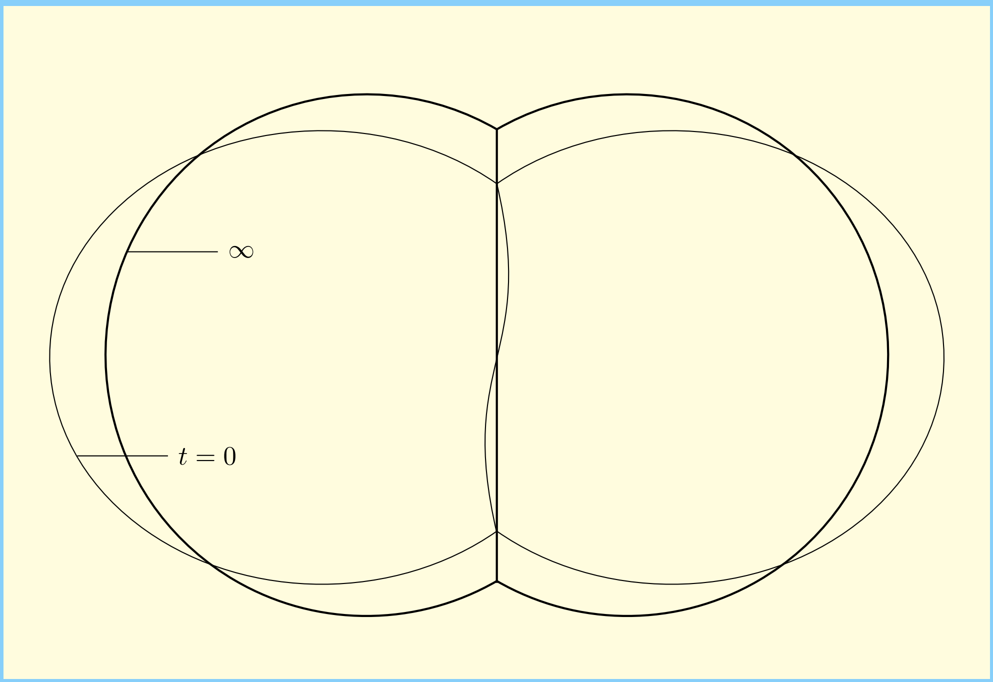
Dynamical Stability in Relation to Variational Stability: Double Bubbles in $\mathbb{R}^2$



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Dynamical Stability in Relation to Variational Stability: Double Bubbles in $\mathbb{R}^2$

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES
DER NATURWISSENSCHAFTEN (DR. RER. NAT.)
DER FAKULTÄT FÜR MATHEMATIK
DER UNIVERSITÄT REGENSBURG

vorgelegt von

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aus Oroumieh, Iran

im Jahr 2015

Promotionsgesuch eingereicht am:	12. November 2015
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Acknowledgements

What to say when existence of this work itself is an acknowledgement. Dear God, thanks. I personally would like to thank all the people who contribute in this existence. They recognize themselves by their beautiful hearts.

Abstract

This thesis is devoted to the investigation of the dynamical stability of standard planar double bubbles. By presenting connections between the dynamical stability and variational stability, we prove that standard planar double bubbles are dynamically stable under the surface diffusion flow.

This investigation leads us to extend a practical tool the so-called *generalized principle of linearized stability* (GPLS) to a more general setting. More precisely, convergence to stationary solutions in fully nonlinear parabolic systems with general nonlinear boundary conditions is proved in situations where the set of stationary solutions creates a C^2-manifold of finite dimension which is normally stable. We apply the parabolic Hölder setting which allows to deal with nonlocal terms including highest order point evaluation.

In this direction a couple of other useful results on linear parabolic systems are also extended. In addition, as an application of our extended version of GPLS, we prove also that the lens-shaped networks generated by circular arcs are dynamically stable under the surface diffusion flow.

Contents

Chapter 1

Introduction

Investigating the dynamical stability of planar double bubbles is a subject of the present work. Among our main results is the presentation of connections between the dynamical stability and variational stability. Therefore, we start with an introduction to these connections.

1.1 The concepts of stability

Let us illustrate the stability concept in differential geometry, the stability concept in dynamical system, and their connections by giving a simple, but fundamental, example from ordinary differential equation (ODE) with Euclidean inner product.

1.1.1 A fundamental example

Consider a cost function $f : \mathbb{R}^n \to \mathbb{R}$. Let us assume $f \in C^2(\mathbb{R}^n)$. For the first variation δf of the cost function f at a point $v \in \mathbb{R}^n$ in a direction $u \in \mathbb{R}^n$ we get easily

$$\delta f(v)\,(u) =< \nabla f(v), u > , \tag{1.1.1}$$

where $< \cdot, \cdot >$ denotes the Euclidean inner product in $\mathbb{R}^n$, and where the norm induced by the inner product (the Euclidean norm) is denoted by $\|\cdot\|$. Therefore, we obtain for the gradient flow of f with respect to the Euclidean inner product

$$x'(t) = -\nabla f(x(t))\,. \tag{1.1.2}$$

We say a point $v \in \mathbb{R}^n$ is stationary for any (possible) variation if

$$\delta f(v)\,(u) = 0 \qquad \forall u \in \mathbb{R}^n\,.$$

In other words, at a stationary point the first variation must vanish for all possible directions. Of course, in view of the identity (1.1.1), this can only happen if $\nabla f = 0$ there. Thus a point is stationary for any possible variation if and only if it is stationary for the gradient flow (1.1.2).

Now let us denote the bilinear form associated to the second variation of the function f by I, i.e.,

$$\delta^2 f(v)(u,w) = I(v)(u,w)\,.$$

To simplify the notation, we usually drop the v-dependence in the bilinear form I. Obviously, this bilinear form is symmetric.

Definition 1.1.1 (The concept of stability in differential geometry)**.** *A stationary point v is variationally stable if*

$$\big(I(v)(u,u) = \big) I(u,u) \geq 0 \qquad \forall u \in \mathbb{R}^n\,.$$

Remark 1.1.2. *An extremum (a maximum or minimum) of f must be stationary and indeed a minimum is variationally stable.*

Definition 1.1.3 (The concept of stability in dynamical system)**.** *A stationary point x_* is dynamically stable under the flow* (1.1.2) *if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0) - x_*\| < \delta \implies x(t) \text{ exists and } \|x(t) - x_*\| < \epsilon\,, \quad \forall t \geq 0\,,$$

and in addition δ can be chosen such that

$$\|x(0) - x_*\| < \delta \implies \lim_{t\to\infty} x(t) = x_\infty$$

at an exponential rate, with x_∞ being some stationary state.

Remark 1.1.4. *In general the stationary state x_∞ (long time limit) differs from the (non perturbed) stationary states x_*. And this is highly likely to happen when the set of stationary solutions creates a manifold near x_*.*

Definition 1.1.5. *Given a stationary point x_*, we define the linear operator*

$$A := \delta\big(\nabla f(x_*)\big) : \mathbb{R}^n \to \mathbb{R}^n$$

as the linearization of the flow (1.1.2) *at x_*.*

A practical and powerful tool to show dynamical stability in situations where the set of stationary solutions creates a manifold is *the generalized principle of linearized stability*:

Theorem 1.1.6. *Let x_* be a stationary solution of* (1.1.2)*. Suppose x_* is normally stable, i.e., assume that*

(i) *near x_* the set of stationary solutions $\mathcal{E}$ is a C^1-manifold of dimension $m \in \mathbb{N}$,*

(ii) $T_{x_*}\mathcal{E} = N(A)$,

(iii) 0 *is a semi-simple eigenvalue of* A*, i.e.,* $N(A) \oplus R(A) = \mathbb{R}^n$*,*

(iv) $\sigma(A) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : Re\ z > 0\}$.

Then x_* *is dynamically stable under the flow* (1.1.2).

We refer the reader to [34, Theorem 10.3.1] for the proof. This principle is introduced for partial differential equations in [35, 36] in certain settings.

Variational stability vs. dynamical stability

Now the following question arises: Suppose a stationary point x_* is variationally stable. Does this imply that x_* is dynamically stable? Of course it is well known that the answer is negative in general, see e.g. Remark 1.1.13 below. But let us find some clear relations between these two concepts by deriving the following important identity.

Lemma 1.1.7. *Let* $v, w \in \mathbb{R}^n$*. Then*

$$I(v,w) =< Av, w > . \tag{1.1.3}$$

Proof. Let us differentiate

$$\delta f(x_* + \epsilon v)\,(w) =< \nabla f(x_* + \epsilon v), w >$$

with respect to ϵ and evaluate it at $\epsilon = 0$. This leads to

$$\begin{aligned} I(x_*)(v,w) &= \frac{d}{d\epsilon}\delta f(x_* + \epsilon v)\,(w)\big|_{\epsilon=0} = \frac{d}{d\epsilon} < \nabla f(x_* + \epsilon v), w > \big|_{\epsilon=0} \\ &=< \frac{d}{d\epsilon}\nabla f(x_* + \epsilon v)\big|_{\epsilon=0}, w >=< \delta\big(\nabla f(x_*)\big)(v), w > \\ &=< Av, w >, \end{aligned}$$

which finishes the proof. □

Remark 1.1.8. *Of course, if the inner product depends on a point, we may get an extra term coming from the variation of the inner product itself. However, it is expected that the contribution from the inner product vanishes, i.e., the inner product can be differentiated by the usual "product rule", for appropriate inner products. We will see that in the case of the surface diffusion flow the contribution indeed vanishes.*

This reminds us of the Levi-Civita (or Riemannian) connection, where one is able to differentiate the inner product by the usual "product rule", i.e.,

$$\frac{d}{dt} < V, W >=< \frac{DV}{dt}, W > + < V, \frac{DW}{dt} >, \quad t \in I\,.$$

Here $\frac{DV}{dt}$ *is the covarient derivative of the vector field* V *along the differentiable curve* $c : I \to M$ *in a Riemannian manifold* M*.*

Corollary 1.1.9. *Let $u \in \mathbb{R}^n$. Then*

$$< A^2 u, u >= I(Au, u) = \|Au\|^2 . \tag{1.1.4}$$

Proof. Set $v = Au$, $w = u$ in the identity (1.1.3) to derive the first identity and the second identity follows by setting $v = u$ and $w = Au$. □

As an important corollary, we obtain

Corollary 1.1.10. *The eigenvalue 0 is semi-simple.*

Proof. The semi-simplicity condition (iii) is equivalent to the condition

$$N(A) = N(A^2) ,$$

which is an immediate consequence of the identity (1.1.4). □

Note that we have not assumed variational stability of the equilibria to show the semi-simplicity condition. We get the semi-simplicity condition for free, if we have an appropriate metric (inner product).

Corollary 1.1.11. *If u is an eigenvector of A with respect to the eigenvalue λ, i.e., $Au = \lambda u$ then*

$$\lambda I(u, u) = \|Au\|^2. \tag{1.1.5}$$

Next we prove:

Lemma 1.1.12. *Assume x_* is variationally stable. Then assertion (iv) of Theorem 1.1.6 is valid.*

Proof. Let $\lambda \in \sigma(A) \setminus \{0\}$. According to the identity (1.1.5), λ is real. Now let λ be an eigenvalue with a corresponding eigenvector $u \in \mathbb{R}^n$. Since x_* is variationally stable we get $(I(x_*)(u, u) =)I(u, u) \geq 0$. Now assume $I(u, u) = 0$. In view of the identity (1.1.5), we obtain $Au = 0$. Therefore $u \in N(A)$, i.e., $\lambda = 0$, a contradiction. Thus $I(u, u) > 0$ for the eigenvector u. Now $\lambda > 0$ by (1.1.5). This proves (iv).

□

We will also prove in an analogous manner assertions (iii) and (iv) for double bubble problems. By proving the other two assertions of the generalized principle of linearized stability we are able to prove the dynamical stability of planar double bubbles as a main result of this thesis, see the introduction below.

Remark 1.1.13. *Another question is: Are local minima of the cost function f dynamically stable under the gradient flow (1.1.2)? In general, the answer is negative. Indeed, counter examples are provided in [3, Proposition 2]. However it is also shown that under the analyticity assumption, local minimality becomes a necessary and sufficient condition for dynamical stability. The proof relies on an inequality by Łojasiewicz.*

1.2 The problem, objectives and main results

In this thesis, we consider as a cost function the length functional of a network of curves (creating a Banach manifold). Moreover, we consider a specific inner product (varying smoothly from point to point) ensuring that some area constraints are satisfied. The main objective is to prove that the stationary solutions of the form "standard planar double bubbles" are dynamically stable under the gradient flow "the surface diffusion flow".

To achieve this we first need to extend the generalized principle of linearized stability (GPLS) to the setting which allows us to deal with double bubbles. Afterwards, as a first application of this extended version of GPLS, we prove the dynamical stability of lens-shaped networks under the surface diffusion flow. See Section 1.2.1 for an introduction to these works.

Finally, we apply our extended version of GPLS to show that the standard planar double bubbles are stable under the surface diffusion flow. Exactly at this point we observed that the non-negativity of the bilinear form (variational stability) and having appropriate inner product (See Remark 1.1.8) play an important role in verifying assertions of GPLS as illustrated in our fundamental example above. We then conjecture that the standard planar double bubbles are stable under smooth enough appropriate gradient flows. See Section 1.2.2 for an introduction to this work.

1.2.1 Generalized principle of linearized stability

This work is motivated by the appearance of nonlocal, nonlinear terms (highest order point evaluations) together with general nonlinear boundary conditions when studying the stability for a fourth-order geometric flow, the surface diffusion flow, with triple junctions.

There are several questions arising doing this study: Which setting for function spaces can be used for the system of partial differential equations (PDE) arising from such geometric problems? Which class of nonlinear parabolic systems can model it? Having in mind that we should also take care of nonlinear boundary conditions, finally do the well-known theorems about stability cover such a general problem?

Let us first look closely to the nature of geometric problems. In most geometric flows, the stationary solutions are invariant under translation and under dilation. (This is the case for example for the volume preserving mean curvature flow and for the surface diffusion flow.) Therefore, typically, we are in a situation where the set of stationary solutions creates locally a smooth finite-dimensional manifold. A simple approach for proving stability for such problems is the generalized principle of linearized stability (GPLS).

Such an approach was introduced by Prüss, Simonett and Zacher [35] for abstract quasilinear problems and also for vector-valued quasilinear parabolic systems with vector-valued nonlinear boundary conditions in the framework of L_p-optimal regularity. This approach was extended in [36] to cover a wider range of settings and a wider range of classes of nonlinear parabolic equations, including fully nonlinear equations but just for abstract evolution equations, i.e., without nonlinear boundary conditions.

However, for geometric flows with triple junctions, because of the highest-order point evaluation in the corresponding parabolic system (due to the movement of the triple junction), one cannot work in a standard L_p-framework, as e.g. in [35]. Moreover, the general nonlinear boundary conditions (due to the contact, angle, curvature and flux conditions) prevent an application of the results of Prüss et al. in [36], which deal with abstract evolution equations in general function spaces.

The purpose is to extend the approaches given in [35, 36] to cover fully nonlinear parabolic systems with general nonlinear boundary conditions in parabolic Hölder spaces. Within this classical setting, i.e., the parabolic Hölder setting, we are allowed to deal with those nonlocal terms.

We have achieved our desired objective which we summarize here: Suppose that for a fully nonlinear parabolic system with general nonlinear boundary conditions we have a finite-dimensional C^2-*manifold of equilibria* $\mathcal{E}$ such that at a point $u_* \in \mathcal{E}$, the null space $N(A_0)$ of the linearization A_0 is given by the tangent space of $\mathcal{E}$ at u_*, zero is a semi-simple eigenvalue of A_0, and the rest of the spectrum of A_0 is stable. Under these assumptions our main result states that the solution with initial data close to u_* exist globally in the classical sense and converges towards the manifold of equilibria, i.e., to some point on $\mathcal{E}$ as time tends to infinity, at an exponential rate. In other words, it is dynamically stable. We published this result in [1].

It is worth noting that for the surface diffusion flow for closed hypersurfaces Escher, Mayer and Simonett [16] used *center manifold theory* to deal with this situation. In fact they showed that the dimension of the set of equilibria coincides with the dimension of the center manifold which then implies that both sets have to coincide. This then implies dynamical stability. Typically it is difficult to apply the theory of center manifolds and this is in particular true for parabolic equations involving highly nonlinear boundary conditions.

Outline. In Section 3.1 we formulate the problem and in Section 3.2 we state and prove our main result, i.e., Theorem 3.2.1. The proof depends upon results for the asymptotic behavior of linear systems which are given in Chapter 4. In this direction, extending the result stated in [28], we construct explicitly an extension operator for the case of vector-valued unknowns (see Section 4.2).

As a first application of this extended version of GPLS we show in Chapter 5 that the lens-shaped networks generated by circular arcs are stable under the surface diffusion flow. Indeed the lens-shaped networks are the simplest examples of the more general triple junctions where the resulting PDE has nonlocal terms in the highest order derivatives, see (5.2.8) and

(5.2.17). Therefore we work in function spaces which yield classical solutions.

The proof of the main theorem follows [35, 36], i.e., it is based on reducing the system to its "normal form" by means of spectral projections. However, there are differences mainly coming from the different natures of the function spaces used: Obviously, the assumption (A_2) in [36], used to get the estimates on functions T and R, see (3.2.7) below, needed for applying the assumption (A_4) in [36], is not satisfied in the parabolic Hölder setting. To overcome this difficulty we have derived these estimates directly from the smoothness assumptions on the nonlinearities, see Proposition 3.2.9 below (cf. [25, Proposition 10]). Moreover, in the parabolic Hölder setting we have

$$\mathbb{E}_1(J) = C^{1+\frac{\alpha}{2m}}(J, X) \cap B(J, X_1) ,$$

which is clearly not continuously embedded in $C(J, X_1)$, i.e., the condition (A_1) in [36] is violated. As a result we have to give more arguments in step (f) of our proof, based on the existence theorem on an arbitrary large time interval, see Proposition 3.2.10 below. Furthermore, as mentioned before, we need to show the asymptotic behavior for linear inhomogeneous systems in parabolic Hölder spaces whose counterpart is available in the L_p-setting.

1.2.2 Planar double bubbles

The standard double bubble is stable in the sense that the second variation of the area functional is non-negative, i.e., it is variationally stable. This follows for example from the fact that it is a local minimum of the area functional under volume constraints. It is however an open problem whether the standard double bubble is dynamically stable under volume conserving gradient flows such as the surface diffusion flow.

The related problem for one bubble has been studied by Escher, Mayer and Simonett, see [16, 17], who showed that spheres are dynamically stable under the surface diffusion flow and the volume preserving mean curvature flow. In this work we show that the standard double bubble in $\mathbb{R}^2$ is dynamically stable under the surface diffusion flow. In case of equal areas the result is illustrated in Figure 1.1. This result is submitted for publication, see [2].

Before moving on to define the problem more precisely, let us make one point clear: Consider a (cost) functional having local minimizers. Even though minimizers exist it is not clear that an associated gradient flow will converge to these minimizers, see [3] for ODE examples. In other words, if a stationary state of the associated gradient flow is a local minimum, this in general does not imply dynamical stability of this equilibrium under the flow.

As just mentioned, the surface diffusion flow is the volume preserving gradient flow of the area functional. Indeed, it is the fastest way to decrease

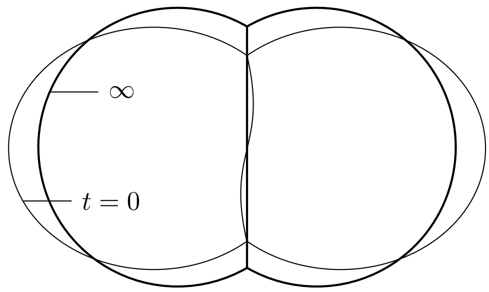

Figure 1.1: An illustration of the dynamical stability of standard planar double bubbles, possibly up to isometries. (cf. the cover page to G. Prokert's PhD thesis [33])

area while preserving the volume w.r.t. the H^{-1}-inner product; see e.g. [30, 40, 19] and the discussion in Section 2.2.3. Let us now define the flow precisely. A surface is evolving in time under *the surface diffusion flow* if its normal velocity is equal to the negative surface Laplacian of its mean curvature at each point, that is, if a surface $\Gamma(t)$ satisfies

$$V(t) = -\Delta_{\Gamma(t)} H_{\Gamma(t)} \,. \tag{1.2.1}$$

Here V stands for the normal velocity, H is the mean curvature, and Δ is the Laplace-Beltrami operator, of the surface $\Gamma(t)$. Surfaces with constant mean curvature are stationary solutions of the flow (2.1.1). This flow leads to a fourth order PDE. Thereby one may try to use PDE theories to answer the question of the dynamical stability. Indeed we employ the extended version of GPLS.

We will see that the non-negativity of the second variation of the area functional and having an appropriate inner product play an important role in verifying at least two of these assertions for the double bubble problem.

Let us note that the center manifold theory is used in [16, 17] to prove the dynamical stability of spheres under the surface diffusion flow and the volume preserving mean curvature flow. We remark that sofar no center manifold theory exists in the case of non-homogenous boundary conditions. Due to the triple junctions, we indeed get nonlinear boundary conditions in the corresponding PDE.

Outline. In Section 6.1 we precisely define the problem which we summarize here: Let Γ^0 be an initial planar double bubble. We suppose that Γ^0 moves according to the surface diffusion flow including certain boundary conditions on the triple junctions. We continue by observing that the set of stationary solutions consists precisely of all standard planar double bubbles.

Next we transfer, via suitable parameterization, this geometric problem to a system of fully nonlinear and nonlocal partial differential equations with nonlinear boundary conditions defined on fixed domains. We then linearize this nonlinear system. This is done in Section 6.2.

In Section 6.3.1 we rewrite this nonlinear system as a perturbation of the linearized problem. We then see how suitably the problem fits to the extended version of GPLS setting which is summarized in Section 3.1.

It then remains to check the conditions of normal stability. Let us note here that understanding the geometric interpretations of the problem was of great help. Semi-simplicity is proved in Section 6.3.5 in an analogous way to the ODE case by driving an identity similar to the identity (1.1.3). Lemma 6.3.13 proves assertion (iv). Like our ODE example, the non-negativity of the second variation is the main ingredient in the proof. We prove assertion (i) in Section 6.3.4 and Corollary 6.3.26 proves assertion (ii). By applying the extended version of GPLS we then complete the proof of the stability, as summarized in Section 6.4.

We continue in Section 6.4.1 to discuss general area preserving geometric flows. We then conjecture that the standard planar double bubbles are dynamically stable under any smooth appropriate gradient flow, see Conjecture 6.4.2.

In addition, Appendix A.1 shows that the second variation is negative for two elements of the basis of the null space which correspond to non-area preserving perturbations.

Chapter 2

Preliminaries

In this relative short chapter we give some basic concepts and background for the research presented in this thesis. I have tried to explain the ideas and the proofs within simple context.

2.1 The surface diffusion flow

A nice reference for the material presented in this section is the PhD thesis of Depner [11, Chapter 2]. See the references given there too.

A planar curve is evolving in time under *the surface diffusion flow* if the normal component of its velocity is equal to the negative surface Laplacian of its curvature at each point and time. Let us be more precise. A curve $\Gamma(t) \subset \mathbb{R}^2$ evolves due to the *surface diffusion flow* if

$$V(t) = -\Delta_{\Gamma(t)}\kappa_{\Gamma(t)} \qquad \Gamma(0) = \Gamma^0, \tag{2.1.1}$$

for Γ^0 being an initial planar curve. Here $V(t)$ stands for the normal velocity, $\kappa_{\Gamma(t)}$ is the curvature, $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator, of the curve $\Gamma(t)$.

Remark 2.1.1. *Throughout this thesis, we say that the curve has positive curvature if it is curved in the direction of the normal. In other words, our sign convention is that κ is negative for circles for which we choose the outer unit normal.*

For a reader's convenience we remind:

Definition 2.1.2. *Let us fix a point $p \in \Gamma(t)$ and consider the curve*

$$c : (t-\epsilon, t+\epsilon) \to \mathbb{R}^2$$

with $c(\tau) \in \Gamma(\tau)$ and $c(t) = p$. Then we define the normal velocity of the evolving curve $\big(\Gamma(t)\big)_{t\in\mathbb{R}}$ at (t,p) by

$$V(t,p) := n(t,p) \cdot \frac{d}{d\tau}c(\tau)\Big|_{\tau=t},$$

where $n(t,p)$ is the unit normal vector of $\Gamma(t)$ at the point $p \in \Gamma(t)$.

Observe that the normal velocity and the curvature depend on the choice of the normal but the flow (2.1.1) does not. Therefore, we have the freedom to choose whichever normal we like. Furthermore, note that the Laplace-Beltrami operator on a curve is just the second derivative operator based on the arc-length parameterization of the curve.

Remark 2.1.3. *In general dimension, κ must be understood as the mean curvature of a hypersurface. Moreover the surface diffusion flow is also called curve diffusion flow in the case of curves.*

Geometric properties

Obviously every closed (without boundary) stationary solution of the flow (2.1.1), i.e., the solution for which $V = 0$, has constant curvature. Hence a stationary solution of the flow (2.1.1) is either a circular arc or a line segment.

Before showing that the motion by surface diffusion flow is length decreasing and area preserving we need some preparation:

Definition 2.1.4. *Take a point $p \in \partial\Gamma(t)$ and let $c : (t-\epsilon, t+\epsilon) \to \mathbb{R}^2$ be a curve with $c(t) = p$ and $c(\tau) \in \partial\Gamma(\tau)$. Then the normal boundary velocity $\nu_{\partial\Gamma}$ is defined as*

$$\nu_{\partial\Gamma}(t,p) := n_{\partial\Gamma}(t,p) \cdot \frac{d}{d\tau} c(\tau)\Big|_{\tau=t},$$

where $n_{\partial\Gamma}(t,p)$ is the outer unit conormal of $\Gamma(t)$ at $p \in \partial\Gamma(t)$.

See Figure 2.1 for the illustration of $n_{\partial\Gamma}(t,p)$.

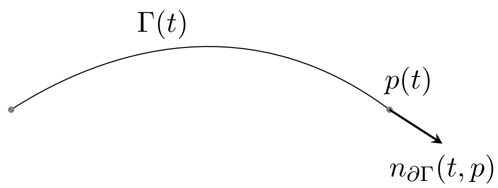

Figure 2.1: The outer unit conormal $n_{\partial\Gamma}$

Lemma 2.1.5 (Formulas for the time derivative of length)**.** *Let $\Gamma(t) \subset \mathbb{R}^2$ be an evolving curve. Then*

$$\frac{d}{dt} \int_{\Gamma(t)} 1 \, \mathrm{d}s = -\int_{\Gamma(t)} V(t)\, \kappa_{\Gamma(t)} \, \mathrm{d}s + \int_{\partial\Gamma(t)} \nu_{\partial\Gamma} .$$

Proof. Apply the Transport theorem (see e.g. [11, Theorem 2.44]) to the function $f \equiv 1$. □

Throughout this work, the integral over $\partial\Gamma = \{p_-, p_+\}$ should be understood as a sum over its elements.

Lemma 2.1.6 (Formulas for the time derivative of area)**.** *Suppose an evolving curve $\Gamma(t) \subset \mathbb{R}^2$ encloses the connected region $\Omega(t)$. Then*

$$\frac{d}{dt}\int_{\Omega(t)} 1 \, \mathrm{d}x = \int_{\Gamma(t)} V(t) \, \mathrm{d}s,$$

where the normal is chosen to be the outer unit normal.

The proof can be found for instance in [11, Lemma 2.46]. Throughout this work we need an identity which can be derived directly from the divergence theorem on curves (see e.g. [11, Theorem 2.29]):

Lemma 2.1.7 (Greens formula)**.** *Let $\Gamma \subset \mathbb{R}^2$ be a bounded curve with the boundary $\partial\Gamma$. Then it holds*

$$\int_\Gamma g \, \Delta_\Gamma f \, \mathrm{d}s = -\int_\Gamma \nabla_\Gamma \, g \cdot \nabla_\Gamma f \, \mathrm{d}s + \int_{\partial\Gamma} g \, (\nabla_\Gamma f \cdot n_{\partial\Gamma}) \tag{2.1.2}$$

for smooth functions $f, g : \Gamma \to \mathbb{R}$.

Now assume that $\Gamma(t) \subset \mathbb{R}^2$ is a smooth, closed, immersed solution to the flow (2.1.1). Let $l(t)$ denote the length of the curve $\Gamma(t)$. Then

$$\begin{aligned}\frac{d}{dt}l(t) &= \frac{d}{dt}\int_{\Gamma(t)} 1 \, \mathrm{d}s = -\int_{\Gamma(t)} V(t) \, \kappa_{\Gamma(t)} \, \mathrm{d}s \\ &= \int_{\Gamma(t)} [\Delta_{\Gamma(t)}\kappa_{\Gamma(t)}]\kappa_{\Gamma(t)} \, \mathrm{d}s = -\int_{\Gamma(t)} |\nabla_{\Gamma(t)}\kappa_{\Gamma(t)}|^2 \mathrm{d}s \leq 0 \, ,\end{aligned}$$

where we used Lemmas 2.1.5 and 2.1.7.

Let us assume further that $\Gamma(t)$ is embedded and encloses the region $\Omega(t)$. Then, denoting by $A(t)$ the area of the region $\Omega(t)$, we get

$$\frac{d}{dt}A(t) = \frac{d}{dt}\int_{\Omega(t)} 1 \, \mathrm{d}x = \int_{\Gamma(t)} V(t) \, \mathrm{d}s = -\int_{\Gamma(t)} \Delta_{\Gamma(t)}\kappa_{\Gamma(t)} \, \mathrm{d}s = 0 \, ,$$

where we used Lemmas 2.1.6 and 2.1.7.

2.2 Gradient flows of length functional

In preparing this section the two papers by Mayer [29, 30] and the survey article by Garcke [19] were used.

2.2.1 Gradient flows in a (pre-)Hilbert space

Let H be a pre-Hilbert space with the inner product $< \cdot, \cdot >_H$. Consider a smooth function (cost functional) $\Phi : H \to \mathbb{R}$ and take any point $x \in H$.

Definition 2.2.1. *Suppose that for any $y \in B_H(x, r)$ we have*

(i) $d_y\Phi : H \to \mathbb{R}$ *is a bounded linear functional on H, and*

(ii) *there exists a unique element* $\mathrm{grad}_H\Phi(y) \in H$ *such that*

$$d_y\Phi(v) = < \mathrm{grad}_H\Phi(y), v >_H \qquad \text{for all } v \in H\,. \tag{2.2.1}$$

Then we say $x : [0, T] \to H$ is a solution of the H-gradient flow equation to the cost functional Φ if

$$x'(t) = -\mathrm{grad}_H\Phi(x(t)) \tag{2.2.2}$$

holds for all $t \in [0, T]$.

Here $B_H(x, r)$ is the open ball of radius $r > 0$ centered at $x \in H$.

Remark 2.2.2. *If H is complete, i.e., if it is a Hilbert space, then by the Riesz representation theorem the statement (ii) automatically holds.*

Note that

$$d_x\Phi(v) = \frac{d}{dt}\Phi(x(t))|_{t=0}\,,$$

where $x : I \to \mathbb{R}$ is a smooth curve in H starting at the point x at $t = 0$ with the initial velocity vector v, that is, $\frac{d}{dt}x(t)|_{t=0} = v$ and $x(0) = x$.

Now obviously we get for the gradient flow equation (2.2.2) that

$$\begin{aligned}\frac{d}{dt}\Phi(x(t)) &= d_{x(t)}\Phi\,(x'(t)) \\ &= < \mathrm{grad}_H\Phi(x(t)), x'(t) >_H \\ &= -\|\mathrm{grad}_H\Phi(x(t))\|^2 \leq 0\,.\end{aligned}$$

That is, the cost functional Φ decreases along the solution curve $x(t)$ of the gradient flow equation (2.2.2) as time evolves. On the other hand for any curve $y(t)$ in H with $\|y'(0)\| = \|\mathrm{grad}_H\Phi(x(0))\|$ and $y(0) = x(0)$ we get, using the Cauchy-Schwarz inequality

$$\begin{aligned}\frac{d}{dt}\Phi(y(0)) &= < \mathrm{grad}_H\Phi(y(0)), y'(0) >_H \\ &= < \mathrm{grad}_H\Phi(x(0)), y'(0) >_H \\ &\geq\ -\|\mathrm{grad}_H\Phi(x(0))\|^2\,,\end{aligned}$$

and the equality holds only when

$$y'(0) = -\mathrm{grad}_H\Phi(x(0))\,.$$

That is, the gradient direction $-\mathrm{grad}_H\Phi(x(0))$ decreases the cost functional Φ most efficiently among all possible directions with respect to the inner product $< \cdot, \cdot >_H$.

2.2.2 L^2-gradient flow of the length functional

Do we need to extend Definition 2.2.1 to Hilbert Manifolds? In the following we try to answer this question. Fix a smooth simple closed curve Γ and consider a pre-Hilbert space

$$H = C^\infty(\Gamma) \subset L^2(\Gamma)\,,$$

equipped with the L^2-inner product

$$< v_1, v_2 >_H := < v_1, v_2 >_{L^2(\Gamma)} = \int_\Gamma v_1 v_2\,.$$

Remind that

$$L^2(\Gamma) = \Big\{ u : \Gamma \to \mathbb{R} : u \text{ is measurable and } \int_\Gamma u^2 < \infty \Big\}.$$

Note that the space H is endowed with the inner product $L^2(\Gamma)$, that is why we have not denoted it as $C^\infty(\Gamma)$. Take as a cost function the length functional, i.e.,

$$\Phi : H \to \mathbb{R}, \quad \Phi(u) = \int_{\Gamma_u} 1\,.$$

Here Γ_u is a closed curve defined as

$$\Gamma_u := \{\, \sigma + u(\sigma)\, n_\Gamma(\sigma) : \sigma \in \Gamma \,\}\,,$$

where notice that $\Gamma_{u\equiv 0} = \Gamma$.

Then we get for any curve $u(t)$ in H with $u(0) \equiv 0 \in H$, and the initial velocity vector $u'(t)\big|_{t=0} = v \in H$, using Lemma 2.1.5, that

$$\begin{aligned} d_{u(0)}\Phi(v) &= \frac{d}{dt}\Phi(u(t))\big|_{t=0} = \frac{d}{dt}\int_{\Gamma_{u(t)}} 1 \,\Big|_{t=0} \\ &= -\int_\Gamma \kappa_{\Gamma_{u(0)}} V_{\Gamma_{u(0)}} = - < \kappa_{\Gamma_{u(0)}}, V_{\Gamma_{u(0)}} >_{L^2(\Gamma_{u(0)})}\,. \end{aligned}$$

Here $V_{\Gamma_{u(0)}}$ denotes the normal velocity of $\{\Gamma_{u(t)};\, t \geq 0\}$ at time $t = 0$ and $\kappa_{\Gamma_{u(0)}}$ is the curvature of the curve $\Gamma_{u(0)} = \Gamma$.

On the other hand, using the fact that $V_{\Gamma_{u(0)}} = u'(t)\big|_{t=0} = v$, we obtain

$$d_{u(0)}\Phi(v) = - < \kappa_\Gamma, v >_H \qquad \text{for all } v \in H\,.$$

We similarly get at any other points $u(t)$,

$$d_{u(t)}\Phi(v) = - < \kappa_{\Gamma_{u(t)}}, v >_{L^2(\Gamma(t))} \text{ (for all } v \in C^\infty(\Gamma(t)))\,.$$

As you can see now the inner product changes from point to point. In other words, one may need to work with Hilbert manifolds where the inner product depends smoothly on the point.

If one allows the dependency of the inner product on the point, then one may get that the L^2-gradient flow of the length functional reads as

$$V_{\Gamma_{u(t)}} = u'(t) = -\mathrm{grad}_{L^2(\Gamma(t))}\Phi(u(t)) = \kappa_{\Gamma_{u(t)}} .$$

One may arrive at the well known *mean curvature flow*

$$V_{\Gamma(t)} = \kappa_{\Gamma(t)}$$

as the L^2-gradient flow of the length functional by using the fact that locally any closed curve $\Gamma(t)$ has the form $\Gamma_{u(t)}$ for some $u(t)$.

Remark 2.2.3. *Of course some works has been done to clarify this issue, see e.g. [29, 30, 19]. But the precise analysis of the problem is lacking. For instance one should justify that the set of all smooth closed curves is a Hilbert manifold and determine its tangent space.*

We do not touch this problem here as this topic exceeds the scope of this thesis.

2.2.3 H^{-1}-gradient flow of the length functional

Now let us again fix a closed curve Γ and consider this time the pre-Hilbert space defined as

$$H = \left\{ u : \Gamma \to \mathbb{R} : u \text{ is smooth and } \int_\Gamma u = 0 \right\}$$

equipped with the H^{-1}-inner product

$$(v_1, v_2)_{H^{-1}} := \int_\Gamma v_1 (-\Delta_\Gamma)^{-1} v_2 .$$

Here the linear operator $(-\Delta_\Gamma)^{-1} : H \to C^\infty(\Gamma)$ is defined by

$$(-\Delta_\Gamma)^{-1} v = u \qquad \text{if } -\Delta_\Gamma u = v .$$

Remark 2.2.4. *Due to Green's formula 2.1.7, we get*

$$\int_\Gamma \Delta_\Gamma u = 0$$

since Γ has no boundary. Therefore as the solvability condition for the equation $-\Delta_\Gamma u = v$, we obtain $\int_\Gamma v = 0$ which is fulfilled since $v \in H$.

Take as a cost function the length functional, i.e.,

$$\Phi : H \to \mathbb{R}, \quad \Phi(u) = \int_{\Gamma_u} 1\,.$$

where Γ_u is defined as before.

Similarly we get for any curve $u(t)$ in H with $u(0) \equiv 0 \in H$, and the initial velocity vector $u'(t)\big|_{t=0} = v \in H$, using Lemma 2.1.5, that

$$\begin{aligned} d_{u(0)}\Phi(v) &= \frac{d}{dt}\Phi(u(t))\big|_{t=0} = \frac{d}{dt}\int_{\Gamma_{u(t)}} 1\,\Big|_{t=0} \\ &= -\int_{\Gamma} \kappa_{\Gamma_{u(0)}} V_{\Gamma_{u(0)}} = <\Delta_{\Gamma_{u(0)}}\kappa_{\Gamma_{u(0)}}, V_{\Gamma_{u(0)}}>_{H^{-1}(\Gamma_{u(0)})}\,. \end{aligned}$$

On the other hand, using the fact that $V_{\Gamma_{u(0)}} = u'(t)\big|_{t=0} = v$, we obtain

$$d_{u(0)}\Phi(v) = <\Delta_\Gamma \kappa_\Gamma, v>_H \qquad \text{for all } v \in H\,.$$

We similarly get at any other points $u(t)$,

$$d_{u(t)}\Phi(v) = <\Delta_{\Gamma_{u(t)}}\kappa_{\Gamma_{u(t)}}, v>_{L^2(\Gamma(t))} \qquad \text{for all } v \in C^\infty(\Gamma_{u(t)})\,.$$

Following the discussion in the previous section, one may arrive at the well known *surface diffusion flow*

$$V_{\Gamma(t)} = -\Delta_{\Gamma(t)}\kappa_{\Gamma(t)}$$

as the H^{-1}-gradient flow of the length functional.

2.3 Functional analysis

Let X, Y, and Z are Banach spaces over $\mathbb{R}$. For the proof of the following theorem we refer the reader to Zeidler [42, Theorem 4.B].

Theorem 2.3.1 (Implicit function theorem of Hildebrandt and Graves (1927))**.** *Suppose that:*

(i) *The mapping $F : U(x_0, y_0) \subseteq X \times Y \to Z$ is defined on an open neighborhood $U(x_0, y_0)$ of (x_0, y_0), and $F(x_0, y_0) = 0$.*

(ii) *F_y exists as a partial Fréchet derivative on $U(x_0, y_0)$ and the inverse operator, $F_y(x_0, y_0)^{-1} : Z \to Y$, exists as a continuous linear operator.*

(iii) *F is a C^m-map, $1 \le m \le \infty$ on a neighborhood of (x_0, y_0).*

Then there exist neighborhoods $U = B_X(x_0, r_1)$ and $V = B_Y(y_0, r_2)$ and a C^m-function

$$\begin{aligned} y : U &\to V \\ x &\mapsto y(x)\,, \end{aligned}$$

such that $F(x, y(x)) = 0$ and for all $(x, y) \in U \times V$ it holds $F(x, y) = 0$ if and only if $y = y(x)$.

2.3.1 Semi-simple eigenvalue

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator. Moreover we denote by $\mathcal{L}(X)$ the space of bounded linear operators on the Banach space X.

Definition 2.3.2. *The resolvent set of A is*

$$\rho(A) = \{ \lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is one-to-one, onto, and } (\lambda I - A)^{-1} \in \mathcal{L}(X) \},$$

and the spectrum of A is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

In other words, $\lambda \in \rho(A)$ if and only if there exists an inverse operator $(\lambda I - A)^{-1} : X \to D(A) \subset X$ bounded as an operator on X. Note that, to verify the boundedness of the operator

$$(\lambda I - A)^{-1} : X \to D(A) \subset X \,,$$

one has to show that there exists a constant $c > 0$ such that

$$\|(\lambda I - A)^{-1} f\|_X \leq c\|f\|_X \quad \forall f \in X$$

or equivalently

$$\|u\|_X \leq c\|(\lambda I - A)u\|_X \quad \forall u \in D(A) \subset X \,.$$

We set

$$R(\lambda, A) := (\lambda I - A)^{-1} \qquad \text{for } \lambda \in \rho(A)$$

and we call $R(\lambda, A)$ the *resolvent operator* or simply *resolvent*.

Remark 2.3.3. *Note that we have not assumed in Definition 2.3.2 that A is a closed operator. But of course, if $\rho(A) \neq \emptyset$, then A is a closed operator.*

A complex number $\lambda \in \sigma(A)$ is said to be an *eigenvalue* of A if $(\lambda I - A)$ is not one-to-one, i.e., if $N(\lambda I - A) \neq \{0\}$.

Definition 2.3.4. *An isolated eigenvalue $\lambda \in \sigma(A)$ is called a semi-simple eigenvalue if*

$$X = N(\lambda I - A) \oplus R(\lambda I - A) \,.$$

Theorem 2.3.5 (Spectral theory of compact operators)**.** *Assume* $\dim X = \infty$*. Let $T : X \to X$ be compact. Then*

(i) $0 \in \sigma(T)$ *and $\sigma(T)$ is at most countably infinite.*

(ii) *Every nonzero $\lambda \in \sigma(T)$ is an eigenvalue of T.*

(iii) *For $\lambda \in \sigma(T) \setminus \{0\}$ the dimension of $N(\lambda I - T)$ is finite.*

(iv) *The eigenvalues can only accumulate at 0.*

(v) *For all nonzero* $\lambda \in \sigma(T)$,

$$1 \leq n_\lambda := max\{n \in \mathbb{N} \ \mid N\big((\lambda I - T)^{n-1}\big) \neq N\big((\lambda I - T)^n\big)\} < \infty .$$

(vi) *For all nonzero* $\lambda \in \sigma(T)$,

$$X = N\big((\lambda I - T)^{n_\lambda}\big) \oplus R\big((\lambda I - T)^{n_\lambda}\big) .$$

For the proof we refer the reader to [7, Theorem 9.9]. We are now ready to prove:

Lemma 2.3.6. *Let* $A : D(A) \subset X \to X$ *be a linear operator with compact resolvent and let zero be an eigenvalue of* A. *Then*

$$0 \textit{ is a semi-simple eigenvalue of } A \iff N(A) = N(A^2)$$

Proof. If zero is semi-simple, then by definition $X = N(A) \oplus R(A)$. This immediately implies $R(A) \cap N(A) = \{0\}$ or equivalently $N(A) = N(A^2)$. Now let us suppose $N(A) = N(A^2)$. As the operator A has compact resolvent, there exists $0 \neq \lambda \in \rho(A)$ such that $R(\lambda, A)$ is compact. Moreover the compactness implies that zero is an isolated eigenvalue of A. On the other hand we have

$$\begin{aligned} N\Big(\big(\frac{1}{\lambda}I - R(\lambda, A)\big)^i\Big) &= N\Big((\lambda I - A)^i\big(\frac{1}{\lambda}I - R(\lambda, A)\big)^i\Big) \\ &= N\Big(\big[(\lambda I - A)\big(\frac{1}{\lambda}I - R(\lambda, A)\big)\big]^i\Big) \\ &= N\Big(\big(\frac{1}{\lambda}(\lambda I - A) - I\big)^i\Big) \\ &= N(A^i) \qquad i = 1, 2, \dots , \end{aligned}$$

where we used the fact that $(\lambda I - A)$ is invertible and commutes with the operator $\big(\frac{1}{\lambda}I - R(\lambda, A)\big)$. Thus

$$N\Big(\frac{1}{\lambda}I - R(\lambda, A)\Big) = N(A) = N(A^2) = N\Big(\big(\frac{1}{\lambda}I - R(\lambda, A)\big)^2\Big) .$$

Now applying Theorem 2.3.5 (vi) we obtain

$$X = N\Big(\frac{1}{\lambda}I - R(\lambda, A)\Big) \oplus R\Big(\frac{1}{\lambda}I - R(\lambda, A)\Big) = N(A) \oplus R(A) ,$$

where we used the fact that $R\Big(\frac{1}{\lambda}I - R(\lambda, A)\Big) = R(A)$ which is easy to check. This finishes the proof. $\square$

Remark 2.3.7. *It is a well-known fact that the operators with compact resolvent are Fredholm with index* 0. *On the other hand, note that if* 0 *is a semi-simple eigenvalue of* A, *then* A *is a Fredholm operator of index* 0. *However, the converse is not in general true.*

A good source for more on semi-simple eigenvalues is the Appendix of Lunardi [26].

2.3.2 Finite-dimensional manifolds on Banach spaces

Let X be a Banach space and let N be a k-dimensional subspace of X.

Lemma 2.3.8. *There exists a linear continuous projection*

$$P : X \to N\,.$$

Proof. Apply the Hahn-Banach theorem to the bounded linear map

$$\alpha_i : N \to \mathbb{R}\,, \quad \alpha_i(v_j) = \delta_{ij} \qquad \text{for } i, j = 1, \dots, k\,,$$

where $\{v_1, \dots, v_k\}$ is a basis of N and δ_{ij} is the Kronecker delta.

Now define the projection $P : X \to N$ by

$$Px = \sum_{i=1}^{k} \alpha_i(x) v_i\,, \quad x \in X\,,$$

where $\alpha_i : X \to \mathbb{R}$ are now the extensions obtained by the Hahn-Banach theorem. □

There are several equivalent precise formulations of a finite-dimensional manifold on a Banach space X:

Definition 2.3.9. *Let $\mathcal{E}$ be a subset of a Banach space X. Then $\mathcal{E}$ is a k-dimensional manifold of class C^m if for every point $u \in \mathcal{E}$ there exists an open neighbourhood $V \subseteq X$ of u such that one of the following equivalent statements holds:*

1. *Diffeomorphism: There is an open neighbourhood $U \subseteq X$ of u, and a C^m-diffeomorphism $F : U \to V$ such that $F\big(N \cap (U - u)\big) = \mathcal{E} \cap V$.*
2. *Parameterization: There is an open neighbourhood $U \subseteq \mathbb{R}^k$ of 0 and a C^m-function $\Psi :\ U \to X$, such that $\Psi(U) = \mathcal{E} \cap V$, $\Psi(0) = u$, and the rank of $\Psi'(0)$ equals k (i.e., $Im(\Psi'(0)) \cong N$).*
3. *Graph: There is an open neighborhood $U \subseteq N$ of 0 and a C^m-map $\phi : (u + U) \to (I - P)X$ such that $\mathcal{E} \cap V$ is the graph of ϕ, where we assume without loss of generality $\phi'(u) = 0$.*

Indeed, in the following we prove that the statements (1)–(3) are equivalent, where without loss of generality we assume $u \equiv 0$.

Proof 2 $\implies$ 3. Since N and $Im(\Psi'(0))$ are isomorphic, we take

$$N = Im(\Psi'(0)) \qquad (\dim N = k).$$

Apply the projection $P : X \to N$ to define the mapping

$$g := P\Psi : U \subset \mathbb{R}^k \to N\,.$$

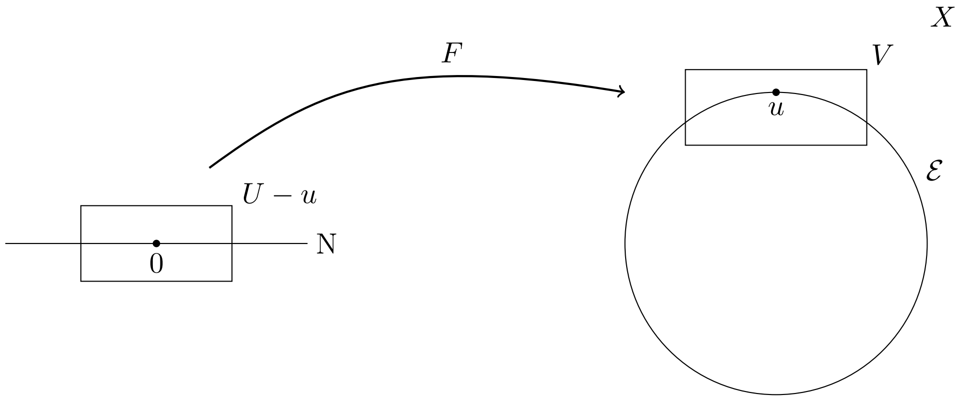

Figure 2.2: Definition of a manifold based on parameterization.

Obviously $g'(0) = P\Psi'(0) : \mathbb{R}^k \to N$ is surjective. Now it follows immediately that $g'(0)$ is bijective as $\dim \mathbb{R}^k = \dim N = k$. Thus, applying the inverse function theorem, we conclude g is a C^m-diffeomorphism of a neighborhood of 0 in $\mathbb{R}^k$ onto $\widetilde{U}$a neighborhood of 0 in N. See the diagram below, roughly illustrating the idea.

$$\begin{tikzcd} U \arrow[r, "\Psi"] & \mathcal{E} \arrow[d, "P"] \\ & N \arrow[ul, "g^{-1}"] \end{tikzcd}$$

Figure 2.3: The diagram used in the proof of 2 $\implies$ 3.

Next we define $\Phi(v) := \Psi(g^{-1}(v))$ for $v \in \widetilde{U}$. Therefore

$$\{\, \Phi(v) : \; v \in \widetilde{U} \,\} = \mathcal{E} \cap W \,,$$

for some neighborhood W of $u \equiv 0$ in X. Now

$$P\Phi(v) = \big((P \circ \Psi) \circ g^{-1}\big)\,(v) = (g \circ g^{-1})(v) = v, \quad v \in \widetilde{U},$$

Hence $\Phi(v) = P\Phi(v) + (I - P)\Phi(v) = v + (I - P)\Phi(v)$ for all $v \in \widetilde{U}$. If we finally define

$$\phi(v) := (I - P)\Phi(v)$$

and use the fact that $\Psi'(0)(\mathbb{R}^k) = N$, we obtain

$$\phi \in C^m(\widetilde{U}, (I - P)X), \quad \phi(0) = \phi'(0) = 0,$$

and

$$\{\, v + \phi(v) : \; v \in \widetilde{U} \,\} = \mathcal{E} \cap W \,, \tag{2.3.1}$$

where
$$\phi'(0) = (I - P)\Phi'(0) = (I - P)\Psi'(0)\big(g^{-1}\big)'(0) = 0$$
as the image of $\Psi'(0)$ is equal to $N = PX$.

Hence we have established our assertion, i.e., near $u(\equiv 0)$ the manifold $\mathcal{E}$ can be represented as the graph over its tangent space $T_u\mathcal{E} = N$ via the function ϕ. $\square$

Proof: 3 $\implies$ 2. Trivial. Graph is a special kind of parameterization. $\square$

Proof: 3 $\implies$ 1. Let us define

$$\begin{aligned} F : X &\to X \\ z &\mapsto Pz + \phi(Pz) + (I - P)z\,. \end{aligned}$$

Now
$$F'(0) = P + \phi'(0)P + I - P = I \qquad \text{as } \phi'(0) = 0.$$

Therefore applying the inverse function theorem we get that F is locally a diffeomorphism of $\widehat{U}$ a neighborhood of 0 in X onto a neighborhood of 0 in X. Furthermore, obviously, $F(N \cap \widehat{U}) \subseteq \mathcal{E} \cap W$ by (2.3.1). $\square$

Proof: 1 $\implies$ 3. See [7, Theorem 5.5] and adapt it to Banach spaces. $\square$

Remark 2.3.10. *Note that the function Ψ defined in the statement 2 above is an immersion. Furthermore, in this work, we will use parameterizations to describe a manifold.*

See [41, Theorem 43.c] for more on manifold in Banach spaces.

Chapter 3

Generalized Principle of Linearized Stability

In this chapter we investigate the dynamical stability of normally stable equilibria for fully nonlinear parabolic systems with nonlinear boundary conditions in a parabolic Hölder spaces. We prove the so-called generalized principle of linearized stability in the parabolic Hölder setting in Theorem 3.2.1.

3.1 Fully nonlinear parabolic systems with general nonlinear boundary conditions in a parabolic Hölder setting

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2m+\alpha}$ with boundary $\partial\Omega$, where $m \in \mathbb{N}$ and $0 < \alpha < 1$. Let also $\nu(x)$ denote the outer normal of $\partial\Omega$ at $x \in \partial\Omega$. We consider the nonlinear boundary value problem

$$\begin{cases} \partial_t u(t,x) + A(u(t,\cdot))(x) = F(u(t,.))(x), & x \in \overline{\Omega}\,, \quad t > 0\,, \\ B_j(u(t,\cdot))(x) = G_j(u(t,.))(x), & x \in \partial\Omega\,, \quad j = 1,\ldots,mN\,, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}\,, \end{cases} \tag{3.1.1}$$

where $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}^N$ and A is a linear $2m$th-order differential operator of the form

$$(Au)(x) = \sum_{|\gamma|\le 2m} a_\gamma(x)\nabla^\gamma u(x)\,, \quad x \in \overline{\Omega}\,.$$

Moreover, B_j are linear differential operators of order m_j,

$$(B_j u)(x) = \sum_{|\beta|\le m_j} b^j_\beta(x)\nabla^\beta u(x)\,, \quad x \in \partial\Omega\,, \quad j = 1,\ldots,mN\,.$$

Here the coefficients $a_\gamma(x) \in \mathbb{R}^{N\times N}$, $b^j_\beta(x) \in \mathbb{R}^N$ and

$$0 \le m_1 \le m_2 \le \cdots \le m_{mN} \le 2m-1\,.$$

Furthermore $n_j \ge 0$ denotes the number of jth-order boundary conditions for $j = 0, \dots, 2m-1$.

We now follow [10, 27] in making the following assumptions on the fully nonlinear terms F and G_j as well as on the smoothness of the coefficients:

(H1) $F : B(0,R) \subset C^{2m}(\overline{\Omega}) \to C(\overline{\Omega})$ is C^1 with Lipschitz continuous derivative, $F(0) = 0, F'(0) = 0$, and the restriction of F to $B(0,R) \subset C^{2m+\alpha}(\overline{\Omega})$ has values in $C^\alpha(\overline{\Omega})$ and is continuously differentiable.

$G_j : B(0,R) \subset C^{m_j}(\overline{\Omega}) \to C(\partial\Omega)$ is C^2 with Lipschitz continuous second-order derivative, $G_j(0) = 0, G_j'(0) = 0$, and the restriction of G_j to $B(0,R) \subset C^{2m+\alpha}(\overline{\Omega})$ has values in $C^{2m+\alpha-m_j}(\partial\Omega)$ and is continuously differentiable.

(H2) The elements of the matrix $a_\gamma(x)$ belong to $C^\alpha(\overline{\Omega})$.

The elements of the matrix $b^j_\beta(x)$ belong to $C^{2m+\alpha-m_j}(\partial\Omega)$.

In assumption (H1) we have written for simplicity $C^s(K)$ instead of $C^s(K)^N$ for $K = \overline{\Omega}, \partial\Omega$. In the same way all function spaces in the following will be vector-valued with a dimension that is determined by the context.

Finally, let $B = (B_1, \dots, B_{mN})$ and $G = (G_1, \dots, G_{mN})$.

Remark 3.1.1. *Note that assumption (H1) allows for very general nonlinearities; for instance, F can depend on $D^\alpha u(x_0)$, where x_0 is a point in $\overline{\Omega}$ with $|\alpha| = 2m$, which is a nonlocal dependence.*

As one guesses from our assumptions above, we are interested in classical solutions and therefore we use the following setting:

$$X = C(\overline{\Omega}), \quad X_0 = C^\alpha(\overline{\Omega}), \quad X_1 = C^{2m+\alpha}(\overline{\Omega})\,.$$

Note that $X_1 \hookrightarrow X_0 \hookrightarrow X$. We write $|\cdot|_j$ for the norm on X_j $(j = 0, 1)$ and $|\cdot|$ for the norm on X. Additionally, let Y be a normed vector space. Then the open ball of radius $r > 0$ centered at $u \in Y$ will be denoted by $B_Y(u, r)$.

Let us now denote by $\mathcal{E} \subset B_{X_1}(0,R)$ the set of stationary solutions (equilibria) of (3.1.1), i.e.,

$$u \in \mathcal{E} \iff u \in B_{X_1}(0,R)\,,\; Au = F(u) \quad \text{in } \Omega \quad \text{and} \quad Bu = G(u) \quad \text{on } \partial\Omega\,. \tag{3.1.2}$$

It follows from assumption (H1) that $u_* \equiv 0$ belongs to $\mathcal{E}$. Although u_* is zero, we will often write u_* instead of 0 to emphasize that we deal with an equilibrium.

We follow [35] in assuming that u_* is contained in a k-dimensional manifold of equilibria, i.e., we assume that there is a neighborhood $U \subset \mathbb{R}^k$ of $0 \in U$, and a C^2-function $\Psi : U \to X_1$, such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u_* \equiv 0$,
- the rank of $\Psi'(0)$ equals k,

In addition we finally require that there are no other stationary solutions near u_* in X_1 than those given by $\Psi(U)$, i.e., for some $r_1 > 0$,

$$\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)\,.$$

Note that the condition $\Psi(U) \subset \mathcal{E}$ is equivalent to the identities

$$A\Psi(\zeta) = F(\Psi(\zeta)) \quad \text{in } \Omega, \quad \text{for all } \zeta \in U, \tag{3.1.3}$$

$$B\Psi(\zeta) = G(\Psi(\zeta)) \quad \text{on } \partial\Omega, \quad \text{for all } \zeta \in U. \tag{3.1.4}$$

The linearization of (3.1.1) at u_* is given by the operator A_0 which is the realization of A with homogeneous boundary conditions in $X = C(\overline{\Omega})$, i.e., the operator with the domain

$$D(A_0) = \Big\{ u \in C(\overline{\Omega}) \cap \bigcap_{1<p<+\infty} W^{2m,p}(\Omega) : \quad Au \in X, \quad Bu = 0 \text{ on } \partial\Omega \Big\}, \tag{3.1.5}$$

$$A_0 u = Au, \quad u \in D(A_0)\,,$$

where we used the fact that $F'(0) = G'(0) = 0$. Note that by assumption (H2), we have

$$A_0|_{C^{2m+\alpha}(\overline{\Omega})} : C^{2m+\alpha}(\overline{\Omega})\big|_{N(B)} \to C^\alpha(\overline{\Omega})\,.$$

Remark 3.1.2. *Since Ω is bounded, $D(A_0)$ is compactly embedded into $C(\overline{\Omega})$, the resolvent operators $(\lambda I - A_0)^{-1}$ are compact for all $\lambda \in \rho(A_0)$, and the spectrum $\sigma(A_0)$ consists of a sequence of isolated eigenvalues.*

Next we turn to the property of the optimal regularity in the parabolic Hölder spaces. To this end it is just enough to take care of the principal parts of the linear operators A and B, i.e.,

$$A_*(x, D) = \sum_{|\gamma|=2m} i^{2m} a_\gamma(x) D^\gamma\,,$$

$$B_{j*}(x, D) = \sum_{|\beta|=m_j} i^{m_j} b^j_\beta(x) D^\beta\,, \quad (j = 1, \ldots, mN)$$

where $D = -i\nabla$. With this notation we have $\nabla^\beta = i^{|\beta|} D^\beta$. Based on the results of V.A. Solonnikov [39], the following conditions, i.e., strong parabolicity of A_* and the Lopatinskii-Shapiro condition for (A_*, B_*) are sufficient for Hölder-optimal regularity of A_0, see Theorem VI.21 in [14]:

(SP) A is strongly parabolic: For all $x \in \overline{\Omega},\ \xi \in \mathbb{R}^n, |\xi| = 1$,

$$\sigma(A_*(x,\xi)) \subset \mathbb{C}_+ .$$

(LS) (Lopatinskii-Shapiro condition) For all $x \in \partial\Omega,\ \xi \in \mathbb{R}^n$, with $\xi \cdot \nu(x) = 0, \lambda \in \overline{\mathbb{C}_+},\ \lambda \neq 0$, and $h \in \mathbb{C}^{mN}$, the system of ordinary differential equations on the half-line

$$\begin{aligned} \lambda v(y) + A_*(x,\xi + i\nu(x)\partial_y)v(y) =& 0\,, && y > 0,\\ B_{j*}(x,\xi + i\nu(x)\partial_y)v(0) =& h_j\,, && j = 1,\dots,mN\,, \end{aligned}$$

admits a unique solution $v \in C_0\left(\mathbb{R}_0^+;\mathbb{C}^N\right)$,

where $C_0\left(\mathbb{R}_0^+;\mathbb{C}^N\right)$ is the space of continuous functions which vanish at infinity.

Remark 3.1.3. *The strong parabolicity condition, i.e., (SP) implies the root condition (cf. Amann [8, Lemma 6.1] or Morrey [32, P. 255]). Concerning the Complementing Condition (LS), here it is formulated in a non-algebraic way but one can find the equivalence of this formulation to the algebraic formulation in Eidelman and Zhitarashu [14, Chapter I.2]. See also Lemma 6.2 in [8].*

We continue by collecting the following basic results on generation of analytic semigroups, the characterization of related interpolation spaces and elliptic regularity in Hölder spaces for the associated elliptic systems:

Theorem 3.1.4. *Under the conditions (H2),(SP) and (LS) the following statements hold.*

(i) *The operator $-A_0$ is sectorial.*

(ii) *For each $\theta \in (0,1)$ such that $2m\theta \notin \mathbb{N}$, we have*

$$D_{-A_0}(\theta,\infty) = \left\{ \varphi \in C^{2m\theta}(\overline{\Omega}) : B_j\varphi = 0 \text{ if } m_j \leq [2m\theta] \right\}$$

and the $C^{2m\theta}$-norm is equivalent to the $D_{-A_0}(\theta,\infty)$-norm.

(iii) *For each $k = 1,\dots,2m-1$ we have*

$$C^k_{\mathcal{B}}(\overline{\Omega}) := \left\{ \varphi \in C^k(\overline{\Omega}) : B_j\varphi = 0 \text{ if } m_j < k \right\} \hookrightarrow D_{-A_0}(\frac{k}{2m},\infty)\,,$$

where $C^k_{\mathcal{B}}(\overline{\Omega})$ is given the norm of $C^k(\overline{\Omega})$.

(iv) *We have the inclusion*

$$\Big\{ \varphi \in \bigcap_{p>1} W^{2m,p}(\Omega) : A\varphi \in C^{\alpha}(\overline{\Omega}),\;\; B_j\varphi \in C^{2m+\alpha-m_j}(\partial\Omega)\,,$$

$$j = 1, \dots, mN \Big\} \subset C^{2m+\alpha}(\overline{\Omega})$$

and there exist a constant C such that

$$\|\varphi\|_{C^{2m+\alpha}(\overline{\Omega})} \le C\bigg(\|A\varphi\|_{C^{\alpha}(\overline{\Omega})} + \|\varphi\|_{C(\overline{\Omega})} + \sum_{j=1}^{mN} \|B_j\varphi\|_{C^{2m+\alpha-m_j}(\partial\Omega)} \bigg). \tag{3.1.6}$$

Proof. The proof is an adaptation of the proof of [28, Theorem 5.2], where the case of a single elliptic equation is proved. Concerning (i) and (ii), see [5, Remark 5.1]. (iii) follows from the characterization of $D_{-A_0}(\frac{k}{2m}, \infty)$ provided in [4], see precisely Remark 5.1 in [4]. In order to prove (iv) one uses that the results of [6] imply the estimate (3.1.6). Moreover, the inclusion in $C^{2m+\alpha}(\overline{\Omega})$ is a consequence of the existence theorems in [23, Section 5]. □

Let us now differentiate (3.1.3) and (3.1.4) w.r.t. ζ and evaluate them at $\zeta = 0$ to obtain

$$\begin{cases} A\Psi'(0) = 0 & \text{in } \Omega\,, \\ B\Psi'(0) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.7}$$

We therefore see that the range $R(\Psi'(0))$ is contained in the null space $N(A_0)$ of A_0. In other words,

$$T_{u_*}(\mathcal{E}) \subseteq N(A_0)\,, \tag{3.1.8}$$

where $T_{u_*}(\mathcal{E})$ represents the tangent space of $\mathcal{E}$ at the point u_*.

Finally we make an additional assumption on the coefficient b^j_β known as 'normality condition', which will be used in the construction of the extension operator presented in Section 4.2:

$$\begin{cases} \text{for each } x \in \partial\Omega, \text{ the matrix } \begin{pmatrix} \sum_{|\beta|=k} b^{j_1}_\beta(x)(\nu(x))^\beta \\ \vdots \\ \sum_{|\beta|=k} b^{j_{n_k}}_\beta(x)(\nu(x))^\beta \end{pmatrix} \text{ is surjective,} \\ \text{where } \{\, j_i : i = 1, \dots, n_k \,\} = \{\, j : m_j = k \,\}\,. \end{cases} \tag{3.1.9}$$

Note that $b^j_\beta(x) \in \mathbb{R}^N$ for all $x \in \partial\Omega$.

Remark 3.1.5. *In general, the normality condition* (3.1.9) *is not implied by the (L-S) condition, see e.g. [4, Remark 1.1].*

In the following, the compatibility conditions read as follows. For j such that $m_j = 0$ and $x \in \partial\Omega$

$$\begin{cases} Bu_0 = G(u_0)\,, \\ B_j(Au_0 - F(u_0)) = G_j'(u_0)(Au_0 - F(u_0))\,. \end{cases} \tag{3.1.10}$$

3.2 Generalized principle of linearized stability in parabolic Hölder spaces

This section is devoted to the statement and proof of our main theorem in this chapter on dynamical stability of stationary solutions of the nonlinear system (3.1.1) in the parabolic Hölder spaces.

Theorem 3.2.1. *Let $u_* \equiv 0 \in X_1$ be a stationary solution of* (3.1.1)*, and assume that the regularity conditions (H1), (H2), Lopatinskii-Shapiro condition (LS), strong parabolicity (SP) and finally the normality condition* (3.1.9) *are satisfied. Moreover let A_0 denote the linearization of* (3.1.1) *at $u_* \equiv 0$ defined in* (3.1.5)*, and require that u_* is normally stable, i.e., suppose that*

(i) *near u_* the set of equilibria $\mathcal{E}$ is a C^2-manifold in X_1 of dimension $k \in \mathbb{N}$,*

(ii) *the tangent space of $\mathcal{E}$ at u_* is given by $N(A_0)$,*

(iii) *the eigenvalue 0 of A_0 is semi-simple, i.e., $R(A_0) \oplus N(A_0) = X$,*

(iv) *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : Re\ z > 0\}$.*

Then the stationary solution u_* is stable in X_1. Moreover, if u_0 is sufficiently close to u_* in X_1 and satisfies the compatibility conditions (3.1.10)*, then the unique solution $u(t)$ of* (3.1.1) *exists globally and approaches some $u_\infty \in \mathcal{E}$ exponentially fast in X_1 as $t \to \infty$. In other words, u_* is dynamically stable.*

Proof. We follow the strategy of [35, 36], i.e., to reduce the system (3.1.1) to its normal form by means of a near-identity, nonlinear transformation of variables. This in turn makes it easier to analyze the system. The proof will be done in steps (a)-(g) and some intermediate results will be formulated as lemmas and propositions.

(a) According to Remark 3.1.2, $0 \in \sigma(A_0)$ is isolated in $\sigma(A_0)$ which together with assumption (iv) gives the following decomposition of

$$\sigma(A_0) = \{0\} \cup \sigma_s, \quad \sigma_s \subset \mathbb{C}_+ = \{\, z \in \mathbb{C} : \mathrm{Re}\ z > 0\,\}.$$

into two disjoint pieces.

Let P^l, $l \in \{c, s\}$, be the spectral projections associated to $\sigma_c = \{0\}$ and σ_s, i.e.,

$$P^c = \frac{1}{2\pi i}\int_\gamma R(\lambda, A_0)\,\mathrm{d}\lambda \qquad \text{and} \qquad P^s = I - P^c \tag{3.2.1}$$

(see [26, Definition A.1.1]). We set $X^l_j := P^l X_j$ and $X^l := P^l X$ for $l \in \{c,s\}$ and $j \in \{0,1\}$, equipped with the norms $|\cdot|_j$ and $|\cdot|$ respectively for $j \in \{0,1\}$. Moreover we define the part of A_0 in X^l by

$$A_l = P^l A_0 P^l \quad \text{ for } l \in \{c,s\}\,.$$

Lemma 3.2.2. $P^c|_{C^\alpha(\overline{\Omega})} \in \mathcal{L}(C^\alpha(\overline{\Omega}), C^{2m+\alpha}(\overline{\Omega}))$

Proof. At first we show $R(\lambda, A_0)|_{C^\alpha(\overline{\Omega})} : C^\alpha(\overline{\Omega}) \to C^{2m+\alpha}(\overline{\Omega})$ for $\lambda \in \rho(A_0)$. If we take $f \in C^\alpha(\overline{\Omega})$ and define $u := R(\lambda, A_0)f$, then $u \in D(A_0)$ and u solves

$$\begin{cases} (\lambda I - A)u = f \in C^\alpha(\overline{\Omega})\,, \\ \qquad\quad Bu = 0\,. \end{cases}$$

By the elliptic regularity theory precisely Theorem 3.1.4 (iv) we get $u \in C^{2m+\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{2m+\alpha}(\overline{\Omega})} \le C(\|f\|_{C^\alpha(\overline{\Omega})} + \|u\|_{C(\overline{\Omega})})\,.$$

In other words,

$$\|R(\lambda, A_0)f\|_{C^{2m+\alpha}(\overline{\Omega})} \le C(\|f\|_{C^\alpha(\overline{\Omega})} + \|R(\lambda, A_0)f\|_{C(\overline{\Omega})})\,.$$

And now by (3.2.1) and the fact that $R(\lambda, A_0) \in \mathcal{L}(X, X)$, the claim follows. $\square$

Note that Lemma 3.2.2 in particular implies $P^l\big|_{C^{2m+\alpha}(\overline{\Omega})} \subset C^{2m+\alpha}(\overline{\Omega})$ for $l \in \{c,s\}$. Since 0 is a semi-simple eigenvalue of A_0, we have $X^c = N(A_0)$ and $X^s = R(A_0)$ (see [26, Proposition A.2.2]) and so P^c and P^s are the projections onto $N(A_0)$ respectively $R(A_0)$. Consequently $A_c \equiv 0$ which is equivalent to say $AP^c \equiv 0$ and $BP^c \equiv 0$. Note that $N(A_0) \subset X_1$ by elliptic regularity precisely Theorem 3.1.4 (iv).

Since $X^c_0 \hookrightarrow X^c \hookrightarrow X_1$, we get $X^c_0 = X^c_1 = X^c = N(A)$. As X^c is a finite-dimensional vector space, all the norms are equivalent. Therefore we choose $|\cdot|$ as a norm on X^c. Furthermore, we take as a norm on X_j and X

$$\begin{cases} |u|_j := |P^c u| + |P^s u|_j \quad \text{ for } j = 0,1\,, \\ \;|u| := |P^c u| + |P^s u|\,. \end{cases} \tag{3.2.2}$$

(b) Next let us demonstrate that near u_*, the manifold $\mathcal{E}$ is the graph of a function $\phi : B_{X^c}(0, \rho_0) \to X^s_1$. To this end we define the mapping

$$g : U \subset \mathbb{R}^k \to X^c, \quad g(\zeta) := P^c\Psi(\zeta)\,, \quad \zeta \in U\,.$$

Taking into account the fact the $\dim X^c = \dim \mathbb{R}^k = k$, It can be easily seen by our assumptions that $g'(0) = P^c\Psi'(0) : \mathbb{R}^k \to X^c$ is bijective.

Thus, we can apply the inverse function theorem to conclude that g is a C^2-diffeomorphism of a neighborhood of 0 in $\mathbb{R}^k$ onto a neighborhood of 0 in X^c, which we choose as $B_{X^c}(0,\rho_0)$ for some $\rho_0 > 0$. Hence the inverse $g^{-1} : B_{X^c}(0,\rho_0) \to U$ is C^2 and $g^{-1}(0) = 0$. If we define $\Phi(v) := \Psi(g^{-1}(v))$ for $v \in B_{X^c}(0,\rho_0)$, we obtain $\Phi \in C^2(B_{X^c}(0,\rho_0), X_1)$, $\Phi(0) = 0$ as well as

$$\{ u_* + \Phi(v) : \ v \in B_{X^c}(0,\rho_0) \} = \mathcal{E} \cap W$$

It is easy to observe that,

$$P^c\Phi(v) = \left((P^c \circ \Psi) \circ g^{-1}\right)(v) = (g \circ g^{-1})(v) = v, \quad v \in B_{X^c}(0,\rho_0),$$

Hence

$$\Phi(v) = P^c\Phi(v) + P^s\Phi(v) = v + P^s\Phi(v) \quad \text{for all } v \in B_{X^c}(0,\rho_0).$$

If we finally define $\phi(v) := P^s\Phi(v)$ and use the fact that $\Psi'(0)(\mathbb{R}^k) \subseteq N(A_0)$, we obtain

$$\phi \in C^2(B_{X^c}(0,\rho_0), X_1^s), \quad \phi(0) = \phi'(0) = 0, \tag{3.2.3}$$

and

$$\{ u_* + v + \phi(v) : \ v \in B_{X^c}(0,\rho_0) \} = \mathcal{E} \cap W\,, \tag{3.2.4}$$

for some neighborhood W of u_* in X_1.

Hence we have established our assertion, i.e., near u_* the manifold $\mathcal{E}$ can be represented as the graph over its tangent space $T_{u_*}\mathcal{E} = N(A_0) = X^c$ via the function ϕ. Now applying P^c and P^s, equations for the equilibria of (3.1.1), i.e., (3.1.3) and (3.1.4) is equivalent to the system

$$\begin{gathered} P^cA\phi(v) = P^cF(v + \phi(v)), \\ P^sA\phi(v) = P^sF(v + \phi(v)), \quad B\phi(v) = G(v + \phi(v)), \end{gathered} \tag{3.2.5}$$

where $v \in B_{X^c}(0,\rho_0)$. Here we have used the fact that $v + \phi(v) = \Psi(g^{-1}(v))$ for $v \in B_{X^c}(0,\rho_0)$ as well as $A_c \equiv 0$.

For later convenience we choose ρ_0 so small that

$$|\phi'(v)|_{\mathcal{L}(X^c,X_1^s)} \le 1, \quad |\phi(v)|_1 \le |v|, \quad \text{for all } v \in B_{X^c}(0,\rho_0). \tag{3.2.6}$$

For $r \in (0,\rho_0)$, we set

$$\eta(r) = \sup\{ \|\phi'(\varphi)\|_{\mathcal{L}(X^c,X_1^s)} : \varphi \in B_{X^c}(0,r) \}.$$

Since $\phi'(0) = 0$, $\eta(r)$ tends to 0 as $r \to 0$. Let $L' > 0$ be such that, for all φ, $\psi \in B_{X^c}(0,r)$ with $r \in (0,\rho_0)$

$$\|\phi'(\varphi) - \phi'(\psi)\|_{\mathcal{L}(X^c,X_1^s)} \le L'|\varphi - \psi|.$$

(c) Now we are in a position to reduce the system (3.1.1) to normal form. For this we let

$$v := P^c u, \quad w := P^s u - \phi(P^c u).$$

As an immediate consequence of this change of variables, we see that

$$\mathcal{E} \cap W = B_{X^c}(0, \rho_0) \times \{0\} \subset X^c \times X_1^s\,.$$

Under this nonlinear transformation of variables, (3.1.1) is transformed into the following system

$$\begin{cases} \partial_t v = T(v, w) & \text{in } \Omega, \\ \partial_t w + P^s A P^s w = R(v, w) & \text{in } \Omega, \\ Bw = S(v, w) & \text{on } \partial\Omega, \\ v(0) = v_0, \quad w(0) = w_0 & \text{in } \Omega, \end{cases} \tag{3.2.7}$$

with $v_0 = P^c u_0$ and $w_0 = P^s u_0 - \phi(P^c u_0)$, where the function T, R and S are given by

$$\begin{aligned} T(v, w) &= P^c F\big(v + \phi(v) + w\big) - P^c A\phi(v) - P^c Aw, \\ R(v, w) &= P^s F\big(v + \phi(v) + w\big) - P^s A\phi(v) - \phi'(v) T(v, w), \\ S(v, w) &= G\big(v + \phi(v) + w\big) - B\phi(v). \end{aligned}$$

Now we need to rewrite the expression for T, R and S into the following more useful form:

$$\begin{aligned} T(v, w) &= P^c\Big(F\big(v + \phi(v) + w\big) - F\big(v + \phi(v)\big)\Big) - P^c Aw, \\ R(v, w) &= P^s\Big(F\big(v + \phi(v) + w\big) - F\big(v + \phi(v)\big)\Big) - \phi'(v) T(v, w), \\ S(v, w) &= G\big(v + \phi(v) + w\big) - G\big(v + \phi(v)\big), \end{aligned}$$

where we have benefited from the equilibrium equations in (3.2.5).

Clearly,

$$R(v, 0) = T(v, 0) = S(v, 0) = 0, \quad v \in B_{X^c}(0, \rho_0).$$

(d) We shall use the parabolic Hölder spaces

$$\mathbb{E}_1(a) := C^{1+\frac{\alpha}{2m}, 2m+\alpha}(I_a \times \overline{\Omega}) = C^{1+\frac{\alpha}{2m}}(I_a, X) \cap B(I_a, X_1)\,,$$

$$\mathbb{E}_0(a) := C^{\frac{\alpha}{2m}, \alpha}(I_a \times \overline{\Omega}) = C^{\frac{\alpha}{2m}}(I_a, X) \cap B(I_a, X_0)\,.$$

Here $0 < a \leq \infty$,

$$I_a := \begin{cases} [0, a] & \text{for } a > 0, \\ [0, \infty) & \text{for } a = \infty\,, \end{cases}$$

and $B(I_a, X_j)$ is a space of all bounded functions $f : I_a \to X_j$ equipped with the supremum norm.

Similarly we introduce the following spaces for functions defined on the boundary

$$\begin{aligned}\mathbb{F}_j(a) :&= C^{1+\frac{\alpha}{2m}-\frac{m_j}{2m},2m+\alpha-m_j}(I_a \times \partial\Omega)\\ &= C^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}}\left(I_a, C(\partial\Omega)\right) \cap B\left(I_a, C^{2m+\alpha-m_j}(\partial\Omega)\right),\end{aligned}$$

and

$$\mathbb{F}(a) = \prod_{j=1}^{mN} \mathbb{F}_j(a).$$

By (3.2.2) you can easily show that $\|p_l u\|_{\mathbb{E}_i(a)} \leq \|u\|_{\mathbb{E}_i(a)}$ for $i = 0, 1$ and $l \in \{c, s\}$, which we will use several times without mention it. The proof of the following Lemma is given in [28, Theorem 2.2].

Lemma 3.2.3. *The following continuous embedding holds with an embedding constant independent of a, with $0 < \theta < 2m + \alpha$.*

$$\mathbb{E}_1(a) \hookrightarrow C^{\frac{\theta}{2m}}(I_a, C^{2m+\alpha-\theta}(\overline{\Omega}))\,.$$

We now state the optimal regularity theorem for the linear system

$$\begin{cases} \partial_t u + Au = f(t) & \text{in } \Omega, \quad t \in (0, a)\,, \\ Bu = g(t) & \text{on } \partial\Omega, \quad t \in (0, a)\,, \\ u(0) = u_0 & \text{in } \Omega\,, \end{cases} \tag{3.2.8}$$

in the parabolic Hölder setting. See Theorem VI.21 in [14].
In the following we need the compatibility conditions

$$\begin{cases} Bu_0 = g(0)\,, \\ B_j f(0) - B_j A u_0 = \partial_t g_j(t)\big|_{t=0} \quad \text{for all } j \text{ such that } m_j = 0\,. \end{cases} \tag{3.2.9}$$

Proposition 3.2.4. *Fix $a < \infty$. The linear system (3.2.8) has a unique solution $u \in \mathbb{E}_1(a)$ if and only if $f \in \mathbb{E}_0(a)$, $g \in \mathbb{F}(a)$, $u_0 \in X_1$, and the compatibility conditions (3.2.9) are satisfied. Moreover there exist $\widetilde{C} = \widetilde{C}(a) > 0$ such that*

$$\|u\|_{\mathbb{E}_1(a)} \leq \widetilde{C}\big(|u_0|_1 + \|f\|_{\mathbb{E}_0(a)} + \|g\|_{\mathbb{F}(a)}\big).$$

We turn next to the problem of global in time existence for the system

$$\begin{cases} \partial_t w + P^s A P^s w = f(t) & \text{in } \Omega\,, \quad t > 0, \\ Bw = g(t) & \text{on } \partial\Omega\,, \quad t > 0, \\ w(0) = w_0 & \text{in } \Omega\,, \end{cases} \tag{3.2.10}$$

where $t \in (0, \infty]$.

Proposition 3.2.5. *Let $0 < a \leq \infty$ and $0 < \sigma < \omega$, where $\omega = inf\,\{Re\lambda : \lambda \in \sigma_s\}$. The linear problem (3.2.10) has a unique solution w such that $e^{\sigma t}w \in \mathbb{E}_1(a)$ if and only if $e^{\sigma t}f \in C^{\frac{\alpha}{2m}}(I_a; X) \cap B(I_a; X_0^s)$, $e^{\sigma t}g \in \mathbb{F}(a)$, $w_0 \in X_1^s$, and the compatibility conditions (3.2.9) are satisfied. Moreover there exists a constant C_0, independent of a, such that*

$$\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} \leq C_0\Big(|w_0|_1 + \|e^{\sigma t}f\|_{\mathbb{E}_0(a)} + \|e^{\sigma t}g\|_{\mathbb{F}(a)}\Big)\,.$$

Proof. To show the "only if" part, use the system of equations (3.2.10). Let us prove the "if" part. First observe that if u solves (3.2.8) with $u_0 = w_0$ then the function $w = P^s u$ solves problem (3.2.10). Now let us denote by u_1 the solution of (3.2.8) with $A+1$ replacing A. Since $\sigma(A_0+1) \subset \mathbb{C}_+$, applying Corollary 4.3.2 below, we get a uniform bound for $e^{\sigma t}u_1$ in $\mathbb{E}_1(\infty)$.

Setting $u_2 = u - u_1$ we find that $z = P^s u_2$ solves the problem

$$\partial_t z + P^s A P^s z = P^s u_1, \quad Bz = 0, \quad z(0) = 0. \tag{3.2.11}$$

Let u_3 denote the solution of

$$\partial_t z + Az = P^s u_1, \quad Bz = 0, \quad z(0) = 0. \tag{3.2.12}$$

By applying Theorem 4.3.1 below to (3.2.12) with $f = P^s u_1$, $u_0 = 0$, $g = 0$ we find a uniform bound for $e^{\sigma t}u_3$ in $\mathbb{E}_1(\infty)$ (it is easy to see that (4.3.3) holds). Now using the fact that $P^s u_3$ solves (3.2.11) we also obtain a uniform bound for $e^{\sigma t}P^s u_3 = e^{\sigma t}P^s u_2$ in $\mathbb{E}_1(\infty)$. This finishes the proof. $\square$

(e) Let us turn our attention to the nonlinearities T, R and S. Here we derive estimates which are needed for applying Proposition 3.2.5.

Let $0 < r \leq R$, and set

$$K(r) = \sup\{\, \|F'(\varphi)\|_{\mathcal{L}(C^{2m+\alpha}(\overline{\Omega}), C^{\alpha}(\overline{\Omega}))} : \varphi \in B(0,r) \subset C^{2m+\alpha}(\overline{\Omega})\,\},$$
$$H_j(r) = \sup\{\, \|G_j'(\varphi)\|_{\mathcal{L}(C^{2m+\alpha}(\overline{\Omega}), C^{2m+\alpha-m_j}(\partial\Omega))} : \varphi \in B(0,r) \subset C^{2m+\alpha}(\overline{\Omega})\,\},$$

for $j = 1, \ldots, mN$. Since $F'(0) = 0$ and $G_j'(0) = 0$, $K(r)$ and $H_j(r)$ tend to 0 as $r \to 0$. Let $L > 0$ be such that, for all φ, $\psi \in B(0,r) \subset C^{2m}(\overline{\Omega})$ with small r,

$$\|F'(\varphi) - F'(\psi)\|_{\mathcal{L}(C^{2m}(\overline{\Omega}), C(\overline{\Omega}))} \leq L\|\varphi - \psi\|_{C^{2m}(\overline{\Omega})},$$
$$\|G_j'(\varphi) - G_j'(\psi)\|_{\mathcal{L}(C^{m_j}(\overline{\Omega}), C(\partial\Omega))} \leq L\|\varphi - \psi\|_{C^{m_j}(\overline{\Omega})}.$$
$$\|G_j''(\varphi) - G_j''(\psi)\|_{\mathcal{L}(C^{m_j}(\overline{\Omega}), \mathcal{L}(C^{m_j}(\overline{\Omega}), C(\partial\Omega))} \leq L\|\varphi - \psi\|_{C^{m_j}(\overline{\Omega})}.$$

In the following, we will always assume that $r \leq \min\{R, \rho_0\}$.

Lemma 3.2.6. *There exist a constant C_1 such that*

$$|T(v,w)| \leq C_1|w|_1$$

for any $u \in \overline{B_{X_1}(0,r)}$.

Proof. From (3.2.6) we see

$$|v+\phi(v)+w|_1 = |u|_1 \le r, \quad |v+\phi(v)|_1 \le |v|_1 + |\phi(v)|_1 \le 2r,$$

and now taking $z_1 = v+\phi(v)+w$ and $z_2 = v+\phi(v)$ in the definition of $T(v,w)$ we get

$$\begin{aligned}|T(v,w)| &= |P^c\left(F(z_1)-F(z_2)\right)| + |P^c Aw| \\ &\le |F(z_1)-F(z_2)|_0 + \|P^c A\|_{\mathcal{L}(X_1,X^c)}|w|_1 \\ &\le (K(2r)+C_2)|w|_1,\end{aligned}$$

where $C_2 := \|P^c A\|_{\mathcal{L}(X_1,X^c)}$ which is not necessarily small. □

Proposition 3.2.7. *If $z_1, z_2 \in \overline{B_{\mathbb{E}_1(a)}(0,r)}$, $\sigma \ge 0$ then*

$$\|e^{\sigma t}(F(z_1)-F(z_2))\|_{\mathbb{E}_0(a)} \le D(r)\|e^{\sigma t}(z_1-z_2)\|_{\mathbb{E}_1(a)},$$

$$\|e^{\sigma t}(G(z_1)-G(z_2))\|_{\mathbb{F}(a)} \le D(r)\|e^{\sigma t}(z_1-z_2)\|_{\mathbb{E}_1(a)},$$

where $D(r)\to 0$ as $r\to 0$.

The proof is given in the appendix.

Lemma 3.2.8. *If $u \in \overline{B_{\mathbb{E}_1(a)}(0,r)}$, then $v+\phi(v) \in \overline{B_{\mathbb{E}_1(a)}(0, 4r+L'r^2)}$.*

Proof. For $0 \le t \le a$, again by (3.2.6), we have

$$|v(t)+\phi(v(t))|_1 \le 2|v(t)|_1 \le 2|u(t)|_1 \le 2r$$

while for $0 \le s \le t \le a$,

$$\begin{aligned}&|v'(t)+\phi'(v(t))v'(t) - v'(s) - \phi'(v(s))v'(s)| \\ &\le |v'(t)-v'(s)| + |\phi'(v(t))\left(v'(t)-v'(s)\right)| + |\phi'(v(t))-\phi'(v(s))|_{\mathcal{L}(X^c,X_1)}|v'(s)| \\ &\le 2(t-s)^{\frac{\alpha}{2m}}\|v\|_{\mathbb{E}_1(a)} + L'|v(t)-v(s)||v'(s)| \\ &\le 2(t-s)^{\frac{\alpha}{2m}}\|v\|_{\mathbb{E}_1(a)} + L'(t-s)^{\frac{\alpha}{2m}}\|v\|^2_{\mathbb{E}_1(a)} \\ &\le (t-s)^{\frac{\alpha}{2m}}\left(2r+L'r^2\right)\end{aligned}$$

Note that we have used (3.2.6) and Lemma 3.2.3 to obtain the second inequality. This completes the proof. □

Proposition 3.2.9. *If $u \in \overline{B_{\mathbb{E}_1(a)}(0,r)}$, $\sigma \ge 0$ then*

$$\begin{aligned}(i)&\quad \|e^{\sigma t}T(v,w)\|_{\mathbb{E}_0(a)} \le C_3\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \\ (ii)&\quad \|e^{\sigma t}R(v,w)\|_{\mathbb{E}_0(a)} \le C(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)}, \\ (iii)&\quad \|e^{\sigma t}S(v,w)\|_{\mathbb{F}(a)} \le C(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)},\end{aligned}$$

where $C(r)\to 0$ as r goes to zero.

Proof. Let us prove (i). Setting $z_1 := u = v + \phi(v) + w$ and $z_2 := v + \phi(v)$ by Lemma 3.2.8 we have

$$\|z_1\|_{\mathbb{E}_1(a)}, \|z_2\|_{\mathbb{E}_1(a)} \le 4r + L'r^2 .$$

Hence we can now apply Proposition 3.2.7 to conclude

$$\begin{aligned}\|e^{\sigma t}T(v,w)\|_{\mathbb{E}_0(a)} &\le \|P^c(e^{\sigma t}(F(z_1) - F(z_2)))\|_{\mathbb{E}_0(a)} + \|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)} \\ &\le \|e^{\sigma t}(F(z_1) - F(z_2))\|_{\mathbb{E}_0(a)} + \|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)} \\ &\le D(4r + L'r^2)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} + \|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)} .\end{aligned}$$

Now let us consider $\|e^{\sigma t}P^cAw\|_{\mathbb{E}_0(a)}$.

For $0 \le t \le a$,

$$|e^{\sigma t}P^cAw(t)| \le C_2|e^{\sigma t}w(t)|_1 \le C_2\|e^{\sigma t}w\|_{\mathbb{E}_1(0,a)}$$

while for $0 \le s \le t \le a$,

$$\begin{aligned}|e^{\sigma t}P^cAw(t) - e^{\sigma t}P^cAw(s)| &\le |A(e^{\sigma t}w(t) - e^{\sigma s}w(s))| \le \|e^{\sigma t}w(t) - e^{\sigma s}w(s)\|_{D(A)} \\ &\le \|e^{\sigma t}w(t) - e^{\sigma s}w(s)\|_{C^{2m}(\overline{\Omega})} \\ &\le C'(t-s)^{\frac{\alpha}{2m}}\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} ,\end{aligned}$$

where we have used Lemma 3.2.3 to obtain the last inequality and C' is the corresponding embedding constant. Setting $C_3 := D(4r + L'r^2) + C' + C_2$ we complete the proof of (i).

We now prove (ii). Similarly as in (i) we get for the first term in $e^{\sigma t}R(v,w)$

$$\|P^s(e^{\sigma t}(F(z_1) - F(z_2)))\|_{\mathbb{E}_0(a)} \le D(4r + L'r^2)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} .$$

Let us estimate the second term in $R(v,w)$ namely, $e^{\sigma t}\phi'(v)T(v,w)$.

For $0 \le t \le a$, by (3.2.6) and Lemma 3.2.6 we have

$$\begin{aligned}|e^{\sigma t}\phi'(v(t))T(v(t),w(t))|_0 &\le |e^{\sigma t}\phi'(v(t))T(v(t),w(t))|_1 \\ &\le \|\phi'(v(t))\|_{\mathcal{L}(X^c,X_1^s)}|e^{\sigma t}T(v(t),w(t))| \\ &\le \eta(r)|e^{\sigma t}T(v(t),w(t))| \le C_1\eta(r)|e^{\sigma t}w(t)|_1 \\ &\le C_1\eta(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} ,\end{aligned}$$

while for $0 \le s \le t \le a$,

$$\begin{aligned}&|e^{\sigma t}\phi'(v(t))T(v(t),w(t)) - e^{\sigma s}\phi'(v(s))T(v(s),w(s))| \\ &\le \|\phi'(v(t))\|_{\mathcal{L}(X^c,X_1)}|e^{\sigma t}T(v(t),w(t)) - e^{\sigma s}T(v(s),w(s))| \\ &\quad + \|\phi'(v(t)) - \phi'(v(s))\|_{\mathcal{L}(X^c,X_1)}|e^{\sigma s}T(v(s),w(s))| \\ &\le (t-s)^{\frac{\alpha}{2m}}\eta(r)\|e^{\sigma t}T(v,w)\|_{\mathbb{E}_0(a)} + C_1\|\phi'(v(t)) - \phi'(v(s))\|_{\mathcal{L}(X^c,X_1)}|e^{\sigma t}w(t)|_1 \\ &\le (t-s)^{\frac{\alpha}{2m}}\eta(r)C_3\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} + C_1L'|v(t) - v(s)|\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} \\ &\le (t-s)^{\frac{\alpha}{2m}}\eta(r)C_3\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} + (t-s)^{\frac{\alpha}{2m}}C_1L'\|v\|_{E_1(a)}\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} \\ &\le (t-s)^{\frac{\alpha}{2m}}\left(\eta(r)C_3 + C_1L'r\right)\|e^{\sigma t}w\|_{\mathbb{E}_1(a)} .\end{aligned}$$

Finally by defining $C(r) := \eta(r)C_3 + C_1L'r + C_1\eta(r) + D(4r + L'r^2)$ we complete the proof of (ii). Using Proposition 3.2.7 and Lemma 3.2.8, we easily get (iii). $\square$

(f) We consider now an existence theorem for problem (3.1.1). As a first step, we show existence for large time using the contraction mapping principle.

Proposition 3.2.10. *For every $T > 0$, there are $r > \rho > 0$ such that (3.1.1) has a solution $u \in \mathbb{E}_1(T)$ provided $|u_0 - u_*|_1 \leq \rho$. Moreover, u is the unique solution in $B_{\mathbb{E}_1(T)}(0, r)$.*

Proof. The proof is almost exactly the same as the one in Theorem 4.1 in [27]. However for the convenience of the reader we provide the details.

Let $0 < r \leq R$ and define a nonlinear map

$$\Gamma : \Big\{ w \in \overline{B(0,r)} \subset \mathbb{E}_1(T) : w(\cdot, 0) = u_0 \Big\} \longrightarrow \mathbb{E}_1(T)\,,$$

by $\Gamma w = v$, where v is the solution of

$$\begin{cases} \partial_t v + Av = F(w)\,, & \text{in } \overline{\Omega} \times [0,T]\,, \\ Bv = G(w)\,, & \text{on } \partial\Omega \times [0,T]\,, \\ v|_{t=0} = u_0\,, & \text{in } \overline{\Omega}\,. \end{cases}$$

Proposition 3.2.4, gives the estimate

$$\|v\|_{\mathbb{E}_1(T)} \leq \widetilde{C}\big(|u_0|_1 + \|F(w)\|_{\mathbb{E}_0(T)} + \|G(w)\|_{\mathbb{F}(T)}\big)\,,$$

with $\widetilde{C} = \widetilde{C}(T)$ in which we could assume without loss of generality that $\widetilde{C} > 1$. Hence by Proposition 3.2.7 we have

$$\|\Gamma(w)\|_{\mathbb{E}_1(T)} \leq \widetilde{C}\left(|u_0|_1 + 2D(r)\|w\|_{\mathbb{E}_1(T)}\right).$$

Consequently, if r is so small that

$$2\widetilde{C}D(r) \leq \frac{1}{2}\,, \tag{3.2.13}$$

and u_0 is so small that

$$|u_0|_1 \leq \frac{r}{2\widetilde{C}}\,, \tag{3.2.14}$$

then Γ maps the ball $\overline{B(0,r)}$ into itself. We then continue to show that Γ is a $\frac{1}{2}$-contraction. To prove this let w_1, $w_2 \in B(0,r)$. Then

$$\|\Gamma w_1 - \Gamma w_2\|_{\mathbb{E}_1(T)} \leq \widetilde{C}\Big(\|F(w_1) - F(w_2)\|_{\mathbb{E}_0(T)} + \|G(w_1) - G(w_2)\|_{\mathbb{F}(T)}\Big)\,,$$

and using again Proposition 3.2.7 we have

$$\begin{aligned}\|\Gamma w_1 - \Gamma w_2\|_{\mathbb{E}_1(T)} &\le 2\widetilde{C}D(r)\|w_1 - w_2\|_{\mathbb{E}_1(T)} \\ &\le \frac{1}{2}\|w_1 - w_2\|_{\mathbb{E}_1(T)} .\end{aligned}$$

Now the statement follows by the contraction mapping principle. $\square$

Let us next observe from the inequalities (3.2.13) and (3.2.14) that for a given time T, we can take r as small as we want provided $\rho < r$ is small enough. Now the strategy for proving the global existence is as follows:

We fix a time T and choose $r \le \min\{R, \rho_0\}$ small enough such that

$$2C_0C(r) \le \frac{1}{2} \tag{3.2.15}$$

and that (3.2.13) holds. For such an r we have a corresponding $\rho < r$ (by Proposition 3.2.10). In conclusion by Proposition 3.2.10, problem (3.1.1) admits for $u_0 \in B_{X_1}(0,\rho)$ a unique solution $u \in \overline{B_{\mathbb{E}_1(T)}(0,r)}$. The strategy is now as follows: We will find some $\delta < \rho$ such that the solution $u(t)$ of (3.1.1) with initial value $u_0 \in B_{X_1}(0,\delta)$ defined on its maximal interval of existence always stays in the ball $B_{X_1}(0,\rho)$. Then by Proposition 3.2.10 we see immediately that the maximal interval of existence can not be bounded and this proves the global existence.

Arguing as above there exists some $\delta' < \frac{\rho}{2}$ such that the problem (3.1.1) admits for $u_0 \in B_{X_1}(0,\delta')$ a unique solution

$$u \in \overline{B_{\mathbb{E}_1(T)}(0,\frac{\rho}{2})} . \tag{3.2.16}$$

Suppose that $u_0 \in B_{X_1}(0,\delta)$, where $\delta \le \delta' < \rho$ is a number to be selected later. Let $[0,t_*)$ be the maximal interval of existence of the solution $u(t)$ of (3.1.1) with initial value u_0. Furthermore let t_1 be the existence time for the ball $B_{X_1}(0,\rho)$, i.e.,

$$t_1 := \sup\Big\{\, t \in (0,t_*) \,:\, |u(\tau)|_1 \le \rho, \quad \tau \in [0,t] \,\Big\}.$$

Suppose also $t_1 < t_*$. Note that $t_1 \ge T$ by (3.2.16).

Lemma 3.2.11. *Under the conditions above we have* $\|u\|_{\mathbb{E}_1(t_1)} \le r$.

Proof. Since $|u_0|_1 < \delta' < \rho$, we have $u \in B_{\mathbb{E}_1(T)}(0,r)$ by Proposition 3.2.10. By the definition of t_1 and the fact that $T \le t_1$ we get $|u(T)|_1 \le \rho$. Therefore we can now start with the initial data $u(T)$ and by finitely often repeating the same process we complete the proof (since T is constant, we will get $u \in B_{\mathbb{E}_1(kT)}(0,r)$ for some k such that $kT > t_1$ and therefore the estimate follows immediately). $\square$

Now we apply (3.2.7), Proposition 3.2.5, Lemma 3.2.11 and Proposition 3.2.9 and derive

$$\begin{aligned}\|e^{\sigma t}w\|_{\mathbb{E}_1(t_1)} &\le C_0\big(|w_0|_1 + \|e^{\sigma t}R(v,w)\|_{\mathbb{E}_0(a)} + \|e^{\sigma t}S(v,w)\|_{\mathbb{F}(a)}\big) \\ &\le C_0|w_0|_1 + 2C_0C(r)\|e^{\sigma t}w\|_{\mathbb{E}_1(t_1)}\,.\end{aligned}$$

Together with (3.2.15) this implies

$$\|e^{\sigma t}w\|_{\mathbb{E}_1(t_1)} \le 2C_0|w_0|_1, \quad \sigma \in [0,\omega). \tag{3.2.17}$$

Hence for $t \in [0,t_1]$

$$|e^{\sigma t}w(t)|_1 \le \|e^{\sigma t}w\|_{\mathbb{E}_1(t_1)} \le 2C_0|w_0|_1\,,$$

and so

$$|w(t)|_1 \le 2C_0e^{-\sigma t}|w_0|_1, \quad t \in [0,t_1], \quad \sigma \in [0,\omega). \tag{3.2.18}$$

Using the equation for v in (3.2.7) and Lemma 3.2.6 we obtain

$$\begin{aligned}|v(t)| &\le |v_0| + \int_0^t |T(v(s),w(s)|\,\mathrm{d}s \\ &\le |v_0| + C_1\int_0^t |w(s)|_1\,\mathrm{d}s \\ &\le |v_0| + C_1\int_0^\infty e^{-\sigma s}\,\mathrm{d}s\,\|e^{\sigma t}w\|_{\mathbb{E}_1(t_1)} \\ &\le |v_0| + \frac{C_1}{\sigma}\|e^{\sigma t}w\|_{\mathbb{E}_1(t_1)} \\ &\le |v_0| + C_4|w_0|_1, \quad t \in [0,t_1],\end{aligned}$$

where $C_4 = 2C_0\frac{C_1}{\sigma}$. Combining the last two estimates and taking into account (3.2.6) we find

$$|u(t)|_1 \le C_5|u_0|_1, \quad t \in [0,t_1].$$

for some constant $C_5 \ge 1$. In particular this inequality is satisfied for $t = t_1$. Therefore choosing $\delta \le \frac{\delta'}{2C_5}$, we find $|u(t_1)|_1 \le \delta'/2$. But this is a contradiction to the definition of t_1 since by (3.2.16) we could start with t_1 and continue further and still being in the ball $B_{X_1}(0,\rho)$ hence $t_1 = t_*$. By Lemma 3.2.11 we get then uniform bounds $\|u\|_{\mathbb{E}_1(a)} \le r$, for all $a < t_*$. As a result of Proposition 3.2.10, we conclude $t_* = \infty$.

(g) Finally, we repeat the estimates above on the interval $[0,\infty)$. This yields

$$|v(t)| \le |v_0| + C_4|w_0|_1, \quad |w(t)|_1 \le 2C_0e^{-\sigma t}|w_0|_1, \quad t \in [0,\infty),$$

for $u_0 \in B_{X_1}(0, \delta)$. Furthermore, the limit

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \left(v_0 + \int_0^\infty T(v(s), w(s))\,\mathrm{d}s \right) =: v_\infty$$

exists in X since the integral converges absolutely. Hence

$$u_\infty := \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) + \phi(v(t)) + w(t) = v_\infty + \phi(v_\infty).$$

exists too and u_∞ is a stationary solution of (3.1.1) due to (3.2.4). Moreover, Lemma 3.2.6 and (3.2.17) imply

$$\begin{aligned}
|v(t) - v_\infty| &= \left| \int_t^\infty T(v(t), w(t))\,\mathrm{d}s \right| \\
&\le C_1 \int_t^\infty |w(s)|_1\,\mathrm{d}s \\
&\le C_1 \int_t^\infty e^{-\sigma s}\,\mathrm{d}s\, \|e^{\sigma t} w\|_{\mathbb{E}_1(\infty)} \\
&\le C_4 e^{-\sigma t} |w_0|_1 \quad t \ge 0.
\end{aligned}$$

Hence $v(t) \to v_\infty$ in X at an exponential rate as $t \to \infty$. Finally, using (3.2.6) and (3.2.18) we conclude

$$\begin{aligned}
|u(t) - u_\infty|_1 &= |v(t) + \phi(v(t)) + w(t) - u_\infty|_1 \\
&\le |v(t) - v_\infty| + |\phi(v(t)) - \phi(v_\infty)|_1 + |w(t)|_1 \\
&\le (2C_4 + 2C_0) e^{-\sigma t} |w_0|_1 \\
&\le C e^{-\sigma t} |P^s u_0 - \phi(P^c u_0)|_1,
\end{aligned}$$

which proves the second part of Theorem 3.2.1. Note that by Lemma 3.2.11 it follows that by choosing $0 < \delta \le \rho$ sufficiently small, the solution starting in $B_{X_1}(u_*, \delta)$ exists for all times and stays within $B_{X_1}(u_*, r)$. This implies stability of u_*. □

Remark 3.2.12. *Note that the assumption $u_* \equiv 0$ in Theorem 3.2.1 is not a restriction. The case of a general stationary solution $\bar{u}$ can be reduced to the case of the zero stationary solution by considering*

$$U(t) = u(t) - \bar{u}.$$

Chapter 4

Asymptotic Behavior for Linear Problems

In this chapter we prove a general result (in the parabolic Hölder setting) on the asymptotic behaviour of solutions to linear parabolic systems with nonhomogeneous boundary conditions, see Theorem 4.3.1. This is done by reducing the problem to the scalar case treated in Section 4.1. We also needed to construct extension operators for the boundary operators in Section 4.2.

Throughout this chapter we follow the notation of the previous chapter, except that here u' denotes the derivative of a function u with respect to time. Let $\sigma^-(-A_0)$ denote the subset of $\sigma(-A_0)$ consisting of elements with negative real parts. Note that $\sigma^-(-A_0)$ is a spectral set due to Remark 3.1.2. Clearly $\sigma^-(-A_0) = -\sigma_s$ and $P^- = P^s$, where P^- is the spectral projection associated to $\sigma^-(-A_0)$.

4.1 Asymptotic behavior for linear scalar equations

Such a result is proven in [10, Theorem. 0.1] for a single equation of second order with first-order boundary condition. Here we extend this result to a single equation of order $2m$ with m boundary conditions (e.g. zeroth-order boundary conditions are included). Precisely we consider the linear problem (3.2.8) with $\boldsymbol{N} = \mathbf{1}$, i.e.,

$$\begin{cases} \partial_t u + Au = f(t) & \text{in } \Omega\,, \quad t \geq 0\,, \\ Bu = g(t) & \text{on } \partial\Omega\,, \quad t \geq 0\,, \\ u(0) = u_0 & \text{in } \Omega\,, \end{cases} \tag{4.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $C^{2m+\alpha}$ boundary, $0 < \alpha < 1$, $g = (g_1, \ldots, g_m), B = (B_1, \ldots, B_m)$, $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ and the operators A and B satisfy the conditions (H2), (L-S), (SP), and the normality condition

(3.1.9). Note that the normality condition in particular implies that

$$0 \le m_1 < m_2 < \cdots < m_m \le 2m-1\,.$$

For convenience, we set

$$\mathcal{L} = -A \quad \text{ and } \quad L = -A_0\,.$$

The realisation L of $\mathcal{L}$ with homogeneous boundary conditions in $X = C(\overline{\Omega})$, defined similarly as (3.1.5), is a sectorial operator by Theorem 3.1.4. Furthermore, if $f \in \mathbb{E}_0(T)$, $g \in \mathbb{F}(T)$ and $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ satisfying the compatibility condition (3.2.9), the unique solution of (4.1.1) belongs to $\mathbb{E}_1(T)$ for all T and in addition it is given by the extension of the Balakrishnan formula with some adaptations (see (37)-(40) of §7 in [28])

$$\begin{aligned}
u(\cdot,t) &= e^{tL}(u_0 - n(\cdot,0)) + \int_0^t e^{(t-s)L}[f(\cdot,s) + \mathcal{L}n(\cdot,s) - n_1'(\cdot,0)]\,\mathrm{d}s \\
&\quad + n_1(\cdot,t) - \int_0^t e^{(t-s)L}(n_1'(\cdot,s) - n_1'(\cdot,0))\,\mathrm{d}s \\
&\quad + n_2(\cdot,0) - L\int_0^t e^{(t-s)L}[n_2(\cdot,s) - n_2(\cdot,0)]\,\mathrm{d}s \qquad (4.1.2)\\
&= e^{tL}u_0 + \int_0^t e^{(t-s)L}[f(\cdot,s) + \mathcal{L}n(\cdot,s)]\,\mathrm{d}s \\
&\quad - L\int_0^t e^{(t-s)L}n(\cdot,s)\,\mathrm{d}s\,, \quad 0 \le t \le T\,. \qquad (4.1.3)
\end{aligned}$$

Here

$$n(t) = \mathcal{N}(g_1(t),\ldots,g_m(t)) = \sum_{s=1}^m \mathcal{N}_s\mathcal{M}_s(g_1(t),\ldots,g_s(t))\,,$$

$$n_1(t) = \begin{cases} 0 & \text{if } m_1 > 0 \\ \mathcal{N}_1\mathcal{M}_1(g_1(t)) & \text{if } m_1 = 0 \end{cases} \quad \text{ and } \quad n_2(t) = n(t) - n_1(t)\,, \qquad (4.1.4)$$

where the operator $\mathcal{N}$ given in the following theorem is a lifting operator with an explicit construction such that

$$\begin{cases} \mathcal{N} \in L(\prod_{j=1}^m C^{2m+\theta'-m_j}(\partial\Omega), C^{2m+\theta'}(\overline{\Omega})), \quad \forall\,\theta' \in [0,\alpha]\,, \\ B_j(\mathcal{N}(g_1,\ldots,g_m))(x) = g_j(x), \quad x \in \partial\Omega, \quad j = 1,\ldots,m\,. \end{cases} \qquad (4.1.5)$$

Theorem 4.1.1. *Given $s = 1,\ldots,m$, there exist*

$$\mathcal{M}_s \in L(\prod_{j=1}^s C^{\theta-m_j}(\partial\Omega), C^{\theta-m_s}(\partial\Omega)), \quad \forall\theta \in [m_s, 2m+\alpha],$$

and

$$\mathcal{N}_s \in L(C^r(\partial\Omega); C^{r+m_s}(\overline{\Omega})), \quad \forall r \in [0, 2m+\alpha-m_j]$$

such that, setting

$$\mathcal{N}(\psi_1, \dots, \psi_m) = \sum_{s=1}^{m} \mathcal{N}_s \mathcal{M}_s(\psi_1, \dots, \psi_s)\,,$$

we have

$$\mathcal{N} \in L(\prod_{j=1}^{m} C^{2m+\theta'-m_j}(\partial\Omega), C^{2m+\theta'}(\overline{\Omega})), \quad \forall \theta' \in [0, \alpha]\,, \tag{4.1.6}$$

and

$$B_j(\mathcal{N}(\psi_1, \dots, \psi_m))(x) = \psi_j(x), \quad x \in \partial\Omega, \quad j = 1, \dots, m\,.$$

Moreover, for each $u \in C(\partial\Omega)$*,*

$$D_x^l \mathcal{N}_s u(x) = 0\,, \quad x \in \partial\Omega\,, \, l \in \mathbb{N}^n\,, \, |l| < m_s\,, \tag{4.1.7}$$

which in particular implies that

$$(B_j \mathcal{N}_s u)(x) \equiv 0, \quad x \in \partial\Omega, \text{ for } j < s\,.$$

Proof. The proof is given in [28, Theorem 6.3]. □

Theorem 4.1.2. *Let* $0 < \omega < -\max\{\operatorname{Re}\lambda : \lambda \in \sigma^-(-A_0)\}$*. Suppose* f *and* g *are such that* $(\sigma, t) \mapsto e^{\omega t} f(\sigma, t) \in \mathbb{E}_0(\infty)$ *and* $(\sigma, t) \mapsto e^{\omega t} g(\sigma, t) \in \mathbb{F}(\infty)$*. Suppose further that* $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ *satisfy the compatibility condition* (3.2.9)*. Let* u *be the solution of* (4.1.1)*. Then* $v(\sigma, t) := e^{\omega t} u(\sigma, t)$ *is bounded in* $[0, +\infty) \times \overline{\Omega}$ *if and only if*

$$\begin{aligned} (I - P^-)u_0 = &- \int_0^{+\infty} e^{-sL}(I - P^-)[f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)]\,\mathrm{d}s \\ &+ L \int_0^{\infty} e^{-sL}(I - P^-)\mathcal{N}g(\cdot, s)\,\mathrm{d}s\,. \end{aligned} \tag{4.1.8}$$

If this is so, the function u *is given by*

$$\begin{aligned} u(\cdot, t) = e^{tL}P^-u_0 &+ \int_0^t e^{(t-s)L}P^-[f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)]\,\mathrm{d}s \\ &- L \int_0^t e^{(t-s)L}P^-\mathcal{N}g(\cdot, s)\,\mathrm{d}s \\ &- \int_t^{+\infty} e^{(t-s)L}(I - P^-)[f(\cdot, s) + \mathcal{L}\mathcal{N}g(\cdot, s)]\,\mathrm{d}s \\ &+ L \int_t^{+\infty} e^{(t-s)L}(I - P^-)\mathcal{N}g(\cdot, s)\,\mathrm{d}s, \end{aligned} \tag{4.1.9}$$

and the function $v = e^{\omega t}u$ *belongs to* $\mathbb{E}_1(\infty)$*, with the estimate*

$$\|v\|_{\mathbb{E}_1(\infty)} \le C(\|u_0\|_{C^{2m+\alpha}(\overline{\Omega})} + \|e^{\omega t} f\|_{\mathbb{E}_0(\infty)} + \|e^{\omega t} g\|_{\mathbb{F}(\infty)}) \tag{4.1.10}$$

for some $c > 0$ *independent of* (u_0, f, g)*.*

Proof. The proof follows the arguments of [10, Theorem 0.1]. The novelty with respect to [10] is the appearance of systems of m boundary conditions (including possibly zeroth-order boundary conditions) which is treated with the method introduced in [28, Section 7].

Taking into account the estimates (see [26, Proposition 2.3.3]) (which hold for small $\epsilon > 0$ and for $t > 0$)

$$\begin{aligned} \|P^- e^{tL}\|_{L(X)} &\leq Ce^{-(\omega+\epsilon)t}\,, \\ \|LP^- e^{tL}\|_{L(X)} &\leq \frac{Ce^{-(\omega+\epsilon)t}}{t}\,, \\ \|e^{-tL}(I-P^-)\|_{L(X)} &\leq Ce^{-(\omega-\epsilon)t}\,, \end{aligned}$$

and arguing as in [26], one can easily verify that the function given by the right hand side of (4.1.9) is bounded by $Ce^{-\omega t}$.

In view of (4.1.3), we have $u = u_1 + u_2$, where u_1 is the function on the right hand side of (4.1.9) and

$$\begin{aligned} u_2(\cdot,t) = e^{tL}\bigg((I-P^-)u_0 &+ \int_0^\infty e^{-sL}(I-P^-)(f(\cdot,s) + \mathcal{L}\mathcal{N}g(\cdot,s))\,\mathrm{d}s \\ &- L\int_0^\infty e^{-sL}(I-P^-)\mathcal{N}g(\cdot,s)\,\mathrm{d}s\bigg) \\ \equiv e^{tL}y\,, \quad t \geq 0\,. \end{aligned}$$

From our assumption on ω, it follows that

$$\sigma(L+\omega I) \cap i\mathbb{R} = \emptyset \tag{4.1.11}$$

and the projection $(I - P^-)$ is the spectral projection associated to the unstable part of $\sigma(L+\omega I)$. Therefore due to

$$e^{\omega t}u_2(\cdot,t) = e^{t(L+\omega I)}y$$

and the fact that y is an element of $(I-P^-)(X)$, $e^{\omega t}u_2(\cdot,t)$ is bounded in $[0,\infty)$ with values in X (i.e., v is bounded) if and only if $y = 0$, i.e., iff (4.1.8) holds.

We now prove that $v = e^{\omega t}u \in \mathbb{E}_1(\infty)$. First note that v solves (4.1.1) with $\mathcal{L}$ replacing $\tilde{\mathcal{L}} = \mathcal{L}+\omega I$, and f and g replacing $\widetilde{f} = fe^{\omega t}$ and $\widetilde{g} = ge^{\omega t}$ respectively. Due to the regularity of the data and the compatibility condition (3.2.9), by Proposition 3.2.4, v belongs to $\mathbb{E}_1(1) = C^{2m+\alpha,1+\frac{\alpha}{2m}}(\overline{\Omega}\times[0,1])$ and

$$\|v\|_{\mathbb{E}_1(a)} \leq C\big(|u_0|_1 + \|\widetilde{f}\|_{\mathbb{E}_0(\infty)} + \|\widetilde{g}\|_{\mathbb{F}(\infty)}\big)\,.$$

Hence it remains to show that $v \in C^{2m+\alpha,1+\frac{\alpha}{2m}}(\overline{\Omega}\times[1,\infty))$. As a result of (4.1.11), we have the following estimates for some $\gamma > 0$:

$$\begin{aligned} \|\widetilde{L}^k e^{t\widetilde{L}}P^-\|_{L(X)} &\leq \frac{C_k e^{-\gamma t}}{t^k}\,, \quad t > 0\,, \\ \|\widetilde{L}^k e^{-t\widetilde{L}}(I-P^-)\|_{L(X)} &\leq C_k e^{-\gamma t}, \quad t > 0\,, \quad k \in \mathbb{N}\,. \end{aligned} \tag{4.1.12}$$

Let us define

$$\widetilde{n}(t) := \mathcal{N}(\widetilde{g}_1(t), \ldots, \widetilde{g}_m(t)) = \sum_{s=1}^{m} \mathcal{N}_s \mathcal{M}_s(\widetilde{g}_1(t), \ldots, \widetilde{g}_s(t)) \tag{4.1.13}$$

and

$$\widetilde{n}_1(t) := \begin{cases} 0 & \text{if } m_1 > 0 \\ \mathcal{N}_1 \mathcal{M}_1(\widetilde{g}_1) & \text{if } m_1 = 0 \end{cases} \quad \text{and} \quad \widetilde{n}_2(t) := \widetilde{n}(t) - \widetilde{n}_1(t)\,. \tag{4.1.14}$$

By decomposing v as $v = P^- v + (I - P^-)v$, using the equality (4.1.2) for the term $P^- v$, the equality (4.1.3) for the term $(I - P^-)v$ and taking into account (4.1.8), we can split $v(t) = v(\cdot, t)$ as $v = \sum_{i=1}^{5} v_i$, where

$$\begin{aligned}
v_1(t) &= e^{t\widetilde{L}} P^-(u_0 - \widetilde{n}(0)) + \int_0^t e^{(t-s)\widetilde{L}} P^- [\widetilde{f}(s) + \widetilde{\mathcal{L}}\widetilde{n}(s) - \widetilde{n}_1'(0)]\,\mathrm{d}s\,, \\
v_2(t) &= P^- \widetilde{n}_1(t) - \int_0^t e^{(t-s)\widetilde{L}} P^-(\widetilde{n}_1'(s) - \widetilde{n}_1'(0))\,\mathrm{d}s\,, \\
v_3(t) &= P^- \widetilde{n}_2(0) - \widetilde{L} \int_0^t e^{(t-s)\widetilde{L}} P^-(\widetilde{n}_2(s) - \widetilde{n}_2(0))\,\mathrm{d}s\,, \\
v_4(t) &= -\int_t^\infty e^{(t-s)\widetilde{L}} (I - P^-)[\widetilde{f}(s) + \widetilde{\mathcal{L}}\widetilde{n}(s)]\,\mathrm{d}s\,, \\
v_5(t) &= \widetilde{L} \int_t^\infty e^{(t-s)\widetilde{L}} (I - P^-)\widetilde{n}(s)\,\mathrm{d}s\,.
\end{aligned}$$

Furthermore, we need the following facts about the regularity of $\widetilde{n}$, which are proven in [28], see (5),(11)-(13) of §7 in this paper:

$$\begin{cases}
\widetilde{n} & \in B([0,\infty);\, C^{2m+\alpha}(\overline{\Omega})) \cap C^{\frac{\alpha}{2m}}([0,\infty);\, C^{2m}(\overline{\Omega}))\,, \\
\widetilde{\mathcal{L}}\widetilde{n} & \in B([0,\infty);\, C^{\alpha}(\overline{\Omega})) \cap C^{\frac{\alpha}{2m}}([0,\infty); X)\,, \\
\widetilde{n}_1 & \in B([0,\infty);\, C^{2m+\alpha}(\overline{\Omega}))\,, \\
\widetilde{\mathcal{L}}\widetilde{n}_1 & \in B([0,\infty);\, C^{\alpha}(\overline{\Omega}))\,, \\
\widetilde{n}_1' & \in C^{\frac{\alpha}{2m}}([0,\infty); X) \cap B([0,\infty); C^{\alpha}(\overline{\Omega}))\,.
\end{cases}$$

Let us first consider v_1. Since $t \to \widetilde{f}(\cdot, t)$, $t \to \widetilde{\mathcal{L}}\widetilde{n}(s)$ and $\widetilde{n}_1'(0)$ belong to $C^{\frac{\alpha}{2m}}([0,\infty); X)$ by [26, Proposition 4.4.1(ii)], we have

$$\begin{cases}
v_1 \in C^{1+\frac{\alpha}{2m}}([1,\infty); X), \\
v_1(t) \in D(A_0) \subseteq \bigcap_{p>1} W^{2m,p}(\Omega) \qquad t \in [1,\infty)\,, \\
v_1' \in B([1,\infty); C^{\alpha}(\overline{\Omega})),
\end{cases} \tag{4.1.15}$$

and

$$\begin{cases}
v_1'(t) = \widetilde{L} v_1(t) + P^-[\widetilde{f}(t) + \widetilde{\mathcal{L}}\widetilde{n}(t) - \widetilde{n}_1'(0)]\,, \quad t \geq 0\,, \\
v_1(0) = P^-(u_0 - \widetilde{n}(0))\,,
\end{cases} \tag{4.1.16}$$

where we have used the fact that $D_{\widetilde{L}}(\frac{\alpha}{2m},\infty) \simeq C^\alpha(\overline{\Omega})$ (by Theorem 3.1.4 (ii)). On the other hand, since $\widetilde{f}$, $\widetilde{\mathcal{L}}\widetilde{n}$, $\widetilde{n}_1'(0)$ and v_1' belong to $B([1,\infty);C^\alpha(\overline{\Omega}))$, by (4.1.16) we conclude that $\widetilde{\mathcal{L}}v_1 \in B([1,\infty);C^\alpha(\overline{\Omega}))$.

Summing up we obtain

$$\begin{cases} v_1 \in C^{1+\frac{\alpha}{2m}}([1,\infty);X), & v_1' \in B([1,\infty);C^\alpha(\overline{\Omega})), \\ \widetilde{\mathcal{L}}v_1 \in B([1,\infty);C^\alpha(\overline{\Omega})), & v_1(t) \in \bigcap\limits_{p>1} W^{2m,p}(\Omega), \quad t \in [1,\infty)\,. \end{cases} \tag{4.1.17}$$

Considering v_2, since $n_1'(t)-n_1'(0) \in C^{\frac{\alpha}{2m}}([0,\infty);X)$, by [26, Proposition 4.4.1(ii)] v_2 satisfies the same properties as v_1 stated in (4.1.15) and $v_2(t) = P^-\widetilde{n}_1(t) + y(t)$, where $y(t)$ is a classical solution of

$$\begin{cases} y'(t) = \widetilde{\mathcal{L}}y(t) - P^-[n_1'(t) - \widetilde{n}_1'(0)], \quad t \geq 0\,, \\ y(0) = 0\,, \\ B_j y(t) = 0, \quad j = 1,\dots,m, \quad t \geq 0\,. \end{cases} \tag{4.1.18}$$

Moreover, since $\widetilde{\mathcal{L}}\widetilde{n}_1(t), n_1'(t) \in B([1,\infty);C^\alpha(\overline{\Omega}))$, we obtain similarly that v_2 satisfies the same properties as v_1 (see (4.1.17)).

Let us consider v_3. We set for each

$$\begin{cases} s = 1,\dots,m & \text{if } m_1 = 0\,, \\ s = 2,\dots,m & \text{if } m_1 > 0\,, \end{cases}$$

$$\psi_s(t) = P^-\mathcal{N}_s\mathcal{M}_s(\widetilde{g}_1(t) - \widetilde{g}_1(0),\dots,\widetilde{g}_s(t) - \widetilde{g}_s(0)), \quad t \in [0,\infty)\,,$$

and

$$v_{3s}(t) = \int_0^t e^{(t-s')\widetilde{L}} P^-\psi_s(s')\,\mathrm{d}s'\,.$$

Therefore

$$v_3(t) = P^-n_2(0) - \widetilde{L} \sum_{s=1 \text{ or } 2}^{m} v_{3s}(t)\,. \tag{4.1.19}$$

We have

$$\psi_s \in C^{\frac{2m+\alpha-m_s}{2m}}([0,\infty);\; D_{\widetilde{L}}(\tfrac{m_s}{2m},\infty)) \tag{4.1.20}$$

because of the fact that $B_j\mathcal{N}_s = 0$ for $j < s$. See (32) of §7 in [28] for more details. Applying [26, Theorem 4.3.16] with $\theta = \frac{2m+\alpha-m_s}{2m}$, $\beta = \frac{m_s}{2m}$, we obtain for every T>0

$$\widetilde{L}v_{3s} \in C^{1+\frac{\alpha}{2m}}([0,T];X), \qquad v_{3s}' \in B([0,T];D_{\widetilde{L}}(1+\tfrac{\alpha}{2m},\infty)).$$

By looking at the proof of Theorem 4.3.16 and Theorem 4.3.1(iii) in [26], we see that

$$\|\widetilde{L}v_{3s}\|_{C^{1+\frac{\alpha}{2m}}([0,T];X)} + \|\widetilde{L}v_{3s}'\|_{B([0,T];D_{\widetilde{L}}(\frac{\alpha}{2m},\infty))} \leq C\|\psi_s\|_{C^{\frac{2m+\alpha-m_s}{2m}}([0,\infty);\; C^{m_s}(\overline{\Omega}))}\,,$$

with the constant C independent of T and hence by (4.1.19) we get

$$v_3' \in B([1,\infty); C^\alpha(\overline{\Omega})) \cap C^{\frac{\alpha}{2m}}([1,\infty); X)$$

and $v_3(t) = P^- \widetilde{n}_2(0) - \widetilde{L} z(t)$, where $z(t)$ is a classical solution of

$$\begin{cases} z'(t) = \widetilde{L} z(t) + P^- [\widetilde{n}_2(t) - \widetilde{n}_2(0)], & t \geq 0\,, \\ z(0) = 0\,. \end{cases} \tag{4.1.21}$$

Moreover by (4.1.21) we easily check that

$$v_3' = \widetilde{\mathcal{L}} v_3 - \widetilde{\mathcal{L}} P^- n_2$$

and therefore $\widetilde{\mathcal{L}} v_3 \in B([1,\infty); C^\alpha(\overline{\Omega}))$. Summing up we obtain that v_3 satisfies the same properties as v_1 (see (4.1.17)).

We now consider v_4. Since again $t \to \widetilde{f}(\cdot, t)$, $t \to \widetilde{\mathcal{L}} \widetilde{n}(s)$ belong to $C^{\frac{\alpha}{2m}}([0,\infty); X)$, by [26, Proposition 4.4.2(ii)] we obtain that it satisfies the same properties as v_1 (see (4.1.17)).

Finally we consider v_5. Due to the estimates (4.1.12), v_5 is clearly bounded with values in $D(L^k)$ for every $k \in \mathbb{N}$. Moreover, Because $L(I - P^-)\widetilde{n} \in B([1,\infty); C^\alpha(\overline{\Omega}))$, $v_5' = \widetilde{L} v_5 - \widetilde{L}(I - P^-)\widetilde{n}$ is Hölder continuous with exponent $\frac{\alpha}{2m}$ with value in X and is bounded with value in $C^\alpha(\overline{\Omega})$. Hence v_5 satisfies the same properties as v_1 (see (4.1.17)).

Since $v = \sum_{i=1}^5 v_i$, we have

$$\begin{cases} v \in C^{1+\frac{\alpha}{2m}}([1,\infty); X), & v' \in B([1,\infty); C^\alpha(\overline{\Omega})), \\ \widetilde{\mathcal{L}} v \in B([1,\infty); C^\alpha(\overline{\Omega})), & v(t) \in \bigcap\limits_{p>1} W^{2m,p}(\Omega), \quad t \in [1,\infty)\,. \end{cases} \tag{4.1.22}$$

Now what is left is to prove that

$$v \in B([1,\infty);\, C^{2m+\alpha}(\overline{\Omega}))$$

and this can be done by using (iv) of Theorem 3.1.4, by means of (4.1.22) and the fact that $B_j v = \widetilde{g}_j \in B([1,\infty); C^{2m+\alpha-m_j}(\partial\Omega))$.

It follows that $v \in C^{2m+\alpha, 1+\frac{\alpha}{2m}}(\overline{\Omega} \times [1,\infty))$, and

$$\|v\|_{C^{2m+\alpha,1+\frac{\alpha}{2m}}(\overline{\Omega}\times[1,\infty))} \leq C(\|u_0\|_X + \|\widetilde{f}\|_{\mathbb{E}_0(\infty)} + \|\widetilde{g}\|_{\mathbb{F}(\infty)})\,,$$

which finishes the proof. □

4.2 An extension operator

In order to apply the semigroup theory, similarly as in the previous section, to obtain results for the asymptotic behavior of linear systems (see next section), we need to construct explicitly an extension operator for the case of vector-valued unknowns.

Let us recall our linear boundary problem:

$$(B_j u)(x) = \sum_{|\beta| \le m_j} b^j_\beta(x) \nabla^\beta u(x)\,, \quad x \in \partial\Omega, \quad j = 1, \ldots, mN\,. \tag{4.2.1}$$

Here $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}^N$, b^j_β are N-dimensional row-vectors and

$$0 \le m_1 \le m_2 \le \cdots \le m_{mN} \le 2m - 1\,.$$

Our goal is to construct explicitly a linear and bounded operator E such that for all $\theta' \in [0, \alpha]$,

$$\begin{cases} g_j \in C^{2m+\theta'-m_j}(\partial\Omega), \quad j = 1, \ldots, mN \implies E(g_1, \ldots, g_{mN}) \in C^{2m+\theta'}(\overline{\Omega}), \\ B_j E(g_1, \ldots, g_{mN}) = g_j, \quad j = 1, \ldots, mN\,. \end{cases} \tag{4.2.2}$$

Note that the case $N = 1$ is treated in [28, Theorem 6.3], i.e., Theorem 4.1.1.

The strategy for proving the existence of the extension operator E satisfying (4.2.2) is as follows: At first, by using the normality condition (3.1.9), we will reduce our linear system to an uncoupled linear system and then with the help of the scalar result, i.e., Theorem 4.1.1, applying it to each component, we finish the proof.

In the following, we set γ_j for the j^{th}-order normal derivatives precisely, for $j = 0, \ldots, 2m - 1$

$$\gamma_j u := D^j u \overbrace{[\nu, \cdots, \nu]}^{j-\text{times}} |_{\partial\Omega}\,,$$

which should be understood component-wise. Remind that $\nu(x)$ is the unit outer normal to $\partial\Omega$ at the point x and $n_k \ge 0$ are the number of k^{th}-order boundary conditions for $k = 0, \ldots, 2m - 1$.

Theorem 4.2.1. *Assume the operators B_j satisfy the regularity condition (H2) and the normality condition* (3.1.9)*. Then there exists a linear bounded operator E satisfying* (4.2.2)*.*

Proof. Without loss of generality we assume that $n_k \ne 0$ for all k between 0 and $2m - 1$, i.e., we have here included all orders k between 0 and $2m - 1$. Indeed, if $n_k = 0$ for some k, we could simply add the boundary conditions $\gamma_k u = 0$.

Let E be defined by

$$
\begin{aligned}
&E(g_1, \ldots, g_{mN}) \qquad\qquad (4.2.3)\\
&\quad := \Big(\mathcal{N}(\psi_{01}, \psi_{11}, \ldots, \psi_{2m-1,1}), \ldots, \mathcal{N}(\psi_{0N}, \psi_{1N}, \ldots, \psi_{2m-1,N})\Big),
\end{aligned}
$$

where the operator

$$\mathcal{N}(\psi_{0i}, \psi_{1i}, \ldots, \psi_{2m-1,i}) = \sum_{s=1}^{2m} \mathcal{N}_s \mathcal{M}_s(\psi_{0i}, \ldots, \psi_{s-1,i})$$

is the extension operator given in Theorem 4.1.1 for the boundary operators $B_j = \gamma_{j-1}$, for $j = 1, \ldots, 2m$. More precisely, $u = E(g_1, \ldots, g_{mN})$ solves the following uncoupled linear system of normal boundary conditions:

$$
\begin{cases}
\gamma_0 u = \psi_0\,, \\
\gamma_1 u = \psi_1\,, \\
\qquad \vdots \\
\gamma_{2m-1} u = \psi_{2m-1}\,,
\end{cases}
\tag{4.2.4}
$$

where

$$\psi_k(x) = \begin{pmatrix} \psi_{k1}(x) \\ \vdots \\ \psi_{kN}(x) \end{pmatrix}$$

will be defined below. Note that by looking at the proof of Theorem 4.1.1 or equivalently Theorem 6.3 in [28], one sees that the number of boundary conditions in Theorem 6.3 in [28] can be replaced by any m' as far as the normality condition is satisfied and $m_j \le 2m-1$ for all $j = 1, \ldots, m'$, which is definitely the case in our situation.

Setting $u = E(g_1, \ldots, g_{mN})$ in (4.2.2) and decomposing derivatives into normal and tangential derivatives, the last condition in (4.2.2) can be rewritten as

$$\sum_{i=0}^{j} S_{j,i} \gamma_i u = \varphi_j\,, \tag{4.2.5}$$

where $S_{j,i}$ are tangential differential operator of order at most $j - i$ and

$$\varphi_0 := \begin{pmatrix} g_1 \\ \vdots \\ g_{n_0} \end{pmatrix}_{n_0 \times 1}, \qquad \varphi_{k+1} := \begin{pmatrix} g_{\sum_{i=0}^{k} n_i + 1} \\ \vdots \\ g_{\sum_{i=0}^{k+1} n_i} \end{pmatrix}_{n_{k+1} \times 1}.$$

In particular for all $k = 0, \ldots, 2m-1$

$$S_{kk}(x) = \begin{pmatrix} \sum_{|\beta|=k} b_\beta^{j_1}(x)(\nu(x))^\beta \\ \vdots \\ \sum_{|\beta|=k} b_\beta^{j_{n_k}}(x)(\nu(x))^\beta \end{pmatrix}_{n_k \times N},$$

where $\{j_i : i = 1, \dots, n_k\} = \{j : m_j = k\}$ and for $j = 0, 1$ in (4.2.5) we have

$$\begin{cases} S_{00}(x)\gamma_0 u = \varphi_0\,, \\ S_{11}(x)\gamma_1 u + \text{ tangential derivatives} + \text{zeroth order normal derivatives} = \varphi_1\,. \end{cases} \tag{4.2.6}$$

By the normality condition, S_{kk} are surjective and therefore there exist matrices R_{kk} which have the same regularity as S_{kk} such that

$$S_{kk}R_{kk} = I \quad \text{ on } \mathbb{R}^{n_k}\,. \tag{4.2.7}$$

Now we are in a position to define ψ_k such that (4.2.5) holds. Define $\psi_0 := R_{00}\varphi_0$. Then

$$S_{00}\gamma_0 u = S_{00}\psi_0 = S_{00}R_{00}\varphi_0 = \varphi_0\,, \tag{4.2.8}$$

that is, (4.2.5) is satisfied for $j = 0$. Let us now consider $j = 1$ which corresponds to the first-order boundary conditions. Using the fact that $\gamma_0 u = \psi_0 = R_{00}\varphi_0$ all tangential derivatives and of course all zeroth-order normal derivatives can be calculated in terms of $R_{00}\varphi_0$. Consequently the condition (4.2.5) for $j = 1$ can be rewritten as

$$S_{11}(x)\gamma_1 u = \varphi_1(x) + \eta_1(x)\,,$$

for some $\eta_1(x)$ which can be calculated in terms of $R_{00}\varphi_0$ or precisely in terms of $(g_1, \dots, g_{n_0})$. Therefore, by defining $\psi_1 := R_{11}(\varphi_1 + \eta_1)$ we are done with the case $j = 1$. By iteration, we define

$$\psi_k := R_{kk}(\varphi_k + \eta_k)$$

for some η_k which can be calculated in terms of $\psi_0, \dots, \psi_{k-1}$. Moreover, by (4.1.7) for each $v \in C(\partial\Omega)$

$$(B_j(\mathcal{N}_s v_1, \dots, \mathcal{N}_s v_N))(x) \equiv 0\,, \quad x \in \partial\Omega\,, \quad m_j < s - 1\,. \tag{4.2.9}$$

And finally the regularity condition in (4.2.2) comes from the fact that the operator $\mathcal{N}$ has a similar regularity property, see (4.1.6), and this finishes the proof. $\square$

4.3 Asymptotic behavior for linear systems

Here we extend the result of Section 4.1 to the systems of mN boundary conditions for a linear system. Precisely we consider the linear problem (3.2.8), i.e.,

$$\begin{cases} \partial_t u + Au = f(t) & \text{in } \Omega, \quad t \geq 0, \\ Bu = g(t) & \text{on } \partial\Omega, \quad t \geq 0, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.3.1}$$

where $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}^N$, Ω is a bounded domain in $\mathbb{R}^n$ with $C^{2m+\alpha}$ boundary, $0 < \alpha < 1$, $g = (g_1, \ldots, g_{mN}), B = (B_1, \ldots, B_{mN})$, $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ and the operators A and B satisfy the conditions (H2), (L-S), (SP) and the normality condition (3.1.9).

Theorem 3.1.4 (i) states that the realisation $-A_0$ of $-A$ with homogeneous boundary conditions in $C(\overline{\Omega})$, defined in (3.1.5), is a sectorial operator.

Furthermore if $f \in \mathbb{E}_0(T)$, $g \in \mathbb{F}(T)$ and $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ satisfying the compatibility condition (3.2.9), the unique solution of (4.3.1) belongs to $\mathbb{E}_1(T)$ for all T and in addition it is given by the extension of the Balakrishnan formula with some adaptations. Indeed, by our explicit construction of the extension operator (see (4.2.3)), we simply can extend Theorem 4.1 in [28] to cover the linear systems (using the same technique). Therefore the following representation formula holds for each $t \in [0, T]$:

$$
\begin{aligned}
u(\cdot, t) = e^{tL}(u_0 - n(0)) &+ \int_0^t e^{(t-s)L}[f(\cdot, s) + \mathcal{L}n(s) - n_1'(0)]\mathrm{d}s \\
&+ n_1(t) - \int_0^t e^{(t-s)L}(n_1'(s) - n_1'(0))\,\mathrm{d}s \\
&- L\int_0^t e^{(t-s)L}[n_2(s) - n_2(0)]\,\mathrm{d}s + n_2(0) \\
= e^{tL}u_0 &+ \int_0^t e^{(t-s)L}[f(\cdot, s) + \mathcal{L}n(s)]\,\mathrm{d}s \\
&- L\int_0^t e^{(t-s)L}n(s)\,\mathrm{d}s\,,
\end{aligned} \tag{4.3.2}
$$

with $\mathcal{L} = -A$ and $L = -A_0$. Here

$$
n(t) = E(g_1(t), \ldots, g_{mN}(t))
$$

and similarly as before

$$
n_1(t) = \begin{cases} 0 & \text{if } m_1 > 0\,, \\ (\mathcal{N}_1\mathcal{M}_1(\psi_{0,1}), \ldots, \mathcal{N}_1\mathcal{M}_1(\psi_{0,N})) & \text{if } m_1 = 0 \end{cases}
$$

and $n_2(t) = n(t) - n_1(t)$, where $\psi_0 = (\psi_{0,1}, \ldots, \psi_{0,N})^T = R_{00}\varphi_0$ which can be written in terms of $g_1, \ldots, g_{n_0}$.

Theorem 4.3.1. *Let $0 < \omega < -\max\{\mathrm{Re}\,\lambda : \lambda \in \sigma^-(-A_0)\}$. Suppose f and g are such that $(\sigma, t) \to e^{\omega t}f(\sigma, t) \in \mathbb{E}_0(\infty)$ and $(\sigma, t) \to e^{\omega t}g(\sigma, t) \in \mathbb{F}(\infty)$. Suppose further that $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ satisfy the compatibility condition (3.2.9). Let u be the solution of (4.3.1). Then $v(\sigma, t) = e^{\omega t}u(\sigma, t)$ is* *bounded in* $[0,+\infty)\times\overline{\Omega}$ *if and only if*

$$(I-P^-)u_0 = -\int_0^{+\infty} e^{-sL}(I-P^-)[f(\cdot,s)+\mathcal{L}Eg(\cdot,s)]\,\mathrm{d}s \\ + L\int_0^{\infty} e^{-sL}(I-P^-)Eg(\cdot,s)\,\mathrm{d}s\,. \tag{4.3.3}$$

In this case, the function u *is given by*

$$\begin{aligned} u(\cdot,t) = e^{tL}P^-u_0 &+ \int_0^t e^{(t-s)L}P^-[f(\cdot,s)+\mathcal{L}Eg(\cdot,s)]\,\mathrm{d}s \\ &- L\int_0^t e^{(t-s)L}P^-Eg(\cdot,s)\,\mathrm{d}s \\ &- \int_t^{+\infty} e^{(t-s)L}(I-P^-)[f(\cdot,s)+\mathcal{L}Eg(\cdot,s)]\,\mathrm{d}s \\ &+ L\int_t^{+\infty} e^{(t-s)L}(I-P^-)Eg(\cdot,s)\,\mathrm{d}s, \end{aligned} \tag{4.3.4}$$

and the function $v = e^{\omega t}u$ *belongs to* $\mathbb{E}_1(\infty)$, *with the estimate*

$$\|v\|_{\mathbb{E}_1(\infty)} \le C(\|u_0\|_{C^{2m+\alpha}(\overline{\Omega})} + \|e^{\omega t}f\|_{\mathbb{E}_0(\infty)} + \|e^{\omega t}g\|_{\mathbb{F}(\infty)}).$$

Proof. The proof is exactly the same as the one of Theorem 4.1.2. More precisely, as you have seen, we used the abstract theories in the proof, i.e., the theory of semigroups of linear operators, except for the part related to the function v_3. Due to our explicit construction of the extension operator (see (4.2.3)) and taking into account (4.1.7) (in order to obtain the same result as (4.1.20)), we can work component-wise and get the same estimate for the function v_3. This finishes the proof. □

In the stable case, i.e, when $\sigma(-A_0) = \sigma^-(-A_0)$, We immediately get the following corollary of Theorem 4.3.1.

Corollary 4.3.2. *Let* $\omega_A := \sup\{Re\lambda : \lambda \in \sigma(-A_0)\} < 0$ *and* $\omega \in (0,-\omega_A)$. *Assume* f *and* g *are such that* $(\sigma,t) \to e^{\omega t}f(\sigma,t) \in \mathbb{E}_0(\infty)$ *and* $(\sigma,t) \to e^{\omega t}g(\sigma,t) \in \mathbb{F}(\infty)$ *and let* $u_0 \in C^{2m+\alpha}(\overline{\Omega})$ *satisfy the compatibility condition* (3.2.9). *Let* u *be the solution of* (4.3.1), *where* $u \in \mathbb{E}_1(T)$ *for all* $T < \infty$. *Then* $v(\sigma,t) = e^{\omega t}u(\sigma,t)$ *belongs to* $\mathbb{E}_1(\infty)$ *and*

$$\|v\|_{\mathbb{E}_1(\infty)} \le C(\|u_0\|_{C^{2m+\alpha}(\overline{\Omega})} + \|e^{\omega t}f\|_{\mathbb{E}_0(\infty)} + \|e^{\omega t}g\|_{\mathbb{F}(\infty)}).$$

Chapter 5

Lens-shaped Networks

In this chapter we show that the lens-shaped networks generated by circular arcs are dynamically stable under the surface diffusion flow. We will see how well the generalized principle of linearized stability in the parabolic Hölder spaces, i.e., Theorem 3.2 can be used as a tool to show the dynamical stability. Indeed, the set of equilibria forms a finite-dimensional smooth manifold and the resulting PDE has nonlocal terms in the highest order derivatives.

5.1 The geometric setting

Remind that the surface diffusion flow is a geometric evolution equation for an evolving hypersurface $\Gamma = \{\Gamma(t)\}_{t>0}$ in which

$$V = -\Delta_{\Gamma(t)}\kappa\,, \tag{5.1.1}$$

where V is the normal velocity, κ is the sum of the principle curvatures, and $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator of the hypersurface $\Gamma(t)$. Furthermore, we remind our sign convention that κ is negative for spheres for which we choose the outer unit normal.

Constant-mean-curvature surfaces are stationary solutions of (5.1.1). Now, it is natural to ask whether these solutions are dynamically stable under the flow. Indeed, Elliott and Garcke [15] showed the dynamical stability of circles in the plane and one year later Escher, Mayer and Simonett [16] proved the dynamical stability of spheres in higher dimensions. In general, the surfaces will meet an outer boundary or they might intersect at triple or multiple lines.

A lens-shaped network consists of two smooth curves and two rays arranged as in Figure 5.1, that is to say, we assume that the network has reflection symmetry across the x^1-axis and that the two curves meet the two rays with a constant angle $\pi - \theta$, where $0 < \theta < \pi$. Note that $\theta = \frac{\pi}{3}$ corresponds to symmetric angles at the triple junction.

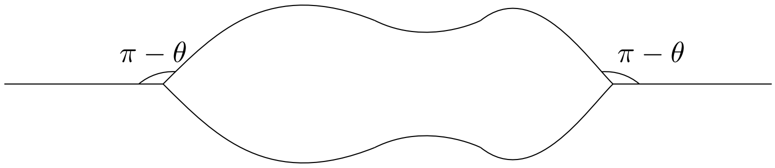

Figure 5.1: Lens-shaped network

More precisely, a lens-shaped network is determined by a curve Γ with the following property:

$$\begin{cases} \partial\Gamma \subset \{(x,y)\in\mathbb{R}^2 : y=0\}\,, \\ \sphericalangle(n,e_2)|_{\partial\Gamma}=\theta\,, \end{cases}$$

where n is the unit normal to Γ pointing outwards of the bubble, see e.g. Figure 5.2.

Then the entire lens-shaped network is defined by four curves: $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$, where Γ_1 is the curve Γ describe above, Γ_2 is the reflection of Γ_1 across the x^1 -axis and Γ_3, Γ_4 are the rays contained in the x^1-axis meeting Γ_1 and Γ_2 at triple junctions.

We study the following problem introduced by Garcke and Novick-Cohen [22]: Find evolving lens-shaped networks $\Gamma_1(t),\dots,\Gamma_4(t)$ as described above with the following properties:

$$\begin{cases} V_i = -\Delta_{\Gamma_i}\kappa_i & \text{on } \Gamma_i(t), \quad t>0, \quad (i=1,2,3,4), \\ \nabla_{\Gamma_1}\kappa_1\cdot n_{\partial\Gamma_1} = \nabla_{\Gamma_2}\kappa_2\cdot n_{\partial\Gamma_2} & \\ \qquad = \nabla_{\Gamma_i}\kappa_i\cdot n_{\partial\Gamma_i} & \text{on } \partial\Gamma_i(t), \quad t>0, \quad (i=3,4), \\ \Gamma_i(t)|_{t=0} = \Gamma_i^0 & \qquad\qquad\qquad (i=1,2,3,4), \end{cases} \tag{5.1.2}$$

where Γ_i^0 $(i=1,2,3,4)$ form a given initial lens-shaped network fulfilling the balance of flux condition, i.e., the second condition in (5.1.2). Here V_i and κ_i are the normal velocity and mean curvature of $\Gamma_i(t)$, respectively, $n_{\partial\Gamma_i}$ is the outer unit conormal of Γ_i at boundary points and ∇_{Γ_i} denotes the surface gradient of the curve $\Gamma_i(t)$.

We choose the unit normal $n_2(\cdot,t)$ of $\Gamma_2(t)$ to be pointed inwards of the bubble. Then with this choice of normals we observe that $\kappa_2=-\kappa_1$ at the boundary points and therefore we get

$$\kappa_1+\kappa_2+\kappa_i=0 \quad \text{on } \partial\Gamma_i(t) \quad \text{for } i=3,4\,,$$

which must hold at the triple junctions for more general triple junctions (non-symmetric, non-flat) with 120 degree angles. We refer to Garcke, Novick-Cohen [22] for the precise setting of the general problem.

Let us note that solutions to (5.1.2) preserve the enclosed area. Indeed, by Lemma 4.22 in [11], we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega(t)} 1\,\mathrm{d}x &= -\int_{\Gamma_2(t)} V_2\,\mathrm{d}s + \int_{\Gamma_1(t)} V_1\,\mathrm{d}s \\
&= \int_{\Gamma_2(t)} \Delta_{\Gamma_2}\kappa_2 \mathrm{d}s - \int_{\Gamma_1(t)} \Delta_{\Gamma_1}\kappa_1 \mathrm{d}s \\
&= \int_{\partial\Gamma_2(t)} \nabla_{\Gamma_2}\kappa_2 \cdot n_{\partial\Gamma_2}\,\mathrm{d}s - \int_{\partial\Gamma_1(t)} \nabla_{\Gamma_1}\kappa_1 \cdot n_{\partial\Gamma_1}\,\mathrm{d}s \\
&= 0\,,
\end{aligned}$$

where $\Omega(t)$ is defined as the region bounded by $\Gamma_1(t)$ and $\Gamma_2(t)$.

Using the fact that the curvature of $\Gamma_3(t)$ and $\Gamma_4(t)$ are zero, it is easy to verify that the family of lens-shaped networks $\Gamma_1(t), \ldots, \Gamma_4(t)$ evolves according to (5.1.2) if $\Gamma(t) := \Gamma_1(t)$ satisfies

$$\begin{cases}
V = -\Delta_\Gamma \kappa & \text{on } \Gamma(t)\,, \quad t > 0\,, \\
\partial\Gamma(t) \subset \{(x,y) \in \mathbb{R}^2 : y = 0\} & \qquad\qquad t > 0\,, \\
n \cdot e_2 = \cos\theta & \text{on } \partial\Gamma(t)\,, \quad t > 0\,, \\
\nabla_\Gamma \kappa \cdot n_{\partial\Gamma} = 0 & \text{on } \partial\Gamma(t)\,, \quad t > 0\,, \\
\Gamma(t)|_{t=0} = \Gamma_0\,,
\end{cases} \tag{5.1.3}$$

where Γ_0 is a given initial curve which fulfills the contact, angle and no-flux condition as above.

Remark 5.1.1. *The equation $V = -\Delta_\Gamma \kappa$ written in a local parameterization is a fourth-order parabolic equation and above we prescribe three boundary conditions. This is due to the fact that (5.1.3) is a free boundary problem because the points in $\partial\Gamma$ can move in the set $\{y = 0\}$, see [9] for a related second-order problem. Moreover, we would like to refer to the work of Schnürer and co-authors [38], where they consider the evolution of symmetric convex lens-shaped networks under the curve shortening flow.*

Let us look at equilibria of the problem (5.1.3). It is easy to verify that the curvature of the stationary solutions is constant and so the set of the stationary solutions of (5.1.3) consists precisely of all circular arcs that intersect the x-axis with $\pi - \theta$ degree angles denoted by $CA_r(a_1, -r\cos\theta)$, where $|r|$ is the radius and $(a_1, -r\cos\theta)$ are the coordinates of the center with $a_1 \in \mathbb{R}$, $r \in \mathbb{R}\backslash\{0\}$ (see Figure 5.2 for the justification of the coordinates of the center). Therefore the set of equilibria forms a 2-parameter family, the parameters are the radius of the circular arc and the first component of the center.

It is a goal of this section to prove the dynamical stability of such stationary solutions (see Theorem 6.4.1) using the generalized principle of linearized stability in parabolic Hölder spaces, i.e., Theorem 3.2.1.

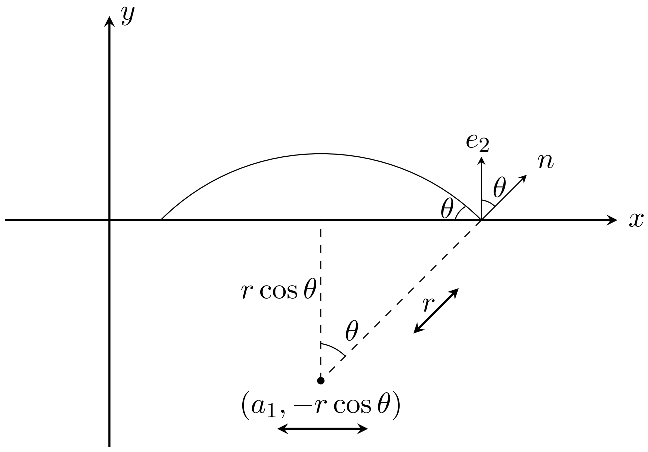

Figure 5.2: Circular arcs $CA_r(a_1, -r\cos\theta)$ for $r > 0$

Let us briefly outline how we proceed. At first we parameterize the curves around a stationary curve with the help of a modified distance function introduced in Depner and Garcke [12]. Note that the linearization in the case of a triple junction with boundary contact is calculated in [12] and the calculations can be easily modified to the present situation. We then formulate the evolution problem with the help of this parameterization and derive a highly nonlinear, nonlocal problem (5.2.17).

In Section 5.3, after deriving the linearization around the stationary solution, we see how our nonlinear, nonlocal problem fits well into our general evolution system (3.1.1). We then continue by checking the assumption (H1), (H2), (LS), (SP) and the normality condition (3.1.9).

Finally, in order to apply Theorem 3.2.1, it remains to check the assumption that the stationary solution is normally stable which is done in Section 5.4.

5.2 Parameterization and PDE formulation

5.2.1 Parameterization

In this section we introduce the mathematical setting in order to reformulate our geometric evolution law, i.e., (5.1.3) as a partial differential equation for an unknown function defined on a fixed domain. To this end, we use a parameterization with two parameters corresponding to a movement in tangential and normal direction, introduced in Depner and Garcke [12], see also [13].

Let us describe $\Gamma(t)$ with the help of a function $\rho : \Gamma^* \times [0, T) \to \mathbb{R}$ as graphs over some fixed stationary solution Γ^*. Note that the curvature κ^*

of Γ^* is constant and negative and the length of Γ^* is $2l^*$, where

$$-\kappa^* l^* = \theta \,.$$

Let x be the arc-length parameter of Γ^*. Then an arc-length parameterization of Γ^* is defined as

$$\Gamma^* = \{\Phi^*(x) : x \in [-l^*, l^*]\} \,.$$

For $\sigma \in \Gamma^*$, we set $(\Phi^*)^{-1}(\sigma) = x(\sigma) \in \mathbb{R}$. From now on, for simplicity, we set

$$\partial_\sigma w(\sigma) := \partial_x (w \circ \Phi^*)(x), \quad \sigma = \Phi^*(x), \tag{5.2.1}$$

i.e., we omit the parameterization. In particular, we use the slight abuse of the notation

$$w(\sigma) = w(x) \quad (\sigma \in \Gamma^*) \,. \tag{5.2.2}$$

In order to parameterize a curve close to Γ^*, we define

$$\begin{aligned} \Psi : \Gamma^* \times (-\epsilon, \epsilon) \times (-\delta, \delta) &\longrightarrow \mathbb{R}^2 \,, \\ (\sigma, w, r) &\mapsto \Psi(\sigma, w, r) := \sigma + w\, n^*(\sigma) + r\, \tau^*(\sigma) \,, \end{aligned} \tag{5.2.3}$$

where τ^* is a tangential vector field on Γ^* with support in a neighborhood of $\partial\Gamma^*$, which equals the outer unit conormal $n_{\partial\Gamma^*}$ at $\partial\Gamma^*$.

We define $\Phi = \Phi_{\rho,\mu}$ (we often omit the subscript (ρ, μ) for shortness) by

$$\Phi : \Gamma^* \times [0, T) \to \mathbb{R}^2 \,, \quad \Phi(\sigma, t) := \Psi(\sigma, \rho(\sigma, t), \mu(\mathrm{pr}(\sigma), t)) \,, \tag{5.2.4}$$

where

$$\rho : \Gamma^* \times [0, T) \to (-\epsilon, \epsilon) \,, \quad \mu : \partial\Gamma^* = \{a^*, b^*\} \times [0, T) \to (-\delta, \delta) \,. \tag{5.2.5}$$

The projection $\mathrm{pr} : \Gamma^* \to \partial\Gamma^* = \{a^*, b^*\}$ is defined by imposing the following condition: The point $\mathrm{pr}(\sigma) \in \partial\Gamma^*$ has the shortest distance on Γ^* to σ. Of course, in a small neighborhood of $\partial\Gamma^*$, the projection pr is well-defined and smooth. And this is enough for our purpose since we need this projection just near $\partial\Gamma^*$ because it is used in the product $\mu(\mathrm{pr}(\sigma), t)\tau^*(\sigma)$, where the second term vanishes outside a (small) neighborhood of $\partial\Gamma^*$. Finally, by setting for small $\epsilon, \delta > 0$ and fixed t

$$(\Phi)_t : \Gamma^* \to \mathbb{R}^2, \quad (\Phi)_t(\sigma) := \Phi(\sigma, t) \quad \forall \sigma \in \Gamma^* \,,$$

we define a new curve through

$$\Gamma_{\rho,\mu}(t) := \mathrm{image}((\Phi)_t) \,. \tag{5.2.6}$$

Note that for $\rho \equiv 0$ and $\mu \equiv 0$ the resulting curve coincides with a stationary curve Γ^*.

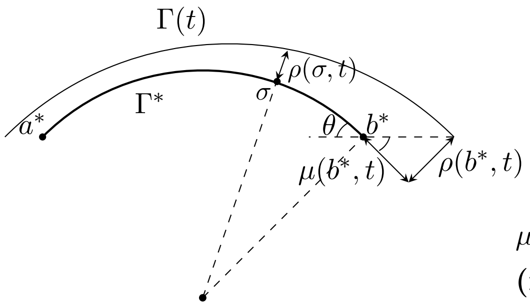

Figure 5.3: parameterizing of an evolving curve over a fixed stationary curve

As Figure 5.3 nicely illustrates, apart from the normal movement, close to the boundary points the parameter μ allows for tangential movement. Therefore the resulting curve not only have the possibility to meet the x-axis at its boundary points but also have the opportunity to be parameterized as a graph over the fixed stationary curve Γ^*. The price to pay is the appearance of nonlocal terms explained explicitly below.

Let us formulate the condition, that the curve $\Gamma(t)$ meets the x-axis at its boundary by

$$\langle \Phi(\sigma, t), e_2 \rangle = 0 \quad \text{ for } \sigma \in \partial\Gamma^*, \quad t \geq 0. \tag{5.2.7}$$

Here and hereafter, $\langle \cdot, \cdot \rangle$ means the inner product in $\mathbb{R}^2$. The following lemma shows that this condition leads to a linear dependency between μ and ρ at the boundary points and as a result, nonlocal terms will enter into formulations.

Lemma 5.2.1. *Equivalent to the equation (5.2.7) is the following condition*

$$\mu = (\cot\theta)\rho \quad \textit{on } \partial\Gamma^*. \tag{5.2.8}$$

Proof. Using the definition of Φ, the fact that $\langle \sigma, e_2 \rangle = 0$ on $\partial\Gamma^*$ and the angle condition on $\partial\Gamma^*$, we easily get

$$\mu = -\frac{\langle n^*, e_2 \rangle}{\langle n_{\partial\Gamma^*}, e_2 \rangle}\rho = -\Big(\frac{\cos\theta}{\cos(\frac{\pi}{2}+\theta)}\Big)\rho = (\cot\theta)\rho \quad \text{ on } \partial\Gamma^*$$

and vice versa. $\square$

We assume that the initial curve Γ_0 from (5.1.3) is also given as a graph over Γ^*, i.e.,

$$\Gamma_0 = \{\Psi(\sigma, \rho_0(\sigma), \mu_0(\mathrm{pr}(\sigma))) : \sigma \in \Gamma^*\}.$$

Furthermore, In order to apply our main result we make the assumption that $\rho_0 \in C^{4+\alpha}(\Gamma^*)$ with $\|\rho_0\|_{C^{4+\alpha}} \leq \epsilon$ for some small $\epsilon > 0$. Note that since Γ_0 is assumed to satisfy the contact condition, $\mu_0 = (\cot\theta)\rho_0$ at $\partial\Gamma^*$.

5.2.2 The nonlocal, nonlinear parabolic boundary value problem

First we derive evolution equation for ρ and μ which has to hold in the case that Γ in (5.2.6) solves (5.1.3). Note that the following calculations are adapted from [21]. The normal velocity V of $\Gamma(t)$ is given as

$$\begin{aligned} V(\sigma,t) &= \langle \Phi_t(\sigma,t), n(\sigma,t)\rangle \\ &= \frac{1}{J(\sigma,\rho(\sigma,t),\mu(\mathrm{pr}(\sigma),t)}\langle \Psi_w, R\Psi_\sigma\rangle \rho_t(\sigma,t) + \langle \Psi_r, n(\sigma,t)\rangle \mu_t(\mathrm{pr}(\sigma),t) \end{aligned}$$

where

$$\begin{aligned} n(\sigma,t) &= \frac{1}{J(\sigma,\rho(\sigma,t),\mu(\mathrm{pr}(\sigma),t)} R\Phi_\sigma(\sigma,t) \\ &= \frac{1}{J(\sigma,\rho(\sigma,t),\mu(\mathrm{pr}(\sigma),t)}\big(R\Psi_\sigma + R\Psi_w \rho_\sigma(\sigma,t)\big)\,. \end{aligned} \tag{5.2.9}$$

Here

$$J = J(\sigma,\rho,\mu) := |\Phi_\sigma| = \sqrt{|\Psi_\sigma|^2 + 2\langle \Psi_\sigma, \Psi_w\rangle \rho_\sigma + |\Psi_w|^2|\rho_\sigma|^2}\,, \tag{5.2.10}$$

and R denotes the anti-clockwise rotation by $\pi/2$ (remember our convention (5.2.1)). In addition, the curvature $\kappa(=\kappa(\sigma,\rho,\mu))$ of $\Gamma(t)$ is computed as

$$\begin{aligned} \kappa &= \frac{1}{(J(\sigma,\rho,\mu))^3}\langle \Phi_{\sigma\sigma}, R\Phi_\sigma\rangle \\ &= \frac{1}{(J(\sigma,\rho,\mu))^3}\Big[\langle \Psi_w, R\Psi_\sigma\rangle \rho_{\sigma\sigma} + \big\{2\langle \Psi_{\sigma w}, R\Psi_\sigma\rangle + \langle \Psi_{\sigma\sigma}, R\Psi_w\rangle\big\}\rho_\sigma \\ &\quad + \{\langle \Psi_{ww}, R\Psi_\sigma\rangle + 2\langle \Psi_{\sigma w}, R\Psi_w\rangle + \langle \Psi_{ww}, R\Psi_w\rangle \rho_\sigma\}(\rho_\sigma)^2 + \langle \Psi_{\sigma\sigma}, R\Psi_\sigma\rangle\Big]. \end{aligned} \tag{5.2.11}$$

Thus the surface diffusion equation can be formulated as

$$\rho_t = a(\sigma,\rho,\mu)\Delta(\sigma,\rho,\mu)\kappa(\sigma,\rho,\mu) + b(\sigma,\rho,\mu)\mu_t\,, \tag{5.2.12}$$

where

$$\begin{aligned} a(\sigma,\rho,\mu) &:= \frac{J(\sigma,\rho,\mu)}{\langle \Psi_w, R\Psi_\sigma\rangle}\,, \quad b(\sigma,\rho,\mu) := -\frac{\langle \Psi_r, R\Psi_\sigma\rangle + \langle \Psi_r, R\Psi_w\rangle \rho_\sigma}{\langle \Psi_w, R\Psi_\sigma\rangle}\,, \\ \Delta(\sigma,\rho,\mu)v &:= \frac{1}{J(\sigma,\rho,\mu)}\partial_\sigma\Big(\frac{1}{J(\sigma,\rho,\mu)}\partial_\sigma v\Big). \end{aligned}$$

Note that we omitted the mapping pr in the function μ as well as the term $(\sigma,\rho(\sigma,t),\mu(\mathrm{pr}(\sigma),t))$ in Ψ_u with $u \in \{\sigma, w, \mu\}$ for reasons of shortness.

Now we will write (5.2.12) as an evolution equation, which is nonlocal in space, just for the mapping ρ, using the linear dependence (5.2.8) on $\partial\Gamma^*$. To do this, with the help of (5.2.8), we rewrite (5.2.12) into

$$\partial_t \rho = \mathfrak{F}(\rho, \rho\circ \mathrm{pr}) + \mathfrak{b}(\rho,\rho\circ\mathrm{pr})\partial_t\left((\cot\theta)\rho\circ\mathrm{pr}\right) \quad \text{in } \Gamma^*\,, \tag{5.2.13}$$

where for $\sigma \in \Gamma^*$

$$\mathfrak{F}(\rho, \rho \circ \mathrm{pr})(\sigma) = a(\sigma, \rho, (\cot\theta)\rho \circ \mathrm{pr})\Delta(\sigma, \rho, (\cot\theta)\rho \circ \mathrm{pr})\kappa(\sigma, \rho, (\cot\theta)\rho \circ \mathrm{pr})\,,$$
$$\mathfrak{b}(\rho, \rho \circ \mathrm{pr})(\sigma) = b(\sigma, \rho, (\cot\theta)\rho \circ \mathrm{pr})\,.$$

By writing (5.2.13) on $\partial\Gamma^*$ and rearranging it we are led to

$$\big(1 - (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr})\big)\partial_t \rho = \mathfrak{F}(\rho, \rho \circ \mathrm{pr}) \quad \text{ on } \partial\Gamma^*\,.$$

Then, it follows that

$$\partial_t \rho = \frac{\mathfrak{F}(\rho, \rho \circ \mathrm{pr})}{1 - (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr})} \quad \text{ on } \partial\Gamma^*\,. \tag{5.2.14}$$

Note that since $\rho \circ \mathrm{pr} = \rho$ on $\partial\Gamma^*$, (5.2.14) is purely an equation for $\rho(\sigma)$ with $\sigma \in \partial\Gamma^* = \{a^*, b^*\}$. Near $\partial\Gamma^*$, where the projection pr is well-defined, the equation (5.2.14) leads to

$$\partial_t \mu(\mathrm{pr}(\sigma)) = (\cot\theta)\partial_t \rho(\mathrm{pr}(\sigma)) = (\cot\theta)\Big\{\frac{\mathfrak{F}(\rho, \rho \circ \mathrm{pr})}{1 - (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr})}\Big\} \circ \mathrm{pr}(\sigma).$$

Therefore the final equation for ρ is

$$\partial_t \rho = \mathfrak{F}(\rho, \rho \circ \mathrm{pr}) + (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr})\Big(\Big\{\frac{\mathfrak{F}(\rho, \rho \circ \mathrm{pr})}{1 - (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr})}\Big\} \circ \mathrm{pr}\Big) \quad \text{ on } \Gamma^*. \tag{5.2.15}$$

We emphasized that the second term on the right hand side of this equation contains nonlocal terms including the highest order (i.e, the fourth-order) point evaluation.

Furthermore, the boundary conditions on $\partial\Gamma^* = \{a^*, b^*\}$ can be written as

$$\begin{aligned}
\mathfrak{G}_1(\rho)(\sigma) &:= \langle n, e_2 \rangle - \cos\theta \\
&= \frac{1}{J(\sigma, \rho, (\cot\theta)\rho)}\big\langle R\Psi_\sigma + R\Psi_w \rho_\sigma\,,\, e_2 \big\rangle - \cos\theta = 0\,, \\
\mathfrak{G}_2(\rho)(\sigma) &:= \partial_\sigma(\kappa(\sigma, \rho, (\cot\theta)\rho)) = 0\,.
\end{aligned} \tag{5.2.16}$$

Note that the operators $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are completely local as the projection pr acts as the identity on its image $\partial\Gamma^*$.

Altogether, by recalling the parameterization (see (5.2.1)), we are led to the following nonlinear, nonlocal problem (see [13, Equation (20)] for the analogous result obtained for the mean curvature flow):

$$\begin{cases} \partial_t \rho(x,t) = \mathcal{F}\Big(x, \rho(x,t), \partial_x^1 \rho(x,t), \dots, \partial_x^4 \rho(x,t), \dots \\ \qquad \dots \rho(\pm l^*, t), \partial_x^1 \rho(\pm l^*, t), \dots, \boldsymbol{\partial_x^4 \rho(\pm l^*, t)}\Big) & \text{for } x \in [-l^*, l^*], \\ \qquad 0 = \mathcal{G}_1(x, \rho(x,t), \partial_x^1 \rho(x,t)) & \text{at } x = \pm l^*, \\ \qquad 0 = \mathcal{G}_2(x, \rho(x,t), \partial_x^1 \rho(x,t), \dots, \partial_x^3 \rho(x,t)) & \text{at } x = \pm l^*, \\ \rho(x,0) = \rho_0(x) & \text{for } x \in [-l^*, l^*], \end{cases} \tag{5.2.17}$$

where the term $\pm l^*$ should be understood in a sense that $+l^*$ is taken in (5.2.17) for the values of x in the neighborhood of l^* and $-l^*$ is taken in (5.2.17) for the values of x in the neighborhood of $-l^*$.

Note that the functions $\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2$ are smooth with respect to the ρ-dependent variables in some neighborhood of $\rho \equiv 0$ as well as the first variable. Indeed as you have seen above, these are rational functions with smooth coefficients in the ρ-dependent variables (possibly inside of square roots which are equal to 1 at $\rho \equiv 0$, see (5.2.10)) with nonzero denominator at $\rho \equiv 0$.

Remark 5.2.2. *Exactly at this point one needs to use the classical setting, e.g. the parabolic Hölder setting rather than the standard L_p-setting (which is a natural choice), i.e.,*

$$W^{1,p}\big((0,T); L_p((-l^*, l^*))\big) \cap L_p\big((0,T); W^{4,p}((-l^*, l^*)\big)$$

because of the nonlocal term $\boldsymbol{\partial_x^4 \rho(\pm l^*, t)}$, see (5.2.17), which can not be defined in this L_p-setting.

5.3 Linearization and general setting

For the linearization of (5.2.17) around $\rho \equiv 0$, that is around the stationary solution Γ^*, we refer to [12] (see also [13]). More precisely, the linearization of the surface diffusion equation is done in [12, Lemma 3.2] and a similar argument as in [12, Lemma 3.4] gives the following linearization of the angle condition

$$\partial_{n_{\partial\Gamma^*}} \rho + \kappa_{n_{\partial\Gamma^*}} \mu = 0 \quad \text{ on } \partial\Gamma^* .$$

Altogether, using the following facts (remind that x is the arc-length parameter of Γ^* and let T^* denote the unit tangential vector of Γ^*)

$$\begin{aligned} \Delta_{\Gamma^*} \rho &= \partial_x^2 \rho & &\text{for } x \in [-l^*, l^*], \\ \partial_{n_{\partial\Gamma^*}} \rho &= \nabla_{\Gamma^*} \rho \cdot n_{\partial\Gamma^*} = \partial_x \rho \, (T^* \cdot n_{\partial_{\Gamma^*}}) = \pm \partial_x \rho & &\text{at } x = \pm l^*, \\ \kappa_{n_{\partial\Gamma^*}} &= \kappa^* & &\text{at } x = \pm l^*, \\ \mu &= \cot\theta \rho & &\text{at } x = \pm l^*. \end{aligned}$$

we get for the linearization of (5.2.17) around $\rho \equiv 0$ the following linear equation for ρ

$$\begin{cases} \partial_t \rho + \partial_x^2(\partial_x^2 + (\kappa^*)^2)\rho = f & \text{for } x \in [-l^*, l^*]\,, \\ \pm\partial_x \rho + \kappa^*(\cot\theta)\rho = g_1 & \text{at } x = \pm l^*\,, \\ \partial_x(\partial_x^2 + (\kappa^*)^2)\rho = g_2 & \text{at } x = \pm l^*\,. \end{cases}$$

Remark 5.3.1. *Note that the linearization does not have any nonlocal term particulary because of the fact that we linearized around stationary solutions.*

Now the nonlinear, nonlocal problem (5.2.17) can be restated as a perturbation of a linearized problem, that is of the form (3.1.1), where $\Omega = (-l^*, l^*)$, the operator A is given by

$$(Au)(x) = \partial_x^2(\partial_x^2 + (\kappa^*)^2)u(x)\,, \quad x \in [-l^*, l^*]\,,$$

and the B_j's are given by

$$(B_1 u)(x) = \pm\partial_x u(x) + \kappa^*(\cot\theta)u(x)\,, \quad x = \pm l^*\,,$$

$$(B_2 u)(x) = \partial_x(\partial_x^2 + (\kappa^*)^2)u(x)\,, \quad x = \pm l^*\,.$$

If we write (5.2.17) in the form of (3.1.1), the corresponding F is a regular function defined in a neighborhood of 0 in $C^4(\overline{\Omega})$ with values in $C(\overline{\Omega})$. Indeed, it is Frechet-differentiable of arbitrary order in a neighborhood of zero (using the differentiability of composition operators, see e.g. Theorem 1 and 2 of [37, Section 5.5.3]) and a similar argument works for the corresponding functions G_1 and G_2. In particular, the assumption (H1) is satisfied with $R = R'$ for sufficiently small R'.

Clearly, the operators A, B_1, B_2 satisfy the assumption (H2), the operator A is uniformly strongly parabolic and the operators $B = (B_1, B_2)$ satisfy the normality condition (3.1.9).

Let us verify that the linearized problem satisfies the complementarity condition, i.e., (LS). For $x = \pm l^*$ and $\lambda \in \overline{\mathbb{C}_+}$, $\lambda \neq 0$ we should consider the following ODE

$$\begin{cases} \lambda v(y) + \partial_y^4 v(y) = 0\,, \quad y > 0\,, \\ \partial_y v(0) = 0\,, \quad \partial_y^3 v(0) = 0\,, \end{cases} \tag{5.3.1}$$

and prove that $v = 0$ is the only solution which vanishes at infinity. This can be done by the energy method. Testing the first line in (5.3.1) with $\bar{v}$ and using the boundary conditions and the fact that v and therefore its derivatives vanish at infinity (since solutions of (5.3.1) are the linear combinations of exponential functions) we obtain

$$
\begin{aligned}
0 &= \lambda \int_0^\infty |v|^2 \,\mathrm{d}y + \int_0^\infty \bar{v}\, \partial_y^4 v \,\mathrm{d}y \\
&= \lambda \int_0^\infty |v|^2 \,\mathrm{d}y - \int_0^\infty \partial_y \bar{v}\, \partial_y^3 v \,\mathrm{d}y \\
&= \lambda \int_0^\infty |v|^2 \,\mathrm{d}y + \int_0^\infty |\partial_y^2 v|^2 \,\mathrm{d}y \,.
\end{aligned}
$$

Since $0 \neq \lambda \in \overline{\mathbb{C}_+}$, the function v has to be zero and so the claim follows.

Concerning the compatibility condition, as we have assumed that the initial curve satisfies the contact, angle, and no-flux conditions, we get at $x = \pm l^*$

$$
\begin{cases}
\mathcal{G}_1(x, \rho_0(x,t), \partial_x^1 \rho_0(x,t)) = 0 \,, \\
\mathcal{G}_2(x, \rho_0(x,t), \partial_x^1 \rho_0(x,t), \partial_x^2 \rho_0(x,t), \partial_x^3 \rho_0(x,t)) = 0 \,,
\end{cases}
\tag{5.3.2}
$$

which is equivalent to the corresponding compatibility condition (3.1.10) since we do not have zeroth-order boundary conditions.

5.4 $\rho \equiv 0$ is normally stable

In this section, we will show that $\rho \equiv 0$, which corresponds to Γ^*, is normally stable, i.e., it satisfies the assumption (i)-(iv) in Theorem 3.2.1.

To begin with, let us consider the eigenvalue problem for the linearized operator A_0 (see (3.1.5) for the precise definition of A_0) which reads as follows

$$
\begin{cases}
\lambda u - \partial_x^2 (\partial_x^2 + (\kappa^*)^2) u = 0 & \text{in } [-l^*, l^*] \,, \\
\pm \partial_x u + \kappa^* \cot\theta \, u = 0 & \text{at } x = \pm l^* \,, \\
\partial_x (\partial_x^2 + (\kappa^*)^2) u = 0 & \text{at } x = \pm l^* \,,
\end{cases}
\tag{5.4.1}
$$

where $u \in D(A_0)$. Multiplying the first line in (5.4.1) with $\big(\partial_x^2 \overline{u} + (\kappa^*)^2 \overline{u}\big)$ and using integration by parts we get

$$
-\lambda I(u,u) + \int_{-l^*}^{l^*} |\partial_x (\partial_x^2 + (\kappa^*)^2) u|^2 \mathrm{d}x = 0 \,,
\tag{5.4.2}
$$

where

$$
I(u,u) = \int_{-l^*}^{l^*} |\partial_x u|^2 - (\kappa^*)^2 |u|^2 \,\mathrm{d}x + \kappa^* \cot\theta \left(|u(l^*)|^2 + |u(-l^*)|^2 \right) .
$$

Note that the same bilinear form appears in [20, p. 1040] (taking $h_+ = h_- = \kappa^* \cot\theta$ in [20]). Furthermore, we refer to [24, Proposition 3.3], where a related bilinear form appears as the second variation of the area functional for double bubbles.

We first consider the case where $\lambda \neq 0$. The positivity of $I(u,u)$ is shown in [20, Section 7], indeed we have

$$h = \kappa^* \cot\theta = \kappa^* \cot(-\kappa^* l^*) = -\frac{\kappa^*}{\tan(\kappa^* l^*)}\,,$$

which is the same equality as in [20, p. 1053]. Now (5.4.2) implies that all eigenvalues are real and positive except zero, in particular the operator A_0 satisfies the assumption (iv) in Theorem 3.2.1.

For $\lambda = 0$, the bilinear form (5.4.2) implies $\partial_x^2 u + (\kappa^*)^2 u = \tilde{c}$, where $\tilde{c}$ is a constant. It follows that $u = a\sin(\kappa^* x) + b\cos(\kappa^* x) + c$, where a, b and c are constants. Applying the boundary conditions we get $b = -c\cos\theta$ and therefore we obtain a 2-dimensional eigenspace for the eigenvalue $\lambda = 0$. In fact we compute

$$N(A_0) = \operatorname{span}\{\sin(\kappa^* x)\,,\, 1 - (\cos\theta)\cos(\kappa^* x)\}\,.$$

Next, let us verify that the eigenvalue 0 of A_0 is semi-simple. Since the operator A_0 has a compact resolvent (see Remark 3.1.2), the semi-simplicity condition is equivalent to the condition that $N(A_0) = N(A_0^2)$ according to Lemma 2.3.6. To show that $N(A_0) = N(A_0^2)$, it can easily be seen that it is sufficient to prove the existence of a projection

$$P : X \to \mathcal{R}(P) = N(A_0)$$

such that P commutes with A_0, that is, $PA_0u = A_0Pu (= 0)$ for all $u \in D(A_0)$.

Indeed we can construct such a projection in the following way:

$$P : X \to N(A_0) : u \mapsto Pu := \alpha_1(u)v_1 + \alpha_2(u)v_2\,, \tag{5.4.3}$$

where

$$v_1 = 1 - \cos\theta\cos(\kappa^* x)\,, \quad v_2 = \sin(\kappa^* x)\,,$$

$$\alpha_1(u) = \frac{\int_{-l^*}^{l^*} u(x)\,\mathrm{d}x}{\int_{-l^*}^{l^*} v_1(x)\mathrm{d}x}\,, \quad \alpha_2(u) = \frac{(u - \alpha_1(u)v_1, v_2)_{-1}}{(v_2, v_2)_{-1}}\,.$$

Here, the inner product is defined as

$$(\rho_1, \rho_2)_{-1} := \int_{-l^*}^{l^*} \partial_x u_{\rho_1} \partial_x u_{\rho_2}\,\mathrm{d}x,$$

where $u_{\rho_i} \in H^1(-l^*, l^*)$ for a given $\rho_i \in H^{-1}(-l^*, l^*) := (H^1(-l^*, l^*))'$ with $\langle \rho_i, 1\rangle_{H^{-1},H^1} = 0$ satisfies

$$\langle \rho_i, \varphi \rangle_{H^{-1},H^1} = \int_{-l^*}^{l^*} \partial_x u_{\rho_i} \partial_x \varphi \,\mathrm{d}x$$

for all $\varphi \in H^1(-l^*, l^*)$ (see [20, Section 4] for more details). Here we denote by $\langle \cdot, \cdot \rangle_{H^{-1},H^1}$ the duality paring between $H^{-1}(-l^*, l^*)$ and $H^1(-l^*, l^*)$.

Since

$$\int_{-l^*}^{l^*} v_1(x)\,\mathrm{d}x \neq 0, \quad \int_{-l^*}^{l^*} v_2(x)\,\mathrm{d}x \neq 0 \quad \text{and} \quad \int_{-l^*}^{l^*} u(x) - \alpha_1(u(x)) v_1(x)\,\mathrm{d}x = 0\,,$$

the coefficients $\alpha_1(u), \alpha_2(u)$ are well defined and moreover $\alpha_i(v_j) = \delta_{ij}$. Therefore P acts as identity on its image $N(A_0)$ or equivalently we get $P^2 = P$ and $R(P) = N(A_0)$.

Furthermore, for $u \in D(A_0)$ we have

$$\begin{aligned}
\alpha_1(A_0 u) &= \frac{\int_{-l^*}^{l^*} A_0 u(x)\,\mathrm{d}x}{\int_{-l^*}^{l^*} v_1(x)\,\mathrm{d}x} \\
&= \frac{\int_{-l^*}^{l^*} \partial_x^2(\partial_x^2 + (\kappa^*)^2) u\,\mathrm{d}x}{\int_{-l^*}^{l^*} v_1(x)\,\mathrm{d}x} = \frac{\partial_x(\partial_x^2 + (\kappa^*)^2) u|_{-l^*}^{l^*}}{\int_{-l^*}^{l^*} v_1(x)\,\mathrm{d}x} = 0\,, \\
\alpha_2(A_0 u) &= \frac{(A_0 u, v_2)_{-1}}{(v_2, v_2)_{-1}} = \frac{(u, A_0 v_2)_{-1}}{(v_2, v_2)_{-1}} = 0\,,
\end{aligned}$$

where we have used the facts that $v_2 \in N(A_0)$ and the operator A_0 is symmetric with respect to the inner product $(\cdot,\cdot)_{-1}$ (see [20, Lemma 5.1]). Therefore

$$P A_0 u = \alpha_1(A_0 u) v_1 + \alpha_2(A_0 u) v_2 = 0\,.$$

This completes the proof of the existence of the desired projection. Consequently the assumption (iii) in Theorem 3.2.1 is verified.

We continue by proving the assumption (i) in Theorem 3.2.1, i.e., near $\rho \equiv 0$, which corresponds to Γ^*, the set $\mathcal{E}$ of equilibria of (5.2.15), (5.2.16) creates a C^2-manifold of dimension 2. According to (3.1.2), $\rho \in \mathcal{E}$ if and only if

$$\begin{cases}
0 = \mathfrak{F}(\rho, \rho \circ \mathrm{pr}) & \\
\qquad + (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr}) \Big(\Big\{ \dfrac{\mathfrak{F}(\rho, \rho \circ \mathrm{pr})}{1 - (\cot\theta)\mathfrak{b}(\rho, \rho \circ \mathrm{pr})} \Big\} \circ \mathrm{pr} \Big) & \text{on } \Gamma^*\,, \\
0 = \mathfrak{G}_1(\rho) & \text{on } \partial\Gamma^*\,, \\
0 = \mathfrak{G}_2(\rho) & \text{on } \partial\Gamma^*\,.
\end{cases} \tag{5.4.4}$$

Here and in what follows we omit the condition $\rho \in B_{X_1}(0, R)$ from the right hand side for reasons of shortness. Similarly as before, by writing the first line in (5.4.4) on $\partial\Gamma^*$ we get $\mathfrak{F}(\rho, \rho \circ \mathrm{pr}) = 0$ on $\partial\Gamma^*$ and hence

$$\rho \in \mathcal{E} \Leftrightarrow \begin{cases} 0 = \mathfrak{F}(\rho, \rho \circ \mathrm{pr}) & \text{on } \Gamma^*\,, \\ 0 = \mathfrak{G}_1(\rho) & \text{on } \partial\Gamma^*\,, \\ 0 = \mathfrak{G}_2(\rho) & \text{on } \partial\Gamma^*\,. \end{cases}$$

Using the definition of $\mathfrak{F}$ and no-flux condition $\mathfrak{G}_2$, by applying Gauss's theorem it follows that

$$\rho \in \mathcal{E} \Leftrightarrow \begin{cases} \rho \in B_{X_1}(0, R)\,, & \\ \kappa\big(\rho, (\cot\theta)\rho \circ \mathrm{pr}\big) \text{ is constant} & \text{on } \Gamma^*\,, \\ \mathfrak{G}_1(\rho) = \langle n, e_2 \rangle - \cos\theta = 0 & \text{on } \partial\Gamma^*\,. \end{cases}$$

Therefore, by taking into account Lemma 5.2.1 we conclude that

$$\mathcal{E} = \Big\{\rho : \rho \text{ parameterizes an element of } CA_r(a_1, -r\cos\theta) \text{ sufficiently close to } \Gamma^*\Big\}.$$

Clearly $\mathcal{E} \neq \emptyset$ as $\rho \equiv 0$ parameterizes $\Gamma^* = CA_{r^*}(0, -r^*\cos\theta)$.

The following lemma demonstrates that actually, all the circular arcs $CA_r(a_1, -r\cos\theta)$ sufficiently close to Γ^* can be parameterized by a unique function ρ depending smoothly on a_1 and r. The idea is to use the implicit function theorem 2.3.1.

Lemma 5.4.1. *There exist positive numbers ϵ and R'' such that each of the circular arcs $CA_r(a_1, -r\cos\theta)$ with $(a_1, r) \in B_{\mathbb{R}^2}((0, r^*), \epsilon)$ is parameterized by a unique $\rho \in B_{X_1}(0, R'')$. Moreover the set $\mathcal{E}$ creates a C^2-manifold of dimension 2 in $X_1 = C^{4+\alpha}([-l^*, l^*])$.*

Proof. Without loss of generality we may assume that Γ^* is centered at the origin of $\mathbb{R}^2$. We use the implicit function theorem with

$$X = \mathbb{R}^2\,, \quad Y = Z = C^{4+\alpha}([-l^*, l^*])\,, \quad (x_0, y_0) = ((0, r^*), 0)$$

and

$$F : X \times Y \longrightarrow Z$$

where (remember our abuse of notation (5.2.2))

$$F(a_1, r, \rho)(\sigma) := \|\Psi(\sigma, \rho(\sigma), \mu(\mathrm{pr}(\sigma)) - (a_1, r\cos\theta)\|^2 - r^2\,, \tag{5.4.5}$$

for all $(a_1, r) \in X$, $\rho \in Y$ and $\sigma \in [-l^*, l^*]$. Here

$$\Psi(\sigma, \rho(\sigma), \mu(\mathrm{pr}(\sigma)) = \sigma + \rho(\sigma)n^*(\sigma) + \mu(\mathrm{pr}(\sigma))\tau^*(\sigma), \quad \mu \circ \mathrm{pr} = (\cot\theta)\rho \circ \mathrm{pr}\,.$$

The derivative $F_\rho(0, r^*, 0)$ is given by

$$F_\rho(0, r^*, 0)(v)(\sigma) \ = \big\langle vn^*(\sigma) + (\cot\theta)(v \circ \mathrm{pr})\tau^*(\sigma)\,,\, \sigma - (0, -r^*\cot\theta)\big\rangle\,.$$

Using the fact that $\sigma - (0, -r^* \cot\theta) = r^* n^*(\sigma)$ (see Figure 5.4) and that τ^* is a tangential vector field, we get

$$F_\rho(0, r^*, 0)(v) = r^* v$$

which implies that $F_\rho(0, r^*, 0)$ is bijective. Furthermore, it is easy to see that F is a smooth map on a neighborhood of $(0, r^*, 0)$.

Hence there exist positive numbers ϵ and R'' such that, for every $(a_1, r) \in B_{\mathbb{R}^2}((0, r^*), \epsilon)$, there is exactly one $\rho(a_1, r) \in X_1$ for which $\rho \in B_{X_1}(0, R'')$ and $F(a_1, r, \rho(a_1, r)) = 0$, i.e.,

$$\|\Psi\big(\sigma, \rho(\sigma, a_1, r), \mu(\mathrm{pr}(\sigma))\big) - (a_1, r\cos\theta)\|^2 - r^2 = 0 \qquad \text{for } \sigma \in \Gamma^* . \quad (5.4.6)$$

In addition the mapping $(a_1, r) \mapsto \rho(a_1, r)$ is smooth on a neighborhood of $x_0 = (0, r^*)$. Finally it is not hard to see that the curve Γ parameterized by $\rho = \rho(a_1, r)$ (i.e., the solution to $F = 0$) belongs to $CA_r(a_1, -r\cos\theta)$. Indeed, the contact condition is satisfied as we have already included here the linear dependency (5.2.8) and now taking into account the relationship between the center and the radius (see (5.4.5)), we find easily that the curve Γ satisfies the desired angle condition (see Figure 5.2). This proves the first assertion of the lemma.

Define a function

$$\begin{aligned} \Upsilon : U &\longrightarrow X_1 \\ (a_1, r) &\mapsto \rho(a_1, r) , \end{aligned}$$

where $U = B_{\mathbb{R}^2}((0, r^*), \epsilon)$. Clearly, the function Υ is smooth and so in particular C^2. Furthermore, $\Upsilon(U) = \mathcal{E}$ with the constant R in defining relation (3.1.2) replaced by R''; and $\Upsilon((0, r^*)) = 0$. Now to prove that the set $\mathcal{E}$ creates a C^2-manifold of dimension 2 in $X_1 = C^{4+\alpha}([-l^*, l^*])$ we only need to verify that the rank of $\Upsilon'((0, r^*))$ is equal to 2. (See the definition of a manifold on page 24.)

Differentiating (5.4.6) with respect to r and evaluating it at $(a_1, r) = (0, r^*)$, we get

$$\begin{aligned} &\big\langle \partial_r \rho(\sigma, 0, r^*) n^*(\sigma) + \partial_r \mu(\mathrm{pr}(\sigma), 0, r^*) \tau^*(\sigma) - (0, -\cos\theta) , \\ &\qquad\qquad\qquad\qquad \sigma - (0, -r^* \cos\theta) \big\rangle - r^* = 0. \end{aligned}$$

Again using the fact that $\sigma - (0, -r^* \cos\theta) = r^* n^*(\sigma)$ and that τ^* is a tangential vector field, we get

$$r^* \partial_r \rho(\sigma, 0, r^*) + \cos\theta(\sigma_2 + r^* \cos\theta) = r^* .$$

By writing it in spherical coordinates, i.e.,

$$\sigma = (\sigma_1, \sigma_2) = \Phi^*(x) = \Big(r^* \sin(\frac{x}{r^*}) , \, r^* \cos(\frac{x}{r^*}) - r^* \cos\theta\Big)$$

we obtain

$$\partial_r \rho(x,0,r^*) = 1 - \cos\theta\cos(\kappa^* x)\,.$$

Analogously, we get $\partial_{a_1}\rho(x,0,r^*) = -\sin(\kappa^* x)$, which finishes the proof. □

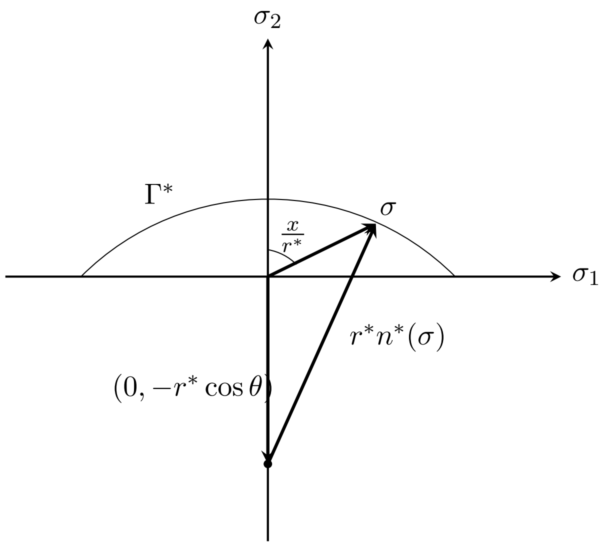

Figure 5.4: The stationary solution Γ^*

Finally it remains to prove the assumption (ii). This is an immediate consequence of the facts that $T_0\mathcal{E} \subseteq N(A_0)$, see (3.1.8), and that $\dim(\mathcal{E}) = \dim(N(A_0))$.

5.5 Lens-shaped networks generated by circular arcs are dynamically stable under surface diffusion flow

In summary, all the assumptions of Theorem 3.2.1 for $R = \min\{R', R''\}$ are satisfied. Thus applying Theorem 3.2.1, we obtain

Theorem 5.5.1. *Suppose Γ^* is an arbitrary circular arc intersecting the x^1-axis with an angle θ. Then $\rho \equiv 0$ is a stable equilibrium of (5.2.17) in the class of all initial values $\rho_0 \in X_1 = C^{4+\alpha}([-l^*, l^*])$ satisfies the compatibility condition (5.3.2). Moreover there exists a $\delta > 0$ such that if $\|\rho_0\|_{X_1} < \delta$ then the corresponding solution of (5.2.17) exists globally in $C^{1+\frac{\alpha}{4},4+\alpha}([0,\infty) \times [-l^*, l^*])$ and converges at an exponential rate in X_1 to some equilibrium ρ_∞ as $t \to \infty$.*

In this sense, the lens-shaped network generated by Γ^* is stable. In addition, every lens-shaped solution of (5.1.2) that starts sufficiently close to the one generated by Γ^* and satisfies the angle condition and the balance of flux condition at $t = 0$ exists globally and converges to some lens-shaped network generated by a circular arc at an exponential rate as $t \to \infty$. In other words, we have shown the dynamical stability under the surface diffusion flow.

Chapter 6

Planar Double Bubbles

6.1 The geometric setting

A planar double bubble $\Gamma \subset \mathbb{R}^2$ consists of three curves $\Gamma_1, \Gamma_2, \Gamma_3$ meeting two common points p_+, p_- (triple junctions) at their boundaries such that Γ_1 and Γ_2 (resp. Γ_2 and Γ_3) enclose the connected region R_1 (resp. R_2). Hence the curve Γ_2 is the curve separating R_1 and R_2, see Figure 6.1.

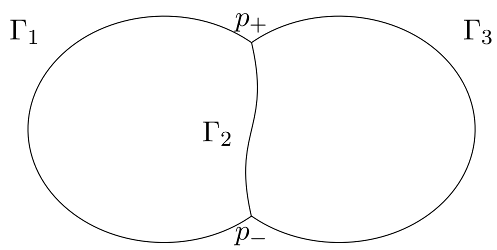

Figure 6.1: A good example of a planar double bubble $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$

We study the following problem introduced by Garcke and Novick-Cohen [22]: Find evolving planar double bubbles $\Gamma(t) = \{\Gamma_1(t), \Gamma_2(t), \Gamma_3(t)\}$ with the following properties:

$$\left.\begin{aligned}
&V_i = -\Delta_{\Gamma_i}\kappa_i && \text{on } \Gamma_i(t)\,,\\
&\sphericalangle(\Gamma_1(t),\Gamma_2(t)) = \sphericalangle(\Gamma_2(t),\Gamma_3(t)) = \sphericalangle(\Gamma_3(t),\Gamma_1(t)) = \tfrac{2\pi}{3} && \text{on } \Sigma(t)\,,\\
&\kappa_1 + \kappa_2 + \kappa_3 = 0 && \text{on } \Sigma(t)\,,\\
&\nabla_{\Gamma_1}\kappa_1 \cdot n_{\partial\Gamma_1} = \nabla_{\Gamma_2}\kappa_2 \cdot n_{\partial\Gamma_2} = \nabla_{\Gamma_3}\kappa_3 \cdot n_{\partial\Gamma_3} && \text{on } \Sigma(t)\,,\\
&\Gamma_i(t)|_{t=0} = \Gamma_i^0\,,
\end{aligned}\right\} \tag{6.1.1}$$

where $i = 1, 2, 3$, $\Gamma_i(t) \subset \mathbb{R}^2$, and

$$\partial\Gamma_1(t) = \partial\Gamma_2(t) = \partial\Gamma_3(t)\,\big(= \{p_+(t), p_-(t)\} =: \Sigma(t)\big)\,.$$

Here V_i is the normal velocity, κ_i is the curvature, and Δ_{Γ_i} is the Laplace-Beltrami operator of the curve Γ_i ($i = 1, 2, 3$). Also ∇_{Γ_i} denotes the surface gradient and $n_{\partial\Gamma_i}$ denotes the outer unit conormal of Γ_i at $\partial\Gamma_i$ ($i = 1, 2, 3$).

Moreover $\Gamma^0 = \{\Gamma_1^0, \Gamma_2^0, \Gamma_3^0\}$ is a given initial planar double bubble, which fulfills the angle $(6.1.1)_2$, the curvature $(6.1.1)_3$ and the balance of flux condition $(6.1.1)_4$ as above and satisfies the compatibility condition

$$\Delta_{\Gamma_1^0}\kappa_1^0 + \Delta_{\Gamma_2^0}\kappa_2^0 + \Delta_{\Gamma_3^0}\kappa_3^0 = 0 \qquad \text{on } \Sigma(0)\,. \tag{6.1.2}$$

Furthermore, the choice of unit normals $n_i(t)$ of $\Gamma_i(t)$ is illustrated in Figure 6.2, which in particular determines the sign of curvatures κ_1, κ_2 and κ_3. We remind again our sign convention: We say that the curve has positive curvature if it is curved in the direction of the normal.

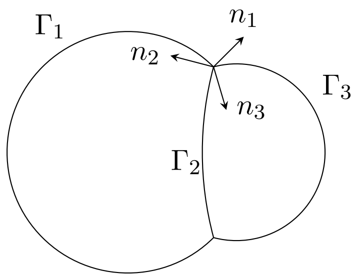

Figure 6.2: The choice of the normals

Let us give a motivation for assuming the condition (6.1.2) on initial planar double bubble.

Lemma 6.1.1. *For a classical solution of the surface diffusion flow* (6.1.1) *we have*

$$\sum_{i=1}^{3} \Delta_{\Gamma_i}\kappa_i = 0 \qquad \textit{on } \Sigma(t)\,. \tag{6.1.3}$$

Proof. At the triple junctions $p_\pm(t)$ we can write for the normal velocities

$$V_i = \Big\langle \frac{d}{d\tau} p_\pm(\tau)\Big|_{\tau=t}, n_i(t) \Big\rangle\,.$$

Now the angle condition implies

$$\sum_{i=1}^{3} V_i = \sum_{i=1}^{3} \Big\langle \frac{d}{d\tau} p_\pm(\tau)\Big|_{\tau=t}, n_i(t) \Big\rangle = \Big\langle \frac{d}{d\tau} p_\pm(\tau)\Big|_{\tau=t}, \sum_{i=1}^{3} n_i(t) \Big\rangle = 0\,.$$

As $V_i = \Delta_{\Gamma_i}\kappa_i$, we obtain (6.1.3). $\square$

Therefore if one seeks for a classical solution which is continuous up to the time $t = 0$, one should impose the condition (6.1.3) on the initial data.

After introducing the problem, let us see its interesting geometric properties:

Lemma 6.1.2. *A classical solution to the surface diffusion flow* (6.1.1) *decreases the total length and preserves the enclosed areas.*

Proof. Assume $\Gamma(t)$ is a solution to the flow (6.1.1) and let

$$l(t) = \sum_{i=1}^{3} \int_{\Gamma_i(t)} 1 \,\mathrm{d}s$$

denote the total length. A transport theorem (see Lemma 2.1.5) gives:

$$\begin{aligned}
\frac{d}{dt} l(t) &= -\sum_{i=1}^{3} \int_{\Gamma_i(t)} V_i \,\kappa_i \,\mathrm{d}s + \int_{\Sigma(t)} \overbrace{\sum_{i=1}^{3} \nu_{\partial\Gamma_i}}^{=0} = \sum_{i=1}^{3} \int_{\Gamma_i(t)} (\Delta_{\Gamma_i(t)}\kappa_i)\kappa_i \,\mathrm{d}s \\
&= -\sum_{i=1}^{3} \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)}\kappa_i|^2 \mathrm{d}s + \int_{\Sigma(t)} \sum_{i=1}^{3} (\nabla_{\Gamma_i(t)}\kappa_i \cdot n_{\partial_{\Gamma_i}})\kappa_i \\
&= -\sum_{i=1}^{3} \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)}\kappa_i|^2 \mathrm{d}s + \int_{\Sigma(t)} (\nabla_{\Gamma_1(t)}\kappa_1 \cdot n_{\partial_{\Gamma_1}}) \underbrace{\sum_{i=1}^{3} \kappa_i}_{=0} \\
&= -\sum_{i=1}^{3} \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)}\kappa_i|^2 \mathrm{d}s \le 0 \,,
\end{aligned} \tag{6.1.4}$$

where we used all the boundary conditions. Note that the sum of the normal boundary velocities $\nu_{\partial\Gamma_i}$ vanishes due to the angle condition, more precisely,

$$\sum_{i=1}^{3} \nu_{\partial\Gamma_i}(t, p_\pm(t)) = \Big(\frac{d}{d\tau} p_\pm(\tau)\Big|_{\tau=t} \Big) \sum_{i=1}^{3} n_{\partial\Gamma_i}(t, p_\pm(t)) = 0 \,.$$

Moreover, the integral over $\Sigma(t) = \{p_+(t), p_-(t)\}$ should be understood as a sum over its elements.

Next, let us prove that the enclosed areas are preserved: It is a standard fact that (see e.g. [24, equation (3.1)])

$$\begin{aligned}
\frac{d}{dt} \int_{R_1(t)} 1 \,\mathrm{d}x &= \int_{\Gamma_1(t)} V_1 \,\mathrm{d}s - \int_{\Gamma_2(t)} V_2 \,\mathrm{d}s \\
&= -\int_{\Gamma_1(t)} \Delta_{\Gamma_1(t)}\kappa_1 \,\mathrm{d}s + \int_{\Gamma_2(t)} \Delta_{\Gamma_2(t)}\kappa_2 \,\mathrm{d}s \\
&= -\int_{\Sigma(t)} \nabla_{\Gamma_1(t)}\kappa_1 \cdot n_{\partial_{\Gamma_1(t)}} + \int_{\Sigma(t)} \nabla_{\Gamma_2(t)}\kappa_2 \cdot n_{\partial_{\Gamma_2(t)}} = 0 \,.
\end{aligned}$$

Similarly, we get $\frac{d}{dt} \int_{R_2(t)} 1 \,\mathrm{d}x = 0$, which completes the proof. □

Let us mention that, via formally matched asymptotic expansions, the flow (6.1.1) is derived as a singular limit of a system of degenerate Cahn-Hilliard equations in [22], where in particular the boundary conditions at each triple junction are derived.

6.1.1 Equilibria

Let a planar double bubble $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ be a stationary solution of the flow (6.1.1), i.e., Γ satisfies (6.1.1) with $V_i = 0$ for $i = 1, 2, 3$. As a consequence

$$\Delta_{\Gamma_i} \kappa_i = 0 \quad (i = 1, 2, 3)\,.$$

By the same arguments used in (6.1.4) we get

$$0 = \sum_{i=1}^{3} \int_{\Gamma_i} (\Delta_{\Gamma_i} \kappa_i) \kappa_i \, \mathrm{d}s = - \sum_{i=1}^{3} \int_{\Gamma_i} |\nabla_{\Gamma_i} \kappa_i|^2 \mathrm{d}s\,.$$

Thus $\nabla_{\Gamma_i} \kappa_i = 0$ on Γ_i. Therefore $\kappa_1, \kappa_2, \kappa_3$ are constant. Summing up, a planar double bubble Γ is a stationary solution of the flow (6.1.1) if and only if

(i) the curvatures κ_i are constant, with $\kappa_1 + \kappa_2 + \kappa_3 = 0$, and

(ii) $\sphericalangle(\Gamma_i, \Gamma_j) = \frac{2\pi}{3}$ on Σ or equivalently $\sum_{i=1}^{3} n_{\partial\Gamma_i} = 0$ on Σ.

It will turn out that the set of stationary solutions consists precisely of all standard planar double bubbles:

Definition 6.1.3. *A standard planar double bubble consists of three circular arcs meeting at their boundaries at* 120 *degree angles. (Here, we interpret a line segment as a circular arc too.)*

We refer to Figure 6.3 for an example. Indeed, as circular arcs and line

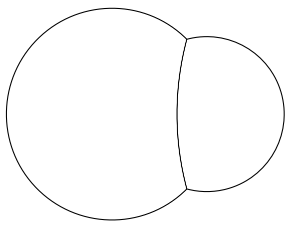

Figure 6.3: The standard planar double bubble

segments are the only curves with constant curvature, it just remains to verify the condition on curvatures. This is done in the following proposition given in [24, Proposition 2.1]:

Proposition 6.1.4. *There is a unique standard planar double bubble (up to rigid motions, i.e., translations and rotations) for given areas in* $\mathbb{R}^2$. *The curvatures satisfy* $\kappa_1 + \kappa_2 + \kappa_3 = 0$.

Remark 6.1.5. *As the choice of the normals in [24] differs from ours, some sign differences particularly for the curvature quantities can occur.*

Therefore the set of all standard planar double bubbles $DB_{r,\gamma,\theta}(a_1, a_2)$ forms a 5-parameter family (see Figure 6.4), where

(i) $r > 0$ is the radius of Γ_1, corresponding to scaling,

(ii) (a_1, a_2) is the center of Γ_1, corresponding to translation,

(iii) the angle θ corresponds to counterclockwise rotation around the center of Γ_1,

(iv) the angle $0 < \gamma < \frac{2\pi}{3}$ corresponds to the curvature ratio.

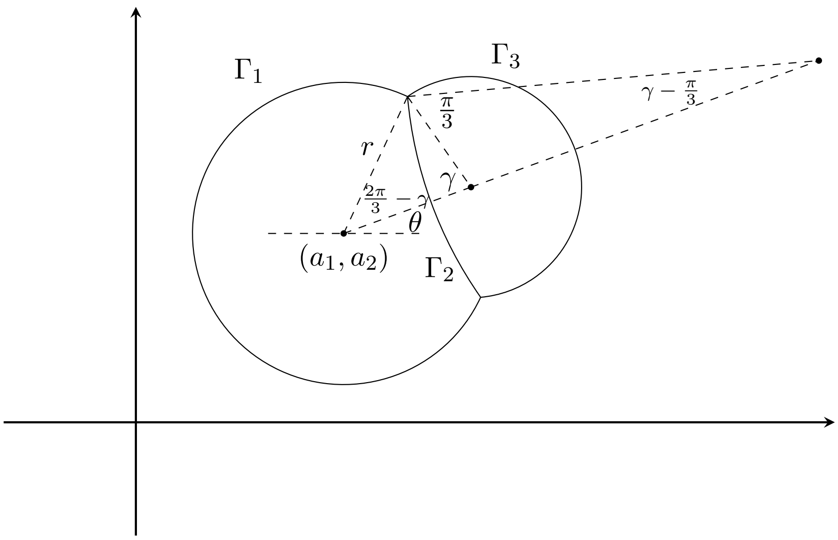

Figure 6.4: The standard planar double bubble $\Gamma = DB_{r,\gamma,\theta}(a_1, a_2)$

Indeed, by the law of sines we have for $\gamma \neq \frac{\pi}{3}$

$$\frac{\kappa_1}{\sin(\gamma + \frac{\pi}{3})} = \frac{\kappa_2}{\sin(\gamma - \frac{\pi}{3})} = \frac{\kappa_3}{\sin(\gamma - \pi)} \tag{6.1.5}$$

and in case $\gamma = \frac{\pi}{3}$ we observe $\kappa_2 = 0$ and $\kappa_1 = -\kappa_3$. Note that due to our choice of normals we always have $\kappa_1 < 0$ and $\kappa_3 > 0$. Moreover,

$$\begin{cases} \kappa_2 > 0 & \text{for } \gamma < \frac{\pi}{3}\,, \\ \kappa_2 < 0 & \text{for } \gamma > \frac{\pi}{3}\,. \end{cases}$$

For later use we define the constants q_i as follows:

$$q_i := -\frac{1}{\sqrt{3}}(\kappa_j - \kappa_k)$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. Then the following result is true.

Lemma 6.1.6. *We have*

$$q_1 = \cot(\gamma + \tfrac{\pi}{3})\kappa_1, \quad q_2 = \begin{cases} \cot(\gamma - \tfrac{\pi}{3})\kappa_2 & \gamma \neq \tfrac{\pi}{3}\,, \\ \dfrac{\kappa_1}{\sin(\frac{\pi}{3})} & \gamma = \tfrac{\pi}{3}\,, \end{cases} \quad q_3 = \cot(\gamma - \pi)\kappa_3\,.$$

Proof. We calculate

$$\begin{aligned} q_2 &= -\frac{1}{\sqrt{3}}(\kappa_3 - \kappa_1) = -\frac{1}{\sqrt{3}}\Big(\frac{-\sin(\gamma) - \sin(\gamma + \frac{\pi}{3})}{\sin(\gamma - \frac{\pi}{3})}\Big)\kappa_2 \\ &= \frac{2}{\sqrt{3}}\Big(\frac{\sin(\gamma + \frac{\pi}{6})\cos(\frac{\pi}{6})}{\sin(\gamma - \frac{\pi}{3})}\Big)\kappa_2 = \cot(\gamma - \tfrac{\pi}{3})\kappa_2 \quad \text{for } \gamma \neq \tfrac{\pi}{3}\,, \end{aligned}$$

and obviously $q_2 = \frac{2}{\sqrt{3}}\kappa_1 = \frac{\kappa_1}{\sin(\frac{\pi}{3})}$ for $\gamma = \frac{\pi}{3}$. The continuity follows from the formula (6.1.5). The proof for q_1 and q_3 is similar. □

Moreover, using the sum-to-product trigonometric identity, we get

$$\begin{cases} \sin(\gamma + \dfrac{\pi}{3}) + \sin(\gamma - \dfrac{\pi}{3}) + \sin(\gamma - \pi) = 0\,, \\ \cos(\gamma + \dfrac{\pi}{3}) + \cos(\gamma - \dfrac{\pi}{3}) + \cos(\gamma - \pi) = 0\,. \end{cases} \tag{6.1.6}$$

One strategy to deal with geometric flows on hypersurfaces is to parameterize the evolving hypersurfaces with respect to a fixed reference hypersurface. This eventually leads to a PDE on a fixed domain allowing us to employ PDE theories.

6.2 PDE formulation and linearization

In this section we introduce the proper setting to reformulate the geometric flow (6.1.1) as a system of partial differential equations for unknown functions defined on fixed domains. For this, we employ a parameterization with two parameters. The parameters correspond to a movement in normal and tangential directions. This parameterization is adapted for two triple junctions from Depner and Garcke [12], see also [13].

6.2.1 Parameterization of planar double bubbles

Let us describe $\Gamma_i(t)$ as a graph over some fixed stationary solution Γ_i^* using functions

$$\rho_i : \Gamma_i^* \times [0, T) \to \mathbb{R} \qquad (i = 1, 2, 3)\,.$$

The precise way how ρ_i defines $\Gamma_i(t)$ will be derived in what follows.

Fix any stationary solution

$$\Gamma^* = DB_{r^*,\gamma^*,\theta^*}(a_1^*, a_2^*) \quad \big(r^* > 0,\ (a_1^*, a_2^*) \in \mathbb{R}^2,\ 0 < \gamma^* < \tfrac{2\pi}{3},\ 0 \leq \theta^* < 2\pi\big)\,.$$

Then we observe

$$
\begin{aligned}
l_1^* &= -\tfrac{1}{\kappa_1^*}(\gamma^* + \tfrac{\pi}{3}),\\
l_2^* &= \begin{cases} -\tfrac{1}{\kappa_2^*}(\gamma^* - \tfrac{\pi}{3}) = -\tfrac{1}{\kappa_1^*}\tfrac{(\gamma^* - \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}\sin(\gamma^* + \tfrac{\pi}{3}) & \text{if } \gamma^* \neq \tfrac{\pi}{3}\,,\\ -\tfrac{1}{\kappa_1^*}\sin(\tfrac{\pi}{3}) & \text{if } \gamma^* = \tfrac{\pi}{3}\,,\end{cases}\\
l_3^* &= -\tfrac{1}{\kappa_3^*}(\gamma^* - \pi) = -\tfrac{1}{\kappa_1^*}\tfrac{(\gamma^* - \pi)}{\sin(\gamma^* - \pi)}\sin(\gamma^* + \tfrac{\pi}{3})\,,
\end{aligned}
$$

where $2l_i^*$ is the length of Γ_i^* $(i = 1, 2, 3)$ and of course $\kappa_1^* = -\frac{1}{r^*}$.

Let $\Phi_i^* : [-l_i^*, l_i^*] \to \mathbb{R}^2$ be an arc-length parameterization of Γ_i^*. Hence

$$\Gamma_i^* = \{\Phi_i^*(x) : x \in [-l_i^*, l_i^*]\}\,.$$

Furthermore, set $(\Phi_i^*)^{-1}(\sigma) = x(\sigma) \in \mathbb{R}$, for $\sigma \in \Gamma_i^*$. To simplify the presentation, we hereafter set

$$\partial_\sigma w(\sigma) := \partial_x (w \circ \Phi_i^*)(x), \quad \sigma = \Phi_i^*(x), \tag{6.2.1}$$

that is, we do not state the parameterization explicitly. We also slightly abuse notation and write

$$w(\sigma) = w(x) \quad (\sigma \in \Gamma_i^*)\,. \tag{6.2.2}$$

To parameterize a curve nearby Γ_i^*, define

$$
\begin{aligned}
\Psi_i : \Gamma_i^* \times (-\epsilon, \epsilon) \times (-\delta, \delta) &\longrightarrow \mathbb{R}^2\,, \qquad (6.2.3)\\
(\sigma, w, r) &\mapsto \Psi_i(\sigma, w, r) := \sigma + w\, n_i^*(\sigma) + r\, \tau_i^*(\sigma)\,.
\end{aligned}
$$

Here τ_i^* denotes a tangential vector field on Γ_i^* having support in a neighborhood of $\partial\Gamma_i^*$, which is equal to the outer unit conormal $n_{\partial\Gamma_i^*}$ at $\partial\Gamma_i^*$.

Define then $\Phi_i = (\Phi_i)_{\rho_i,\mu_i}$ (we often omit for shortness the subscript (ρ_i, μ_i)) by

$$\Phi_i : \Gamma_i^* \times [0, T) \to \mathbb{R}^2\,, \quad \Phi_i(\sigma, t) := \Psi_i(\sigma, \rho_i(\sigma, t), \mu_i(\mathrm{pr}_i(\sigma), t))\,, \tag{6.2.4}$$

for the functions

$$\rho_i : \Gamma_i^* \times [0, T) \to (-\epsilon, \epsilon)\,, \quad \mu_i : \Sigma^* \times [0, T) \to (-\delta, \delta)\,, \tag{6.2.5}$$

where, similarly as before, $\Sigma^* = \partial\Gamma_i^* = \{p_+^*, p_-^*\}$.

The projection $\mathrm{pr}_i : \Gamma_i^* \to \Sigma^*$ is defined by imposing the following condition: The point $\mathrm{pr}_i(\sigma) \in \partial\Gamma_i^*$ has the shortest distance on Γ_i^* to σ. Clearly, in a small neighborhood of $\partial\Gamma_i^*$, the projection pr_i is well-defined and this is sufficient for us since this projection is just used in the product $\mu_i(\mathrm{pr}_i(\sigma), t)\tau_i^*(\sigma)$, where the second term vanishes outside a (small) neighborhood of $\partial\Gamma_i^*$.

Now let us set, for small $\epsilon, \delta > 0$ and fix t,

$$(\Phi_i)_t : \Gamma_i^* \to \mathbb{R}^2, \quad (\Phi_i)_t(\sigma) := \Phi_i(\sigma, t) \quad \forall \sigma \in \Gamma_i^*$$

to finally define a new curve

$$\Gamma_{\rho_i,\mu_i}(t) := \mathrm{image}((\Phi_i)_t)\,. \tag{6.2.6}$$

Observe that for $\rho_i \equiv 0$ and $\mu_i \equiv 0$, the curve $\Gamma_{\rho_i,\mu_i}(t)$ coincides with Γ_i^* for all t.

At each triple junction, we have prepared for a movement in normal and tangential direction, allowing for an evolution of the triple junctions. Therefore, we can now formulate the condition, that the curves $\Gamma_i(t)$ meet at the triple junctions at their boundary by

$$\Phi_1(\sigma, t) = \Phi_2(\sigma, t) = \Phi_3(\sigma, t) \quad \text{ for } \sigma \in \Sigma^*,\ t \geq 0\,. \tag{6.2.7}$$

Next we prove that this condition leads to a linear dependency at the boundary points. As a result, nonlocal terms will eventually enter into PDE-formulations of the geometric evolution problem.

Lemma 6.2.1. *Equivalent to the equations (6.2.7) are the following conditions*

$$\begin{cases} \text{(i)} & 0 = \rho_1 + \rho_2 + \rho_3 \qquad \textit{on } \Sigma^*, \\ \text{(ii)} & \mu_i = -\frac{1}{\sqrt{3}}(\rho_j - \rho_k) \qquad \textit{on } \Sigma^*, \end{cases} \tag{6.2.8}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ *and* $(3, 1, 2)$.

Here the linear dependency (ii) can be recast as the matrix equation

$$\mu = \mathcal{J}\rho \quad \text{ on } \Sigma^*, \tag{6.2.9}$$

with the notations $\mu = (\mu_1, \mu_2, \mu_3)$, $\rho = (\rho_1, \rho_2, \rho_3)$ and the matrix

$$\mathcal{J} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Proof. First we prove that (6.2.7) implies (6.2.8). Using the definition of Φ_i, (6.2.7) can be rewritten as

$$\rho_i\, n_i^* + \mu_i\, n_{\partial\Gamma_i^*} = \rho_j\, n_j^* + \mu_j\, n_{\partial\Gamma_j^*} \quad \text{ on } \Sigma^* \tag{6.2.10}$$

for $(i, j) = (1, 2), (2, 3)$. By setting

$$q := \rho_1\, n_1^* + \mu_1\, n_{\partial\Gamma_1^*} = \rho_2\, n_2^* + \mu_2\, n_{\partial\Gamma_2^*} = \rho_3\, n_3^* + \mu_3\, n_{\partial\Gamma_3^*} \quad \text{ on } \Sigma^*$$

we obtain $\rho_i = \langle q, n_i^* \rangle$ for $i = 1, 2, 3$. Thus the angle condition for Γ^* gives

$$\sum_{i=1}^{3} \rho_i = \sum_{i=1}^{3} \langle q, n_i^* \rangle = \langle q, \sum_{i=1}^{3} n_i^* \rangle = 0 \,.$$

This proves (i). As a result of (6.2.10) we see further

$$\rho_i \langle n_i^*, n_j^* \rangle + \mu_i \langle n_{\partial \Gamma_i^*}, n_j^* \rangle = \rho_j \qquad \text{on } \Sigma^* \,.$$

On the other hand the angle condition implies

$$\langle n_i^*, n_j^* \rangle = \cos(\tfrac{2\pi}{3}) \,, \quad \langle n_{\partial \Gamma_i^*}, n_j^* \rangle = \cos(2\pi - (\tfrac{2\pi}{3} + \tfrac{\pi}{2})) = -\sin(\tfrac{2\pi}{3}) \qquad \text{on } \Sigma^*$$

for $(i, j) = (1, 2), (2, 3), (3, 1)$. Therefore using (i) we conclude

$$\mu_i = -\tfrac{1}{s}(\rho_j - c\rho_i) = -\tfrac{1}{s}((1 + c)\rho_j + c\rho_k) = \tfrac{c}{s}(\rho_j - \rho_k) \,,$$

where $s := \sin(\frac{2\pi}{3})$ and $c := \cos(\frac{2\pi}{3}) = -\frac{1}{2}$ and this yields assertion (ii). The proof of the converse statement is explicitly given in [12, Lemma 2.3]. $\square$

Note that we followed [21] in proving statement (i), while an easier proof is given here for assertion (ii). Notice further that (6.2.8) easily implies

$$\mu_1 + \mu_2 + \mu_3 = 0 \qquad \text{on } \Sigma^* \,. \tag{6.2.11}$$

Remark 6.2.2. *Let us now note that it is within this set, i.e., the set of all planar double bubbles which can be described as the graph over Γ^*, that we will seek a solution to the problem* (6.1.1).

Naturally, we assume also that the initial double bubble Γ^0 from (6.1.1) is given as a graph over Γ^*, i.e.,

$$\Gamma_i^0 = \{\Psi_i\big(\sigma, \rho_i^0(\sigma), \mu_i^0(\mathrm{pr}(\sigma))\big) : \sigma \in \Gamma_i^*\} \quad (i = 1, 2, 3)$$

for some function ρ^0. Here $\mu^0 = \mathcal{J}\rho^0$ on Σ^* as Γ^0 is assumed to be a double bubble, i.e., the curves Γ_i^0 meet two triple junctions at their boundaries.

6.2.2 Nonlocal, nonlinear parabolic boundary-value PDE

The idea is to first derive evolution equations for ρ_i and μ_i which have to hold if the Γ_i $(i = 1, 2, 3)$ in (6.2.6) satisfy the condition (6.2.7) and solve the surface diffusion flow (6.1.1) and then to make use of the linear dependency (6.2.9) in deriving evolution equations solely for the functions ρ_i.

As you may have noticed, nonlocal terms will appear in the formulations since this linear dependency (6.2.9) just holds at the boundary points.

Appendix B.1 provides for the reader's convenience the derivation in detail. Indeed a similar derivation is done in Chapter 5 , which is originally given in [21], [13]. Therefore, let us present the final system of fourth-order nonlinear, nonlocal PDEs for $t > 0$, $i = 1, 2, 3$ and $j = 1, 2, \ldots, 6$:

$$\begin{cases} \partial_t \rho_i = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) & \\ \qquad\quad + \mathfrak{B}_i(\rho_i, \rho|_{\Sigma^*})\big(\big\{\mathcal{J}\big(I - \mathfrak{B}(\rho, \rho|_{\Sigma^*})\mathcal{J}\big)^{-1}\mathfrak{F}(\rho, \rho|_{\Sigma^*})\big\} \circ \mathrm{pr}_i\big)_i & \text{on } \Gamma_i^*, \\ \quad 0 = \mathfrak{G}_j(\rho) & \text{on } \Sigma^*, \end{cases} \tag{6.2.12}$$

with the initial conditions

$$\rho_i(\cdot, 0) = \rho_i^0 \text{ on } \Gamma_i^*\,,$$

where in particular $\mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*})$ is a fourth-order nonlinear equation in ρ_i.

Remark 6.2.3. *Note that the price to pay for obtaining equations solely for functions ρ_i is the appearance of nonlocal terms, in particular the nonlocal terms of highest-order (fourth-order) $\mathfrak{F}(\rho, \rho|_{\Sigma^*}) \circ \mathrm{pr}_\mathrm{i}$, into the formulation.*

As demonstrated at the beginning of Appendix B.1, the functions $\mathfrak{F}_i, \mathfrak{B}_i, \mathfrak{G}_j$ are rational functions in the ρ-dependent variables, with nonzero denominators in some neighborhood of $\rho \equiv 0$ in $C^1(\Gamma^*)$ (can be inside of square roots equalling to 1 in some neighborhood of $\rho \equiv 0$ in $C^1(\Gamma^*)$, see the term $\frac{1}{J_i}$).

6.2.3 Linearization around the stationary solution

The linearization of the surface diffusion equations and the angle conditions around the stationary solution $\rho \equiv 0$ are done in [12, Lemma 3.2] and [12, Lemma 3.4] respectively.

Remark 6.2.4. *Note that the situation in [12] is slightly different from ours, but nevertheless the results obtained there are applicable to our problem. More precisely, the authors in [12] consider the situation where, apart from the appearance of a triple junction, one has to deal with a fixed boundary. However, as they assume that the triple junction will not touch the outer fixed boundary, they can use an explicit parameterization, exactly as ours, around a triple junction and another parameterization near the fixed boundary and finally they compose them with the help of a cut-off function. Thus we can use their result for each triple junction.*

Therefore, taking into account the linear dependency (ii) from Lemma 6.2.1, we get for the linearization of the nonlinear problem (6.2.12) around $\rho \equiv 0$ (that is, around the stationary solution Γ^*) the following linear system for $i = 1, 2, 3$

$$\partial_t \rho_i + \Delta_{\Gamma_i^*}\big(\Delta_{\Gamma_i^*}\rho_i + (\kappa_i^*)^2\rho_i\big) = 0 \quad \text{in } \Gamma_i^*\,, \tag{6.2.13}$$

with the boundary conditions on Σ^*

$$\begin{cases} \rho_1 + \rho_2 + \rho_3 = 0\,, & \\ q_i^*\rho_i + \partial_{n_{\partial\Gamma_i^*}}\rho_i = q_j^*\rho_j + \partial_{n_{\partial\Gamma_j^*}}\rho_j & (i,j) = (1,2),(2,3), \\ \sum_{i=1}^3 \Delta_{\Gamma_i^*}\rho_i + (\kappa_i^*)^2\rho_i = 0, & \\ \partial_{n_{\partial\Gamma_i^*}}\big(\Delta_{\Gamma_i^*}\rho_1 + (\kappa_i^*)^2\rho_i\big) = \partial_{n_{\partial\Gamma_j^*}}\big(\Delta_{\Gamma_j^*}\rho_j + (\kappa_j^*)^2\rho_j\big) & (i,j) = (1,2),(2,3), \end{cases} \tag{6.2.14}$$

where

$$q_i^* = -\frac{1}{\sqrt{3}}(\kappa_j^* - \kappa_k^*)$$

for $(i,j,k) = (1,2,3),(2,3,1)$ and $(3,1,2)$.

Let us recall the parameterization (remember our abuse of notation (6.2.2)) and employ the following facts

$$\begin{aligned} \Delta_{\Gamma_i^*}\rho_i &= \partial_x^2\rho_i && \text{for } x \in [-l_i^*, l_i^*]\,, \\ \partial_{n_{\partial\Gamma_i^*}}\rho_i &= \nabla_{\Gamma_i^*}\rho_i \cdot n_{\partial\Gamma_i^*} && \\ &= \partial_x\rho_i\,(T_i^* \cdot n_{\partial_{\Gamma_i^*}}) = \pm\partial_x\rho_i && \text{at } x = \pm l_i^*\,, \\ \kappa_{n_{\partial\Gamma_i^*}} &= \kappa_i^* && \text{at } x = \pm l_i^*\,, \end{aligned}$$

where x is the arc length parameter of Γ_i^* and denote by T_i^* the tangential vector of Γ_i^*. We can then rewrite the linearized problem in terms of functions $\rho_i : [-l_i^*, l_i^*] \times [0,T) \to \mathbb{R}$ as

$$\partial_t\rho_i + \partial_x^2\big(\partial_x^2 + (\kappa_i^*)^2\big)\rho_i = 0 \quad \text{for } x \in [-l_i^*, l_i^*]$$

with the boundary conditions

$$\begin{cases} \rho_1 + \rho_2 + \rho_3 = 0\,, \\ q_1^*\rho_1 \pm \partial_x\rho_1 = q_2^*\rho_2 \pm \partial_x\rho_2 = q_3^*\rho_3 \pm \partial_x\rho_3\,, \\ \sum_{i=1}^3\big(\partial_x^2\rho_i + (\kappa_i^*)^2\rho_i\big) = 0\,, \\ \partial_x\big(\partial_x^2 + (\kappa_1^*)^2\big)\rho_1 = \partial_x\big(\partial_x^2 + (\kappa_2^*)^2\big)\rho_2 = \partial_x\big(\partial_x^2 + (\kappa_3^*)^2\big)\rho_3\,. \end{cases} \tag{6.2.15}$$

In the boundary conditions (6.2.15) we have omitted the terms $\pm l_i^*$ in the functions ρ_i. That is, for instance the boundary condition $\rho_1 + \rho_2 + \rho_3 = 0$ should be read as

$$\rho_1(\pm l_1^*) + \rho_2(\pm l_2^*) + \rho_3(\pm l_3^*) = 0\,.$$

Furthermore, notice that the linearized problem is completely local as, in particular, we linearized around a stationary solution.

6.3 Verifying the hypotheses of Theorem 3.2.1

In order to apply this theorem to prove dynamical stability, we must first show that our nonlinear, nonlocal problem (6.2.12) has the form (3.1.1). We then devote the rest to show that the problem (6.2.12) verifies all hypothesis of Theorem 3.2.1.

6.3.1 General setting

If we change the variables by setting for each $i = 2, 3$

$$x = \frac{\tilde{x} + l_1^*}{2l_1^*} l_i^* + \frac{\tilde{x} - l_1^*}{2l_1^*} l_i^* \qquad \tilde{x} \in [-l_1^*, l_1^*]\,,$$

then we easily can restate the nonlinear, nonlocal system (6.2.12) as a perturbation of a linearized problem, that is of the form (3.1.1), with $\Omega = [-l_1^*, l_1^*]$,

$$A\rho = \begin{bmatrix} (\mathbf{l}_1)^4 & 0 & 0 \\ 0 & (\mathbf{l}_2)^4 & 0 \\ 0 & 0 & (\mathbf{l}_3)^4 \end{bmatrix} \partial_x^4 \rho + \begin{bmatrix} (\mathbf{l}_1 \kappa_1^*)^2 & 0 & 0 \\ 0 & (\mathbf{l}_2 \kappa_2^*)^2 & 0 \\ 0 & 0 & (\mathbf{l}_3 \kappa_3^*)^2 \end{bmatrix} \partial_x^2 \rho\,,$$

and

$$\begin{aligned}
B_1\rho &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \rho\,, \\
B_2\rho &= \pm \begin{bmatrix} \mathbf{l}_1 & -\mathbf{l}_2 & 0 \end{bmatrix} \partial_x \rho + \begin{bmatrix} q_1^* & -q_2^* & 0 \end{bmatrix} \rho\,, \\
B_3\rho &= \pm \begin{bmatrix} 0 & \mathbf{l}_2 & -\mathbf{l}_3 \end{bmatrix} \partial_x \rho + \begin{bmatrix} 0 & q_2^* & -q_3^* \end{bmatrix} \rho\,, \\
B_4\rho &= \begin{bmatrix} (\mathbf{l}_1)^2 & (\mathbf{l}_2)^2 & (\mathbf{l}_3)^2 \end{bmatrix} \partial_x^2 \rho + \begin{bmatrix} (\kappa_1^*)^2 & (\kappa_2^*)^2 & (\kappa_3^*)^2 \end{bmatrix} \rho\,, \\
B_5\rho &= \begin{bmatrix} (\mathbf{l}_1)^3 & -(\mathbf{l}_2)^3 & 0 \end{bmatrix} \partial_x^3 \rho + \begin{bmatrix} \mathbf{l}_1 (\kappa_1^*)^2 & -\mathbf{l}_2 (\kappa_2^*)^2 & 0 \end{bmatrix} \partial_x \rho\,, \\
B_6\rho &= \begin{bmatrix} 0 & (\mathbf{l}_2)^3 & -(\mathbf{l}_3)^3 \end{bmatrix} \partial_x^3 \rho + \begin{bmatrix} 0 & \mathbf{l}_2 (\kappa_2^*)^2 & -\mathbf{l}_3 (\kappa_3^*)^2 \end{bmatrix} \partial_x \rho\,.
\end{aligned}$$

To simplify the presentation, we have dropped the tilde. Here

$$\rho : [-l_1^*, l_1^*] \times [0, \infty) \to \mathbb{R}^3, \quad \rho = (\rho_1, \rho_2, \rho_3)^T$$

and the constants are given as $\mathbf{l}_i := \dfrac{l_1^*}{l_i^*}$ $(i = 1, 2, 3)$.

When we write (6.2.12) in the form of (3.1.1), the corresponding F is a smooth function defined in some neighborhood of 0 in $C^4(\overline{\Omega})$ having values in $C(\overline{\Omega})$. The reason is that, F is Fréchet-differentiable of arbitrary order in some neighborhood of 0 (using the differentiability of composition operators, see e.g. Theorem 1 and 2 of [37, Section 5.5.3]). The same argument works for the corresponding functions G_i. We have obtained that assumption (H1) is satisfied.

Obviously, the operators A and B_j satisfy the smoothness assumption (H2) and the operator A is strongly parabolic. Now let us check that the Lopatinskii-Shapiro condition (LS) holds. To verify this, for $\lambda \in \overline{\mathbb{C}_+}, \lambda \neq 0$, we consider the following ODE

$$\begin{cases} \lambda v_i(y) + (\mathbf{l}_i)^4 \partial_y^4 v_i(y) = 0, \quad (y > 0)\,, \\ v_1(0) + v_2(0) + v_3(0) = 0\,, \\ \qquad \mathbf{l}_1 \partial_y v_1(0) = \mathbf{l}_2 \partial_y v_2(0) = \mathbf{l}_3 \partial_y v_3(0)\,, \\ \sum_{i=1}^3 (\mathbf{l}_i)^2 \partial_y^2 v_i(0) = 0\,, \\ \qquad (\mathbf{l}_1)^3 \partial_y^3 v_1(0) = (\mathbf{l}_2)^3 \partial_y^3 v_2(0) = (\mathbf{l}_3)^3 \partial_y^3 v_3(0) \end{cases} \tag{6.3.1}$$

and we show that $v \equiv 0$ is the only classical solution that vanishes at infinity. The energy methods provide a simple proof: We test the first line of the equation (6.3.1) with the function $\dfrac{1}{\mathbf{l}_i}\overline{v_i}$ and sum for $i = 1, 2, 3$ to find

$$\begin{aligned} \sum_{i=1}^3 \frac{\lambda}{\mathbf{l}_i} \int_0^\infty |v_i|^2 \,\mathrm{d}y &= -\sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty \overline{v_i}\, \partial_y^4 v_i \,\mathrm{d}y \\ &= \sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty \partial_y \overline{v_i}\, \partial_y^3 v_i \,\mathrm{d}y + \overbrace{\sum_{i=1}^3 \overline{v_i} \Big[(\mathbf{l}_i)^3 \partial_y^3 v_i\Big]\Big|_0^\infty}^{=0} \\ &= -\sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty |\partial_y^2 v_i|^2 \,\mathrm{d}y + \overbrace{\sum_{i=1}^3 (\mathbf{l}_i)^2 \partial_y^2 v_i \Big[\mathbf{l}_i \partial_y \overline{v_i}\Big]\Big|_0^\infty}^{=0} \\ &= -\sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty |\partial_y^2 v_i|^2 \,\mathrm{d}y\,. \end{aligned}$$

Here we have used all boundary conditions at $y = 0$ and the fact that the functions v_i and consequently all their derivatives vanish exponentially at infinity. The latter holds due to the fact that the solutions of the above equations are linear combinations of exponential functions. The facts that $0 \neq \lambda \in \overline{\mathbb{C}_+}$ and $\mathbf{l}_i > 0$ enforce $v \equiv 0$. This verifies the claim.

Furthermore, the matrices

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{l}_1 & -\mathbf{l}_2 & 0 \\ 0 & \mathbf{l}_2 & -\mathbf{l}_3 \end{bmatrix},$$

$$\begin{bmatrix} (\mathbf{l}_1)^2 & (\mathbf{l}_2)^2 & (\mathbf{l}_3)^2 \end{bmatrix}, \qquad \begin{bmatrix} (\mathbf{l}_1)^3 & -(\mathbf{l}_2)^3 & 0 \\ 0 & (\mathbf{l}_2)^3 & -(\mathbf{l}_3)^3 \end{bmatrix}$$

are surjective and hence the normality condition (3.1.9) is satisfied.

Compatibility condition

We next turn our attention to the corresponding compatibility condition (3.1.10). As we have assumed the initial planar double bubble Γ^0 fulfills the contact, angle, the curvature and the balance of flux condition, we see $\mu^0 = \mathcal{J}\rho^0$ and $\mathfrak{G}_j(\rho^0) = 0$ for $j = 1, 2, \ldots, 6$. This is exactly the first condition in (3.1.10).

Concerning the second equation in the compatibility condition (3.1.10), the following lemma shows that it is equivalent to the geometric compatibility condition (6.1.2) if the existence of triple junctions and the angle condition for the initial data are already assumed.

Lemma 6.3.1. *Under the conditions $\mathfrak{G}_j(\rho^0) = 0$ $(j = 1, 2, 3)$ and $\mu^0 = \mathcal{J}\rho^0$ on Σ^*, the second equation in the corresponding compatibility condition (3.1.10) and the geometric compatibility condition (6.1.2) are equivalent, provided ρ^0 is sufficiently small in the C^1-norm.*

Proof. The second equation in the corresponding first-order compatibility condition (3.1.10) reads as

$$\sum_{i=1}^{3} \mathfrak{F}_i(\rho_i^0, \rho^0) + \mathcal{B}_i(\rho_i, \rho^0)\big(\mathcal{J}\ \overbrace{\big(I - \mathcal{B}(\rho^0, \rho^0)\mathcal{J}\big)^{-1}\mathfrak{F}(\rho^0, \rho^0)}^{z:=}\ \big)_i = 0 \qquad (6.3.2)$$

on Σ^*. Here we have used the facts that the zeroth-order boundary operator $B_1 u = \sum_{i=1}^{3} u_i$ and $G_1 \equiv 0$. Let us remind that

$$\mathfrak{F}_i(\rho_i^0, \rho^0) = \frac{1}{\langle n_i^*, n_i^0\rangle} \Delta\big(\,.\,, \rho_i^0, (\mathcal{J}\rho^0)_i\big)\kappa_i\big(\,.\,, \rho_i^0, (\mathcal{J}\rho^0)_i\big),$$

$$\mathfrak{B}_i(\rho_i^0, \rho^0) = \frac{\big\langle n_{\partial\Gamma_i^*}, n_i^0\big\rangle}{\big\langle n_i^*, n_i^0\big\rangle},$$

and $\tau_i^* = n_{\partial\Gamma_i^*}$ on Σ^*.

On the other hand, the angle condition implies

$$\big\langle n_i^*, n_i^0\big\rangle = \big\langle n_j^*, n_j^0\big\rangle, \quad \big\langle n_{\partial\Gamma_i^*}, n_i^0\big\rangle = \big\langle n_{\partial\Gamma_j^*}, n_j^0\big\rangle \qquad \text{on } \Sigma^*.$$

Thus (6.3.2) can be rewritten as

$$\frac{1}{\langle n_1^*, n_1^0\rangle} \sum_{i=1}^{3} \Delta\big(\,.\,, \rho_i^0, (\mathcal{J}\rho^0)_i\big)\kappa_i\big(\,.\,, \rho_i^0, (\mathcal{J}\rho^0)_i\big) + \mathfrak{B}_1 \sum_{i=1}^{3} (\mathcal{J}z)_i = 0 \qquad \text{on } \Sigma^*,$$

where $\langle n_1^*, n_1^0\rangle \neq 0$ if Γ^0 is close enough to Γ^* in C^1-norm, that is if ρ^0 is sufficiently small in the C^1-norm.

Moreover, due to the definition of the matrix $\mathcal{J}$, we have

$$\sum_{i=1}^{3} \big(\mathcal{J}y\big)_i = 0 \quad \forall y \in \mathbb{R}^3 .$$

Hence the compatibility condition (6.3.2) is equivalent to

$$\sum_{i=1}^{3} \Delta\big(\sigma, \rho_i^0, (\mathcal{J}\rho^0)_i\big)\kappa_i\big(\sigma, \rho_i^0, (\mathcal{J}\rho^0)_i\big) = 0\,,$$

which is exactly the geometric compatibility condition (6.1.2) written in a parameterization. This finishes the proof. $\square$

6.3.2 Spectrum of A_0 : Double bubble conjecture

Since $\Omega = [-l_1^*, l_1^*] \subset \mathbb{R}$, the linearized operator A_0 (see (3.1.5)) is defined as $A_0 u = Au$ with domain

$$D(A_0) = \Big\{ u \in C^4(\overline{\Omega}) : Bu = 0 \text{ on } \partial\Omega \Big\},$$

where A and B is defined in Section 6.3.1. Due to Remark 3.1.2, the spectrum of the linearized operator A_0 consists entirely of eigenvalues. As the analysis of the eigenvalue problem is invariant under the change of variables, we switch to the setting where the functions u_i ($i = 1, 2, 3$) have different domains.

Now, the eigenvalue problem for the linearized operator A_0 reads as follows: For $i = 1, 2, 3$,

$$\Delta_{\Gamma_i^*}\big(\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i\big) = \lambda u_i \quad \text{in } \Gamma_i^* \quad (i = 1, 2, 3)\,, \tag{6.3.3}$$

subject to the boundary conditions on Σ^*

$$\begin{cases} u_1 + u_2 + u_3 = 0\,, \\ q_i^* u_i + \partial_{n_{\partial\Gamma_i^*}} u_i = q_j^* u_j + \partial_{n_{\partial\Gamma_j^*}} u_j\,, \\ \sum_{i=1}^{3} \Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i = 0\,, \\ \partial_{n_{\partial\Gamma_i^*}}\big(\Delta_{\Gamma_i^*} u_1 + (\kappa_i^*)^2 u_i\big) = \partial_{n_{\partial\Gamma_j^*}}\big(\Delta_{\Gamma_j^*} u_j + (\kappa_j^*)^2 u_j\big)\,, \end{cases} \tag{6.3.4}$$

where $(i, j) = (1, 2), (2, 3)$.

To derive a bilinear form associated with this eigenvalue problem, let us multiply the equation (6.3.3) by $-\big(\Delta_{\Gamma_i^*}\overline{u_i} + (\kappa_i^*)^2\overline{u_i}\big)$ and then integrate by parts and sum over $i = 1, 2, 3$ to find

$$\sum_{i=1}^{3} \int_{\Gamma_i^*} \big|\nabla_{\Gamma_i^*}\big(\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i\big)\big|^2 \mathrm{d}s \quad = -\lambda \sum_{i=1}^{3} \int_{\Gamma_i^*} u_i\big(\Delta_{\Gamma_i^*}\overline{u_i} + (\kappa_i^*)^2\overline{u_i}\big)\,\mathrm{d}s\,.$$

Here, as usual, we have used the last two boundary conditions. We observe further

$$-\sum_{i=1}^{3} \int_{\Gamma_i^*} u_i\big(\Delta_{\Gamma_i^*}\overline{u_i} + (\kappa_i^*)^2\overline{u_i}\big)\,ds = \sum_{i=1}^{3} \int_{\Gamma_i^*} \big|\nabla_{\Gamma_i^*} u_i\big|^2 - (\kappa_i^*)^2|u_i|^2 \ \mathrm{d}s$$
$$- \sum_{i=1}^{3} \int_{\Sigma^*} u_i\,\partial_{n_{\partial\Gamma_i^*}} u_i \overline{u_i}\,.$$

On the other hand

$$
\begin{aligned}
\sum_{i=1}^{3}\int_{\Sigma^*} u_i\,\partial_{n_{\partial\Gamma_i^*}}\overline{u_i} &= \sum_{i=1}^{3}\int_{\Sigma^*}\left(u_i\,\partial_{n_{\partial\Gamma_i^*}}\overline{u_i} + q_i^*|u_i|^2 - q_i^*|u_i|^2\right)\\
&= \sum_{i=1}^{3}\int_{\Sigma^*}\left(\partial_{n_{\partial\Gamma_i^*}}u_i + q_i^* u_i\right)\overline{u_i} - \sum_{i=1}^{3}\int_{\Sigma^*} q_i^*|u_i|^2\\
&= \int_{\Sigma^*}\left(\partial_{n_{\partial\Gamma_1^*}}u_1 + q_1^* u_1\right)\underbrace{\sum_{i=1}^{3}\overline{u_i}}_{=0} - \sum_{i=1}^{3}\int_{\Sigma^*} q_i^*|u_i|^2\\
&= -\sum_{i=1}^{3}\int_{\Sigma^*} q_i^*|u_i|^2\,.
\end{aligned}
$$

We now combine the three equalities above to discover

$$
\sum_{i=1}^{3}\int_{\Gamma_i^*}\left|\nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}u_i + (\kappa_i^*)^2 u_i\right)\right|^2 \mathrm{d}s = \lambda I(u,u)\,, \tag{6.3.5}
$$

where

$$
\begin{aligned}
I(u,u) := &\sum_{i=1}^{3}\int_{\Gamma_i^*}\left|\nabla_{\Gamma_i^*}u_i\right|^2 - (\kappa_i^*)^2|u_i|^2\ \mathrm{d}s + \sum_{i=1}^{3}\int_{\Sigma^*} q_i^*|u_i|^2\\
&= -\sum_{i=1}^{3}\int_{\Gamma_i^*}\overline{u_i}\left(\Delta_{\Gamma_i^*}u_i + (\kappa_i^*)^2 u_i\right)\mathrm{d}s + \sum_{i=1}^{3}\int_{\Sigma^*}\left(\partial_{n_{\partial\Gamma_i^*}}u_i + q_i^* u_i\right)\overline{u_i}\,.
\end{aligned} \tag{6.3.6}
$$

Note carefully that in (6.3.6) we just used integration by parts to obtain the second equality. It is interesting now to see that although (due to the linearized angle condition and the fact that on the boundary $u_1+u_2+u_3=0$) we have

$$
\sum_{i=1}^{3}\int_{\Sigma^*}\left(\partial_{n_{\partial\Gamma_i^*}}u_i + q_i^* u_i\right)\overline{u_i} = 0\,, \tag{6.3.7}
$$

but nevertheless this does not effect the value of $I(u,u)$ (cf. [24, Remark 3.7]).

Remark 6.3.2. *The identity (6.3.5) in particular shows that $\lambda\in\mathbb{R}$.*

Remark 6.3.3. *Indeed as one may have expected, the linearized problem (6.2.13), (6.2.14) is the gradient flow of the energy functional*

$$
E(u) = \frac{I(u,u)}{2}\,,
$$

with respect to the H^{-1}-inner product, see for instance [21].

Related problem: Double bubble conjecture

The goal of this section is to prove that, a part from zero, the spectrum of the linearized problem lies in $\mathbb{R}_+$. We do this by considering the bilinear form $I(\,,\,)$.

In the following we state the second variation formula proved in general dimension by Morgan and co-authors:

Proposition 6.3.4. ([24, Proposition 3.3]). *Let Γ^* be a stationary planar double bubble and let φ_t be a one-parameter variation which preserves the areas of enclosed regions. Furthermore denote by $L(t)$ the length of $\varphi_t(\Gamma^*)$. Then*

$$\frac{d^2}{dt^2}L(t)\big|_{t=0} = I(u,u)\,,$$

where $u_i = \langle \frac{d}{dt}\varphi_t, n_i^* \rangle$.

Here and hereafter, by (one-parameter) variations $\{\varphi_t\}_{|t|<\epsilon} : \Gamma \to \mathbb{R}^2$ of a double bubble $\Gamma \subset \mathbb{R}^2$ we mean the variations which are smooth (up to the triple junctions) having equal values along triple junctions.

Remark 6.3.5. *Notice that in (6.3.6) we have used outer unit conormals where inner unit conormals are used in [24]. In addition, the constants q_i^* and their corresponding ones in [24] are also opposite in signs due to the different choice of normals. This explains the sign differences.*

Remark 6.3.6. *Of course, a double bubble is stationary for any variation preserving the area of the enclosed regions if and only if it is stationary for the surface diffusion flow (6.1.1), see Section 6.1.1 and [24, page 465].*

Following [24], we denote by $\mathcal{F}(\Gamma)$ the space of functions $u \in H^1(\Gamma)$ satisfying

$$\begin{cases} u_1 + u_2 + u_3 = 0 \qquad \text{on } \Sigma\,, \\ \displaystyle\int_{\Gamma_1} u_1 = \int_{\Gamma_2} u_2 = \int_{\Gamma_3} u_3\,. \end{cases}$$

Lemma 6.3.7. ([24, Lemma 3.2]). *Let Γ^* be a stationary double bubble. Then for any smooth $u \in \mathcal{F}(\Gamma^*)$ there is an area preserving variation $\{\varphi_t\}$ of Γ^* such that the normal components of the associated infinitesimal vector field are the functions u_i, i.e., $u_i = \langle \frac{d}{dt}\varphi_t, n_i^* \rangle$, $i = 1, 2, 3$.*

We are now ready to present:

Definition 6.3.8 (The concept of stability in differential geometry)**.** *A double bubble Γ^* is said to be variationally stable if it is stationary and*

$$I(u,u) \geq 0 \qquad \forall\, u \in \mathcal{F}(\Gamma^*)\,.$$

Indeed it is an open problem whether for double bubbles this concept of stability in differential geometry is equivalent to the concept of stability in dynamical systems. There are several evidences in this work which show how closely these two concepts are, starting from Lemma 6.3.13 below.

Remark 6.3.9. *Note that the concept of stability in differential geometry is called stable in [24].*

Corollary 6.3.10. *A perimeter-minimizing double bubble for prescribed areas is variationally stable.*

Proof. Let Γ be a perimeter-minimizing double bubble. As a minimizer, the second derivative of length is nonnegative along all variations which preserve the area. In other words, by Proposition 6.3.4 $I(u,u) \geq 0$ for all functions u given by normal components of volume preserving variations. On the other hand, by Lemma 6.3.7 we know that every smooth element of $\mathcal{F}(\Gamma)$ is of this form. Therefore $I(u,u) \geq 0$ for all $u \in \mathcal{F}(\Gamma)$, which finishes the proof. $\square$

Theorem 6.3.11. ([18, Theorem 2.9]). *The standard planar double bubble is the unique perimeter-minimizing double bubble enclosing and separating two given regions of prescribed areas.*

As an important corollary, one gets: (see also [31, Theorem 3.2])

Corollary 6.3.12. *The standard planar double bubble is variationally stable.*

We are now ready to see the first evidence.

Lemma 6.3.13. $\sigma(A_0) \setminus \{0\} \subset \mathbb{R}_+$.

Proof. Let $\lambda \in \sigma(A_0) \setminus \{0\}$. As mentioned before the spectrum consists entirely of eigenvalues. In addition, according to Remark 6.3.2, λ is real.

Therefore, let λ be an eigenvalue with a corresponding eigenvector $u \in C^{4+\alpha}(\Gamma^*)$. This means u solves the eigenvalue problem (6.3.3) subject to the boundary conditions (6.3.4) for λ. Since $\lambda \neq 0$, we deduce after integrating (6.3.3):

$$\int_{\Gamma_1^*} u_1 = \int_{\Gamma_2^*} u_2 = \int_{\Gamma_3^*} u_3 \,,$$

where we employed the divergence theorem and the last boundary condition. This together with the first boundary condition implies that $u \in \mathcal{F}(\Gamma^*)$. Therefore $I(u,u) \geq 0$ by Corollary 6.3.12.

Now assume $I(u,u) = 0$. In view of the equation (6.3.5), we obtain

$$\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i = c_i$$

for some constants c_i (cf. [24, Lemma 3.8]). This together with the equation (6.3.3) immediately implies $u \in N(A_0)$, i.e., $\lambda = 0$, a contradiction. Thus $I(u,u) > 0$ for the eigenvector u. Now $\lambda > 0$ by (6.3.3). This finishes the proof. $\square$

The bilinear form $I(\ ,\ )$ is further discussed in Appendix A.1.

6.3.3 Null space of the linearized operator

We next determine the null space of the linearized operator A_0. That is, we consider the case $\lambda = 0$ in the eigenvalue problem (6.3.3),(6.3.4).

Using the identity (6.3.5), we easily get $u \in N(A_0)$ if and only if there exists a constant vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ such that

$$\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i = c_i \quad \text{on } \Gamma_i^* \quad (i = 1, 2, 3), \tag{6.3.8}$$

subject to the conditions

$$\begin{cases} u_1 + u_2 + u_3 = 0 & \text{on } \Sigma^*, \\ q_1^* u_1 + \partial_{n_{\partial\Gamma_1^*}} u_1 = q_2^* u_2 + \partial_{n_{\partial\Gamma_2^*}} u_2 = q_3^* u_3 + \partial_{n_{\partial\Gamma_3^*}} u_3 & \text{on } \Sigma^*, \\ c_1 + c_2 + c_3 = 0\,. \end{cases} \tag{6.3.9}$$

Notice that the constant vector $c = c(u)$ depends linearly on u by (6.3.8).

Definition 6.3.14. *Following [24], we define the space of Jacobi functions*

$$\mathcal{J}(\Gamma^*) := \left\{ u \in N(A_0) : c = c(u) = 0 \right\}.$$

We need, for later use, an identity that relates the null space $N(A_0)$ to the bilinear form $I(\,,\,)$.

Lemma 6.3.15. *Assume $u \in N(A_0)$. Then*

$$I(u, u) = -\sum_{i=1}^{3} c_i \int_{\Gamma_i^*} u_i\,,$$

where the constants c_i, satisfying $\sum_{i=1}^{3} c_i = 0$, depend linearly on u by (6.3.8).

Proof. By inserting (6.3.8) into the definition of the bilinear form (6.3.6) and taking into account the equation (6.3.7) coming from the first two boundary conditions in (6.3.9), we get the desired identity. □

As a corollary we get

Corollary 6.3.16. *If $u \in N(A_0) \cap \mathcal{F}(\Gamma^*)$, then $I(u, u) = 0$.*

Let us rewrite the linear equations (6.3.8) as a system of linear nonhomogeneous second order ordinary differential equations with constant coefficients

$$\partial_x^2 u_i + (\kappa_i^*)^2 u_i = c_i \quad \text{for } x \in [-l_i^*, l_i^*] \quad (i = 1, 2, 3)\,,$$

with the conditions

$$\begin{cases} u_1 + u_2 + u_3 = 0 & \text{on } \Sigma^*, \\ q_1^* u_1 \pm \partial_x u_1 = q_2^* u_2 \pm \partial_x u_2 = q_3^* u_3 \pm \partial_x u_3 & \text{on } \Sigma^*, \\ c_1 + c_2 + c_3 = 0 \end{cases}$$

for the functions $u_i : [-l_i^*, l_i^*] \to \mathbb{R}$.

Determination of Jacobi functions

Let us first consider the case $\kappa_2^* \neq 0$. The general solution of the linearized problem is then

$$u_i(x) = a_i \sin(\kappa_i^* x) + b_i \cos(\kappa_i^* x) \quad (i = 1, 2, 3)\,. \tag{6.3.10}$$

We calculate at $x = \pm l_1^*$

$$\begin{aligned} q_1^* u_1 &= \mp \cot(\gamma^* + \tfrac{\pi}{3})\kappa_1^* a_1 \sin(\gamma^* + \tfrac{\pi}{3}) + \cot(\gamma^* + \tfrac{\pi}{3})\kappa_1^* b_1 \cos(\gamma^* + \tfrac{\pi}{3}) \\ &= \mp a_1 \kappa_1^* \cos(\gamma^* + \tfrac{\pi}{3}) + b_1 \kappa_1^* \cot(\gamma^* + \tfrac{\pi}{3}) \cos(\gamma^* + \tfrac{\pi}{3})\,, \end{aligned}$$

$$\pm \partial_x u_1 = \pm a_1 \kappa_1^* \cos(\gamma^* + \tfrac{\pi}{3}) + b_1 \kappa_1^* \sin(\gamma^* + \tfrac{\pi}{3})\,.$$

Therefore

$$q_1^* u_1 \pm \partial_x u_1 = b_1 \frac{\kappa_1^*}{\sin(\gamma^* + \frac{\pi}{3})} \qquad \text{at } x = \pm l_1^*\,.$$

Similarly we get

$$\begin{aligned} q_2^* u_2 \pm \partial_x u_2 &= b_2 \frac{\kappa_2^*}{\sin(\gamma^* - \frac{\pi}{3})} \qquad \text{at } x = \pm l_2^*\,, \\ q_3^* u_3 \pm \partial_x u_3 &= b_3 \frac{\kappa_3^*}{\sin(\gamma^* - \pi)} \qquad \text{at } x = \pm l_3^*\,. \end{aligned}$$

Thus we conclude

$$b_1 \frac{\kappa_1^*}{\sin(\gamma^* + \frac{\pi}{3})} = b_2 \frac{\kappa_2^*}{\sin(\gamma^* - \frac{\pi}{3})} = b_3 \frac{\kappa_3^*}{\sin(\gamma^* - \pi)}\,.$$

Furthermore, $u_1(\pm l_1^*) + u_2(\pm l_2^*) + u_3(\pm l_3^*) = 0$ reads as

$$\begin{aligned} &\mp a_1 \sin(\gamma^* + \frac{\pi}{3}) + b_1 \cos(\gamma^* + \frac{\pi}{3}) \\ &\qquad \mp a_2 \sin(\gamma^* - \frac{\pi}{3}) + b_2 \cos(\gamma^* - \frac{\pi}{3}) \\ &\qquad\qquad \mp a_3 \sin(\gamma^* - \pi) + b_3 \cos(\gamma^* - \pi) = 0\,. \end{aligned}$$

Altogether, we have to find solutions to the following system

$$\begin{cases} a_1 \sin(\gamma^* + \dfrac{\pi}{3}) + a_2 \sin(\gamma^* - \dfrac{\pi}{3}) + a_3 \sin(\gamma^* - \pi) = 0\,, \\ b_1 \cos(\gamma^* + \dfrac{\pi}{3}) + b_2 \cos(\gamma^* - \dfrac{\pi}{3}) + b_3 \cos(\gamma^* - \pi) = 0\,, \\ b_1 \dfrac{\kappa_1^*}{\sin(\gamma^* + \frac{\pi}{3})} = b_2 \dfrac{\kappa_2^*}{\sin(\gamma^* - \frac{\pi}{3})} = b_3 \dfrac{\kappa_3^*}{\sin(\gamma^* - \pi)}\,. \end{cases}$$

Due to the identities (6.1.5) and (6.1.6) we get

$$\begin{cases} a_1 \sin(\gamma^* + \dfrac{\pi}{3}) + a_2 \sin(\gamma^* - \dfrac{\pi}{3}) + a_3 \sin(\gamma^* - \pi) = 0\,, \\ b_1 = b_2 = b_3\,. \end{cases}$$

Therefore, in view of the formula (6.1.6), we obtain

$$(a_1, a_2, a_3) \in \operatorname{span}\Big\{(1,1,1), \big(0, -\tfrac{\sin(\gamma^*-\pi)}{\sin(\gamma^*-\frac{\pi}{3})}, 1\big)\Big\}, \quad (b_1, b_2, b_3) \in \operatorname{span}\{(1,1,1)\}\,.$$

This shows the following lemma:

Lemma 6.3.17. *Assume $\kappa_2^* \neq 0$. Then the space of Jacobi functions is a three-dimensional vector space whose basis consists of*

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ \cos(\kappa_2^* x) \\ \cos(\kappa_3^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \frac{\sin(\gamma^*)}{\sin(\gamma^*-\frac{\pi}{3})} \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}.$$

We now consider the case $\kappa_2^* = 0$. The general solution of the linearized problem is then

$$\begin{aligned} &u_1 = a_1 \sin(\kappa_1^* x) + b_1 \cos(\kappa_1^* x)\,, \quad u_2 = a_2 x + b_2\,, \\ &u_3 = a_3 \sin(\kappa_3^* x) + b_3 \cos(\kappa_3^* x)\big(= -a_3 \sin(\kappa_1^* x) + b_3 \cos(\kappa_1^* x)\big)\,, \end{aligned}$$

where we used the fact that $\kappa_3^* = -\kappa_1^*$ in case $\gamma^* = \frac{\pi}{3}$. Let us also remind that for $\gamma^* = \frac{\pi}{3}$ we have

$$q_2^* = \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} \qquad \text{and} \qquad l_2^* = -\frac{\sin(\frac{\pi}{3})}{\kappa_1^*}$$

and so $q_2^* \, l_2^* = -1$. Therefore,

$$q_2^* u_2 \pm \partial_x u_2 = \mp a_2 + \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} b_2 \pm a_2 = \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} b_2 \qquad \text{at } x = \pm l_2^*\,.$$

Taking into account the calculation done previously for u_1 and u_3, the condition $q_1^* u_1 \pm \partial_x u_1 = q_2^* u_2 \pm \partial_x u_2 = q_3^* u_3 \pm \partial_x u_3$ reads as

$$b_1 \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} = b_2 \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} = b_3 \frac{\kappa_3^*}{\sin(-\frac{2\pi}{3})} \Big(= b_3 \frac{-\kappa_1^*}{-\sin(\frac{\pi}{3})} = b_3 \frac{\kappa_1^*}{\sin(\frac{\pi}{3})}\Big)\,.$$

Therefore, we conclude $b_1 = b_2 = b_3$. Furthermore, $u_1(\pm l_1^*) + u_2(\pm l_2^*) + u_3(\pm l_3^*) = 0$ reads as

$$\mp a_1 \sin(\tfrac{\pi}{3}) + b_1 \cos(\tfrac{2\pi}{3}) \mp a_2 \frac{\sin(\frac{\pi}{3})}{\kappa_1^*} + b_2 \pm a_3 \sin(\tfrac{\pi}{3}) + b_3 \cos(\tfrac{2\pi}{3}) = 0\,.$$

Moreover, using the facts that $b_1 = b_2 = b_3$ and $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$, we see that

$$b_1 \cos(\tfrac{2\pi}{3}) + b_2 + b_3 \cos(\tfrac{2\pi}{3}) = 0\,.$$

In summary, we have to find solutions to the following system

$$\begin{cases} a_1 + \dfrac{a_2}{\kappa_1^*} - a_3 = 0\,, \\ b_1 = b_2 = b_3. \end{cases}$$

Therefore,

$$(a_1, a_2, a_3) \in \operatorname{span}\big\{(1,0,1),(0,\kappa_1^*,1)\big\}, \quad (b_1, b_2, b_3) \in \operatorname{span}\{(1,1,1)\}\,.$$

Lemma 6.3.18. *Assume $\kappa_2^* = 0$. Then the space of Jacobi functions is* 3-| *and its basis is given by*

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ 1 \\ \cos(\kappa_1^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ 0 \\ -\sin(\kappa_1^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \kappa_1^* x \\ -\sin(\kappa_1^* x) \end{pmatrix}.$$

The null space $N(A_0)$ is at most five-dimensional

Next we try to get an upper bound on the dimension of the null space.

Lemma 6.3.19. *The null space $N(A_0)$ of the linearized operator A_0 is at most five-dimensional.*

Proof. We have already shown that the space of Jacobi functions is three-’. Therefore it is enough to show that there exist at most two independent vectors in the null space $N(A_0)$ for which $c \neq 0$.

Take any three vector functions $u^{(1)}, u^{(2)}, u^{(3)} \in N(A_0)$ for which the vector constants $c^{(i)} = c^{(i)}(u^{(i)}) \neq 0$, $i = 1,2,3$. Then as

$$c^{(i)} \in \big\{\, c = (c_1, c_2, c_3) \in \mathbb{R}^3 : c_1 + c_2 + c_3 = 0 \big\}$$

which is a two-dimensional subspace of $\mathbb{R}^3$, there exist scalars a_1, a_2, a_3, not all zero, such that

$$0 = \sum_{j=1}^{3} a_i c^{(i)} = \sum_{i}^{3} a_i T u^{(i)} = T\Big(\sum_{i=1}^{3} a_i u^{(i)}\Big)\,.$$

Here T is the linear operator defined by the left hand side of (6.3.8). Thus we get $\sum_{i=1}^{3} a_i u^{(i)} \in \mathcal{J}(\Gamma^*)$, in other words,

$$\sum_{i=1}^{3} a_i u^{(i)} = \sum_{j=1}^{3} b_j v^{(j)},$$

where $\{v^{(1)}, v^{(2)}, v^{(3)}\}$ is a basis of $J(\Gamma^*)$. This means that the vectors

$$u^{(1)}, u^{(2)}, u^{(3)}, v^{(1)}, v^{(2)}, v^{(3)}$$

are linearly dependent and completes the proof. □

Indeed, we will prove in Corollary 6.3.26 below that the dimension of the null space is exactly five.

6.3.4 Manifold of equilibria

Our goal in this section is to prove that near $\rho \equiv 0$, which corresponds to Γ^*, the set $\mathcal{E}$ of equilibria of the nonlinear system (6.2.12) creates a smooth manifold of dimension 5.

Equilibria of the nonlinear system

Let us first identify the set of equilibria $\mathcal{E}$ of the nonlinear system (6.2.12). According to (3.1.2), $\rho \in \mathcal{E}$ if and only if for $i = 1, 2, 3$ and $j = 1, 2, ..., 6$,

$$\begin{cases} \rho \in B_{X_1}(0, R)\,, & \\ 0 = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) & \\ \qquad + \mathfrak{B}_i(\rho_i, \rho|_{\Sigma^*})\Big(\Big\{\mathcal{J}\big(I - \mathfrak{B}(\rho, \rho|_{\Sigma^*})\mathcal{J}\big)^{-1}\mathfrak{F}(\rho, \rho|_{\Sigma^*})\Big\} \circ \mathrm{pr}_i\Big)_i & \text{on } \Gamma_i^*\,, \\ 0 = \mathfrak{G}_j(\rho) & \text{on } \Sigma^*. \end{cases}$$

Similarly as done in Section 6.2.2, we can write the first three equations as a vector identity on Σ^* and thereby obtain $\mathfrak{F}(\rho, \rho|_{\Sigma^*}) = 0$. Thus

$$\rho \in \mathcal{E} \Leftrightarrow \begin{cases} \rho \in B_{X_1}(0, R)\,, & & \\ 0 = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) & \text{on } \Gamma_i^*\,, & i = 1, 2, 3\,, \\ 0 = \mathfrak{G}_j(\rho) & \text{on } \Sigma^*\,, & j = 1, \ldots, 6\,. \end{cases}$$

Taking into account (B.1.4), the definition of $\mathfrak{F}_i$, the balance of flux conditions $\mathfrak{G}_5, \mathfrak{G}_6$ and the condition on curvature $\mathfrak{G}_4$, by applying the Gauss theorem, we see

$$\rho \in \mathcal{E} \Leftrightarrow \begin{cases} \rho \in B_{X_1}(0, R)\,, & & \\ \kappa_i\big(\rho_i, (\mathcal{J}\rho \circ \mathrm{pr})_i\big) \text{ are constant,} & \text{on } \Gamma^*\,, & \\ \mathfrak{G}_j(\rho) = 0 & \text{on } \Sigma^*\,, & j = 1, 2, 3, 4\,. \end{cases}$$

Therefore, using Lemma 6.2.1, we conclude:

$$\mathcal{E} = \Big\{\, \rho \in B_{X_1}(0, R) : \rho \text{ parameterizes a standard planar double bubble} \,\Big\}.$$

Level set representation of standard double bubbles

Next we represent standard planar double bubbles as a subset of the zero level sets of some smooth functions. Let $S_{r_i}(O_i)$, $i = 1, 2, 3$, be the corresponding circles to standard planar double bubble $\Gamma = \{\,\Gamma_1, \Gamma_2, \Gamma_3\,\}$. In other words, $\Gamma_i \subset S_{r_i}(O_i)$, where r_i and O_i are the radius and the center of Γ_i respectively.

Lemma 6.3.20. *Let* $\Gamma = DB_{r,\gamma,0}(0, 0)$. *Then*

$$\big\{\, \sigma \in \mathbb{R}^2 : G_i(\sigma, r, \gamma) = 0 \,\big\} = S_{r_i}(O_i) \supset \Gamma_i \qquad (i = 1, 2, 3),$$

where $G_i : \mathbb{R}^2 \times (0,\infty) \times (0, \frac{2\pi}{3}) \to \mathbb{R}$ *are smooth functions defined by*

$$
\begin{aligned}
r\sin(\gamma+\tfrac{\pi}{3})G_1(\sigma,r,\gamma) &= \tfrac{1}{2}\sin(\gamma+\tfrac{\pi}{3})\Big(|\sigma|^2 - r^2\Big)\,,\\
r\sin(\gamma+\tfrac{\pi}{3})G_2(\sigma,r,\gamma) &= \tfrac{1}{2}\Big(\sin(\gamma-\tfrac{\pi}{3})|\sigma|^2 - 2r\sin(\tfrac{\pi}{3})\big\langle\sigma,(1,0)\big\rangle - r^2\sin(\gamma-\pi)\Big)\,,\\
r\sin(\gamma+\tfrac{\pi}{3})G_3(\sigma,r,\gamma) &= \tfrac{1}{2}\Big(\sin(\gamma-\pi)|\sigma|^2 + 2r\sin(\tfrac{\pi}{3})\big\langle\sigma,(1,0)\big\rangle - r^2\sin(\gamma-\tfrac{\pi}{3})\Big)\,,
\end{aligned}
$$

with the property that

$$G_1 + G_2 + G_3 = 0\,. \tag{6.3.11}$$

The proof is given in Appendix A.2. Next let us look at the gradient of the functions G_i.

Lemma 6.3.21. *Let* $\Gamma = DB_{r,\gamma,0}(0,0)$. *Then*

$$\nabla_\sigma G_i(\sigma,r,\gamma) = n_i(\sigma) \qquad \textit{for } \sigma\in\Gamma_i \;\; (i=1,2,3).$$

Proof. It is easy to see that

$$\sigma - O_i = -\frac{1}{\kappa_i} n_i(\sigma) \qquad \text{for } \sigma\in\Gamma_i \quad (i=1,2,3)\,.$$

Using this, we calculate

$$
\begin{aligned}
r\sin(\gamma+\tfrac{\pi}{3})\nabla_\sigma G_2(\sigma,r,\gamma) &= \sin(\gamma-\tfrac{\pi}{3})\sigma - r\sin(\tfrac{\pi}{3})(1,0)\\
&= \sin(\gamma-\tfrac{\pi}{3})\big(\sigma - O_2\big)\\
&= -\tfrac{\sin(\gamma-\frac{\pi}{3})}{\kappa_2} n_2(\sigma) \qquad\qquad \text{for } \sigma\in\Gamma_2\,.
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
r\sin(\gamma+\tfrac{\pi}{3})\nabla_\sigma G_1(\sigma,r,\gamma) &= -\tfrac{\sin(\gamma+\frac{\pi}{3})}{\kappa_1} n_1(\sigma) \qquad \text{for } \sigma\in\Gamma_1,\\
r\sin(\gamma+\tfrac{\pi}{3})\nabla_\sigma G_3(x,r,\gamma) &= -\tfrac{\sin(\gamma-\pi)}{\kappa_3} n_3(\sigma) \qquad \text{for } \sigma\in\Gamma_3\,.
\end{aligned}
$$

Since $r\sin(\gamma+\frac{\pi}{3}) = -\frac{\sin(\gamma+\frac{\pi}{3})}{\kappa_1}$, by the identity (6.1.5) we complete the proof. □

Furthermore, the following result holds.

Proposition 6.3.22. *Let* $\Gamma = DB_{r,\gamma,0}(0,0)$. *Then*

$$
\begin{cases}
\partial_r G_1(\sigma,r,\gamma) = -1 & \textit{for } \sigma\in\Gamma_1,\\
\partial_r G_2(\sigma,r,\gamma) = -\frac{1}{\sin(\gamma+\frac{\pi}{3})}\Big(\frac{\sin(\frac{\pi}{3})}{r}\sigma_1 + \sin(\gamma-\pi)\Big) & \textit{for } \sigma\in\Gamma_2,\\
\partial_r G_3(\sigma,r,\gamma) = +\frac{1}{\sin(\gamma+\frac{\pi}{3})}\Big(\frac{\sin(\frac{\pi}{3})}{r}\sigma_1 - \sin(\gamma-\frac{\pi}{3})\Big) & \textit{for } \sigma\in\Gamma_3.
\end{cases}
$$

Proof. According to Lemma 6.3.20, we have

$$G_i(\sigma, r, \gamma) = 0 \qquad \text{for } \sigma \in \Gamma_i\,.$$

Therefore, differentiating with respect to r in the definitions of functions G_i, we observe

$$\begin{aligned} -r\sin(\gamma+\tfrac{\pi}{3})\partial_r G_1(\sigma,r,\gamma) &= \sin(\gamma+\tfrac{\pi}{3})r && \text{for } \sigma\in\Gamma_1,\\ -r\sin(\gamma+\tfrac{\pi}{3})\partial_r G_2(\sigma,r,\gamma) &= \sin(\tfrac{\pi}{3})\big\langle\sigma,(1,0)\big\rangle+\sin(\gamma-\pi)r && \text{for } \sigma\in\Gamma_2\,,\\ r\sin(\gamma+\tfrac{\pi}{3})\partial_r G_3(\sigma,r,\gamma) &= \sin(\tfrac{\pi}{3})\big\langle\sigma,(1,0)\big\rangle-\sin(\gamma-\tfrac{\pi}{3})r && \text{for } \sigma\in\Gamma_3\,, \end{aligned}$$

which finishes the proof. □

Similarly we get

Proposition 6.3.23. *Let* $\Gamma = DB_{r,\gamma,0}(0,0)$. *Then*

$$\begin{cases} \partial_\gamma G_1(\sigma,r,\gamma) = 0 & \textit{for } \sigma\in\Gamma_1,\\ \partial_\gamma G_2(\sigma,r,\gamma) = \frac{1}{2r\sin(\gamma+\frac{\pi}{3})}\Big(\cos(\gamma-\tfrac{\pi}{3})|\sigma|^2 - r^2\cos(\gamma-\pi)\Big) & \textit{for } \sigma\in\Gamma_2,\\ \partial_\gamma G_3(\sigma,r,\gamma) = \frac{1}{2r\sin(\gamma+\frac{\pi}{3})}\Big(\cos(\gamma-\pi)|\sigma|^2 - r^2\cos(\gamma-\tfrac{\pi}{3})\Big) & \textit{for } \sigma\in\Gamma_3. \end{cases}$$

Five-dimensional smooth manifold

Throughout this section, without loss of generality, we may assume that the center of Γ_1^* is at the origin of $\mathbb{R}^2$ and that the angle $\theta^* = 0$, that is

$$\Gamma^* = DB_{r^*,\gamma^*,0}(0,0)\,.$$

Clearly, $\mathcal{E} \neq \emptyset$ as $\rho \equiv 0$ parameterizes $\Gamma^* = DB_{r^*,\gamma^*,0}(0,0)$. First we demonstrate, by applying the implicit function theorem, that every standard planar double bubble $DB_{r,\gamma,\theta}(a_1,a_2)$ sufficiently close to $\Gamma^* = DB_{r^*,\gamma^*,0}(0,0)$ can be parameterized by some unique vector function $\rho = (\rho_1,\rho_2,\rho_3)$ depending smoothly on the parameters a_1, a_2, r, γ and θ. We continue then to verify that the set $\mathcal{E}$ of equilibria is actually a smooth manifold of dimension five.

Theorem 6.3.24. *Any standard planar double bubble* $DB_{r,\gamma,\theta}(a_1,a_2)$ *sufficiently close to* Γ^**, i.e.,* $(a_1,a_2,r,\gamma,\theta) \in B_\epsilon(0,0,r^*,\gamma^*,0)$ *for sufficiently small* ϵ*, can be parameterized by some unique smooth vector function* $\rho = \rho(a_1,a_2,r,\gamma,\theta) \in B_{X_1}(0,R)$.

Proof. Let us use the implicit function theorem 2.3.1 with

$$\begin{aligned} &(x_0,y_0) = \big((0,0,r^*,\gamma^*,0),0\big),\\ &\qquad X = \mathbb{R}^2\times B_{\delta_1}(r^*)\times B_{\delta_2}(\gamma^*)\times\mathbb{R}\,,\quad Z = Y,\\ &\qquad Y = \Big\{\rho\in C^{4+\alpha}(\Gamma_1^*)\times C^{4+\alpha}(\Gamma_2^*)\times C^{4+\alpha}(\Gamma_3^*) : \rho_1+\rho_2+\rho_3 = 0 \text{ on } \Sigma^*\Big\}, \end{aligned}$$

$$F : X \times Y \to Z\,,$$
$$\big((a_1, a_2, r, \gamma, \theta), \rho\big) \mapsto (F_1, F_2, F_3)$$

with

$$F_i(a_1, a_2, r, \gamma, \theta, \rho) := G_i\Big(Q_\theta T_{\vec{a}} \Psi_i(\cdot, \rho_i, \mu_i \circ \mathrm{pr}_i), r, \gamma\Big) \qquad (i = 1, 2, 3)\,.$$

Here G_i are the functions stated in Lemma 6.3.20 and

$$\Psi_i(\cdot, \rho_i, \mu_i \circ \mathrm{pr}_i)(\sigma) = \sigma + \rho_i(\sigma) n_i^*(\sigma) + \mu_i(\mathrm{pr}_i(\sigma)) \tau_i^*(\sigma) \qquad \text{for } \sigma \in \Gamma_i^*\,,$$

where $\mu = \mathcal{J}\rho$ on Σ^*. Furthermore,

$$Q_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad T_{\vec{a}} v = v - \vec{a}$$

are the clockwise rotation matrix and the translation operator respectively.

Indeed, the image of the function F lies in $Z = Y$, that is

$$F_1 + F_2 + F_3 = 0 \quad \text{on } \Sigma^*. \tag{6.3.12}$$

To see this, note that for $\sigma \in \Sigma^*$,

$$\Psi_1(\cdot, \rho_1, \mu_1 \circ \mathrm{pr}_1)(\sigma) = \Psi_2(\cdot, \rho_2, \mu_2 \circ \mathrm{pr}_2)(\sigma) = \Psi_3(\cdot, \rho_3, \mu_3 \circ \mathrm{pr}_3)(\sigma)\,,$$

by Lemma 6.2.1. This together with the identity (6.3.11) proves (6.3.12).

Moreover, since $\Psi_i|_{\rho=0} = I$, according to Lemma 6.3.20 we have

$$F_i(x_0, y_0)(\sigma) = F_i\big((0, 0, r^*, \gamma^*, 0), 0\big)(\sigma) = G_i(\sigma, r^*, \gamma^*) = 0 \qquad \text{for } \sigma \in \Gamma_i^*.$$

Thus $F(x_0, y_0) = 0$. Now let us compute the derivative $\partial_\rho F(x_0, y_0)$:

$$\partial_\rho F_i(x_0, y_0)(v)(\sigma) = \nabla_\sigma G_i(\sigma, r^*, \gamma^*) \cdot \big(v_i\, n_i^*(\sigma) + \big(\mathcal{J}\, v(\mathrm{pr}_i(\sigma))\big)_i\, \tau_i^*(\sigma)\big) = v_i\,,$$

where we used Lemma 6.3.21. Thus

$$\partial_\rho F(x_0, y_0) = I\,. \tag{6.3.13}$$

Furthermore, F is a smooth map on a neighborhood of (x_0, y_0).

Therefore, according to the implicit function theorem, there exist neighborhoods $U = B_\epsilon(x_0)$ of x_0 and $V = B_{X_1}(0, R)$ of $y_0 = 0$ and a smooth function

$$\rho : U \longrightarrow V$$
$$(a_1, a_2, r, \gamma, \theta) \mapsto \rho(a_1, a_2, r, \gamma, \theta)\,,$$

such that $\rho(x_0) = 0$ and for every $(a_1, a_2, r, \gamma, \theta) \in B_\epsilon(0, 0, r^*, \frac{\pi}{3}, 0)$ we have

$$F\big((a_1, a_2, r, \gamma, \theta), \rho(a_1, a_2, r, \gamma, \theta)\big) = 0\,. \tag{6.3.14}$$

Moreover if $(x, y) \in U \times V$ and $F(x, y) = 0$ then $y = \rho(x)$.

We now claim that $\Gamma_\rho = \{\Gamma_{\rho_1}, \Gamma_{\rho_2}, \Gamma_{\rho_3}\}$ parameterized by the function $\rho = \rho(a_1, a_2, r, \gamma, \theta)$ is the standard planar double bubble $DB_{r,\gamma,\theta}(a_1, a_2)$. To see this, note

$$\begin{aligned}
&F_i\big((a_1, a_2, r, \gamma, \theta), \rho(a_1, a_2, r, \gamma,\theta)\big) = 0 \\
&\qquad\qquad\qquad \iff G_i\Big(Q_\theta T_{\vec{a}} \Psi_i(\cdot, \rho_i, \mu_i \circ \mathrm{pr}_i), r, \gamma\Big) = 0 \\
&\qquad\qquad\qquad \iff Q_\theta T_{\vec{a}} \Gamma_{\rho_i} \subset S_{r_i}(O_i) \quad \text{by Lemma 6.3.20.}
\end{aligned}$$

Therefore, since Lemma 6.2.1 guaranties that the curves $\Gamma_{\rho_1}, \Gamma_{\rho_2}, \Gamma_{\rho_3}$ meet at their boundaries, we end up with two choices: Either $\Gamma_{\rho_i} = \Gamma_i$, where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ is a standard double bubble $DB_{r,\gamma,\theta}(a_1, a_2)$ or Γ_{ρ_i} is the complementary part of Γ_i in $S_{r_i}(O_i)$. But the latter can not happen since the norm of ρ is small. Hence

$$\Gamma_{\rho(a_1,a_2,r,\gamma,\theta)} = DB_{r,\gamma,\theta}(a_1, a_2)\,,$$

as required. □

Theorem 6.3.25. *The set of equilibria $\mathcal{E}$ is in a neighborhood of zero a C^2-manifold in X_1 of dimension 5.*

Proof. Remind that we have shown

$$\begin{aligned}
\mathcal{E} \cap U &= \Big\{\rho \in B_{X_1}(0, R) : \rho \text{ parameterizes a} \\
&\qquad\qquad\qquad\qquad \text{standard planar double bubble}\Big\} \cap U \\
&= \Big\{\, \rho(a_1, a_2, r, \gamma, \theta) : (a_1, a_2, r, \gamma, \theta) \in U = B_\epsilon(0, 0, r^*, \gamma^*, 0) \,\Big\}\,,
\end{aligned}$$

where the function

$$\begin{aligned}
\rho :\ U &\longrightarrow X_1 = C^{4+\alpha}(\Gamma_1^*) \times C^{4+\alpha}(\Gamma_2^*) \times C^{4+\alpha}(\Gamma_3^*) \\
(a_1, a_2, r, \gamma, \theta) &\mapsto \rho\ (a_1, a_2, r, \gamma, \theta)
\end{aligned}$$

is smooth, in particular C^2 and $\rho(U) = \mathcal{E}$, $\rho(x_0) = \rho(0, 0, r^*, \gamma^*, 0) = 0$.

Therefore, it is left to check that the rank of $\rho'(x_0)$ equals five (see the definition of a manifold on page 23). To do this, we differentiate (6.3.14) with respect to $\iota \in \{a_1, a_2, r, \gamma, \theta\}$ and evaluate at x_0 to get

$$\partial_\iota F(x_0, 0) + \partial_\rho F(x_0, 0) \partial_\iota \rho(x_0) = 0\,.$$

Therefore, (6.3.13) gives

$$\partial_\iota \rho(x_0) = -\partial_\iota F(x_0, 0) \qquad (\iota \in \{a_1, a_2, r, \gamma, \theta\}).$$

We now calculate

$$\partial_{a_1} F_i(x_0, 0) = \nabla_\sigma G_i(\sigma, r^*, \gamma^*) \cdot (-1, 0) = n_i^*(\sigma) \cdot (-1, 0) = \cos(\kappa_i^* x),$$

where we used the fact $n_i^*(\sigma) = -(\cos(\kappa_i^* x), \sin(\kappa_i^* x))$, $i = 1, 2, 3$. Thus

$$\partial_{a_1} \rho(x_0) = \big(\cos(\kappa_1^* x), \cos(\kappa_2^* x), \cos(\kappa_3^* x)\big).$$

Similarly, we get $\partial_{a_2} \rho(x_0) = \big(\sin(\kappa_1^* x), \sin(\kappa_2^* x), \sin(\kappa_3^* x)\big)$.

Next we calculate

$$\begin{aligned} \partial_\theta F_i(x_0, 0) &= \nabla_\sigma G_i(\sigma, r^*, \gamma^*) \cdot \Big(\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \cdot \sigma \Big) \\ &= n_i^*(\sigma) \cdot \sigma^\perp = n_i^*(\sigma) \cdot \big(-\frac{1}{\kappa_i^*} n_i^*(\sigma) + O_i^* \big)^\perp \\ &= n_i^*(\sigma) \cdot O_i^{*\perp} \end{aligned}$$

and so

$$\partial_\theta \rho(x_0) = \tfrac{\sin(\frac{\pi}{3})}{\sin(\gamma^*)} r^* \begin{pmatrix} 0 \\ \frac{\sin(\gamma^*)}{\sin(\gamma^* - \frac{\pi}{3})} \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}.$$

We now compute the derivative $\partial_r F(x_0, 0) = \partial_r G(\sigma, r^*, \gamma^*)$. According to Proposition 6.3.22

$$\partial_r G_2(\sigma, r^*, \gamma^*) = -\frac{1}{\sin(\gamma^* + \frac{\pi}{3})} \Big(\frac{\sin(\frac{\pi}{3})}{r^*} \sigma_1 + \sin(\gamma^* - \pi) \Big).$$

First we consider the case $\kappa_2^* \neq 0$. Employing the arc-length parameterization of Γ_2^* derived in Proposition A.3.1 we obtain

$$\begin{aligned} \partial_r G_2(\sigma, r^*, \gamma^*) &= -\frac{1}{\sin(\gamma^* + \frac{\pi}{3})} \Big(\frac{\sin(\frac{\pi}{3})}{r^*} \sigma_1 + \sin(\gamma^* - \pi) \Big) \\ &= \frac{\kappa_1^* \sin(\frac{\pi}{3})}{\kappa_2^* \sin(\gamma^* + \frac{\pi}{3})} \cos(\kappa_2^* x) - \frac{1}{\sin(\gamma^* + \frac{\pi}{3})} \Big(\frac{\sin^2(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} + \sin(\gamma^* - \pi) \Big) \\ &= \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x) - \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}, \end{aligned}$$

where we applied the formula $\sin^2(x) - \sin^2(y) = \sin(x + y)\sin(x - y)$.

A similar argument works for $\partial_r G_3(\sigma, r^*, \gamma^*)$. Altogether we derive in case $\gamma^* \neq \frac{\pi}{3}$,

$$\partial_r \rho(x_0) = \begin{pmatrix} 1 \\ -\dfrac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x) + \dfrac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \\ \dfrac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \pi)} \cos(\kappa_3^* x) + \dfrac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \pi)} \end{pmatrix}.$$

Next we consider the case $\kappa_2^* = 0$: We calculate

$$\partial_r G_2(\sigma, r^*, \tfrac{\pi}{3}) = -\frac{1}{\sin(\frac{2\pi}{3})}\Big(\frac{\sin(\frac{\pi}{3})}{r^*}\frac{r^*}{2} + \sin(-\frac{2\pi}{3})\Big) = \frac{1}{2}\,.$$

Therefore, we derive in case $\kappa_2^* = 0$, i.e., when $x_0 = (0, 0, r^*, \frac{\pi}{3}, 0)$,

$$\partial_r \rho(x_0) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\cos(\kappa_1^* x) - 1 \end{pmatrix}.$$

Finally let us calculate $\partial_\gamma F(x_0, 0)$. We have $\partial_\gamma F(x_0, 0) = \partial_\gamma G(\sigma, r^*, \gamma^*)$. We first consider the case $\kappa_2^* \neq 0$: Employing the arc length parameterization of Γ^* to the formulas derived in Proposition 6.3.23 we derive in case $\kappa_2^* \neq 0$ that

$$\partial_\gamma \rho(x_0) = \begin{pmatrix} 0 \\ a_2 \cos(\kappa_2^* x) + b_2 \\ a_3 \cos(\kappa_3^* x) + b_3 \end{pmatrix}$$

for some constants a_i, b_i (see the Appendix for the explicit form of the constants). This immediately implies that $\partial_\gamma \rho(x_0)$ is independent from the other elements of $\rho'(x_o)$.

However, we give the explicit formula in case $\kappa_2^* = 0$. Using Proposition 6.3.23 we see

$$\begin{aligned}
\partial_\gamma G_2(\sigma, r^*, \tfrac{\pi}{3}) &= \frac{1}{2r^* \sin(\frac{2\pi}{3})}\Big(\frac{1}{4}(r^*)^2 + x^2 + \frac{1}{2}(r^*)^2\Big) \\
&= -\frac{\kappa_1^*}{\sin(\frac{2\pi}{3})}\Big(\frac{1}{2}x^2 + \frac{3}{8}\frac{1}{(\kappa_1^*)^2}\Big),
\end{aligned}$$

$$\begin{aligned}
\partial_\gamma G_3(\sigma, r^*, \tfrac{\pi}{3}) &= \frac{1}{2r^* \sin(\frac{2\pi}{3})}\Big(\cos(-\tfrac{2\pi}{3})|\sigma|^2 - (r^*)^2\Big) \\
&= \frac{-1}{2r^* \sin(\frac{2\pi}{3})}(\frac{1}{2}|\sigma|^2 + (r^*)^2) \\
&= \frac{1}{\kappa_1^* \sin(\frac{2\pi}{3})}(1 + \frac{1}{2}\cos(\kappa_1^* x))\,.
\end{aligned}$$

In summary, we have proved that the rank of $\rho'(x_0)$ is equal to five and we have shown that the set of equilibria $\mathcal{E}$ is a C^2-manifold in X_1 of dimension five. Moreover

$$T_0\mathcal{E} = \operatorname{span}\big\{v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}\big\},$$

where

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ \cos(\kappa_2^* x) \\ \cos(\kappa_3^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \frac{\sin(\gamma^*)}{\sin(\gamma^* - \frac{\pi}{3})}\sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix},$$

$$v^{(4)} = \begin{pmatrix} 1 \\ \frac{-\sin(\frac{\pi}{3})}{\sin(\gamma^*-\frac{\pi}{3})}\cos(\kappa_2^* x) + \frac{\sin(\gamma^*+\frac{\pi}{3})}{\sin(\gamma^*-\frac{\pi}{3})} \\ \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^*-\pi)}\cos(\kappa_3^* x) + \frac{\sin(\gamma^*+\frac{\pi}{3})}{\sin(\gamma^*-\pi)} \end{pmatrix}, \quad v^{(5)} = \begin{pmatrix} 0 \\ a_2\cos(\kappa_2^* x) + b_2 \\ a_3\cos(\kappa_3^* x) + b_3 \end{pmatrix}.$$

Although $v^{(i)}$ are continuous in particular at $\kappa_2^* = 0$, for convenience we state them in case $\kappa_2^* = 0$:

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ 1 \\ \cos(\kappa_1^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ 0 \\ -\sin(\kappa_1^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \kappa_1^* x \\ -\sin(\kappa_1^* x) \end{pmatrix},$$

$$v^{(4)} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\cos(\kappa_1^* x) - 1 \end{pmatrix}, \quad v^{(5)} = \begin{pmatrix} 0 \\ \frac{\kappa_1^*}{\sin(\frac{\pi}{3})}(\frac{1}{2}x^2 + \frac{3}{8}\frac{1}{(\kappa_1^*)^2}) \\ \frac{-1}{\kappa_1^*\sin(\frac{\pi}{3})}(\frac{1}{2}\cos(\kappa_1^* x) + 1) \end{pmatrix}.$$

□

Geometric interpretation of the null space

As an immediate corollary of Theorem 6.3.25 we get

Corollary 6.3.26. *The null space $N(A_0)$ is five dimensional. Furthermore,*

$$T_0\mathcal{E} = N(A_0)\,.$$

Proof. It always holds

$$T_0\mathcal{E} \subseteq N(A_0),$$

see equation (3.1.8). Thus, according to Theorem 6.3.25 and Lemma 6.3.19,

$$5 = \dim(T_0\mathcal{E}) \leq \dim(N(A_0)) \leq 5\,.$$

It follows that $\dim(N(A_0)) = 5$ and moreover $T_0\mathcal{E} = N(A_0)$. □

Variations preserving areas and curvatures

We easily see, using formula (6.1.5), that

$$\begin{cases} \int_{\Gamma_1^*} {v^{(1)}}_1 = \int_{\Gamma_2^*} {v^{(1)}}_2 = \int_{\Gamma_3^*} {v^{(1)}}_3 = -2\dfrac{\sin(\gamma^*+\frac{\pi}{3})}{\kappa_1^*}\,, \\ \int_{\Gamma_1^*} {v^{(2)}}_1 = \int_{\Gamma_2^*} {v^{(2)}}_2 = \int_{\Gamma_3^*} {v^{(2)}}_3 = 0\,, \\ \int_{\Gamma_1^*} {v^{(3)}}_1 = \int_{\Gamma_2^*} {v^{(3)}}_2 = \int_{\Gamma_3^*} {v^{(3)}}_3 = 0. \end{cases}$$

In other words,

$$\mathcal{J}(\Gamma^*) \subseteq \mathcal{F}(\Gamma^*). \tag{6.3.15}$$

By Lemma 6.3.7, each of the $v^{(i)}$ $(i = 1, 2, 3)$ corresponds to a first variation of Γ^* which preserves the areas, and the curvatures to first order. Indeed, we have demonstrated in the proof of Theorem 6.3.25 that $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ correspond to the first variations of the double bubble Γ^* associated with translation along x-axis, translation along y-axis and rotation around the center of Γ_1^*, respectively.

Variations not preserving areas and curvatures

It is shown in the proof of Theorem 6.3.25 that $v^{(4)}$ corresponds to the first variations of the double bubble Γ^* associated with uniform scaling (with the scale factor $\frac{r}{r^*}$). Let $A_i(r)$ denote the area of the regions $R_i(r)$ corresponding to the double bubble $DB_{r,\gamma^*,\theta^*}(a_1^*, a_2^*)$. Then (see equation (3.1) in [24])

$$\partial_r A_1 = \int_{\Gamma_1^*} {v^{(4)}}_1 - \int_{\Gamma_2^*} {v^{(4)}}_2 > 0\,, \quad \partial_r A_2 = \int_{\Gamma_2^*} {v^{(4)}}_2 - \int_{\Gamma_3^*} {v^{(4)}}_3 > 0 \tag{6.3.16}$$

according to Lemma A.4.1 (ii).

Again remember from the proof of Theorem 6.3.25 that $v^{(5)}$ corresponds to the first variation of Γ^* with respect to the angle γ, that is w.r.t. the curvature ratio. Similarly we denote by $A_i(\gamma)$ the area of the regions $R_i(\gamma)$ corresponding to the double bubble $DB_{r^*,\gamma,\theta^*}(a_1^*, a_2^*)$. Then

$$\partial_\gamma A_1 = \int_{\Gamma_1^*} {v^{(5)}}_1 - \int_{\Gamma_2^*} {v^{(5)}}_2 > 0\,, \qquad \partial_\gamma A_2 = \int_{\Gamma_2^*} {v^{(5)}}_2 - \int_{\Gamma_3^*} {v^{(5)}}_3 < 0 \tag{6.3.17}$$

according to Lemma A.4.2 (ii).

We now define the matrix

$$D := \begin{pmatrix} \partial_r A_1 & \partial_\gamma A_1 \\ \partial_r A_2 & \partial_\gamma A_2 \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_1^*} {v^{(4)}}_1 - \int_{\Gamma_2^*} {v^{(4)}}_2 & \int_{\Gamma_1^*} {v^{(5)}}_1 - \int_{\Gamma_2^*} {v^{(5)}}_2 \\ \int_{\Gamma_2^*} {v^{(4)}}_2 - \int_{\Gamma_3^*} {v^{(4)}}_3 & \int_{\Gamma_2^*} {v^{(5)}}_2 - \int_{\Gamma_3^*} {v^{(5)}}_3 \end{pmatrix}.$$

Lemma 6.3.27. *The matrix D is invertible for each $0 < \gamma^* < \frac{2\pi}{3}$.*

Proof. Let us calculate its determinant. Inequalities (6.3.16) and (6.3.17) imply

$$\det D = \partial_r A_1 \partial_\gamma A_2 - \partial_\gamma A_1 \partial_r A_2 < 0$$

Now as the determinant of the matrix D is strictly negative, we conclude that the matrix D is for each $0 < \gamma^* < \frac{2\pi}{3}$ invertible, □

As a further result of Lemma A.4.1 and A.4.2 (ii), we get $v^{(4)}, v^{(5)} \notin \mathcal{F}(\Gamma^*)$. Therefore, we conclude from Lemma 6.3.7 that the corresponding variations do not preserve areas to first order. Indeed we will show below in Lemma A.1.2 that

$$I(u,u) < 0 \qquad \text{for } u = v^{(4)}, v^{(5)} .$$

In addition they do not preserve the curvatures to first order too as the constant vectors $c(v^{(4)})$ and $c(v^{(5)})$ are nonzero.

6.3.5 Semi-simplicity

We need to show two small propositions. The first one is stated and proved in the proof of Lemma 3.8 in [24].

Proposition 6.3.28. *If $u \in \mathcal{F}(\Gamma^*)$ satisfies $I(u,u) = 0$, then*

$$I(u,v) = 0 \qquad \forall\, v \in \mathcal{F}(\Gamma^*) .$$

Proof. According to Corollary 6.3.12, $I(v+tu, v+tu) \geq 0$ for all $v \in \mathcal{F}(\Gamma^*)$ and $t \in \mathbb{R}$. Therefore

$$\begin{aligned} I(v+tu, v+tu) &= I(v,v) + 2tI(u,v) + t^2 I(u,u) \\ &= I(v,v) + 2tI(u,v) . \end{aligned}$$

This forces $I(u,v) = 0$ as t can take arbitrary negative values. □

Proposition 6.3.29. *Let $z \in R(A_0)$. Then there exists $u \in \mathcal{F}(\Gamma^*) \cap D(A_0)$ such that $Au = z$.*

Proof. Clearly, there exists $\widetilde{u} \in D(A_0)$ such that $A\widetilde{u} = z$. The actual task is to find two constants $\alpha(\widetilde{u})$, $\beta(\widetilde{u})$ such that

$$u := \widetilde{u} + \alpha(\widetilde{u})v^{(4)} + \beta(\widetilde{u})v^{(5)}$$

satisfies

$$\int_{\Gamma_1^*} u_1 = \int_{\Gamma_2^*} u_2 = \int_{\Gamma_3^*} u_3 .$$

(This will finish the proof since $v^{(4)}, v^{(5)} \in N(A_0)$ implies $Au = A\widetilde{u} = z$.) To do so, let us recast this integral constraint into the matrix form

$$D \begin{pmatrix} \alpha(\widetilde{u}) \\ \beta(\widetilde{u}) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\Gamma_1^*} \widetilde{u}_2 - \int_{\Gamma_2^*} \widetilde{u}_1 \\ \displaystyle\int_{\Gamma_1^*} \widetilde{u}_3 - \int_{\Gamma_3^*} \widetilde{u}_2 \end{pmatrix} ,$$

where the matrix D is given above. The invertibility of this matrix proved in Lemma 6.3.27 finishes the proof. □

We are now ready to prove:

Lemma 6.3.30. *The eigenvalue* 0 *of* A_0 *is semi-simple.*

Proof. Since the operator A_0 has a compact resolvent, the semi-simplicity condition is equivalent to the condition that $N(A_0) = N(A_0^2)$, see Lemma 2.3.6. In other words, to prove semi-simplicity it suffices to check that

$$R(A_0) \cap N(A_0) = \{0\} .$$

To prove this, let $z \in R(A_0) \cap N(A_0)\ \big(\subset D(A_0)\big)$. According to Proposition 6.3.29 there exists $u \in \mathcal{D}(A_0) \cap \mathcal{F}(\Gamma^*)$ such that $Au = z$. From this, exactly as done in Section 6.3.2, we derive the identity

$$\sum_{i=1}^{3} \int_{\Gamma_i^*} \left|\nabla_{\Gamma_i^*}\big(\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i\big)\right|^2 \mathrm{d}s = I(z, u)\,, \tag{6.3.18}$$

where we used only the facts that $u, z \in D(A_0)$.

Moreover, similarly as before, an integration and application of the divergence theorem using the fact that $u \in D(A_0)$ gives

$$\int_{\Gamma_1^*} z_1 = \int_{\Gamma_2^*} z_2 = \int_{\Gamma_3^*} z_3\,,$$

and so $z \in \mathcal{F}(\Gamma^*)$.

Now since $z \in N(A_0) \cap \mathcal{F}(\Gamma^*)$, Corollary 6.3.16 tells us $I(z, z) = 0$. Therefore, according to Proposition 6.3.28,

$$I(z, u) = 0$$

as $u \in \mathcal{F}(\Gamma^*)$. In view of the identity (6.3.18), we obtain $u \in N(A_0)$. Consequently $z = Au = 0$, which finishes the proof. $\square$

Remark 6.3.31. *The main ingredient in the proof is the identity 6.3.18. Here of course we slightly use the non-negativity of the bilinear form. However, one could write the identity 6.3.18 in terms of the linearized operator* A_0 *and the* H^{-1}*-inner product. This would give immediately, i.e., without assuming the non-negativity of the bilinear form, the semi-simplicity condition.*

6.4 Standard planar double bubbles are dynamically stable under surface diffusion flow

Summing up, we have shown that all the hypotheses of Theorem 3.2.1 are satisfied. Thereby applying Theorem 3.2.1 we conclude:

Theorem 6.4.1. *Let Γ^* be a standard planar double bubble. Then $\rho^* \equiv 0$ is a stable equilibrium of (6.2.12) in $X_1 = C^{4+\alpha}(\overline{\Omega}\,, \mathbb{R}^3)$. Moreover, if ρ^0 is sufficiently close to $\rho^* \equiv 0$ in X_1 and satisfies the corresponding compatibility conditions (3.1.10), then the problem (6.2.12) has a unique solution*

$$\rho \in C^{1+\frac{\alpha}{2m},2m+\alpha}\big([0,\infty) \times \overline{\Omega}\,, \mathbb{R}^3\big)$$

and approaches some $\rho^\infty \in \mathcal{E}$, parameterizing some standard planar double bubble, exponentially fast in X_1 as $t \to \infty$.

In this sense, the standard planar double bubble Γ^* is stable under the surface diffusion flow. In addition, every planar double bubble that starts sufficiently close to Γ^* and satisfies the angle, curvature, balance of flux condition and the condition on the Laplace of the curvatures, see (6.1.2), at $t = 0$ exists globally and converges to some standard planar double bubble, enclosing the same areas as its initial data, at an exponential rate as $t \to \infty$. In other words, they are dynamically stable under the surface diffusion flow. We illustrate this result in Figure 1.1.

6.4.1 Open problem: General area preserving gradient flows

It is to be expected that for any appropriate, see Remark 1.1.8, sufficiently smooth area preserving gradient flow

$$V = -\mathrm{grad}_{\mathcal{H}(\Gamma)}\mathrm{Length}$$

one obtains the following identity

$$\|z\|^2_{\mathcal{H}(\Gamma^*)} = I(z,u)\,, \tag{6.4.1}$$

where $z := \delta\big(\mathrm{grad}_{\mathcal{H}(\Gamma)}\mathrm{Length}\big)(u)$. Here $\mathcal{H}(\Gamma)$ denotes a (pre-)Hilbert manifold with some area constraints.

In particular, if u is a eigenvector of the operator $\delta\big(\mathrm{grad}_{\mathcal{H}(\Gamma)}\mathrm{Length}\big)$ with respect to the eigenvalue λ, then we get

$$\Big\|\delta\big(\mathrm{grad}_{\mathcal{H}(\Gamma)}\mathrm{Length}\big)(u)\Big\|^2_{\mathcal{H}(\Gamma^*)} = \lambda I(u,u)\,. \tag{6.4.2}$$

Comparing the identifies (6.4.2) and (6.4.1) with the identities (6.3.5) and (6.3.18) respectively, we expect that our approach can be used for other area preserving gradient flows. Therefore we conjecture that

Conjecture 6.4.2. *Standard planar double bubbles are dynamically stable under appropriate sufficiently smooth area preserving gradient flows.*

It would be desirable to analyze the problem systematically.

It is worth noting that our strategy can also be applied to the higher dimensional cases. The possible difficulties would be verifying that the set of equilibria only consists of the standard double bubbles and determining the dimension of the space of Jacobi functions.

Appendix A

A.1 More about the bilinear form $I(\,,\,)$

Lemma A.1.1. *Within the class of functions u satisfying the linearized angle condition, we have*

$$\{\, u : I(u,u) = 0 \,\} \cap \mathcal{F}(\Gamma^*) = N(A_0) \cap \mathcal{F}(\Gamma^*).$$

Proof. We first assume $u \in \mathcal{F}(\Gamma^*)$ such that $I(u,u) = 0$. Then by Lemma 3.8 in [24] and the fact that u satisfies the linearized angle condition, we conclude that $u \in N(A_0)$. The converse statement is Corollary 6.3.16. □

Note that we have already shown in Section 6.3.4 that

$$N(A_0) \cap \mathcal{F}(\Gamma^*) = \mathcal{J}(\Gamma^*) = \operatorname{span}\{v^{(1)}, v^{(2)}, v^{(3)}\}.$$

On the other hand we obtain:

Lemma A.1.2. *For the bilinear form I it holds*

$$I(u,u) < 0 \quad \textit{for } u = v^{(4)}, v^{(5)} \qquad (0 < \gamma^* < \tfrac{2\pi}{3})\,.$$

Proof. According to Lemma 6.3.15,

$$I(u,u) = -\sum_{i=1}^{3} c_i(u) \int_{\Gamma_i^*} u_i \qquad \text{for } v^{(4)}, v^{(5)}\big(\in N(A_0)\big)\,.$$

Now assertion (iii) in Lemma A.4.1 and Lemma A.4.2 below proves the lemma. □

A.2 The proof of Lemma 6.3.20

Consider the standard planar double bubble $\Gamma = DB_{r,\gamma,0}(0,0)$. That is the left circular arc Γ_1 has radius $r_1 = r$ centered at $O_1 = (0,0)$ and all the other centers also lie on the x-axis, for some $r > 0$, $0 < \gamma < \frac{2\pi}{3}$, see Figure A.1.

It follows directly from the law of sines that in case $\gamma \neq \frac{\pi}{3}$

$$O_2 = \Big(\frac{\sin(\frac{2\pi}{3})}{\sin(\gamma - \frac{\pi}{3})}\, r\,,\, 0\Big), \qquad r_2 = \Big|\frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma - \frac{\pi}{3})}\Big|\, r\,,$$

$$O_3 = \Big(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma)}\, r\,,\, 0\Big)\,, \qquad r_3 = \frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma)} r\,.$$

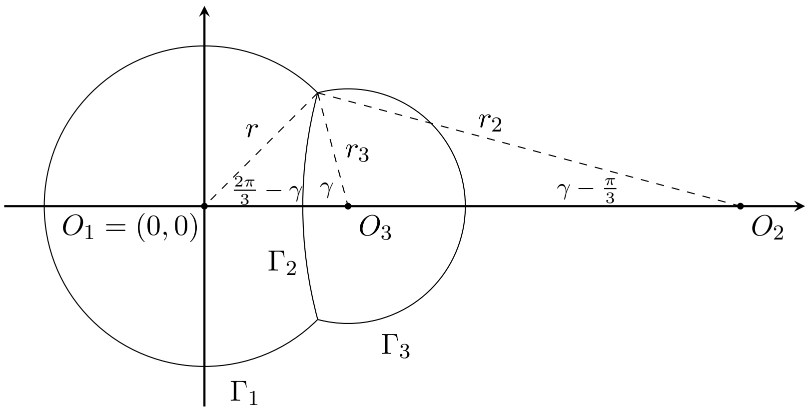

Figure A.1: Standard planar double bubble $\Gamma = DB_{r,\gamma,0}(0,0)$

Therefore, for $\sigma = (\sigma_1, \sigma_2) \in \Gamma_2$, $\gamma \neq \frac{\pi}{3}$, we have

$$\begin{aligned}
0 &= \Big|\sigma \; - \Big(\frac{\sin(\frac{2\pi}{3})}{\sin(\gamma - \frac{\pi}{3})} r, 0\Big)\Big|^2 - \Big(\frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma - \frac{\pi}{3})} r\Big)^2 \\
&= |\sigma|^2 - 2\sigma \cdot \Big(\frac{\sin(\frac{2\pi}{3})}{\sin(\gamma - \frac{\pi}{3})} r, 0\Big) + \frac{\sin^2(\frac{2\pi}{3}) - \sin^2(\frac{2\pi}{3} - \gamma)}{\sin^2(\gamma - \frac{\pi}{3})} r^2 \\
&= \frac{2}{\sin(\gamma - \frac{\pi}{3})} r \sin(\gamma + \tfrac{\pi}{3}) G_2(\sigma, r, \gamma)\,,
\end{aligned}$$

where we applied the formula $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$.

Similarly, for $\sigma \in \Gamma_3$ we obtain

$$0 = \Big|\sigma - \Big(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma)} r, 0\Big)\Big|^2 - \Big(\frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma)} r\Big)^2 = \frac{2}{\sin(\gamma - \pi)} r \sin(\gamma + \tfrac{\pi}{3}) G_3(\sigma, r, \gamma)$$

and obviously for $\sigma \in \Gamma_1$ we have

$$0 = |\sigma|^2 - r^2 = \frac{2}{\sin(\gamma + \frac{\pi}{3})} r \sin(\gamma + \tfrac{\pi}{3}) G_1(\sigma, r, \gamma)\,.$$

Furthermore, we see for $\sigma \in \Gamma_2$, $\gamma = \frac{\pi}{3}$ that $0 = \frac{r}{2} - \sigma_1 = G_2(\sigma, r, \frac{\pi}{3})$.

Finally, the identity (6.1.6) easily verifies (6.3.11). This finishes the proof as the coefficients appearing above are all nonzero and well-defined.

A.3 Arc-length parameterization of Γ^*

Proposition A.3.1. *An arc-length parameterization of Γ_i^*, $i = 1,2,3$ is given as follows: For $\gamma^* \neq \frac{\pi}{3}$,*

$$(\sigma_1, \sigma_2) = \sigma = \begin{cases} \frac{1}{\kappa_1^*}\big(\cos(\kappa_1^* x), \sin(\kappa_1^* x)\big) & \text{for } \sigma \in \Gamma_1^*\,, \\ \Big(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} r^* + \frac{1}{\kappa_2^*}\cos(\kappa_2^* x)\ ,\ \frac{1}{\kappa_2^*}\sin(\kappa_2^* x)\Big) & \text{for } \sigma \in \Gamma_2^*\,, \\ \Big(\frac{-\sin(\frac{\pi}{3})}{\sin(\gamma^* - \pi)} r^* + \frac{1}{\kappa_3^*}\cos(\kappa_3^* x)\ ,\ \frac{1}{\kappa_3^*}\sin(\kappa_3^* x)\Big) & \text{for } \sigma \in \Gamma_3^*\,, \end{cases}$$

Moreover this arc-length parameterization is continuous at $\gamma^* = \frac{\pi}{3}$ and in particular $\sigma = (\frac{r^*}{2}\,, x)$ for $\sigma \in \Gamma_2^*$, $\gamma^* = \frac{\pi}{3}$.

Proof. We give the proof for Γ_2^*. We observe

$$\begin{aligned}(\sigma_1, \sigma_2) = \sigma &= O_2^* - \frac{1}{\kappa_2^*} n_2^* = O_2^* + \frac{1}{\kappa_2^*}\big(\cos(\kappa_2^* x), \sin(\kappa_2^* x)\big) \\ &= \left(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} r^* + \frac{1}{\kappa_2^*}\cos(\kappa_2^* x)\ ,\ \frac{1}{\kappa_2^*}\sin(\kappa_2^* x)\right) \qquad \text{for } \sigma \in \Gamma_2^*\,.\end{aligned}$$

The proof of the continuity can be done using the identity (6.1.5) and the L'Hôpital's rule. □

A.4 The signs of the integrals

Lemma A.4.1. *Let $0 < \gamma^* < \frac{2\pi}{3}$. Then*

(i) $\displaystyle\int_{\Gamma_1^*} v^{(4)}{}_1 > 0\,, \quad \int_{\Gamma_2^*} v^{(4)}{}_2 < 0\,, \quad \int_{\Gamma_3^*} v^{(4)}{}_3 < 0\,,$

(ii) $\displaystyle\int_{\Gamma_1^*} v^{(4)}{}_1 - \int_{\Gamma_2^*} v^{(4)}{}_2 > 0\,, \quad \int_{\Gamma_2^*} v^{(4)}{}_2 - \int_{\Gamma_3^*} v^{(4)}{}_3 > 0\,,$

(iii) $\displaystyle\sum_{i=1}^{3} c_i(v^4) \int_{\Gamma_i^*} v^{(4)}{}_i > 0\,.$

Proof. In order to easily see the general strategy of the proof, let us first verify the assertions for $\gamma^* = \frac{\pi}{3}$:

$$\begin{aligned}
&\int_{-l_1^*}^{l_1^*} v^{(4)}{}_1 = \int_{-l_1^*}^{l_1^*} 1 = 2l_1^* > 0\,, && c_1(v^{(4)}) = (\kappa_1^*)^2 > 0\,,\\
&\int_{-l_2^*}^{l_2^*} v^{(4)}{}_2 = -\int_{-l_2^*}^{l_2^*} \frac{1}{2} = -l_2^* < 0\,, && c_2(v^{(4)}) = 0\,,\\
&\int_{-l_3^*}^{l_3^*} v^{(4)}{}_3 = -\int_{-l_3^*}^{l_3^*} 1+\cos(\kappa_1^* x) < 0\,, && c_3(v^{(4)}) = -(\kappa_3^*)^2 < 0\,.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{3} c_i(v^{(4)}) \int_{\Gamma_i^*} v^{(4)}{}_i > 0 \qquad (\gamma^* = \tfrac{\pi}{3})\,.$$

Next we calculate

$$\begin{aligned}
\int_{\Gamma_2^*} v^{(4)}{}_2 - \int_{\Gamma_3^*} v^{(4)}{}_3 &= -l_2^* + 2l_1^* + \int_{-l_1^*}^{l_1^*} \cos(\kappa_1^* x) = -l_2^* + 2l_1^* + 2\frac{\sin(\kappa_1^* l_1^*)}{\kappa_1^*}\\
&= \frac{\sin(\frac{\pi}{3})}{\kappa_1^*} + 2l_1^* - 2\frac{\sin(\frac{\pi}{3})}{\kappa_1^*} = 2l_1^* + r^*\sin(\tfrac{\pi}{3}) > 0\,,
\end{aligned}$$

where we have used the facts that

$$l_1^* = l_3^*, \quad l_2^* = -\frac{\sin(\frac{\pi}{3})}{\kappa_1^*}, \quad \kappa_1^* l_1^* = -\tfrac{2\pi}{3} \qquad (\gamma^* = \tfrac{\pi}{3})\,.$$

Obviously $\int_{\Gamma_1^*} v^{(4)}{}_1 - \int_{\Gamma_2^*} v^{(4)}{}_2 > 0$ which completes the proof of assertions (i)-(iii) in case $\gamma^* = \frac{\pi}{3}$.

Assume now $\gamma^* \neq \frac{\pi}{3}$. Then we calculate $\int_{\Gamma_1^*} v^{(4)}{}_1 = \int_{\Gamma_1^*} 1 = 2l_1^* > 0$ and

$$\begin{aligned}
\frac{1}{2}\int_{-l_2^*}^{l_2^*} v^{(4)}{}_2 &= \frac{1}{2}\int_{-l_2^*}^{l_2^*} \Big(\frac{-\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}\cos(\kappa_2^* x) + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}\Big)\\
&= \frac{-\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}\frac{\sin(\kappa_2^* l_2^*)}{\kappa_2^*} + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} l_2^* = \frac{\sin(\frac{\pi}{3})}{\kappa_2^*} + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} l_2^*\\
&= l_2^*\Big(\frac{-\sin(\frac{\pi}{3})}{(\gamma^* - \frac{\pi}{3})} + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}\Big) =: l_2^* f(\gamma^*)\,,
\end{aligned}$$

where we used the fact that $\kappa_2^* l_2^* = -(\gamma^* - \frac{\pi}{3})$. Similarly

$$\int_{-l_3^*}^{l_3^*} v^{(4)}{}_3 = 2l_3^*\Big(\frac{\sin(\frac{\pi}{3})}{(\gamma^* - \pi)} + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \pi)}\Big) =: 2l_3^*\, g(\gamma^*)\,.$$

Obviously the function g is negative on $(0, \frac{2\pi}{3})$. Taking into account the fact that $\sin(x) < x$ for $0 < x < \pi$, it is easy to check that $f(0) < 0$, $f' < 0$ and so $f < 0$ on $(0, \frac{2\pi}{3})$ too. Thus

$$\int_{-l_2^*}^{l_2^*} v^{(4)}{}_2 = 2l_2^* f(\gamma^*) < 0, \quad \int_{-l_3^*}^{l_3^*} v^{(4)}{}_3 = 2l_3^* g(\gamma^*) < 0\,.$$

Assertion (i) follows.

Similar argument shows that $g' > 0$ and so $f' - g' < 0$ on $(0, \frac{2\pi}{3})$. This together with $(f-g)(\frac{2\pi}{3}) = 0$ implies $f - g > 0$. Observe further that

$$\frac{\sin(\gamma^* - \pi)}{(\gamma^* - \pi)} \leq \frac{\sin(-\frac{\pi}{3})}{(-\frac{\pi}{3})} = \frac{\sin(\frac{\pi}{3})}{(\frac{\pi}{3})} < \frac{\sin(\gamma^* - \frac{\pi}{3})}{(\gamma^* - \frac{\pi}{3})} \qquad \text{on } (0, \tfrac{2\pi}{3})$$

as the function $\frac{\sin(x)}{x}$ is strictly increasing and decreasing on intervals $(-\pi, 0)$ and $(0, \pi)$ respectively. Thus we conclude

$$\frac{l_3^*}{l_2^*} = \frac{(\gamma^* - \pi)}{(\gamma^* - \frac{\pi}{3})} \frac{\kappa_2^*}{\kappa_3^*} = \frac{(\gamma^* - \pi)}{\sin(\gamma^* - \pi)} \frac{\sin(\gamma^* - \frac{\pi}{3})}{(\gamma^* - \frac{\pi}{3})} > 1 \qquad (0 < \gamma^* < \tfrac{2\pi}{3})\,.$$

We are now ready to estimate

$$\begin{aligned} \int_{-l_2^*}^{l_2^*} {v^{(4)}}_2 - \int_{-l_3^*}^{l_3^*} {v^{(4)}}_3 &= 2l_2^* f(\gamma^*) - 2l_3^* g(\gamma^*) \\ &> 2l_2^*\big(f(\gamma^*) - g(\gamma^*)\big) > 0\,. \end{aligned}$$

Moreover, $\int_{\Gamma_1^*} {v^{(4)}}_1 - \int_{\Gamma_2^*} {v^{(4)}}_2 > 0$ by assertion (i). This proves assertion (ii).

To prove assertion (iii) we observe:

$$c_2(v^{(4)}) = (\kappa_2^*)^2 \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} = \kappa_1^* \kappa_2^*\,, \quad c_3(v^{(4)}) = (\kappa_3^*)^2 \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \pi)} = \kappa_1^* \kappa_3^*\,,$$

and $c_1(v^{(4)}) = (\kappa_1^*)^2$. Therefore, taking into account that $\kappa_1^* < 0$

$$\begin{aligned} \sum_{i=1}^{3} c_i(v^4) \int_{\Gamma_i^*} {v^{(4)}}_i &= 2l_1^*(\kappa_1^*)^2 + 2l_2^*\kappa_1^*\kappa_2^* f(\gamma^*) + 2l_3^*\kappa_1^*\kappa_3^* g(\gamma^*) \\ &> 2l_2^*\kappa_1^*\kappa_2^* f(\gamma^*) + 2l_3^*\kappa_1^*\kappa_3^* f(\gamma^*) = 2\kappa_1^* f(\gamma^*)\big(l_2^*\kappa_2^* + l_3^*\kappa_3^*\big) \\ &= 2\kappa_1^* f(\gamma^*)(-(\gamma^* - \tfrac{\pi}{3}) - (\gamma^* - \pi)) = 4\kappa_1^* f(\gamma^*)(-\gamma^* + \tfrac{2\pi}{3}) \\ &> 0\,. \end{aligned}$$

□

Lemma A.4.2. *Let $0 < \gamma^* < \frac{2\pi}{3}$. Then*

(i) $\displaystyle\int_{\Gamma_1^*} {v^{(5)}}_1 = 0\,, \quad \int_{\Gamma_2^*} {v^{(5)}}_2 < 0\,, \quad \int_{\Gamma_3^*} {v^{(5)}}_3 > 0\,,$

(ii) $\displaystyle\int_{\Gamma_1^*} {v^{(5)}}_1 - \int_{\Gamma_2^*} {v^{(5)}}_2 > 0\,, \quad \int_{\Gamma_2^*} {v^{(5)}}_2 - \int_{\Gamma_3^*} {v^{(5)}}_3 < 0\,,$

(iii) $\displaystyle\sum_{i=1}^{3} c_i(v^5) \int_{\Gamma_i^*} {v^{(5)}}_i > 0\,.$

Proof. Let us first consider the case $\gamma^* = \frac{\pi}{3}$. Then

$$\begin{aligned}
&\int_{-l_1^*}^{l_1^*} v^{(5)}{}_1 = 0, && c_1(v^{(4)}) = 0\,,\\
&\int_{-l_2^*}^{l_2^*} v^{(5)}{}_2 = \tfrac{\kappa_1^*}{\sin(\frac{\pi}{3})} \int_{-l_2^*}^{l_2^*} \frac{1}{2}x^2 + \frac{3}{8}\frac{1}{(\kappa_1^*)^2} < 0\,, && c_2(v^{(5)}) = \tfrac{\kappa_1^*}{\sin(\frac{\pi}{3})} < 0,\\
&\int_{-l_3^*}^{l_3^*} v^{(5)}{}_3 = \frac{-1}{\kappa_1^* \sin(\frac{\pi}{3})} \int_{-l_3^*}^{l_3^*} \frac{1}{2}\cos(\kappa_1^* x) + 1 > 0\,, && c_3(v^{(5)}) = \frac{1}{2}\frac{-(\kappa_1^*)^2}{\kappa_1^* \sin(\frac{\pi}{3})} > 0\,.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{3} c_i(v^{(5)}) \int_{\Gamma_i^*} v^{(5)}{}_i > 0 \qquad (\gamma^* = \tfrac{\pi}{3})\,.$$

Assertion (ii) is an immediate consequence of assertion (i). This proves (i)-(iii) in case $\gamma^* = \frac{\pi}{3}$.

Now assume $\gamma^* \neq \frac{\pi}{3}$. Clearly $\int_{\Gamma_1^*} v^{(5)}{}_1 = 0$ as $v^{(5)}{}_1 = 0$. Next we compute

$$\begin{aligned}
|\sigma|^2 &= \frac{1}{(\kappa_2^*)^2} + \frac{\sin^2(\frac{\pi}{3})}{\sin^2(\gamma^* - \frac{\pi}{3})}(r^*)^2 + 2r^* \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x)\frac{1}{\kappa_2^*}\\
&= \frac{\sin^2(\gamma^* + \frac{\pi}{3})}{\sin^2(\gamma^* - \frac{\pi}{3})}(r^*)^2 + \frac{\sin^2(\frac{\pi}{3})}{\sin^2(\gamma^* - \frac{\pi}{3})}(r^*)^2 - 2(r^*)^2 \frac{\sin(\frac{\pi}{3})\sin(\gamma^* + \frac{\pi}{3})}{\sin^2(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x)\\
&\geq (r^*)^2 \Big(\frac{\sin(\gamma^* + \frac{\pi}{3}) - \sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}\Big)^2 \qquad \text{for } \sigma \in \Gamma_2^*\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\int_{\Gamma_2^*} v^{(5)}{}_2 &= \int_{\Gamma_2^*} \partial_\gamma G_2(\sigma, r^*, \gamma^*)\\
&= \tfrac{1}{r^* \sin(\gamma^* + \frac{\pi}{3})} \int_{\Gamma_2^*} \cos(\gamma^* - \tfrac{\pi}{3})|\sigma|^2 - (r^*)^2 \cos(\gamma^* - \pi)\\
&\geq \tfrac{2l_2^*}{r^* \sin(\gamma^* + \frac{\pi}{3})} (r^*)^2 f(\gamma^*)\,,
\end{aligned}$$

where

$$f(x) := \cos(x - \tfrac{\pi}{3})\Big(\frac{\sin(x + \frac{\pi}{3}) - \sin(\frac{\pi}{3})}{\sin(x - \frac{\pi}{3})}\Big)^2 + \cos(x)\,.$$

It is not hard to show that the function f is strictly decreasing on $(0, \frac{2\pi}{3})$. Together with the fact that this function vanishes at $\gamma^* = \frac{2\pi}{3}$ we conclude $f > 0$ and so $\int_{\Gamma_2^*} v^{(5)}{}_2 < 0$. A similar proof works for $v^{(5)}{}_3$. This completes the proof of (i).

The statement (ii) is an immediate consequence of assertion (i). Similarly you can check that $c_2(v^{(5)}) < 0$ and $c_3(v^{(5)}) > 0$ which easily gives (iii). $\square$

Appendix B

B.1 Deriving the parabolic system

For the normal velocity V_i of $\Gamma_i(t) := \Gamma_{\rho_i,\mu_i}(t)$ we obtain with the convention (6.2.1)

$$\begin{aligned} V_i(\sigma,t) &= \langle \partial_t \Phi_i(\sigma,t), n_i(\sigma,t)\rangle \\ &= \frac{1}{J_i}\big\langle \partial_w \Psi_i, R\partial_\sigma \Psi_i\big\rangle \partial_t \rho_i(\sigma,t) + \big\langle \partial_r \Psi_i, n_i(\sigma,t)\big\rangle \partial_t \mu_i(\mathrm{pr}_i(\sigma),t) \quad \sigma \in \Gamma_i^*, \end{aligned}$$

where the unit normal n_i of $\Gamma_i(t) := \Gamma_{\rho_i,\mu_i}(t)$ is given by

$$\begin{aligned} n_i(\sigma,t) &= \frac{1}{J_i} R\partial_\sigma \Phi_i(\sigma,t) \\ &= \frac{1}{J_i}\big(R\partial_\sigma \Psi_i + R\partial_w \Psi_i\, \partial_\sigma \rho_i(\sigma,t)\big) \quad \sigma \in \Gamma_i^*. \end{aligned} \tag{B.1.1}$$

Here

$$J_i = J_i(\sigma,\rho_i,\mu_i) := |\partial_\sigma \Phi_i| = \sqrt{|\partial_\sigma \Psi_i|^2 + 2\langle \partial_\sigma \Psi_i, \partial_w \Psi_i\rangle \partial_\sigma \rho_i + |\partial_w \Psi_i|^2 (\partial_\sigma \rho_i)^2},$$

and R denotes the anti-clockwise rotation by $\pi/2$. Next computing the curvature $\kappa_i (= \kappa_i(\sigma,\rho_i,\mu_i))$ of $\Gamma_i(t) := \Gamma_{\rho_i,\mu_i}(t)$ we get

$$\begin{aligned} \kappa_i &= \frac{1}{{J_i}^3}\big\langle \partial_\sigma^2 \Phi_i, R\partial_\sigma \Phi_i\big\rangle \qquad \text{(B.1.2)} \\ &= \frac{1}{{J_i}^3}\Big[\langle \partial_w \Psi_i, R\partial_\sigma \Psi_i\rangle \partial_\sigma^2 \rho_i + \big\{2\big\langle \partial_{\sigma w}\Psi_i, R\partial_\sigma \Psi_i\big\rangle + \big\langle \partial_\sigma^2 \Psi_i, R\partial_w \Psi_i\big\rangle\big\}\partial_\sigma \rho_i \\ &\qquad + \big\{\big\langle \partial_{ww}\Psi_i, R\partial_\sigma \Psi_i\big\rangle + 2\big\langle \partial_{\sigma w}\Psi_i, R\partial_w \Psi_i\big\rangle \\ &\qquad\qquad + \big\langle \partial_{ww}\Psi_i, R\partial_w \Psi_i\big\rangle \partial_\sigma \rho_i\big\}(\partial_\sigma \rho_i)^2 + \langle \Psi_{\sigma\sigma}, R\Psi_\sigma\rangle\Big]. \end{aligned}$$

Therefore, the surface diffusion flow equations can be reformulated as

$$\partial_t \rho_i = a_i(\sigma,\rho_i,\mu_i)\Delta(\sigma,\rho_i,\mu_i)\kappa_i(\sigma,\rho_i,\mu_i) + b_i(\sigma,\rho_i,\mu_i)\partial_t \mu_i\,, \tag{B.1.3}$$

where

$$a_i(\sigma,\rho_i,\mu_i) := \frac{J_i(\sigma,\rho_i,\mu_i)}{\big\langle \partial_w\Psi_i, R\partial_\sigma\Psi_i\big\rangle}\ \Big(=\frac{1}{\big\langle n_i^*(\sigma), n_i(\sigma,t)\big\rangle}\Big)\,,$$

$$b_i(\sigma,\rho_i,\mu_i) := \frac{\big\langle \partial_r\Psi_i, R\partial_\sigma\Psi_i\big\rangle + \big\langle \partial_r\Psi_i, R\partial_w\Psi_i\big\rangle\partial_\sigma\rho_i}{-\big\langle \partial_w\Psi_i, R\partial_\sigma\Psi_i\big\rangle}\Big(=\frac{\big\langle \tau_i^*(\sigma), n_i(\sigma,t)\big\rangle}{\big\langle n_i^*(\sigma), n_i(\sigma,t)\big\rangle}\ \Big)\,,$$

$$\Delta(\sigma,\rho_i,\mu_i)v := \frac{1}{J_i(\sigma,\rho_i,\mu_i)}\partial_\sigma\Big(\frac{1}{J_i(\sigma,\rho_i,\mu_i)}\partial_\sigma v\Big).$$

Note that we have omitted the projection pr_i in the functions μ_i and the term $(\sigma,\rho_i(\sigma,t),\mu_i(\mathrm{pr}_i(\sigma),t))$ in $\partial_u\Psi_i$ with $u\in\{\sigma,w,\mu\}$ to shorten the formulas. Furthermore note

$$b_i|_{\rho_i\equiv 0} = -\langle\tau_i^*,\pm n_i^*\rangle = 0\,,\qquad a_i|_{\rho_i\equiv 0} = 1\,. \tag{B.1.4}$$

We will now make use of the linear dependency (6.2.9) to derive from the equations (B.1.3) evolution equations solely for the functions ρ_i. For this, let us rewrite (B.1.3) into

$$\partial_t\rho_i = \mathfrak{F}_i(\rho_i,\rho|_{\Sigma^*}) + \mathfrak{B}_i(\rho_i,\rho|_{\Sigma^*})\partial_t\big(\mathcal{J}\rho\circ\mathrm{pr}_i\big)_i \quad \text{in } \Gamma_i^*\,, \tag{B.1.5}$$

where for $\sigma\in\Gamma_i^*$

$$\begin{aligned}\mathfrak{F}_i(\rho_i,\rho|_{\Sigma^*})(\sigma) &= a_i\big(\sigma,\rho_i,(\mathcal{J}\rho|_{\Sigma^*})_i\big)\Delta\big(\sigma,\rho_i,(\mathcal{J}\rho|_{\Sigma^*})_i\big)\kappa_i\big(\sigma,\rho_i,(\mathcal{J}\rho|_{\Sigma^*})_i\big),\\ \mathfrak{B}_i(\rho_i,\rho|_{\Sigma^*})(\sigma) &= b_i\big(\sigma,\rho_i,(\mathcal{J}\rho|_{\Sigma^*})_i\big),\end{aligned}$$

and where we used the linear dependency (6.2.9). By writing (B.1.5) as a vector identity on Σ^* we get

$$\partial_t\rho = \mathfrak{F}(\rho,\rho|_{\Sigma^*}) + \mathfrak{B}(\rho,\rho|_{\Sigma^*})\mathcal{J}(\partial_t\rho) \quad \text{on } \Sigma^*\,, \tag{B.1.6}$$

where we employed the following notations

$$\begin{aligned}\mathfrak{F}(\rho,\rho|_{\Sigma^*})(\sigma) &:= \Big(\mathfrak{F}_i(\rho_i,\rho|_{\Sigma^*})(\sigma)\Big)_{i=1,2,3} \qquad \text{for } \sigma\in\Sigma^*\,,\\ \mathfrak{B}(\rho,\rho|_{\Sigma^*})(\sigma) &:= \mathrm{diag}\Big(\big(\mathfrak{B}_i(\rho_i,\rho|_{\Sigma^*})(\sigma)\big)_{i=1,2,3}\Big) \qquad \text{for } \sigma\in\Sigma^*\,.\end{aligned}$$

We rearrange to find

$$\big(I-\mathfrak{B}(\rho,\rho|_{\Sigma^*})\mathcal{J}\big)\partial_t\rho = \mathfrak{F}(\rho,\rho|_{\Sigma^*}) \quad \text{on } \Sigma^*\,. \tag{B.1.7}$$

Consequently we get

$$\partial_t\rho = \big(I-\mathfrak{B}(\rho,\rho|_{\Sigma^*})\mathcal{J}\big)^{-1}\mathfrak{F}(\rho,\rho|_{\Sigma^*}) \quad \text{on } \Sigma^*.$$

According to (B.1.4), in some neighborhood of $\rho \equiv 0$ in $C^1(\Gamma^*)$ the inverse $\big(I - \mathfrak{B}(\rho, \rho|_{\Sigma^*})\mathcal{J}\big)^{-1}$ exists. Inserting the above equation into the equation (B.1.6) we can finally reformulate the surface diffusion flow equations

$$V_i = -\Delta_{\Gamma_i}\kappa_i \quad \text{ on } \Gamma_i(t)$$

as a system of the evolution equations for functions ρ_i defined on fixed domains Γ_i^* (or equivalently on $[-l_i^*, l_i^*]$)

$$\partial_t\rho_i = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) + \mathfrak{B}_i(\rho_i, \rho|_{\Sigma^*})\Big(\big\{\mathcal{J}\big(I - \mathfrak{B}(\rho, \rho|_{\Sigma^*})\mathcal{J}\big)^{-1}\mathfrak{F}(\rho, \rho|_{\Sigma^*})\big\} \circ \mathrm{pr}_i\Big)_i.$$

Finally, we rewrite the boundary conditions at $\sigma \in \Sigma^*$ as

$$\begin{aligned}
\mathfrak{G}_1(\rho)(\sigma) &:= \rho_1(\sigma) + \rho_2(\sigma) + \rho_3(\sigma) = 0\,,\\
\mathfrak{G}_2(\rho)(\sigma) &:= \langle n_1(\sigma), n_2(\sigma)\rangle - \cos\tfrac{2\pi}{3}\\
&= \Big\langle \frac{1}{J_1}(\partial_\sigma\Psi_1 + \partial_w\Psi_1\,\partial_\sigma\rho_1)\,,\ \frac{1}{J_2}(\partial_\sigma\Psi_2 + \partial_w\Psi_2\,\partial_\sigma\rho_2)\Big\rangle - \cos\tfrac{2\pi}{3} = 0\,,\\
\mathfrak{G}_3(\rho)(\sigma) &:= \langle n_2(\sigma), n_3(\sigma)\rangle - \cos\tfrac{2\pi}{3}\\
&= \Big\langle \frac{1}{J_2}(\partial_\sigma\Psi_2 + \partial_w\Psi_2\,\partial_\sigma\rho_2)\,,\ \frac{1}{J_3}(\partial_\sigma\Psi_3 + \partial_w\Psi_3\,\partial_\sigma\rho_3)\Big\rangle - \cos\tfrac{2\pi}{3} = 0\,,\\
\mathfrak{G}_4(\rho)(\sigma) &:= \sum_{i=1}^{3}\kappa_i\big(\sigma, \rho_i, (\mathcal{J}\rho|_{\Sigma^*})_i\big) = 0\,,\\
\mathfrak{G}_5(\rho)(\sigma) &:= \frac{1}{J_1}\partial_\sigma\big(\kappa_1\big(\sigma, \rho_1, (\mathcal{J}\rho|_{\Sigma^*})_1\big) - \frac{1}{J_2}\partial_\sigma\big(\kappa_2\big(\sigma, \rho_2, (\mathcal{J}\rho|_{\Sigma^*})_2\big) = 0\,,\\
\mathfrak{G}_6(\rho)(\sigma) &:= \frac{1}{J_2}\partial_\sigma\big(\kappa_2\big(\sigma, \rho_2, (\mathcal{J}\rho|_{\Sigma^*})_2\big) - \frac{1}{J_3}\partial_\sigma\big(\kappa_3\big(\sigma, \rho_3, (\mathcal{J}\rho|_{\Sigma^*})_3\big) = 0\,.
\end{aligned}$$

We emphasize that the operators $\mathfrak{G}_i$ $(i = 1, \dots, 5)$ are purely local due to the fact that the projections pr_i act as the identity on their image Σ^*.

B.2 Proof of Proposition 3.2.7

In fact we are following the steps in the proof of Theorem 4.1 in [27].
For $0 \le t \le a$,

$$\begin{aligned}
\|e^{\sigma t}F\left(z_1\left(t,\cdot\right)\right) - e^{\sigma t}F(z_2(t,\cdot))\|_{C^\alpha(\overline{\Omega})} &\le K(r)\|e^{\sigma t}(z_1(t,\cdot) - z_2(t,\cdot))\|_{C^{2m+\alpha}(\overline{\Omega})}\\
&\le K(r)\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)},
\end{aligned}$$

$$\begin{aligned}
\|e^{\sigma t}G_j\left(z_1\left(t,\cdot\right)\right) - e^{\sigma t}G_j(z_2(t,\cdot))\|_{C^{2m+\alpha-m_j}(\partial\Omega)} &\le H_j(r)\|e^{\sigma t}(z_1(t,\cdot) - z_2(t,\cdot))\|_{X_1}\\
&\le H_j(r)\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)},
\end{aligned}$$

while for $0 \le s \le t \le a$,

$$
\begin{aligned}
&\|e^{\sigma t}F(z_1(t,\cdot)) - e^{\sigma t}F(z_2(t,\cdot)) - e^{\sigma s}F(z_1(s,\cdot)) + e^{\sigma s}F(z_2(s,\cdot))\|_{C(\overline{\Omega})} \\
&= \Big\| \int_0^1 e^{\sigma t}F'\left(\lambda z_1(t,\cdot) + (1-\lambda)z_2(t,\cdot)\right)\left(z_1(t,\cdot) - z_2(t,\cdot)\right) \\
&\qquad - e^{\sigma s}F'\left(\lambda z_1(s,\cdot) + (1-\lambda)z_2(s,\cdot)\right)\left(z_1(s,\cdot) - z_2(s,\cdot)\right) \mathrm{d}\lambda \Big\|_{C(\overline{\Omega})}
\end{aligned}
$$

$$
\begin{aligned}
\le\ & \int_0^1 \Big\| \left(F'\big(\lambda z_1(t,\cdot) + (1-\lambda)z_2(t,\cdot)\big) - F'\big(\lambda z_1(s,\cdot) + (1-\lambda)z_2(s,\cdot)\big)\right) \cdot \\
& \quad \cdot e^{\sigma t}(z_1(t,\cdot) - z_2(t,\cdot))\Big\|_{C(\overline{\Omega})} \mathrm{d}\lambda \\
& + \int_0^1 \big\|F'\big(\lambda z_1(s,\cdot) + (1-\lambda)z_2(s,\cdot)\big) \cdot \\
& \qquad \cdot \big(e^{\sigma t}(z_1(t,\cdot) - z_2(t,\cdot)) - e^{\sigma s}(z_1(s,\cdot) - z_2(s,\cdot))\big)\big\| \ \mathrm{d}\lambda
\end{aligned}
$$

$$
\begin{aligned}
\le\ & \frac{L}{2}\Big(\|z_1(t,\cdot) - z_1(s,\cdot)\|_{C^{2m}(\overline{\Omega})} + \|z_2(t,\cdot) - z_2(s,\cdot)\|_{C^{2m}(\overline{\Omega})}\Big) e^{\sigma t}\|z_1(t,\cdot) - z_2(t,\cdot)\|_{C^{2m}(\overline{\Omega})} \\
& + Lr\|e^{\sigma t}\left(z_1(t,\cdot) - z_2(t,\cdot)\right) - e^{\sigma s}(z_1(s,\cdot) - z_2(s,\cdot))\|_{C^{2m}(\overline{\Omega})} \\
\le\ & \frac{L}{2}(t-s)^{\frac{\alpha}{2m}}(\|z_1\|_{C^{\frac{\alpha}{2m}}((0,a),C^{2m}(\overline{\Omega}))} + \|z_2\|_{C^{\frac{\alpha}{2m}}((0,a),C^{2m}(\overline{\Omega}))})\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)} \\
& + Lr(t-s)^{\frac{\alpha}{2m}}\|e^{\sigma t}\left(z_1 - z_2\right)\|_{C^{\frac{\alpha}{2m}}((0,a),C^{2m}(\overline{\Omega}))} \\
\le\ & 2Lr(t-s)^{\frac{\alpha}{2m}}\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)} .
\end{aligned}
$$

The last inequality is a consequence of Lemma 3.2.3 and the fact that $z_1, z_2 \in B_{\mathbb{E}_1(a)}(0,r)$.

Since $1 + \frac{\alpha}{2m} - \frac{m_j}{2m} < 1$ for j with $m_j \ge 1$, we get similarly

$$
\begin{aligned}
&\|e^{\sigma t}G_j(z_1(t,\cdot)) - e^{\sigma t}G_j(z_2(t,\cdot)) - e^{\sigma s}G_j(z_1(s,\cdot)) + e^{\sigma s}G_j(z_2(s,\cdot))\|_{C(\partial\Omega)} \\
&\quad \le 2Lr(t-s)^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}}\|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a),}
\end{aligned}
$$

where we have used the embedding

$$\mathbb{E}_1(a) \hookrightarrow C^{1+\frac{\alpha}{2m}-\frac{m_j}{2m}}((0,a), C^{m_j}(\overline{\Omega})),$$

which is a consequence of Lemma 3.2.3.

For j such that $m_j = 0$, we have to estimate the complete norm, i.e.,

$$\|e^{\sigma t}(G_j(z_1) - G_j(z_2))\|_{C^{1+\frac{\alpha}{2m}}(I,C(\partial\Omega))},$$

which includes the time derivative. The proof is again similar, but for the convenience we give some details of the main part of it namely, estimating

$$\|e^{\sigma t}\frac{d}{dt}(G_j(z_1) - G_j(z_2))\|_{C^{\frac{\alpha}{2m}}(I,C(\partial\Omega))}.$$

Note that exactly at this point one needs C^2-regularity for G_j.

For $0 \le s \le t \le a$, we have

$$\begin{aligned}
&\big\| e^{\sigma t} G_j'(z_1(t,\cdot)) z_1'(t,\cdot) - e^{\sigma t} G_j'(z_2(t,\cdot)) z_2'(t,\cdot) - \\
&\qquad\qquad - e^{\sigma s} G_j'(z_1(s,\cdot)) z_1'(s,\cdot) + e^{\sigma s} G_j'(z_2(s,\cdot)) z_2'(s,\cdot) \big\|_{C(\partial\Omega)} \\
\le\ & \Big\| \int_0^1 e^{\sigma t} G_j''\big(\lambda z_1(t,\cdot) + (1-\lambda) z_2(t,\cdot)\big) \big(z_1(t,\cdot) - z_2(t,\cdot)\big) \cdot \\
&\quad \cdot \big(\lambda z_1'(t,\cdot) + (1-\lambda) z_2'(t,\cdot)\big) - e^{\sigma s} G_j''\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big) \cdot \\
&\qquad\qquad \cdot \big(z_1(s,\cdot) - z_2(s,\cdot)\big) \big(\lambda z_1'(s,\cdot) + (1-\lambda) z_2'(s,\cdot)\big)\, \mathrm{d}\lambda \Big\|_{C(\partial\Omega)} \\
&+ \Big\| \int_0^1 e^{\sigma t} G_j'\big(\lambda z_1(t,\cdot) + (1-\lambda) z_2(t,\cdot)\big) \big(z_1'(t,\cdot) - z_2'(t,\cdot)\big) \\
&\quad - e^{\sigma s} G_j'\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big) \big(z_1'(s,\cdot) - z_2'(s,\cdot)\big)\, \mathrm{d}\sigma \big\|_{C(\partial\Omega)} \\
\le\ & \int_0^1 \big\| \big(G_j''\big(\lambda z_1(t,\cdot) + (1-\lambda) z_2(t,\cdot)\big) - G_j''\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big)\big) \cdot \\
&\quad \cdot e^{\sigma t} \big(z_1(t,\cdot) - z_2(t,\cdot)\big) \big(\lambda z_1'(t,\cdot) + (1-\lambda) z_2'(t,\cdot)\big) \big\|_{C(\partial\Omega)}\, \mathrm{d}\lambda \\
&+ \int_0^1 \big\| G_j''\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big) \big(e^{\sigma t}\big(z_1(t,\cdot) - z_2(t,\cdot)\big) - \\
&\qquad - e^{\sigma s}\big(z_1(s,\cdot) - z_2(s,\cdot)\big)\big) \times \big(\lambda z_1'(t,\cdot) + (1-\lambda) z_2'(t,\cdot)\big) \big\|_{C(\partial\Omega)}\, \mathrm{d}\lambda \\
&+ \int_0^1 \big\| G_j''\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big)\, e^{\sigma s} \big(z_1(s,\cdot) - z_2(s,\cdot)\big) \cdot \\
&\quad \cdot \big(\lambda z_1'(t,\cdot) + (1-\lambda) z_2'(t,\cdot) - \lambda z_1'(s,\cdot) - (1-\lambda) z_2'(s,\cdot)\big) \big\|_{C(\partial\Omega)}\, \mathrm{d}\lambda \\
&+ \int_0^1 \big\| \big(G_j'\big(\lambda z_1(t,\cdot) + (1-\lambda) z_2(t,\cdot)\big) - G_j'\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big)\big) \cdot \\
&\quad \cdot e^{\sigma t} \big(z_1'(t,\cdot) - z_2'(t,\cdot)\big) \big\|_{C(\partial\Omega)}\, \mathrm{d}\lambda \\
&+ \int_0^1 \big\| G_j'\big(\lambda z_1(s,\cdot) + (1-\lambda) z_2(s,\cdot)\big) \cdot \\
&\quad \cdot \big(e^{\sigma t}\big(z_1'(t,\cdot) - z_2'(t,\cdot)\big) - e^{\sigma s}\big(z_1'(s,\cdot) - z_2'(s,\cdot)\big)\big) \big\|_{C(\partial\Omega)}\, \mathrm{d}\lambda \\
\le\ & \big(12Lr^2 + 2Lr\big) (t-s)^{\frac{\alpha}{2m}} \|e^{\sigma t}(z_1 - z_2)\|_{\mathbb{E}_1(a)},
\end{aligned}$$

where we have used the fact that $\lambda, 1-\lambda \le 1$ and

$$\mathbb{E}_1(a) \hookrightarrow C^{\frac{\alpha}{2m}}(I, C^{2m}(\overline{\Omega})) \hookrightarrow C^{\frac{\alpha}{2m}}(I, C(\overline{\Omega})).$$

Summing up, by adding all constants we get a constant $D(r)$ such that $D(r)$ goes to zero as $r \longrightarrow 0$ and so the statement follows.

List of notations

Γ	A planar curve or networks of curves
$\partial\Gamma$	Boundary of Γ
V	Normal velocity of Γ
$\nu_{\partial\Gamma}$	Normal boundary velocity of Γ
κ	Curvature
∇_Γ	Surface gradient of Γ
Δ_Γ	Laplace-Beltrami Operator of Γ
n_Γ	Unit normal vector of Γ
$n_{\partial\Gamma}$	Outer unit conormal of Γ at $\partial\Gamma$
$V = -\Delta_\Gamma \kappa$	Surface diffusion flow
Γ^*	Stationary configuration
Γ^0	Initial configuration
ρ	$\rho : \Gamma^* \times [0, T) \to \mathbb{R}$
$DB_{r,\gamma,\theta}(a_1, a_2)$	Set of all standard planar double bubbles
$CA_r(a_1, -r\cos(\theta))$	Set of all circular arcs with $\pi - \theta$ contact angles
T	Unit tangential vector field
τ^*	A specific tangential vector field on Γ^*

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