Universität Regensburg
Mathematik

Stable variational approximations of boundary value problems for Willmore flow with Gaussian curvature

John W. Barrett, Harald Garcke and Robert Nürnberg

Preprint Nr. 01/2016

Stable variational approximations of boundary value problems for Willmore flow with Gaussian curvature

John W. Barrett[†] Harald Garcke[‡] Robert Nürnberg[†]

Abstract

We study numerical approximations for geometric evolution equations arising as gradient flows for energy functionals that are quadratic in the principal curvatures of a two-dimensional surface. Beside the well-known Willmore and Helfrich flows we will also consider flows involving the Gaussian curvature of the surface. Boundary conditions for these flows are highly nonlinear, and we use a variational approach to derive weak formulations, which naturally can be discretized with the help of a mixed finite element method. Our approach uses a parametric finite element method, which can be shown to lead to good mesh properties. We prove stability estimates for a semidiscrete (discrete in space, continuous in time) version of the method and show existence and uniqueness results in the fully discrete case. Finally, several numerical results are presented involving convergence tests as well as the first computations with Gaussian curvature and/or free or semi-free boundary conditions.

Key words. Willmore flow, parametric finite elements, tangential movement, spontaneous curvature, clamped boundary conditions, Navier boundary conditions, Gaussian curvature energy, line energy.

AMS subject classifications. 65M60, 65M12, 35K55, 53C44

1 Introduction

Energies involving the principal curvatures of a two-dimensional surface in the three dimensional Euclidean space play an important role in geometry, physics, biology and imaging. The Willmore energy given as the integrated square of the mean curvatures is of great interest in geometry, cf. Willmore (1993). However, also more general functionals involving the principal curvatures appear in the theory of elastic plates and shells, and go back to work of Poisson (1812), Germain (1821) and Kirchhoff (1850). In the theory of biological membranes the work of Helfrich (1973) used generalized curvature functionals, which lead to a huge interest for curvature functionals in the field of biophysics. Boundary

[†]Department of Mathematics, Imperial College London, London, SW7 2AZ, UK
[‡]Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

value problems involving curvature functionals also play an important role in imaging, for example in problems involving surface restoration and image inpainting, cf. Clarenz et al. (2004); Bobenko and Schröder (2005). Analytical and numerical work on static and evolutionary questions in the context of curvature functionals so far have been mainly focused on the case of closed surfaces, and we refer to Simon (1993); Kuwert and Schätzle (2001); Rivière (2008); Marques and Neves (2014) for analytical results, and to Mayer and Simonett (2002); Rusu (2006); Dziuk (2008); Barrett et al. (2008) for numerical results.

Much less is known for boundary value problems involving functionals that include curvature quantities. Analytical results often need small data assumptions, use symmetries or consider the graph case. We refer to Nitsche (1993); Bergner et al. (2009); Dall'Acqua et al. (2008); Deckelnick and Grunau (2009); Schätzle (2010); Deckelnick et al. (2015) for the static case and to Abels et al. (2016) for an evolution problem for the Willmore energy with boundary conditions. Numerical approaches to problems involving the Willmore energy and boundary conditions are discussed in Peres Hari et al. (2001); Clarenz et al. (2004); Bobenko and Schröder (2005); Deckelnick et al. (2015). In this context we refer to Wang and Du (2008), see also Du (2011), who used a phase field approach to study open membranes numerically.

Experimental observations for open membranes are reported in Saitoh et al. (1998). Capovilla and Guven (2004); Tu and Ou-Yang (2003) and Biria et al. (2013) used variational calculus to derive equilibrium equations for a bilayer membrane having an edge, and also provided physical interpretations of the equations obtained.

To the knowledge of the authors, no results are available in the literature so far for numerical approaches of evolution problems that involve also the Gaussian curvature and/or free or semi-free boundary conditions. It is the goal of this paper to derive and analyze a finite element approximation of L^2–gradient flows for curvature functionals of Willmore and Helfrich type that allow also for Gaussian curvature and (semi-)free boundary conditions. We are interested in discretizations which allow to treat the highly nonlinear boundary conditions in a variational way, which will then make it possible to derive stability estimates. In order to do so, it is necessary to generalize work of Dziuk (2008) and Barrett et al. (2016) on computational Willmore flow for closed surfaces to the case of open surfaces. Due to the highly nonlinear boundary conditions, this is a nontrivial task.

In order to formulate the governing problems in more detail, we parameterize the surfaces over a fixed oriented, compact, smooth reference manifold $\Upsilon \subset \mathbb{R}^3$ with boundary $\partial\Upsilon$. We now consider a hypersurface Γ with boundary $\partial\Gamma$ parameterized by $\vec{x} : \Upsilon \to \mathbb{R}^3$ with normal $\vec{\nu}$ given by the orientation. Denoting by $\nabla_s = (\partial_{s_1}, \partial_{s_2}, \partial_{s_3})$ the surface gradient on Γ we define $\nabla_s \vec{\chi} = \left(\partial_{s_j} \chi_i\right)_{i,j=1}^3$.

We then define the second fundamental tensor for Γ as $\nabla_s \vec{\nu}$, and we recall that $-\nabla_s \vec{\nu}(\vec{z})$, for any $\vec{z} \in \Gamma$, is a symmetric linear map that has a zero eigenvalue with eigenvector $\vec{\nu}$. The remaining two eigenvalues, $\varkappa_1, \varkappa_2$, are the principal curvatures of Γ at $\vec{z}$; see e.g. (Deckelnick et al., 2005, p. 152). Hence $-\nabla_s \vec{\nu}(\vec{z})$ induces a linear map $\mathcal{S} : T_{\vec{z}}\Gamma \to T_{\vec{z}}\Gamma$ on the tangent space $T_{\vec{z}}\Gamma$ for any $\vec{z} \in \Gamma$. The map $-\mathcal{S}$ is called the

Weingarten map or shape operator. The mean curvature $\varkappa$ and the Gaussian curvature $\mathcal{K}$ can now be stated as

$$\varkappa = \operatorname{tr} \mathcal{S} = \varkappa_1 + \varkappa_2 \qquad \text{and} \qquad \mathcal{K} = \det(\mathcal{S}) = \varkappa_1\, \varkappa_2\,, \tag{1.1}$$

where we note that unit spheres with outer unit normal have mean curvature $\varkappa = -2$. It then follows that $|\nabla_s \vec{\nu}|^2 = \varkappa_1^2 + \varkappa_2^2 = \varkappa^2 - 2\,\mathcal{K}$. The mean curvature vector is given as

$$\Delta_s \, \vec{\mathrm{id}} = \varkappa\, \vec{\nu} =: \vec{\varkappa} \qquad \text{on } \Gamma\,, \tag{1.2}$$

where $\Delta_s = \nabla_s \,.\, \nabla_s$ is the Laplace–Beltrami operator on Γ.

The Willmore energy is now given as

$$E_0(\Gamma) := \tfrac{1}{2} \int_\Gamma \varkappa^2 \,\mathrm{d}\mathcal{H}^2 = \tfrac{1}{2} \int_\Gamma |\vec{\varkappa}|^2 \,\mathrm{d}\mathcal{H}^2\,, \tag{1.3}$$

see e.g. Willmore (1993) for details. Here and throughout $\mathcal{H}^d$, $d = 1, 2$, denotes the d-dimensional Hausdorff measure. Realistic models for biological cell membranes lead to energies more general than (1.3). In the original derivation of Helfrich (1973) a possible asymmetry in the membrane, originating e.g. from a different chemical environment, was taken into account. This lead Helfrich to the energy

$$E_{\overline{\varkappa}}(\Gamma) = \tfrac{1}{2} \int_\Gamma (\varkappa - \overline{\varkappa})^2 \,\mathrm{d}\mathcal{H}^2 = \tfrac{1}{2} \int_\Gamma |\vec{\varkappa} - \overline{\varkappa}\, \vec{\nu}|^2 \,\mathrm{d}\mathcal{H}^2\,, \tag{1.4}$$

where $\overline{\varkappa} \in \mathbb{R}$ is the given so-called spontaneous curvature. Similarly to Barrett et al. (2016), we will also consider the energy

$$E_{\overline{\varkappa},\beta}(\Gamma) := E_{\overline{\varkappa}}(\Gamma) + \tfrac{\beta}{2} \left(M(\Gamma) - M_0\right)^2 \tag{1.5a}$$

with

$$M(\Gamma) = \int_\Gamma \varkappa \,\mathrm{d}\mathcal{H}^2 = \int_\Gamma \vec{\varkappa} \,.\, \vec{\nu} \,\mathrm{d}\mathcal{H}^2 \tag{1.5b}$$

and given constants $\beta \in \mathbb{R}_{\geq 0}$, $M_0 \in \mathbb{R}$. Models employing the energy (1.5a) are often called area-difference elasticity (ADE) models, see Seifert (1997). We note that for present models, choosing $\beta > 0$ does not have a physically meaningful interpretation for surfaces with boundary.

For open surfaces also contributions taking Gaussian curvature and line energy into account are relevant. We hence consider

$$E(\Gamma) := E_{\overline{\varkappa},\beta}(\Gamma) + \alpha_G \int_\Gamma \mathcal{K} \,\mathrm{d}\mathcal{H}^2 + \gamma\, \mathcal{H}^1(\partial\Gamma)\,, \tag{1.6}$$

for given $\alpha_G \in \mathbb{R}$ and $\gamma \in \mathbb{R}_{\geq 0}$.

Similarly to (1.2), fundamental to many approaches, which numerically approximate evolving curves in a parametric way, is the identity

$$\vec{\mathrm{id}}_{ss} = \vec{\varkappa}_{\partial\Gamma} \quad \text{on } \partial\Gamma\,, \tag{1.7}$$

where $\vec{\varkappa}_{\partial\Gamma}$ is the curvature vector on $\partial\Gamma$. Here we choose the arclength s of the curve $\partial\Gamma$ such that $(\vec{\mathrm{id}}_s, \vec{\mu}, \vec{\nu})$, where

$$\vec{\mu} = \vec{\nu} \times \vec{\mathrm{id}}_s \quad \text{on} \quad \partial\Gamma \tag{1.8}$$

denotes the conormal to Γ on $\partial\Gamma$, form a positively oriented orthonormal basis of $\mathbb{R}^3$. Note that $\vec{\mu}$ is a vector that is perpendicular to the unit tangent $\vec{\mathrm{id}}_s$ on $\partial\Gamma$ and lies in the tangent space of Γ. Now (1.7) can be rewritten as

$$\vec{\mathrm{id}}_{ss} = \vec{\varkappa}_{\partial\Gamma} = \varkappa_\mu\,\vec{\mu} + \varkappa_\nu\,\vec{\nu} \quad \text{on} \quad \partial\Gamma\,, \tag{1.9}$$

where $\varkappa_\mu$ is the geodesic curvature and $\varkappa_\nu$ is the normal curvature. It then follows from the Gauß–Bonnet theorem,

$$\int_\Gamma \mathcal{K}\,\mathrm{d}\mathcal{H}^2 = 2\,\pi\,m(\Gamma) + \int_{\partial\Gamma} \varkappa_\mu\,\mathrm{d}\mathcal{H}^1, \tag{1.10}$$

where $m(\Gamma) \in \mathbb{Z}$ denotes the Euler characteristic of Γ, that the energy (1.6), is equivalent to

$$E(\Gamma) := E_{\overline{\varkappa},\beta}(\Gamma) + \alpha_G \left[\int_{\partial\Gamma} \vec{\varkappa}_{\partial\Gamma}\,.\,\vec{\mu}\,\mathrm{d}\mathcal{H}^1 + 2\,\pi\,m(\Gamma)\right] + \gamma\,\mathcal{H}^1(\partial\Gamma)\,. \tag{1.11}$$

It turns out that the first variation of the energy is given by, compare (A.49) in the appendix, Nitsche (1993) and Barrett et al. (2016),

$$\Delta_s\,\varkappa - (\tfrac{1}{2}\,(\varkappa - \overline{\varkappa})^2 + \beta\,(M(\Gamma) - M_0)\,\varkappa)\,\varkappa + (\varkappa - \overline{\varkappa} + \beta\,(M(\Gamma) - M_0))\,|\nabla_s\,\vec{\nu}|^2\,,$$

and the gradient flow dynamics hence moves a point on the surface Γ with a normal velocity which is the negative of the above expression.

The gradient flow is hence given as a family $(\Gamma(t))_{t\in[0,T]}$ of evolving surfaces with boundary $\partial\Gamma(t)$ that are parameterized by $\vec{x}(\cdot,t) : \Upsilon \to \mathbb{R}^3$ for which

$$\mathcal{V} = -\Delta_s\,\varkappa + (\tfrac{1}{2}\,(\varkappa - \overline{\varkappa})^2 + \beta\,A\,\varkappa)\,\varkappa - (\varkappa - \overline{\varkappa} + \beta\,A)\,|\nabla_s\,\vec{\nu}|^2 \tag{1.12}$$

holds, where

$$A = M(\Gamma(t)) - M_0\,. \tag{1.13}$$

Here

$$\vec{\mathcal{V}}(\vec{z},t) := \vec{x}_t(\vec{q},t) \qquad \forall\ \vec{z} = \vec{x}(\vec{q},t) \in \Gamma(t) \tag{1.14}$$

defines the velocity of $\Gamma(t)$, and $\mathcal{V} := \vec{\mathcal{V}}\,.\,\vec{\nu}$ is the normal velocity of the evolving hypersurface $\Gamma(t)$. The flow (1.12) is of fourth order (taking into account that $\varkappa$ involves two derivatives of the parameterization).

In this paper, we consider four different types of boundary conditions on $\partial\Gamma(t)$. The boundary $\partial\Gamma(t)$ can either move freely, or move along the boundary of a fixed domain Ω, or it will be fixed, $\partial\Gamma(t) = \partial\Gamma(0)$. For the latter case two types of boundary conditions arise: clamped and Navier. As noted earlier, the flow (1.12) is a highly nonlinear fourth order parabolic partial differential equation for the parameterization $\vec{x}$. Hence, if the boundary of $\Gamma(t)$ is fixed, two boundary conditions are needed in order to yield a well-posed problem. If the boundary $\Gamma(t)$ can move, however, then an additional boundary condition is needed to close the system. Similarly to Barrett et al. (2012, Remark 2.1), we may write $\Gamma(t)$ locally near the boundary as a graph over a time-dependent domain $\mathcal{D}(t)$. The fact that $\partial\mathcal{D}(t)$ can move shows the need for three boundary conditions to obtain a well-posed problem. In the free boundary case, the three necessary boundary conditions are given by

$$(\nabla_s \varkappa) \,.\, \vec\mu + \gamma\,\varkappa_\nu - \alpha_G\,\tau_s = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{1.15a}$$
$$-\tfrac12\,(\varkappa-\overline\varkappa)^2 - \beta\,A\,\varkappa + \gamma\,\varkappa_\mu - \alpha_G\,\mathcal{K} = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{1.15b}$$
$$\varkappa - \overline\varkappa + \beta\,A + \alpha_G\,\varkappa_\nu = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{1.15c}$$

where τ denotes the torsion of the curve $\partial\Gamma(t)$, see the appendix for a derivation. We note that in the case $\beta=\gamma=\alpha_G=0$ the condition (1.15b) collapses to (1.15c), and so we conjecture that for this choice of parameters the evolution problem is not well posed. For the partially free case, when $\partial\Gamma(t)\subset\partial\Omega$ for all $t\in[0,T]$, where $\partial\Omega$ is the boundary of a fixed open domain $\Omega\subset\mathbb{R}^3$, we let $\partial\Omega$ be given by a function $F\in C^1(\mathbb{R}^3)$ such that

$$\partial\Omega = \{\vec z\in\mathbb{R}^3 : F(\vec z)=0\} \qquad \text{and} \qquad |\nabla F(\vec z)| = 1 \quad \forall\ \vec z\in\partial\Omega\,,$$

and we denote the normal to Ω on $\partial\Omega$ by $\vec{\mathrm{n}}_{\partial\Omega} = \nabla F$. The necessary boundary conditions are then

$$\partial\Gamma(t)\subset\partial\Omega \tag{1.16a}$$
$$[(\nabla_s\varkappa)\,.\,\vec\mu + \gamma\,\varkappa_\nu - \alpha_G\,\tau_s]\,(\vec\mu\,.\,\vec{\mathrm{n}}_{\partial\Omega}) - [-\tfrac12\,(\varkappa-\overline\varkappa)^2 - \beta\,A\,\varkappa + \gamma\,\varkappa_\mu - \alpha_G\,\mathcal{K}]\,(\vec\nu\,.\,\vec{\mathrm{n}}_{\partial\Omega}) = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{1.16b}$$
$$\varkappa - \overline\varkappa + \beta\,A + \alpha_G\,\varkappa_\nu = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{1.16c}$$

see the appendix for a derivation. Clamped boundary conditions are given by

$$\partial\Gamma(t) = \partial\Gamma(0) \quad \text{and} \quad \vec\mu(t) = \vec\zeta(t) \quad \text{on } \partial\Gamma(0)\,, \tag{1.17}$$

where $\vec\zeta\in C([0,T],C(\partial\Gamma(0),\mathbb{S}^{d-1}))$. Similarly, Navier boundary conditions are given by

$$\partial\Gamma(t) = \partial\Gamma(0) \quad \text{and} \quad \varkappa = \overline\varkappa - \beta\,A - \alpha_G\,\varkappa_\nu \quad \text{on } \partial\Gamma(0)\,, \tag{1.18}$$

see the appendix for a derivation. Of course, for the two fixed boundary conditions, when $\partial\Gamma(t)=\partial\Gamma(0)$ for $t\geq 0$, the line energy contributions in (1.11) play no role. Similarly, for clamped boundary conditions, (1.17), the last integral in (1.11) is fully determined by the data, and so Gaussian curvature plays no role in this case.

In some cases, in particular in applications for biomembranes, cf. Tu (2013), the surface area of Γ needs to stay constant during the evolution. In this case one can consider

$$E_\lambda(\Gamma) = E(\Gamma) + \lambda\,\mathcal{H}^2(\Gamma) \tag{1.19}$$

has to be considered. Here, λ is a Lagrange multiplier for the area constraint, which can be interpreted as a surface tension. In this case (1.12) is replaced by

$$\mathcal{V} = -\Delta_s\,\varkappa + (\tfrac12\,(\varkappa-\overline\varkappa)^2 + \beta\,A\,\varkappa)\,\varkappa - (\varkappa - \overline\varkappa + \beta\,A)\,|\nabla_s\,\vec\nu|^2 + \lambda\,\varkappa\,, \tag{1.20}$$

and (1.15b) is replaced by

$$-\tfrac{1}{2}\,(\varkappa - \overline{\varkappa})^2 - \beta\, A\, \varkappa + \gamma\, \varkappa_\mu - \alpha_G\, \mathcal{K} = \lambda \quad \text{on} \ \ \partial\Gamma(t)\,. \tag{1.21}$$

In Section 2 we will derive a weak formulation for the continuous problem. This will be the basis for the semidiscrete finite element approximation introduced in Section 3 for which we can show a stability result. In Section 4 we formulate a fully discrete finite element approximation for which we can show that a unique solution exists. After a discussion on how to solve the fully discrete linear algebra problem in Section 5, we present in Section 6 several numerical results, many of them for situations for which no computations were available beforehand.

2 Weak formulations/Formal calculus of PDE constrained optimization

On recalling (1.14), we define the following time derivative that follows the parameterization $\vec{x}(\cdot, t)$ of $\Gamma(t)$. Let

$$\partial_t^\circ\, \phi = \phi_t + \vec{\mathcal{V}}\,.\,\nabla\,\phi \qquad \forall\ \phi \in H^1(\mathcal{G}_T)\,, \tag{2.1}$$

where we have defined the space-time surface

$$\mathcal{G}_T := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}\,. \tag{2.2}$$

Here we stress that this definition is well-defined, even though ϕ_t and $\nabla\,\phi$ do not make sense separately for a function $\phi \in H^1(\mathcal{G}_T)$. For later use we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \psi, \phi\rangle_{\Gamma(t)} = \langle \partial_t^\circ\,\psi, \phi\rangle_{\Gamma(t)} + \langle \psi, \partial_t^\circ\,\phi\rangle_{\Gamma(t)} + \left\langle \psi\,\phi, \nabla_s\,.\,\vec{\mathcal{V}}\right\rangle_{\Gamma(t)} \qquad \forall\ \psi,\phi \in H^1(\mathcal{G}_T)\,, \tag{2.3}$$

see Lemma 5.2 in Dziuk and Elliott (2013). Here $\langle\cdot,\cdot\rangle_{\Gamma(t)}$ denotes the L^2–inner product on $\Gamma(t)$. It immediately follows from (2.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{H}^2(\Gamma(t)) = \left\langle \nabla_s\,.\,\vec{\mathcal{V}}, 1\right\rangle_{\Gamma(t)} = \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\mathcal{V}}\right\rangle_{\Gamma(t)}\,. \tag{2.4}$$

In this section we would like to derive a weak formulation for the L^2–gradient flow of $E(\Gamma(t))$. To this end, we need to consider variations of the energy with respect to $\Gamma(t) = \vec{x}(\Upsilon, t)$. For any given $\vec{\chi} \in [H^1(\Gamma(t))]^3$ and for any $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 \in \mathbb{R}_{>0}$, let

$$\Gamma_\varepsilon(t) := \{\vec{\Psi}(\vec{z},\varepsilon) : \vec{z} \in \Gamma(t)\}\,, \quad \text{where} \ \ \vec{\Psi}(\vec{z},0) = \vec{z} \ \text{and} \ \tfrac{\partial\vec{\Psi}}{\partial\varepsilon}(\vec{z},0) = \vec{\chi}(\vec{z}) \quad \forall\ \vec{z} \in \Gamma(t)\,. \tag{2.5}$$

We note that in the case of a fixed boundary, we choose variations $\vec{\chi} \in [H^1_0(\Gamma(t))]^3$, and so $\partial\Gamma_\varepsilon(t) = \partial\Gamma(t) = \partial\Gamma(0)$. The first variation of $\mathcal{H}^2(\Gamma(t))$ with respect to $\Gamma(t)$ in the direction $\vec{\chi} \in [H^1(\Gamma(t))]^3$ is given by

$$\left[\frac{\delta}{\delta\Gamma}\,\mathcal{H}^2(\Gamma(t))\right](\vec{\chi}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\mathcal{H}^2(\Gamma_\varepsilon(t))\,|_{\varepsilon=0}$$
$$= \lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\left[\mathcal{H}^2(\Gamma_\varepsilon(t)) - \mathcal{H}^2(\Gamma(t))\right] = \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\chi}\right\rangle_{\Gamma(t)}, \tag{2.6}$$

see e.g. the proof of Lemma 1 in Dziuk (2008). For later use we note that generalized variants of (2.6) also hold. Namely, we have that

$$\left[\frac{\delta}{\delta\Gamma}\,\langle w, 1\rangle_{\Gamma(t)}\right](\vec{\chi}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\langle w_\varepsilon, 1\rangle_{\Gamma_\varepsilon(t)}\,|_{\varepsilon=0} = \left\langle w\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\chi}\right\rangle_{\Gamma(t)} \qquad \forall\ w \in L^\infty(\Gamma(t))\,, \tag{2.7}$$

where $w_\varepsilon \in L^\infty(\Gamma_\varepsilon(t))$, for any $w \in L^\infty(\Gamma(t))$, is defined by

$$w_\varepsilon(\vec{\Psi}(\vec{z},\varepsilon)) = w(\vec{z}) \qquad \forall\ \vec{z} \in \Gamma(t)\,, \tag{2.8}$$

and similarly for $\vec{w} \in [L^\infty(\Gamma(t))]^3$. This definition of w_ε yields that $\partial^0_\varepsilon\,w = 0$, where

$$\partial^0_\varepsilon\,w(\vec{z}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} w_\varepsilon(\vec{\Psi}(\vec{z},\varepsilon))\,|_{\varepsilon=0} \qquad \forall\ \vec{z} \in \Gamma(t). \tag{2.9}$$

Of course, (2.7) is the first variation analogue of (2.3) with $w = \psi\,\phi$ and $\partial^0_\varepsilon\,\psi = \partial^0_\varepsilon\,\phi = 0$. Similarly, it holds that

$$\left[\frac{\delta}{\delta\Gamma}\,\langle \vec{w}, \vec{\nu}\rangle_{\Gamma(t)}\right](\vec{\chi}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\langle \vec{w}_\varepsilon, \vec{\nu}_\varepsilon\rangle_{\Gamma_\varepsilon(t)}\,|_{\varepsilon=0} = \left\langle (\vec{w}\,.\,\vec{\nu})\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\chi}\right\rangle_{\Gamma(t)} + \left\langle \vec{w}, \partial^0_\varepsilon\,\vec{\nu}\right\rangle_{\Gamma(t)}$$
$$\forall\ \vec{w} \in [L^\infty(\Gamma(t))]^3\,, \tag{2.10}$$

where $\partial^0_\varepsilon\,\vec{w} = \vec{0}$ and $\vec{\nu}_\varepsilon(t)$ denotes the unit normal on $\Gamma_\varepsilon(t)$. In this regard, we note the following result concerning the variation of $\vec{\nu}$, with respect to $\Gamma(t)$, in the direction $\vec{\chi} \in [H^1(\Gamma(t))]^3$:

$$\partial^0_\varepsilon\,\vec{\nu} = -[\nabla_s\,\vec{\chi}]^T\,\vec{\nu} \quad \text{on} \quad \Gamma(t) \quad \Rightarrow \quad \partial^\circ_t\,\vec{\nu} = -[\nabla_s\,\vec{\mathcal{V}}]^T\,\vec{\nu} \quad \text{on} \quad \Gamma(t)\,, \tag{2.11}$$

see Schmidt and Schulz (2010, Lemma 9). Finally, we note that for $\vec{\eta} \in [H^1(\Gamma(t))]^3$ it holds that

$$\left[\frac{\delta}{\delta\Gamma}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\eta}\right\rangle_{\Gamma(t)}\right](\vec{\chi}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\eta}_\varepsilon\right\rangle_{\Gamma_\varepsilon(t)}\,|_{\varepsilon=0} = \langle \nabla_s\,.\,\vec{\eta}, \nabla_s\,.\,\vec{\chi}\rangle_{\Gamma(t)}$$
$$+ \sum_{l,\,m=1}^{3}\left[\langle (\vec{\nu})_l\,(\vec{\nu})_m\,\nabla_s\,(\vec{\eta})_m, \nabla_s\,(\vec{\chi})_l\rangle_{\Gamma(t)} - \langle (\nabla_s)_m\,(\vec{\eta})_l, (\nabla_s)_l\,(\vec{\chi})_m\rangle_{\Gamma(t)}\right]$$
$$= \langle \nabla_s\,\vec{\eta}, \nabla_s\,\vec{\chi}\rangle_{\Gamma(t)} + \langle \nabla_s\,.\,\vec{\eta}, \nabla_s\,.\,\vec{\chi}\rangle_{\Gamma(t)} - \left\langle (\nabla_s\,\vec{\eta})^T, \underline{\underline{D}}(\vec{\chi})\,(\nabla_s\,\vec{\mathrm{id}})^T\right\rangle_{\Gamma(t)}, \tag{2.12}$$

where $\partial_\varepsilon^0\,\vec\eta = \vec 0$, see Lemma 2 and the proof of Lemma 3 in Dziuk (2008). Here

$$\underline{\underline{D}}(\vec\chi) := \nabla_s\,\vec\chi + (\nabla_s\,\vec\chi)^T\,, \tag{2.13}$$

and we note that our notation is such that $\nabla_s\,\vec\chi = (\nabla_\Gamma\,\vec\chi)^T$, with $\nabla_\Gamma\,\vec\chi = (\partial_{s_i}\,\chi_j)_{i,j=1}^3$ defined as in Dziuk (2008). It follows from (2.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta\right\rangle_{\Gamma(t)} = \left\langle \nabla_s\,\vec\eta, \nabla_s\,\vec{\mathcal{V}}\right\rangle_{\Gamma(t)} + \left\langle \nabla_s\,.\,\vec\eta, \nabla_s\,.\,\vec{\mathcal{V}}\right\rangle_{\Gamma(t)} - \left\langle (\nabla_s\,\vec\eta)^T, \underline{\underline{D}}(\vec{\mathcal{V}})\,(\nabla_s\,\vec{\rm id})^T\right\rangle_{\Gamma(t)}$$
$$\forall\ \vec\eta \in \{\vec\xi \in H^1(\mathcal{G}_T) : \partial_t^\circ\,\vec\xi = \vec 0\}\,. \tag{2.14}$$

For closed surfaces, in the seminal work Dziuk (2008), the author introduced a stable semidiscrete finite element approximation of Willmore flow, which is based on the discrete analogue of the identity $\frac12\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\vec\varkappa,\vec\varkappa\rangle_{\Gamma(t)} = -\left\langle \vec f_\Gamma, \vec{\mathcal{V}}\right\rangle_{\Gamma(t)}$, where

$$\left\langle \vec f_\Gamma, \vec\chi\right\rangle_{\Gamma(t)} = \langle \nabla_s\,\vec\varkappa, \nabla_s\,\vec\chi\rangle_{\Gamma(t)} + \langle \nabla_s\,.\,\vec\varkappa, \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)} - \left\langle (\nabla_s\,\vec\varkappa)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T\right\rangle_{\Gamma(t)}$$
$$+ \tfrac12\left\langle |\vec\varkappa|^2\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi\right\rangle_{\Gamma(t)} \qquad \forall\ \vec\chi \in [H^1(\Gamma(t))]^3\,. \tag{2.15}$$

In the recent paper Barrett et al. (2015a) the present authors were able to extend (2.15), and the corresponding semidiscrete approximation, to the case of nonzero β and $\overline\varkappa$ in (1.5a). The approximation is based on a suitable weak formulation, which can be obtained by considering the first variation of (1.5a) subject to the side constraint, the weak formulation of (1.2),

$$\langle\vec\varkappa,\vec\eta\rangle_{\Gamma(t)} + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta\right\rangle_{\Gamma(t)} = 0 \qquad \forall\ \vec\eta \in [H^1(\Gamma(t))]^3\,. \tag{2.16}$$

To this end, one defines the Lagrangian

$$\widetilde L(\Gamma(t),\vec\varkappa,\vec y) = \tfrac12\left\langle |\vec\varkappa - \overline\varkappa\,\vec\nu|^2, 1\right\rangle_{\Gamma(t)} + \tfrac{\beta}{2}\left(\langle\vec\varkappa,\vec\nu\rangle_{\Gamma(t)} - M_0\right)^2 - \langle\vec\varkappa,\vec y\rangle_{\Gamma(t)} - \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec y\right\rangle_{\Gamma(t)} \tag{2.17}$$

with $\vec y \in [H^1(\Gamma(t))]^3$ being a Lagrange multiplier for (2.16). Then, on using ideas from the formal calculus of PDE constrained optimization, see e.g. Tröltzsch (2010), one can compute the direction of steepest descent $\vec f_\Gamma$ of $E_{\overline\varkappa,\beta}(\Gamma(t))$, under the constraint (2.16). In particular, we formally require that

$$\left[\frac{\delta}{\delta\Gamma}\,\widetilde L\right](\vec\chi) = \lim_{\varepsilon\to0}\tfrac{1}{\varepsilon}\left[\widetilde L(\Gamma_\varepsilon(t),\vec\varkappa,\vec y) - \widetilde L(\Gamma(t),\vec\varkappa,\vec y)\right] = -\left\langle \vec f_\Gamma, \vec\chi\right\rangle_{\Gamma(t)}\,, \tag{2.18a}$$

$$\left[\frac{\delta}{\delta\vec\varkappa}\,\widetilde L\right](\vec\xi) = \lim_{\varepsilon\to0}\tfrac{1}{\varepsilon}\left[\widetilde L(\Gamma(t),\vec\varkappa+\varepsilon\,\vec\xi,\vec y) - \widetilde L(\Gamma(t),\vec\varkappa,\vec y)\right] = 0\,, \tag{2.18b}$$

$$\left[\frac{\delta}{\delta\vec y}\,\widetilde L\right](\vec\eta) = \lim_{\varepsilon\to0}\tfrac{1}{\varepsilon}\left[\widetilde L(\Gamma(t),\vec\varkappa,\vec y+\varepsilon\,\vec\eta) - \widetilde L(\Gamma(t),\vec\varkappa,\vec y)\right] = 0\,. \tag{2.18c}$$

On recalling (2.7)–(2.12), this yields that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_{\overline{\varkappa},\beta}(\Gamma(t)) = -\left\langle \vec{f}_\Gamma, \vec{\mathcal{V}}\right\rangle_{\Gamma(t)}$, where

$$\left\langle \vec{f}_\Gamma, \vec{\chi}\right\rangle_{\Gamma(t)} = \langle \nabla_s \vec{y}, \nabla_s \vec{\chi}\rangle_{\Gamma(t)} + \langle \nabla_s . \vec{y}, \nabla_s . \vec{\chi}\rangle_{\Gamma(t)} - \left\langle (\nabla_s \vec{y})^T, \underline{\underline{D}}(\vec{\chi})\,(\nabla_s \vec{\mathrm{id}})^T\right\rangle_{\Gamma(t)}$$
$$- \tfrac{1}{2}\left\langle [|\vec{\varkappa} - \overline{\varkappa}\,\vec{\nu}|^2 - 2\,(\vec{y}\,.\,\vec{\varkappa})]\,\nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi}\right\rangle_{\Gamma(t)} + (\beta\,A - \overline{\varkappa})\,\langle \vec{\varkappa}, [\nabla_s \vec{\chi}]^T\,\vec{\nu}\rangle_{\Gamma(t)}$$
$$- \beta\,A\left\langle (\vec{\varkappa}\,.\,\vec{\nu})\,\nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi}\right\rangle_{\Gamma(t)} = 0 \qquad \forall\ \vec{\chi} \in [H^1(\Gamma(t))]^3\,, \tag{2.19a}$$

$$\left\langle \vec{\varkappa} + (\beta\,A - \overline{\varkappa})\,\vec{\nu} - \vec{y}, \vec{\xi}\right\rangle_{\Gamma(t)} = 0 \qquad \forall\ \vec{\xi} \in [H^1(\Gamma(t))]^3\,, \tag{2.19b}$$

$$\langle \vec{\varkappa}, \vec{\eta}\rangle_{\Gamma(t)} + \left\langle \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\eta}\right\rangle_{\Gamma(t)} = 0 \qquad \forall\ \vec{\eta} \in [H^1(\Gamma(t))]^3\,, \tag{2.19c}$$

where

$$A(t) = \langle \vec{\varkappa}, \vec{\nu}\rangle_{\Gamma(t)} - M_0\,. \tag{2.19d}$$

Clearly, (2.19b) implies that $\vec{\varkappa} + (\beta\,A - \overline{\varkappa})\,\vec{\nu} = \vec{y}$. In the case $\overline{\varkappa} = \beta = 0$, this collapses to $\vec{y} = \vec{\varkappa}$, and so (2.19a) collapses to (2.15). In the context of the numerical approximation of the L^2–gradient flow of $E_{\overline{\varkappa},\beta}(\Gamma(t))$, (2.19a–d) gives rise to the following weak formulation, where we recall (1.14). Given $\Gamma(0)$, for all $t \in (0, T]$ find $\Gamma(t) = \vec{x}(\Upsilon, t)$, with $\vec{\mathcal{V}}(t) \in [H^1(\Gamma(t))]^3$, and $\vec{y}(t) \in [H^1(\Gamma(t))]^3$ such that (2.19a) holds with $\vec{\varkappa} = \vec{y} - (\beta\,A - \overline{\varkappa})\,\vec{\nu}$ and $A(t) = \langle \vec{\varkappa}, \vec{\nu}\rangle_{\Gamma(t)} - M_0$, and such that

$$\left\langle \vec{\mathcal{V}} - \vec{f}_\Gamma, \vec{\chi}\right\rangle_{\Gamma(t)} = 0 \quad \forall\ \vec{\chi} \in [H^1(\Gamma(t))]^3\,, \tag{2.20a}$$

$$\langle \vec{y}, \vec{\eta}\rangle_{\Gamma(t)} + \left\langle \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\eta}\right\rangle_{\Gamma(t)} = (\beta\,A - \overline{\varkappa})\,\langle \vec{\nu}, \vec{\eta}\rangle_{\Gamma(t)} \quad \forall\ \vec{\eta} \in [H^1(\Gamma(t))]^3\,. \tag{2.20b}$$

Under discretization, (2.20a,b) does not have good mesh properties. That is because the discretizations will exhibit mesh movements that are almost exclusively in the normal direction, which in general leads to bad meshes. To see this, we note that (2.20a,b) is the weak formulation of

$$\vec{\mathcal{V}} = \left[-\Delta_s\,\varkappa + (\tfrac{1}{2}\,(\varkappa - \overline{\varkappa})^2 + \beta\,A\,\varkappa)\,\varkappa - |\nabla_s\,\vec{\nu}|^2\,(\varkappa - \overline{\varkappa} + \beta\,A)\right]\vec{\nu}\,, \tag{2.21}$$

which agrees with Barrett et al. (2008, (1.12)). A derivation of (2.21) is given in the appendix. In order to overcome the undesirable mesh effects for a discretization of (2.20a,b), the authors in Barrett et al. (2016) replaced the side constraint (2.16) with the more general side constraint

$$\left\langle \underline{\underline{Q}}_\theta\,\vec{\varkappa}, \vec{\eta}\right\rangle_{\Gamma(t)} + \left\langle \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\eta}\right\rangle_{\Gamma(t)} = 0 \qquad \forall\ \vec{\eta} \in [H^1(\Gamma(t))]^3\,, \tag{2.22}$$

where $\theta \in [0, 1]$ is a fixed parameter, and where $\underline{\underline{Q}}_\theta$ is defined by

$$\underline{\underline{Q}}_\theta = \theta\,\underline{\underline{\mathrm{Id}}} + (1 - \theta)\,\vec{\nu} \otimes \vec{\nu} \quad \text{on}\ \ \Gamma(t)\,. \tag{2.23}$$

On recalling (1.2), we note that on the continuous level (2.22) trivially holds independently of the choice of $\theta \in [0, 1]$. However, on the discrete level (2.22), for $\theta < 1$, leads to an induced tangential motion and good meshes, in general. See e.g. (3.39) in Section 3 below for more details.

From now on we consider open surfaces. Then, similarly to (2.22), we consider the side constraint

$$\left\langle \underline{\underline{Q}}_\theta\, \vec\varkappa, \vec\eta \right\rangle_{\Gamma(t)} + \left\langle \nabla_s \vec{\rm id}, \nabla_s \vec\eta \right\rangle_{\Gamma(t)} = \langle \vec{\rm m}, \vec\eta \rangle_{\partial\Gamma(t)} \qquad \forall\, \vec\eta \in [H^1(\Gamma(t))]^3 \,, \tag{2.24}$$

which again holds trivially on the continuous level for $\vec{\rm m}$ being the conormal $\vec\mu$. Here $\langle\cdot,\cdot\rangle_{\partial\Gamma(t)}$ denotes the L^2–inner product on $\partial\Gamma(t)$. We stress that in the clamped case, $\partial\Gamma(t) = \partial\Gamma(0)$ and $\vec{\rm m} = \vec\mu = \vec\zeta$ in (2.24) are fixed given data, recall (1.17). For the other three types of boundary conditions, (2.24) weakly defines the conormal $\vec\mu(t)$ to $\Gamma(t)$ on $\partial\Gamma(t)$. In the discrete setting, the discrete analogue of (2.24) will weakly define a discrete conormal $\vec{\rm m}^h(t)$, which will in general be different from the true conormal $\vec\mu^h(t)$ to $\Gamma^h(t)$ on $\partial\Gamma^h(t)$, defined via the discrete analogue of (1.8), where $\Gamma^h(t)$ is a discrete approximation of $\Gamma(t)$.

Similarly to (2.16), and for later use, we introduce the weak formulation of (1.7): Find $\vec\varkappa_{\partial\Gamma} \in [H^1(\partial\Gamma(t))]^3$ such that

$$\langle \vec\varkappa_{\partial\Gamma}, \vec\eta \rangle_{\partial\Gamma(t)} + \left\langle \vec{\rm id}_s, \vec\eta_s \right\rangle_{\partial\Gamma(t)} = 0 \quad \forall\, \vec\eta \in [H^1(\partial\Gamma(t))]^3 \,. \tag{2.25}$$

Similarly to (2.7) it holds that

$$\left[\frac{\delta}{\delta\Gamma}\, \langle w, 1\rangle_{\partial\Gamma(t)}\right](\vec\chi) = \frac{\rm d}{{\rm d}\varepsilon}\, \langle w_\varepsilon, 1\rangle_{\partial\Gamma_\varepsilon(t)} \mid_{\varepsilon=0} = \left\langle w\, \vec{\rm id}_s, \vec\chi_s \right\rangle_{\partial\Gamma(t)}$$
$$\forall\, w \in L^\infty(\partial\Gamma(t)),\ \vec\chi \in [H^1_{\partial\Gamma}(\Gamma(t))]^3 \,, \tag{2.26}$$

where $\partial_\varepsilon^0\, \vec w = \vec 0$, and where

$$H^1_{\partial\Gamma}(\Gamma(t)) := \{\eta \in H^1(\Gamma(t)) : \eta \mid_{\partial\Gamma(t)} \in H^1(\partial\Gamma(t))\} \,. \tag{2.27}$$

Moreover, similarly to (2.12), we note that for $\vec\eta \in [H^1_{\partial\Gamma}(\Gamma(t))]^3$ it holds that

$$\left[\frac{\delta}{\delta\Gamma} \left\langle \vec{\rm id}_s, \vec\eta_s \right\rangle_{\partial\Gamma(t)}\right](\vec\chi) = \langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}\, \vec\eta_s, \vec\chi_s \rangle_{\partial\Gamma(t)} \,, \tag{2.28}$$

where $\partial_\varepsilon^0\, \vec\eta = \vec 0$, and where

$$\underline{\underline{\mathcal{P}}}_{\partial\Gamma} = \underline{\underline{\rm Id}} - \vec{\rm id}_s \otimes \vec{\rm id}_s \qquad \text{on } \ \partial\Gamma(t) \,. \tag{2.29}$$

For notational convenience, we also define

$$[H^1_{\nabla F}(\Gamma(t))]^3 = \{\vec\eta \in [H^1_{\partial\Gamma}(\Gamma(t))]^3 : \vec\eta \,.\, \nabla F = 0 \text{ on } \ \partial\Gamma(t)\} \tag{2.30}$$

and

$$\mathbb{X}(\Gamma(t)) = \begin{cases} [H^1_{\partial\Gamma}(\Gamma(t))]^3 & \text{free boundary conditions,} \\ [H^1_{\nabla F}(\Gamma(t))]^3 & \text{semi-free boundary conditions,} \\ [H^1_0(\Gamma(t))]^3 & \text{fixed boundary conditions.} \end{cases} \tag{2.31}$$

We first consider the three types of boundary conditions that do not involve fixing the conormal on $\partial\Gamma(t)$, i.e. free, semi-free and Navier. We recall the energy (1.11) and the fact that $m(\Gamma(t))$ is a topological invariant, which does not change its value under continuous deformations of the surface $\Gamma(t)$. We hence define the Lagrangian omitting the term $m(\Gamma(t))$ as follows. Let

$$\begin{aligned} L(\Gamma(t), \vec\varkappa, \vec{\mathrm{m}}, \vec\varkappa_{\partial\Gamma}, \vec y, \vec z) = \tfrac12 \left\langle |\vec\varkappa - \overline{\varkappa}\,\vec\nu|^2, 1\right\rangle_{\Gamma(t)} + \tfrac{\beta}{2}\left(\langle \vec\varkappa, \vec\nu\rangle_{\Gamma(t)} - M_0\right)^2 + \gamma\,\mathcal{H}^1(\partial\Gamma(t)) \\ + \alpha_G \left[\langle \vec\varkappa_{\partial\Gamma}, \vec{\mathrm{m}}\rangle_{\partial\Gamma(t)} - \langle \vec\varkappa_{\partial\Gamma}, \vec z\rangle_{\partial\Gamma(t)} - \left\langle \vec{\mathrm{id}}_s, \vec z_s\right\rangle_{\partial\Gamma(t)}\right] \\ - \left\langle \underline{\underline{Q}}_\theta\,\vec\varkappa, \vec y\right\rangle_{\Gamma(t)} - \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec y\right\rangle_{\Gamma(t)} + \langle \vec{\mathrm{m}}, \vec y\rangle_{\partial\Gamma(t)}\,, \end{aligned} \tag{2.32}$$

where $\vec y \in [H^1(\Gamma(t))]^3$ and $\vec z \in [H^1(\partial\Gamma(t))]^3$ are Lagrange multipliers for (2.24) and (2.25), respectively. We now want to compute the direction of steepest descent $\vec f_\Gamma$ of $E(\Gamma(t))$, where the curvature vector, $\vec\varkappa$, and the conormal $\vec{\mathrm{m}} = \vec\mu$, satisfy (2.24), and the curve curvature vector, $\vec\varkappa_{\partial\Gamma}$, satisfies (2.25). This means that $\vec f_\Gamma$ needs to fulfill

$$\left\langle \vec f_\Gamma, \vec\chi\right\rangle_{\Gamma(t)} = -\left[\frac{\delta}{\delta\Gamma}\,E(\Gamma(t))\right](\vec\chi) \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,. \tag{2.33}$$

Using (2.7)–(2.12) and (2.18a–c), with $\widetilde L$ replaced by L, as well as

$$\left[\frac{\delta}{\delta\vec{\mathrm{m}}}\,L\right](\vec\varphi) = \lim_{\varepsilon\to0}\tfrac{1}{\varepsilon}\,[L(\Gamma(t), \vec\varkappa, \vec{\mathrm{m}} + \varepsilon\,\vec\varphi, \vec\varkappa_{\partial\Gamma}, \vec y, \vec z) - L(\Gamma(t), \vec\varkappa, \vec{\mathrm{m}}, \vec\varkappa_{\partial\Gamma}, \vec y, \vec z)] = 0\,, \tag{2.34}$$

and similarly for $\frac{\delta}{\delta\vec\varkappa_{\partial\Gamma}}\,L = 0$, which yields that $\vec z = \vec{\mathrm{m}}$, and $\frac{\delta}{\delta\vec z}\,L = 0$, one computes, on noting (2.26) and (2.28), that

$$\begin{aligned} \left\langle \vec f_\Gamma, \vec\chi\right\rangle_{\Gamma(t)} &= \langle \nabla_s\,\vec y, \nabla_s\,\vec\chi\rangle_{\Gamma(t)} + \langle \nabla_s\,.\,\vec y, \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)} - \left\langle (\nabla_s\,\vec y)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\mathrm{id}})^T\right\rangle_{\Gamma(t)} \\ &\quad - \tfrac12\left\langle [|\vec\varkappa - \overline{\varkappa}\,\vec\nu|^2 - 2\,\underline{\underline{Q}}_\theta\,\vec y\,.\,\vec\varkappa]\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\chi\right\rangle_{\Gamma(t)} \\ &\quad - \beta\,A\left\langle (\vec\varkappa\,.\,\vec\nu)\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\chi\right\rangle_{\Gamma(t)} - (\beta\,A - \overline{\varkappa})\,\langle \vec\varkappa, \partial_\varepsilon^0\,\vec\nu\rangle_{\Gamma(t)} \\ &\quad + \left\langle \partial_\varepsilon^0\,[\underline{\underline{Q}}_\theta\,\vec\varkappa], \vec y\right\rangle_{\Gamma(t)} - \gamma\left\langle \vec{\mathrm{id}}_s, \vec\chi_s\right\rangle_{\partial\Gamma(t)} \\ &\quad + \alpha_G\left[\left\langle \vec\varkappa_{\partial\Gamma}\,.\,\vec{\mathrm{m}}, \vec{\mathrm{id}}_s\,.\,\vec\chi_s\right\rangle_{\partial\Gamma(t)} + \langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}\,\vec{\mathrm{m}}_s, \vec\chi_s\rangle_{\partial\Gamma(t)}\right] \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,, \end{aligned} \tag{2.35a}$$

$$\left\langle \vec\varkappa + (\beta\,A - \overline{\varkappa})\,\vec\nu - \underline{\underline{Q}}_\theta\,\vec y, \vec\xi\right\rangle_{\Gamma(t)} = 0 \qquad \forall\ \vec\xi \in [H^1(\Gamma(t))]^3\,, \tag{2.35b}$$

$$\langle \alpha_G\,\vec\varkappa_{\partial\Gamma} + \vec y, \vec\varphi\rangle_{\partial\Gamma(t)} = 0 \qquad \forall\ \vec\varphi \in [H^1(\partial\Gamma(t))]^3\,, \tag{2.35c}$$

with (2.24) and (2.25). As $\partial_\varepsilon^0\,\vec\varkappa = 0$, we have that

$$\partial_\varepsilon^0\,[\underline{\underline{Q}}_\theta\,\vec\varkappa] = (1-\theta)\left[(\vec\varkappa\,.\,\partial_\varepsilon^0\,\vec\nu)\,\vec\nu + (\vec\varkappa\,.\,\vec\nu)\,\partial_\varepsilon^0\,\vec\nu\right]. \tag{2.36}$$

We observe that (2.35b,c) imply that

$$\underline{\underline{Q}}_\theta\,\vec{y} = \vec{\varkappa} + (\beta\,A - \overline{\varkappa})\,\vec{\nu} \quad \text{on} \quad \Gamma(t) \qquad \text{and} \qquad \vec{y} = -\alpha_G\,\vec{\varkappa}_{\partial\Gamma} \quad \text{on} \quad \partial\Gamma(t)\,. \tag{2.37}$$

If $\theta = 0$, then it follows from (2.37), together with (1.2), (1.7) and (1.9), that $\varkappa = \overline{\varkappa} - \beta\,A - \alpha_G\,\varkappa_\nu$ holds on $\partial\Gamma(t)$. However, if $\theta \in (0,1]$, then the two conditions in (2.37) are incompatible if $\alpha_G \neq 0$, unless the geodesic curvature $\varkappa_\mu$ vanishes on $\partial\Gamma(t)$, since the first condition in (2.37) yields that $\vec{y} = (\varkappa + \beta\,A - \overline{\varkappa})\,\vec{\nu}$. Hence for general boundaries $\partial\Gamma(t)$ and $\alpha_G \neq 0$ we need to take $\theta = 0$, at least locally at the boundary. Therefore we need to consider a variable $\theta \in L^\infty(\Gamma(t))$. The calculation (2.35a–c) remains valid provided that $\partial^0_\varepsilon\,\theta = 0$. We will make this more rigorous on the discrete level, see (3.18) below.

It follows from (2.35a), (2.36) and (2.11) that

$$\begin{aligned}
\left\langle \vec{f}_\Gamma, \vec{\chi} \right\rangle_{\Gamma(t)} &= \langle \nabla_s\,\vec{y}, \nabla_s\,\vec{\chi}\rangle_{\Gamma(t)} + \langle \nabla_s\,.\,\vec{y}, \nabla_s\,.\,\vec{\chi}\rangle_{\Gamma(t)} - \left\langle (\nabla_s\,\vec{y})^T, \underline{\underline{D}}(\vec{\chi})\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma(t)} \\
&\quad - \tfrac12 \left\langle [|\vec{\varkappa} - \overline{\varkappa}\,\vec{\nu}|^2 - 2\,\vec{y}\,.\,\underline{\underline{Q}}_\theta\,\vec{\varkappa}]\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\chi} \right\rangle_{\Gamma(t)} - \beta\,A \left\langle (\vec{\varkappa}\,.\,\vec{\nu})\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\chi} \right\rangle_{\Gamma(t)} \\
&\quad + (\beta\,A - \overline{\varkappa})\,\langle \vec{\varkappa}, (\nabla_s\,\vec{\chi})^T\,\vec{\nu}\rangle_{\Gamma(t)} - (1-\theta)\,\langle \vec{\varkappa}\,.\,([\nabla_s\,\vec{\chi}]^T\,\vec{\nu})\,\vec{\nu}, \vec{y}\rangle_{\Gamma(t)} \\
&\quad - (1-\theta)\,\langle (\vec{\varkappa}\,.\,\vec{\nu})\,[\nabla_s\,\vec{\chi}]^T\,\vec{\nu}, \vec{y}\rangle_{\Gamma(t)} - \gamma \left\langle \vec{\rm id}_s, \vec{\chi}_s \right\rangle_{\partial\Gamma(t)} \\
&\quad + \alpha_G \left[\left\langle \vec{\varkappa}_{\partial\Gamma}\,.\,\vec{\rm m}, \vec{\rm id}_s\,.\,\vec{\chi}_s \right\rangle_{\partial\Gamma(t)} + \langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}\,\vec{\rm m}_s, \vec{\chi}_s\rangle_{\partial\Gamma(t)} \right] \quad \forall\ \vec{\chi} \in \mathbb{X}(\Gamma(t))\,;
\end{aligned} \tag{2.38}$$

see Barrett et al. (2016) for a similar computation.

For the case of clamped boundary conditions, (1.17), the "unknown" $\vec{\rm m}$ in (2.24) and (2.32) is replaced by the given data $\vec{\zeta}$. Then there is no variation in $\vec{\rm m}$, so that we no longer obtain (2.35c) and, of course, the terms involving $\partial\Gamma(t)$ in (2.38) play no role as $\vec{\chi} \in [H^1_0(\Gamma(t))]^3$. Hence in this case it is not necessary to take $\theta = 0$ in the vicinity of $\partial\Gamma(t) = \partial\Gamma(0)$.

If the surface area of $\Gamma(t)$ has to be preserved during the evolution, cf. (1.19)–(1.21), the right hand side of (2.38) has an additional term $-\lambda\,\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\chi}\rangle_{\Gamma(t)}$, on recalling (2.6).

3 Semidiscrete finite element approximation

The parametric finite element spaces are defined as follows, see also Barrett et al. (2008). Let $\Upsilon^h \subset \mathbb{R}^3$ be a two-dimensional *polyhedral surface*, i.e. a union of nondegenerate triangles with no hanging vertices (see Deckelnick et al. (2005, p. 164)), approximating the reference manifold Υ. In particular, let $\Upsilon^h = \bigcup_{j=1}^J \overline{o^h_j}$, where $\{o^h_j\}_{j=1}^J$ is a family of mutually disjoint open triangles. Then let $\underline{V}^h(\Upsilon^h) := \{\vec{\chi} \in C(\Upsilon^h, \mathbb{R}^3) : \vec{\chi}\,|_{o^h_j}$ is linear, $j = 1, \ldots, J\}$. We consider a family of parameterizations $\vec{X}^h(\cdot, t) \in \underline{V}^h(\Upsilon^h)$ with $\vec{X}^h(\Upsilon^h, t) = \Gamma^h(t)$ and with $\Gamma^h(0)$ an approximation of $\Gamma(0)$. In particular, let $\Gamma^h(t) = \bigcup_{j=1}^J \overline{\sigma^h_j(t)}$, where $\{\sigma_j^h(t)\}_{j=1}^J$ is a family of mutually disjoint open triangles with vertices $\{\vec{q}_k^h(t)\}_{k=1}^K$. Then let

$$\underline{V}^h(\Gamma^h(t)) := \{\vec{\chi} \in [C(\Gamma^h(t))]^3 : \vec{\chi}|_{\sigma_j^h} \text{ is linear}, j = 1, \ldots, J\}$$
$$=: [W^h(\Gamma^h(t))]^3 \subset [H^1(\Gamma^h(t))]^3 ,$$

where $W^h(\Gamma^h(t)) \subset H^1(\Gamma^h(t))$ is the space of scalar continuous piecewise linear functions on $\Gamma^h(t)$, with $\{\chi_k^h(\cdot, t)\}_{k=1}^K$ denoting the standard basis of $W^h(\Gamma^h(t))$, i.e.

$$\chi_k^h(\vec{q}_l^h(t), t) = \delta_{kl} \qquad \forall\, k, l \in \{1, \ldots, K\}\,,\ t \in [0, T]\,. \tag{3.1}$$

For later purposes, we also introduce $\pi^h(t) : C(\Gamma^h(t)) \to W^h(\Gamma^h(t))$, the standard interpolation operator at the nodes $\{\vec{q}_k^h(t)\}_{k=1}^K$, and similarly $\vec{\pi}^h(t) : [C(\Gamma^h(t))]^3 \to \underline{V}^h(\Gamma^h(t))$. Let

$$\underline{V}_0^h(\Gamma^h(t)) := \underline{V}^h(\Gamma^h(t)) \cap [H_0^1(\Gamma^h(t))]^3 \tag{3.2a}$$

and

$$\underline{V}^h(\partial\Gamma^h(t)) := \{\vec{\psi} \in [C(\partial\Gamma^h(0))]^3 : \exists\, \vec{\chi} \in \underline{V}^h(\Gamma^h(t))\ \vec{\chi}|_{\partial\Gamma^h(0)} = \vec{\psi}\}\,. \tag{3.2b}$$

For later use, we introduce the decomposition

$$\underline{V}^h(\Gamma^h(t)) = \underline{V}_{\partial\Gamma}^h(\Gamma^h(t)) \oplus \underline{V}_0^h(\Gamma^h(t))\,, \tag{3.3}$$

where we note that $\underline{V}_{\partial\Gamma}^h(\Gamma^h(t))$ is clearly isomorphic to $\underline{V}^h(\partial\Gamma^h(t))$. We also introduce

$$\underline{V}_{\nabla F}^h(\Gamma^h(t)) := \{\vec{\chi} \in \underline{V}^h(\Gamma^h(t)) : (\vec{\chi} \,.\, \nabla F)(\vec{q}_k^h(t)) = 0 \quad \forall\ \vec{q}_k^h(t) \in \partial\Gamma^h(t)\}\,. \tag{3.4}$$

In order to treat all four boundary conditions in a compact way, we also define

$$\mathbb{X}(\Gamma^h(t)) = \begin{cases} \underline{V}^h(\Gamma^h(t)) & \text{free boundary conditions,} \\ \underline{V}_{\nabla F}^h(\Gamma^h(t)) & \text{semi-free boundary conditions,} \\ \underline{V}_0^h(\Gamma^h(t)) & \text{fixed boundary conditions,} \end{cases} \tag{3.5}$$

where fixed boundary conditions can be either clamped or Navier.

We denote the L^2–inner products on $\Gamma^h(t)$ and $\partial\Gamma^h(t)$ by $\langle\cdot,\cdot\rangle_{\Gamma^h(t)}$ and $\langle\cdot,\cdot\rangle_{\partial\Gamma^h(t)}$, respectively. In addition, for piecewise continuous functions, with possible jumps across the edges of $\{\sigma_j^h\}_{j=1}^J$, we also introduce the mass lumped inner product

$$\langle\eta, \phi\rangle_{\Gamma^h(t)}^h := \tfrac{1}{3} \sum_{j=1}^J \mathcal{H}^2(\sigma_j^h) \sum_{k=1}^3 (\eta \,.\, \phi)((\vec{q}_{j_k}^h)^-)\,,$$

where $\{\vec{q}_{j_k}^h\}_{k=1}^3$ are the vertices of σ_j^h, and where we define $\eta((\vec{q}_{j_k}^h)^-) := \lim_{\sigma_j^h \ni \vec{p} \to \vec{q}_{j_k}^h} \eta(\vec{p})$. We naturally extend this definition to vector and tensor functions. We also define the mass lumped inner product $\langle\cdot,\cdot\rangle_{\partial\Gamma^h(t)}^h$ in the obvious way.

Following Dziuk and Elliott (2013, (5.23)), we define the discrete material velocity for $\vec{z} \in \Gamma^h(t)$ by

$$\vec{\mathcal{V}}^h(\vec{z}, t) := \sum_{k=1}^{K} \left[\frac{\mathrm{d}}{\mathrm{d}t}\, \vec{q}^h_k(t)\right] \chi^h_k(\vec{z}, t)\,. \tag{3.6}$$

Then, similarly to (2.1), we define

$$\partial_t^{\circ,h}\, \phi = \phi_t + \vec{\mathcal{V}}^h \,.\, \nabla\, \phi \qquad \forall\ \phi \in H^1(\mathcal{G}^h_T)\,, \quad \text{where} \quad \mathcal{G}^h_T := \bigcup_{t\in[0,T]} \Gamma^h(t) \times \{t\}\,. \tag{3.7}$$

On differentiating (3.1) with respect to t, it immediately follows that

$$\partial_t^{\circ,h}\, \chi^h_k = 0 \quad \forall\ k \in \{1, \ldots, K\}\,, \tag{3.8}$$

see also Dziuk and Elliott (2013, Lem. 5.5). It follows directly from (3.8) that

$$\partial_t^{\circ,h}\, \phi(\cdot, t) = \sum_{k=1}^{K} \chi^h_k(\cdot, t)\, \frac{\mathrm{d}}{\mathrm{d}t}\, \phi_k(t) \quad \text{on } \Gamma^h(t) \tag{3.9}$$

for $\phi(\cdot, t) = \sum_{k=1}^{K} \phi_k(t)\, \chi^h_k(\cdot, t) \in W^h(\Gamma^h(t))$.

For later use, we also introduce the finite element spaces

$$W(\mathcal{G}^h_T) := \{\chi \in C(\mathcal{G}^h_T) : \chi(\cdot, t) \in W^h(\Gamma^h(t)) \quad \forall\ t \in [0, T]\}\,,$$
$$W_T(\mathcal{G}^h_T) := \{\chi \in W(\mathcal{G}^h_T) : \partial_t^{\circ,h}\, \chi \in C(\mathcal{G}^h_T)\}\,.$$

We recall from Dziuk and Elliott (2013, Lem. 5.6) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\sigma^h_j(t)} \phi \;\mathrm{d}\mathcal{H}^2 = \int_{\sigma^h_j(t)} \partial_t^{\circ,h}\, \phi + \phi\, \nabla_s \,.\, \vec{\mathcal{V}}^h \;\mathrm{d}\mathcal{H}^2 \quad \forall\ \phi \in H^1(\sigma^h_j(t))\,, j \in \{1, \ldots, J\}\,, \tag{3.10}$$

which immediately implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \eta, \phi \rangle_{\Gamma^h(t)} = \langle \partial_t^{\circ,h}\, \eta, \phi \rangle_{\Gamma^h(t)} + \langle \eta, \partial_t^{\circ,h}\, \phi \rangle_{\Gamma^h(t)} + \langle \eta\, \phi, \nabla_s \,.\, \vec{\mathcal{V}}^h \rangle_{\Gamma^h(t)} \qquad \forall\ \eta, \phi \in W_T(\mathcal{G}^h_T)\,. \tag{3.11}$$

Similarly, we recall from Barrett et al. (2015b, Lem. 3.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \eta, \phi \rangle^h_{\Gamma^h(t)} = \langle \partial_t^{\circ,h}\, \eta, \phi \rangle^h_{\Gamma^h(t)} + \langle \eta, \partial_t^{\circ,h}\, \phi \rangle^h_{\Gamma^h(t)} + \langle \eta\, \phi, \nabla_s \,.\, \vec{\mathcal{V}}^h \rangle^h_{\Gamma^h(t)} \quad \forall\ \eta, \phi \in W_T(\mathcal{G}^h_T)\,. \tag{3.12}$$

Moreover, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \eta, \phi \rangle^h_{\partial\Gamma^h(t)} = \langle \partial_t^{\circ,h}\, \eta, \phi \rangle^h_{\partial\Gamma^h(t)} + \langle \eta, \partial_t^{\circ,h}\, \phi \rangle^h_{\partial\Gamma^h(t)} + \langle \eta\, \phi, \vec{\mathrm{id}}_s \,.\, \vec{\mathcal{V}}^h_s \rangle^h_{\partial\Gamma^h(t)} \quad \forall\ \eta, \phi \in W_T(\mathcal{G}^h_T)\,. \tag{3.13}$$

We also note the discrete version of (2.14),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\eta \right\rangle_{\Gamma^h(t)} = \left\langle \nabla_s\,\vec\eta, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)} + \left\langle \nabla_s\,.\,\vec\eta, \nabla_s\,.\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)} \\ - \left\langle (\nabla_s\,\vec\eta)^T, \underline{\underline{D}}(\vec{\mathcal{V}}^h)\,(\nabla_s\,\vec{\mathrm{id}})^T \right\rangle_{\Gamma^h(t)} \quad \forall\ \vec\eta \in \{\vec\xi \in [W_T(\mathcal{G}^h_T)]^3 : \partial_t^{\circ,h}\,\vec\xi = \vec 0\}\,, \tag{3.14}$$

as well as the corresponding version for $\partial\Gamma^h(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \vec{\mathrm{id}}_s, \vec\eta_s \right\rangle_{\partial\Gamma^h(t)} = \left\langle \underline{\underline{\mathcal{P}}}^h_{\partial\Gamma}\,\vec\eta_s, \vec{\mathcal{V}}^h_s \right\rangle_{\partial\Gamma^h(t)} \quad \forall\ \vec\eta \in \{\vec\xi \in [W_T(\mathcal{G}^h_T)]^3 : \partial_t^{\circ,h}\,\vec\xi = \vec 0\}\,, \tag{3.15}$$

which follows similarly to (2.28). Here, similarly to (2.29), we have defined

$$\underline{\underline{\mathcal{P}}}^h_{\partial\Gamma} = \underline{\underline{\mathrm{Id}}} - \vec{\mathrm{id}}_s \otimes \vec{\mathrm{id}}_s \quad \text{on} \quad \partial\Gamma^h(t)\,. \tag{3.16}$$

For later use, we introduce the vertex normal function $\vec\omega^h(\cdot,t) \in \underline{V}^h(\Gamma^h(t))$ with

$$\vec\omega^h(\vec q^h_k(t),t) := \frac{1}{\mathcal{H}^2(\Lambda^h_k(t))} \sum_{j\in\Theta^h_k} \mathcal{H}^2(\sigma^h_j(t))\,\vec\nu^h\,|_{\sigma^h_j(t)}\,,$$

where for $k = 1,\ldots,K$ we define $\Theta^h_k := \{j : \vec q^h_k(t) \in \overline{\sigma^h_j(t)}\}$ and set $\Lambda^h_k(t) := \cup_{j\in\Theta^h_k}\overline{\sigma^h_j(t)}$. Here we note that

$$\left\langle \vec z, w\,\vec\nu^h \right\rangle^h_{\Gamma^h(t)} = \left\langle \vec z, w\,\vec\omega^h \right\rangle^h_{\Gamma^h(t)} \qquad \forall\ \vec z \in \underline{V}^h(\Gamma^h(t))\,,\ w \in W^h(\Gamma^h(t))\,. \tag{3.17}$$

In addition, for a given parameter $\theta \in [0,1]$ we introduce $\theta^h \in W^h(\Gamma^h(t))$ such that

$$\theta^h(\vec q^h_k(t),t) = \begin{cases} 1 & \vec q^h_k(t) \in \partial\Gamma^h(t)\,, \\ \theta & \vec q^h_k(t) \notin \partial\Gamma^h(t)\,, \end{cases} \quad \text{for clamped boundary conditions,}$$
$$\theta^h(\vec q^h_k(t),t) = \begin{cases} 0 & \vec q^h_k(t) \in \partial\Gamma^h(t)\,, \\ \theta & \vec q^h_k(t) \notin \partial\Gamma^h(t)\,, \end{cases} \quad \text{for all other boundary conditions.} \tag{3.18}$$

Then we introduce $\underline{\underline{Q}}^h_{\theta^h} \in [W^h(\Gamma^h(t))]^{3\times 3}$ by setting, for $k \in \{1,\ldots,K\}$,

$$\underline{\underline{Q}}^h_{\theta^h}(\vec q^h_k(t),t) = \theta^h(\vec q^h_k(t),t)\,\underline{\underline{\mathrm{Id}}} + (1-\theta^h(\vec q^h_k(t),t))\,\frac{\vec\omega^h(\vec q^h_k(t),t)\otimes\vec\omega^h(\vec q^h_k(t),t)}{|\vec\omega^h(\vec q^h_k(t),t)|^2}\,, \tag{3.19}$$

where here and throughout we assume that $\vec\omega^h(\vec q^h_k(t),t) \neq \vec 0$ for $k = 1,\ldots,K$ and $t \in [0,T]$. Only in pathological cases could this assumption be violated, and in practice this never occurred. We note that

$$\left\langle \underline{\underline{Q}}^h_{\theta^h}\,\vec z, \vec v \right\rangle^h_{\Gamma^h(t)} = \left\langle \vec z, \underline{\underline{Q}}^h_{\theta^h}\,\vec v \right\rangle^h_{\Gamma^h(t)} \quad \text{and} \quad \left\langle \underline{\underline{Q}}^h_{\theta^h}\,\vec z, \vec\omega^h \right\rangle^h_{\Gamma^h(t)} = \left\langle \vec z, \vec\omega^h \right\rangle^h_{\Gamma^h(t)} \tag{3.20}$$

for all $\vec{z}, \vec{v} \in \underline{V}^h(\Gamma^h(t))$.

Moreover, in the case of clamped boundary conditions, we let $\vec{\zeta}^h(t) \in \underline{V}^h(\partial\Gamma^h(0))$ be a suitable approximation of $\vec{\zeta}(t)$ on $\partial\Gamma(0)$. On recalling from the introduction that at present $\beta > 0$ does not make sense from a modelling point of view for open surfaces, we set $\beta = 0$ from now on for simplicity. Mathematically the case $\beta > 0$ may be considered, and the resulting terms can then treated as in the closed surface case, see Barrett et al. (2016) for details. Similarly to the continuous setting, recall (1.11), (2.24), (2.25), we consider the first variation of the discrete energy

$$E^h(\Gamma^h(t)) := \tfrac{1}{2}\left\langle|\vec{\kappa}^h - \overline{\varkappa}\,\vec{\nu}^h|^2, 1\right\rangle^h_{\Gamma^h(t)} + \alpha_G\left[\left\langle\vec{\kappa}^h_{\partial\Gamma}, \vec{\mathrm{m}}^h\right\rangle^h_{\partial\Gamma^h(t)} + 2\,\pi\, m(\Gamma^h(t))\right] + \gamma\,\mathcal{H}^1(\partial\Gamma^h(t)) \tag{3.21}$$

subject to the side constraints

$$\left\langle\underline{\underline{Q}}^h_{\theta^h}\,\vec{\kappa}^h, \vec{\eta}\right\rangle^h_{\Gamma^h(t)} + \left\langle\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\eta}\right\rangle_{\Gamma^h(t)} = \left\langle\vec{\mathrm{m}}^h, \vec{\eta}\right\rangle^h_{\partial\Gamma^h(t)} \qquad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma^h(t)) \tag{3.22}$$

and

$$\left\langle\vec{\kappa}^h_{\partial\Gamma}, \vec{\chi}\right\rangle^h_{\partial\Gamma^h(t)} + \left\langle\vec{\mathrm{id}}_s, \vec{\chi}_s\right\rangle_{\partial\Gamma^h(t)} = 0 \qquad \forall\ \vec{\chi} \in \underline{V}^h(\partial\Gamma^h(t))\,. \tag{3.23}$$

Of course, for clamped boundary conditions we set $\vec{\mathrm{m}}^h = \vec{\zeta}^h$, whereas for the other three boundary conditions $\vec{\mathrm{m}}^h(t) \in \underline{V}^h(\partial\Gamma^h(t))$ is an unknown. When taking variations of (3.22), we need to compute variations of the discrete vertex normal $\vec{\omega}^h$. To this end, for any given $\vec{\chi} \in \mathbb{X}(\Gamma^h(t))$ we introduce $\Gamma^h_\varepsilon(t)$ as in (2.5) and $\partial^{0,h}_\varepsilon$ defined by (2.9), both with $\Gamma(t)$ replaced by $\Gamma^h(t)$. We then observe that it follows from (3.17) with $w = 1$ and the discrete analogue of (2.10) that

$$\left\langle\vec{z}, \partial^{0,h}_\varepsilon\,\vec{\omega}^h\right\rangle^h_{\Gamma^h(t)} = \left\langle\vec{z}, \partial^{0,h}_\varepsilon\,\vec{\nu}^h\right\rangle^h_{\Gamma^h(t)} + \left\langle(\vec{z}\,.\,(\vec{\nu}^h - \vec{\omega}^h))\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\chi}\right\rangle^h_{\Gamma^h(t)} \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma^h(t)),\ \vec{\chi} \in \mathbb{X}(\Gamma^h(t))\,. \tag{3.24}$$

An immediate consequence is that

$$\left\langle\vec{z}, \partial^{\circ,h}_t\,\vec{\omega}^h\right\rangle^h_{\Gamma^h(t)} = \left\langle\vec{z}, \partial^{\circ,h}_t\,\vec{\nu}^h\right\rangle^h_{\Gamma^h(t)} + \left\langle(\vec{z}\,.\,(\vec{\nu}^h - \vec{\omega}^h))\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\mathcal{V}}^h\right\rangle^h_{\Gamma^h(t)} \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma^h(t))\,. \tag{3.25}$$

In addition, we note that for all $\vec{\xi},\ \vec{\eta} \in \underline{V}^h(\Gamma^h(t))$ with $\partial^{0,h}_\varepsilon\,\vec{\xi} = \partial^{0,h}_\varepsilon\,\vec{\eta} = \vec{0}$ it holds that

$$\partial^{0,h}_\varepsilon\,\pi^h\left[\left(\vec{\xi}\,.\,\frac{\vec{\omega}^h}{|\vec{\omega}^h|}\right)\left(\vec{\eta}\,.\,\frac{\vec{\omega}^h}{|\vec{\omega}^h|}\right)\right] = \pi^h\left[\vec{G}^h(\vec{\xi},\vec{\eta})\,.\,\partial^{0,h}_\varepsilon\,\vec{\omega}^h\right] \quad \text{on } \Gamma^h(t)\,, \tag{3.26}$$

where

$$\vec{G}^h(\vec{\xi},\vec{\eta}) = \vec{\pi}^h\left[\frac{1}{|\vec{\omega}^h|^2}\left((\vec{\xi}\,.\,\vec{\omega}^h)\,\vec{\eta} + (\vec{\eta}\,.\,\vec{\omega}^h)\,\vec{\xi} - 2\,\frac{(\vec{\eta}\,.\,\vec{\omega}^h)\,(\vec{\xi}\,.\,\vec{\omega}^h)}{|\vec{\omega}^h|^2}\,\vec{\omega}^h\right)\right]. \tag{3.27}$$

It follows that

$$\vec{G}^h(\vec{\xi},\vec{\eta})\,.\,\vec{\omega}^h = 0 \quad \forall\ \vec{\xi},\ \vec{\eta} \in \underline{V}^h(\Gamma^h(t))\,. \tag{3.28}$$

Considering at first all the boundary conditions that do not involve fixing the conormal, i.e. free, semi-free and Navier, we have the discrete analogue of (2.32) and define the Lagrangian

$$\begin{aligned} L^h(\Gamma^h(t), \vec\kappa^h, \vec{\mathrm{m}}^h, \vec\kappa^h_{\partial\Gamma}, \vec Y^h, \vec Z^h) &= \tfrac12 \left\langle |\vec\kappa^h - \overline{\varkappa}\,\vec\nu^h|^2, 1\right\rangle^h_{\Gamma^h(t)} + \gamma\,\mathcal{H}^1(\partial\Gamma^h(t)) \\ &\quad + \alpha_G \left[\left\langle \vec\kappa^h_{\partial\Gamma}, \vec{\mathrm{m}}^h \right\rangle^h_{\partial\Gamma^h(t)} - \left\langle \vec\kappa^h_{\partial\Gamma}, \vec Z^h \right\rangle^h_{\partial\Gamma^h(t)} - \left\langle \vec{\mathrm{id}}_s, \vec Z_s \right\rangle_{\partial\Gamma^h(t)} \right] \\ &\quad - \left\langle \underline{\underline{Q}}^h_{\theta^h}\, \vec\kappa^h, \vec Y^h \right\rangle^h_{\Gamma^h(t)} - \left\langle \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec Y^h \right\rangle_{\Gamma^h(t)} + \left\langle \vec{\mathrm{m}}^h, \vec Y^h \right\rangle^h_{\partial\Gamma^h(t)}, \end{aligned} \tag{3.29}$$

where $\vec\kappa^h \in \underline{V}^h(\Gamma^h(t))$ and $\vec\kappa^h_{\partial\Gamma} \in \underline{V}^h_{\partial\Gamma}(\Gamma^h(t))$ satisfy (3.22) and (3.23), respectively, with $\vec Y^h \in \underline{V}^h(\Gamma^h(t))$ and $\vec Z^h \in \underline{V}^h_{\partial\Gamma}(\Gamma^h(t))$ being the corresponding Lagrange multipliers. Similarly to (2.35a–c) with (2.24), (2.25), on recalling the formal calculus of PDE constrained optimization, we obtain an L^2–gradient flow of $E^h(\Gamma^h(t))$ subject to the side constraint (3.22) by setting $[\frac{\delta}{\delta\Gamma^h} L^h](\vec\chi) = -\left\langle \underline{\underline{Q}}^{h,\star}_\theta \vec{\mathcal{V}}^h, \vec\chi \right\rangle^h_{\Gamma^h(t)}$ for $\vec\chi \in \mathbb{X}(\Gamma^h(t))$, $[\frac{\delta}{\delta\vec\kappa^h} L^h](\vec\xi) = 0$ for $\vec\xi \in \underline{V}^h(\Gamma^h(t))$, $[\frac{\delta}{\delta\vec Y^h} L^h](\vec\eta) = 0$ for $\vec\eta \in \underline{V}^h(\Gamma^h(t))$, $[\frac{\delta}{\delta\vec{\mathrm{m}}^h} L^h](\vec\varphi) = 0$ for $\vec\varphi \in \underline{V}^h(\partial\Gamma^h(t))$, $[\frac{\delta}{\delta\vec\kappa^h_{\partial\Gamma}} L^h](\vec\phi) = 0$ for $\vec\phi \in \underline{V}^h(\partial\Gamma^h(t))$, yielding $\vec Z^h = \vec{\mathrm{m}}^h$, and $[\frac{\delta}{\delta\vec Z^h} L^h](\vec\phi) = 0$ for $\vec\phi \in \underline{V}^h(\partial\Gamma^h(t))$. Here we have defined

$$\underline{\underline{Q}}^{h,\star}_\theta(\vec q^h_k(t), t) = \begin{cases} \underline{\underline{\mathrm{Id}}} & \vec q^h_k(t) \in \partial\Gamma^h(t)\,, \\ \underline{\underline{Q}}^h_{\theta^h} & \vec q^h_k(t) \notin \partial\Gamma^h(t)\,. \end{cases} \tag{3.30}$$

Here we consider $[\frac{\delta}{\delta\Gamma^h} L^h](\vec\chi) = -\left\langle \underline{\underline{Q}}^{h,\star}_\theta \vec{\mathcal{V}}^h, \vec\chi \right\rangle^h_{\Gamma^h(t)}$ in place of $[\frac{\delta}{\delta\Gamma^h} L^h](\vec\chi) = -\left\langle \vec{\mathcal{V}}^h, \vec\chi \right\rangle^h_{\Gamma^h(t)}$ in order to allow implicit tangential motion of vertices. In particular, we will show in Theorem 3.1, below, that for $\theta = 0$ good meshes are enforced via the equation (3.31d). But these meshes can only be realized, if the motion of the vertices is not constrained to be in normal direction only. On the other hand, we must not allow an implicit tangential motion at the boundary nodes of $\partial\Gamma^h(t)$, as we wish to reparameterize $\Gamma^h(t)$ and not change the shape of $\Gamma^h(t)$ via this tangential motion. Hence, for the boundary nodes we replace θ with 1 in the definition (3.30).

Overall this gives rise to the following semidiscrete finite element approximation, where we note that $\partial^{0,h}_\varepsilon\,\theta^h = 0$. Given $\Gamma^h(0)$, for all $t \in (0, T]$ find $\Gamma^h(t)$, with $\vec{\mathcal{V}}^h \in \mathbb{X}(\Gamma^h(t))$, $\vec\kappa^h(t) \in \underline{V}^h(\Gamma^h(t))$, $\vec Y^h(t) \in \underline{V}^h(\Gamma^h(t))$, $\vec\kappa^h_{\partial\Gamma}(t) \in \underline{V}^h(\partial\Gamma^h(t))$ and $\vec{\mathrm{m}}^h(t) \in \underline{V}^h(\partial\Gamma^h(t))$ such that

$$\left\langle \underline{\underline{Q}}_\theta^{h,\star}\,\vec{\mathcal{V}}^h, \vec\chi \right\rangle_{\Gamma^h(t)}^h - \left\langle \nabla_s\,\vec Y^h, \nabla_s\,\vec\chi \right\rangle_{\Gamma^h(t)} - \left\langle \nabla_s\,.\,\vec Y^h, \nabla_s\,.\,\vec\chi \right\rangle_{\Gamma^h(t)} + \gamma \left\langle \vec{\rm id}_s, \vec\chi_s \right\rangle_{\partial\Gamma^h(t)}$$
$$+ \left\langle (\nabla_s\,\vec Y^h)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma^h(t)} + \tfrac12 \left\langle \left[|\vec\kappa^h - \varkappa\,\vec\nu^h|^2 - 2\,\vec Y^h\,.\,\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h\right] \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi \right\rangle_{\Gamma^h(t)}^h$$
$$+ \varkappa \left\langle \vec\kappa^h, [\nabla_s\,\vec\chi]^T\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h - \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h, \vec\kappa^h)\,.\,\vec\nu^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi \right\rangle_{\Gamma^h(t)}^h$$
$$+ \left\langle (1-\theta^h)\,\vec G^h(\vec Y^h, \vec\kappa^h), [\nabla_s\,\vec\chi]^T\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h$$
$$- \alpha_G \left[\left\langle \vec\kappa_{\partial\Gamma}^h\,.\,\vec{\rm m}^h, \vec{\rm id}_s\,.\,\vec\chi_s \right\rangle_{\partial\Gamma^h(t)}^h + \left\langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}^h\,\vec{\rm m}_s^h, \vec\chi_s \right\rangle_{\partial\Gamma^h(t)} \right] = 0 \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma^h(t))\,, \tag{3.31a}$$
$$\left\langle \vec\kappa^h - \varkappa\,\vec\nu^h - \underline{\underline{Q}}_{\theta^h}^h\,\vec Y^h, \vec\xi \right\rangle_{\Gamma^h(t)}^h = 0 \qquad \forall\ \vec\xi \in \underline{V}^h(\Gamma^h(t))\,, \tag{3.31b}$$
$$\left\langle \alpha_G\,\vec\kappa_{\partial\Gamma}^h + \vec Y^h, \vec\varphi \right\rangle_{\partial\Gamma^h(t)}^h = 0 \qquad \forall\ \vec\varphi \in \underline{V}^h(\partial\Gamma^h(t))\,, \tag{3.31c}$$
$$\left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h, \vec\eta \right\rangle_{\Gamma^h(t)}^h + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta \right\rangle_{\Gamma^h(t)} = \left\langle \vec{\rm m}^h, \vec\eta \right\rangle_{\partial\Gamma^h(t)}^h \qquad \forall\ \vec\eta \in \underline{V}^h(\Gamma^h(t))\,, \tag{3.31d}$$
$$\left\langle \vec\kappa_{\partial\Gamma}^h, \vec\phi \right\rangle_{\partial\Gamma^h(t)}^h + \left\langle \vec{\rm id}_s, \vec\phi_s \right\rangle_{\partial\Gamma^h(t)} = 0 \qquad \forall\ \vec\phi \in \underline{V}^h(\partial\Gamma^h(t))\,, \tag{3.31e}$$

where $\vec G^h(\vec Y^h, \vec\kappa^h) \in \underline{V}^h(\Gamma^h(t))$ is defined as in (3.27). Of course, in the case of fixed boundary conditions we have $\mathbb{X}(\Gamma^h(t)) = \underline{V}_0^h(\Gamma^h(t))$, and so the terms involving $\partial\Gamma^h(t)$ in (3.31a) drop out. In addition, for fixed boundary conditions (3.31e) is invariant in time. Moreover, in the case of clamped boundary conditions, (1.17), $\vec{\rm m}^h(t) = \vec\zeta^h(t)$ on $\partial\Gamma^h(0)$ is fixed, and so the Lagrangian (3.29) simplifies. The semidiscrete finite element approximation is then given by (3.31a,b,d), with $\vec{\rm m}^h$ in (3.31d) replaced by $\vec\zeta^h$. For later use we also observe that combining (3.31b) and (3.31d), on recalling (3.20), yields that

$$\left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec Y^h, \underline{\underline{Q}}_{\theta^h}^h\,\vec\eta \right\rangle_{\Gamma^h(t)}^h + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta \right\rangle_{\Gamma^h(t)} = \left\langle \vec{\rm m}^h, \vec\eta \right\rangle_{\partial\Gamma^h(t)}^h - \varkappa \left\langle \vec\omega^h, \vec\eta \right\rangle_{\Gamma^h(t)}^h \qquad \forall\ \vec\eta \in \underline{V}^h(\Gamma^h(t))\,. \tag{3.32}$$

In deriving (3.31a–d) from the six variations of L^h mentioned above, we have made use of the obvious discrete variants of (2.7)–(2.12), (2.26), (2.28) and recalled (3.24), (3.26) and (3.28), which requires (3.16). We note that (3.31b,c) and (3.17) imply that

$$\vec\pi^h\,[\underline{\underline{Q}}_{\theta^h}^h\,\vec Y^h] = \vec\kappa^h - \varkappa\,\vec\omega^h \quad \text{and} \quad \vec Y^h\,|_{\partial\Gamma^h(t)} = -\alpha_G\,\vec\kappa_{\partial\Gamma}^h\,, \tag{3.33}$$

which is the discrete analogue of (2.37).

In order to be able to consider surface area conserving variants of (3.31a–d), we introduce a Lagrange multiplier $\lambda^h(t) \in \mathbb{R}$ for the constraint

$$\frac{\rm d}{{\rm d}t}\,\mathcal{H}^2(\Gamma^h(t)) = \left\langle \nabla_s\,.\,\vec{\mathcal{V}}^h, 1 \right\rangle_{\Gamma^h(t)} = \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)} = 0\,, \tag{3.34}$$

where we recall (2.4). Now, on writing (3.31a) as

$$\left\langle \underline{\underline{Q}}_\theta^{h,\star}\,\vec{\mathcal{V}}^h, \vec\chi \right\rangle_{\Gamma^h(t)}^h = \left\langle \nabla_s\,\vec{Y}^h, \nabla_s\,\vec\chi \right\rangle_{\Gamma^h(t)} + \left\langle \vec{f}^h, \vec\chi \right\rangle_{\Gamma^h(t)}^h \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma^h(t))\,,$$

we consider

$$\left\langle \underline{\underline{Q}}_\theta^{h,\star}\,\vec{\mathcal{V}}^h, \vec\chi \right\rangle_{\Gamma^h(t)}^h = \left\langle \nabla_s\,\vec{Y}^h, \nabla_s\,\vec\chi \right\rangle_{\Gamma^h(t)} + \left\langle \vec{f}^h, \vec\chi \right\rangle_{\Gamma^h(t)}^h - \lambda^h \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi \right\rangle_{\Gamma^h(t)} \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma^h(t))\,, \tag{3.35}$$

where

$$\lambda^h(t) = \Big(- \left\langle \nabla_s\,\vec{Y}^h, \nabla_s\,[\vec\Pi_0^h\,\vec\kappa^h] \right\rangle_{\Gamma^h(t)} - \left\langle \vec{f}^h, \vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h + \left\langle \vec\Pi_0^h\,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h + \left\langle \vec{\rm m}^h, \vec{\mathcal{V}}^h \right\rangle_{\partial\Gamma^h(t)}^h \Big) \Big/ \left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h, \vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h\,, \tag{3.36}$$

with $\vec\Pi_0^h : \underline{V}^h(\Gamma^h(t)) \to \underline{V}_0^h(\Gamma^h(t))$ being the projection onto $\underline{V}_0^h(\Gamma^h(t))$. Here we note that

$$\left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h, \vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h = \left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec\Pi_0^h\,\vec\kappa^h, \vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h \geq 0\,, \tag{3.37}$$

with strict inequality for $\theta \in (0,1]$ unless $\vec\Pi_0^h\,\vec\kappa^h = \vec 0$, and for $\theta = 0$ unless $\vec\kappa^h(\vec q_k^h(t),t)\,.\,\vec\omega^h(\vec q_k^h(t),t) = 0$ for all $\vec q_k^h(t) \in \Gamma^h(t) \setminus \partial\Gamma^h(t)$. In order to motivate (3.36) we note, on recalling (3.31d), (3.30) and (3.20) that

$$\left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)} = - \left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h, \vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h \tag{3.38a}$$

and

$$\begin{aligned} \left\langle \underline{\underline{Q}}_\theta^{h,\star}\,\vec{\mathcal{V}}^h, \vec\Pi_0^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h &= \left\langle \vec\Pi_0^h\,\vec{\mathcal{V}}^h, \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h \\ &= \left\langle \vec\Pi_0^h\,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h \right\rangle_{\Gamma^h(t)}^h + \left\langle \vec{\rm m}^h, \vec{\mathcal{V}}^h \right\rangle_{\partial\Gamma^h(t)}^h - \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}. \end{aligned} \tag{3.38b}$$

Hence (3.35) with $\vec\chi = \vec\Pi_0^h\,\vec\kappa^h$ and (3.36), (3.38a,b) yield that (3.34) is satisfied. Of course, in the case of fixed boundary conditions, the terms involving $\vec{\mathcal{V}}^h \in \underline{V}_0^h(\Gamma^h(t))$ in (3.36) drop out, and the last term on the right hand side of (3.35) can be equivalently written as $\lambda^h \left\langle \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h, \vec\chi \right\rangle_{\Gamma^h(t)}^h$, on noting (3.31d), since $\vec\chi \in \underline{V}_0^h(\Gamma^h(t))$.

The following theorem establishes that (3.31a–e) is indeed a weak formulation for the L^2–gradient flow of $E^h(\Gamma^h(t))$, recall (3.21), subject to the side constraints (3.22) and (3.23). We will also show that for $\theta = 0$ the scheme produces *conformal polyhedral surfaces*. Here we recall from Barrett et al. (2008, §4.1) that the surface $\Gamma^h(t)$ is a conformal polyhedral surfaces if

$$\left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta \right\rangle_{\Gamma^h(t)} = 0 \quad \forall\ \vec\eta \in \left\{ \vec\xi \in \underline{V}_0^h(\Gamma^h(t)) : \vec\xi(\vec q_k^h(t))\,.\,\vec\omega^h(\vec q_k^h(t),t) = 0, k = 1,\ldots,K \right\}. \tag{3.39}$$

Note that the definition in Barrett et al. (2008, §4.1) is for closed surfaces, and that it implies that $\vec{\omega}^h$ is parallel to the discrete Laplacian of $\vec{\mathrm{id}}\,|_{\Gamma^h(t)}$. Hence (3.39) is a natural generalization of that definition, since we only enforce this constraint at the interior nodes of $\Gamma^h(t)$. We recall from Barrett et al. (2008) that conformal polyhedral surfaces exhibit good meshes. In particular, coalescence of vertices cannot occur. Moreover, we recall that the two-dimensional analogue of conformal polyhedral surfaces are equidistributed polygonal curves, see Barrett et al. (2007, 2011). Now introducing the parameter $\theta \in [0, 1]$, we obtain a family of schemes that interpolate between the choices $\theta = 0$ and $\theta = 1$, with the latter meaning that all vertices are transported approximately only in the normal direction. This corresponds to the original approach in Dziuk (2008), and so the choice $\theta = 1$ can be interpreted as a natural generalization of the approximation in Dziuk (2008) to surfaces with boundary.

We now present a stability proof for the semidiscrete scheme (3.31a–e), where in the case of clamped boundary conditions we assume that $\vec{\zeta}^h \in C(\partial\Gamma^h(0), \mathbb{S}^{d-1})$ does not vary in time.

THEOREM. 3.1. *Let* $\theta \in [0, 1]$ *and let* $\{(\Gamma^h, \vec{\kappa}^h, \vec{\mathrm{m}}^h, \vec{\kappa}^h_{\partial\Gamma}, \vec{Y}^h)(t)\}_{t\in[0,T]}$ *be a solution to* (3.31a–e)*, where in the clamped case we fix* $\vec{\mathrm{m}}^h(t) = \vec{\zeta}^h$ *and do not require* (3.31c)*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E^h(\Gamma^h(t)) = -\left\langle \underline{\underline{Q}}^{h,\star}_{\theta}\, \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle^h_{\Gamma^h(t)} \leq 0\,. \tag{3.40}$$

Moreover, if $\theta = 0$ *then* $\Gamma^h(t)$ *is a conformal polyhedral surface for all* $t \in (0, T]$.

Proof. First we consider the cases where the boundary is not clamped. Taking the time derivative of (3.31d) with $\partial_t^{\circ,h}\, \vec{\eta} = \vec{0}$, yields that

$$\begin{aligned}
&\left\langle \partial_t^{\circ,h}\, (\underline{\underline{Q}}^h_{\theta^h}\, \vec{\kappa}^h), \vec{\eta} \right\rangle^h_{\Gamma^h(t)} + \left\langle (\underline{\underline{Q}}^h_{\theta^h}\, \vec{\kappa}^h \,.\, \vec{\eta})\, \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec{\mathcal{V}}^h \right\rangle^h_{\Gamma^h(t)} + \left\langle \nabla_s\, \vec{\mathcal{V}}^h, \nabla_s\, \vec{\eta} \right\rangle_{\Gamma^h(t)} \\
&\qquad + \left\langle \nabla_s \,.\, \vec{\mathcal{V}}^h, \nabla_s \,.\, \vec{\eta} \right\rangle_{\Gamma^h(t)} - \left\langle (\nabla_s\, \vec{\eta})^T, \underline{\underline{D}}(\vec{\mathcal{V}}^h)\, (\nabla_s\, \vec{\mathrm{id}})^T \right\rangle_{\Gamma^h(t)} \\
&= \left\langle \partial_t^{\circ,h}\, \vec{\mathrm{m}}^h, \vec{\eta} \right\rangle^h_{\partial\Gamma^h(t)} + \left\langle \vec{\mathrm{m}}^h \,.\, \vec{\eta}, \vec{\mathrm{id}}_s \,.\, \vec{\mathcal{V}}^h_s \right\rangle_{\partial\Gamma^h(t)}\,,
\end{aligned} \tag{3.41}$$

where we have noted (3.12), (3.13) and (3.14). Similarly, taking the time derivative of (3.31e) with $\partial_t^{\circ,h}\, \vec{\phi} = \vec{0}$ yields, on noting (3.13) and (3.15), that

$$\left\langle \partial_t^{\circ,h}\, \vec{\kappa}^h_{\partial\Gamma}, \vec{\phi} \right\rangle^h_{\partial\Gamma^h(t)} + \left\langle \vec{\kappa}^h_{\partial\Gamma} \,.\, \vec{\phi}, \vec{\mathrm{id}}_s \,.\, \vec{\mathcal{V}}^h_s \right\rangle^h_{\partial\Gamma^h(t)} + \left\langle \underline{\underline{\mathcal{P}}}^h_{\partial\Gamma}\, \vec{\phi}_s, \vec{\mathcal{V}}^h_s \right\rangle_{\partial\Gamma^h(t)} = 0\,. \tag{3.42}$$

Choosing $\vec{\chi} = \vec{\mathcal{V}}^h \in \mathbb{X}(\Gamma^h(t))$ in (3.31a), $\vec{\eta} = \vec{Y}^h \in \underline{V}^h(\Gamma^h(t))$ in (3.41) and combining yields, on noting the discrete variant of (2.11), that

$$\begin{aligned}
&\left\langle \underline{\underline{Q}}_{\theta}^{h,\star}\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h + \tfrac{1}{2}\left\langle [|\vec\kappa^h - \overline{\varkappa}\,\vec\nu^h|^2 - 2\,\vec Y^h \,.\, \underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h]\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h \\
&\quad - \overline{\varkappa}\left\langle \vec\kappa^h, \partial_t^{\circ,h}\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h), \vec Y^h \right\rangle_{\Gamma^h(t)}^h + \left\langle (\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h \,.\, \vec Y^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h \\
&\quad - \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h, \vec\kappa^h)\,.\,\vec\nu^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h \\
&\quad + \left\langle (1-\theta^h)\,\vec G^h(\vec Y^h, \vec\kappa^h), [\nabla_s\,\vec{\mathcal{V}}^h]^T\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h + \gamma\left\langle \vec{\rm id}_s, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)} \\
&\quad - \alpha_G\left[\left\langle \vec\kappa_{\partial\Gamma}^h \,.\, \vec{\rm m}^h, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)}^h + \left\langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}^h\,\vec{\rm m}_s^h, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)}\right] \\
&= \left\langle \partial_t^{\circ,h}\,\vec{\rm m}^h, \vec Y^h \right\rangle_{\partial\Gamma^h(t)}^h + \left\langle \vec{\rm m}^h \,.\, \vec Y^h, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)}^h . \qquad (3.43)
\end{aligned}$$

Choosing $\vec\phi = \vec{\rm m}^h$ in (3.42), it follows from (3.43), on recalling (3.17) and (3.33), that

$$\begin{aligned}
&\left\langle \underline{\underline{Q}}_{\theta}^{h,\star}\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h + \tfrac{1}{2}\left\langle |\vec\kappa^h - \overline{\varkappa}\,\vec\nu^h|^2\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h - \overline{\varkappa}\left\langle \vec\kappa^h, \partial_t^{\circ,h}\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h \\
&\quad + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h), \vec Y^h \right\rangle_{\Gamma^h(t)}^h - \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h, \vec\kappa^h)\,.\,\vec\nu^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h \\
&\quad + \left\langle (1-\theta^h)\,\vec G^h(\vec Y^h, \vec\kappa^h), [\nabla_s\,\vec{\mathcal{V}}^h]^T\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h + \gamma\left\langle \vec{\rm id}_s, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)} \\
&\quad = -\alpha_G\left[\left\langle \vec\kappa_{\partial\Gamma}^h, \partial_t^{\circ,h}\,\vec{\rm m}^h \right\rangle_{\partial\Gamma^h(t)}^h + \left\langle \partial_t^{\circ,h}\,\vec\kappa_{\partial\Gamma}^h, \vec{\rm m}^h \right\rangle_{\partial\Gamma^h(t)}^h + \left\langle \vec\kappa_{\partial\Gamma}^h \,.\, \vec{\rm m}^h, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)}^h\right] \\
&\quad = -\alpha_G \frac{\rm d}{{\rm d}t}\left\langle \vec\kappa_{\partial\Gamma}^h, \vec{\rm m}^h \right\rangle_{\partial\Gamma^h(t)}^h . \qquad (3.44)
\end{aligned}$$

We have from (3.20), (3.33) and (3.17) that

$$\begin{aligned}
&\left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h), \vec Y^h \right\rangle_{\Gamma^h(t)}^h - \overline{\varkappa}\left\langle \vec\kappa^h, \partial_t^{\circ,h}\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h \\
&\qquad = \left\langle \partial_t^{\circ,h}\,\vec\kappa^h, \underline{\underline{Q}}_{\theta^h}^h\,\vec Y^h \right\rangle_{\Gamma^h(t)}^h - \overline{\varkappa}\left\langle \vec\kappa^h - \overline{\varkappa}\,\vec\nu^h, \partial_t^{\circ,h}\,\vec\nu^h \right\rangle_{\Gamma^h(t)}^h \\
&\qquad\quad + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h) - \underline{\underline{Q}}_{\theta^h}^h\,\partial_t^{\circ,h}\,\vec\kappa^h, \vec Y^h \right\rangle_{\Gamma^h(t)}^h \\
&\qquad = \tfrac{1}{2}\left\langle \partial_t^{\circ,h}\,|\vec\kappa^h - \overline{\varkappa}\,\vec\nu^h|^2, 1 \right\rangle_{\Gamma^h(t)}^h + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{\theta^h}^h\,\vec\kappa^h) - \underline{\underline{Q}}_{\theta^h}^h\,\partial_t^{\circ,h}\,\vec\kappa^h, \vec Y^h \right\rangle_{\Gamma^h(t)}^h . \qquad (3.45)
\end{aligned}$$

Combining (3.44) and (3.45), on noting (3.13), (3.21), $\partial_t^{\circ,h}\,\theta^h = 0$ (which follows from (3.9) and (3.18)) and the invariance of $m(\Gamma^h(t))$ under continuous deformations, yields that

$$\left\langle \underline{\underline{Q}}_{\theta}^{h,\star}\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h + \frac{\rm d}{{\rm d}t}\,E^h(\Gamma^h(t)) + P = 0\,,$$

where

$$P := \left\langle (1-\theta^h)\,\vec\kappa^h\,.\,\partial_t^{\circ,h}\,\vec\omega^h, \frac{\vec Y^h\,.\,\vec\omega^h}{|\vec\omega^h|^2}\right\rangle^h_{\Gamma^h(t)} + \left\langle (1-\theta^h)\,\vec Y^h\,.\,\partial_t^{\circ,h}\,\vec\omega^h, \frac{\vec\kappa^h\,.\,\vec\omega^h}{|\vec\omega^h|^2}\right\rangle^h_{\Gamma^h(t)}$$
$$- 2\left\langle (1-\theta^h)\,(\vec\kappa^h\,.\,\vec\omega^h)\,(\vec Y^h\,.\,\vec\omega^h), \frac{\vec\omega^h\,.\,\partial_t^{\circ,h}\,\vec\omega^h}{|\vec\omega^h|^4}\right\rangle^h_{\Gamma^h(t)}$$
$$- \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h,\vec\kappa^h)\,.\,\vec\nu^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal V}^h\right\rangle^h_{\Gamma^h(t)}$$
$$+ \left\langle (1-\theta^h)\,\vec G^h(\vec Y^h,\vec\kappa^h), [\nabla_s\,\vec{\mathcal V}^h]^T\,\vec\nu^h\right\rangle^h_{\Gamma^h(t)}. \tag{3.46}$$

It remains to show that P as defined in (3.46) vanishes. To see this, we observe that it follows from (3.28), (3.27), the discrete variant of (2.11) and (3.25) that

$$P = \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h,\vec\kappa^h), \partial_t^{\circ,h}\,\vec\omega^h\right\rangle^h_{\Gamma^h(t)}$$
$$+ \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h,\vec\kappa^h)\,.\,(\vec\omega^h-\vec\nu^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal V}^h\right\rangle^h_{\Gamma^h(t)}$$
$$- \left\langle (1-\theta^h)\,(\vec G^h(\vec Y^h,\vec\kappa^h), \partial_t^{\circ,h}\,\vec\nu^h\right\rangle^h_{\Gamma^h(t)} = 0\,. \tag{3.47}$$

This proves the desired result (3.40) when the boundary is not clamped.

In the case of clamped boundary conditions we have that $\vec{\rm m}^h(t) = \vec\zeta$ and $\vec{\mathcal V}^h \in \underline{V}^h_0(\Gamma^h(t))$, and so the right hand side of (3.41) is zero, which means that we do not need (3.42). Hence the right hand side of (3.44) is zero, and so the desired result (3.40) follows for clamped boundary conditions.

If $\theta = 0$ then it immediately follows from (3.31d) that (3.39) holds. Hence $\Gamma^h(t)$ is a conformal polyhedral surface. ◻

REMARK. 3.1. *It is clear from the above proof that on replacing* $\langle \underline{\underline{Q}}^{h,\star}_\theta\,\vec{\mathcal V}^h, \vec\chi\rangle^h_{\Gamma^h(t)}$ *in* (3.31a) *with* $\langle \underline{\underline{Q}}^{h,\star}_\varsigma\,\vec{\mathcal V}^h, \vec\chi\rangle^h_{\Gamma^h(t)}$, *for* $\varsigma \in [0,1]$, *we obtain a slightly different family of schemes that are also stable. I.e. solutions to these schemes satisfy* $\frac{\rm d}{{\rm d}t}\,E^h(\Gamma^h(t)) = -\langle \underline{\underline{Q}}^{h,\star}_\varsigma\,\vec{\mathcal V}^h, \vec{\mathcal V}^h\rangle^h_{\Gamma^h(t)}$ *in place of* (3.40). *In view of the desired tangential motion for* $\theta = 0$, *it would be natural to choose* $\varsigma = 0$ *in this case, or at least to choose* $\varsigma \in [0,1)$, *in order to allow for nonzero tangential motion in* (3.31a). *In fact, in practice we observe that for* $\varsigma = 1$ *the corresponding fully discrete finite element approximation yields unsatisfactory results. Moreover, the proof of the following theorem demonstrates that in order to satisfy the conservation property* (3.34), *it is desirable to keep the left hand side of* (3.35) *as stated, i.e. to choose* $\varsigma = \theta$.

THEOREM. 3.2. *Let* $\theta \in [0,1]$ *and let* $\{(\Gamma^h, \vec\kappa^h, \vec{\rm m}, \vec\kappa^h_{\partial\Gamma}, \vec Y^h, \lambda^h)(t)\}_{t\in[0,T]}$ *be a solution to* (3.35), (3.31b–e) *and* (3.36), *where in the clamped case we fix* $\vec{\rm m}^h(t) = \vec\zeta^h$ *and do not* *require* (3.31c)*. Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E^h(\Gamma^h(t)) = -\left\langle \underline{\underline{Q}}^{h,\star}_{\theta}\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle^h_{\Gamma^h(t)} \leq 0, \tag{3.48}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{H}^2(\Gamma^h(t)) = 0\,. \tag{3.49}$$

Moreover, if $\theta = 0$ then $\Gamma^h(t)$ is a conformal polyhedral surface for all $t \in (0,T]$.

Proof. Choosing $\vec{\chi} = \vec{\Pi}^h_0\,\vec{\kappa}^h$ in (3.35) yields, on noting (3.36), (3.37) and (3.38a) that (3.34) holds, which yields the desired result (3.49). The stability result (3.48) directly follows from the proof of Theorem 3.1. In particular, choosing $\vec{\chi} = \vec{\mathcal{V}}^h$ in (3.35), on noting (3.34), yields that

$$\left\langle \underline{\underline{Q}}^{h,\star}_{\theta}\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle^h_{\Gamma^h(t)} = \left\langle \nabla_s\,\vec{Y}^h, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)} + \left\langle \vec{f}^h, \vec{\mathcal{V}}^h \right\rangle^h_{\Gamma^h(t)}.$$

Combining this with (3.41) yields that (3.43) holds, and the rest of the proof proceeds as that of Theorem 3.1. Finally, as in the proof of Theorem 3.1, for $\theta = 0$ it follows from (3.31d) that $\Gamma^h(t)$ is a conformal polyhedral surface. □

Remark. 3.2. *Similarly to Remark 3.3 in Barrett et al. (2016), we can also consider a natural alternative to the scheme* (3.31a–e)*, which does not use the normalization of the discrete vertex normal $\vec{\omega}^h$ as in* (3.19)*. In particular, on letting $\underline{\underline{\mathcal{Q}}}^h_{\theta^h} \in [W^h(\Gamma^h(t))]^{3\times 3}$ be defined by*

$$\underline{\underline{\mathcal{Q}}}^h_{\theta^h}(\vec{q}^h_k(t),t) = \theta^h(\vec{q}^h_k(t),t)\,\underline{\underline{\mathrm{Id}}} + (1-\theta^h(\vec{q}^h_k(t),t))\,\vec{\omega}^h(\vec{q}^h_k(t),t)\otimes\vec{\omega}^h(\vec{q}^h_k(t),t)$$

for all $k \in \{1,\ldots,K\}$, and on replacing $\underline{\underline{Q}}^h_{\theta^h}$ and $\underline{\underline{Q}}^{h,\star}_{\theta}$ in (3.31a–e) *by $\underline{\underline{\mathcal{Q}}}^h_{\theta^h}$ and $\underline{\underline{\mathcal{Q}}}^{h,\star}_{\theta}$, respectively, as well as adjusting the terms involving $(1-\theta^h)$ in* (3.31a)*, we obtain a new scheme that can be shown to satisfy all the properties of* (3.31a–e)*. In fact, in the case of closed surfaces this new scheme collapses to the scheme* (3.41a–c) *from Barrett et al. (2016) in the case $\beta = 0$. However, in the interest of consistency and continuity, we concentrate on the scheme* (3.31a–e) *in this paper, as we did in Barrett et al. (2016).*

4 Fully discrete finite element approximation

In this section we consider a fully discrete variant of the scheme (3.35), (3.31b–e) and (3.36) from Section 3. To this end, let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_m := t_{m+1} - t_m$, $m = 0,\ldots,M-1$. Let Γ^m be a polyhedral surface, approximating $\Gamma^h(t_m)$, $m = 0,\ldots,M$, with boundary $\partial\Gamma^m$. Following Dziuk (1991), we now parameterize the new surface Γ^{m+1} over Γ^m. Hence, we introduce the following finite element spaces. Let $\Gamma^m = \bigcup_{j=1}^J \overline{\sigma^m_j}$, where $\{\sigma^m_j\}_{j=1}^J$

is a family of mutually disjoint open triangles with vertices $\{\vec{q}^m_k\}_{k=1}^K$. Then for $m = 0, \ldots, M-1$, let

$$\underline{V}^h(\Gamma^m) := \{\vec{\chi} \in [C(\Gamma^m)]^3 : \vec{\chi}\,|_{\sigma^m_j} \text{ is linear } \forall\ j = 1, \ldots, J\} =: [W^h(\Gamma^m)]^3 \subset [H^1(\Gamma^m)]^3\,,$$

for $m = 0, \ldots, M-1$. We denote the standard basis of $W^h(\Gamma^m)$ by $\{\chi^m_k\}_{k=1}^K$. In addition, similarly to (3.2a,b), we also introduce $\underline{V}^h_0(\Gamma^m)$ and $\underline{V}^h(\partial\Gamma^m)$. We also introduce $\pi^m : C(\Gamma^m) \to W^h(\Gamma^m)$, the standard interpolation operator at the nodes $\{\vec{q}^m_k\}_{k=1}^K$, and similarly $\vec{\pi}^m : [C(\Gamma^m)]^3 \to \underline{V}^h(\Gamma^m)$. Throughout this paper, we will parameterize the new closed surface Γ^{m+1} over Γ^m, with the help of a parameterization $\vec{X}^{m+1} \in \underline{V}^h(\Gamma^m)$, i.e. $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$. Similarly to (3.5), let

$$\mathbb{X}(\Gamma^m) = \begin{cases} \underline{V}^h(\Gamma^m) & \text{free boundary conditions,} \\ \underline{V}^h_{\nabla F}(\Gamma^m) & \text{semi-free boundary conditions,} \\ \underline{V}^h_0(\Gamma^m) & \text{fixed boundary conditions,} \end{cases} \tag{4.1}$$

where

$$\underline{V}^h_{\nabla F}(\Gamma^m) := \{\vec{\chi} \in \underline{V}^h(\Gamma^m) : (\vec{\chi}\,.\,\nabla F)(\vec{q}^m_k) = 0 \quad \forall\ \vec{q}^m_k \in \partial\Gamma^m\}\,. \tag{4.2}$$

We also introduce the L^2–inner products $\langle\cdot,\cdot\rangle_{\Gamma^m}$ and $\langle\cdot,\cdot\rangle_{\partial\Gamma^m}$, as well as their mass lumped inner variants $\langle\cdot,\cdot\rangle^h_{\Gamma^m}$ and $\langle\cdot,\cdot\rangle^h_{\partial\Gamma^m}$. Similarly to (3.17), we note that

$$\langle \vec{z}, w\,\vec{\nu}^m\rangle^h_{\Gamma^m} = \langle \vec{z}, w\,\vec{\omega}^m\rangle^h_{\Gamma^m} \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma^m)\,,\ w \in W^h(\Gamma^m)\,,$$

where $\vec{\omega}^m := \sum_{k=1}^K \chi^m_k\,\vec{\omega}^m_k \in \underline{V}^h(\Gamma^m)$, and where for $k = 1, \ldots, K$ we let $\Theta^m_k := \{j : \vec{q}^m_k \in \overline{\sigma^m_j}\}$ and set $\Lambda^m_k := \cup_{j\in\Theta^m_k}\overline{\sigma^m_j}$ and $\vec{\omega}^m_k := \frac{1}{\mathcal{H}^2(\Lambda^m_k)} \sum_{j\in\Theta^m_k} \mathcal{H}^2(\sigma^m_j)\,\vec{\nu}^m_j$.

We make the following very mild assumption.

($\mathcal{A}$) We assume for $m = 0, \ldots, M-1$ that $\mathcal{H}^2(\sigma^m_j) > 0$ for all $j = 1, \ldots, J$ and that $\vec{0} \notin \{\vec{\omega}^m_k\}_{k=1}^K$, for all $m = 0, \ldots, M-1$.

In addition, and similarly to (3.18) and (3.19), we first introduce $\theta^m \in W^h(\Gamma^m)$, and then $\underline{\underline{Q}}^m_{\theta^m} \in [W^h(\Gamma^m)]^{3\times3}$ by setting $\underline{\underline{Q}}^m_{\theta^m}(\vec{q}^m_k) = \theta^m(\vec{q}^m_k)\,\vec{\mathrm{Id}} + (1 - \theta^m(\vec{q}^m_k))\,|\vec{\omega}^m_k|^{-2}\,\vec{\omega}^m_k \otimes \vec{\omega}^m_k$ for $k = 1, \ldots, K$. We also define $\underline{\underline{Q}}^{m,\star}_\theta \in [W^h(\Gamma^m)]^{3\times3}$ similarly to (3.30) in terms of $\underline{\underline{Q}}^m_{\theta^m}$. Similarly to (3.27) and (3.16), we let

$$\vec{G}^m(\vec{\xi},\vec{\eta}) = \vec{\pi}^m\left[\frac{1}{|\vec{\omega}^m|^2}\left((\vec{\xi}\,.\,\vec{\omega}^m)\,\vec{\eta} + (\vec{\eta}\,.\,\vec{\omega}^m)\,\vec{\xi} - 2\,\frac{(\vec{\eta}\,.\,\vec{\omega}^m)\,(\vec{\xi}\,.\,\vec{\omega}^m)}{|\vec{\omega}^m|^2}\,\vec{\omega}^m\right)\right] \tag{4.3}$$

and

$$\underline{\underline{\mathcal{P}}}^m_{\partial\Gamma} = \underline{\underline{\mathrm{Id}}} - \vec{\mathrm{id}}_s \otimes \vec{\mathrm{id}}_s \quad \text{on} \quad \partial\Gamma^m\,. \tag{4.4}$$

Given Γ^0 and $\vec{\kappa}^0$, $\vec{Y}^0 \in \underline{V}^h(\Gamma^0)$, $\vec{\mathrm{m}}^0 \in \underline{V}^h(\partial\Gamma^0)$, let $\vec{\kappa}^0_{\partial\Gamma} \in \underline{V}^h(\partial\Gamma^0)$ be such that

$$\left\langle \vec{\kappa}^0_{\partial\Gamma}, \vec{\eta}\right\rangle^h_{\partial\Gamma^0} + \left\langle \vec{\mathrm{id}}_s, \vec{\eta}_s\right\rangle_{\partial\Gamma^0} = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h(\partial\Gamma^0)\,. \tag{4.5}$$

On recalling (3.32) and (3.17), we consider the following fully discrete approximation of (3.35), (3.31b–e) and (3.36). For $m = 0, \ldots, M-1$, find $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \mathbb{X}(\Gamma^m) \times \underline{V}^h(\Gamma^m)$, with $\vec{X}^{m+1} = \vec{\mathrm{id}}\,|_{\Gamma^m} + \delta\vec{X}^{m+1}$, and $(\vec\kappa_{\partial\Gamma}^{m+1}, \vec{\mathrm{m}}^{m+1}) \in [\underline{V}^h(\partial\Gamma^m)]^2$ such that

$$
\begin{aligned}
&\left\langle \underline{\underline{Q}}_\theta^{m,\star}\, \frac{\vec{X}^{m+1} - \vec{\mathrm{id}}}{\tau_m}, \vec\chi \right\rangle_{\Gamma^m}^h - \left\langle \nabla_s\, \vec{Y}^{m+1}, \nabla_s\, \vec\chi \right\rangle_{\Gamma^m} + \gamma \left\langle \vec{X}_s^{m+1}, \vec\chi_s \right\rangle_{\partial\Gamma^m} \\
&\quad + \alpha_G \left\langle \vec{\mathrm{m}}_s^{m+1}, \vec\chi_s \right\rangle_{\partial\Gamma^m} = \left\langle \nabla_s \,.\, \vec{Y}^m, \nabla_s \,.\, \vec\chi \right\rangle_{\Gamma^m} \\
&\quad - \left\langle (\nabla_s\, \vec{Y}^m)^T, \underline{\underline{D}}(\vec\chi)\, (\nabla_s\, \vec{\mathrm{id}})^T \right\rangle_{\Gamma^m} - \overline{\varkappa} \left\langle \vec\kappa^m, [\nabla_s\, \vec\chi]^T\, \vec\nu^m \right\rangle_{\Gamma^m}^h \\
&\quad - \tfrac12 \left\langle \left[|\vec\kappa^m - \overline{\varkappa}\, \vec\nu^m|^2 - 2\, \vec{Y}^m \,.\, \underline{\underline{Q}}_{\theta^m}^m\, \vec\kappa^m \right] \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec\chi \right\rangle_{\Gamma^m}^h \\
&\quad + \left\langle (1-\theta^m)\, (\vec{G}^m(\vec{Y}^m, \vec\kappa^m) \,.\, \vec\nu^m)\, \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec\chi \right\rangle_{\Gamma^m}^h - \left\langle (1-\theta^m)\, \vec{G}^m(\vec{Y}^m, \vec\kappa^m), [\nabla_s\, \vec\chi]^T\, \vec\nu^m \right\rangle_{\Gamma^m}^h \\
&\quad + \alpha_G \left\langle \vec\kappa_{\partial\Gamma}^m \,.\, \vec{\mathrm{m}}^m, \vec{\mathrm{id}}_s \,.\, \vec\chi_s \right\rangle_{\partial\Gamma^m}^h + \alpha_G \left\langle (\underline{\underline{\mathrm{Id}}} + \underline{\underline{\mathcal{P}}}_{\partial\Gamma}^m)\, \vec{\mathrm{m}}_s^m, \vec\chi_s \right\rangle_{\partial\Gamma^m} \\
&\quad - \lambda^m \left\langle \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec\chi \right\rangle_{\Gamma^m} \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma^m)\,, \qquad\qquad (4.6\mathrm{a})
\end{aligned}
$$

$$
\left\langle \underline{\underline{Q}}_{\theta^m}^m\, \vec{Y}^{m+1}, \underline{\underline{Q}}_{\theta^m}^m\, \vec\eta \right\rangle_{\Gamma^m}^h + \left\langle \nabla_s\, \vec{X}^{m+1}, \nabla_s\, \vec\eta \right\rangle_{\Gamma^m} = \left\langle \vec{\mathrm{m}}^{m+1}, \vec\eta \right\rangle_{\partial\Gamma^m}^h - \overline{\varkappa} \left\langle \vec\omega^m, \vec\eta \right\rangle_{\Gamma^m}^h \qquad \forall\ \vec\eta \in \underline{V}^h(\Gamma^m)\,, \qquad (4.6\mathrm{b})
$$

$$
\left\langle \alpha_G\, \vec\kappa_{\partial\Gamma}^{m+1} + \vec{Y}^{m+1}, \vec\varphi \right\rangle_{\partial\Gamma^m}^h = 0 \qquad \forall\ \vec\varphi \in \underline{V}^h(\partial\Gamma^m)\,, \qquad (4.6\mathrm{c})
$$

$$
\left\langle \vec\kappa_{\partial\Gamma}^{m+1}, \vec\eta \right\rangle_{\partial\Gamma^m}^h + \left\langle \vec{X}_s^{m+1}, \vec\eta_s \right\rangle_{\partial\Gamma^m} = 0 \quad \forall\ \vec\eta \in \underline{V}^h(\partial\Gamma^m) \qquad (4.6\mathrm{d})
$$

and set $\vec\kappa^{m+1} = \vec\pi^m\, [\underline{\underline{Q}}_{\theta^m}^m\, \vec{Y}^{m+1}] + \overline{\varkappa}\, \vec\omega^m$ and $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$. We note that for $\alpha_G = 0$ the scheme simplifies, as we no longer need (4.6d). In addition, for clamped conditions, we replace $\vec{\mathrm{m}}^{m+1}$ in (4.6b) with $\vec\zeta^m$, an approximation of $\vec\zeta(t_m)$, and do not require (4.6c). Finally, for $m \geq 1$ we note that here and throughout, as no confusion can arise, we denote by $\vec\kappa^m$ the function $\vec z \in \underline{V}^h(\Gamma^m)$, defined by $\vec z(\vec q_k^m) = \vec\kappa^m(\vec q_k^{m-1})$, $k = 1 \to K$, where $\vec\kappa^m \in \underline{V}(\Gamma^{m-1})$ is given, and similarly for $\vec{Y}^m$, $\vec{\mathrm{m}}^m$ and $\vec\kappa_{\partial\Gamma}^m$.

Of course, (4.6a–d) with $\lambda^m = 0$ corresponds to a fully discrete approximation of (3.31a–c,e), (3.32). For a fully discrete approximation of surface area preserving Willmore flow, on recalling (3.36), we let

$$
\begin{aligned}
\lambda^m = \Bigg(&- \left\langle \nabla_s\, \vec{Y}^m, \nabla_s\, [\vec\Pi_0^m\, \vec\kappa^m] \right\rangle_{\Gamma^m} - \left\langle \vec f^m, \vec\Pi_0^m\, \vec\kappa^m \right\rangle_{\Gamma^m}^h \\
&+ \left\langle (\vec\Pi_0^m - \underline{\underline{\mathrm{Id}}})\, \frac{\vec{\mathrm{id}} - \vec{X}^{m-1}}{\tau_m}, \underline{\underline{Q}}_{\theta^m}^m\, \vec\kappa^m \right\rangle_{\Gamma^m}^h + \left\langle \vec{\mathrm{m}}^m, \frac{\vec{\mathrm{id}} - \vec{X}^{m-1}}{\tau_m} \right\rangle_{\partial\Gamma^m}^h \Bigg) \\
&\Big/ \left\langle \underline{\underline{Q}}_{\theta^m}^m\, \vec\kappa^m, \vec\Pi_0^m\, \vec\kappa^m \right\rangle_{\Gamma^m}^h, \qquad (4.7)
\end{aligned}
$$

where for convenience we have re-written (4.6a) as

$$\left\langle \underline{\underline{Q}}_{\theta}^{m,\star}\,\frac{\vec{X}^{m+1}-\vec{\rm id}}{\tau_m},\vec\chi\right\rangle_{\Gamma^m}^h - \left\langle \nabla_s\,\vec{Y}^{m+1},\nabla_s\,\vec\chi\right\rangle_{\Gamma^m} = \left\langle \vec{f}^m,\vec\chi\right\rangle_{\Gamma^m}^h - \lambda^m\left\langle \nabla_s\,\vec{\rm id},\nabla_s\,\vec\chi\right\rangle_{\Gamma^m} \quad \forall\,\vec\chi\in\mathbb{X}(\Gamma^m)\,, \tag{4.8}$$

and where $\vec\Pi_0^m : \underline{V}^h(\Gamma^m) \to \underline{V}_0^h(\Gamma^m)$ is the projection onto $\underline{V}_0^h(\Gamma^m)$. We also define $\vec{X}^{-1} = \vec{X}^0 = \vec{\rm id}\,|_{\Gamma^0}$. Similarly to (3.37), we note that the denominator in (4.7) is always nonzero for $\theta\in(0,1]$ unless $\vec\Pi_0^m\,\vec\kappa^m = \vec0$, and for $\theta = 0$ unless $\vec\kappa^m(\vec{q}_k^m)\,.\,\vec\omega^m(\vec{q}_k^m) = 0$ for all $\vec{q}_k^m \in \Gamma^m\setminus\partial\Gamma^0$.

4.1 Fixed cases

In the case of fixed boundary, the scheme (4.6a–d) simplifies dramatically. First of all, we note that the equation (4.6d) is not needed, since $\vec\kappa_{\partial\Gamma}^{m+1} = \vec\kappa_{\partial\Gamma}^0$ is fixed given data, recall (4.5), and the terms involving $\partial\Gamma^m$ in (4.6a) disappear. Moreover, in the case of Navier boundary conditions, it is also possible to eliminate the unknown $\vec{\rm m}^{m+1}$ from the finite element approximation (4.6a–d). Overall, for Navier boundary conditions we obtain: Given Γ^0 and $\vec\kappa^0$, $\vec{Y}^0\in\underline{V}^h(\Gamma^0)$, let $\vec\kappa_{\partial\Gamma}^0\in\underline{V}^h(\partial\Gamma^0)$ be defined by (4.5). Then, for $m = 0,\ldots,M-1$ find $(\delta\vec{X}^{m+1},\vec{Y}^{m+1})\in\underline{V}_0^h(\Gamma^m)\times\underline{V}^h(\Gamma^m)$, with $\vec{X}^{m+1} = \vec{\rm id}\,|_{\Gamma^m} + \delta\vec{X}^{m+1}$ and $\vec{Y}^{m+1}\,|_{\partial\Gamma^0} = -\alpha_G\,\vec\kappa_{\partial\Gamma}^0$, such that

$$\begin{aligned}
&\left\langle \underline{\underline{Q}}_{\theta}^{m,\star}\,\frac{\vec{X}^{m+1}-\vec{\rm id}}{\tau_m},\vec\chi\right\rangle_{\Gamma^m}^h - \left\langle \nabla_s\,\vec{Y}^{m+1},\nabla_s\,\vec\chi\right\rangle_{\Gamma^m} = \left\langle \nabla_s\,.\,\vec{Y}^m,\nabla_s\,.\,\vec\chi\right\rangle_{\Gamma^m}\\
&- \left\langle (\nabla_s\,\vec{Y}^m)^T,\underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T\right\rangle_{\Gamma^m} - \varkappa\left\langle\vec\kappa^m,[\nabla_s\,\vec\chi]^T\,\vec\nu^m\right\rangle_{\Gamma^m}^h\\
&- \tfrac12\left\langle\left[|\vec\kappa^m-\varkappa\,\vec\nu^m|^2 - 2\,\vec{Y}^m\,.\,\underline{\underline{Q}}_{\theta^m}^m\,\vec\kappa^m\right]\nabla_s\,\vec{\rm id},\nabla_s\,\vec\chi\right\rangle_{\Gamma^m}^h\\
&+\left\langle(1-\theta^m)\,(\vec{G}^m(\vec{Y}^m,\vec\kappa^m)\,.\,\vec\nu^m)\,\nabla_s\,\vec{\rm id},\nabla_s\,\vec\chi\right\rangle_{\Gamma^m}^h - \left\langle(1-\theta^m)\,\vec{G}^m(\vec{Y}^m,\vec\kappa^m),[\nabla_s\,\vec\chi]^T\,\vec\nu^m\right\rangle_{\Gamma^m}^h\\
&-\lambda^m\left\langle\nabla_s\,\vec{\rm id},\nabla_s\,\vec\chi\right\rangle_{\Gamma^m} \qquad \forall\,\vec\chi\in\underline{V}_0^h(\Gamma^m)\,,
\end{aligned} \tag{4.9a}$$

$$\left\langle \underline{\underline{Q}}_{\theta^m}^m\,\vec{Y}^{m+1},\underline{\underline{Q}}_{\theta^m}^m\,\vec\eta\right\rangle_{\Gamma^m}^h + \left\langle\nabla_s\,\vec{X}^{m+1},\nabla_s\,\vec\eta\right\rangle_{\Gamma^m} = -\varkappa\left\langle\vec\omega^m,\vec\eta\right\rangle_{\Gamma^m}^h \quad \forall\,\vec\eta\in\underline{V}_0^h(\Gamma^m)\,, \tag{4.9b}$$

and set $\vec\kappa^{m+1} = \vec\pi^m\,[\underline{\underline{Q}}_{\theta^m}^m\,\vec{Y}^{m+1}] + \varkappa\,\vec\omega^m$ and $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$.

For clamped boundary conditions, on the other hand, we let $\vec\zeta^m\in\underline{V}^h(\partial\Gamma^m)$ be an approximation to $\vec\zeta(t_m)\in[C(\partial\Gamma(t_m))]^3$, and then consider: Given Γ^0 and $\vec\kappa^0$, $\vec{Y}^0\in\underline{V}^h(\Gamma^0)$, for $m = 0,\ldots,M-1$ find $(\delta\vec{X}^{m+1},\vec{Y}^{m+1})\in\underline{V}_0^h(\Gamma^m)\times\underline{V}^h(\Gamma^m)$, with $\vec{X}^{m+1} = \vec{\rm id}\,|_{\Gamma^m} + \delta\vec{X}^{m+1}$, such that (4.9a) holds as well as

$$\left\langle \underline{\underline{Q}}_{\theta^m}^m\,\vec{Y}^{m+1},\underline{\underline{Q}}_{\theta^m}^m\,\vec\eta\right\rangle_{\Gamma^m}^h + \left\langle\nabla_s\,\vec{X}^{m+1},\nabla_s\,\vec\eta\right\rangle_{\Gamma^m} = \left\langle\vec\zeta^m,\vec\eta\right\rangle_{\partial\Gamma^0}^h - \varkappa\left\langle\vec\omega^m,\vec\eta\right\rangle_{\Gamma^m}^h \quad \forall\,\vec\eta\in\underline{V}^h(\Gamma^m)\,. \tag{4.10}$$

Then set $\vec\kappa^{m+1} = \vec\pi^m\,[\underline{\underline{Q}}_{\theta^m}^m\,\vec{Y}^{m+1}] + \varkappa\,\vec\omega^m$ and $\Gamma^{m+1} = \vec{X}^{m+1}(\Gamma^m)$ as before.

4.2 Implicit treatment of area conservation

In practice it can be advantageous to consider an implicit Lagrange multiplier λ^{m+1} in order to obtain a better discrete surface area conservation. In particular, we replace (4.8) with

$$\left\langle \underline{\underline{Q}}_{\theta}^{m,\star}\, \frac{\vec{X}^{m+1} - \vec{\mathrm{id}}}{\tau_m}, \vec{\chi} \right\rangle_{\Gamma^m}^{h} - \left\langle \nabla_s\, \vec{Y}^{m+1}, \nabla_s\, \vec{\chi} \right\rangle_{\Gamma^m} = \left\langle \vec{f}^m, \vec{\chi} \right\rangle_{\Gamma^m}^{h} - \lambda^{m+1} \left\langle \nabla_s\, \vec{X}^{m+1}, \nabla_s\, \vec{\chi} \right\rangle_{\Gamma^m} \quad \forall\ \vec{\chi} \in \mathbb{X}(\Gamma^m) \tag{4.11}$$

and require the coupled solution $(\vec{X}^{m+1}, \vec{Y}^{m+1}, \vec{\kappa}^{m+1}) \in [\underline{V}^h(\Gamma^m)]^3$ and $\lambda^{m+1} \in \mathbb{R}$ to satisfy the nonlinear system (4.11), (4.6b–d) as well as an adapted variant of (4.7), where the superscript m is replaced by $m+1$ in all occurrences of $\vec{\kappa}^m$, $\vec{Y}^m$ and λ^m. In addition, $\frac{\vec{\mathrm{id}} - \vec{X}^{m-1}}{\tau_m}$ in (4.7) is replaced by $\frac{\vec{X}^{m+1} - \vec{\mathrm{id}}}{\tau_m}$. In practice this nonlinear system can be solved with a fixed point iteration as follows. Let $\lambda^{m+1,0} = \lambda^m$ and $\vec{\kappa}^{m+1,0} = \vec{\kappa}^m$. Then, for $i \geq 0$, find a solution $(\vec{X}^{m+1,i+1}, \vec{Y}^{m+1,i+1}, \vec{\kappa}_{\partial\Gamma}^{m+1,i+1}, \vec{\mathrm{m}}^{m+1,i+1}) \in \mathbb{X}(\Gamma^m) \times \underline{V}^h(\Gamma^m) \times [\underline{V}^h(\partial\Gamma^m)]^2$ to the linear system (4.11), (4.6b–d), where any superscript $m+1$ on left hand sides is replaced by $m+1, i+1$, and by $m+1, i$ on the right hand side of (4.11). Then let $\vec{\kappa}^{m+1,i+1} = \vec{\pi}^m\, [\underline{\underline{Q}}_{\theta^m}^m\, \vec{Y}^{m+1,i+1}] + \overline{\varkappa}\, \vec{\omega}^m$ be defined as usual, and compute $\lambda^{m+1,i+1}$ as the unique solution to

$$\lambda^{m+1,i+1} = \Bigg(- \left\langle \nabla_s\, \vec{Y}^{m+1,i+1}, \nabla_s\, [\vec{\Pi}_0^m\, \vec{\kappa}^{m+1,i+1}] \right\rangle_{\Gamma^m} - \left\langle \vec{f}^m, \vec{\Pi}_0^m\, \vec{\kappa}^{m+1,i+1} \right\rangle_{\Gamma^m}^{h} + \left\langle (\vec{\Pi}_0^m - \underline{\underline{\mathrm{Id}}})\, \frac{\vec{X}^{m+1,i+1} - \vec{\mathrm{id}}}{\tau_m}, \underline{\underline{Q}}_{\theta^m}^m\, \vec{\kappa}^{m+1,i+1} \right\rangle_{\Gamma^m}^{h} + \left\langle \vec{\mathrm{m}}^{m+1,i+1}, \frac{\vec{X}^{m+1,i+1} - \vec{\mathrm{id}}}{\tau_m} \right\rangle_{\partial\Gamma^m}^{h} \Bigg) \Big/ \left\langle \underline{\underline{Q}}_{\theta^m}^m\, \vec{\kappa}^{m+1,i+1}, \vec{\Pi}_0^m\, \vec{\kappa}^{m+1,i+1} \right\rangle_{\Gamma^m}^{h}$$

and continue the iteration until $|\lambda^{m+1,i+1} - \lambda^{m+1,i}| < 10^{-8}$. In practice this iteration always converged in fewer than ten steps, and at little extra computational cost compared to the linear scheme (4.6a–d), since the linear system (4.8), (4.6b–d) can be easily factorized with the help of sparse factorization packages such as UMFPACK, see Davis (2004).

4.3 Existence and uniqueness

THEOREM. 4.1. *Let the assumptions* ($\mathcal{A}$) *hold, and let* $\theta \in [0, 1]$. *Then there exists a unique solution* $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}, \vec{\kappa}_{\partial\Gamma}^{m+1}, \vec{\mathrm{m}}^{m+1}) \in \mathbb{X}(\Gamma^m) \times \underline{V}^h(\Gamma^m) \times [\underline{V}^h(\partial\Gamma^m)]^2$ *to* (4.6a–d) *in all the situations where the boundary* $\partial\Gamma^m$ *is not clamped. In the case of clamped boundary conditions, there exists a unique solution* $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}_0^h(\Gamma^m) \times \underline{V}^h(\Gamma^m)$ *to* (4.9a), (4.10).

Proof. We first consider the three situations where the surface is not clamped at the boundary. As this system is linear, existence follows from uniqueness. To investigate the latter, we consider the system: Find $(\vec{X}, \vec{Y}, \vec{\kappa}_{\partial\Gamma}, \vec{\mathrm{m}}) \in \mathbb{X}(\Gamma^m) \times \underline{V}^h(\Gamma^m) \times [\underline{V}^h(\partial\Gamma^m)]^2$ such that

$$\frac{1}{\tau_m} \left\langle \underline{\underline{Q}}_\theta^{m,\star} \vec{X}, \vec{\chi} \right\rangle_{\Gamma^m}^h - \left\langle \nabla_s \vec{Y}, \nabla_s \vec{\chi} \right\rangle_{\Gamma^m} + \gamma \left\langle \vec{X}_s, \vec{\chi}_s \right\rangle_{\partial\Gamma^m} + \alpha_G \left\langle \vec{\mathrm{m}}_s, \vec{\chi}_s \right\rangle_{\partial\Gamma^m} = 0 \qquad \forall\ \vec{\chi} \in \mathbb{X}(\Gamma^m)\,, \tag{4.12a}$$

$$\left\langle \underline{\underline{Q}}_{\theta^m}^m \vec{Y}, \underline{\underline{Q}}_{\theta^m}^m \vec{\eta} \right\rangle_{\Gamma^m}^h + \left\langle \nabla_s \vec{X}, \nabla_s \vec{\eta} \right\rangle_{\Gamma^m} - \langle \vec{\mathrm{m}}, \vec{\eta} \rangle_{\partial\Gamma^m}^h = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma^m)\,, \tag{4.12b}$$

$$\left\langle \alpha_G\, \vec{\kappa}_{\partial\Gamma} + \vec{Y}, \vec{\varphi} \right\rangle_{\partial\Gamma^m}^h = 0 \qquad \forall\ \vec{\varphi} \in \underline{V}^h(\partial\Gamma^m)\,, \tag{4.12c}$$

$$\langle \vec{\kappa}_{\partial\Gamma}, \vec{\eta} \rangle_{\partial\Gamma^m}^h + \left\langle \vec{X}_s, \vec{\eta}_s \right\rangle_{\partial\Gamma^m} = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h(\partial\Gamma^m)\,. \tag{4.12d}$$

Choosing $\vec{\chi} = \vec{X} \in \mathbb{X}(\Gamma^m)$ in (4.12a) and $\vec{\eta} = \vec{Y} \in \underline{V}^h(\Gamma^m)$ in (4.12b) yields that

$$\frac{1}{\tau_m} \left\langle \underline{\underline{Q}}_\theta^{m,\star} \vec{X}, \vec{X} \right\rangle_{\Gamma^m}^h + \gamma \left\langle \vec{X}_s, \vec{X}_s \right\rangle_{\partial\Gamma^m} + \left\langle Q_{\theta^m}^m \vec{Y}, Q_{\theta^m}^m \vec{Y} \right\rangle_{\Gamma^m}^h + \alpha_G \left\langle \vec{\mathrm{m}}_s, \vec{X}_s \right\rangle_{\partial\Gamma^m} - \left\langle \vec{\mathrm{m}}, \vec{Y} \right\rangle_{\partial\Gamma^m}^h = 0\,. \tag{4.13}$$

Combining (4.13) with choosing $\vec{\eta} = \alpha_G\, \vec{\mathrm{m}}$ in (4.12d) and $\vec{\varphi} = \vec{\mathrm{m}}$ in (4.12c) yields that

$$\frac{1}{\tau_m} \left\langle \underline{\underline{Q}}_\theta^{m,\star} \vec{X}, \vec{X} \right\rangle_{\Gamma^m}^h + \gamma \left\langle \vec{X}_s, \vec{X}_s \right\rangle_{\partial\Gamma^m} + \left\langle Q_{\theta^m}^m \vec{Y}, Q_{\theta^m}^m \vec{Y} \right\rangle_{\Gamma^m}^h = 0\,, \tag{4.14}$$

and hence $\vec{X} \in \underline{V}_0^h(\Gamma^m)$, on recalling the definition of $\underline{\underline{Q}}_\theta^{m,\star}$, i.e. the fully discrete version of (3.30). Together with (4.12d) we obtain that $\vec{\kappa}_{\partial\Gamma} = \vec{0}$, and so (4.12c) yields $\vec{Y} \in \underline{V}_0^h(\Gamma^m)$. It follows from $Q_{\theta^m}^m \vec{Y} = \vec{0}$ and (4.12b) with $\vec{\eta} = \vec{X}$ that $\langle \nabla_s \vec{X}, \nabla_s \vec{X} \rangle_{\Gamma^m} = 0$, and so $\vec{X} = 0$. Similarly, combining $\vec{X} = \vec{0}$ and $\vec{\mathrm{m}} = \vec{0}$ and (4.12a) with $\vec{\chi} = \vec{Y} \in \underline{V}_0^h(\Gamma^m) \subset \mathbb{X}(\Gamma^m)$ gives $\vec{Y} = 0$. Hence there exists a unique solution $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}, \vec{\kappa}_{\partial\Gamma}^{m+1}, \vec{\mathrm{m}}^{m+1}) \in \underline{V}_0^h(\Gamma^m) \times \underline{V}^h(\Gamma^m) \times [\underline{V}^h(\partial\Gamma^m)]^2$ to (4.6a–d).

The proof for clamped boundary conditions is analogous. In particular, we obtain first (4.12a,b), and then (4.14), without the boundary terms. It then follows from (4.14) that $\vec{X} = \vec{Y} = \vec{0}$, and so there exists a unique solution $(\delta\vec{X}^{m+1}, \vec{Y}^{m+1}) \in \underline{V}_0^h(\Gamma^m) \times \underline{V}^h(\Gamma^m)$ to (4.9a), (4.10). □

5 Solution of the algebraic equations

We recall that $\{\chi_k^m\}_{k=1}^K$ denotes the standard basis of $W^h(\Gamma^m)$. Similarly, let $\{\chi_{\partial,k}^m\}_{k=1}^{K_\partial}$ be the standard basis of $W^h(\partial\Gamma^m)$. Then, on recalling the rewrite of $\langle(\nabla_s\,\vec\eta)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T\rangle_{\Gamma(t)}$ in (2.12), which we now apply to Γ^m, we introduce the matrices $M, A, A_\gamma, A_\theta \in \mathbb{R}^{K\times K}$, $\vec M, \vec{\mathcal{M}}_{Q^\star}, \vec{\mathcal{M}}_{Q^2}, \vec A, \vec A_\gamma, \vec A_\theta, \vec{\mathcal{B}}, \vec{\mathcal{R}} \in (\mathbb{R}^{3\times3})^{K\times K}$, as well as $M_{\partial\Gamma}, A_{\partial\Gamma} \in \mathbb{R}^{K_\partial\times K_\partial}$, $\vec M_{\partial\Gamma}, \vec A_{\partial\Gamma} \in (\mathbb{R}^{3\times3})^{K_\partial\times K_\partial}$ and $M_{\partial\Gamma,\Gamma}, A_{\partial\Gamma,\Gamma} \in \mathbb{R}^{K\times K_\partial}$, $\vec M_{\partial\Gamma,\Gamma}, \vec A_{\partial\Gamma,\Gamma} \in (\mathbb{R}^{3\times3})^{K\times K_\partial}$, with entries

$$
\begin{aligned}
&M_{kl} := \langle \chi_l^m, \chi_k^m\rangle_{\Gamma^m}^h\,, \quad [\vec{\mathcal{M}}_{Q^\star}]_{kl} := \left\langle \chi_l^m, \chi_k^m\,\underline{\underline{Q}}_\theta^{m,\star}\right\rangle_{\Gamma^m}^h, \quad [\vec{\mathcal{M}}_{Q^2}]_{kl} := \left\langle \chi_l^m, \chi_k^m\,(\underline{\underline{Q}}_{\theta^m}^m)^2\right\rangle_{\Gamma^m}^h \\
&A_{kl} := \langle \nabla_s\,\chi_l^m, \nabla_s\,\chi_k^m\rangle_{\Gamma^m}\,, \quad [A_\gamma]_{kl} := \gamma\,\langle [\chi_l^m]_s, [\chi_k^m]_s\rangle_{\partial\Gamma^m}\,, \\
&[M_{\partial\Gamma}]_{kl} := \left\langle \chi_{\partial,l}^m, \chi_{\partial,k}^m\right\rangle_{\partial\Gamma^m}^h\,, \quad [A_{\partial\Gamma}]_{kl} := \left\langle [\chi_{\partial,l}^m]_s, [\chi_{\partial,k}^m]_s\right\rangle_{\partial\Gamma^m}\,, \\
&[M_{\partial\Gamma,\Gamma}]_{kl} := \left\langle \chi_{\partial,l}^m, \chi_k^m\right\rangle_{\partial\Gamma^m}^h\,, \quad [A_{\partial\Gamma,\Gamma}]_{kl} := \left\langle [\chi_{\partial,l}^m]_s, [\chi_k^m]_s\right\rangle_{\partial\Gamma^m}\,, \\
&\vec{\mathcal{B}}_{kl} := \left(\langle [\nabla_s]_j\,\chi_l^m, [\nabla_s]_i\,\chi_k^m\rangle_{\Gamma^m}\right)_{i,j=1}^3\,, \quad \vec{\mathcal{R}}_{kl} := \left\langle \nabla_s\,\chi_l^m\,.\,\nabla_s\,\chi_k^m, (\underline{\underline{\rm Id}} - \vec\nu^m\otimes\vec\nu^m)\right\rangle_{\Gamma^m}\,, \\
&[A_\theta]_{kl} := \tfrac12\left\langle \left[|\vec\kappa^m - \overline\varkappa\,\vec\nu^m|^2 - 2\,\vec Y^m\,.\,\underline{\underline{Q}}_{\theta^m}^m\,\vec\kappa^m\right]\nabla_s\,\chi_l^m, \nabla_s\,\chi_k^m\right\rangle_{\Gamma^m}^h \\
&\qquad\qquad - \left\langle (1-\theta^m)\,(\vec G^m(\vec Y^m,\vec\kappa^m)\,.\,\vec\nu^m)\,\nabla_s\,\chi_l^m, \nabla_s\,\chi_k^m\right\rangle_{\Gamma^m}^h\,,
\end{aligned}
$$

and $\vec M_{kl} := M_{kl}\,\underline{\underline{\rm Id}}$, $\vec A_{kl} := A_{kl}\,\underline{\underline{\rm Id}}$, $[\vec A_\gamma]_{kl} := [A_\gamma]_{kl}\,\underline{\underline{\rm Id}}$, $[\vec A_\theta]_{kl} := [A_\theta]_{kl}\,\underline{\underline{\rm Id}}$, $[\vec M_{\partial\Gamma}]_{kl} := [M_{\partial\Gamma}]_{kl}\,\underline{\underline{\rm Id}}$, $[\vec A_{\partial\Gamma}]_{kl} := [A_{\partial\Gamma}]_{kl}\,\underline{\underline{\rm Id}}$, $[\vec M_{\partial\Gamma,\Gamma}]_{kl} := [M_{\partial\Gamma,\Gamma}]_{kl}\,\underline{\underline{\rm Id}}$, $[\vec A_{\partial\Gamma,\Gamma}]_{kl} := [A_{\partial\Gamma,\Gamma}]_{kl}\,\underline{\underline{\rm Id}}$. It holds that $(\vec{\mathcal{B}}_{kl})^T = \vec{\mathcal{B}}_{lk} =: [\vec{\mathcal{B}}^\star]_{kl}$. In the above we have used the convention that the subscripts in the matrix notations refer to the test and trial domains, respectively. A single subscript is used where the two domains are the same, and if that single domain is Γ^m, then we omit the subscript completely. In addition, we define $\vec b_\theta \in (\mathbb{R}^3)^K$ and $\vec b_\alpha \in (\mathbb{R}^3)^K$ with entries

$$
\begin{aligned}
[\vec b_\theta]_k &= \left\langle \left[\overline\varkappa\,\vec\kappa^m + (1-\theta^m)\,\vec G^m(\vec Y^m,\vec\kappa^m)\right].\,\nabla_s\,\chi_k^m, \vec\nu^m\right\rangle_{\Gamma^m}^h\,, \\
[\vec b_\alpha]_k &= \alpha_G\left\langle (\vec\kappa_{\partial\Gamma}^m\,.\,\vec{\rm m}^m)\,\vec{\rm id}_s, [\chi_k^m]_s\right\rangle_{\partial\Gamma^m}^h + \alpha_G\left\langle (\underline{\underline{\rm Id}} + \underline{\underline{\mathcal{P}}}_{\partial\Gamma}^m)\,\vec{\rm m}_s^m, [\chi_k^m]_s\right\rangle_{\partial\Gamma^m}.
\end{aligned}
$$

Then the linear system (4.6a–d), in situations where the boundary $\partial\Gamma^m$ is not clamped, can be reformulated as follows. Find $(\vec Y^{m+1},\,\delta\vec X^{m+1}, \vec\kappa_{\partial\Gamma}^{m+1}, \vec{\rm m}^{m+1}) \in (\mathbb{R}^3)^{2K+2K_\partial}$ such that

$$
\begin{pmatrix}
\vec A & -\frac{1}{\tau_m}\,\vec{\mathcal{M}}_{Q^\star} - \vec A_\gamma & 0 & -\alpha_G\,\vec A_{\partial\Gamma,\Gamma} \\
\vec{\mathcal{M}}_{Q^2} & \vec A & 0 & -\vec M_{\partial\Gamma,\Gamma} \\
(\vec M_{\partial\Gamma,\Gamma})^T & 0 & \alpha_G\,\vec M_{\partial\Gamma} & 0 \\
0 & (\vec A_{\partial\Gamma,\Gamma})^T & \vec M_{\partial\Gamma} & 0
\end{pmatrix}
\begin{pmatrix}
\vec Y^{m+1} \\ \delta\vec X^{m+1} \\ \vec\kappa_{\partial\Gamma}^{m+1} \\ \vec{\rm m}^{m+1}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
[\vec{\mathcal{B}}^\star - \vec{\mathcal{B}} + \vec{\mathcal{R}}]\,\vec Y^m + (\vec A_\theta + \vec A_\gamma + \lambda^m\,\vec A)\,\vec X^m + \vec b_\theta - \vec b_\alpha \\
-\vec A\,\vec X^m - \overline\varkappa\,\vec M\,\vec\omega^m \\
\vec 0 \\
-(\vec A_{\partial\Gamma,\Gamma})^T\,\vec X^m
\end{pmatrix}
=: \begin{pmatrix} \vec g_X \\ \vec g_2 \\ \vec 0 \\ \vec g_3 \end{pmatrix}, \qquad (5.1)
$$

where, with the obvious abuse of notation, $\delta\vec{X}^{m+1} = (\delta\vec{X}_1^{m+1}, \ldots, \delta\vec{X}_K^{m+1})^T$, $\vec{Y}^{m+1} = (\vec{Y}_1^{m+1}, \ldots, \vec{Y}_K^{m+1})^T$ and $\vec{\kappa}_{\partial\Gamma}^{m+1} = ([\vec{\kappa}_{\partial\Gamma}^{m+1}]_1, \ldots, [\vec{\kappa}_{\partial\Gamma}^{m+1}]_{K_\partial})^T$, $\vec{\mathrm{m}}^{m+1} = (\vec{\mathrm{m}}_1^{m+1}, \ldots, \vec{\mathrm{m}}_{K_\partial}^{m+1})^T$, are the vectors of coefficients with respect to the standard bases for $\vec{X}^{m+1} - \vec{X}^m$, $\vec{Y}^{m+1}$ and $\vec{\kappa}_{\partial\Gamma}^{m+1}$, $\vec{\mathrm{m}}^{m+1}$, respectively.

In order to account for the desired boundary condition for $\delta\vec{X}^{m+1} \in \mathbb{X}(\Gamma^m)$, we replace any entries in $\vec{g}_X$, that correspond to the constraints $\delta\vec{X}^{m+1}(\vec{q}_k^m) = \vec{0}$ for Navier boundary conditions (and clamped boundary conditions, see below), or to the constraints $\delta\vec{X}^{m+1}(\vec{q}_k^m) \,.\, \vec{\mathrm{e}}_3 = 0$ for the semi-free conditions with $\Omega = \mathbb{R}^2 \times \mathbb{R}_{>0}$, with zero, and replace the corresponding rows in $\vec{A}$, $\vec{A}_\gamma$, $\vec{A}_{\partial\Gamma,\Gamma}$ with zero rows, and the corresponding rows in $-\frac{1}{\tau_m}\vec{\mathcal{M}}_{Q^\star}$ with rows from the identity matrix.

5.1 Fixed cases

In the case of a fixed boundary, the above linear system simplifies. In particular, the linear systems for (4.9a,b), in the Navier case, and for (4.9a), (4.10), in the clamped case, can be formulated with the help of the following auxiliary system: Find $(\vec{Y}^{m+1}, \delta\vec{X}^{m+1}) \in (\mathbb{R}^3)^{2K}$ such that

$$\begin{pmatrix} \vec{A} & -\frac{1}{\tau_m}\vec{\mathcal{M}}_{Q^\star} \\ \vec{\mathcal{M}}_{Q^2} & \vec{A} \end{pmatrix} \begin{pmatrix} \vec{Y}^{m+1} \\ \delta\vec{X}^{m+1} \end{pmatrix}$$
$$= \begin{pmatrix} [\vec{\mathcal{B}}^\star - \vec{\mathcal{B}} + \vec{\mathcal{R}}]\,\vec{Y}^m + (\vec{A}_\theta + \lambda^m\,\vec{A})\,\vec{X}^m + \vec{b}_\theta \\ -\vec{A}\,\vec{X}^m - \overline{\varkappa}\,\vec{M}\,\vec{\omega}^m + \vec{M}_{\partial\Gamma,\Gamma}\,\vec{\zeta}^m \end{pmatrix} =: \begin{pmatrix} \vec{g}_X \\ \vec{g}_Y \end{pmatrix}. \tag{5.2}$$

In the Navier case, we also account for the boundary condition $\vec{Y}^{m+1}\mid_{\partial\Gamma^0} = -\alpha_G\,\vec{\kappa}_{\partial\Gamma}^0$ by replacing the entries in $\vec{g}_Y$, that correspond to a boundary degree of freedom k, with $-\alpha_G\,\vec{\kappa}_{\partial\Gamma}^0(\vec{q}_k^m)$, and by replacing the corresponding rows in $\vec{\mathcal{M}}_{Q^2}$ and $\vec{A}$ with rows from the identity matrix and zero rows, respectively.

6 Numerical results

We implemented our fully discrete finite element approximations within the finite element toolbox ALBERTA, see Schmidt and Siebert (2005). The arising systems of linear equations were solved with the help of the sparse factorization package UMFPACK, see Davis (2004). For the computations involving surface area preserving Willmore flow, we always employ the implicit Lagrange multiplier formulation discussed in §4.2.

The fully discrete schemes (4.6a–d), as well as (4.9a,b) and (4.9a), (4.10), need initial data $\vec{\kappa}^0$, $\vec{Y}^0$, $\vec{\kappa}_{\partial\Gamma}^0$, $\vec{\mathrm{m}}^0$. Given the initial triangulation Γ^0, we let $\vec{\mathrm{m}}^0 \in \underline{V}^h(\partial\Gamma^0)$ be such that

$$\left\langle \vec{\mathrm{m}}^0, \vec{\eta} \right\rangle_{\partial\Gamma^0}^h = \left\langle \vec{\mu}^0, \vec{\eta} \right\rangle_{\partial\Gamma^0} \qquad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma^0)\,,$$

with $\vec{\mu}^0$ denoting the conormal on $\partial\Gamma^0$. In the case of clamped boundary conditions, we let $\vec{\rm m}^0 = \vec{\zeta}^0$. In addition, we let

$$\vec{\kappa}^0 = -\tfrac{2}{R}\,\vec{\omega}^0 \tag{6.1}$$

for simulations where $\Gamma(0)$ is part of the a sphere of radius R, i.e. $\Gamma(0) \subset \partial B_R(\vec{0})$, and otherwise define $\vec{\kappa}^0 \in \underline{V}^h(\Gamma^0)$ to be the solution of

$$\left\langle \vec{\kappa}^0, \vec{\eta} \right\rangle^h_{\Gamma^0} + \left\langle \nabla_s \,\vec{\rm id}, \nabla_s \,\vec{\eta} \right\rangle_{\Gamma^0} = \left\langle \vec{\rm m}^0, \vec{\eta} \right\rangle^h_{\partial\Gamma^0} \qquad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma^0)\,. \tag{6.2}$$

Then we define

$$\vec{Y}^0 = \vec{\kappa}^0 - \overline{\varkappa}\,\vec{\omega}^0\,. \tag{6.3}$$

Moreover, we let $\vec{\kappa}^0_{\partial\Gamma} \in \underline{V}^h(\partial\Gamma^0)$ be such that

$$\left\langle \vec{\kappa}^0_{\partial\Gamma}, \vec{\eta} \right\rangle^h_{\partial\Gamma^0} + \left\langle \vec{\rm id}_s, \vec{\eta}_s \right\rangle_{\partial\Gamma^0} = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h(\partial\Gamma^0)\,.$$

Often in our numerical experiments we will choose as initial data a segment of a sphere. In order to conveniently create a triangulation of such segments, we define the following lifting from the unit disc $\overline{B_1(\vec{0}) \cap \mathbb{R}^2 \times \{0\}}$ to $\Gamma(0) \subset \overline{\partial B_R(\vec{0})}$ as follows, where $R > 0$ is given. Let $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be given. Then $(x_1, x_2, 0)^T \in \overline{B_1(\vec{0})}$ is mapped to

$$R(\tfrac{\cos\vartheta}{r}\,x_1, \tfrac{\cos\vartheta}{r}\,x_2, \sin\vartheta)^T\,, \quad \text{where} \quad r = (x_1^2 + x_2^2)^{\frac12}\,,\ \vartheta = (\tfrac{\pi}{2} - \alpha)\,(1 - r) + \alpha\,. \tag{6.4}$$

Throughout this section we use uniform time steps $\tau_m = \tau$, $m = 0, \ldots, M-1$, and set $\tau = 10^{-3}$ unless stated otherwise. In addition, unless stated otherwise, we fix $\overline{\varkappa} = \alpha_G = \gamma = 0$. At times we will discuss the discrete energy of the numerical solutions, which, similarly to (3.21), is defined by

$$\begin{aligned} E^{m+1}(\Gamma^m) := {}& \tfrac12 \left\langle |\vec{\kappa}^{m+1} - \overline{\varkappa}\,\vec{\nu}^m|^2, 1 \right\rangle^h_{\Gamma^m} + \alpha_G \left[\left\langle \vec{\kappa}^{m+1}_{\partial\Gamma}, \vec{\rm m}^{m+1} \right\rangle^h_{\partial\Gamma^m} + 2\,\pi\, m(\Gamma^m) \right] \\ & + \gamma\, \mathcal{H}^1(\partial\Gamma^m)\,. \end{aligned}$$

6.1 Convergence experiments for fixed boundary

Here we consider the following exact solution from Deckelnick et al. (2015). Let

$$\Gamma(t) = \{(x_1, x_2, \varphi(x_1, x_2, t))^T : (x_1, x_2, 0)^T \in \overline{B_1(\vec{0})}\}\,, \tag{6.5}$$

where $\varphi(x_1, x_2, t) = (R^2(t) - x_1^2 - x_2^2)^{\frac12} - (R^2(t) - 1)^{\frac12}$ and $R(t) = 2 - 0.7\,\sin(2\,\pi\,t)$. We note that $\Gamma(t) \subset \partial B_{R(t)}(\vec{c}(t))$, $\vec{c}(t) = (0, 0, -(R^2(t) - 1)^{\frac12})^T$, and that $\partial\Gamma(t) = \partial\Gamma(0) = \mathbb{S}^1$. Moreover, it holds that $\varkappa = -\frac{2}{R(t)}$ on $\Gamma(t)$ and

$$\vec{\mu}(t) = \frac{(R^2(t) - 1)^{\frac12}}{R(t)}\,[\vec{\rm id} - (R^2(t) - 1)^{-\frac12}\,\vec{\rm e}_3] \quad \text{on } \partial\Gamma(0) = \mathbb{S}^1\,. \tag{6.6}$$

		$\theta = 1$		$\theta = 0$	
K	h_{Γ^0}	$\|\Gamma^h - \Gamma\|_{L^\infty}$	EOC	$\|\Gamma^h - \Gamma\|_{L^\infty}$	EOC
71	3.1765e-01	3.5536e-03	–	4.3730e-03	–
220	1.9962e-01	1.4718e-03	1.897544	1.1270e-03	2.918814
1188	9.0341e-02	4.0234e-04	1.635853	3.3587e-04	1.526934
4539	4.5382e-02	1.0919e-04	1.894341	9.0679e-05	1.901880

Table 1: Errors for the convergence experiment with Navier boundary conditions.

		$\theta = 1$		$\theta = 0$	
K	h_{Γ^0}	$\|\Gamma^h - \Gamma\|_{L^\infty}$	EOC	$\|\Gamma^h - \Gamma\|_{L^\infty}$	EOC
71	3.1765e-01	2.3693e-03	–	2.5225e-03	–
220	1.9962e-01	1.6378e-03	0.794862	1.6048e-03	0.973558
1188	9.0341e-02	3.2161e-04	2.053128	3.1401e-04	2.057618
4539	4.5382e-02	8.7943e-05	1.883364	8.6446e-05	1.873566

Table 2: Errors for the convergence experiment with clamped boundary conditions.

It is then not difficult to show that, for clamped boundary conditions, (1.17), with $\vec\zeta = \vec\mu$ as in (6.6), (6.5) is a solution to (2.21) with $\overline{\varkappa} = \beta = 0$ and with the additional right hand side term $g(\cdot, t)\,\vec\nu$, where

$$g(\vec z, t) = \varphi_t\,\vec e_3\,.\,\vec\nu = \frac{(R^2(t) - z_1^2 - z_2^2)^{\frac12}}{R(t)}\left[(R^2(t) - z_1^2 - z_2^2)^{-\frac12} - (R^2(t) - 1)^{-\frac12}\right] R(t)\,R'(t)$$
$$= \left[1 - \left(\frac{R^2(t) - z_1^2 - z_2^2}{R^2(t) - 1}\right)^{\frac12}\right] R'(t) \quad \text{for } \vec z \in \Gamma(t)\,. \tag{6.7}$$

Similarly, (2.21) with $\beta = 0$, with the time-dependent spontaneous curvature $\overline{\varkappa}(t) = -\frac{2}{R(t)}$, and with the same additional right hand side, is solved by (6.5). On the fully discrete level we add the term $\langle g(\cdot, t_{m+1})\,\vec\omega^m, \vec\chi\rangle^h_{\Gamma^m}$ to the right hand side of (4.9a). For the initial data $\Gamma(0)$ from (6.5) we adapt (6.4) to

$$(2\,\tfrac{\cos\vartheta}{r}\,x_1, 2\,\tfrac{\cos\vartheta}{r}\,x_2, 2\,\sin\vartheta - \sqrt{3})^T \quad \text{where} \quad r = (x_1^2+x_2^2)^{\frac12}\,,\ \vartheta = (\tfrac{\pi}{2} - \tfrac{\pi}{3})\,(1-r) + \tfrac{\pi}{3}\,. \tag{6.8}$$

The remaining initial data for the scheme (4.9a,b) is chosen as in (6.1) and (6.3).

For the convergence experiments we take $T = 0.5$ and choose $\tau = 0.125\,h^2_{\Gamma^0}$. See Table 1 for the two convergence experiments for Navier boundary conditions for $\theta = 0$ and $\theta = 1$. Here, in order to approximate $\overline{\varkappa}(t) = -\frac{2}{R(t)}$, we replace $\overline{\varkappa}$ in (4.9a,b) with $\overline{\varkappa}^{m+1} = -\frac{2}{R(t_{m+1})}$. Here and in what follows we always compute the error $\|\Gamma^h - \Gamma\|_{L^\infty} := \max_{m=1,\ldots,M} \|\Gamma^m - \Gamma(t_m)\|_{L^\infty}$, where $\|\Gamma^m - \Gamma(t_m)\|_{L^\infty} := \max_{k=1,\ldots,K} \operatorname{dist}(\vec q^m_k, \Gamma(t_m))$ between the discrete surfaces Γ^m, $m = 1, \ldots, M$, and the true solution on the interval $[0, T]$. The same table for clamped boundary conditions, with $\vec\zeta^m = \vec\mu(t_{m+1})$ as in (6.6), can be found in Table 2. Here we again use the initial data (6.1) and (6.3), now for the scheme (4.9a), (4.10).

It is clear from Tables 1 and 2 that for these experiments there is little difference between the choices $\theta = 0$ and $\theta = 1$.

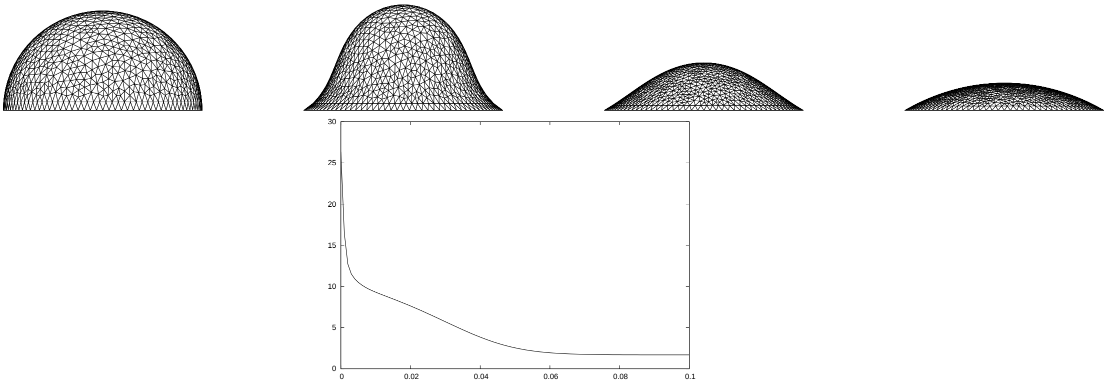

Figure 1: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 120° = \frac{2}{3}\,\pi$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

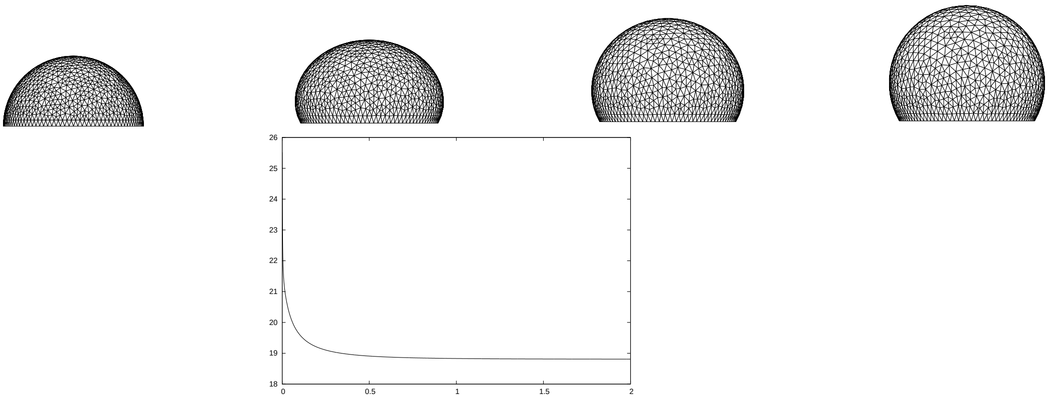

Figure 2: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 210° = \frac{7}{6}\,\pi$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.5,\ 2$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

6.2 Clamped boundary conditions

In this subsection, when $\partial\Gamma(0) = \mathbb{S}^1$, we will often choose

$$\vec{\zeta}^m = \sin\rho\,\vec{\rm id} + \cos\rho\,\vec{\rm e}_3 \quad \text{on } \partial\Gamma^0\,, \tag{6.9}$$

for a given $\rho \in [0, 2\,\pi)$. For the first numerical experiment for clamped boundary conditions, we take half of a unit sphere as initial data. We choose (6.9) with $\rho = 120° = \frac{2}{3}\,\pi$ and $\rho = 210° = \frac{7}{6}\,\pi$. The initial triangulation is such that $K = 1188$ and $J = 2274$. See the evolutions for the scheme (4.9a), (4.10) with $\theta = 0$ in Figures 1 and 2. We repeat the last experiment also for the choice $\theta = 1$. Then a severe deterioration in the mesh can be observed, see Figure 3. It is for this reason that from now on we only consider our schemes with $\theta = 0$. Here we recall from (3.39) and Theorem 3.1, that for the semidiscrete scheme, the choice $\theta = 0$ leads to conformal polyhedral surfaces. In practice, the discrete surfaces for the fully discrete scheme with $\theta = 0$ in general also exhibit good mesh properties.

We also perform a computation with nonzero spontaneous curvature $\overline{\varkappa}$. To this end,

Figure 3: ($\theta = 1$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 210^\circ = \frac{7}{6}\,\pi$. A plot of Γ^m at times $t = 0,\ 1,\ 2,\ 5$.

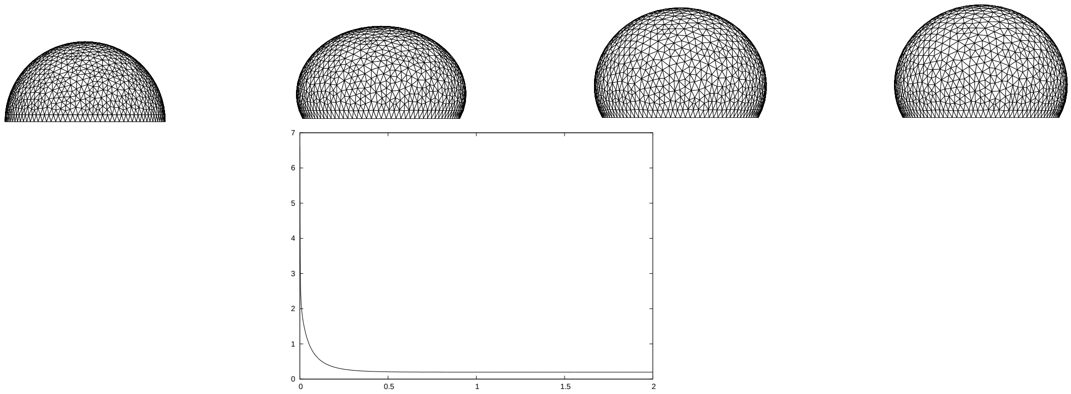

Figure 4: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 210^\circ = \frac{7}{6}\,\pi$ and $\overline{\varkappa} = -2$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.5,\ 1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

we repeat the simulation in Figure 2, but now set $\overline{\varkappa} = -2$. See Figure 4 for the numerical results.

Inspired by Deckelnick et al. (2015, Fig. 3), we also present an experiment for the following graphs. Let $\Gamma_{0,i} = \{(x, y, u_i(x, y))^T : (x, y, 0)^T \in \overline{B_1(\vec{0})}\}$, $i = 1, 2$, where

$$u_1(x, y) = \lambda\ \left(1 - 8\,x^2\,y^2\,(2 - x^2 - y^2)^2 + \tfrac{1}{2}\,(1 - \cos(2\,\pi\,(x^2 + y^2)))\right)$$

and $u_2(x, y) = -u_1(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}})$. We set $K = 1188$ and $J = 2274$, and let $\rho = 90^\circ = \frac{\pi}{2}$ in (6.9). See Figure 5 for a computation with $\lambda = 0.2$ for $\Gamma(0) = \Gamma_{0,1}$, and Figure 6 for a computation with $\lambda = 0.2$ for $\Gamma(0) = \Gamma_{0,2}$. Both simulations settle on the same stationary solution. For larger values of λ, however, the two different initial data lead to different steady state solutions, as already observed in Deckelnick et al. (2015). We can confirm this behaviour with the simulations shown in Figures 7 and 8, where we now choose $\lambda = 0.5$ and $\tau = 5 \times 10^{-4}$.

An experiment for a half-torus is shown in Figure 9. Here the large radius is $R = 2$, while the small radius is $r = 1$. Moreover, we have $K = 800$ and $J = 1536$. For the clamped condition we fix $\vec{\zeta}^m = \vec{\mu}^0 = \vec{e}_3$ on $\partial\Gamma^0$.

6.3 Navier boundary conditions

We take half of a unit sphere, with boundary $\mathbb{S}^1$. We choose $\overline{\varkappa} = \pm 1$. We set $K = 1188$ and $J = 2274$. See the evolutions in Figures 10 and 11. Clearly the evolutions in Figures 10

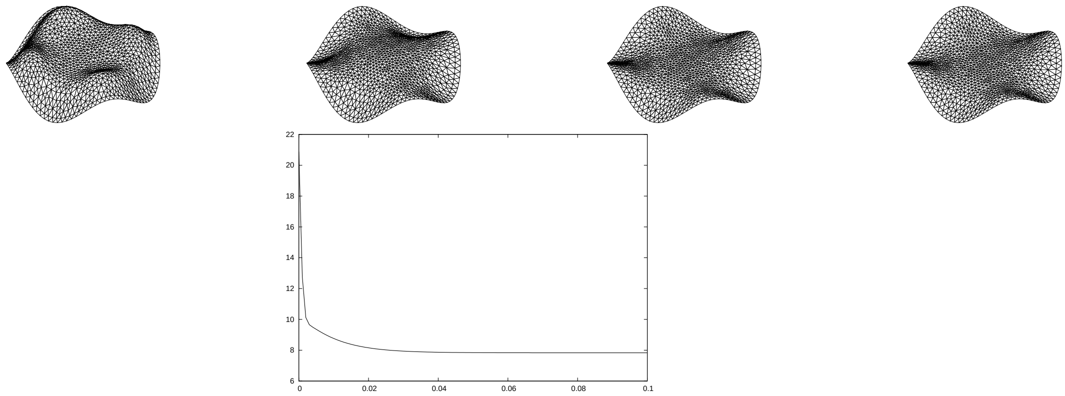

Figure 5: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 90° = \frac{\pi}{2}$ for $\Gamma_{0,1}$ with $\lambda = 0.2$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

and 11 settle on a small cap of a sphere of radius 2. In order to obtain the larger segment of that sphere, we start the same experiment from a different initial surface. Here we take the lift (6.4) with $\alpha = -0.5$ and $R = 1$. Then we obtain the results shown in Figure 12.

Some experiments for half-tori are shown in the next figures. Here the large radius is $R = 2$, while the small radius is $r = 1$. Moreover, we have $K = 800$ and $J = 1536$.

Next we discuss some numerical experiments for a half sphere with $\overline{\varkappa} = 0$ and nonzero α_G. We observe that for $\alpha_G > -2$, the half sphere evolves to a flat disk. As an example, we show this behaviour for $\alpha_G = -1$ in Figure 15. For $\alpha_G = -2$ the half sphere appears to be stationary, while for $\alpha_G < -2$ the half sphere expands. As an example, we show this behaviour for $\alpha_G = -3$ in Figure 16. Of course, if we enforce a constraint on the surface area, then the half sphere can no longer evolve to a flat disk, even for $\alpha_G > -2$. As an example, we present a simulation for the surface area preserving flow for $\alpha_G = 2$ in Figure 17. Similarly, the experiment for $\alpha_G = -5$ with conserved surface area is shown in Figure 18.

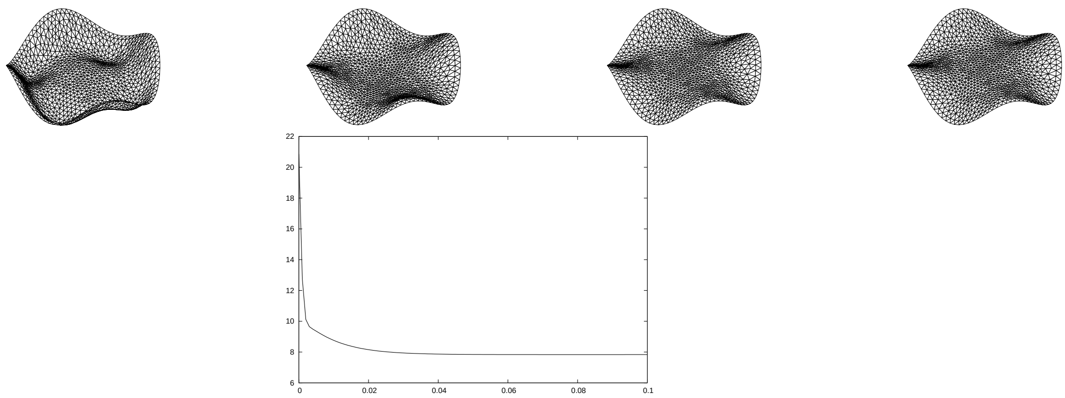

Figure 6: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 90° = \frac{\pi}{2}$ for $\Gamma_{0,2}$ with $\lambda = 0.2$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

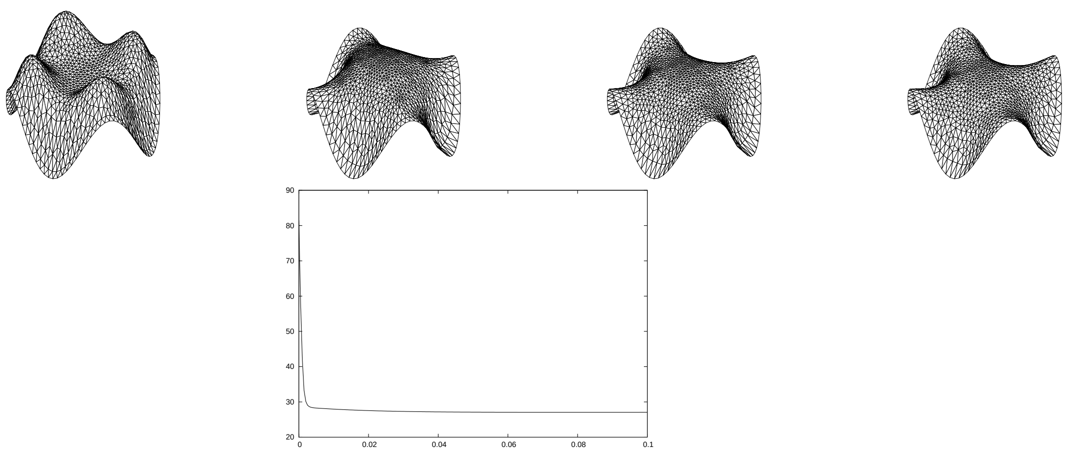

Figure 7: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 90° = \frac{\pi}{2}$ for $\Gamma_{0,1}$ with $\lambda = 0.5$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

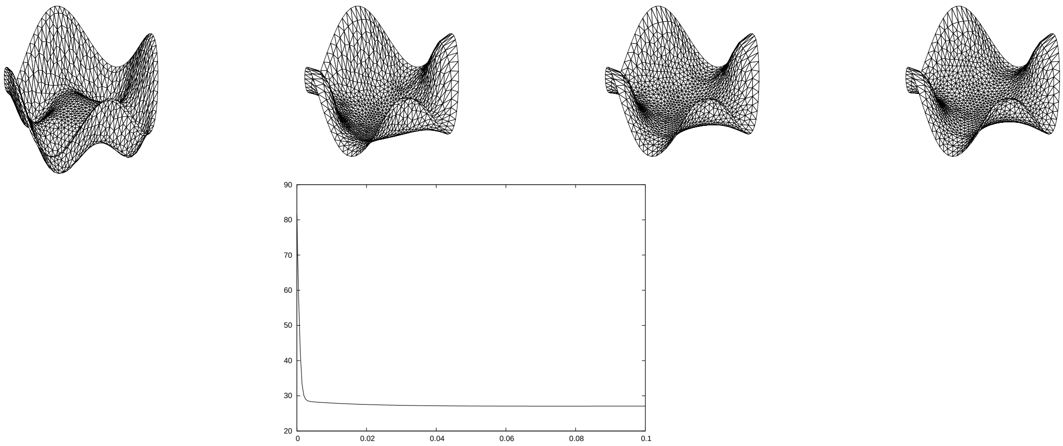

Figure 8: ($\theta = 0$) Willmore flow for clamped boundary conditions (6.9) with $\rho = 90^\circ = \frac{\pi}{2}$ for $\Gamma_{0,2}$ with $\lambda = 0.5$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

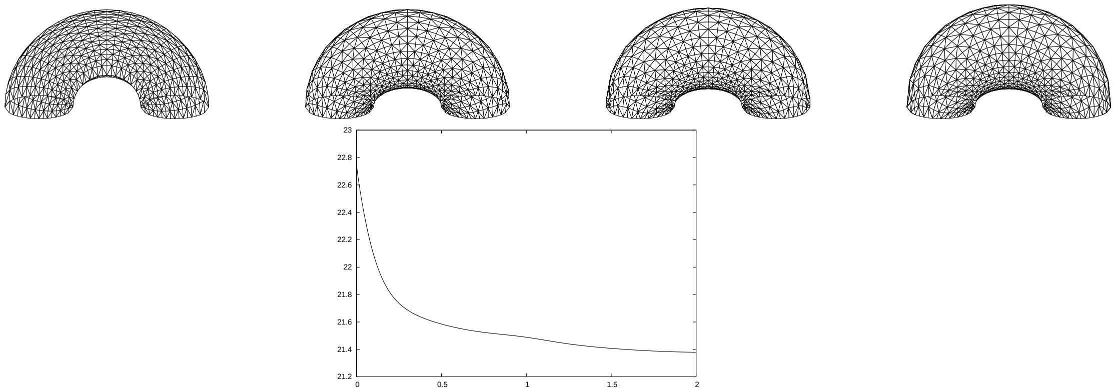

Figure 9: ($\theta = 0$) Willmore flow for clamped boundary conditions with $\vec{\zeta}^m = \vec{\mu}^0 = \vec{e}_3$. A plot of Γ^m at times $t = 0,\ 0.5,\ 1,\ 2$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

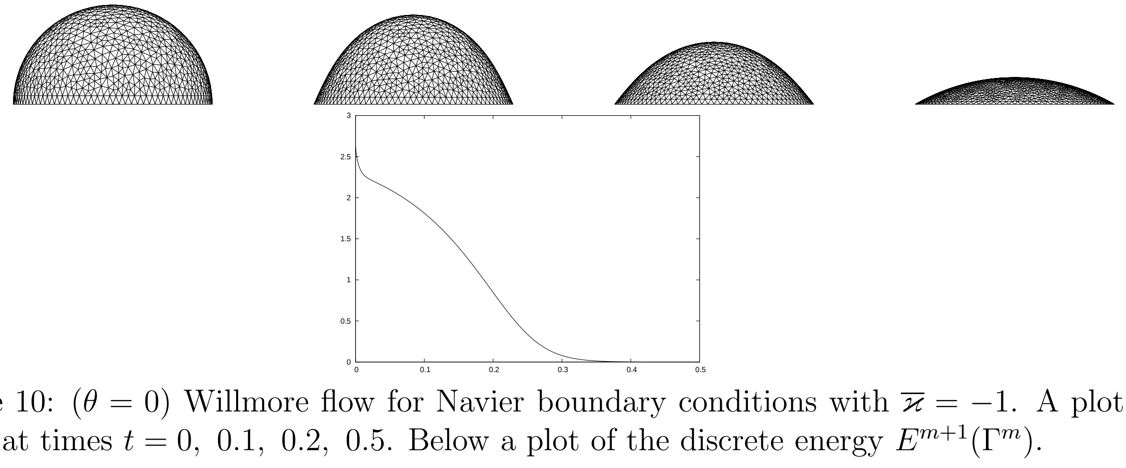

Figure 10: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = -1$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.2,\ 0.5$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

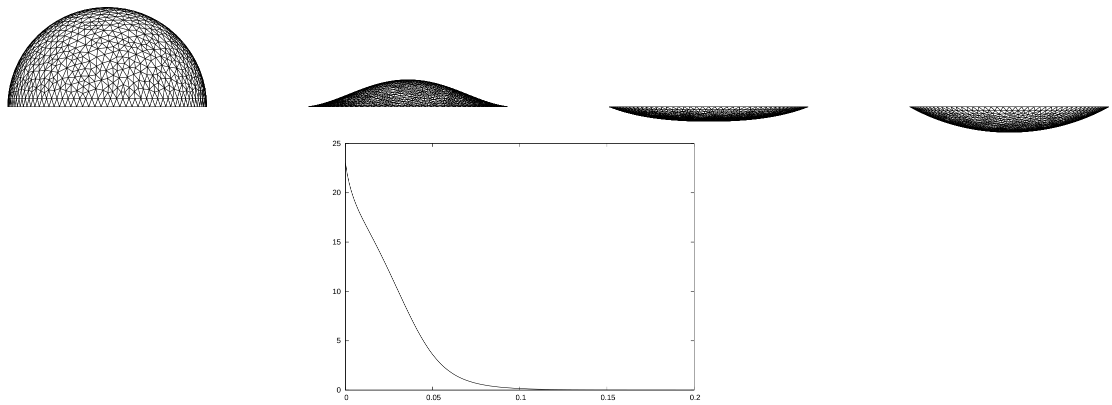

Figure 11: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = 1$. A plot of Γ^m at times $t = 0,\ 0.05,\ 0.1,\ 0.2$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

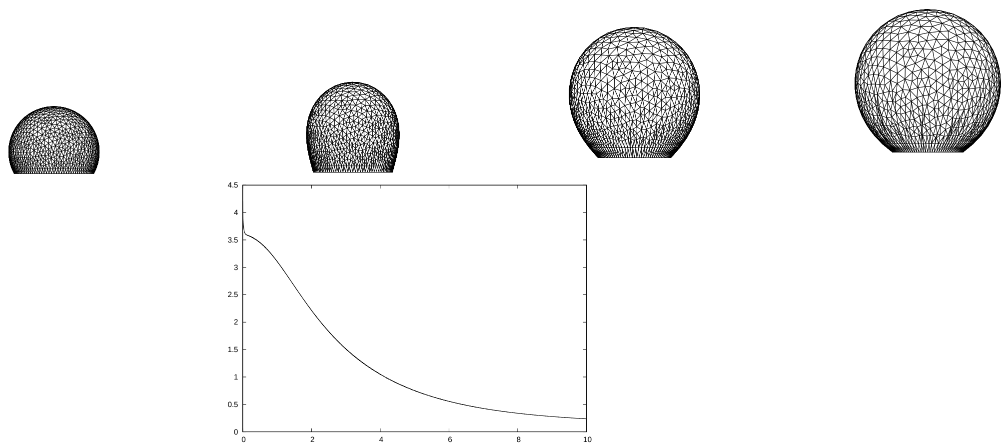

Figure 12: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = -1$. A plot of Γ^m at times $t = 0,\ 1,\ 5,\ 10$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

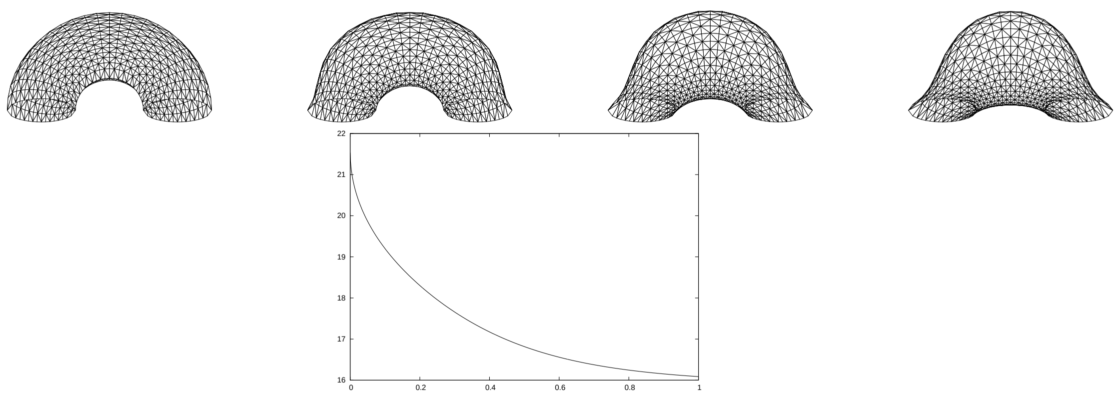

Figure 13: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = 0$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.5,\ 1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

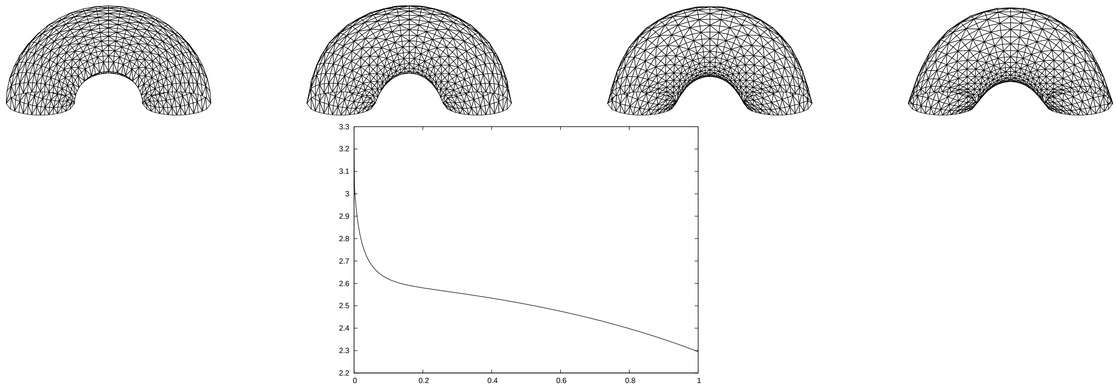

Figure 14: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = -1$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.5,\ 1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

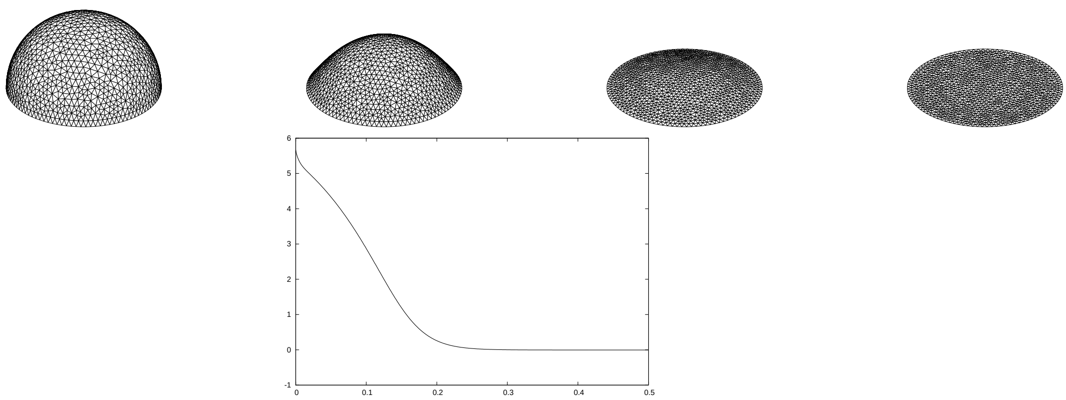

Figure 15: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = 0$ and $\alpha_G = -1$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.2,\ 0.5$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

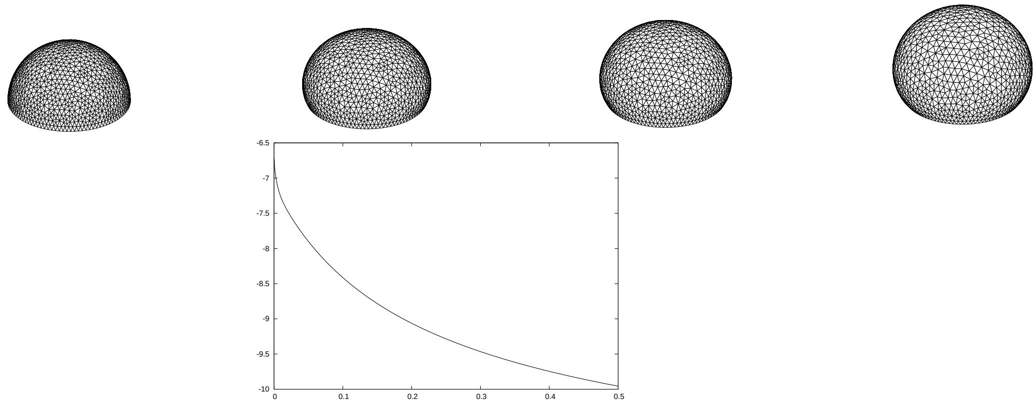

Figure 16: ($\theta = 0$) Willmore flow for Navier boundary conditions with $\overline{\varkappa} = 0$ and $\alpha_G = -3$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.2,\ 0.5$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

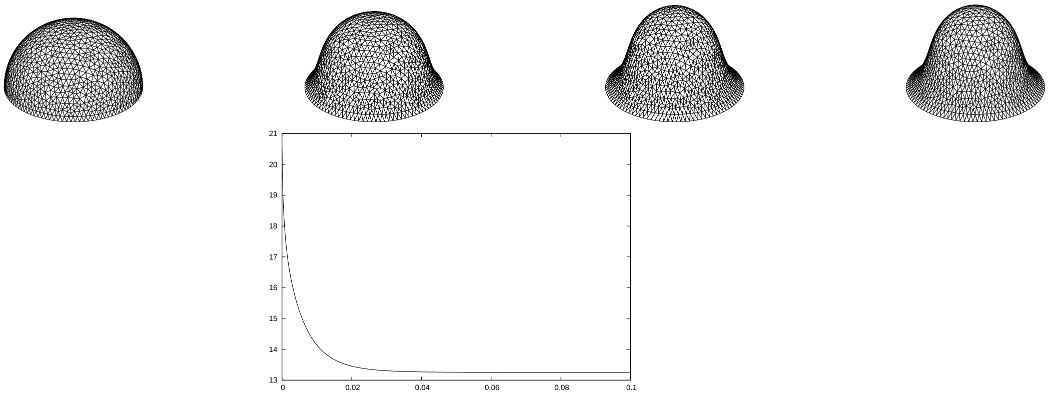

Figure 17: ($\theta = 0$) Area preserving Willmore flow for Navier boundary conditions with $\overline{\varkappa} = 0$ and $\alpha_G = 2$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

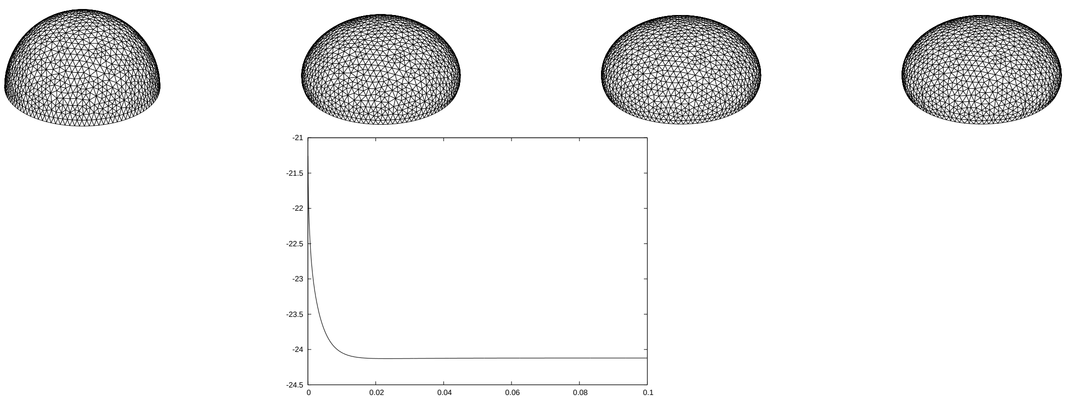

Figure 18: ($\theta = 0$) Area preserving Willmore flow for Navier boundary conditions with $\overline{\varkappa} = 0$ and $\alpha_G = -5$. A plot of Γ^m at times $t = 0,\ 0.01,\ 0.05,\ 0.1$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

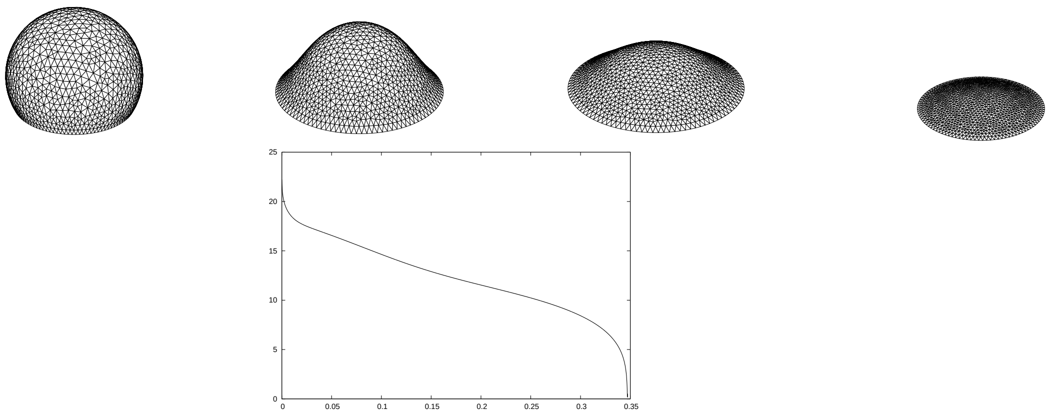

Figure 19: ($\theta = 0$) Willmore flow for semi-free boundary conditions with $\overline{\varkappa} = 0$ and $\gamma = 1$. A plot of Γ^m at times $t = 0,\ 0.1,\ 0.2,\ 0.33$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

6.4 Semi-free boundary conditions

We present some numerical results for surfaces that are attached to the x-y plane. For the first experiment we take as initial data a segment of the unit sphere that is smaller than a half sphere. The evolution for the parameters $\overline{\varkappa} = 0$ and $\gamma = 1$ can be seen in Figure 19. The same evolution for $\overline{\varkappa} = -1$ and $\gamma = 1$ is shown in Figure 20. Due to the presence of line energy, and due to $\overline{\varkappa} = 0$ in Figure 19, the surface Γ^m will shrink to a point eventually. Whereas in Figure 20 this is no longer the case, since $\overline{\varkappa} = -1$. We also compare these evolutions to a run for $\overline{\varkappa} = 0$, $\gamma = 1$ and $\alpha_G = -1$, see Figure 21. In all three experiments we have used the smaller time step size $\tau = 10^{-4}$.

A computation for surface area preserving Willmore flow for a half sphere, with $\overline{\varkappa} = -2$ can be seen in Figure 22. As the initial surface we take a half sphere that is stretched in the interior, so as to satisfy the boundary conditions without being a steady state. For this experiment we use $\tau = 2 \times 10^{-5}$. The observed relative loss of surface area is 0.46%, while the same run without the surface area conservation loses 6.8% of the original surface area. We omit a visualization of that experiment, as it is very close to the evolution in Figure 22.

6.5 Free boundary conditions

For the first experiment for free boundary conditions we use as initial data a cap of a sphere that is slightly smaller than a halfsphere. An experiment for $\overline{\varkappa} = -2$ and $\gamma = 1$ is shown in Figure 23. Repeating the same experiment with a larger initial cap of the unit sphere gives the results in Figure 24, where in this we have used $\tau = 10^{-4}$. In both experiments it can be observed that the surface tries to close up to a unit sphere due to the presence of line energy.

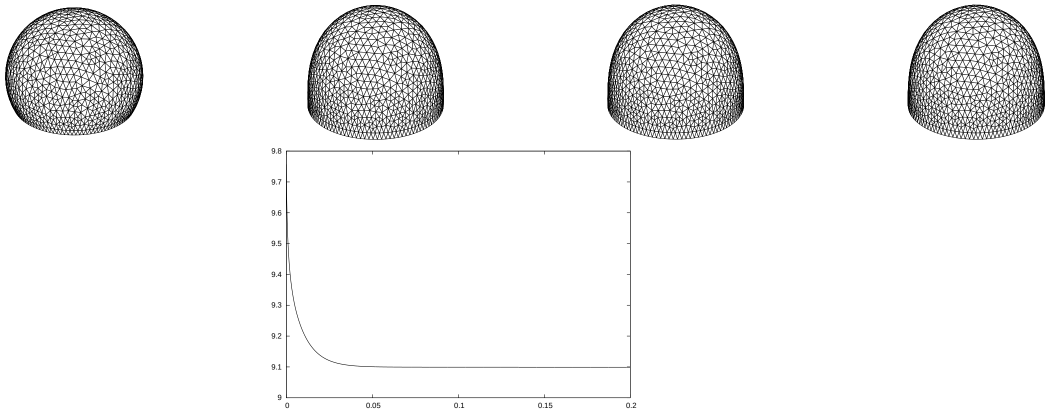

Figure 20: ($\theta = 0$) Willmore flow for semi-free boundary conditions with $\overline{\varkappa} = -1$ and $\gamma = 1$. A plot of Γ^m at times $t = 0,\ 0.05,\ 0.1,\ 0.2$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

A Derivation of strong formulation and boundary conditions

We recall from Section 2 that our numerical method is based on the weak formulation (2.20a), (2.35a), (2.24), (2.25) and (2.37) of the L^2–gradient flow of the energy $E(\Gamma(t))$, see (1.11). It follows from (1.2), (2.37) and (2.11) that

$$\vec{\varkappa} \,.\, \partial_\varepsilon^0\, \vec{\nu} = 0\,, \qquad \partial_\varepsilon^0\, (\underline{\underline{Q}}_\theta\, \vec{\varkappa}) = -(1-\theta)\, \varkappa\, [\nabla_s\, \vec{\chi}]^T\, \vec{\nu}\,,$$
$$\tfrac{1}{2} \left[|\vec{\varkappa} - \overline{\varkappa}\, \vec{\nu}|^2 - 2\, \underline{\underline{Q}}_\theta\, \vec{y} \,.\, \vec{\varkappa} \right] + \beta\, A\, \vec{\varkappa} \,.\, \vec{\nu} = -\tfrac{1}{2}\, (\varkappa^2 - \overline{\varkappa}^2)\,. \tag{A.1}$$

We recall that on the continuous level $\vec{\mathrm{m}} = \vec{\mu}$ and that $\theta \in [0, 1]$ is a fixed parameter. Here we need to choose $\theta = 0$ if $\alpha_G \neq 0$ for the free, semi-free or Navier boundary conditions, as otherwise the two conditions in (2.37) are incompatible in general. Then this weak formulation can be formulated as follows. Given $\Gamma(0)$, for all $t \in (0, T]$ find $\Gamma(t)$ and $\vec{y}(t) \in [H^1(\Gamma(t))]^3$ such that

$$\begin{aligned} \left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_{\Gamma(t)} &= \langle \nabla_s\, \vec{y}, \nabla_s\, \vec{\chi} \rangle_{\Gamma(t)} + \langle \nabla_s \,.\, \vec{y}, \nabla_s \,.\, \vec{\chi} \rangle_{\Gamma(t)} - \left\langle (\nabla_s\, \vec{y})^T, \underline{\underline{D}}(\vec{\chi})\, (\nabla_s\, \vec{\mathrm{id}})^T \right\rangle_{\Gamma(t)} \\ &\quad + \tfrac{1}{2} \left\langle (\varkappa^2 - \overline{\varkappa}^2)\, \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec{\chi} \right\rangle_{\Gamma(t)} - (1-\theta) \left\langle \varkappa\, \vec{y}, [\nabla_s\, \vec{\chi}]^T\, \vec{\nu} \right\rangle_{\Gamma(t)} \\ &\quad - \gamma \left\langle \vec{\mathrm{id}}_s, \vec{\chi}_s \right\rangle_{\partial\Gamma(t)} + \alpha_G \left[\left\langle \vec{\varkappa}_{\partial\Gamma} \,.\, \vec{\mu}, \vec{\mathrm{id}}_s \,.\, \vec{\chi}_s \right\rangle_{\partial\Gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}\, \vec{\mu}_s, \vec{\chi}_s \right\rangle_{\partial\Gamma(t)} \right] \\ &\qquad\qquad \forall\ \vec{\chi} \in \mathbb{X}(\Gamma(t))\,, \end{aligned} \tag{A.2a}$$

with $\vec{y} = y\, \vec{\nu} + \vec{u}$, where $y = \varkappa - \overline{\varkappa} + \beta\, A$, $A(t) = \langle \varkappa, 1 \rangle_{\Gamma(t)} - M_0$ and $\vec{u} \,.\, \vec{\nu} = 0$. Of course, the first equation in (2.37) implies that $\vec{u} = \vec{0}$ if $\theta \in (0, 1]$. Here the mean curvature $\varkappa$ is defined by (1.2), the curve curvature vector $\vec{\varkappa}_{\partial\Gamma}$ is given by (1.7), and the conormal $\vec{\mu}(t)$

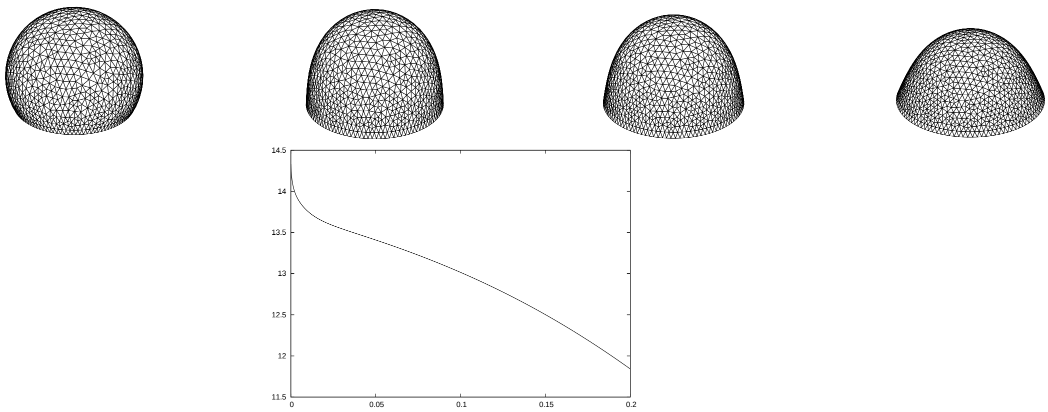

Figure 21: ($\theta = 0$) Willmore flow for semi-free boundary conditions with $\overline{\varkappa} = 0$, $\gamma = 1$ and $\alpha_G = -1$. A plot of Γ^m at times $t = 0,\ 0.05,\ 0.1,\ 0.2$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

is defined by (1.8). In addition, we either fix

$$\vec{y} = -\alpha_G\,\vec{\varkappa}_{\partial\Gamma} \quad \text{on} \ \ \partial\Gamma(t) \tag{A.3}$$

or

$$\vec{\mu} = \vec{\zeta} \quad \text{on} \ \ \partial\Gamma(t) = \partial\Gamma(0)\,. \tag{A.4}$$

For later use, we introduce the second fundamental form $\mathbb{I}$ of $\Gamma(t)$, which is given as

$$\mathbb{I}(\vec{\tau}_1, \vec{\tau}_2) = -[\partial_{\vec{\tau}_1}\,\vec{\nu}]\,.\,\vec{\tau}_2 = -[(\nabla_s\,\vec{\nu})\,\vec{\tau}_1]\,.\,\vec{\tau}_2 \qquad \text{on} \ \ \Gamma(t)\,, \tag{A.5}$$

for all tangential vectors $\vec{\tau}_i$, $i = 1, 2$. We note that $\mathbb{I}(\cdot,\cdot)$ is a symmetric bilinear form, as $\nabla_s\,\vec{\nu}$ is symmetric. Now it holds for the Gaussian curvature $\mathcal{K}$ of $\Gamma(t)$ that $\mathcal{K} = \det\,(\mathbb{I}(\vec{\tau}_i, \vec{\tau}_j))_{i,j=1}^2$, where $\vec{\tau}_1$, $\vec{\tau}_2$ are two orthonormal tangential vectors. Hence on $\partial\Gamma(t)$ we can compute the Gaussian curvature as

$$\mathcal{K} = \mathbb{I}(\vec{\mathrm{id}}_s, \vec{\mathrm{id}}_s)\,\mathbb{I}(\vec{\mu}, \vec{\mu}) - \mathbb{I}(\vec{\mathrm{id}}_s, \vec{\mu})\,\mathbb{I}(\vec{\mu}, \vec{\mathrm{id}}_s) = \mathbb{I}(\vec{\mathrm{id}}_s, \vec{\mathrm{id}}_s)\,\mathbb{I}(\vec{\mu}, \vec{\mu}) - [\mathbb{I}(\vec{\mathrm{id}}_s, \vec{\mu})]^2 \quad \text{on} \ \ \partial\Gamma(t)\,, \tag{A.6}$$

where we recall from (1.8) that $(\vec{\mathrm{id}}_s, \vec{\mu}, \vec{\nu})$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. Moreover, it follows from (1.9) and (1.1) that

$$\varkappa_\nu = \vec{\mathrm{id}}_{ss}\,.\,\vec{\nu} = -\vec{\mathrm{id}}_s\,.\,\vec{\nu}_s = \mathbb{I}(\vec{\mathrm{id}}_s, \vec{\mathrm{id}}_s) \quad \text{and} \quad \varkappa = \mathbb{I}(\vec{\mathrm{id}}_s, \vec{\mathrm{id}}_s) + \mathbb{I}(\vec{\mu}, \vec{\mu}) \quad \text{on} \ \ \partial\Gamma(t)\,. \tag{A.7}$$

In addition, let the torsion τ of the curve $\partial\Gamma(t)$ be defined by

$$\vec{\nu}_s \times \vec{\mathrm{id}}_s = \tau\,\vec{\nu}\,, \tag{A.8}$$

where we have observed that $\vec{\nu}$ is perpendicular to both $\vec{\nu}_s$ and $\vec{\mathrm{id}}_s$. For later use, we also note from (A.8) and (1.9) that

$$\vec{\mu}_s = \vec{\nu}_s \times \vec{\mathrm{id}}_s + \vec{\nu} \times \vec{\mathrm{id}}_{ss} = \vec{\nu}_s \times \vec{\mathrm{id}}_s + \varkappa_\mu\,\vec{\nu} \times \vec{\mu} = \vec{\nu}_s \times \vec{\mathrm{id}}_s - \varkappa_\mu\,\vec{\mathrm{id}}_s = \tau\,\vec{\nu} - \varkappa_\mu\,\vec{\mathrm{id}}_s\,. \tag{A.9}$$

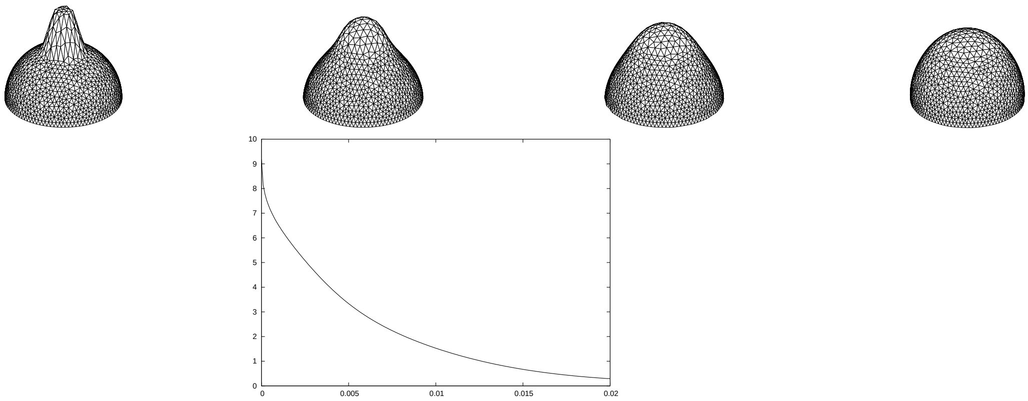

Figure 22: ($\theta = 0$) Area preserving Willmore flow for semi-free boundary conditions with $\overline{\varkappa} = -1$ and $\gamma = 0$. A plot of Γ^m at times $t = 0,\ 0.005,\ 0.01,\ 0.02$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

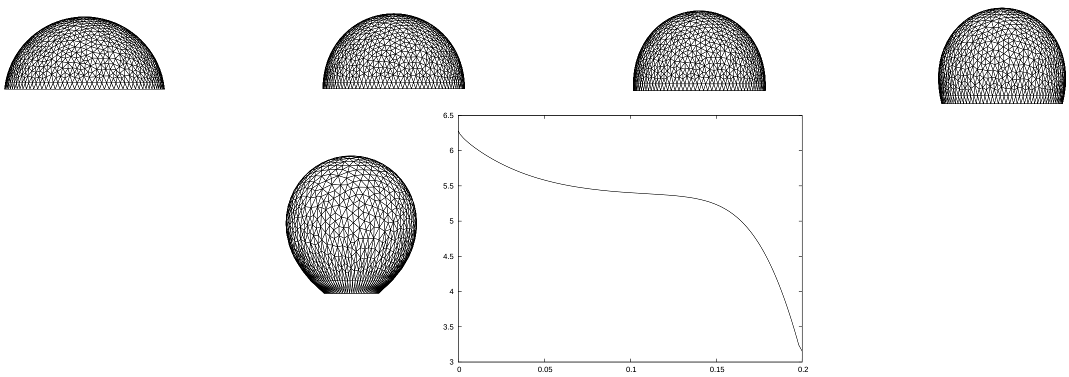

Figure 23: ($\theta = 0$) Willmore flow for free boundary conditions with $\overline{\varkappa} = -2$ and $\gamma = 1$. A plot of Γ^m at times $t = 0,\ 0.05,\ 0.1,\ 0.15,\ 0.2$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

Similarly, since $\vec{\nu} = \vec{\mathrm{id}}_s \times \vec{\mu}$, it follows from (1.9) and (A.8) that

$$\vec{\nu}_s = \vec{\mathrm{id}}_{ss} \times \vec{\mu} + \vec{\mathrm{id}}_s \times \vec{\mu}_s = \varkappa_\nu\, \vec{\nu} \times \vec{\mu} + \tau\, \vec{\mathrm{id}}_s \times \vec{\nu} = -\varkappa_\nu\, \vec{\mathrm{id}}_s - \tau\, \vec{\mu}\,, \tag{A.10}$$

and hence, recall (A.5), that

$$\mathrm{I\!I}(\vec{\mathrm{id}}_s, \vec{\mu}) = \tau\,. \tag{A.11}$$

For later use we note that

$$\int_{\Gamma(t)} \nabla_s\, g \,\mathrm{d}\mathcal{H}^2 = -\int_{\Gamma(t)} g\, \varkappa\, \vec{\nu} \,\mathrm{d}\mathcal{H}^2 + \int_{\partial\Gamma(t)} g\, \vec{\mu} \,\mathrm{d}\mathcal{H}^1\,, \tag{A.12}$$

see e.g. Theorem 2.10 in Dziuk and Elliott (2013).

Starting from the weak formulation (A.2a), we will now recover the corresponding strong formulation together with the boundary conditions that are enforced by it. It

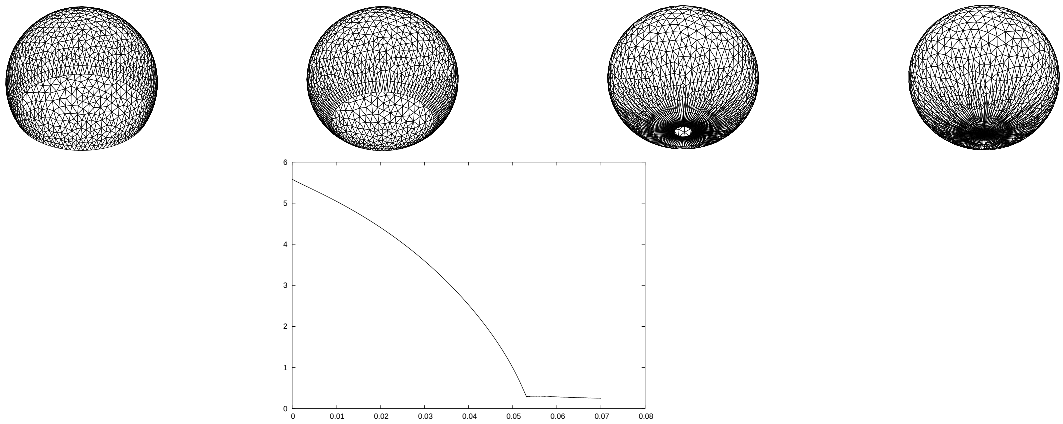

Figure 24: ($\theta = 0$) Willmore flow for free boundary conditions with $\overline{\varkappa} = -2$ and $\gamma = 1$. A plot of Γ^m at times $t = 0,\ 0.02\, 0.05,\ 0.07$. Below a plot of the discrete energy $E^{m+1}(\Gamma^m)$.

follows from (A.2a) and (1.9) that

$$\begin{aligned}
\left\langle \vec{\mathcal{V}}, \vec\chi \right\rangle_{\Gamma(t)} &= \langle \nabla_s\,(y\,\vec\nu), \nabla_s\,\vec\chi\rangle_{\Gamma(t)} + \langle \nabla_s\,.\,(y\,\vec\nu), \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)} - \left\langle [\nabla_s\,(y\,\vec\nu)]^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma(t)} \\
&\quad + \tfrac12 \left\langle (\varkappa^2 - \overline{\varkappa}^2), \nabla_s\,.\,\vec\chi \right\rangle_{\Gamma(t)} - (1-\theta)\left\langle \varkappa\, y\,\vec\nu, [\nabla_s\,\vec\chi]^T\,\vec\nu \right\rangle_{\Gamma(t)} + \langle \nabla_s\,\vec u, \nabla_s\,\vec\chi\rangle_{\Gamma(t)} \\
&\quad + \langle \nabla_s\,.\,\vec u, \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)} - \left\langle (\nabla_s\,\vec u)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma(t)} - (1-\theta)\left\langle \varkappa\,\vec u, [\nabla_s\,\vec\chi]^T\,\vec\nu \right\rangle_{\Gamma(t)} \\
&\quad + \gamma\,\langle \varkappa_\mu\,\vec\mu + \varkappa_\nu\,\vec\nu, \vec\chi\rangle_{\partial\Gamma(t)} + \alpha_G \left[\left\langle \varkappa_\mu, \vec{\rm id}_s\,.\,\vec\chi_s \right\rangle_{\partial\Gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}\,\vec\chi_s, \vec\mu_s \right\rangle_{\partial\Gamma(t)} \right] \\
&=: \sum_{\ell=1}^{9} T_\ell + \gamma\,\langle \varkappa_\mu\,\vec\mu + \varkappa_\nu\,\vec\nu, \vec\chi\rangle_{\partial\Gamma(t)} + \alpha_G \left[\left\langle \varkappa_\mu, \vec{\rm id}_s\,.\,\vec\chi_s \right\rangle_{\partial\Gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_{\partial\Gamma}\,\vec\mu_s, \vec\chi_s \right\rangle_{\partial\Gamma(t)} \right] \\
&\qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,. \qquad \text{(A.13)}
\end{aligned}$$

Of course, we have that $\vec\nu\,.\,(\nabla_s\,\vec\chi)^T\,\vec\nu = ([\nabla_s\,\vec\chi]\,\vec\nu)\,.\,\vec\nu = ([\nabla_s\,\vec\chi]\,\vec\nu)\,.\,\vec\nu = \vec 0\,.\,\vec\nu = 0$, and so the term T_5 on the right hand side of (A.13) vanishes. For the term T_2 on the right hand side of (A.13) we obtain, on recalling that $[\nabla_s\,(y\,\vec\nu)]\,.\,\vec\nu = 0$ and $\nabla_s\,.\,\vec\nu = -\varkappa$, that

$$T_2 = \langle \nabla_s\,.\,(y\,\vec\nu), \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)} = -\,\langle y\,\varkappa, \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)} = -\,\langle (\varkappa - \overline{\varkappa} + \beta\,A)\,\varkappa, \nabla_s\,.\,\vec\chi\rangle_{\Gamma(t)}\,. \quad \text{(A.14)}$$

Hence noting (A.14) and (A.12) yields that

$$\begin{aligned}
\sum_{\ell=1}^{5} T_\ell &= \langle \nabla_s\,(y\,\vec\nu), \nabla_s\,\vec\chi\rangle_{\Gamma(t)} - \left\langle [\nabla_s\,(y\,\vec\nu)]^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma(t)} \\
&\qquad - \left\langle \tfrac12\,(\varkappa - \overline{\varkappa})^2 + \beta\,A\,\varkappa, \nabla_s\,.\,\vec\chi \right\rangle_{\Gamma(t)} \\
&= \langle \nabla_s\,(y\,\vec\nu), \nabla_s\,\vec\chi\rangle_{\Gamma(t)} - \left\langle [\nabla_s\,(y\,\vec\nu)]^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma(t)} \\
&\qquad + \left\langle \nabla_s\,[\tfrac12\,(\varkappa - \overline{\varkappa})^2 + \beta\,A\,\varkappa], \vec\chi \right\rangle_{\Gamma(t)} + \left\langle [\tfrac12\,(\varkappa - \overline{\varkappa})^2 + \beta\,A\,\varkappa]\,\varkappa\,\vec\nu, \vec\chi \right\rangle_{\Gamma(t)} \\
&\qquad - \left\langle \tfrac12\,(\varkappa - \overline{\varkappa})^2 + \beta\,A\,\varkappa, \vec\chi\,.\,\vec\mu \right\rangle_{\partial\Gamma(t)} \quad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,. \qquad \text{(A.15)}
\end{aligned}$$

In order to deal with the first two terms on the right hand side of (A.15), similarly to Dziuk (2008, p. 64), it is not difficult to prove that

$$\nabla_s (y\,\vec\nu) : \nabla_s \vec\chi - [\nabla_s (y\,\vec\nu)]^T : (\underline{\underline{D}}(\vec\chi)\,(\nabla_s \vec{\rm id})^T) = \nabla_s (y\,\vec\nu) : ((\vec\nu \otimes \vec\nu)\,\nabla_s \vec\chi) - y\,\nabla_s \vec\nu : \nabla_s \vec\chi\,. \tag{A.16}$$

In addition, one can also show that

$$\nabla_s (y\,\vec\nu) : ((\vec\nu \otimes \vec\nu)\,\nabla_s \vec\chi) - y\,\nabla_s \vec\nu : \nabla_s \vec\chi = \nabla_s y \,.\, \nabla_s (\vec\chi \,.\, \vec\nu) - \nabla_s \,.\, (y\,(\nabla_s \vec\nu)^T\,\vec\chi) \\ - y\,(|\nabla_s \vec\nu|^2\,\vec\nu + \nabla_s y) \,.\, \vec\chi\,. \tag{A.17}$$

We give the proofs of (A.16) and (A.17) below, see (A.20) and (A.21). Combining (A.15), (A.16) and (A.17), on noting that $\frac12\,\nabla_s y^2 = \nabla_s [\frac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,(\varkappa - \overline\varkappa) + \frac12\,\beta^2\,A^2] = \nabla_s [\frac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa]$ and (A.12), yields that

$$\begin{aligned}\sum_{\ell=1}^5 T_\ell &= \langle \nabla_s y, \nabla_s (\vec\chi \,.\, \vec\nu)\rangle_{\Gamma(t)} - \left\langle \nabla_s (y\,(\nabla_s \vec\nu)^T\,\vec\chi), 1\right\rangle_{\Gamma(t)} - \left\langle y\,|\nabla_s \vec\nu|^2\,\vec\nu + \tfrac12\,\nabla_s y^2, \vec\chi\right\rangle_{\Gamma(t)} \\ &\quad + \left\langle \nabla_s [\tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa], \vec\chi\right\rangle_{\Gamma(t)} + \left\langle [\tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa]\,\varkappa\,\vec\nu, \vec\chi\right\rangle_{\Gamma(t)} \\ &\quad - \left\langle \tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa, \vec\chi \,.\, \vec\mu\right\rangle_{\partial\Gamma(t)} \\ &= - \langle \Delta_s y, \vec\chi \,.\, \vec\nu\rangle_{\Gamma(t)} + \langle (\nabla_s y) \,.\, \vec\mu, \vec\chi \,.\, \vec\nu\rangle_{\partial\Gamma(t)} - \left\langle y\,(\nabla_s \vec\nu)^T\,\vec\chi, \vec\mu\right\rangle_{\partial\Gamma(t)} \\ &\quad + \left\langle [\tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa]\,\varkappa - y\,|\nabla_s \vec\nu|^2, \vec\chi \,.\, \vec\nu\right\rangle_{\Gamma(t)} \\ &\quad - \left\langle \tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa, \vec\chi \,.\, \vec\mu\right\rangle_{\partial\Gamma(t)} \quad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,. \end{aligned} \tag{A.18}$$

Finally, on noting that $\nabla_s y = \nabla_s \varkappa$, it follows that

$$\begin{aligned}\sum_{i=\ell}^5 T_\ell = &- \langle \Delta_s \varkappa, \vec\chi \,.\, \vec\nu\rangle_{\Gamma(t)} \\ &+ \left\langle [\tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa]\,\varkappa - (\varkappa - \overline\varkappa + \beta\,A)\,|\nabla_s \vec\nu|^2, \vec\chi \,.\, \vec\nu\right\rangle_{\Gamma(t)} \\ &+ \langle (\nabla_s \varkappa) \,.\, \vec\mu, \vec\chi \,.\, \vec\nu\rangle_{\partial\Gamma(t)} - \langle (\varkappa - \overline\varkappa + \beta\,A)\,(\nabla_s \vec\nu)\,\vec\mu, \vec\chi\rangle_{\partial\Gamma(t)} \\ &- \left\langle \tfrac12\,(\varkappa - \overline\varkappa)^2 + \beta\,A\,\varkappa, \vec\chi \,.\, \vec\mu\right\rangle_{\partial\Gamma(t)} \quad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,. \end{aligned} \tag{A.19}$$

For completeness, we present short proofs of (A.16) and (A.17). For the former, we define $\underline{\underline{P}} = \underline{\underline{\rm Id}} - \vec\nu \otimes \vec\nu = \nabla_s \vec{\rm id}$, so that $\underline{\underline{P}}\,(\nabla_s \vec\chi)^T = (\nabla_s \vec\chi)^T$ and $(\nabla_s \vec\chi)\,\underline{\underline{P}} = \nabla_s \vec\chi$. Then, on noting $(\nabla_s \vec\nu)^T = \nabla_s \vec\nu$ and $\underline{\underline{P}}\,\vec\nu = \vec\nu$, it holds that

$$\begin{aligned}&\nabla_s (y\,\vec\nu) : \nabla_s \vec\chi - [\nabla_s (y\,\vec\nu)]^T : (\underline{\underline{D}}(\vec\chi)\,(\nabla_s \vec{\rm id})^T) \\ &\quad = \nabla_s (y\,\vec\nu) : \nabla_s \vec\chi - [\nabla_s (y\,\vec\nu)]^T : (\nabla_s \vec\chi + (\nabla_s \vec\chi)^T)\,\underline{\underline{P}} \\ &\quad = {\rm tr}((y\,\nabla_s \vec\nu + \nabla_s y \otimes \vec\nu)\,\nabla_s \vec\chi) - {\rm tr}((y\,\nabla_s \vec\nu + \vec\nu \otimes \nabla_s y)\,\nabla_s \vec\chi\,\underline{\underline{P}}) \\ &\qquad\quad - {\rm tr}((y\,\nabla_s \vec\nu + \vec\nu \otimes \nabla_s y)\,(\nabla_s \vec\chi)^T\,\underline{\underline{P}}) \\ &\quad = {\rm tr}((\nabla_s y \otimes \vec\nu)\,\nabla_s \vec\chi) - y\ {\rm tr}((\nabla_s \vec\nu)^T\,(\nabla_s \vec\chi)) \\ &\quad = (\vec\nu \otimes \nabla_s y) : \nabla_s \vec\chi - y\,\nabla_s \vec\nu : \nabla_s \vec\chi = [(\vec\nu \otimes \vec\nu)\,\nabla_s (y\,\vec\nu)] : \nabla_s \vec\chi - y\,\nabla_s \vec\nu : \nabla_s \vec\chi \\ &\quad = \nabla_s (y\,\vec\nu) : ((\vec\nu \otimes \vec\nu)\,\nabla_s \vec\chi) - y\,\nabla_s \vec\nu : \nabla_s \vec\chi\,. \end{aligned} \tag{A.20}$$

In order to prove (A.17), we note that $\nabla_s (\vec\chi \,.\, \vec\nu) = (\nabla_s \vec\chi)^T \vec\nu + (\nabla_s \vec\nu)^T \vec\chi$, and similarly $\nabla_s y = \nabla_s ((y\,\vec\nu) \,.\, \vec\nu) = [\nabla_s (y\,\vec\nu)]^T \vec\nu + y\,(\nabla_s \vec\nu)^T \vec\nu = [\nabla_s (y\,\vec\nu)]^T \vec\nu$. Hence it follows that

$$\begin{aligned}
\nabla_s y \,.\, \nabla_s (\vec\chi \,.\, \vec\nu) &= ((\nabla_s (y\,\vec\nu))^T \vec\nu) \,.\, ((\nabla_s \vec\chi)^T \vec\nu + (\nabla_s \vec\nu)^T \vec\chi) \\
&= \nabla_s (y\,\vec\nu) : ((\vec\nu \otimes \vec\nu)\,\nabla_s \vec\chi) + \nabla_s y \,.\, [(\nabla_s \vec\nu)^T \vec\chi] \\
&= \nabla_s (y\,\vec\nu) : ((\vec\nu \otimes \vec\nu)\,\nabla_s \vec\chi) + \nabla_s \,.\, [y\,(\nabla_s \vec\nu)^T \vec\chi] - y\,\nabla_s \,.\, [(\nabla_s \vec\nu)^T \vec\chi] \,. \qquad \text{(A.21)}
\end{aligned}$$

Moreover, it holds that

$$\nabla_s \,.\, [(\nabla_s \vec\nu)^T \vec\chi] = (\Delta_s \vec\nu) \,.\, \vec\chi + \nabla_s \vec\nu : \nabla_s \vec\chi = -(|\nabla_s \vec\nu|^2\,\vec\nu + \nabla_s y) \,.\, \vec\chi + \nabla_s \vec\nu : \nabla_s \vec\chi\,, \qquad \text{(A.22)}$$

where we have used the fact that, see Appendix A in Ecker (2004) for a proof,

$$\Delta_s \vec\nu = -|\nabla_s \vec\nu|^2\,\vec\nu - \nabla_s \varkappa\,. \qquad \text{(A.23)}$$

Combining (A.21) and (A.22) gives the desired result (A.17).

We now deal with the terms involving $\vec u$, where we recall that $\vec u \,.\, \vec\nu = 0$, and that $\vec u = \vec 0$ if $\theta \in (0,1]$, which implies that $T_9 = -(1-\theta)\,\left\langle \varkappa\,\vec u, [\nabla_s \vec\chi]^T \vec\nu \right\rangle_{\Gamma(t)} = -\left\langle \varkappa\,\vec u, [\nabla_s \vec\chi]^T \vec\nu \right\rangle_{\Gamma(t)}$ for all $\theta \in [0,1]$. Using the standard summation notation, we have that

$$\begin{aligned}
\sum_{\ell=6}^{9} T_\ell &= \langle \partial_{s_k} u_j, \partial_{s_k} \chi_j \rangle_{\Gamma(t)} + \langle \partial_{s_k} u_k, \partial_{s_k} \chi_k \rangle_{\Gamma(t)} - \left\langle (\partial_{s_j} u_i)\,(\partial_{s_j} \chi_k + \partial_{s_k} \chi_j), \delta_{ik} - \nu_i\,\nu_k \right\rangle_{\Gamma(t)} \\
&\qquad - \langle \varkappa\,u_i\,\partial_{s_i} \chi_j, \nu_j \rangle_{\Gamma(t)} \\
&= \left\langle \partial_{s_k} u_k, \partial_{s_j} \chi_j \right\rangle_{\Gamma(t)} - \left\langle \partial_{s_j} u_k, \partial_{s_k} \chi_j \right\rangle_{\Gamma(t)} + \left\langle (\partial_{s_j} u_i)\,\partial_{s_j} \chi_k, \nu_i\,\nu_k \right\rangle_{\Gamma(t)} \\
&\qquad - \langle \varkappa\,u_i\,\partial_{s_i} \chi_j, \nu_j \rangle_{\Gamma(t)} =: \sum_{\ell=1}^{4} S_4 \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,. \qquad \text{(A.24)}
\end{aligned}$$

On noting (A.12), it holds that

$$S_1 = -\left\langle \partial_{s_j}\,\partial_{s_k} u_k, \chi_j \right\rangle_{\Gamma(t)} - \langle \varkappa\,\partial_{s_k} u_k, \chi_j\,\nu_j \rangle_{\Gamma(t)} + \langle \partial_{s_k} u_k, \chi_j\,\mu_j \rangle_{\partial\Gamma(t)}\,, \qquad \text{(A.25a)}$$

$$S_2 = \left\langle \partial_{s_k}\,\partial_{s_j} u_k, \chi_j \right\rangle_{\Gamma(t)} + \left\langle \varkappa\,\partial_{s_j} u_k, \chi_j\,\nu_k \right\rangle_{\Gamma(t)} - \left\langle \partial_{s_j} u_k, \chi_j\,\mu_k \right\rangle_{\partial\Gamma(t)}\,, \qquad \text{(A.25b)}$$

$$S_4 = \langle \varkappa\,\partial_{s_i} u_i, \chi_j\,\nu_j \rangle_{\Gamma(t)} + \langle \partial_{s_i} \varkappa, u_i\,\chi_j\,\nu_j \rangle_{\Gamma(t)} + \langle \varkappa\,u_i, \chi_j\,\partial_{s_i} \nu_j \rangle_{\Gamma(t)} - \langle \varkappa\,u_i, \chi_j\,\nu_j\,\mu_i \rangle_{\partial\Gamma(t)}\,. \qquad \text{(A.25c)}$$

Combining (A.25a–c) yields that

$$\begin{aligned}
&S_1 + S_2 + S_4 \\
&\quad = -\left\langle \partial_{s_j}\,\partial_{s_k} u_k - \partial_{s_k}\,\partial_{s_j} u_k, \chi_j \right\rangle_{\Gamma(t)} + \left\langle \varkappa\,\partial_{s_j} u_k, \chi_j\,\nu_k \right\rangle_{\Gamma(t)} + \langle u_i\,\partial_{s_i} \varkappa, \chi_j\,\nu_j \rangle_{\Gamma(t)} \\
&\qquad\quad + \langle \varkappa\,u_i, \chi_j\,\partial_{s_i} \nu_j \rangle_{\Gamma(t)} + \left\langle \mu_j\,\partial_{s_k} u_k - \mu_k\,\partial_{s_j} u_k - \varkappa\,\mu_i\,\nu_j\,u_i, \chi_j \right\rangle_{\partial\Gamma(t)} \\
&\quad = -\langle \nu_j\,(\partial_{s_i} \nu_k)\,\partial_{s_i} u_k - \nu_k\,(\partial_{s_i} \nu_j)\,\partial_{s_i} u_k, \chi_j \rangle_{\Gamma(t)} + \left\langle \varkappa\,\partial_{s_j} u_k, \chi_j\,\nu_k \right\rangle_{\Gamma(t)} \\
&\qquad\quad + \langle u_i\,\partial_{s_i} \varkappa, \chi_j\,\nu_j \rangle_{\Gamma(t)} + \langle \varkappa\,u_i, \chi_j\,\partial_{s_i} \nu_j \rangle_{\Gamma(t)} \\
&\qquad\quad + \left\langle \mu_j\,\partial_{s_k} u_k - \mu_k\,\partial_{s_j} u_k - \varkappa\,\mu_i\,\nu_j\,u_i, \chi_j \right\rangle_{\partial\Gamma(t)} \qquad \forall\ \vec\chi \in \mathbb{X}(\Gamma(t))\,, \qquad \text{(A.26)}
\end{aligned}$$

where we have recalled from Dziuk and Elliott (2013, Lemma 2.6) that

$$\partial_{s_i}\partial_{s_j}\phi - \partial_{s_j}\partial_{s_i}\phi = [(\nabla_s\vec\nu)\,\nabla_s\phi]_j\,\nu_i - [(\nabla_s\vec\nu)\,\nabla_s\phi]_i\,\nu_j \qquad \forall\, i,j\in\{1,\ldots,d\}\,.$$

Moreover, on noting (A.12) and (A.23), we have that

$$\begin{aligned}
S_3 &= \left\langle(\partial_{s_j}u_i)\,\partial_{s_j}\chi_k, \nu_i\,\nu_k\right\rangle_{\Gamma(t)}\\
&= -\left\langle\partial_{s_j}\partial_{s_j}u_i, \chi_k\,\nu_i\,\nu_k\right\rangle_{\Gamma(t)} - \left\langle(\partial_{s_j}u_i)\,\partial_{s_j}(\nu_i\,\nu_k), \chi_k\right\rangle_{\Gamma(t)} + \left\langle\nu_i\,\nu_k\,\mu_j\,\partial_{s_j}u_i, \chi_k\right\rangle_{\partial\Gamma(t)}\\
&= \left\langle\partial_{s_j}\partial_{s_j}\nu_i, u_i\,\chi_k\,\nu_k\right\rangle_{\Gamma(t)} + 2\left\langle(\partial_{s_j}u_i)\,\partial_{s_j}\nu_i, \chi_k\,\nu_k\right\rangle_{\Gamma(t)} - \left\langle(\partial_{s_j}u_i)\,\partial_{s_j}(\nu_i\,\nu_k), \chi_k\right\rangle_{\Gamma(t)}\\
&\qquad + \left\langle\nu_i\,\nu_k\,\mu_j\,\partial_{s_j}u_i, \chi_k\right\rangle_{\partial\Gamma(t)}\\
&= -\left\langle\partial_{s_i}\varkappa, u_i\,\chi_k\,\nu_k\right\rangle_{\Gamma(t)} + \left\langle(\partial_{s_j}u_i)\,\partial_{s_j}\nu_i, \chi_k\,\nu_k\right\rangle_{\Gamma(t)} - \left\langle(\partial_{s_j}u_i)\,\partial_{s_j}\nu_k, \chi_k\,\nu_i\right\rangle_{\Gamma(t)}\\
&\qquad + \left\langle\nu_i\,\nu_k\,\mu_j\,\partial_{s_j}u_i, \chi_k\right\rangle_{\partial\Gamma(t)} \qquad \forall\,\vec\chi\in\mathbb{X}(\Gamma(t))\,.
\end{aligned}\tag{A.27}$$

Summing (A.26) and (A.27), on recalling (A.24), $\vec u\,.\,\vec\nu = 0$ and the symmetry of the Weingarten map, yields that

$$\begin{aligned}
\sum_{\ell=6}^{9} T_\ell = \sum_{\ell=1}^{4} S_\ell &= \left\langle\varkappa\,\partial_{s_j}u_k, \chi_j\,\nu_k\right\rangle_{\Gamma(t)} + \left\langle\varkappa\,u_i\,\partial_{s_i}\nu_j, \chi_j\right\rangle_{\Gamma(t)} + B\\
&= \left\langle\varkappa\,\partial_{s_j}u_k, \chi_j\,\nu_k\right\rangle_{\Gamma(t)} - \left\langle\varkappa\,\partial_{s_j}u_i, \chi_j\,\nu_i\right\rangle_{\Gamma(t)} + B = B \quad \forall\,\vec\chi\in\mathbb{X}(\Gamma(t))\,,
\end{aligned}\tag{A.28}$$

where

$$B = \left\langle(\mu_j\,\partial_{s_k} - \mu_k\,\partial_{s_j})\,u_k - \varkappa\,\mu_i\,\nu_j\,u_i, \chi_j\right\rangle_{\partial\Gamma(t)} + \left\langle\nu_i\,\nu_k\,\mu_j\,\partial_{s_j}u_i, \chi_k\right\rangle_{\partial\Gamma(t)}\,. \tag{A.29}$$

We will now deal with the boundary terms arising in (A.13), (A.19) and (A.28). Collecting these gives:

$$\begin{aligned}
\sum_{\ell=1}^{6} B_\ell &= \langle(\nabla_s\varkappa)\,.\,\vec\mu + \gamma\,\varkappa_\nu, \vec\chi\,.\,\vec\nu\rangle_{\partial\Gamma(t)} - \left\langle\tfrac12\,(\varkappa-\overline\varkappa)^2 + \beta\,A\,\varkappa - \gamma\,\varkappa_\mu, \vec\chi\,.\,\vec\mu\right\rangle_{\partial\Gamma(t)}\\
&\qquad - \langle(\varkappa-\overline\varkappa+\beta\,A)\,(\nabla_s\vec\nu)\,\vec\mu, \vec\chi\rangle_{\partial\Gamma(t)} + \alpha_G\left\langle\varkappa_\mu, \vec{\mathrm{id}}_s\,.\,\vec\chi_s\right\rangle_{\partial\Gamma(t)} + \alpha_G\left\langle\underline{\underline{P}}\,\vec\chi_s, \vec\mu_s\right\rangle_{\partial\Gamma(t)}\\
&\qquad + B\,.
\end{aligned}\tag{A.30}$$

Here we note that in total we will consider four different types of boundary conditions.

(i). $\partial\Gamma(t)$ is free, see (1.15a–c).

(ii). $\partial\Gamma(t)\subset\partial\Omega$ is semi-free, see (1.16a–c).

(iii). $\partial\Gamma(t) = \partial\Gamma(0)$ is fixed and clamped, see (1.17).

(iv). $\partial\Gamma(t) = \partial\Gamma(0)$ is fixed and Navier, see (1.18).

We will derive (1.15a–c) and (1.16a–c) below.

Boundary terms in (A.30) only play a role in cases (i) and (ii), as $\vec{\chi} = \vec{0}$ on $\partial\Gamma(t) = \partial\Gamma(0)$ in cases (iii) and (iv), recall (2.31). We recall from (A.3) that $\vec{y} = -\alpha_G\,\vec{\varkappa}_{\partial\Gamma} = -\alpha_G\,[\varkappa_\mu\,\vec{\mu} + \varkappa_\nu\,\vec{\nu}]$ on $\partial\Gamma(t)$, which implies that

$$\vec{u} = -\alpha_G\,\varkappa_\mu\,\vec{\mu} \quad \text{on} \quad \partial\Gamma(t)\,, \tag{A.31a}$$

and

$$-\alpha_G\,\varkappa_\nu = \varkappa - \overline{\varkappa} - \beta\,A \quad \text{on} \quad \partial\Gamma(t)\,. \tag{A.31b}$$

It follows from (A.5) that

$$(\nabla_s\,\vec{\nu})\,\vec{\mu}\,.\,\vec{\chi} = -\mathrm{I\!I}(\vec{\mu},\vec{\mathrm{id}}_s)\,(\vec{\chi}\,.\,\vec{\mathrm{id}}_s) - \mathrm{I\!I}(\vec{\mu},\vec{\mu})\,(\vec{\chi}\,.\,\vec{\mu})\,, \tag{A.32}$$

and so we obtain from (A.31b) that

$$B_3 = \alpha_G\,\langle \varkappa_\nu\,(\nabla_s\,\vec{\nu})\,\vec{\mu},\vec{\chi}\rangle_{\partial\Gamma(t)} = -\alpha_G\left\langle \varkappa_\nu, \mathrm{I\!I}(\vec{\mu},\vec{\mathrm{id}}_s)\,(\vec{\chi}\,.\,\vec{\mathrm{id}}_s) + \mathrm{I\!I}(\vec{\mu},\vec{\mu})\,(\vec{\chi}\,.\,\vec{\mu})\right\rangle_{\partial\Gamma(t)}\,. \tag{A.33}$$

Moreover, we have from (1.9) that

$$\begin{aligned} B_4 &= \alpha_G\left\langle \varkappa_\mu\,\vec{\mathrm{id}}_s, \vec{\chi}_s\right\rangle_{\partial\Gamma(t)} = -\alpha_G\left\langle (\varkappa_\mu)_s\,\vec{\mathrm{id}}_s, \vec{\chi}\right\rangle_{\partial\Gamma(t)} - \alpha_G\left\langle \varkappa_\mu\,\vec{\mathrm{id}}_{ss}, \vec{\chi}\right\rangle_{\partial\Gamma(t)} \\ &= -\alpha_G\left\langle (\varkappa_\mu)_s\,\vec{\mathrm{id}}_s + (\varkappa_\mu)^2\,\vec{\mu} + \varkappa_\mu\,\varkappa_\nu\,\vec{\nu}, \vec{\chi}\right\rangle_{\partial\Gamma(t)}\,. \end{aligned} \tag{A.34}$$

It follows from (A.9) and (A.10) that

$$\begin{aligned} B_5 &= \alpha_G\,\langle \underline{\underline{P}}\,\vec{\mu}_s, \vec{\chi}_s\rangle_{\partial\Gamma(t)} = \alpha_G\,\langle \tau\,\vec{\nu}, \vec{\chi}_s\rangle_{\partial\Gamma(t)} = -\alpha_G\,\langle \tau_s\,\vec{\nu} + \tau\,\vec{\nu}_s, \vec{\chi}\rangle_{\partial\Gamma(t)} \\ &= \alpha_G\left\langle -\tau_s\,\vec{\nu} + \varkappa_\nu\,\tau\,\vec{\mathrm{id}}_s + \tau^2\,\vec{\mu}, \vec{\chi}\right\rangle_{\partial\Gamma(t)}\,. \end{aligned} \tag{A.35}$$

Combining (A.34) and (A.35) yields that

$$B_4 + B_5 = \alpha_G\left\langle (\varkappa_\nu\,\tau - (\varkappa_\mu)_s)\,\vec{\mathrm{id}}_s + (\tau^2 - (\varkappa_\mu)^2)\,\vec{\mu} - (\tau_s + \varkappa_\mu\,\varkappa_\nu)\,\vec{\nu}, \vec{\chi}\right\rangle_{\partial\Gamma(t)}\,. \tag{A.36}$$

It remains to consider the term $B_6 = B$, recall (A.29). Of course, $B_6 = 0$ if $\alpha_G = 0$, as then $\vec{u} = \vec{0}$ on $\partial\Gamma(t)$, recall (A.31a). Let

$$\begin{aligned} B_6 = B &= \langle \mu_j\,\partial_{s_k}\,u_k, \chi_j\rangle_{\partial\Gamma(t)} - \langle \mu_k\,\partial_{s_j}\,u_k, \chi_j\rangle_{\partial\Gamma(t)} - \langle \varkappa\,\mu_i\,\nu_j\,u_i, \chi_j\rangle_{\partial\Gamma(t)} \\ &\quad + \langle \nu_i\,\nu_k\,\mu_j\,\partial_{s_j}\,u_i, \chi_k\rangle_{\partial\Gamma(t)} =: \sum_{\ell=1}^{4} D_\ell\,. \end{aligned} \tag{A.37}$$

It follows from (A.31a) and (A.9) that

$$\nabla_s\,.\,\vec{u} = -\alpha_G\,[(\nabla_s\,\varkappa_\mu)\,.\,\vec{\mu} + \varkappa_\mu\,\nabla_s\,.\,\vec{\mu}] = -\alpha_G\left[\partial_{\vec{\mu}}\,\varkappa_\mu + \varkappa_\mu\,\vec{\mathrm{id}}_s\,.\,\vec{\mu}_s\right] = \alpha_G\left[(\varkappa_\mu)^2 - \partial_{\vec{\mu}}\,\varkappa_\mu\right]. \tag{A.38}$$

Hence we have that

$$D_1 = \langle \nabla_s \,.\, \vec{u}, \vec{\mu} \,.\, \vec{\chi} \rangle_{\partial\Gamma(t)} = \alpha_G \left\langle (\varkappa_\mu)^2 - \partial_{\vec{\mu}}\, \varkappa_\mu, \vec{\mu} \,.\, \vec{\chi} \right\rangle_{\partial\Gamma(t)} . \tag{A.39}$$

Similarly, we obtain that

$$D_2 = \alpha_G \, \langle \nabla_s \, \varkappa_\mu, \vec{\chi} \rangle_{\partial\Gamma(t)} = \alpha_G \, \langle \partial_{\vec{\mu}}\, \varkappa_\mu, \vec{\mu} \,.\, \vec{\chi} \rangle_{\partial\Gamma(t)} + \alpha_G \left\langle (\varkappa_\mu)_s, \vec{\mathrm{id}}_s \,.\, \vec{\chi} \right\rangle_{\partial\Gamma(t)} . \tag{A.40}$$

Combining (A.39) and (A.40) yields that

$$D_1 + D_2 = \alpha_G \left\langle (\varkappa_\mu)^2 \, \vec{\mu} + (\varkappa_\mu)_s \, \vec{\mathrm{id}}_s, \vec{\chi} \right\rangle_{\partial\Gamma(t)} . \tag{A.41}$$

In addition, we see that

$$D_3 = - \langle \varkappa \, \mu_i \, \nu_j \, u_i, \chi_j \rangle_{\partial\Gamma(t)} = \alpha_G \, \langle \varkappa \, \varkappa_\mu \, \vec{\nu}, \vec{\chi} \rangle_{\partial\Gamma(t)} . \tag{A.42}$$

Finally, we compute that

$$\begin{aligned} D_4 &= \left\langle \nu_i \, \nu_k \, \mu_j \, \partial_{s_j} \, u_i, \chi_k \right\rangle_{\partial\Gamma(t)} = -\alpha_G \left\langle \varkappa_\mu \, \nu_i \, \nu_k \, \mu_j \, \partial_{s_j} \, \mu_i, \chi_k \right\rangle_{\partial\Gamma(t)} \\ &= -\alpha_G \, \langle \varkappa_\mu \, [(\vec{\mu} \,.\, \nabla_s) \, \vec{\mu}] \,.\, \vec{\nu}, \vec{\chi} \,.\, \vec{\nu} \rangle_{\partial\Gamma(t)} = -\alpha_G \, \langle \varkappa_\mu \, \mathrm{I\!I}(\vec{\mu}, \vec{\mu}) \, \vec{\nu}, \vec{\chi} \rangle_{\partial\Gamma(t)} . \end{aligned} \tag{A.43}$$

It remains to collect all the contributions from the tangential, co-normal and normal parts of $\vec{\chi}$ in B_1, B_2 in (A.30), (A.33), (A.36), (A.41), (A.42) and (A.43), where we also recall (A.37). Beginning with the tangential terms, we observe, on noting (A.11), that

$$\begin{aligned} & - \alpha_G \left\langle \varkappa_\nu \, \mathrm{I\!I}(\vec{\mathrm{id}}_s, \vec{\mu}), \vec{\chi} \,.\, \vec{\mathrm{id}}_s \right\rangle_{\partial\Gamma(t)} + \alpha_G \left\langle \varkappa_\nu \, \tau - (\varkappa_\mu)_s, \vec{\chi} \,.\, \vec{\mathrm{id}}_s \right\rangle_{\partial\Gamma(t)} + \alpha_G \left\langle (\varkappa_\mu)_s, \vec{\chi} \,.\, \vec{\mathrm{id}}_s \right\rangle_{\partial\Gamma(t)} \\ & \quad = \left\langle 0, \vec{\chi} \,.\, \vec{\mathrm{id}}_s \right\rangle_{\partial\Gamma(t)} . \end{aligned} \tag{A.44}$$

Repeating the same for the co-normal terms, we obtain, on recalling (A.7), (A.6) and (A.11), that

$$\begin{aligned} & B_2 - \alpha_G \, \langle \varkappa_\nu \, \mathrm{I\!I}(\vec{\mu}, \vec{\mu}), \vec{\chi} \,.\, \vec{\mu} \rangle_{\partial\Gamma(t)} + \alpha_G \left\langle \tau^2 - (\varkappa_\mu)^2, \vec{\chi} \,.\, \vec{\mu} \right\rangle_{\partial\Gamma(t)} + \alpha_G \left\langle (\varkappa_\mu)^2, \vec{\chi} \,.\, \vec{\mu} \right\rangle_{\partial\Gamma(t)} \\ & \quad = B_2 - \alpha_G \, \langle \mathcal{K}, \vec{\chi} \,.\, \vec{\mu} \rangle_{\partial\Gamma(t)} = \left\langle -\tfrac{1}{2} \, (\varkappa - \overline{\varkappa})^2 - \beta \, A \, \varkappa + \gamma \, \varkappa_\mu - \alpha_G \, \mathcal{K}, \vec{\chi} \,.\, \vec{\mu} \right\rangle_{\partial\Gamma(t)} . \end{aligned} \tag{A.45}$$

Of course, (A.45) will give rise to the boundary condition

$$-\tfrac{1}{2} \, (\varkappa - \overline{\varkappa})^2 - \beta \, A \, \varkappa + \gamma \, \varkappa_\mu - \alpha_G \, \mathcal{K} = 0 \quad \text{on} \;\; \partial\Gamma(t) . \tag{A.46}$$

Finally, for the normal terms we obtain, on recalling (A.7), that

$$\begin{aligned} & B_1 - \alpha_G \, \langle \tau_s + \varkappa_\mu \, \varkappa_\nu, \vec{\chi} \,.\, \vec{\nu} \rangle_{\partial\Gamma(t)} + \alpha_G \, \langle \varkappa \, \varkappa_\mu - \varkappa_\mu \, \mathrm{I\!I}(\vec{\mu}, \vec{\mu}), \vec{\chi} \,.\, \vec{\nu} \rangle_{\partial\Gamma(t)} \\ & \quad = B_1 - \alpha_G \, \langle \tau_s, \vec{\chi} \,.\, \vec{\nu} \rangle_{\partial\Gamma(t)} = \langle (\nabla_s \, \varkappa) \,.\, \vec{\mu} + \gamma \, \varkappa_\nu - \alpha_G \, \tau_s, \vec{\chi} \,.\, \vec{\nu} \rangle_{\partial\Gamma(t)} . \end{aligned} \tag{A.47}$$

Of course, (A.47) will give rise to the boundary condition

$$(\nabla_s \, \varkappa) \,.\, \vec{\mu} + \gamma \, \varkappa_\nu - \alpha_G \, \tau_s = 0 \quad \text{on} \;\; \partial\Gamma(t) . \tag{A.48}$$

Hence the corresponding strong formulation, on recalling (A.13), (A.19) and (A.28), is

$$\vec{\mathcal{V}} = \left[-\Delta_s \varkappa + (\tfrac{1}{2}(\varkappa - \overline{\varkappa})^2 + \beta\, A\, \varkappa)\, \varkappa - (\varkappa - \overline{\varkappa} + \beta\, A)\, |\nabla_s \vec{\nu}|^2\right] \vec{\nu} \quad \text{on} \quad \Gamma(t)\,. \tag{A.49}$$

Moreover, when $\partial\Gamma(t)$ is nonempty, and if $\partial\Gamma(t)$ is not fixed, then the following natural boundary conditions arise on recalling (A.46), (A.48) and (A.31b):

$$(\nabla_s \varkappa)\,.\,\vec{\mu} + \gamma\, \varkappa_\nu - \alpha_G\, \tau_s = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{A.50a}$$
$$-\tfrac{1}{2}(\varkappa - \overline{\varkappa})^2 - \beta\, A\, \varkappa + \gamma\, \varkappa_\mu - \alpha_G\, \mathcal{K} = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{A.50b}$$
$$\varkappa - \overline{\varkappa} + \beta\, A + \alpha_G\, \varkappa_\nu = 0 \quad \text{on} \quad \partial\Gamma(t)\,. \tag{A.50c}$$

In the special case $\beta = 0$, the conditions (A.50a–c) agree with the ones stated in Tu et al. (2006, Eq. (6), (7), (8)), where we note that our sign convention for the conormal is such that $\vec{\mu}$ on $\partial\Gamma(t)$ points out of $\Gamma(t)$, as opposed to into $\Gamma(t)$ as is assumed in Tu et al. (2006).

We now consider the partially free case, when $\partial\Gamma(0) \subset \partial\Omega$, where $\partial\Omega$ is a fixed boundary of an open domain $\Omega \subset \mathbb{R}^3$ with normal $\vec{\mathrm{n}}_{\partial\Omega}$, and $\partial\Gamma(t)$ is required to remain on $\partial\Omega$ throughout the evolution. Hence our variation $\vec{\chi}$ in (A.2a) and beyond is such that

$$\vec{\chi}\,.\,\vec{\mathrm{n}}_{\partial\Omega} = 0 \quad \text{on} \quad \partial\Gamma(t)\,. \tag{A.51}$$

As $\vec{\mathrm{id}}_s\,.\,\vec{\mathrm{n}}_{\partial\Omega} = 0$ on $\partial\Gamma(t)$, it follows from (A.51) that

$$(\vec{\chi}\,.\,\vec{\mu})\,(\vec{\mu}\,.\,\vec{\mathrm{n}}_{\partial\Omega}) + (\vec{\chi}\,.\,\vec{\nu})\,(\vec{\nu}\,.\,\vec{\mathrm{n}}_{\partial\Omega}) = 0 \quad \text{on} \quad \partial\Gamma(t)\,. \tag{A.52}$$

Hence (A.52), (A.45) and (A.47) yield the boundary condition

$$[(\nabla_s \varkappa)\,.\,\vec{\mu} + \gamma\, \varkappa_\nu - \alpha_G\, \tau_s]\,(\vec{\mu}\,.\,\vec{\mathrm{n}}_{\partial\Omega}) - [-\tfrac{1}{2}(\varkappa - \overline{\varkappa})^2 - \beta\, A\, \varkappa + \gamma\, \varkappa_\mu - \alpha_G\, \mathcal{K}]\,(\vec{\nu}\,.\,\vec{\mathrm{n}}_{\partial\Omega}) = 0 \quad \text{on} \quad \partial\Gamma(t)\,, \tag{A.53}$$

in place of (A.50a,b), provided that $\vec{\mu}\,.\,\vec{\mathrm{n}}_{\partial\Omega} \neq 0$. In the case $\beta = \alpha_G = 0$ the boundary conditions (A.50c) and (A.53) are stated as (4.7) and (4.8) in Abels et al. (2016).

In the case of surface area preservation, there is an extra term $-\lambda\, \langle \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec{\chi} \rangle_{\Gamma(t)}$ on the right hand side of (A.2a), on recalling (1.19), (2.6) and (2.33). On noting (A.12), we have that

$$-\lambda \left\langle \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec{\chi} \right\rangle_{\Gamma(t)} = -\lambda\, \langle \nabla_s\,.\,\vec{\chi}, 1 \rangle_{\Gamma(t)} = \lambda\, \langle \varkappa\, \vec{\nu}, \vec{\chi} \rangle_{\Gamma(t)} - \lambda\, \langle 1, \vec{\chi}\,.\,\vec{\mu} \rangle_{\partial\Gamma(t)}\,.$$

Hence we obtain the desired changes (1.20) and (1.21) to (A.49) and (A.50c), respectively, on recalling (A.45).

References

H. Abels, H. Garcke, and L. Müller. Local well-posedness for volume-preserving mean curvature and Willmore flows with line tension. *Math. Nachr.*, 289(2–3):136–174, 2016.

J. W. Barrett, H. Garcke, and R. Nürnberg. A parametric finite element method for fourth order geometric evolution equations. *J. Comput. Phys.*, 222(1):441–462, 2007.

J. W. Barrett, H. Garcke, and R. Nürnberg. On the parametric finite element approximation of evolving hypersurfaces in $\mathbb{R}^3$. *J. Comput. Phys.*, 227(9):4281–4307, 2008.

J. W. Barrett, H. Garcke, and R. Nürnberg. Parametric approximation of Willmore flow and related geometric evolution equations. *SIAM J. Sci. Comput.*, 31(1):225–253, 2008.

J. W. Barrett, H. Garcke, and R. Nürnberg. The approximation of planar curve evolutions by stable fully implicit finite element schemes that equidistribute. *Numer. Methods Partial Differential Equations*, 27(1):1–30, 2011.

J. W. Barrett, H. Garcke, and R. Nürnberg. Elastic flow with junctions: Variational approximation and applications to nonlinear splines. *Math. Models Methods Appl. Sci.*, 22(11):1250037, 2012.

J. W. Barrett, H. Garcke, and R. Nürnberg. Finite element approximation for the dynamics of asymmetric fluidic biomembranes, 2015a. Preprint No. 03/2015, University Regensburg, Germany.

J. W. Barrett, H. Garcke, and R. Nürnberg. On the stable numerical approximation of two-phase flow with insoluble surfactant. *M2AN Math. Model. Numer. Anal.*, 49(2): 421–458, 2015b.

J. W. Barrett, H. Garcke, and R. Nürnberg. Computational parametric Willmore flow with spontaneous curvature and area difference elasticity effects. *SIAM J. Numer. Anal.*, 2016. (accepted for publication).

M. Bergner, A. Dall'Acqua, and S. Fröhlich. Symmetric Willmore surfaces of revolution satisfying natural boundary conditions, 2009. Preprint 14/2009, University Magdeburg, Germany.

A. Biria, M. Maleki, and E. Fried. Continuum theory for the edge of an open lipid bilayer. *Adv. Appl. Mech.*, 46:1–68, 2013.

A. I. Bobenko and P. Schröder. Discrete Willmore flow. In *ACM SIGGRAPH 2005 Courses*, SIGGRAPH '05, New York, NY, 2005. ACM.

R. Capovilla and J. Guven. Stress and geometry of lipid vesicles. *J. Phys. Condens. Matter*, 16(22):2187–2191, 2004.

U. Clarenz, U. Diewald, G. Dziuk, M. Rumpf, and R. Rusu. A finite element method for surface restoration with smooth boundary conditions. *Comput. Aided Geom. Design*, 21 (5):427–445, 2004.

A. Dall'Acqua, K. Deckelnick, and H.-C. Grunau. Classical solutions to the Dirichlet problem for Willmore surfaces of revolution. *Adv. Calc. Var.*, 1(4):379–397, 2008.

T. A. Davis. Algorithm 832: UMFPACK V4.3—an unsymmetric-pattern multifrontal method. *ACM Trans. Math. Software*, 30(2):196–199, 2004.

K. Deckelnick, G. Dziuk, and C. M. Elliott. Computation of geometric partial differential equations and mean curvature flow. *Acta Numer.*, 14:139–232, 2005.

K. Deckelnick and H.-C. Grunau. A Navier boundary value problem for Willmore surfaces of revolution. *Analysis*, 29(3):229–258, 2009.

K. Deckelnick, H.-C. Grunau, and M. Röger. Minimising a relaxed Willmore functional for graphs subject to boundary conditions, 2015. http://arxiv.org/abs/1503.01275.

K. Deckelnick, J. Katz, and F. Schieweck. A C^1-finite element method for the Willmore flow of two-dimensional graphs. *Math. Comp.*, 84(296):2617–2643, 2015.

Q. Du. Phase field calculus, curvature-dependent energies, and vesicle membranes. *Philos. Mag.*, 91(1):165–181, 2011.

G. Dziuk. An algorithm for evolutionary surfaces. *Numer. Math.*, 58(6):603–611, 1991.

G. Dziuk. Computational parametric Willmore flow. *Numer. Math.*, 111(1):55–80, 2008.

G. Dziuk and C. M. Elliott. Finite element methods for surface PDEs. *Acta Numer.*, 22:289–396, 2013.

K. Ecker. *Regularity Theory for Mean Curvature Flow*. Birkhäuser, Boston, 2004.

S. Germain. Recherches sur la théorie des surfaces élastiques. *Imprimerie de Huzard-Courcier*, 1821.

W. Helfrich. Elastic properties of lipid bilayers: Theory and possible experiments. *Z. Naturforsch.*, 28c:693–703, 1973.

G. R. Kirchhoff. Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. *J. Reine Angew. Math.*, 40:51–88, 1850.

E. Kuwert and R. Schätzle. The Willmore flow with small initial energy. *J. Differential Geom.*, 57(3):409–441, 2001.

F. C. Marques and A. Neves. Min-max theory and the Willmore conjecture. *Ann. of Math.*, 179(2):683–782, 2014.

U. F. Mayer and G. Simonett. A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow. *Interfaces Free Bound.*, 4(1):89–109, 2002.

J. C. C. Nitsche. Boundary value problems for variational integrals involving surface curvatures. *Quart. Appl. Math.*, 51(2):363–387, 1993.

L. Peres Hari, D. Givoli, and J. Rubinstein. Computation of open Willmore-type surfaces. *Appl. Numer. Math.*, 37(1-2):257–269, 2001.

S. D. Poisson. Mémoire sur les surfaces élastiques. *Cl. Sci. Mathém. Phys. Inst. de France*, pages 167–225, 1812.

T. Rivière. Analysis aspects of Willmore surfaces. *Invent. Math.*, 174(1):1–45, 2008.

R. E. Rusu. *Numerische Analysis für den Krümmungsfluß und den Willmorefluß*. PhD thesis, University Freiburg, Freiburg, 2006.

A. Saitoh, K. Takiguchi, Y. Tanaka, and H. Hotani. Opening-up of liposomal membranes by talin. *PNAS*, 95(3):1026–1031, 1998.

R. Schätzle. The Willmore boundary problem. *Calc. Var. Partial Differential Equations*, 37(3-4):275–302, 2010.

A. Schmidt and K. G. Siebert. *Design of Adaptive Finite Element Software: The Finite Element Toolbox ALBERTA*, volume 42 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2005.

S. Schmidt and V. Schulz. Shape derivatives for general objective functions and the incompressible Navier–Stokes equations. *Control Cybernet.*, 39(3):677–713, 2010.

U. Seifert. Configurations of fluid membranes and vesicles. *Adv. Phys.*, 46:13–137, 1997.

L. Simon. Existence of surfaces minimizing the Willmore functional. *Comm. Anal. Geom.*, 1(2):281–326, 1993.

F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010.

Z.-C. Tu. Challenges in theoretical investigations of configurations of lipid membranes. *Chin. Phys. B*, 22(2):28701, 2013.

Z. C. Tu, L. Q. Ge, and Z. C. Ou-Yang. From fluid membranes to polymer membranes. *Thin Solid Films*, 509(1–2):58–63, 2006.

Z. C. Tu and Z. C. Ou-Yang. Lipid membranes with free edges. *Phys. Rev. E*, 68(6): 061915, 2003.

X. Wang and Q. Du. Modelling and simulations of multi-component lipid membranes and open membranes via diffuse interface approaches. *J. Math. Biol.*, 56(3):347–371, 2008.

T. J. Willmore. *Riemannian Geometry*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1993.