}[\underline{X}] \to \mathbb{Z}[\underline{X}]$ with $\partial^\alpha_{\mathbb{Z}}(X^\beta) = (\beta!/(\beta-\alpha)!)X^{\beta-\alpha}$, $\beta \in \mathbb{N}^n$. One has $\partial^\alpha_{\mathbb{Z}} = \prod_{i=1}^n (\partial^{e_i}_{\mathbb{Z}})^{\alpha_i}$ where we write $\alpha = (\alpha_1,\dots,\alpha_n)$ and *where* $e_1, e_2, \ldots, e_n$ *denote the elements* $(1,0,\ldots,0)$, $(0,1,0,\ldots,0)$, $\ldots$, $(0,\ldots,0,1)$ *of* $\mathbb{N}^n$. *With an induction on* $|\alpha| = \alpha_1 + \ldots + \alpha_n$ *one can prove for all* $z, z' \in \mathbb{Z}[\underline{X}]$ *the equality*

$$\partial_{\mathbb{Z}}^{\alpha}(z \cdot z') = \sum_{\beta,\beta' \in \mathbb{N}^n,\ \beta+\beta'=\alpha} \binom{\alpha}{\beta} (\partial_{\mathbb{Z}}^{\beta} z)(\partial_{\mathbb{Z}}^{\beta'} z').$$

Using the identity

$$D_\alpha = ((\alpha!)^{-1}\partial_{\mathbb{Z}}^{\alpha}) \otimes_{\mathbb{Z}} k : S = \mathbb{Z}[\underline{X}] \otimes_{\mathbb{Z}} k \to \mathbb{Z}[\underline{X}] \otimes_{\mathbb{Z}} k = S$$

one gets for all $s, s' \in S$

$$D_\alpha(s \cdot s') = \sum_{\beta,\beta' \in \mathbb{N}^n,\ \beta+\beta'=\alpha} (D_\beta s) \cdot (D_{\beta'} s').$$

c) *For an additive element* a *of* S *and for a multi-index* $\alpha \in \mathbb{N}^n$ *one has* $D_\alpha a \in \{a\} \cup k$.

Definition (3.7.3). *A* ***monomial order on S*** *is a total order* $\le$ *on* $\mathbb{N}^n$ *such that the induced order (also denoted by* $\le$*) on the image of the injective map* $\mathbb{N}^n \to S : \alpha \mapsto X^\alpha$ *is compatible with the divisibility relation and with the product of in* S*, i.e. for all multi-indices* $\alpha, \beta\, \alpha', \beta' \in \mathbb{N}^n$

i) $X^\alpha | X^\beta$ *implies* $X^\alpha \le X^\beta$ *and*

ii) $X^\alpha \le X^\beta$ *and* $X^{\alpha'} \le X^{\beta'}$ *imply* $X^\alpha \cdot X^{\alpha'} \le X^\beta \cdot X^{\beta'}$.

Example (3.7.4). a) For $\alpha, \beta \in \mathbb{N}^n$ define

$$\alpha \le_{\mathrm{lex}} \beta \quad :\Leftrightarrow \quad \alpha = \beta \quad \text{or} \quad \beta_i - \alpha_i > 0 = \beta_{i-1} - \alpha_{i-1} = \ldots = \beta_1 - \alpha_1 \text{ for some } i \in \{1,\ldots,n\}.$$

We call $\le_{\mathrm{lex}}$ **the lexicographical order on** $\mathbb{N}^n$. It is a monomial order on S.

b) For $\alpha, \beta \in \mathbb{N}^n$ define

$$\alpha \le_{\mathrm{grlex}} \beta \quad :\Leftrightarrow \quad (|\alpha|, \alpha) \le' (|\beta|, \beta)$$

for the the lexicographical order $\le'$ on $\mathbb{N} \times \mathbb{N}^n$. We call $\le_{\mathrm{grlex}}$ **the graded lexicographical order on** $\mathbb{N}^n$. It is a monomial order on S.

c) Let $\le$ be a monomial order on S. Then for every permutation of the basis $(X_1, \ldots, X_n)$, i.e. for every automorphism σ of $\{1, \ldots, n\}$, we get a monomial order $\le_\sigma$ on S as follows. Write $(\alpha_1, \ldots, \alpha_n)_\sigma := (\alpha_{\sigma 1}, \ldots, \alpha_{\sigma n})$. Define $\alpha \le_\sigma \beta :\Leftrightarrow \alpha_\sigma \le \beta_\sigma$.

Remark (3.7.5). *Not to confuse monomial orders with the product order on* $\mathbb{N}^n$ *we write* $\le_c$ *for the product order, i.e.*

$$\alpha \le_c \beta \quad \Leftrightarrow \quad \beta - \alpha \in \mathbb{N}^n.$$

Definition (3.7.6). *Let* $\le$ *be a monomial order on* S.

a) *For a polynomial* $f \in S\backslash\{0\}$ *we define* ***the multidegree of f*** *and* ***the leading term of f (w.r.t.*** $\le$*) by*

$$\mathrm{multideg}\, f := \max\{\alpha \in \mathbb{N}^n \mid f_\alpha \ne 0\}, \quad \mathrm{LT} f := f_{\mathrm{multideg}\, f}$$

and call f ***monic*** *if* $\mathrm{LT} f = X^{\mathrm{multideg}\, f}$. *We set* $\mathrm{LT}0 := 0$.

b) We write $\langle \mathrm{LT}(I)\rangle$ *for the ideal* $\langle \mathrm{LT} f \mid f \in I\rangle$ *of* S*.*

c) A ***Gröbner basis of I (w.r.t. ≤)*** *is a finite subset* $F \subseteq S$ *which generates* I *and has the property* $\langle \mathrm{LT}(I)\rangle = \langle \mathrm{LT} f \mid f \in F\rangle$*. Such a set is called* ***reduced Gröbner basis of I (w.r.t. ≤)*** *if* $0 \notin F$*, the elements of* F *are monic and for every two distinct elements* f, g *of* F *and every* $\alpha \in \mathbb{N}^n$ *with* $\mathrm{LT} g | f_\alpha$ *one has* $f_\alpha = 0$*.*

Remark (3.7.7). *For elements* $f, g \in S\backslash\{0\}$ *we have*

$$\mathrm{LT}(fg) = \mathrm{LT} f \cdot \mathrm{LT} g, \quad \mathrm{multideg}\,(fg) = \mathrm{multideg}\, f + \mathrm{multideg}\, g.$$

The Buchberger algorithm gives a method to compute a reduced Gröbner basis of an ideal of S (see theorem (3.7.11)).

Definition (3.7.8). *Fix a monomial order on* S *and let* $F \subseteq S$ *be a finite subset.*

a) An element $f \in F\backslash\{0\}$ *is called* ***reducible by*** $\boldsymbol{F\backslash\{f\}}$ *if there are a multi-index* $\alpha \in \mathbb{N}^n$ *with* $f_\alpha \neq 0$ *and an element* $g \in F\backslash\{f\}$ *with* $\mathrm{LT} g | f_\alpha$*.*

b) Let $f \in F\backslash\{0\}$ *be reducible by* $F\backslash\{f\} =: G$*. A* ***one-step reduction of f by G*** *is an assignment*

$$G \cup \{f\} \mapsto G \cup \{f'\} \quad \text{where} \quad f' = c\big(f - \frac{f_\alpha}{\mathrm{LT} g} g\big)$$

for some $\alpha \in \mathbb{N}^n$*,* $g \in G\backslash\{0\}$ *with* $\mathrm{LT} g | f_\alpha \neq 0$ *and for some* $c \in k\backslash\{0\}$ *such that* f' *is monic or* $f' = 0$*.*

c) Let $f^{(0)} \in F\backslash\{0\}$ *and* $F\backslash\{f^{(0)}\} =: G$*. A* ***reduction of*** $\boldsymbol{f^{(0)}}$ ***by G*** *is a sequence of one-step reductions*

$$G \cup \{f^{(0)}\} \mapsto G \cup \{f^{(1)}\} \mapsto G \cup \{f^{(2)}\} \mapsto \dots$$

such that for some $k \in \mathbb{N}$ *the element* $f^{(k)}$ *is zero or not reducible by* G*.*

d) A ***reduction of F*** *is a sequence*

$$F = F^{(0)} \mapsto F^{(1)} \mapsto F^{(2)} \mapsto \dots$$

where for every $i = 0, 1, 2, \dots$ *the assignment* $F^{(i)} \mapsto F^{(i+1)}$ *is a reduction of an element* $f \in F^{(i)}$ *by* $F^{(i)}\backslash\{f\}$ *and for some* $k \in \mathbb{N}$ *no* $h \in F^{(k)}\backslash\{0\}$ *is reducible by* $F^{(k)}\backslash\{h\}$*.*

e) For $f, g \in F\backslash\{0\}$ *we define the* ***s-polynomial of f and g***

$$s(f, g) := f\frac{m}{\mathrm{LT} f} - g\frac{m}{\mathrm{LT} g}$$

where m *is the monic smallest common multiple of* $\mathrm{LT} f$ *and* $\mathrm{LT} g$*.*

f) A ***Buchberger step on F*** *is a sequence*

$$F \mapsto F' \mapsto F'' \mapsto F'''$$

where $F' := F \cup \{s(f, g)\}$ *for some* $f, g \in F\backslash\{0\}$ *with* $s(f, g) \notin F \cup \{0\}$*,* $F' \mapsto F''$ *is a reduction of* $s(f, g)$ *by* F *and* $F''' := F''\backslash\{0\}$*.*

g) *A **Buchberger algorithm on F** is a sequence of Buchberger steps*

$$F = F^{(0)} \mapsto F^{(1)} \mapsto F^{(2)} \mapsto \dots$$

such that for some $k \in \mathbb{N}$ the system $F^{(k)}$ is a Gröbner basis of $\langle F \rangle$.

Remark (3.7.9). *a) Let $F \mapsto F'$ be a reduction of an element $f \in F$ by $G = F \backslash \{f\}$ or a Buchberger algorithm. Then F and F' generate the same ideal in S.*

b) A reduction of a Gröbner basis F of $\langle F \rangle$ is a Gröbner basis of $\langle F \rangle$ again. To see this it is enough to assume that the reduction consists of a single reduction step. A subset $H \subseteq S$ is a Gröbner basis of $\langle H \rangle$ if and only if for each $h \in \langle H \rangle$ there is some $q \in H$ with $\mathrm{LT}\, q \,|\, \mathrm{LT}\, h$. Let $F = G \cup \{f\} \mapsto G \cup \{f'\}$ be a one step reduction and $f' = c(f - g f_\alpha / \mathrm{LT}\, g)$, G, f, g, α as in definition (3.7.8)b). Let h be a n element of $\langle F \rangle$. By assumption on F there is some $q \in F$ with $\mathrm{LT}\, q \,|\, \mathrm{LT}\, h$. If $q \in G$ then $q \in G \cup \{f'\}$ and we are done. Assume $q = f$. If $\alpha = \mathrm{multideg}\, f$ then $\mathrm{LT}\, g \,|\, \mathrm{LT}\, f \,|\, \mathrm{LT}\, h$. Thus $g \in G \subseteq G \cup \{f'\}$ has the wished property. If $\alpha \neq \mathrm{multideg}\, f$ then $\mathrm{multideg}\, f = \mathrm{multideg}\, f'$ and therefore $\mathrm{LT}\, f' \,|\, \mathrm{LT}\, h$.

The following theorem is known as Buchberger's Criterion.

Theorem (3.7.10). *As in the definition above fix a monomial order $\leq$ on S and let $F \subseteq S$ be a finite subset. The set F is a Gröbner basis of $\langle F \rangle$ if and only if for all pairs $f, g \in F \backslash \{0\}$ with $s(f,g) \neq 0$ one can write*

$$s(f,g) = \sum_{h \in F} a_h \cdot h$$

for elements $a_h \in S$ with $\mathrm{multideg}\, s(f,g) \geq \mathrm{multideg}(a_h \cdot h)$ if $a_h \cdot h \neq 0$.

Proof. [CLO], theorem 6 on page 85. □

Theorem (3.7.11). *Fix a monomial order on S and let $F \subseteq S$ be a finite subset.*

a) Every element $0 \neq f \in F$ has a reduction by $F \backslash \{f\}$.

b) There is a Buchberger algorithm on F.

c) If F is a Gröbner basis of $\langle F \rangle$, then there is a reduction of F.

d) There is a Buchberger algorithm $F \mapsto F'$ of F and a reduction $F' \mapsto F''$ of F'. Then F'' is a reduced Gröbner basis of $\langle F \rangle$.

Proof. a) [CLO], proof of theorem 3 on page 64.

b) [CLO], proof of theorem 2 on page 90.

c) [CLO], proof of proposition 6 on page 92.

d) Follows from b) and c) and remark (3.7.9).

□

Remark (3.7.12). *Fix a monomial order on S and let $F \subseteq S$ be a finite subset which generates I. Let $F \mapsto F'$ be a Buchberger algorithm and $F' \mapsto F''$ be a reduction as in theorem (3.7.11). Let M be a subset of S with $F \subseteq M$ with the property that M is stable under one-step reductions and under forming the s-polynomial $(f,g) \mapsto s(f,g)$, i.e.*

- *If $f, g \in M$, $\alpha \in \mathbb{N}^n$ with $f \neq g$, $0 \neq f_\alpha \in \langle \mathrm{LT} g\rangle$ and $c \in k\backslash\{0\}$ such that $f' = c(f - g f_\alpha/\mathrm{LT} g)$ is monic or $= 0$ then $f' \in M$,*

- *If $f, g \in M\backslash\{0\}$ and m is the monic smallest common multiple of $\mathrm{LT} f$ and $\mathrm{LT} g$ then $s(f,g) = fm/\mathrm{LT} f - gm/\mathrm{LT} g \in M$.*

Then by definition of one-step reductions and of the s-polynomial the Gröbner basis F' of I and the reduced Gröbner basis F'' of I lie in M.

Definition (3.7.13). *Let I be a homogeneous ideal of S. Consider the graded lexicographical order on S. A* ***Giraud basis of I*** *is a homogeneous finite subset F of S which generates I and has the property that for every multi-index $\alpha \in \{\mathrm{multideg}\, g \,|\, g \in I\backslash\{0\}\}$ and every $f \in F$ with $|\alpha| < \mathrm{deg} f$ one has $D_\alpha f = 0$.*

Lemma (3.7.14). *Let F be a reduced Gröbner basis of I w.r.t. the graded lexicographical order and assume that the elements of F are homogeneous. Then F is a Giraud basis of I.*

Proof. Let f be an element of F and let g be an element of $I\backslash\{0\}$ with $|\alpha| < \deg f$ for $\alpha :=$ multideg g. Since F is a Gröbner basis of I we have $\mathrm{LT} g \in \langle \mathrm{LT} h \,|\, h \in F\rangle$. This yields the inequality multideg $h \leq_c \alpha$ for some $h \in F$. We have $h \neq f$ because $\deg h \leq |\alpha| < \deg f$. Let $\gamma \in \mathbb{N}^n$. If multideg $h \nleq_c \gamma$, then $\alpha \nleq_c \gamma$ and therefore $D_\alpha f_\gamma = 0$. If multideg $h \leq_c \gamma$, then multideg $h \leq \gamma$ (property of monomial orders) which implies $f_\gamma = 0$ (the Gröbner basis is reduced), in particular $D_\alpha f_\gamma = 0$. We get $D_\alpha f = \sum_{\gamma \in \mathbb{N}^n} D_\alpha f_\gamma = 0$. □

Corollary (3.7.15). *Let I be generated by a homogeneous finite subset F of S. For the graded lexicographical order let $F \mapsto F'$ be a Buchberger algorithm and $F' \mapsto F''$ a reduction. Then F'' is a Giraud basis of I.*

Proof. The set M of all homogenous elements of S has the properties from remark (3.7.12). Thus the elements of F'' are homogeneous. Since F'' is a reduced Gröbner basis of I (see theorem (3.7.11)) it is a Giraud basis of I (see lemma (3.7.14)). □

Theorem (3.7.16). *Let I be a homogeneous ideal of S and let F be a Giraud basis of I. Then $\mathcal{U}_S(F)$ is the ring of invariants of (I,S) (see definitions (2.5.15) and (3.7.1)).*

Proof. [BHM], corollary 2.3. □

Corollary (3.7.17). *For a cone C over k and any field extension $K|k$ the additive groups $\mathrm{Rid}(C) \times_k K$ and $\mathrm{Rid}(C \times_k K)$ over K are isomorphic.*

Proof. Let V be a vector group over k which has C as a subcone. Set $S := \Gamma V$, $I := \ker(\Gamma V \to \Gamma C)$. Choose a k-basis $x_1, \ldots, x_n$ of S_1. Let F be a Giraud basis of I (it exists by theorem (3.7.11) and lemma (3.7.14)). Then F is a Giraud basis of $I \otimes_k K$ w.r.t. the K-basis $x_1 \otimes 1, \ldots, x_n \otimes 1$ of $(S \otimes_k K)_1$ where we identify F with its image in $S \otimes_k K$. Then by theorem (3.7.16) the graded subalgebra

$$\mathcal{U}_{S \otimes_k K}(F) = \mathcal{U}_S(F) \otimes_k K$$

of $S \times_k K$ over K is the ring of invariants of $(\mathrm{Rid}(C \times_k K), V \times_k K)$. The isomorphism $(S/\langle \mathcal{U}_S(F)_+\rangle) \otimes_k K \cong (S \otimes_k K)/\langle \mathcal{U}_S(F)_+ \otimes_k K\rangle$ of graded K-algebras induces an isomorphism of additive groups over K

$$\mathrm{Rid}(C)\times_k K = \mathrm{Spec}\big((S/\langle\mathcal{U}_S(F)_+\rangle)\otimes_k K\big) \cong \mathrm{Spec}\big((S\otimes_k K)/\langle\mathcal{U}_S(F)_+\otimes_k K\rangle\big) = \mathrm{Rid}(C\times_k K).$$

□

3.8 Proof of theorem (3.2.6)

In this section we prove theorem (3.2.6). This will complete the proof of the main theorem. First we need some lemmata. Until end of this section let k be a field of characteristic $p > 0$ and let $S = k[x_1,\dots,x_m,y_1,\dots,y_n,z_1,\dots,z_l]$ be a graded polynomial ring over k, i.e. S is a symmetric algebra over k and

$$(x_1,\dots,x_m,y_1,\dots,y_n,z_1,\dots,z_l) =: (\underline{x},\underline{y},\underline{z})$$

is a k-basis of S_1. As in the section above a finite subset of $\cup_{n\in\mathbb{N}}S_n$ will be called homogeneous finite subset of S. For a subset A of S or a tuple $\underline{v} := (v_1,\dots,v_s)$ of elements of S we write $\langle A\rangle$ resp. $\langle\underline{v}\rangle$ for the ideal of S generated by A resp. $v_1,\dots,v_s$.

In the proof of the inequality $\dim\mathcal{U}_{\langle F\rangle}+2\le\dim\mathcal{U}_{\langle\psi F\rangle}$ from theorem (3.2.6) we will write $\mathcal{U}_{\langle F\rangle} = K[\underline{u}]$ for a family k-algebraically independent polynomials $\underline{u} = (u_1,\dots,u_s)$ and show that there is a K-algebraically independent family of polynomials $\underline{u}' = (u'_1,\dots,u'_s)$ in $\mathcal{U}_{\langle\psi F\rangle}$, which assures $\dim\mathcal{U}_{\langle F\rangle}\le\dim\mathcal{U}_{\langle\psi F\rangle}$. For this argument we need lemma (3.8.2). Further we will show that $\underline{u}'$ can be extended to a K-algebraically independent family in $\mathcal{U}_{\langle\psi F\rangle}$ by at least two elements. This will be achieved with lemma (3.8.14).

Lemma (3.8.1). *Let F be a finite subset of S. Set $\nu := m+n+l$.*

a) For all $u\in\mathcal{U}_S(F)$ and all $\alpha\in\mathbb{N}^\nu$ one has $D_\alpha u\in\mathcal{U}_S(F)$ (see definition (3.7.1)),

b) For every finite subset G of $\mathcal{U}_S(F)$ one has $\mathcal{U}_S(G)\subseteq\mathcal{U}_S(F)$.

Proof. a) The element u is a k-linear combination of elements of the form $D_{\alpha_1}g_1\cdot\ldots\cdot D_{\alpha_t}g_t$ for $t\in\mathbb{N}$, $\alpha_1,\dots,\alpha_t\in\mathbb{N}^\nu$ and $g_1,\dots,g_t\in F$. We show with an induction on t that $D_\alpha u\in\mathcal{U}_S(F)$. If $t=0$ then $u=1$ and $D_\alpha u\in k\subseteq U_S(F)$. If $t=1$ then $u=D_{\alpha_1}g_1$ and $D_\alpha u = \binom{\alpha+\alpha_1}{\alpha}D_{\alpha+\alpha_1}g_1\in\mathcal{U}_S(F)$ (see remark (3.7.2)). Assume $t\ge 2$. Write $h := D_{\alpha_1}g_1\cdot\ldots\cdot D_{\alpha_{t-1}}g_{t-1}$, $h' := D_{\alpha_t}g_t$. For all $\beta\le_c\alpha$ one has $D_\beta h\in\mathcal{U}_S(F)$ and $D_{\alpha-\beta}h'\in\mathcal{U}_S(F)$ by induction hypothesis. Then with remark (3.7.2) we get $D_\alpha u = \sum_{\beta\le_c\alpha}D_\beta h\cdot D_{\alpha-\beta}h'\in\mathcal{U}_S(F)$.

b) The k-subalgebra $\mathcal{U}_S(G)$ of S is generated by elements $D_\alpha(g)$ for $g\in G$, $\alpha\in\mathbb{N}^\nu$. By a) these elements lie in $\mathcal{U}_S(F)$.

□

Lemma (3.8.2). *Let ϕ denote the morphism $S\to S$ of $k[\underline{y},\underline{z}]$-algebras with $\phi(x_1)=\ldots=\phi(x_m)=0$. Let $q\in S\backslash k$ be a homogeneous element. There are homogeneous additive elements $c_1,\dots,c_l$ of S with*

$$\mathcal{U}_S(\phi(q)) = k[\phi(c_1),\dots,\phi(c_l)],\quad \mathcal{U}_S(q)\supseteq k[c_1,\dots,c_l]$$

such that $\phi(c_1),\dots,\phi(c_l)$ are k-algebraically independent.

Proof. We may assume $(\underline{y}, \underline{z}) = \underline{y}$ and $\phi(q) \neq 0$.
First case: $\phi(q)$ additive. Then we have $\mathcal{U}_S(\phi(q)) = k[\phi(q)]$. The element q is a Giraud basis of the principal ideal $\langle q \rangle$ of S. Then by theorem (3.7.16) $\mathcal{U}_S(q)$ is the ring of invariants of $(\langle q \rangle, S)$. Thus there is a set of homogeneous additive elements $\mathcal{A} \subseteq S$ with $\mathcal{U}_S(q) = k[\mathcal{A}]$. Let $\mathcal{B}$ be a homogeneous additive basis of $\phi(\mathcal{A})$ (see definition (2.3.5)). By lemma (2.3.8) there are coefficients $\lambda_b \in k$ and p-powers e_b, $b \in \mathcal{B}$, with $\phi(q) = \sum_{b \in \mathcal{B}} \lambda_b b^{e_b}$ because $\phi(q) \in \phi(\mathcal{U}_S(q)) = K[\phi(\mathcal{A})]$. Choose elements $a_b \in \mathcal{A}$ with $\phi(a_b) = b$ for all $b \in \mathcal{B}$. Set $c := \sum_{b \in \mathcal{B}} \lambda_b a_b^{e_b}$. Then c is a homogeneous additive element of S with $c \in k[\mathcal{A}] = \mathcal{U}_S(q)$ and we have $\mathcal{U}_S(\phi(q)) = k[\phi(q)] = k[\phi(c)]$.
Second case = the general case. An arbitrary element of $\mathcal{U}_S(\phi(q))$ is a finite sum

$$u = \sum_i \lambda_i \cdot (D_{\alpha_1^i} \phi(q)) \cdot \ldots \cdot (D_{\alpha_{j_i}^i} \phi(q))$$

for suitable coefficients $\lambda_i \in k$ and multi-indices $\alpha_\nu^i \in \mathbb{N}^m$. Then for a suitable $r \in \langle \underline{x} \rangle$ we have

$$u = r + s \quad \text{for} \quad s = \sum_i \lambda_i \cdot (D_{\alpha_1^i} q) \cdot \ldots \cdot (D_{\alpha_{j_i}^i} q) \in \mathcal{U}_S(q)$$

which implies $u = \phi(s)$. As above by theorem (3.7.16) there are k-algebraically independent homogeneous additive elements $u_1, \ldots, u_l \in k[\underline{y}, \underline{z}]$ with $\mathcal{U}_S(\phi(q)) = k[u_1, \ldots, u_l]$. Then we find homogeneous elements $s_1, \ldots, s_l \in \mathcal{U}_S(q)$ with $\phi(s_1) = u_1, \ldots, \phi(s_l) = u_l$. By the argument of the first case for every $i \in \{1, \ldots, l\}$ we find a homogeneous additive element $c_i \in \mathcal{U}_S(s_i)$ with $\phi(c_i) = u_i$. Then by lemma (3.8.1) the c_i lie in $\mathcal{U}_S(q)$. □

Lemma (3.8.3). *Let F be a homogeneous finite subset of S and let g be a homogeneous element of S with $g \in \langle F \rangle \backslash \{0\}$. The g lies in $\langle f \in F \,|\, \deg f \leq \deg g \rangle$.*

Proof. Write $g = \sum_{f \in F} \lambda_f f$ for polynomials $\lambda_f \in S$. Let $\pi_j : S \to S_j \subseteq S$, $j \in \mathbb{N}$, denote the projection on the jth homogeneous component. We have

$$g = \pi_{\deg g} g = \sum_{f \in F, \deg f \leq \deg g} \pi_{\deg g - \deg f}(\lambda_f) \cdot f.$$

□

Lemma (3.8.4). *Let $\phi : S \to S$ denote the $k[\underline{y}, \underline{z}]$-algebra homomorphism with $\phi(x_1) = \ldots = \phi(x_m) = 0$. Let $\underline{a} = (a_1, \ldots, a_s)$ be a finite family in $k[\underline{x}]$ and $\underline{a}' = (a_1', \ldots, a_{s'}')$ a finite family in $k[\underline{x}] + k[\underline{y}, \underline{z}]$ such that $\underline{a}$ and $\underline{\phi(a')} = (\phi(a_1'), \ldots, \phi(a_{s'}'))$ are k-algebraically independent families in S respectively. Then the family $(\underline{a}, \underline{a}')$ is k-algebraically independent.*

Proof. Let $k[\underline{T}, \underline{T}']$ denote the symmetric algebra over k such that

$$(\underline{T}, \underline{T}') = (T_1, \ldots, T_s, T_1', \ldots, T_{s'}')$$

is a k-basis of $k[\underline{T}, \underline{T}']_1$. We have to show that the k-algebra homomorphism $k[\underline{T}, \underline{T}'] \to S$ with $T_i \mapsto a_i$, $T_{i'}' \mapsto a_{i'}'$ is injective. This map is the composition

$$k[\underline{T}, \underline{T}'] = k[\underline{T}] \otimes_k k[\underline{T}'] \xrightarrow{\varphi} k[\underline{x}] \otimes_k k[\underline{T}'] \xrightarrow{\theta} k[\underline{x}] \otimes_k k[\underline{T}'] \xrightarrow{\epsilon} S$$

where φ is the $k[\underline{T}']$-algebra homomorphism with $\varphi(T_i) = a_i$, θ is the $k[\underline{x}]$-algebra homomorphism with $T_{i'}' \to T_{i'}' + a_{i'}' - \phi(a_{i'}')$ and ϵ is the $k[\underline{x}]$-algebra homomorphism with $T_{i'}' \to \phi(a_{i'}')$. The map φ is injective because $\underline{a}$ is k-algebraically independent, θ is a isomorphism and ϵ is injective because $\underline{\phi(a')}$ is k-algebraically independent. □

Definition (3.8.5). *a) For a homogeneous ideal I of S with $I \neq \langle k[\underline{z}] \cap I\rangle$ define*

$$d_I := \max\{\min\{\deg f \mid f \in F\backslash k[\underline{z}]\} \mid F \text{ homogeneous finite subset of } S \text{ with } \langle F\rangle = I\}.$$

b) A homogeneous finite subset F of S with $\langle F\rangle \neq \langle k[\underline{z}] \cap \langle F\rangle\rangle$ is called ***k[z]-prepared*** *if*

$$d_{\langle F\rangle} = \min\{\deg f \mid f \in F\backslash k[\underline{z}]\}.$$

Example (3.8.6). For $F := \{z_1, z_2^2 + x_1 z_1, z_2^3 x_1, x_1^4\}$, $G := \{z_1, z_2^2, x_1^4\}$, $I := \langle F\rangle$ we have $d_I = 4$, $\langle G\rangle = I$, G is $k[\underline{z}]$-prepared and F is not $k[\underline{z}]$-prepared.

Lemma (3.8.7). *Let F be a homogeneous finite subset of S with $\langle F\rangle \neq \langle k[\underline{z}] \cap \langle F\rangle\rangle$. Then there is a homogeneous finite subset G of S with $G\backslash F \subseteq k[\underline{z}]$ which satisfies $\langle F\rangle = \langle G\rangle$ and which is $k[\underline{z}]$-prepared.*

Proof. Choose a $k[\underline{z}]$-prepared homogeneous finite subset H of S with $\langle H\rangle = \langle F\rangle =: I$. Set

$$G_1 := \{f \in F \mid \deg f \geq d_I\}, \quad G_2 := \{h \in H \mid \deg h < d_I\}.$$

and set $G := G_1 \cup G_2$. Then we have $G\backslash F \subseteq G_2 \subseteq k[\underline{z}]$. The identity $\langle G\rangle = \langle F\rangle$ follows from the inclusions

$$\langle G\rangle \subseteq I = \langle F\rangle \subseteq \langle G_1\rangle + \langle f \in F \mid \deg f < d_I\rangle \subseteq \langle G_1\rangle + \langle G_2\rangle = \langle G\rangle$$

where the last inclusion "'$\subseteq$"' holds by lemma (3.8.3). As H is $k[\underline{z}]$-prepared, G is it, too. □

Lemma (3.8.8). *Let E be a $k[\underline{z}]$-subalgebra of S such that for all elements f of S and all multi-indices $\alpha \in \mathbb{N}^{n+m+l}$, $\beta \in \mathbb{N}^l$ one has the implication*

$$f \in E \text{ and } \underline{z}^{-\beta} f_\alpha \in S \quad \Rightarrow \quad \underline{z}^{-\beta} f_\alpha \in E.$$

Let I be a homogeneous ideal of S with $I \neq \langle k[\underline{z}] \cap I\rangle$. Let F be a $k[\underline{z}]$-prepared homogeneous finite subset of S with $\langle F\rangle = I$ and assume $F^{=} \subseteq E$, where we define

$$F^{=} := \{f \in F \mid \deg f = d_I\}.$$

Fix a monomial order on S. Let $F \mapsto F'$ be a Buchberger algorithm and let $F' \mapsto F''$ be a reduction (see definition (3.7.8)). Then F'' has the same property as F, i.e. F'' is a $k[\underline{z}]$-prepared homogeneous finite subset of S with $\langle F''\rangle = I$ and $F''^{=} \subseteq E$.

Example (3.8.9). The k-subalgebra $k[\underline{y}, \underline{z}][x_i x_j \mid 1 \leq i < j \leq m]$ of S satisfies the assumption on E of lemma (3.8.8).

Proof of lemma (3.8.8). By definition a Buchberger algorithm is a finite sequence of assignments $G \mapsto G'$, where every assignment (see definition (3.7.8))

i) is a one-step reduction or

ii) has the form $G \mapsto G \cup \{s(f, g)\}$ for $f, g \in G\backslash\{0\}$ or

iii) has the from $G \mapsto G\backslash\{0\}$.

The reduction $F' \mapsto F''$ is a finite sequence of one-step reductions. Thus we may assume that $F \mapsto F''$ is of the type i), ii) or iii). The claim is clear if $F \mapsto F''$ is of the type iii).
Assume that $F \mapsto F''$ is of type i). Then $F \mapsto F''$ has the form $F = G \cup \{f\} \mapsto F'' = G \cup \{f'\}$, where $f \in F \backslash \{0\}$, $G = F \backslash \{f\}$ and $f' = c(f - gf\alpha/g_\beta)$ for an element $g \in G$, a coefficient $c \in k\backslash\{0\}$ and multi-indices $\alpha, \beta \in \mathbb{N}^{n+m+l}$ with $0 \neq g_\beta | f_\alpha$. Then we have $I = \langle F \rangle = \langle F'' \rangle$ and the element f' is homogeneous. For $f' = 0$ we have $F''^{=} \subseteq E$ and F'' is $k[\underline{z}]$-prepared. Assume $f' \neq 0$. Then we have $\deg f' = \deg f \geq \deg g$. We have to show that $f' \in k[\underline{z}]$, if $\deg f' < d_I$, and that $f' \in E$, if $\deg f' = d_I$. For $\deg f' < d_I$ the elements f, g lie in $k[\underline{z}]$, which implies $f' \in k[\underline{z}]$. For $\deg f' = \deg g = d_I$ we have $f_\alpha / g_\beta \in k$, which implies $f' \in E$. For $\deg f' = d_I > \deg g$ the monomial g_β has the form $c' z^{\beta'}$ for some $c' \in k\backslash\{0\}$, $\beta' \in \mathbb{N}^l$ and by the property of E the element f_α / g_β lies in E, which implies $f' \in E$. Thus, if $F \mapsto F''$ is of type i), F'' has the claimed properties.
Assume that $F \mapsto F''$ is of the type ii), i.e. $F'' = F \cup \{s(f,g)\}$, where $s(f,g)$ is the s-polynomial of some $f, g \in F$, i.e. $s(f,g) = fm/f_\alpha - gm/g_\beta$ for some $\alpha, \beta \in \mathbb{N}^n$ with $f_\alpha, g_\beta \neq 0$ and for the monic smallest common multiple m of f_α, g_β. Then we have $I = \langle F \rangle = \langle F'' \rangle$ and the s-polynomial $s(f,g)$ is homogeneous. For $s(f,g) = 0$ we have $F''^{=} \subseteq E$ and F'' is $k[\underline{z}]$-prepared. Assume $s(f,g) \neq 0$. Then we have $\deg s(f,g) \geq \deg f, \deg g$. As in the case of type i) we have to show that $s(f,g) \in k[\underline{z}]$, if $\deg s(f,g) < d_I$, and that $s(f,g) \in E$, if $\deg s(f,g) = d_I$. For $\deg g, \deg f < d_I$ the elements g, f lie in $k[\underline{z}]$, which implies $s(f,g) \in k[\underline{z}]$. For $\deg s(f,g) = d_I = \deg f = \deg g$ we have $m/f_\alpha, m/g_\beta \in k$ and thus $s(f,g) \in E$. For $\deg s(f,g) = d_I = \deg f > \deg g$ we have $g_\beta \in k[\underline{z}]$ and $m = c'' f_\alpha$ for some $c'' \in k$. Thus the property of E yields $m/g_\beta \in E$, which implies $s(f,g) \in E$. This completes the prove of lemma (3.8.8). □

Next we introduce "'lexicographically prepared"' families. They are only relevant in the case of type 5-*.

Definition (3.8.10). *A **lexicographically prepared family of S** is finite family $\mathcal{F} = (h_1, \ldots, h_s)$ of homogeneous elements of S of degree two such that*

- $h_1, \ldots, h_s \neq 0$,
- $\operatorname{multideg} h_1 <_{\text{lex}} \ldots <_{\text{lex}} \operatorname{multideg} h_s$ *where* $\leq_{\text{lex}}$ *denotes the lexicographical monomial order on* S *and*
- *for every choice* $1 \leq i \neq j \leq s$ *the monomial* $(h_j)_{\operatorname{multideg} h_i}$ *is zero.*

Write $|\mathcal{F}|$ *for the set* $\{h_1, \ldots, h_s\}$.

Lemma (3.8.11). *a) Every k-subvector space V of S with $V \subseteq \sum_{1 \leq i < j \leq m} x_i x_j \cdot k$ has a lexicographically prepared family as a basis.*

b) Let $\mathcal{F}$ be a lexicographically prepared family in S with $|\mathcal{F}| \subseteq k[x_i x_j \,|\, 1 \leq i < j \leq m]$. Then for every element $h \in |\mathcal{F}|$ and every exponent $e \in \mathbb{N}_{\geq 1}$ one has $h^e \notin \langle |\mathcal{F}| \backslash \{h\} \rangle$.

Proof. a) The vector space V lies in the k-vector space with the basis $(x_i x_j)_{1 \leq i < j \leq m}$. Write

$$\{x_i x_j \,|\, 1 \leq i < j \leq m\} = \{v_1, \ldots, v_t\} \quad \text{with} \quad v_1 <_{\text{lex}} \ldots <_{\text{lex}} v_t.$$

Choose a basis $\mathcal{B}$ of V and apply the Gaussian elimination on $\mathcal{B}$ with respect to $(v_1, \ldots, v_s)$.

b) There are $i_0 < j_0$ and $\lambda \in k\backslash\{0\}$ with $\mathrm{LT}h = \lambda x_{i_0} x_{j_0}$. Set $J := \langle x_i \,|\, i \in \{1,\dots,m\}\backslash\{i_0, j_0\}\rangle$. Then we have $\mathrm{LT}h \notin J$ and $|\mathcal{F}|\backslash\{h\} \subseteq J$, which implies $h^e \notin J \supseteq \langle|\mathcal{F}|\backslash\{h\}\rangle$.

□

Lemma (3.8.12). *Let $h_1, \dots, h_n$ be elements of $k[\underline{x}]$ and assume*

1) $\mathrm{char}k = p = 3$, $n = 1$ *and* $h_1 = x_1 x_2^2$ *or*

2) $\mathrm{char}k = p = 2$ *and* $(h_1, \dots, h_n)$ *is a lexicographically prepared family of S consisting of elements of* $k[x_i x_j \,|\, 1 \le i < j \le m]$.

Let J be an ideal of S with $J = \langle J \cap k[\underline{z}]\rangle$. Then we have the inclusion

$$k[y_1^p + h_1, \dots, y_n^p + h_n, \underline{z}] \cap \big(k[\underline{y}, \underline{z}] + J\big) \subseteq k[\underline{z}] + J.$$

Proof. Write $\underline{y^p + h}$ for $(y_1^p + h_1, \dots, y_n^p + h_n)$. Let g be an element of $k[\underline{y^p + h}, \underline{z}] \cap \big(k[\underline{y}, \underline{z}] + J\big)$. Write $g = \sum_{\gamma \in \mathbb{N}^n} (\underline{y^p + h})^\gamma g_\gamma$ for suitable $g_\gamma \in k[\underline{z}]$. We show that, for $\epsilon \neq 0$, g_ϵ lies in J. Then we have $g - g_0 \in J$, which implies $g \in k[\underline{z}] + J$. Let $\epsilon \in \mathbb{N}^n\backslash\{0\}$. Choose $i \in \{1, \dots, m\}$ with $\epsilon_i \neq 0$ and write $\epsilon_i = sq$ for a suitable p-power q and an element $s \in \mathbb{Z}\backslash p\mathbb{Z}$. Set $\epsilon' := \epsilon - q e_i$ where $e_1, e_2, \dots, e_{n+m+l}$ denote the elements $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, $\dots$, $(0,\dots,0,1)$ of $\mathbb{N}^{n+m+l}$. Let N resp. N' denote the k-subvector space of $k[\underline{x}]$ generated by the element h_i^q resp. by the set $\{\underline{h}^\alpha \,|\, \alpha \in \mathbb{N}^n\backslash\{qe_i\}\}$. Then we have

$$N \cap N' = N \cap \{\underline{h}^\alpha \,|\, \alpha \in \mathbb{N}^n\backslash\{qe_i\}, |\alpha| = q\} \cdot k \subseteq N \cap \langle h_j \,|\, j \in \{1,\dots,n\}\backslash\{i\}\rangle = \{0\},$$

where the last identity is trivial in case 1) and holds by lemma (3.8.11) in case 2). Thus the k-linear map $N \oplus N' \to k[\underline{x}]$ is injective. Choose some k-subvector space N'' of $k[\underline{x}]$ such that $N \oplus N' \oplus N'' \to k[\underline{x}]$ is an isomorphism. Then for the $k[\underline{z}]$-submodules of $k[\underline{x}, \underline{z}] \cong k[\underline{x}] \otimes_k k[\underline{z}]$

$$M := N \otimes_k k[\underline{z}], \quad M' := N' \otimes_k k[\underline{z}], \quad M'' := N'' \otimes_k k[\underline{z}]$$

the induced map $\phi : M \oplus M' \oplus M'' \to k[\underline{x}, \underline{z}]$ is an isomorphism of $k[\underline{z}]$-modules. Let ρ denote the composition of $k[\underline{z}]$-linear maps

$$\begin{array}{ll} S = k[\underline{x}, \underline{y}, \underline{z}] \cong \bigoplus_{\delta \in \mathbb{N}^n} k[\underline{x}, \underline{z}] \overset{\mathrm{pr}_{p\cdot\epsilon'}}{\to} k[\underline{x}, \underline{z}] \overset{\phi^{-1}}{\to} M \oplus M' \oplus M'' \overset{\mathrm{pr}_M}{\to} M & \overset{\sigma}{\cong} k[\underline{z}] \overset{\cdot s^{-1}}{\to} k[\underline{z}] \subseteq S. \\ \sum_{\delta \in \mathbb{N}^n} r_\delta \underline{y}^\delta \leftarrow\!\shortmid (r_\delta)_{\delta \in \mathbb{N}^n} & r h_i^q \leftarrow\!\shortmid r \end{array}$$

where $pr_{p\cdot\epsilon'}$, pr_M are the obvious projections. Then we have $\rho(k[\underline{y}, \underline{z}]) = \{0\}$ and $\rho(J) \subseteq J$, since J is generated by elements of $k[\underline{z}]$. We get

$$J \supseteq \rho(k[\underline{y}, \underline{z}] + J) \ni \rho(g) = s^{-1} \cdot \sigma \circ \mathrm{pr_M} \circ \phi^{-1}\Big[\sum_{\epsilon' \le \gamma \in \mathbb{N}^n} \binom{\gamma}{\epsilon'} \underline{h}^{\gamma - \epsilon'} g_\gamma\Big] = s^{-1} \cdot \binom{\epsilon}{\epsilon'} g_\epsilon = s^{-1} \cdot \binom{sq}{q} g_\epsilon = g_\epsilon$$

where $s = \binom{sq}{q}$ comes from the identity

$$\sum_{1 \le j \le s} \binom{s}{j} X^{qj} Y^{q(s-j)} = (X^q + Y^q)^s = (X + Y)^{qs} = \sum_{0 \le i \le qs} \binom{qs}{i} X^i Y^{qs-i}.$$

□

Lemma (3.8.13). *Let f be a non-zero element of S of total degree d, i.e. the maximal degree of the non-vanishing homogeneous parts of f is d, and let $\alpha \in \mathbb{N}^{n+m+l}$ be a multi-index with $|\alpha| = \alpha_1 + \ldots + \alpha_{n+m+l} \geq d$. Then we have $f_\alpha = D_\alpha f \cdot (\underline{x}, \underline{y}, \underline{z})^\alpha$.*

Proof. Write $s := n+m+l$. Write $f = \sum_{\gamma \in \mathbb{N}^s} \lambda_\gamma (\underline{x}, \underline{y}, \underline{z})^\gamma$ for suitable $\lambda_\gamma \in k$. We have

$$D_\alpha f = \binom{\alpha}{\alpha} \lambda_\alpha (\underline{x}, \underline{y}, \underline{z})^{\alpha - \alpha} + \sum_{\gamma \in \mathbb{N}^s \setminus \{\alpha\}} \binom{\gamma}{\alpha} \lambda_\gamma (\underline{x}, \underline{y}, \underline{z})^{\gamma - \alpha} = \lambda_\alpha + 0$$

because, if $\gamma \neq \alpha$, we have $\binom{\gamma}{\alpha} = 0$ for $\gamma - \alpha \in \mathbb{Z}^s \setminus \mathbb{N}^s$ and $\lambda_\gamma = 0$ for $\gamma - \alpha \in \mathbb{N}^s$. □

Lemma (3.8.14). *Let J denote the ideal $\langle x_i x_j \,|\, 1 \leq i < j \leq m \rangle$ of S. Let f be an element of $(J + k[\underline{y}, \underline{z}]) \setminus k[\underline{y}, \underline{z}]$. Then there are two k-algebraically independent additive elements $a_1, a_2 \in \mathcal{U}_S(f) \cap k[\underline{x}]$.*

Proof. Write $f = \sum_{\alpha \in \mathbb{N}^m} c_\alpha \underline{x}^\alpha$ for suitable $c_\alpha \in k[\underline{y}, \underline{z}]$. Choose a multi-index $\beta \in \mathbb{N}^{n+l}$ such that there is some $\alpha' \in \mathbb{N}^m \setminus \{0\}$ with $(c_{\alpha'})_\beta \neq 0$ and such that $|\beta|$ is equal to the highest total degree of all $c_\alpha \neq 0$, $\alpha \in \mathbb{N}^m \setminus \{0\}$. As above $(\_)_\beta$ is the projection $k[\underline{y}, \underline{z}] \to (\underline{y}, \underline{z})^\beta \cdot k$. Then by lemma (3.8.13) the $D_\beta c_\alpha$, for $\alpha \neq 0$, lie in k and $D_\beta c_{\alpha'}$ is not zero. We get

$$D_\beta f = \sum_{\alpha \in \mathbb{N}^m} (D_\beta c_\alpha) \underline{x}^\alpha \in \big(J \cap k[\underline{x}] + k[\underline{y}, \underline{z}]\big) \setminus k[\underline{y}, \underline{z}]. \tag{3.8.14.A}$$

For an index $i \in \{1, \ldots, m\}$ write $\underline{x}_i$ for $(x_1, \ldots, \widehat{x_i}, \ldots, x_m)$.

Claim: For every $i \in \{1, \ldots, m\}$ with $D_\beta f \notin k[\underline{x}_i, \underline{y}, \underline{z}]$ there is an additive element a of S with $a \in \mathcal{U}_S(D_\beta f) \cap k[\underline{x}_i]$.

We will prove this claim below. As $D_\beta f \notin k[\underline{y}, \underline{z}]$, by the claim there is an additive element $a_1 \in \mathcal{U}_S(D_\beta f) \cap k[\underline{x}]$. Since $a_1 \notin k = \cap_i k[\underline{x}_i]$ (a is additive), there is some $j \in \{1, \ldots, m\}$ with $a_1 \notin k[\underline{x}_j]$. Then we have $D_\beta f \notin k[\underline{x}_j, \underline{y}, \underline{z}]$. Then by the claim there is an additive element a_2 of S with $a_2 \in \mathcal{U}_S(D_\beta f) \cap k[\underline{x}_j]$. By lemma (3.8.1) we have $\mathcal{U}_S(D_\beta f) \subseteq D_S(f)$, which implies $a_1, a_2 \in D_S(f) \cap k[\underline{x}]$. By $a_1 \notin k[\underline{x}_j] \ni a_2 \notin k$ the elements a_1, a_2 are k-algebraically independent.

Proof of the claim: Write $D_\beta f = \sum_{s=0}^t h_s x_i^s + r$ for suitable $h_s \in k[\underline{x}_i]$, $r \in k[\underline{y}, \underline{z}]$ with $h_t \neq 0$. From (3.8.14.A) we get $h_t \notin k$. Then for the multi-index $\gamma := te_i$ we have $D_\gamma D_\beta f = h_t \in k[\underline{x}_i] \setminus k$, which implies $k \neq \mathcal{U}_S(D_\gamma D_\beta f) \subseteq k[\underline{x}_i]$. by remark (2.3.12) and theorem (3.7.16) we find an additive element $a \in \mathcal{U}_S(D_\gamma D_\beta f)$. From $\mathcal{U}_S(D_\gamma D_\beta f) \subseteq \mathcal{U}_S(D_\beta f)$ we get $a \in k[\underline{x}_i] \cap \mathcal{U}_S(D_\beta f)$. This completes the proof of the claim and the proof of lemma (3.8.14). □

Proof of theorem (3.2.6). We reformulate the statement of theorem (3.2.6) in our setting. Let $k = K$ be a perfect field. Let $h_1, \ldots, h_n$ be homogeneous elements of $k[\underline{x}]$ of degree p with $\{h_1, \ldots, h_n\} \neq \{0\}$ and assume

- $p = 3$, $n = 1$ and $h_1 = x_1 x_2^2$ or
- $p = 2$ and $\{h_1, \ldots, h_n\} \subseteq k[x_i x_j \,|\, 1 \leq i < j \leq m]$.

Let F be a finite subset of $k[\underline{y}^p, \underline{z}]$ of homogeneous elements of S such that $S/\langle F \rangle$ has Krull dimension m. Let ψ denote the $k[\underline{z}]$-algebra homomorphism

$$\psi : k[\underline{y}^p, \underline{z}] \to k[\underline{y^p + h}, \underline{z}] \quad \text{with} \quad (\psi(y_1^p), \ldots, \psi(y_n^p)) = (y_1^p + h_1, \ldots, y_n^p + h_n).$$

Let $\mathcal{U}_{\langle F\rangle}$ resp. $\mathcal{U}_{\langle\psi F\rangle}$ denote the ring of invariants of $(\langle F\rangle, S)$ resp. $(\langle\psi F\rangle, S)$. We have to prove the inequality

$$\dim\mathcal{U}_{\langle F\rangle} + 2 \le \dim\mathcal{U}_{\langle\psi F\rangle}.$$

Claim. For $p = 2$ we may assume, that the family $\underline{h} = (h_1,\dots,h_n)$ is lexicographically prepared.

Proof of claim. We replace $(\underline{y},\underline{h},\underline{z},\psi)$ by $(\underline{y}',\underline{h}',\underline{z}',\psi') = (y'_1,\dots,y'_{n'},h'_1,\dots,h'_{n'},z'_1,\dots,z'_{l'},\psi')$ as follows. For a matrix M over K the expression M^2 resp. $M^{1/2}$ denotes the entry wise square resp. square root of M. For a family $\mathcal{F}$ of elements of S the expression $\langle\mathcal{F}\rangle_K$ denotes the by the family generated K-subvector space of S. By lemma (3.8.11) there is a lexicographically prepared basis $\underline{h}' = (h'_1,\dots,h'_{n'})$ of $\langle\underline{h}\rangle_K$. Then there is a $n'\times n$-matrix B over K with $\underline{h}' = B\underline{h}$. Then for $\underline{y}' := (y'_1,\dots,y'_{n'}) := B^{1/2}\underline{y}$ one has $B(\underline{y}^2 + \underline{h}) = \underline{y}'^2 + \underline{h}'$. There is a unique $n\times n'$-matrix B' with $\underline{h} = B'\underline{h}'$. We show that $\underline{y}'$ is K-linearly independent. Let C be a $1\times n'$-matrix over K with $0 = C\underline{y}'$. Then one has $0 = CB^{1/2}\underline{y}$, which implies $CB^{1/2} = 0$, $C^2B = 0$, $C^2\underline{h}' = C^2B\underline{h} = 0$ and therefore $C = 0$. Thus $\underline{y}'$ is K-linearly independent. Further we show

$$\langle\underline{y}'\rangle_K \cap \langle\underline{y} + B'^{1/2}\underline{y}'\rangle_K = \{0\}.$$

Let s be an element of this intersection. Then there is a $1\times n'$-matrix L and a $1\times n$-matrix N with $L\underline{y}' = s = N(\underline{y} + B'^{1/2}\underline{y}')$. This implies $0 = (L^2B + N^2 + N^2B'B)\underline{y}^2$, which implies $0 = L^2B + N^2 + N^2B'B$, which implies $0 = (L^2B + N^2 + N^2B'B)\underline{h} = (L^2 + N^2B' + N^2B')\underline{h}' = L^2\underline{h}'$, which implies $0 = L\underline{y}' = s$. Thus the intersection is zero.
Choose a K-basis $z'_{l+1},\dots,z'_{l'}$ of $\langle\underline{y}+B'^{1/2}\underline{y}'\rangle_K$ and set $\underline{z}' := (z'_1,\dots,z'_{l'}) := (z_1,\dots,z_l,z'_{l+1},\dots,z'_{l'})$. Then $(\underline{x},\underline{y}',\underline{z}')$ is a K-basis of S_1. Define the $K[\underline{z}']$-algebra homomorphism

$$\psi' : K[\underline{y}'^2,\underline{z}'] \to K[\underline{y}'^2 + \underline{h}',\underline{z}'],\quad \underline{y}'^2 \mapsto \underline{y}'^2 + \underline{h}'.$$

Then the restriction of ψ' to $K[\underline{y}^2 + B'^{1/2}\underline{y}']$ is the identity. The restriction of ψ' to the subset

$$K[\underline{y}^2,\underline{z}] = K[y'^2,(\underline{y} - B'^{1/2}\underline{y}')^2,\underline{z}] \quad\text{of}\quad K[\underline{y}'^2,\underline{y} - B'^{1/2}\underline{y}',\underline{z}] = K[\underline{y}'^2,\underline{z}']$$

is equal to ψ, since we have $\psi(\underline{z}) = \psi'(\underline{z})$ and

$$\psi'(\underline{y}^2) = \psi'(B'\underline{y}'^2) + \psi'(\underline{y}^2 + B'\underline{y}'^2) = B'(\underline{y}'^2 + \underline{h}') + \underline{y}^2 + B'\underline{y}'^2 = \underline{y}^2 + \underline{h} = \psi(\underline{y}^2).$$

In particular we have $\psi'(F) = \psi(F)$. Altogether we see that the tuple $(\underline{y}',\underline{z}',\underline{h}',\psi')$ satisfies the same assumptions as $(\underline{y},\underline{z},\underline{h},\psi)$ and that $\underline{h}'$ is lexicographically prepared. This completes the proof of the claim.

We go on with the proof of theorem (3.2.6). By the claim we may assume that the family $\underline{h}$ is lexicographically prepared if $p = 2$. For the $K[\underline{y},\underline{z}]$-algebra homomorphism $\phi : S \to S$ with $\phi(\underline{x}) = 0$ we have $F = \phi\psi F$. The ideal $\langle\psi F\rangle$ of S is not generated by elements of $K[\underline{z}]$, i.e. $\langle\psi F\rangle \ne \langle K[\underline{z}] \cap \langle\psi F\rangle\rangle$. Otherwise we would have

$$F = \phi\psi F \subseteq \phi\Big(\big\langle K[\underline{z}] \cap \langle\psi F\rangle\big\rangle\Big) \subseteq \Big\langle\phi\big(K[\underline{z}] \cap \langle\psi F\rangle\big)\Big\rangle \subseteq \big\langle K[\underline{z}] \cap \langle\phi\psi F\rangle\big\rangle \subseteq \langle\underline{z}\rangle$$

which would imply

$$m = \dim S/\langle F\rangle \ge \dim K[\underline{x},\underline{y}] = m + n > m.$$

By lemma (3.8.7) there is a homogeneous finite subset G of S with $G\backslash\psi F \subseteq K[\underline{z}]$ which satisfies $\langle\psi F\rangle = \langle G\rangle$ and which is $k[\underline{z}]$-prepared. Then we have $G = (G \cap \psi F) \cup (G\backslash\psi F) \subseteq K[\underline{y^p + h}, \underline{z}]$. Let G'' be a reduced Gröbner basis of $\langle\psi F\rangle$ defined as follows. Let $\leq$ denote the graded lexicographical order on S with respect to the ordering $(\underline{y}, \underline{z}, \underline{x})$ (see example (3.7.4)), i.e.

$$\underline{x}^{\alpha}\underline{y}^{\beta}\underline{z}^{\gamma} \leq \underline{x}^{\alpha'}\underline{y}^{\beta'}\underline{z}^{\gamma'} \Leftrightarrow (|\alpha| + |\beta| + |\gamma|, \beta, \gamma, \alpha) \leq_{\text{lex}} (|\alpha'| + |\beta'| + |\gamma'|, \beta', \gamma', \alpha')$$

for the lexicographical order $\leq_{\text{lex}}$ on $\mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^l \times \mathbb{N}^n$. W.r.t. $\leq$ let $G \mapsto G'$ denote a Buchberger algorithm and let $G' \mapsto G''$ denote a reduction (see theorem (3.7.11)). Then G'' is a reduced Gröbner basis of $\langle\psi F\rangle$. We have

$$\langle\psi F\rangle = \langle G\rangle = \langle G''\rangle =: I.$$

The set $\phi G''\backslash\{0\}$ is a reduced Gröbner basis of $\langle F\rangle$ w.r.t. $\leq$ by the following four facts:

- The ideal $\langle\phi G''\rangle$ of S generated by $\phi G''$ is equal to $\langle F\rangle$:

$$\phi G'' \subseteq \phi(\langle\psi F\rangle) \subseteq \langle\phi\psi F\rangle = \langle F\rangle, \quad F = \phi\psi F \subseteq \phi(\langle G''\rangle) \subseteq \langle\phi G''\rangle.$$

- For an arbitrary $g'' \in G''$ with $\phi(g'') \neq 0$ it has a monomial lying in $K[\underline{y}, \underline{z}]$. As g'' is homogeneous, this implies $\operatorname{multideg}\phi(g'') = \operatorname{multideg} g''$ and $\operatorname{LT}\phi(g'') = \operatorname{LT} g''$.

- The monomials m_α, $\alpha \in \mathbb{N}^{m+n+l}$, of elements $m \in \phi G''\backslash\{0\}$ are monomials of elements of G''.

- We show that $\phi G''\backslash\{0\}$ is a Gröbner basis of $\langle F\rangle$ with Buchberger's criterion, theorem (3.7.10): Let f'', g'' be elements of G'' with $\phi(f''), \phi(g'') \neq 0$ and $s(\phi(f''), \phi(g'')) \neq 0$. Then we have $\operatorname{LT} f'' = \operatorname{LT}\phi(f'')$, $\operatorname{LT} g'' = \operatorname{LT}\phi(g'')$ and $\phi(s(f'', g'')) = s(\phi(f''), \phi(g''))$. Since G'' is a Gröbner basis of $\langle G''\rangle$ by theorem (3.7.10) there are elements $a_h \in S$, for $h \in G''$, with $s(f'', g'') = \sum_{h \in G''} a_h \cdot h$ such that for all $h \in G''$ with $a_h \cdot h \neq 0$ one has $\operatorname{multideg} s(f'', g'') \geq \operatorname{multideg}(a_h \cdot h)$. Then we get $s(\phi(f''), \phi(g'')) = \sum_{h \in G''} \phi(a_h)\phi(h)$ and, if $\phi(a_h)\phi(h) \neq 0$,

$$\operatorname{multideg} s(\phi(f''), \phi(g'')) = \operatorname{multideg} s(f'', g'') \geq \operatorname{multideg}(a_h \cdot h) = \operatorname{multideg}(\phi(a_h)\phi(h)).$$

For an arbitrary homogeneous finite subset $M \subseteq S\backslash\{0\}$ write

$$M^{=} := \{m \in M \mid \deg m = d_I\}.$$

We define the $K[\underline{z}]$-subalgebra of S

$$\mathcal{E} := K[\underline{y}, \underline{z}][x_i x_j \mid 1 \leq i < j \leq m] \quad \text{resp.} \quad \mathcal{E} := K[\underline{y}, \underline{z}][x_1 \cdot x_2^2]$$

if $p = 2$ resp. $p = 3$. We have $G \subseteq K[\underline{y^p + h}, \underline{z}] \subseteq \mathcal{E}$. By lemma (3.8.8) G'' is $K[\underline{z}]$-prepared with $G''^{=} \subseteq \mathcal{E}$. For the ideal $J := \langle g \in G \mid \deg g < d_I\rangle$ of S we have $J = \langle K[\underline{z}] \cap J\rangle$. We have the inclusions

$$G''^{=} \cap K[\underline{y}, \underline{z}] \subseteq \big((\psi F)^{=} \cdot K + J\big) \cap K[\underline{y}, \underline{z}] \subseteq (\psi F)^{=} \cdot K \cap (K[\underline{y}, \underline{z}] + J) + J \subseteq$$
$$\subseteq K[\underline{y^p + h}, \underline{z}] \cap (K[\underline{y}, \underline{z}] + J) + J \subseteq K[\underline{z}] + J$$

where the first inclusion holds by lemma (3.8.3) and the last inclusion holds by lemma (3.8.12). Further we have

$$G''^{=} \cap (K[\underline{z}] + J) = G''^{=} \cap (K[\underline{z}] \cap I + J) \subseteq G''^{=} \cap \langle K[\underline{z}] \cap I \rangle \neq G''^{=}$$

where "'≠"' holds because G'' is $K[\underline{z}]$-prepared. Thus there is an element $f \in G''^{=} \backslash K[\underline{y}, \underline{z}]$. It lies in $\mathcal{E}$. Then by lemma (3.8.14) there are two K-algebraically independent additive elements $a_1', a_2' \in \mathcal{U}_S(f) \cap K[\underline{x}] \subseteq \mathcal{U}_S(G'') \cap K[\underline{x}]$.

For an arbitrary element $g \in G''$ by lemma (3.8.2) there are homogeneous additive elements $c_1^g, \ldots, c_{s_g}^g \in S$ with

$$\mathcal{U}_S(\phi(g)) = K[\phi(c_1^g), \ldots, \phi(c_{s_g}^g)], \quad \mathcal{U}_S(g) \supseteq K[c_1^g, \ldots, c_{s_g}^g].$$

Since G'' resp. $\phi G'' \backslash \{0\}$ is a Giraud basis of $\langle \psi F \rangle$ resp. $\langle F \rangle$ (see lemma (3.7.14)), by theorem (3.7.16) we have

$$\mathcal{U}_{\langle F \rangle} = \mathcal{U}_S(\phi G'' \backslash \{0\}) = K[\phi(c_i^g) \mid g \in G'', 1 \leq i \leq s_g], \quad \mathcal{U}_{\langle \psi F \rangle} = \mathcal{U}_S(G'') \supseteq K[c_i^g \mid g \in G'', 1 \leq i \leq s_g].$$

Thus there are homogeneous additive elements $c_1, \ldots, c_s \in S$ with

$$\mathcal{U}_{\langle F \rangle} = K[\phi(c_1), \ldots, \phi(c_s)], \quad \mathcal{U}_{\langle \psi F \rangle} \supseteq K[c_1, \ldots, c_s].$$

By lemma (2.3.4) we can assume, that the family $(\phi(c_1), \ldots, \phi(c_s))$ is K-algebraically independent. Then by lemma (3.8.4) the family $\underline{c}' := (c_1, \ldots, c_s, a_1', a_2')$ is K-algebraically independent and we get

$$\dim \mathcal{U}_{\langle F \rangle} + 2 = s + 2 = \dim K[\underline{c}'] = \operatorname{trdeg}(\operatorname{Quot}(K[\underline{c}'])/K) \leq \operatorname{trdeg}(\operatorname{Quot}(\mathcal{U}_{\langle \psi F \rangle})/K) = \dim \mathcal{U}_{\langle \psi F \rangle}$$

which completes the proof of theorem (3.2.6). □

Chapter 4

A variation of blow-up strategies

In this chapter we show that, for resolving singularities of finite-dimensional excellent noetherian reduced schemes, it is enough to achieve an improvement of the invariant i (see definition (2.6.4)) which is a combination of the Hilbert-Samuel-function and the dimension of the ridge. By an improvement of i we mean an i^N-decrease (see definition (4.1.21)). This is a finite sequence of permissible blow-ups such that the invariant i^N of the "'worst"' points decreases. More precisely we show the following. Let $\mathcal{C}$ be a subcategory of the category of finite-dimensional excellent noetherian reduced schemes and let s be a strategy which associates a sequence

$$X = s(X)_0 \leftarrow s(X)_1 \leftarrow s(X)_2 \leftarrow \dots$$

of permissible blow-ups to every scheme X of $\mathcal{C}$. For example, this could be the strategy from [CJS]. We define a new strategy $i^N(s)$, depending on s, with the property that $i^N(s)$ yields a resolution of singularities for each scheme X of $\mathcal{C}$ if and only if s yields an i^N-decrease for each X (see lemma (4.2.3) and corollary (4.2.6)). One application of our main theorem is a criterion, in terms of Dir-i-near points, for the fact that a sequence

$$X = i^N(s)(X)_0 \leftarrow i^N(s)(X)_1 \leftarrow i^N(s)(X)_2 \leftarrow \dots$$

is a resolution of singularities, provided $\dim X \le 5$, see corollary (4.2.6). The strategy $i^N(s)$ has a good functoriality property if the functoriality of the given strategy s is good. More precisely we show that, if s has the property that for all schemes X of $\mathcal{C}$ the base change of $X = s(X)_0 \leftarrow s(X)_1 \leftarrow \dots$ with a surjective regular morphism $Y \to X$ is isomorphic to $Y = s(X \times_X Y)_0 \leftarrow s(X \times_X Y)_1 \leftarrow \dots$ up to contraction (see definition (4.3.3)) then the strategy $i^N(s)$ has also this property (see corollary (4.3.8)).
Convention: In the whole chapter by a blow-up we mean the blow-up of a scheme X in a center D such that no generic point of X lies in D.

4.1 Blow-up sequences for excellent schemes

Motivated by the variant of the Hilbert-Samuel-function for finite-dimensional, excellent schemes (see definition (4.1.5)) from [CJS] we introduce the refined invariant i^N (see definition (4.1.8)) and list some properties. Using a result about $\Sigma^{\max}$-eliminations, theorem 5.17 in [CJS], which we call $\max H^N$-eliminations, we deduce that every sequence of i^N-decreases results in a regular scheme (see corollary (4.1.22)).

Definition (4.1.1). *a) A scheme X over a field k is* ***geometrically regular*** *if for every field extension K of k the scheme $X \times_k K$ is regular. A morphism of schemes $S \to T$* ***has regular fibers*** *resp.* ***has geometrically regular fibers*** *if for each point t of T the fiber $S \times_T t$ over the residue field of t is regular resp. geometrically regular.*

b) A locally noetherian scheme is ***quasi-excellent*** *if*

i) for each $x \in X$ all the completion morphism $\operatorname{Spec}(\widehat{\mathcal{O}_{X,x}}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$ has geometrically regular fibers and

ii) for every X-scheme Y of finite type the set of all regular points of Y is open in Y.

c) A locally noetherian scheme X is ***catenary*** *if for every pair of closed irreducible subschemes $Y \subseteq Z$ of X every maximal chain $Y = Y_0 \subseteq Y_1 \subseteq \ldots Y_l = Z$ of closed irreducible subschemes of X has the same length, and X is* ***universally catenary*** *if every X-scheme of finite type is catenary.*

d) A locally noetherian schemes X is ***excellent*** *if it is quasi-excellent and universally catenary.*

Remark (4.1.2). *Each base change $S' = S \times_T T' \to T'$ of a morphism $S \to T$ with geometrically regular fibers has geometrically regular fibers: Let t' be a point of T' and let t denote its image in T. Let k', k denote the residue fields of t', t and let L be a field extension of k'. Then the scheme $(S' \times_{T'} t') \times_{k'} L$ is regular because it is isomorphic to the regular scheme $(S \times_T t) \times_k L$.*

The following theorem shows that many schemes are excellent.

Theorem (4.1.3). *a) The spectrum $\operatorname{Spec}(R)$ of a complete local noetherian ring R (e.g. a field) or of a Dedekind ring R with $\operatorname{charQuot}(\mathrm{R}) = 0$ (e.g. $R = \mathbb{Z}$) is excellent.*

b) A scheme locally of finite type over an excellent scheme is excellent.

c) For an excellent affine scheme $\operatorname{Spec}(A)$ and a multiplicative set S of the ring A the scheme $\operatorname{Spec}(S^{-1}A)$ is excellent.

Proof. [EGAIV], section 7.8. □

Corollary (4.1.4). *Let $X' \to X$ be a blow-up of an excellent scheme X. Then X' is excellent.*

Proof. By proposition (2.1.5) $X' \to X$ is locally of finite type. Then the claim follows from theorem (4.1.3). □

Setting: Until end of the section we fix a finite-dimensional excellent scheme X and a natural number N with $\dim X \leq N$.

In [CJS] the following variant of the Hilbert-Samuel-function is introduced. As in section 2.2 the set $\mathbb{N}^{\mathbb{N}}$ is partially ordered with

$$(\nu_0, \nu_1, \ldots) \leq (\nu'_0, \nu'_1, \ldots) \Leftrightarrow \text{for all } j \in \mathbb{N} \text{ one has } \nu_j \leq \nu'_j \text{ in } \mathbb{N}.$$

Definition (4.1.5). *a) We define the map*

$$H_X^N : X \to \mathbb{N}^{\mathbb{N}} : x \to H_{X,x}^{(\phi_X^N(x))}$$

for $\phi_X^N(x) := \max\{N - \dim \mathcal{O}_{Z,x} \,|\, Z$ *irreducible component of* X *with* $x \in Z\}$.

b) We denote the ordered subset $\{H_X^N(x) \,|\, x \in X\}$ *of* $\mathbb{N}^{\mathbb{N}}$ *by* $\mathrm{im} H_X^N$ *and denote the subset of* $\mathrm{im} H_X^N$ *of all maximal elements by* $\max \mathrm{im} H_X^N$.

c) For $\nu \in \mathbb{N}^{\mathbb{N}}$ *we define*

$$\{H_X^N \geq \nu\} := \{x \in X \,|\, H_X^N(x) \geq \nu\},$$
$$\{H_X^N = \nu\} := \{x \in X \,|\, H_X^N(x) = \nu\},$$
$$\{H_X^N = \max\} := \{x \in X \,|\, H_X^N(x) \in \max \mathrm{im} H_X^N\},$$
$$\{H_X^N < \max\} := \{x \in X \,|\, x \notin \{H_X^N = \max\}\}.$$

Lemma (4.1.6). *Let* x *be a point of* X*. Then one has* $H_X^N(x) \geq H^{(0)}(\kappa(x)[T_1, \ldots, T_N])$*. The equality holds if and only if* X *is regular at* x*.*

Proof. We have $\phi_X^N(x) \geq N - \dim \mathcal{O}_{X,x}$ and equality holds if x lies on a unique irreducible component of X. Let m be a natural number and set $d := \dim \mathcal{O}_{X,x}$. By lemma (2.2.2) we have $H_{X,x}^{(m)} \geq H^{(m)}(\kappa(x)[T_1, \ldots, T_d])$ and equality holds if and only if X is regular at x. We get

$$H_X^N(x) = H_{X,x}^{(\phi_X^N(x))} \geq H_{X,x}^{(N-d)} \geq H^{(N-d)}(\kappa(x)[T_1, \ldots, T_d]) = H^{(0)}(\kappa(x)[T_1, \ldots, T_N]). \quad (4.1.6.A)$$

If X is regular at x, then x lies on a unique irreducible component of X and both inequalities in (4.1.6.A) are equalities. If we have $H_X^N(x) = H^{(0)}(\kappa(x)[T_1, \ldots, T_N])$ then we get $H_{X,x}^{(m)} = H^{(m)}(\kappa(x)[T_1, \ldots, T_d])$ for $m = N - d$, i.e. X is regular at x. □

Remark (4.1.7). *As a conclusion the set of all regular points of* X *is the set* $\{H_X^N = \nu_{\mathrm{reg}}^N\}$ *for* $\nu_{\mathrm{reg}}^N := \left(1, \binom{N}{1}, \binom{N+1}{2}, \binom{N+2}{3}, \ldots\right) \in \mathbb{N}^{\mathbb{N}}$.

Similarly we do this for the following refined invariant. As in section 2.6 let $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ have the lexicographical order, i.e. for $\nu, \nu' \in \mathbb{N}^{\mathbb{N}}$ and $r, r' \in \mathbb{N}$ one has

$$(\nu, r) \leq (\nu', r') \quad \Leftrightarrow \quad \nu = \nu' \text{ and } r \leq r' \quad \text{or} \quad \nu < \nu' \text{ in } \mathbb{N}^{\mathbb{N}}.$$

Definition (4.1.8). *a) Define*

$$i_X^N : X \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N} : x \to i_{X,x}^{(\phi_X^N(x))} = (H_{X,x}^{(\phi_X^N(x))}, \dim \mathrm{Rid}_{X,x} + \phi_X^N(x))$$

for $\phi_X^N(x)$ *as above.*

b) As above define the ordered subset $\mathrm{im}\, i_X^N$ *of* $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ *and the subset* $\max \mathrm{im}\, i_X^N$.

c) For $\mu \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ *define* $\{i_X^N \geq \mu\}$, $\{i_X^N = \mu\}$, $\{i_X^N = \max\}$ *analogously to c) above.*

Lemma (4.1.9). *Let x be a point of X. Then one has $i_X^N(x) \geq (\nu_{\text{reg}}^N, N)$ and equality holds if and only if X is regular at x.*

Proof. We show that we have $\dim \text{Rid}_{X,x} + \phi_X^N(x) = N$ if X regular at x. The rest follows from lemma (4.1.6) and remark (4.1.7). Assume that x is a regular point. Then we have $\phi_X^N(x) = N - d$ for $d := \dim \mathcal{O}_{X,x}$ since x lies on a unique irreducible component of X. Since $C_{X,x} = \text{Spec}(\text{gr}\mathcal{O}_{X,x})$ is a vector group of dimension d over $\kappa(x)$ we have $C_{X,x} = \text{Rid}_{X,x}$ and $\dim \text{Rid}_{X,x} = d = N - \phi_X^N(x)$. □

By the following theorem the map H_X^N is upper semi-continuous.

Theorem (4.1.10). *For each $\nu \in \mathbb{N}^{\mathbb{N}}$ the set $\{H_X^N \geq \nu)$ is closed in X. Further the subset $\{H_X^N = max\}$ of X is closed.*

Proof. [CJS], theorem 1.33 and lemma 1.36. □

Theorem (4.1.11). *Let D be an irreducible permissible closed subscheme of X. Then for all points x, y of D one has $H_X^N(x) = H_Y^N(x)$.*

Proof. [CJS], theorem 2.3. □

Remark (4.1.12). *Recall that by proposition (2.1.5) for a blow-up $Y' \to Y$ (in a center which contains no generic point of Y, by our convention for this chapter)*

- *$Y' \to Y$ is locally of finite type and $\dim Y = \dim Y'$ if Y is locally noetherian and*
- *Y' is reduced if Y is reduced.*

Theorem (4.1.13). *Let $\pi : X' \to X$ be a permissible blow-up of X. Let $x' \in X'$ be a point. Write $x := \pi(x')$ and $d := \text{trdeg}(\kappa(x')/\kappa(x))$. Then we have*

a) $H_{X'}^N(x') \leq H_X^N(x)$ and equality holds if and only if x' is near to x, i.e. $H_{X',x'}^{(d)} = H_{X,x}^{(0)}$,

b) $\phi_{X'}^N(x') \leq \phi_X^N(x) + d$ and equality holds if x' is near to x,

c) $i_{X'}^N(x') \leq i_X^N(x)$ and

d) if x is regular, then x' is regular.

Proof. We may assume that x lies in the blow-center. a) and b) are [CJS], theorem 2.10. d) follows from a) and lemma (4.1.6). By corollary (2.6.5) one has $i_{X',x'}^{(d)} \leq i_{X,x}^{(0)}$. With b) we get

$$i_{X'}^N(x') = i_{X',x'}^{(\phi_{X'}^N(x'))} \leq i_{X',x'}^{(\phi_X^N(x)+d)} \leq i_{X,x}^{(\phi_X^N(x))} = i_X^N(x)$$

which shows c). □

Definition (4.1.14). *Let $X \overset{\pi}{\leftarrow} X'$ be a composition of permissible blow-ups. Let x, x' be points of X, X' .*

*a) We say **x' is near to x** if $\pi(x') = x$ and $H_{X'}^N(x') = H_X^N(x)$.*

b) *We say* $\boldsymbol{x'}$ ***is i-near to*** $\boldsymbol{x}$ *if* $\pi(x') = x$ *and* $i^N_{X'}(x') = i^N_X(x)$.

Remark (4.1.15). *By theorem (4.1.13)* x' *is i-near to* x *if and only if one has* $H^{(d)}_{X',x'} = H^{(0)}_{X,x}$ *and* $\phi^N_{X'}(x') = \phi^N_X(x) + d$ *and*

$$0 = \dim \mathrm{Rid}_{X',x'} + \phi^N_{X'}(x') - (\dim \mathrm{Rid}_{X,x} + \phi^N_X(x)) = \dim \mathrm{Rid}_{X',x'} + d - \dim \mathrm{Rid}_{X,x}.$$

In particular definition (4.1.14) does not depend on the choice of N *and we have a coincidence with the definitions (2.2.7) and (3.2.2).*

Definition (4.1.16). *Let* X *be reduced. A* $\mathbf{max}\,\boldsymbol{H^N}$***-elimination for*** $\boldsymbol{X}$ *is a finite composition* $X' \to X$ *of permissible blow-ups such that for every connected component* U *of* X *and for the induced morphism* $U' = U \times_X X' \to U$ *one has*

a) *either* U *is regular and* $U' \to U$ *is an isomorphism*

b) *or* U *is not regular, the induced morphism* $\{H^N_U < \max\} \times_X X' \to \{H^N_U < \max\}$ *is an isomorphism and one has* $\mathrm{im} H^N_{U'} \cap \max \mathrm{im} H^N_U = \varnothing$.

Theorem (4.1.17). *Let* X *be noetherian and reduced and let* $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$ *be a sequence of* $\max H^N$*-eliminations. Then there is some* $n \in \mathbb{N}$ *such that* X_n *is regular.*

Proof. [CJS], Theorem 5.17 □

We consider a weaker form of an elimination.

Definition (4.1.18). *Let* X *be reduced. A* ***weak*** $\mathbf{max}\,\boldsymbol{H^N}$***-elimination (for*** $\boldsymbol{X}$***)*** *is a finite composition* $X' \to X$ *of permissible blow-ups such that for each singularity* x *of* X *with* $x \in \{H^N_X = \max\}$ *there is no point* $x' \in X'$ *near to* x.

Corollary (4.1.19). *Let* X *be reduced.*

a) *If* X *is connected, for each weak* $\max H^N$*-elimination* $\rho : X' \to X$ *there is a* $\max H^N$*-elimination* $Y' \to Y = X$ *and a composition of permissible blow-ups* $X' \to Y'$ *such that* ρ *is the composition* $X' \to Y' \to X$.

b) *Let* X *be noetherian and let* $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$ *be a sequence of weak* $\max H^N$*-eliminations. Then there is some* $n \in \mathbb{N}$ *such that* X_n *is regular.*

Proof. a) If X is regular the claim is clear. Assume that X is not regular. Write $X = X_0 \overset{\pi_0}{\leftarrow} X_1 \overset{\pi_1}{\leftarrow} \dots \overset{\pi_{n-1}}{\leftarrow} X_n = X'$ for blow-ups $\pi_0, \pi_1, \dots, \pi_{n-1}$ in permissible centers $D_0, D_1, \dots, D_{n-1}$. For $i \in \{0, \dots, n-1\}$ the center D_i may have one of the following types.

(type 1) $D_i \subseteq (X_i \to X)^{-1}(\{H_X = \max\})$

(type 2) $D_i \subseteq (X_i \to X)^{-1}(\{H_X < \max\})$

(type 3) neither type 1 nor type 2.

Step 1. Fix some $i \in \{0, \dots, n-1\}$. By theorem (4.1.11) the map $H^N_{X_i}$ is constant on each irreducible component of D_i. Thus D_i is a disjoint union of two permissible subschemes D_i^{t1}, D_i^{t2} of type 1 and type 2. For $U_i^{t1} := X \backslash D_i^{t1}$, $U_i^{t2} := X \backslash D_i^{t2}$ we have $D_i^{t1} \subseteq U_i^{t2}$, $D_i^{t2} \subseteq U_i^{t1}$. The blow-up $X_i \leftarrow \mathrm{Bl}_{D_i^{t1}} X_i =: \widetilde{X}_i$ is of type 1. The induced morphism $U_i^{t1} \leftarrow \widetilde{X}_i \times_{X_i} U_i^{t1}$ is an isomorphism. The blow-up $\widetilde{X}_i \leftarrow \mathrm{Bl}_{D_i^{t2}} \widetilde{X}_i =: \widetilde{\widetilde{X}}_i$ is permissible and of type 2. The base change of π_i with U_i^{t1} resp. U_i^{t2} is the base change of $X_i \leftarrow \widetilde{\widetilde{X}}_i$ with U_i^{t1} resp. U_i^{t2}. Then X_{i+1} and $\widetilde{\widetilde{X}}_i$ are isomorphic as X_i-schemes. Thus π_i is a composition of a permissible blow-up $X_i \leftarrow \widetilde{X}_i$ of type 1 with a permissible blow-up $\widetilde{X}_i \leftarrow X_{i+1}$ of type 2.

Step 2. By step 1 we may assume that each blow-up π_i is of type 1 or of type 2. Fix some $i \in \{0, \dots, n-1\}$. Assume that π_i is of type 2 and π_{i+1} is of type 1. Define $U_i := X_i \backslash D_i$. The induced morphism $U_i \leftarrow U_i \times_{X_i} X_{i+1}$ is an isomorphism. We have $D_{i+1} \subseteq U_i \times_{X_i} X_{i+1}$. For $V_i := (X_i \to X_0)^{-1}(\{H_X < \max\})$ the intersection $V_i \cap D_{i+1}$ is empty where we identify D_{i+1} with its image in U_i. Since U_i and V_i cover X_i, D_{i+1} is closed in X_i. Then the blow-up $\mathrm{Bl}_{D_i \cup D_{i+1}} \to X_i$ is equal to $\pi_{i+1} \circ \pi_i$. By step 1 $\pi_{i+1} \circ \pi_i$ is a composition of a permissible type 1 blow-up $X_i \leftarrow \widehat{X}_i$ with a permissible type 2 blow-up $\widehat{X}_i \leftarrow X_{i+1}$.

Step 3. By step 1 we may assume that each blow-up π_i is of type 1 or type 2. By step 2 we may assume that for some $j \in \{0, \dots, n\}$ the blow-ups $\pi_0, \dots, \pi_j$ are of type 1 and $\pi_{j+1}, \dots, \pi_{n-1}$ are of type 2. Set $Y' := X_{j+1}$. For the morphism $\pi_j \circ \dots \circ \pi_0 : Y' \to X$ the induced morphism $\{H^N_X < \max\} \times_X Y' \to \{H^N_X < \max\}$ is an isomorphism. We show $\mathrm{im} H^N_{Y'} \cap \max \mathrm{im} H^N_X = \varnothing$. Assume that there is some point y' of Y' with $H^N_{Y'}(y') \in \max \mathrm{im} H^N_X$. By theorem (4.1.13) for the image x in X of y' we have $H^N_{Y'}(y') \le H^N_X(x)$ which implies $H^N_{Y'}(y') = H^N_X(x)$ and $y \in (Y' \to X)^{-1}(\{H^N_X = \max\})$. Since the blow-ups $\pi_{j+1}, \dots, \pi_{n-1}$ are of type 2 there is a unique point $x' \in X'$ with image y' in Y' and we have $H^N_{X'}(x') = H^N_{Y'}(y')$. Then x' is near to x in contradiction to the assumption. Thus $X \leftarrow Y'$ is a $\max \mathrm{im} H^N_X$-elimination.

b) Let X have $m \in \mathbb{N}_{\ge 1}$ irreducible components. By proposition (2.1.5) each scheme X_i has m irreducible components. In particular each X_i has at most m connected components. Then for some $l \in \mathbb{N}$ the number of the connected components of $X_l, X_{l+1}, X_{l+2}, \dots$ is the same. We may assume $X = X_l$. Treating each connected components of X separately we may assume that X is connected. Thus we are reduced to the case that all schemes $X = X_0, X_1, X_2, \dots$ are connected.

Inductively we construct a sequence of $\max H^N$-eliminations $X_0 = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots$ and for each $n \in \mathbb{N}$ we construct a finite composition $Y_n \leftarrow X_n$ of permissible blow-ups such that

$$\begin{array}{ccccccc} X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \dots \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\ Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \dots \end{array}$$

commutes. Then by theorem (4.1.17) for some n the scheme Y_n is regular and theorem (4.1.13) implies that X_n is regular.

For $n \in \mathbb{N}$ let $Y_0 \leftarrow Y_1 \leftarrow \dots \leftarrow Y_n$ and $Y_0 \overset{=}{\leftarrow} X_0, Y_1 \leftarrow X_1, \dots, Y_n \leftarrow X_n$ be already constructed. Then $Y_0, Y_1, \dots, Y_n$ are connected. The composition $Y_n \leftarrow X_n \leftarrow X_{n+1}$ is a weak $\max H^N$-elimination by lemma (4.1.20) below. By a) there is a $\max H^N$-elimination $Y_n \leftarrow Y_{n+1}$ and a finite composition of permissible blow-ups $Y_{n+1} \leftarrow X_{n+1}$ such that

$$\begin{array}{ccc} X_n & \longleftarrow & Y_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longleftarrow & Y_{n+1} \end{array}$$

commutes. This completes the proof of the corollary.

□

Lemma (4.1.20). *Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3$ be a sequence of schemes where $X_0 \leftarrow X_1$ and $X_2 \leftarrow X_3$ are permissible blow-ups and where $X_1 \leftarrow X_2$ is a weak $\max \mathrm{im} H^N$-elimination. Then the composition $X \leftarrow X_3$ is a weak $\max \mathrm{im} H^N$-elimination.*

Proof. Assume that there is a point x_3 of X_3 which is near to it image x_0 in X_0 and x_0 is a singularity with $x_0 \in \{H^N_{X_0} = \max\}$. By theorem (4.1.13) we have

$$H^N_{X_0}(x_0) = H^N_{X_1}(x_1) = H^N_{X_2}(x_2) = H^N_{X_3}(x_3)$$

for the images x_1, x_2 of x_3 in X_1, X_2. By the same theorem for each point x'_1 of X_1 and its image x'_0 in X_0 we have

$$H^N_{X_1}(x_1) = H^N_{X_0}(x_0) \not< H^N_{X_0}(x'_0) \geq H^N_{X_1}(x'_1).$$

Thus x_1 lies in $\{H^N_{X_1} = \max\}$. Since x_2 is near to x_1 this is a contradiction. □

Definition (4.1.21). *Let X be reduced.*

*a) An **i^N-decrease (for X)** is a finite sequence of permissible blow-ups $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n$, for $n \geq 1$, such that for each singularity x of X with $x \in \{i^N_X = \max\}$ there is no point $x_n \in X_n$ i-near to x. It is called **short** if $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_{n-1}$ is not an i^N-decrease or $n = 1$.*

*b) A sequence $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$ of permissible blow-ups **yields an i^N-decrease** if for some $n \in \mathbb{N}$ the sequence $X = X_0 \leftarrow \ldots \leftarrow X_n$ is an i^N-decrease.*

Corollary (4.1.22). *Let X be noetherian and reduced. Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$ a sequence of schemes where each morphism $X_i \leftarrow X_{i+1}$ is given by an i^N-decrease. Then for some $n \in \mathbb{N}$ the scheme X_n is regular.*

Proof. We show that the composition $X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_{2N+1}$ is a weak $\max H^N$-elimination. Then we get a sequence $X_0 \leftarrow X_{2N+1} \leftarrow X_{4N+2} \leftarrow \ldots$ of weak $\max H^N$-eliminations and with corollary (4.1.19) the claim follows. We have $\mathrm{im}\, i^N_{X_n} \subseteq \mathbb{N}^{\mathbb{N}} \times \{0, 1, 2, \ldots, 2N\}$. Let x_{2N+1} be a point of X_{2N+1}. Assume that x_{2N+1} is near to a singularity $\widetilde{x}_0 \in X_0$ with $\widetilde{x}_0 \in \{H^N_X = \max\}$. Denote the image of x_{2N+1} in X_{2N} by $\widetilde{x}_{2N}$. Then by theorem (4.1.13) we have $i^N_{X_{2N}}(\widetilde{x}_{2N}) \geq i^N_{X_{2N+1}}(x_{2N+1})$ and $\widetilde{x}_{2N}$ is a singularity. Choose some point $x_{2N} \in X_{2N}$ with $\max \mathrm{im}\, i^N_X \ni i^N_{X_{2N}}(x_{2N}) \geq i^N_{X_{2N}}(\widetilde{x}_{2N})$. Then we have $i^N_{X_{2N}}(x_{2N}) > i^N_{X_{2N+1}}(x_{2N+1})$ because otherwise $\widetilde{x}_{2N}$ would by a point of $\{i^N_{X_{2N}} = \max\}$ and x_{2N+1} would be a i-near point of $\widetilde{x}_{2N}$ in contradiction to the assumption. Inductively one shows that there is a sequence of elements $x_0 \in X_0$, $x_1 \in X_1$, …,$x_{2N} \in X_{2N}$ with

$$i^N_{X_0}(x_0) > i^N_{X_1}(x_1) > \ldots > i^N_{X_{2N}}(x_{2N}) > i^N_{X_{2N+1}}(x_{2N+1}). \tag{4.1.22.A}$$

Then we have $\nu := H^N_{X_0}(x_0) > H^N_{X_{2N+1}}(x_{2N+1})$ because otherwise (4.1.22.A) would be a strictly decreasing sequence in $\{\nu\} \times \{0, 1 \ldots, 2N\}$ which is not possible. This is a contradiction to $H^N_{X_{2N+1}}(x_{2N+1}) = H^N_{X_0}(\widetilde{x}_0)$ and $\widetilde{x}_0 \in \{i^N_{X_0} = \max\}$. This completes the proof. □

4.2 The i^N-iterated variation of blow-up strategies

Let N be a natural number. Let $\mathcal{C}$ denote a subcategory of the category of schemes such that each scheme of $\mathcal{C}$ is noetherian, excellent and reduced with dimension at most N.
For a given strategy on $\mathcal{C}$ (see the definition below) we define the i^N-iterated variation $i^N(s)$. For a scheme X of $\mathcal{C}$ the blow-up sequence $X = i^N(s)(X)_0 \leftarrow i^N(s)(X)_1 \leftarrow \ldots$ is, roughly spoken, applying the strategy s on X until for some n the composition

$$X = s(X)_0 \leftarrow s(X)_1 \leftarrow \ldots s(X)_n =: Y$$

is an i^N-decrease and then applying the strategy s on Y and so on. If s is a desingularization then $i^N(s)$ has this property. With our main theorem we find an equivalent description for $i^N(s)$ to be a desingularization (for dimension up to five).

Definition (4.2.1). *a) A **(permissible) strategy s on $\mathcal{C}$ (to resolve singularities)** is the datum of a sequence $s(X) = (X = s(X)_0 \leftarrow s(X)_1 \leftarrow s(x)_2 \leftarrow \ldots)$ morphisms of $\mathcal{C}$ for each scheme X of $\mathcal{C}$ where each morphism in the sequence is a permissible blow-up.*

*b) For a scheme X of $\mathcal{C}$ and a strategy s on $\mathcal{C}$ **s is a desingularization of X** if there is a $n \in \mathbb{N}$ such that $s(X)_n$ is regular.*

*c) A strategy on $\mathcal{C}$ is a **desingularization** if it is a desingularization of each scheme of $\mathcal{C}$.*

*d) A strategy on $\mathcal{C}$ is an **i^N-decrease** if for each scheme X of $\mathcal{C}$ the sequence $s(X)$ yield an i^N-decrease.*

Definition (4.2.2). *Let s be a strategy on $\mathcal{C}$. **The i^N-iterated variation $i^N(s)$ of s** is a strategy on $\mathcal{C}$ which is defined by the following two properties*

a) If for a scheme X of $\mathcal{C}$ the sequence $s(X)$ does not yield an i^N-decrease then one has $s(X) = i^N(s)(X)$.

b) If for a scheme X of $\mathcal{C}$ and some $n \in \mathbb{N}$ the sequence $X = s(X)_0 \leftarrow s(X)_1 \leftarrow \ldots \leftarrow s(X)_n =: Y$ is a short i^N-decrease then $i^N(s)(X)$ is the sequence

$$X = s(X)_0 \leftarrow \ldots \leftarrow s(X)_n \leftarrow i^N(s)(Y)_1 \leftarrow i^N(s)(Y)_2 \leftarrow \ldots.$$

Lemma (4.2.3). *Let s be a strategy on $\mathcal{C}$.*

a) If s is a desingularization then s is an i^N-decrease.

b) The following are equivalent.

i) s is an i^N-decrease.

ii) $i^N(s)$ is an i^N-decrease.

iii) $i^N(s)$ is a desingularization.

Proof. a) Let X be a scheme of $\mathcal{C}$. There is some $n \in \mathbb{N}$ such that $s(X)_n$ is regular, in particular the sequence $X = s(X)_0 \leftarrow \ldots \leftarrow s(X)_n$ is an i^N-decrease.

i)⇒ ii) Let X be a scheme of $\mathcal{C}$. There is some $n \in \mathbb{N}$ such that $s(X)_0 \leftarrow \ldots \leftarrow s(X)_n$ is a short i^N-decrease. Then this sequence is equal to $i^N(s)(X)_0 \leftarrow \ldots \leftarrow i^N(s)(X)_n$. In particular $i^N(s)(X)$ yields an i^N-decrease.

ii)⇒ i) Assume that there is a scheme X of $\mathcal{C}$ such that $s(X)$ does not yield an i^N-decrease. Then we have $s(X) = i^N(s)(X)$ and $i^N(s)(X)$ does not yield an i^N-decrease.

ii)⇒iii) Let X be a scheme of $\mathcal{C}$. There are integers $0 = n_0 \leq n_1 \leq n_2 \leq \ldots$ such that for each $j \in \mathbb{N}$ the sequence $i^N(s)(X)_{n_j} \leftarrow i^N(s)(X)_{n_j+1} \leftarrow \ldots \leftarrow i^N(s)(X)_{n_{j+1}}$ is a short i^N-decrease. Then by corollary (4.1.22) $i^N(s)(X)_{n_m}$ is regular for some $m \in \mathbb{N}$ which implies that $i^N(s)$ is a desingularization of X.

iii)⇒ii) Follows from a).

□

Definition (4.2.4). *Let X be a finite-dimensional excellent reduced scheme and let $X = X_0 \leftarrow X_1 \leftarrow \ldots$ be a sequence of permissible blow-ups in centers $D_i \subseteq X_i$ and let $x_n \in X_n$, $x_0 \in X_0$ be points.*

*a) Let $x_1, \ldots, x_{n-1}$ denote the images of x_n in $X_1, \ldots, X_{n-1}$. We say **x_n is Dir-near to x_0** if x_n is near to x_0 and for each $j \in \{0, 1 \ldots, n-1\}$ with $x_j \in D_j$ the point x_{j+1} lies in $\mathbb{P}(\mathrm{Dir}(C_{X_j,D_j,x_j}))$.*

*b) We say **x_n is Dir-i-near to x_0** if x_n is Dir-near and i-near to x_0 (see definition (4.1.14)).*

We have the following deduction from the main theorem (3.2.1).

Theorem (4.2.5). *Let X be a finite-dimensional excellent reduced scheme with $\dim X \leq 5$. Let $X = X_0 \leftarrow \ldots \leftarrow X_n$ be a sequence of permissible blow-ups. A point x_n of X_n is i-near to a point x_0 of X_0 if and only if it is Dir-i-near to x_0.*

Let $\mathcal{C}_{\leq 5}$ denote the full subcategory of $\mathcal{C}$ of all schemes X of dimension at most five.

Corollary (4.2.6). *a) A i^N-iterated variation $i^N(s)$ of a strategy s on $\mathcal{C}$ is a desingularization if and only if for each scheme X of $\mathcal{C}$ there is some $n \in \mathbb{N}$ such that there is no point x_n of $i^N(s)(X)_n$ which is i-near to a singularity x of X with $x \in \{i_X^N = \max\}$.*

b) A i^N-iterated variation $i^N(s)$ of a strategy s on $\mathcal{C}_{\leq 5}$ is a desingularization if and only if for each scheme X of $\mathcal{C}_{\leq 5}$ there is some $n \in \mathbb{N}$ such that there is no point x_n of $i^N(s)(X)_n$ which is Dir-i-near to a singularity x of X with $x \in \{i_X^N = \max\}$.

4.3 Functoriality

As above let N be a natural number and let $\mathcal{C}$ denote a subcategory of the category of schemes such that each scheme of $\mathcal{C}$ is noetherian, excellent and reduced with dimension at most N.
In this section we show that a i^N-iterated variation $i^N(s)$ of a strategy s (see definition (4.2.2)) is functorial (see definition (4.3.3)) with respect to surjective regular morphisms if s has this property.

Definition (4.3.1). *A morphism of schemes with locally noetherian fibers is* ***regular****, if it is flat and it has geometrically regular fibers.*

Remark (4.3.2). *a) Smooth morphism are regular, see [Stacks], Tag 07R6.*

b) Flat morphisms are closed under base change. Then by remark (4.1.2) regular morphisms are closed under base change.

Definition (4.3.3). *Let E be a class of scheme morphisms. A strategy s on $\mathcal{C}$ is* ***functorial in E*** *if for each pair of schemes X, Y of $\mathcal{C}$ and each morphism $Y \to X$ of E the sequences $s(X) \times_X Y$ and $s(Y)$ are isomorphic up to contraction, i.e. there is a map $\phi : \mathbb{N} \to \mathbb{N}$ with $\phi(0) = 0$ and $\phi(n) \leq \phi(n+1) \leq \phi(n) + 1$ for all $n \in \mathbb{N}$ and there is a commutative diagram*

$$\begin{array}{ccccccc} Y = s(X)_0 \times_X Y & \longleftarrow & s(X)_1 \times_X Y & \longleftarrow & s(X)_2 \times_X Y & \longleftarrow & \dots \\ \downarrow = & & \downarrow \cong & & \downarrow \cong & & \\ Y = s(Y)_{\phi(0)} & \longleftarrow & s(Y)_{\phi(1)} & \longleftarrow & s(Y)_{\phi(2)} & \longleftarrow & \dots \end{array}$$

with vertical isomorphisms where for all $n \in \mathbb{N}$ the morphism $s(Y)_{\phi(n)} \leftarrow s(Y)_{\phi(n+1)}$ is the identity if $\phi(n) = \phi(n+1)$.

Remark (4.3.4). *a) By the definition of blow-ups the vertical isomorphisms in definition (4.3.3) are unique and the morphisms in the sequence $s(X) \times_X Y$ are permissible blow-ups.*

b) For a desingularization on $\mathcal{C}$ one can define the following weaker form of functoriality. We call a desingularization s on $\mathcal{C}$ composition-functorial in E if for each pair of schemes X, Y of $\mathcal{C}$ and each morphism $Y \to X$ of E there are numbers $n, m \in \mathbb{N}$ such that $s(X)_n \times_X Y$ and $s(Y)_m$ are regular and there is an isomorphism $s(X)_n \times_X Y \cong s(Y)_m$ of Y-schemes.

Proposition (4.3.5). *Let X, Y be locally noetherian finite-dimensional excellent reduced schemes with $\dim X, \dim Y \leq N$. Let $f : Y \to X$ be a regular morphism.*

a) For each point $y \in Y$ one has $H_Y^N(y) = H_X^N(f(y))$.

b) If X is regular then Y is regular.

c) For each point $y \in Y$ one has $i_Y^N(y) = i_X^N(f(y))$.

d) If D is a permissible subscheme of X, then $D \times_X Y$ is a permissible subscheme of Y.

Proof. We may assume that X, Y are noetherian. Let y be a point of Y and write $x := f(y)$. For $d := \dim \mathcal{O}_{Y \times_X x, y}$ by [CJS], lemma 1.37 (1), we have $\phi_Y^N(y) = \phi_X^N(x) - d$ and $H_Y^N(y) = H_X^N(x)$. Thus a) holds.

b) Let y be a point of Y. With a) and with remark (4.1.7) we have $H_Y^N(y) = H_X^N(f(y)) = \nu_{\mathrm{reg}}^N$ and therefore y is regular.

c) For x and d as above, we have a non-canonical isomorphism $C_{Y,y} \cong C_{X,x} \times_{\kappa(x)} \mathbb{A}^d_{\kappa(y)}$ of cones over $\kappa(y)$, by [CJS], lemma 1.27. With corollary (3.7.17) we get

$$\mathrm{Rid}(C_{Y,y}) \cong \mathrm{Rid}(C_{X,x} \times_{\kappa(x)} \kappa(y)) \times_{\kappa(y)} \mathbb{A}^d_{\kappa(y)} \cong \mathrm{Rid}(C_{X,x}) \times_{\kappa(x)} \kappa(y) \times_{\kappa(y)} \mathbb{A}^d_{\kappa(y)}$$

which implies $\dim \mathrm{Rid}_{Y,y} = \dim \mathrm{Rid}_{X,x} + d$. Together with $\phi_Y^N(y) = \phi_X^N(x) - d$ and $H_Y^N(y) = H_X^N(x)$ we get $i_Y^N(y) = i_X^N(x)$.

d) Let $\mathcal{J}$ denote the quasi-coherent ideal sheaf $\mathcal{O}_Y$ which is associated to the closed immersion $D \times_X Y \to Y$. Then we have $\mathcal{J} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ which implies $\mathrm{gr}_{\mathcal{J}} \mathcal{O}_Y = \oplus_{n \in \mathbb{N}} \mathcal{J}^n \otimes_{\mathcal{O}_Y} \mathcal{O}_{D \times_X Y} = (\mathrm{gr}_{\mathcal{I}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Thus the morphism $C_{Y, D \times_X Y} \to D$ is flat as a base change of $C_{X,D} \to D$ with Y over X. The scheme D is regular and the morphism $D \times_X Y \to D$ is regular as a base change of a regular morphism, see remark (4.3.2). Then by b) $D \times_X Y$ is regular.

□

Lemma (4.3.6). *Let X, Y be schemes of $\mathcal{C}$. Let*

$$\begin{array}{ccccccc}
Y = X_0 \times_X Y & \longleftarrow & X_1 \times_X Y & \longleftarrow & X_2 \times_X Y & \longleftarrow & \dots \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\
X = X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \dots
\end{array}$$

be a commutative diagram of schemes with cartesian squares where f_0 is a surjective regular morphism and where $X_0 \leftarrow X_1 \leftarrow \dots$ is a sequence of permissible blow-ups and $X_0 \times_X Y \leftarrow X_1 \times_X Y \leftarrow \dots$ is the induced sequence.

a) All squares are cartesian, the morphisms $f_1, f_2, \dots$ are surjective and regular and all horizontal morphisms are permissible blow-ups.

b) For each $n \in \mathbb{N}$ the sequence $X_0 \times_X Y \leftarrow \dots \leftarrow X_n \times_X Y$ is an i^N-decrease if and only if $X_0 \leftarrow \dots \leftarrow X_n$ is an i^N-decrease.

Proof. a) One has $X_{j+1} \times_{X_j} (X_j \times_X Y) = X_{j+1} \times_X Y$. The morphisms $f_1, f_2, \dots$ are surjective and regular since they are a base change of f_0 (see remark (4.3.2)). Let $D_0, D_1, D_2, \dots$ denote permissible centers of the blow-ups $X_0 \leftarrow X_1 \leftarrow \dots$. For each j the morphism $X_j \times_X Y \leftarrow X_{j+1} \times_X Y$ is the blow-up in $D_j \times_{X_j} (X_j \times_X Y) = D_j \times_X Y$ since f_j is flat. By proposition (4.3.5) the subscheme $D_j \times_X Y$ of $X_j \times_X Y$ is permissible.

b) Since f_0 is surjective, by proposition (4.3.5) we have $\max \mathrm{im}\, i_Y^N = \max \mathrm{im}\, i_X^N$. Write $Y_n := X_n \times_X Y$. Let y, y_n, x, x_n be points of Y, Y_n, X, X_n with

$$\begin{array}{ccc}
y & \longleftarrow\!\shortmid & y_n \\
\downarrow & & \downarrow \\
x & \longleftarrow\!\shortmid & x_n
\end{array}$$

By proposition (4.3.5) we have

$$H_Y^N(y) = H_X^N(x), \quad i_Y^N(y) = i_X^N(x), \quad i_{Y_n}^N(y_n) = i_{X_n}^N(x_n).$$

Then

- y is a singularity if and only if x is a singularity,
- $y \in \{i_Y^N = \max\}$ if and only if $x \in \{i_X^N = \max\}$ and
- y_n is i-near to y if and only if x_n is i-near to x.

The surjectivity of f_n yields the claim.

□

We can not drop the assumption that f_0 is surjective, as the following example shows.

Example (4.3.7). We give an example for a commutative diagram of schemes

$$\begin{array}{ccc} U & \longleftarrow & X' \times_X U \\ \downarrow & & \downarrow \\ X & \longleftarrow & X' \end{array}$$

where X is a noetherian two-dimensional excellent reduced scheme, $X \leftarrow X'$ is a permissible blow-up and the vertical morphisms are open immersions (in particular regular morphisms) such that $X \leftarrow X'$ is not an i^N-decrease but $U \leftarrow X' \times_X U$ is an i^N-decrease.
Let X be the closed subscheme $V(f)$, for $f = x^2 + y^4 z$, of the affine space $\operatorname{Spec}(k[x,y,z]) =: Z$ for a field k with $\operatorname{char} k \neq 2$. Similarly to example (2.1.4) one sees that the singular points of X are the closed subscheme $V(\langle x,y\rangle)$ of Z. Let ξ be a closed point of $V(\langle x,y\rangle) \subseteq Z$ and $\eta := \langle x,y\rangle \in Z$. Identify x,y with their image in the regular local ring $\mathcal{O}_{Z,\xi}$ resp. $\mathcal{O}_{Z,\eta}$. The family (x,y) is a system of regular parameters of $\mathcal{O}_{Z,\eta}$. There is some $q \in \mathcal{O}_{Z,\xi}$ such that (x,y,q) is a system of regular parameters of $\mathcal{O}_{Z,\xi}$. Let $\mathcal{X},\mathcal{Y},\mathcal{Q}$ resp. $\mathcal{X},\mathcal{Y}$ denote the initial forms of x,y,q in $\operatorname{gr}\mathcal{O}_{Z,\xi}$ resp. of x,y in $\operatorname{gr}\mathcal{O}_{Z,\eta}$. The graded $\kappa(\xi)$-algebra $\operatorname{gr}\mathcal{O}_{Z,\xi}$ is a polynomial ring with variables $\mathcal{X},\mathcal{Y},\mathcal{Q}$. The graded $\kappa(\eta)$-algebra $\operatorname{gr}\mathcal{O}_{Z,\eta}$ is a polynomial ring with variables $\mathcal{X},\mathcal{Y}$. We have $\operatorname{in}_\xi f = \mathcal{X}^2$, $\operatorname{in}_\eta f = \mathcal{X}^2$. This implies $\operatorname{gr}\mathcal{O}_{X,\xi} = \operatorname{gr}\mathcal{O}_{Z,\xi}/\langle\mathcal{X}^2\rangle = \operatorname{gr}\mathcal{O}_{X,\eta}[\mathcal{Q}]$. We have $\phi_X^N(\xi) = N-2$ and $\phi_X^N(\eta) = N-1$. Then we get

$$H_X^N(\xi) = H^{(\phi_X^N(\xi))}(\operatorname{gr}\mathcal{O}_{X,\xi}) = H^{(\phi_X^N(\eta)-1)}(\operatorname{gr}\mathcal{O}_{X,\eta}[\mathcal{Q}]) = H^{(\phi_X^N(\eta))}(\operatorname{gr}\mathcal{O}_{X,\eta}) = H_X^N(\eta) =: \nu.$$

Thus we have $V(\langle x,y\rangle) = \{H_X^N = \nu\} = \{H_X^N = \max\}$. By lemma (2.2.2) for all $n \in \mathbb{N}$ we have

$$\nu_n = \binom{N+n}{n} - \binom{N+n-2}{n-2}.$$

The ridge $\operatorname{Rid}_{X,\xi}$ resp. $\operatorname{Rid}_{X,\eta}$ at ξ resp. η is the closed subgroup $V(\mathcal{X})$ of the vector group $C_{Z,\xi} = \operatorname{Spec}(\operatorname{gr}\mathcal{O}_{Z,\xi})$ resp. $C_{Z,\eta}$. Thus we have

$$\dim \operatorname{Rid}_{X,\xi} + \phi_X^N(\xi) = 2 + N - 2 = 1 + N - 1 = \dim \operatorname{Rid}_{X,\eta} + \phi_X^N(\eta)$$

and we have $V(\langle x,y\rangle) = \{i_X = (\nu, N)\} = \{i_X^N = \max\}$. Thus we have

$$\dim \operatorname{Rid}_{X,\xi} + \phi_X^N(\xi) = 2 + N - 2 = 1 + N - 1 = \dim \operatorname{Rid}_{X,\eta} + \phi_X^N(\eta)$$

and we have $V(\langle x,y\rangle) = \{i_X = (\nu,N)\} = \{i_X^N = \max\}$.
Let $X \leftarrow X'$ be the blow-up of X in the closed subscheme $D = V(\langle x,y\rangle)$ of Z. Then X' is covered by the open subschemes $X_1' := \operatorname{Spec}(k[x,y,z]/\langle x^2+y^2z\rangle)$ and $X_2' := \operatorname{Spec}(k[x,y,z]/\langle 1+x^2y^4z\rangle)$. The scheme X_2' is regular because it is smooth over k. The scheme X_1' is a closed subscheme of $\operatorname{Spec}(k[x,y,z]) =: Z_1'$. The singular points of X_1' are the closed subscheme $S' := V(\langle x,y\rangle)$ of Z_1'. Let m' resp. η' denote the point $\langle x,y,z\rangle$ resp. $\langle x,y\rangle$ of S' and let ξ' be a point of $S'\backslash\{m',\eta'\}$. As above we see that

- $\operatorname{gr}\mathcal{O}_{Z_1',m'}$ is a graded polynomial ring over $\kappa(m')$ with variables $\mathcal{X} := \operatorname{in}_{m'}(x)$, $\mathcal{Y} := \operatorname{in}_{m'}(y)$, $\mathcal{Z} := \operatorname{in}_{m'}(z)$ and one has $\operatorname{in}_{m'}(x^2+y^2z) = \mathcal{X}^2$,
- $\operatorname{gr}\mathcal{O}_{Z_1',\eta'}$ is a graded polynomial ring over $\kappa(\eta')$ with variables $\mathcal{X} := \operatorname{in}_{\eta'}(x)$, $\mathcal{Y} := \operatorname{in}_{\eta'}(y)$ and one has $\operatorname{in}_{\eta'}(x^2+y^2z) = \mathcal{X}^2 + c\cdot\mathcal{Y}^2$ for $c := \operatorname{in}_{\eta'}(z) \in \kappa(\eta')\backslash\{0\}$,
- $\operatorname{gr}\mathcal{O}_{Z_1',\xi'}$ is a graded polynomial ring over $\kappa(\xi')$ with variables $\mathcal{X} := \operatorname{in}_{\xi'}(x)$, $\mathcal{Y} := \operatorname{in}_{\xi'}(y)$ and some third variable $\mathcal{Q}$ and one has $\operatorname{in}_{\xi'}(x^2+y^2z) = \mathcal{X}^2 + c\cdot\mathcal{Y}^2$ for $c := \operatorname{in}_{\xi'}(z) \in \kappa(\xi')\backslash\{0\}$.

Then as above and with lemma (2.2.2) we get $H_{X'}^N(\xi') = H_{X'}^N(m') = H_{X'}^N(\eta') = \nu$ for the same $\nu \in \mathbb{N}^{\mathbb{N}}$ as above. The ridge $\operatorname{Rid}_{X',m'}$ is the closed subgroup $V(\mathcal{X})$ of the vector group $C_{Z_1',m'}$. The ridge $\operatorname{Rid}_{X',\xi'}$ resp. $\operatorname{Rid}_{X',\eta'}$ is the closed subgroup $V(\mathcal{X},\mathcal{Y})$ of the vector group $C_{Z_1',\xi'}$ resp. $C_{Z_1',\eta'}$. Then we have

$$i_{X'}^N(m') = (\nu,N), \quad i_{X'}^N(\eta') = (\nu,N-1) = i_{X'}^N(\xi').$$

The image of m' in X is the point $m := \langle x,y,z\rangle$. Thus m' is i-near to $m \in \{i_X^N = \max\}$ and $X \leftarrow X'$ is not an i^N-decrease. On the other hand for the open subscheme $U := X\backslash\{m\}$ we have $\max\operatorname{im} i_U^N = \{(\nu,N)\}$ and the blow-up $X' \times_X U$ of U does not contain m' (which is the only point of X' with $i_{X'}^N = (\nu,N)$). Thus $U \leftarrow X' \times_X U$ is an i^N-decrease.

Corollary (4.3.8). *Let E be a class of scheme morphisms such that*

- *each morphism of E is surjective and regular,*
- *E contains the class of all isomorphisms of schemes and*
- *E is stable under base change and compositions.*

Let s be an in E functorial strategy on $\mathcal{C}$. Then $i^N(s)$ is functorial in E.

Proof. Let X,Y be a pair of schemes of $\mathcal{C}$ and let $Y \to X$ be a morphism of E. Let

$$\begin{array}{ccccccc}
Y = s(X)_0 \times_X Y & \longleftarrow & s(X)_1 \times_X Y & \longleftarrow & s(X)_2 \times_X Y & \longleftarrow & \dots \\
\downarrow = & & \downarrow \cong & & \downarrow \cong & & \\
Y = s(Y)_{\phi(0)} & \longleftarrow & s(Y)_{\phi(1)} & \longleftarrow & s(Y)_{\phi(2)} & \longleftarrow & \dots
\end{array}$$

be a commutative diagram of schemes as in definition (4.3.3) for a map $\phi : \mathbb{N} \to \mathbb{N}$ with $\phi(0) = 0$ and $\phi(n) \le \phi(n+1) \le \phi(n)+1$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}_{\ge 1}$ be arbitrary such that $X = s(X)_0 \leftarrow s(X)_1 \leftarrow \dots \leftarrow s(X)_{m-1}$ is not an i^N-decrease then by proposition (4.3.5) both sequences $s(X)_0 \times_X Y \leftarrow \ldots \leftarrow s(X)_{m-1} \times_X Y$ and $s(Y)_{\phi(0)} \leftarrow \ldots \leftarrow s(Y)_{\phi(m-1)}$ are not i^N-decreases. Then by definition of $i^N(s)$ we have a commutative diagram

(4.3.8.A)
$$\begin{array}{ccccccc}
Y = i^N(s)(X)_0 \times_X Y & \longleftarrow & i^N(s)(X)_1 \times_X Y & \longleftarrow & \ldots & \longleftarrow & i^N(s)(X)_m \times_X Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
Y = s(X)_0 \times_X Y & \longleftarrow & s(X)_1 \times_X Y & \longleftarrow & \ldots & \longleftarrow & s(X)_m \times_X Y \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} \\
Y = s(Y)_{\phi(0)} & \longleftarrow & s(Y)_{\phi(1)} & \longleftarrow & \ldots & \longleftarrow & s(Y)_{\phi(m)} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
Y = i^N(s)(Y)_{\phi(0)} & \longleftarrow & i^N(s)(Y)_{\phi(1)} & \longleftarrow & \ldots & \longleftarrow & i^N(s)(Y)_{\phi(m)}
\end{array}$$

If for each choice of m the sequence $X = s(X)_0 \leftarrow s(X)_1 \leftarrow \ldots \leftarrow s(X)_m$ is not an i^N-decrease then we are done. Assume that for some m the sequence is a short i^N-decrease. Then set $X' := s(X)_m$ and $Y' := s(Y)_{\phi(m)}$. Since $Y' \cong X' \times_X Y \to X'$ is a composition of a base change of a morphism of E with an isomorphism, the morphism $Y' \to X'$ lies in E. We have a commutative diagram of schemes

$$\begin{array}{ccccccc}
X' \times_X Y = s(X')_0 \times_X Y & \longleftarrow & s(X')_1 \times_X Y & \longleftarrow & s(X')_2 \times_X Y & \longleftarrow & \ldots \\
\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} & & \\
Y' = s(X')_0 \times_{X'} Y' & \longleftarrow & s(X')_1 \times_{X'} Y' & \longleftarrow & s(X')_2 \times_{X'} Y' & \longleftarrow & \ldots \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
Y' = s(Y')_{\phi'(0)} & \longleftarrow & s(Y')_{\phi'(1)} & \longleftarrow & s(Y')_{\phi'(2)} & \longleftarrow & \ldots
\end{array}$$

where the upper vertical squares are cartesian and the upper isomorphisms are induced by the isomorphism $Y' \cong X' \times_X Y$ and where the lower diagram is given by the functoriality of s. Let $m' \in \mathbb{N}_{\geq 1}$ be arbitrary such that $X' = s(X')_0 \leftarrow \ldots \leftarrow s(X')_{m'-1}$ is not an i^N-decrease. Then as above we get a commutative diagram of schemes

(4.3.8.B)
$$\begin{array}{ccccccc}
i^N(s)(X)_{m+0} \times_X Y & \longleftarrow & i^N(s)(X)_{m+1} \times_X Y & \longleftarrow & \ldots & \longleftarrow & i^N(s)(X)_{m+m'} \times_X Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
s(X')_0 \times_X Y & \longleftarrow & s(X')_1 \times_X Y & \longleftarrow & \ldots & \longleftarrow & s(X')_{m'} \times_X Y \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} \\
s(Y')_{\phi'(0)} & \longleftarrow & s(Y')_{\phi'(1)} & \longleftarrow & \ldots & \longleftarrow & s(Y')_{\phi'(m')} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
i^N(s)(Y)_{\phi(m)+\phi'(0)} & \longleftarrow & i^N(s)(Y)_{\phi(m)+\phi'(1)} & \longleftarrow & \ldots & \longleftarrow & i^N(s)(Y)_{\phi(m)+\phi'(m')}
\end{array}$$

The diagrams (4.3.8.A) and (4.3.8.B) yield a commutative diagram

$$\begin{array}{ccccccc} Y = i^N(s)(X)_0 \times_X Y & \longleftarrow & i^N(s)(X)_1 \times_X Y & \longleftarrow & \dots & \longleftarrow & i^N(s)(X)_{m+m'} \times_X Y \\ \downarrow = & & \downarrow \cong & & & & \downarrow \cong \\ Y = i^N(s)(Y)_{\psi(0)} & \longleftarrow & i^N(s)(Y)_{\psi(1)} & \longleftarrow & \dots & \longleftarrow & i^N(s)(Y)_{\psi(m+m')} \end{array}$$

for $\psi(j) = \phi(j)$, if $j \leq m$, and $\psi(j) = \phi(m) + \phi'(j-m)$, if $j \geq m$. If for each m' the sequence $X' = s(X')_0 \leftarrow \dots \leftarrow s(X')_{m'}$ is not an i^N-decrease, then we are done. Otherwise go on as above. Inductively we find a map $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(0) = 0$ and $\psi(n) \leq \psi(n+1) \leq \psi(n) + 1$ for all $n \in \mathbb{N}$ and a commutative diagram

$$\begin{array}{ccccccc} Y = i^N(s)(X)_0 \times_X Y & \longleftarrow & i^N(s)(X)_1 \times_X Y & \longleftarrow & i^N(s)(X)_2 \times_X Y & \longleftarrow & \dots \\ \downarrow = & & \downarrow \cong & & \downarrow \cong & & \\ Y = i^N(s)(Y)_{\psi(0)} & \longleftarrow & i^N(s)(Y)_{\psi(1)} & \longleftarrow & i^N(s)(Y)_{\psi(2)} & \longleftarrow & \dots \end{array}$$

as wished. □

Remark (4.3.9). *If E contains non-trivial open immersions and s is functorial in E we can not expect that $i^N(s)$ is functorial in E. The problem is that for a $n \in \mathbb{N}_{\geq 1}$ and a scheme X of $\mathcal{C}$ the sequence*

$$s(X)_0 \leftarrow \dots \leftarrow s(X)_n$$

can be not an i^N-decrease while the base change with an open subscheme U of X

$$s(X)_0 \times_X U \leftarrow \dots \leftarrow s(X)_n \times_X U$$

can be an i^N-decrease, see example (4.3.7). We can not exclude the case that the blow-up $U' := s(X)_n \times_X U \leftarrow s(X)_{n+1} \times_X U$ is neither an isomorphism nor the morphism $U' = s(U')_0 \leftarrow s(U')_1$. If such a case appears then the strategy $i^N(s)$ is not functorial in E.

Bibliography

[Ab] S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces.* Pure and Applied Mathematics, Vol. 24, Academic Press (1966).

[BHM] J. Berthomieu, P. Hivert, H. Mourtada, *Computing Hironaka's invariants: ridge and directrix.* Arithmetic, geometry, cryptography and coding theory 2009, Contemp. Math. **521** (2010), 9–20.

[CJS] V. Cossart, U. Jannsen, S. Saito, *Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes.* preprint arXiv:math.AG/0905.2191v1 (2009).

[CLO] D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms.* Undergraduate Texts in Mathematics, 3rd ed., An introduction to computational algebraic geometry and commutative algebra, Springer, New York, (2007).

[Co] I. S. Cohen, *On the structure and ideal theory of complete local rings.* Trans. Amer. Math. Soc. **59** (1946), 54–106.

[CP] V. Cossart, O. Piltant, *Resolution of Singularities of Arithmetical Threefolds II.* preprint arXiv:1412.0868 (2014).

[dJ] A. J. de Jong, *Smoothness, semi-stability and alterations.* Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51–93.

[Di] B. Dietel, *A refinement of Hironaka's additive group schemes for an extended invariant.* Ph.D. thesis, Regensburg http://epub.uni-regensburg.de/31359/ (2015).

[EGAIV] A. Grothendieck, J.A. Dieudonné, *Eléments de Géométrie Algébrique IV, Seconde partie.* Publ. Math. IHÉS **24** (1965).

[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry I: Schemes with Examples and Exercises.* Wiesbaden: Vieweg+Teubner Verlag, 2010

[Hi1] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. **79** (1964), 109–203.

[Hi2] H. Hironaka, *On the characters ν^* and τ^* of singularities.* J. Math. Kyoto Univ. **7** (1967), 19–43.

[Hi3] H. Hironaka, *Additive groups associated with points of a projective space.* Ann. of Math. **92** (1970), 327–334.

[Hi4] H. Hironaka, *Certain numerical characters of singularities.* J. Math. Kyoto Univ. **10** (1970), 151–187.

[Li] Q. Liu, *Algebraic Geometry and Arithmetic Curves.* Oxford Graduate Texts in Mathematics **6** Oxford University Press (2006).

[Ma] H. Matsumura, *Commutative ring theory.* Cambridge Studies in Advanced Mathematics **8** Translated from the Japanese by M. Reid, Cambridge University Press (1986).

[Mi] H. Mizutani, *Hironaka's additive group schemes.* Nagoya Math. J. **52** (1973), 85–95.

[Od] T. Oda, *Hironaka's additive group scheme.* Number theory, algebraic geometry and commutative algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo (1973), 181–219.

[Si] B. Singh, *Effect of a permissible blowing-up on the local Hilbert functions.* Invent. Math. **26** (1974), 201–212.

[Stacks] *Stacks Project.* `http://stacks.math.columbia.edu` (2015)

[Za] O. Zariski, *Reduction of the singularities of algebraic three dimensional varieties.* Ann. of Math. (2) **45** (1944), 472–542.