Heights of Toric Varieties

Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat.) der Fakultät für Mathematik der Universität Regensburg

vorgelegt von

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im Jahr 2015

Promotionsgesuch eingereicht am 23.11.2015.

Die Arbeit wurde angeleitet von Prof. Dr. Walter Gubler.

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Abstract

We show that the toric local height of a toric variety with respect to a toric semipositive metrized line bundle over an arbitrary non-Archimedean field can be written as the integral over a polytope of a concave function. For discrete non-Archimedean fields, this was proved by Burgos–Philippon–Sombra in [BPS14a]. To show this statement, we first prove an induction formula for the non-Archimedean local height of a variety, generalizing a theorem of Chambert-Loir–Thuillier. Then, in analogy to [BPS14a], we translate arithmetic-geometric objects like toric divisors over arbitrary valuation rings of rank one and toric semipositive metrics over non-discrete non-Archimedean fields, in terms of convex analysis like piecewise affine and concave functions.

Furthermore, we prove that the global height of a variety over a finitely generated field can be expressed as an integral of local heights over a set of places of this field. In contrast to a similar statement in [BPS14b], it allows arbitrary non-Archimedean places. Combining this expression with our results on toric geometry, we get an interesting formula for the global height. This formula will be illustrated in a final natural example where not all relevant non-Archimedean places are discrete.

Contents

Introduction

Height Theory

The height of rational points of a variety is a real-valued function which behaves well under algebraic operations and which is a helpful tool to control the number and distribution of these rational points. Therefore, it plays a fundamental role in the proof of finiteness results in Diophantine geometry like the theorems of Mordell–Weil and Faltings (see, for instance, [BG06]).

In [Fal91], Faltings generalized the height of points to the height of (sub-)varieties using arithmetic intersection theory by Gillet–Soulé [GS90]. We sketch his definition which points out that the height of a variety is the arithmetic analogue of the degree in the classical intersection theory. Let X be an n-dimensional smooth projective variety over $\mathbb{Q}$ equipped with a regular proper $\mathbb{Z}$-model $\mathscr{X}$. Then, by [GS90], there is an arithmetic Chow ring $\widehat{\mathrm{CH}}^*(\mathscr{X})_\mathbb{Q}$ and an arithmetic degree map $\widehat{\deg}\colon \widehat{\mathrm{CH}}^{n+1}(\mathscr{X})_\mathbb{Q} \to \mathbb{R}$. Let L be a line bundle on X endowed with a $\mathbb{Z}$-model $\mathscr{L}$ of L on $\mathscr{X}$ and a smooth metric $\|\cdot\|$ on its analytification $L(\mathbb{C})$ on $X(\mathbb{C})$. To each Hermitian line bundle $\overline{\mathscr{L}} = (\mathscr{L}, \|\cdot\|)$, one can associate its first arithmetic Chern class $\hat{\mathrm{c}}_1(\overline{\mathscr{L}}) \in \widehat{\mathrm{CH}}^*(\mathscr{X})$. The height of X with respect to $\overline{\mathscr{L}}$ is defined as

$$\mathrm{h}_{\overline{\mathscr{L}}}(X) = \widehat{\deg}\left(\hat{\mathrm{c}}_1(\overline{\mathscr{L}})^{n+1}\right). \tag{0.1}$$

In [BGS94], Bost–Gillet–Soulé proved important properties of this height, for example an arithmetic Bézout theorem.

This definition has the disadvantage that it only works for smooth projective varieties and smooth metrics. Moreover, it depends on the existence of models. It is more general and flexible to use the adelic language by Zhang [Zha95], equipping the line bundle with a metric at each place of $\mathbb{Q}$ instead of a model and allowing uniform limits of semipositive metrics. A remarkable application of Zhang's height of varieties is his proof of the Bogomolov conjecture for Abelian varieties over a number field in [Zha98].

From the adelic point of view, it is more convenient to define the height as a sum of local heights. Here, "local" means that we fix a place of $\mathbb{Q}$ and work over the corresponding completion $\mathbb{Q}_v$. Local heights can be studied for any field with absolute value which was systematically done by Gubler [Gub97], [Gub98], [Gub03].

In the following, we outline the case of a local height over a field K which is complete with respect to an arbitrary non-trivial non-Archimedean absolute value K. Let X be a proper variety over K and denote by X^{an} its analytification in the sense of Berkovich. On a line bundle L on X, every model of some positive tensor power $L^{\otimes e}$ induces an algebraic metric on L. A semipositive metric is the uniform limit of algebraic metrics that satisfy a certain positivity property. Let $\overline{L}$ be a semipositive metrized line bundle on X and Z a

t-dimensional cycle of X. Let $s_0, \ldots, s_t$ be non-zero meromorphic sections of $\overline{L}$ satisfying

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset. \tag{0.2}$$

Then, Gubler showed the existence of a local height

$$\lambda_{(\overline{L},s_0),\ldots,(\overline{L},s_t)}(Z) \in \mathbb{R}\,,$$

using refined intersection theory and, since the valuation ring K° is not necessarily noetherian, methods from formal and rigid geometry. If K° is discrete, hence Noetherian, and the metric is induced by an algebraic K°-model, then this local height is the usual intersection product of the Cartier divisors $\mathrm{div}(s_0), \ldots, \mathrm{div}(s_t)$ on the model.

In [Cha06], Chambert-Loir introduced a measure $\mathrm{c}_1(\overline{L})^{\wedge t} \wedge \delta_Z$ on X^{an} such that for algebraic metrics an induction formula as in the Archimedean case holds. An important statement of my thesis is the following corresponding formula (cf. Theorem 1.4.3) which generalizes a result of Chambert-Loir and Thuillier [CT09, Théorème 4.1].

Theorem 1 (Induction formula). *Let notation be as above. For simplicity, we assume that Z is a subvariety. If $Z \not\subseteq |\,\mathrm{div}(s_t)|$, then let $s_{t,Z} := s_t|_Z$, otherwise we choose any non-zero meromorphic section $s_{t,Z}$ of $L_t|_Z$.*

Then, the function $\log\|s_t\|$ is integrable with respect to $\mathrm{c}_1(\overline{L})^{\wedge t} \wedge \delta_Z$ and we have

$$\lambda_{(\overline{L},s_0),\ldots,(\overline{L},s_t)}(Z) = \lambda_{(\overline{L},s_0),\ldots,(\overline{L},s_{t-1})}(\mathrm{cyc}(s_{t,Z})) - \int_{X^{\mathrm{an}}} \log\|s_{t,Z}\|\ \mathrm{c}_1(\overline{L})^{\wedge t} \wedge \delta_Z.$$

The proof is based on [CT09] where the formula is demonstrated under the additional assumptions that K is a completion of a number field and $s_0, \ldots, s_t$ are global sections such that their Cartier divisors intersect properly on Z. The heart of the proof is an approximation theorem saying that $\log\|s_t\|$ can be approximated by suitable functions $\log\|1\|_n$, where $\|\cdot\|_n$ are formal metrics on the trivial bundle $\mathcal{O}_X$. To show this, we use techniques from analytic and formal geometry.

In the case of Archimedean fields, local heights can be handled in a similar way. We will recall this in section 1.5.

Now, we come back to (global) heights. In [Gub97], Gubler introduced the notion of an M-field. In this thesis, this is a field K together with a measured set M of absolute values on K satisfying the product formula (Definition 3.1.1). The easiest example is $\mathbb{Q}$ together with the set of standard normalized absolute values $M_{\mathbb{Q}}$, equipped with the counting measure. But the notion of M-fields also includes number fields, function fields and finitely generated fields.

Let us consider a projective variety X over an M-field K and a line bundle L on X. A semipositive M-metric on L is a family of semipositive metrics $\|\cdot\|_v$ on L_v, $v \in M$. Write $\overline{L} = (L, (\|\cdot\|_v)_v)$ and $\overline{L}_v = (L_v, \|\cdot\|_v)$ for each $v \in M$. Let Z be a t-dimensional cycle such that the function

$$M \longrightarrow \mathbb{R}, \quad v \longmapsto \lambda_{(\overline{L}_v,s_0),\ldots,(\overline{L}_v,s_t)}(Z_v)$$

is μ-integrable for any choice of sections $s_0, \ldots, s_t$ of L which satisfy condition (0.2). For example, if we consider the $M_{\mathbb{Q}}$-field $\mathbb{Q}$, the μ-integrability is satisfied for every cycle Z and a quasi-algebraic metrized line bundle $\overline{L}$, i. e. almost all metrics of $\overline{L}$ are induced by a common model over $\mathbb{Z}$. The (global) height of Z is defined as

$$\mathrm{h}_{\overline{L}}(Z) = \int_M \lambda_{(\overline{L}_v, s_0),\ldots,(\overline{L}_v, s_t)}(Z_v)\,\mathrm{d}\mu(v). \tag{0.3}$$

By the product formula, this definition is independent of the choice of sections. Note that all the mentioned heights can be also defined for $t+1$ distinct line bundles.

In [Mor00], Moriwaki defined the height of a variety over a finitely generated field K over $\mathbb{Q}$ as an arithmetic intersection number as in (0.1) and generalized the Bogomolov conjecture to such fields. As observed by Gubler [Gub03, Example 11.22], this finitely generated extension has a $\mathfrak{M}$-field structure for a natural set of places $\mathfrak{M}$ related to the normal variety B with $K = \mathbb{Q}(B)$. Burgos–Philippon–Sombra proved in [BPS14b, Theorem 2.4] that the height of Moriwaki can be written as an integral of local heights over $\mathfrak{M}$. In this thesis, their result is generalized in a certain way as outlined in the following.

Let B be a b-dimensional normal proper variety over a global field F. We denote by K the function field of B, which is a finitely generated extension of F. Choosing nef quasi-algebraic metrized line bundles $\overline{H}_1, \ldots, \overline{H}_b$ on B, we can equip K with a natural structure $(\mathfrak{M}, \mu)$ of an $\mathfrak{M}$-field (see 3.2.4). Let $\pi\colon \mathcal{X} \to B$ be a dominant morphism of proper varieties over F of relative dimension n and denote by X the generic fiber of π. Let $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n$ be semipositive quasi-algebraic line bundles on $\mathcal{X}$ and choose any invertible meromorphic sections $s_0, \ldots, s_n$ of $\mathcal{L}_0, \ldots, \mathcal{L}_n$ respectively, which satisfy (0.2). These line bundles induce $\mathfrak{M}$-metrized line bundles $\overline{L}_0, \ldots, \overline{L}_n$ on X. We prove in Theorem 3.3.4:

Theorem 2. *The function $\mathfrak{M} \to \mathbb{R}$, $w \mapsto \lambda_{(\overline{L}_{0,w}, s_0),\ldots,(\overline{L}_{n,w}, s_n)}(X)$, is μ-integrable and we have*

$$\mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_0,\ldots,\overline{\mathcal{L}}_n}(\mathcal{X}) = \int_{\mathfrak{M}} \lambda_{(\overline{L}_{0,w}, s_0),\ldots,(\overline{L}_{n,w}, s_n)}(X)\,\mathrm{d}\mu(w).$$

Burgos–Philippon–Sombra have shown this formula in the case when $F = \mathbb{Q}$ and the varieties $\mathcal{X}$, B and the occuring metrized line bundles are induced by models over $\mathbb{Z}$ similarly as in (0.1). The main difficulty in their proof appears at the Archimedean place, where well-known techniques from complex geometry as the Ehresmann's fibration theorem are used. In our proof, we can just copy the Archimedean part, but at the non-Archimedean places, we integrate over Berkovich spaces and we use methods from algebraic and formal geometry instead.

Toric Geometry

Toric varieties are a special class of varieties that have a nice description through combinatorial data from convex geometry. So they are well-suited for testing conjectures and for computations in algebraic geometry. Let K be any field, then a complete fan Σ of polyhedral cones in a vector space $N_\mathbb{R} \simeq \mathbb{R}^n$ corresponds to a proper toric variety X_Σ over K with torus $\mathbb{T} \simeq \operatorname{Spec} K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The torus $\mathbb{T}$ acts on X_Σ and hence, every toric object should have a certain invariance property with respect to this action, in order to describe it in terms of convex geometry.

A support function on Σ, i. e. a concave function $\Psi\colon N_\mathbb{R} \to \mathbb{R}$ which is linear on each cone of Σ and has integral slopes, corresponds to a base-point-free toric line bundle L on X_Σ together with a toric section s. Moreover, one can associate to Ψ a polytope

$\Delta_\Psi = \{m \in M_\mathbb{R} \mid m \geq \Psi\}$ in the dual space $M_\mathbb{R}$ of $N_\mathbb{R}$. Then a famous result in classical toric geometry is the degree formula:

$$\deg_L(X_\Sigma) = n! \operatorname{vol}_M(\Delta_\Psi),$$

where vol_M is the Haar measure on $M_\mathbb{R}$ such that the underlying lattice $M \simeq \mathbb{Z}^n$ has covolume one. As mentioned above, the arithmetic analogue of the degree of a variety with respect to a line bundle is the height of a variety with respect to a metrized line bundle. So it is a natural question if one can find an analogous formula for the height. This problem was tackled by Burgos, Philippon and Sombra in the monograph [BPS14a] and they have shown the following.

Assume that the pair (X_Σ, L) lies over $\mathbb{Q}$ and let $M_\mathbb{Q}$ be the set of places of $\mathbb{Q}$. To a family $(\vartheta_v)_{v\in M_\mathbb{Q}}$ of concave functions on Δ_Ψ with $\vartheta_v \equiv 0$ for allmost all v, one can associate an $M_\mathbb{Q}$-metrized lined bundle $\overline{L} = (L, (\|\cdot\|_v)_v)$. Then the height of X_Σ with respect to $\overline{L}$ is given by

$$\mathrm{h}_{\overline{L}}(X_\Sigma) = (n+1)! \sum_{v\in M_\mathbb{Q}} \int_{\Delta_\Psi} \vartheta_v \operatorname{dvol}_M.$$

Indeed, to state and prove this formula, Burgos–Philippon–Sombra systematically studied in [BPS14a] the arithmetic geometry of toric varieties in terms of convex geometry. In particular, they described models of toric divisors over discrete valuation rings by piecewise affine functions on polyhedral complexes. Furthermore, for a field which is complete with respect to an Archimedean or discrete non-Archimedean absolute value, they classified semipositive toric metrized line bundles and their associated measures and local heights, by concave functions and their associated Monge-Ampère measures and Legendre-Fenchel duals.

As mentioned before, metrized line bundles and their associated measures and local heights can be also studied for non-Archimedean fields with non-necessarily discrete valuation. So it is a quite natural question if the results in [BPS14a] extend to arbitrary non-Archimedean fields. This issue is handled in my thesis.

In analogy to [BPS14a, § 3.6], we describe toric divisors on toric schemes over arbitrary valuation rings of rank one (see Theorem 2.3.3). This description is based on the theory of toric schemes over valuation rings of rank one by Gubler [Gub13] and the classification of these schemes by admissible fans by Gubler and Soto [GS13].

Furthermore, we study metrics, measures and local heights over a non-necessarily discrete non-Archimedean field K, following the ideas of [BPS14a, § 4]. Let L be a toric line bundle on a proper toric variety X_Σ over K together with any toric section s, and let Ψ be the corresponding support function on the complete fan Σ. A continuous metric $\|\cdot\|$ on L is toric if the function $p \mapsto \|s(p)\|$ is invariant under the action of a certain closed analytic subgroup of $\mathbb{T}^{\mathrm{an}}$ (Definition 2.4.1). We will give the following classification of toric metrics over algebraically closed non-Archimedean fields (Theorem 2.5.8):

Theorem 3. *There is a bijective correspondence between the sets of*

(i) semipositive toric metrics on L;

(ii) concave functions ψ on $N_\mathbb{R}$ such that the function $|\psi - \Psi|$ is bounded;

(iii) continuous concave functions ϑ on Δ_Ψ.

For the first bijection, one associates to the toric metric $\|\cdot\|$ the function ψ on $N_{\mathbb{R}}$ given by $\psi(u) = \log\|s \circ \operatorname{trop}^{-1}(u)\|$, where $\operatorname{trop}\colon N_{\mathbb{R}} \to \mathbb{T}^{\mathrm{an}}$ is the tropicalization map from tropical geometry (see 2.4.5). The second bijection is given by the Legendre-Fenchel dual from convex analysis (see A.7). Essential for the proof are characterizations of semipositive formal metrics developed in [GK15]. Note that the concave function $\psi = \Psi$ defines a distinguished metric on L, called canonical.

Next, we show that the measure $\mathrm{c}_1(\overline{L})^{\wedge n} \wedge \delta_{X_\Sigma}$ induced by a semipositive toric metrized line bundle $\overline{L} = (L, \|\cdot\|)$, satisfies the following formula

$$\operatorname{trop}_*(\mathrm{c}_1(\overline{L})^{\wedge n} \wedge \delta_{X_\Sigma}|_{\mathbb{T}^{\mathrm{an}}}) = n!\mathcal{M}_M(\psi),$$

where ψ is the concave function given by $\|\cdot\|$ and $\mathcal{M}_M(\psi)$ is the Monge-Ampère measure of ψ (see A.17).

Now, all ingredients are developed to state and show a formula for the local height in the toric setting as proved in [BPS14a, Theorem 5.1.6] for a discrete non-Archimedean field. Let X_Σ be an n-dimensional projective toric variety over K and $\overline{L}$ a semipositive toric metrized line bundle, and denote by $\overline{L}^{\mathrm{can}}$ the same line bundle equipped with the canonical metric. The toric local height of X_Σ with respect to $\overline{L}$ is defined as

$$\lambda^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = \lambda_{(\overline{L},s_0),\ldots,(\overline{L},s_n)}(X_\Sigma) - \lambda_{(\overline{L}^{\mathrm{can}},s_0),\ldots,(\overline{L}^{\mathrm{can}},s_n)}(X_\Sigma),$$

where $s_0, \ldots, s_n$ are any invertible meromorphic sections of L satisfying the intersection condition (0.2). We show the following main result (Theorem 2.6.6):

Theorem 4. *Let notation be as above. Then we have*

$$\lambda^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = (n+1)! \int_{\Delta_\Psi} \vartheta \,\mathrm{dvol}_M,$$

where $\vartheta\colon \Delta_\Psi \to \mathbb{R}$ *is the concave function associated to* $(\overline{L}, s)$ *given by Theorem 3.*

The proof is analogous to [BPS14a]. It is based on induction relative to n and uses the induction formula (Theorem 1) in an essential way.

The formula in Theorem 4 has the following application as suggested to me by José Burgos Gil. In the setting of Theorem 2, let $\pi\colon \mathcal{X} \to B$ be a dominant morphism of varieties over a global field F such that its generic fiber X is an n-dimensional toric variety over the function field $K = F(B)$. This field is equipped with the $\mathfrak{M}$-field structure induced by the metrized line bundles $\overline{H}_1, \ldots, \overline{H}_b$. Assume that $\overline{\mathcal{L}_0} = \cdots = \overline{\mathcal{L}_n} = \overline{\mathcal{L}}$ and that the induced semipositive $\mathfrak{M}$-metrized line bundle $\overline{L}$ is toric. Let s be any toric section of L and Ψ the associated support function. Then $\overline{L}$ defines, for each $w \in \mathfrak{M}$, a concave function $\vartheta_w\colon \Delta_\Psi \to \mathbb{R}$.

Note that in this setting a non-Archimedean place $w \in \mathfrak{M}$ is not necessarily discrete. So, we cannot use only the formula for toric local heights from [BPS14a]. However, combining theorems 2 and 4 (resp. its Archimedean analogue), we obtain

$$\mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}},\ldots,\overline{\mathcal{L}}}(\mathcal{X}) = (n+1)! \int_{\mathfrak{M}} \int_{\Delta_\Psi} \vartheta_w(x) \,\mathrm{dvol}(x) \,\mathrm{d}\mu(w). \tag{0.4}$$

This formula allows us to compute the height of a non-toric variety coming from a fibration with toric generic fiber. It generalizes Corollary 3.1 in [BPS14b] where the global field is

$\mathbb{Q}$ and the metrized line bundles are induced by models over $\mathbb{Z}$, i. e. where the left-hand side is an arithmetic intersection number as in (0.1). In this setting only Archimedean and discrete non-Archimedean places occur.

In [BPS14b], the formula corresponding to (0.4) is considered in the special case that X is a translate of a subtorus in the projective space and canonical metrics. This can be imitated in our setting and we further particularize to the case when B is an elliptic curve leading to a natural example to illustrate our theory.

Acknowledgements

First of all, I express my deepest thanks for my advisor Walter Gubler without whose time, patience and advice, none of this thesis would have been possible. He always had an open door for my questions, so I could profit much from his mathematical knowledge.

Next, I am very grateful to José Burgos Gil for his support and for generously sharing his ideas, especially for chapter 3.

It is a pleasure to thank Jascha Smacka for both his proofreading efforts and his numerous answers to my questions. Further, I thank Bernhard Dietel, Klaus Künnemann, Hartwig Mayer, Lukas Pottmeyer, Johannes Sprang, Alexander Voitovitch and Philipp Vollmer for helpful discussions and remarks.

Special thanks go to my parents and Ute Deigendesch for always encouraging me and enriching my non-mathematical life.

This research was partially supported by the DFG grant: SFB 1085 "Higher invariants".

Conventions and Notations

$\mathbb{N}$ is the set of natural numbers containing zero. All occuring rings and algebras are commutative with unity. For a ring R, the group of units is denoted by $R^\times$.

A variety over a field k is an irreducible and reduced scheme which is separated and of finite type over k. The function field of a variety X over k is denoted by $k(X)$ or $\mathrm{K}(X)$. For a proper scheme Y over a field, we denote by $Y^{(n)}$ the set of subvarieties of codimension n. A prime cycle on Y is just a subvariety of Y.

By a line bundle we mean a locally free sheaf of rank one. For an invertible meromorphic section s of a line bundle, we denote by $\mathrm{div}(s)$ the associated Cartier divisor and by $\mathrm{cyc}(s)$ the associated Weil divisor. The support of $\mathrm{div}(s)$ is denoted by $|\mathrm{div}(s)|$.

A measure is a signed measure, i. e. it is not necessarily non-negative. A non-Archimedean field is a field which is complete with respect to a non-trivial non-Archimedean absolute value $|\cdot|$.

For the notations used from convex geometry, we refer to Appendix A. Furthermore, notations and terminology defined in this thesis are listed in the index.

Chapter 1.

Metrics, Local Heights and Measures over Non-Archimedean Fields

In this chapter, we prove an induction formula for the local height of a variety over a non-Archimedean field with respect to DSP metrized pseudo-divisors (Theorem 1.4.3), generalizing a result of Chambert-Loir and Thuillier [CT09, Théorème 4.1]. This formula is important for our work on toric varieties since it serves as definition for local heights in our key source [BPS14a].

Before that, we recall the theory of local heights over non-Archimedean fields from [Gub98] and [Gub03], and the theory of measures associated to metrized line bundles introduced in [Cha06] and developed in [Gub07b].

In section 1.5, we give a short overview of local heights and measures over complex varieties.

Let K be a *non-Archimedean field*, i. e. a field which is complete with respect to a non-trivial non-Archimedean absolute value $|\cdot|$. Its valuation ring is denoted by K°, the associated maximal ideal by $K^{\circ\circ}$ and the residue field by $\tilde{K} = K^\circ/K^{\circ\circ}$.

1.1. Analytic and Formal Geometry

In this section, we recall some facts about the (Berkovich-) analytification of schemes over K and of formal schemes over K°. In the analytic part we follow [BPS14a, § 1.2]. See also [Ber90] and [Ber93] for further informations. The basic references for formal geometry are [Gub98, § 1] and [Gub07b, § 2] and, for details, [Bos14].

Let X be a scheme of finite type over K.

1.1.1. First let $X = \operatorname{Spec}(A)$ be affine. Then the *(Berkovich-) analytic space* X^{an} associated to X is the set of multiplicative seminorms on A extending the absolute value $|\cdot|$ on K. We endow it with the coarsest topology such that the functions $X^{\mathrm{an}} \to \mathbb{R}$, $p \mapsto p(f)$ are continuous for every $f \in A$.

Next we will define a sheaf of rings $\mathcal{O}_{X^{\mathrm{an}}}$ on X^{an}: Each $p \in X^{\mathrm{an}}$ induces a multiplicative norm on the integral domain $A/\ker(p)$ and therefore a non-Archimedean absolute value on its quotient field extending $|\cdot|$ on K. We write $\mathscr{H}(p)$ for the completion of this field with respect to that absolute value. The image of $f \in A$ in $\mathscr{H}(p)$ is denoted by $f(p)$ and we write also $|\cdot|$ for the absolute value in $\mathscr{H}(p)$. Then we have $p(f) = |f(p)|$ for each $f \in A$.

An *analytic function* s on an open set U of X^{an} is a function

$$s\colon U \to \coprod_{p\in U} \mathscr{H}(p)\,,$$

such that, for each $p \in U$, we have $s(p) \in \mathscr{H}(p)$ and there is an open neighborhood $p \in V \subseteq U$ with the property that, for all $\varepsilon > 0$, there are elements $f, g \in A$ with $|g(q)| \neq 0$ and $|s(q) - f(q)/g(q)| < \varepsilon$ for all $q \in V$. These functions form a sheaf of rings $\mathcal{O}_{X^{\mathrm{an}}}$ and we get a locally ringed space $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$.

1.1.2. For any scheme X of finite type over K we define the *analytic space* X^{an} by gluing the affine analytic spaces obtained from an open affine cover of X. For a morphism $\varphi\colon X \to Y$ of schemes of finite type over K we have a canonical map $\varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ defined by $\varphi^{\mathrm{an}}(p) := p \circ \varphi^\sharp$ on suitable affine open subsets.

The analytification functor preserves many properties of schemes and their morphisms. So an analytic space X^{an} is Hausdorff (resp. compact) if and only if X is separated (resp. proper). On the category of proper schemes over K this functor is fully faithful and induces an equivalence between the categories of coherent $\mathcal{O}_X$- and $\mathcal{O}_{X^{\mathrm{an}}}$-modules. The proofs and more such GAGA theorems can be found in [Ber90, § 3.4].

The analytification of a formal scheme is more difficult because at first we need arbitrary analytic spaces. Here we only give an overview and not the precise definition of these spaces.

1.1.3. The *Tate algebra* $K\langle x_1, \dots, x_n\rangle$ consists of the formal power series $f = \sum_\nu a_\nu x^\nu$ in $K[[x_1, \dots, x_n]]$ such that $|a_\nu| \to 0$ as $|\nu| \to \infty$. This K-algebra is the completion of $K[x_1, \dots, x_n]$ with respect to the Gauß norm $\|f\| = \max_\nu |a_\nu|$.

A *K-affinoid algebra* is an algebra $\mathcal{A}$ over K which is isomorphic to $K\langle x_1, \dots, x_n\rangle/I$ for an ideal I. We may use the quotient norm from $K\langle x_1, \dots, x_n\rangle$ to define a K-Banach algebra $(\mathcal{A}, \|\cdot\|)$. The presentation and hence the induced norm of an affinoid algebra is not unique but two norms on $\mathcal{A}$ are equivalent and thus, define the same concept of boundedness.

1.1.4. The *Berkovich spectrum* $\mathscr{M}(\mathcal{A})$ of a K-affinoid algebra $\mathcal{A}$ is defined as the set of multiplicative seminorms p on $\mathcal{A}$ satisfying $p(f) \leq \|f\|$ for all $f \in \mathcal{A}$. It only depends on the algebraic structure on $\mathcal{A}$. As above, we endow it with the coarsest topology such that the maps $p \mapsto p(f)$ are continuous for all $f \in \mathcal{A}$. Then $\mathscr{M}(\mathcal{A})$ is a non-empty compact space.

1.1.5. A *rational subdomain* of $\mathscr{M}(\mathcal{A}) = \mathscr{M}(K\langle x_1, \dots, x_n\rangle/I)$ is defined by

$$\mathscr{M}(\mathcal{A})\left(\frac{f_1}{g}, \dots, \frac{f_m}{g}\right) := \{p \in \mathscr{M}(\mathcal{A}) \mid |f_i(p)| \leq |g(p)|, i = 1, \dots, m\},$$

where $f_1, \dots, f_m, g \in \mathcal{A}$ generate the unit ideal in $\mathcal{A}$. It is the Berkovich spectrum of the affinoid algebra

$$\mathcal{A}\left\langle\frac{f_1}{g}, \dots, \frac{f_m}{g}\right\rangle := K\left\langle x_1, \dots, x_n, y_1, \dots, y_m\right\rangle / \left(I, g y_i - f_i \mid i = 1, \dots, m\right).$$

More generally one defines an *affinoid subdomain* in $\mathscr{M}(\mathcal{A})$ as the Berkovich spectrum of an affinoid algebra defined by a certain universal property (see [BGR84, 7.2.2]). Such a domain is a finite union of rational domains by the theorem of Gerritzen-Grauert ([BGR84, 7.3.5/3]).

A *(Berkovich) analytic space* over K is given by an atlas of affinoid subdomains $\mathscr{M}(\mathcal{A})$. The difficulties in this construction arise because the charts $\mathscr{M}(\mathcal{A})$ are not open. Analytic functions on such a chart are given by the elements of $\mathcal{A}$. The precise definition can be found in [Ber90, § 3] where such spaces are called strictly analytic spaces.

1.1.6. We say that a K°-algebra A is *admissible* if it is isomorphic to $K^\circ\langle x_1,\ldots,x_n\rangle/I$ for an ideal I and A has no K°-torsion. If A is admissible, then I is finitely generated (see [BL93a, Proposition 1.1]). A formal scheme $\mathfrak{X}$ over K° is called *admissible* if there is a locally finite covering of open subsets isomorphic to formal affine schemes $\mathrm{Spf}(A)$ for admissible K°-algebras A.

In this case, the *generic fiber* $\mathfrak{X}^{\mathrm{an}}$ of $\mathfrak{X}$ is the analytic space locally defined by the Berkovich spectrum of the K-affinoid algebra $\mathcal{A} = A\otimes_{K^\circ} K$. Moreover we define the *special fiber* $\widetilde{\mathfrak{X}}$ of $\mathfrak{X}$ as the $\tilde{K}$-scheme locally given by $\mathrm{Spec}(A/K^{\circ\circ}A)$, i. e. $\widetilde{\mathfrak{X}}$ is a scheme of locally finite type over $\tilde{K}$ with the same topological space as $\mathfrak{X}$ and the structure sheaf $\mathcal{O}_{\widetilde{\mathfrak{X}}} := \mathcal{O}_{\mathfrak{X}}\otimes_{K^\circ}\tilde{K}$.

There is a *reduction map* $\mathrm{red}\colon \mathfrak{X}^{\mathrm{an}}\to\widetilde{\mathfrak{X}}$ assigning each seminorm p in a neighborhood $\mathscr{M}(A\otimes_{K^\circ}K)$ to the prime ideal $\{a\in A \mid p(a\otimes 1)<1\}/K^{\circ\circ}A$. This map is surjective and anti-continuous. If $\widetilde{\mathfrak{X}}$ is reduced, then red coincides with the reduction map in [Ber90, 2.4]. In this case, for every irreducible component V of $\widetilde{\mathfrak{X}}$, there is a unique point $\xi_V\in\mathfrak{X}^{\mathrm{an}}$ such that $\mathrm{red}(\xi_V)$ is the generic point of V (see [Ber90, Proposition 2.4.4]).

1.1.7. Assume that K is algebraically closed and let $\mathfrak{X}=\mathrm{Spf}(A)$ be an admissible formal affine scheme over K° with reduced generic fiber $\mathfrak{X}^{\mathrm{an}}$. Let $\mathcal{A}=A\otimes_{K^\circ}K$ be the associated K-affinoid algebra and let $\mathcal{A}^\circ$ be the K°-subalgebra of power bounded elements in $\mathcal{A}$. Then $\mathfrak{X}' := \mathrm{Spf}(\mathcal{A}^\circ)$ is an admissible formal scheme over K° with $\mathfrak{X}'^{\mathrm{an}}=\mathfrak{X}^{\mathrm{an}}$ and with reduced special fiber $\widetilde{\mathfrak{X}'}$. We have a canonical morphism $\mathfrak{X}'\to\mathfrak{X}$ whose restriction to the special fibers is finite and surjective. By gluing, these assertions also hold for non-necessarily affine schemes. For details, we refer to [Gub98, Proposition 1.11 and 8.1].

1.1.8. Let $\mathscr{X}$ be a flat scheme of finite type over K° with generic fiber X and let π be some non-zero element in $K^{\circ\circ}$. Locally we can replace the coordinate ring $\mathscr{A}$ by the π-adic completion of $\mathscr{A}$ and get an admissible formal scheme $\hat{\mathscr{X}}$ over K° with special fiber equal to the special fiber $\widetilde{\mathscr{X}}$ of $\mathscr{X}$. The generic fiber $\hat{\mathscr{X}}^{\mathrm{an}}$, denoted by X°, is an analytic subdomain of X^{an} and is locally given by

$$\{p\in(\mathrm{Spec}\,\mathscr{A}\otimes_{K^\circ}K)^{\mathrm{an}} \mid p(a)\le 1\ \forall\, a\in\mathscr{A}\}\,.$$

Then the surjective reduction map $\mathrm{red}\colon X^\circ\to\widetilde{\mathscr{X}}$ is locally given by

$$p\longmapsto \{a\in\mathscr{A}\mid p(a)<1\}/K^{\circ\circ}\mathscr{A}.$$

If $\mathscr{X}$ is proper over K°, then $X^\circ = X^{\mathrm{an}}$ and the reduction map is defined on the whole of X^{an}. If $\widetilde{\mathscr{X}}$ is reduced, then each maximal point of $\widetilde{\mathscr{X}}$ has a unique inverse image in X°. We refer to [Gub13, 4.9–4.13] for details.

If K is algebraically closed and X is reduced, then the construction in 1.1.7 gives us a formal admissible scheme $\mathfrak{X}$ over K° with generic fiber $\mathfrak{X}^{\mathrm{an}}=X^\circ$ and with reduced special fiber $\widetilde{\mathfrak{X}}$ such that the canonical morphism $\widetilde{\mathfrak{X}}\to\widetilde{\mathscr{X}}$ is finite and surjective.

1.2. Metrics, Local Heights and Measures

From now on, we assume that the non-Archimedean field K is algebraically closed. This is no serious restriction because we can always perform base change to the completion of the algebraic closure of any non-Archimedean field and local heights and measures do not depend on the choice of the base field.

Let X be a reduced proper scheme over K and L a line bundle on X. This defines a line bundle L^{an} on the compact space X^{an}.

Definition 1.2.1. A *metric* $\|\cdot\|$ on L is the datum, for any section s of L^{an} on a open subset $U \subseteq X^{\mathrm{an}}$, of a continuous function $\|s(\cdot)\| \colon U \longrightarrow \mathbb{R}_{\geq 0}$, such that

(i) it is compatible with the restriction to smaller open subsets;

(ii) for all $p \in U$, $\|s(p)\| = 0$ if and only if $s(p) = 0$;

(iii) for any $\lambda \in \mathcal{O}_{X^{\mathrm{an}}}(U)$ and for all $p \in U$, $\|(\lambda s)(p)\| = |\lambda(p)| \cdot \|s(p)\|$.

On the set of metrics on L we define the distance function

$$\mathrm{d}(\|\cdot\|, \|\cdot\|') := \sup_{p \in X^{\mathrm{an}}} \left|\log(\|s(p)\|/\|s(p)\|')\right|,$$

where s is any local section of L^{an} not vanishing at p. Clearly, this definition is independent of the choice of s. The pair $\overline{L} := (L, \|\cdot\|)$ is called a *metrized line bundle*. Operations on line bundles like tensor product, dual and pullback extend to metrized line bundles.

Definition 1.2.2. A *formal (K°-)model* of X is an admissible formal scheme $\mathfrak{X}$ over K° with a fixed isomorphism $\mathfrak{X}^{\mathrm{an}} \simeq X^{\mathrm{an}}$. Note that we identify $\mathfrak{X}^{\mathrm{an}}$ with X^{an} via this isomorphism.

A *formal (K°-)model* of (X, L) is a triple $(\mathfrak{X}, \mathfrak{L}, e)$ consisting of a formal model $\mathfrak{X}$ of X, a line bundle $\mathfrak{L}$ on $\mathfrak{X}$ and an integer $e \geq 1$, together with an isomorphism $\mathfrak{L}^{\mathrm{an}} \simeq (L^{\otimes e})^{\mathrm{an}}$. When $e = 1$, we write $(\mathfrak{X}, \mathfrak{L})$ instead of $(\mathfrak{X}, \mathfrak{L}, 1)$.

Definition 1.2.3. To a formal K°-model $(\mathfrak{X}, \mathfrak{L}, e)$ of (X, L) we associate a metric $\|\cdot\|$ on L in the following way: If $\mathfrak{U}$ is a formal trivialization of $\mathfrak{L}$ and if s is a section of L^{an} on $\mathfrak{U}^{\mathrm{an}}$ such that $s^{\otimes e}$ corresponds to $\lambda \in \mathcal{O}_{X^{\mathrm{an}}}(\mathfrak{U}^{\mathrm{an}})$ with respect to this trivialization, then

$$\|s(p)\| = |\lambda(p)|^{1/e}$$

for all $p \in \mathfrak{U}^{\mathrm{an}}$. A metric on L obtained in this way is called a $\mathbb{Q}$-*formal metric* and, if $e = 1$, a *formal metric*.

Such a $\mathbb{Q}$-formal metric is said to be *semipositive* if the reduction $\widetilde{\mathfrak{L}}$ of $\mathfrak{L}$ on the special fiber $\widetilde{\mathfrak{X}}$ is nef, i.e. we have $\deg_{\widetilde{\mathfrak{L}}}(C) \geq 0$ for every closed integral curve C in $\widetilde{\mathfrak{X}}$.

Remark 1.2.4. In the literature, $\mathbb{Q}$-formal metrics are often just called formal metrics (e.g. in [Cha06] and [CT09]). In Definition 1.2.3, we basically follow the notation of [CD12] and the papers by W. Gubler.

1.2.5. The dual, the tensor product and the pullback of ($\mathbb{Q}$-)formal metrics are again ($\mathbb{Q}$-)formal metrics. Furthermore, the tensor product and the pullback of semipositive $\mathbb{Q}$-formal metrics are semipositive.

1.2.6. Every line bundle L on X has a formal K°-model $(\mathfrak{X}, \mathfrak{L})$ and hence a formal metric $\|\cdot\|$. For proofs of this and the following statements we refer to [Gub98, § 7]. Since K is algebraically closed and X is reduced, we may always assume that $\mathfrak{X}$ has reduced special fiber (see 1.1.7). Then the formal metric determines the K°-model $\mathfrak{L}$ on $\mathfrak{X}$ up to isomorphisms, more precisely we have canonically

$$\mathfrak{L}(\mathfrak{U}) \cong \{s \in L^{\mathrm{an}}(\mathfrak{U}^{\mathrm{an}}) \mid \|s(p)\| \leq 1 \ \forall p \in \mathfrak{U}^{\mathrm{an}}\} \tag{1.1}$$

for each formal open subset $\mathfrak{U}$ of $\mathfrak{X}$.

A formal metric is characterized by the property that there exists an admissible covering $\{U_i\}_{i\in I}$ of X^{an} by affinoid domains and non-vanishing regular sections $s_i \in L^{\mathrm{an}}(U_i)$ such that $\|s_i(x)\| = 1$ for all $x \in U_i$.

Definition 1.2.7. An *algebraic K°-model* $\mathscr{X}$ of X is a flat and proper scheme over K° together with an isomorphism of the generic fiber of $\mathscr{X}$ onto X. An *algebraic K°-model* $(\mathscr{X}, \mathscr{L}, e)$ of (X, L) consists of a line bundle $\mathscr{L}$ on an algebraic K°-model $\mathscr{X}$ of X and a fixed isomorphism $\mathscr{L}|_X \cong L^e$.

As in Definition 1.2.3, an algebraic model $(\mathscr{X}, \mathscr{L}, e)$ of (X, L) induces a metric $\|\cdot\|$ on L, called *algebraic metric*. Such a metric is said to be *semipositive* if, for every closed integral curve C in the special fiber $\widetilde{\mathscr{X}}$, we have $\deg_{\mathscr{L}}(C) \geq 0$.

The following relatively recent result shows that, on algebraic varieties, it is always possible to work with algebraic in place of $\mathbb{Q}$-formal metrics.

Proposition 1.2.8. *Let L be a line bundle on a proper variety X over K and let $\|\cdot\|$ be a metric on L. Then, $\|\cdot\|$ is $\mathbb{Q}$-formal if and only if $\|\cdot\|$ is algebraic.*

Proof. The fact that every algebraic metric is $\mathbb{Q}$-formal follows easily from 1.1.8. The other direction is [GK15, Corollary 5.12]. □

1.2.9. A *metrized pseudo-divisor* $\hat{D}$ on X is a triple $\hat{D} := (\overline{L}, Y, s)$ where $\overline{L}$ is a metrized line bundle, Y is a closed subset of X and s is a nowhere vanishing section of L on $X \setminus Y$. Then $(\mathcal{O}(D), |D|, s_D) := (L, Y, s)$ is a pseudo-divisor in the sense of [Ful98, 2.2]. We can always define the pullback of a metrized pseudo-divisor $\hat{D}$ on X by a proper morphism $\varphi\colon X' \to X$, namely

$$\varphi^*\hat{D} := (\varphi^*\overline{\mathcal{O}}(D), \varphi^{-1}|D|, \varphi^* s_D).$$

Note that this is an advantage over Cartier divisors in order to formulate intersection theory.

Example 1.2.10. Let $\overline{L}$ be a metrized line bundle on X and s an *invertible meromorphic section* of L, i. e. there is an open dense subset U of X such that s restricts to a non-vanishing local section of L on U. Then the pair $(\overline{L}, s)$ determines a pseudo-divisor

$$\widehat{\operatorname{div}}(s) := \left(\overline{L}, |\operatorname{div}(s)|, s|_{X\setminus|\operatorname{div}(s)|}\right),$$

where $|\operatorname{div}(s)|$ is the support of the Cartier divisor $\operatorname{div}(s)$.

Every real-valued continuous function φ on X^{an} defines a metric on the trivial line bundle $\mathcal{O}_X$ given by $\|1\| = e^{-\varphi}$. We denote this metrized line bundle by $\mathcal{O}(\varphi)$. Then we get a metrized pseudo-divisor $\widehat{\mathcal{O}}(\varphi) := (\mathcal{O}(\varphi), \emptyset, 1)$.

1.2.11. Let $\hat{D}_0, \ldots, \hat{D}_t$ be metrized pseudo-divisors with $\mathbb{Q}$-formal metrics and let Z be a t-dimensional cycle on X with

$$|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset. \tag{1.2}$$

Note that condition (1.2) is much weaker than the usual assumption that $\hat{D}_0, \ldots, \hat{D}_t$ *intersect properly* on Z, that is, for all $I \subseteq \{0, \ldots, t\}$, each irreducible component of $Z \cap \bigcap_{i \in I} |D_i|$ has dimension $t - |I|$.

For $\mathbb{Q}$-formal metrized pseudo-divisors there is a refined intersection product with cycles on X developed by Gubler (see [Gub98, §8] and [Gub03, §5]). By means of this product, one can define the *local height* $\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$ as the real intersection number of $\hat{D}_0, \ldots, \hat{D}_t$ and Z on a joint formal K°-model. For details, we refer to [Gub98, §9] and [Gub03, §9]. If K° is a discrete valuation ring, hence Noetherian, and all the K°-models are algebraic, then we can use the usual intersection product.

Proposition 1.2.12. *The local height $\lambda(Z) := \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$ is characterized by the following properties:*

(i) It is multilinear and symmetric in $\hat{D}_0, \ldots, \hat{D}_t$ and linear in Z.

(ii) For a proper morphism $\varphi\colon X' \to X$ and a t-dimensional cycle Z' on X' satisfying $|D_0| \cap \cdots \cap |D_t| \cap |\varphi(Z)| = \emptyset$, we have

$$\lambda_{\varphi^*\hat{D}_0,\ldots,\varphi^*\hat{D}_t}(Z') = \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(\varphi_* Z').$$

(iii) Let $\lambda'(Z)$ be the local height obtained by replacing the metric $\|\cdot\|$ of $\hat{D}_0$ by another $\mathbb{Q}$-formal metric $\|\cdot\|'$. If the $\mathbb{Q}$-formal metrics of $\hat{D}_1, \ldots, \hat{D}_t$ are semipositive and if Z is effective, then

$$|\lambda(Z) - \lambda'(Z)| \leq \mathrm{d}(\|\cdot\|, \|\cdot\|') \cdot \deg_{\mathcal{O}(D_1),\ldots,\mathcal{O}(D_t)}(Z). \tag{1.3}$$

Proof. The properties (i) and (ii) are mentioned in [Gub03, Remark 9.3] for formal metrics and easily extend to $\mathbb{Q}$-formal metrics by multilinearity. The last property follows from the metric change formula in [Gub03, Remark 9.5]. □

1.2.13. If $\mathscr{X}$ is an algebraic K°-model of X, then there is a K°-model $\mathscr{Y}$ of X with reduced special fiber and a proper K°-morphism $\mathscr{Y} \to \mathscr{X}$ which is the identity on X. This follows from [BLR95, Theorem 2.1'].

Moreover, let $\overline{L}$, $\overline{L}'$ be algebraic metrized line bundles on X induced by algebraic K°-models $(\mathscr{X}, \mathscr{L}, e)$ and $(\mathscr{X}', \mathscr{L}', e')$ respectively. Taking the closure $\mathscr{X}''$ of X in $\mathscr{X} \times_{K^\circ} \mathscr{X}'$ and pulling back $\mathscr{L}$, $\mathscr{L}'$ to $\mathscr{X}''$, we obtain models inducing the given metrics on L and L' but living on the same model $\mathscr{X}''$.

Hence, we can always assume that $\mathscr{L}$ and $\mathscr{L}'$ live on a common model with reduced special fiber.

The same holds for formal models. Note that in the formal case it is much easier to obtain a model with reduced special fiber, see 1.1.7.

For global heights and Archimedean local heights of subvarieties there is an induction formula which can be taken as definition for the heights (see [BGS94, (3.2.2)] and [Gub03, Proposition 3.5]). A. Chambert-Loir has introduced the following measure on X^{an} such that an analogous induction formula holds for non-Archimedean local heights (cf. [Cha06, 2.3]).

Definition 1.2.14. Let X be a reduced proper scheme over K of dimension n and let $\overline{L}_i$, $i = 1, \ldots, n$, be $\mathbb{Q}$-formal metrized line bundles on X. By 1.2.13, there is a formal K°-model $\mathfrak{X}$ of X with reduced special fiber and, for each i, a formal K°-model $(\mathfrak{X}, \mathfrak{L}_i, e_i)$ of (X, L_i) inducing the metric of $\overline{L}_i$. We denote by $\widetilde{\mathfrak{X}}^{(0)}$ the set of irreducible components of the special fiber $\widetilde{\mathfrak{X}}$. Then we define a discrete (signed) measure on X^{an} by

$$\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_n) = \frac{1}{e_1 \cdots e_n} \sum_{V \in \widetilde{\mathfrak{X}}^{(0)}} \deg_{\widetilde{\mathfrak{L}}_1, \ldots, \widetilde{\mathfrak{L}}_n}(V) \cdot \delta_{\xi_V},$$

where δ_{ξ_V} is the Dirac measure in the unique point $\xi_V \in X^{\mathrm{an}}$ such that $\mathrm{red}(\xi_V)$ is the generic point of V (see 1.1.6).

More generally, let Y be a t-dimensional subvariety of X, then we define

$$\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Y = i_* \left(\mathrm{c}_1(\overline{L}_1|_Y) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t|_Y) \right),$$

where $i\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is the induced immersion. We also write shortly $\mathrm{c}_1(\overline{L}_1) \cdots \mathrm{c}_1(\overline{L}_t) \delta_Y$. This measure extends by linearity to t-dimensional cycles.

1.2.15. The measure in Definition 1.2.14 is multilinear and symmetric in metrized line bundles. Moreover, the total mass of $\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Y$ equals the degree $\deg_{L_1, \ldots, L_t}(Y)$. If the metrics of the $\overline{L}_i$ are semipositive, then it is a positive measure.

Proposition 1.2.16 (Induction formula)**.** *Let $\hat{D}_0, \ldots, \hat{D}_t$ be $\mathbb{Q}$-formal metrized pseudo-divisors and let Z be a t-dimensional prime cycle with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. If $|Z| \not\subseteq |D_t|$, then let $s_{D_t,Z} := s_{D_t}|_Z$, otherwise we choose any non-zero meromorphic section $s_{D_t,Z}$ of $\mathcal{O}(D_t)|_Z$. Let Y be the Weil divisor of $s_{D_t,Z}$ considered as a cycle on X. Then we have*

$$\lambda_{\hat{D}_0, \ldots, \hat{D}_t}(Z) = \lambda_{\hat{D}_0, \ldots, \hat{D}_{t-1}}(Y) - \int_{X^{\mathrm{an}}} \log \|s_{D_t,Z}\| \cdot \mathrm{c}_1(\overline{\mathcal{O}}(D_0)) \wedge \cdots \wedge \mathrm{c}_1(\overline{\mathcal{O}}(D_{t-1})) \wedge \delta_Z.$$

Proof. This follows from [Gub03, Remark 9.5] and Definition 1.2.14. □

1.3. Semipositive Metrics, Local Heights and Measures

It would be nice if we could extend local heights to all continuous metrics. Although the $\mathbb{Q}$-formal metrics are dense in the space of continuous metrics, this is not possible because the continuity property (1.3) in Proposition 1.2.12 only holds for semipositive $\mathbb{Q}$-formal metrics. So the best we can do here is to extend the heights to limits of semipositive $\mathbb{Q}$-formal metrics. Then the canonical local heights for subvarieties of an abelian variety are contained in this theory (see [Gub10, Ex. 3.7] for details).

In this and the following section, let X be a reduced proper scheme over the algebraically closed field K.

Definition 1.3.1. Let $\overline{L} = (L, \|\cdot\|)$ be a metrized line bundle on X. The metric $\|\cdot\|$ is called *semipositive* if there exists a sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of semipositive $\mathbb{Q}$-formal metrics on L such that
$$\lim_{n\to\infty} \mathrm{d}(\|\cdot\|_n, \|\cdot\|) = 0.$$
In this case we say that $\overline{L} = (L, \|\cdot\|)$ is a *semipositive (metrized) line bundle.* The metric is said to be *DSP* (for "difference of semipositive") if there are semipositive metrized line bundles $\overline{M}$, $\overline{N}$ on X such that $\overline{L} = \overline{M} \otimes \overline{N}^{-1}$. Then $\overline{L}$ is called *DSP (metrized) line bundle* as well.

Remark 1.3.2. If $\|\cdot\|$ is a $\mathbb{Q}$-formal metric, then [GK15, Proposition 7.2] says that $\|\cdot\|$ is semipositive in the sense of Definition 1.2.3 if and only if $\|\cdot\|$ is semipositive as defined in Definition 1.3.1. So there is no ambiguity in the use of the term semipositive metric. This answers the question raised in [BPS14a, Remark 1.4.2].

Remark 1.3.3. W. Gubler works with slightly more general metrics, called semipositive admissible or $\hat{\mathfrak{g}}_X^+$-metrics (cf. [Gub03, 10.2, 10.3]). We have choosen the same definition of semipositive metrics as in [BPS14a] and the papers by A. Chambert-Loir because it suffices for our purposes and is more suitable for toric geometry.

1.3.4. The tensor product and the pullback (with respect to a proper morphism) of semipositive metrics are again semipositive. The tensor product, the dual and the pullback of DSP metrics are also DSP.

1.3.5. By means of Proposition 1.2.12, we can easily extend the local heights to DSP metrics. Concretely, let Y be a t-dimensional prime cycle and $\hat{D}_i = (L_i, \|\cdot\|_i, |D_i|, s_i)$, $i = 0, \dots, t$, a collection of semipositive metrized pseudo-divisors on X with
$$|D_0| \cap \cdots \cap |D_t| \cap Y = \emptyset.$$
By Definition 1.3.1, there is, for each i, an associated sequence of semipositive $\mathbb{Q}$-formal metrics $\|\cdot\|_{i,n}$ on L_i such that $\mathrm{d}(\|\cdot\|_{i,n}, \|\cdot\|_i) \to 0$ for $n \to \infty$. Then we define the *local height* of Y with respect to $\hat{D}_0, \dots, \hat{D}_t$ as
$$\lambda_{\hat{D}_0,\dots,\hat{D}_t}(Y) := \lim_{n\to\infty} \lambda_{(L_0,\|\cdot\|_{0,n},|D_0|,s_0),\dots,(L_t,\|\cdot\|_{t,n},|D_t|,s_t)}(Y). \tag{1.4}$$
This extends obviously to cycles and DSP metrics. For details see [Gub97, § 1] or [Gub02, Theorem 5.1.8].

Let Z be a t-dimensional cycle of X and $(\overline{L}_i, s_i)$, $i = 0, \dots, t$, DSP metrized line bundles on X with invertible meromorphic sections such that
$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| \cap |Z| = \emptyset.$$
By Example 1.2.10, we obtain DSP metrized pseudo-divisors $\widehat{\mathrm{div}}(s_i)$, $i = 0, \dots, t$. We define the *local height* of Z with respect to $(\overline{L}_0, s_0), \dots, (\overline{L}_t, s_t)$ as
$$\lambda_{(\overline{L}_0,s_0),\dots,(\overline{L}_t,s_t)}(Z) := \lambda_{\widehat{\mathrm{div}}(s_0),\dots,\widehat{\mathrm{div}}(s_t)}(Z)\,. \tag{1.5}$$

The local heights with respect to DSP metrics have the expected properties as stated in the following proposition.

Proposition 1.3.6. *Let Z be a t-dimensional cycle of X and $\hat{D}_0, \ldots, \hat{D}_t$ DSP metrized pseudo-divisors on X with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. Then there is a unique local height $\lambda(Z) := \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$ satisfying the following properties:*

(i) If $\hat{D}_0, \ldots, \hat{D}_t$ are $\mathbb{Q}$-formal metrized, then $\lambda(Z)$ is the local height of 1.2.11.

(ii) $\lambda(Z)$ is multilinear and symmetric in $\hat{D}_0, \ldots, \hat{D}_t$ and linear in Z.

(iii) For a proper morphism $\varphi\colon X' \to X$ and a t-dimensional cycle Z' on X' satisfying $|D_0| \cap \cdots \cap |D_t| \cap |\varphi(Z)| = \emptyset$, we have

$$\lambda_{\varphi^*\hat{D}_0,\ldots,\varphi^*\hat{D}_t}(Z') = \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(\varphi_* Z').$$

In particular, $\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$ does not change when restricting the metrized pseudo-divisors to the prime cycle Z.

(iv) Let $\lambda'(Z)$ be the local height obtained by replacing the metric $\|\cdot\|$ of $\hat{D}_0$ by another DSP metric $\|\cdot\|'$. If the metrics of $\hat{D}_1, \ldots, \hat{D}_t$ are semipositive and if Z is effective, then

$$|\lambda(Z) - \lambda'(Z)| \leq \mathrm{d}(\|\cdot\|, \|\cdot\|') \cdot \deg_{\mathcal{O}(D_1),\ldots,\mathcal{O}(D_t)}(Z).$$

(v) Let f be a rational function on X and let $\hat{D}_0 = \widehat{\mathrm{div}}(f)$ be endowed with the trivial metric on $\mathcal{O}(D_0) = \mathcal{O}_X$. If $Y = \sum_P n_P P$ is a cycle representing $D_1.\cdots.D_t.Z \in \mathrm{CH}_0\left(|D_1| \cap \cdots \cap |D_t| \cap |Z|\right)$, then

$$\lambda(Z) = \sum_P n_P \cdot \log|f(P)|.$$

Proof. This follows immediately from Proposition 1.2.12 and the construction in 1.3.5, and is established in [Gub03, Theorem 10.6]. □

In the same manner we can generalize Chambert Loir's measures to semipositive and DSP line bundles:

Proposition 1.3.7. *Let Y be a t-dimensional subvariety of X and $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 1, \ldots, t$, semipositive line bundles. For each i, let $(\|\cdot\|_{i,n})_{n\in\mathbb{N}}$ be the corresponding sequence of $\mathbb{Q}$-formal semipositive metrics on L_i converging to $\|\cdot\|_i$. Then the measures*

$$\mathrm{c}_1(L_1, \|\cdot\|_{1,n}) \wedge \cdots \wedge \mathrm{c}_1(L_t|, \|\cdot\|_{t,n}) \wedge \delta_Y$$

converge weakly to a regular Borel measure on X^{an}. This measure is independent of the choice of the sequences.

Proof. This follows from [Gub07b, Proposition 3.12]. □

Definition 1.3.8. Let Y be a t-dimensional subvariety of X and $\overline{L}_i = (L_i, \|\cdot\|_i)$, $i = 1, \ldots, t$, semipositive line bundles. We denote the limit measure in 1.3.7 by $\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Y$ or shortly by $\mathrm{c}_1(\overline{L}_1) \ldots \mathrm{c}_1(\overline{L}_t)\delta_Y$. By multilinearity, this notion extends to a t-dimensional cycle Y of X and DSP line bundles $\overline{L}_1, \ldots, \overline{L}_t$.

Chambert Loir's measure is uniquely determined by the following property which is taken as definition in [Gub07b, 3.8].

Proposition 1.3.9. *Let $\overline{L}_1, \dots, \overline{L}_t$ be DSP line bundles on X and let Z be a t-dimensional cycle. For $j = 1, \dots, t$ we choose any metrized pseudo-divisor $\hat{D}_j$ with $\overline{\mathcal{O}}(D_j) = \overline{L}_j$, for example $\hat{D}_j = (\overline{L}_j, X, \emptyset)$.*

If g is any continuous real-valued function on X^{an}, then there is a sequence of $\mathbb{Q}$-formal metrics $(\|\cdot\|_n)_{n\in\mathbb{N}}$ on $\mathcal{O}_X$ such that $\log\|1\|_n^{-1}$ tends uniformly to g for $n \to \infty$ and

$$\int_{X^{\mathrm{an}}} g \cdot \mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Z = \lim_{n\to\infty} \lambda_{(\mathcal{O}_X, \|\cdot\|_n, \emptyset, 1), \hat{D}_1, \dots, \hat{D}_t}(Z).$$

Proof. By [Gub07b, Proposition 3.3], the $\mathbb{Q}$-formal metrics are dense in the space of continuous metrics on $\mathcal{O}_X$. This implies the existence of the sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$. The second part follows from [Gub07b, Proposition 3.8]. □

Corollary 1.3.10. *Let Z be a cycle on X of dimension t and let $\hat{D}_0, \dots, \hat{D}_t$ be DSP metrized pseudo-divisors with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. Replacing the metric $\|\cdot\|$ on $\mathcal{O}(D_0)$ by another DSP metric $\|\cdot\|'$, we obtain a metrized pseudo-divisor $\hat{E}$. Then, $g := \log(\|s_{D_0}\|/\|s_{D_0}\|')$ extends to a continuous function on X and*

$$\lambda_{\hat{D}_0, \dots, \hat{D}_t}(Z) - \lambda_{\hat{E}, \hat{D}_1, \dots, \hat{D}_t}(Z) = \int_{X^{\mathrm{an}}} g \cdot \mathrm{c}_1(\overline{\mathcal{O}}(D_1)) \wedge \cdots \wedge \mathrm{c}_1(\overline{\mathcal{O}}(D_t)) \wedge \delta_Z.$$

Proof. Clearly g defines a continuous function on X. By means of Proposition 1.3.9,

$$\begin{aligned} &\int_{X^{\mathrm{an}}} g \cdot \mathrm{c}_1(\overline{\mathcal{O}}(D_1)) \dots \mathrm{c}_1(\overline{\mathcal{O}}(D_t))\delta_Z \\ &= \lambda_{(\mathcal{O}_X, \|\cdot\|/\|\cdot\|', \emptyset, 1), \hat{D}_1, \dots, \hat{D}_t}(Z) \\ &= \lambda_{(\mathcal{O}(D_0), \|\cdot\|, |D_0|, s_0), \hat{D}_1, \dots, \hat{D}_t}(Z) - \lambda_{(\mathcal{O}(D_0), \|\cdot\|', |D_0|, s_0), \hat{D}_1, \dots, \hat{D}_t}(Z), \end{aligned}$$

proving the statement. □

Proposition 1.3.11. *Let Z be a t-dimensional cycle of X and $\overline{L}_1, \dots, \overline{L}_t$ DSP line bundles. Then the measure $\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Z$ has the following properties:*

(i) It is multilinear and symmetric in $\overline{L}_1, \dots, \overline{L}_t$ and linear in Z.

(ii) Let $\varphi\colon X' \to X$ be a morphism of proper schemes over K and Z' a t-dimensional cycle of X', then

$$\varphi_* \left(\mathrm{c}_1(\varphi^*\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\varphi^*\overline{L}_t) \wedge \delta_{Z'}\right) = \mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_{\varphi_* Z'}.$$

(iii) If the metrics of $\overline{L}_1, \dots, \overline{L}_t$ are semipositive, then $\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Z$ is a positive measure with total mass $\deg_{L_1, \dots, L_t}(Z)$.

Proof. We refer to Corollary 3.9 and Proposition 3.12 in [Gub07b]. □

Remark 1.3.12. With the previous notation, let K' be an algebraically closed extension of K equipped with a complete absolute value extending $|\cdot|$, and denote by $\pi\colon X_{K'} \to X$ the base change. Then, by [Gub07b, Remark 3.10],

$$\pi_* \left(\mathrm{c}_1(\pi^*\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\pi^*\overline{L}_t) \wedge \delta_{Y_{K'}}\right) = \mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Y.$$

Thus, by base change to the completion of the algebraic closure and using linearity in the irreducible components, we may extend the measures in Definition 1.3.8 to all proper schemes X and cycles Y over an arbitrary non-Archimedean field.

1.4. Induction Formula for Local Heights

Now we generalize the induction formula from Proposition 1.2.16 to DSP metrized line bundles. This formula enables us to define the local height inductively. Our proof is based on [CT09, Théorème 4.1] where the formula is shown under the additional assumptions that X is projective over a completion of a number field and $s_0, \dots, s_t$ are global sections such that their Cartier divisors intersect properly.

In this section let X be a reduced proper scheme over an algebraically closed non-Archimedean field K.

At first, we prove the following approximation theorem corresponding to [CT09, Théorème 3.1]. In contrast to [CT09] we show it in a more analytic fashion.

Proposition 1.4.1 (Approximation theorem)**.** *Let $(L, \|\cdot\|)$ be a semipositive formal metrized line bundle on X with a global section s which is invertible as a meromorphic section. Then there is a sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of formal metrics on the trivial bundle $\mathcal{O}_X$ with the following properties:*

(i) The sequence $\left(\log\|1\|_n^{-1}\right)_{n\in\mathbb{N}}$ converges pointwise to $\log\|s\|^{-1}$ and it is monotonically increasing.

(ii) For each $n \in \mathbb{N}$, the metric $\|\cdot\|/\|\cdot\|_n$ on $L\otimes\mathcal{O}_X^{-1} = L$ is semipositive.

Proof. We fix some non-zero element π in $K^{\circ\circ}$ and define, for each $n \in \mathbb{N}$, the closed sets

$$A_n := \{x \in X^{\mathrm{an}} \mid \|s(x)\| \geq |\pi^n|\} \quad \text{and} \quad B_n := \{x \in X^{\mathrm{an}} \mid \|s(x)\| \leq |\pi^n|\}\,. \tag{1.6}$$

By 1.2.6, the formal metric $\|\cdot\|$ on L is given by an admissible covering $\{U_i\}_{i\in I}$ of X^{an} by affinoid domains, and non-vanishing regular sections $t_i \in L^{\mathrm{an}}(U_i)$ with $\|t_i\| \equiv 1$. Let $g_{ij} = t_j/t_i \in \mathcal{O}(U_i \cap U_j)^\times$ be the transition functions. Then the non-vanishing $s|_{U_i\cap A_n}$ may be identified with regular functions $f_i \in \mathcal{O}(U_i\cap A_n)^\times$ satisfying $f_i = g_{ij}f_j$ on $U_i\cap U_j\cap A_n$. Since the functions $f_i^{-1} \in \mathcal{O}(U_i\cap A_n)$, $\pi^{-n} \in \mathcal{O}(U_i \cap B_n)$ are local frames of $\mathcal{O}_{X^{\mathrm{an}}}$ on affinoid domains, we get by 1.2.6 a formal metric $\|\cdot\|_n$ on $\mathcal{O}_X$ given by

$$\|1\|_n = |f_i| \text{ on } U_i\cap A_n \quad\text{and}\quad \|1\|_n = |\pi^n| \text{ on } U_i \cap B_n.$$

Consider, for each $n \in \mathbb{N}$, the function

$$\log\|1\|_n^{-1} = \begin{cases} \log|f_i|^{-1} & \text{on } U_i\cap A_n \\ \log|\pi^n|^{-1} & \text{on } U_i\cap B_n \end{cases} = \begin{cases} \log\|s\|^{-1} & \text{on } A_n \\ n\log|\pi|^{-1} & \text{on } B_n \end{cases} = \min\{\log\|s\|^{-1}, n\log|\pi|^{-1}\}.$$

Clearly, the sequence $\left(\log\|1\|_n^{-1}\right)_{n\in\mathbb{N}}$ tends pointwise to $\log\|s\|^{-1}$ and is monotonically increasing.

Moreover, we have to show that, for each $n\in\mathbb{N}$, the formal metric $\|\cdot\|'_n := \|\cdot\|/\|\cdot\|_n$ is semipositive on $L\otimes\mathcal{O}_X^{-1} = L$. For the admissible covering $\{U_i\cap A_n, U_i\cap B_n\}_{i\in I}$ by affinoid domains, there exists a formal K°-model $\mathfrak{X}_n$ of X^{an} and a formal open covering

$\{\mathfrak{U}_{i,n}, \mathfrak{V}_{i,n}\}_{i\in I}$ of $\mathfrak{X}_n$ such that $\mathfrak{U}_{i,n}^{\mathrm{an}} = U_i \cap A_n$ and $\mathfrak{V}_{i,n}^{\mathrm{an}} = U_i \cap B_n$ (see [BL93b, Theorem 5.5]). We may assume that $\mathfrak{X}_n$ has reduced special fiber (cf. 1.1.7). Then, by 1.2.6, the formal metric $\|\cdot\|_n'$ is associated to the formal K°-model $(\mathfrak{L}_n', \mathfrak{X}_n)$ of (X, L) given by

$$\mathfrak{L}_n'(\mathfrak{U}) = \{r \in L^{\mathrm{an}}(\mathfrak{U}^{\mathrm{an}}) \mid \|r(x)\|_n' \leq 1 \ \forall x \in \mathfrak{U}^{\mathrm{an}}\} \tag{1.7}$$

on a formal open subset $\mathfrak{U}$ of $\mathfrak{X}_n$. Therefore, we can consider s as a global section of $\mathfrak{L}_n'$ since we have

$$\|s\|_n' = \frac{\|s\|}{\|1\|_n} = \begin{cases} 1 & \text{on } A_n \\ \|s\| \cdot |\pi^{-n}| & \text{on } B_n \end{cases} \leq 1.$$

Let $C \subseteq \widetilde{\mathfrak{X}}_n$ be a closed integral curve. If s doesn't vanish identically on C, then

$$\deg_{\widetilde{\mathfrak{L}}_n'}(C) = \deg(\mathrm{c}_1(\widetilde{\mathfrak{L}}_n').C) = \deg(\mathrm{div}(s|_C)) \geq 0.$$

If s vanishes identically on C, let $\mathfrak{B}_n$ be the union of the formal open $\mathfrak{V}_{i,n}$'s. Then it follows by (1.6) that $\widetilde{\mathfrak{B}}_n = \mathrm{red}(B_n)$ contains C.

By refining the above trivialization $\{U_i, t_i\}$ to $\{U_i \cap A_n, U_i \cap B_n\}$, the metric $\|\cdot\|$ is induced by a formal model $\mathfrak{L}_n$ which also lives on $\mathfrak{X}_n$ and which is given similarly as in (1.7). This implies $\mathfrak{L}_n|_{\mathfrak{B}_n} \cong \mathfrak{L}_n'|_{\mathfrak{B}_n}$ given by $r \mapsto \pi^n \cdot r$. Since $\widetilde{\mathfrak{B}}_n$ is a neighborhood of C and $\widetilde{\mathfrak{L}}_n$ is nef, we obtain

$$\deg_{\widetilde{\mathfrak{L}}_n'}(C) = \deg_{\widetilde{\mathfrak{L}}_n}(C) \geq 0,$$

which implies the semipositivity of $\|\cdot\|/\|\cdot\|_n$. □

Corollary 1.4.2. *We use the notations from the approximation theorem and in addition, we consider a t-dimensional cycle Z and semipositive line bundles $\overline{L}_1, \ldots, \overline{L}_{t-1}$ on X. Let μ be a (signed) finite measure on X^{an} such that, for every $\mathbb{Q}$-formal metric $\|\cdot\|'$ on $\mathcal{O}_X$,*

$$\lim_{n\to\infty} \int_{X^{\mathrm{an}}} \log\|1\|' \cdot \mathrm{c}_1(\mathcal{O}_X, \|\cdot\|_n)\, \mathrm{c}_1(\overline{L}_1) \ldots \mathrm{c}_1(\overline{L}_{t-1})\delta_Z = \int_{X^{\mathrm{an}}} \log\|1\|' \cdot \mu\ .$$

Then the sequence $\Big(\mathrm{c}_1(\mathcal{O}_X, \|\cdot\|_n)\, \mathrm{c}_1(\overline{L}_1) \ldots \mathrm{c}_1(\overline{L}_{t-1})\delta_Z\Big)_{n\in\mathbb{N}}$ of measures on X^{an} converges weakly to μ.

Proof. Let $\nu := \mathrm{c}_1(L, \|\cdot\|)\, \mathrm{c}_1(\overline{L}_1) \ldots \mathrm{c}_1(\overline{L}_{t-1})\delta_Z$ and $\mu_n := \mathrm{c}_1(\mathcal{O}_X, \|\cdot\|_n)\, \mathrm{c}_1(\overline{L}_1) \ldots \mathrm{c}_1(\overline{L}_{t-1})\delta_Z$ for each $n \in \mathbb{N}$. Then, by the approximation theorem 1.4.1 and Proposition 1.3.11 (iii), the measures

$$\nu - \mu_n = \mathrm{c}_1\Big(L, \tfrac{\|\cdot\|}{\|\cdot\|_n}\Big)\, \mathrm{c}_1(\overline{L}_1) \ldots \mathrm{c}_1(\overline{L}_{t-1})\delta_Z$$

are positive with finite total mass $\deg_{L, L_1, \ldots, L_{t-1}}(Z)$, independent of n.

Let $\varepsilon > 0$ and let f be any continuous function on X^{an}. By [Gub07b, Proposition 3.3], the set of $\mathbb{Q}$-formal metrics on $\mathcal{O}_X$ is embedded into a dense subset of $C(X^{\mathrm{an}})$, i. e. there is a $\mathbb{Q}$-formal metric $\|\cdot\|'$ such that for $g := -\log\|1\|'$ we have

$$\sup_{x\in X^{\mathrm{an}}} |g(x) - f(x)| \cdot \deg_{L, L_1, \ldots, L_{t-1}}(Z) < \varepsilon/3$$

and

$$\sup_{x\in X^{\mathrm{an}}} |g(x) - f(x)| \cdot |\nu - \mu|\,(X^{\mathrm{an}}) < \varepsilon/3.$$

Moreover, there is by assumption an $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\left| \int g\mu - \int g\mu_n \right| < \varepsilon/3 \,.$$

Finally, we obtain, for all $n \geq N$,

$$\begin{aligned}
&\left| \int f \cdot \mu - \int f \cdot \mu_n \right| \\
= &\left| \int (f-g)(\nu - \mu_n) + \int g(\mu - \mu_n) + \int (g-f)(\nu - \mu) \right| \\
\leq &\ \sup |f-g| \cdot \deg_{L,L_1,\ldots,L_{t-1}}(Z) + \left| \int g(\mu - \mu_n) \right| + \sup |g-f| \cdot |\nu - \mu| \, (X^{\mathrm{an}}) \\
< &\ \varepsilon \,.
\end{aligned}$$

This proves the result. □

Theorem 1.4.3 (Induction formula)**.** *Let Z be a t-dimensional prime cycle on X and let $\hat{D}_i = (\overline{L}_i, |D_i|, s_i)$, $i = 0, \ldots, t$, be DSP pseudo-divisors with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. If $|Z| \not\subseteq |D_t|$, then let $s_{t,Z} := s_t|_Z$, otherwise we choose any non-zero meromorphic section $s_{t,Z}$ of $L_t|_Z$. Let $\mathrm{cyc}(s_{t,Z})$ be the Weil divisor of $s_{t,Z}$ considered as a cycle on X.*

Then the function $\log \|s_{t,Z}\|$ is integrable with respect to $\mathrm{c}_1(\overline{L}_0) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{t-1}) \wedge \delta_Z$ and we have

$$\begin{aligned}
\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z) = \lambda_{\hat{D}_0,\ldots,\hat{D}_{t-1}}&(\mathrm{cyc}(s_{t,Z})) \\
&- \int_{X^{\mathrm{an}}} \log \|s_{t,Z}\| \cdot \mathrm{c}_1(\overline{L}_0) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{t-1}) \wedge \delta_Z \,.
\end{aligned} \tag{1.8}$$

Remark 1.4.4. If $\overline{L}_0, \ldots, \overline{L}_t$ have $\mathbb{Q}$-formal metrics, then this result is just Proposition 1.2.16. It is also evident if L_t is the trivial bundle and hence, $\log \|s_{t,Z}\|$ is a continuous function on Z. The difficulties of the general case arise from the relation between the limit process defining the measure, and the poles of the function $\log \|s_{t,Z}\|$.

Proof of the induction formula 1.4.3. By Proposition 1.3.6 (iii), we may assume that $X = Z$, especially $s_t = s_{t,Z}$. Furthermore, we can suppose that X is projective by Chow's lemma (see, for instance, [GW10, Theorem 13.100]) and functoriality of the height (Proposition 1.3.6). Multiplying the metric $\| \cdot \|$ on L_t by a constant, changes both sides of the equality (1.8) by the same additive constant (see Corollary 1.3.10). Hence, we can assume that

$$\sup_{x \in X^{\mathrm{an}}} \|s_t(x)\| \leq 1. \tag{1.9}$$

Step 1: Reduction to the case of a global section s_t of L_t and properly intersecting supports $|D_0|, \ldots, |D_t|$ on Z. Since X is projective, there is a very ample line bundle H (provided with some semipositive metric) and a non-trivial global section r of H such that $L_t \otimes H$ is also very ample and $s_t \otimes r$ is a global section of $L_t \otimes H$. By the moving lemma (see for example [Liu06, Exercise 9.1.2]) we find invertible meromorphic sections s'_j of $\overline{L}_j$, $j = 0, \ldots, t-1$, such that $|\operatorname{div}(s'_0)|, \ldots, |\operatorname{div}(s'_{t-1})|$ and $|\operatorname{div}(s_t)| \cup |\operatorname{div}(r)|$ intersect properly on Z. Then we have

$$\lambda_{\hat{D}_0,\dots,\hat{D}_t}(Z)-\lambda_{\hat{D}_0,\dots,\hat{D}_{t-1}}(\operatorname{cyc}(s_t)) = \lambda_{\widehat{\operatorname{div}}(s'_0),\dots,\widehat{\operatorname{div}}(s'_{t-1}),\widehat{\operatorname{div}}(s_t)}(Z)-\lambda_{\widehat{\operatorname{div}}(s'_0),\dots,\widehat{\operatorname{div}}(s'_{t-1})}(\operatorname{cyc}(s_t)),$$

because both sides are given by a limit process as in (1.4) that only depends on $\overline{L}_0,\dots,\overline{L}_{t-1}$ and $\hat{D}_t$ by Proposition 1.2.16. Now, we may replace $\widehat{\operatorname{div}}(s_t)$ by

$$(\overline{L}_t\otimes\overline{H}, |\operatorname{div}(s_t)|\cup|\operatorname{div}(r)|, s_t\otimes r) - (\overline{H}, |\operatorname{div}(r)|, r)$$

and the first step follows from the multilinearity of the local heights.

Step 2: Reduction to the case of semipositive metrized pseudo-divisors $\hat{D}_0,\dots,\hat{D}_t$. Because, for $i=0,\dots,t$, the line bundle $\overline{L}_i$ is DSP metrized, we have $\overline{L}_i=\overline{M}_i\otimes\overline{N}_i^{-1}$ for semipositive metrized bundles $\overline{M}_i$ and $\overline{N}_i$. There is, for each i, a very ample line bundle H_i (provided with some semipositive metric) such that $N_i\otimes H_i$ is also very ample. By the first step, $|D_0|,\dots,|D_t|$ intersect properly and so, we find hyperplane sections r_i of $N_i\otimes H_i$, $i=0,\dots,t$, such that $|\operatorname{div}(r_0)|,\dots,|\operatorname{div}(r_t)|, |D_0|,\dots,|D_t|$ intersect properly, too. Especially,

$$(|D_0|\cup|\operatorname{div}(r_0)|)\cap\cdots\cap(|D_0|\cup|\operatorname{div}(r_0)|)=\emptyset.$$

Hence, for $i=0,\dots,t$, we may replace $\hat{D}_i$ by $(\overline{L}_i, |D_i|\cup|\operatorname{div}(r_i)|, s_i)$. Because

$$(\overline{L}_i, |D_i|\cup|\operatorname{div}(r_i)|, s_i) = (\overline{M}_i\otimes\overline{H}_i, |D_i|\cup|\operatorname{div}(r_i)|, s_i\otimes r_i) - (\overline{N}_i\otimes\overline{H}_i, |\operatorname{div}(r_i)|, r_i)$$

is the difference of two semipositive metrized pseudo-divisors, the second step follows from the multilinearity of the local heights.

In the following we fix, for each $i=0,\dots,t-1$, a semipositive $\mathbb{Q}$-formal metric $\|\cdot\|'$ on L_i and a semipositive formal metric $\|\cdot\|'$ on L_t. For each $i=0,\dots,t$, we denote the corresponding metrized line bundle by $\overline{M}_i$ and the metrized pseudo-divisor $(\overline{M}_i, |D_i|, s_i)$ by $\hat{E}_i$. Then we can extend $\varphi_i := \log\|s_i\|' - \log\|s_i\|$ to a continuous function on X^{an} and $\mathcal{O}_X(\varphi_i) := \overline{L}_i\otimes\overline{M}_i^{-1}$ is a DSP line bundle.

Step 3: Reduction to the case where the metric of $\overline{L}_t$ is formal. If the theorem holds for $\log\|s_t\|'$, then $\log\|s_t\| = \log\|s_t\|' - \varphi_t$ is integrable with respect to $\mathrm{c}_1(\overline{L}_0)\dots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z$ and we get

$$\begin{aligned}
&\int_{X^{\mathrm{an}}} \log\|s_t\|\cdot \mathrm{c}_1(\overline{L}_0)\dots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z\\
&= \int_{X^{\mathrm{an}}} \log\|s_t\|'\cdot \mathrm{c}_1(\overline{L}_0)\dots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z - \int_{X^{\mathrm{an}}} \varphi_t\cdot \mathrm{c}_1(\overline{L}_0)\dots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z\\
&= \lambda_{\hat{D}_0,\dots,\hat{D}_{t-1}}(\operatorname{cyc}(s_t)) - \lambda_{\hat{D}_0,\dots,\hat{D}_{t-1},\hat{E}_t}(Z) - \int_{X^{\mathrm{an}}} \varphi_t\cdot \mathrm{c}_1(\overline{L}_0)\dots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z.
\end{aligned}$$

By the metric change formula 1.3.10, we have

$$\int_{X^{\mathrm{an}}} \varphi_t\cdot \mathrm{c}_1(\overline{L}_0)\dots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z = \lambda_{\hat{D}_0,\dots,\hat{D}_t}(Z) - \lambda_{\hat{D}_0,\dots,\hat{D}_{t-1},\hat{E}_t}(Z)$$

and hence, the theorem is proved. Thus, we may assume that $\hat{D}_t = (\overline{L}_t, |D_t|, s_t)$ is a semipositive formal metrized pseudo-divisor.

Step 4: We prove by induction on $k\in\{0,\dots,t\}$ that the theorem holds if $\overline{L}_i$ is a $\mathbb{Q}$-formal *metrized line bundle for $i \geq k$.* The base case $k = 0$ is just the induction formula for $\mathbb{Q}$-formal metrics (see Proposition 1.2.16). We assume that the statement holds for k and show it for $k+1$. Since $\overline{M}_k, \dots, \overline{M}_{t-1}$ are $\mathbb{Q}$-formal metrized line bundles, we have

$$\begin{aligned} &\int_{X^{\mathrm{an}}} \log \|s_t\| \cdot \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_k) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ &= \lambda_{\hat{D}_0,\dots,\hat{D}_{k-1},\hat{E}_k,\dots,\hat{E}_{t-1}}(\mathrm{cyc}(s_t)) - \lambda_{\hat{D}_0,\dots,\hat{D}_{k-1},\hat{E}_k,\dots,\hat{E}_t}(Z). \end{aligned} \tag{1.10}$$

Let $\overline{L}_i$ be $\mathbb{Q}$-formal for $i \geq k+1$, that means we may assume that $\overline{L}_i = \overline{M}_i$. Since $\overline{L}_k = \overline{M}_k \otimes \mathcal{O}(\varphi_k)$, we obtain, by Proposition 1.3.11 (i),

$$\begin{aligned} &\mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_k)\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ =\ &\mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_k) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ &+ \mathrm{c}_1(\mathcal{O}(\varphi_k))\, \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z\ . \end{aligned} \tag{1.11}$$

By the metric change formula 1.3.10, we get

$$\begin{aligned} &\lambda_{\hat{D}_0,\dots,\hat{D}_k,\hat{E}_{k+1},\dots,\hat{E}_t}(Z) \\ &= \lambda_{\hat{D}_0,\dots,\hat{D}_{k-1},\hat{E}_k,\dots,\hat{E}_t}(Z) + \int_{X^{\mathrm{an}}} \varphi_k \cdot \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_t) \delta_Z \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} &\lambda_{\hat{D}_0,\dots,\hat{D}_k,\hat{E}_{k+1},\dots,\hat{E}_{t-1}}(\mathrm{cyc}(s_t)) \\ &= \lambda_{\hat{D}_0,\dots,\hat{D}_{k-1},\hat{E}_k,\dots,\hat{E}_{t-1}}(\mathrm{cyc}(s_t)) + \int_{X^{\mathrm{an}}} \varphi_k \cdot \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_{\mathrm{cyc}(s_t)}. \end{aligned} \tag{1.13}$$

The function $\log \|s_t\|$ is measurable and, by (1.9), non-positive. Hence, we can compute the following integrals, where infinite values are allowed,

$$\begin{aligned} &\int_{X^{\mathrm{an}}} \log \|s_t\| \cdot \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_k)\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ \overset{(1.11)}{=} &\int_{X^{\mathrm{an}}} \log \|s_t\| \cdot \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_k) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ &+ \int_{X^{\mathrm{an}}} \log \|s_t\| \cdot \mathrm{c}_1(\mathcal{O}(\varphi_k))\, \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ \overset{(1.10)}{=} &\lambda_{\hat{D}_0,\dots,\hat{D}_{k-1},\hat{E}_k,\dots,\hat{E}_{t-1}}(\mathrm{cyc}(s_t)) - \lambda_{\hat{D}_0,\dots,\hat{D}_{k-1},\hat{E}_k,\dots,\hat{E}_t}(Z) \\ &+ \int_{X^{\mathrm{an}}} \log \|s_t\| \cdot \mathrm{c}_1(\mathcal{O}(\varphi_k))\, \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z \\ \overset{(1.12),(1.13)}{=} &\lambda_{\hat{D}_0,\dots,\hat{D}_k,\hat{E}_{k+1},\dots,\hat{E}_{t-1}}(\mathrm{cyc}(s_t)) - \int_{X^{\mathrm{an}}} \varphi_k\, \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1}) \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_{\mathrm{cyc}(s_t)} \\ &- \lambda_{\hat{D}_0,\dots,\hat{D}_k,\hat{E}_{k+1},\dots,\hat{E}_t}(Z) + \int_{X^{\mathrm{an}}} \varphi_k\, \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_t) \delta_Z \\ &+ \int_{X^{\mathrm{an}}} \log \|s_t\| \cdot \mathrm{c}_1(\mathcal{O}(\varphi_k))\, \mathrm{c}_1(\overline{L}_0) \dots \mathrm{c}_1(\overline{L}_{k-1})\, \mathrm{c}_1(\overline{M}_{k+1}) \dots \mathrm{c}_1(\overline{M}_{t-1}) \delta_Z\ . \end{aligned}$$

Thus, we have to prove

$$\begin{aligned}&\int_{X^{\mathrm{an}}} \varphi_k \cdot \mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_t)\delta_Z\\ =&\int_{X^{\mathrm{an}}} \varphi_k \cdot \mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_{\mathrm{cyc}(s_t)}\\ &-\int_{X^{\mathrm{an}}} \log\|s_t\| \cdot \mathrm{c}_1(\mathcal{O}(\varphi_k))\,\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_Z\ .\end{aligned}$$

By step 1–3, we can apply the approximation theorem 1.4.1: Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of formal metrics on $\mathcal{O}_X$ such that the functions $g_n := \log\|1\|_n^{-1}$ tend pointwise to $\log\|s_t\|^{-1}$, the sequence $(g_n)_{n\in\mathbb{N}}$ is monotonically increasing and $(\mathcal{O}_X, \|\cdot\|_n)$ is a DSP line bundle. Additionally, we may assume that the functions g_n are non-negative by (1.9) and by their construction in the approximation theorem. Applying Lebesgue's monotone convergence theorem and using Proposition 1.3.9 and 1.3.11 (i), we obtain

$$\begin{aligned}&\int_{X^{\mathrm{an}}} \log\|s_t\|^{-1} \cdot \mathrm{c}_1(\mathcal{O}(\varphi_k))\,\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_Z\\ =&\lim_{n\to\infty}\int_{X^{\mathrm{an}}} g_n \cdot \mathrm{c}_1(\mathcal{O}(\varphi_k))\,\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_Z\\ =&\lim_{n\to\infty}\lambda_{\widehat{\mathcal{O}}(g_n),\widehat{\mathcal{O}}(\varphi_k),\hat{D}_0,\ldots,\hat{D}_{k-1},\hat{E}_{k+1},\ldots,\hat{E}_{t-1}}(Z)\\ =&\lim_{n\to\infty}\lambda_{\widehat{\mathcal{O}}(\varphi_k),\widehat{\mathcal{O}}(g_n),\hat{D}_0,\ldots,\hat{D}_{k-1},\hat{E}_{k+1},\ldots,\hat{E}_{t-1}}(Z)\\ =&\lim_{n\to\infty}\int_{X^{\mathrm{an}}} \varphi_k \cdot \mathrm{c}_1(\mathcal{O}_X, \|\cdot\|_n)\,\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_Z\ .\end{aligned}$$

Finally, we must show the following equation for the continuous function $\varphi_k = \log\Big(\frac{\|\cdot\|_k'}{\|\cdot\|_k}\Big)$:

$$\begin{aligned}&\lim_{n\to\infty}\int_{X^{\mathrm{an}}} \varphi_k \cdot \mathrm{c}_1(\mathcal{O}_X, \|\cdot\|_n)\,\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_Z\\ =&\int_{X^{\mathrm{an}}} \varphi_k \cdot \Big(\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_t)\delta_Z \qquad (1.14)\\ &\qquad - \mathrm{c}_1(\overline{L}_1)\ldots\mathrm{c}_1(\overline{L}_{k-1})\,\mathrm{c}_1(\overline{M}_{k+1})\ldots\mathrm{c}_1(\overline{M}_{t-1})\delta_{\mathrm{cyc}(s_t)}\Big)\ .\end{aligned}$$

The induction hypothesis implies that equation (1.14) always holds if $\frac{\|\cdot\|_k'}{\|\cdot\|_k}$ is a $\mathbb{Q}$-formal metric. But then Corollary 1.4.2 (under the assumption of step 2) says that this equation is also true if φ_k is only continuous. This shows the induction formula (1.8) and hence, the integrability of $\log\|s_t\|$ with respect to $\mathrm{c}_1(\overline{L}_0)\cdots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z$. □

Corollary 1.4.5. *With the same notations as in Theorem 1.4.3, any proper closed subset of Z has measure zero with respect to $\mathrm{c}_1(\overline{L}_0)\wedge\ldots\wedge\mathrm{c}_1(\overline{L}_{t-1})\wedge\delta_Z$.*

Proof. We may assume that $X = Z$ and, by Chow's lemma and Proposition 1.3.11 (ii), that Z is projective. Then any proper closed subset A of Z is contained in the support of an effective pseudo-divisor $(L, |\mathrm{div}(s)|, s)$ on Z. By the induction formula, the function $\log\|s\|^{-1}$ is integrable with respect to $\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z$, but it takes the value $+\infty$ on $|\mathrm{div}(s)|$. Thus the support $|\mathrm{div}(s)|$ and also the subset A have measure zero with respect to $\mathrm{c}_1(\overline{L}_0)\ldots\mathrm{c}_1(\overline{L}_{t-1})\delta_Z$. □

1.5. Metrics, Local heights and Measures over Archimedean fields

Following [Gub03, § 2,3 and 10], we recall some definitions and statements about Archimedean local heights. More details can be found in [Gub02]. Additionally we prove the Archimedean counterpart of the induction formula (Theorem 1.4.3) which generalizes slightly the Archimedean part of [CT09, Théorème 4.1]. This theory is used later for the study of global heights.

Let K be a field which is complete with respect to an Archimedean absolute value. As before, we assume for simplicity that K is algebraically closed. Indeed, by Ostrowski's theorem, we have $K = \mathbb{C}$.

In this section, let X be a reduced proper scheme over $\mathbb{C}$ and $X^{\mathrm{an}} = X(\mathbb{C})$ the associated compact complex analytic space. Let L be an algebraic line bundle on X and L^{an} its analytification.

1.5.1. By Bloom and Herrera [BH69], *differential forms* on X^{an} are defined as follows. There is an open covering $\{U_i\}_i$ of X^{an} such that U_i is a closed analytic subset of an open complex ball. On each U_i, the differential forms are given by restriction of smooth complex-valued differential forms defined on such balls. Two forms on U_i are identified if they coincide on the non-singular locus of U_i. We write $\mathcal{A}^*(U_i)$ for the complex of differential forms on U_i. By gluing, we obtain a sheaf $\mathcal{A}^*_{X^{\mathrm{an}}}$. On this sheaf, we have differential operators $\partial, \overline{\partial}$, an exterior product and pullbacks with respect to analytic morphisms. These operations are defined locally on $\mathcal{A}^*(U_i)$ by extending the forms to a ball as above and applying the corresponding constructions for complex manifolds. We denote by $\mathcal{O}_{X^{\mathrm{an}}}$ the sheaf of analytic functions.

1.5.2. A *metric* on L, a *metrized line bundle* on X and a *metrized pseudo-divisor* on X are defined as in Definition 1.2.1 and 1.2.9. A metric $\|\cdot\|$ on L is called *smooth* if, for each local section s of L^{an}, the function $\|s(\cdot)\|^2$ is smooth.

Let $\|\cdot\|$ be a smooth metric on L. The *first Chern form* of $\overline{L} = (L, \|\cdot\|)$, denoted $\mathrm{c}_1(\overline{L})$, is the differential form on X^{an} defined, for any non-vanishing local section s of L^{an} on an open subset U, as

$$\mathrm{c}_1(\overline{L})|_U = \frac{1}{2\pi i}\partial\overline{\partial}\log\|s\|^2.$$

Indeed, the first Chern form does not depend on the choice of s and it is a real and closed $(1,1)$-form. Moreover, c_1 is linear in $\overline{L}$ and commutes with pullback.

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \le 1\}$. A smooth metric $\|\cdot\|$ on L is called *semipositive* if, for each holomorphic map $\varphi\colon \mathbb{D} \to X^{\mathrm{an}}$,

$$\int_{\mathbb{D}} \varphi^*\,\mathrm{c}_1(\overline{L}) \ge 0.$$

The pullback of a semipositive metrized line bundle by any analytic morphism is still semipositive.

1.5.3. An arbitrary metric $\|\cdot\|$ on L is *semipositive* if there is a sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of semipositive smooth metrics on L that converges uniformly to $\|\cdot\|$. A metric $\|\cdot\|$ on L is *DSP* if $(L, \|\cdot\|)$ is the quotient of two semipositive metrized line bundles.

Note that, for a smooth metric, the definitions of the term "semipositive" in 1.5.2 and 1.5.3 are equivalent. So there is no ambiguity in the use of this notion.

1.5.4. A *current* of degree r on X^{an} is a linear functional T on the space of compactly supported forms in $\mathcal{A}^r(X^{\mathrm{an}})$ with the following property: For each point in X^{an}, there is an open neighborhood $U \subseteq X$, which is a closed analytic subset of an open complex ball B, and a current T_U on B such that $T(\omega|_U) = T_U(\omega)$ for every $\omega \in \mathcal{A}^r(B)$ with compact support. As in the smooth case, the complex of currents on X^{an} is equipped with a bigrading, differential operators $\partial, \overline{\partial}$, pushforwards and an exterior product with differential forms. Moreover, we have the current of integration δ_Y along an analytic subvariety Y^{an} and the current $[\eta]$ associated to a L^1-form η. We refer to [Kin71] for details and to [Gub02, 2.1.1] for an overview about currents on analytic varieties.

Definition 1.5.5. A *Green current* for a t-dimensional cycle Z on X is a $(t{+}1, t{+}1)$-current g_Z on X^{an} such that

$$\frac{i}{2\pi}\partial\overline{\partial} g_Z = [\omega_Z] - \delta_Z$$

for a smooth differential form ω_Z on X^{an}.

Example 1.5.6. Let $\overline{L} = (L, \|\cdot\|)$ be a smooth metrized line bundle and s an invertible meromorphic section of L. Then the Poincaré-Lelong formula says

$$\frac{i}{2\pi}\partial\overline{\partial}\left[\log\|s\|^{-2}\right] = [\mathrm{c}_1(\overline{L})] - \delta_{\mathrm{cyc}(s)}.$$

Hence, $\left[\log\|s\|^{-2}\right]$ is a Green current for $\mathrm{cyc}(s)$.

Definition 1.5.7. Let $\hat{D} = (\overline{L}, |D|, s)$ be a smooth metrized pseudo-divisor and g_Z a Green current for a prime cycle Z on X. If $Z \nsubseteq |D|$, then let $s_Z := s|_Z$ and if $Z \subseteq |D|$, we choose any non-zero meromorphic section s_Z of $L|_Z$. Then we define the $*$-product by

$$\hat{D} * g_Z := i_*\left[\log\|s_Z\|^{-2}\right] + \mathrm{c}_1(\overline{L}) \wedge g_Z,$$

where $i\colon Z \hookrightarrow X$. We extend this definition to cycles by linearity.

Remark 1.5.8. The current $\hat{D} * g_Z$ is only well-defined up to $\sum_W \left[\log|f_W|^{-2}\right]$, where W ranges over finitely many subvarieties of $|D| \cap |Z|$ and f_W is a non-zero rational function on W. When $|D|$ intersects $|Z|$ properly, the current is well-defined. In any case, $\hat{D} * g_Z$ is a Green current for a cycle representing $D.Z \in \mathrm{CH}\left(|D| \cap |Z|\right)$.

Let s be an invertible meromorphic section of a smooth metrized line bundle $\overline{L}$ and $\widehat{\mathrm{div}}(s)$ the associated metrized pseudo-divisor (cf. Example 1.2.10). If X is smooth and $|\mathrm{div}(s)|$ intersects Z properly, then $\widehat{\mathrm{div}}(s) * g_Z = \log\|s\|^{-2} * g_Z$ is the $*$-product of [GS90, § 2].

1.5.9. Let $i\colon Z \hookrightarrow X$ be the embedding of a prime cycle and 0_Z the zero current on Z. For smooth metrized pseudo-divisors $\hat{D}_1, \dots, \hat{D}_k$ on X, we set

$$\hat{D}_1 * \cdots * \hat{D}_k \wedge \delta_Z := i_*\left(i^*\hat{D}_1 * \cdots * i^*\hat{D}_k * 0_Z\right).$$

This is a well-defined current up to $\sum_W \left[\log|f_W|^{-2}\right]$, where W ranges over the prime cycles of $|D_1| \cap \cdots \cap |D_k| \cap Z$ and $f_W \in K(W)^\times$. By linearity, it extends to arbitrary cycles Z.

Definition 1.5.10. Let Z be a t-dimensional cycle on X and $\hat{D}_0, \dots, \hat{D}_t$ smooth metrized pseudo-divisors such that

$$|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset.$$

Then we define the *local height* of Z with respect to $\hat{D}_0, \ldots, \hat{D}_t$ as

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z) := (\hat{D}_0 * \cdots * \hat{D}_t \wedge \delta_Z)\,(1/2)\,.$$

1.5.11. The Archimedean local heights with respect to smooth metrized pseudo-divisors have the properties listed in Proposition 1.2.12 for non-Archimedean local heights with respect to $\mathbb{Q}$-formal metrized pseudo-divisors. This is proved in [Gub03, § 3].

Thus, we can extend the Archimedean local heights to semipositive and DSP metrized pseudo-divisors as in 1.3.5. By [Gub03, Theorem 10.6], they satisfy the same properties stated in Proposition 1.3.6. for the non-Archimedean case.

1.5.12. Let Y be a t-dimensional subvariety of X and let $\overline{L}_1, \ldots, \overline{L}_t$ be smooth metrized line bundles on X. We denote by δ_Y the current of integration along the analytic subvariety Y^{an}. Then the current

$$\mathrm{c}_1(\overline{L}_1) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_t) \wedge \delta_Y$$

defines a (signed) measure on X^{an}. This notion extends linearly to a cycle Y.

Analogously to Proposition 1.3.7, we extend this measure to semipositive and DSP metrized line bundles. It has the same properties as in Proposition 1.3.11.

Now we state and prove an induction formula similarly to Theorem 1.4.3. This formula was proved in [CT09, Théorème 4.1] under the stronger assumptions that X is projective and that the supports of the Cartier divisors of the occuring sections intersect properly.

Theorem 1.5.13 (Induction formula)**.** *Let Z be a t-dimensional prime cycle on X and let $\hat{D}_i = (\overline{L}_i, |D_i|, s_i)$, $i = 0, \ldots, t$, be DSP pseudo-divisors with*

$$|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset.$$

If $|Z| \not\subseteq |D_t|$, then let $s_{t,Z} := s_t|_Z$, otherwise we choose any non-zero meromorphic section $s_{t,Z}$ of $L_t|_Z$. Let $\mathrm{cyc}(s_{t,Z})$ be the Weil divisor of $s_{t,Z}$ considered as a cycle on X.

Then the function $\log\|s_{t,Z}\|$ is integrable with respect to $\mathrm{c}_1(\overline{L}_0) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{t-1}) \wedge \delta_Z$ and we have

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z) = \lambda_{\hat{D}_0,\ldots,\hat{D}_{t-1}}(\mathrm{cyc}(s_{t,Z})) - \int_{X^{\mathrm{an}}} \log\|s_{t,Z}\| \cdot \mathrm{c}_1(\overline{L}_0) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{t-1}) \wedge \delta_Z\,.$$

Proof. We get an Archimedean version of the approximation theorem 1.4.1 just by copying the proof of the Archimedean part of [CT09, Théorème 3.1]. Then, replacing $\mathbb{Q}$-formal metrics by smooth metrics and using the corresponding properties of the Archimedean local heights and measures, we can prove this theorem similarly as the non-Archimedean induction formula 1.4.3. □

Chapter 2.

Metrics and Local Heights of Toric Varieties

We show a formula to compute the local height of a toric variety over an arbitrary non-Archimedean field (Theorem 2.6.6). For discrete non-Archimedean fields, this was proved by Burgos–Philippon–Sombra in [BPS14a, Theorem 5.1.6]. To state and prove this formula, we study toric divisors over arbitrary valuation rings of rank one (section 2.3) and toric semipositive metrics over non-discrete non-Archimedean fields (section 2.5).

In this chapter, let M be a free Abelian group of rank n and $N := M^\vee := \operatorname{Hom}(M,\mathbb{Z})$ its dual group. The natural pairing between $m \in M$ and $u \in N$ is denoted by $\langle m, u\rangle := u(m)$. We have the split torus $\mathbb{T} := \operatorname{Spec}(K[M])$ over a field K of rank n. Then M can be considered as the character lattice of $\mathbb{T}$ and N as the lattice of one-parameter subgroups. For $m \in M$ we will write χ^m for the corresponding character. If G is an Abelian group, we set $N_G = N \otimes_{\mathbb{Z}} G$. In particular, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is an n-dimensional real vector space with dual space $M_{\mathbb{R}}$.

The needed notions and statements of convex geometry are summarized in Appendix A.

2.1. Toric Varieties

We give a short overview of the theory of (normal) toric varieties over a field K following [BPS14a, 3.1–3.4], especially in the notation. For details and proofs, we also refer to [KKMS73], [Ful93] and [CLS11].

The notations concerning polyhedra and their properties can be found in the appendix A.

Definition 2.1.1. Let K be a field and $\mathbb{T}$ a split torus over K. A ($\mathbb{T}$-)*toric variety* is a normal irreducible variety X over K containing $\mathbb{T}$ as an open subset such that the translation action of $\mathbb{T}$ on itself extends to an algebraic action $\mu\colon \mathbb{T} \times X \to X$.

2.1.2. There is a nice description of toric varieties in combinatorial data. At first we have a bijection between the sets of

(i) strongly convex rational polyhedral cones σ in $N_{\mathbb{R}}$,

(ii) isomorphism classes of affine $\mathbb{T}$-toric varieties X over K.

This correspondence is given by $\sigma \mapsto U_\sigma = \operatorname{Spec}(K[M_\sigma])$, where $K[M_\sigma]$ is the semigroup algebra of

$$M_\sigma = \sigma^\vee \cap M = \{m \in M \mid \langle m, u\rangle \geq 0 \ \forall u \in \sigma\}.$$

The action of $\mathbb{T}$ on U_σ is induced by

$$K[M_\sigma] \to K[M] \otimes K[M_\sigma], \qquad \chi^m \mapsto \chi^m \otimes \chi^m.$$

More generally, we consider a fan Σ in $N_\mathbb{R}$ (Definition A.4). If $\sigma, \sigma' \in \Sigma$, then U_σ and $U_{\sigma'}$ glue together along the open subset $U_{\sigma \cap \sigma'}$. So, we obtain a $\mathbb{T}$-toric variety

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma.$$

This construction induces a bijection between the set of fans Σ in $N_\mathbb{R}$ and the set of isomorphism classes of toric varieties X_Σ with torus $\mathbb{T}$.

2.1.3. Many properties of toric varieties are encoded in their fans, for example:

(i) A toric variety X_Σ is proper if and only if the fan is complete, i. e. $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = N_\mathbb{R}$.

(ii) A toric variety X_Σ is smooth if and only if the minimal generators of each cone $\sigma \in \Sigma$ are part of a $\mathbb{Z}$-basis of N.

2.1.4. Let X_Σ be the toric variety of the fan Σ in $N_\mathbb{R}$. Then there is a bijective correspondence between the cones in Σ and the $\mathbb{T}$-orbits in X_Σ. The closures of the orbits in X_Σ have a structure of toric varieties which we describe in the following: For $\sigma \in \Sigma$ we set

$$N(\sigma) = N / \langle N \cap \sigma \rangle, \quad M(\sigma) = N(\sigma)^\vee = M \cap \sigma^\perp, \quad O(\sigma) = \operatorname{Spec}(K[M(\sigma)]),$$

where $\sigma^\perp$ denotes the orthogonal space to σ. Then $O(\sigma)$ is a torus over K of dimension $n - \dim(\sigma)$ which can be identified with a $\mathbb{T}$-orbit in X_Σ via the surjection

$$K[M_\sigma] \longrightarrow K[M(\sigma)], \quad \chi^m \longmapsto \begin{cases} \chi^m & \text{if } m \in \sigma^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $V(\sigma)$ the closure of $O(\sigma)$ in X_Σ. Then $V(\sigma)$ can be identified with the $O(\sigma)$-toric variety $X_{\Sigma(\sigma)}$, which is given by the fan

$$\Sigma(\sigma) = \{\tau + \langle N \cap \sigma \rangle_\mathbb{R} \mid \tau \in \Sigma,\, \tau \supseteq \sigma\} \tag{2.1}$$

in $N(\sigma)_\mathbb{R} = N_\mathbb{R} / \langle N \cap \sigma \rangle_\mathbb{R}$.

Definition 2.1.5. Let X_i, $i = 1, 2$, be toric varieties with torus $\mathbb{T}_i$. We say that a morphism $\varphi \colon X_1 \to X_2$ is *toric* if φ maps $\mathbb{T}_1$ into $\mathbb{T}_2$ and $\varphi|_{\mathbb{T}_1} \colon \mathbb{T}_1 \to \mathbb{T}_2$ is a morphism of group schemes.

2.1.6. Any toric morphism $\varphi \colon X_1 \to X_2$ is *equivariant*, i. e. we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T}_1 \times X_1 & \xrightarrow{\mu_1} & X_1 \\ \Big\downarrow{\scriptstyle \varphi|_{\mathbb{T}_1} \times \varphi} & & \Big\downarrow{\scriptstyle \varphi} \\ \mathbb{T}_1 \times X_1 & \xrightarrow{\mu_2} & X_2 \end{array},$$

where μ_1, μ_2 denote the torus actions.

Toric morphisms can be described in combinatorial terms.

2.1.7. For $i = 1, 2$, let N_i be a lattice with associated torus $\mathbb{T}_i = \operatorname{Spec} K[N_i^\vee]$ and let Σ_i be a fan in $N_{i,\mathbb{R}}$. Let $H\colon N_1 \to N_2$ be a linear map which is *compatible* with Σ_1 and Σ_2. That is, for each cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ with $H(\sigma_1) \subseteq \sigma_2$. Then H induces a group morphism $\mathbb{T}_1 \to \mathbb{T}_2$ of tori and, by the compatibility of H, this group morphism extends to a toric morphism $\varphi_H\colon X_{\Sigma_1} \to X_{\Sigma_2}$.

We fix N_i, $\mathbb{T}_i$ and Σ_i, $i = 1, 2$, as above. Then the assignment $H \mapsto \varphi_H$ induces a bijection between the sets of

(i) linear maps $H\colon N_1 \to N_2$, which are compatible with Σ_1 and Σ_2;

(ii) toric morphisms $\varphi\colon X_{\Sigma_1} \to X_{\Sigma_2}$.

A toric morphism $\varphi_H\colon X_{\Sigma_1} \to X_{\Sigma_2}$ is proper if and only if $H^{-1}(|\Sigma_2|) = |\Sigma_1|$.

Definition 2.1.8. A $\mathbb{T}$-*Cartier divisor* on a $\mathbb{T}$-toric variety X is a Cartier divisor D on X which is invariant under the action of $\mathbb{T}$ on X, i. e. we have $\mu^*D = p_2^*D$ denoting by $\mu\colon \mathbb{T} \times X \to X$ the toric action and by $p_2\colon \mathbb{T} \times X \to X$ the second projection.

Torus-invariant Cartier divisors can be described in terms of support functions:

Definition 2.1.9. A continuous function $\Psi\colon |\Sigma| \longrightarrow \mathbb{R}$ is called a *virtual support function* on Σ, if there exists a set $\{m_\sigma\}_{\sigma\in\Sigma}$ of elements in M such that, for each cone $\sigma \in \Sigma$, we have $\Psi(u) = \langle m_\sigma, u\rangle$ for all $u \in \sigma$. It is said to be *strictly concave* if, for different maximal cones $\sigma, \tau \in \Sigma$, we have $m_\sigma \neq m_\tau$. A *support function* is a concave virtual support function on a complete fan.

2.1.10. Let Ψ be a virtual support function given by the data $\{m_\sigma\}_{\sigma\in\Sigma}$. Then Ψ determines a $\mathbb{T}$-Cartier divisor

$$D_\Psi := \big\{ (U_\sigma, \chi^{-m_\sigma}) \big\}_{\sigma\in\Sigma}$$

on X_Σ. The map $\Psi \mapsto D_\Psi$ is an isomorphism between the group of virtual support functions on Σ and the group of $\mathbb{T}$-Cartier divisors on X_Σ. The divisors D_{Ψ_1} and D_{Ψ_2} are rationally equivalent if and only if $\Psi_1 - \Psi_2$ is linear.

Definition 2.1.11. Let X be a toric variety. A *toric line bundle* on X is a pair (L, z) consisting of a line bundle L on X and a non-zero element z in the fiber L_{x_0} of the unit point x_0 of $U_0 = \mathbb{T}$. A *toric section* is a meromorphic section s of a toric line bundle which is regular and non-vanishing on the torus U_0 and such that $s(x_0) = z$.

2.1.12. Let D be a $\mathbb{T}$-Cartier divisor on a toric variety X_Σ. Then there is an associated line bundle $\mathcal{O}(D)$ and a meromorphic section s_D such that $\operatorname{div}(s_D) = D$. Since the support of D lies in the complement of $\mathbb{T}$, the section s_D is regular and non-vanishing on $\mathbb{T}$. Thus, D corresponds to a toric line bundle $(\mathcal{O}(D), s_D(x_0))$ with toric section s_D. This assignment determines an isomorphism between the group of $\mathbb{T}$-Cartier divisors on X_Σ and the group of isomorphism classes of toric line bundles with toric sections.

Let Ψ be a virtual support function on Σ. By 2.1.10, this function corresponds bijectively to the isomorphism class of the toric line bundle with toric section $((\mathcal{O}(D_\Psi), s_{D_\Psi}(x_0)), s_{D_\Psi})$, which we simply denote by (L_Ψ, s_Ψ).

2.1.13. Let X_Σ be a $\mathbb{T}$-toric variety. We denote by $\mathrm{Pic}(X_\Sigma)$ the Picard group of X_Σ and by $\mathrm{Div}_\mathbb{T}(X_\Sigma)$ the group of $\mathbb{T}$-Cartier divisors. Then we have a exact sequence of Abelian groups

$$M \longrightarrow \mathrm{Div}_\mathbb{T}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0,$$

where the first morphism is given by $m \mapsto \mathrm{div}(\chi^m)$. In particular, every toric line bundle admits a toric section and, if s and s' are two toric sections, then there is an $m \in M$ such that $s' = \chi^m s$.

2.1.14. Let D_Ψ be a $\mathbb{T}$-Cartier divisor on a toric variety X_Σ. Then the associated Weil divisor $\mathrm{cyc}(s_\Psi)$ is invariant under the torus action. Indeed, let $\Sigma^{(1)}$ be the set of one-dimensional cones in Σ. Each ray $\tau \in \Sigma^{(1)}$ gives a minimal generator $v_\tau \in \tau \cap N$ and a corresponding $\mathbb{T}$-invariant prime divisor $V(\tau)$ on X_Σ (see 2.1.4). Then we have

$$\mathrm{cyc}(s_\Psi) = \sum_{\tau \in \Sigma^{(1)}} -\Psi(v_\tau) V(\tau). \tag{2.2}$$

2.1.15. We describe the intersection of a $\mathbb{T}$-Cartier divisor with the closure of an orbit. Let Σ be a fan in $N_\mathbb{R}$ and Ψ a virtual support function on Σ given by the defining vectors $\{m_\tau\}_{\tau \in \Sigma}$. Let σ be a cone of Σ and $V(\sigma)$ the corresponding orbit closure. Each cone $\tau \succeq \sigma$ corresponds to a cone $\overline{\tau}$ of the fan $\Sigma(\sigma)$ defined in (2.1). Since $m_\tau - m_\sigma|_\sigma = 0$, we have $m_\tau - m_\sigma \in M(\sigma) = M \cap \sigma^\perp$. Thus, the defining vectors $\{m_\tau - m_\sigma\}_{\overline{\tau} \in \Sigma(\sigma)}$ gives us a well-defined virtual support function $(\Psi - m_\sigma)(\sigma)$ on $\Sigma(\sigma)$.

When $\Psi|_\sigma \neq 0$, then D_Ψ and $V(\sigma)$ do not intersect properly. But D_Ψ is rationally equivalent to $D_{\Psi - m_\sigma}$ and the latter divisor properly intersects $V(\sigma)$. Moreover, we have $D_{\Psi - m_\sigma}|_{V(\sigma)} = D_{(\Psi - m_\sigma)(\sigma)}$. For details, we refer to [BPS14a, Proposition 3.3.14].

We end this section with some positivity statements about $\mathbb{T}$-Cartier divisors. For this, we consider a complete fan Σ in $N_\mathbb{R}$ and a virtual support function Ψ on Σ given by the defining vectors $\{m_\sigma\}_{\sigma \in \Sigma}$.

2.1.16. Many properties of the associated toric line bundle $\mathcal{O}(D_\Psi)$ are encoded in its support function.

(i) $\mathcal{O}(D_\Psi)$ is generated by global sections if and only if Ψ is concave;

(ii) $\mathcal{O}(D_\Psi)$ is ample if and only if Ψ is strictly concave.

If Ψ is concave, then the stability set Δ_Ψ from A.7 is a lattice polytope and $\{\chi^m\}_{m \in M \cap \Delta_\Psi}$ is a basis of the K-vector space $\Gamma(X_\Sigma, \mathcal{O}(D_\Psi))$. Moreover, we have in this case

$$\deg_{\mathcal{O}(D_\Psi)}(X_\Sigma) = n! \,\mathrm{vol}_M(\Delta_\Psi). \tag{2.3}$$

2.1.17. Assume that Ψ is strictly concave or equivalently that D_Ψ is ample. We use the notations and statements from A.20. Then the stability set $\Delta := \Delta_\Psi$ is a full dimensional lattice polytope and Σ coincides with the normal fan Σ_Δ of Δ. Thus, a facet F of Δ correspond to a ray σ_F of Σ and we can reformulate (2.2),

$$\mathrm{cyc}(s_\Psi) = \sum_F - \langle F, v_F \rangle \, V(\sigma_F),$$

where the sum is over the facets F of Δ and v_F is the minimal inner facet normal of F (see A.21).

2.1.18. Assume that Ψ is concave or D_Ψ is generated by global sections. Then $\Delta = \Delta_\Psi$ is a (not necessarily full dimensional) lattic polytope. We set

$$M(\Delta) = M \cap \mathbb{L}_\Delta, \quad N(\Delta) = M(\Delta)^\vee = N / (N \cap \mathbb{L}_\Delta^\perp),$$

where $\mathbb{L}_\Delta$ denotes the linear subspace of $M_\mathbb{R}$ associated to the affine hull $\operatorname{aff}(\Delta)$ of Δ. We choose any $m \in \operatorname{aff}(\Delta) \cap M$. Then, the lattic polytope $\Delta - m$ is full dimensional in $\mathbb{L}_\Delta = M(\Delta)_\mathbb{R}$. Let Σ_Δ be the normal fan of $\Delta - m$ in $N(\Delta)_\mathbb{R}$ (see A.20). The projection $H\colon N \to N(\Delta)$ is compatible with Σ and Σ_Δ and so, by 2.1.7, it induces a proper toric morphism $\varphi\colon X_\Sigma \to X_{\Sigma_\Delta}$. We set $\Delta' = \Delta - m$ and consider the function

$$\Psi_{\Delta'}\colon N(\Delta)_\mathbb{R} \longrightarrow \mathbb{R}, \quad u \longmapsto \min_{m' \in \Delta'} \langle m', u \rangle.$$

This is a strictly concave support function on Σ_Δ. By 2.1.16, the divisor $D_{\Psi_{\Delta'}}$ is ample, and

$$D_\Psi = \varphi^* D_{\Psi_{\Delta'}} + \operatorname{div}(\chi^{-m}). \tag{2.4}$$

2.2. Toric Schemes over Valuation Rings of Rank One

In this section we summarize some facts from the theory of toric schemes over valuation rings of rank one developed in [Gub13] and [GS13].

Let K be a field equipped with a non-Archimedean absolute value $|\cdot|$ and denote by K° its valuation ring. Then we have a valuation $\operatorname{val} := -\log|\cdot|$ of rank one and a value group $\Gamma := \operatorname{val}(K^\times) \subseteq \mathbb{R}$. As usual, we fix a free Abelian group M of rank n with dual N. Let $\mathbb{T}_S$ bet the split torus $\mathbb{T}_S = \operatorname{Spec}(K^\circ[M])$ over $S = \operatorname{Spec}(K^\circ)$ with generic fiber $\mathbb{T} = \operatorname{Spec}(K[M])$ and special fiber $\mathbb{T}_{\widetilde{K}} = \operatorname{Spec}(\widetilde{K}[M])$.

Definition 2.2.1. A ($\mathbb{T}_S$-)*toric scheme* is a normal integral separated S-scheme $\mathscr{X}$ of finite type, such that the generic fiber $\mathscr{X}_\eta$ contains $\mathbb{T}$ as an open subset and the translation action of $\mathbb{T}$ on itself extends to an algebraic action $\mathbb{T}_S \times_S \mathscr{X} \to \mathscr{X}$ over S.

Remark 2.2.2. In [Gub13] and [GS13], a $\mathbb{T}_S$-toric scheme is not necessarily normal and of finite type over S where such a scheme is called normal $\mathbb{T}_S$-toric variety. Here, we follow the definition in [BPS14a].

Definition 2.2.3. Let X be a $\mathbb{T}$-toric variety and let $\mathscr{X}$ be a $\mathbb{T}_S$-toric scheme. Then $\mathscr{X}$ is called a ($\mathbb{T}_S$-)*toric model* of X if $\mathscr{X}$ is an algebraic model of X over S such that the fixed isomorphism $\mathscr{X}_\eta \simeq X$ identifies $(\mathbb{T}_S)_\eta$ with $\mathbb{T}$.

If $\mathscr{X}$ and $\mathscr{X}'$ are toric models of X and $\alpha\colon \mathscr{X} \to \mathscr{X}'$ is an S-morphism, we say that α is a *morphism of toric models* if its restriction to $\mathbb{T}$ is the identity.

2.2.4. By [Gub13, Lemma 4.2], a toric scheme $\mathscr{X}$ is flat over S and the generic fiber $\mathscr{X}_\eta$ is a $\mathbb{T}$-toric variety over K. Thus, $\mathscr{X}$ is a $\mathbb{T}_S$-toric model of $\mathscr{X}_\eta$.

In analogy to toric varieties over K, we can describe toric schemes over K° with torus $\mathbb{T}_S$ in terms of convex geometry:

2.2.5. A Γ-*admissible cone* σ in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ is a strongly convex cone which is of the form

$$\sigma = \bigcap_{i=1}^{k} \{(u,r) \in N_{\mathbb{R}} \times \mathbb{R}_{\geq 0} \mid \langle m_i, u\rangle + l_i \cdot r \geq 0\} \quad \text{with } m_i \in M,\ l_i \in \Gamma, i = 1, \dots, k.$$

For such a cone σ, we define

$$K[M]^\sigma := \Big\{ \sum_{m \in M} \alpha_m \chi^m \in K[M] \mid \langle m, u\rangle + \mathrm{val}(\alpha_m) \cdot r \geq 0\ \forall\, (u,r) \in \sigma \Big\}.$$

This is an M-graded K°-subalgebra of $K[M]$ which is an integrally closed domain. It is finitely generated as a K°-algebra if and only if the following condition (F) is fulfilled:

(F) The value group Γ is discrete or the vertices of $\sigma \cap (N_{\mathbb{R}} \times \{1\})$ are contained in $N_\Gamma \times \{1\}$.

Hence, we get an affine $\mathbb{T}_S$-toric scheme $\mathscr{U}_\sigma := \mathrm{Spec}(K[M]^\sigma)$ over S if and only if (F) holds. If Γ is discrete or divisible, then (F) is always correct.

2.2.6. A fan in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ is called Γ-*admissible* if it consists of Γ-admissible cones. Given such a fan $\widetilde{\Sigma}$, the affine $\mathbb{T}_S$-toric schemes $\mathscr{U}_\sigma, \sigma \in \widetilde{\Sigma}$, glue together along the open subschemes corresponding to the common faces as in the case of toric varieties. So we obtain a scheme

$$\mathscr{X}_{\widetilde{\Sigma}} = \bigcup_{\sigma \in \widetilde{\Sigma}} \mathscr{U}_\sigma \tag{2.5}$$

over S. By [GS13, Theorem 3], $\widetilde{\Sigma} \mapsto \mathscr{X}_{\widetilde{\Sigma}}$ defines a bijection between the sets of

(i) Γ-admissible fans in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ whose cones satisfy condition (F),

(ii) isomorphism classes of $\mathbb{T}_S$-toric schemes over S.

In this case, $\mathscr{X}_{\widetilde{\Sigma}}$ is proper over S if and only if $\widetilde{\Sigma}$ is complete, i. e. $|\widetilde{\Sigma}| = N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ (see [Gub13, Proposition 11.8]).

It is also possible to describe toric schemes in terms of polyhedra in $N_{\mathbb{R}}$.

2.2.7. Let σ be a cone in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. For $r \in \mathbb{R}_{\geq 0}$, we set

$$\sigma_r := \{u \in N_{\mathbb{R}} \mid (u,r) \in \sigma\}.$$

Then $\sigma \mapsto \sigma_1$ defines a bijection between the set of Γ-admissible cones in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, which are not contained in $N_{\mathbb{R}} \times \{0\}$, and the set of strongly convex Γ-rational polyhedra in $N_{\mathbb{R}}$. The inverse map is given by $\Lambda \mapsto \mathrm{c}(\Lambda)$, where $\mathrm{c}(\Lambda)$ is the closure of $\mathbb{R}_{>0}(\Lambda \times \{1\})$ in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$.

2.2.8. Let $\widetilde{\Sigma}$ be a Γ-admissible fan, whose cones satisfy (F). Then we have two kinds of cones σ in $\widetilde{\Sigma}$:

(i) If σ is contained in $N_{\mathbb{R}} \times \{0\}$, then $K[M]^\sigma = K[M_{\sigma_0}]$. Hence, $\mathscr{U}_\sigma$ is equal to the toric variety U_{σ_0} associated to σ_0 (see 2.1.2) and it is contained in the generic fiber of $\mathscr{X}_{\widetilde{\Sigma}}$.

(ii) If σ is not contained in $N_{\mathbb{R}} \times \{0\}$, then $\Lambda := \sigma_1$ is a strongly convex Γ-rational polyhedron in $N_{\mathbb{R}}$. It easily follows that $K[M]^\sigma$ is equal to

$$K[M]^\Lambda := \Big\{ \sum_{m\in M} \alpha_m \chi^m \in K[M] \mid \langle m, u\rangle + \mathrm{val}(\alpha_m) \geq 0 \ \forall\, u \in \Lambda \Big\}.$$

Thus, $\mathscr{U}_\sigma$ equals the $\mathbb{T}_S$-toric scheme $\mathscr{U}_\Lambda := \mathrm{Spec}(K[M]^\Lambda)$. The generic fiber of $\mathscr{U}_\Lambda = \mathscr{U}_\sigma$ is identified with the $\mathbb{T}$-toric variety $U_{\sigma_0} = U_{\mathrm{rec}(\Lambda)}$, where $\mathrm{rec}\,(\Lambda)$ is the recession cone of Λ (see A.3).

We set $\Sigma := \{\sigma_0 \mid \sigma \in \widetilde{\Sigma}\}$ and $\Pi := \{\sigma_1 \mid \sigma \in \widetilde{\Sigma}\}$. Then Σ is a fan in $N_{\mathbb{R}}$ and Π is a Γ-rational polyhedral complex in $N_{\mathbb{R}}$ (see A.4 for the definition). Now we can rewrite the open cover (2.5) as

$$\mathscr{X}_{\widetilde{\Sigma}} = \bigcup_{\sigma\in\Sigma} U_\sigma \cup \bigcup_{\Lambda\in\Pi} \mathscr{U}_\Lambda$$

using the same gluing data. The generic fiber of this toric scheme is the $\mathbb{T}$-toric variety X_Σ associated to Σ, i. e. $\mathscr{X}_{\widetilde{\Sigma}}$ is a toric model of X_Σ.

2.2.9. If the value group Γ is discrete, then the special fiber $\widetilde{\mathscr{X}_{\widetilde{\Sigma}}}$ is reduced if and only if the vertices of all $\Lambda \in \Pi$ are contained in N_Γ. If the valuation is not discrete, then $\widetilde{\mathscr{X}_{\widetilde{\Sigma}}}$ is always reduced (see [Gub13, Proposition 7.11 and 7.12]).

2.2.10. Conversely, if we start with an arbitrary Γ-rational polyhedral complex Π, we can't expect that the *cone*

$$\mathrm{c}(\Pi) := \{\mathrm{c}(\Lambda) \mid \Lambda \in \Pi\} \cup \{\mathrm{rec}(\Lambda) \times \{0\} \mid \Lambda \in \Pi\}$$

is a fan in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. However, the correspondence $\Pi \mapsto \mathrm{c}(\Pi)$ gives a bijection between *complete* Γ-rational polyhedral complexes in $N_{\mathbb{R}}$ and *complete* Γ-admissible fans in $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ (see [BS11, Corollary 3.11]).

We will consider Γ-rational polyhedral complexes Π in $N_{\mathbb{R}}$ that satisfy the following condition:

(F') The value group Γ is discrete or, for each $\Lambda \in \Pi$, the vertices of Λ are contained in N_Γ.

Proposition 2.2.11. *The correspondence* $\Pi \mapsto \mathscr{X}_{\mathrm{c}(\Pi)}$ *gives a bijection between the sets of*

(i) complete Γ*-rational polyhedral complexes* Π *in* $N_{\mathbb{R}}$ *which satisfy condition (F');*

(ii) isomorphism classes of proper $\mathbb{T}_S$*-toric schemes over* S.

Proof. This follows from the results in [BS11], [GS13] and [Gub13] mentioned in 2.2.10 and 2.2.6. □

Corollary 2.2.12. *Let* Σ *be a complete fan in* $N_{\mathbb{R}}$*. Then there is a bijective correspondence between the sets of*

(i) complete Γ*-rational polyhedral complexes* Π *in* $N_{\mathbb{R}}$ *with* $\mathrm{rec}(\Pi) = \Sigma$ *(see A.5) and satisfying condition (F');*

(ii) isomorphism classes of proper $\mathbb{T}_S$-toric models of X_Σ over S.

We end this section with a description of the orbits of a toric scheme. We assume that Π is a complete Γ-rational polyhedral complex in $N_\mathbb{R}$ which satisfies condition (F'). This gives us a complete Γ-admissible fan $\mathrm{c}(\Pi)$ in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ and a complete fan $\mathrm{rec}(\Pi)$ in $N_\mathbb{R}$. We set $\mathscr{X}_\Pi := \mathscr{X}_{\mathrm{c}(\Pi)}$ and we identify the generic fiber $\mathscr{X}_{\Pi,\eta}$ with the toric variety $X_{\mathrm{rec}(\Pi)}$.

Notation 2.2.13. For $\Lambda \in \Pi$, let $\mathbb{L}_\Lambda$ be the $\mathbb{R}$-linear subspace of $N_\mathbb{R}$ associated to the affine space $\mathrm{aff}(\Lambda)$. We set

$$N(\Lambda) = N/(N \cap \mathbb{L}_\Lambda), \quad M(\Lambda) = N(\Lambda)^\vee = M \cap \mathbb{L}_\Lambda^\perp,$$

generalizing the notation in 2.1.4. Furthermore, we define

$$\widetilde{M}(\Lambda) = \{m \in M(\Lambda) \mid \langle m, u\rangle \in \Gamma \ \forall\, u \in \Lambda\}, \quad \widetilde{N}(\Lambda) = \widetilde{M}(\Lambda)^\vee .$$

Because of the Γ-rationality of Λ, the lattice $\widetilde{M}(\Lambda)$ is of finite index in $M(\Lambda)$. We define the *multiplicity* of a polyhedron $\Lambda \in \Pi$ by

$$\mathrm{mult}(\Lambda) = [M(\Lambda) : \widetilde{M}(\Lambda)]. \tag{2.6}$$

Let $\Lambda' \in \Pi$ and Λ a face of Λ'. The *local cone* (or *angle*) of Λ' at Λ is defined as

$$\angle(\Lambda, \Lambda') := \{t(u - v) \mid u \in \Lambda', v \in \Lambda, t \geq 0\} .$$

This is a polyhedral cone.

There is a bijection between torus orbits of $\mathscr{X}_\Pi$ and the two kinds of cones in $\mathrm{c}(\Pi)$ corresponding to cones in $\mathrm{rec}(\Pi)$ and polyhedra in Π.

First, the cones in $\mathrm{rec}(\Pi)$ correspond to the $\mathbb{T}$-orbits on the generic fiber $\mathscr{X}_{\Pi,\eta} = X_{\mathrm{rec}(\Pi)}$ via $\sigma \mapsto O(\sigma)$ as in 2.1.4. We denote by $\mathscr{V}(\sigma)$ the Zariski closure of $O(\sigma)$ in $\mathscr{X}_\Pi$. Then $\mathscr{V}(\sigma)$ is a scheme of relative dimension $n - \dim(\sigma)$ over S. Moreover, we have $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \mathscr{V}(\tau)$.

Proposition 2.2.14. *There is a canonical isomorphism from $\mathscr{V}(\sigma)$ to the* $\mathrm{Spec}(K^\circ[M(\sigma)])$-*toric scheme $\mathscr{X}_{\Pi(\sigma)}$ over K° which is given by the Γ-rational polyhedral complex*

$$\Pi(\sigma) = \{\Lambda + \langle N \cap \sigma\rangle_\mathbb{R} \mid \Lambda \in \Pi,\, \mathrm{rec}(\Lambda) \supseteq \sigma\}$$

in $N(\sigma)_\mathbb{R} = N_\mathbb{R}/\langle N \cap \sigma\rangle_\mathbb{R}$.

Proof. This follows from [Gub13, Proposition 7.14]. □

Second, the polyhedra of Π correspond to the $\mathbb{T}_{\widetilde{K}}$-orbits on the special fiber $\widetilde{\mathscr{X}_\Pi}$. This bijective correspondence is given by

$$O\colon \Lambda \longmapsto \mathrm{red}(\mathrm{trop}^{-1}(\mathrm{ri}\,\Lambda)),$$

where red is the reduction map from 1.1.8, trop is the tropicalization map from 2.4.5 and $\mathrm{ri}(\Lambda)$ is the relative interior of Λ from A.1. For details, we refer to [Gub13, Proposition 6.22 and 7.9]. For $\Lambda \in \Pi$, we denote by $V(\Lambda)$ the Zariski closure of $O(\Lambda)$ in $\mathscr{X}_\Pi$. Then

$V(\Lambda)$ is contained in the special fiber $\widetilde{\mathscr{X}_\Pi}$ and has dimension $n - \dim(\Lambda)$. Moreover, we have

$$\Lambda \preceq \Lambda' \iff O(\Lambda') \subseteq V(\Lambda) \quad \text{and} \quad \sigma \preceq \operatorname{rec}(\Lambda) \iff O(\Lambda) \subseteq \mathscr{V}(\sigma)\ . \tag{2.7}$$

Proposition 2.2.15. *The variety $V(\Lambda)$ is equivariantly (but non-canonically) isomorphic to the* $\operatorname{Spec}(\widetilde{K}[\widetilde{M}(\Lambda)])$*-toric variety $X_{\Pi(\Lambda)}$ over $\widetilde{K}$ which is given by the fan*

$$\Pi(\Lambda) = \{\angle(\Lambda, \Lambda') + \mathbb{L}_\Lambda \mid \Lambda' \in \Pi,\ \Lambda' \supseteq \Lambda\} \tag{2.8}$$

in $\widetilde{N}(\Lambda)_\mathbb{R} = N(\Lambda)_\mathbb{R} = N_\mathbb{R}/\mathbb{L}_\Lambda$.

Proof. This is [Gub13, Proposition 7.15]. □

2.2.16. In particular, there is a bijection between vertices of Π and the irreducible components of the special fiber $\widetilde{\mathscr{X}_\Pi}$. For each $v \in \Pi^0$, the associated component $V(v)$ is a toric variety over $\widetilde{K}$ with torus associated to the character lattice $\{m \in M \mid \langle m, v\rangle \in \Gamma\}$ and given by the fan $\Pi(v) = \{\mathbb{R}_{\geq 0}(\Lambda' - v) \mid \Lambda' \in \Pi, \Lambda' \ni v\}$ in $N_\mathbb{R}$.

2.3. $\mathbb{T}_S$-Cartier Divisors on Toric Schemes

We extend the theory of $\mathbb{T}$-Cartier divisors to toric schemes over a valuation ring of rank one. This generalizes [KKMS73, § IV.3] and [BPS14a, § 3.6] where the case of discrete valuation is handled and which we use as a guideline.

We keep the notations of the previous section. Furthermore, we only consider Γ-rational polyhedral complexes which satisfy the following condition:

(F') The value group Γ is discrete or, for each $\Lambda \in \Pi$, the vertices of Λ are contained in N_Γ.

This ensures that the regarded schemes are of finite type over K° and hence, they are models of their generic fiber in the sense of Definition 1.2.7. In principle we could work without this assumption. But it is no restriction because we can always perform base change to the algebraic closure of K. Then the value group of this algebraically closed field is divisible and the second condition of (F') is always satisfied.

Definition 2.3.1. A $\mathbb{T}_S$*-Cartier divisor* on a $\mathbb{T}_S$-toric scheme $\mathscr{X}$ is a Cartier divisor D on $\mathscr{X}$ which is invariant under the action of $\mathbb{T}_S$ on $\mathscr{X}$, i. e. we have $\mu^*D = p_2^*D$ denoting by $\mu\colon \mathbb{T}_S \times \mathscr{X} \to \mathscr{X}$ the toric action and by $p_2\colon \mathbb{T}_S \times \mathscr{X} \to \mathscr{X}$ the second projection.

For simplicity we only study $\mathbb{T}_S$-Cartier divisors on proper schemes.

2.3.2. Let Π be a complete Γ-rational polyhedral complex in $N_\mathbb{R}$ satisfying (F') and $\mathscr{X}_\Pi$ the associated proper $\mathbb{T}_S$-toric scheme. Let ψ be a Γ-lattice function on Π given by defining vectors $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi}$ in $M \times \Gamma$ (see A.11). These vectors have to satisfy the condition

$$(\langle m_\Lambda, \cdot\rangle + l_\Lambda)\,|_{\Lambda\cap\Lambda'} = (\langle m_{\Lambda'}, \cdot\rangle + l_{\Lambda'})\,|_{\Lambda\cap\Lambda'} \quad \text{for all } \Lambda, \Lambda' \in \Pi. \tag{2.9}$$

On each open subset $\mathscr{U}_\Lambda$, the vector (m_Λ, l_Λ) determines a rational function $\alpha_\Lambda^{-1}\chi^{-m_\Lambda}$, where $\alpha_\Lambda \in K^\times$ is any element with $\operatorname{val}(\alpha_\Lambda) = l_\Lambda$. For $\Lambda, \Lambda' \in \Pi$, condition (2.9) implies that

$$\mathrm{val}(\alpha_{\Lambda'}/\alpha_\Lambda) + \langle m_{\Lambda'} - m_\Lambda, u\rangle = 0 \quad \text{for all } u \in \Lambda \cap \Lambda',$$

and therefore, $\alpha_{\Lambda'}\chi^{m_{\Lambda'}}/\alpha_\Lambda\chi^{m_\Lambda}$ is regular and non-vanishing on $\mathscr{U}_\Lambda \cap \mathscr{U}_{\Lambda'} = \mathscr{U}_{\Lambda\cap\Lambda'}$. Since Π is complete, the set $\{\mathscr{U}_\Lambda\}_{\Lambda\in\Pi}$ is an open cover of $\mathscr{X}_\Pi$. Thus, ψ defines a Cartier divisor

$$D_\psi = \left\{\left(\mathscr{U}_\Lambda, \alpha_\Lambda^{-1}\chi^{-m_\Lambda}\right)\right\}_{\Lambda\in\Pi}, \tag{2.10}$$

where $\alpha_\Lambda \in K^\times$ is any element with $\mathrm{val}(\alpha_\Lambda) = l_\Lambda$. The divisor D_ψ only depends on ψ and not on the particular choice of defining vectors and elements α_Λ. It is easy to see that D_ψ is $\mathbb{T}_S$-invariant.

We can classify $\mathbb{T}_S$-Cartier divisors in terms of Γ-lattice functions:

Theorem 2.3.3. *Let Π be a complete Γ-rational polyhedral complex in $N_\mathbb{R}$ satisfying (F') and let $\mathscr{X}_\Pi$ be the corresponding proper $\mathbb{T}_S$-toric scheme.*

(i) The assignment $\psi \mapsto D_\psi$ is an isomorphism between the group of Γ-lattice functions on Π and the group of $\mathbb{T}_S$-Cartier divisors on $\mathscr{X}_\Pi$.

(ii) The divisors D_{ψ_1} and D_{ψ_2} are rationally equivalent if and only if $\psi_1 - \psi_2$ is affine.

For the proof, we need the following helpful lemma.

Lemma 2.3.4. *Let $\Lambda \in \Pi$. Then, for each $\mathbb{T}_S$-Cartier divisor D on $\mathscr{U}_\Lambda$, we have*

$$D = \mathrm{div}(\alpha\chi^m)$$

for some $m \in M$ and $\alpha \in K^\times$.

Proof. Let us consider the K°-algebra $A := \mathcal{O}_{\mathscr{U}_\Lambda}(\mathscr{U}_\Lambda) = K[M]^\Lambda$ and the fractional ideal $I := \Gamma(\mathscr{U}_\Lambda, \mathcal{O}_{\mathscr{U}_\Lambda}(-D))$ of A. Since D is $\mathbb{T}_S$-invariant, the K°-module I is graded by M, i. e. we can write $I = \bigoplus_{m\in M} I_m$, where I_m is a K°-submodule contained in $K\chi^m$. Because K° is a valuation ring of rank one, either $I_m = (0)$ or $I_m = K^{\circ\circ}\alpha_m\chi^m$ or $I_m = K^\circ\alpha_m\chi^m$ or $I_m = K\chi^m$ for some $m \in M$, $\alpha_m \in K^\times$. Since I is finitely generated as an A-module, we deduce

$$I = \bigoplus_{\alpha_m\chi^m\in I} K^\circ\alpha_m\chi^m\ . \tag{2.11}$$

Now we fix a point $p \in O(\Lambda)$. Then D is principal on an open neighborhood U of p in $\mathscr{U}_\Lambda$. We may assume that $U = \mathrm{Spec}(A_h)$ for some $h \in A$ with $h(p) \neq 0$. Hence, $D|_U = \mathrm{div}(f)|_U$ for some $f \in K(M)^\times = \mathrm{Quot}(A)^\times$. This implies

$$I_h = \mathcal{O}_{\mathscr{U}_\Lambda}(-D)(U) = f \cdot \mathcal{O}_{\mathscr{U}_\Lambda}(U) = f \cdot A_h\ .$$

In particular, $f \in I_h$, and by (2.11), we can write

$$f = \sum_i \frac{c_i}{h^k}\alpha_{m_i}\chi^{m_i} \quad \text{with } c_i \in K^\circ \setminus \{0\}, k \in \mathbb{N}_0.$$

Since $\alpha_{m_i}\chi^{m_i}/f \in \mathcal{O}_{\mathscr{U}_\Lambda}(U)$ and $p \in U$, we deduce $(\alpha_{m_i}\chi^{m_i}/f)\,(p) \neq 0$ for some i. There exists an open neighborhood $W \subseteq U$ of p on which $\alpha_{m_i}\chi^{m_i}/f$ is non-vanishing and thus,

$$\operatorname{div}(\alpha_{m_i}\chi^{m_i})|_W = \operatorname{div}(f)|_W = D|_W. \tag{2.12}$$

By [GS13, Corollary 2.12 (c)], we have an injective homomorphism $D \mapsto \operatorname{cyc}(D)$ from the group of Cartier divisors on $\mathscr{U}_\Lambda$ to the group of Weil divisors on $\mathscr{U}_\Lambda$, which restricts to a homomorphism of the corresponding groups of $\mathbb{T}_S$-invariant divisors. The $\mathbb{T}_S$-invariant prime (Weil) divisors are exactly the $\mathbb{T}_S$-orbit closures of codimension one. By (2.7),

$$p \in O(\Lambda) \subseteq \bigcap_{\substack{v\in\Pi^0,\\ v\preceq\Lambda}} V(v) \cap \bigcap_{\substack{\tau\in\operatorname{rec}(\Pi)^1,\\ \tau\preceq\operatorname{rec}(\Lambda)}} \mathscr{V}(\tau),$$

and therefore, W meets each $\mathbb{T}_S$-invariant prime divisor of $\mathscr{U}_\Lambda$. Thus, equation (2.12) implies $\operatorname{cyc}(D) = \operatorname{cyc}(\operatorname{div}(\alpha_{m_i}\chi^{m_i}))$ and hence, $D = \operatorname{div}(\alpha_{m_i}\chi^{m_i})$. □

Proof of Theorem 2.3.3. (i) Let ψ be a Γ-lattice function on Π given by defining vectors $\{(m_\Lambda, \operatorname{val}(\alpha_\Lambda))\}_{\Lambda\in\Pi}$. Then, by the construction in 2.3.2, D_ψ is a well-defined $\mathbb{T}_S$-Cartier divisor on $\mathscr{X}_\Pi$. It is easy to see that this assignment defines a group homomorphism.

To prove injectivity, we assume that ψ maps to the zero divisor $(\mathscr{X}_\Pi, 1)$. Then, for each $\Lambda \in \Pi$, the function $\alpha_\Lambda^{-1}\chi^{-m_\Lambda}$ is invertible on $\mathscr{U}_\Lambda$ or equivalently,

$$\psi(u) = \langle m_\Lambda, u\rangle + \operatorname{val}(\alpha_\Lambda) = 0 \quad \text{for all } u \in \Lambda.$$

Therefore, ψ is identically zero and we proved injectivity.

For surjectivity, let D be an arbitrary $\mathbb{T}_S$-Cartier divisor on $\mathscr{X}_\Pi$. By Lemma 2.3.4, there exist, for each $\Lambda \in \Pi$, elements $\alpha_\Lambda \in K^\times$ and $m_\Lambda \in M$, such that $D|_{\mathscr{U}_\Lambda} = \operatorname{div}(\alpha_\Lambda\,\chi^{m_\Lambda})|_{\mathscr{U}_\Lambda}$. Since D is a Cartier divisor, we have, for $\Lambda, \Lambda' \in \Pi$,

$$\operatorname{div}(\alpha_\Lambda\,\chi^{m_\Lambda})|_{\mathscr{U}_{\Lambda\cap\Lambda'}} = \operatorname{div}(\alpha_{\Lambda'}\,\chi^{m_{\Lambda'}})|_{\mathscr{U}_{\Lambda'\cap\Lambda}}\ ,$$

which implies that

$$\operatorname{val}(\alpha_\Lambda) + \langle m_\Lambda, u\rangle = \operatorname{val}(\alpha_{\Lambda'}) + \langle m_{\Lambda'}, u\rangle \quad \text{for all } u \in \Lambda\cap\Lambda'. \tag{2.13}$$

For each $\Lambda \in \Pi$, we set $\psi(u) := \langle -m_\Lambda, u\rangle - \operatorname{val}(\alpha_\Lambda)$ for all $u \in \Lambda$. By (2.13), this determines a well-defined Γ-lattice function $\psi\colon N_\mathbb{R} \to \mathbb{R}$ and, by (2.10), ψ maps to D.

(ii) We claim that a $\mathbb{T}_S$-Cartier divisor on $\mathscr{X}_\Pi$ is principal if and only if it has the form $\operatorname{div}(\alpha\chi^m)$ for $\alpha \in K^\times, m \in M$. Indeed, let D be any principal $\mathbb{T}_S$-Cartier divisor on $\mathscr{X}_\Pi$, i. e. $D = \operatorname{div}(f)$ for some $f \in K(\mathscr{X}_\Pi)^\times$. The support of D is disjoint from the torus $\mathbb{T}$. Therefore, when regarded as an element of $K(\mathbb{T})^\times$, f has zero divisor on $\mathbb{T}$. This implies $f \in K[M]^\times$ and thus, $f = \alpha\chi^m$ for some $\alpha \in K^\times$ and $m \in M$.

Using this equivalence, statement (ii) follows easily from (i). □

2.3.5. Let $\mathscr{X}$ be a toric scheme over S. A *toric line bundle* on $\mathscr{X}$ is a pair $(\mathscr{L}, z)$ consisting of a line bundle $\mathscr{L}$ on $\mathscr{X}$ and a non-zero element z in the fiber $\mathscr{L}_{x_0}$ of the unit point $x_0 \in \mathscr{X}_\eta$. A *toric section* is a meromorphic section s of a toric line bundle which is regular and non-vanishing on the torus $\mathbb{T} \subseteq \mathscr{X}_\eta$ and such that $s(x_0) = z$.

As in 2.1.12, each $\mathbb{T}$-Cartier divisor D on $\mathscr{X}$ defines a toric line bundle $(\mathcal{O}(D), s_D(x_0))$ with toric section s_D as well as each Γ-lattice function ψ defines a toric line bundle with toric section $((\mathcal{O}(D_\psi), s_{D_\psi}(x_0)), s_{D_\psi})$, which we simply denote by $(\mathscr{L}_\psi, s_\psi)$.

Let (X_Σ, D_Ψ) be a proper toric variety with a $\mathbb{T}$-Cartier divisor. A *toric model* of (X_Σ, D_Ψ) is a triple $(\mathscr{X}, D, e)$ consisting of a $\mathbb{T}_S$-toric model $\mathscr{X}$ of X_Σ, a $\mathbb{T}_S$-Cartier divisor D on $\mathscr{X}$ and an integer $e > 0$ such that $D|_{X_\Sigma} = eD_\Psi$.

Clearly, every toric model $(\mathscr{X}, D, e)$ of (X_Σ, D_Ψ) induces an algebraic model $(\mathscr{X}, \mathcal{O}(D), e)$ of (X_Σ, L_Ψ) in the sense of Definition 1.2.7 such that the toric section $s_D|_{X_\Sigma}$ of $\mathcal{O}(D)|_{X_\Sigma}$ is identified with the toric section $s_\Psi^{\otimes e}$ of $L_\Psi^{\otimes e}$. Such algebraic models are called *toric models*.

Theorem 2.3.6. *Let Σ be a complete fan in $N_\mathbb{R}$ and Ψ a virtual support function on Σ. Then the assignment $(\Pi, \psi) \mapsto (\mathscr{X}_\Pi, D_\psi)$ gives a bijection between the sets of*

(i) pairs (Π, ψ), where Π is a complete Γ-rational polyhedral complex in $N_\mathbb{R}$ satisfying (F') and $\mathrm{rec}(\Pi) = \Sigma$, and ψ is a Γ-lattice function on Π with $\mathrm{rec}(\psi) = \Psi$;

(ii) isomorphism classes of toric models $(\mathscr{X}, D, 1)$ of (X_Σ, D_Ψ).

Proof. Let (Π, ψ) be a pair as in (i) and let $\{(m_\Lambda, \mathrm{val}(\alpha_\Lambda))\}_{\Lambda\in\Pi}$ be defining vectors of ψ. Then

$$D_\psi|_{X_\Sigma} = \{(\mathscr{U}_\Lambda, \alpha_\Lambda^{-1}\chi^{-m_\Lambda})\}|_{X_{\mathrm{rec}(\Pi)}} = \{(U_{\mathrm{rec}(\Lambda)}, \chi^{-m_\Lambda})\} = D_{\mathrm{rec}(\psi)} = D_\Psi \ .$$

Hence, $(\mathscr{X}_\Pi, D_\psi, 1)$ is a toric model of (X_Σ, D_Ψ). The statement follows from Corollary 2.2.12 and Theorem 2.3.3. □

Now we describe the restriction of $\mathbb{T}_S$-Cartier divisors to closures of orbits. But we are only interested in the case of orbits lying in the special fiber. The other case can be handled analogously to [BPS14a, Proposition 3.6.12].

Let Π be a complete Γ-rational polyhedral complex in $N_\mathbb{R}$ satisfying (F') and $\mathscr{X}_\Pi$ the associated proper $\mathbb{T}_S$-toric scheme. Let ψ be a Γ-lattice function on Π given by defining vectors $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi}$ and let D_ψ be the associated $\mathbb{T}_S$-Cartier divisor.

Let $\Lambda \in \Pi$ be a polyhedron. We assume that $\psi|_\Lambda = 0$. Using Notation 2.2.13 and (2.8), we define a virtual support function $\psi(\Lambda)$ on the rational fan $\Pi(\Lambda)$ in $N(\Lambda)_\mathbb{R}$ given by the following defining vectors $\{m_\sigma\}_{\sigma\in\Pi(\Lambda)}$:

For each cone $\sigma \in \Pi(\Lambda)$, let $\Lambda_\sigma \in \Pi$ be the unique polyhedron with $\Lambda \preceq \Lambda_\sigma$ and $\angle(\Lambda, \Lambda_\sigma) + \mathbb{L}_\Lambda = \sigma$. The condition $\psi|_\Lambda = 0$ implies that $m_{\Lambda_\sigma} \in \mathbb{L}_\Lambda^\perp$ and $\langle m_{\Lambda_\sigma}, u\rangle = -l_\Lambda \in \Gamma$ for all $u \in \Lambda$. Therefore, m_{Λ_σ} lies in $\widetilde{M}(\Lambda)$. We set $m_\sigma := m_{\Lambda_\sigma}$.

Proposition 2.3.7. *Let notation be as above. If $\psi|_\Lambda = 0$, then D_ψ properly intersects the orbit closure $V(\Lambda)$. Moreover, the restriction of D_ψ to $V(\Lambda)$ is the divisor $D_{\psi(\Lambda)}$.*

Proof. The $\mathbb{T}_S$-Cartier divisor D_ψ is given by $\{(\mathscr{U}_\Lambda, \alpha_\Lambda^{-1}\chi^{-m_\Lambda})\}_{\Lambda\in\Pi}$, where $\alpha_\Lambda \in K^\times$ is any element of $K^\times$ with $\mathrm{val}(\alpha_\Lambda) = l_\Lambda$. If $\psi|_\Lambda = 0$, then $\mathrm{val}(\alpha_\Lambda) + \langle m_\Lambda, u\rangle = 0$ for all $u \in \Lambda$. Thus, the local equation $\alpha_\Lambda^{-1}\chi^{-m_\Lambda}$ of D_ψ in $\mathscr{U}_\Lambda$ is a unit in $\mathcal{O}_{\mathscr{X}_\Pi}(\mathscr{U}_\Lambda) = K[M]^\Lambda$. Hence, the orbit $O(\Lambda) \subseteq \mathscr{U}_\Lambda$ does not meet the support of D_ψ and so, $V(\Lambda)$ and D_ψ intersect properly. Furthermore,

$$D_\psi|_{V(\Lambda)} = \left\{\left(\mathscr{U}_{\Lambda_\pi} \cap V(\Lambda), \alpha_{\Lambda_\pi}^{-1}\chi^{-m_{\Lambda_\pi}}|_{\mathscr{U}_{\Lambda_\pi}\cap V(\Lambda)}\right)\right\}_{\pi\in\Pi(\Lambda)} .$$

Using the non-canonical isomorphism $\widetilde{K}[U_\pi] \simeq \widetilde{K}[\mathscr{U}_{\Lambda_\pi} \cap V(\Lambda)]$, we get

$$D_\psi|_{V(\Lambda)} = \{(U_\pi, \chi^{-m_\pi})\}_{\pi\in\Pi(\Lambda)} = D_{\phi(\Lambda)},$$

proving the claim. □

Proposition 2.3.8. *Let* Π *be a complete* Γ*-rational polyhedral complex in* $N_\mathbb{R}$ *and* ψ *a concave* Γ*-lattice function on* Π*. Let* $\Lambda \in \Pi$ *be a* k*-dimensional polyhedron and* $v \in \mathrm{ri}(\Lambda)$*. Then,*

$$\mathrm{mult}(\Lambda)\deg_{D_\psi}(V(\Lambda)) = (n-k)!\,\mathrm{vol}_{M(\Lambda)}(\partial\psi(v)), \tag{2.14}$$

where $\mathrm{mult}(\Lambda)$ *is the multiplicity of* Λ *(see (2.6)) and* $\partial\psi(v)$ *is the sup-differential of* ψ *at* v *(see A.15). Note that the affine space of* $\partial\psi(v)$ *is associated to the linear space* $M(\Lambda)_\mathbb{R}$ *and hence, the measure* $\mathrm{vol}_{M(\Lambda)}$ *is also defined on* $\mathrm{aff}(\partial\psi(v))$ *(see A.16).*

Proof. Let $(m_\Lambda, l_\Lambda) \in M \times \Gamma$ be a defining vector of ψ on Λ. Then D_ψ is rationally equivalent to $D_{\psi-m_\Lambda-l_\Lambda}$ and $\partial(\psi - m_\Lambda - l_\Lambda)(v) = \partial\psi(v) - m_\Lambda$. Thus, replacing ψ by $\psi - m_\Lambda - l_\Lambda$ does not change both sides of equation (2.14) and we may assume that $\psi|_\Lambda = 0$.

By Proposition 2.3.7 and (2.3),

$$\deg_{D_\psi}(V(\Lambda)) = \deg_{D_{\psi(\Lambda)}}(X_{\Pi(\Lambda)}) = (n-k)!\,\mathrm{vol}_{\widetilde{M}(\Lambda)}(\Delta_{\psi(\Lambda)}).$$

It is easy to see that $\partial\psi(v) = \partial\psi(\Lambda)(\overline{0}) \subset \widetilde{M}(\Lambda)_\mathbb{R}$. So we deduce from Proposition A.19,

$$\mathrm{vol}_{\widetilde{M}(\Lambda)}(\Delta_{\psi(\Lambda)}) = \mathrm{vol}_{\widetilde{M}(\Lambda)}\left(\partial\psi(\Lambda)(\overline{0})\right) = \mathrm{vol}_{\widetilde{M}(\Lambda)}(\partial\psi(v)) = \frac{\mathrm{vol}_{M(\Lambda)}(\partial\psi(v))}{[M(\Lambda):\widetilde{M}(\Lambda)]},$$

proving the result. □

2.4. Toric Metrics

In this section, we recall the basic facts about toric metrics from [BPS14a, § 4.3]. These are metrics on a toric line bundle that satisfy a certain invariance property with respect to the torus action, and they can be classified by a certain class of continuous functions on $N_\mathbb{R}$. Note that in [BPS14a, § 4.1–4.3] the non-Archimedean fields are not assumed to be discrete, in contrast to the rest of this chapter § 4.

We fix the following notation. Let K be either $\mathbb{C}$ or an algebraically closed field which is complete with respect to a non-trivial non-archimedean absolute value $|\cdot|$. Then we have a valuation $\mathrm{val} := -\log|\cdot|$ and a divisible value group $\Gamma := \mathrm{val}(K^\times)$ of rank one. The theory could be developed for arbitrary non-Archimedean fields, but it is no serious restriction to assume that K is algebraically closed because this theory is stable under base change and in the classical setting, the analysis is also done over $\mathbb{C}$.

We fix a free Abelian group M of rank n with dual N and denote by $\mathbb{T} = \mathrm{Spec}(K[M])$ the n-dimensional split torus over K. Let Σ be a complete fan in $N_\mathbb{R}$ and X_Σ the corresponding proper toric variety. Furthermore, let Ψ be a virtual support function on Σ and (L, s) the associated toric line bundle with toric section.

If $K = \mathbb{C}$, then $X_\Sigma^{\mathrm{an}} = X_\Sigma(\mathbb{C})$ is the associated complex analytic space with complex torus $\mathbb{T}^{\mathrm{an}} \simeq (\mathbb{C}^\times)^n$. If K is non-Archimedean, then X_Σ^{an} is the Berkovich analytic space associated to X_Σ as defined in 1.1.2. In both cases, the algebraic line bundle L defines an analytic line bundle L^{an} on X_Σ^{an}.

Definition 2.4.1. A metric $\|\cdot\|$ on L is called *toric* if, for all $p, q \in \mathbb{T}^{\mathrm{an}}$ satisfying $|\chi^m(p)| = |\chi^m(q)|$ for each $m \in M$, we have $\|s(p)\| = \|s(q)\|$.

It easily follows from 2.1.13 that this definition is independent of the choice of the toric section s.

Remark 2.4.2. In [BPS14a, 4.2], the authors study the action of the analytic group $\mathbb{T}^{\mathrm{an}}$ on X_Σ^{an} and in particular, the action of the compact analytic subgroup

$$\mathbb{S} = \{p \in \mathbb{T}^{\mathrm{an}} \mid |\chi^m(p)| = 1 \text{ for all } m \in M\},$$

called *compact torus*. By [BPS14a, (4.2.1) and Proposition 4.2.15], we have for $p \in \mathbb{T}^{\mathrm{an}}$,

$$\mathbb{S} \cdot p = \{q \in \mathbb{T}^{\mathrm{an}} \mid |\chi^m(p)| = |\chi^m(q)| \text{ for all } m \in M\}.$$

Hence, a metric $\|\cdot\|$ is toric if and only if the function $p \mapsto \|s(p)\|$ is invariant under the action of $\mathbb{S}$.

2.4.3. Given an arbitrary metric $\|\cdot\|$ on L, we can associate to it a toric metric in the following way: For $\sigma \in \Sigma$, let s_σ be a toric section of L which is regular and non-vanishing in U_σ.

If $K = \mathbb{C}$, then we set, for $p \in U_\sigma^{\mathrm{an}}$,

$$\|s_\sigma(p)\|_{\mathbb{S}} := \exp\left(\int_{\mathbb{S}} \log \|s_\sigma(t \cdot p)\| \, \mathrm{d}\mu_{\mathrm{Haar}}(t)\right),$$

where μ_{Haar} denotes the Haar measure on $\mathbb{S}$ of total mass 1.

If K is non-Archimedean, we set, for $p \in U_\sigma^{\mathrm{an}}$,

$$\|s_\sigma(p)\|_{\mathbb{S}} := \|s_\sigma(\tilde{p})\|,$$

where $\tilde{p} \in U_\sigma^{\mathrm{an}}$ is given by

$$\sum_{m \in M_\sigma} \alpha_m \chi^m \longmapsto \max_m |\alpha_m||\chi^m(p)|.$$

We easily deduce that these assignments define a toric metric $\|\cdot\|_{\mathbb{S}}$ on L. This process is called *torification* of $\|\cdot\|$.

Proposition 2.4.4. *Toric metrics are invariant under torification. Moreover, torification is multiplicative with respect to products of metrized line bundles and continuous with respect to uniform convergence of metrics.*

Proof. This is established in [BPS14a, Proposition 4.3.4] and follows easily from the definition. □

2.4.5. We have the *tropicalization map* trop: $\mathbb{T}^{\mathrm{an}} \to N_{\mathbb{R}}$, $p \mapsto \mathrm{trop}(p)$, where $\mathrm{trop}(p)$ is the element of $N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R})$ given by

$$\langle m, \mathrm{trop}(p) \rangle := -\log |\chi^m(p)|.$$

This defines a proper surjective continuous map. For details, we refer to [Pay09, § 3].

Let $\|\cdot\|$ be a toric metric on L. Then consider the following diagram

$$\begin{array}{ccc} \mathbb{T}^{\mathrm{an}} & \xrightarrow{\log\|s(\cdot)\|} & \mathbb{R} \\ & {}_{\mathrm{trop}}\searrow \quad \nearrow & \\ & N_{\mathbb{R}} & \end{array} .$$

Since $\|\cdot\|$ is toric, the function $\log \|s(\cdot)\|$ is constant along the fibers of trop. Moreover, trop is surjective and closed, and hence, there exists a unique continuous function on $N_{\mathbb{R}}$ making the above diagram commutative. This causes the following definition.

Definition 2.4.6. Let $\overline{L} = (L, \|\cdot\|)$ be a metrized toric line bundle on X_Σ and s a toric section of L. We define the function

$$\psi_{\overline{L},s} \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad u \longmapsto \log \|s(p)\|_{\mathbb{S}},$$

where $p \in \mathbb{T}^{\mathrm{an}}$ is any element with $\mathrm{trop}(p) = u$. The line bundle and the toric section are usually clear from the context and we alternatively denote this function by $\psi_{\|\cdot\|}$.

2.4.7. For an alternative description of $\psi_{\overline{L},s}$ in the non-Archimedean case, we consider the canonical section $\rho \colon N_{\mathbb{R}} \to \mathbb{T}^{\mathrm{an}}$ which is given, for each $u \in N_{\mathbb{R}}$, by the multiplicative norm

$$\rho(u) \colon K[M] \longrightarrow \mathbb{R}_{\geq 0}, \quad \sum_{m \in M} \alpha_m \chi^m \longmapsto \max_{m \in M} |\alpha_m| \exp(-\langle m, u \rangle).$$

By [Ber90, Example 5.2.12], we deduce that this section is a homeomorphism of $N_{\mathbb{R}}$ onto a closed subset of $\mathbb{T}^{\mathrm{an}}$. It is easy to see that $\psi_{\overline{L},s}(u) = \log \|s(\rho(u))\|$ for all $u \in N_{\mathbb{R}}$.

Proposition 2.4.8. *Let notation be as in Definition 2.4.6 and let K' be a complete valued field extension of K. Let $(\overline{L}', s')$ be the metrized toric line bundle with toric section obtained by base change to K'. Then*

$$\psi_{\overline{L}',s'} = \psi_{\overline{L},s}\,.$$

Proof. This follows from the definition of $\psi_{\overline{L},s}$ and propositions 4.1.5 and 4.2.16 in [BPS14a]. □

Proposition 2.4.9. *Let $\overline{L} = (L, \|\cdot\|)$ and $\overline{L}'$ be metrized toric line bundles on X_Σ with toric sections s and s', respectively. Let $\varphi \colon X_{\Sigma'} \to X_\Sigma$ be a toric morphism with corresponding linear map H as in 2.1.7. Then*

$$\psi_{\overline{L}\otimes\overline{L}',s\otimes s'} = \psi_{\overline{L},s} + \psi_{\overline{L}',s'}\,, \quad \psi_{\overline{L}^{-1},s^{-1}} = -\psi_{\overline{L}} \quad \textit{and} \quad \psi_{\varphi^*\overline{L},\varphi^*s} = \varphi_{\overline{L},s} \circ H\,.$$

Moreover, if $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is a sequence of metrics on L that converges to $\|\cdot\|$, then $\left(\psi_{\|\cdot\|_n}\right)_{n\in\mathbb{N}}$ converges uniformly to $\psi_{\|\cdot\|}$.

Proof. This is established in propositions 4.3.14 and 4.3.19 in [BPS14a] and follows easily from the definitions. □

2.4.10. In order to characterize toric metrics by functions on $N_{\mathbb{R}}$, we need the *Kajiwara-Payne tropicalization* of X_Σ introduced by [Kaj08] and [Pay09]. This is a topological space N_Σ together with a tropicalization map $X_\Sigma^{\mathrm{an}} \to N_\Sigma$. As a set, N_Σ is a disjoint union of linear spaces

$$N_\Sigma = \coprod_{\sigma\in\Sigma} N(\sigma)_{\mathbb{R}}\,,$$

where $N(\sigma) = N/\langle N\cap\sigma\rangle$ is the quotient lattice as in 2.1.4. Following [Pay09, Remark 3.4], the topology on X_Σ is determined by the following basis. Let σ be a cone in Σ and τ a face of σ. We choose a finite set of generators $m_1,\dots,m_r$ for the semigroup $M_\sigma = M\cap\sigma^\vee$. If $m_i\in\tau^\perp$, then m_i can be evaluated on $N(\tau)_{\mathbb{R}}$. For each open set $U\subseteq N(\sigma)_{\mathbb{R}}$ and real number $\lambda>0$, let $C(U,\lambda)$ be the truncated cylinder

$$C(U,\lambda) = \bigcup_{\tau\preceq\sigma}\{u\in N(\tau)_{\mathbb{R}} \mid \pi(u)\in U \text{ and } \langle m_i,u\rangle>\lambda \text{ for } m_i\in\tau^\perp\setminus\sigma^\perp, i=1,\dots,r\},$$

where $\pi\colon N(\tau)_{\mathbb{R}}\to N(\sigma)_{\mathbb{R}}$ is the canonical projection. Then these truncated cylinders define a basis for the topology on N_Σ. A sequence of points in $N(\tau)_{\mathbb{R}}$ tends to a point $u\in N(\sigma)_{\mathbb{R}}$ if and only if their images under π tend to u in $N(\sigma)_{\mathbb{R}}$ and they move toward infinity in the image of the cone σ in $N(\rho)_{\mathbb{R}}$ for all cones ρ such that $\tau\preceq\rho\npreceq\sigma$.

The toric variety X_Σ is the disjoint union of tori $\mathbb{T}_{N(\sigma)} = \operatorname{Spec} K[M(\sigma)], \sigma\in\Sigma$. Hence, we can define the *tropicalization map*

$$\mathrm{trop}\colon X_\Sigma^{\mathrm{an}} \longrightarrow N_\Sigma$$

as the disjoint union of tropicalization maps $\mathrm{trop}\colon \mathbb{T}_{N(\sigma)}^{\mathrm{an}}\to N(\sigma)_{\mathbb{R}}$ as defined in 2.4.5. This is also a proper surjective continuous map. Especially, $N_\Sigma = \mathrm{trop}(X_\Sigma^{\mathrm{an}})$ is a compact space.

Proposition 2.4.11. *Let Σ be a complete fan in $N_{\mathbb{R}}$ and Ψ a virtual support function on Σ. We set $L = L_\Psi$.*

Then, for any metric $\|\cdot\|$ on L, the function $\psi_{\|\cdot\|}-\Psi$ extends to a continuous function on N_Σ. In particular, the function $|\psi_{\|\cdot\|}-\Psi|$ is bounded.

Moreover, the assignment $\|\cdot\|\mapsto\psi_{\|\cdot\|}$ is a bijection between the sets of

(i) toric metrics on L;

(ii) continuous functions $\psi\colon N_{\mathbb{R}}\to\mathbb{R}$ sucht that $\psi-\Psi$ can be extended to a continuous function on N_Σ.

Proof. This is proved in Proposition 4.3.10 and Corollary 4.3.13 in [BPS14a]. The inverse map is given as follows: Let ψ be a function as in (ii) and let $\{m_\sigma\}$ be a set of defining vectors of Ψ. For each cone $\sigma\in\Sigma$, the section $s_\sigma = \chi^{m_\sigma}s$ is a non-vanishing regular section on U_σ. Then we obtain a toric metric $\|\cdot\|_\psi$ on L characterized by

$$\|s_\sigma(p)\|_\psi := \exp\big((\psi-m_\sigma)(\mathrm{trop}(p))\big) \tag{2.15}$$

on U_σ. □

Definition 2.4.12. Let L be a toric line bundle on X_Σ with toric section s and let Ψ be the associated virtual support function on Σ. By Proposition 2.4.11, the function $\psi := \Psi$ defines a toric metric on L. This metric is called the *canonical metric* of L. We denoted it by $\|\cdot\|_{\mathrm{can}}$ and write $\overline{L}^{\mathrm{can}} = (L, \|\cdot\|_{\mathrm{can}})$.

Remark 2.4.13. By [BPS14a, Proposition 4.3.15], the canonical metric only depends on the structure of toric line bundle of L and not on the choice of s.

Proposition 2.4.14. *Let L, L' be toric line bundles on X_Σ and let $\varphi\colon X'_\Sigma \to X_\Sigma$ be a toric morphism. Let $\sigma \in \Sigma$ and $\iota\colon V(\sigma) \to X_\Sigma$ the closed immersion of 2.1.4. Then*

$$\overline{L \otimes L'}^{\mathrm{can}} = \overline{L}^{\mathrm{can}} \otimes \overline{L'}^{\mathrm{can}}, \quad \overline{L^{-1}}^{\mathrm{can}} = (\overline{L}^{\mathrm{can}})^{-1}, \quad \overline{\varphi^* L}^{\mathrm{can}} = \varphi^* \overline{L}^{\mathrm{can}} \quad \textit{and} \quad \overline{\iota^* L}^{\mathrm{can}} = \iota^* \overline{L}^{\mathrm{can}}.$$

Proof. The first two statements are established in [BPS14a, Proposition 4.3.16]. The last two statements are the corollaries 4.3.20 and 4.3.18 in [BPS14a]. □

2.5. Semipositive Toric Metrics and Measures over Non-Archimedean Fields

In the case of an algebraically closed non-Archimedean field, we study algebraic metrics induced by toric models. Then we classify semipositive toric metrics in terms of concave functions (Theorem 2.5.8). Moreover, we characterize the measures associated to semipositive metrics (Corollary 2.5.11). These results are proved in [BPS14a, § 4.5–4.8] in the case of a discrete non-Archimedean and an Archimedean field. We follow their ideas of the proofs using in particular our theory of $\mathbb{T}_S$-Cartier divisors developed in section 2.3.

In this section, let K be an algebraically closed field which is complete with respect to a non-trivial non-Archimedean absolute value $|\cdot|$. Then we have a valuation $\mathrm{val} := -\log|\cdot|$ and a divisible value group $\Gamma := \mathrm{val}(K^\times) \subseteq \mathbb{R}$. We fix a free Abelian group M of rank n with dual N and denote by $\mathbb{T} = \mathrm{Spec}(K[M])$ the n-dimensional split torus over K.

Let Σ be a complete fan in $N_\mathbb{R}$ and X_Σ the corresponding proper toric variety. Furthermore, let Ψ be a virtual support function on Σ and (L, s) the associated toric line bundle with toric section.

2.5.1. Let Π be a complete Γ-rational polyhedral complex in $N_\mathbb{R}$ with $\mathrm{rec}(\Pi) = \Sigma$, and let ψ be a Γ-rational piecewise affine function on Π with $\mathrm{rec}(\psi) = \Psi$. Let $e > 0$ be an integer such that $e\psi$ is a Γ-lattice function given by the defining vectors $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi}$ in $M \times \Gamma$. Then $e\psi$ defines a $\mathbb{T}_S$-Cartier divisor

$$D_{e\psi} = \left\{ \left(\mathscr{U}_\Lambda, \alpha_\Lambda^{-1} \chi^{-m_\Lambda} \right) \right\}_{\Lambda\in\Pi},$$

where $\alpha_\Lambda \in K^\times$ with $\mathrm{val}(\alpha_\Lambda) = l_\Lambda$, and the pair $(\Pi, e\psi)$ defines a toric model $(\mathscr{X}_\Pi, D_{e\psi}, e)$ of (X_Σ, D_Ψ) (see Theorem 2.3.6). We write $\mathscr{L} = \mathcal{O}(D_{e\psi})$ and $L = \mathcal{O}(D_\Psi)$ for the corresponding toric line bundles. By Definition 1.2.7, the model $(\mathscr{X}_\Pi, \mathscr{L}, e)$ induces an algebraic metric $\|\cdot\|_\mathscr{L}$ on L.

Proposition 2.5.2. *Let notation be as above. Then the metric $\|\cdot\|_\mathscr{L}$ is toric. Moreover, the equalities $\psi_{\|\cdot\|_\mathscr{L}} = \psi$ and $\|\cdot\|_\mathscr{L} = \|\cdot\|_\psi$ hold.*

Proof. Let $\Lambda \in \Pi$. Recall that $\mathscr{U}_\Lambda := \operatorname{Spec}(K[M]^\Lambda)$ is an algebraic K°-model of $U_{\operatorname{rec}(\Lambda)}$. By 1.1.8, the associated formal scheme has generic fiber

$$U^\circ_{\operatorname{rec}(\Lambda)} := \left\{ p \in U^{\mathrm{an}}_{\operatorname{rec}(\Lambda)} \mid p(f) \leq 1 \ \forall f \in K[M]^\Lambda \right\}.$$

Then $\mathscr{U}_\Lambda$ is a trivialization of $\mathscr{L}$ on which $s_\Psi^{\otimes e}$, considered as a meromorphic section of $\mathscr{L}$, corresponds to the rational function $\alpha_\Lambda^{-1}\chi^{-m_\Lambda}$. Hence, by Definition 1.2.7, we have

$$\|s_\Psi(p)\|_{\mathscr{L}} = |\alpha_\Lambda^{-1}\chi^{-m_\Lambda}(p)|^{1/e}$$

for all $p \in U^\circ_{\operatorname{rec}(\Lambda)}$. Let $u \in \Lambda$ and $p \in \mathbb{T}^{\mathrm{an}}$ with $\operatorname{trop}(p) = u$. The below-mentioned Lemma 2.5.3 implies that $p \in U^\circ_{\operatorname{rec}(\Lambda)}$ and we obtain

$$\log \|s_\Psi(p)\|_{\mathscr{L}} = \log |\alpha_\Lambda^{-1}\chi^{-m_\Lambda}(p)|^{1/e} = \frac{1}{e}\left(\langle m_\Lambda, u\rangle + l_\Lambda\right) = \psi(u).$$

This shows that the metric $\|\cdot\|_{\mathscr{L}}$ is toric. We deduce, by Definition 2.4.6, that $\psi_{\|\cdot\|_{\mathscr{L}}} = \psi$ and, by Proposition 2.4.11, that $\|\cdot\|_{\mathscr{L}} = \|\cdot\|_\psi$. ☐

Lemma 2.5.3. *Let* Π *be a complete* Γ*-rational polyhedral complex in* $N_\mathbb{R}$ *with* $\operatorname{rec}(\Pi) = \Sigma$ *and let* $\operatorname{red}\colon X_\Sigma^\circ \to \widetilde{\mathscr{X}_\Pi}$ *be the reduction map from 1.1.8. Let* $\Lambda \in \Pi$ *and* $p \in \mathbb{T}^{\mathrm{an}}$*. Then*

$$\operatorname{trop}(p) \in \Lambda \iff p \in U^\circ_{\operatorname{rec}(\Lambda)} \iff \operatorname{red}(p) \in \widetilde{\mathscr{U}_\Lambda}.$$

Proof. By [Gub13, Lemma 6.21], we have $\operatorname{trop}(p) \in \Lambda$ if and only if $p \in \mathbb{T}^{\mathrm{an}}$ satisfies $|p(f)| \leq 1$ for all $f \in K[M]^\Lambda$ or, in other words, $p \in U^\circ_{\operatorname{rec}(\Lambda)}$. By the description of the reduction map in 1.1.8, this is equivalent to $\operatorname{red}(p) \in \widetilde{\mathscr{U}_\Lambda}$. ☐

Corollary 2.5.4. *Let* ψ *be a* Γ*-rational piecewise affine concave function on* $N_\mathbb{R}$ *with* $\operatorname{rec}(\psi) = \Psi$*. Then the metric* $\|\cdot\|_\psi$ *is induced by a toric model.*

Proof. As in the proof of [BPS14a, Theorem 3.7.3], we can show that there exists a complete Γ-rational polyhedral complex Π in $N_\mathbb{R}$ such that $\operatorname{rec}(\Pi) = \Sigma$ and ψ is piecewise affine on Π. Since Γ is divisible, the complex Π gives a proper toric scheme $\mathscr{X}_\Pi$. Then Proposition 2.5.2 says that $\|\cdot\|_\psi$ is induced by a toric model $(\mathscr{X}_\Pi, D_{e\psi}, e)$ of (X_Σ, D_Ψ). ☐

Proposition 2.5.5. *Let* $\|\cdot\|$ *be an algebraic metric on* L*. Then the function* $\psi_{\|\cdot\|}$ *is* Γ*-rational piecewise affine.*

Proof. There exists a proper K°-model $(\mathscr{X}, \mathscr{L}, e)$ of (X_Σ, L) inducing the metric $\|\cdot\|$. Let $\{\mathscr{U}_i\}_{i\in I}$ be a trivialization of $\mathscr{L}$. Then the subsets $U_i^\circ = \operatorname{red}^{-1}(\mathscr{U}_i \cap \widetilde{\mathscr{X}})$ form a finite closed cover of X_Σ^{an}. On $\mathscr{U}_i$ the meromorphic section $s^{\otimes e}$ corresponds to a rational function $\lambda_i \in K(M)^\times$ such that on U_i° we have

$$\|s(p)\| = |\lambda_i(p)|^{1/e}.$$

We write $\lambda_i = \frac{\sum_{m\in M}\alpha_m\chi^m}{\sum_{m\in M}\beta_m\chi^m}$. Using the continuous map $\rho\colon N_\mathbb{R} \to \mathbb{T}^{\mathrm{an}}$ from 2.4.7, we have on the closed subset $\Lambda_i := \rho^{-1}(U_i^\circ \cap \mathbb{T}^{\mathrm{an}}) \subseteq N_{\mathbb{R}}$,

$$\begin{aligned}\psi_{\|\cdot\|}(u) &= \log \|s(\rho(u))\| \\ &= \log |\lambda_i(\rho(u))|^{1/e} \\ &= \frac{1}{e}\log\left(\max_{m\in M} |\alpha_m| \exp(-\langle m,u\rangle)\right) - \frac{1}{e}\log\left(\max_{m\in M} |\beta_m| \exp(-\langle m,u\rangle)\right) \\ &= \frac{1}{e}\min_{m\in M}\left(\langle m,u\rangle + \mathrm{val}(\beta_m)\right) - \frac{1}{e}\min_{m\in M}\left(\langle m,u\rangle + \mathrm{val}(\alpha_m)\right).\end{aligned}$$

We see that $\psi_{\|\cdot\|}|_{\Lambda_i}$ is the difference of two Γ-rational piecewise affine concave functions. Since $\{\Lambda_i\}_{i\in I}$ is a finite closed cover of $N_{\mathbb{R}}$, we deduce that $\psi_{\|\cdot\|}$ is Γ-rational piecewise affine. □

Next we study semipositive toric metrics on L.

Proposition 2.5.6. *Let $\|\cdot\|$ be an algebraic metric on L.*

(i) If $\|\cdot\|$ is semipositive, then $\psi_{\|\cdot\|}$ is concave.

(ii) We assume that $\|\cdot\|$ is toric. Then $\|\cdot\|$ is semipositive if and only if $\psi_{\|\cdot\|}$ is concave.

Proof. (ii) Because each algebraic metric is $\mathbb{Q}$-formal (see Proposition 1.2.8), this follows from [GK15, Corollary 8.12].

(i) For $\|\cdot\|$ semipositive, we have to show that $\psi_{\|\cdot\|}$ is concave along any affine line. By a density argument, we may assume that the line is Γ-rational. Similarly as in the proof of [BPS14a, Proposition 4.7.1], we use pullback with respect to a suitable equivariant morphism to reduce the concavity on the affine line to the case of $\mathbb{P}^1_K$. By [GH15, Corollary B.18], the torification of a semipositive algebraic metric on $\mathbb{P}^1_K$ is semipositive. Hence, the claim follows from (ii). □

Corollary 2.5.7. *Let $\|\cdot\|$ be a semipositive algebraic metric on L. Then the toric metric $\|\cdot\|_{\mathbb{S}}$ is also algebraic and semipositive.*

Proof. By the propositions 2.5.6 (i), 2.5.5 and 2.4.11, the function $\psi = \psi_{\|\cdot\|}$ is concave Γ-rational piecewise affine with $\mathrm{rec}(\psi) = \Psi$. Then Corollary 2.5.4 says that the metric $\|\cdot\|_{\mathbb{S}} = \|\cdot\|_\psi$ is algebraic and Proposition 2.5.6 (ii) implies that it is semipositive. □

Now, we can characterize semipositive toric metrics.

Theorem 2.5.8. *Let Ψ be a virtual support function on the complete fan Σ in $N_{\mathbb{R}}$ and set $L = L_\Psi$. Then there is a bijection between the sets of*

(i) semipositive toric metrics on L;

(ii) concave functions ψ on $N_{\mathbb{R}}$ such that the function $|\psi - \Psi|$ is bounded;

(iii) continuous concave functions on Δ_Ψ.

The bijections are given by $\|\cdot\| \mapsto \psi_{\|\cdot\|} \mapsto \psi^\vee_{\|\cdot\|}$.

Proof. The bijection between (ii) and (iii) follows from Proposition A.9. To prove the bijection between (i) and (ii), let $\|\cdot\|$ be a semipositive toric metric on L. By Proposition 2.4.11, the function $|\psi_{\|\cdot\|} - \Psi|$ is bounded. Furthermore, there exists a sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of semipositive algebraic metrics converging to the toric metric $\|\cdot\|$. Proposition 2.5.6 (i) says that the functions $\psi_{\|\cdot\|_n}$ are concave. By Proposition 2.4.9, the sequence $(\psi_{\|\cdot\|_n})_{n\in\mathbb{N}}$ converges uniformly to $\psi_{\|\cdot\|}$ and hence, the latter is also concave.

Conversely, let ψ be a concave function on $N_\mathbb{R}$ such that $|\psi - \Psi|$ is bounded. Then Ψ is also concave and, by Proposition A.14, there is a sequence of Γ-rational piecewise affine concave functions $(\psi_k)_{k\in\mathbb{N}}$ with $\operatorname{rec}(\psi_k) = \Psi$, that uniformly converges to ψ. Because ψ_k is a piecewise affine concave function with $\operatorname{rec}(\psi_k) = \Psi$, the function $\psi_k - \Psi$ continuously extends on N_Σ. Thus, $\psi - \Psi$ extends to a continuous function on N_Σ, too. By Proposition 2.4.11, we obtain toric metrics $\|\cdot\|_\psi$ and $\|\cdot\|_{\psi_k}, k \in \mathbb{N}$, given as in (2.15). Then the sequence of metrics $(\|\cdot\|_{\psi_k})_{k\in\mathbb{N}}$ converges to $\|\cdot\|_\psi$. By Proposition 2.5.2, the metric $\|\cdot\|_{\psi_k}$ is algebraic and therefore, by Proposition 2.5.6 (ii), semipositive. Thus, the metric $\|\cdot\|_\psi$ is also semipositive. □

Remark 2.5.9. Theorem 2.5.8 also holds in the Archimedean setting of the sections 1.5 and 2.4. This is proved side by side to the discrete non-Archimedean case in [BPS14a, Theorem 4.8.1].

We characterize Chambert-Loir's measure associated to a semipositive toric metrized line bundle. Let $\psi\colon N_\mathbb{R} \to \mathbb{R}$ be a concave function. We extend the Monge-Ampère measure $\mathcal{M}_M(\psi)$ on $N_\mathbb{R}$ (Definition A.17) to a measure $\overline{\mathcal{M}}_M(\psi)$ on N_Σ by setting

$$\overline{\mathcal{M}}_M(\psi)(E) = \mathcal{M}_M(\psi)\,(E \cap N_\mathbb{R})$$

for any Borel subset E of N_Σ.

Theorem 2.5.10. *Let $\|\cdot\|$ be a semipositive algebraic toric metric on L and $\psi = \psi_{\|\cdot\|}$ the associated function on $N_\mathbb{R}$. Then*

$$\operatorname{trop}_*\left(\mathrm{c}_1\left(L, \|\cdot\|\right)^n\right) = n!\,\overline{\mathcal{M}}_M(\psi).$$

Proof. By the propositions 2.5.5, 2.5.6 (i) and 2.4.11, the function ψ is Γ-rational piecewise affine concave with $\operatorname{rec}(\psi) = \Psi$. Then Corollary 2.5.4 implies that the metric $\|\cdot\|_\psi$ is defined by a toric K°-model $(\mathscr{X}_\Pi, D_{e\psi}, e)$ of (X_Σ, L).

By 2.2.16, the vertices of Π correspond bijectively to the irreducible components of the special fiber $\widetilde{\mathscr{X}_\Pi}$. Since the valuation of K is not discrete, the special fiber $\widetilde{\mathscr{X}_\Pi}$ is reduced (see 2.2.9). For each $v \in \Pi^0$, let $V(v)$ be the corresponding component and ξ_v the unique point of $(X_\Sigma)^{\mathrm{an}}$ such that $\operatorname{red}(\xi_v)$ is the generic point of $V(v)$ (see 1.1.8). Then, by Definition 1.2.14,

$$\mathrm{c}_1(\overline{L})^n = \frac{1}{e^n}\sum_{v\in\Pi^0} \deg_{D_{e\psi}}(V(v))\,\delta_{\xi_v}\,.$$

Since $\operatorname{red}(\xi_v)$ is the generic point of the n-dimensional irreducible component $V(v)$, it is clear that $\xi_v \in \mathbb{T}^{\mathrm{an}}$. We have $\operatorname{red}(\xi_v) \in V(v) = \widetilde{\mathscr{U}_v}$ and hence, by Lemma 2.5.3, $\operatorname{trop}(\xi_v) = v$. Therefore,

$$\operatorname{trop}_*(\mathrm{c}_1(\overline{L})^n) = \frac{1}{e^n}\sum_{v\in\Pi^0} \deg_{D_{e\psi}}(V(v))\,\delta_v\,.$$

On the other hand, by Proposition A.19 and Proposition 2.3.8,

$$\begin{aligned}\mathcal{M}_M(\psi) &= \frac{1}{e^n}\mathcal{M}_M(e\psi)\\ &= \frac{1}{e^n}\sum_{v\in\Pi^0}\operatorname{vol}_M(\partial(e\psi)(v))\,\delta_v\\ &= \frac{1}{n!\,e^n}\sum_{v\in\Pi^0}\operatorname{mult}(v)\deg_{D_{e\psi}}(V(v))\,\delta_v\,.\end{aligned}$$

Since the value group of K is divisible, the multiplicity $\operatorname{mult}(v)$ of a vertex v is one. The statement follows from the definition of $\overline{\mathcal{M}}_M(\psi)$. □

Corollary 2.5.11. *Let $\|\cdot\|$ be a semipositive toric metric on L and $\psi = \psi_{\|\cdot\|}$ the associated concave function on $N_\mathbb{R}$. Then*

$$\operatorname{trop}_*\left(\operatorname{c}_1\left(L,\|\cdot\|\right)^n\right) = n!\,\overline{\mathcal{M}}_M(\psi).$$

Proof. Let $(\|\cdot\|_k)_{k\in\mathbb{N}}$ be a sequence of semipositive algebraic metrics that converges to $\|\cdot\|$. Taking the torifications and using Proposition 2.4.4 and Corollary 2.5.7, we may assume that the $\|\cdot\|_k$, $k\in\mathbb{N}$, are also toric. By Proposition 1.3.7, the measures $\operatorname{trop}_*\left(\operatorname{c}_1(L,\|\cdot\|_k)^n\right)$ converge weakly to $\operatorname{trop}_*\left(\operatorname{c}_1(L,\|\cdot\|)^n\right)$ on N_Σ.

By Proposition 2.4.9, the functions $\psi_{\|\cdot\|_k}$ converge uniformly to ψ. Thus, by Proposition A.18, the measures $\mathcal{M}_M(\psi_k)$ converge weakly to $\mathcal{M}_M(\psi)$. Theorem 2.5.10 implies that

$$\operatorname{trop}_*\left(\operatorname{c}_1(L,\|\cdot\|)\right)|_{N_\mathbb{R}} = n!\,\mathcal{M}_M(\psi). \tag{2.16}$$

By Corollary 1.4.5, the set $X_\Sigma^{\mathrm{an}}\setminus\mathbb{T}^{\mathrm{an}}$ has measure zero with respect to $\operatorname{c}_1(L,\|\cdot\|)^n$ and so, $N_\Sigma\setminus N_\mathbb{R}$ has measure zero with respect to $\operatorname{trop}_*(\operatorname{c}_1(L,\|\cdot\|)^n)$. Since the $\overline{\mathcal{M}}_M(\psi)$-measure of $N_\Sigma\setminus N_\mathbb{R}$ is also zero, the statement follows from equation (2.16). □

At the end of this section, we quote a result about the restriction of semipositive metrics to toric orbits which will be useful in the proof of the local height formula. Let Ψ be a support function on Σ and (L,s) the associated toric line bundle with toric section. Let σ be a cone of Σ and $V(\sigma)$ the corresponding orbit closure with the structure of a toric variety (cf. 2.1.4). We denote by $\iota\colon V(\sigma)\to X_\Sigma$ the closed immersion. Let $m_\sigma\in M$ be a defining vector of Ψ at σ and set $s_\sigma=\chi^{m_\sigma}s$. By 2.1.15, the divisor $D_{\Psi-m_\sigma}=\operatorname{div}(s_\sigma)$ properly intersects $V(\sigma)$ and we can restrict s_σ to $V(\sigma)$ to obtain a toric section ι^*s_σ of the toric line bundle $\mathcal{O}(D_{(\Psi-m_\sigma)(\sigma)})\simeq\iota^*L$.

Proposition 2.5.12. *Let notation be as above and denote by F_σ the face of Δ_Ψ associated to σ (see A.20). Let $\|\cdot\|$ be a semipositive toric metric on L. Then, for all $m\in F_\sigma-m_\sigma$,*

$$\psi^\vee_{\iota^*\overline{L},\iota^*s_\sigma}(m)=\psi^\vee_{\overline{L},s}(m+m_\sigma)\,.$$

Proof. We can prove the statement as in [BPS14a, Proposition 4.8.8] since the discreteness of the valuation doesn't play a role in that proof. □

2.6. Toric Local Heights over Non-Archimedean Fields

Now, all ingredients are developed to state and prove a formula for the local height of a toric variety over an arbitrary non-Archimedean field. This generalizes work by Burgos–Philippon–Sombra who showed this formula under the additional assumption that the field is discretely valued (see [BPS14a, Theorem 5.1.6]).

Let K be an algebraically closed field which is complete with respect to a non-Archimedean non-trivial absolute value $|\cdot|$ and denote by $\Gamma = -\log|K^\times|$ the associated divisible value group. As explained before, the algebraic closedness of K is no restriction since local heights are stable under base change. We fix a lattice $M \simeq \mathbb{Z}^n$ with dual $M^\vee = N$ and denote by $\mathbb{T} = \operatorname{Spec}(K[M])$ the n-dimensional split torus over K. Let Σ be a complete fan on $N_\mathbb{R}$ and X_Σ the associated proper $\mathbb{T}$-toric variety.

Following [BPS14a, § 5.1], we define a local height for toric metrized line bundles that does not depend on the choice of sections. Even though it differs from the definition of a local height, we can use it to compute global heights of the toric variety X_Σ and, more generally, of orbit closures and images under toric morphisms (cf. Proposition 3.4.2).

Definition 2.6.1. Let $\overline{L}_i$, $i = 0, \dots, t$, be toric line bundles on X_Σ equipped with DSP toric metrics. We denote by $\overline{L}_i^{\mathrm{can}}$ the same toric line bundle endowed with the canonical metric. Let Y be a t-dimensional prime cycle of X_Σ and let $\varphi\colon Y' \to Y$ be a birational morphism such that Y' is projective. Recall the definition of local heights in 1.3.5. Then the *toric local height* of Y with respect to $\overline{L}_0, \dots, \overline{L}_t$ is defined as

$$\lambda^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_t}(Y) = \lambda_{(\varphi^*\overline{L}_0,s_0),\dots,(\varphi^*\overline{L}_t,s_t)}(Y') - \lambda_{(\varphi^*\overline{L}_0^{\mathrm{can}},s_0),\dots,(\varphi^*\overline{L}_t^{\mathrm{can}},s_t)}(Y'),$$

where $s_0 \dots, s_t$ are any invertible meromorphic sections with

$$|\operatorname{div}(s_0)| \cap \dots \cap |\operatorname{div}(s_t)| \cap Y = \emptyset. \tag{2.17}$$

This definition extends to cycles by linearity. When $\overline{L}_0 = \dots = \overline{L}_t = \overline{L}$, we write shortly $\lambda^{\mathrm{tor}}_{\overline{L}}(Y) = \lambda^{\mathrm{tor}}_{\overline{L}_0,\dots,\overline{L}_t}(Y)$.

Remark 2.6.2. Proposition 1.3.6 (iii, v) implies that the toric local height does not depend on the choice of φ and Y' nor on the choice of sections. When $\operatorname{div}(s_0), \dots, \operatorname{div}(s_t)$ intersect properly on Y, then condition (2.17) is fullfilled.

Proposition 2.6.3. *The toric local height is symmetric and multilinear in the metrized line bundles.*

Proof. This follows easily from Proposition 1.3.6 (ii). □

Definition 2.6.4. Let $\overline{L} = (L, \|\cdot\|)$ be a semipositive metrized toric line bundle with a toric section s. Let Ψ be the corresponding support function on Σ and $\psi_{\overline{L},s}$ the associated concave function on $N_\mathbb{R}$. The *roof function* associated to $(\overline{L}, s)$ is the concave function $\vartheta_{\overline{L},s}\colon \Delta_\Psi \to \mathbb{R}$ given by

$$\vartheta_{\overline{L},s} = \psi^\vee_{\overline{L},s},$$

where $\psi^\vee_{\overline{L},s}$ denotes the Legendre-Fenchel dual (see A.7). We will denote $\vartheta_{\overline{L},s}$ by $\vartheta_{\|\cdot\|}$ if the line bundle and section are clear from the context.

2.6.5. Let notation be as above. By Proposition 2.4.8, the roof function $\vartheta_{\overline{L},s}$ is invariant under complete valued field extensions. If $\|\cdot\|$ is an algebraic metric, then, by Proposition 2.5.5 and A.12, the roof function $\vartheta_{\|\cdot\|}$ is piecewise affine concave.

Theorem 2.6.6. *Let Σ be a complete fan on $N_{\mathbb{R}}$. Let $\overline{L}=(L,\|\cdot\|)$ be a toric line bundle on X_Σ equipped with a semipositive toric metric. We choose any toric section s of L and denote by Ψ the corresponding support function on Σ. Then, the toric local height of X_Σ with respect to $\overline{L}$ is given by*

$$\lambda_{\overline{L}}^{\mathrm{tor}}(X_\Sigma)=(n+1)!\int_{\Delta_\Psi}\vartheta_{\overline{L},s}\,\mathrm{d}\,\mathrm{vol}_M\,, \tag{2.18}$$

where Δ_Ψ is the stability set of Ψ and vol_M is the Haar measure on $M_{\mathbb{R}}$ such that M has covolume one.

Proof. We prove this theorem analogously to [BPS14a, Theorem 5.1.6]. Since the metric $\|\cdot\|$ is semipositive, the functions $\psi_{\|\cdot\|}$ and $\Psi=\mathrm{rec}(\psi_{\|\cdot\|})$ are concave. We set $\Delta=\Delta_\Psi$, $\psi=\psi_{\|\cdot\|}$ and $\vartheta=\vartheta_{\|\cdot\|}$.

First, we reduce to the case of an ample line bundle L. Let Σ_Δ be the normal fan of Δ in $N(\Delta)_{\mathbb{R}}$ (see A.20). We choose any $m\in\mathrm{aff}(\Delta)\cap M$ and set $\Delta'=\Delta-m$. By 2.1.18, there is a proper toric morphism $\varphi\colon X_\Sigma\to X_{\Sigma_\Delta}$ and an ample divisor $D_{\Psi_{\Delta'}}$ on X_{Σ_Δ} such that $D_\Psi=\varphi^*D_{\Psi_{\Delta'}}+\mathrm{div}(\chi^{-m})$. The function $(\psi-m)^\vee$ lives on $\Delta'=\Delta_{\Psi_{\Delta'}}\subseteq M(\Delta)_{\mathbb{R}}$ and so, by Theorem 2.5.8, it defines a semipositive metric $\|\cdot\|_{\Delta'}$ on the the line bundle $\mathcal{O}(D_{\Psi_{\Delta'}})$ on X_{Σ_Δ}. Set $\overline{L}_{\Delta'}=(\mathcal{O}(D_{\Psi_{\Delta'}}),\|\cdot\|_{\Delta'})$. Using Proposition 2.4.9, we obtain an isometry

$$\overline{L}=(\mathcal{O}(D_\Psi),\|\cdot\|_\psi)\xrightarrow{\sim}(\mathcal{O}(D_{\Psi-m}),\|\cdot\|_{\psi-m})=\varphi^*(\overline{L}_{\Delta'}).$$

By Proposition 2.4.14, there is also an isometry between $\overline{L}^{\mathrm{can}}$ and $\varphi^*(\overline{L}_{\Delta'}^{\mathrm{can}})$. Thus, by the functoriality of the local height (Proposition 1.3.6 (iii)),

$$\lambda_{\overline{L}}^{\mathrm{tor}}(X_\Sigma)=\lambda_{\varphi^*\overline{L}_{\Delta'}}^{\mathrm{tor}}(X_\Sigma)=\lambda_{\overline{L}_{\Delta'}}^{\mathrm{tor}}(\varphi_*X_\Sigma).$$

If $\dim(\Delta)<n$, then on the one hand, the integral in (2.18) is zero. On the other hand, $\dim(X_\Sigma)=n>\dim(X_{\Sigma_\Delta})$ implies $\varphi_*X_\Sigma=0$ and hence, $\lambda_{\overline{L}}^{\mathrm{tor}}(X_\Sigma)$ is also zero. If $\dim(\Delta)=n$, then φ is a birational morphism and $\varphi_*X_\Sigma=X_{\Sigma_\Delta}$. Moreover,

$$(n+1)!\int_\Delta\psi^\vee\,\mathrm{dvol}_M=(n+1)!\int_{\Delta-m}(\psi-m)^\vee\,\mathrm{dvol}_M=(n+1)!\int_{\Delta_{\Psi_{\Delta'}}}\psi^\vee_{\|\cdot\|_{\Delta'}}\,\mathrm{dvol}_{M(\Delta)}.$$

So it is enough to prove the theorem for the ample line bundle $L_{\Delta'}$ on the projective variety X_{Σ_Δ}. Hence, we may assume that L is ample and X_Σ is projective.

We prove the theorem by induction on $n=\dim(X_\Sigma)$. If $n=0$, then $X_\Sigma=\operatorname{Spec}K=\mathbb{P}^0$, $\Psi=0$, $\Delta=\{0\}$ and $L=\mathcal{O}(D_0)=\mathcal{O}_{\mathbb{P}^0}$. By the induction formula (Theorem 1.4.3) and Definition 2.4.6, we obtain

$$\lambda_{(\overline{L},s)}(X_\Sigma)=-\log\|s\|=-\psi(0)\quad\text{and}\quad\lambda_{(\overline{L}^{\mathrm{can}},s)}(X_\Sigma)=-\log\|s\|_{\mathrm{can}}=-\Psi(0)=0\,.$$

Therefore,

$$\lambda_{\overline{L}}^{\mathrm{tor}}(X_\Sigma)=-\psi(0)=\vartheta(0)=1!\int_\Delta\vartheta\,\mathrm{dvol}_M\,.$$

Let $n \geq 1$ and let $s_0, \dots, s_{n-1}$ be invertible meromorphic sections of L such that $|\mathrm{div}(s_0)| \cap \dots \cap |\mathrm{div}(s_{n-1})| \cap |\mathrm{div}(s)| = \emptyset$. By the induction formula 1.4.3,

$$\lambda_{(\overline{L},s),\dots,(\overline{L},s)}(X_\Sigma) = \lambda_{(\overline{L},s_0),\dots,(\overline{L},s_{n-1})}(\mathrm{cyc}(s)) - \int_{X_\Sigma^{\mathrm{an}}} \log \|s\| \ \mathrm{c}_1(\overline{L})^n \,. \tag{2.19}$$

For each facet F of Δ, let $v_F \in N$ be the minimal inner facet normal of F (see A.21) and $\sigma_F = \mathbb{R}_{\geq 0} v_F$ the corresponding ray in Σ. Since L is ample, we obtain by 2.1.17,

$$\lambda_{(\overline{L},s_0),\dots,(\overline{L},s_{n-1})}(\mathrm{cyc}(s)) = \sum_F - \langle F, v_F\rangle \ \lambda_{(\overline{L},s_0),\dots,(\overline{L},s_{n-1})} \left(V(\sigma_F)\right), \tag{2.20}$$

where the sum is over the facets F of Δ. By functoriality, the local height of $V(\sigma_F)$ with respect to $\overline{L}$ coincides with the local height with respect to $\overline{L}|_{V(\sigma_F)}$. Moreover, by Proposition 2.4.14, the restriction of the canonical metric of L to the toric variety $V(\sigma_F)$ coincides with the canonical metric of $L|_{V(\sigma_F)}$. Subtracting from equation (2.20) the analogous formula for the canonical metric, we get

$$\begin{aligned} \sum_F - \langle F, v_F\rangle \ \lambda^{\mathrm{tor}}_{\overline{L}|_{V(\sigma_F)}}(V(\sigma_F)) = {} & \lambda_{(\overline{L},s_0),\dots,(\overline{L},s_{n-1})} \left(\mathrm{cyc}(s)\right) \\ & - \lambda_{(\overline{L}^{\mathrm{can}},s_0),\dots,(\overline{L}^{\mathrm{can}},s_{n-1})} \left(\mathrm{cyc}(s)\right) \,. \end{aligned} \tag{2.21}$$

Corollary 1.4.5 says that the measure of $X_\Sigma^{\mathrm{an}} \setminus \mathbb{T}^{\mathrm{an}}$ with respect to $\mathrm{c}_1(\overline{L})^n$ is zero. Since the tropicalization map is continuous and, by Definition 2.4.6, $\log \|s\| = \mathrm{trop}^* \psi$, we deduce

$$\int_{X_\Sigma^{\mathrm{an}}} \log \|s\| \ \mathrm{c}_1(\overline{L})^n = \int_{\mathbb{T}^{\mathrm{an}}} \mathrm{trop}^*(\psi) \ \mathrm{c}_1(\overline{L})^n = \int_{N_\mathbb{R}} \psi \ \mathrm{trop}_*(\mathrm{c}_1(\overline{L})^n) \,.$$

By Corollary 2.5.11, $\mathrm{trop}_*(\mathrm{c}_1(\overline{L})^n) = n! \, \mathcal{M}_M(\psi)$ and therefore,

$$\int_{X_\Sigma^{\mathrm{an}}} \log \|s\| \ \mathrm{c}_1(\overline{L})^n = n! \int_{N_\mathbb{R}} \psi \ \mathrm{d}\mathcal{M}_M(\psi) \,. \tag{2.22}$$

By Proposition A.19, we have $\mathcal{M}_M(\Psi) = \mathrm{vol}_M(\Delta)\delta_0$. Hence, in the case of the canonical metric, equation (2.22) is reduced to

$$\int_{X_\Sigma^{\mathrm{an}}} \log \|s\|_{\mathrm{can}} \ \mathrm{c}_1\big(\overline{L}^{\mathrm{can}}\big)^n = n! \, \mathrm{vol}_M(\Delta)\Psi(0) = 0. \tag{2.23}$$

Subtracting from (2.19) the analogous induction formula for the canonical metric and using (2.21), (2.22) and (2.23), we obtain

$$\lambda^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = \sum_F - \langle F, v_F\rangle \ \lambda^{\mathrm{tor}}_{\overline{L}|_{V(\sigma_F)}}(V(\sigma_F)) - n! \int_{N_\mathbb{R}} \psi \ \mathrm{d}\mathcal{M}_M(\psi) \,. \tag{2.24}$$

We set temporarily $\sigma = \sigma_F$ and denote by $\iota \colon V(\sigma) \to X_\Sigma$ the closed immersion. Choose any element m_σ in $F \cap M$, i. e. m_σ is a defining vector of Ψ at σ, and set $s_\sigma = \chi^{m_\sigma} s$. By 2.1.15, $\iota^* s_\sigma$ is a toric section of the toric line bundle $\iota^* L \simeq \mathcal{O}\big(D_{(\Psi - m_\sigma)(\sigma)}\big)$. Hence, by the induction hypothesis,

$$\lambda^{\mathrm{tor}}_{\iota^*\overline{L}}\left(V\left(\sigma\right)\right) = n! \int_{\Delta_{(\Psi-m_\sigma)(\sigma)}} \psi^\vee_{\iota^*\overline{L},\iota^* s_\sigma} \operatorname{dvol}_{M(\sigma)} .$$

By Proposition 2.5.12, the function $\psi^\vee_{\iota^*\overline{L},\iota^* s_\sigma}$ is the translate of $\psi^\vee|_F$ by $-m_\sigma$ and we have $\Delta_{(\Psi-m_\sigma)(\sigma)} = F - m_\sigma$. Since Δ is of dimension n, we get $M(\sigma_F) = M(F)$ and therefore,

$$\lambda^{\mathrm{tor}}_{\overline{L}|_{V(\sigma_F)}}\left(V\left(\sigma_F\right)\right) = n! \int_{F-m_\sigma} \psi^\vee(m+m_\sigma) \operatorname{dvol}_{M(F)}(m) = n! \int_F \vartheta \operatorname{dvol}_{M(F)} . \tag{2.25}$$

Inserting (2.25) into (2.24) and applying Proposition A.22, we obtain

$$\lambda^{\mathrm{tor}}_{\overline{L}}(X_\Sigma) = -n! \sum_F \langle F, v_F\rangle \int_F \vartheta \operatorname{dvol}_{M(F)} - n! \int_{N_\mathbb{R}} \psi \operatorname{d}\mathcal{M}_M(\psi) = (n+1)! \int_\Delta \vartheta \operatorname{dvol}_M ,$$

proving the theorem. □

Remark 2.6.7. In the Archimedean case, we define toric local heights and roof functions in the exact same manner as above, using the notions of sections 1.5 and 2.4. Then Theorem 2.6.6 also holds in the Archimedean setting. This is proved in the same way as the discrete non-Archimedean case in [BPS14a, Theorem 5.1.6]. Note that here we implicitly make use of the induction formula 1.5.13.

The following two corollaries correspond to the propositions 5.1.11 and 5.1.13 in [BPS14a].

Corollary 2.6.8. *Let notation be as in Theorem 2.6.6. Let $\sigma \in \Sigma$ be a cone of codimension d and $V(\sigma)$ the corresponding orbit closure. Then*

$$\lambda^{\mathrm{tor}}_{\overline{L}}(V(\sigma)) = (d+1)! \int_{F_\sigma} \vartheta_{\overline{L},s} \operatorname{dvol}_{M(\sigma)} ,$$

where F_σ is the face of Δ_Ψ associated to σ (see A.20) and $\mathrm{vol}_{M(\sigma)}$ is the Haar measure with respect to the lattice $M(\sigma) = M \cap \sigma^\perp$ on the affine space containing F_σ (see A.16).

Proof. The propositions 2.4.14 and 1.3.6 (iii) imply $\lambda^{\mathrm{tor}}_{\overline{L}}(V(\sigma)) = \lambda^{\mathrm{tor}}_{\overline{L}|_{V(\sigma)}}(V(\sigma))$. The result can be proved similarly to (2.25) using Theorem 2.6.6 instead of the induction hypothesis. □

Corollary 2.6.9. *Let Let N' be a lattice of rank d and Σ' a complete fan on $N'_\mathbb{R}$. Let $H\colon N' \to N$ be a linear map which is compatible with Σ' and Σ, and let $\varphi\colon X_{\Sigma'} \to X_\Sigma$ be the corresponding proper toric morphism (see 2.1.7). We denote by $H^\vee\colon M \to M'$ the dual map and by $H(N')^{\mathrm{sat}}$ the saturation of the lattice $H(N')$ in N.*

Let $\overline{L}$ be a toric line bundle on X_Σ with a semipositive toric metric. Choose any toric section s of L and let Ψ be the associated support function.

(i) If H is not injective, then $\lambda^{\mathrm{tor}}_{\varphi^*\overline{L}}(X_{\Sigma'}) = 0$.

(ii) If H is injective, then

$$\lambda^{\mathrm{tor}}_{\varphi^*\overline{L}}(X_{\Sigma'}) = \left[H(N')^{\mathrm{sat}} : H(N')\right] \lambda^{\mathrm{tor}}_{\overline{L}}(\varphi(X_{\Sigma'})) = (d+1)! \int_{H^\vee(\Delta_\Psi)} \left(\psi_{\|\cdot\|} \circ H\right)^\vee \operatorname{dvol}_{M'} .$$

Proof. This result can be proved analogously to Corollary [BPS14a, 5.1.13] using the corresponding results from this thesis. □

Remark 2.6.10. In [BPS14a, § 5.1], the formula corresponding to Theorem 2.6.6 is extended to toric local heights with respect to distinct line bundles. Moreover, the toric local height of a translated toric subvariety and its behavior with respect to equivariant morphisms is studied. For arbitrary non-Archimedean fields, these results can be stated and proved analogously using the herein developed theory.

Chapter 3.

Global Heights of Varieties over Finitely Generated Fields

In [Mor00], Moriwaki defined the height of a variety over a finitely generated field over $\mathbb{Q}$ with respect to Hermitian line bundles as an arithmetic intersection number in the sense of Gillet–Soulé [GS90]. Then Burgos–Philippon–Sombra showed in [BPS14b] that this height can be written as an integral of local heights over a measured set of places of the finitely generated field. Furthermore, they applied their formulas for local heights of toric varieties from [BPS14a] to compute some arithmetic intersection numbers of non-toric arithmetic varieties coming from a fibration with toric generic fiber.

In this chapter, we extend these results to finitely generated fields over a global field and quasi-algebraic metrized line bundles. Note that in this setting non-discrete non-Archimedean places occur. Hence, we actually need our theory developed in Chapter 1 and 2. This generalization was suggested to me by José Burgos Gil. At the end, we particularize to the case of the function field of an elliptic curve leading to a natural example to illustrate our theory.

3.1. Global Heights of Varieties over an M-Field

First we explain the notion of M-fields introduced by Gubler in [Gub97, Definition 2.1]. Theses fields include global fields and more generally, finitely generated fields over global fields. Then we construct global heights of subvarieties by integrating local heights over M. Note that Gubler's definition of an M-field is more general than ours.

Definition 3.1.1. Let K be a field and M a family of inequivalent absolute values on K together with a positive measure μ on M. Then K is called an *M-field* if, for each $f \in K^\times$,

(i) the function $M \to \mathbb{R}$, $v \mapsto \log|f|_v$, is μ-integrable;

(ii) the *product formula* $\int_M \log|f|_v \, \mathrm{d}\mu(v) = 0$ holds.

Example 3.1.2. A *global field* F is either a number field or the function field of a smooth projective curve over a countable field. We endow F with the following structure of an M_F-field.

If $F = \mathbb{Q}$, then let $M_\mathbb{Q}$ be the set consisting of the Archimedean and the p-adic absolute values, normalized in the standard way, and equip $M_\mathbb{Q}$ with the counting measure.

If $F = k(C)$ is the function field of a smooth projective curve C over a countable field k, let $M_{k(C)}$ be the set of absolute values $|\cdot|_v$, indexed by the closed points $v \in C$, which are given, for $\alpha \in k(C)^\times$, by

$$|\alpha|_v = c_k^{-\operatorname{ord}_v(\alpha)}, \quad c_k = \begin{cases} \mathrm{e} & \text{if } |k| = \infty \\ |k| & \text{if } |k| < \infty \end{cases},$$

where ord_v is the discrete valuation of the local ring $\mathcal{O}_{C,v}$. We endow $M_{k(C)}$ with the point measure μ given by $\mu(v) = [k(v) : k]$.

Let F_0 denote either $\mathbb{Q}$ or $k(C)$. If F is a finite extension of F_0, let M_F be the set of absolute values $|\cdot|_v$ extending an absolute value $|\cdot|_{v_0}$ on F_0. We equip M_F with the point measure μ given by

$$\mu(v) = \frac{[F_v : F_{0,v_0}]}{[F : F_0]}\mu(v_0), \tag{3.1}$$

where F_v denotes the completion of F with respect to $|\cdot|_v$, and similarly for F_{0,v_0}.

In all cases, it can be shown that F together with (M_F, μ) is an M_F-field. For details, we refer to [BPS14c, 2.1] and, for more advanced examples, to [Gub97, § 2].

Remark 3.1.3. In the above definition of a global field we assumed in the case of a function field $k(C)$ that k is countable to ensure the construction of the $\mathfrak{M}$-field in 3.2.4. This assumption is just made for simplicity. In general, we are concerned with finitely many varieties, metrized line bundles or meromorphic sections and hence, we can find a countable finitely generated subfield over which all these objects are defined.

Definition 3.1.4. Let K be an M-field and let $\mathbb{K}_v$ be the completion of an algebraic closure of the completion of K with respect to $v \in M$. Let X be a proper variety over K and L a line bundle on X. We set $X_v = X \times_K \operatorname{Spec}(\mathbb{K}_v)$ and $L_v = L \otimes_K \mathbb{K}_v$. If v is Archimedean, then we denote by $X_v^{\mathrm{an}} = X_v(\mathbb{K}_v)$ the complex analytic space associated to X. If v is non-Archimedean, then X_v^{an} is the Berkovich analytic space associated to X_v over $\mathbb{K}_v$ as defined in 1.1.2. We call X_v^{an} the *analytification* of X with respect to v (or $|\cdot|_v$).

An *(M-)metric* on L is a family of metrics $\|\cdot\|_v$, $v \in M$, where $\|\cdot\|_v$ is a metric on L_v^{an}. The corresponding *(M-)metrized line bundle* is denoted by $\overline{L} = (L, (\|\cdot\|_v)_v)$. An (M-)metric on L is said to be *semipositive* if $\|\cdot\|_v$ is semipositive for all $v \in M$ (cf. Definition 1.3.1 and 1.5.3). Moreover, a metrized line bundle $\overline{L}$ is *DSP* if there are semipositive metrized line bundles $\overline{M}$, $\overline{N}$ on X such that $\overline{L} = \overline{M} \otimes \overline{N}^{-1}$.

Let Z be a t-dimensional cycle on X and $(\overline{L}_i, s_i)$, $i = 0, \ldots, t$, DSP metrized line bundles on X with invertible meromorphic sections such that $|\operatorname{div}(s_0)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset$. For $v \in M$, we set for the local height at v,

$$\lambda_{(\overline{L}_0,s_0),\ldots,(\overline{L}_t,s_t)}(Z, v) := \lambda_{\widehat{\operatorname{div}}(s_0)_v,\ldots,\widehat{\operatorname{div}}(s_t)_v}(Z_v),$$

where $\widehat{\operatorname{div}}(s_i)_v$ is the pseudo-divisor on X_v induced by $\widehat{\operatorname{div}}(s_i)$ (cf. Example 1.2.10).

Definition 3.1.5. Let K be an M-field and X a proper variety over K. A t-dimensional prime cycle Y of X is called *integrable* with respect to DSP metrized line bundles $\overline{L}_i$, $i = 0, \ldots, t$, on X if there is a birational proper map $\varphi\colon Y' \to Y$ with Y' projective, and invertible meromorphic sections s_i of $\varphi^* L_i$, $i = 0, \ldots, t$, meeting Y' properly, such that the function

$$M \longrightarrow \mathbb{R}, \quad v \longmapsto \lambda_{(\varphi^*\overline{L}_0,s_0),\dots,(\varphi^*\overline{L}_t,s_t)}(Y',v) \tag{3.2}$$

is μ-integrable on M. A t-dimensional cycle is *integrable* if its components are integrable.

3.1.6. For an integrable cycle Y, the μ-integrability of (3.2) holds for any choice of a morphism φ, a cycle Y' and invertible meromorphic sections $s_0,\dots,s_t$ satisfying only

$$|\mathrm{div}(s_0)| \cap \cdots \cap |\mathrm{div}(s_t)| = \emptyset.$$

Moreover, the notion of integrability of cycles is closed under tensor product and pullback of DSP metrized line bundles. This can be proved as in [BPS14a, Proposition 1.5.8] by means of [Gub03, Proposition 11.5].

Definition 3.1.7. Let X be a proper variety over an M-field K and Y a t-dimensional prime cycle on X which is integrable with respect to DSP metrized line bundles $\overline{L}_0,\dots,\overline{L}_t$ on X. Let Y' and $s_0,\dots,s_t$ be as in Definition 3.1.5. Then the *global height* of Y with respect to $\overline{L},\dots,\overline{L}_t$ is defined as

$$\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_t}(Y) = \int_M \lambda_{(\varphi^*\overline{L}_0,s_0),\dots,(\varphi^*\overline{L}_t,s_t)}(Y',v)\ \mathrm{d}\mu(v).$$

By linearity, we extend this definition to all t-dimensional cycles on X.

Using Corollary 1.3.6 (iii), the Archimedean analogon mentioned in 1.5.11 and the product formula of K, we see that this definition is independent of the choice of the sections.

Proposition 3.1.8. *The global height of integrable cycles has the following basic properties:*

(i) It is symmetric and multilinear with respect to tensor products of DSP metrized line bundles.

(ii) Let $\varphi\colon X' \to X$ be a morphism of proper varieties over K and let Z' be a t-dimensional cycle such that φ_*Z' is integrable with respect to DSP metrized line bundles $\overline{L}_0,\dots,\overline{L}_t$ on X. Then we have

$$\mathrm{h}_{\varphi^*\overline{L}_0,\dots,\varphi^*\overline{L}_t}(Z') = \mathrm{h}_{\overline{L}_0,\dots,\overline{L}_t}(\varphi_*Z').$$

Proof. Using 3.1.6, we get the results by integrating the corresponding formulas stated in Proposition 1.3.6 (non-Archimedean case) and in 1.5.11 (Archimedean case). □

We consider the special case of the global height over a global field.

Definition 3.1.9. Let F be a global field with the structure (M_F,μ) of an M_F-field as in Example 3.1.2. Let $\mathcal{X}$ be a proper variety over a global field F and $\mathcal{L}$ a line bundle on $\mathcal{X}$. We call an M_F-metric on $\mathcal{L}$ *quasi-algebraic* if there exist a finite subset $S \subseteq M_F$ containing the Archimedean places and a proper algebraic model $(\mathscr{X},\mathscr{L},e)$ of $(\mathcal{X},\mathcal{L})$ over the ring

$$F_S^\circ = \{\alpha \in F \mid |\alpha|_v \le 1\,\forall v \notin S\},$$

such that, for each $v \notin S$, the metric $\|\cdot\|_v$ is induced by the localization

$$(\mathscr{X} \times_{F_S^\circ} \operatorname{Spec}\mathbb{F}_v^\circ, \mathscr{L} \otimes_{F_S^\circ} \mathbb{F}_v^\circ, e).$$

Proposition 3.1.10. *Let $\mathcal{X}$ be a proper variety over a global field F. Then every cycle of $\mathcal{X}$ is μ-integrable with respect to DSP quasi-algebraic M_F-metrized line bundles on $\mathcal{X}$.*

Proof. This is [BPS14a, Proposition 1.5.14]. □

Proposition 3.1.11 (Global induction formula)**.** *Let $\mathcal{X}$ be a d-dimensional proper variety over a global field F and $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d$ quasi-algebraic DSP metrized line bundles on $\mathcal{X}$. If s_d is any invertible meromorphic section of $\overline{\mathcal{L}}_d = (\mathcal{L}_d, (\|\cdot\|_{d,v})_v)$, then there is only a finite number of $v \in M_F$ such that*

$$\int_{\mathcal{X}_v^{\mathrm{an}}} \log \|s_d\|_{d,v} \, \mathrm{c}_1(\overline{\mathcal{L}}_{0,v}) \wedge \cdots \wedge \mathrm{c}_1(\overline{\mathcal{L}}_{d-1,v}) \neq 0$$

and we have

$$\begin{aligned}\mathrm{h}_{\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_d}(\mathcal{X}) = \mathrm{h}_{\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_{d-1}}&(\mathrm{cyc}(s_d)) \\ &- \sum_{v \in M_F} \mu(v) \int_{\mathcal{X}_v^{\mathrm{an}}} \log \|s_d\|_{d,v} \, \mathrm{c}_1(\overline{\mathcal{L}}_{0,v}) \wedge \cdots \wedge \mathrm{c}_1(\overline{\mathcal{L}}_{d-1,v}) \,,\end{aligned}$$

with $\mu(v)$ as in (3.1)*.*

Proof. The first part follows from the proof of [BPS14a, Proposition 1.5.14]. For the second part, we use Proposition 3.1.10 and integrate the local induction formulas (theorems 1.4.3 and 1.5.13) over M_F. □

Proposition 3.1.12. *Let F be a global field and F' a finite extension of F with the induced structure of an $M_{F'}$-field (see Example 3.1.2). Let $\mathcal{X}$ be an F-variety, $\overline{\mathcal{L}}_i$, $i = 0, \dots, t$, quasi-algebraic DSP metrized line bundles on $\mathcal{X}$ and $\mathcal{Z}$ a t-dimensional cycle on $\mathcal{X}$. We denote by $\pi\colon \mathcal{X}' \to \mathcal{X}$ the morphism, by $\mathcal{Z}'$ the cycle and by $\pi^*\overline{\mathcal{L}}_i$ the $M_{F'}$-metrized line bundles obtained by base change to F'. Then*

$$\mathrm{h}_{\pi^*\overline{\mathcal{L}}_0,\dots,\pi^*\overline{\mathcal{L}}_t}(\mathcal{Z}') = \mathrm{h}_{\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_t}(\mathcal{Z}).$$

Proof. This follows from [BPS14a, Proposition 1.5.10]. □

3.2. $\mathfrak{M}$-Fields from Varieties over a Global Field

Let F be a global field with the canonical M_F-field structure from Example 3.1.2. Let B be a b-dimensional normal proper variety over F with function field $K = F(B)$.

In this section, we endow the field K with the structure of an $\mathfrak{M}$-field where $\mathfrak{M}$ is a natural set of places induced by nef quasi-algebraic M_F-metrized line bundles on B. This generalizes the $\mathfrak{M}$-fields obtained by Moriwaki's construction in [Mor00, § 3] where the function field of an arithmetic variety and a family of nef Hermitian line bundles are considered (see also [Gub03, Example 11.22]).

Definition 3.2.1. Let $\overline{L}$ be a quasi-algebraic M_F-metrized line bundle on B. We say that $\overline{L}$ is *nef* if $\|\cdot\|$ is semipositive and, for each point $p \in B(\overline{F})$, the global height $\mathrm{h}_{\overline{L}}(p)$ is non-negative.

Example 3.2.2. Let $\overline{L} = (L, (\|\cdot\|_v)_v)$ be a semipositive quasi-algebraic metrized line bundle. We assume that $\overline{L}$ is generated by small global sections, i. e. for each point $p \in B(\overline{F})$, there exists a global section s such that $p \notin |\mathrm{div}(s)|$ and $\sup_{x\in B_v^{\mathrm{an}}} \|s(x)\|_v \leq 1$ for all $v \in M_F$. Then $\overline{L}$ is nef.

The idea of the following proof was suggested to me by José Burgos Gil.

Lemma 3.2.3. *Let V be a d-dimensional subvariety of B and let $\overline{L}_1, \dots, \overline{L}_d$ be nef quasi-algebraic M_F-metrized line bundles on B. Then,*

$$\mathrm{h}_{\overline{L}_1,\dots,\overline{L}_d}(V) \geq 0.$$

Proof. We may assume that $V = B$ and, by Chow's Lemma and Proposition 3.1.8 (ii), that there is a closed immersion $\varphi\colon B \hookrightarrow \mathbb{P}^m_F$. Consider the line bundle $\varphi^*\mathcal{O}_{\mathbb{P}^m_F}(1)$ on B, equipped with the metric $\frac{1}{2}\varphi^*\|\cdot\|_{\mathrm{can},v_0}$ at one place $v_0 \in M_F$ and with the metric $\varphi^*\|\cdot\|_{\mathrm{can},v}$ at all other places $v \neq v_0$. This M_F-metrized line bundle is denoted by $\overline{L}$. For each point $p \in B(\overline{F})$ with function field $F(p)$, there exists a homogeneous coordinate x_j, considered as a global section of $\mathcal{O}_{\mathbb{P}^m_F}(1)$, such that $p \notin |\mathrm{div}(\varphi^* x_j)|$ and hence,

$$\mathrm{h}_{\overline{L}}(p) = -\sum_{w\in M_{F(p)}} \mu(w) \log \|x_j \circ \varphi(p)\|_{\mathrm{can},w} + \sum_{\substack{w\in M_{F(p)}\\ w|v_0}} \mu(w)\log 2 \geq \log 2 > 0. \tag{3.3}$$

We extend the group of isomorphism classes of M_F-metrized line bundles on B by $\mathbb{Q}$-coefficients and write its group structure additively. For $i = 1, \dots, d$, and a positive rational number ε, we set $\overline{L}_{i,\varepsilon} := \overline{L}_i + \varepsilon\overline{L}$. Since $\overline{L}_i$ is nef, we obtain, by (3.3) and the multilinearity of the heights, for each point $p \in B(\overline{F})$,

$$\mathrm{h}_{\overline{L}_{i,\varepsilon}}(p) = \mathrm{h}_{\overline{L}_i}(p) + \varepsilon\,\mathrm{h}_{\overline{L}}(p) \geq \varepsilon \log 2 > 0. \tag{3.4}$$

Now, we distinguish between number fields and function fields. First, let F be a number field. Since $\overline{L}_{i,\varepsilon}$ is semipositive quasi-algebraic, there exists a sequence $(\overline{L}_{i,\varepsilon,k})_{k\in\mathbb{N}}$ that converges to $\overline{L}_{i,\varepsilon}$ and that consists of M_F-metrized line bundles which are induced by vertically nef smooth Hermitian $\mathbb{Q}$-line bundles $\overline{\mathscr{L}}_{i,\varepsilon,k}$, $k \in \mathbb{N}$, on a common model $\mathscr{B}_{\varepsilon,k}$ over the ring of integers $\mathcal{O}_F$. By propositions 1.3.6 (iv) and 1.5.11, we have, for all $k \in \mathbb{N}$ and all $p \in B(\overline{F})$,

$$\left|\mathrm{h}_{\overline{L}_{i,\varepsilon,k}}(p) - \mathrm{h}_{\overline{L}_{i,\varepsilon}}(p)\right| \leq \sum_{w\in M_{F(p)}} \mu(w)\,\mathrm{d}\left(\|\cdot\|_{i,\varepsilon,k,w}, \|\cdot\|_{i,\varepsilon,w}\right).$$

Note that the sum is finite and does not depend on p. Hence, by (3.4), there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $p \in B(\overline{F})$,

$$\mathrm{h}_{\overline{\mathscr{L}}_{i,\varepsilon,k}}(\overline{\{p\}}) = \mathrm{h}_{\overline{L}_{i,\varepsilon,k}}(p) \geq 0.$$

Thus, for all $k \geq k_0$, we have nef smooth Hermitian $\mathbb{Q}$-line bundles $\overline{\mathscr{L}}_{1,\varepsilon,k}, \dots, \overline{\mathscr{L}}_{d,\varepsilon,k}$ in the sense of Moriwaki [Mor00, § 2]. So we can apply [Mor00, Proposition 2.3 (1)], which also holds for number fields, to get

$$\mathrm{h}_{\overline{L}_{1,\varepsilon,k},\ldots,\overline{L}_{d,\varepsilon,k}}(B) = \mathrm{h}_{\overline{\mathscr{L}}_{1,\varepsilon,k},\ldots,\overline{\mathscr{L}}_{d,\varepsilon,k}}(\overline{B}) \geq 0. \tag{3.5}$$

Next, let F be the function field of a smooth projective curve C over any field. Since $\overline{L}_{i,\varepsilon}$ is semipositive quasi-algebraic, there exists a sequence $(\overline{L}_{i,\varepsilon,k})_{k\in\mathbb{N}}$ that converges to $\overline{L}_{i,\varepsilon}$ and that consists of M_F-metrized line bundles which are induced by vertically nef $\mathbb{Q}$-line bundles $\mathscr{L}_{i,\varepsilon,k}$, $k \in \mathbb{N}$, on a common model $\pi_{\varepsilon,k} \colon \mathscr{B}_{\varepsilon,k} \to C$. As in the number field case, we can deduce, for sufficiently large k's and for all $p \in B(\overline{F})$,

$$\mathrm{h}_{\overline{L}_{i,\varepsilon,k}}(p) \geq 0. \tag{3.6}$$

By [Gub08, Theorem 3.5 (d)], the height with respect to such algebraic metrized line bundles is given as an algebraic intersection number of the associated models. So, the inequality (3.6) just says that the line bundles $\mathscr{L}_{1,\varepsilon,k}, \ldots, \mathscr{L}_{d,\varepsilon,k}$ on the model $\mathscr{B}_{\varepsilon,k}$ are horizontally nef. Using that they are also vertically nef, it follows from Kleiman's Theorem [Kle66, Theorem III.2.1] that

$$\mathrm{h}_{\overline{L}_{1,\varepsilon,k},\ldots,\overline{L}_{d,\varepsilon,k}}(B) = \deg_C\left((\pi_{\varepsilon,k})_*(\mathrm{c}_1(\mathscr{L}_{1,\varepsilon,k})\ldots\mathrm{c}_1(\mathscr{L}_{d,\varepsilon,k}))\right) \geq 0. \tag{3.7}$$

Finally, by (3.5) for number fields and by (3.7) for function fields, we obtain, by continuity of heights in metrized line bundles,

$$\mathrm{h}_{\overline{L}_1,\ldots,\overline{L}_n}(B) = \lim_{\varepsilon\to 0}\mathrm{h}_{\overline{L}_{1,\varepsilon},\ldots,\overline{L}_{d,\varepsilon}}(B) = \lim_{\varepsilon\to 0}\lim_{k\to\infty}\mathrm{h}_{\overline{L}_{1,\varepsilon,k},\ldots,\overline{L}_{d,\varepsilon,k}}(B) \geq 0,$$

proving the lemma. □

Now, we equip the field $K = F(B)$ with the structure of an $\mathfrak{M}$-field induced by nef quasi-algebraic metrized line bundles.

3.2.4. Let $\overline{H}_1, \ldots, \overline{H}_b$ be nef quasi-algebraic line bundles on B. Let $B^{(1)}$ denote the set of one-codimensional subvarieties of B. By Lemma 3.2.3, each $V \in B^{(1)}$ induces a non-Archimedean absolute value on K given, for $f \in K$, by

$$|f|_V = \mathrm{e}^{-\mathrm{h}_{\overline{H}_1,\ldots,\overline{H}_b}(V)\,\mathrm{ord}_V(f)}, \tag{3.8}$$

where ord_V is the discrete valuation associated to the regular local ring $\mathcal{O}_{B,V}$. We equip $B^{(1)}$ with the counting measure μ_{fin}.

Let us fix a place $v \in M_F$. Then we define the *generic points* of B_v^{an} as

$$B_v^{\mathrm{gen}} = B_v^{\mathrm{an}} \setminus \bigcup_{V\in B^{(1)}} V_v^{\mathrm{an}}.$$

Since each $V \in B^{(1)}$ is contained in the support of the divisor of a rational function, a point $p \in B_v^{\mathrm{an}}$ lies in B_v^{gen} if and only if, for each $f \in K^\times$, p does not lie in the analytification (with respect to v) of the support of $\mathrm{div}(f)$. Thus, each $p \in B_v^{\mathrm{gen}}$ defines a well-defined absolute value on K given by

$$|f|_{v,p} = |f(p)|. \tag{3.9}$$

If v is non-Archimedean, then this absolute value is just p. On B_v^{an} we have the positive measure

$$\mu_v = \mathrm{c}_1(\overline{H}_{1,v}) \wedge \cdots \wedge \mathrm{c}_1(\overline{H}_{b,v}),$$

as defined in Definition 1.3.8 (non-Archimedean case) and 1.5.12 (Archimedean case). Each V_v^{an}, $V \in B^{(1)}$, has measure zero with respect to μ_v by Corollary 1.4.5 (non-Archimedean case) and by [CT09, Corollaire 4.2] (Archimedean case). Since F is countable, $B^{(1)}$ is also countable and therefore $B_v^{\mathrm{an}} \setminus B_v^{\mathrm{gen}}$ has measure zero with respect to μ_v. So we get a positive measure on B_v^{gen}, which we also denote by μ_v.

In conclusion, we obtain a measure space

$$(\mathfrak{M}, \mu) = (B^{(1)}, \mu_{\mathrm{fin}}) \sqcup (\bigsqcup_{v \in M_F} B_v^{\mathrm{gen}}, \bigsqcup_{v \in M_F} \mu_v), \tag{3.10}$$

which is in bijection with a set of absolute values on K.

The following shows that $(K, \mathfrak{M}, \mu)$ satisfies the product formula and so it is an $\mathfrak{M}$-field:

Proposition 3.2.5. *Let $f \in K^\times$, then the function $\mathfrak{M} \to \mathbb{R}$, $w \mapsto \log|f|_w$ is integrable with respect to μ and we have the product formula*

$$\int_{\mathfrak{M}} \log|f|_w \, \mathrm{d}\mu(w) = 0.$$

Proof. Let $f \in K^\times$ be a non-zero rational function on B. Then, for almost every $V \in B^{(1)}$, we have $f \in \mathcal{O}_{B,V}^\times$. Hence, the function on $B^{(1)}$ given by $V \mapsto \log|f|_V$ is μ_{fin}-integrable.

For each $v \in M_F$, the function on B_v^{gen} given by $p \mapsto \log|f(p)|$ is μ_v-integrable (see theorems 1.4.3 and 1.5.13). Since the trivially metrized line bundle $\overline{\mathcal{O}}_B$ and $\overline{H}_1, \dots, \overline{H}_b$ are quasi-algebraic, there is, by Proposition 3.1.11, only a finite number of $v \in M_F$ such that

$$\int_{B_v^{\mathrm{gen}}} \log|f(p)| \, \mathrm{d}\mu_v(p) \neq 0.$$

Summing up, the function $\mathfrak{M} \to \mathbb{R}$, $w \mapsto \log|f|_w$, is μ-integrable.

By the global induction formula 3.1.11, we obtain

$$\begin{aligned}
\int_{\mathfrak{M}} \log|f|_w \, \mathrm{d}\mu(w) &= \sum_{V \in B^{(1)}} -\mathrm{ord}_V(f)\, \mathrm{h}_{\overline{H}_1,\dots,\overline{H}_b}(V) + \sum_{v \in M_F} \mu(v) \int_{B_v^{\mathrm{an}}} \log|f(p)| \, \mathrm{d}\mu_v(p) \\
&= -\mathrm{h}_{\overline{H}_1,\dots,\overline{H}_b}(\mathrm{cyc}(f)) + \sum_{v \in M_F} \mu(v) \int_{B_v^{\mathrm{an}}} \log|f(p)| \, \mathrm{d}\mu_v(p) \\
&= -\mathrm{h}_{\overline{\mathcal{O}}_B,\overline{H}_1,\dots,\overline{H}_b}(B) \\
&= 0,
\end{aligned}$$

which concludes the proof. □

3.3. Relative Varieties over a Global Field

Let B be a normal proper variety over a global field F and let $\pi\colon \mathcal{X} \to B$ be a dominant morphism of proper varieties over F. We denote by $K = F(B)$ the function field of B and by X the generic fiber of π, that means $X = \mathcal{X} \times_B \operatorname{Spec}(K)$ is a proper variety over K. We assume that K is equipped with the structure of an $\mathfrak{M}$-field induced by nef quasi-algebraic metrized line bundles $\overline{H}_1, \dots, \overline{H}_b$ on B as in (3.10).

In this section, we prove the main result of this chapter (Theorem 3.3.4) showing that the height $\mathrm{h}_{\pi^*\overline{H}_1,\dots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_n}(\mathcal{X})$ with respect to DSP quasi-algebraic M_F-metrized line bundles $\overline{\mathcal{L}}_i$ is equal to the height $\mathrm{h}_{\overline{L}_0,\dots,\overline{L}_n}(X)$ with respect to induced $\mathfrak{M}$-metrized line bundles $\overline{L}_i$. Note that the first height is a sum of local heights over M_F whereas the second is an integral over $\mathfrak{M}$. This generalizes Theorem 2.4 in [BPS14a] where the global field is $\mathbb{Q}$ and the metrized line bundles are induced by models over $\mathbb{Z}$.

3.3.1. Let $\overline{\mathcal{L}} = (\mathcal{L}, (\|\cdot\|_v)_v)$ be an M_F-metrized line bundle on $\mathcal{X}$. Then $\overline{\mathcal{L}}$ induces an $\mathfrak{M}$-metric on the line bundle $L = \mathcal{L} \otimes K$ on X given as follows:

For each $V \in B^{(1)}$, consider the non-Archimedean absolute value $|\cdot|_V$ on K from (3.8) and let $\mathbb{K}_V$ be the completion of an algebraic closure of the completion of K with respect to $|\cdot|_V$. We get a proper $\mathbb{K}_V^\circ$-model

$$(\mathcal{X}_V, \mathcal{L}_V) := (\mathcal{X} \times_B \operatorname{Spec} \mathbb{K}_V^\circ, \mathcal{L} \otimes \mathbb{K}_V^\circ)$$

of (X, L). By Definition 1.2.7, the model $(\mathcal{X}_V, \mathcal{L}_V)$ induces a metric $\|\cdot\|_V$ on the analytification L_V^{an} over X_V^{an} with respect to $|\cdot|_V$.

Let us fix a place $v \in M_F$. By (3.9), a generic point $p \in B_v^{\mathrm{gen}}$ induces an absolute value $|\cdot|_{v,p}$ on K. We denote by $\mathbb{K}_{v,p}$ the completion of an algebraic closure of the completion of K with respect to $|\cdot|_{v,p}$ and by $X_{v,p}^{\mathrm{an}}$ the analytification of X with respect to $|\cdot|_{v,p}$. Then the projection $\mathcal{X}_v \times_{B_v} \operatorname{Spec} \mathbb{K}_{v,p} \to \mathcal{X}_v$ induces a morphism

$$i_p \colon X_{v,p}^{\mathrm{an}} \to \mathcal{X}_v^{\mathrm{an}}. \tag{3.11}$$

Note that i_p is injective if v is an Archimedean place (cf. [BPS14b, (2.1)]), but not necessarily in the non-Archimedean case. The analytification $L_{v,p}^{\mathrm{an}}$ of L with respect to $|\cdot|_{v,p}$ can be identified with the line bundle $i_p^*\mathcal{L}_v^{\mathrm{an}}$ and we equip it with the metric $\|\cdot\|_{v,p} := i_p^*\|\cdot\|_v$.

Summing up, we obtain an $\mathfrak{M}$-metrized line bundle

$$\overline{L} = (L, (\|\cdot\|_w)_{w \in \mathfrak{M}}) \tag{3.12}$$

on X.

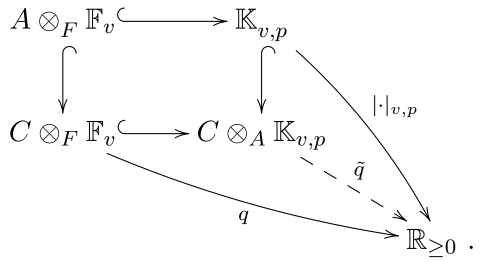
Lemma 3.3.2. *Let $i_p \colon X_{v,p}^{\mathrm{an}} \to \mathcal{X}_v^{\mathrm{an}}$ be the morphism from (3.11) and $\pi_v \colon \mathcal{X}_v^{\mathrm{an}} \to B_v^{\mathrm{an}}$ the morphism of $\mathbb{F}_v$-analytic spaces induced by $\pi \colon \mathcal{X} \to B$. Then we have*

$$i_p(X_{v,p}^{\mathrm{an}}) = \pi_v^{-1}(p).$$

Proof. We only show this for a non-Archimedean place v, the Archimedean case is established at the beginning of [BPS14b, § 2]. We may assume that $B = \operatorname{Spec}(A)$ resp. $\mathcal{X} = \operatorname{Spec}(C)$ for finitely generated F-algebras A and C. Then π corresponds to an injective F-algebra homomorphism $A \hookrightarrow C$ and we have $X = \operatorname{Spec}(C \otimes_A K)$ with $K = F(B) = \operatorname{Quot}(A)$.

Let $q \in \mathcal{X}_v^{\mathrm{an}}$, that means q is a multiplicative seminorm on $C \otimes_F \mathbb{F}_v$ satisfying $q|_{\mathbb{F}_v} = |\cdot|_v$. Then q lies in $i_p(X_{v,p}^{\mathrm{an}})$ if and only if it extends to a multiplicative seminorm $\tilde{q}$ on $C \otimes_A \mathbb{K}_{v,p}$

with $\tilde{q}|_{\mathbb{K}_{v,p}} = |\cdot|_{v,p}$. This is illustrated in the following diagram,



On the one hand, if we have such a commutative diagram, then

$$\pi_v(q) = q|_{A\otimes\mathbb{F}_v} = |\cdot|_{v,p}|_{A\otimes\mathbb{F}_v} = p.$$

On the other hand, if $\pi_v(q) = p$, then we have a multiplicative seminorm $\tilde{q}$ given by

$$C \otimes_A \mathbb{K}_{v,p} = (C \otimes_F \mathbb{F}_v) \otimes_{(A\otimes\mathbb{F}_v)} \mathbb{K}_{v,p} \longrightarrow \mathscr{H}(q) \,\widehat{\otimes}_{\mathscr{H}(p)} \mathbb{K}_{v,p} \xrightarrow{y} \mathbb{R}_{\geq 0}\,,$$

where y is some element of the non-empty Berkovich spectrum $\mathscr{M}\left(\mathscr{H}(q)\,\widehat{\otimes}_{\mathscr{H}(p)}\mathbb{K}_{v,p}\right)$ (cf. [Duc09, 0.3.2]). It follows easily that we obtain a commutative diagram as above. This proves the result. □

We need the following projection formula for heights in the proof of the main theorem.

Proposition 3.3.3. *Let $\pi\colon \mathcal{W} \to V$ be a morphism of proper varieties over a global field F of dimensions $n+b-1$ and $b-1$ respectively, with $b, n \geq 0$. Let $\overline{H}_i$, $i = 1, \ldots, b$, and $\overline{\mathcal{L}}_j$, $j = 1, \ldots, n$, be DSP quasi-algebraic line bundles on V and $\mathcal{W}$ respectively. Then*

$$\mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_1,\ldots,\overline{\mathcal{L}}_n}(\mathcal{W}) = \deg_{\mathcal{L}_1,\ldots,\mathcal{L}_n}(\mathcal{W}_\eta)\,\mathrm{h}_{\overline{H}_1,\ldots,\overline{H}_b}(V),$$

where $\mathcal{W}_\eta$ denotes the generic fiber of π. In particular, if $\dim(\pi(\mathcal{W})) \leq b-2$, then $\mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_1,\ldots,\overline{\mathcal{L}}_n}(\mathcal{W}) = 0$.

Proof. By continuity of the height, we may assume that the metrics in $\overline{H}_i$ and $\overline{\mathcal{L}}_j$ are smooth or algebraic for all i, j. We prove this proposition by induction on n. If $n = 0$, then we obtain by functoriality of the height (Proposition 3.1.8),

$$\mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b}(\mathcal{W}) = \mathrm{h}_{\overline{H}_1,\ldots,\overline{H}_b}(\pi_*(\mathcal{W})) = \deg(\mathcal{W}_\eta)\,\mathrm{h}_{\overline{H}_1,\ldots,\overline{H}_b}(V).$$

Let $n \geq 1$. We choose any invertible meromorphic section s_n of $\mathcal{L}_n$ and denote by $\|\cdot\|_n = (\|\cdot\|_{n,v})_v$ the metric of $\overline{\mathcal{L}}_n$. Then the global induction formula 3.1.11 implies

$$\begin{aligned}\mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_1,\ldots,\overline{\mathcal{L}}_n}(\mathcal{W}) = {} & \mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_1,\ldots,\overline{\mathcal{L}}_{n-1}}(\mathrm{cyc}(s_n)) \\ & - \sum_{v\in M_F} \mu(v) \int_{\mathcal{W}_v^{\mathrm{an}}} \log\|s_n\|_{n,v} \bigwedge_{i=1}^{b} \mathrm{c}_1(\pi^*\overline{H}_{i,v}) \wedge \bigwedge_{j=1}^{n-1} \mathrm{c}_1(\overline{\mathcal{L}}_{j,v}).\end{aligned}$$

If v is Archimedean, then $\bigwedge_{i=1}^{b} \mathrm{c}_1(\overline{H}_{i,v})$ is the zero measure on V_v^{an} since $\dim(V_v^{\mathrm{an}}) = b-1$. Thus, the measure in the above integral vanishes and so the integral is zero.

If v is non-Archimedean, then the metrics in $\overline{H}_{i,v}$, $i = 1, \ldots, b$, are induced by models $\mathscr{H}_i$ of $H_{i,v}^{e_i}$ on a common model $\mathscr{V}$ of V_v over $\operatorname{Spec} \mathbb{F}_v^\circ$. By linearity, we may assume that $e_i = 1$ for all i. Analogously, the metrics in $\overline{\mathcal{L}}_{j,v}$, $j = 1, \ldots, n$, are induced by models $\mathscr{L}_j$ of $\mathcal{L}_{j,v}$ on a common model $\mathscr{W}$ of $\mathcal{W}_v$.

We may assume that the morphism $\pi_v \colon \mathcal{W}_v \to V_v$ extends to a morphism $\tau \colon \mathscr{W} \to \mathscr{V}$ over $\operatorname{Spec} \mathbb{F}_v^\circ$. Indeed, let $\mathscr{W}'$ be the closure of the image of π_v in $\mathscr{W} \times_{\mathbb{F}_v^\circ} \mathscr{V}$. This is a proper model of $\mathcal{W}_v$ equipped with morphisms $\tau' \colon \mathscr{W}' \to \mathscr{V}$ and $f \colon \mathscr{W}' \to \mathscr{W}$ such that $\tau'|_{\mathcal{W}_v} = \pi_v$ and $\| \cdot \|_{\mathscr{L}_j} = \| \cdot \|_{f^* \mathscr{L}_j}$. Then replace $\mathscr{W}$ by $\mathscr{W}'$ and $\mathscr{L}_j$ by $f^* \mathscr{L}_j$.

Since the special fiber $\widetilde{\mathscr{V}}$ of $\mathscr{V}$ has dimension $b-1$, the degree with respect to $\mathscr{H}_1, \ldots, \mathscr{H}_b$ of a cycle of $\widetilde{\mathscr{V}}$ is zero. Hence, for every irreducible component Y of the special fiber of $\mathscr{W}$, we have by means of the projection formula,

$$\deg_{\tau^* \mathscr{H}_1, \ldots, \tau^* \mathscr{H}_b, \mathscr{L}_1, \ldots, \mathscr{L}_{n-1}}(Y) = \deg_{\mathscr{H}_1, \ldots, \mathscr{H}_b}(\tau_*(\mathrm{c}_1(\mathscr{L}_1) \ldots \mathrm{c}_1(\mathscr{L}_{n-1}).Y)) = 0.$$

Therefore, for each $v \in M_F$, the measure in the above integral vanishes and so the integral is zero.

Finally, we obtain by the induction hypothesis,

$$\begin{aligned} \mathrm{h}_{\pi^* \overline{H}_1, \ldots, \pi^* \overline{H}_b, \overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_n}(\mathcal{W}) &= \mathrm{h}_{\pi^* \overline{H}_1, \ldots, \pi^* \overline{H}_b, \overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{n-1}}(\mathrm{cyc}(s_n)) \\ &= \deg_{\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}}(\mathrm{cyc}(s_n)_\eta) \, \mathrm{h}_{\overline{H}_1, \ldots, \overline{H}_b}(V) \\ &= \deg_{\mathcal{L}_1, \ldots, \mathcal{L}_n}(\mathcal{W}_\eta) \, \mathrm{h}_{\overline{H}_1, \ldots, \overline{H}_b}(V), \end{aligned}$$

proving the result. □

Theorem 3.3.4. *Let B be a b-dimensional normal proper variety over a global field F and let $\overline{H}_1, \ldots, \overline{H}_b$ be nef quasi-algebraic line bundles on B. Let $K = F(B)$ be the function field of B and $(\mathfrak{M}, \mu)$ the associated structure of an $\mathfrak{M}$-field on K as in (3.10).*

Let $\pi \colon \mathcal{X} \to B$ be a dominant morphism of proper varieties over F and X the generic fiber of π. Let Y be an n-dimensional prime cycle of X and $\mathcal{Y}$ its closure in $\mathcal{X}$. For $j = 0, \ldots, n$, let $\overline{L}_j$ be an $\mathfrak{M}$-metrized line bundle on X which is induced by a DSP quasi-algebraic line bundle $\overline{\mathcal{L}}_j$ on $\mathcal{X}$ as in (3.12).

Then Y is integrable with respect to $\overline{L}_0, \ldots, \overline{L}_n$ and we have

$$\mathrm{h}_{\overline{L}_0, \ldots, \overline{L}_n}(Y) = \mathrm{h}_{\pi^* \overline{H}_1, \ldots, \pi^* \overline{H}_b, \overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n}(\mathcal{Y}). \tag{3.13}$$

Proof. By Chow's lemma (see, for instance, [GW10, Theorem 13.100]) and functoriality of the height (Proposition 3.1.8 (ii)), we reduce to the case when the proper varieties are projective over F. Then π is also projective. By (multi-)linearity of the height (Proposition 3.1.8 (i)), we may assume that the line bundles $\mathcal{L}_j$ are very ample and their M_F-metrics are semipositive. Making a finite base change and using Proposition 3.1.12, we may suppose that B and $\mathcal{X}$ are geometrically integral.

We prove this theorem by induction on the dimension of Y. If $\dim(Y) = -1$, thus $Y = \emptyset$, then Y is integrable since the local heights of Y are zero. Equation (3.13) holds in this case because $\mathcal{Y}$ is empty as well.

From now on we suppose that $\dim(Y) = n \geq 0$. Then the restriction $\pi|_{\mathcal{Y}} \colon \mathcal{Y} \to B$ is dominant. By Proposition 3.1.8 (ii), the height does not change if we restrict the corresponding metrized line bundles to $\mathcal{Y}$. So we may assume that $\mathcal{Y} = \mathcal{X}$, $Y = X$ and $n = \dim(Y) = \dim(X)$.

Let $s_0, \ldots, s_n$ be global sections of $\mathcal{L}_0, \ldots, \mathcal{L}_n$ respectively, whose Cartier divisors intersect properly on X, and consider the function

$$\rho \colon \mathfrak{M} \longrightarrow \mathbb{R}, \quad w \longmapsto \lambda_{(\overline{L}_0, s_0|_X), \ldots, (\overline{L}_n, s_n|_X)}(X, w).$$

We must show that ρ is μ-integrable and that

$$\int_{\mathfrak{M}} \rho(w) \, \mathrm{d}\mu(w) = \mathrm{h}_{\pi^*\overline{H}_1, \ldots, \pi^*\overline{H}_b, \overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n}(\mathcal{X}).$$

By the induction formula of local heights (Theorem 1.4.3 in the non-Archimedean and Theorem 1.5.13 in the Archimedean case), there is a decomposition $\rho = \rho_1 + \rho_2$ into well-defined functions $\rho_1, \rho_2 \colon \mathfrak{M} \to \mathbb{R}$ given by

$$\rho_1(w) = \lambda_{(\overline{L}_0, s_0|_X), \ldots, (\overline{L}_{n-1}, s_{n-1}|_X)}(\mathrm{cyc}(s_n|_X), w)$$

and

$$\rho_2(w) = \int_{X_w^{\mathrm{an}}} \log \|s_n|_{X_w}\|_{n,w}^{-1} \, \mathrm{c}_1(\overline{L}_{0,w}) \wedge \cdots \wedge \mathrm{c}_1(\overline{L}_{n-1,w}).$$

Moreover, we can write the cycle $\mathrm{cyc}(s_n)$ in $\mathcal{X}$ as

$$\mathrm{cyc}(s_n) = \mathrm{cyc}(s_n)_{\,\mathrm{hor}/B} + \mathrm{cyc}(s_n)_{\mathrm{vert}/B},$$

where $\mathrm{cyc}(s_n)_{\,\mathrm{hor}/B}$ contains the components which are dominant over B and $\mathrm{cyc}(s_n)_{\mathrm{vert}/B}$ contains the components not meeting the generic fiber X of π.

By the induction hypothesis, the function ρ_1 is μ-integrable and

$$\begin{aligned} \int_{\mathfrak{M}} \rho_1(w) \, \mathrm{d}\mu(w) &= \mathrm{h}_{\overline{L}_0, \ldots, \overline{L}_{n-1}}(\mathrm{cyc}(s_n|_X)) \\ &= \mathrm{h}_{\pi^*\overline{H}_1, \ldots, \pi^*\overline{H}_b, \overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_{n-1}}(\mathrm{cyc}(s_n)_{\,\mathrm{hor}/B}). \end{aligned} \tag{3.14}$$

If $w = V \in B^{(1)}$, then we just can copy the corresponding part of the proof of [BPS14b, Theorem 2.4]. In this case we obtain

$$\rho_2(V) = \sum_{\substack{\mathcal{W} \in \mathcal{X}^{(1)} \\ \pi(\mathcal{W}) = V}} \mathrm{h}_{\overline{H}_1, \ldots, \overline{H}_b}(V) \, \mathrm{ord}_{\mathcal{W}}(s_n) \deg_{\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}}(\mathcal{W}_V), \tag{3.15}$$

where $\mathcal{W}_V$ denotes the generic fiber of $\pi|_{\mathcal{W}} \colon \mathcal{W} \to V$. This formula implies the integrability of ρ_2 on $B^{(1)}$ with respect to the counting measure μ_{fin} because there are only finitely many $\mathcal{W} \in \mathcal{X}^{(1)}$ such that $\operatorname{ord}_{\mathcal{W}}(s_n) \neq 0$. By (3.15) and Proposition 3.3.3,

$$\begin{aligned}
\int_{B^{(1)}} \rho_2(w)\, \mathrm{d}\mu_{\mathrm{fin}}(w) &= \sum_{V \in B^{(1)}} \rho_2(V) \\
&= \sum_{V \in B^{(1)}} \sum_{\substack{\mathcal{W} \in \mathcal{X}^{(1)} \\ \pi(\mathcal{W}) = V}} \mathrm{h}_{\overline{H}_1, \dots, \overline{H}_b}(V) \operatorname{ord}_{\mathcal{W}}(s_n) \deg_{\mathcal{L}_0, \dots, \mathcal{L}_{n-1}}(\mathcal{W}_V) \\
&= \sum_{V \in B^{(1)}} \sum_{\substack{\mathcal{W} \in \mathcal{X}^{(1)} \\ \pi(\mathcal{W}) = V}} \operatorname{ord}_{\mathcal{W}}(s_n)\, \mathrm{h}_{\pi^*\overline{H}_1, \dots, \pi^*\overline{H}_b, \overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_{n-1}}(\mathcal{W}) \\
&\quad + \sum_{\substack{\mathcal{W} \in \mathcal{X}^{(1)} \\ \dim(\pi(\mathcal{W})) \leq b-2}} \operatorname{ord}_{\mathcal{W}}(s_n) \underbrace{\mathrm{h}_{\pi^*\overline{H}_1, \dots, \pi^*\overline{H}_b, \overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_{n-1}}(\mathcal{W})}_{=0} \\
&= \mathrm{h}_{\pi^*\overline{H}_1, \dots, \pi^*\overline{H}_b, \overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_{n-1}}(\operatorname{cyc}(s_n)_{\mathrm{vert}/B}).
\end{aligned} \tag{3.16}$$

Now, let v be a place of M_F and p a generic point of B_v^{an}. We have to show that the function

$$\rho_2(p) = \int_{X_{v,p}^{\mathrm{an}}} \log i_p^* \|s_n\|_{n,v}^{-1} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(i_p^* \overline{\mathcal{L}}_{j,v})$$

is integrable with respect to $\mu_v = \mathrm{c}_1(\overline{H}_{1,v}) \wedge \dots \wedge \mathrm{c}_1(\overline{H}_{b,v})$. Furthermore, we have to prove that

$$\int_{B_v^{\mathrm{gen}}} \rho_2(p)\, \mathrm{d}\mu_v(p) = \int_{\mathcal{X}_v^{\mathrm{an}}} \log \|s_n\|_{n,v}^{-1} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(\overline{\mathcal{L}}_{j,v}) \wedge \bigwedge_{i=1}^{b} \mathrm{c}_1(\pi^* \overline{H}_{i,v}) \tag{3.17}$$

and that this integral is zero for all but finitely many $v \in M_F$.

If $v \in M_F$ is an Archimedean place, then the proof of [BPS14b, Theorem 2.4] shows that ρ_2 is μ_v-integrable on B_v^{gen} and that the equation (3.17) holds.

From now on, we consider the case where $v \in M_F$ is non-Archimedean. We first assume that, for each $j = 0, \dots, n-1$ and $i = 1, \cdots, b$, the metrics on $\mathcal{L}_{j,v}$ and $H_{i,v}$ are algebraic. Then the function ρ_2 is μ_v-integrable because μ_v is a discrete finite measure.

We choose, for each j, a proper model $(\mathscr{X}_j, \mathscr{L}_j, e_j)$ of $(\mathcal{X}_v, \mathcal{L}_{j,v})$ over $\operatorname{Spec} \mathbb{F}_v^\circ$ that induces the metric of $\overline{\mathcal{L}}_{j,v}$. Note that we omit the place v in the notation of the models in order not to burden the notation. By linearity, we may assume that $e_j = 1$ for all j. Furthermore, we can suppose that the models $\mathscr{X}_j$ agree with a common model $\mathscr{X}$ with reduced special fiber (cf. Remark 1.2.13). In the same way, we have a proper $\mathbb{F}_v^\circ$-model $\mathscr{B}$ of B_v with reduced special fiber and, for each $i = 1, \dots, b$, a model $\mathscr{H}_i$ of $H_{i,v}$ on $\mathscr{B}$ inducing the corresponding metric. As in the proof of Proposition 3.3.3, we can asume that the morphism $\pi_v \colon \mathcal{X}_v \to B_v$ extends to a morphism $\tau \colon \mathscr{X} \to \mathscr{B}$ over $\mathbb{F}_v^\circ$.

To construct a suitable model of $X_{v,p} = X \times_K \operatorname{Spec} \mathbb{K}_{v,p}$ over $\mathbb{K}_{v,p}^\circ$, we consider the commutative diagram

$$\begin{array}{ccc}
\operatorname{Spec} \mathbb{K}_{v,p} & \longrightarrow & B \\
\downarrow & & \downarrow \\
\operatorname{Spec} \mathbb{F}_v & \longrightarrow & \operatorname{Spec} F
\end{array} .$$

The universal property of the fiber product induces a unique morphism $\operatorname{Spec}\mathbb{K}_{v,p} \to B_v$. Because $\mathscr{B}$ is proper over $\mathbb{F}_v^\circ$ and by the valuative criterion, this morphism extends to $\operatorname{Spec}\mathbb{K}_{v,p}^\circ \to \mathscr{B}$. Let $\mathscr{X}_p$ be the fiber product $\mathscr{X} \times_{\mathscr{B}} \operatorname{Spec}\mathbb{K}_{v,p}^\circ$. This is a model of $X_{v,p}$ over $\mathbb{K}_{v,p}^\circ$, indeed

$$\mathscr{X}_p \times_{\mathbb{K}_{v,p}^\circ} \operatorname{Spec}\mathbb{K}_{v,p} = \mathscr{X} \times_{\mathscr{B}} \underbrace{\mathscr{B} \times_{\mathbb{F}_v^\circ} \operatorname{Spec}\mathbb{F}_v}_{=B_v} \times_{B_v} \operatorname{Spec}\mathbb{K}_{v,p} = \mathcal{X}_v \times_{B_v} \operatorname{Spec}\mathbb{K}_{v,p} = X_{v,p}.$$

We denote the special fibers of $\mathscr{B}$, $\mathscr{X}$ and $\mathscr{X}_p$ by $\widetilde{\mathscr{B}}$, $\widetilde{\mathscr{X}}$ and $\widetilde{\mathscr{X}_p}$ respectively. By 1.1.8, there exists a formal admissible scheme $\mathfrak{X}_p$ over $\mathbb{K}_{v,p}^\circ$ with generic fiber $\mathfrak{X}_p^{\mathrm{an}} = X_{v,p}^{\mathrm{an}}$ and with reduced special fiber $\widetilde{\mathfrak{X}}_p$ such that the canonical morphism $\iota_p\colon \widetilde{\mathfrak{X}}_p \to \widetilde{\mathscr{X}_p}$ is finite and surjective. We obtain the following commutative diagram

$$\begin{array}{ccccccc}
X_{v,p}^{\mathrm{an}} & \xrightarrow{=} & X_{v,p}^{\mathrm{an}} & \xrightarrow{i_p} & \mathcal{X}_v^{\mathrm{an}} & \xrightarrow{\pi_v} & B_v^{\mathrm{an}} \\
\Big\downarrow{\scriptstyle \mathrm{red}} & & \Big\downarrow{\scriptstyle \mathrm{red}} & & \Big\downarrow{\scriptstyle \mathrm{red}} & & \Big\downarrow{\scriptstyle \mathrm{red}} \\
\widetilde{\mathfrak{X}}_p & \xrightarrow{\iota_p} & \widetilde{\mathscr{X}_p} & \xrightarrow{j_p} & \widetilde{\mathscr{X}} & \xrightarrow{\tilde{\tau}} & \widetilde{\mathscr{B}}
\end{array},$$

where red is the reduction map from 1.1.6 and 1.1.8. Note that $\widetilde{\mathscr{X}_p} = \widetilde{\mathscr{X}} \times_{\widetilde{\mathscr{B}}} \widetilde{\mathbb{K}}_{v,p}$.

By Definition 1.2.14, the left-hand side of equation (3.17) is equal to

$$\begin{aligned}
&\int_{B_v^{\mathrm{gen}}} \left(\int_{X_{v,p}^{\mathrm{an}}} \log i_p^* \|s_n\|_{n,v}^{-1} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(i_p^* \overline{\mathcal{L}}_{j,v}) \right) \bigwedge_{i=1}^{b} \mathrm{c}_1(\overline{H}_{i,v})(p) \qquad (3.18)\\
&= \sum_{Z \in \widetilde{\mathscr{B}}^{(0)}} \left(\sum_{V \in \widetilde{\mathfrak{X}}_{\xi_Z}^{(0)}} \log \|s_n(i_{\xi_Z}(\xi_V))\|_{n,v}^{-1} \deg_{\iota_{\xi_Z}^* j_{\xi_Z}^* \widetilde{\mathscr{L}}_0, \ldots, \iota_{\xi_Z}^* j_{\xi_Z}^* \widetilde{\mathscr{L}}_{n-1}}(V) \right) \deg_{\mathscr{H}_1, \ldots, \mathscr{H}_b}(Z),
\end{aligned}$$

where ξ_Z (resp. ξ_V) denotes the unique point whose reduction is the generic point of Z (resp. V).

First, we consider the inner sum. Let Z be an irreducible component of $\widetilde{\mathscr{B}}$ with generic point $\eta_Z = \mathrm{red}(\xi_Z)$. For $W \in \widetilde{\mathscr{X}}_{\xi_Z}^{(0)}$, let $\xi_W = \xi_V$ for any $V \in \widetilde{\mathfrak{X}}_{\xi_Z}^{(0)}$ with $\iota_{\xi_Z}(V) = W$. Then $i_{\xi_Z}(\xi_W)$ does not depend on the particular choice of V. Hence, Lemma 3.3.5 below implies

$$\begin{aligned}
&\sum_{V \in \widetilde{\mathfrak{X}}_{\xi_Z}^{(0)}} \log \|s_n(i_{\xi_Z}(\xi_V))\|_{n,v}^{-1} \deg_{\left(\iota_{\xi_Z}^* j_{\xi_Z}^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(V) \\
&\qquad = \sum_{W \in \widetilde{\mathscr{X}}_{\xi_Z}^{(0)}} \log \|s_n(i_{\xi_Z}(\xi_W))\|_{n,v}^{-1} \, \mathrm{m}(W, \widetilde{\mathscr{X}}_{\xi_Z}) \, \deg_{\left(j_{\xi_Z}^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(W), \qquad (3.19)
\end{aligned}$$

where $\mathrm{m}(W, \widetilde{\mathscr{X}}_{\xi_Z})$ denotes the multiplicity of W in $\widetilde{\mathscr{X}}_{\xi_Z}$.

By [EGAI, Ch. 0, (2.1.8)], there is a bijective map

$$\left\{ Y \in \widetilde{\mathscr{X}}^{(0)} \mid \tilde{\tau}(Y) = Z \right\} \longrightarrow \widetilde{\mathscr{X}}_{\eta_Z}^{(0)}, \quad Y \longmapsto Y_{\eta_Z}. \qquad (3.20)$$

The special fiber of $\mathscr{B}$ is reduced and hence, applying [Ber90, 2.4.4(ii)] and using the compatibility of reduction and algebraic closure, we deduce $\widetilde{\mathbb{K}}_{v,\xi_Z} = \widetilde{\overline{\mathscr{H}(\xi_Z)}} = \overline{\kappa(\eta_Z)}$. Therefore, $\widetilde{\mathscr{X}}_{\xi_Z} = \widetilde{\mathscr{X}} \times_{\widetilde{\mathscr{B}}} \widetilde{\mathbb{K}}_{v,\xi_Z}$ is the base change of the fiber $\widetilde{\mathscr{X}}_{\eta_Z} = \widetilde{\mathscr{X}} \times_{\widetilde{\mathscr{B}}} \kappa(\eta_Z)$ by $\kappa(\eta_Z) \to \overline{\kappa(\eta_Z)}$. Thus, by [Sta15, Lemma 32.6.10], we obtain a surjective map

$$\widetilde{\mathscr{X}}^{(0)}_{\xi_Z} \longrightarrow \widetilde{\mathscr{X}}^{(0)}_{\eta_Z}. \tag{3.21}$$

Composing the maps (3.20) and (3.21), we get a canonical surjective map

$$\widetilde{\mathscr{X}}^{(0)}_{\xi_Z} \longrightarrow \Big\{ Y \in \widetilde{\mathscr{X}}^{(0)} \mid \tilde{\tau}(Y) = Z \Big\}$$

with finite fibers. More precisely, for each irreducible component Y in $\widetilde{\mathscr{X}}$ with $\tilde{\tau}(Y) = Z$, the scheme $Y_{\xi_Z} = Y \times_Z \operatorname{Spec} \widetilde{\mathbb{K}}_{v,\xi_Z}$ is a finite union of (non-necessarily reduced) irreducible components of $\widetilde{\mathscr{X}}^{(0)}_{\xi_Z}$. Since $i_{\xi_Z}(\xi_W) = \xi_Y$ for $W \in Y^{(0)}_{\xi_Z}$, we deduce

$$\begin{aligned} &\sum_{W \in \widetilde{\mathscr{X}}^{(0)}_{\xi_Z}} \log \|s_n(i_{\xi_Z}(\xi_W))\|^{-1}_{n,v} \ \mathrm{m}(W, \widetilde{\mathscr{X}}_{\xi_Z}) \ \deg_{\left(j^*_{\xi_Z}\widetilde{\mathscr{L}}_k\right)_{k=0,\dots,n-1}}(W) \\ &\qquad = \sum_{\substack{Y \in \widetilde{\mathscr{X}}^{(0)} \\ \tilde{\tau}(Y) = Z}} \log \|s_n(\xi_Y)\|^{-1}_{n,v} \ \deg_{\left(j^*_{\xi_Z}\widetilde{\mathscr{L}}_k\right)_{k=0,\dots,n-1}}(Y_{\xi_Z}). \end{aligned} \tag{3.22}$$

Let Y be an irreducible component of $\widetilde{\mathscr{X}}$ with generic point η_Y such that $\tilde{\tau}(Y) = Z$. Then Lemma 3.3.6 below shows that

$$\deg_{\mathscr{L}_0,\dots,\mathscr{L}_{n-1},\tau^*\mathscr{H}_1,\dots,\tau^*\mathscr{H}_b}(Y) = \deg_{j^*_{\xi_Z}\widetilde{\mathscr{L}}_0,\dots,j^*_{\xi_Z}\widetilde{\mathscr{L}}_{n-1}}(Y_{\xi_Z}) \deg_{\mathscr{H}_1,\dots,\mathscr{H}_b}(Z). \tag{3.23}$$

Combining the equations (3.18), (3.19), (3.22) and (3.23), we obtain

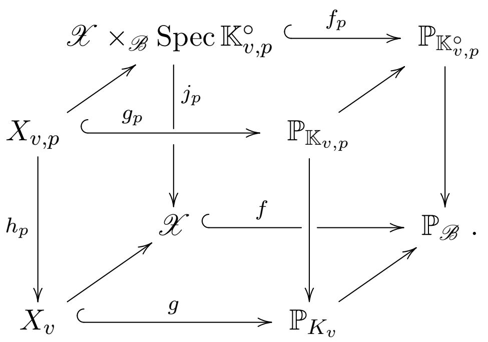
$$\begin{aligned} &\int_{B^{\mathrm{gen}}_v} \Big(\int_{X^{\mathrm{an}}_{v,p}} \log i^*_p \|s_n\|^{-1}_{n,v} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(i^*_p \overline{\mathcal{L}}_{j,v}) \Big) \bigwedge_{i=1}^{b} \mathrm{c}_1(\overline{H}_{i,v})(p) \\ &= \sum_{Z \in \widetilde{\mathscr{B}}^{(0)}} \sum_{\substack{Y \in \widetilde{\mathscr{X}}^{(0)} \\ \tilde{\tau}(Y) = Z}} \log \|s_n(\xi_Y)\|^{-1}_{n,v} \ \deg_{\mathscr{L}_0,\dots\mathscr{L}_{n-1},\tau^*\mathscr{H}_1,\dots,\tau^*\mathscr{H}_b}(Y) \\ &= \sum_{Y \in \widetilde{\mathscr{X}}^{(0)}} \log \|s_n(\xi_Y)\|^{-1}_{n,v} \ \deg_{\mathscr{L}_0,\dots\mathscr{L}_{n-1},\tau^*\mathscr{H}_1,\dots,\tau^*\mathscr{H}_b}(Y) \\ &= \int_{\mathcal{X}^{\mathrm{an}}_v} \log \|s_n\|^{-1}_{n,v} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(\overline{\mathcal{L}}_{j,v}) \wedge \bigwedge_{i=1}^{b} \mathrm{c}_1(\pi^* \overline{H}_{i,v}) \ , \end{aligned}$$

using in the next-to-last equality that, for an irreducible component Y of $\widetilde{\mathscr{X}}$ with $\dim(\tilde{\tau}(Y)) \leq b-1$, the degree is zero. This proves equation (3.17) in the algebraic case.

We next assume that, for each $j = 0, \dots, n$, the metric $\|\cdot\|_{j,v}$ on $\mathcal{L}_{j,v}$ is algebraic, but that the metrics on $H_{i,v}$, $i = 1, \dots, b$, are not necessarily algebraic. For this case, we once again show that ρ_2 is μ_v-integrable and that the equality (3.17) holds.

As in the previous case, we may assume that, for each $j = 0, \ldots, n$, there is a proper model $(\mathscr{L}_j, \mathscr{X})$ of $(\mathcal{L}_{j,v}, \mathcal{X}_v)$ over $\mathbb{F}_v^\circ$ inducing the corresponding metric. We choose any projective model $\mathscr{B}$ over $\mathbb{F}_v^\circ$ of the projective variety B_v and suppose, as in the previous case, that $\pi_v\colon \mathcal{X}_v \to B_v$ extends to a proper morphism $\tau\colon \mathscr{X} \to \mathscr{B}$. Because $\mathcal{X}_v$ is projective over $\mathbb{F}_v$ and by [Gub03, Proposition 10.5], we may assume that $\mathscr{X}$ is projective over $\mathbb{F}_v^\circ$ and thus, τ is projective. Using Serre's theorem (see [GW10, Theorem 13.62]), the line bundle $\mathscr{L}_j$ is the difference of two very ample line bundles relative to τ. By multilinearity of the height, we reduce to the case where $\mathscr{L}_j$ is very ample relative to τ. Because $\mathscr{B}$ is projective over $\mathbb{F}_v^\circ$, we deduce by [GW10, Summary 13.71 (3)] that there is a closed immersion $f_j\colon \mathscr{X} \hookrightarrow \mathbb{P}^{N_j}_{\mathscr{B}}$ such that $\mathscr{L}_j \simeq f_j^* \mathcal{O}_{\mathbb{P}^{N_j}_{\mathscr{B}}}(1)$.

For projective spaces $\mathbb{P}^{N_j}$, $j = 0, \ldots, n$, let $\mathbb{P} := \mathbb{P}^{N_0} \times \cdots \times \mathbb{P}^{N_n}$ be the multiprojective space and let $\mathcal{O}_{\mathbb{P}}(e_j)$ be the pullback of $\mathcal{O}_{\mathbb{P}^{N_j}}(1)$ by the j-th projection. Since B is geometrically integral, we have the function field $K_v = \mathbb{F}_v(B_v)$ and we define $X_v = \mathcal{X}_v \times_{B_v} K_v$ and $L_{j,v} = \mathcal{L}_{j,v} \otimes K_v$. We obtain the following commutative diagram



Note that each horizontal arrow is a closed immersion because f is a closed immersion and the other morphisms are obtained by base change.

Let $p \in B_v^{\mathrm{gen}}$. Then the metric $\|\cdot\|_{v,p} = i_p^* \|\cdot\|_v$ on $L_{j,v,p} = g_p^* \mathcal{O}_{\mathbb{P}_{\mathbb{K}_{v,p}}}(e_j)$ is induced by

$$j_p^* \mathscr{L}_j = j_p^* f^* \mathcal{O}_{\mathbb{P}_{\mathscr{B}}}(e_j) = f_p^* \mathcal{O}_{\mathbb{P}_{\mathbb{K}_{v,p}^\circ}}(e_j).$$

Hence, $\overline{L}_{j,v,p} = g_p^* \overline{\mathcal{O}}_{\mathbb{P}_{\mathbb{K}_{v,p}}}(e_j)$, where $\overline{\mathcal{O}}_{\mathbb{P}_{\mathbb{K}_{v,p}}}(e_j)$ is endowed with the canonical metric. By Proposition 3.2.5, the field K_v together with $(B_v^{\mathrm{gen}}, \mu_v)$ is a B_v^{gen}-field in the sense of [Gub02, 5.2]. Therefore, [Gub02, Proposition 5.3.7(d)] says that every n-dimensional cycle on $\mathbb{P}_{K_v}$ is μ_v-integrable on B_v^{gen} with respect to $\overline{\mathcal{O}}_{\mathbb{P}_{K_v}}(e_0), \ldots, \overline{\mathcal{O}}_{\mathbb{P}_{K_v}}(e_n)$. Since integrability is closed under tensor product and pullback (see 3.1.6), the local height ρ is μ_v-integrable on B_v^{gen}. By the induction hypothesis, we deduce that $\rho_2 = \rho - \rho_1$ is also μ_v-integrable on B_v^{gen}.

For proving the equality (3.17), we study ρ in more detail. We can choose global sections t_j of $\mathcal{O}_{\mathbb{P}_{K_v}}(e_j)$, $j = 0, \ldots, n$, such that

$$(|\mathrm{div}(g^* t_0)| \cup |\mathrm{div}(s_{0,v})|) \cap \cdots \cap (|\mathrm{div}(g^* t_n)| \cup |\mathrm{div}(s_{n,v})|) \cap X_v = \emptyset.$$

Then we get, by Proposition 1.3.6 (iii) and (v),

$$\begin{aligned}\rho(p) &= \lambda_{(\overline{L}_0,s_0),\ldots,(\overline{L}_n,s_n)}(X,p)\\ &= \lambda_{(\overline{\mathcal{O}}_{\mathbb{P}_{\mathbb{K}_v}}(e_0),t_0),\ldots,(\overline{\mathcal{O}}_{\mathbb{P}_{\mathbb{K}_v}}(e_n),t_n)}(g_*(X_v),p) + \sum_{j=0}^{n} \log\left|\frac{g^*t_j}{s_{j,v}}(Y_j)\right|_{v,p}, \end{aligned} \tag{3.24}$$

where Y_j is any zero dimensional representative of the refined intersection

$$\operatorname{div}(g^*t_0)\ldots\operatorname{div}(g^*t_{j-1}).\operatorname{div}(s_{j+1,v})\ldots\operatorname{div}(s_{n,v}).X_v\ .$$

We can express ρ in terms of the Chow form of the n-dimensional subvariety X_v of the multiprojective space $\mathbb{P}_{K_v}$. This is a multihomogenous polynomial $F_{X_v}(\boldsymbol{\xi}_0,\ldots,\boldsymbol{\xi}_n)$ with coefficients in K_v and in the variables $\boldsymbol{\xi}_j = (\xi_{j0},\ldots,\xi_{jN_j})$ viewed as dual coordinates on $\mathbb{P}^{N_j}_{K_v}$ (see [Gub02, Remark 2.4.17] for details). By (3.24) and [Gub02, Example 4.5.16], we obtain

$$\rho(p) = \log|F_{X_v}|_{v,p} - \log|F_{X_v}(\mathbf{t}_0,\ldots,\mathbf{t}_n)|_{v,p} + \sum_{j=0}^{n}\log\left|\frac{g^*t_j}{s_{j,v}}(Y_j)\right|_{v,p}, \tag{3.25}$$

where in the first term we use the Gauss norm and in the second term $\mathbf{t}_j$ denotes the dual coordinate of t_j.

The next goal is to express $\rho(p)$ in the form $\log\|s\|$ in order to apply the induction formula (Theorem 1.4.3). The last two summands in (3.25) already have this form since a rational function is an invertible meromorphic section on the trivial bundle. For the first term, let $F_{X_v}(\boldsymbol{\xi}) = \sum_{\mathbf{m}} a_{\mathbf{m}}\boldsymbol{\xi}^{\mathbf{m}}$, where $a_{\mathbf{m}} \in K_v$, $\boldsymbol{\xi} = (\boldsymbol{\xi}_0,\ldots,\boldsymbol{\xi}_n)$ and we use the usual multi-index notation. Since F_{X_v} is only unique up to multiples of ${K_v}^{\times}$, we may assume that there exists an $\mathbf{m}'$ such that $a_{\mathbf{m}'} = 1$. Let N be the number of the multi-indices $\mathbf{m}$ with $a_{\mathbf{m}} \neq 0$. We consider the rational map

$$\phi\colon B_v \dashrightarrow \mathbb{P}^{N-1}_{\mathbb{F}_v}, \quad x \longmapsto (a_{\mathbf{m}}(x))_{\mathbf{m}}.$$

Using a blow-up of B_v as in [Har77, Example II.7.17.3] and functoriality of the measure μ_v (Proposition 1.3.11 (ii)), we may assume that ϕ is a morphism. Let $\|\cdot\|$ be the pullback of the canonical metric on $\mathcal{O}_{\mathbb{P}^{N-1}_{\mathbb{F}_v}}(1)$ and let $x_{\mathbf{m}'}$ be the coordinate of $\mathbb{P}^{N-1}_{\mathbb{F}_v}$ corresponding to $\mathbf{m}'$, considered as a global section of $\mathcal{O}(1)$. Then we have

$$\log\|\phi^*x_{\mathbf{m}'}(p)\|^{-1} = \log\max_{\mathbf{m}}\left|\frac{a_{\mathbf{m}}}{a_{\mathbf{m}'}}(p)\right| = \log\max_{\mathbf{m}}|a_{\mathbf{m}}(p)| = \log|F_{X_v}|_{v,p}\,.$$

Hence, $\rho(p)$ is of the form $\log\|s\|$ for a suitable DSP metrized line bundle on B_v and an invertible meromorphic section s.

Now, for each $i = 1,\ldots,n$, we choose a sequence of algebraic semipositive metrics $(\|\cdot\|_{i,v,k})_{k\in\mathbb{N}}$ on $H_{i,v}$ that converges to the semipositive metric $\|\cdot\|_{i,v}$ on H_i. Denote $\overline{H}_{i,v,k} = (H_{i,v},\|\cdot\|_{i,v,k})$ and set

$$\mu_{v,k} = \mathrm{c}_1(\overline{H}_{1,v,k})\wedge\cdots\wedge\mathrm{c}_1(\overline{H}_{b,v,k}).$$

Applying, for each $k\in\mathbb{N}$, the induction formula (Theorem 1.4.3) to $\int_{B_v^{\mathrm{gen}}}\rho(p)\,\mathrm{d}\mu_{v,k}(p)$, then using the continuity of local heights with respect to metrics (see 1.3.5) and applying the induction formula again, we obtain

$$\lim_{k\to\infty} \int_{B_v^{\mathrm{gen}}} \rho(p)\, \mathrm{d}\mu_{v,k}(p) = \int_{B_v^{\mathrm{gen}}} \rho(p)\, \mathrm{d}\mu_v(p). \tag{3.26}$$

Analogously we can show this for the local height ρ_1 and hence, we get

$$\lim_{k\to\infty} \int_{B_v^{\mathrm{gen}}} \rho_2(p)\, \mathrm{d}\mu_{v,k}(p) = \int_{B_v^{\mathrm{gen}}} \rho_2(p)\, \mathrm{d}\mu_v(p). \tag{3.27}$$

On the other hand, using Theorem 1.4.3 as above,

$$\begin{aligned} &\lim_{k\to\infty} \int_{\mathcal{X}_v^{\mathrm{an}}} \log \|s_n\|_{n,v} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(\overline{\mathcal{L}}_{j,v}) \wedge \bigwedge_{i=1}^{b} \mathrm{c}_1(\pi^*\overline{H}_{i,v,k}) \\ &= \int_{\mathcal{X}_v^{\mathrm{an}}} \log \|s_n\|_{n,v} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(\overline{\mathcal{L}}_{j,v}) \wedge \bigwedge_{i=1}^{b} \mathrm{c}_1(\pi^*\overline{H}_{i,v}). \end{aligned} \tag{3.28}$$

Thus, the equality (3.17) for semipositive metrics on $H_{i,v}$ and algebraic metrics on $\mathcal{L}_{j,v}$ follows by (3.27), (3.28) and the algebraic case.

In the last step, we assume that the metrics on $H_{i,v}$ and $\mathcal{L}_{j,v}$ are semipositive and not necessarily algebraic. We can proceed similarly to the corresponding part in [BPS14b, Theorem 2.4] and choose, for each $j = 0, \dots, n$, a sequence of algebraic semipositive metrics $(\|\cdot\|_{j,v,k})_{k\in\mathbb{N}}$ on $\mathcal{L}_{j,v}$ that converges to $\|\cdot\|_{j,v}$. For $p \in B_v^{\mathrm{gen}}$, we set

$$\rho_{2,k}(p) := \int_{X_{v,p}^{\mathrm{an}}} \log i_p^*\|s_n\|_{n,v,k}^{-1} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(i_p^*\overline{\mathcal{L}}_{j,v,k}).$$

By the induction formula 1.2.16 and Proposition 1.2.12 (iii), we obtain for each $k, l \in \mathbb{N}$,

$$\begin{aligned} |\rho_{2,k}(p) - \rho_{2,l}(p)| &= \Big| \lambda_{(\overline{L}_{0,k},s_0),\dots,(\overline{L}_{n,k},s_n)}(X,p) - \lambda_{(\overline{L}_{0,k},s_0),\dots,(\overline{L}_{n-1,k},s_{n-1})}(\mathrm{cyc}(s_n|_X),p) \\ &\quad - \lambda_{(\overline{L}_{0,l},s_0),\dots,(\overline{L}_{n,l},s_n)}(X,p) + \lambda_{(\overline{L}_{0,l},s_0),\dots,(\overline{L}_{n-1,l},s_{n-1})}(\mathrm{cyc}(s_n|_X),p) \Big| \\ &\leq \sum_{j=0}^{n} \mathrm{d}(\|\cdot\|_{j,v,k}, \|\cdot\|_{j,v,l}) \deg_{L_0,\dots,L_{j-1},L_{j+1},\dots,L_n}(X) \\ &\quad + \sum_{j=0}^{n-1} \mathrm{d}(\|\cdot\|_{j,v,k}, \|\cdot\|_{j,v,l}) \deg_{L_0,\dots,L_{j-1},L_{j+1},\dots,L_{n-1}}(\mathrm{cyc}(s_n|_X)). \end{aligned}$$

Hence, the sequence $(\rho_{2,k})_{k\in\mathbb{N}}$ converges uniformly to ρ_2. The measure μ_v has finite total mass and, by the previous case, the functions $\rho_{2,k}$ are μ_v-integrable. So, we deduce that ρ_2 is μ_v-integrable and that

$$\lim_{k\to\infty} \int_{B_v^{\mathrm{gen}}} \rho_{2,k}\, \mathrm{d}\mu_v(p) = \int_{B_v^{\mathrm{gen}}} \rho_2(p)\, \mathrm{d}\mu_v(p).$$

Thus, using (3.17) for the functions $\rho_{2,k}$ and applying the induction formula 1.4.3, the equality (3.17) also holds in the case when all the metrics are semipositive.

By Proposition 3.1.11, the integral in (3.17) is zero for all but finitely many $v \in M_F$ because the line bundles $\pi^*\overline{H}_1, \ldots, \pi^*\overline{H}_b, \overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n$ are quasi-algebraic.

In conclusion, the function $\rho = \rho_1 + \rho_2$ is μ-integrable and we obtain, by using the induction hypothesis (3.14), (3.16), (3.17) and the global induction formula 3.1.11,

$$\begin{aligned}
\mathrm{h}_{\overline{L}_0,\ldots,\overline{L}_n}(X) &= \int_M \rho_1(w)\,\mathrm{d}\mu(w) + \int_{B^{(1)}} \rho_2(w)\,\mathrm{d}\mu_{\mathrm{fin}}(w) + \sum_{v\in M_F} \mu(v) \int_{B_v^{\mathrm{gen}}} \rho_2(p)\,\mathrm{d}\mu_v(p) \\
&= \mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_0,\ldots,\overline{\mathcal{L}}_{n-1}}(\mathrm{cyc}(s_n)_{\ \mathrm{hor}/B}) \\
&\quad + \mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_0,\ldots,\overline{\mathcal{L}}_{n-1}}(\mathrm{cyc}(s_n)_{\mathrm{vert}/B}) \\
&\quad + \sum_{v\in M_F} \mu(v) \int_{\mathcal{X}_v^{\mathrm{an}}} \log\|s_n\|_{n,v}^{-1} \bigwedge_{j=0}^{n-1} \mathrm{c}_1(\overline{\mathcal{L}}_{j,v}) \wedge \bigwedge_{i=1}^{b} \mathrm{c}_1(\pi^*\overline{H}_{i,v}) \\
&= \mathrm{h}_{\pi^*\overline{H}_1,\ldots,\pi^*\overline{H}_b,\overline{\mathcal{L}}_0,\ldots,\overline{\mathcal{L}}_n}(\mathcal{X}),
\end{aligned}$$

proving the theorem. □

Lemma 3.3.5. *Let notation be as in the proof of Theorem 3.3.4, in particular $W \in \widetilde{\mathscr{X}}_{\xi_Z}^{(0)}$. Then,*

$$\sum_{\substack{V\in\widetilde{\mathfrak{X}}_{\xi_Z}^{(0)} \\ \iota_{\xi_Z}(V)=W}} \deg_{\left(\iota_{\xi_Z}^* j_{\xi_Z}^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(V) = \mathrm{m}(W, \widetilde{\mathscr{X}}_{\xi_Z}) \cdot \deg_{\left(j_{\xi_Z}^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(W),$$

where $\mathrm{m}(W, \widetilde{\mathscr{X}}_{\xi_Z})$ *denotes the multiplicity of W in $\widetilde{\mathscr{X}}_{\xi_Z}$.*

Proof. In order not to burden the notation, we omit each ξ_Z. For $V \in \widetilde{\mathfrak{X}}^{(0)}$, the projection formula says

$$\deg_{\left(\iota^* j^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(V) = [\mathrm{K}(V) : \mathrm{K}(\iota(V))] \deg_{\left(j^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(\iota(V)). \tag{3.29}$$

Let π be any non-zero element in the maximal ideal $\mathbb{K}_{v,\xi_Z}^{\circ\circ}$. Applying the projection formula in [Gub98, Proposition 4.5] to the Cartier divisor $\mathrm{div}(\pi)$, we get $\iota_*(\widetilde{\mathfrak{X}}) = \mathrm{cyc}(\widetilde{\mathscr{X}})$. This implies

$$\sum_{\substack{V\in\widetilde{\mathfrak{X}}^{(0)} \\ \iota(V)=W}} [\mathrm{K}(V) : \mathrm{K}(W)] \deg_{\left(\iota^* j^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(W) = \mathrm{m}(W, \widetilde{\mathscr{X}}) \deg_{\left(j^* \widetilde{\mathscr{L}}_k\right)_{k=0,\ldots,n-1}}(W). \tag{3.30}$$

The statement follows from (3.29) and (3.30). □

Lemma 3.3.6. *We keep the notations of the proof of Theorem 3.3.4. Then,*

$$\deg_{\mathscr{L}_0,\ldots,\mathscr{L}_{n-1},\tau^*\mathscr{H}_1,\ldots,\tau^*\mathscr{H}_b}(Y) = \deg_{j_{\xi_Z}^*\widetilde{\mathscr{L}}_0,\ldots,j_{\xi_Z}^*\widetilde{\mathscr{L}}_{n-1}}(Y_{\xi_Z}) \deg_{\mathscr{H}_1,\ldots,\mathscr{H}_b}(Z).$$

Proof. Let $\sum_{W\in\widetilde{\mathscr{X}}^{(n)}} n_W\, W$ be any cycle representing $\mathrm{c}_1(\mathscr{L}_0)\ldots\mathrm{c}_1(\mathscr{L}_{n-1}).Y \in \mathrm{CH}_b(\widetilde{\mathscr{X}})$

and let j_{η_Z} be the projection $\widetilde{\mathscr{X}}_{\eta_Z} \to \widetilde{\mathscr{X}}$. Since $\tilde{\tau}_*(W) = 0$ if $\dim(\tau(W)) \leq b-1$, we obtain

$$\begin{aligned}\tilde{\tau}_*\left(\sum_{W\in\widetilde{\mathscr{X}}^{(n)}} n_W\, W\right) &= \sum_{\substack{W\in\widetilde{\mathscr{X}}^{(n)}\\ \dim(\tilde{\tau}(W))=n}} n_W\,[\mathrm{K}(W):\mathrm{K}(\tilde{\tau}(W))]\cdot\tilde{\tau}(W)\\ &= \sum_{\substack{W\in\widetilde{\mathscr{X}}^{(n)}\\ \tilde{\tau}(W)=Z}} n_W\,[\mathrm{K}(W):\mathrm{K}(Z)]\cdot Z\\ &= \deg\left(\sum_{W\in\widetilde{\mathscr{X}}^{(n)}} n_W\, W_{\eta_Z}\right)\cdot Z\\ &= \deg\left(\mathrm{c}_1(j_{\eta_Z}^*\widetilde{\mathscr{L}_0})\ldots\mathrm{c}_1(j_{\eta_Z}^*\widetilde{\mathscr{L}}_{n-1}).Y_{\eta_Z}\right)\cdot Z\\ &= \deg_{j_{\eta_Z}^*\widetilde{\mathscr{L}}_0,\ldots,j_{\eta_Z}^*\widetilde{\mathscr{L}}_{n-1}}(Y_{\eta_Z})\cdot Z\ .\end{aligned}$$

Since the degree is stable under base change, we deduce

$$\begin{aligned}&\deg_{\mathscr{L}_0,\ldots,\mathscr{L}_{n-1},\tau^*\mathscr{H}_1,\ldots,\tau^*\mathscr{H}_b}(Y)\\ &= \deg_{\mathscr{H}_1,\ldots,\mathscr{H}_b}(\tilde{\tau}_*(\mathrm{c}_1(\mathscr{L}_0)\ldots\mathrm{c}_1(\mathscr{L}_{n-1}).Y))\\ &= \deg_{j_{\eta_Z}^*\widetilde{\mathscr{L}}_0,\ldots,j_{\eta_Z}^*\widetilde{\mathscr{L}}_{n-1}}(Y_{\eta_Z})\deg_{\mathscr{H}_1,\ldots,\mathscr{H}_b}(Z)\\ &= \deg_{j_{\xi_Z}^*\widetilde{\mathscr{L}}_0,\ldots,j_{\xi_Z}^*\widetilde{\mathscr{L}}_{n-1}}(Y_{\xi_Z})\deg_{\mathscr{H}_1,\ldots,\mathscr{H}_b}(Z)\ ,\end{aligned}$$

proving the result. □

3.4. Global Heights of Toric Varieties over Finitely Generated Fields

Following [BPS14b, § 3] closely, we apply the theory of toric varieties developed in [BPS14a] and Chapter 2, to get some combinatorial formulas for heights of non-toric varieties over global fields. Indeed, our non-discrete non-Archimedean toric geometry is necessary since the measure space $\mathfrak{M}$ from (3.10) contains arbitrary non-Archimedean absolute values, in contrast to the measure space considered in [BPS14b, § 1].

As usual, we fix a lattice $M \simeq \mathbb{Z}^n$ with dual $M^\vee = N$ and use the respective notations from Chapter 2.

At first, we consider an arbitrary M-field K with associated set of absolute values M and positive measure μ on M. Let Σ be a complete fan in $N_\mathbb{R}$ and let X_Σ be the associated proper toric variety over K with torus $\mathbb{T} = \operatorname{Spec} K[M]$.

3.4.1. Let L be a toric line bundle on X_Σ. An M-metric $\|\cdot\| = (\|\cdot\|_v)_{v\in M}$ on L is *toric* if, for each $v \in M$, the metric $\|\cdot\|_v$ on L_v is toric (see Definition 2.4.1). The *canonical M-metric* on L, denoted $\|\cdot\|_{\mathrm{can}}$, is given, for each $v \in M$, by the canonical metric on L_v (see Definition 2.4.12). We will write $\overline{L}^{\mathrm{can}} = (L, \|\cdot\|_{\mathrm{can}})$.

Let s be a toric section on L and Ψ the associated virtual support function. Then a toric M-metric $(\|\cdot\|_v)_v$ on L induces a family $(\psi_{\overline{L},s,v})_{v\in M}$ of real-valued functions on $N_\mathbb{R}$ as in Definition 2.4.6. If $\|\cdot\|$ is semipositive, then each $\psi_{\overline{L},s,v}$ is concave and we obtain a family $(\vartheta_{\overline{L},s,v})_{v\in M}$ of concave functions on Δ_Ψ called *v-adic roof functions* (cf. Definition

2.6.4 and Remark 2.6.7). When $\overline{L}$ and s are clear from the context, we also denote $\psi_{\overline{L},s,v}$ by ψ_v and $\vartheta_{\overline{L},s,v}$ by ϑ_v.

Proposition 3.4.2. *For each $i = 0, \dots, t$, let $\overline{L}_i$ be a toric line bundle on X_Σ equipped with a DSP toric M-metric and denote by $\overline{L}_i^{\text{can}}$ the same toric line bundle endowed with the canonical M-metric. Let Y be either the closure of an orbit or the image of a proper toric morphism, of dimension t. Then Y is integrable with respect to $\overline{L}_0^{\text{can}}, \dots, \overline{L}_t^{\text{can}}$ and*

$$\mathrm{h}_{\overline{L}_0^{\text{can}}, \dots, \overline{L}_t^{\text{can}}}(Y) = 0. \tag{3.31}$$

Furthermore, if Y is integrable with respect to $\overline{L}_0, \dots, \overline{L}_t$, then the global height is given by

$$\mathrm{h}_{\overline{L}_0, \dots, \overline{L}_t}(Y) = \int_M \lambda^{\text{tor}}_{\overline{L}_0, \dots, \overline{L}_t}(Y, v)\, \mathrm{d}\mu(v), \tag{3.32}$$

where $\lambda^{\text{tor}}_{\overline{L}_0, \dots, \overline{L}_t}(Y, v) = \lambda^{\text{tor}}_{\overline{L}_{0,v}, \dots, \overline{L}_{t,v}}(Y_v)$ is the toric local height from Definition 2.6.1.

Proof. The first statement and equation (3.31) can be shown using the same arguments as in [BPS14a, Proposition 5.2.4]. Reducing to $Y = X_\Sigma$ and $L_0 = \dots = L_n = L$, the proof is based on an inductive argument over the dimension of X_Σ, using, for each $v \in M$, the local induction formula and the fact that, for a toric section s of L, we have as in (2.23),

$$\int_{X^{\text{an}}_{\Sigma,v}} \log \|s\|_{\text{can},v}\, \mathrm{c}_1(\overline{L}_v^{\text{can}})^n = 0.$$

The second equation follows easily from the first one. ☐

Corollary 3.4.3. *Let $\overline{L} = (L, (\|\cdot\|_v)_v)$ be a toric line bundle on X_Σ equipped with a semipositive toric M-metric. Choose any toric section s of L and denote by Ψ the corresponding support function on Σ. If X_Σ is integrable with respect to $\overline{L}$, then*

$$\mathrm{h}_{\overline{L}}(X_\Sigma) = (n+1)! \int_M \int_{\Delta_\Psi} \vartheta_{\overline{L},s,v}\, \mathrm{dvol}_M\, \mathrm{d}\mu(v).$$

Proof. This is a direct consequence of Proposition 3.4.2 and the formulas for the toric local height (Theorem 2.6.6 if v is non-Archimedean, and [BPS14a, Theorem 5.1.6] else). ☐

Now we consider the particular case of an $\mathfrak{M}$-field which is induced by a variety over a global field as in section 3.2. Let B be a b-dimensional normal proper variety over a global field F and let $\overline{H}_1, \dots, \overline{H}_b$ be nef quasi-algebraic metrized line bundles on B. This provides the function field $K = F(B)$ with the structure $(\mathfrak{M}, \mu)$ of an $\mathfrak{M}$-field as in (3.10). Let X be an n-dimensional proper toric variety over K with torus $\mathbb{T} = \operatorname{Spec} K[M]$, described by a complete fan Σ in $N_\mathbb{R}$. We choose a base-point-free toric line bundle L on X together with a toric section s and denote by Ψ the associated support function on Σ.

Let $\pi\colon \mathcal{X} \to B$ be a dominant morphism of proper varieties over F such that X is the generic fiber of π. We equip L with a toric $\mathfrak{M}$-metric $\|\cdot\|$ such that $\overline{L} = (L, \|\cdot\|)$ is induced by a semipositive quasi-algebraic M_F-metrized line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ as in (3.12). Then it follows easily that $\overline{L}$ is also semipositive and so, for each $v \in \mathfrak{M}$, the function ψ_v is concave.

The following result generalizes Corollary 3.1 in [BPS14b], where the global field is $\mathbb{Q}$ and the metrized line bundles are induced by models over $\mathbb{Z}$. It is essentially based on our main theorems 2.6.6 and 3.3.4.

Corollary 3.4.4. *Let notation be as above. Then the function*

$$\mathfrak{M} \longrightarrow \mathbb{R}, \quad w \longmapsto \int_{\Delta_\Psi} \vartheta_{\overline{L},s,w}(m)\,\mathrm{dvol}_M(m) \tag{3.33}$$

is μ-integrable and,

$$\mathrm{h}_{\pi^*\overline{H}_1,\dots,\pi^*\overline{H}_b,\overline{\mathcal{L}},\dots,\overline{\mathcal{L}}}(\mathcal{X}) = \mathrm{h}_{\overline{L}}(X) = (n+1)! \int_{\mathfrak{M}} \int_{\Delta_\Psi} \vartheta_w(m)\,\mathrm{dvol}_M(m)\,\mathrm{d}\mu(w). \tag{3.34}$$

Proof. By Theorem 2.6.6 (non-Archimedean case) and [BPS14a, Theorem 5.1.6] (Archimedean case), we have

$$(n+1)! \int_{\Delta_\Psi} \vartheta_w\,\mathrm{dvol}_M = \lambda^{\mathrm{tor}}_{\overline{L}_{0,w},\dots,\overline{L}_{n,w}}(X_w). \tag{3.35}$$

Hence, Theorem 3.3.4 implies the μ-integrability of the function (3.33). The first equality of (3.34) is Theorem 3.3.4. The second follows readily from (3.32) and (3.35). □

Proposition 3.4.5. *We use the same notation as above.*

(i) For each $m \in \Delta_\Psi$, the function $\mathfrak{M} \longrightarrow \mathbb{R}$, $w \longmapsto \vartheta_w(m)$ is μ-integrable.

(ii) The function

$$\vartheta_{\overline{L},s} \colon \Delta_\Psi \longrightarrow \mathbb{R}, \quad m \longmapsto \int_{\mathfrak{M}} \vartheta_{\overline{L},s,w}(m)\,\mathrm{d}\mu(w)$$

is continuous and concave.

(iii) The function $\mathfrak{M} \times \Delta_\Psi \longrightarrow \mathbb{R}$, $(w,m) \longmapsto \vartheta_w(m)$ is $(\mu \times \mathrm{vol}_M)$-integrable.

(iv) We have

$$\mathrm{h}_{\pi^*\overline{H}_1,\dots,\pi^*\overline{H}_b,\overline{\mathcal{L}},\dots,\overline{\mathcal{L}}}(\mathcal{X}) = \mathrm{h}_{\overline{L}}(X) = (n+1)! \int_{\Delta_\Psi} \vartheta_{\overline{L},s}(m)\,\mathrm{dvol}_M(m),$$

where $\vartheta_{\overline{L},s}$ is the function in (ii).

Proof. The proof of (i)–(iii) respectively (iv) is analogous to [BPS14b, Theorem 3.2 respectively Corollary 3.4] using Corollary 3.4.4 in place of [BPS14b, Corollary 3.1]. It utilizes in an essential way that ϑ_w is concave (see Theorem 2.5.8 and Remark 2.5.9). □

3.5. Heights of Translates of Subtori over the Function Field of an Elliptic Curve

In [BPS14b, § 4], the corresponding formulas in section 3.4 are particularized to the case when X is the normalization of a translate of a subtorus in the projective space and canonical metrics. We will recall their statements in our setting and apply these to the case of the function field of an elliptic curve.

Let B be a b-dimensional normal proper variety over a global field F and let $\overline{H}_1, \dots, \overline{H}_b$ be nef quasi-algebraic M_F-metrized line bundles on B. We equip $K = F(B)$ with the structure $(\mathfrak{M}, \mu)$ of an $\mathfrak{M}$-field as in (3.10).

For $r \geq 1$, let us consider the projective space $\mathbb{P}^r_B = \mathbb{P}^r_F \times_F B$ over B and the universal line bundle $\mathcal{O}_{\mathbb{P}^r_B}(1)$. We equip $\mathcal{O}_{\mathbb{P}^r_B}(1)$ with the metric obtained by pulling back the canonical M_F-metric of $\mathcal{O}_{\mathbb{P}^r_F}(1)$ and denote this by $\overline{\mathcal{O}(1)} = \overline{\mathcal{O}_{\mathbb{P}^r_B}(1)}$.

For $\boldsymbol{m}_j \in \mathbb{Z}^n$ and $f_j \in K^\times$, $j = 0, \dots, r$, we consider the morphism

$$\mathbb{G}^n_{\mathrm{m},K} \longrightarrow \mathbb{P}^r_K, \quad \boldsymbol{t} \longmapsto (f_0 \boldsymbol{t}^{\boldsymbol{m}_0} : \dots : f_r \boldsymbol{t}^{\boldsymbol{m}_r}),$$

where $f_j \boldsymbol{t}^{\boldsymbol{m}_j} = f_j t_1^{m_{j,1}} \cdots t_n^{m_{j,n}}$. For simplicity, we assume that $\boldsymbol{m}_0 = 0$, $f_0 = 1$ and that $\boldsymbol{m}_0, \dots, \boldsymbol{m}_r$ generate $\mathbb{Z}^n$ as an abelian group. Denote by Y the closure of the image of this morphism. Then Y is a translated toric subvariety of $\mathbb{P}^r_K$ (cf. [BPS14a, Definition 3.2.6]), but not a toric variety over K since it is not necessarily normal.

Let $\mathcal{Y}$ be the closure of Y in $\mathbb{P}^r_B$ and let $\pi \colon \mathcal{Y} \to B$ be the morphism obtained by restricting $\mathbb{P}^r_B \to B$. Our goal is to compute the arithmetic intersection number $\mathrm{h}_{\pi^*\overline{H}_1, \dots, \pi^*\overline{H}_b, \overline{\mathcal{O}(1)}, \dots, \overline{\mathcal{O}(1)}}(\mathcal{Y})$ using formula (3.34). Since Y is not necessarily normal, we consider the normalization $\mathcal{X}$ of $\mathcal{Y}$ and the induced dominant morphism $\mathcal{X} \to B$ which we also denote by π. Then the generic fiber $X = \mathcal{X} \times_B K$ is a $\mathbb{G}^n_{\mathrm{m},K}$-toric variety over K. Let $\overline{\mathcal{L}}$ be the pullback of $\overline{\mathcal{O}(1)}$ to X and $\overline{L}$ the associated $\mathfrak{M}$-metrized line bundle on X as in (3.12). Then $\overline{L}$ is a toric semipositive $\mathfrak{M}$-metrized line bundle on X.

Analogously to [BPS14b, Proposition 4.1], we have the following description of the associated w-adic roof functions.

Proposition 3.5.1. *Let notation be as above and let s be the toric section of L induced by the section x_0 of $\mathcal{O}(1)$. The polytope associated to (L, s) is given by*

$$\Delta = \mathrm{conv}(\boldsymbol{m}_0, \dots, \boldsymbol{m}_r)$$

and, for $w \in \mathfrak{M}$, the w-adic roof function $\vartheta_w \colon \Delta \to \mathbb{R}$ is the upper envelope of the extended polytope $\Delta_w \subseteq \mathbb{R}^n \times \mathbb{R}$ given by

$$\Delta_w = \begin{cases} \mathrm{conv}\big((\boldsymbol{m}_j, -\mathrm{h}_{\overline{H}_1, \dots, \overline{H}_b}(V) \operatorname{ord}_V(f_j))_{j=0,\dots,r}\big), & \text{if } w = V \in B^{(1)}, \\ \mathrm{conv}\big((\boldsymbol{m}_j, \log |f_j(p)|_v)_{j=0,\dots,r}\big), & \text{if } w = p \in B^{\mathrm{gen}}_v, v \in M_F. \end{cases}$$

Now we differ from the setting in [BPS14b, § 4] and consider the special case of the function field of an elliptic curve equipped with a canonical metrized line bundle. Note that in this case non-discrete non-Archimedean absolute values naturally occur.

3.5.2. Let E be an elliptic curve over the global field F and let H be an ample symmetric line bundle on E. We choose any rigidification ρ of H, i. e. $\rho \in H_0(F) \setminus \{0\}$. By the theorem of the cube, we have, for each $m \in \mathbb{Z}$, a canonical identification $[m]^*H = H^{\otimes m^2}$ of rigidified line bundles. Then there exists a unique M_F-metric $\| \cdot \|_\rho = (\| \cdot \|_{\rho,v})_v$ on H such that, for all $v \in M_F$, $m \in \mathbb{Z}$,

$$[m]^* \| \cdot \|_{\rho,v} = \| \cdot \|_{\rho,v}^{\otimes m^2}.$$

For details, see [BG06, Theorem 9.5.7]. We call such an M_F-metric *canonical* because it is canonically determined by H up to $(|a|_v)_{v \in M_F}$ for some $a \in F^\times$. By [Gub07a, 3.5], the canonical metric $\| \cdot \|_\rho$ is quasi-algebraic and, since H is ample and symmetric, it is semipositive.

The global height associated to $\overline{H} = (H, \| \cdot \|_\rho)$ is equal to the Néron-Tate height $\hat{\mathrm{h}}_H$ (see

[BG06, Corollary 9.5.14]). In particular, it does not depend on the choice of the canonical metric. Since H is ample, we have $\mathrm{h}_{\overline{H}} = \hat{\mathrm{h}}_H \geq 0$.

For each $v \in M_F$, the canonically metrized line bundle $\overline{H}$ induces the *canonical measure* $\mathrm{c}_1(\overline{H}_v) = \mathrm{c}_1(H_v, \|\cdot\|_{\rho,v})$ which does not depend on the choice of the canonical metric (see [Gub07b, 3.15]) and which is positive. It has the properties

$$\mathrm{c}_1(\overline{H}_v)(E_v^{\mathrm{an}}) = \deg_H(E) \quad \text{and} \quad [m]^* \,\mathrm{c}_1(\overline{H}_v) = m^2 \,\mathrm{c}_1(\overline{H}_v) \text{ for all } m \in \mathbb{Z}.$$

For a detailed description of these measures, we have to consider three kinds of places $v \in M_F$.

(i) The set of Archimedean places in M_F is denoted by M_F^∞. For v Archimedean, $E_v^{\mathrm{an}} = E(\mathbb{C})$ is a complex analytic space which is biholomorphic to a complex torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\Im\tau > 0$. The canonical measure $\mathrm{c}_1(\overline{H}_v)$ corresponds to the Haar measure on this torus with total mass $\deg_H(E)$.

(ii) The set of non-Archimedean places v with E of good reduction at v is denoted by M_F^{g}. For such a v, the canonical measure $\mathrm{c}_1(\overline{H}_v)$ is a Dirac measure at a single point of E_v^{an}. Indeed, let $\mathscr{E}_v$ be the Néron model of E_v over $\mathbb{F}_v^\circ$. Since E has good reduction at v, the scheme $\mathscr{E}_v$ is proper and smooth, and its special fiber $\widetilde{\mathscr{E}_v}$ is an elliptic curve over $\widetilde{\mathbb{F}}_v$. Let ξ_v be the unique point of E_v^{an} such that $\mathrm{red}(\xi_v)$ is the generic point of $\widetilde{\mathscr{E}_v}$. Then $\mathrm{c}_1(\overline{H}_v) = \deg_H(E)\,\delta_{\xi_v}$.

(iii) The set of non-Archimedean places v with E of bad reduction at v is denoted by M_F^{b}. Let $v \in M_F^{\mathrm{b}}$, then E_v^{an} is a Tate elliptic curve over $\mathbb{F}_v$, i. e. E_v^{an} is isomorphic as an analytic group to $\mathbb{G}_{\mathrm{m},v}^{\mathrm{an}}/q^{\mathbb{Z}}$, where $\mathbb{G}_{\mathrm{m},v}$ is the multiplicative group over $\mathbb{F}_v$ with fixed coordinate x and q is an element of $\mathbb{G}_{\mathrm{m},v}(\mathbb{F}_v) = \mathbb{F}_v^\times$ with $|q|_v < 1$ (see, for instance, [BGR84, 9.7.3]). Denote by $\mathrm{trop}\colon \mathbb{G}_{\mathrm{m},v}^{\mathrm{an}} \to \mathbb{R}$, $p \mapsto -\log p(x)$, the tropicalization map and set $\Lambda_v := -\log|q|_v\mathbb{Z}$. Then we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{G}_{\mathrm{m},v}^{\mathrm{an}} & \xrightarrow{\ \mathrm{trop}\ } & \mathbb{R} \\ \downarrow & & \downarrow \\ E_v^{\mathrm{an}} & \xrightarrow{\ \overline{\mathrm{trop}}\ } & \mathbb{R}/\Lambda_v. \end{array}$$

Consider the continuous section $\rho\colon \mathbb{R} \to \mathbb{G}_{\mathrm{m},v}^{\mathrm{an}}$ of trop, where $\rho(u)$ is given by

$$\sum_{m\in\mathbb{Z}} \alpha_m x^m \longmapsto \max_{m\in\mathbb{Z}} |\alpha_m| \exp(-m \cdot u) \tag{3.36}$$

as in 2.4.7. Using $E_v^{\mathrm{an}} = \mathbb{G}_{\mathrm{m},v}^{\mathrm{an}}/q^{\mathbb{Z}}$, this section ρ descends to a continuous section $\bar{\rho}\colon \mathbb{R}/\Lambda_v \to E_v^{\mathrm{an}}$ of $\overline{\mathrm{trop}}$. The image of $\bar{\rho}$ is a canonical subset $S(E_v^{\mathrm{an}})$ of E_v^{an} which is called the *skeleton* of E_v^{an}. By [Ber90, Ex. 5.2.12 and Thm. 6.5.1], this is a closed subset of E_v^{an} and $\overline{\mathrm{trop}}$ restricts to a homeomorphism from $S(E_v^{\mathrm{an}})$ onto $\mathbb{R}/\Lambda_v$. By [Gub07b, Corollary 9.9], the canonical measure $\mathrm{c}_1(\overline{H}_v)$ on E_v^{an} is supported on the skeleton $S(E_v^{\mathrm{an}})$ and corresponds to the unique Haar measure on $\mathbb{R}/\Lambda_v$ with total mass $\deg_H(E)$.

Recall that we consider the morphism

$$\mathbb{G}_{\mathrm{m},K}^n \longrightarrow \mathbb{P}_K^r, \quad \boldsymbol{t} \longmapsto (1 : f_1\boldsymbol{t}^{\boldsymbol{m}_1} : \cdots : f_r\boldsymbol{t}^{\boldsymbol{m}_r})$$

with $\boldsymbol{m}_1 \dots, \boldsymbol{m}_r \in \mathbb{Z}^n$ generating $\mathbb{Z}^n$ as a group and $f_1, \dots f_r \in K^\times = F(B)^\times$. The closure of the image of this morphism in $\mathbb{P}^r_B$ is denoted by $\mathcal{Y}$.

Corollary 3.5.3. *With notations as above, we particularize to the case where the variety B is an elliptic curve E over F and $\overline{H}$ is an ample symmetric line bundle on E together with a canonical M_F-metric as in 3.5.2.*

Then $\mathrm{h}_{\pi^*\overline{H},\overline{\mathcal{O}(1)},\dots,\overline{\mathcal{O}(1)}}(\mathcal{Y})$ *is equal to*

$$(n+1)!\deg_H(E)\Bigg(\frac{1}{\deg_H(E)}\sum_{P\in C}\int_\Delta \vartheta_P(x)\,\mathrm{dvol}(x) + \sum_{v\in M_F^\infty}\int_{E(\mathbb{C})}\int_\Delta \vartheta_p(x)\,\mathrm{dvol}(x)\,\mathrm{d}\mu_{\mathrm{Haar}}(p)$$
$$+\sum_{v\in M_F^{\mathrm{g}}}\int_\Delta \vartheta_{\xi_v}(x)\,\mathrm{dvol}(x) + \sum_{v\in M_F^{\mathrm{b}}}\int_{\mathbb{R}/\Lambda_v}\int_\Delta \vartheta_{\bar\rho(u)}(x)\,\mathrm{dvol}(x)\,\mathrm{d}\mu_{\mathrm{Haar}}(\overline{u})\Bigg),$$

where $C \subset E^{(1)}$ is the set of irreducible components of the divisors $\mathrm{cyc}(f_j), j = 0, \dots, r$, vol *is the Lebesgue measure on $\mathbb{R}^n$ and μ_{Haar} is the Haar probability measure of the respective space.*

Proof. Since the height is invariant under normalization, we have $\mathrm{h}_{\pi^*\overline{H},\overline{\mathcal{O}(1)},\dots,\overline{\mathcal{O}(1)}}(\mathcal{Y}) = \mathrm{h}_{\pi^*\overline{H},\overline{\mathcal{L}},\dots,\overline{\mathcal{L}}}(\mathcal{X})$. We get the result by Theorem 3.3.4, Corollary 3.4.3, Proposition 3.5.1 and the description in 3.5.2. □

Example 3.5.4. Let $F = \mathbb{Q}$ and let E be an elliptic curve over $\mathbb{Q}$ with origin O and j-invariant j. For simplicity, we assume that E has good reduction at 2 and 3. Then E is given by an (affine) Weierstraß equation

$$g(x,y) := y^2 - (x^3 + Ax + B) = 0 \tag{3.37}$$

with coefficients A, B in $\mathbb{Z}$, which is minimal at each place $v \neq 2, 3$ (cf. [Sil92, Proposition VIII.8.7]). This Weierstraß equation also defines a model $\mathscr{E} \subset \mathbb{P}^2_{\mathbb{Z}}$ of E over $\mathbb{Z}$. We set $\mathcal{A} := \mathcal{O}(\mathscr{E} \setminus \{O\}) = \mathbb{Z}[x,y]/(g)$. Then we have $K = \mathbb{Q}(E) = \mathrm{Quot}(\mathcal{A})$.

Furthermore, we consider the case when $n = 1$ and $m_i = i$, $i = 0, \dots, r$, and we choose a family $f_0, \dots, f_r \in \mathcal{A} \subset K$ of pairwise coprime polynomials with $f_0 = 1$ as before. Then $\Delta = [0, r]$. Let $w \in \mathfrak{M}$ and $\vartheta_w \colon [0, r] \to \mathbb{R}$ the w-adic roof function. We have to consider four cases corresponding to the four summands in Corollary 3.5.3.

(i) Let $w = P \in E^{(1)}$. If $P = O$, then $\hat{\mathrm{h}}_H(P) = 0$ and thus $\vartheta_O \equiv 0$. Otherwise, there is at most one $i \in \{0, \dots, r\}$ such that $\mathrm{ord}_P(f_i) \neq 0$ because $f_0, \dots, f_r$ are pairwise coprime. Since $\mathrm{ord}_P(f_0) = 0$, $\hat{\mathrm{h}}_H(P) \geq 0$ and $\mathrm{ord}_P(f_i) \geq 0$, Proposition 3.5.1 implies

$$\int_0^r \vartheta_P(x)\,\mathrm{d}x = -\frac{1}{2}\hat{\mathrm{h}}_H(P)\,\mathrm{ord}_P(f_r). \tag{3.38}$$

(ii) Let $w = p \in E(\mathbb{C})^{\mathrm{gen}}$. By Proposition 3.5.1, we obtain

$$\vartheta_p \colon [0,r] \to \mathbb{R},\ x \mapsto \max_{\substack{0\leq j\leq x\leq k\leq r\\ j\neq k}}\left(\frac{\log|f_k(p)| - \log|f_j(p)|}{k-j}(x-j) + \log|f_j(p)|\right). \tag{3.39}$$

In particular, $\vartheta_p(0) = \log|f_0(p)| = 0$. We deduce

$$\int_0^r \vartheta_p(x)\,\mathrm{d}x = \sum_{i=1}^{r-1} \vartheta_p(i) + \frac{\vartheta_p(r)}{2}.$$

By [Sil94, Corollary I.4.3], there exists a unique lattice $\Lambda \subset \mathbb{C}$ such that the map

$$\mathbb{C}/\Lambda \longrightarrow E(\mathbb{C}), \quad z \longmapsto \left(\wp(z), \tfrac{1}{2}\wp'(z)\right),$$

is a complex analytic isomorphism of complex Lie groups, where $\wp$ is the Weierstraß $\wp$-function associated to Λ. Then,

$$\int_{E(\mathbb{C})^{\mathrm{gen}}} \int_0^r \vartheta_p(x)\,\mathrm{d}x\,\mathrm{d}\mu_{\mathrm{Haar}}(p)$$
$$= \frac{1}{\mathrm{vol}(\Lambda)} \int_{\mathbb{C}/\Lambda} \sum_{i=1}^{r-1} \vartheta_{(\wp(z),\frac{1}{2}\wp'(z))}(i) + \frac{\vartheta_{(\wp(z),\frac{1}{2}\wp'(z))}(r)}{2}\,\mathrm{d}z \tag{3.40}$$

with $\vartheta_{(\wp(z),\frac{1}{2}\wp'(z))}$ as in (3.39) and where $\mathrm{vol}(\Lambda)$ denotes the covolume of Λ.

(iii) Let $w = p \in E_v^{\mathrm{gen}}$, $v \in M_{\mathbb{Q}}^{\mathrm{g}}$. Then the scheme $\mathscr{E}_v = \mathscr{E} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}_v^{\circ}$ is proper and smooth over $\mathbb{C}_v^{\circ}$. Thus, the special fiber $\widetilde{\mathscr{E}_v}$ is an elliptic curve over $\widetilde{\mathbb{C}}_v = \overline{\mathbb{F}}_v$. By Corollary 3.5.3, we have to study the unique point $\xi_v \in E_v^{\mathrm{an}}$ such that $\mathrm{red}(\xi_v)$ is the generic point of $\widetilde{\mathscr{E}_v}$. By [Kna00, Theorem 2.4], the local ring $\mathcal{O}_{\mathscr{E}_v,\mathrm{red}(\xi_v)}$ admits a real-valued valuation ord_v. Then ξ_v is given by $-\log \xi_v = \mathrm{ord}_v$. In particular, $\log|f_i(\xi_v)| \leq 0$ and $\log|f_i(\xi_v)| < 0$ if and only if $f_i \in \mathbb{C}_v^{\circ\circ}\mathcal{A}$. Since $f_0, \dots, f_r$ are pairwise coprime, we deduce as in the first case,

$$\int_0^r \vartheta_{\xi_v}(x)\,\mathrm{d}x = -\frac{\mathrm{ord}_v(f_r)}{2}\ . \tag{3.41}$$

(iv) Let $w = p \in E_v^{\mathrm{gen}}$, $v \in M_{\mathbb{Q}}^{\mathrm{b}}$. Since E has bad reduction at v, we have for the j-invariant $|j|_v > 1$. Thus, by [Sil94, Theorem V.5.3], there is a unique $q_v = q \in \mathbb{Q}_v^{\times}$ with $|q|_v = |j|_v^{-1} < 1$ such that E_v is isomorphic over $\mathbb{C}_v$ to the Tate curve E_q defined by

$$y'^2 + x'y' = x'^3 + A'(q)x' + B'(q), \tag{3.42}$$

where $A'(q)$ and $B'(q)$ are universal integral power series in q that converge in $\mathbb{Q}_v^{\circ}$. By [Sil94, Theorem V.3.1], we get an isomorphism of analytic groups $\mathbb{G}_{\mathrm{m},v}^{\mathrm{an}}/q^{\mathbb{Z}} \to E_v^{\mathrm{an}}$ given on the $\mathbb{C}_v$-rational points by

$$w \cdot q^{\mathbb{Z}} \longmapsto \begin{cases} (x'(w), y'(w))\,, & \text{if } w \notin q^{\mathbb{Z}}, \\ O, & \text{if } w \in q^{\mathbb{Z}}, \end{cases}$$

where

$$x'(w) = \sum_{n=-\infty}^{\infty} \frac{q^n w}{(1-q^n w)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \tag{3.43}$$

and

$$y'(w) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}w^2}{(1-q^n w)^3} + \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}\,. \tag{3.44}$$

By the change of coordinates $(\tilde{x}, \tilde{y}) = \left(x' + \frac{1}{12},\, y' + \frac{1}{2}x'\right)$, the elliptic curve E_q can be written in the form

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{A}\,\tilde{x} + \tilde{B} \tag{3.45}$$

where $\tilde{A} = \tilde{A}(q) = A'(q) - \frac{3}{144}$ and $\tilde{B} = \tilde{B}(q) = B'(q) - \frac{1}{12}A'(q) + \frac{1}{864}$ are power series in q with coefficients in $\mathbb{Z}[1/6]$. Let $\alpha_v = \alpha = (A\tilde{B}/\tilde{A}B)^{1/2}$. Using the proof of [Sil92, Proposition III.1.4 (b)], we have the following relations between the coordinates,

$$(x,y) = \left(\alpha_v^2\tilde{x},\, \alpha_v^3\tilde{y}\right) = \left(\alpha_v^2\left(x' + \tfrac{1}{12}\right),\, \alpha_v^3\left(y' + \tfrac{1}{2}x'\right)\right). \tag{3.46}$$

As in the Archimedean case (ii), we get

$$\int_{\mathbb{R}/-\log|q|_v\mathbb{Z}} \int_0^r \vartheta_{\bar{\rho}(\overline{u})}(x)\,\mathrm{dvol}(x)\;\mathrm{d}\mu_{\mathrm{Haar}}(\overline{u}) = \frac{1}{\log|j|_v}\int_0^{\log|j|_v} \sum_{i=1}^{r-1}\vartheta_{\bar{\rho}(\overline{u})}(i) + \frac{1}{2}\vartheta_{\bar{\rho}(\overline{u})}(r)\;\mathrm{d}u\,, \tag{3.47}$$

where $\bar{\rho}(\overline{u})$ is defined after (3.36) and $\vartheta_{\bar{\rho}(\overline{u})}$ is given as in (3.39) with the values

$$\log|f_i(\bar{\rho}(\overline{u}))| = \log\left|f_i\left(\alpha^2(x'(w) + \tfrac{1}{12}),\, \alpha^3(y'(w) + \tfrac{1}{2}x'(w))\right)\right|_{\bar{\rho}(\overline{u})}. \tag{3.48}$$

Conclusion: Inserting (3.38), (3.40), (3.41) and (3.47) into the formula of Corollary 3.5.3, the arithmetic intersection number $\mathrm{h}_{\pi^*\overline{H},\overline{\mathcal{O}(1)},\overline{\mathcal{O}(1)}}(\mathcal{Y})$ is equal to

$$-\sum_{P\in|\mathrm{div}(f_r)|}\hat{\mathrm{h}}_H(P)\,\mathrm{ord}_P(f_r) + \frac{2\deg_H(E)}{\mathrm{vol}(\Lambda)}\int_{\mathbb{C}/\Lambda}\sum_{i=1}^{r-1}\vartheta_{(\wp(z),\frac{1}{2}\wp'(z))}(i) + \frac{\vartheta_{(\wp(z),\frac{1}{2}\wp'(z))}(r)}{2}\;\mathrm{d}z$$
$$-\deg_H(E)\sum_{v\in M_{\mathbb{Q}}^{\mathrm{g}}}\mathrm{ord}_v(f_r) + \sum_{v\in M_{\mathbb{Q}}^{\mathrm{b}}}\frac{2\deg_H(E)}{\log|j|_v}\int_0^{\log|j|_v}\sum_{i=1}^{r-1}\vartheta_{\bar{\rho}(\overline{u})}(i) + \frac{\vartheta_{\bar{\rho}(\overline{u})}(r)}{2}\;\mathrm{d}u,$$

where, for $p = \bar{\rho}(\overline{u}) \in E_v^{\mathrm{gen}}$ and $p = (\wp(z), \frac{1}{2}\wp'(z)) \in E(\mathbb{C})^{\mathrm{gen}}$,

$$\vartheta_p(i) = \max_{\substack{0\le j\le i\le k\le r\\ j\ne k}}\left(\frac{\log|f_k(p)| - \log|f_j(p)|}{k-j}(i-j) + \log|f_j(p)|\right), \tag{3.49}$$

and $\log|f_i(\bar{\rho}(\overline{u}))|$, $i = 1,\dots,r$, is given by (3.48) and the series (3.43), (3.44).

Example 3.5.5. We keep the assumptions and notations from Example 3.5.4, choosing now $r = 2$ and the specific functions $f_0 = 1$, $f_1 = y$ and $f_2 = px$ for a prime p. For simplicity, we assume that E has good or multiplicative reduction at each place of $\mathbb{Q}$ (cf.

Remark 3.5.6). Then the concluding formula in Example 3.5.4 can be further simplified:

The zeros of $f_2 = px$ are $(0, \pm\sqrt{B})$ and $\operatorname{ord}_{(0,\pm\sqrt{B})}(f_2) = 1$. Furthermore, if $v = p$, then $\operatorname{ord}_v(px) = -\log|p|_p = \log(p)$, and otherwise, $\operatorname{ord}_v(px) = 0$.

In the case of bad reduction, i.e. $v \in M_{\mathbb{Q}}^{\mathrm{b}}$, we have to compute $\log|f_i(x(w), y(w))|_{\bar{\rho}(\overline{u})}$. Recall that in this case $v \neq 2, 3$. First, we consider $|y(w)|_{\rho(u)}$ for $u \in (0, -\log|q|_v)$. For $\alpha_v = \alpha$ as in (3.46), we obtain, by (3.46), (3.43) and (3.44),

$$y(w,q) = \alpha^3 \left(\tfrac{1}{2}x'(w,q) + y'(w,q)\right) = \frac{\alpha^3}{2} \sum_{n=-\infty}^{\infty} \frac{q^n w + q^{2n} w^2}{(1-q^n w)^3}. \tag{3.50}$$

For $n \geq 0$, we have $|q^n w|_{\rho(u)} = |q|_v^n \exp(-u) < 1$. Thus, for each $n \geq 0$,

$$\left| \frac{q^n w + q^{2n} w^2}{(1-q^n w)^3} \right|_{\rho(u)} = \left| q^n w + q^{2n} w^2 \right|_{\rho(u)} = \max\left(|q^n w|_{\rho(u)}, \left|q^{2n} w^2\right|_{\rho(u)}\right) = |q|_v^n \exp(-u).$$

Since $|q|_v^k \exp(-u) < |q|_v^l \exp(-u)$ for $k > l \geq 0$, we obtain

$$\left| \sum_{n=0}^{\infty} \frac{q^n w + q^{2n} w^2}{(1-q^n w)^3} \right|_{\rho(u)} = \exp(-u). \tag{3.51}$$

For $n < 0$ and $u \in (0, \log|q|_v^{-1})$, we have $|q^{3n} w^3|_{\rho(u)} > |q^{2n} w^2|_{\rho(u)} > |q^n w|_{\rho(u)} > 1$, and therefore,

$$\left| \frac{q^n w + q^{2n} w^2}{(1-q^n w)^3} \right|_{\rho(u)} = \frac{|q^{2n} w^2|_{\rho(u)}}{|q^{3n} w^3|_{\rho(u)}} = |q^n w|_{\rho(u)}^{-1}.$$

Since $|q^k w|_{\rho(u)}^{-1} < |q^l w|_{\rho(u)}^{-1}$ for $k < l \leq -1$, we deduce

$$\left| \sum_{n=1}^{\infty} \frac{q^{-n} w + q^{-2n} w^2}{(1-q^{-n} w)^3} \right|_{\rho(u)} = |q^{-1} w|_{\rho(u)}^{-1} = |q|_v \exp(u). \tag{3.52}$$

Using (3.48), (3.50), (3.51), (3.52) and $\left|\frac{1}{2}\right|_v = 1$, we conclude

$$|f_1(\bar{\rho}(\overline{u}))| = |y(w,q)|_{\rho(u)} = \begin{cases} |\alpha|_v^3 \exp(-u), & \text{if } 0 < u < -\frac{1}{2}\log|q|_v, \\ |\alpha|_v^3 |q|_v \exp(u), & \text{if } -\frac{1}{2}\log|q|_v < u < -\log|q|_v. \end{cases} \tag{3.53}$$

Since $u \mapsto |y(w,q)|_{\rho(u)}$ is continuous, we can replace "$<$" by "$\leq$" in (3.53). Analogously, one can show that

$$|x'(w,q)|_{\rho(u)} = \begin{cases} \exp(-u), & \text{if } 0 \leq u \leq -\frac{1}{2}\log|q|_v, \\ |q|_v \exp(u), & \text{if } -\frac{1}{2}\log|q|_v \leq u \leq -\log|q|_v. \end{cases} \tag{3.54}$$

Hence, (3.48) and (3.54) imply, for almost all $u \in [0, -\log|q|_v]$,

$$\begin{aligned} |f_2(\bar{\rho}(\overline{u}))| = |px(w)|_{\rho(u)} &= |\alpha^2|_v |p|_v \left|x'(w) + \tfrac{1}{12}\right|_{\rho(u)} \\ &= |\alpha^2|_v |p|_v \max(|x'(w)|_{\rho(u)}, 1) = |\alpha|_v^2 |p|_v. \end{aligned} \tag{3.55}$$

Since $u \mapsto |f_2(\bar{\rho}(\overline{u}))|$ is continuous, the equality (3.55) holds for all $u \in [0, -\log|q|_v]$.

Recall that $|j|_v = |q|_v^{-1}$. Using (3.49) and the formulas (3.53) and (3.55), the roof functions in the case of bad reduction are given by $\vartheta_{\bar{\rho}(\overline{u})}(0) = 0$, $\vartheta_{\bar{\rho}(\overline{u})}(2) = \log\left(|\alpha|_v^2|p|_v\right)$ and

$$\vartheta_{\bar{\rho}(\overline{u})}(1) = \begin{cases} \max\left(\frac{\log\left(|\alpha|_v^2|p|_v\right)}{2}, 3\log|\alpha|_v - u\right), & \text{if } 0 \le u \le \frac{1}{2}\log|j|_v, \\ \max\left(\frac{\log\left(|\alpha|_v^2|p|_v\right)}{2}, 3\log|\alpha|_v - \log|j|_v + u\right), & \text{if } \frac{1}{2}\log|j|_v \le u \le \log|j|_v. \end{cases} \tag{3.56}$$

Now, we use the assumption that the bad reduction of E at v is multiplicative. Let $L = \mathbb{Q}_v(\alpha)$. Then we have either $L = \mathbb{Q}_v$ (split case) or $L/\mathbb{Q}_v$ is unramified of degree 2 (non-split case), see [Sil94, Exercise 5.11]. By the proof of [Sil92, Proposition VII.5.4 (a)], the Weierstraß equation (3.37) is still minimal over L. Moreover, the Weierstraß equation (3.42) is minimal over L (see [Tat74, Theorem 5]) and thus, since $v \neq 2, 3$, the equation (3.45) is minimal over L as well. We deduce by [Sil92, VII.1.3 (b)] that $|\alpha|_v = 1$.

In particular, if $v \neq p$, the function $\vartheta_{\bar{\rho}(\overline{u})}$ is identically zero. If $v = p$, an easy computation shows that

$$\int_0^{\log|j|_p} \vartheta_{\bar{\rho}(\overline{u})}(1) + \frac{\vartheta_{\bar{\rho}(\overline{u})}(2)}{2}\,\mathrm{d}u = \begin{cases} -\frac{1}{4}\left(\log|j|_p\right)^2 - \frac{1}{2}\log(p)\log|j|_p, & \text{if } \log|j|_p \le \log(p), \\ \frac{1}{4}(\log(p))^2 - \log(p)\log|j|_p, & \text{if } \log|j|_p \ge \log(p). \end{cases}$$

The invariant j is a rational number with integral p-adic valuation $\nu_p(j) < 0$. Hence, $\log|j|_p/\log(p) = -\nu_p(j) \ge 1$, and we can omit the first case.

Conclusion: The height $\mathrm{h}_{\pi^*\overline{H},\overline{\mathcal{O}(1)},\overline{\mathcal{O}(1)}}(\mathcal{Y})$ in our specific Example 3.5.5 is given by

$$\begin{aligned} &\frac{\deg_H(E)}{\mathrm{vol}(\Lambda)} \int_{\mathbb{C}/\Lambda} \max\left(\log|p\cdot\wp(z)|, 2\log\left|\tfrac{1}{2}\wp'(z)\right|\right) + \log|p\cdot\wp(z)|\,\mathrm{d}z \\ &- 2\hat{\mathrm{h}}_H(0,\sqrt{B}) - \deg_H(E)\cdot\log(p)\cdot b(p), \end{aligned} \tag{3.57}$$

where

$$b(p) = \begin{cases} 1, & \text{if } p \in M_{\mathbb{Q}}^{\mathrm{g}}, \\ 2 + \frac{1}{2\nu_p(j)}, & \text{if } p \in M_{\mathbb{Q}}^{\mathrm{b}}, \end{cases}$$

denoting by ν_p the usual p-adic valuation.

We see that this height is, at the Archimedean place, an integral of terms including the Weierstraß $\wp$-function, and with concrete terms at the non-Archimedean places.

Example 3.5.6. In the previous example it is important that, for each place $v \in M_{\mathbb{Q}}^{\mathrm{b}}$, the reduction is multiplicative in order to ensure that $|\alpha_v|_v = 1$ for $\alpha_v = (A\tilde{B}/\tilde{A}B)^{1/2}$ as in (3.46). Keeping the notations of Example 3.5.5, we consider a concrete elliptic curve, now allowing additive reduction at some place: Let E be the elliptic curve over $\mathbb{Q}$ given by

$$y^2 = x^3 - 2^6 3^3 5^2 \cdot x + 2^4 3^3 5^3 71.$$

Then E is described by the invariants

$$\Delta = -2^{12}3^{12}5^7 7, \quad j = 2^{18}5^{-1}7^{-1}, \quad c_4 = 2^{10}3^4 5^2.$$

Using [Sil92, Proposition VII.5.1], we see that E has additive reduction at the place $v = 5$, multiplicative reduction at $v = 7$ and good reduction at the other places. We have $|\alpha_7|_7 = 1$. Let us compute $|\alpha_5|_5 = \sqrt{|A\tilde{B}(q_5)/\tilde{A}(q_5)B|_5}$. On the one hand, we have

$$|A/B|_5 = \left|-2^2 5^{-1} 71^{-1}\right|_5 = 5.$$

On the other hand, we consider $\left|\tilde{B}(q_5)/\tilde{A}(q_5)\right|$ where, as in (3.45),

$$\tilde{A}(q_5) = -\frac{3}{144} - 5\sum_{n\geq 1}^{\infty} \frac{n^3 q_5^n}{1-q_5^n} \quad \text{and} \quad \tilde{B}(q_5) = \frac{1}{864} - \frac{7}{12}\sum_{n\geq 1}^{\infty} \frac{n^5 q_5^n}{1-q_5^n}.$$

Using $|q_5|_5 = |j|_5^{-1} = \frac{1}{5}$, an easy computation shows that $|\tilde{A}(q_5)|_5 = \max(|\frac{3}{144}|_5, |5|_5|q_5|_5) = 1$ and $|\tilde{B}(q_5)|_5 = \max(|\frac{1}{864}|_5, |q_5|_5) = 1$. Hence, we obtain

$$|\alpha_5|_5 = \left(\left|A\tilde{B}(q_5)/\tilde{A}(q_5)B\right|_5\right)^{1/2} = \sqrt{5}.$$

By (3.56), we get $\vartheta_{\bar{\rho}(\overline{u})}(0) = 0$, $\vartheta_{\bar{\rho}(\overline{u})}(2) = \log(5 \cdot |p|_5)$ and

$$\vartheta_{\bar{\rho}(\overline{u})}(1) = \begin{cases} \frac{3}{2}\log 5 - u, & \text{if } 0 \leq u \leq \frac{1}{2}\log 5, \\ \frac{1}{2}\log 5 + u, & \text{if } \frac{1}{2}\log 5 \leq u \leq \log 5. \end{cases}$$

Thus,

$$\int_0^{\log|j|_p} \vartheta_{\bar{\rho}(\overline{u})}(1) + \frac{\vartheta_{\bar{\rho}(\overline{u})}(2)}{2}\,\mathrm{d}u = \begin{cases} \frac{5}{4}(\log 5)^2, & \text{if } p = 5, \\ \frac{7}{4}(\log 5)^2, & \text{if } p = 7. \end{cases}$$

Then the height $\mathrm{h}_{\pi^*\overline{H},\overline{\mathcal{O}(1)},\overline{\mathcal{O}(1)}}(\mathcal{Y})$ in this concrete Example 3.5.6 is given by

$$\begin{aligned} &\frac{\deg_H(E)}{\mathrm{vol}(\Lambda)}\int_{\mathbb{C}/\Lambda} \max\left(\log|p\cdot\wp(z)|, 2\log\left|\tfrac{1}{2}\wp'(z)\right|\right) + \log|p\cdot\wp(z)|\,\mathrm{d}z \\ &- 2\hat{\mathrm{h}}_H(0,\sqrt{B}) - \deg_H(E)\cdot\log(p)\cdot b(p), \end{aligned}$$

where

$$b(p) = \begin{cases} -\frac{5}{2}, & \text{if } p = 5, \\ -\frac{7}{2}, & \text{if } p = 7, \\ 1, & \text{else.} \end{cases}$$

Note that this result differs from the formula (3.57) if $p = 5$ or $p = 7$.

This example was constructed by means of [SAGE] and the table on page 108 in [BK75].

Appendix A.

Convex Geometry

In this appendix, we collect the notions and statements of convex geometry that we need for the study of toric geometry. We follow the notation of [BPS14a, § 2] which is based on the classical book [Roc70].

Let M be a free Abelian group of rank n and $N := M^\vee := \operatorname{Hom}(M, \mathbb{Z})$ its dual group. The natural pairing between $m \in M$ and $u \in N$ is denoted by $\langle m, u\rangle := u(m)$. If G is an Abelian group, we set $N_G := N \otimes_{\mathbb{Z}} G = \operatorname{Hom}(M, G)$. In particular, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is an n-dimensional real vector space with dual space $M_{\mathbb{R}} = \operatorname{Hom}(N, \mathbb{R})$. We denote by Γ a subgroup of $\mathbb{R}$.

A.1. A *polyhedron* Λ in $N_{\mathbb{R}}$ is a non-empty set defined as the intersection of finitely many close half-spaces, i. e.

$$\Lambda = \bigcap_{i=1}^{r} \{u \in N_{\mathbb{R}} \mid \langle m_i, u\rangle \geq l_i\}, \quad \text{where } m_i \in M_{\mathbb{R}},\ l_i \in \mathbb{R},\ i = 1, \ldots, r. \tag{A.1}$$

A *polytope* is a bounded polyhedron. A *face* Λ' of a polyhedron Λ, denoted by $\Lambda' \preceq \Lambda$, is either Λ itself or of the form $\Lambda \cap H$ where H is the boundary of a closed half-space containing Λ. A face of Λ of codimension 1 is called a *facet*, a face of dimension 0 is a *vertex*. The *relative interior* of Λ, denoted by $\operatorname{ri} \Lambda$, is the interior of Λ in its affine hull.

A.2. Let Λ be a polyhedron in $N_{\mathbb{R}}$. We call Λ *strongly convex* if it does not contain any affine line. We say that Λ is Γ-*rational* if there is a representation as (A.1) with $m_i \in M$ and $l_i \in \Gamma$. If $\Gamma = \mathbb{Q}$, we just say Λ is *rational*. We say that a polytope in $M_{\mathbb{R}}$ is *lattice* if its vertices lie in M.

A.3. A *polyhedral cone* in $N_{\mathbb{R}}$ is a polyhedron σ such that $\lambda\sigma = \sigma$ for all $\lambda \geq 0$. Its *dual* is defined as

$$\sigma^\vee := \{m \in M_{\mathbb{R}} \mid \langle m, u\rangle \geq 0 \ \forall u \in \sigma\}.$$

We denote by $\sigma^\perp$ the set of $m \in M_{\mathbb{R}}$ with $\langle m, u\rangle = 0$ for all $u \in \sigma$. The *recession cone* of a polyhedron Λ is defined as

$$\operatorname{rec}(\Lambda) := \{u \in N_{\mathbb{R}} \mid u + \Lambda \subseteq \Lambda\}.$$

If Λ has a representation as (A.1), the recession cone can be written as

$$\operatorname{rec}(\Lambda) = \bigcap_{i=1}^{r} \{u \in N_{\mathbb{R}} \mid \langle m_i, u\rangle \geq 0\}.$$

A.4. A *polyhedral complex* Π in $N_{\mathbb{R}}$ is a non-empty finite set of strongly convex polyhedra such that

(i) every face of $\Lambda \in \Pi$ lies also in Π;

(ii) if $\Lambda, \Lambda' \in \Pi$, then $\Lambda \cap \Lambda'$ is empty or a face of Λ and Λ'.

Note that, in contrast to the notion in [BPS14a, Definition 2.1.4], a polyhedral complex only contains strongly convex polyhedra.

A polyhedral complex Π is called Γ-*rational* (resp. *rational*) if each $\Lambda \in \Pi$ is Γ-rational (resp. rational). The *support* of Π is defined as the set $|\Pi| := \bigcup_{\Lambda \in \Pi} \Lambda$. We say that Π is *complete* if $|\Pi| = N_{\mathbb{R}}$. We will denote by Π^k the subset of k-dimensional polyhedra of Π.

A *fan* in $N_{\mathbb{R}}$ is a polyhedral complex in $N_{\mathbb{R}}$ consisting of (strongly convex) rational polyhedral cones.

A.5. Let Π be a polyhedral complex in $N_{\mathbb{R}}$. The *recession* $\operatorname{rec}(\Pi)$ of Π is defined as

$$\operatorname{rec}(\Pi) = \{\operatorname{rec}(\Lambda) \mid \Lambda \in \Pi\}.$$

If Π is a complete Γ-rational polyhedral complex, then $\operatorname{rec}(\Pi)$ is a complete fan in $N_{\mathbb{R}}$. This follows from [BS11, Theorem 3.4].

A.6. Let C be a convex set in a real vector space. A function $f \colon C \to \mathbb{R}$ is *concave* if

$$f(tu_1 + (1-t)u_2) \geq tf(u_1) + (1-t)f(u_2) \tag{A.2}$$

for all $u_1, u_2 \in C$ and $0 < t < 1$.

Note that we use the same terminology as in convex analysis. In the classical books on toric varieties [KKMS73], [Ful93], [CLS11], our concave functions are called "convex".

A.7. Let f be a function on $N_{\mathbb{R}}$. We define the *stability set* of f as

$$\Delta_f := \{m \in M_{\mathbb{R}} \mid \langle m, \cdot \rangle - f \text{ is bounded below}\}.$$

This is a convex set in $M_{\mathbb{R}}$. The definition is only useful in case of a concave function as otherwise $\Delta_f = \emptyset$. The *(Legendre-Fenchel) dual* of f is the function

$$f^\vee \colon \Delta_f \longrightarrow \mathbb{R}, \quad m \longmapsto \inf_{u \in N_{\mathbb{R}}} (\langle m, u \rangle - f(u)).$$

It is a continuous concave function.

A.8. Let $f \colon N_{\mathbb{R}} \to \mathbb{R}$ be a concave function. The *recession function* $\operatorname{rec}(f)$ of f is defined as

$$\operatorname{rec}(f) : N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad u \longmapsto \lim_{\lambda \to \infty} \frac{f(\lambda u)}{\lambda}.$$

By [Roc70, Theorem 13.1], $\operatorname{rec}(f)$ is the support function of the stability set Δ_f, i.e. it is given by

$$\operatorname{rec}(f)(u) = \inf_{m \in \Delta_f} \langle m, u \rangle$$

for $u \in N_{\mathbb{R}}$.

Proposition A.9. *Let Σ be a complete fan in $N_{\mathbb{R}}$ and let $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ be a virtual support function on Σ, i. e. it is a function given by $\Psi|_\sigma = \langle m_\sigma, \cdot\rangle$ where $m_\sigma \in M$, $\sigma \in \Sigma$ (Definition 2.1.9). Then the assignment $\psi \mapsto \psi^\vee$ gives a bijection between the sets of*

(i) concave functions ψ on $N_{\mathbb{R}}$ such that $|\psi - \Psi|$ is bounded,

(ii) continuous concave functions on Δ_Ψ.

Proof. If Ψ is concave, this follows from the propositions 2.5.20 (2) and 2.5.23 in [BPS14a]. If Ψ is not concave, both sets are empty ☐

A.10. A continuous function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ is *piecewise affine* if there is a finite cover $\{\Lambda_i\}_{i\in I}$ of $N_{\mathbb{R}}$ by closed subsets such that $f|_{\Lambda_i}$ is an affine function.

Let Π be a complete polyhedral complex in $N_{\mathbb{R}}$. We say that f is a piecewise affine function *on* Π if f is affine on each polyhedron of Π.

A.11. Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a piecewise affine function on $N_{\mathbb{R}}$. Then there is a complete polyhedral complex Π in $N_{\mathbb{R}}$ such that, for each $\Lambda \in \Pi$,

$$f|_\Lambda(u) = \langle m_\Lambda, u\rangle + l_\Lambda \quad \text{with } (m_\Lambda, l_\Lambda) \in M_{\mathbb{R}} \times \mathbb{R}\ . \tag{A.3}$$

The set $\{(m_\Lambda, l_\Lambda)\}_{\Lambda\in\Pi}$ is called a set of *defining vectors* of f. We call f a Γ-*lattice* function if it has a representation as (A.3) with $(m_\Lambda, l_\Lambda) \in M \times \Gamma$ for each $\Lambda \in \Pi$. We say that f is a Γ-*rational* piecewise affine function if there is an integer $e > 0$ such that ef is a Γ-lattice function.

A.12. Let f be a concave piecewise affine function f on $N_{\mathbb{R}}$. Then there are $m_i \in M_{\mathbb{R}}$, $l_i \in \mathbb{R}$, $i = 1, \ldots, r$, such that f is given by

$$f(u) = \min_{i=1,\ldots,r} \langle m_i, u\rangle + l_i \quad \text{for } u \in N_{\mathbb{R}}.$$

The stability set Δ_f is a polytope in $M_{\mathbb{R}}$ which is the convex hull of $m_1, \ldots, m_r$. The function f is piecewise affine concave if and only if $f^\vee$ is a piecewise affine concave function on Δ_f. The recession function of f is given by

$$\operatorname{rec}(f)\colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad u \longmapsto \min_{i=1,\ldots,r} \langle m_i, u\rangle.$$

The function $\operatorname{rec}(f)$ has integral slopes if and only if the stability set Δ_f is a lattice polytope.

A.13. Let f be a piecewise affine function on $N_{\mathbb{R}}$. Then we can write $f = g - h$, where g and h are concave piecewise affine functions on $N_{\mathbb{R}}$. The *recesssion function* of f is defined as $\operatorname{rec}(f) = \operatorname{rec}(g) - \operatorname{rec}(h)$.

In Theorem 2.5.8 we need the following assertion.

Proposition A.14. *Let Γ be a non-trivial subgroup of $\mathbb{R}$. Let Ψ be a support function on a complete fan in $N_{\mathbb{R}}$ (Definition 2.1.9) and ψ a concave function on $N_{\mathbb{R}}$ such that $|\psi - \Psi|$ is bounded. Then there is a sequence of Γ-rational piecewise affine concave functions $(\psi_k)_{k\in\mathbb{N}}$, with $\operatorname{rec}(\psi_k) = \Psi$, that uniformly converges to ψ.*

Proof. Since Ψ is a support function with $|\psi - \Psi|$ bounded, the stability set Δ_Ψ is a lattice polytope in $M_\mathbb{R}$ with $\Delta_\Psi = \Delta_\psi$. Thus, by Proposition [BPS14a, Proposition 2.5.23 (2)], there is a sequence of piecewise affine concave functions $(\psi_k)_{k\in\mathbb{N}}$ with $\Delta_{\psi_k} = \Delta_\Psi$, that converges uniformly to ψ. Because the divisible hull of Γ lies dense in $\mathbb{R}$, we may assume that the ψ_k's are Γ-rational. Finally, Proposition 2.3.10 in [BPS14a] says that $\Delta_{\psi_k} = \Delta_\Psi$ implies $\operatorname{rec}(\psi_k) = \Psi$. □

A.15. Let f be a concave function on $N_\mathbb{R}$. The *sup-differential* of f at $u \in N_\mathbb{R}$ is defined as

$$\partial f(u) := \{m \in M_\mathbb{R} \mid \langle m, v - u\rangle \geq f(v) - f(u) \text{ for all } v \in N_\mathbb{R}\}.$$

For each $u \in N_\mathbb{R}$, the sup-differential $\partial f(u)$ is a non-empty compact convex set. For a subset E of $N_\mathbb{R}$, we set

$$\partial f(E) := \bigcup_{u\in E} \partial f(u).$$

A.16. Let L be a lattice. We denote by vol_L the unique Haar measure on $L_\mathbb{R}$ such that L has covolume one. If A is an affine space with associated vector space $L_\mathbb{R}$, then vol_L induces a measure on A which we also denote by vol_L.

A.17. Let f be a concave function on $N_\mathbb{R}$. The *Monge-Ampère measure* of f with respect to M is defined, for any Borel subset E of $N_\mathbb{R}$, as

$$\mathcal{M}_M(f)(E) := \operatorname{vol}_M(\partial f(E)),$$

where vol_M is the measure from A.16. Then the total mass is $\mathcal{M}_M(f)(N_\mathbb{R}) = \operatorname{vol}_M(\Delta_f)$.

Proposition A.18. *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of concave functions on $N_\mathbb{R}$ that converges uniformly to a function f. Then the Monge-Ampère measures $\mathcal{M}_M(f_k)$ converge weakly to $\mathcal{M}_M(f)$.*

Proof. This follows from [BPS14a, Proposition 2.7.2]. □

Proposition A.19. *Let f be a piecewise affine concave function on a complete polyhedral complex Π in $N_\mathbb{R}$. Then*

$$\mathcal{M}_M(f) = \sum_{v\in\Pi^0} \operatorname{vol}_M(\partial f(v))\,\delta_v,$$

where δ_v is the Dirac measure supported on v. In particular, if Ψ is a support function on a complete fan in $N_\mathbb{R}$, then

$$\mathcal{M}_M(\Psi) = \operatorname{vol}_M(\Delta_\Psi)\delta_0.$$

Proof. This is Proposition 2.7.4 and Example 2.7.5 in [BPS14a]. □

A.20. Let Δ be an n-dimensional lattice polytope in $M_\mathbb{R}$ and let F be a face of Δ. Then we set

$$\sigma_F := \{u \in N_\mathbb{R} \mid \langle m - m', u\rangle \geq 0 \text{ for all } m \in \Delta,\, m' \in F\}.$$

This is a strongly convex rational polyhedral cone which is normal to F. By setting $\Sigma_\Delta := \{\sigma_F \mid F \preceq \Delta\}$, we obtain a complete fan in $N_\mathbb{R}$. We call Σ_Δ the *normal fan* of Δ.

The assignment $F \mapsto \sigma_F$ defines a bijective order reversing correspondence between faces of Δ and cones of Σ_Δ. The inverse map sends a cone σ to the face

$$F_\sigma := \{m \in \Delta \mid \langle m' - m, u\rangle \geq 0 \text{ for all } m' \in \Delta,\ u \in \sigma\}. \tag{A.4}$$

For details, we refer to [CLS11, § 2.3].

We also use the notation F_σ in the following situation. Let Σ be a fan in $N_\mathbb{R}$ and Ψ a support function on Σ with associated lattice polytope Δ_Ψ. For $\sigma \in \Sigma$, we denote by F_σ the face of Δ_Ψ given as in (A.4).

A.21. Let F be a lattice polytope in $M_\mathbb{R}$. We denote by $\mathrm{aff}(F)$ the affine hull of F and by $\mathbb{L}_F$ the linear subspace of $M_\mathbb{R}$ associated to $\mathrm{aff}(F)$. Then $M(F) := M \cap \mathbb{L}_F$ defines a lattice in $\mathbb{L}_F$. By A.16, we have a measure $\mathrm{vol}_{M(F)}$ on $\mathbb{L}_F = M(F)_\mathbb{R}$ as well as an induced measure on $\mathrm{aff}(F)$ which we also denote by $\mathrm{vol}_{M(F)}$.

If Δ is a full dimensional lattice polytope in $M_\mathbb{R}$ and F is a facet of Δ, we denote by $v_F \in N$ the unique minimal generator of the ray $\sigma_F \in \Sigma_\Delta$ (see A.20). We call v_F the *minimal inner facet normal* of F.

Proposition A.22. *Let f be a concave function on $N_\mathbb{R}$ such that the stability set Δ_f is a lattice polytope of dimension n. With the notations in A.21 we have*

$$-\int_{N_\mathbb{R}} f \,\mathrm{d}\mathcal{M}_M(f) = (n+1)\int_{\Delta_f} f^\vee \,\mathrm{dvol}_M + \sum_F \langle F, v_F\rangle \int_F f^\vee \,\mathrm{dvol}_{M(F)},$$

where the sum is over the facets F of Δ_f.

Proof. This is [BPS14a, Corollary 2.7.10]. □

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