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Dirac propagator in Kerr geometry in
Eddington-Finkelstein-type coordinates

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Preprint Nr. 03/2016

An Integral Representation for the Massive Dirac Propagator in Kerr Geometry in Eddington-Finkelstein-Type Coordinates

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(Dated: June 2016)

ABSTRACT. The massive Dirac equation is considered in the non-extreme Kerr geometry in horizon-penetrating Eddington-Finkelstein-type coordinates. We derive an integral representation for the Dirac propagator involving the solutions of the ODEs which arise in Chandrasekhar's separation of variables. This integral representation describes the dynamics of Dirac waves outside and across the event horizon, up to the Cauchy horizon.

For the proof, we write the Dirac equation in Hamiltonian form. One of the main difficulties is that the time evolution is not unitary, because the wave may "hit" the singularity. This problem is resolved by imposing suitable Dirichlet-type boundary conditions inside the Cauchy horizon, having no effect on the outside dynamics. Another main difficulty is that the Dirac Hamiltonian fails to be elliptic at the horizons. Combining the theory of symmetric hyperbolic systems with elliptic methods near the boundary, we construct a self-adjoint extension of the resulting Hamiltonian. We finally apply Stone's formula to the spectral measure of the Hamiltonian and express the resolvent in terms of solutions of the separated ODEs.

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I. INTRODUCTION

In [9], an integral representation for the propagator of the massive Dirac equation in the non-extreme Kerr geometry outside the event horizon is derived in Boyer-Lindquist coordinates. This integral representation has been used to study the long-time behavior including decay rates as well as the probability that the Dirac particle escapes to infinity [10]. The shortcoming of this integral representation is that it gives a solution of the Cauchy problem only outside the event horizon. In the present paper, an integral

representation is constructed which completely describes the dynamics of Dirac waves outside, across, and inside the event horizon, up to the Cauchy horizon.

The methods for deriving our integral representation are quite different from those used in [9], as is now outlined. We work with horizon-penetrating advanced Eddington-Finkelstein-type coordinates (τ, r, θ, ϕ) [17], i.e., an analytic extension of Boyer-Lindquist coordinates which covers both the exterior and interior black hole regions without exhibiting poles at the horizons and which has a proper coordinate time τ as needed for the formulation of the Cauchy problem. Moreover, we work with a regular Carter tetrad, that is, a symmetric Newman-Penrose null tetrad frame which makes use of, on the one hand, the discrete time and angle reversal isometries of Kerr geometry and, on the other hand, its Petrov type. After computing the corresponding spin coefficients, the massive Dirac equation is given in Hamiltonian form

$$i\partial_\tau\psi(\tau) = H\psi(\tau). \quad (1)$$

We introduce a scalar product on the solution space and show explicitly that the Dirac Hamiltonian is symmetric with respect to this scalar product (on smooth and compactly supported wave functions). Furthermore, it is established that this scalar product coincides with the canonical scalar product obtained by integrating the normal component of the Dirac current. We point out that in this setting, the Dirac equation and the scalar product are smooth at the horizons.

In order to apply the spectral theorem, we need to construct a self-adjoint extension of the Dirac Hamiltonian. First, to have a unitary time evolution, we must prevent that the Dirac wave “hits” the curvature singularity. To this end, we “shield” the singularity by imposing *Dirichlet-type boundary conditions* on a time-like surface inside the Cauchy horizon. Clearly, these boundary conditions change the dynamics of the Dirac wave because the wave is reflected on the boundary surface. However, the dynamics *outside the Cauchy horizon* is not affected (because the reflected wave cannot cross the Cauchy horizon and reenter the region outside the Cauchy horizon; see FIG. 1 on page 12). Since the Dirac Hamiltonian is not elliptic at the horizons, we employ a general method for non-uniformly elliptic boundary value problems introduced in [13]. This gives a self-adjoint extension, making it possible to solve the Cauchy problem by

$$\psi(\tau) = e^{-i\tau H} \psi_0 = \int_{\mathbb{R}} e^{-i\omega\tau} \psi_0 dE_\omega,$$

where dE_ω is the spectral measure of H and $\psi_0 := \psi(0)$ is the initial data. By means of Stone’s formula, we can represent the spectral measure via the resolvent $\text{Res}_{\omega_c} := (H - \omega_c)^{-1}$ with $\omega_c \in \mathbb{C} \setminus \mathbb{R}$. The final step is to express the resolvent in terms of the solutions of the ODEs obtained from Chandrasekhar’s separation of variables. Employing this separation ansatz in (1) and using the angular ODEs, the Dirac Hamiltonian becomes a first-order ordinary differential operator in the radial variable. The resolvent of this operator can be described in terms of the Green’s matrix of the radial ODEs. Performing the sum over all angular momentum modes yields the resolvent in separated form. We thus obtain an integral representation

$$\psi(\tau, r, \theta, \phi) = \frac{1}{2\pi i} \sum_{k,l \in \mathbb{Z}} e^{-ik\phi} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} e^{-i\omega\tau} [\text{Res}_{\omega+i\epsilon}^{k,l} - \text{Res}_{\omega-i\epsilon}^{k,l}] \psi_{0,k}(r, \theta) d\omega,$$

where k denotes the azimuthal mode and l labels the eigenvalues of the angular operator (for details see Theorem IV.3). Because the system of radial ODEs cannot be solved analytically without making suitable approximations or by considering asymptotics, here, the asymptotic radial solutions at infinity and at the event and Cauchy horizons are used for guidance in the implicit construction of the radial solutions needed for the computation of the Green’s matrix.

The article is organized as follows. Section II states the settings for Kerr geometry and for the Dirac equation. Moreover, required results from the asymptotic and spectral analysis are recalled without giving proofs. Section III contains the Hamiltonian formulation of the Dirac equation in Kerr geometry in horizon-penetrating coordinates and a derivation of a suitable scalar product for the Hilbert space of solutions of the Cauchy problem. Also, the symmetry of the Dirac Hamiltonian with respect to this scalar product is proven explicitly. In Section IV, it is shown that the Dirac Hamiltonian has an essentially self-adjoint extension. Finally, the integral representation for the Dirac propagator is constructed.

II. PRELIMINARIES

In this section, we briefly recall the relevant basics on the non-extreme Kerr geometry in horizon-penetrating advanced Eddington-Finkelstein-type coordinates, the Dirac equation in the Newman-Penrose formalism, Chandrasekhar's separation of variables, and asymptotic and spectral results on solutions to the corresponding radial and angular first-order ODE systems.

The non-extreme Kerr geometry is an asymptotically flat Lorentzian 4-manifold $(\mathfrak{M}, \mathbf{g})$ with topology $S^2 \times \mathbb{R}^2$. It consists of a differential manifold \mathfrak{M} and a stationary, axisymmetric Lorentzian metric \mathbf{g} , which is given in horizon-penetrating advanced Eddington-Finkelstein-type coordinates (τ, r, θ, ϕ) with $\tau \in \mathbb{R}, r \in \mathbb{R}_{>0}, \theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$ by [17]

$$\begin{aligned} \mathbf{g} = & \left(1 - \frac{2Mr}{\Sigma}\right) d\tau \otimes d\tau - \frac{2Mr}{\Sigma} \left([dr - a \sin^2(\theta) d\phi] \otimes d\tau + d\tau \otimes [dr - a \sin^2(\theta) d\phi] \right) \\ & - \left(1 + \frac{2Mr}{\Sigma}\right) (dr - a \sin^2(\theta) d\phi) \otimes (dr - a \sin^2(\theta) d\phi) - \Sigma d\theta \otimes d\theta - \Sigma \sin^2(\theta) d\phi \otimes d\phi, \end{aligned} \quad (2)$$

where M is the mass and aM the angular momentum of the black hole with $0 \leq a < M$, and $\Sigma(r, \theta) := r^2 + a^2 \cos^2(\theta)$. The event and Cauchy horizons are located at $r_{\pm} := M \pm \sqrt{M^2 - a^2}$, respectively. The advanced Eddington-Finkelstein-type coordinates are an analytic extension of the standard Boyer-Lindquist coordinates (t, r, θ, φ) [3], covering both the exterior and interior black hole regions while being smooth at the horizons. In terms of the Boyer-Lindquist coordinates, the advanced Eddington-Finkelstein-type time and azimuthal angle coordinates read

$$\tau = t + r_{\star} - r = t + \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-} \ln |r - r_-| \quad (3)$$

$$\phi = \varphi + \int \frac{a}{\Delta} dr = \varphi + \frac{a}{r_+ - r_-} \ln \left| \frac{r - r_+}{r - r_-} \right|,$$

where

$$r_{\star} = r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-} \ln |r - r_-|$$

is the Regge-Wheeler coordinate and $\Delta := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2$ is the horizon function. This coordinate system possesses a proper coordinate time unlike in the case of the original advanced Eddington-Finkelstein null coordinates [7, 8].

It is advantageous to describe Kerr geometry in the Newman-Penrose formalism using a regular Carter tetrad [4, 17]

$$\begin{aligned} \mathbf{l} &= \frac{1}{\sqrt{2\Sigma r_+}} \left([\Delta + 4Mr] \partial_{\tau} + \Delta \partial_r + 2a \partial_{\phi} \right) \\ \mathbf{n} &= \frac{r_+}{\sqrt{2\Sigma}} (\partial_{\tau} - \partial_r) \\ \mathbf{m} &= \frac{1}{\sqrt{2\Sigma}} \left(ia \sin(\theta) \partial_{\tau} + \partial_{\theta} + i \csc(\theta) \partial_{\phi} \right) \\ \overline{\mathbf{m}} &= -\frac{1}{\sqrt{2\Sigma}} \left(ia \sin(\theta) \partial_{\tau} - \partial_{\theta} + i \csc(\theta) \partial_{\phi} \right) \end{aligned} \quad (4)$$

because this frame is adapted to the two principal null directions of the Weyl tensor and to the discrete time and angle reversal isometries. Thus, since Kerr geometry is algebraically special and of Petrov type

D, one has the computational advantage that the four spin coefficients κ, σ, λ , and ν as well as the four Weyl scalars Ψ_0, Ψ_1, Ψ_3 , and Ψ_4 vanish [15], and that certain spin coefficients are linearly dependent. Solving the first Maurer-Cartan equation of structure, the spin coefficients corresponding to the tetrad frame (4) read [17]

$$\begin{aligned} \kappa = \sigma = \lambda = \nu = 0, \quad \alpha = -\beta = -\frac{1}{(2\Sigma)^{3/2}} \left([r^2 + a^2] \cot(\theta) - ira \sin(\theta) \right) \\ \pi = -\tau = \frac{ia \sin(\theta)}{\sqrt{2\Sigma}(r - ia \cos(\theta))}, \quad \mu = -\frac{r_+}{\sqrt{2\Sigma}(r - ia \cos(\theta))}, \quad \varrho = -\frac{\Delta}{\sqrt{2\Sigma}r_+(r - ia \cos(\theta))} \quad (5) \\ \gamma = -\frac{r_+}{2^{3/2}\sqrt{\Sigma}(r - ia \cos(\theta))}, \quad \epsilon = \frac{r^2 - a^2 - 2ia \cos(\theta)(r - M)}{2^{3/2}\sqrt{\Sigma}r_+(r - ia \cos(\theta))}. \end{aligned}$$

Introducing a spin bundle $S\mathfrak{M} = \mathfrak{M} \times \mathbb{C}^4$ on \mathfrak{M} with fibers $S_{\mathbf{x}}\mathfrak{M} \simeq \mathbb{C}^4$, $\mathbf{x} \in \mathfrak{M}$, we can formulate the general relativistic, massive Dirac equation without an external potential

$$(\gamma^\mu \nabla_\mu + im)\psi(x^\mu) = 0,$$

where ∇ is the metric connection on $S\mathfrak{M}$, the γ^μ , $\mu \in \{0, 1, 2, 3\}$, are the Dirac matrices, ψ is the Dirac 4-spinor defined on the fibers $S_{\mathbf{x}}\mathfrak{M}$, and m is the fermion rest mass. In the Newman-Penrose formalism for spinors, i.e., a formulation with an internal (local) orthonormal dyad spinor frame analogous to the tetrad formulation, this equation becomes the coupled first-order PDE system

$$\begin{aligned} (n^\mu \partial_\mu + \bar{\mu} - \bar{\gamma})\mathcal{G}_1 - (\bar{m}^\mu \partial_\mu + \bar{\beta} - \bar{\tau})\mathcal{G}_2 = i\mu_\star \mathcal{F}_1 \\ (l^\mu \partial_\mu + \bar{\epsilon} - \bar{\varrho})\mathcal{G}_2 - (m^\mu \partial_\mu + \bar{\pi} - \bar{\alpha})\mathcal{G}_1 = i\mu_\star \mathcal{F}_2 \quad (6) \\ (l^\mu \partial_\mu + \epsilon - \varrho)\mathcal{F}_1 + (\bar{m}^\mu \partial_\mu + \pi - \alpha)\mathcal{F}_2 = i\mu_\star \mathcal{G}_1 \\ (n^\mu \partial_\mu + \mu - \gamma)\mathcal{F}_2 + (m^\mu \partial_\mu + \beta - \tau)\mathcal{F}_1 = i\mu_\star \mathcal{G}_2 \end{aligned}$$

with $\psi = (\mathcal{F}_1, \mathcal{F}_2, -\mathcal{G}_1, -\mathcal{G}_2)^T$ and $\mu_\star := m/\sqrt{2}$ [5]. Substituting the Carter tetrad (4) and the corresponding spin coefficients (5) into the system (6), and applying the spinor transformation

$$\psi' = \mathcal{P}\psi = (\mathcal{H}_1, \mathcal{H}_2, -\mathcal{I}_1, -\mathcal{I}_2)^T, \quad \gamma'^\mu = \mathcal{P}\gamma^\mu \mathcal{P}^{-1}, \quad (7)$$

where

$$\mathcal{P} = \text{diag}\left(\sqrt{r - ia \cos(\theta)}, \sqrt{r - ia \cos(\theta)}, \sqrt{r + ia \cos(\theta)}, \sqrt{r + ia \cos(\theta)}\right),$$

one obtains

$$\begin{aligned} r_+(\partial_\tau - \partial_r)\mathcal{I}_1 - (-ia \sin(\theta)\partial_\tau + \partial_\theta - i \csc(\theta)\partial_\phi + 2^{-1} \cot(\theta))\mathcal{I}_2 = \sqrt{2}i\mu_\star(r + ia \cos(\theta))\mathcal{H}_1 \\ r_+^{-1}([\Delta + 4Mr]\partial_\tau + \Delta\partial_r + 2a\partial_\phi + r - M)\mathcal{I}_2 - (ia \sin(\theta)\partial_\tau + \partial_\theta + i \csc(\theta)\partial_\phi + 2^{-1} \cot(\theta))\mathcal{I}_1 \\ = \sqrt{2}i\mu_\star(r + ia \cos(\theta))\mathcal{H}_2 \quad (8) \\ r_+^{-1}([\Delta + 4Mr]\partial_\tau + \Delta\partial_r + 2a\partial_\phi + r - M)\mathcal{H}_1 + (-ia \sin(\theta)\partial_\tau + \partial_\theta - i \csc(\theta)\partial_\phi + 2^{-1} \cot(\theta))\mathcal{H}_2 \\ = \sqrt{2}i\mu_\star(r - ia \cos(\theta))\mathcal{I}_1 \end{aligned}$$

$$r_+(\partial_\tau - \partial_r)\mathcal{H}_2 + (ia \sin(\theta)\partial_\tau + \partial_\theta + i \csc(\theta)\partial_\phi + 2^{-1} \cot(\theta))\mathcal{H}_1 = \sqrt{2}i\mu_\star(r - ia \cos(\theta))\mathcal{I}_2,$$

which is the starting point for the Hamiltonian formulation of the massive Dirac equation on a Kerr background geometry derived in the next section. Note that the system (8) corresponds to a transformed Dirac equation of the form

$$-\sqrt{\Sigma}\gamma^0 \mathcal{P}^\dagger \mathcal{P}^{-1} \left(\gamma'^\mu [\nabla_\mu + \mathcal{P}\partial_\mu(\mathcal{P}^{-1})] + im \right) \psi' = 0, \quad (9)$$

where $\gamma^0 = \text{diag}(1, 1, -1, -1)$. This will become relevant in the constructions of the Hamiltonian formulation and the scalar product.

For the computation of the resolvent of the essentially self-adjoint extension of the Dirac Hamiltonian, the asymptotics of the radial ODEs at infinity and at the inner horizon boundaries as well as spectral properties of the angular ODE system are required. These follow from the separation of the system (8) by means of Chandrasekhar's product ansatz employing plane wave τ - and ϕ -modes. Here, the results are recalled. For a detailed analysis and proofs see [17]. Substituting the mode separation ansatz

$$\begin{aligned}\mathcal{H}_1 &= e^{-i(\omega\tau+k\phi)} \mathcal{R}_+(r) \mathcal{T}_+(\theta) \\ \mathcal{H}_2 &= e^{-i(\omega\tau+k\phi)} \mathcal{R}_-(r) \mathcal{T}_-(\theta) \\ \mathcal{J}_1 &= e^{-i(\omega\tau+k\phi)} \mathcal{R}_-(r) \mathcal{T}_+(\theta) \\ \mathcal{J}_2 &= e^{-i(\omega\tau+k\phi)} \mathcal{R}_+(r) \mathcal{T}_-(\theta),\end{aligned}$$

where $\omega \in \mathbb{R}$ and $k \in \mathbb{Z} + 1/2$, into (8) yields the first-order radial and angular ODE systems

$$\widehat{O}_{r_*} \widetilde{\mathcal{R}} = \frac{\sqrt{|\Delta|}}{r^2 + a^2} \begin{pmatrix} 0 & 1 \\ \text{sign}(\Delta) & 0 \end{pmatrix} \xi \widetilde{\mathcal{R}} \quad (10)$$

$$\widehat{O}_\theta \mathcal{T} = \xi \mathcal{T} \quad (11)$$

with matrix-valued operators

$$\widehat{O}_{r_*} := \mathbf{1}_{\mathbb{C}^2} \partial_{r_*} + \frac{i}{r^2 + a^2} \begin{pmatrix} -\omega(\Delta + 4Mr) - 2ak & -\sqrt{2|\Delta|} \mu_* r \\ \sqrt{2|\Delta|} \text{sign}(\Delta) \mu_* r & \omega\Delta \end{pmatrix} \quad (12)$$

$$\widehat{O}_\theta := \begin{pmatrix} \sqrt{2}\mu_* a \cos(\theta) & -\partial_\theta - 2^{-1} \cot(\theta) + a\omega \sin(\theta) + kcsc(\theta) \\ \partial_\theta + 2^{-1} \cot(\theta) + a\omega \sin(\theta) + kcsc(\theta) & -\sqrt{2}\mu_* a \cos(\theta) \end{pmatrix}, \quad (13)$$

functions $\widetilde{\mathcal{R}} = (\sqrt{|\Delta|} \mathcal{R}_+, r_* \mathcal{R}_-)^T$, $\mathcal{T} = (\mathcal{T}_+, \mathcal{T}_-)^T$, and constant of separation ξ . The asymptotics and decay properties of the radial ODE system at infinity, the event horizon, and the Cauchy horizon are specified in the following lemmas:

Lemma II.1. *Every nontrivial solution $\widetilde{\mathcal{R}}$ of (10) is asymptotically as $r \rightarrow \infty$ of the form*

$$\widetilde{\mathcal{R}}(r_*) = \widetilde{\mathcal{R}}_\infty(r_*) + E_\infty(r_*) = D_\infty \begin{pmatrix} f_\infty^{(1)} \exp(i\phi_+(r_*)) \\ f_\infty^{(2)} \exp(-i\phi_-(r_*)) \end{pmatrix} + E_\infty(r_*)$$

with the asymptotic diagonalization matrix

$$D_\infty := \begin{cases} \begin{pmatrix} \cosh(\Omega) & \sinh(\Omega) \\ \sinh(\Omega) & \cosh(\Omega) \end{pmatrix} & \text{for } \omega^2 \geq 2\mu_*^2 \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \cosh(\Omega) + i \sinh(\Omega) & \sinh(\Omega) + i \cosh(\Omega) \\ \sinh(\Omega) + i \cosh(\Omega) & \cosh(\Omega) + i \sinh(\Omega) \end{pmatrix} & \text{for } \omega^2 < 2\mu_*^2, \end{cases}$$

where

$$\Omega := \begin{cases} \frac{1}{4} \ln \left(\frac{\omega - \sqrt{2}\mu_*}{\omega + \sqrt{2}\mu_*} \right) & \text{for } \omega^2 \geq 2\mu_*^2 \\ \frac{1}{4} \ln \left(\frac{\sqrt{2}\mu_* - \omega}{\sqrt{2}\mu_* + \omega} \right) & \text{for } \omega^2 < 2\mu_*^2, \end{cases}$$

the asymptotic phases

$$\phi_{\pm}(r_{\star}) := \text{sign}(\omega) \times \begin{cases} -\sqrt{\omega^2 - 2\mu_{\star}^2} r_{\star} + 2M \left(\pm\omega - \frac{\mu_{\star}^2}{\sqrt{\omega^2 - 2\mu_{\star}^2}} \right) \ln(r_{\star}) & \text{for } \omega^2 \geq 2\mu_{\star}^2 \\ \sqrt{2\mu_{\star}^2 - \omega^2} i r_{\star} + 2M \left(\pm\omega - \frac{i\mu_{\star}^2}{\sqrt{2\mu_{\star}^2 - \omega^2}} \right) \ln(r_{\star}) & \text{for } \omega^2 < 2\mu_{\star}^2, \end{cases}$$

the constants $\mathfrak{f}_{\infty} = (\mathfrak{f}_{\infty}^{(1)}, \mathfrak{f}_{\infty}^{(2)})^T \neq 0$, and the error $\|E_{\infty}(r_{\star})\| = \|\tilde{\mathcal{R}}(r_{\star}) - \tilde{\mathcal{R}}_{\infty}(r_{\star})\| \leq a/r_{\star}$ for a suitable constant $a \in \mathbb{R}_{>0}$.

Lemma II.2. Every nontrivial solution $\tilde{\mathcal{R}}$ of (10) is asymptotically as $r \rightarrow r_{\pm}$ of the form

$$\tilde{\mathcal{R}}(r_{\star}) = \tilde{\mathcal{R}}_{r_{\pm}}(r_{\star}) + E_{r_{\pm}}(r_{\star}) = \begin{pmatrix} \mathfrak{g}_{r_{\pm}}^{(1)} \exp\left(2i \left[\omega + k\Omega_{\text{Kerr}}^{(\pm)}\right] r_{\star}\right) \\ \mathfrak{g}_{r_{\pm}}^{(2)} \end{pmatrix} + E_{r_{\pm}}(r_{\star})$$

with the constants $\mathfrak{g}_{r_{\pm}} = (\mathfrak{g}_{r_{\pm}}^{(1)}, \mathfrak{g}_{r_{\pm}}^{(2)})^T \neq 0$ and $\Omega_{\text{Kerr}}^{(\pm)} := a/(2Mr_{\pm})$, as well as an error with exponential decay $\|E_{r_{\pm}}(r_{\star})\| \leq p_{\pm} \exp(\pm q_{\pm} r_{\star})$ for r sufficiently close to r_{\pm} and suitable constants $p_{\pm}, q_{\pm} \in \mathbb{R}_{>0}$.

For the angular ODE system, the following proposition holds:

Proposition II.3. For any $\omega \in \mathbb{R}$ and $k \in \mathbb{Z} + 1/2$, the differential operator (13) has a complete set of orthonormal eigenfunctions $(\mathcal{T}_l)_{l \in \mathbb{Z}}$ in $L^2((0, \pi), \sin(\theta) d\theta)^2$. The corresponding eigenvalues ξ_l are real-valued and non-degenerate, and can thus be ordered as $\xi_l < \xi_{l+1}$. Moreover, the eigenfunctions are pointwise bounded and smooth away from the poles,

$$\mathcal{T}_l \in L^{\infty}((0, \pi))^2 \cap C^{\infty}((0, \pi))^2.$$

Both the eigenfunctions \mathcal{T}_l and the eigenvalues ξ_l depend smoothly on ω .

III. HAMILTONIAN FORMULATION AND SCALAR PRODUCT

To derive the Hamiltonian formulation of the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington-Finkelstein-type coordinates, it is advantageous to first rewrite the system (8) in the form

$$(\mathcal{R} + \mathcal{A})\psi' = 0,$$

where

$$\mathcal{R} := \begin{pmatrix} -\sqrt{2i}\mu_{\star}r & 0 & -\mathcal{D}_- & 0 \\ 0 & -\sqrt{2i}\mu_{\star}r & 0 & -\mathcal{D}_+ \\ \mathcal{D}_+ & 0 & \sqrt{2i}\mu_{\star}r & 0 \\ 0 & \mathcal{D}_- & 0 & \sqrt{2i}\mu_{\star}r \end{pmatrix} \quad (14)$$

and

$$\mathcal{A} := \begin{pmatrix} \sqrt{2}\mu_{\star}a \cos(\theta) & 0 & 0 & \overline{\mathcal{L}} \\ 0 & \sqrt{2}\mu_{\star}a \cos(\theta) & \mathcal{L} & 0 \\ 0 & \overline{\mathcal{L}} & \sqrt{2}\mu_{\star}a \cos(\theta) & 0 \\ \mathcal{L} & 0 & 0 & \sqrt{2}\mu_{\star}a \cos(\theta) \end{pmatrix} \quad (15)$$

are radial and angular matrix-valued differential operators with

$$\mathcal{D}_+ := r_+^{-1} ([\Delta + 4Mr]\partial_\tau + \Delta\partial_r + 2a\partial_\phi + r - M)$$

$$\mathcal{D}_- := r_+(\partial_\tau - \partial_r)$$

$$\mathcal{L} := ia \sin(\theta)\partial_\tau + \partial_\theta + icsc(\theta)\partial_\phi + \frac{1}{2} \cot(\theta).$$

Separating the τ -derivative and multiplying by the inverse of the matrix (cf. Eq.(9))

$$\tilde{\gamma}'^\tau := -\sqrt{\Sigma}\gamma^0\mathcal{P}^\dagger\mathcal{P}^{-1}\gamma'^\tau = \begin{pmatrix} 0 & 0 & -r_+ & -ia \sin(\theta) \\ 0 & 0 & ia \sin(\theta) & -r_+^{-1}[\Delta + 4Mr] \\ r_+^{-1}[\Delta + 4Mr] & -ia \sin(\theta) & 0 & 0 \\ ia \sin(\theta) & r_+ & 0 & 0 \end{pmatrix} \quad (16)$$

as well as the imaginary unit leads to

$$i\partial_\tau\psi' = -i(\tilde{\gamma}'^\tau)^{-1}(\mathcal{R}|_{\mathfrak{N}} + \mathcal{A}|_{\mathfrak{N}})\psi' =: H\psi', \quad (17)$$

where the restrictions $\mathcal{R}|_{\mathfrak{N}} = P^*\mathcal{R}$ and $\mathcal{A}|_{\mathfrak{N}} = P^*\mathcal{A}$ are the pullbacks of the radial and angular operators (14) and (15) to the family of space-like hypersurfaces $\mathfrak{N} = (\mathfrak{N}_\tau)_{\tau \in \mathbb{R}}$ with $\mathfrak{N}_\tau := \{\tau = \text{const.}\}$. The Dirac Hamiltonian H can be recast in the more convenient form

$$H = \alpha^j \partial_j + \mathcal{V}, \quad j \in \{r, \theta, \phi\}, \quad (18)$$

with the matrix-valued functions

$$\alpha^r := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} i\Delta & r_+ a \sin(\theta) & 0 & 0 \\ r_+^{-1}\Delta a \sin(\theta) & -i(\Delta + 4Mr) & 0 & 0 \\ 0 & 0 & -i(\Delta + 4Mr) & r_+^{-1}\Delta a \sin(\theta) \\ 0 & 0 & r_+ a \sin(\theta) & i\Delta \end{pmatrix} \quad (19)$$

$$\alpha^\theta := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} -a \sin(\theta) & ir_+ & 0 & 0 \\ ir_+^{-1}[\Delta + 4Mr] & a \sin(\theta) & 0 & 0 \\ 0 & 0 & -a \sin(\theta) & -ir_+^{-1}[\Delta + 4Mr] \\ 0 & 0 & -ir_+ & a \sin(\theta) \end{pmatrix} \quad (20)$$

$$\alpha^\phi := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} ia & r_+ \csc(\theta) & 0 & 0 \\ r_+^{-1}\csc(\theta)(\Delta - 2\Sigma) & -ia & 0 & 0 \\ 0 & 0 & -ia & r_+^{-1}\csc(\theta)(\Delta - 2\Sigma) \\ 0 & 0 & r_+ \csc(\theta) & ia \end{pmatrix} \quad (21)$$

and the potential

$$\mathcal{V} := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_3 & \mathcal{B}_4 \end{pmatrix}, \quad (22)$$

where the \mathcal{B}_k , $k \in \{1, 2, 3, 4\}$, are the (2×2) -blocks

$$\mathcal{B}_1 := \begin{pmatrix} i(r - M) - 2^{-1} a \cos(\theta) & 2^{-1} i r_+ \cot(\theta) \\ r_+^{-1} (a \sin(\theta)(r - M) + 2^{-1} i \cot(\theta)(\Delta + 4Mr)) & 2^{-1} a \cos(\theta) \end{pmatrix}$$

$$\begin{aligned}
\mathcal{B}_2 &:= \begin{pmatrix} -\sqrt{2}\mu_* r_+(r - ia \cos(\theta)) & -\sqrt{2}i\mu_* a \sin(\theta)(r - ia \cos(\theta)) \\ \sqrt{2}i\mu_* a \sin(\theta)(r - ia \cos(\theta)) & -\sqrt{2}r_+^{-1}\mu_* (\Delta + 4Mr)(r - ia \cos(\theta)) \end{pmatrix} \\
\mathcal{B}_3 &:= \begin{pmatrix} -\sqrt{2}r_+^{-1}\mu_* (\Delta + 4Mr)(r + ia \cos(\theta)) & \sqrt{2}i\mu_* a \sin(\theta)(r + ia \cos(\theta)) \\ -\sqrt{2}i\mu_* a \sin(\theta)(r + ia \cos(\theta)) & -\sqrt{2}\mu_* r_+(r + ia \cos(\theta)) \end{pmatrix} \\
\mathcal{B}_4 &:= \begin{pmatrix} -2^{-1}a \cos(\theta) & r_+^{-1}(a \sin(\theta)(r - M) - 2^{-1}i \cot(\theta)(\Delta + 4Mr)) \\ -2^{-1}i r_+ \cot(\theta) & i(r - M) + 2^{-1}a \cos(\theta) \end{pmatrix}.
\end{aligned} \tag{23}$$

On the space-like hypersurface \mathfrak{N}_τ , one can introduce the scalar product [9]

$$(\psi|\phi)|_{\mathfrak{N}_\tau} := \int_{\mathfrak{N}_\tau} \langle \psi | \boldsymbol{\psi} \phi \rangle d\mu_{\mathfrak{N}_\tau}, \tag{24}$$

where

$$\langle \cdot | \cdot \rangle : S_{\mathbf{x}}\mathfrak{M} \times S_{\mathbf{x}}\mathfrak{M} \rightarrow \mathbb{C}, \quad (\psi, \phi) \mapsto \psi^* \phi \tag{25}$$

denotes the indefinite spin scalar product of signature (2, 2), $\psi^* := \psi^\dagger \mathcal{S}$ is the adjoint Dirac spinor, $\boldsymbol{\psi} = \gamma^\mu \nu_\mu$ the Clifford contraction of the future-directed, time-like normal $\boldsymbol{\nu}$, and $d\mu_{\mathfrak{N}_\tau} = \sqrt{|\det(\mathbf{g}_{|\mathfrak{N}_\tau})|} d\phi d\theta dr$ with the induced metric $\mathbf{g}_{|\mathfrak{N}_\tau}$ is the invariant measure on \mathfrak{N}_τ . Note that the scalar product is independent of the choice of the specific space-like hypersurface. The matrix \mathcal{S} is defined via the relation [19]

$$\gamma^{\mu\dagger} := \mathcal{S} \gamma^\mu \mathcal{S}^{-1}. \tag{26}$$

With (16) and the spinor transformation (7), we find

$$\gamma^\mu = -\frac{1}{\sqrt{\Sigma}} (\mathcal{P}^\dagger)^{-1} \gamma^0 \tilde{\gamma}^\mu \mathcal{P}$$

and, thus, with the defining relation (26), we obtain for the corresponding matrix \mathcal{S} the expression

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The future-directed vector field $\boldsymbol{\nu}$, which is normal to the space-like hypersurface \mathfrak{N}_τ , is determined by means of the conditions

$$\langle \boldsymbol{\nu} | \partial_r \rangle_{|\mathbf{g}} = \langle \boldsymbol{\nu} | \partial_\theta \rangle_{|\mathbf{g}} = \langle \boldsymbol{\nu} | \partial_\phi \rangle_{|\mathbf{g}} = 0 \quad \text{and} \quad \langle \boldsymbol{\nu} | \boldsymbol{\nu} \rangle_{|\mathbf{g}} = 1,$$

where $\langle \cdot | \cdot \rangle_{|\mathbf{g}} = \mathbf{g}(\cdot, \cdot)$ is the space-time inner product on the manifold \mathfrak{M} given by (2), yielding

$$\boldsymbol{\nu} = \left(1 + \frac{2Mr}{\Sigma}\right)^{1/2} \partial_\tau - \frac{2Mr}{\Sigma} \left(1 + \frac{2Mr}{\Sigma}\right)^{-1/2} \partial_r.$$

The corresponding dual co-vector reads

$$\boldsymbol{\nu} = \left(1 + \frac{2Mr}{\Sigma}\right)^{-1/2} d\tau. \tag{27}$$

Further, the induced spatial metric $\mathbf{g}_{|\mathfrak{N}_\tau}$ on \mathfrak{N}_τ is the restriction of (2) from \mathfrak{M} to the submanifold \mathfrak{N}_τ

$$\mathbf{g}_{|\mathfrak{N}_\tau} = -\left(1 + \frac{2Mr}{\Sigma}\right) (dr - a \sin^2(\theta) d\phi) \otimes (dr - a \sin^2(\theta) d\phi) - \Sigma d\theta \otimes d\theta - \Sigma \sin^2(\theta) d\phi \otimes d\phi.$$

This gives rise to the expression

$$\sqrt{|\det(\mathbf{g}_{|\mathfrak{N}_\tau})|} = \Sigma \sin(\theta) \left(1 + \frac{2Mr}{\Sigma}\right)^{1/2} \quad (28)$$

as the factor in the volume measure $d\mu_{|\mathfrak{N}_\tau}$. Since the scalar product (24) is invariant under spinor transformations, we obtain with (27) and (28) for the primed wave functions (7) used in (17)

$$(\psi'|\phi')_{|\mathfrak{N}_\tau} = \iiint \psi'^\dagger \mathcal{S}' \gamma'^\tau \phi' \Sigma \sin(\theta) d\phi d\theta dr. \quad (29)$$

Again using the matrix (16), i.e., $\gamma'^\tau = -\mathcal{P}(\mathcal{P}^\dagger)^{-1} \gamma^0 \tilde{\gamma}'^\tau / \sqrt{\Sigma}$, this scalar product becomes

$$\begin{aligned} (\psi'|\phi')_{|\mathfrak{N}_\tau} &= - \iiint \psi'^\dagger \mathcal{S}' \mathcal{P}(\mathcal{P}^\dagger)^{-1} \gamma^0 \tilde{\gamma}'^\tau \phi' \sqrt{\Sigma} \sin(\theta) d\phi d\theta dr \\ &= - \iiint \psi'^\dagger \mathcal{P} \mathcal{P}^\dagger \mathcal{S}' \mathcal{P}(\mathcal{P}^\dagger)^{-1} \gamma^0 \tilde{\gamma}'^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &= - \iiint \psi'^\dagger \mathcal{P} \mathcal{S}(\mathcal{P}^\dagger)^{-1} \gamma^0 \tilde{\gamma}'^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &= - \iiint \psi'^\dagger \mathcal{S} \mathcal{P}^\dagger(\mathcal{P}^\dagger)^{-1} \gamma^0 \tilde{\gamma}'^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &= \iiint \psi'^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr, \end{aligned} \quad (30)$$

where

$$\Gamma^\tau := -\mathcal{S} \gamma^0 \tilde{\gamma}'^\tau = \begin{pmatrix} r_+^{-1}[\Delta + 4Mr] & -ia \sin(\theta) & 0 & 0 \\ ia \sin(\theta) & r_+ & 0 & 0 \\ 0 & 0 & r_+ & ia \sin(\theta) \\ 0 & 0 & -ia \sin(\theta) & r_+^{-1}[\Delta + 4Mr] \end{pmatrix}. \quad (31)$$

Note that in the latter derivation, we have first employed the relation $\sqrt{\Sigma} \mathbf{1}_{C^4} = \mathcal{P} \mathcal{P}^\dagger$, then the transformation law for the matrix \mathcal{S}' , namely $\mathcal{S} = \mathcal{P}^\dagger \mathcal{S}' \mathcal{P}$, and finally the fact that both \mathcal{S} and the product $\mathcal{P} \mathcal{S}$ are self-adjoint leading to the relation $\mathcal{P} \mathcal{S} = \mathcal{S} \mathcal{P}^\dagger$. The integration limits are suppressed for ease of notation. They are explicitly given when necessary. Moreover, the eigenvalues λ_1, λ_2 of the matrix (31) are positive and with algebraic multiplicities $\mu_A(\lambda_1) = \mu_A(\lambda_2) = 2$

$$\begin{aligned} \lambda_1 &= \frac{1}{2} \left(r_+ + \frac{\Delta + 4Mr}{r_+} + \sqrt{\left(r_+ - \frac{\Delta + 4Mr}{r_+} \right)^2 + 4a^2 \sin^2(\theta)} \right) > 0 \\ \lambda_2 &= \frac{1}{2} \left(r_+ + \frac{\Delta + 4Mr}{r_+} - \sqrt{\left(r_+ + \frac{\Delta + 4Mr}{r_+} \right)^2 - 4(\Sigma + 2Mr)} \right) > 0, \end{aligned}$$

showing that (30) is indeed positive-definite.

Theorem III.1. *The Dirac Hamiltonian (18) is symmetric with respect to the scalar product (30).*

Proof. To establish the symmetry, namely that

$$(\psi' | H \phi')_{|\mathfrak{N}_r} = (H \psi' | \phi')_{|\mathfrak{N}_r},$$

we begin by splitting the potential \mathcal{V} given in (22) into mass-independent and mass-dependent parts

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{\mu_*},$$

where

$$\mathcal{V}_0 := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} \mathcal{B}_1 & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & \mathcal{B}_4 \end{pmatrix} \quad \text{and} \quad \mathcal{V}_{\mu_*} := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} 0_{\mathbb{C}^2} & \mathcal{B}_2 \\ \mathcal{B}_3 & 0_{\mathbb{C}^2} \end{pmatrix}$$

with the (2×2) -blocks \mathcal{B}_k defined in (23), to obtain anti-self-adjoint and self-adjoint matrices $\Gamma^\tau \mathcal{V}_0$ and $\Gamma^\tau \mathcal{V}_{\mu_*}$, where $\Gamma^\tau = \Gamma^{\tau\dagger}$ is defined in (31), i.e.,

$$\Gamma^\tau \mathcal{V}_0 = -\mathcal{V}_0^\dagger \Gamma^\tau \quad \text{and} \quad \Gamma^\tau \mathcal{V}_{\mu_*} = \mathcal{V}_{\mu_*}^\dagger \Gamma^\tau. \quad (32)$$

This leads to

$$\begin{aligned} (\psi' | H \phi')_{|\mathfrak{N}_r} &= \iiint \psi'^\dagger \Gamma^\tau H \phi' \sin(\theta) \, d\phi \, d\theta \, dr \\ &= \iiint \psi'^\dagger \Gamma^\tau \alpha^j \partial_j (\phi') \sin(\theta) \, d\phi \, d\theta \, dr + \iiint \psi'^\dagger \Gamma^\tau \mathcal{V}_0 \phi' \sin(\theta) \, d\phi \, d\theta \, dr \\ &\quad + \iiint \psi'^\dagger \Gamma^\tau \mathcal{V}_{\mu_*} \phi' \sin(\theta) \, d\phi \, d\theta \, dr. \end{aligned}$$

Integration by parts of the first triple integral and application of the relations (32) in the remaining two triple integrals yields

$$\begin{aligned} (\psi' | H \phi')_{|\mathfrak{N}_r} &= - \iiint \partial_j (\psi'^\dagger \Gamma^\tau \alpha^j \sin(\theta)) \phi' \, d\phi \, d\theta \, dr - \iiint \psi'^\dagger \mathcal{V}_0^\dagger \Gamma^\tau \phi' \sin(\theta) \, d\phi \, d\theta \, dr \\ &\quad + \iiint \psi'^\dagger \mathcal{V}_{\mu_*}^\dagger \Gamma^\tau \phi' \sin(\theta) \, d\phi \, d\theta \, dr \\ &= - \iiint \partial_j (\psi'^\dagger) \Gamma^\tau \alpha^j \phi' \sin(\theta) \, d\phi \, d\theta \, dr \\ &\quad - \iiint \psi'^\dagger [\partial_j (\Gamma^\tau) \alpha^j + \Gamma^\tau \partial_j (\alpha^j) + \Gamma^\tau \alpha^\theta \cot(\theta)] \phi' \sin(\theta) \, d\phi \, d\theta \, dr \\ &\quad - \iiint (\mathcal{V}_0 \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) \, d\phi \, d\theta \, dr + \iiint (\mathcal{V}_{\mu_*} \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) \, d\phi \, d\theta \, dr. \end{aligned} \quad (33)$$

Note that in the integration by parts, the angular derivatives do not give rise to boundary terms as the two-dimensional submanifold S^2 is compact without boundary. The radial boundary term on the other hand vanishes due to Dirichlet-type boundary conditions imposed on the Dirac wave functions. More precisely, the radial boundary term reads

$$\iint_{S^2} \psi'^\dagger \Gamma^\tau \alpha^r \phi' \sin(\theta) \, d\phi \, d\theta \Big|_{r_1}^{r_2}. \quad (34)$$

Direct computation of the matrix $\Gamma^\tau \alpha^r$ gives

$$\Gamma^\tau \alpha^r = i \operatorname{diag} \left(-\frac{\Delta}{r_+}, r_+, r_+, -\frac{\Delta}{r_+} \right) \quad (35)$$

and, thus, the radial boundary term becomes

$$ir_+ \iint_{S^2} \left(-\frac{\Delta}{r_+^2} \bar{\psi}'_1 \phi'_1 + \bar{\psi}'_2 \phi'_2 + \bar{\psi}'_3 \phi'_3 - \frac{\Delta}{r_+^2} \bar{\psi}'_4 \phi'_4 \right) \sin(\theta) d\phi d\theta \Big|_{r_1}^{r_2}.$$

In order for this expression to vanish, the radial Dirichlet-type boundary condition on the Dirac wave functions

$$\left(-\frac{\Delta}{r_+^2} \bar{\psi}'_1 \phi'_1 + \bar{\psi}'_2 \phi'_2 + \bar{\psi}'_3 \phi'_3 - \frac{\Delta}{r_+^2} \bar{\psi}'_4 \phi'_4 \right) \Big|_{r_i} = 0, \quad (36)$$

where $i \in \{1, 2\}$, is imposed at each boundary r_1, r_2 . This condition can be brought into a more suitable form as follows. In terms of the spin scalar product (25) and using the relation $\mathcal{S}' \gamma'^r = i \Gamma^\tau \alpha^r / \Sigma$, condition (36) can be stated as

$$\langle \psi' | \gamma'^r \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = 0.$$

Introducing \mathbf{n} as the unit normal to the hypersurface $\{\tau\} \times \partial \mathfrak{N}_\tau \simeq \{\tau\} \times S^2$ which is tangential to \mathfrak{N}_τ , we can write the radial Dirichlet-type boundary condition as

$$\langle \psi' | \mathfrak{n} \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = 0 \quad \Leftrightarrow \quad (\mathfrak{n} - i) \psi' \Big|_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = 0, \quad (37)$$

where the slash again denotes Clifford multiplication. With $\mathfrak{n}^2 = -1$, the implication can be easily verified by

$$\langle \mathfrak{n} \psi' | \mathfrak{n} \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = \langle \psi' | \mathfrak{n}^2 \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = -\langle \psi' | \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = -\langle -i \psi' | -i \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau}$$

$$\Leftrightarrow \langle (\mathfrak{n} - i) \psi' | (\mathfrak{n} - i) \phi' \rangle_{|\{\tau\} \times \partial \mathfrak{N}_\tau} = 0.$$

We point out that the mixed terms in the last line cancel each other. This boundary condition has the effect that Dirac waves are reflected on the $\{\tau\} \times \partial \mathfrak{N}_\tau$ hypersurface. Next, the explicit calculation of the square bracket in (33) gives

$$\partial_j (\Gamma^\tau \alpha^j) + \Gamma^\tau \partial_j (\alpha^j) + \Gamma^\tau \alpha^\theta \cot(\theta) = -2 \mathcal{V}_0^\dagger \Gamma^\tau.$$

Moreover, all three matrix products $\Gamma^\tau \alpha^j$ are anti-self-adjoint

$$\Gamma^\tau \alpha^j = -\alpha^{j\dagger} \Gamma^\tau.$$

Therefore, one immediately finds that

$$\begin{aligned} (\psi' | H \phi')_{|\mathfrak{N}_\tau} &= \iiint \partial_j (\psi'^\dagger) \alpha^{j\dagger} \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr + 2 \iiint (\mathcal{V}_0 \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &\quad - \iiint (\mathcal{V}_0 \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr + \iiint (\mathcal{V}_{\mu_*} \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &= \iiint (\alpha^j \partial_j \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr + \iiint (\mathcal{V}_0 \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &\quad + \iiint (\mathcal{V}_{\mu_*} \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr \\ &= \iiint (H \psi')^\dagger \Gamma^\tau \phi' \sin(\theta) d\phi d\theta dr = (H \psi' | \phi')_{|\mathfrak{N}_\tau}. \end{aligned}$$

□

IV. SPECTRAL THEORY OF THE DIRAC HAMILTONIAN AND INTEGRAL REPRESENTATION FOR THE PROPAGATOR

In this section, we show that the Dirac Hamiltonian in the non-extreme Kerr geometry in horizon-penetrating coordinates has an essentially self-adjoint extension. Moreover, applying Stone's formula, the resolvent of this extension can be used in the construction of an integral representation for the propagator, which yields the dynamics of Dirac waves outside, across, and inside the event horizon, up to the Cauchy horizon. In more detail, the aim is to derive an expression for the spectral measure dE_ω arising in the formal spectral decomposition representation of the Dirac propagator

$$U(\tau) = e^{-i\tau H} = \int_{\mathbb{R}} e^{-i\omega\tau} dE_\omega. \quad (38)$$

Since this spectral decomposition only applies to self-adjoint operators, we first require an essentially self-adjoint extension of the Dirac Hamiltonian. This problem involves the technical difficulty that the Dirac Hamiltonian in Kerr geometry is not elliptic at the event and Cauchy horizons, which can be easily seen from the determinant of the principal symbol

$$P(\mathbf{x}, \boldsymbol{\xi}) = -i(\gamma'^\tau)^{-1} \gamma'^j \xi_j,$$

where $\mathbf{x} = (\tau, r, \theta, \phi)$ and $\boldsymbol{\xi} \in T_{\mathbf{x}}^* \mathfrak{M}$. A short computation yields

$$\det(P(\mathbf{x}, \boldsymbol{\xi})) = \left(\frac{g^{ij} \xi_i \xi_j}{g^{\tau\tau}} \right)^2.$$

The Hamiltonian fails to be elliptic if the determinant vanishes, i.e., if $g^{ij} \xi_i \xi_j = 0$ whenever $\boldsymbol{\xi}$ is non-zero. Since

$$g_{|\mathfrak{M}_\tau} = g^{ij} \partial_i \otimes \partial_j = -\frac{1}{\Sigma} \left(\Delta \partial_r \otimes \partial_r + a(\partial_r \otimes \partial_\phi + \partial_\phi \otimes \partial_r) + \partial_\theta \otimes \partial_\theta + \csc^2(\theta) \partial_\phi \otimes \partial_\phi \right),$$

this holds only true at the event and Cauchy horizons. Thus, in the proof, a general method for non-uniformly elliptic boundary value problems [13] is employed. This method combines results from the theory of symmetric hyperbolic systems with near-boundary elliptic methods [2, 6, 14, 18]. Subsequently, we directly compute the resolvent of the essentially self-adjoint extension using the Green's matrix of the radial first-order ODE system and the angular projector corresponding to the angular first-order ODE system, which arise in Chandrasekhar's separation of variables. Then, by means of Stone's formula, we can express the spectral measure in terms of the resolvent. In the following, in order to formulate the Cauchy problem and the domain of definition of the essentially self-adjoint extension of the Dirac Hamiltonian, the geometrical and functional-analytic setting is stated.

Let (\mathfrak{M}, g) be the non-extreme Kerr geometry with the metric (2) in horizon-penetrating Eddington-Finkelstein-type coordinates (3). Choosing a radius $r_0 < r_-$, where r_- is again the Cauchy horizon, we consider the subset $\mathcal{M} := \{\tau, r > r_0, \theta, \phi\} \subset \mathfrak{M}$. Moreover, let $\partial\mathcal{M} := \{\tau, r = r_0, \theta, \phi\}$ be a time-like inner boundary of \mathfrak{M} and $\mathcal{N} = (\mathcal{N}_\tau)_{\tau \in \mathbb{R}}$,

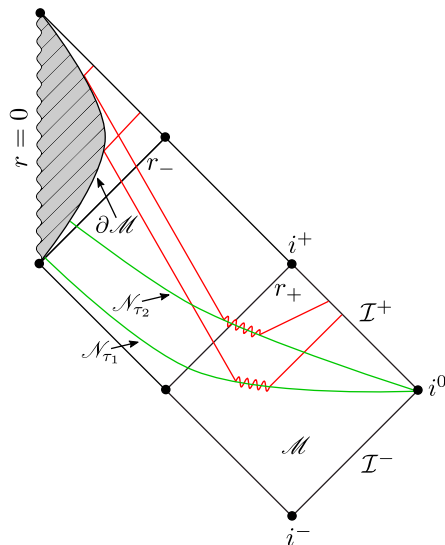


FIG. 1: Carter-Penrose diagram of the region \mathcal{M} of Kerr geometry with constant- τ hypersurfaces $\mathcal{N}_{\tau_1}, \mathcal{N}_{\tau_2}$ cut-off at the boundary $\partial\mathcal{M}$. Cauchy data is propagated in τ -direction along the constant- τ hypersurfaces. Radial Dirichlet-type boundary conditions imposed on $\partial\mathcal{M}$ reflect the wave functions such that unitarity is preserved.

with $\mathcal{N}_\tau := \{\tau = \text{const.}, r > r_0, \theta, \phi\}$, be a family of space-like hypersurfaces with boundaries $\partial\mathcal{N}_\tau := \partial\mathcal{M} \cap \mathcal{N}_\tau \simeq S^2$. These hypersurfaces constitute a foliation of \mathcal{M} along the time direction characterized by the parameter τ (see FIG. 1). At $\partial\mathcal{M}$, we assume the radial Dirichlet-type boundary conditions (37) to obtain a unitary time evolution of the Dirac waves. Near $\partial\mathcal{M}$, we have a locally time-like Killing vector field K which is a linear combination of the Killing fields describing the stationarity and axisymmetry of Kerr geometry $\partial_\tau, \partial_\phi$. This Killing field is given by $K = \partial_\tau + b\partial_\phi$, where $b = b(r_0) \in \mathbb{R} \setminus \{0\}$ is a constant [15]. It corresponds to the Killing field $K = \partial_t$ of [13] which is represented by a coordinate system that describes an observer who is co-moving along the flow lines of the Killing field. Further, using the results of the previous section, one can set up a Hilbert space $(\mathcal{H}, (\cdot|\cdot)|_{\mathcal{N}_\tau})$ with $\mathcal{H} = L^2(\mathcal{N}_\tau, S\mathcal{M})$, where $S\mathcal{M}$ denotes the spin bundle of \mathcal{M} , and the scalar product (30).

A. Essentially Self-Adjoint Extension of the Dirac Hamiltonian

Below, a lemma showing the existence of a unique global solution of the Cauchy problem of the massive Dirac equation in the above setting in the class $C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M})$ and a theorem giving an essentially self-adjoint extension of the Dirac Hamiltonian H defined by (18) and (19)-(23) in a specific domain of the Hilbert space are stated. The lemma is fundamental as a technical tool for the proof of theorem. The proofs of both the lemma and the theorem are given in detail in [13]. Note that in this work, the construction of a self-adjoint extension of the Dirac Hamiltonian is discussed for a general class of non-uniformly elliptic mixed initial/boundary value problems for space-times with dimension $d \geq 3$. Since Kerr geometry is a four-dimensional special case of this framework, all the results apply.

Lemma IV.1. *The Cauchy problem of the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington-Finkelstein-type coordinates*

$$\begin{cases} i\partial_\tau\psi = H\psi \\ \psi|_{\tau_0} = \psi_0 \in C_0^\infty(\mathcal{N}_{\tau_0}, S\mathcal{M}) \end{cases}$$

with radial Dirichlet-type boundary conditions at $\partial\mathcal{M}$ given by

$$(\not{\mathbf{r}} - i)\psi|_{\partial\mathcal{M}} = 0,$$

where the initial data is smooth, compactly supported outside, across, and inside the event horizon, up to the Cauchy horizon, and is compatible with the boundary conditions, i.e.,

$$(\not{\mathbf{r}} - i)(H^p\psi_0) = 0 \quad \forall p \in \mathbb{N}_0,$$

has a unique global solution in the class of smooth wave functions with spatially compact support

$$\{\psi \in C_{\text{sc}}^\infty(\mathcal{M}, S\mathcal{M}) \mid (\not{\mathbf{r}} - i)(H^p\psi)|_{\partial\mathcal{M}} = 0 \quad \forall p \in \mathbb{N}_0\}.$$

Evaluating this solution at subsequent times τ and τ' gives rise to a unique unitary propagator

$$U^{\tau', \tau} : C_{\text{sc}}^\infty(\mathcal{N}_\tau, S\mathcal{M}) \rightarrow C_{\text{sc}}^\infty(\mathcal{N}_{\tau'}, S\mathcal{M}).$$

Theorem IV.2. *The massive Dirac Hamiltonian H in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington-Finkelstein-type coordinates with domain of definition*

$$\mathcal{D}(H) = \{\psi \in C_0^\infty(\mathcal{N}_\tau, S\mathcal{M}) \mid (\not{\mathbf{r}} - i)(H^p\psi)|_{\partial\mathcal{N}_\tau} = 0 \quad \forall p \in \mathbb{N}_0\} \subset \mathcal{H}$$

is essentially self-adjoint.

B. Resolvent of the Dirac Hamiltonian and Integral Representation for the Propagator

In order to construct the integral representation for the Dirac propagator (38), which is a spectral decomposition representation of the exponential of the Dirac Hamiltonian, a resolvent method is used. To this end, it was shown that the Dirac Hamiltonian H has an essentially self-adjoint extension (see Theorem IV.2). The spectrum $\sigma(H) \subseteq \mathbb{R}$ of this extension is on the real line. Thus, for all $\omega_c \in \mathbb{C} \setminus \mathbb{R}$ with real part $\Re(\omega_c) = \omega \in \sigma(H)$, the resolvent $\text{Res}_{\omega_c} = (H - \omega_c)^{-1} \in L(\mathcal{H})$ exists and is given uniquely. Having computed the resolvent, we can apply Stone's formula for unbounded self-adjoint operators [16]

$$\frac{1}{2} (E_{[a,b]} + E_{(a,b)}) = \text{s-lim}_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b [(H - \omega - i\epsilon)^{-1} - (H - \omega + i\epsilon)^{-1}] d\omega,$$

where $E_I := \chi_I(H)$ is the spectral projection of H onto the interval I , s-lim denotes the strong limit of operators, and $a < b \in \mathbb{R}$, to obtain

$$\frac{1}{2} e^{-i\tau H} (E_{[a,b]} + E_{(a,b)}) \psi = \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b e^{-i\omega\tau} [(H - \omega - i\epsilon)^{-1} - (H - \omega + i\epsilon)^{-1}] \psi d\omega. \quad (39)$$

This leads to the following theorem:

Theorem IV.3. *The massive Dirac propagator in the non-extreme Kerr geometry in advanced Eddington-Finkelstein-type coordinates can be expressed via the integral representation*

$$\psi'(\tau) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} e^{-ik\phi} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} e^{-i\omega\tau} [(H_k - \omega - i\epsilon)^{-1} - (H_k - \omega + i\epsilon)^{-1}] \psi'_{0,k}(r, \theta) d\omega,$$

where the resolvents $(H_k - \omega \mp i\epsilon)^{-1}$ for fixed k -mode are unique and of the form

$$(H_k - \omega \mp i\epsilon)^{-1} = - \sum_{l \in \mathbb{Z}} \left[\int_{r_0}^{\infty} \mathcal{E} \begin{pmatrix} G(r, r')_{l,k,\omega \pm i\epsilon} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & G(r, r')_{l,k,\omega \pm i\epsilon} \end{pmatrix} \mathcal{E}^{-1}(r', \theta) dr' \right] Q_l$$

with Q_l being the spectral projector onto a finite-dimensional invariant eigenspace of the angular operator (13) corresponding to the spectral parameter ξ_l , $G(r, r')_{l,k,\omega \pm i\epsilon}$ is the two-dimensional Green's matrix of the radial first-order ODE system (10), and

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{E}^{-1}(r', \theta) = - \begin{pmatrix} i(\Delta(r') + 4Mr') & r_+ a \sin(\theta) & 0 & 0 \\ 0 & 0 & -ir_+ & a \sin(\theta) \\ 0 & 0 & r_+ a \sin(\theta) & i(\Delta(r') + 4Mr') \\ a \sin(\theta) & -ir_+ & 0 & 0 \end{pmatrix}.$$

Proof. For the computation of the resolvent of the operator $H_k - \omega_c$, where the imaginary part of the complex-valued frequency ω_c is now restricted to the values $\Im(\omega_c) = \pm\epsilon$ with ϵ sufficiently small so that it can be considered as a slightly non-self-adjoint perturbation (and denoting the restriction by ω_ϵ), we first employ the mode ansatz

$$\psi' = e^{-i(\omega_\epsilon \tau + k\phi)} \psi'_{\text{sep.}} \quad \text{with} \quad \psi'_{\text{sep.}} = \begin{pmatrix} \mathcal{R}_+(r) \mathcal{T}_+(\theta) \\ \mathcal{R}_-(r) \mathcal{T}_-(\theta) \\ -\mathcal{R}_-(r) \mathcal{T}_+(\theta) \\ -\mathcal{R}_+(r) \mathcal{T}_-(\theta) \end{pmatrix}$$

in Eq.(17). This yields

$$(H_k - \omega_\epsilon) \psi'_{\text{sep.}} = \mathbf{0}, \quad (40)$$

where $H_k = \alpha^r \partial_r + \alpha^\theta \partial_\theta - ik\alpha^\phi + \mathcal{V}$ with the α^j and \mathcal{V} given in (19)-(21) and (22). Next, we introduce the spectral projector Q_l onto a finite-dimensional invariant eigenspace of the angular operator (13)

with spectral parameter ξ_l , $l \in \mathbb{Z}$. By means of the family of spectral projectors $(Q_l)_{l \in \mathbb{Z}}$, one can express the angular operator as

$$\widehat{O}_\theta = \sum_{l \in \mathbb{Z}} \xi_l Q_l.$$

This family is complete, i.e., $\sum_{l \in \mathbb{Z}} Q_l = \mathbf{1}$. Moreover, the spectral projectors are idempotent, that is, $Q_l^2 = Q_l$. Generally speaking, these projectors make sure that the functions $\mathcal{T}_\pm(\theta)$ are solutions of the angular ODE system (11). Note that the projection operator Q_l is an integral operator

$$(Q_l f)(\theta) = \int Q_l(\theta, \theta') f(\theta') d(\cos(\theta')). \quad (41)$$

Applying the completeness constraint to Eq.(40) and substituting (11), we obtain

$$-\sum_{l \in \mathbb{Z}} Q_l (\Sigma + 2Mr)^{-1} \mathcal{M}(\partial_r; r, \theta)_{l, k, \omega_\epsilon} \psi'_{\text{sep.}} = \mathbf{0}, \quad (42)$$

where

$$\mathcal{M}(\partial_r; r, \theta)_{l, k, \omega_\epsilon} := \begin{pmatrix} i O_{k, \omega_\epsilon} & a \sin(\theta) U_{\omega_\epsilon} & ir_+ S_l & a \sin(\theta) \bar{S}_l \\ \frac{a \sin(\theta)}{r_+} O_{k, \omega_\epsilon} & -\frac{i(\Delta + 4Mr)}{r_+} U_{\omega_\epsilon} & a \sin(\theta) S_l & -\frac{i(\Delta + 4Mr)}{r_+} \bar{S}_l \\ -\frac{i(\Delta + 4Mr)}{r_+} \bar{S}_l & a \sin(\theta) S_l & -\frac{i(\Delta + 4Mr)}{r_+} U_{\omega_\epsilon} & \frac{a \sin(\theta)}{r_+} O_{k, \omega_\epsilon} \\ a \sin(\theta) \bar{S}_l & ir_+ S_l & a \sin(\theta) U_{\omega_\epsilon} & i O_{k, \omega_\epsilon} \end{pmatrix}$$

and the differential operators O_{k, ω_ϵ} , U_{ω_ϵ} , and the function S_l are defined by

$$O_{k, \omega_\epsilon} := \Delta \partial_r + r - M - i\omega_\epsilon(\Delta + 4Mr) - 2iak$$

$$U_{\omega_\epsilon} := r_+(\partial_r + i\omega_\epsilon)$$

$$S_l := \xi_l + \sqrt{2}i\mu_* r.$$

The problem of computing the resolvent of the operator in (42) can essentially be reduced to finding the two-dimensional Green's matrix of the first-order radial ODE system (10). For this purpose, it is advantageous to rewrite Eq.(42) in the separated form

$$-\sum_{l \in \mathbb{Z}} Q_l (\Sigma + 2Mr)^{-1} \mathcal{B}(r, \theta) \mathcal{R}(\partial_r; r)_{l, k, \omega_\epsilon} \psi'_{\text{sep.}} = \mathbf{0}, \quad (43)$$

where the matrix $\mathcal{B}(r, \theta)$ and the matrix-valued operator $\mathcal{R}(\partial_r; r)_{l, k, \omega_\epsilon}$ read

$$\mathcal{B}(r, \theta) := \begin{pmatrix} i & a \sin(\theta) & 0 & 0 \\ \frac{a \sin(\theta)}{r_+} & -\frac{i(\Delta + 4Mr)}{r_+} & 0 & 0 \\ 0 & 0 & -\frac{i(\Delta + 4Mr)}{r_+} & \frac{a \sin(\theta)}{r_+} \\ 0 & 0 & a \sin(\theta) & i \end{pmatrix}$$

and

$$\mathcal{R}(\partial_r; r)_{l, k, \omega_\epsilon} := \begin{pmatrix} O_{k, \omega_\epsilon} & 0 & r_+ S_l & 0 \\ 0 & U_{\omega_\epsilon} & 0 & \bar{S}_l \\ \bar{S}_l & 0 & U_{\omega_\epsilon} & 0 \\ 0 & r_+ S_l & 0 & O_{k, \omega_\epsilon} \end{pmatrix}.$$

Then, using the constant matrix

$$\mathcal{C} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

Eq.(43) can be brought into block diagonal form

$$\begin{aligned} & \sum_{l \in \mathbb{Z}} Q_l (H_{l,k,\omega_\epsilon} - \omega_\epsilon) \psi'_{\text{sep.}} \\ &= - \sum_{l \in \mathbb{Z}} Q_l \mathcal{E}(r, \theta) \begin{pmatrix} \mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & \mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} \end{pmatrix} \mathcal{C}^{-1} \psi'_{\text{sep.}} = \mathbf{0} \end{aligned} \quad (44)$$

with $\mathcal{E}(r, \theta) := (\Sigma + 2Mr)^{-1} \mathcal{B}(r, \theta) \mathcal{C}$ and

$$\begin{pmatrix} \mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & \mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} \end{pmatrix} = \mathcal{C}^{-1} \mathcal{R}(\partial_r; r)_{l,k,\omega_\epsilon} \mathcal{C},$$

where

$$\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} := \begin{pmatrix} O_{k,\omega_\epsilon} & r_+ S_l \\ \bar{S}_l & U_{\omega_\epsilon} \end{pmatrix}.$$

Note that the equation

$$\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} \mathcal{R}(r) = \mathbf{0}, \quad (45)$$

for $\mathcal{R} = (\mathcal{R}_+, \mathcal{R}_-)^T$, is equivalent to the radial first-order ODE system (10). Hence, the key quantity in the determination of the resolvent of the operator $H_k - \omega_\epsilon$ is the Green's matrix of this equation, i.e., the solution $G(r, r')_{l,k,\omega_\epsilon}$ to the distributional equation

$$\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} G(r, r')_{l,k,\omega_\epsilon} = \delta(r - r') \mathbf{1}_{\mathbb{C}^2}. \quad (46)$$

To compute the Green's matrix, one has to consider weak solutions of the Dirac equation, i.e., solutions ψ' that satisfy

$$(\psi' | (H - \omega) \phi')_{|\mathcal{N}_r} = 0 \quad \forall \phi' \in \mathcal{D}(H), \quad (47)$$

because of the singular behavior of the wave functions at the event and Cauchy horizons (cf. Lemma II.2). Assuming that $\text{supp } \phi' \subset (r_\pm - \epsilon, r_\pm + \epsilon) \times S^2$, where $\epsilon > 0$ defines small neighborhoods around these horizons, only the radial term (35) arising in the integration by parts of (47) is of relevance for the radial ODE system (45). Evaluating this expression at the horizons shows that in order to have a weak solution, the components ψ'_1 and ψ'_4 can be chosen arbitrarily, whereas the components ψ'_2 and ψ'_3 have to be continuous. Further, let $\Phi_1 = (\Phi_{1,1}, \Phi_{1,2})^T$ and $\Phi_2 = (\Phi_{2,1}, \Phi_{2,2})^T$ be functions which

- are linearly independent weak solutions of the homogeneous equation $\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} \Phi(r, r') = \mathbf{0}$ for $r \neq r'$,
- have jump discontinuities at $r = r'$,
- are square integrable, that is, $\|\Phi_{1/2}(r, r')\|_2^2 = \int_{r_0}^\infty \|\Phi_{1/2}(r, r')\|^2 dr < \infty$,
- have exponential decay at infinity,

$$\begin{aligned} \lim_{r \rightarrow \infty} |\exp(i\omega_\epsilon r)| \|\Phi_{1/2}(r, r')\| &= 1 \quad \text{for } \Im(\omega_\epsilon) < 0 \\ \lim_{r \rightarrow \infty} |\exp(-i\omega_\epsilon r)| \|\Phi_{1/2}(r, r')\| &= 1 \quad \text{for } \Im(\omega_\epsilon) > 0, \end{aligned}$$

- satisfy the Dirichlet boundary conditions (37) at $r = r_0$,
- have the asymptotics

$$\Phi_{1/2} \simeq \frac{c_{1,\infty}}{\sqrt{|\Delta|}} \exp(i\phi_+(r_\star)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } r \rightarrow \infty \quad \text{and} \quad \Im(\omega_\epsilon) < 0$$

$$\Phi_{1/2} \simeq c_{2,\infty} \exp(-i\phi_-(r_\star)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } r \rightarrow \infty \quad \text{and} \quad \Im(\omega_\epsilon) > 0$$

$$\Phi_{1/2} \simeq \begin{pmatrix} \frac{c_{1,r_\pm}}{\sqrt{|\Delta|}} \exp\left(2i\left[\omega_\epsilon + k\Omega_{\text{Kerr}}^{(\pm)}\right]r_\star\right) \\ c_{2,r_\pm} \end{pmatrix} \quad \begin{array}{l} \text{for } r \rightarrow r_+ \quad \text{and} \quad \Im(\omega_\epsilon) < 0 \\ r \rightarrow r_- \quad \text{and} \quad \Im(\omega_\epsilon) > 0 \end{array}$$

$$\Phi_{1/2} \simeq c_{2,r_\pm} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{array}{l} \text{for } r \rightarrow r_+ \quad \text{and} \quad \Im(\omega_\epsilon) > 0 \\ r \rightarrow r_- \quad \text{and} \quad \Im(\omega_\epsilon) < 0 \end{array}$$

according to Lemma II.1 and Lemma II.2.

In the appendix, the solutions Φ_1 and Φ_2 are specified, and their existence is shown. It turns out that their r' -dependence can be described purely by Heaviside step functions Θ . For clarity, in what follows, we write out these Heaviside step functions, making it possible to consider Φ_1 and Φ_2 as functions only of r . Next, if $\Im(\omega_\epsilon) < 0$, we make the ansatz

$$G(r, r')_{l,k,\omega_\epsilon} = \begin{cases} \Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_+ < r' \quad \text{and} \quad r_0 < r' < r_- \\ \Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r - r')\Phi_2(r)P_2(r') & \text{for } r_- < r' < r_+ \end{cases} \quad (48)$$

and, if $\Im(\omega_\epsilon) > 0$,

$$G(r, r')_{l,k,\omega_\epsilon} = \begin{cases} \Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_+ < r' \quad \text{and} \quad r_0 < r' < r_- \\ \Theta(r' - r)\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_- < r' < r_+ \end{cases} \quad (49)$$

Applying the radial operator $\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon}$ to (48) and (49), we obtain

$$\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} G(r, r')_{l,k,\omega_\epsilon} = \begin{pmatrix} \Delta & 0 \\ 0 & r_+ \end{pmatrix} \delta(r - r') \begin{cases} [\Phi_1(r')P_1(r') \mp \Phi_2(r')P_2(r')] & \text{for } \Im(\omega_\epsilon) < 0 \\ [\pm\Phi_1(r')P_1(r') - \Phi_2(r')P_2(r')] & \text{for } \Im(\omega_\epsilon) > 0 \end{cases}$$

Identifying this with (46) yields

$$\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & r_+^{-1} \end{pmatrix} = \begin{cases} \Phi_1(r')P_1(r') \mp \Phi_2(r')P_2(r') & \text{for } \Im(\omega_\epsilon) < 0 \\ \pm\Phi_1(r')P_1(r') - \Phi_2(r')P_2(r') & \text{for } \Im(\omega_\epsilon) > 0 \end{cases}$$

The solutions $P_{1/2}(r')$ of these systems read for $\Im(\omega_\epsilon) < 0$

$$P_{1,1}(r') = \frac{\Phi_{2,2}(r')}{\Delta(r')W(r')}, \quad P_{1,2}(r') = -\frac{\Phi_{2,1}(r')}{r_+W(r')}, \quad P_{2,1}(r') = \pm\frac{\Phi_{1,2}(r')}{\Delta(r')W(r')}, \quad P_{2,2}(r') = \mp\frac{\Phi_{1,1}(r')}{r_+W(r')}$$

and for $\Im(\omega_\epsilon) > 0$

$$P_{1,1}(r') = \pm\frac{\Phi_{2,2}(r')}{\Delta(r')W(r')}, \quad P_{1,2}(r') = \mp\frac{\Phi_{2,1}(r')}{r_+W(r')}, \quad P_{2,1}(r') = \frac{\Phi_{1,2}(r')}{\Delta(r')W(r')}, \quad P_{2,2}(r') = -\frac{\Phi_{1,1}(r')}{r_+W(r')}$$

where $W(r') := \Phi_{1,1}(r')\Phi_{2,2}(r') - \Phi_{1,2}(r')\Phi_{2,1}(r')$ is the Wronskian. Substitution into (48) and (49) leads, on the one hand, for the cases $r_+ < r'$ and $r_0 < r' < r_-$ for both $\Im(\omega_\epsilon) < 0$ and $\Im(\omega_\epsilon) > 0$ to the same Green's matrix

$$G(r, r')_{l,k,\omega_\epsilon} = \frac{\Theta(r-r')}{W(r')} \begin{pmatrix} \frac{\Phi_{1,1}(r)\Phi_{2,2}(r')}{\Delta(r')} - \frac{\Phi_{1,1}(r)\Phi_{2,1}(r')}{r_+} \\ \frac{\Phi_{1,2}(r)\Phi_{2,2}(r')}{\Delta(r')} - \frac{\Phi_{1,2}(r)\Phi_{2,1}(r')}{r_+} \end{pmatrix} \\ + \frac{\Theta(r'-r)}{W(r')} \begin{pmatrix} \frac{\Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} - \frac{\Phi_{2,1}(r)\Phi_{1,1}(r')}{r_+} \\ \frac{\Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} - \frac{\Phi_{2,2}(r)\Phi_{1,1}(r')}{r_+} \end{pmatrix}$$

and, on the other hand, for the case $r_- < r' < r_+$ to the Green's matrices

$$G(r, r')_{l,k,\omega_\epsilon} = \begin{pmatrix} \frac{\Phi_{1,1}(r)\Phi_{2,2}(r') - \Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{2,1}(r)\Phi_{1,1}(r') - \Phi_{1,1}(r)\Phi_{2,1}(r')}{r_+} \\ \frac{\Phi_{1,2}(r)\Phi_{2,2}(r') - \Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{2,2}(r)\Phi_{1,1}(r') - \Phi_{1,2}(r)\Phi_{2,1}(r')}{r_+} \end{pmatrix} \\ \times \begin{cases} \frac{\Theta(r-r')}{W(r')} & \text{for } \Im(\omega_\epsilon) < 0 \\ -\frac{\Theta(r'-r)}{W(r')} & \text{for } \Im(\omega_\epsilon) > 0. \end{cases}$$

From the separated, block-diagonalized representation of $H_k - \omega_\epsilon$ given in (44), we can directly read off the resolvent as

$$(H_k - \omega_\epsilon)^{-1} = - \sum_{l \in \mathbb{Z}} \left[\int_{r_0}^{\infty} \mathcal{E} \begin{pmatrix} G(r, r')_{l,k,\omega_\epsilon} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & G(r, r')_{l,k,\omega_\epsilon} \end{pmatrix} \mathcal{E}^{-1}(r', \theta) dr' \right] Q_l. \quad (50)$$

In order to show that this operator is actually the resolvent of $H_k - \omega_\epsilon$, one verifies that $(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1}\psi' = \psi'$ holds. Hence, applying (50) to ψ' and, subsequently, (44) to the resultant, we obtain in a first step

$$(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1}\psi' \\ = \sum_{l,m \in \mathbb{Z}} \int_{-1}^1 Q_l(\theta, \theta'') \mathcal{E}(r, \theta'') \begin{pmatrix} \mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & \mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega_\epsilon} \end{pmatrix} \\ \times \mathcal{E}^{-1} \left[\int_{r_0}^{\infty} \mathcal{E} \begin{pmatrix} G(r, r')_{l,k,\omega_\epsilon} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & G(r, r')_{l,k,\omega_\epsilon} \end{pmatrix} \mathcal{E}^{-1}(r', \theta') dr' \right] \\ \times \left[\int_{-1}^1 Q_m(\theta'', \theta') \psi'(\theta') d(\cos(\theta')) \right] d(\cos(\theta'')),$$

where we have used that the spectral projectors Q_l are integral operators (cf. Eq.(41)). Next, rearrang-

ing the integrals and employing (46) leads to

$$\begin{aligned}
(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1} \psi' &= \sum_{l, m \in \mathbb{Z}} \int_{-1}^1 \int_{-1}^1 \int_{r_0}^\infty Q_l(\theta, \theta'') \mathcal{E}(r, \theta'') \delta(r - r') \mathbb{1}_{\mathbb{C}^4} \\
&\quad \times \mathcal{E}^{-1}(r', \theta') Q_m(\theta'', \theta') \psi'(\theta') dr' d(\cos(\theta')) d(\cos(\theta'')) \quad (51) \\
&= \sum_{l, m \in \mathbb{Z}} \int_{-1}^1 \int_{-1}^1 Q_l(\theta, \theta'') Q_m(\theta'', \theta') \psi'(\theta') d(\cos(\theta')) d(\cos(\theta'')).
\end{aligned}$$

Since the spectral projectors are idempotent, that is, $Q_l Q_m = \delta_{lm} Q_l$, one finds for their integral kernels

$$\sum_{m \in \mathbb{Z}} Q_l(\theta, \theta'') Q_m(\theta'', \theta') = \sum_{m \in \mathbb{Z}} \delta_{lm} \delta(\cos(\theta) - \cos(\theta'')) Q_m(\theta'', \theta') = \delta(\cos(\theta) - \cos(\theta'')) Q_l(\theta, \theta').$$

Substituting this into (51) gives

$$\begin{aligned}
(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1} \psi' &= \sum_{l \in \mathbb{Z}} \int_{-1}^1 \int_{-1}^1 \delta(\cos(\theta) - \cos(\theta'')) Q_l(\theta, \theta') \psi'(\theta') d(\cos(\theta')) d(\cos(\theta'')) \\
&= \sum_{l \in \mathbb{Z}} \int_{-1}^1 Q_l(\theta, \theta') \psi'(\theta') d(\cos(\theta')) = \sum_{l \in \mathbb{Z}} Q_l \psi' = \psi',
\end{aligned}$$

where in the second line the θ'' -integration was performed, and in the third line the completeness relation for the spectral projectors was applied.

Using the spectral projectors $E_{[-n, n]}$ and $E_{(-n, n)}$ on the finite intervals $[-n, n]$ and $(-n, n)$, respectively, the Dirac propagator $U(\tau)$ can be expressed as

$$\psi'(\tau) = U(\tau) \psi'_0 = e^{-i\tau H} \psi'_0 = e^{-i\tau H} \lim_{n \rightarrow \infty} E_{(-n, n)} \psi'_0 = \lim_{n \rightarrow \infty} e^{-i\tau H} \frac{1}{2} (E_{[-n, n]} + E_{(-n, n)}) \psi'_0.$$

Employing Stone's formula (39) finally yields

$$\psi'(\tau) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} e^{-ik\phi} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} e^{-i\omega\tau} [(H - \omega - i\epsilon)^{-1} - (H - \omega + i\epsilon)^{-1}] \psi'_{0, k} d\omega$$

with the resolvents $(H - \omega \mp i\epsilon)^{-1}$ given by (50). □

Appendix: Choice of Radial Solutions

Here, we choose specific solutions $\Phi_1(r, r')$ and $\Phi_2(r, r')$ that satisfy the properties listed in the proof of Theorem IV.3. We must distinguish the cases $\Im(\omega_\epsilon) < 0$ and $\Im(\omega_\epsilon) > 0$. Moreover, r' can be outside the event horizon, between the event and Cauchy horizons, or inside the Cauchy horizon. Since the explicit forms of these solutions are not known, their behavior is described by asymptotic expansions. In preparation, we first specify functions with suitable decay at infinity given by

$$\begin{aligned}
\Phi_{<}^{(\infty)}(r) &= \frac{a_{<}}{\sqrt{|\Delta|}} \exp(i\phi_+(r_\star)) \left[\mathbb{1}_{\mathbb{C}^2} + \mathcal{O}\left(\frac{1}{r_\star}\right) \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \Im(\omega_\epsilon) < 0 \\
\Phi_{>}^{(\infty)}(r) &= a_{>} \exp(-i\phi_-(r_\star)) \left[\mathbb{1}_{\mathbb{C}^2} + \mathcal{O}\left(\frac{1}{r_\star}\right) \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \Im(\omega_\epsilon) > 0.
\end{aligned}$$

Second, we define functions which are square integrable at the event and Cauchy horizons and obey the proper asymptotics

$$\Phi_{<}^{(+)}(r) = \frac{b_{<}}{\sqrt{|\Delta|}} \exp\left(2i\left[\omega_\epsilon + k\Omega_{\text{Kerr}}^{(+)}\right]r_\star\right) \left[\mathbb{1}_{\mathbb{C}^2} + \mathcal{O}(\exp(qr_\star))\right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \Im(\omega_\epsilon) < 0$$

$$\Phi_{>}^{(+)}(r) = b_{>} \left[\mathbb{1}_{\mathbb{C}^2} + \mathcal{O}(\exp(qr_\star))\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \Im(\omega_\epsilon) > 0$$

and

$$\Phi_{<}^{(-)}(r) = c_{<} \left[\mathbb{1}_{\mathbb{C}^2} + \mathcal{O}(\exp(-qr_\star))\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \Im(\omega_\epsilon) < 0$$

$$\Phi_{>}^{(-)}(r) = \frac{c_{>}}{\sqrt{|\Delta|}} \exp\left(2i\left[\omega_\epsilon + k\Omega_{\text{Kerr}}^{(-)}\right]r_\star\right) \left[\mathbb{1}_{\mathbb{C}^2} + \mathcal{O}(\exp(-qr_\star))\right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \Im(\omega_\epsilon) > 0.$$

The quantities a_{\geq} , b_{\geq} , c_{\geq} are real-valued constants. To clarify our notation, we point out that the superscripts (∞) , $(+)$, and $(-)$ designate the asymptotic expansions at infinity, the event horizon, and the Cauchy horizon, respectively. The existence of ODE solutions with the asymptotics as described above follows from the construction of Jost solutions (to this end, one rewrites the radial Dirac equation as a second-order scalar equation and proceeds as in [1] or [12]). We also point out that our asymptotic expansions ensure that the solutions are square integrable near the horizons. For example, at the event horizon, the Regge-Wheeler coordinate r_\star tends to minus infinity. As a consequence, the exponential factor $\exp(2i[\omega_\epsilon + k\Omega_{\text{Kerr}}^{(+)}]r_\star)$ tends to zero if $\Im(\omega_\epsilon) < 0$. However, this exponential factor would not be square integrable if $\Im(\omega_\epsilon) > 0$. Last, a function that satisfies the Dirichlet boundary conditions at $r = r_0$ is introduced

$$\Phi_{\partial\mathcal{M}}(r) = c_0 \Phi_{\partial\mathcal{M}}^{(2)}(r) \begin{pmatrix} -r_+/\sqrt{|\Delta|} \\ 1 \end{pmatrix},$$

where c_0 is a real-valued constant and $\Phi_{\partial\mathcal{M}}^{(2)}$ denotes the second component of $\Phi_{\partial\mathcal{M}}$. Then, for $\Im(\omega_\epsilon) < 0$, the radial solutions can be expressed as

$$\Phi_1(r, r_+ < r') = \Theta(r - r') \Phi_{<}^{(\infty)}(r)$$

$$\Phi_2(r, r_+ < r') = \Theta(r' - r) \Theta(r - r_+) \Phi_{<}^{(+)}(r)$$

$$\Phi_1(r, r_- < r' < r_+) = \Theta(r - r') \Theta(r_+ - r) \Phi_{>}^{(+)}(r) + \Theta(r - r_+) \Phi_{<}^{(\infty)}(r)$$

$$\Phi_2(r, r_- < r' < r_+) = \Theta(r - r') \Theta(r_+ - r) \Phi_{<}^{(+)}(r)$$

$$\Phi_1(r, r_0 < r' < r_-) = \Theta(r - r') \Theta(r_+ - r) \Phi_{<}^{(-)}(r) + \Theta(r - r_+) \Phi_{<}^{(\infty)}(r)$$

$$\Phi_2(r, r_0 < r' < r_-) = \Theta(r' - r) \Phi_{\partial\mathcal{M}}(r),$$

whereas for $\Im(\omega_\epsilon) > 0$, we choose them as

$$\Phi_1(r, r_+ < r') = \Theta(r - r') \Phi_{>}^{(\infty)}(r)$$

$$\Phi_2(r, r_+ < r') = \Theta(r' - r) \Theta(r - r_-) \Phi_{>}^{(+)}(r) + \Theta(r_- - r) \Phi_{\partial\mathcal{M}}(r)$$

$$\Phi_1(r, r_- < r' < r_+) = \Theta(r' - r) \Theta(r - r_-) \Phi_{<}^{(-)}(r) + \Theta(r_- - r) \Phi_{\partial\mathcal{M}}(r)$$

$$\Phi_2(r, r_- < r' < r_+) = \Theta(r' - r) \Theta(r - r_-) \Phi_{>}^{(-)}(r)$$

$$\begin{aligned}\Phi_1(r, r_0 < r' < r_-) &= \Theta(r - r') \Theta(r_- - r) \Phi_{>}^{(-)}(r) \\ \Phi_2(r, r_0 < r' < r_-) &= \Theta(r' - r) \Phi_{\partial\mathcal{M}}(r).\end{aligned}$$

A case-by-case analysis shows that these solutions are uniquely determined by the conditions listed in the proof of Theorem IV.3.

Acknowledgments

The authors are grateful to Niky Kamran for useful discussions and comments. This work was supported by the DFG research grant “Dirac Waves in the Kerr Geometry: Integral Representations, Mass Oscillation Property and the Hawking Effect.”

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