Étale duality of semistable schemes over local rings of positive characteristic

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES
DER NATURWISSENSCHAFTEN (DR. RER. NAT.)
AN DER FAKULTÄT FÜR MATHEMATIK
DER UNIVERSITÄT REGENSBURG

vorgelegt von

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im Jahr 2016

Promotionsgesuch eingerieicht am:	28.04.2016
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Zusammenfassung

In dieser Dissertation studieren wir Dualitätssatze für relative logarithmische de Rham-Witt Garben auf semi-stabilen Schemata X über einem lokalen Ring $\mathbb{F}_q[[t]]$, wobei $\mathbb{F}_q$ ein endlicher Körper ist. Als Anwendung erhalten wir eine neue Filtrierung auf dem maximalen abelschen Quotienten $\pi_1^{\mathrm{ab}}(U)$ der étalen Fundamentalgruppe auf einem offenen Unterschema $U \subseteq X$, die ein Maß für die Verzweigung entlang eines Divisoren D mit normalen Überkreuzungen und $\mathrm{Supp}(D) \subseteq X - U$ gibt. Diese Filtrierung stimmt im Falle von relativer Dimension Null mit der Brylinski-Kato-Matsuda Filtrierung überein.

Schlüsselwörte: étale Dualität, relative logarithmische de Rham-Witt Garben, purity, semi-stabile Schemata, Verzweigung, Klassenkörpertheorie.

Abstract

In this thesis, we study duality theorems for the relative logarithmic de Rham-Witt sheaves on semi-stable schemes X over a local ring $\mathbb{F}_q[[t]]$, for a finite field $\mathbb{F}_q$. As an application, we obtain a new filtration on the maximal abelian quotient $\pi_1^{\mathrm{ab}}(U)$ of the étale fundamental groups $\pi_1(U)$ of an open subscheme $U \subseteq X$, which gives a measure of ramification along a divisor D with normal crossing and $\mathrm{Supp}(D) \subseteq X - U$. This filtration coincides with the Brylinski-Kato-Matsuda filtration in the relative dimension zero case.

Keywords: étale duality, relative logarithmic de Rham-Witt sheaves, purity, semi-stable schemes, ramification, class field theory.

Contents

Introduction

The étale fundamental group, which classifies the finite étale coverings of a scheme, was introduced into algebraic geometry by Grothendieck in the 1960's in [SGA1, [Gro71]]. According to loc.cit., given a geometric point $\bar{x}$ on a connected scheme X, the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ is defined as the inverse limit of automorphisms $Aut_X(Y(\bar{x}))$ of the fiber set $Y(\bar{x})$ over the geometrical point $\bar{x}$, where Y ranges over all finite étale coverings of X. If X is a variety over the complex number field $\mathbb{C}$, then the étale fundamental group is the profinite completion of the topological fundamental group. As the étale fundamental group of a field is its absolute Galois group, we may view the theory of étale fundamental groups as a generalization of the Galois theory of fields to the case of schemes.

An important subject in number theory is the class field theory of global and local fields k, which describes the abelianization G_k^{ab} of the absolute Galois groups in terms of objects associated to the field k. Analogously, the higher dimensional class field theory of a scheme X, which was developed by Kato[Kat90], Parshin[Par75], Saito [KS86] et al. in the later 1970's and 1980's, does the same for the abelianization $\pi_1^{ab}(X)$ of the étale fundamental group.

There are many delicate results in class field theory; ramification theory is one of them. The aim of this thesis is to study ramification theory for higher-dimensional schemes of characteristic $p > 0$. As in the classical case, we want to define a filtration on the abelianized fundamental group of an open subscheme U of a regular scheme X over a finite field $\mathbb{F}_q$, which measures the ramification of a finite étale covering of U along the complement $D = X - U$. More precisely, let $D = \bigcup_{i=1}^{s} D_i$ be a reduced effective Cartier divisor on X such that $\text{Supp}(D)$ has simple normal crossing, where $D_1, \cdots, D_s$ are the irreducible components of D, and let U be its complement in X. We want to define a quotient group $\pi_1^{\text{ab}}(X, mD)/p^n$ of $\pi_1^{\text{ab}}(U)/p^n$, for a divisor $mD = \sum_{i=1}^{s} m_i D_i$ with each $m_i \geq 1$, which classifies the finite étale coverings of degree p^n over U with ramification bounded by mD along the divisor D.

We define the quotient group $\pi_1^{\text{ab}}(X, mD)/p^n$ by using the relationship between $\pi_1^{\text{ab}}(U)/p^n$ and $H^1(U, \mathbb{Z}/p^n\mathbb{Z})$, and by then applying a duality theorem for certain cohomology groups. For this we assume some finiteness conditions on the scheme X. First assume that X is smooth and proper of dimension d over the finite field $\mathbb{F}_q$. For a finite étale covering of U

of degree ℓ^n, where ℓ is a prime different from p, this was already done by using duality theory in ℓ-adic cohomology [SGA1 [Gro71]]. The Poincaré-Pontrjagin duality theorem gives isomorphisms

$$\pi_1^{\mathrm{ab}}(U)/\ell^n \cong \mathrm{Hom}(H^1(U, \mu_\ell^{\otimes n}), \mathbb{Q}/\mathbb{Z}) \cong H_c^d(U, \mu_\ell^{\otimes n}).$$

The case of degree p^n coverings is more subtle, as we deal with wild ramification and there is no obvious analogue of cohomology with compact support for logarithmic de Rham-Witt sheaves. In [JS15], Jannsen and Saito proposed a new approach. Their duality theorem, based on Serre's coherent duality and Milne's duality theorems, together with Pontrjagin duality give isomorphisms

$$\pi_1^{ab}(U)/p^n \cong \mathrm{Hom}(H^1(U, \mathbb{Z}/p^n\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \varprojlim_m H^d(X, W_n\Omega^d_{X|mD,\log}),$$

where $W_n\Omega^d_{X|mD,\log}$ (see Definition 3.3.1) is the logarithmic de Rham-Witt sheaf twisted by some divisor mD. Using these isomorphisms, they defined a quotient $\pi_1^{\mathrm{ab}}(X, mD)/p^n$ of $\pi_1^{ab}(U)/p^n$, ramified of order mD where m is the smallest value such that the above isomorphism factors through $H^d(X, W_n\Omega^d_{X|mD,\log})$. We may think of $\pi_1^{\mathrm{ab}}(X, mD)/p^n$ as the quotient of $\pi_1^{ab}(U)$ classifying abelian étale coverings of U of degree p^n with ramification bounded by mD. In [KS14][KS15], Kerz and Saito also defined a similar quotient group by using curves on X.

In this thesis we assume that X is proper (or projective) over a discrete valuation ring R. More precisely, we may assume that X is a proper semi-stable scheme over $\mathrm{Spec}(R)$. Then there are two cases: mixed and equi-characteristic. In the mixed characteristic case, instead of logarithmic de Rham-Witt sheaves, Sato [Sat07b] defined the p-adic Tate twists, and proved an arithmetic duality theorem for X. In [Uzn13], Uznu proved that over the p-adic field $\pi_1^{ab}(U)/n$ is isomorphic to some motivic homology groups, for all $n > 0$.

In this thesis, we treat the equi-characteristic case, where the wildly ramified case has not been considered before. We follow the approach suggested by Jannsen and Saito in [JS15]. Note that the situation is different from their case, as most of the cohomology groups we meet are not finite. This requires us to use some suitable topological structure on them. The main result of this thesis is the following theorem.

Theorem A (Theorem 3.4.5). *Let $X \to Spec(\mathbb{F}_q[[t]])$ be a projective strictly semistable scheme of relative dimension d, and let X_s be its special fiber. Let D be an effective Cartier divisor on X such that $Supp(D)$ has simple normal crossing, and let U be its open complement. Then there is a perfect pairing of topological $\mathbb{Z}/p^n\mathbb{Z}$-modules*

$$H^i(U, W_n\Omega^r_{U,log}) \times \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}) \to H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,log}) \xrightarrow{Tr} \mathbb{Z}/p^n\mathbb{Z},$$

where the first term is endowed with the discrete topology, and the second term is endowed *with the profinite topology.*

Therefore, we can define a filtration $\mathrm{Fil}_\bullet$ on $H^i(U, W_n\Omega^r_{U,log})$ via the inverse limit (see Definition 3.4.12). This theorem and Pontrjagin duality give isomorphisms

$$\pi_1^{ab}(U)/p^n \cong \mathrm{Hom}(H^1(U, \mathbb{Z}/p^n\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \varprojlim_m H^{d+1}_{X_s}(X, W_n\Omega^{d+1}_{X|mD,\log}),$$

and so we may define $\pi_1^{\mathrm{ab}}(X, mD)/p^n$ as the dual of $\mathrm{Fil}_m H^1(U, \mathbb{Z}/p^n\mathbb{Z})$ (see Definition 3.4.12).

This thesis is organized as follows.

In the first chapter, we will prove a new purity theorem on certain regular schemes. Its cohomological version will be used later for the trace map in the above duality theorem. We recall some basic facts on étale sheaves and logarithmic de Rham-Witt sheaves in the first three sections. In the fourth section, we collect the known purity results. In the fifth section, we prove the new purity result theorem and study the compatibility with previously known results.

Theorem B (Theorem 1.5.4). *Assume X is as before, and $i : X_s \hookrightarrow X$ is the special fiber, which is a reduced divisor and has simple normal crossing. Then there is a canonical isomorphism*

$$Gys^{log}_{i,n} : \nu^d_{n,X_s}[-1] \xrightarrow{\cong} Ri^! W_n\Omega^{d+1}_{X,log}$$

in $D^+(X_s, \mathbb{Z}/p^n\mathbb{Z})$.

Our goal in the second chapter is to develop an absolute duality on X. This can be achieved by combining an absolute coherent duality on the local ring $B = \mathrm{Spec}(\mathbb{F}_q[[t]])$ and a relative duality for f. For the former, we use the Grothendieck local duality, and the latter is following theorem.

Theorem C (Theorem 2.3.1). *Let $f : X \to B = Spec(\mathbb{F}_q[[t]])$ be a projective strictly semistable scheme. Then there is a canonical trace isomorphism*

$$Tr_f : \Omega^{d+1}_X[d] \xrightarrow{\cong} f^!\Omega^1_B.$$

In the third chapter, we study the duality theorems of logarithmic de Rham-Witt sheaves on our projective semistable scheme. In fact, we will prove two duality theorems. The first one is for $H^i(X, W_n\Omega^j_{X,\log})$, which we call unramified duality:

Theorem D (Theorem 3.1.1). *The natural pairing*

$$H^i(X, W_n\Omega^j_{X,\log}) \times H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-j}_{X,\log}) \to H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,\log}) \xrightarrow{Tr} \mathbb{Z}/p^n\mathbb{Z}$$

is a non-degenerated pairing of finite $\mathbb{Z}/p^n\mathbb{Z}$-modules.

The second duality theorem is the above main Theorem A for $H^i(U, W_n\Omega^j_{U,\log})$. We call it ramified duality. To define the pairing, we do further studies on the sheaves $W_n\Omega^r_{X|mD,\log}$ in the middle two sections.

In the last chapter, we will compare this filtration with previously known filtrations in some special cases. The first interesting case would be the filtration in local ramification theory. We can show that for the local field $K = \mathbb{F}_q((t))$ our filtration agree with the non-log version of Brylinski-Kato filtration $\mathrm{fil}_\bullet H^1(K, \mathbb{Z}/p^n\mathbb{Z})$ [Bry83] [Kat89] defined by Matsuda[Mat97].

Proposition E (Proposition 4.2.3)**.** *For any integer $m \geq 1$, we have $Fil_m H^1(K, \mathbb{Z}/p^n\mathbb{Z}) = fil_m H^1(K, \mathbb{Z}/p^n\mathbb{Z})$.*

Notations and Conventions

In this thesis, sheaves and cohomology will be taken with respect to the étale topology unless indicated otherwise. For a variety Y over a perfect field k of characteristic $p > 0$, the differential sheaf Ω^1_Y is the relative differential sheaf $\Omega^1_{Y/k}$. For a regular scheme X, we write the absolute differential sheaf $\Omega^1_{X/\mathbb{Z}}$ as Ω^1_X for short. For a scheme X and $u \in \mathbb{N}_0$, we denote X^u as the set of points of codimension u in X. The finite field $\mathbb{F}_q$ is a finite field of characteristic $p > 0$, i.e. q is a finite power of the prime p.

Acknowledgment

During the time of my Ph.D study, too many people to mention them individually have contributed in personal and scientific respects in one or another way. I would like to thank some of them in particular.

First and Foremost, I would like to thank my doctoral supervisor Prof. Dr. Uwe Jannsen for all his support, patience and confidence in me during the past four years. Without his warm encouragement and thoughtful guidance this thesis would not have been achievable.

The topic of this thesis emerged from suggestion and conjecture by Prof. Shuji Saito, which is now our main theorem. I would like to thank him for his insightful discussions during his stay at Regensburg.

I also thank Prof. Dr. Moritz Kerz for his scientific advice and numerous helpful discussions.

Many people helped me before I came to Regensburg. I would like to extend my thanks to Prof. Kezheng Li, Prof. Ye Tian, Prof. Song Wang, Prof. Xiandong Wang, Prof. Fei Xu and Prof. Kejian Xu.

I am thankful to my former and current colleagues Ivan Barrientos, Veronika Ertl, Patrick Forré, Morten Lüders, Xu Shen, Florian Strunk, Georg Tamme, Minh-Hoang Tran, Shanwen Wang, Yitao Wu for helpful discussions on mathematical problems, to Tobias Sitte, Martin Ruderer, Yuning Liu for their help during the time in Regensburg.

I also want to thank the financial support from Graduiertenkolleg GRK 1692 "Curvature, Cycles, and Cohomology" and SFB 1085 "Higher Invariants" in previous years.

Last but not least, my deep and sincere gratitude to my family for their continuous and unparalleled love, help and support.

Chapter 1

Purity

1.1 Preliminaries on étale sheaves

Let X be a scheme, let $i : Z \hookrightarrow X$ be a closed immersion, and let $j : U = X - Z \hookrightarrow X$ be the open complement. Let $Sh(X)$ be the category of étale abelian sheaves on X, and let $D^*(X)$ be its derived category with boundedness conditions, $* = b, +, -$.

Definition 1.1.1. *The category $T(X)$ is defined by the following data:*

Objects are triples $(\mathscr{G}, \mathscr{H}, \phi)$, with $\mathscr{G} \in Sh(Z)$, $\mathscr{H} \in Sh(U)$, and a morphism $\phi : \mathscr{G} \to i^* j_* \mathscr{H}$ in $Sh(Z)$.

Morphisms are pairs $(\xi, \eta) : (\mathscr{G}_1, \mathscr{H}_1, \phi_1) \to (\mathscr{G}_2, \mathscr{H}_2, \phi_2)$, with morphisms $\xi : \mathscr{G}_1 \to \mathscr{G}_2$ and $\eta : \mathscr{H}_1 \to \mathscr{H}_2$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{G}_1 & \xrightarrow{\phi_1} & i^* j_* \mathscr{H}_1 \\ \downarrow \xi & & \downarrow i^* j_* \eta \\ \mathscr{G}_2 & \xrightarrow{\phi_2} & i^* j_* \mathscr{H}_2 \end{array}$$

Theorem 1.1.2. *([Fu11, Prop.5.4.2])*

(i) There is an equivalence of the categories

$$\begin{array}{ccc} Sh(X) & \longrightarrow & T(X) \\ \mathscr{F} & \longmapsto & (i^*\mathscr{F}, j^*\mathscr{F}, \phi_{\mathscr{F}}) \end{array}$$

where $\phi_{\mathscr{F}}$ is the pullback of the adjunction $\mathscr{F} \to j_* j^* \mathscr{F}$ under the closed immersion $i : Z \hookrightarrow X$.

(ii) Under above identification, we have the following six functors

$$Sh(Z) \underset{i^!}{\overset{i^*}{\leftleftarrows}} \!\!\xrightarrow{i_*}\! Sh(X) \underset{j_*}{\overset{j_!}{\leftleftarrows}} \!\!\xrightarrow{j^*}\! Sh(U)$$

which are given by

$$i^* : \mathscr{G} \longleftarrow\!\!\shortmid (\mathscr{G}, \mathscr{H}, \phi); \qquad j_! : (0, \mathscr{H}, 0) \longleftarrow\!\!\shortmid \mathscr{H}$$

$$i_* : \mathscr{G} \longmapsto (\mathscr{G}, 0, 0); \qquad j^* : (\mathscr{G}, \mathscr{H}, \phi) \longmapsto \mathscr{H}$$

$$i^! : Ker(\phi) \longleftarrow\!\!\shortmid (\mathscr{G}, \mathscr{H}, \phi); \qquad j_* : (i^* j_* \mathscr{H}, \mathscr{H}, id) \longleftarrow\!\!\shortmid \mathscr{H}.$$

(iii) The pairs (i^*, i_*),$(i_*, i^!)$, $(j_!, j^*)$, $(j^*, j_*)$are pairs of adjoint functors, so that each functor in (ii) is left adjoint to the functor below it.

(iv) The functors $i^*, i_*, j^*, j_!$ are exact, and $j_*, i^!$ are left exact.

(v) The composites $i^* j_!, i^! j_!, i^! j_*, j^* i_*$ are zero.

Definition 1.1.3. *The functor $Ri^! : D^+(X) \to D^+(Z)$ is the right derived functors of $i^!$, and the functor $Rj_* : D^+(U) \to D^+(X)$ be the right derived functor of j_*. The sheaf $R^q i^!(\mathscr{F})$(resp. $R^q j_*(\mathscr{F})$) is the q-th cohomology sheaf of $Ri^!(\mathscr{F})$ (resp. $Rj_*(\mathscr{F})$).*

Proposition 1.1.4. *([Fu11, Prop.5.6.11]) For $\mathscr{F} \in Sh(X)$, we have an exact sequence(so called localization sequence)*

$$0 \to i_* i^! \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} \to i_* R^1 i^! \mathscr{F} \to 0,$$

and isomorphisms

$$R^q j_*(j^* \mathscr{F}) \cong i_*(R^{q+1} i^! \mathscr{F}). \quad (q \geq 1)$$

1.2 Local-global spectral sequences

For the convenience of the reader, we repeat the relevant material from [JSS14].

Let X, Z, U be as before, and let m be a non-negative integer. Let $D^+(X, \mathbb{Z}/m\mathbb{Z})$ be the full subcategory of étale $\mathbb{Z}/m\mathbb{Z}$-sheaves on X, whose objects are the lower bounded cohomology sheaves. For any $\mathscr{F} \in D^+(X, \mathbb{Z}/m\mathbb{Z})$, by taking an injective resolution $I^\bullet \to \mathscr{F}$, we have an exact sequence

$$0 \to i_* i^! I^\bullet \to I^\bullet \to j_* j^* I^\bullet \to 0.$$

The connecting morphism gives a map

$$\delta^{\mathrm{loc}}_{U,Z}(\mathscr{F}) : Rj_* j^* \mathscr{F} \longrightarrow i_* Ri^! \mathscr{F}[1].$$

in $D^+(X, \mathbb{Z}/m\mathbb{Z})$, which is functorial in $\mathscr{F}$, and for any integer r,

$$\delta^{\mathrm{loc}}_{U,Z}(\mathscr{F})[r] = (-1)^r \delta^{\mathrm{loc}}_{U,Z}(\mathscr{F}[r]).$$

We want to study the connection morphism δ^{loc} locally for a local ring of dimension 1. Before that, we introduce the following notations.

For $x \in X$, we have the following natural morphisms

$$\begin{array}{ccc} \{x\} & \xrightarrow{i'_x} & \overline{\{x\}} \\ & \searrow^{i_x} & \downarrow{\scriptstyle i_{\overline{\{x\}}}} \\ & & X \end{array}$$

Definition 1.2.1. *For $x \in X$, we define a functor*

$$Ri^!_x\mathscr{F} : D^+(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow D^+(x, \mathbb{Z}/m\mathbb{Z})$$

as the composition $(i'_x)^* \circ Ri^!_{\overline{x}}\mathscr{F}$, where $(i'_x)^*$ is the (topological) inverse image functor for étale sheaf.

Let x, y be points in X such that x has codimension 1 in $T := \overline{\{y\}}$. Let i_T(resp. i_x, i_y, φ) be the natural map $T \hookrightarrow X$ (resp. $x \to X, y \to X, \mathrm{Spec}(O_{T,x}) \to T$). Note that x is closed in $\mathrm{Spec}(O_{T,x})$, and its complement is y, so we have already defined $\delta^{\mathrm{loc}}_{y,x}(\mathscr{G})$, for any étale sheaf $\mathscr{G}$ on $\mathrm{Spec}(O_{T,x})$.

Definition 1.2.2. *([JSS14, 0.5]) For $\mathscr{F} \in D^+(X, \mathbb{Z}/m\mathbb{Z})$, we define the morphism*

$$\delta^{loc}_{y,x}(\mathscr{F}) : Ri_{y*}Ri^!_y\mathscr{F} \longrightarrow Ri_{x*}Ri^!_x\mathscr{F}$$

as $Ri_{T*}R\varphi_*(\delta^{loc}_{y,x}(\varphi^* Ri^!_T\mathscr{F}))$ in $D^+(X, \mathbb{Z}/m\mathbb{Z})$.

These connecting morphisms for all points on X give rise to a local-global spectral sequence [Sat07a, §1.12] of étale sheaves on X

$$E_1^{u,v} = \bigoplus_{x \in X^u} R^{u+v} i_{x*}(Ri^!_x\mathscr{F}) \Longrightarrow \mathcal{H}^{u+v}(\mathscr{F}).$$

For a closed immersion $i : Z \hookrightarrow X$, there is a localized variant

$$E_1^{u,v} = \bigoplus_{x \in X^u \cap Z} R^{u+v} i_{x*}(Ri^!_x\mathscr{F}) \Longrightarrow i_* R^{u+v} i^!(\mathscr{F}).$$

For $x \in X^u \cap Z$, the natural map $i_x : x \to X$ can be written as $i_x = i \circ \imath_x : x \xrightarrow{s} Z \xrightarrow{i} X$ and i_* is exact, so we have the following result.

Proposition 1.2.3. *For any closed immersion $i : Z \hookrightarrow X$, there is a local-global spectral sequence*

$$E_1^{u,v} = \bigoplus_{x \in Z^u} R^{u+v} \imath_{x*}(R i_x^! \mathscr{F}) \Longrightarrow R^{u+v} i^!(\mathscr{F}).$$

where $\imath_x : x \to Z$ and $i_x : x \to X$ are natural maps.

1.3 The Logarithmic de Rham-Witt sheaves

1.3.1 Basic properties

Let X be a scheme of dimension d over a perfect field k of characteristic $p > 0$, and let $W_n(k)$ be the ring of Witt vectors of length n.

Based on ideas of Lubkin, Bloch and Deligne, Illusie defined the de Rham-Witt complex [Ill79]. Recall the de Rham-Witt complex $W\Omega^\bullet_{X/k}$ is the inverse limit of an an inverse system $(W_n\Omega^\bullet_{X/k})_{n\geq 1}$ of complexes

$$W_n\Omega^\bullet_{X/k} := (W_n\Omega^0_{X/k} \xrightarrow{d} W_n\Omega^1_{X/k} \to \cdots \xrightarrow{d} W_n\Omega^i_{X/k} \xrightarrow{d} \cdots)$$

of sheaves of $W_n\mathcal{O}_X$-modules on the Zariski site of X. The complex $W_n\Omega^\bullet_{X/k}$ is called the de Rham-Witt complex of level n.

This complex $W_n\Omega^\bullet_{X/k}$ is a strictly anti-commutative differential graded $W_n(k)$-algebra. In the rest of this section, we will omit the subscript $/k$ to simplify the notation.

We have the following operators on the de Rham-Witt complex ([Ill79, I]):

(i) The projection $R : W_n\Omega^\bullet_X \to W_{n-1}\Omega^\bullet_X$, which is a surjective homomorphism of differential graded algebras.

(ii) The Verschiebung $V : W_n\Omega^\bullet_X \to W_{n+1}\Omega^\bullet_X$, which is an additive homomorphism.

(iii) The Frobenius $F : W_n\Omega^\bullet_X \to W_{n-1}\Omega^\bullet_X$, which is a homomorphism of differential graded algebras.

Proposition 1.3.1. *([Ill79, I 1.13,1.14])*

(i) For each $n \geq 1$, and each i, $W_n\Omega^i_X$ is a quasi-coherent $W_n\mathcal{O}_X$-module.

(ii) For any étale morphism $f : X \to Y$, $f^*W_n\Omega^i_Y \to W_n\Omega^i_X$ is an isomorphism of $W_n\mathcal{O}_X$-modules.

Remark 1.3.2. *Let $\mathscr{F}$ be a quasi-coherent on X, we denote its associated sheaf on $X_{ét}$ by $\mathscr{F}_{ét}$, then we have $H^i(X_{Zar}, \mathscr{F}) = H^i(X_{ét}, \mathscr{F}_{ét})$, for all $i \geq 0$[Mil80, III 3.7]. By the above proposition, we may also denote $W_n\Omega^i_X$ as sheaf on $X_{ét}$, and its étale and Zariski cohomology groups are agree.*

Cartier operators are another type of operators on the de Rham-Witt complex. Before stating the theorem, we set

$$ZW_n\Omega^i_X := \mathrm{Ker}(d : W_n\Omega^i_X \to W_n\Omega^{i+1}_X);$$

$$BW_n\Omega^i_X := \mathrm{Im}(d : W_n\Omega^{i-1}_X \to W_n\Omega^i_X);$$

$$\mathscr{H}^i(W_n\Omega^\bullet_X) := ZW_n\Omega^i_X / BW_n\Omega^i_X;$$

$$\begin{aligned} Z_1W_n\Omega^i_X : \quad &= \mathrm{Im}(F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X) \\ &= \mathrm{Ker}(F^{n-1}d : W_n\Omega^i_X \xrightarrow{d} W_n\Omega^{i+1}_X \xrightarrow{F^{n-1}} \Omega^{i+1}_X). \end{aligned}$$

Since $W_1\Omega^i_X \cong \Omega^i_X$, $ZW_1\Omega^i_X$(resp. $BW_1\Omega^i_X$) is also denoted by $Z\Omega^i_X$(resp. $B\Omega^i_X$). Note that $Z\Omega^i_X$, $B\Omega^i_X$, and $\mathscr{H}^i(\Omega^\bullet_X)$ can be given $\mathcal{O}_X$-module structures via the absolute Frobenius morphism F on $\mathcal{O}_X$.

Theorem 1.3.3. *(Cartier, [Kat70, Thm. 7.2], [Ill96, Thm. 3.5]) Suppose X is of finite type over k. Then there exists a unique p-linear homomorphism of graded $\mathcal{O}_X$-algebras*

$$C^{-1} : \bigoplus \Omega^i_X \longrightarrow \bigoplus \mathscr{H}^i(\Omega^\bullet_X)$$

satisfying the following two conditions:

(i) For $a \in \mathcal{O}_X$, $C^{-1}(a) = a^p$;

(ii) For $dx \in \Omega^1_X$, $C^{-1}(dx) = x^{p-1}dx$.

If X is moreover smooth over k, then C^{-1} is an isomorphism. It is called inverse Cartier isomorphism. The inverse of C^{-1} is called Cartier operator, and is denoted by C.

Higher Cartier operators can be defined as follows, which comes back to the above theorem in the case $n = 1$.

Proposition 1.3.4. *([IR83, III], [Kat85, § 4]) If X is smooth over k, then there is a unique higher Cartier morphism $C : Z_1W_{n+1}\Omega^i_X \to W_n\Omega^i_X$ such that the diagram*

$$\begin{array}{ccc} Z_1W_n\Omega^i_X & \xrightarrow{\quad V \quad} & W_{n+1}\Omega^q_X \\ & {\scriptstyle C}\searrow \quad \nearrow{\scriptstyle p} & \\ & W_n\Omega^i_X & \end{array}$$

is commutative. We have an isomorphism

$$W_n\Omega^i_X \underset{\cong}{\xrightarrow{F}} Z_1W_{n+1}\Omega^i_X / dV^{n-1}\Omega^{i-1}_X,$$

and an exact sequence

$$0 \to dV^{n-1}\Omega^{i-1}_X \to Z_1W_{n+1}\Omega^i_X \xrightarrow{C} W_n\Omega^i_X.$$

For a smooth variety X over k, the composite morphism

$$W_{n+1}\Omega^i_X \xrightarrow{F} W_n\Omega^i_X \twoheadrightarrow W_n\Omega^i_X/dV^{n-1}\Omega^{i-1}_X$$

is trivial on $\mathrm{Ker}(R : W_{n+1}\Omega^i_X \to W_n\Omega^i_X) = V^n\Omega^i_X + dV^n\Omega^{i-1}_X$. Therefore F induces a morphism

$$F : W_n\Omega^i_X \to W_n\Omega^i_X/dV^{n-1}\Omega^{i-1}_X.$$

Definition 1.3.5. *Let X be a smooth variety over k. For any positive integer n, and any non-negative integer i, we define the i-th logarithmic de Rham-Witt sheaf of length n as*

$$W_n\Omega^i_{X,log} := Ker(W_n\Omega^i_X \xrightarrow{1-F} W_n\Omega^i_X/dV^{n-1}\Omega^{i-1}_X).$$

For any $x \in X$, we denote $W_n\Omega^i_{x,\log} := W_n\Omega^i_{\kappa(x),\log}$, where $\kappa(x)$ is the residue field at x.

Remark 1.3.6. *(Local description [Ill79, I 1.3]) The i-th logarithmic de Rham-Witt sheaf $W_n\Omega^i_{X,\log}$ is the additive subsheaf of $W_n\Omega^i_X$, which is étale locally generated by sections $d\log[x_1]_n \cdots d\log[x_i]_n$, where $x_i \in \mathcal{O}_X^\times$, $[x]_n$ is the Teichmüller representative of x in $W_n\mathcal{O}_X$, and $d\log[x]_n := \frac{d[x]_n}{[x]_n}$. In other words, it is the image of*

$$\begin{array}{rccc} d\log : & (\mathcal{O}_X^\times)^{\otimes i} & \longrightarrow & W_n\Omega^i_X \\ & (x_1, \cdots, x_i) & \longmapsto & d\log[x_1]_n \cdots d\log[x_i]_n \end{array}$$

Proposition 1.3.7 ([CTSS83],[Ill79])**.** *For a smooth variety X over k, we have the following exact sequences of étale sheaves on X:*

(i) $0 \to W_n\Omega^i_{X,log} \xrightarrow{p^m} W_{n+m}\Omega^i_{X,log} \xrightarrow{R} W_m\Omega^i_{X,log} \to 0$;

(ii) $0 \to W_n\Omega^i_{X,log} \to W_n\Omega^i_X \xrightarrow{1-F} W_n\Omega^i_X/dV^{n-1}\Omega^{i-1}_X \to 0$;

(iii) $0 \to W_n\Omega^i_{X,log} \to Z_1W_n\Omega^i_X \xrightarrow{C-1} W_n\Omega^i_X \to 0.$

Proof. The first assertion is Lemma 3 in [CTSS83], and the second is Lemma 2 in loc.cit.. The last one is Lemma 1.6 in [GS88a], which can easily be deduce from (ii). In particular, for $n = 1$, (iii) can be also found in [Ill79]. □

The logarithmic de Rham-Witt sheaves $W_n\Omega^i_{X,\log}$ are $\mathbb{Z}/p^n\mathbb{Z}$-sheaves, which have a similar duality theory as the $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaves $\mu_\ell^{\otimes n}$ with $\ell \neq p$ for a smooth proper variety:

Theorem 1.3.8. *(Milne duality [Mil86, 1.12]) Let X be a smooth proper variety over k of dimension d, and let n be a positive integer. Then the following holds:*

(i) There is a canonical trace map $tr_X : H^{d+1}(X, W_n\Omega^d_{X,log}) \to \mathbb{Z}/p^n\mathbb{Z}$. It is bijective if X is connected;

(ii) For any integers i and r with $0 \le r \le d$, the natural pairing

$$H^i(X, W_n\Omega^r_{X,log}) \times H^{d+1-i}(X, W_n\Omega^{d-r}_{X,log}) \to \mathbb{Z}/p^n\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^n\mathbb{Z}$-modules.

Remark 1.3.9. *The proof can be obtained in the following way: using the exact sequence (i) in Proposition 1.3.7, we reduce to the case $n = 1$, which can be obtained from Serre's coherent duality via the exact sequence (ii) and (iii) in the same proposition.*

1.3.2 Boundary maps

The results in this subsection can be found in [JSS14, 0.7], for convenience of the reader, we briefly summarize it. The boundary (residue) maps on logarithmic de Rham-Witt sheaves were first defined by Kato [Kat86].

Since we will only use this in the characteristic $p > 0$ case, we may assume that X is a noetherian excellent scheme over $\mathbb{F}_q$. Let $x, y \in X$ be two points with $x \in \overline{\{y\}} =: Z$ of codimension 1 in Z. Let $i_x : x \to X$ and $i_y : y \to X$ be the natural maps. We may further replace Z with $\mathrm{Spec}(\mathcal{O}_{Z,x})$.

Case (I): Regular case. We assume that $\mathcal{O}_{Z,x}$ is regular, so that it is a discrete valuation ring. Let $K = \kappa(y)$ be its fraction field and $k = \kappa(x)$ be its residue field. Both of them are fields of characteristic p. We know that $H^i(k, W_n\Omega^r_{k,\mathrm{log}}) = 0$ for $i \neq 0, 1$.

(I.1) Sub-case $i = 0$. The boundary map $\partial^{\mathrm{val}}_{y,x}$ is defined as the composition

$$H^0(K, W_n\Omega^{r+1}_{K,\mathrm{log}}) \xleftarrow[\cong]{d\mathrm{log}} K^M_{r+1}(K)/p^n \xrightarrow{\partial} K^M_r(k)/p^n \xrightarrow[\cong]{d\mathrm{log}} H^0(k, W_n\Omega^r_{k,\mathrm{log}}),$$

where $K^M_r(F)$ is the r-th Milnor K-group of a field F, $d\mathrm{log}$ is the symbol map, and ∂ is the residue map for Milnor K-theory.

(I.2) Sub-case $i = 1$. In this case, we need assume $[k : k^p] \le r$. The boundary map $\partial^{\mathrm{val}}_{y,x}$ is defined as the composition

$$\begin{array}{ccc} H^1(K, W_n\Omega^{r+1}_{K,\mathrm{log}}) & \xrightarrow{\partial^{\mathrm{val}}_{y,x}} & H^1(k, W_n\Omega^r_{k,\mathrm{log}}) \\ \downarrow \cong & & \uparrow \cong \\ H^1(k, H^0(K^{\mathrm{sh}}, W_n\Omega^{r+1}_{K^{\mathrm{sh}},\mathrm{log}})) & \xrightarrow{\partial} & H^1(k, H^0(\overline{k}, W_n\Omega^r_{\overline{k},\mathrm{log}})) \end{array}$$

where the vertical isomorphisms come from Hochschild-Serre spectral sequence and the fact that $cd_p(k) \le 1$ and

$$H^i(K^{\mathrm{sh}}, W_n\Omega^{r+1}_{K^{\mathrm{sh}},\mathrm{log}}) = 0 = H^i(\overline{k}, W_n\Omega^r_{\overline{k},\mathrm{log}}) \qquad \text{for} \quad i > 0.$$

The map ∂ is induced by the boundary map defined in (I.1).

Case (II): General case. In this case, we take the normalization $\pi : Z' \to Z$ of $Z = \mathrm{Spec}(\mathcal{O}_{Z,x})$. It is a finite morphism since Z is excellent . Then we define

$$\partial^{\mathrm{val}}_{y,x}(\cdot) = \sum_{x'|x} Cor_{\kappa(x')/\kappa(x)}(\partial^{\mathrm{val}}_{y,x'}(\cdot))$$

where the sum is taken over all points $x' \in Z'$ lying over x, $\partial^{\mathrm{val}}_{y,x'}$ is the boundary map defined in Case (I) for the discrete valuation ring $\mathcal{O}_{Z',x'}$, and

$$Cor_{\kappa(x')/\kappa(x)} : H^i(x', W_n\Omega^r_{x',\log}) \to H^i(x, W_n\Omega^r_{x,\log})$$

is the corestriction map, which is defined in two different cases as in (I).

(II.1) Sub-case $i = 0$. The corestriction map is defined as the composition

$$H^0(x', W_n\Omega^{r+1}_{x',\log}) \xleftarrow[\cong]{d\log} K^M_{r+1}(\kappa(x'))/p^n \xrightarrow{Nr_{x'/x}} K^M_r(\kappa(x))/p^n \xrightarrow[\cong]{d\log} H^0(x, W_n\Omega^r_{x,\log}),$$

where $Nr_{x'/x}$ is the norm map in Milnor K-theory.

(II.2) Sub-case $i = 1$. We sheafified the corestriction map in (II.1), and get an induced corestriction or trace map

$$\mathrm{tr}_{x'/x} : \pi_* W_n\Omega^r_{x',\log} \to W_n\Omega^r_{x,\log}.$$

Then we define the map $Cor_{\kappa(x')/\kappa(x)}$ as the map induced by $\mathrm{tr}_{x'/x}$ on H^1.

Furthermore, we define the boundary maps for sheaves by sheafification. i.e., we want to define a morphism of étale sheaves on X:

$$i_{y*}W_n\Omega^{r+1}_{y,\log} \to i_{x*}W_n\Omega^r_{x,\log}.$$

By adjunction, it is enough to define

$$i_x^* i_{y*}W_n\Omega^{r+1}_{y,\log} \to W_n\Omega^r_{x,\log}$$

in the category of étale sheaves on x. Let $Z_1, \cdots, Z_a$ be the distinct irreducible components of $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{Z,x})$. Note that the affine coordinate ring of each Z_i is a strict henselian local domain of dimension 1 with residue field $\kappa(\overline{x})$. Let η_i be the generic point of Z_i, by looking at stalks, it is enough to construct

$$\bigoplus_{i=1}^{a} H^0(\eta_i, W_n\Omega^{r+1}_{y,\log}) \to H^0(\overline{x}, W_n\Omega^r_{x,\log}).$$

Now we define this map as the sum of maps that defined on cohomology groups for Z_i as before.

1.3.3 Normal crossing varieties

In [Sat07a], Sato generalized the definition of logarithmic de Rham-Witt sheaves from smooth varieties to more general varieties, and proved that they share similar properties on normal crossing varieties.

Let Z be a variety over k of dimension d. For a non-negative integer m and a positive integer $n > 0$, we denote by $C_n^\bullet(Z, m)$ the following complex of étale sheaves on X

$$\bigoplus_{x\in Z^0} i_{x*}W_n\Omega^m_{x,\log} \xrightarrow{(-1)^m\cdot\partial} \bigoplus_{x\in Z^1} i_{x*}W_n\Omega^{m-1}_{x,\log} \xrightarrow{(-1)^m\cdot\partial} \cdots \xrightarrow{(-1)^m\cdot\partial} \bigoplus_{x\in Z^p} i_{x*}W_n\Omega^{m-p}_{x,\log} \to \cdots$$

where i_x is the natural map $x \to Z$, and ∂ denotes the sum of Kato's boundary maps (see §1.3.2).

Definition 1.3.10. *(i) The mth homological logarithmic Hodge-Witt sheaf is defined as the 0-th cohomology sheaf $\mathscr{H}^0(C_n^\bullet(Z,m))$ of the complex $C_n^\bullet(Z,m)$, and denoted by $\nu^m_{n,Z}$.*

(ii) The mth cohomological Hodge-Witt sheaf is the image of

$$d\log : (\mathcal{O}_Z^\times)^{\otimes m} \longrightarrow \bigoplus_{x\in Z^0} i_{x*}W_n\Omega^m_{x,log},$$

and denoted by $\lambda^m_{n,Z}$.

Remark 1.3.11. *If Z is smooth, then $\nu^m_{n,Z} = \lambda^m_{n,Z} = W_n\Omega^m_{Z,\log}$, but in general $\lambda^m_{n,Z} \subsetneq \nu^m_{n,Z}$ ([Sat07a], Rmk. 4.2.3).*

Definition 1.3.12. *The variety Z is called normal crossing variety if it is everywhere étale locally isomorphic to*

$$Spec(k[x_0, \cdots, x_d]/(x_0\cdots x_a))$$

for some integer $a \in [0, d]$, where $d = dim(Z)$. A normal crossing variety is called simple if every irreducible component is smooth.

Proposition 1.3.13. *([Sat07a, Cor. 2.2.5(1)]) For a normal crossing variety Z, the natural map $\nu^m_{n,Z} \longrightarrow C_n^\bullet(Z,m)$ is a quasi-isomorphism of complexes.*

Theorem 1.3.14. *([Sat07a, Thm. 1.2.2]) Let Z be a normal crossing variety over a finite field, and proper of dimension d. Then the following holds:*

(i) There is a canonical trace map $tr_Z : H^{d+1}(Z, \nu^d_{n,Z}) \to \mathbb{Z}/p^n\mathbb{Z}$. It is bijective if Z is connected.

(ii) For any integers i and j with $0 \le j \le d$, the natural pairing

$$H^i(Z, \lambda^j_{n,Z}) \times H^{d+1-i}(Z, \nu^{d-j}_{n,Z}) \to H^{d+1}(Z, \nu^d_{n,Z}) \xrightarrow{tr_Z} \mathbb{Z}/p^n\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^n\mathbb{Z}$-modules.

1.4 The known purity results

In the ℓ-adic world, the sheaf $\mu_\ell^{\otimes n}$ on a regular scheme has purity. This was called Grothendieck's absolute purity conjecture, and it was proved by Gabber and can be found in [Fuj02]. In the p-adic world, one may ask if purity holds for the logarithmic de Rham-Witt sheaves, but those sheaves only have semi-purity(see Remark 1.4.3 below).

Proposition 1.4.1. *([Gro85]) Let $i : Z \hookrightarrow X$ be a closed immersion of smooth schemes of codimension c over a perfect field k of characteristic $p > 0$. Then, for $r \geq 0$ and $n \geq 1$, $R^m i^! W_n\Omega^r_{X,log} = 0$ if $m \neq c, c+1$.*

For the logarithmic de Rham-Witt sheaf at top degree, i.e., $W_n\Omega^d_{X,\log}$ where $d = \dim(X)$, the following theorem tells us $R^{c+1}i^! W_n\Omega^d_{X,\log} = 0$.

Theorem 1.4.2. *([GS88b],[Mil86]) Assume $i : Z \hookrightarrow X$ is as above. Let $d = dim(X)$. Then, for $n \geq 1$, there is a canonical isomorphism (called Gysin morphism)*

$$Gys^d_i : W_n\Omega^{d-c}_{Z,log}[-c] \xrightarrow{\cong} Ri^! W_n\Omega^d_{X,log}$$

in $D^+(Z, \mathbb{Z}/p^n\mathbb{Z})$.

Remark 1.4.3. *Note that the above theorem is only for the d-th logarithmic de Rham-Witt sheaf. For $m < d$, the $R^{c+1}i^! W_n\Omega^m_{X,log}$ is non zero in general [Mil86, Rem. 2.4]. That's the reason why we say they only have semi-purity.*

Sato generalized the above theorem to normal crossing varieties.

Theorem 1.4.4. *([Sat07a, Thm. 2.4.2]) Let X be a normal crossing varieties of dimension d, and $i : Z \hookrightarrow X$ be a closed immersion of pure codimension $c \geq 0$. Then, for $n \geq 1$, there is a canonical isomorphism(also called Gysin morphism)*

$$Gys^d_i : \nu^{d-c}_{n,Z}[-c] \xrightarrow{\cong} Ri^! \nu^d_{n,X}$$

in $D^+(Z, \mathbb{Z}/p^n\mathbb{Z})$.

Remark 1.4.5. *The second Gysin morphism coincides with the first one, when X and Z are smooth [Sat07a, 2.3,2.4]. In loc.cit., Sato studied the Gysin morphism of $\nu^r_{n,X}$ for $0 \leq r \leq d$. In fact the above isomorphism for $\nu^d_{n,X}$ was already proved by Suwa [Suw95, 2.2] and Morse [Mos99, 2.4].*

Corollary 1.4.6. *If $i : Z \hookrightarrow X$ is a normal crossing divisor(i.e normal crossing subvariety of codimension 1) and X is smooth, then we have*

$$Gys^d_i : \nu^{d-1}_{n,Z}[-1] \xrightarrow{\cong} Ri^! W_n\Omega^d_{X,log}$$

in $D^+(Z, \mathbb{Z}/p^n\mathbb{Z})$.

Proof. This follows from the fact that $W_n\Omega^d_{X,\log} = \nu^d_{n,X}$, when X is smooth. $\square$

We want to generalize this corollary to the case where X is regular. For this, we need a purity result of Shiho [Shi07], which is a generalization of Theorem 1.4.2 for smooth schemes to regular schemes.

Definition 1.4.7. *Let X be a scheme over $\mathbb{F}_q$, and $i \in \mathbb{N}_0, m \in \mathbb{N}$. Then we define the i-th logarithmic de Rham-Witt sheaf $W_n\Omega^i_{X,log}$ as the image of*

$$d\log : (\mathcal{O}_X^\times)^{\otimes i} \longrightarrow W_n\Omega^i_X,$$

where $d\log$ is defined by

$$d\log(x_1 \otimes \cdots \otimes x_i) = d\log[x_1]_n \cdots d\log[x_i]_n,$$

and $[x]_n$ is the Teichmüller representative of x in $W_n\mathcal{O}_X$.

Remark 1.4.8. *Note that this definition is a simple generalization of the classical definition for smooth X, by comparing with the local description of logarithmic de Rham-Witt sheaves in Remark 1.3.6.*

As in Theorem 1.3.3, we can define the inverse Cartier operator similarly for a scheme over $\mathbb{F}_q$. Using the Néron-Popescu approximation theorem [Swa98](see Theorem 3.2.10 below) and Grothendieck's limit theorem (SGA 4 [AGV72, VII, Thm. 5.7]), Shiho showed the following results.

Proposition 1.4.9. *([Shi07, Prop. 2.5]) If X is a regular scheme over $\mathbb{F}_q$, the inverse Cartier homomorphism C^{-1} is an isomorphism.*

Using the same method, we can prove:

Theorem 1.4.10. *The results of Theorem 1.3.4 also hold for a regular scheme over $\mathbb{F}_q$.*

Proposition 1.4.11. *([Shi07, Prop. 2.8, 2.10, 2.12]) Let X be a regular scheme over $\mathbb{F}_q$. Then we have the following exact sequences:*

(i) $0 \to W_n\Omega^i_{X,log} \xrightarrow{p^m} W_{n+m}\Omega^i_{X,log} \xrightarrow{R} W_m\Omega^i_{X,log} \to 0$;

(ii) $0 \to W_n\Omega^i_{X,log} \to W_n\Omega^i_X \xrightarrow{1-F} W_n\Omega^i_X/dV^{n-1}\Omega^{i-1}_X \to 0$;

(iii) $0 \to W_n\Omega^i_{X,log} \to Z_1W_n\Omega^i_X \xrightarrow{C-1} W_n\Omega^i_X \to 0$.

Proof. The claim (iii) is easily obtained from (ii), as in the smooth case. When $n = 1$, this is Proposition 2.10 in loc.cit.. $\square$

Let $\mathscr{C}$ be a category of regular schemes of characteristic $p > 0$, such that, for any $x \in X$, the absolute Frobenius $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is finite. Shiho showed the following cohomological purity result.

Theorem 1.4.12. *([Shi07, Thm. 3.2]) Let X be a regular scheme over $\mathbb{F}_q$, and let $i : Z \hookrightarrow X$ be a regular closed immersion of codimension c. Assume moreover that $[\kappa(x) : \kappa(x)^p] = p^N$ for any $x \in X^0$, where $\kappa(x)$ is the residue field at x. Then there exists a canonical isomorphism*

$$\theta_{i,n}^{q,m,log} : H^q(Z, W_n\Omega^{m-c}_{Z,log}) \xrightarrow{\cong} H^{q+c}_Z(X, W_n\Omega^m_{X,log})$$

if $q = 0$ holds or if $q > 0, m = N, X \in ob(\mathscr{C})$ hold.

Remark 1.4.13. *In [Shi07, Cor. 3.4], Shiho also generalized Proposition 1.4.1 to the case that $Z \hookrightarrow X$ is a regular closed immersion, and without the assumption on the residue fields.*

Corollary 1.4.14. *Let X be as in Theorem 1.4.12, $i_x : x \to X$ be a point of codimension p. Then there exists a canonical isomorphism*

$$\theta^{log}_{i_x,n} : W_n\Omega^{N-p}_{x,log}[-p] \xrightarrow{\cong} Ri^!_x W_n\Omega^N_{X,log}$$

in $D^+(x, \mathbb{Z}/p^n\mathbb{Z})$.

Proof. Let X_x be the localization of X at x. The assertion is a local problem, hence we may assume $X = X_x$. By the above Remark 1.4.13, we have $R^j i^!_x W_n\Omega^N_{X,\log} = 0$ for $j \neq p, p+1$. The natural map $W_n\Omega^{N-p}_{x,\log}[-p] \to R^p i^!_x W_n\Omega^N_{X,\log}[0]$ induces the desired morphism $\theta^{\log}_{i_x,n}$, and the above theorem tells us this morphism induces isomorphisms on cohomology groups. An alternative way is to show $R^{p+1}i^!_x W_n\Omega^N_{X,\log} = 0$ directly as Shiho's arguments in the proof of Theorem 1.4.12. □

1.5 A new result on purity for semistable schemes

We recall the following definitions.

Definition 1.5.1. *For a regular scheme X and a divisor D on X, we say that D has normal crossing if it satisfies the following conditions:*

(i) D is reduced, i.e. $D = \bigcup_{i\in I} D_i$ (scheme-theoretically), where $\{D_i\}_{i\in I}$ is the family of irreducible components of D;

(ii) For any non-empty subset $J \subset I$, the (scheme-theoretically) intersection $\bigcap_{j\in J} D_j$ is a regular scheme of codimension $\#J$ in X, or otherwise empty.

If moreover each D_i is regular, we called D has simple normal crossing.

Let R be a complete discrete valuation ring, with quotient field K, residue field k, and the maximal ideal $\mathfrak{m} = (\pi)$, where π is a uniformizer of R.

Definition 1.5.2. *Let $X \to Spec(R)$ be a scheme of finite type over $Spec(R)$. We call X a semistable (resp. strictly semistable) scheme over $Spec(R)$, if it satisfies the following conditions:*

(i) X is regular, $X \to Spec(R)$ is flat, and the generic fiber $X_\eta := X_K := X \times_{Spec(R)} Spec(K)$ is smooth;

(ii) The special fiber $X_s := X_k := X \times_{Spec(R)} Spec(k)$ is a divisor with normal crossings (resp. simple normal crossings) on X.

Remark 1.5.3 (Local description of semistable schemes)**.** *Let X be a semistable scheme over $Spec(R)$, then it is everywhere étale locally isomorphic to*

$$Spec(R[T_0, \cdots, T_d]/(T_0 \cdots T_a - \pi))$$

for some integer a with $a \in [0, d]$, where d denotes the relative dimension of X over $Spec(R)$. In particular, this implies the special fiber of a semistable(resp. strictly semistable) scheme is a normal (resp. simple normal) crossing variety.

Theorem 1.5.4. *Let $X \to B := Spec(\mathbb{F}_q[[t]])$ be a projective strictly semistable scheme over the formal power series ring $\mathbb{F}_q[[t]]$ of relative dimension d, and $i : X_s \hookrightarrow X$ be the natural morphism. Then, there is a canonical isomorphism*

$$Gys^{log}_{i,n} : \nu^d_{n,X_s}[-1] \xrightarrow{\cong} Ri^! W_n\Omega^{d+1}_{X,log}$$

in $D^+(X_s, \mathbb{Z}/p^n\mathbb{Z})$.

We will use Shiho's cohomological purity result (Theorem 1.4.12) in the proof, and the following lemma guarantees our X satisfies the assumption there.

Lemma 1.5.5. *Let X be as in Theorem 1.5.4.*

(i) Let A be a ring of characteristic $p > 0$. If the absolute Frobenius $F : A \to A, a \mapsto a^p$ is finite, then the same holds for any quotient or localization.

(ii) For any $x \in X$, the absolute Frobenius $F : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is finite. In particular, our X is in the category $\mathscr{C}$.

(iii) For any $x \in X^0$, we have $[\kappa(x) : \kappa(x)^p] = p^{d+1}$.

Proof. (i) By the assumption, there is a surjection of A-modules $\bigoplus_{i=1}^m A \twoheadrightarrow A$ for some m, where the A-module structure in the target is twisted by F. For the quotienting out by an ideal I, then tensoring with A/I, we still have a surjection $\bigoplus_{i=1}^m A/I \twoheadrightarrow A/I$. If S is a multiplicative set, then tensoring with $S^{-1}A$ still gives a surjection $\bigoplus_{i=1}^m S^{-1}A \twoheadrightarrow S^{-1}A$.

(ii) Note that $\mathbb{F}_q[[t]]$ as $\mathbb{F}_q[[t]]^p$-module is free with basis $\{1, t, \cdots, t^{p-1}\}$. The absolute Frobenius on the polynomial ring $\mathbb{F}_q[[t]][x_1, \cdots, x_n]$ is also finite. Now the local ring $\mathcal{O}_{X,x}$ is obtained from a polynomial ring over $\mathbb{F}_q[[t]]$ after passing to a quotient and a localization. Hence the assertion follows by (i).

(iii) For $x \in X^0$, the transcendence degree $\mathrm{tr.deg}_{\mathbb{F}_q((t))}\kappa(x) = d$, and $\kappa(x)$ is a finitely generated extension over $\mathbb{F}_q((t))$. So the p-rank of $\kappa(x)$ is the p-rank of $\mathbb{F}_q((t))$ increased by d, and we know that $[\mathbb{F}_q((t)) : \mathbb{F}_q((t))^p] = p$. □

We need the following result of Moser.

Proposition 1.5.6. *([Mos99, Prop. 2.3]) Let Y be a scheme of finite type over a perfect field of characteristic $p > 0$, let $i_x : x \to Y$ be a point of Y with dim($\overline{\{x\}}$)= p. Then we have $R^q i_{x*} W_n\Omega^p_{x,\log} = 0$, for all $n \geq 1$ and $q \geq 1$.*

With the help of the above preparations, we can now prove the theorem.

Proof of Theorem 1.5.4. By Proposition 1.2.3, we have the following local-global spectral sequence of étele sheaves on $Z := X_s$:

$$E_1^{u,v} = \bigoplus_{x\in Z^u} R^{u+v}\imath_{x*}(Ri^!_x W_n\Omega^{d+1}_{X,\log}) \Longrightarrow R^{u+v}i^!(W_n\Omega^{d+1}_{X,\log})$$

where, for $x \in Z$, $\imath$ (resp. $i_x = i \circ \imath$) denotes the natural map $x \to Z$ (resp. $x \to X$).

For $x \in Z^u$, we have an isomorphism

$$\theta^{\log}_{i_x,n} : W_n\Omega^{d-u}_{x,\log}[-u-1] \xrightarrow{\cong} Ri^!_x W_n\Omega^{d+1}_{X,\log},$$

by Corollary 1.4.14. Then

$$E_1^{u,v} = \bigoplus_{x\in Z^u} R^{v-1}\imath_{x*}W_n\Omega^{d-u}_{x,\log} = 0, \qquad \text{if} \quad v \neq 1.$$

where the last equality follows from Proposition 1.5.6. Hence the local-global spectral sequence degenerates at the E_1-page, i.e., $R^r i^! W_n\Omega^{d+1}_{X,\log}$ is the $(r-1)$-th cohomology sheaf of the following complex

$$\bigoplus_{x\in Z^0} \imath_{x*}R^1 i^!_x W_n\Omega^{d+1}_{X,\log} \xrightarrow{d_1^{00}} \bigoplus_{x\in Z^1} \imath_{x*}R^2 i^!_x W_n\Omega^{d+1}_{X,\log} \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{x\in Z^{r-1}} \imath_{x*}R^r i^!_x W_n\Omega^{d+1}_{X,\log} \xrightarrow{d_1^{r-1,0}} \bigoplus_{x\in Z^r} \imath_{x*}R^{r+1} i^!_x W_n\Omega^{d+1}_{X,\log}$$

$$\cdots \longrightarrow \bigoplus_{x\in Z^d} \imath_{x*}R^{d+1} i^!_x W_n\Omega^{d+1}_{X,\log}.$$

We denote this complex by $B^\bullet_n(Z,d)$. Then Corollary 1.4.14 implies that, $\theta^{\log}_{i_x,n}$ gives an isomorphism $C^s_n(Z,d) \cong B^s_n(Z,d)$, for any s, i.e., a term-wise isomorphism between the complexes $C^\bullet_n(Z,d)$ and $B^\bullet_n(Z,d)$. The following theorem will imply that $\theta^{\log}_{i_x,n}$ induces an isomorphism of complexes.

Theorem 1.5.7. *Let $X, Z := X_s$ be as in Theorem 1.5.4. For $x \in Z^c$ and $y \in Z^{c-1}$ with*

$x \in \overline{\{y\}}$, *then the following diagram*

$$\begin{array}{ccc} H^q(y, W_n\Omega^{r-c}_{y,log}) & \xrightarrow{(-1)^r\partial^{val}_{y,x}} & H^q(x, W_n\Omega^{r-c-1}_{x,log}) \\ \downarrow{\scriptstyle \theta^{log}_{i_y,n}} & & \downarrow{\scriptstyle \theta^{log}_{i_x,n}} \\ H^{c+q}_y(X, W_n\Omega^r_{X,log}) & \xrightarrow[\delta^{loc}_{y,x}(W_n\Omega^r_{X,log})]{} & H^{c+1+q}_x(X, W_n\Omega^r_{X,log}) \end{array}$$

commutes if $q = 0$ *or* $(q, r) = (1, d+1)$.

Theorem 1.5.7 implies 1.5.4. By Theorem 1.5.7, under the Gysin isomorphisms $\eta^{\log}$, the morphisms $d_1^{r,0}$ coincide with the boundary maps of logarithmic de Rham-Witt sheaves. Hence $C_n^\bullet(Z,d)$ coincides with $B_n^\bullet(Z,d)$. And the Proposition 1.3.13 said the complex $C_n^\bullet(Z,d)$ is acyclic at positive degree, and isomorphic to $\nu^d_{n,Z}$ at zero degree. This shows the claim. □

We are now turning to the proof of commutativity of the above diagram .

Proof of Theorem 1.5.7. We may assume $y \in Z_j$ for some irreducible component Z_j of Z. Note that Z_j is smooth by our assumption. Then we have a commutative diagram:

$$\begin{array}{ccc} H^q(y, W_n\Omega^{r-c}_{y,\log}) & \xrightarrow{\theta^{\log}_{i_y,n}} & H^{c+q}_y(X, W_n\Omega^r_{X,\log}) \\ \downarrow{\scriptstyle Gys_{i_y}} & & \uparrow \\ H^{c-1+q}_y(Z_j, W_n\Omega^{r-1}_{Z_j,\log}) & \xrightarrow{\theta_{Z_j\hookrightarrow X}} & H^{c-1+q}_y(Z_j, \mathscr{H}^1_{Z_j}(W_n\Omega^r_{X,\log})), \end{array}$$

where the right-vertical morphism is the edge morphism of Leray spectral sequence.

Hence we have the following diagram:

$$\begin{array}{ccc} H^q(y, W_n\Omega^{r-c}_{y,\log}) & \xrightarrow{(-1)^r\partial^{\mathrm{val}}_{y,x}} & H^q(x, W_n\Omega^{r-c-1}_{x,\log}) \\ \downarrow{\scriptstyle Gys_{y\to Z_j}} & (1) & \downarrow{\scriptstyle Gys_{x\to Z_j}} \\ H^{c-1+q}_y(Z_j, W_n\Omega^{r-1}_{Z_j,\log}) & \xrightarrow{\delta^{loc}_{y,x}(W_n\Omega^{r-1}_{Z_j,\log})} & H^{c+q}_x(Z_j, W_n\Omega^{r-1}_{Z_j,\log}) \\ \downarrow{\scriptstyle \theta_{Z_j\hookrightarrow X}} & (2) & \downarrow{\scriptstyle \theta_{Z_j\hookrightarrow X}} \\ H^{c-1+q}_y(Z_j, \mathscr{H}^1_{Z_j}(W_n\Omega^r_{X,\log})) & \xrightarrow{\delta^{loc}_{y,x}(\mathscr{H}^1_{Z_j}(W_n\Omega^r_{X,\log}))} & H^{c+q}_x(Z_j, \mathscr{H}^1_{Z_j}(W_n\Omega^r_{X,\log})) \\ \downarrow & (3) & \downarrow \\ H^{c+q}_y(X, W_n\Omega^r_{X,\log}) & \xrightarrow{\delta^{loc}_{y,x}(W_n\Omega^r_{X,\log})} & H^{c+1+q}_x(X, W_n\Omega^r_{X,\log}). \end{array}$$

The square (1) is (-1)-commutative if $q = 0$ or $(q, r) = (1, d+1)$. This is Theorem 3.1.1 for $q = 0$ and Corollary 3.4.1 for $(q, r) = (1, d + 1)$ in [JSS14]. The square (2) is commutative by the functoriality of $\delta^{\mathrm{loc}}_{y,x}$. The square (3) is (-1)-commutative by the functoriality of the Leray spectral sequences: here the sign (-1) arise from the difference of degrees. Therefore the desired diagram is commutative. □

Remark 1.5.8. *In [Shi07, Thm. 5.4], Shiho proved this compatibility for more general regular schemes X over $\mathbb{F}_q$ where the residue fields of x, y are not necessarily to be finite or perfect, but assuming that $n = 1$.*

Remark 1.5.9. *From the above proof, we can see that our new purity theorem still holds if we replace the finite field $\mathbb{F}_q$ with a perfect field k of characteristic $p > 0$.*

Corollary 1.5.10 (Cohomological purity)**.** *Assume X and $Z := X_s$ are as before. For any integer $i \leq d + 1$, there is a canonical isomorphism*

$$Gys_{i,n} : H^{d+1-i}(Z, \nu^d_{n,Z}) \xrightarrow{\cong} H^{d+2-i}_Z(X, W_n\Omega^{d+1}_{X,\log})$$

Proof. This follows from the spectral sequence

$$E^{u,v}_1 = H^u(Z, R^v i^! W_n\Omega^{d+1}_{X,\log}) \Rightarrow H^{u+v}_Z(X, W_n\Omega^{d+1}_{X,\log})$$

and the above purity theorem. □

Corollary 1.5.11. *There is a canonical map, called the trace map:*

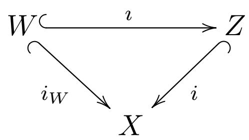
$$Tr : H^{d+2}_Z(X, W_n\Omega^{d+1}_{X,log}) \longrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

It is bijective if Z is connected.

Proof. We define this trace map as $Gys^{-1}_{i,n} \circ tr_Z$, where tr_Z is the trace map in Theorem 1.3.14. □

We conclude this chapter with the following compatibility result.

Proposition 1.5.12. *Let X, Z be as in Theorem 1.5.4, and let W be a smooth closed subscheme of Z of codimension r, giving the following commutative diagram:*



Then the following diagram:

$$\begin{tikzcd} W_n\Omega^{d-r}_{W,log} \arrow[rr, "Gys_{W\hookrightarrow Z}"] \arrow[dr, "\theta_{W\hookrightarrow X}"'] & & R^r\imath^! \nu^d_{n,Z} \arrow[dl, "Gys_{Z\hookrightarrow X}"] \\ & R^{r+1}i^!_W W_n\Omega^{d+1}_{X,log} & \end{tikzcd}$$

commutes. Recall that $\theta_{W\hookrightarrow X}$ is the Gysin morphism defined by Shiho in Theorem 1.4.12, $Gsy_{W\hookrightarrow Z}$ be the Gysin morphism defined by Sato in Theorem 1.4.4, and $Gys_{Z\hookrightarrow X}$ is the Gysin morphism in Theorem 1.5.4.

Proof. We may assume $W \subseteq Z_j$, for some irreducible component Z_j of Z. We denote the natural morphisms as in the following diagram:

$$\begin{tikzcd} W \arrow[r, "\imath_W"] \arrow[dr, "i_W"'] & Z_i \arrow[r, "\imath_{Z_j}"] \arrow[d, "i_{Z_j}"] & Z \arrow[dl, "i"] \\ & X & \end{tikzcd}$$

Then we have

$$\begin{tikzcd} W_n\Omega^{d-r}_W \arrow[rr, "(1)"'] \arrow[dr] \arrow[ddr] & & R^r\imath^!\nu^d_{n,Z} \arrow[ddl] \\ (2) & R^r\imath^!_W W_n\Omega^d_{Z_j} \arrow[ur] \arrow[d] & (3) \\ & R^{r+1}i^!_W W_n\Omega^{d+1}_{X,\mathrm{log}} & \end{tikzcd}$$

Sato's Gysin morphisms satisfy a transitivity property, which implies the square (1) commutes. So does (2) due to Shiho's remark in [Shi07, Rem. 3.13]. It's enough to show the square (3) commute. Hence we reduced to the case $r = 0$. Then we may write $W_n\Omega^d_{W,\mathrm{log}}$ as the kernel of $\oplus_{x\in W^0} i_{x*}W_n\Omega^d_{x,\mathrm{log}} \to \oplus_{y\in W^0} i_{y*}W_n\Omega^{d-1}_{y,\mathrm{log}}$. Then Sato's Gysin morphism will be the identity, and our definition of Gysin morphism locally is exactly that given by Shiho(see the proof of Theorem 1.5.4). □

Chapter 2

Coherent duality

From now on, we fix the notation as in the following diagram.

$$
\begin{array}{ccccc}
X_s & \xhookrightarrow{i} & X & \xhookleftarrow{j} & X_\eta \\
\Big\downarrow{\scriptstyle f_s} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_\eta} \\
s & \xhookrightarrow{i_s} & B = Spec(\mathbb{F}_p[[t]]) & \xhookleftarrow{j_\eta} & \eta
\end{array}
$$

where f is a projective strictly semistable scheme, $Z = X_s$ is the special fiber and X_η is the generic fiber.

2.1 The sheaf Ω^1_X

We recall some lemmas on local algebras.

Lemma 2.1.1. *Let $(A, \mathfrak{m}, k) \to (A', \mathfrak{m}', k')$ be a local morphism of local rings. If A' is regular and flat over A, then A is regular.*

Proof. Note that flat base change commutes with homology. Hence, for $q > \dim(A')$, we have $\mathrm{Tor}_q^A(k, k) \otimes_A A' = \mathrm{Tor}_q^{A'}(k \otimes A', k \otimes A') = 0$. Since A' is faithful flat over A, this implies $\mathrm{Tor}_q^A(k, k) = 0$, for $q > \dim(A')$. Therefore, the global dimension of A is finite. Thanks to Serre's theorem [MR89, Thm. 19.2], A is regular. □

Lemma 2.1.2. *Let A be a regular local ring of characteristic $p > 0$ such that $A^p \to A$ is finite. Then Ω^1_A is a free A-module.*

Proof. By [Kun69, Thm. 2.1], regularity implies that $A^p \to A$ is flat, so faithfully flat. The above lemma implies that A^p is regular as well. Then we use a conjecture of Kunz, which was proved in [KN82], that there exists a p-basis of A. Therefore the assertion follows. □

Proposition 2.1.3. *The absolute differential sheaf* Ω^1_X *is a locally free* $\mathcal{O}_X$*-module of rank* $d+1$.

Proof. Note that we have an exact sequence

$$f^*\Omega^1_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0.$$

Both $f^*\Omega^1_B$ and $\Omega^1_{X/B}$ are coherent, so is Ω^1_X. Then we may reduce to local case, and the assertion is clear by the above lemma.

□

Corollary 2.1.4. *The sheaves* Ω^i_X, $Z\Omega^i_X$ *(via F),* $\Omega^i_X/B\Omega^i_X$ *(via F) are locally free* $\mathcal{O}_X$*-modules.*

Proof. As in the smooth case, using Cartier isomorphisms, we can show this inductively. □

2.2 Grothendieck duality theorem

The Grothendieck duality theorem studies the right adjoint functor of Rf_* in $D^+_{\mathrm{qc}}(X)$, the derived category of $\mathcal{O}_X$-modules with bounded from below quasi-coherent cohomology sheaves. There are several approaches to this functor.

In our case, we follow a more geometric approach, which was given by Hartshorne in [Har66]. Here we only use the Grothendieck duality theorem for projective morphisms.

Definition 2.2.1. *A morphism* $g : M \to N$ *of schemes is called projectively embeddable if it factors as*

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & N \\ & {}_{p}\searrow \quad \nearrow_{q} & \\ & \mathbb{P}^n_N & \end{array}$$

for some $n \in \mathbb{N}$*, where* q *is the natural projection,* p *is a finite morphism.*

Remark 2.2.2. *In our case, the projectivity of the scheme* $f : X \to S = Spec(\mathbb{F}_q[[t]])$ *and the fact that the basis is affine imply that* f *is projectively embeddable [EGA, II 5.5.4 (ii)].*

Theorem 2.2.3 (Grothendieck duality theorem)**.** *Let* $g : M \to N$ *be a projectively embeddable morphism of noetherian schemes of finite Krull dimension. Then, there exists a functor* $f^! : D^+_{qc}(N) \to D^+_{qc}(M)$ *such that the following holds:*

(i) If $h : N \to T$ *is a second projectively embeddable morphism, then* $(g \circ h)^! = h^! \circ g^!$*;*

(ii) If g *is smooth of relative dimension* n*, then* $g^!(\mathscr{G}) = f^*(\mathscr{G}) \otimes \Omega^n_{M/N}[n]$*;*

(iii) If g *is a finite morphism, then* $g^!(\mathscr{G}) = \bar{g}^* R\mathscr{H}om_{\mathcal{O}_N}(g_*\mathcal{O}_M, \mathscr{G})$*, where* $\bar{g}$ *is the induced morphism* $(M, \mathcal{O}_M) \to (N, g_*\mathcal{O}_M)$*;*

(iv) There is an isomorphism

$$\theta_g : Rg_* R\mathscr{H}om_{\mathcal{O}_M}(\mathscr{F}, g^!\mathscr{G}) \xrightarrow{\cong} R\mathscr{H}om_{\mathcal{O}_N}(Rf_*\mathscr{F}, \mathscr{G}),$$

for $\mathscr{F} \in D^-_{qc}(M)$, $\mathscr{G} \in D^+_{qc}(N)$.

2.3 Relative coherent duality

Note that the morphism $Rf_*\Omega^{d+1}_X[d] \to \Omega^1_B$ induces a map $\Omega^{d+1}_X[d] \to f^!\Omega^1_B$ by adjunction between $(Rf_*, f^!)$, which is denoted by Tr_f . In this section, we prove the following relative duality result.

Theorem 2.3.1. *Let $f : X \to B = Spec(\mathbb{F}_q[[t]])$ be a projective strictly semistable scheme. Then we have a canonical isomorphism*

$$Tr_f : \Omega^{d+1}_X[d] \xrightarrow{\cong} f^!\Omega^1_B.$$

This theorem can be obtained by some explicit calculations.

Lemma 2.3.2. *—Let $\iota : X \hookrightarrow \mathbb{P}^N_B$ be a regular closed immersion with the defining sheaf $\mathcal{I}$. Then the sequence of $\mathcal{O}_X$-modules*

$$0 \to \iota^*\mathcal{I}/\mathcal{I}^2 \to \iota^*\Omega^1_{\mathbb{P}^N_B} \to \Omega^1_X \to 0$$

is exact.

Proof. We only need to show that the left morphism is injective. Let $\mathcal{K}$ be the kernel of the canonical morphism $\iota^*\Omega^1_{\mathbb{P}^N_B} \to \Omega^1_X$. Since both $\iota^*\Omega^1_{\mathbb{P}^N_B}$ and Ω^1_X are coherent and locally free, the kernel $\mathcal{K}$ is coherent and flat. Then it follows that $\mathcal{K}$ is also locally free. By counting the ranks, we have rank($\mathcal{I}/\mathcal{I}^2$)=rank($\mathcal{K}$). Thus, the induced surjective morphism $\iota^*\mathcal{I}/\mathcal{I}^2 \to \mathcal{K}$ is an isomorphism. □

Proof of Theorem 2.3.1. As f is projective, we have a decomposition of maps

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \mathbb{P}^N_B \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle p} & \\ B & & \end{array}$$

for some $N \in \mathbb{Z}$. Here ι is a regular closed immersion. Using Koszul resolution of $\mathcal{O}_X$ with respect to the closed immersion ι, we have

$$\iota^!\Omega^{N+1}_{\mathbb{P}^N_B} \cong \iota^*\Omega^{N+1}_{\mathbb{P}^N_B} \otimes \iota^*\det(\mathcal{I}/\mathcal{I}^2)^\vee[d-N].$$

By Lemma 2.3.2, we have

$$\iota^*\Omega^{N+1}_{\mathbb{P}^N_B} \otimes \iota^*\det(\mathcal{I}/\mathcal{I}^2)^\vee \cong \Omega^{d+1}_X.$$

Since p is smooth, we have isomorphisms

$$p^!\Omega^1_B \cong p^!\mathcal{O}_B \otimes p^*\Omega^1_B \cong \Omega^N_{\mathbb{P}^N_B/B}[N] \otimes p^*\Omega^1_B \cong \Omega^{N+1}_{\mathbb{P}^N_B}[N].$$

Noting that

$$f^!\Omega^1_B = \iota^! p^! \Omega^1_B,$$

the theorem follows.

□

Remark 2.3.3. *From the proof, we can see Theorem 2.3.1 is still true in more general situations. In the light of our application, we just proof this simple case and remark that it also holds if we replace the finite field $\mathbb{F}_q$ with a perfect field k of characteristic $p > 0$.*

2.4 Grothendieck local duality

Let $(R, \mathfrak{m})$ be a regular local ring of dimension n with maximal ideal $\mathfrak{m}$, and $R/\mathfrak{m} \cong \mathbb{F}_q$. For any finite R-module M, we have a canonical pairing

$$H^i_{\mathfrak{m}}(\mathrm{Spec}(R), M) \times \mathrm{Ext}^{n-i}_R(M, \Omega^n_R) \to H^n_{\mathfrak{m}}(\mathrm{Spec}(R), \Omega^n_R) \xrightarrow{\mathrm{Res}} \mathbb{F}_q \xrightarrow{tr_{\mathbb{F}_q/\mathbb{F}_p}} \mathbb{Z}/p\mathbb{Z}. \quad (2.4.1)$$

Theorem 2.4.1 (Grothendieck local duality)**.** *For each $i \geq 0$, the pairing (2.4.1) induces isomorphisms*

$$Ext^{n-i}_R(M, \Omega^n_R) \otimes_R \hat{R} \cong Hom_{\mathbb{Z}/p\mathbb{Z}}(H^i_{\mathfrak{m}}(Spec(R), M), \mathbb{Z}/p\mathbb{Z}),$$

$$H^i_{\mathfrak{m}}(Spec(R), M) \cong Hom_{cont}(Ext^{n-i}_R(M, \Omega^n_R), \mathbb{Z}/p\mathbb{Z}),$$

where Hom_{cont} denotes the set of continuous homomorphisms with respect to $\mathfrak{m}$-adic topology on Ext group.

Proof. This is slightly different from the original form of Grothendieck local duality in [GH67, Thm. 6.3]. But, in our case, the dualizing module $I = H^n_{\mathfrak{m}}(Spec(R), \Omega^n_R)$ can be written as $\varinjlim_n \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(A/\mathfrak{m}^n, \mathbb{Z}/p\mathbb{Z}) \subset Hom_{\mathbb{Z}/p\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z})$. Then we identify

$$Hom_A(H^i_{\mathfrak{m}}(\mathrm{Spec}(R), M), I) = Hom_A(H^i_{\mathfrak{m}}(\mathrm{Spec}(R), M), \varinjlim_n \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(A/\mathfrak{m}^n, \mathbb{Z}/p\mathbb{Z}))$$

$$= Hom_A(H^i_{\mathfrak{m}}(\mathrm{Spec}(R), M), Hom_{\mathbb{Z}/p\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z})) = Hom_{\mathbb{Z}/p\mathbb{Z}}(H^i_{\mathfrak{m}}(\mathrm{Spec}(R), M), \mathbb{Z}/p\mathbb{Z}),$$

where the second equality follows from the fact that each element of $H^i_{\mathfrak{m}}(\mathrm{Spec}(R), M)$ is annihilated by some power of $\mathfrak{m}$. The second isomorphism in the theorem follows from the definition of continuity.

□

2.5 Absolute coherent duality

In our case, the base scheme $B = Spec(\mathbb{F}_q[[t]])$ is a complete regular local ring of dimension 1. Combining Grothendieck local duality on the base scheme B with the relative duality theorem 2.3.1, we obtain an absolute duality on X.

Proposition 2.5.1. *Let $\mathscr{F}$ be a locally free $\mathcal{O}_X$-module on X, and let $\mathscr{F}^t$ be the sheaf given by $\mathscr{H}om(\mathscr{F}, \Omega_X^{d+1})$. Then we have*

$$H^i_{\mathfrak{m}}(B, Rf_*\mathscr{F}) \cong H^i_{X_s}(X, \mathscr{F}); \tag{2.5.1}$$

$$Ext^{1-i}_{\mathcal{O}_B}(Rf_*\mathscr{F}, \Omega^1_B) \cong H^{d+1-i}(X, \mathscr{F}^t). \tag{2.5.2}$$

Proof. For the first equation, we have the following canonical identifications:

$$\begin{aligned}
H^i_{\mathfrak{m}}(B, Rf_*\mathscr{F}) &= H^i(B, i_{s*}Ri_s^! Rf_*\mathscr{F}) \\
&= H^i(B, i_{s*}Rf_{s*}Ri^!\mathscr{F}) \\
&= H^i(\mathbb{F}_q, Rf_{s*}Ri^!\mathscr{F}) \qquad (i_{s*} \text{is exact}) \\
&= H^i(X_s, Ri^!\mathscr{F}) \\
&= H^i_{X_s}(X, \mathscr{F})
\end{aligned}$$

where the second equality follows from $Rf_{s*}Ri^! \xrightarrow{\cong} Ri_s^! Rf_*$ in (SGA 4 [AGV72, Tome 3, XVIII, Corollaire 3.1.12.3]) or ([Fu11, Prop. 8.4.9]).

For the second, we have

$$\begin{aligned}
\mathrm{Ext}^{1-i}_{\mathcal{O}_B}(Rf_*\mathscr{F}, \Omega^1_B) &= R^{1-i}\Gamma(B, R\mathscr{H}om_{\mathcal{O}_B}(Rf_*\Omega^j_X, \Omega^1_B)) \\
&= R^{1-i}\Gamma(B, Rf_*R\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, f^!\Omega^1_B)) \qquad (\text{adjunction}) \\
&= R^{1-i}\Gamma(B, Rf_*R\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \Omega^{d+1}_X[d])) \qquad (\text{Theorem 2.3.1}) \\
&= R^{1-i}\Gamma(B, Rf_*\mathscr{F}^t[d]) \qquad (\text{definition of } \mathscr{F}^t) \\
&= H^{d+1-i}(X, \mathscr{F}^t)
\end{aligned}$$

□

By taking different $\mathscr{F}$ in the above theorem and using Grothendieck local duality, we obtain the following corollaries.

Corollary 2.5.2. *The natural pairing*

$$H^i(X, \Omega^j_X) \times H^{d+1-i}_{X_s}(X, \Omega^{d+1-j}_X) \to H^{d+1}_{X_s}(X, \Omega^{d+1}_X) \xrightarrow{tr} \mathbb{Z}/p\mathbb{Z}$$

induces isomorphisms

$$H^i(X,\Omega_X^j) \cong Hom_{\mathbb{Z}/p\mathbb{Z}}(H_{X_s}^{d+1-i}(X,\Omega_X^{d+1-j}),\mathbb{Z}/p\mathbb{Z}),$$

$$H_{X_s}^{d+1-i}(X,\Omega_X^{d+1-j}) \cong Hom_{cont}(H^i(X,\Omega_X^j),\mathbb{Z}/p\mathbb{Z}).$$

i.e., it is a perfect pairing of topological $\mathbb{Z}/p\mathbb{Z}$-modules if we endow $H^i(X,\Omega_X^j)$ with the $\mathfrak{m}$-adic topology and $H_{X_s}^{d+1-i}(X,\Omega_X^{d+1-j})$ with the discrete topology.

Remark 2.5.3. *For the first isomorphism, we can ignore the topological structure on cohomology groups, and view it as an isomorphism of $\mathbb{Z}/p\mathbb{Z}$-modules. In the next chapter, we only use this type of isomorphisms(cf. Proposition 3.4.8).*

Corollary 2.5.4. *The natural pairing*

$$H^i(X,Z\Omega_X^j) \times H_{X_s}^{d+1-i}(X,\Omega_X^{d+1-j}/B\Omega_X^{d+1-j}) \to H_{X_s}^{d+1}(X,\Omega_X^{d+1}) \xrightarrow{tr} \mathbb{Z}/p\mathbb{Z}$$

is a perfect pairing of topological $\mathbb{Z}/p\mathbb{Z}$-modules, if we endow the cohomology groups with the topological structures as in the above corollary.

Furthermore, we can do the same thing for twisted logarithmic Kähler differential sheaves.

Let X be as before , and let $\jmath : U \hookrightarrow X$ be the complement of a reduced divisor D on X with simple normal crossings. Let $D_1,\cdots,D_s$ be the irreducible components of D. For $\underline{m} = (m_1,\cdots,m_s)$ with $m_i \in \mathbb{N}$ let

$$mD = \underline{m}D = \sum_{i=1}^{s} m_i D_i \tag{2.5.3}$$

be the associated divisor.

Definition 2.5.5. *For the above defined D on X and $j \geq 0$, $m = \underline{m} \in \mathbb{N}^s$, we set*

$$\Omega^j_{X|mD} = \Omega^j_X(logD)(-mD) = \Omega^j_X(logD) \otimes \mathcal{O}_X(-mD) \tag{2.5.4}$$

where $\Omega_X^j(logD)$ denotes the sheaf of absolute Kähler differential j-forms on X with logarithmic poles along $|D|$. Similarly, we can define $Z\Omega^j_{X|mD}$, $B\Omega^j_{X|mD}$.

Remark 2.5.6. *Note that $\Omega^{d+1}_{X|D} = \Omega_X^{d+1} \otimes \mathcal{O}_X(D) \otimes \mathcal{O}_X(-D) = \Omega_X^{d+1}$, where d is the relative dimension of X over B.*

Corollary 2.5.7. *The natural pairing*

$$H^i(X,\Omega^j_{X|-mD}) \times H_{X_s}^{d+1-i}(X,\Omega^{d+1-j}_{X|(m+1)D}) \to H_{X_s}^{d+1}(X,\Omega_X^{d+1}) \xrightarrow{tr} \mathbb{Z}/p\mathbb{Z}$$

is a perfect pairing of topological $\mathbb{Z}/p\mathbb{Z}$-modules, if we endow the cohomology groups with the topological structures as before.

Proof. Note that the pairing

$$\Omega^j_X(\log D)(-mD) \otimes \Omega^{d+1-j}_X((m-1)D) \to \Omega^{d+1}_X \tag{2.5.5}$$

is perfect. □

Similarly, we also have the following result.

Corollary 2.5.8. *The natural pairing*

$$H^i(X, Z\Omega^j_{X|-mD}) \times H^{d+1-i}_{X_s}(X, \Omega^{d+1-j}_{X|(m+1)D}/B\Omega^{d+1-j}_{X|(m+1)D}) \to H^{d+1}_{X_s}(X, \Omega^{d+1}_X) \xrightarrow{tr} \mathbb{Z}/p\mathbb{Z}$$

is a perfect pairing of topological $\mathbb{Z}/p\mathbb{Z}$*-modules, if we endow the cohomology groups with the topological structures as before.*

Chapter 3

Logarithmic duality

Recall that $f : X \to B = Spec(\mathbb{F}_q[[t]])$ be a projective strictly semistable scheme of relative dimension d. In this chapter, we will prove two duality theorems. The first one is for $H^i(X, W_n\Omega^j_{X,\log})$, which we call unramified duality. The second is for $H^i(U, W_n\Omega^j_{U,\log})$, where U is the open complement of a reduced effective Cartier divisor with $\mathrm{Supp}(D)$ has simple normal crossing. We call it the ramified duality.

3.1 Unramified duality

The product on logarithmic de Rham-Witt sheaves

$$W_n\Omega^j_{X,\log} \otimes W_n\Omega^{d+1-j}_{X,\log} \to W_n\Omega^{d+1}_{X,\log}$$

induces a pairing

$$i^*W_n\Omega^j_{X,\log} \otimes^{\mathbb{L}} Ri^!W_n\Omega^{d+1-j}_{X,\log} \to Ri^!(W_n\Omega^j_{X,\log} \otimes W_n\Omega^{d+1-j}_{X,\log}) \to Ri^!W_n\Omega^{d+1}_{X,\log},$$

where the first morphism is given by the adjoint map of the diagonal map ϕ in the following diagram

$$\begin{array}{ccc} Ri_*(i^*W_n\Omega^j_{X,\log} \otimes^{\mathbb{L}} Ri^!W_n\Omega^{d+1-j}_{X,\log}) & \xrightarrow{\cong} & W_n\Omega^j_{X,\log} \otimes^{\mathbb{L}} Ri_*Ri^!W_n\Omega^{d+1-j}_{X,\log} \\ & \searrow^{\phi} & \downarrow \text{adj} \\ & & W_n\Omega^j_{X,\log} \otimes W_n\Omega^{d+1-j}_{X,\log}. \end{array}$$

Here the isomorphism is the given by the projection formula.

Applying $R\Gamma(X_s, \cdot)$ and the proper base change theorem(SGA$4\frac{1}{2}$, [D$^+$77, Arcata IV]), we have a pairing

$$H^i(X, W_n\Omega^j_{X,\log}) \times H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-j}_{X,\log}) \to H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,\log}) \xrightarrow{Tr} \mathbb{Z}/p^n\mathbb{Z}, \quad (3.1.1)$$

where the trace map Tr is given by Corollary 1.5.11.

Theorem 3.1.1. *The pairing (3.1.1) is a perfect pairing of finite $\mathbb{Z}/p^n\mathbb{Z}$-modules.*

Proof. By the exact sequence (i) in Proposition 1.4.11, the problem is reduced to the case $n = 1$. For this case, we use the classical method as in [Mil86], i.e., using the exact sequence (ii) and (iii) in Proposition 1.4.11, we reduce the problem to coherent duality. Before we do this, we have to check the compatibility between the trace maps. It is enough to do this on the base scheme B, by the definitions of trace map and residue map, and the following commutative diagram:

$$\begin{array}{ccc} Rf_*\Omega^{d+1}_{X,\log}[d+1] & \longrightarrow & \Omega^1_{B,\log}[1] \\ \uparrow & & \uparrow \\ Rf_*\Omega^{d+1}_X[d] & \longrightarrow & \Omega^1_B. \end{array}$$

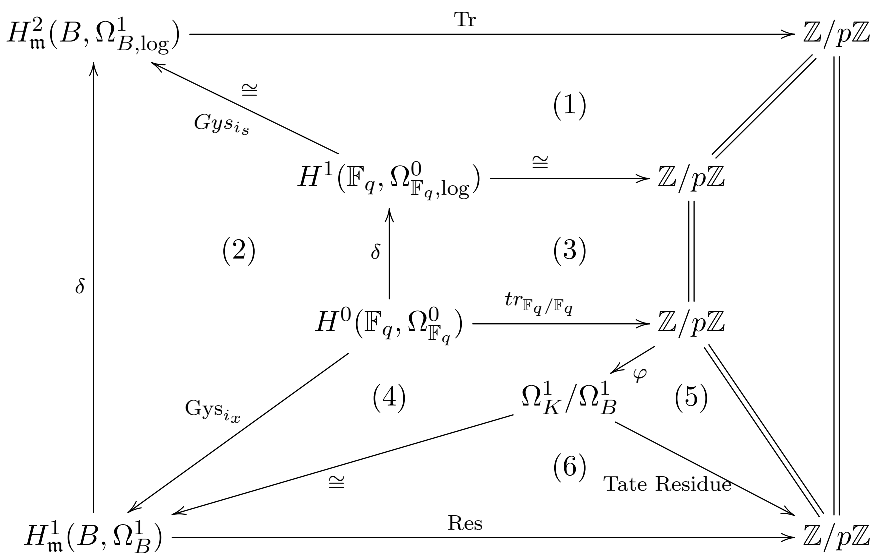
Proposition 3.1.2. *The following diagram*

$$\begin{array}{ccc} H^2_{\mathfrak{m}}(B, \Omega^1_{B,log}) & \xrightarrow{Tr} & \mathbb{Z}/p\mathbb{Z} \\ {\scriptstyle\delta}\uparrow & & \Vert \\ H^1_{\mathfrak{m}}(B, \Omega^1_B) & \xrightarrow{Res} & \mathbb{Z}/p\mathbb{Z} \end{array}$$

commutes, where δ is the connection map induced by the following exact sequence

$$0 \to \Omega^1_{B,log} \to \Omega^1_B \to \Omega^1_B \to 0.$$

Proof. We have the following diagram



where the morphism $\varphi : a \mapsto ad\log(t)$, and $K = \mathbb{F}_q((t))$.

The diagrams (1) and (6) are commutative by the definition of Tr and Res. (2) is commutative by the functoriality of δ. That classical Milne duality is compatible with coherent duality implies (3) is commutative. The local description of the Gysin map will imply that the diagram (4) commutes [Shi07, Lem. 3.5] , [Gro85, II (3.4)]. The diagram (5) commutes by the explicit definition of φ and the definition of the Tate residue map. □

Proof of Theorem 3.1.1(cont.) By taking the cohomolgy groups of the exact sequences (ii) and (iii) in Proposition 1.4.11, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\cdots \to H^i(X, \Omega^j_{X,\log}) & \longrightarrow & H^i(X, Z\Omega^j_X) & \longrightarrow & H^i(X, \Omega^j_X) \to \\
\downarrow & & \downarrow\cong & & \downarrow\cong \\
\cdots \to H^{d+2-i}_{X_s}(X, \Omega^{d+1-j}_{X,\log})^* & \to & H^{d+1-i}_{X_s}(X, \Omega^{d+1-j}_X/B\Omega^{d+1-j}_X)^* & \to & H^{d+1-i}_{X_s}(X, \Omega^{d+1-j}_X)^* \to
\end{array}$$

where M^* means $\mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z})$, for any $\mathbb{Z}/p\mathbb{Z}$-module M.

The isomorphisms for cohomology groups of $Z\Omega^j_X$ and Ω^j_X are from the coherent duality theorem, see Corollary 2.5.4 and 2.5.2. Hence, we have

$$H^i(X, \Omega^j_{X,\log}) \xrightarrow{\cong} H^{d+2-i}_{X_s}(X, \Omega^{d+1-j}_{X,\log})^*.$$

□

Remark 3.1.3. *For the case of $j = 0$, by using the purity theorem 1.5.4, the above pairing agrees with that in [Sat07a, Thm. 1.2.2].*

3.2 The relative Milnor K-sheaf

On a smooth variety over a field, the logarithmic de Rham-Witt sheaves are closely related to the Milnor K-sheaves via the Bloch-Gabber-Kato theorem [BK86]. In this section, we first recall some results on the Milnor K-sheaf on a regular scheme, and then define the relative Milnor K-sheaf with respect to some divisor D as in [RS15, §2.3]. At last, we show the Bloch-Gabber-Kato theorem still holds on a regular scheme over $\mathbb{F}_q$. This result is well known to the experts but due to the lack of reference, we give a detailed proof.

In this section, we fix Y to be a connected regular scheme over $\mathbb{F}_q$ of dimension d (cf. Remark 3.2.2 below).

Definition 3.2.1. *For any integer r, we define the rth-Milnor K-sheaf $\mathcal{K}^M_{r,Y_{Zar}}$ to be the sheaf*

$$T \mapsto Ker\left(i_{\eta*}K^M_r(\kappa(\eta)) \xrightarrow{\partial} \bigoplus_{x\in Y^1\cap T} i_{x*}K^M_{r-1}(\kappa(x)\right)$$

on the Zariski site of Y, where T is any open subset of Y, η is the generic point of Y, $i_x : x \to T$ is the natural map and ∂ is the sheafified residue map of Milnor K-theory of *fields (cf. §1.3.2). The sheaf* $\mathcal{K}^M_{r,Y}$ *is the associated sheaf of* $\mathcal{K}^M_{r,Y_{Zar}}$ *on the (small) étale site of* Y.

In particular, $\mathcal{K}^M_{r,Y} = 0$ for $r < 0$, $\mathcal{K}^M_{0,Y} = \mathbb{Z}$ and $\mathcal{K}^M_{1,Y} = \mathcal{O}^\times_X$.

Remark 3.2.2. *The connectedness assumption on* Y *is just for simplification our notations. In case* Y *is not connected, we write* $Y = \coprod_j Y_j$ *as disjoint union of its connected components. Then define*

$$\mathcal{K}^M_{r,Y} := \bigoplus_j i_{Y_j *}\mathcal{K}^M_{r,Y_j}$$

where $i_{Y_j} : Y_j \to Y$ *be the natural map. The results in this section still hold for non-connected* Y.

We define $\mathcal{K}^{M,\text{naive}}_{r,Y}$ to be the sheaf on the étale site of Y associated to the functor

$$A \mapsto \otimes_{i\geq 0}(A^\times)^{\otimes i}/ < a \otimes b | a + b = 1 >$$

from commutative rings to graded rings.

Theorem 3.2.3. *([Ker09, Thm. 1.3], [Ker10, Thm. 13, Prop. 10]) Let* Y *be a connected regular scheme over* $\mathbb{F}_q$.

(i) The natural homomorphism $\mathcal{K}^{M,naive}_{r,Y} \to \mathcal{K}^M_{r,Y}$ *is surjective;*

(ii) If the residue fields at all point of Y *are infinite, the map* $\mathcal{K}^{M,naive}_{r,Y} \to \mathcal{K}^M_{r,Y}$ *is an isomorphism;*

(iii) The Gersten complex

$$0 \to \mathcal{K}^M_{r,Y} \to i_{\eta *}K^M_r(\kappa(\eta)) \to \bigoplus_{x\in Y^1} i_{x*}K^M_{r-1}(\kappa(x)) \to \bigoplus_{x\in Y^2} i_{x*}K^M_{r-2}(\kappa(x)) \to \cdots$$

is universally exact(see Definition 3.2.4 below) on the étale site of Y.

Definition 3.2.4. *Let* $A' \to A \to A''$ *be a sequence of abelian groups. We say this sequence is universally exact if* $F(A') \to F(A) \to F(A'')$ *is exact for every additive functor* $F : Ab \to \mathcal{B}$ *which commutes with filtering small limit. Here* Ab *is the category of abelian groups and* $\mathcal{B}$ *is an abelian category satisfying AB5 (see [Gro57]).*

Remark 3.2.5.

(i) Another way to define the Milnor K*-sheaf [Ker10] is to first define the naive* K*-sheaf* $\mathcal{K}^{M,naive}_{r,Y}$ *as above, and then the (improved) Milnor* K*-sheaf as the universal continuous functor associated to the naive* K*-sheaf. By Theorem 3.2.3 (iii), this definition agree with ours.*

(ii) Theorem 3.2.3 (i) implies that $\mathcal{K}^M_{r,Y}$ *is étale locally generated by the symbols of the form* $\{x_1, \cdots, x_r\}$ *with all* $x_i \in \mathcal{O}^\times_Y$.

(iii) The functor $-\otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ is an additive functor and it commutes with filtering small limit, so the universal exactness property of Gersten complex implies the following sequence

$$0 \to \mathcal{K}^M_{r,Y}/p^n \to i_{\eta *}K^M_r(\kappa(\eta))/p^n \to \bigoplus_{x\in Y^{(1)}} i_{x*}K^M_{r-1}(\kappa(x))/p^n \to \cdots$$

is exact.

Let $D = \cup_{i=1}^s D_i$ be a reduced effective Cartier divisor on Y such that $\mathrm{Supp}(D)$ has simple normal crossing, let $D_1, \cdots, D_s$ be the irreducible components of D, and let $\jmath : U := Y - D \hookrightarrow Y$ be the open complement. For $\underline{m} = (m_1, \cdots, m_s)$ with $m_i \in \mathbb{N}$ let

$$mD = \underline{m}D = \sum_{i=1}^s m_i D_i$$

be the associated divisor. On $\mathbb{N}^s$, we define a semi-order as follows:

$$\underline{m'} \geq \underline{m'} \quad \text{if} \quad m'_i \geq m_i \quad \text{for all } i.$$

Using this semi-order, we denote

$$\underline{m'}D \geq \underline{m}D \quad \text{if} \quad \underline{m'} \geq \underline{m}.$$

By the above theorem, we may define the relative Milnor K-sheaves with respect to mD using symbols(cf. [RS15, Def. 2.7]):

Definition 3.2.6. *For $n \in \mathbb{Z}, m \in \mathbb{N}^s$, we define the Zariski sheaf $\mathcal{K}^M_{r,Y|mD,Zar}$ to be the image of the following map*

$$\begin{aligned} Ker(\mathcal{O}_Y^\times \to \mathcal{O}_{mD}^\times) \otimes_{\mathbb{Z}} \jmath_*\mathcal{K}^M_{r-1,U_{Zar}} &\to \jmath_*\mathcal{K}^M_{r,U_{Zar}} \\ x \otimes \{x_1, \cdots, x_{n-1}\} &\mapsto \{x, x_1, \cdots, x_{n-1}\} \end{aligned}$$

and define $\mathcal{K}^M_{r,Y|mD}$ to be its associated sheaf on the étale site.

By this definition, it is clear that $\jmath_*\mathcal{K}^M_{r,Y|mD} = \mathcal{K}^M_{r,U}$, for any $m \in \mathbb{N}^s$.

Proposition 3.2.7. *([RS15, Cor. 2.13]) If $m' \geq m$, then we have the inclusions of étale sheaves*

$$\mathcal{K}^M_{r,X|m'D} \subseteq \mathcal{K}^M_{r,X|mD} \subseteq \mathcal{K}^M_{r,X}.$$

Proof. The statement in [RS15] is on the Zariski and Nisnevich sites, and in particular, works on the étale site. Instead of repeating the argument, we take a simple example: $X = \mathrm{Spec}(A)$ is a local ring, $D = (t)$ for some $t \in A$, then $U = \mathrm{Spec}(A[t^{-1}])$. We want to show $\mathcal{K}^M_{r,X|D} \subseteq \mathcal{K}^M_{r,X}$. It suffices to show $\{1 + at, t\} \in \mathcal{K}^M_{2,X}$, for any $a \in A$. Since $1 + tA$ is multiplicatively generated by elements in $1+tA^{\times}$, as

$$1+at=(1+t\frac{1}{1+t(a-1)})(1+t(a-1)).$$

We may assume $a\in A^{\times}$. Note that $0=\{1+at,-at\}=\{1+at,-a\}+\{1+at,t\}$, we have $\{1+at,t\}=-\{1+at,-a\}\in\mathcal{K}^M_{2,X}$. □

Now we recall the Bloch-Gabber-Kato theorem on regular schemes.

Lemma 3.2.8. *There is a natural map* $d\log[-]_n:\mathcal{K}^M_{r,Y}\to W_n\Omega^r_Y$.

Proof. If all the residue fields of Y are infinite, this is clear by Theorem 3.2.3. If not, we can still construct this map via the following local computations. We first define this for any ring A. Recall that the (improved) Milnor K-theory of A can be defined by the first row of the following diagram [Ker10]:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{K}^M_r(A) & \longrightarrow & K^M_r(A(t)) & \xrightarrow{i_{1*}-i_{2*}} & K^M_r(A(t_1,t_2)) \\ & & & & \downarrow{\scriptstyle d\log[-]_n} & & \downarrow{\scriptstyle d\log[-]_n} \\ 0 & \longrightarrow & M_n & \longrightarrow & W_n\Omega^r_{A(t)} & \xrightarrow{i_{1*}-i_{2*}} & W_n\Omega^r_{A(t_1,t_2)} \end{array}$$

Here $A(t)$ is the rational function ring over A, that is $A[t]_S$ the localization of the one variable polynomial ring with respect to the multiplicative set $S=\{\sum_{i\in I}a_it^i|\,\langle a_i\rangle_{i\in I}=A\}$. The assertion follows from the following claim.

Claim $M_n=W_n\Omega^r_A$.

Once we have the $d\log$ map for any ring, we will get a map on the Zariski site by sheafification. The desired map is obtained by taking the associated map on the étale site. Now it suffices to prove the above Claim.

Proof of Claim. We first assume $n=1$, and in this case we have

$$A(t)\otimes_A\Omega^1_A\oplus A(t)\xrightarrow{\cong}\Omega^1_{A(t)}$$

$$(a\otimes w,b)\mapsto a\cdot w+b\cdot dt$$

Using this explicit expression, we can show the kernel of $i_{1*}-i_{2*}$ is Ω^r_A. For general n, we first noted that $W_n\Omega^r_A\subseteq M_n$, and we can prove the claim by induction on n via the exact sequence

$$0\to V^n\Omega^r_A+dV^n\Omega^{r-1}_A\to W_{n+1}\Omega^r_A\xrightarrow{R}W_n\Omega^r_A\to 0$$

as i_{1*} and i_{2*} commute with R and V. □

By Definition 1.4.7, the image of $d\log[-]_n$ contains in $W_n\Omega^r_{Y,\log}$. It is clear that the map $d\log[-]_n$ factors through $\mathcal{K}^M_{r,Y}/p^n$. Therefore we have the following result.

Proposition 3.2.9. *The natural map in Lemma 3.2.8 induces an isomorphism*

$$dlog\,[-] : \mathcal{K}^M_{r,Y}/p^n \xrightarrow{\cong} W_n\Omega^r_{Y,\log}.$$

Proof. It is enough to show this map is injective. This is a local question, so we may assume $Y = \mathrm{Spec}(A)$ is a regular local ring over $\mathbb{F}_q$. The Néron-Popescu desingularization theorem below tells us that we can assume $Y = \mathrm{Spec}(A)$ is smooth over $\mathbb{F}_q$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{K}^M_{r,Y}/p^n & \hookrightarrow & i_{\eta*}K^M_r(\kappa(x))/p^n \\ {\scriptstyle d\log[-]_n}\downarrow & & \downarrow{\scriptstyle\cong} \\ W_n\Omega^r_{X,\log} & \hookrightarrow & i_{\eta*}W_n\Omega^r_{x,\log} \end{array}$$

of étale sheaves on Y. The injection on the first row follows from Theorem 3.2.3(iii), and Remark 1.3.11 implies the injection on the second row. Therefore the assertion follows from the fact that the right vertical map is an isomorphism, which is given by the classical Bloch-Gabber-Kato theorem[BK86]. □

Theorem 3.2.10 (Néron-Popescu, [Swa98])**.** *Any regular local ring A of characteristic p can be written as a filtering colimit $\varinjlim_i A_i$ with each A_i is smooth (of finite type) over $\mathbb{F}_q$.*

3.3 The relative logarithmic de Rham-Witt sheaves

Assume X is as before, i.e., a projective strictly semistable scheme over $B = \mathrm{Spec}(\mathbb{F}_q[[t]])$. Let $D = \cup_{i=1}^s D_i$ be a reduced effective Cartier divisor on Y such that $\mathrm{Supp}(D)$ has simple normal crossing, let $D_1, \cdots, D_s$ be the irreducible components of D, and let $\jmath : U := Y - D \hookrightarrow Y$ be the open complement of D. For $\underline{m} = (m_1, \cdots, m_s)$ with $m_i \in \mathbb{N}$ let

$$mD = \underline{m}D = \sum_{i=1}^s m_i D_i \tag{3.3.1}$$

be the associated divisor. In the previous section, we defined a semi-order on $\mathbb{N}^s$ and denoted

$$\underline{m'}D \geq \underline{m}D \quad \text{if} \quad \underline{m'} \geq \underline{m}. \tag{3.3.2}$$

Definition 3.3.1. *For $r \geq 0, n \geq 1$, we define*

$$W_n\Omega^r_{X|mD,log} \subset \jmath_* W_n\Omega^r_{U,log}$$

to be the étale additive subsheaf generated étale locally by

$$dlog[x_1]_n \wedge \cdots \wedge dlog[x_r]_n \quad \text{with} \quad x_i \in \mathcal{O}_U^\times, \text{ for all } i, \text{ and } x_1 \in 1 + \mathcal{O}_X(-mD),$$

where $[x]_n = (x, 0, \cdots, 0) \in W_n(\mathcal{O}_X)$ *is the Teichmüller representative of* $x \in \mathcal{O}_X$*, and* $d\log[x]_n = \frac{d[x]_n}{[x]_n}$ *as before.*

Corollary 3.3.2. *For any* $m \in \mathbb{N}^s$, $\jmath^* W_n\Omega^r_{X|mD,log} = W_n\Omega^r_{U,log}$. *If* $m' \geq m$*, then we have the inclusions of étale sheaves*

$$W_n\Omega^r_{X|m'D,log} \subseteq W_n\Omega^r_{X|mD,log} \subseteq W_n\Omega^r_{X,log}.$$

Proof. By definition, $W_n\Omega^r_{X|mD,\log}$ is the image of $\mathcal{K}^M_{r,X|mD}$ under the $d\log[-]_n$. Hence the first claim is clear and the second follows from Proposition 3.2.7 and 3.2.9. □

Theorem 3.3.3 ([JS15])**.** *There is an exact sequence of étale sheaves on X:*

$$0 \to W_{n-1}\Omega^r_{X|[m/p]D,log} \xrightarrow{p} W_n\Omega^r_{X|mD,log} \xrightarrow{R} \Omega^r_{X|mD,log} \to 0,$$

where $[m/p]D = \sum_{i=1}^{s}[m_i/p]D_i$*, and* $[m_i/p] = min\{m \in \mathbb{N} | m \geq m_i/p\}$.

Proof. This is a local problem, and the local proof in [JS15] also works in our situation. The idea is reduced to a similar result for Milnor K-groups by the Bloch-Gabber-Kato theorem. Then the graded pieces on Milnor K-groups can be represented as differential forms, which will give the desired exactness. □

If $m'D \geq mD$, then the relation $\mathcal{O}_X(-m'D) \subseteq \mathcal{O}_X(-mD)$ induces a natural transitive map $W_n\Omega^r_{X|m'D,\log} \hookrightarrow W_n\Omega^r_{X|mD,\log}$. This gives us a pro-system of abelian sheaves "$\varprojlim_m$"$W_n\Omega^r_{X|mD,\log}$.

Corollary 3.3.4. *The following sequence is exact*

$$0 \to \text{“}\varprojlim_m\text{”} W_{n-1}\Omega^r_{X|mD,log} \to \text{“}\varprojlim_m\text{”} W_n\Omega^r_{X|mD,log} \to \text{“}\varprojlim_m\text{”} \Omega^r_{X|mD,log} \to 0$$

as pro-objects, where "$\varprojlim_m$" *is the pro-system of sheaves defined by the ordering between the D's, which is defined in (3.3.2).*

In the rest of this section, we want to define a pairing between $W_n\Omega^r_{U,\log}$ and the pro-system "$\varprojlim_m$"$W_n\Omega^{d+1-r}_{X|mD,\log}$. First, we define a pairing between the quasicoherent sheaf $\jmath_* W_n\Omega^r_U$ and this pro-system. Then, using two terms complexes, we obtain the desired pairing.

Theorem 3.3.5 ([JS15])**.** *The wedge product on de Rham-Witt complex induces a natural pairing*

$$\jmath_* W_n\Omega^r_U \times \text{“}\varprojlim_m\text{”} W_n\Omega^{d+1-r}_{X|mD,log} \to W_n\Omega^{d+1}_X.$$

In [JS15], Jannsen and Saito study this pairing for a smooth proper variety over a finite field of characteristic $p > 0$, and their proof still works in our case. The proof only needs some calculations with Witt vectors; we repeat their arguments here.

Lemma 3.3.6 ([JS15]). *Let A be an $\mathbb{F}_q$-algebra, and let $a, t \in A$. Then*

$$[1+ta]_n = (x_0, \cdots, x_{n-1}) + (y_0, \cdots, y_{n-1})$$

with $(x_0, \cdots, x_{n-1}) \in W_n(\mathbb{F}_q)$, and $y_0, \cdots, y_{n-1} \in tA$. Here $[x]_n = (x, 0, \cdots, 0) \in W_n(A)$ is the Teichmüller representative of $x \in A$.

Note that we have $dx = 0$ for $x \in W_n(\mathbb{F}_q)$, so the above lemma implies $d[1+at]_n = d(y_0, \cdots, y_{n-1})$ with $y_i \in tA$.

Lemma 3.3.7 ([JS15]). *For $(a_0, \cdots, a_{n-1}) \in W_n(A)$ and $t \in A$ the following are equivalent:*

(i) $(a_0, a_1, \cdots, a_{n-1}) = (tb_0, t^p b_1, \cdots, t^{p^{n-1}} b_{n-1})$ *with* $b_i \in A$.

(ii) $(a_0, a_1, \cdots, a_{n-1}) = [t]_n \cdot (c_0, c_1, \cdots, c_{n-1})$ *with* $c_i \in A$.

Corollary 3.3.8 ([JS15]). *With the above notations, for $t \in A$, we have the following formula:*

$$\begin{aligned} d[1+t^{p^{m(n-1)}}]_n &= d(t^{p^{m(n-1)}} y_0, \cdots, t^{p^{m(n-1)}} y_{n-1}) = d([t]_n^{p^m} \cdot (c_0, \cdots, c_{n-1})) \\ &= p^m [t]_n^{p^m-1} \cdot (c_0, \cdots, c_{n-1}) d[t]_m + [t]_n^{p^m} \cdot d(c_0, \cdots, c_{n-1}) \end{aligned}$$

In particular, the coefficients of $d[1+t^{p^{m(n-1)}}]_n$ lies in $[t]_n^{p^m-1} W_n(A)$.

Proof of Theorem 3.3.5. It is equivalent to check the wedge product induces a well-defined map

$$\jmath_* W_n\Omega^r_U \to \mathrm{Hom}(\text{“}\varprojlim_m\text{”} W_n\Omega^{d+1-r}_{X|mD,\log}, W_n\Omega^{d+1}_X) = \varinjlim_m \mathrm{Hom}(W_n\Omega^{d+1-r}_{X|mD,\log}, W_n\Omega^{d+1}_X).$$

Let α be a given local section of $W_n\Omega^r_U$, we need to find a suitable m and a local section β of $W_n\Omega^{d+1-r}_{X|mD,\log}$ such that $\alpha \wedge \beta$ is a local section of $W_n\Omega^{d+1}_X$. This is equivalent to show that we can find $a_1 \in 1 + \mathcal{O}_X(-mD)$ such that the coefficient of

$$\alpha \wedge \frac{d[1+a_1]_n}{[1+a_1]_n} \wedge \frac{d[a_2]_n}{[a_2]_n} \wedge \cdots \wedge \frac{d[a_{d+1-r}]_n}{[a_{d+1-r}]_n}$$

lies in $W_n(\mathcal{O}_X)$. By the above corollary, this is possible if we take m big enough to eliminate the “poles” of α along D. □

Remark 3.3.9. *We don't have a map from “$\varprojlim_m$” $W_n\Omega^{d+1-r}_{X|mD,log}$ to Hom$(\jmath_* W_n\Omega^r_U, W_n\Omega^{d+1}_X)$.*

In [Mil86], Milne defined a pairing of two terms complexes as follows:

Let

$$\mathscr{F}^\bullet = (\mathscr{F}^1 \xrightarrow{d_{\mathscr{F}}} \mathscr{F}^2), \quad \mathscr{G}^\bullet = (\mathscr{G}^1 \xrightarrow{d_{\mathscr{G}}} \mathscr{G}^2)$$

and

$$\mathscr{H}^\bullet = (\mathscr{H}^1 \xrightarrow{d_{\mathscr{H}}} \mathscr{H}^2)$$

be two terms complexes. A pairing of two terms complexes

$$\mathscr{F}^\bullet \times \mathscr{G}^\bullet \to \mathscr{H}^\bullet$$

is a system of pairings

$$< \cdot, \cdot >^0_{0,0} \colon \mathscr{F}^0 \times \mathscr{G}^0 \to \mathscr{H}^0;$$

$$< \cdot, \cdot >^1_{0,1} \colon \mathscr{F}^0 \times \mathscr{G}^1 \to \mathscr{H}^1;$$

$$< \cdot, \cdot >^1_{1,0} \colon \mathscr{F}^1 \times \mathscr{G}^0 \to \mathscr{H}^1,$$

such that

$$d_{\mathscr{H}}(< x, y >^0_{0,0}) =< x, d_{\mathscr{G}}(y) >^1_{0,1} + < d_{\mathscr{F}}(x), y >^1_{1,0} \tag{3.3.3}$$

for all $x \in \mathscr{F}^0$, $y \in \mathscr{G}^0$. Such a pairing is the same as a mapping

$$\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet \to \mathscr{H}^\bullet.$$

In our situation, we set

$$W_n\mathscr{F}^\bullet := [\jmath_* Z_1 W_n\Omega^r_U \xrightarrow{1-C} \jmath_* W_n\Omega^r_U];$$

$$W_n\mathscr{G}^\bullet := [\text{“}\varprojlim_m\text{''} W_n\Omega^{d+1-r}_{X|mD,\log} \to 0];$$

$$W_n\mathscr{H}^\bullet := [W_n\Omega^{d+1}_X \xrightarrow{1-C} W_n\Omega^{d+1}_X].$$

Corollary 3.3.10 ([JS15])**.** *We have a natural pairing of complexes of length two*

$$W_n\mathscr{F}^\bullet \times W_n\mathscr{G}^\bullet \to W_n\mathscr{H}^\bullet. \tag{3.3.4}$$

Proof. It is equivalent to define the following three compatible pairings

$$\jmath_* Z_1 W_n\Omega^r_U \times \text{“}\varprojlim_m\text{''} W_n\Omega^{d+1-r}_{X|mD,\log} \to W_n\Omega^{d+1}_X \tag{3.3.5}$$

$$\jmath_* Z_1 W_n\Omega^r_U \times 0 \to W_n\Omega^{d+1}_X \tag{3.3.6}$$

$$\jmath_* W_n\Omega^r_U \times \text{“}\varprojlim_m\text{''} W_n\Omega^{d+1-r}_{X|mD,\log} \to W_n\Omega^{d+1}_X \tag{3.3.7}$$

The pairing (3.3.6) is the zero pairing, (3.3.5) and (3.3.7) are defined similarly as given in Theorem 3.3.5. The compatibility in the sense of (3.3.3) of those pairings is clear by noting that $W_n\Omega^{d+1-r}_{X|mD,\log} \subseteq \operatorname{Ker}(1-C)$.

□

3.4 Ramified duality

By Proposition 1.4.11 (iii), in $D^b(X, \mathbb{Z}/p^n\mathbb{Z})$ we have

$$R\jmath_* W_n\Omega^r_{U,\log} \cong [\jmath_* Z_1 W_n\Omega^r_U \xrightarrow{1-C} \jmath_* W_n\Omega^r_U] = W_n\mathscr{F}^\bullet. \tag{3.4.1}$$

Lemma 3.4.1. *Let D be a normal crossing divisor on X. Then the induced open immersion $\jmath : U := X - |D| \to X$ is affine.*

Proof. To be an affine morphism is étale locally on the target, and étale locally D is given by one equation. □

Then we have $\mathbb{H}^i(X, W_n\mathscr{F}^\bullet) \cong \mathbb{H}^i(U, [Z_1W_n\Omega^r_U \to W_n\Omega^r_U]) \cong H^i(U, W_n\Omega^r_{U,\log})$, by the above lemma and Proposition 1.4.11 (iii). Therefore the pairing (3.3.4) induces a pairing of cohomology groups

$$H^i(U, W_n\Omega^r_{U,\log}) \times \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}) \to H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,\log}) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/p^n\mathbb{Z}. \tag{3.4.2}$$

Note that, since the two left terms in the above pairing are not finite groups, we need to endow them with some suitable topological structures in order to obtain a perfect pairing.

Proposition 3.4.2. *The pairing (3.4.2) is a pairing of topological abelian groups, if we endow the first term with discrete topology and the second with profinite topology.*

Proof. It is equivalent to show that the following two induced morphisms are continuous:

$$H^i(U, W_n\Omega^r_{U,\log}) \to \mathrm{Hom}_{\mathrm{cont}}(\varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}), \mathbb{Z}/p^n\mathbb{Z});$$

$$\varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}) \to \mathrm{Hom}(H^i(U, W_n\Omega^r_{U,\log}), \mathbb{Z}/p^n\mathbb{Z}).$$

Here the target Hom sets are endowed with the compact-open topologies. This first map is automatically continuous, as the topological structure on $H^i(U, W_n\Omega^r_{U,\log})$ is discrete. For the second map the claim is implied by the following two lemmas. □

Lemma 3.4.3. *Let*

$$< \cdot, \cdot >: M \times \varprojlim_i N_i \to S$$

be a pairing of topological abelian groups, where $(N_i)_{i\in I}$ is a filtered inverse system of finite groups, M and S are discrete groups, and the inverse limit is endowed with the limit topology, i.e., the profinite topology. Assume that for each $m \in M$ the induced morphism

$$< m, \cdot >: \varprojlim_i N_i \to S$$

factors through N_i, for some i. Then the pairing is continuous.

Proof. It is enough to prove that the induced map

$$\alpha : \varprojlim_i N_i \to \mathrm{Hom}(M, S)$$

is continuous. Note that the target Hom group is endowed with compact-open topology. The neighborhoods of 0 in $\mathrm{Hom}(M, S)$ is given by $\{f \in \mathrm{Hom}(M, S) | f(K) = 0\}$, for some compact set $K \subset M$. But the topology on M is discrete, so any compact subset is finite. It suffices to show that the inverse image of $O := \{f \in \mathrm{Hom}(M, S) | f(m) = 0\}$, for some $m \in M$, is open. Note that we have $\alpha^{-1}(O) = \mathrm{Ker}(< m, \cdot >)$ follows from the definition. Now by our assumption, we have, for some $i \in I$,

$$\mathrm{Ker}(\varprojlim_i N_i \to N_i) \subseteq \mathrm{Ker}(< m, \cdot >).$$

But $\mathrm{Ker}(\varprojlim_i N_i \to N_i)$ is a fundamental neighborhood of 0 for the profinite topology, which is open. This is the desired conclusion. □

Lemma 3.4.4. *For any $\alpha \in H^i(U, W_n\Omega^r_{U,log})$, the following morphism, which is induced by the pairing (3.4.2),*

$$< \alpha, \cdot >: \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,log}) \to H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,log})$$

factors through $H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,log})$ for some $m \in \mathbb{N}^s$.

Proof. We have the following commutative diagram

$$\begin{array}{ccccc}
H^i(U, W_n\Omega^r_{U,\log}) & \times & \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}) & \to & H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,\log}) \\
\| & & \| & & \| \\
\mathbb{H}^i(X, W_n\mathscr{F}^\bullet) & \times & \mathbb{H}^{d+2-i}_{X_s}(X, W_n\mathscr{G}^\bullet) & \longrightarrow & \mathbb{H}^{d+2}_{X_s}(X, W_n\mathscr{H}^\bullet) \\
\downarrow e & & \| & & \| \\
\mathrm{Ext}^i(W_n\mathscr{G}^\bullet, W_n\mathscr{H}^\bullet) & \times & \mathbb{H}^{d+2-i}_{X_s}(X, W_n\mathscr{G}^\bullet) & \longrightarrow & \mathbb{H}^{d+2}_{X_s}(X, W_n\mathscr{H}^\bullet) \\
\| & & \| & & \| \\
\varinjlim_m \mathrm{Ext}^i(W_n\Omega^{d+1-r}_{X|mD,\log}, W_n\Omega^{d+1}_{X,\log}) & \times & \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}) & \to & H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,\log})
\end{array}$$

The second row is the cup-product pairing, the third row is the Yoneda paring, and the last row the limit of Yoneda pairing. The arrow e is given by the edge morphism of the Ext-spectral sequence. It is well known that the cup-product pairing commutes with Yoneda pairing [GH70]. Note that a morphism to direct limit implies that it factors through some term of the direct system. □

Now our main theorem in this thesis is the following result.

Theorem 3.4.5. *For each i, r, the above pairing (3.4.2) is a perfect pairing of topological $\mathbb{Z}/p^n\mathbb{Z}$-modules.*

In the rest of this section, we focus on the proof of this theorem.

Proof. We can proceed the analogously to the proof of Theorem 3.1.1. First, we reduce this theorem to the case $n = 1$, then using Cartier operator, we will study the relation between this theorem and coherent duality theorems.

Step 1: Reduction to the case $n = 1$. We have a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \to & H^i(U, W_{n-1}\Omega^r_{U,\log}) & \to & H^i(U, W_n\Omega^r_{U,log}) & \to & H^i(U, \Omega^r_{U,\log}) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \varprojlim_m H^{d+2-i}_{X_s}(X, W_{n-1}\Omega^{d+1-r}_{X|mD,\log})^* & \to & \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log})^* & \to & \varprojlim_m H^{d+2-i}_{X_s}(X, \Omega^{d+1-r}_{X|mD,\log})^* & \to \cdots
\end{array}$$

where M^* means $\mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}/\mathbb{Z})$, i.e., its Pontrjagin dual, for any locally compact topological abelian group M. The fist row is the long exact sequence induced by the short exact sequence in Proposition 1.4.11 (i), and the second row is an long exact sequence induced by Corollary 3.3.4. Note that the inverse limit is exact in our case, as $H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log})$ is finite for any m, the inverse systems satisfy the Mittag-Leffler condition.

By the five lemma and induction, our problem is reduced to the case $n = 1$.

Step 2: Proof of the case $n = 1$. In this special case, the relation between the relative logarithmic de Rham-Witt and coherent sheaves can be summarized as follows.

Theorem 3.4.6. *([JS15]) We have the following exact sequence*

$$0 \to \Omega^{d+1-r}_{X|mD,log} \to \Omega^{d+1-r}_{X|mD} \xrightarrow{C^{-1}-1} \Omega^{d+1-r}_{X|mD}/B\Omega^{d+1-r}_{X|mD} \to 0,$$

where $\Omega^j_{X|mD} = \Omega^j_X(logD) \otimes \mathcal{O}_X(-mD)$ is defined in (2.5.4).

Proof. This is again a local problem, and the local proof in [JS15] also works in our situation. The key ingredient is to show $\Omega^i_{X|mD,\log} = \Omega^i_{X,\log} \cap \Omega^i_{X|mD}$. This can be obtained by a refinement of [Kat82, Prop. 1]. □

Lemma 3.4.7. *For any $\underline{m} \in \mathbb{N}^s$, we denote $mD = \underline{m}D$ as in (3.3.1), and*

$$\Omega^j_{X|mD} = \Omega^j_X(logD)(-mD) = \Omega^j_X(logD) \otimes \mathcal{O}_X(-mD)$$

as before. The parings

$$< \alpha, \beta >^0_{0,0} = \alpha \wedge \beta : Z\Omega^r_{X|-mD} \times \Omega^{d+1-r}_{X|(s+1)D} \to \Omega^{d+1}_{X|D};$$
$$< \alpha, \beta >^1_{0,1} = C(\alpha \wedge \beta) : Z\Omega^r_{X|-mD} \times \Omega^{d+1-r}_{X|(m+1)D}/B\Omega^{d+1-r}_{X|(m+1)D} \to \Omega^{d+1}_{X|D};$$
$$< \alpha, \beta >^1_{1,0} = \alpha \wedge \beta : \Omega^r_{X|-mD} \times \Omega^{d+1-r}_{X|(m+1)D} \to \Omega^{d+1}_{X|D},$$

define a paring of (two terms) complexes

$$\mathscr{F}_m^\bullet \times \mathscr{G}_{-m}^\bullet \to \mathscr{H}^\bullet \tag{3.4.3}$$

with

$$\begin{aligned}
\mathscr{F}_m^\bullet &= (Z\Omega^r_{X|-mD} \xrightarrow{1-C} \Omega^r_{X|-mD}) \\
\mathscr{G}_{-m}^\bullet &= (\Omega^{d+1-r}_{X|(m+1)D} \xrightarrow{C^{-1}-1} \Omega^{d+1-r}_{X|(m+1)D}/B\Omega^{d+1-r}_{X|(m+1)D}) \\
\mathscr{H}^\bullet &= (\Omega^{d+1}_{X|D} \xrightarrow{1-C} \Omega^{d+1}_{X|D}) = (\Omega^{d+1}_{X} \xrightarrow{1-C} \Omega^{d+1}_{X}).
\end{aligned}$$

Proof. This is easy to verify. □

By taking hypercohomology, the pairing (3.4.3) induces a pairing of hypercohomology groups:

$$\mathbb{H}^i(X, \mathscr{F}_m^\bullet) \times \mathbb{H}^{d+2-i}_{X_s}(X, \mathscr{G}_{-m}^\bullet) \to \mathbb{H}^{d+2}_{X_s}(X, \mathscr{H}^\bullet) \cong H^{d+2}_{X_s}(X, \Omega^{d+1}_{X,log}) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/p\mathbb{Z}. \tag{3.4.4}$$

Now by Corollary 2.5.7 and 2.5.8, we can show

Proposition 3.4.8. *The pairing (3.4.4) induces the following isomorphism.*

$$\begin{aligned}
\mathbb{H}^i(X, \mathscr{F}_m^\bullet) &\xrightarrow{\cong} Hom_{\mathbb{Z}/p\mathbb{Z}}(\mathbb{H}^{d+2-i}_{X_s}(X, \mathscr{G}_{-m}^\bullet), \mathbb{Z}/p\mathbb{Z}) \\
&\cong Hom_{\mathbb{Z}/p\mathbb{Z}}(H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|(m+1)D,log}), \mathbb{Z}/p\mathbb{Z}).
\end{aligned} \tag{3.4.5}$$

Proof. This can be done by using the hypercohomology spectral sequences

$${}^{I}E_1^{p,q} = H^q(X, \mathscr{F}_m^p) \Rightarrow \mathbb{H}^{p+q}(X, \mathscr{F}_m^\bullet);$$

$${}^{II}E_1^{p,q} = H^q_{X_s}(X, \mathscr{G}_{-m}^p) \Rightarrow \mathbb{H}^{p+q}_{X_s}(X, \mathscr{G}_{-m}^\bullet).$$

Corollary 2.5.7 and 2.5.8 tell us that

$${}^{I}E_1^{p,q} \cong \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}({}^{II}E_1^{d+1-q,1-p}, \mathbb{Z}/p^n\mathbb{Z}),$$

and this isomorphism is compatible with d_1. Hence we still have this kind of duality at the E_2-pages. Note that, by definition, $p \neq 0, 1$, ${}^{I}E_1^{p,q} = {}^{II}E_1^{p,q} = 0$. Hence both spectral sequences degenerate at the E_2-pages. Therefore we have the isomorphism in the claim. □

Up to now, we haven't used any topological structure on the (hyper-)cohomology group. Note that $H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|(m+1)D,\log})$ is a finite group, and we endow it with discrete topology. Hence we endow $\mathbb{H}^i(X, \mathscr{F}_m^\bullet)$ with the discrete topology. Now the Pontragin duality theorem implies:

Proposition 3.4.9. *There is a perfect pairing of topological* $\mathbb{Z}/p\mathbb{Z}$*-modules:*

$$\varinjlim_m \mathbb{H}^i(X, \mathscr{F}_m^\bullet) \times \varprojlim_m H^{d+2-i}_{X_s}(X, \Omega^{d+1-r}_{X|(m+1)D,log}) \to \mathbb{Z}/p\mathbb{Z} \tag{3.4.6}$$

where the first term is endowed with direct limit topology, and the second with the inverse limit topology.

Proof. Note that the Pontrjagin dual $\mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbb{Z}/p\mathbb{Z})$ commutes with direct and inverse limits. Then the proof is straightforward. □

Remark 3.4.10. *The direct limit topology of discrete topological spaces is still discrete, and the inverse limit topololgy of finite discrete topological spaces is profinite.*

We still need to calculate the direct limit term in the above proposition.

Proposition 3.4.11. *The direct limit* $\varinjlim_m \mathbb{H}^i(X, \mathscr{F}_m^\bullet) \cong H^i(U, \Omega^r_{U,log})$.

Proof. First, direct limits commute with (hyper-)cohomology, hence

$$\varinjlim_m \mathbb{H}^i(X, \mathscr{F}_m^\bullet) = \mathbb{H}^i(X, \varinjlim_m \mathscr{F}_m^\bullet).$$

Note that

$$\varinjlim_m \mathscr{F}_m^\bullet = [\jmath_* Z\Omega^r_U \xrightarrow{1-C} \jmath_*\Omega^r_U].$$

For coherent sheaves, the affine morphism $\jmath$ (see Lemma 3.4.1) gives an exact functor $\jmath_*$. Hence we have

$$\mathbb{H}^i(X, [\jmath_* Z\Omega^r_U \xrightarrow{1-C} \jmath_*\Omega^r_U]) = \mathbb{H}^i(U, [Z\Omega^r_U \xrightarrow{1-C} \Omega^r_U]) = H^i(U, \Omega^r_{U,\mathrm{log}}),$$

where the last equality follows from the special case $n = 1$ of Proposition 1.4.11 (iii). □

Proof of ***Step 2****.* The duality theorem in case $n = 1$ directly follows from the above two propositions. □

Now the proof of our main Theorem 3.4.5 is complete.

We denote

$$\begin{aligned}\Phi : H^i(U, W_n\Omega^r_{U,\mathrm{log}}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{cont}}(\varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\mathrm{log}}), \mathbb{Z}/p^n\mathbb{Z}); \\ &\cong \varinjlim_m \mathrm{Hom}(H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\mathrm{log}}), \mathbb{Z}/p^n\mathbb{Z})\end{aligned}$$

$$\Psi : \varprojlim_m H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\mathrm{log}}) \xrightarrow{\cong} \mathrm{Hom}(H^i(U, W_n\Omega^r_{U,\mathrm{log}}), \mathbb{Z}/p^n\mathbb{Z}).$$

Using this duality theorem, we can at last define a filtration as follows:

Definition 3.4.12. *Assume X, X_s, D, U are as before. For any $\chi \in H^i(U, W_n\Omega^r_{U,\log})$, we define the higher Artin conductor*

$$ar(\chi) := min\{m \in \mathbb{N}_0^s \mid \Phi(\chi) \text{ factors through } H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,log})\},$$

For $m \in \mathbb{N}^s$, we define

$$Fil_m H^i(U, W_n\Omega^r_{U,log}) := \{\chi \in H^i(U, W_n\Omega^r_{U,\log}) | \ ar(\chi) \leq m\},$$

and

$$\pi_1^{ab}(X, mD)/p^n := Hom(Fil_m H^1(U, \mathbb{Z}/p^n\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

endowed with the usual profinite topology of the dual.

It is clear that $\mathrm{Fil}_\bullet$ is an increasing filtration with respect to the semi-order on $\mathbb{N}^s$. If $s = 1$, i.e., the semi-order is an order, we have

$$\mathrm{Fil}_m H^i(U, W_n\Omega^r_{U,\log}) = \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|mD,\log}), \mathbb{Z}/p^n\mathbb{Z}).$$

The quotient $\pi_1^{\mathrm{ab}}(X, mD)/p^n$ can be thought of as classifying abelian étale coverings of U of degree p^n with ramification bounded by the divisor mD.

Chapter 4

Comparison with the classical case

In this chapter, we want to compare our filtration with the classical one in the local ramification theory.

4.1 Local ramification theory

Let K be a local field, i.e., a complete discrete valuation field of characteristic $p > 0$, let $\mathcal{O}_K$ be its ring of integers, let k be its residue field, and let ν_K be its valuation. We fix a uniformizer $\pi \in \mathcal{O}_K$, which generates the maximal ideal $\mathfrak{m} \in \mathcal{O}_K$.

The local class field theory [Ser79] gives us an Artin reciprocity homomorphism

$$\mathrm{Art}_K : K^\times \to Gal(K^{\mathrm{ab}}/K),$$

where K^{ab} is the maximal abelian extension of K. Note that both $K^\times$ and $Gal(K^{\mathrm{ab}}/K)$ are topological groups. Recall the topological structure on $K^\times$ is given by the valuation on K, and $Gal(K^{\mathrm{ab}}/K)$ is the natural profinite topology.

For any $m \in \mathbb{N}$, the Atrin map induces an isomorphism of topological groups

$$\mathrm{Art}_K \otimes 1 : K^\times \otimes \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} Gal(K^{\mathrm{ab}}/K) \otimes \mathbb{Z}/m\mathbb{Z}.$$

In particular, take $m = p^n$, it gives:

$$\mathrm{Art}_K \otimes 1 : K^\times/(K^\times)^{p^n} \xrightarrow{\cong} Gal(K^{\mathrm{ab}}/K) \otimes \mathbb{Z}/p^n\mathbb{Z}. \tag{4.1.1}$$

For $n \geq 1$, the Artin-Schreier-Witt theory tells us there is a natural isomorphism

$$\delta_n : W_n(K)/(1-F)W_n(K) \xrightarrow{\cong} H^1(K, \mathbb{Z}/p^n\mathbb{Z}), \tag{4.1.2}$$

where $W_n(K)$ is the ring of Witt vector of length n and F is the Frobenius.

Note that $H^1(K, \mathbb{Z}/p^n\mathbb{Z})$ is dual to $Gal(K^{\mathrm{ab}}/K) \otimes \mathbb{Z}/p^n\mathbb{Z}$, the interplay between (4.1.1) and (4.1.2) gives rise to the following theorem.

Theorem 4.1.1 (Artin-Schreier-Witt). *There is a perfect pairing of topological groups, that we call the Artin-Schreier-Witt symbol*

$$W_n(K)/(1-F)W_n(K) \times K^\times/(K^\times)^{p^n} \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \qquad (4.1.3)$$
$$(a,b) \mapsto [a,b) := (b, L/K)(\alpha) - \alpha$$

where $(1-F)(\alpha) = a$, for some $\alpha \in W_n(K^{sep})$, $L = K(\alpha)$, $(b, L/K)$ is the norm residue of b in L/K, and the topological structure on the first term is discrete, on the second term is induced from $K^\times$.

Proof. This pairing is non-degenerate [Tho05, Prop. 3.2]. Taking the topological structure into account, we get the perfectness by Pontrjagin duality. □

We have filtrations on the two left terms in the pairing (4.1.3). On $W_n(K)$, Brylinski [Bry83] and Kato [Kat89] defined an increasing filtration, called the Brylinski-Kato filtration, using the valuation on K:

$$\mathrm{fil}_m^{\log} W_n(K) = \{(a_{n-1}, \cdots, a_1, a_0) \in W_n(K) |\ p^i \nu_K(a_i) \geq -m\}.$$

We also have its non-log version introduced by Matsuda [Mat97].

$$\mathrm{fil}_m W_n(K) = \mathrm{fil}_{m-1}^{\log} W_n(K) + V^{n-n'} \mathrm{fil}_m^{\log} W_{n'}(K), \qquad (4.1.4)$$

where $n' = \min\{n, \mathrm{ord}_p(m)\}$ and $V : W_{n-1}(K) \to W_n(K)$ is the Verscheibung on Witt vectors.

Both of them induce filtrations on the quotient $W_n(K)/(1-F)W_n(K)$, and we define

$$\mathrm{fil}_m^{\log} H^1(K, \mathbb{Z}/p^n\mathbb{Z}) = \delta_n(\mathrm{fil}_m^{\log}(W_n(K)/(1-F)W_n(K))) = \delta_n(\mathrm{fil}_m^{\log} W_n(K)), \qquad (4.1.5)$$

$$\mathrm{fil}_m H^1(K, \mathbb{Z}/p^n\mathbb{Z}) = \delta_n(\mathrm{fil}_m(W_n(K)/(1-F)W_n(K))) = \delta_n(\mathrm{fil}_m W_n(K)). \qquad (4.1.6)$$

We have the following fact on the relation of two above filtrations.

Lemma 4.1.2. *([Kat89], [Mat97]) For an integer $m \geq 1$, we have*

(i) $\mathrm{fil}_m H^1(K, \mathbb{Z}/p^n\mathbb{Z}) \subset \mathrm{fil}_m^{\log} H^1(K, \mathbb{Z}/p^n\mathbb{Z}) \subset \mathrm{fil}_{m+1} H^1(K, \mathbb{Z}/p^n\mathbb{Z})$,

(ii) $\mathrm{fil}_m H^1(K, \mathbb{Z}/p^n\mathbb{Z}) = \mathrm{fil}_{m-1}^{\log} H^1(K, \mathbb{Z}/p^n\mathbb{Z})$ if $(m,p) = 1$.

Remark 4.1.3. *The non-log version of Brylinski-Kato filtration is closely related to the Kähler differential module Ω_K^1 [Mat97], and it has an higher analogy on $H^1(U)$, where U is an open smooth subscheme of a normal variety X over a perfect field with $(X-U)_{red}$ is the support of an effective Cartier divisor [KS14].*

On $K^\times$, we have a natural decreasing filtration given by:

$$U_K^{-1} = K^\times, U_K^0 = \mathcal{O}_K^\times,\ U_K^m = \{x \in K^\times |\ x \equiv 1 \bmod \pi^m\}.$$

The following theorem says the paring (4.1.3) is compatible with those filtrations.

Theorem 4.1.4. *([Bry83, Thm. 1]) Underling the Artin-Schreier-Witt symbol (4.1.3), the orthogonal complement of* $\mathit{fil}^{log}_{m-1}H^1(K,\mathbb{Z}/p^n\mathbb{Z})$ *is* $U_K^m\cdot(K^\times)^{p^n}/(K^\times)^{p^n}$*, for any integer* $m\geq 1$*.*

Proof. A more complete proof can be found in [Tho05, §5]. □

Corollary 4.1.5. *The Artin-Schreier-Witt symbol (4.1.3) induces a perfect pairing of finite groups*

$$\mathit{fil}_m(W_n(K)/(1-F)W_n(K))\times K^\times/(K^\times)^{p^n}\cdot U_K^m\longrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

Proof. First, note that the filtration $\{U_K^m\}_m$ has no jump greater or equal to 0 that divisible by p, as the residue field of K is perfect. Then, we may assume $(m,p)=1$. By Lemma 4.1.2 (ii) and the above Brylinski's theorem, we have, the orthogonal complement of $\mathrm{fil}_mH^1(K,\mathbb{Z}/p^n\mathbb{Z})=\mathrm{fil}^{log}_{m-1}H^1(K,\mathbb{Z}/p^n\mathbb{Z})$ is $U_K^m\cdot(K^\times)^{p^n}/(K^\times)^{p^n}$. The rest follows easily from the fact that the Pontrjagin dual $H^\wedge$ of an open subgroup of a locally compact group G is isomorphic to $G^\wedge/H^\perp$, where $H^\perp$ is the orthogonal complement of H. □

4.2 Comparison of filtrations

Let $X=B=\mathrm{Spec}\mathbb{F}_q[[t]]$, $D=s=(t)$ be the unique closed point. Then $U=\mathrm{Spec}(\mathbb{F}_q((t)))$. Our duality theorem 3.4.5 in this setting is:

Corollary 4.2.1. *The pairing*

$$H^i(K,W_n\Omega^j_{U,log})\times\varprojlim_m H^{2-i}_s(B,W_n\Omega^{1-j}_{B|mD,\log})\to\mathbb{Z}/p^n\mathbb{Z}$$

is a perfect paring of topological groups.

In particular, we take $i=1,j=0$, and get

$$H^1(U,\mathbb{Z}/p^n\mathbb{Z})\times\varprojlim_m H^1_s(B,W_n\Omega^1_{B|mD,\log})\to\mathbb{Z}/p^n\mathbb{Z}. \tag{4.2.1}$$

We want to compare this pairing (4.2.1) with the Artin-Schreier-Witt symbol (4.1.3).

Lemma 4.2.2. *We have* $H^1_s(B,W_n\Omega^1_{B|mD,log})\cong K^\times/(K^\times)^{p^n}\cdot U_K^m$*. The diagram*

$$\begin{array}{ccc} H^1_s(B,W_n\Omega^1_{B|(m+1)D,log}) & \xleftarrow{\cong} & K^\times/(K^\times)^{p^n}\cdot U_K^{m+1}\\ \downarrow & & \downarrow\\ H^1_s(B,W_n\Omega^1_{B|mD,log}) & \xleftarrow{\cong} & K^\times/(K^\times)^{p^n}\cdot U_K^{m}\end{array}$$

commutes, where the left vertical arrow is induced by the morphism of sheaves, and the right vertical arrow is given by projection. In particular,

$$\varprojlim_m H^1_s(B, W_n\Omega^1_{B|mD,log}) \cong K^\times/(K^\times)^{p^n}.$$

Proof. We prove this by induction on n. If $n = 1$, the localization sequence gives the following exact sequence

$$0 \to H^0_s(B, \Omega^1_{B|mD,\log}) \to H^0(B, \Omega^1_{B|mD,\log}) \to H^0(U, \Omega^1_{U,\log}) \to H^1_s(B, \Omega^1_{B|mD,\log}) \to 0.$$

The Bloch-Gabber-Kato theorem [BK86] says $K^\times/(K^\times)^p \xrightarrow{\cong} H^0(U, \Omega^1_{U,\log})$, and by definition, it is easy to see that $U^m_K \cdot (K^\times)^p/(K^\times)^p \xrightarrow{\cong} H^0(B, \Omega^1_{B|mD,\log})$. For the induction process, we use the exact sequence in Theorem (3.3.3):

$$0 \to W_{n-1}\Omega^1_{B|[m/p]D,\log} \xrightarrow{p} W_n\Omega^1_{B|mD,\log} \xrightarrow{R} \Omega^1_{X|mD,\log} \to 0.$$

Note that the first term involves dividing by p. But for the filtration $\{U^m_K\}_m$, there are no jump greater or equal to 0 that divisible by p, as the residue field of K is perfect. The commutativity of the diagram follows also directly from the above computation. □

Now our main result in this chapter is the following:

Proposition 4.2.3. *The filtration we defined in Definition 3.4.12 is same as the non-log version of Brylinski-Kato filtration, i.e., for any integer $m \geq 1$,*

$$Fil_m H^1(U, \mathbb{Z}/p^n\mathbb{Z}) = fil_m H^1(U, \mathbb{Z}/p^n\mathbb{Z}).$$

Proof. Since $\mathbb{N}$ is a linearly ordered set, we have

$$\begin{aligned} Fil_m H^1(U, \mathbb{Z}/p^n\mathbb{Z}) &= \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(H^1_s(B, \Omega^1_{B|mD,\log}), \mathbb{Z}/p^n\mathbb{Z}) \\ &= \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(K^\times/(K^\times)^{p^n} \cdot U^m_K, \mathbb{Z}/p^n\mathbb{Z}) \\ &= fil_m H^1(U, \mathbb{Z}/p^n\mathbb{Z}) \end{aligned}$$

where the second equality is given by Lemma 4.2.2, and the last is Corollary 4.1.5. □

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